Quantum Algebras and Integrable Boundaries in AdS/CFT

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Abstract

This thesis studies quantum integrable structures such as Yangians and quantum affine algebras that arise in and are inspired by the AdS/CFT duality, with a primary emphasis on the exploration of integrable boundaries deeply hidden in the duality. The main goal of this thesis is to find novel algebraic structures and methods that could lead to new horizons in the theory of quantum groups and in the exploration of boundary effects in the gauge/gravity dualities.

The main thrust of this work is the exploration of the AdS/CFT worlsheet scattering theory and of integrable boundaries that manifest themselves as Dp-branes ($p+1$-dimensional Dirichlet submanifolds) which are a necessary part of the superstring theory. The presence of these objects breaks some of the underlying symmetries and leads to boundary scattering theory governed by coideal subalgebras of the bulk symmetry. Here the boundary scattering theory for $D3$-, $D5$- and $D7$-branes is considered in detail, and the underlying boundary Yangian symmetries are revealed.

The AdS/CFT worldsheet scattering theory is shown to be closely related to that of the deformed Hubbard chain. This similarity allows us to apply the quantum deformed approach to the boundary scattering theory. Such treatment of the system leads to quantum affine symmetries that manifest themselves in a very elegant and compact form. In such a way the symmetries of distinct boundaries that previously seemed to be unrelated to each other emerge in a uniform and coherent form.

The quantum deformed approach also helps us to better understand the phenomena of the so-called secret symmetry. It is called secret due to its peculiar feature of appearing as a level-one generator of the Yangian of the system. However it does not have a Lie algebra (level-zero) analogue. This symmetry is shown to have origins in the quantum deformed model, where it manifest itselfs as two, level-one and level-minus-one, generators of the corresponding quantum affine algebra.

Contents

Preface

This manuscript will take the reader to a magical world of symmetries, where Yangian and quantum affine algebras converge. It will lead on a magnificent journey following a yellow-brick road of a superstring of infinite length and light-cone momentum throughout the $AdS_5 \times S^5$ spacetime. We will encounter various D-branes blocking our way and will find elegant integrable solutions giving a safe bypass from these obstacles.

The experience gained will allow the reader a glimpse into an even more extraordinary world of quantum deformations, where the previously encountered structures emerge in completely new prospects and require more elaborate methods for being conquered.

The final part of the manuscript will lead to a quest for the origins of the secret symmetry. This quest will require to scout through many strange worlds of quantum symmetries that are fearlessly protecting secrets of the AdS/CFT.

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Declaration

This dissertation is the result of my own work and includes nothing which is the outcome of work done in collaboration except where specifically indicated in the text. The research described in this dissertation was carried out at the Department of Mathematics of the University of York between October 2008 and September 2012. Except where reference is made to the work of others, all the results are original and based on the following collaborative works where I have contributed:

- M. de Leeuw and V. Regelskis, *Integrable boundaries in AdS/CFT: revisiting the Z=0 giant graviton and D7-brane* , eprint, [`arXiv:1206.4704`].
- V. Regelskis, *Reflection algebras for sl(2) and gl(1|1)*, eprint, [`arXiv:1206.6498`].
- M. de Leeuw, T. Matsumoto, S. Moriyama, V. Regelskis and A. Torrielli, *Secret Symmetries in AdS/CFT*, Phys. Scr. 86 (2012) 028502, [`arXiv:1204.2366`].
- M. de Leeuw, V. Regelskis and A. Torrielli, *The Quantum Affine Origin of the AdS/CFT Secret Symmetry*, J. Phys. A 45 (2012) 175202, [`arXiv:1112.4989`].
- M. de Leeuw, T. Matsumoto and V. Regelskis, *Coideal Quantum Affine Algebra and Boundary Scattering of the Deformed Hubbard Chain*, J. Phys. A 45 (2012) 065205 [Identified by the referee as an article of particular interest], [`arXiv:1110.4596`].
- M. de Leeuw, T. Matsumoto and V. Regelskis, *The Bound State S-matrix of the Deformed Hubbard Chain*, JHEP 1204 (2012) 021, [`arXiv:1109.1410`].
- V. Regelskis, *The Secret Symmetries of the AdS/CFT Reflection Matrices*, JHEP 1108 (2011) 006, [`arXiv:1105.4497`].
- N. MacKay and V. Regelskis, *Achiral Boundaries and the Twisted Yangian of the D5-brane*, JHEP 1108 (2011) 019, [`arXiv:1105.4128`].
- D. H. Correa, V. Regelskis and C. A. S. Young, *Integrable Achiral D5-brane Reflections and Asymptotic Bethe Equations*, J. Phys. A 44 (2011) 325403, [`arXiv:1105.3707`].
- N. MacKay and V. Regelskis, *Reflection Algebra, Yangian Symmetry and Bound-states in AdS/CFT*, JHEP 1201 (2012) 134, [`arXiv:1101.6062`].
- N. MacKay and V. Regelskis, *Yangian Symmetry of the Y=0 Maximal Giant Graviton*, JHEP 1012 (2010) 076, [`arXiv:1010.3761`].
- N. MacKay and V. Regelskis, *On the Reflection of Magnon Bound States*, JHEP 1008 (2010) 055, [`arXiv:1006.4102`].

None of the original works contained in this dissertation has been submitted by me for any other degree, diploma or similar qualification.

Signed: .... (Vidas Regelskis)
Date: ....

Chapter 1

Introduction

During the last twenty years mathematics and physics have significantly influenced each other and became highly entangled. Theoretical physics was always producing a wide variety of new concepts and problems that became important subjects of mathematical research. The growth of gauge, gravity and string theories have made the relation between these two disciplines closer than ever before. An important driving force was the discovery of quantum groups [1–5] and of gauge/gravity dualities [6–8]. Here the leading role was played by the the so-called AdS/CFT correspondence and the underlying integrable structure of it [9].

Quantum integrable systems constitute a special class of models in both mathematics and physics and are studied principally through the quantum inverse scattering method (QISM) and related methods [10–12]. Their properties allow them to be solved exactly and thus integrable models form a very useful playground for studying various systems. A common feature shared by these models is that they have hidden algebraic structures with a Lie algebra at the core. Furthermore, such systems typically enjoy not only a symmetry of Lie type, but also a much larger and more powerful symmetry, for example of Yangian or quantum affine type.

A far-reaching concept in integrable systems is the effect of boundaries and the corresponding boundary conditions. They are unavoidable in almost all models of physics and are of fundamental importance. The introduction of boundaries into the theory of quantum groups leads to a whole new class of the so-called reflection algebras [13–15]. Such algebras were shown to appear in numerous models of physics and are at the core of the integrable structure of them. However a coherent framework for describing such algebras is not known, and many properties of reflection algebras are still an open question.

When studying integrable models with periodic boundary conditions, the spectrum is governed by the S-matrix and thus indirectly through the underlying symmetry algebra, which is conventionally called the bulk symmetry. However, for integrable systems with open boundaries, there is another object, called the reflection matrix, or K-matrix, which describes the scattering of excitations from the boundary [16]. Generically, boundaries preserve a subalgebra of the bulk Lie algebra and this subalgebra then determines the corresponding reflection matrix. However this is usually not enough to determine the bound state reflection matrix and a coideal subalgebra of the corresponding bulk Yangian or quantum affine algebra is required [17].

A very distinctive algebra arises at the core of the AdS/CFT duality [18–20] and leads to a variety of coideal subalgebras [21–26]. These algebras have very specific properties and hardly fit into the current classification of quantum groups. Thus new algebraic methods must be developed to put these new algebras onto a firm ground.

The goal of my research is to explore the symmetries of the worldsheet scattering in AdS/CFT by building a connection between the theory of quantum groups and the integrable structure of AdS/CFT, in particular by shedding more light on the effects of boundaries and different boundary configurations, and find elegant, exact solutions and methods describing the models that arise from and are inspired by the gauge/gravity dualities. An important part of my work is to link the integrable structures and boundary algebras arising in AdS/CFT to the already known ones, in particular, to those of the principal chiral model defined on a semi-infinite line [17, 27], to the deformed Hubbard chain [28–30], and to the axiomatic theory of the quantum symmetric pairs [31, 32]. In such a way the methods presented in this manuscript can be generalized and applied to other gauge/gravity dualities and relevant models of both mathematics and physics.

In this chapter we will briefly recall the notion of integrability, the link between Hopf algebras, quantum groups and the Yang-Baxter equation, and also between reflection algebras and the reflection equation. We will then make a short glimpse at the integrable structure of AdS/CFT and give an outline of this thesis. Some of the topics briefly covered here will be explored in much more detail in the subsequent chapters and more references to earlier works will be given.

1.1 Integrability and the algebraic Bethe ansatz for spin chains

We will introduce the notion of integrable systems through the Liouville theorem. Such systems have a $2m$-dimensional phase space M, and have m independent conserved quantities (constants of motion) $\{I_j\}_{j=1,\dots,m}$ in involution. Then the Liouville theorem states that such a system can be solved exactly. Examples of such systems are harmonic oscillators in m dimensions, Toda lattices, nonlinear sigma models, Heisenberg spin chain. There are also systems with $k > m$ independent conserved quantities, for example the Kepler system. Such systems are called superintegrable, however in this case not all of the conserved quantities are in involution.

Let us start by recalling some basic definitions. A manifold M equipped with a non-degenerate closed 2-form

$$\omega : TM \to T^*M \ , \tag{1.1.1}$$

between the tangent bundle TM and the cotangent bundle T^*M, is called a symplectic manifold with a symplectic 2-form ω and is conveniently denoted by (M, ω). The non-degeneracy of ω implies that there exists a well-defined inverse

$$\Omega : T^*M \to TM \ , \qquad \Omega = \omega^{-1} \ . \tag{1.1.2}$$

In such a way any one-form df on M can be identified with a vector field and for any differentiable function $\mathcal{H} : M \to \mathbb{R}$, there exists a unique vector field $X_\mathcal{H}$ called the Hamiltonian vector field such that for any vector field Y on M the following identity holds,

$$d\mathcal{H}(Y) = \omega(X_\mathcal{H}, Y) \, . \tag{1.1.3}$$

We will use the notion of a Hamiltonian vector field to define a Poisson manifold.

Consider a bilinear map on M, called the Poisson bracket,

$$\{\cdot\,,\cdot\} : C^\infty(M) \times C^\infty(M) \to C^\infty(M) \, , \tag{1.1.4}$$

such that for any $f, g, h \in C^\infty(M)$

$$\{f, g\} = \omega(X_f, X_g) = df(X_g) = \mathcal{L}_{X_g} f \, , \tag{1.1.5}$$

where $\mathcal{L}_X$ is the Lie derivative along the vector field X with the following properties:

- It is skew-symmetric: $\{f, g\} = -\{g, f\}$.
- It obeys the Jacobi identity: $\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$.
- It is a derivation of $C^\infty(M)$ in its first argument: $\{fg, h\} = f\{g, h\} + \{f, h\}g$.

A Poisson manifold W is a manifold endowed with a Poisson backet satisfying the properties above.

Let us make two remarks. First, any symplectic manifold is a Poisson manifold, but the converse is not true. There are many Poisson manifolds that are not symplectic manifolds. Secondly, a vector space of differentiable functions on a Poisson manifold has the structure of a Lie algebra; the assignment $f \mapsto X_f$ is a Lie algebra homomorphism, whose kernel consists of the (locally) constant functions.

We are now ready to define the notion of integrability. Consider a dynamical system modeled by a symplectic manifold M equipped with a Poisson brackets and the Hamiltonian $\mathcal{H}(M)$. and its time-evolution given by following the integral curves of the Hamiltonian vector field $X_\mathcal{H}$ on M corresponding to $\mathcal{H}$,

$$X_\mathcal{H}(f) = \{\mathcal{H}, f\} \, . \tag{1.1.6}$$

Let $f = f(m(t))$, here $m(t)$ is any integral curve of $X_\mathcal{H}$ parametrized by t. Then (1.1.6) becomes

$$\frac{df}{dt} = \{\mathcal{H}, f\} \, . \tag{1.1.7}$$

The function f is called a conserved quantity (or an integral of motion) if it Poisson-cummutes with $\mathcal{H}$, i.e. $\{\mathcal{H}, f\} = 0$. Two functions f and g are in involution if $\{f, g\} = 0$. This terminology allows to give the following definition of a classical integrable system:

Definition 1.1.1. *A classical dynamical system on a $2m$-dimensional symplectic manifold M with a Hamiltonian $\mathcal{H}$ is Liouville integrable if it possesses m independent conserved quanttities $\{I_j\}_{j=1,\ldots,m}$ in involution.*

Here the independence means that the tangent space of the surface defined by $I_j = f_j$ exists everywhere and is of dimension m. Furthermore, there can be at most m independent quantities in involution, otherwise the Poisson bracket would be degenerate.

There is also a notion of a quantum integrable system. In the quantum setting, functions on the phase space must be replaced by self-adjoint operators on the Hilbert space, and the notion of Poisson-commuting functions is replaced by commuting operators. Consequently, a system is considered to be quantum integrable if there exist operators $\hat{I}_j$ corresponding to conserved charges which can be simultaneously diagonalized together with the Hamiltonian operator $\hat{\mathcal{H}}$,

$$[\hat{\mathcal{H}}, \hat{I}_j] = 0 \,, \qquad [\hat{I}_j, \hat{I}_k] = 0 \,. \tag{1.1.8}$$

In the semiclassical limit, these operators correspond to symbols that are independent Poisson-commuting functions on the phase space.

The importance of Liouville integrability, as the terminology implies, is that system with such properties can be solved exactly. Solving the system is considered as finding the resulting spectrum of $\mathcal{H}$ and I_j. A very simple but widely applied example of an integrable system is the Heisenberg $XXX_{\frac{1}{2}}$ spin chain which can be exactly solved using the Bethe ansatz.

The classical Bethe ansatz is applied to a periodic chain with L sites. At each site the spin variable can be facing either up or down. Consequently the Hilbert space of the spin chain is

$$\mathscr{H}^{(L)} = \prod_{n=1}^{L} \otimes V^{\frac{1}{2}} \,, \tag{1.1.9}$$

where each local space $V^{\frac{1}{2}}$ is a spin-$\frac{1}{2}$ irrep of $\mathfrak{su}(2)$ with a basis $\{\uparrow, \downarrow\}$. Therefore the dimension of the Hilbert space is $dim\, \mathscr{H}^{(L)} = 2^L$. The Hamiltonian for a such system is very simple,

$$H = -\sum_{i=1}^{L} (\vec{\sigma}_i \cdot \vec{\sigma}_{i+1} - 1) \,, \quad \text{with} \quad \vec{\sigma}_{L+1} := \vec{\sigma}_1 \,, \tag{1.1.10}$$

and reflects that the interactions are short ranged, – the nearest neighbours are interacting only; here $\vec{\sigma}_i$ are the Pauli matrices acting on the i-th lattice site. Denote e^{iP} the operator shifting the states by one lattice unit. Then the Hamiltonian is clearly translation invariant

$$[\mathrm{e}^{iP}, H] = 0 \,. \tag{1.1.11}$$

The periodicity of the closed spin-chain gives

$$\mathrm{e}^{iPL} = 1 \,. \tag{1.1.12}$$

Finally, define $\Delta^{(L)}(X) = \sum_{i=1}^{L} X_i$, where $X_i \in \{\sigma_i^x, \sigma_i^y, \sigma_i^z\}$. Then the Hamiltonian H is $\mathfrak{su}(2)$ symmetric,

$$[H, \Delta^{(L)}(X)] = 0 \,. \tag{1.1.13}$$

Let us denote the state with all spins-down as the vacuum state. Next, we divide the Hilbert space into subspaces of states with equal number of spins-up,

$$\mathscr{H}^{(L)} = \sum_{M=0}^{N} \mathscr{H}_M^{(L)} , \tag{1.1.14}$$

where $\mathscr{H}_0^{(L)}$ represents the vacuum state, $\mathscr{H}_1^{(L)}$ represents the subspace of all configurations with one spin-up and etc. The dimension of each subspace is $dim\,\mathscr{H}_M^{(L)} = \binom{L}{M}$.

The natural way to define the wave function for an eigenvector from the first sector is

$$|\Psi_1\rangle = \sum_{x=1}^{L} f(x)\,|x\rangle \,, \tag{1.1.15}$$

where $|x\rangle$ represents the state with spin-up at the site x. The function $f(x)$ describes the probability that a single spin-up is precisely at the site x. Next, translation invariance and periodicity offer a very simple form of $f(x)$, namely it is a plane wave,

$$f(x) = \mathrm{e}^{ikx} \,, \tag{1.1.16}$$

where k is momentum constrained by the periodicity $f(x+L) = f(x)$ and is $k = 2\pi I/L$ with $I = 0, 1, ..., L-1$.

The wave function for an eigenvector from the second sector is very similar,

$$|\Psi_2\rangle = \sum_{x_1,x_2} f(x_1,x_2)\,|x_1,x_2\rangle \,. \tag{1.1.17}$$

Here $|x_1,x_2\rangle$ represents a vector in $\mathscr{H}_2^{(L)}$ with spins-ups at the lattice sites x_1 and x_2. The periodicity condition now reads

$$f(x_1,x_2) = f(x_2,x_1+L) \,. \tag{1.1.18}$$

The solution for the unknown function $f(x_1,x_2)$ was found by Bethe and is [33]

$$f(x_1,x_2) = A_{12}\,\mathrm{e}^{i(k_1x_1+k_2x_2)} + A_{21}\,\mathrm{e}^{i(k_1x_2+k_2x_1)} \,, \tag{1.1.19}$$

It respects the periodicity condition by requiring the following constraints to hold,

$$A_{12} = A_{21}\,\mathrm{e}^{ik_1L} \,, \qquad A_{21} = A_{12}\,\mathrm{e}^{ik_2L} \,, \tag{1.1.20}$$

Finally, the full translational symmetry of the system requires

$$f(x_1+L,x_2+L) = f(x_1,x_2) \,, \qquad \mathrm{e}^{i(k_1+k_2)L} = 1 \,. \tag{1.1.21}$$

The Bethe ansatz is treating the scattering of two spin waves as purely elastic. The only dynamics allowed is the permutation of the quasi-momenta. This is the consequence of integrability.

The physical properties are encoded in the scattering amplitudes,

$$S_{12} = \frac{A_{21}}{A_{12}} \;, \qquad S_{21} = \frac{A_{12}}{A_{21}} \;, \tag{1.1.22}$$

in terms of which Bathe ansatz reads as

$$f(x_1, x_2) = A_{12} \left(\mathrm{e}^{i(k_1 x_1 + k_2 x_2)} + S_{12}\, \mathrm{e}^{i(k_1 x_2 + k_2 x_1)} \right) , \tag{1.1.23}$$

and the scattering amplitudes satisfy

$$\mathrm{e}^{ik_1 L} S_{12} = 1 \;, \qquad \mathrm{e}^{ik_2 L} S_{21} = 1 \;. \tag{1.1.24}$$

Let us continue and generalize these equations for an arbitrary number M of spins-up. The generic form of a wave function is

$$|\Psi_M\rangle = \sum_{1 \leq x_1 < x_2 < \ldots < x_M \leq L} f(x_1, ..., x_M)\, |x_1, ..., x_M\rangle \;, \tag{1.1.25}$$

and the Bethe ansatz acquires the form

$$f(x_1, ..., x_M) = \sum_{p \in \mathcal{P}_M} A_p\, \mathrm{e}^{i(k_{p(1)} x_1 + \ldots + k_{p(M)} x_M)}, \tag{1.1.26}$$

where the sums runs over the $M!$ permutations p of the labels of quasi-momenta k_i. The periodicity condition generalizes to

$$f(x_1, ..., x_M) = f(x_2, ..., x_M, x_1 + L) \;. \tag{1.1.27}$$

And therefore we arrive to the Bethe ansatz equations (BAE)

$$\mathrm{e}^{ik_i L} = \prod_{j=1,\; j \neq i}^{M} S_{ji}(k_j, k_i) \;, \quad \text{for} \quad i = 1, \ldots, M \;. \tag{1.1.28}$$

It is worth considering the case $M = 3$ more explicitly as it is closely related to the Yang-Baxter equation [34,35]. For a scalar S-matrix the ordering of factorization does not play a role. However in the matrix case there are two inequivalent ways of factorizing the three particle scattering. They are shown in figure 1.1 and represent the Yang-Baxter equation,

$$S^{(3)}_{123} = S_{23} S_{13} S_{12} = S_{12} S_{13} S_{23} \;. \tag{1.1.29}$$

The Bethe equations can be derived heuristically if we consider M particles on a circle of circumference L. Transporting the j'th particle around the circle reads as shifting of the particle position l_j by $l_j \to l_j + L$. In the absence of other particles this transportation

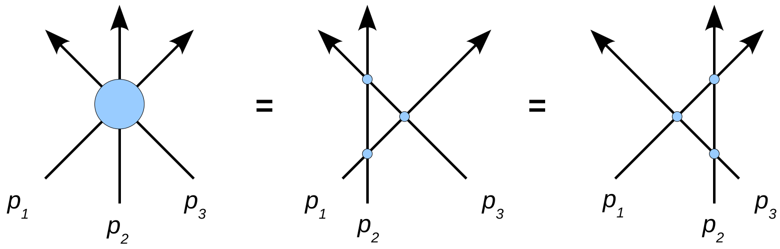

Figure 1.1: Factorization of the three particle scattering.

would produce a phase shift $\exp(ip_jL)$ only. By introducing other particles we must include the scattering of the j'th particle with all the other particles around the circle and therefore the phase picks up factors $S_{j,k}$ for all $k \neq j$. Consequently, the heuristic derivation yields the BAE (1.1.28). In the $L \to \infty$ limit the system becomes infinite dimensional and the number of conserved quantities becomes infinite as well. For more examples and details how to treat infinite dimensional Hamiltonian systems we refer to e.g. [36]. The properties of the S-matrix will be discussed in more detail in Chapter 3, where we will consider the AdS/CFT worldsheet scattering.

Bethe ansatz equations can be easily generalized for a system with open boundaries. In this case the periodicity condition corresponds to transporting the test particle along the chain towards the right boundary, reflecting, translating backwards to the left boundary, reflecting, and then translating it back to the initial site of the chain. In such a way the Bethe ansatz equations become

$$\mathrm{e}^{2ik_iL} = K(k_i)^2 \prod_{j=1,\ j\neq i}^{M} S_{ji}(k_j,k_i)\ S_{ji}(-k_j,k_i)\ , \tag{1.1.30}$$

where $K(k_i)$ is the boundary reflection matrix. This time the $M = 2$ case is related to the boundary Yang-Baxter equation shown in figure 1.2.

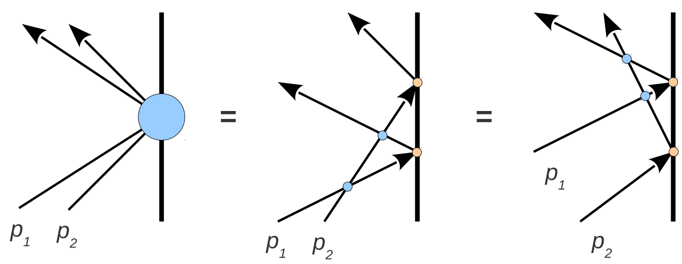

Figure 1.2: Factorization of the boundary scattering.

The statement of exact factorization reads as

$$K^{(3)}_{123} = K_{23}S_{2\underline{1}}K_{13}S_{12} = S_{\underline{21}}K_{13}S_{1\underline{2}}K_{23}\ , \tag{1.1.31}$$

where the underbarred notation denotes reflected states and the subindex $_3$ denotes the boundary. Note that there can be two different boundary conditions, a trivial boundary with no boundary degrees of freedom, and the one with a boundary state located at the extra lattice sites $L = 0$ and $L = M + 1$. In such a way the explicit form of $K(k_i)$ depends on the corresponding boundary conditions.

1.2 Hopf algebras and the YBE

Here we shall give a brief introduction to Hopf algebras and show how these naturally produce the Yang-Baxter equation [34,35]. We will come back to this topic in more detail in Chapter 2. For a comprehensive guide to Hopf algebras and quantum groups we refer to [37].

We start from an associative unital algebra $(\mathcal{A}, m, i)$ over $\mathbb{C}$. The associativity of multiplication map $m: \mathcal{A}\otimes\mathcal{A} \to \mathcal{A}$ for an arbitrary algebra elements a, b, c reads as

$$[m\,(m\otimes 1)]\,(a\otimes b\otimes c) = [m\,(1\otimes m)]\,(a\otimes b\otimes c)\,. \tag{1.2.1}$$

The unit map i is linked to unit element $1 \in \mathcal{A}$ as $i:\ \lambda\in\mathbb{C}\mapsto 1{\cdot}\lambda\in\mathcal{A}$. Next we introduce a comultiplication (coproduct) $\Delta:\ \mathcal{A}\to\mathcal{A}\otimes\mathcal{A}$ which we require to be coassociative

$$(\Delta\otimes id)\,\Delta(a) = (id\otimes\Delta)\,\Delta(a)\ , \tag{1.2.2}$$

where id is the identity map. One more required element for the coalgebra structure is a counit map $\epsilon:\ \mathcal{A}\to\mathbb{C}$ which satisfies

$$(id\otimes\epsilon)\,\Delta(a) = (\epsilon\otimes id)\,\Delta(a) = id\ . \tag{1.2.3}$$

The structure $(\mathcal{A}, m, i, \Delta, \epsilon)$ is an algebra and a coalgebra simultaneously and is called a bi-algebra if comultiplication Δ and counit ϵ are algebra homomorphisms,

$$\epsilon(ab) = \epsilon(a)\epsilon(b)\ , \qquad \Delta(ab) = \Delta(a)\Delta(b)\ . \tag{1.2.4}$$

By introducing an antipode $S:\ \mathcal{A}\to\mathcal{A}$ which is an anti-homomorphism

$$S(ab) = S(b)S(a)\ , \tag{1.2.5}$$

satisfying the following condition

$$m\,(S\otimes id)\,\Delta(a) = m\,(id\otimes S)\,\Delta(a) = \epsilon(a)id\ , \tag{1.2.6}$$

the bialgebra becomes a Hopf algebra. To summarize what was said, let us give an explicit definition.

Definition 1.2.1. *A Hopf algebra over a field K is a K-module $\mathcal{A}$ equipped with K-module maps*

$$\begin{array}{llll} m:\mathcal{A}\otimes\mathcal{A}\to\mathcal{A} & \textit{(multiplication)}, & i:\mathbb{C}\to\mathcal{A} & \textit{(unit map)},\\ \Delta:\mathcal{A}\to\mathcal{A}\otimes\mathcal{A} & \textit{(comultiplication)}, & \epsilon:\mathcal{A}\to\mathbb{C} & \textit{(counit map)},\\ S:\mathcal{A}\to\mathcal{A} & \textit{(antipode)}, & & \end{array} \tag{1.2.7}$$

such that m and Δ are coalgebra homomorphisms, i and ϵ are algebra homorphisms, S is algebra anti-homomorphism, and satisfy the following properties:

$$\begin{aligned}
&m \circ (id \otimes m) = m \circ (m \otimes id) && \textit{(associativity)},\\
&m \circ (id \otimes i) \cong m \circ (i \otimes id) \cong id && \textit{(existence of unit)},\\
&(id \otimes \Delta) \circ \Delta = (\Delta \otimes id) \circ \Delta && \textit{(coassociativity)},\\
&(\epsilon \otimes id) \circ \Delta \cong (id \otimes \epsilon) \circ \Delta \cong id && \textit{(existence of counit)},\\
&m \circ (id \otimes S) \circ \Delta = m \circ (S \otimes id) \circ \Delta = i \circ \epsilon\,, &&\\
&\Delta \circ m = (m \otimes m) \circ (\Delta \otimes \Delta) && \textit{(connection axiom)}.
\end{aligned} \tag{1.2.8}$$

If the multiplication is commutative the algebra is called commutative, otherwise it is a non-commutative algebra. A similar but slightly extended classification applies to coalgebras and Hopf algebras. These can be cocommutative or non-cocommutative. However some of the non-cocommutative Hopf algebras can be endowed with an additional algebraic structure called the *quasi-cocommutativity* property. Such Hopf algebras interpolate between cocommutativity and (completely) non-cocommutativity in a controlled and very special way. To explore such algebras we need to introduce a permutation map,

$$\sigma:\ \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes \mathcal{A}\,, \quad a \otimes b \mapsto b \otimes a\,, \tag{1.2.9}$$

which acts by interchanging the order of the operands only. The Hopf algebra is cocommutative iff

$$\Delta^{op}(a) := \sigma \circ \Delta(a) = \Delta(a) \quad \text{for all} \quad a \in \mathcal{A}\,. \tag{1.2.10}$$

Furthermore, $\Delta^{op}(a)$ is also a comultiplication. In such a way the set $(\mathcal{A}, \mu^{op}, i, \Delta^{op}, \epsilon, S')$, where $S'(a) = S^{-1}(a)$, is also a Hopf algebra and is convienently called as the opposite Hopf algebra and is denoted by $\mathcal{A}^{op}$. It is isomorphic to $\mathcal{A}^{op} \cong \mathcal{A}$ as a Hopf algebra. The algebra of prime interest to us will be a quasitriangular Hopf algebra:

Definition 1.2.2. *A given Hopf algebra is called quasitriangular if there exists an invertible element, called the universal R-matrix $\mathcal{R} = \sum_i a'_i \otimes a''_i \in \mathcal{A} \otimes \mathcal{A}$, such that*

$$\Delta^{op}(a) = \mathcal{R}\, \Delta(a)\, \mathcal{R}^{-1} \quad \textit{for all} \quad a \in \mathcal{A}\,, \tag{1.2.11}$$

and satisfying

$$(id \otimes \Delta)\, \mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{12} = \sum_{i,j} a'_i a'_j \otimes a''_j \otimes a''_i\,, \tag{1.2.12}$$

$$(\Delta \otimes id)\, \mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23} = \sum_{i,j} a'_i \otimes a'_j \otimes a''_i a''_j\,, \tag{1.2.13}$$

$$(S \otimes id)\, \mathcal{R} = \left(id \otimes S^{-1}\right) \mathcal{R} = \mathcal{R}^{-1}\,. \tag{1.2.14}$$

The notation used above is

$$\mathcal{R}_{12} = \sum_i a'_i \otimes a''_i \otimes 1\,, \quad \mathcal{R}_{13} = \sum_i a'_i \otimes 1 \otimes a''_i\,, \quad \mathcal{R}_{23} = \sum_i 1 \otimes a'_i \otimes a''_i\,. \tag{1.2.15}$$

The quasitriangularity means that the comultiplication Δ and its transpose Δ^{op} are related linearly. Heuristically it can be understood as an equivalence between two different ways of 'adding things up'. A trivial example of a quasitriangular cocommutative algebra is when $\mathcal{R} = \mathbb{C} \otimes \mathbb{C}$. A Hopf algebra is called triangular if $\mathcal{R}_{21} = \mathcal{R}^{-1}$, where $\mathcal{R}_{21} = \sum_i a_i'' \otimes a_i'$. A noncocommutative quasitriangular Hopf algebra is called a quantum group. The most interesting feature for us is that a quasitriangular Hopf algebra naturally produces a solution to the Yang-Baxter equation in the so-called universal form.

Proposition 1.2.1. *Let $(\mathcal{A}, \mathcal{R})$ be a quasitriangular Hopf algebra. Then,*

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} \, . \tag{1.2.16}$$

Proof. By (1.2.13) we have

$$[(\sigma \circ \Delta) \otimes id]\, \mathcal{R} = \sigma_{12}\, (\Delta \otimes id)\, \mathcal{R} = \sigma_{12}\, (\mathcal{R}_{13}\mathcal{R}_{23}) = \mathcal{R}_{23}\mathcal{R}_{13} \, . \tag{1.2.17}$$

On the other hand the same expression can be written as

$$\begin{aligned} [(\sigma \circ \Delta) \otimes id]\, \mathcal{R} &= \sum_i \Delta^{op}(a_i') \otimes a_i'' = \sum_i \mathcal{R}_{12}\Delta(a_i')\mathcal{R}_{12}^{-1} \otimes a_i'' \\ &= \mathcal{R}_{12}\big(\sum_i \Delta(a_i') \otimes a_i'' \big)\mathcal{R}_{12}^{-1} = \mathcal{R}_{12}\, [(\Delta \otimes id)\, \mathcal{R}]\, \mathcal{R}_{12}^{-1} \\ &= \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\mathcal{R}_{12}^{-1} \, , \end{aligned} \tag{1.2.18}$$

and therefore (1.2.16) follows.

□

1.3 Reflection algebras and the BYBE

One of the most challenging questions in the scattering theory is the solution to the so-called universal boundary Yang-Baxter equation,

$$\mathcal{K}_{23}\mathcal{R}_{1\underline{2}}\mathcal{K}_{13}\mathcal{R}_{12} = \mathcal{R}_{\underline{12}}\mathcal{K}_{13}\mathcal{R}_{1\underline{2}}\mathcal{K}_{23} \, , \tag{1.3.1}$$

where the underbarred notation is associated with the reflected states; we will explain this point in more detail a little bit further. This equation is also called the reflection equation [15]. Similarly, algebras defining solutions of the reflection equation are conveniently called reflection algebras (see e.g. [14, 38–43]).

There are several approaches leading towards the universal solution of (1.3.1) (see e.g. [44, 45]); however these are valid for specific cases only, and thus there is no canonical approach in solving the reflection equation. In such a way in most cases the refection equation is understood as a matrix equation.

Here we will briefly recall some aspects of the representation theory of the Hopf algebras that will be relevant to us in defining the boundary reflection matrix. We will give more details and references in Chapter 2.

Consider a quasitriangular Hopf algebra $(\mathcal{A}, \mathcal{R})$. Let $\mathcal{V}$ be a finite dimensional vector space and $T_z : \mathcal{A} \to \mathrm{End}(\mathcal{V})$ be a finite dimensional representation of $\mathcal{A}$, where z denotes the evaluation parameter of the representation. Then the R-matrix can be defined as an intertwining matrix

$$R = (T_z \otimes T_w)[\mathcal{R}] \in \mathrm{End}\big(\mathcal{V} \otimes \mathcal{V}\big)\,. \tag{1.3.2}$$

and can be obtained by solving the intertwining equation

$$(T_z \otimes T_w)[\Delta^{op}(a)]\, R - R\, (T_z \otimes T_w)[\Delta(a)] = 0 \quad \text{for all} \quad a \in \mathcal{A}\,, \tag{1.3.3}$$

Note that $(T_z \otimes T_w)[\Delta(a)]$ is required to be irreducible. Such R-matrix then automatically satisfies (1.2.16) defined on the tensor space $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$.

Let $\mathcal{B} \subset \mathcal{A}$ be a left coideal subalgebra,

$$\Delta(b) \in \mathcal{A} \otimes \mathcal{B} \quad \text{for all} \quad b \in \mathcal{B}\,. \tag{1.3.4}$$

Let $\mathcal{W}$ be a finite dimensional vector space and $\bar{T}_s : \mathcal{B} \to \mathrm{End}(\mathcal{W})$ be a finite dimensional representation of $\mathcal{B}$, called boundary representation, where s denotes a boundary spectral parameter. Let $T_z^* : \mathcal{A} \to \mathrm{End}(\mathcal{V})$ be a conjugate representation of $\mathcal{A}$, for example $T_z^* \cong T_{1/z}$ for quantum affine algebras, and $T_z^* \cong T_{-z}$ for Yangian algebras. Then the boundary reflection matrix can be defined as an intertwining matrix

$$K \in \mathrm{End}(\mathcal{V} \otimes \mathcal{W})\,, \tag{1.3.5}$$

and can be obtained by solving the boundary intertwining equation

$$(T_z \otimes \bar{T}_s)[\Delta(b)]\, K - K\, (T_z^* \otimes \bar{T}_s)[\Delta(b)] = 0 \quad \text{for all} \quad b \in \mathcal{B}\,, \tag{1.3.6}$$

Here $(T_z \otimes \bar{T}_s)[\Delta(b)]$ is required to be irreducible. We are now ready to introduce a notion of the reflection algebra. Let us give some necessary preliminaries. Define $R_{1\underline{2}} := (T_z \otimes T_w^* \otimes id)[\mathcal{R}_{12}]$, and similarly for $R_{\underline{1}2}$, $R_{\underline{12}}$. Set $K_{23} = 1 \otimes K$, and similarly for K_{13}. Then:

Definition 1.3.1. *A coideal subalgebra $\mathcal{B} \subset \mathcal{A}$ is called a reflection algebra if the intertwining equation* (1.3.6) *defines a K-matrix $K \in \mathrm{End}(\mathcal{V} \otimes \mathcal{W})$ satisfying the reflection equation*

$$K_{23} R_{\underline{1}2} K_{13} R_{12} = R_{\underline{12}} K_{13} R_{1\underline{2}} K_{23}\,. \tag{1.3.7}$$

The property that reflection algebra must be a coideal subalagebra was first observed in [46]. We will consider such reflection algebras in more detail in Chapter 2. In the remaining part of this section we want to give an approach which could lead to a universal solution of the reflection equation and a universal reflection matrix. This approach is inspired by the algebraic structures of the AdS/CFT duality and will be considered in more detail in Chapters 3 and 4.

We would like to lift the considerations presented above to the algebra level. Suppose there exists an involutive algebra automorphism

$$\kappa : \mathcal{A} \to \mathcal{A}, \quad a \mapsto \underline{a} := \kappa(a)\,, \tag{1.3.8}$$

and $\kappa^2 = id$, such that

$$T_z(\underline{a}) = T^*_z(a) \quad \text{for all} \quad a \in \mathcal{A}\,, \tag{1.3.9}$$

and any pair of mutually compatible representations T_z and T^*_z in the sense as was described above.

Definition 1.3.2. *Let $\kappa : \mathcal{A} \to \mathcal{A}$ be an involutive algebra automorphism as defined above. Then we call κ a reflection automorphism of $\mathcal{A}$.*

The reflection automorphism is a necessary step in the search of a universal reflection matrix. Let us define a modified coproduct,

$$\Delta^{ref} := (\kappa \otimes id) \circ \Delta\,. \tag{1.3.10}$$

Definition 1.3.3. *We call Δ^{ref} defined by* (1.3.10) *the reflected coproduct.*

Now we are ready to introduce a notion of the universal reflection algebra and the universal reflection matrix.

Definition 1.3.4. *We call $\mathcal{B}$ a universal reflection algebra if there exists an invertible element, called a universal K-matrix $\mathcal{K} = \sum_i a'_i \otimes b''_i \in \mathcal{A} \otimes \mathcal{B}$, such that*

$$\Delta^{ref}(b) = \mathcal{K}\,\Delta(b)\,\mathcal{K}^{-1} \quad \textit{for all} \quad b \in \mathcal{B}\,, \tag{1.3.11}$$

and satisfying the universal reflection equation (1.3.1)

$$\mathcal{K}_{23}\mathcal{R}_{\underline{1}2}\mathcal{K}_{13}\mathcal{R}_{12} = \mathcal{R}_{\underline{12}}\mathcal{K}_{13}\mathcal{R}_{1\underline{2}}\mathcal{K}_{23}\ ,$$

where $\mathcal{R}_{12} = \sum_i a'_i \otimes a''_i \otimes 1$, $\mathcal{R}_{1\underline{2}} = (id \otimes \kappa \otimes id)\mathcal{R}_{12} = \sum_i a'_i \otimes \underline{a}''_i \otimes 1$, and similarly for $\mathcal{R}_{\underline{1}2}$, $\mathcal{R}_{\underline{12}}$; and $\mathcal{K}_{13} = \sum_i a'_i \otimes 1 \otimes b''_i$, $\mathcal{K}_{23} = \sum_i 1 \otimes a'_i \otimes b''_i$.

Note that a given Hopf algebra $\mathcal{A}$ may have several inequivalent coideal subalgebras, and thus there can be a family of reflection algebras leading to inequivalent reflection matrices satisfying the reflection equation for a single R-matrix. There are some universal properties that are respected by all $\mathcal{B}$'s:

Definition 1.3.5. *Let $b \in \mathcal{B}$ be such that $\Delta^{ref}(b) = \Delta(b)$. Then we call b coreflective.*

Proposition 1.3.1. *Let $c \in \mathcal{B}$ be a central element of the algebra. Then c is coreflective.*

Proof. This follows directly from (1.3.11). □

The reflection matrix for the left boundary can be obtained by composing with the flip operator σ,

$$\mathcal{K}^{op} := \sigma(\mathcal{K}) \in \mathcal{B} \otimes \mathcal{A}\,. \tag{1.3.12}$$

Define $\Delta^{op.ref}(b) := \sigma \circ \Delta^{ref}$. Then the relation (1.3.11) for the left boundary becomes

$$\Delta^{op.ref}(b) = \mathcal{K}^{op}\,\Delta^{op}(b)\,(\mathcal{K}^{op})^{-1} \quad \text{for all} \quad b \in \mathcal{B}\,. \tag{1.3.13}$$

$$\begin{tikzcd}
\Delta^{op} \arrow[rr, leftrightarrow, "\sigma"'] \arrow[d, leftrightarrow, "id \otimes \kappa"] & & \Delta \arrow[d, leftrightarrow, "\kappa \otimes id"'] \\
\Delta^{op.ref} \arrow[rr, leftrightarrow, "\sigma"] & & \Delta^{ref}
\end{tikzcd}$$

The maps σ and κ lead to the following commutative diagram,

A somewhat similar approach was considered in [44] where the role of the reflected algebra $\mathcal{A}^{ref}$ and the reflected coproduct Δ^{ref} is played by a twisted Hopf algebra and a twisted coproduct, and $\mathcal{K} \in \mathcal{A} \otimes \tilde{\mathcal{A}}^{*}$, where $\tilde{\mathcal{A}}^{*}$ is a twisted-dual algebra to $\mathcal{A}$, and coincides with $\mathcal{A}$ as a linear space.

The key problem of the universal approach to the reflection equation presented above is that it is not possible to construct the reflection automorphism κ for a generic algebra $\mathcal{A}$, thus this approach could be applied to specific algebras only.

An algebra which does have a reflection automorphism is the centrally extended $\mathfrak{psu}(2|2)_{\mathcal{C}}$ algebra playing a key role in the worldsheet scattering theory of the AdS/CFT correspondence. An exclusive feature of this algebra is its braided Hopf algebra structure and an $SL(2)$ outer-automorphism group which accommodates the reflection automorphism. This algebra is very rich in coideal subalgebras which we will explore in Chapters 3 and 4. However, the universal R-matrix is not known for this algebra, and this obscures finding a universal reflection K-matrix. Nevertheless this approach has proved to be very useful in finding worldsheet reflection matrices in AdS/CFT, and played a crucial role in defining the representation of the reflected algebra and determining boundary algebras for the q-deformed worldsheet scattering.

To finalize we want to give a 'dictionary' relating quantum groups and reflection algebras.

Bulk	**Boundary**
Hopf algebra $\mathcal{A}$	Coideal subalgebra $\mathcal{B}$
Coproduct $\Delta(a) \in \mathcal{A} \otimes \mathcal{A},\ \forall a \in \mathcal{A}$	Coproduct $\Delta(b) \in \mathcal{A} \otimes \mathcal{B},\ \forall b \in \mathcal{B}$
Flip map σ	Reflection map κ
Opposite coproduct $\Delta^{op} = \sigma \circ \Delta$	Reflected coproduct $\Delta^{ref} = (\kappa \otimes id) \circ \Delta$
R-matrix $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$	K-matrix $\mathcal{K} \in \mathcal{A} \otimes \mathcal{B}$
$\Delta^{op}(a) = \mathcal{R}\, \Delta(a)\, \mathcal{R}^{-1}$	$\Delta^{ref}(b) = \mathcal{K}\, \Delta(b)\, \mathcal{K}^{-1}$
YBE: $\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$	BYBE (RE): $\mathcal{K}_{23}\mathcal{R}_{\underline{1}2}\mathcal{K}_{13}\mathcal{R}_{12} = \mathcal{R}_{\underline{12}}\mathcal{K}_{13}\mathcal{R}_{1\underline{2}}\mathcal{K}_{23}$
Quantum group	Reflection algebra

1.4 Integrability in AdS/CFT

The AdS/CFT correspondence, as originally conjectured by Maldacena [6], states an equivalence (or duality) between two very different theories:

- $\mathcal{N} = 4$ super Yang-Mills theory in 4-dimensions with the gauge group $SU(N)$ and coupling constant g_{YM} in the conformal phase;
- Type IIB superstring theory on $AdS_5 \times S^5$ where both AdS_5 and S^5 have the same radius and the coupling constant is $g_S = g_{YM}^2$;

The AdS/CFT conjecture states that these theories, including operator observables, states, correlation functions and full dynamics, are equivalent to each other.

In this section we shall briefly review the connection between the type IIB superstring theory compactified on a $AdS_5 \times S^5$ and the $\mathcal{N} = 4$ super Yang-Mills theory. There are currently many nice reviews on the subject [47–52]), the most recent and exhaustive one being the review [9]. Thus we will be rather concise in this section and concentrate more on the geometry, where D-branes live, the subject of our investigations.

Let us start from a flat ten dimensional Minkowski spacetime where the type IIB superstring theory lives. The spacetime will become compatified to $AdS_5 \times S^5$ by introducing a stack of N parallel $D3$-branes that are sitting together very closely to each other and are extended along a $(3 + 1)$ dimensional plane in the $(9 + 1)$ dimensional spacetime. String theory on this configuration contains two kinds of perturbative excitations, namely closed and open strings. The closed strings are excitations of an empty space, while the open strings end on D-branes and describe excitations of the D-branes. Open strings both of whose end points are attached to a single brane can have arbitrary short length and therefore can have massless modes, while the stings attached to different D-branes have mass proportional to the distance between those D-branes and induce a $U(1)^N$ gauge theory. In the limit when all D-branes become coincident, all of the open strings can become arbitrary short and therefore at the low energy (supergravity) limit dominates, i.e. only the massless modes survive giving raise to the full $U(N)$ gauge symmetry on the boundary (fig. 1.3). Although only a $SU(N)$ gauge field theory is usually considered, because the overall $U(1) = U(N)/SU(N)$ factor corresponds to a global translation of the stack of the D-branes thus can be ignored when considering the local dynamics of the brane.

The massless modes of closed strings give a gravity supermultiplet in ten dimensions and in the low energy limit (energies lower than the string scale $1/l_s$) the effective theory is the type IIB supergravity. The low energy effective theory on the brane is the $\mathcal{N} = 4$ super-Yang-Mills theory in its conformal phase with a gauge group $SU(N)$. The complete action of the configuration can be written in the form

$$S = S_{bulk} + S_{brane} + S_{int} \ , \tag{1.4.1}$$

which in the low energy limit becomes an effective action of massless modes and can be read as

$$S_{eff} = S_{sugra} + S_{SYM}, \tag{1.4.2}$$

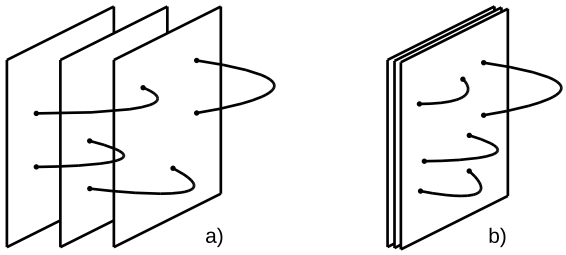

Figure 1.3: a) A stack of N separated $D3$-branes inducing massive $U(1)^N$ gauge theory, b) The stack of N coincident $D3$-branes inducing massless $U(N)$ gauge theory.

because the interaction part of the action $S_{int} \sim g_s (\alpha')^2$, and thus the $\alpha' \to 0$ limit (while keeping g_s fixed) gives $S_{int} \to 0$. Therefore, the supergravity decouples from the brane and in this sense is considered to be free. Note that it is still an interacting theory on its own.

Lets take a closer look to the the $D3$-brane. This supergravity solution is of special interest for several reasons. Its worldbrane has 4-dimensional Poincare invariance, it has a constant axion and dilaton fields, and is self dual and regular at all points. The spacetime metric of such configuration may be written in the following form

$$ds^2 = \left(1 + \frac{L^4}{y^4}\right)^{-\frac{1}{2}} \eta_{ij} dx^i dx^j + \left(1 + \frac{L^4}{y^4}\right)^{\frac{1}{2}} \left(dy^2 + y^2 d\Omega_5^2\right), \tag{1.4.3}$$

where $i, j = 0 \ldots 3$ and the radius L of the $D3$-brane is given by

$$L^4 = 4\pi g_s N \left(\alpha'\right)^2. \tag{1.4.4}$$

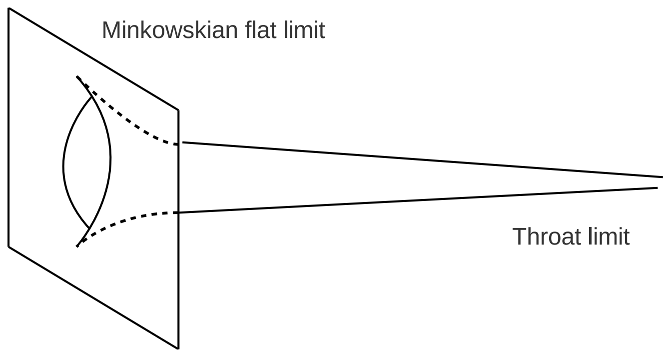

Figure 1.4: Minkowski and throat region of the AdS.

In the limit $y \gg L$ we recover the flat spacetime R^{10}, while $y < L$ corresponds to the geometry that is often referred to as the throat (Fig. 1.4) and would appear to be singular as $y \ll L$. Although, after introducing a coordinate

$$u \equiv \frac{L^2}{y}, \tag{1.4.5}$$

the limit $y \ll L$ reads as large u limit and transforms the metric to the following asymptotic form

$$ds^2 = L^2 \left(\frac{1}{u^2} \eta_{ij} dx^i dx^j + \frac{du^2}{u^2} + d\Omega_5^2 \right), \tag{1.4.6}$$

which is referred to a product geometry of five-sphere S^5 with metric $L^2 d\Omega_5^2$ and the hyperbolic space AdS_5 with a conformal metric $\propto du^2 + \eta_{ij} dx^i dx^j$. It is a space of constant negative curvature and in the general case AdS_d can be defined by a Lobachevsky type embedding in $\mathbb{R}^{d+1}$. Using parametrization $x_0 = R\cosh\rho\cos\tau$, $x_{d+1} = R\cosh\rho\sin\tau$ and $x_i = R\sinh\rho\Omega_i$, where $\sum_i \Omega_i = 1$, the metric of AdS_d space can be written as

$$ds^2 = R^2 \left(-\cosh^2\rho\, d\tau + d\rho^2 + \sinh^2\rho\, d\Omega^2 \right), \tag{1.4.7}$$

with $\rho \geq 0$ and $0 \leq \tau \leq 2\pi$ (fig. 1.5). This metric has a hyperboloid topology of $S^1 \times \mathbb{R}^{p+1}$ with S^1 representing closed timelike curves in the τ direction. By setting $d = 5$ and taking the limit $\tau \to \infty$ we obtain a flat 4-dimensional Minkowski space time on the boundary of AdS_5.

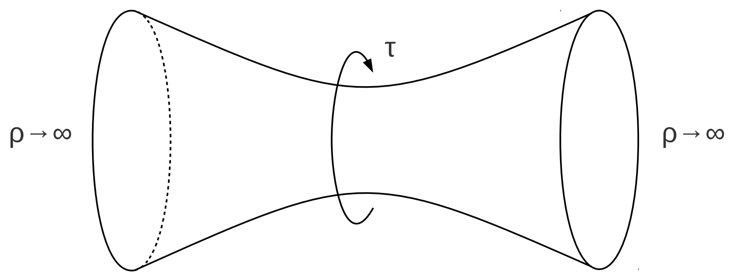

Figure 1.5: Anti de-Sitter spacetime parametrized by ρ and τ.

It is worth noting, that Anti de-Sitter space is a solution of the Einstein equation with a constant negative cosmological constant

$$R_{\mu\nu} - \tfrac{1}{2} g_{\mu\nu} R = 8\pi G \Lambda g_{\mu\nu}. \tag{1.4.8}$$

The Maldacena AdS/CFT duality considers the $\alpha' \to 0$ limit while keeping g_s and N fixed. In such a way only the $AdS_5 \times S^5$ region of the $D3$-brane geometry survives and contributes to the string dynamics of physical processes, while the asymptotically flat region decouples from the theory and does not influence the string dynamics. The conjectures states that the supergravity and super Yang-Mills spectrum coincides. Any field $\phi(\vec{x}, z)$ propagating in the AdS, where z is the distance from the boundary of AdS, is in one-to-one correspondence with some gauge invariant operator $\mathcal{O}(\vec{x})$ in the conformal field theory. The correspondence is realised by a relation between the energy of the field and the scaling dimension of the of the operator in the field theory. The strongest conjecture formulation states, that the generating functional of the correlation functions of the conformal field theory side coincides with the partition function of the superstring theory with the boundary condition stating that the field ϕ has the value ϕ_0 on the boundary of AdS,

$$\left\langle e^{\int d^4x \phi_0(\vec{x})\mathcal{O}(\vec{x})} \right\rangle_{CFT} = Z_{string}\left[\phi(\vec{x},z)|_{z=0} = \phi_0(\vec{x})\right]. \tag{1.4.9}$$

This relation is valid for any field in the theory.

Quantum integrability, the study of exactly-solvable models of quantum physics, has long enjoyed a fertile exchange between physics and mathematics. Most of the quantum groups and associated algebras have their origin in some classical or quantum integrable system of fundamental physics. However, at the early stages of exploration of the AdS/CFT duality, there were no signs of integrability observed. This changed in 2003, when integrability was found to govern certain limits of the duality, including non-local charges [53] in the string sigma model [54], and the spin-chain picture on the gauge side [55–57]. Since then, integrability became an important part of the core structure of the duality and has led to an astonishing range of new results. The complete story of the integrability has been merged into a comprehensive 23-part review, an overview of which is given in [9].

The $\mathcal{N} = 4$ super Yang-Mills is a non-Abelian gauge theory in $3+1$ dimensions with a $G = SU(N)$ gauge group. This theory is unique, it has a vanishing β-function at all values of the coupling constant g_{YM}, and has the largest possible spacetime symmetry. The bosonic part of the global symmetry is $SO(4,2) \times SO(6)$ and coincides exactly with the isometries of the $AdS_5 \times S^5$ background. The first factor is the conformal group of four-dimensional spacetime and includes the Lorentz symmetry as a subgroup, $SO(3,1) \subset SO(4,2)$. The second factor is the so-called R-symmetry, $SO(6) \simeq SU(4)$. These symmetries are enhanced by 32 supercharges thus generating the $PSU(2,2|4)$ supergroup.

The field content of this theory consists of a gauge field $\mathcal{A}_\mu$ which is a singlet $\mathbf{1}$ under the R-symmetry, six massless scalar fields ϕ^I, $I = 1\ldots 6$, transforming in $\mathbf{6}$, four chiral and four anti-chiral fermions, ψ^a_α and $\psi_{\dot{\alpha}a}$, $a = 1\ldots 4$, transforming in $\mathbf{4}$ and $\bar{\mathbf{4}}$ of $SU(4)$; and $\alpha, \dot{\alpha} = 1, 2$ are spinoral indices of two independent $SU(2)$ that constitute the Lorentz algebra.

The natural observables are the correlation functions of local single-trace gauge-invariant operators of ‘words’ composed of all possible ‘letters’, fields ϕ^I, ψ^a_α, $\psi_{\dot{\alpha}a}$ and covariant derivatives $\mathcal{D}_\mu$, all transforming in the adjoint representation of $SU(N)$ and evaluated at a point x. Such operators have a well-defined classical scaling dimension Δ which is simply a sum of the mass dimensions of each individual component.

Such operators can be classified by a sextuplet of charges, $(\Delta, S_1, S_2; J_{12}, J_{34}, J_{56})$ of the global $PSU(2,2|4)$, where Δ is the aforementioned conformal dimension, S_1 and S_2 are the two charges (spins) of the Lorentz group, and J_{ij}'s are three independent R-charges of $SO(6)$. Then, by the AdS/CFT conjecture, operators of scaling dimension Δ are identified with string states of energy $H = \Delta$. More precisely,

- single trace operators are in one-to-one correspondence to single strings, multi-trace operators correspond to multi-string states;
- the charges S_i correspond to angular momenta of strings in AdS_5, the three charges J_i correspond to angular momenta of string in S^5.

Consider an operator composed of fields $Z = \phi^5 + i\phi^6$ only, $\Psi_L = \mathrm{Tr}[Z^L]$, with $L \geq 2$. This operator has a set of charges $(L, 0, 0; 0, 0, L)$ satisfying $\Delta = J_{56}$ and is a chiral primary (BPS) operator. In such a way the classical conformal dimension Δ of Ψ_L is protected against quantum corrections and thus Ψ_L is a good starting point in trying to solve the theory.

Choose L to be very large, $L \to \infty$. Then the operator Ψ_L corresponds to a ligh-cone superstring with an infinite momentum $p_+ \to \infty$ spinning in the maximal $S^2 \subset S^5$ stretched in the 56-plane. The light-cone gauge preserves a subgroup $PSU(2|2)_L \times PSU(2|2)_R \subset PSU(2,2|4)$ of the global group. The same symmetry manifests itself in the gauge theory side of the correspondence. The operator Ψ_L can be identified with a periodic spin chain of length L and vacuum reference state $Z = \downarrow$. In such a way $\Psi_L = |0\rangle$ becomes the BMN vacuum state state of the theory with 'zero energy' $H = \Delta - J_{46} = 0$ [58].

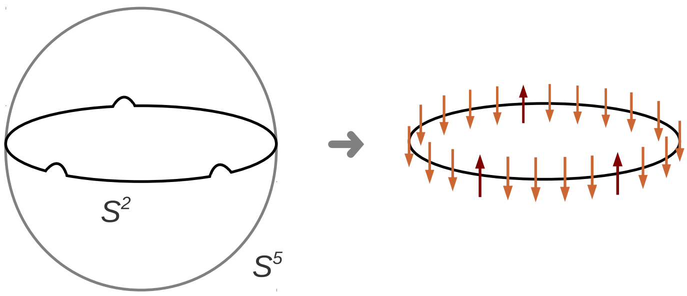

Figure 1.6: Duality between the light-cone superstring stretching a maximal $S^2 \subset S^5$ and an excited periodic spin chain; here $Z = \downarrow$ is the vacuum reference state and $\uparrow$ represents any allowed excitation.

Then by replacing some of Z's in Ψ_L by other super Yang-Mills fields, which in the string side of the duality correspond to worldsheet excitations and in the spin chain language are called magnons, one obtains an excited spin chain configuration with $H = \Delta - J_{56} > 0$ (see figure 1.6) and an underlying centrally extended symmetry algebra,

$$\mathfrak{psu}(2|2)_L \times \mathfrak{psu}(2|2)_R \ltimes \mathbb{R}^3 \,, \tag{1.4.10}$$

where $\mathbb{R}^3 = \{H, C, C^\dagger\}$ and $C \sim g(1 - \mathrm{e}^{ip_{ws}})$ with p_{ws} being the worldsheet momentum of an individual excitation satisfying $\sum p_{ws} = 0$, i.e. the total worldsheet momentum of the string is required to be vanishing [19]. Therefore the theory can be solved by applying the standard Bethe ansatz and R-matrix techniques that we have discussed in the previous sections. The symmetry algebra (1.4.10) implies that the worldsheet S-matrix factorized into two equivalent and independence factors, left and right,

$$S_{ws} = S_L \otimes S_R \,, \tag{1.4.11}$$

each governed by a single copy of $\mathfrak{psu}(2|2) \ltimes \mathbb{R}^3$. In such a way one only needs to consider one sector at a time, which simplifies the model significantly.

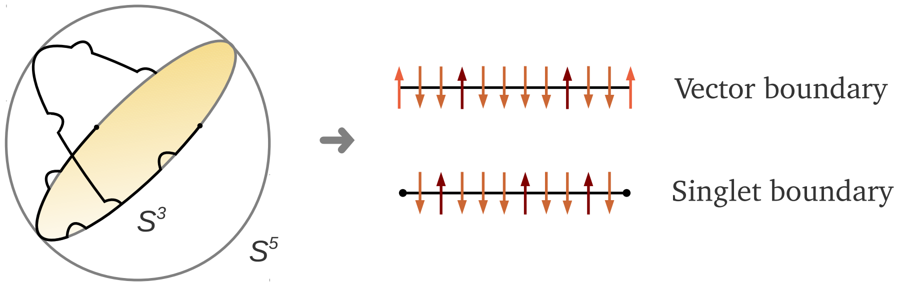

Figure 1.7: Duality between open light-cone superstrings ending on a $D3$-brane wrapping a maximal $S^3 \subset S^5$ and open spin chains. Boundary conditions depend on the relative orientation of the string and the brane leading to two different cases: the vertically oriented open string gives raise to a vector boundary, a boundary with boundary fields, the horizontally oriented string gives raise to a singlet boundary, a boundary without boundary fields.

So far we have only considered closed strings that correspond to periodic spin chains. By introducing D-branes into the type IIB supertring theory one gets a whole family of new configurations that were absent before – open strings ending on the D-branes, the most famous ones being the $D3$-, $D5$- and $D7$-branes [59–65]. Such configurations correspond to spin chains with open boundary conditions (see figure 1.7). Deep in the bulk, i.e. far away from its ends, open strings behave exactly the same as their closed relatives. However, by getting closer to the ends the boundary effects emerge into the theory. These boundary effects are very extensive and depend not only on the type of the D-brane the string is attached to, but also on the type of embedding of the brane inside the $AdS_5 \times S^5$ background and the relative orientation of the string and the brane, thus leading to a vast variety of new phenomena. This requires new mathematical methods to be invented which go far beyond the standard techniques.

1.5 Outline

In this chapter we have given a motivation for this manuscript – integrable structures arising from and inspired by the AdS/CFT duality. In the subsequent chapters we will give an exhaustive description of algebraic methods, quantum algebras, bulk and boundary scattering theories that were developed in the quest for uncovering integrable boundaries in the AdS/CFT.

Each chapter will present an individual topic and, for readers' convenience, will be given in as much as possible self-contained way. However, the topics that will be covered in this manuscript are intimately related to each other and thus each chapter serves as a precursor for the subsequent one.

The second chapter presents the theory of reflection algebras that has been inspired by the algebraic structures and boundary scattering in AdS/CFT. This chapter is written in as much as possible mathematically rigorous way and introduces a framework of reflection algebras that are later used throughout the rest of the manuscript. Here we give a generalization of the axiomatic theory of coideal subalgebras and quantum symmetric pairs for quantum affine algebras. We also present generalized twisted Yangian algebras of two types. We then explicitly construct reflection algebras based on the aforementioned constructions for two most simple Lie algebras, $\mathfrak{sl}(2)$ and $\mathfrak{gl}(1|1)$, for 'singlet' and 'vector' boundary conditions, and show a relation between such quantum affine and Yangian reflection algebras.

In the third chapter we present reflection algebras and boundary scattering theory for integrable boundaries in AdS/CFT. We start by recalling the necessary preliminaries and the underlying symmetries of the light-cone superstring and worldsheet S-matrix. We then proceed by presenting the well-known boundary configurations: $D3$-branes that are also known as the maximal giant gravitons, and the $D3$-$D7$- and $D3$-$D5$-brane systems. We construct generalized twisted Yangian algebras for these boundaries and calculate fundamental and selected bound state reflection matrices.

The fourth chapter deals with a quantum deformed model of the AdS/CFT worldsheet scattering. This approach is also known as the one-dimensional double-deformed Hubbard chain, as the underlying symmetries of both models is the same. Here we construct the bound state representation of the underlying quantum affine algebra of novel type and the corresponding bound state S-matrix for arbitrary bound states. We also construct quantum deformed models of selected boundaries considered in the previous chapter. We show that quantum deformed approach to boundary scattering in AdS/CFT leads to quantum affine reflection algebras that are of a very elegant and symmetric form and fit into the generalized theory of quantum symmetric pairs presented in the second chapter. We calculate fundamental and selected bound state reflection matrices that are quantum analogues of the reflection matrices found in the third chapter.

The final chapter presents the so-called secret symmetry of the worldsheet S-matrix and selected reflection matrices. This is a very distinctive symmetry observed in diverse sectors of AdS/CFT, and thus is one of the most interesting mysteries of this duality. Here we explore the quantum affine origin of this symmetry and build a bridge to its relative of the quantum affine superalgebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$. Nevertheless this does not completely solve the mystery of this symmetry, as there are still quite a few of its properties unknown. Thus we do not say the quest for integrable structures in boundary scattering in AdS/CFT is accomplished, we rather say the quest continues!

Chapter 2

Reflection algebras

Quantum affine algebras and Yangians are the simplest examples of infinite-dimensional quantum groups and play a central role in quantum integrable systems [1, 10, 12]. These algebras were introduced in [1–5] and since then had a significant impact on the development of the quantum inverse scattering method and related quantum integrable models.

The quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$ and the Yangian $\mathcal{Y}(\mathfrak{g})$ are deformations of the universal enveloping algebras $\mathcal{U}(\widehat{\mathfrak{g}})$ and $\mathcal{U}(\mathfrak{g}[u])$ respectively, where $\widehat{\mathfrak{g}}$ is the affine Kac–Moody algebra, a central extension of the Lie algebra of maps $\mathbb{C}^\times \to \mathfrak{g}$ of a finite Lie algebra $\mathfrak{g}$, and $\mathfrak{g}[u]$ is a deformation of the Lie algebra of maps $\mathbb{C} \to \mathfrak{g}$. In such a way, Yangians may be viewed as a specific degenerate limit of the quantum affine algebras [4]. We refer to [37] for complete details on quantum groups.

In this chapter we will concentrate on finite dimensional representations of quantum groups, the so-called *evaluation* representations. These are constructed via the epimorphisms $\mathrm{ev}_a : \mathcal{U}_q(\widehat{\mathfrak{g}}) \to \mathcal{U}_q(\mathfrak{g})$ and $\mathrm{ev}_a : \mathcal{Y}(\mathfrak{g}) \to \mathcal{U}(\mathfrak{g})$ called the evaluation homomorphisms, which evaluate the $\mathfrak{g}$-valued polynomials at a point $a \in \mathbb{C}$. Such representations have wide applications in both mathematics and physics. However they can be constructed for some Lie algebras only.

We will consider integrable models with open boundary conditions, and concentrate on the algebras that define solutions of the reflection equation [15]. These algebras are one-sided coideal subalgebras and are conveniently called reflection algebras. Such algebras have been extensively studied in e.g. [14, 38–43] and more recently in e.g. [66, 67]. For field theoretical applications of the coideal subalgebras and corresponding reflection matrices we refer to [16, 17]. We have selected two simple Lie (super-)algebras, $\mathfrak{sl}(2)$ and $\mathfrak{gl}(1|1)$, and have studied reflection algebras for the corresponding quantum groups: the quantum affine (super-) algebras $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$ and $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$, and the Yangians $\mathcal{Y}(\mathfrak{sl}(2))$ and $\mathcal{Y}(\mathfrak{gl}(1|1))$. We refer to [37] and [68–70] for details on these algebras. For each quantum group we consider *singlet* and *vector* boundary conditions. The singlet boundary forms a singlet (trivial) representation of the boundary (reflection) algebra, thus there are no boundary degrees of freedom in the associated field theory. The vector boundary forms a vector representation of the boundary algebra and has boundary degrees of freedom. Singlet boundaries have been heavily studied and the corresponding boundary algebras and solutions of the reflection equation for most of the semisimple Lie (super-) algebras are well known. However vector boundaries have not been studied as much as the singlet ones. The corresponding reflection matrices are usually constructed via the bootstrap procedure by fusing bulk S-matrices with an appropriate scalar reflection matrix (see e.g. an overview [71]). Thus we hope the reflection algebras we have constructed in this manuscript will serve not only as neat examples but will contribute to the exploration of vector boundaries.

We show that reflection algebras for quantum affine algebras are quantum affine coideal subalgebras that are a generalization of the quantum symmetric pairs of simple Lie algebras considered in [31] and [32], where quantum symmetric pairs and the associated coideal subalgebras for all simple Lie algebras have been classified. Such coideal subalgebras were also considered in [72–74] and recently generalized for Kac-Moody algebras in [75]. Similar quantum affine coideal subalgebra for algebra of type $A_1^{(1)}$ was considered in [76], for $D_n^{(1)}$ in [77], for the double affine Hecke algebras of type $C^\vee C_n$ in [78], for $\mathfrak{o}(n)$ and $\mathfrak{sp}(2n)$ in [79], and for the Sine-Gordon and affine Toda field theories in [46,80,81]. These algebras also play a crucial role in the q-deformed AdS/CFT [25,26]; we will discuss this topic thoroughly in Chapter 4.

For the Yangian case we will consider two types of (generalized) twisted Yangians. For a singlet boundary we will consider the twisted Yangian introduced in [40], while for a vector boundary we will employ the twisted Yangian introduced in [23]. In order to distinguish these two algebras and simplify the notation we name them the twisted Yangian of *type I* and of *type II* respectively. We note that reflection algebras and Yangians for $\mathfrak{gl}(n)$ in various contexts have been extensively studied in [39,42,43], for superalgebras $\mathfrak{sl}(m|n)$ and $\mathfrak{gl}(m|n)$ in [82–85]. We refer to [41] where twisted Yangians of type I and the corresponding reflection matrices for generic classical Lie algebras and for both singlet and vector boundaries were found. See [24] for an 'achiral' extension of such Yangians (and Section 3.5).

We also give some arguments that quantum affine coideal subalgebras in the rational $q \to 1$ limit specialize to the twisted Yangians. This is an important but quite technical question that is worthy of exploration on its own, thus we will be rather concise and heuristic concerning this claim in the present manuscript. A thorough exploration of the Yangian limit of the quantum affine enveloping algebras can be found in [86–89].

This chapter is organized as follows. In Section 2.1 we give the necessary preliminaries. Here we will recall some of the mathematical formalism presented in Sections 1.2 and 1.3 that will be heavily employed in this chapter. We recap the construction of quantum symmetric pairs and coideal subalgebras adhering closely to [31]. We then give a definition of the quantum affine coideal subalgebra, and also definitions of the twisted Yangians introduced in [40] and [23]. In Section 2.2 we construct reflection algebras for the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$. In Section 2.3 we construct twisted Yangians for $\mathcal{Y}(\mathfrak{sl}(2))$. In Sections 2.4 and 2.5 we repeat same derivations for quantum groups $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ and $\mathcal{Y}(\mathfrak{gl}(1|1))$ respectively. Appendix A contains a heuristic Yangian limit $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2)) \to \mathcal{Y}(\mathfrak{sl}(2))$.

2.1 Preliminaries

Quasitriangular Hopf algebras and the Yang-Baxter equation. Let $\mathcal{A}$ be a Hopf algebra over $\mathbb{C}$ equipped with multiplication $\mu : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$, unit $\iota : \mathbb{C} \to \mathcal{A}$, comultiplication $\Delta : A \to A \otimes A$, counit $\epsilon : \mathcal{A} \to \mathbb{C}$ and antipode $S : \mathcal{A} \to \mathcal{A}$. Let $\sigma : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A} \otimes A$ be the $\mathbb{C}$-linear map such that $\sigma(a_1 \otimes a_2) = a_2 \otimes a_1$ for any $a_1,\ a_2 \in \mathcal{A}$. Then $(\mathcal{A}, \mu^{op}, \iota, \Delta^{op}, \epsilon, S^{-1})$, where $\mu^{op} = \mu \circ \sigma$ and $\Delta^{op} = \sigma \circ \Delta$, is called the *opposite* Hopf algebra of $\mathcal{A}$ and denoted $\mathcal{A}^{op}$.

Let $\mathcal{A}$ be a quasitriangular Hopf algebra. Then there exists an invertible element $\mathcal{R} \in \mathcal{A} \otimes \mathcal{A}$ called the *universal* R-matrix such that

$$\Delta^{op}(a) = \mathcal{R}\, \Delta(a) \mathcal{R}^{-1} \quad \text{for any} \quad a \in \mathcal{A}\,, \tag{2.1.1}$$

which satisfies the universal Yang-Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}\,, \tag{2.1.2}$$

where $\mathcal{R}_{12} \in \mathcal{A} \otimes \mathcal{A} \otimes 1$, $\mathcal{R}_{13} \in \mathcal{A} \otimes 1 \otimes \mathcal{A}$ and $\mathcal{R}_{23} \in 1 \otimes \mathcal{A} \otimes \mathcal{A}$.

Let $\mathcal{A}$ be a quantum affine universal enveloping algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$. Let $\mathcal{V}$ be a finite dimensional vector space and $T_z : \mathcal{A} \to End(\mathcal{V})$ be a finite dimensional representation of $\mathcal{A}$, where z denotes the spectral parameter of the representation. Then $(T_z \otimes T_w) : \mathcal{R} \to R(z/w) \in End(\mathcal{V} \otimes \mathcal{V})$ maps the universal R-matrix to a matrix called the trigonometric R-matrix. In such a way (2.1.1) becomes the intertwining equation,

$$(T_z \otimes T_w)[\Delta^{op}(a)]\, R(z/w) = R(z/w)\, (T_z \otimes T_w)[\Delta(a)]\,, \tag{2.1.3}$$

and the Yang-Baxter equation (2.1.2) on the space $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{V}$ becomes

$$R_{12}(z/w)R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}(z/w)\,. \tag{2.1.4}$$

In the case of an irreducible representation $T_z \otimes T_w$ the intertwining equation (2.1.3) defines the R-matrix uniquely up to an overall scalar factor. Furthermore, this R-matrix satisfies (2.1.4) automatically.

Coideal subalgebras and the reflection equation. Consider the reflection equation [15]

$$R_{12}(z/w)K_{13}(z)R_{12}(zw)K_{23}(w) = K_{23}(w)R_{12}(zw)K_{13}(z)R_{12}(z/w) \tag{2.1.5}$$

defined on the tensor space $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{W}$, where $\mathcal{W}$ is the boundary vector space. Here $K_{13}(z)$ and $K_{23}(w)$ are reflection matrices such that $K_{23}(w) = 1 \otimes K$ with $K \in \mathrm{End}(\mathcal{V} \otimes \mathcal{W})$, and a similar relation holds for $K_{13}(z)$.

Let $\mathcal{B} \subset \mathcal{A}$ be a left coideal subalgebra,

$$\Delta(b) \in \mathcal{A} \otimes \mathcal{B} \quad \text{for all} \quad b \in \mathcal{B}\,. \tag{2.1.6}$$

Let $\bar{T}_s : \mathcal{B} \to End(\mathcal{W})$ be a finite dimensional representation of $\mathcal{B}$, called the *boundary* representation; here s denotes the boundary spectral parameter.

Definition 2.1.1. *A coideal subalgebra $\mathcal{B}$ is called a quantum affine reflection algebra if the intertwining equation*

$$(T_{1/z} \otimes \bar{T}_s)[\Delta(b)]\, K(z) = K(z)\, (T_z \otimes \bar{T}_s)[\Delta(b)] \quad \textit{for all} \quad b \in \mathcal{B}\,, \tag{2.1.7}$$

for some representation T_z and $\bar{T}_s$ defines a K-matrix $K(z) \in End(\mathcal{V} \otimes \mathcal{W})$ satisfying the reflection equation (2.1.5).

Let the boundary vector space be one-dimensional, $\mathcal{W} = \mathbb{C}$. Then $\bar{T}_s = \epsilon$ and $K(z) \in End(\mathcal{V})$. Note that $\mathcal{V} \otimes \mathbb{C} \cong \mathcal{V}$ as a vector space. In this case the intertwining equation (2.1.7) becomes

$$(T_{1/z} \otimes \epsilon)[\Delta(b)]\, K(z) = K(z)\, (T_z \otimes \epsilon)[\Delta(b)] \quad \text{for all} \quad b \in \mathcal{B}\,. \tag{2.1.8}$$

For an irreducible representation T_z (resp. $T_z \otimes \bar{T}_s$) of $\mathcal{B}$, the intertwining equation (2.1.8) (resp. (2.1.7)) defines the K-matrix uniquely up to an overall scalar factor. Note that the boundary representation $\bar{T}_s$ may be different from T_z; however, in this paper we will consider the $\bar{T}_s \cong T_z$ case only (i.e. when the representations are isomorphic).

Definition 2.1.2. *Let $\bar{T}_s \cong T_z$ be a non-trivial boundary representation. Then we call* (2.1.7) *the intertwining equation for a vector boundary. We call* (2.1.8) *the intertwining equation for a singlet boundary.*

In the subsequent sections we will consider coideal subalgebras compatible with the reflection equation for the quantum affine enveloping algebras and Yangians.

2.1.1 Coideal subalgebras for quantum deformed algebras

Quantum symmetric pairs and coideal subalgebras. We will begin by introducing the necessary notation and then we will give the definition of the coideal subalgebras of the universal enveloping algebras. We will be adhering closely to [31].

Let $\mathfrak{g}$ be a semisimple Lie algebra of rank n. Let Φ denote the root space of $\mathfrak{g}$, and Φ^+ be the set of the positive roots. Let $\pi = \{\alpha_i\}_{i\in\mathbf{I}}$ be a basis of simple positive roots in Φ^+. Here $\mathbf{I} = \{1, \ldots, n\}$ denotes the set of Dynkin nodes of $\mathfrak{g}$. We will use λ to denote any root in Φ. Let $(\cdot, \cdot)$ denote a non-degenerate Cartan inner product on $\mathbf{h}^*$, the dual of the Cartan subalgebra $\mathbf{h}$ of $\mathfrak{g}$. Then the matrix elements of the Cartan matrix $(a_{ij})_{i,j\in\mathbf{I}}$ are given by $a_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$. There exists a set of coprime positive integers (r_i) such that $(b_{ij}) = (r_i a_{ij})$ is symmetric and is called the symmetrized Cartan matrix.

The triangular decomposition of $\mathfrak{g}$ is given by $\mathbf{n}^- \oplus \mathbf{h} \oplus \mathbf{n}^+$, and the basis for $\mathbf{n}^-$ (resp. $\mathbf{n}^+$) is $\{f_i\}_{i\in\mathbf{I}}$ (resp. $\{e_i\}_{i\in\mathbf{I}}$). Let $h_i = [e_i, f_i]$ for all $i \in \mathbf{I}$. Then $\{e_i, f_i, h_i\}_{i\in\mathbf{I}}$ is a Chevalley basis for $\mathfrak{g}$ satisfying

$$[h_i, h_j] = 0\,, \qquad [e_i, f_j] = \delta_{ij} h_i\,, \qquad [h_i, e_j] = a_{ji} e_j\,, \qquad [h_i, f_j] = -a_{ji} f_j\,, \tag{2.1.9}$$

and the Serre relations

$$(\mathrm{ad}\, e_i)^{1-a_{ji}}\, e_j = 0\,, \qquad (\mathrm{ad}\, f_i)^{1-a_{ji}}\, f_j = 0\,. \tag{2.1.10}$$

Let $\theta : \mathfrak{g} \to \mathfrak{g}$ be a maximally split involutive Lie algebra automorphism (involution) of $\mathfrak{g}$, i.e.

$$\theta(\mathbf{h}) = \mathbf{h}\,, \quad \{\theta(e_i) = e_i,\, \theta(f_i) = f_i \,|\, \theta(h_i) = h_i\}\,, \quad \{\theta(e_i) \in \mathbf{n}^-,\, \theta(f_i) \in \mathbf{n}^+ \,|\, \theta(h_i) \neq h_i\}\,. \tag{2.1.11}$$

It defines a symmetric pair $(\mathfrak{g}, \mathfrak{g}^\theta)$, where $\mathfrak{g}^\theta$ is the θ–fixed subalgebra of $\mathfrak{g}$, and induces an involution Θ of the root space Φ. Let $\pi_\Theta = \{\Theta(\alpha_i) = \alpha_i \,|\, \alpha_i \in \pi\}$ denote the Θ–fixed subset of π. Then, by (2.1.11), $\Theta(-\alpha_j) \in \Phi^+$ for all $\alpha_j \in \pi\backslash\pi_\Theta$.

Let p be a permutation of $\{1, \ldots, n\}$ such that

$$\Theta(\alpha_j) \in -\alpha_{p(j)} - \mathbb{Z}\pi_\Theta \quad \text{for all} \quad \alpha_j \notin \pi_\Theta\,, \tag{2.1.12}$$

and $p(i) = i$ otherwise. Let π^* be a maximal subset of $\pi\backslash\pi_\Theta$ such that $\alpha_j \in \pi^*$ if $p(j) = j$, or only one of the pair α_j, $\alpha_{p(j)}$ is in π^* if $p(j) \neq j$. Then for a given j such that $\alpha_j \in \pi^*$ there exists a sequence $\{\alpha_{j_1}, \ldots, \alpha_{j_r}\}$, where $\alpha_{j_k} \in \pi_\Theta$, and a set of positive integers $\{m_1, \ldots, m_r\}$ such that θ defined by

$$\theta(f_i) = f_i\,, \qquad \theta(e_i) = e_i\,, \qquad \theta(h_i) = h_i \qquad \text{for all} \quad \alpha_i \in \pi_\Theta\,, \tag{2.1.13}$$

and

$$\begin{aligned}
\theta(f_j) &= \big(\mathrm{ad}\, e_{j_1}^{(m_1)} \cdots e_{j_r}^{(m_r)}\big) e_{p(j)}\,, & \theta(f_{p(j)}) &= (-1)^{m_{(j)}}\big(\mathrm{ad}\, e_{j_r}^{(m_r)} \cdots e_{j_1}^{(m_1)}\big) e_j\,,\\
\theta(e_j) &= (-1)^{m_{(j)}}\big(\mathrm{ad}\, f_{j_1}^{(m_1)} \cdots f_{j_r}^{(m_r)}\big) f_{p(j)}\,, & \theta(e_{p(j)}) &= \big(\mathrm{ad}\, f_{j_r}^{(m_r)} \cdots f_{j_1}^{(m_1)}\big) f_j\,,\\
\theta(h_j) &= -m_1 h_{j_1} - \ldots - m_r h_{j_r} - h_{p(j)}\,, & \theta(h_{p(j)}) &= -m_1 h_{j_1} - \ldots + m_r h_{j_r} - h_j\,,
\end{aligned} \tag{2.1.14}$$

for all $\alpha_j \in \pi^*$ is an involution of g (up to a slight adjustment and rescaling of the definition of power (m_j) such that $[\theta(e_j), \theta(f_j)] = \theta(h_j)$ and

$$\begin{aligned}
\big(\mathrm{ad}\, f_{j_1}^{(m_1)} \cdots f_{j_r}^{(m_r)}\big)\big[\big(\mathrm{ad}\, e_{j_r}^{(m_r)} \cdots e_{j_1}^{(m_1)}\big)\, e_j\big] &= e_j\,,\\
\big(\mathrm{ad}\, e_{j_1}^{(m_1)} \cdots e_{j_r}^{(m_r)}\big)\big[\big(\mathrm{ad}\, f_{j_r}^{(m_r)} \cdots f_{j_1}^{(m_1)}\big) f_j\big] &= f_j\,,
\end{aligned} \tag{2.1.15}$$

would hold). Here $(\mathrm{ad}\, a)b = [a, b]$ and $m_{(j)} = m_1 + \ldots + m_r$. Note that the notation used in (2.1.14) corresponds to $\Theta(\alpha_j) = -m_1\alpha_{j_1} - \ldots - m_r\alpha_{j_r} - \alpha_{p(j)}$. The special case $\Theta(\alpha_j) = -\alpha_j$ for all simple roots $\alpha_j \in \pi$ gives Chevalley anti-automorphism $\kappa(f_j) = e_j$, $\kappa(e_j) = f_j$, $\kappa(h_j) = -h_j$.

Let the quantum deformed universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of a semisimple complex Lie algebra $\mathfrak{g}$ of rank n be generated by the elements $\xi_i^\pm$, $k_i^{\pm 1}$ ($k_i = q^{r_i h_i}$, $i \in \mathbf{I}$, and $q \in \mathbb{C}^\times$ is transcendental) that correspond to the standard Chevalley-Serre realization satisfying

$$\begin{aligned}
k_i k_i^{-1} &= k_i^{-1} k_i = 1\,, & k_i k_j &= k_j k_i\,,\\
k_i \xi_j^\pm k_i^{-1} &= q^{\pm b_{ij}} \xi_j^\pm\,, & [\xi_i^+, \xi_j^-] &= \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}\,,
\end{aligned} \tag{2.1.16}$$

and the quantum Serre relations

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1-a_{ij} \\ m \end{bmatrix}_{q_i} (\xi_i^{\pm})^m \xi_j^{\pm} (\xi_i^{\pm})^{1-a_{ij}-m} = 0 \,, \quad \text{for all} \quad i \neq j \,. \tag{2.1.17}$$

The notation used in here is $q_i = q^{r_i}$ and

$$[n]_q = \frac{q^n - q^{-n}}{q - q^{-1}} \,, \qquad [n]_q! = [n]_q [n-1]_q \cdots [1]_q \,, \qquad \begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{[n]_q!}{[n-m]_q! \, [m]_q!} \,. \tag{2.1.18}$$

The algebra $\mathcal{U}_q(\mathfrak{g})$ becomes a Hopf algebra when equipped with the coproduct Δ, antipode S and counit ϵ given by

$$\begin{aligned} &\Delta(k_i) \;= k_i \otimes k_i \,, && S(k_i) \;= k_i^{-1} \,, && \epsilon(k_i) = 1 \,, \\ &\Delta(\xi_i^+) = \xi_i^+ \otimes 1 + k_i \otimes \xi_i^+ \,, && S(\xi_i^+) = -k_i^{-1} \xi_i^+ \,, && \\ &\Delta(\xi_i^-) = \xi_i^- \otimes k_i^{-1} + 1 \otimes \xi_i^- \,, && S(\xi_i^-) = -\xi_i^- k_i \,, && \epsilon(\xi_i^{\pm}) = 0 \,. \end{aligned} \tag{2.1.19}$$

Being a Hopf algebra, $\mathcal{U}_q(\mathfrak{g})$ admits a right adjoint action making $\mathcal{U}_q(\mathfrak{g})$ into a right module. The right adjoint action is defined by

$$\left(\mathrm{ad}_r\, \xi_i^+\right) a = k_i^{-1} a\, \xi_i^+ - k_i^{-1} \xi_i^+ a \,, \quad \left(\mathrm{ad}_r\, \xi_i^-\right) a = a\, \xi_i^- - \xi_i^- k_i\, a\, k_i^{-1} \,, \quad \left(\mathrm{ad}_r\, k_i\right) a = k_i^{-1} a\, k_i \,. \tag{2.1.20}$$

We shall also be using a short-hand notation $\left(\mathrm{ad}_r\, \xi_i^{\pm} \cdots \xi_j^{\pm}\right) a = \left(\mathrm{ad}_r\, \xi_i^+ \cdots \left(\mathrm{ad}_r\, \xi_j^+\right)\right) a$, for any $a \in \mathcal{U}_q(\mathfrak{g})$.

Let $\mathcal{T}$ be an abelian subgroup $\mathcal{T} \subset \mathcal{U}_q(\mathfrak{g})$ generated by $k_i^{\pm}$. Set $Q(\pi)$ to be equal to the integral lattice generated by π, i.e. $Q(\pi) = \sum_{1 \leq i \leq n} \mathbb{Z}\alpha_i$. Then there is an isomorphism τ of abelian groups from $Q(\pi)$ to $\mathcal{T}$ defined by $\tau(\alpha_i) = k_i$, thus for every $\lambda \in \Phi$ there is an image $\tau(\lambda) \in \mathcal{T}$.

Consider the involution θ of $\mathfrak{g}$ defined in (2.1.13) and (2.1.14). It can be lifted to the quantum case in the following sense.

Theorem 2.1.1 (Theorem 7.1 of [31]). *There exists an algebra automorphism $\tilde{\theta}$ of $\mathcal{U}_q(\mathfrak{g})$ such that*

$$\begin{aligned} &\tilde{\theta}(q) = q^{-1} \,, \\ &\tilde{\theta}(\xi_i^{\pm}) = \xi_i^{\pm} \quad \textit{for all} \quad \alpha_i \in \pi_\Theta \,, \\ &\tilde{\theta}(\tau(\lambda)) = \tau(\Theta(-\lambda)) \quad \textit{for all} \quad \tau(\lambda) \in \mathcal{T} \,, \\ &\tilde{\theta}(\xi_j^-) = \left[\left(\mathrm{ad}_r\, \xi_{j_1}^{+\,(m_1)} \cdots \xi_{j_r}^{+\,(m_r)}\right) k_{p(j)}^{-1} \xi_{p(j)}^+\right] \\ &\textit{and} \quad \tilde{\theta}(\xi_{p(j)}^-) = (-1)^{m(j)} \left[\left(\mathrm{ad}_r\, \xi_{j_r}^{+\,(m_r)} \cdots \xi_{j_1}^{+\,(m_1)}\right) k_j^{-1} \xi_j^+\right] \quad \textit{for all} \quad \alpha_j \in \pi^* \,. \end{aligned} \tag{2.1.21}$$

This construction allows us to define a left coideal subalgebra of $\mathcal{U}_q(\mathfrak{g})$ induced by the involution Θ. Let $\mathcal{T}_\Theta = \{\tau(\lambda) \,|\, \Theta(\lambda) = \lambda\}$ be a Θ–fixed subalgebra of $\mathcal{T}$. Let $\mathcal{M}$ be a Hopf subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by $\xi_i^{\pm}$, $k_i^{\pm 1}$ for all $\alpha_i \in \pi_\Theta$. Note that $k_j k_{p(j)}^{-1} \in \mathcal{T}_\Theta$ for all $\alpha_j \in \pi^*$, thus $\mathcal{T}_\mathcal{M} \subseteq \mathcal{T}_\Theta$ where $\mathcal{T}_\mathcal{M} = \{k_i^{\pm 1}\}$ is the Cartan subgroup of $\mathcal{M}$. Furthermore, $\tilde{\theta}^2 = id$ when restricted to $\mathcal{M}$ and to $\mathcal{T}$. Finally, in the $q \to 1$ limit $\tilde{\theta}$ specializes to θ.

Theorem 2.1.2 (Theorem 7.2 of [31]). *The subalgebra $\mathcal{B} \subset \mathcal{U}_q(\mathfrak{g})$ generated by $\mathcal{M}$, $\mathcal{T}_\Theta$ and the elements*

$$B_j^- = \xi_j^- k_j - d_j\, \tilde{\theta}(\xi_j^-)k_j \quad \textit{for all} \quad \alpha_j \in \pi\backslash\pi_\Theta\,, \tag{2.1.22}$$

and suitable $d_j \in \mathbb{C}^\times$ is a left coideal subalgebra of $\mathcal{U}_q(\mathfrak{g})$.

Let U^+ (resp. U^-) be the subalgebra of $\mathcal{U}_q(\mathfrak{g})$ generated by ξ_i^+ (resp. $k_i\xi_i^-$) for all $\alpha_i \in \pi_\Theta$. Set $\mathcal{M}^\pm = U^\pm \cap \mathcal{M}$. By the definition, the elements $\tilde{\theta}(\xi_j^-)k_j$ are such that (see Section 6 and the proof of the Theorem 7.2 of [31]),

$$\Delta(\tilde{\theta}(\xi_j^-)k_j) \in k_j \otimes \tilde{\theta}(\xi_j^-)k_j + \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{M}^+\mathcal{T}_\Theta \subset \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{B}\,. \tag{2.1.23}$$

Hence the coproducts of B_j^- are of the following form,

$$\Delta(B_j^-) \in k_j \otimes B_j^- + \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{M}^+\mathcal{T}_\Theta \subset \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{B}\,. \tag{2.1.24}$$

Corollary 2.1.1. The subalgebra $\mathcal{D} \subset \mathcal{B}$ generated by $\mathcal{M}$, $\mathcal{T}_\Theta$ and the elements B_j^- for any but not all $\alpha_j \in \pi\backslash\pi_\Theta$ is a left coideal subalgebra of $\mathcal{U}_q(\mathfrak{g})$.

The pair ($\mathcal{U}_q(\mathfrak{g})$, $\mathcal{B}$) is called the quantum symmetric pair and is the quantum analog of the pair of enveloping algebras ($\mathcal{U}(\mathfrak{g}), \mathcal{U}(\mathfrak{g}^\theta)$). For more details consult Section 7 of [31]; for explicit $\mathcal{B}$'s and corresponding suitable choices of d_j's for various simple Lie algebras see [32]. Note that the action of $\tilde{\theta}$ on ξ_j^+ is not explicitly defined by Theorem 2.1.1, but is constrained by requiring $\tilde{\theta}$ to be an automorphism of $\mathcal{U}_q(\mathfrak{g})$.

In some cases it is more convenient to work with an equivalent coideal subalgebra $\mathcal{B}'$ which is obtained by interchanging all ξ_i^- and $k_i^{-1}\xi_i^+$, $\{i \in \mathbf{I}\}$. Let us show this explicitly. Consider a $\mathbb{C}$-linear algebra anti-automorphism κ_B of $\mathcal{U}_q(\mathfrak{g})$ given by

$$\kappa_B(\xi_i^-) = c_B^{-1}k_i^{-1}\xi_i^+\,, \quad \kappa_B(\xi_i^+) = c_B\,\xi_i^- k_i\,, \quad \kappa_B(\tau(\lambda)) = \tau(\lambda)\,. \tag{2.1.25}$$

Then there exists $c_B \in \mathbb{C}^\times$ such that $\kappa_B(\mathcal{B}) = \mathcal{B}$ holds. This is easy to check. Firstly,

$$\kappa_B((\mathrm{ad}_r\,\xi_i^+)b) = -c_B(\mathrm{ad}_r\,\xi_i^-)\kappa_B(b)\,, \qquad \kappa_B((\mathrm{ad}_r\,\xi_i^-)b) = -c_B^{-1}(\mathrm{ad}_r\,\xi_i^+)\kappa_B(b)\,. \tag{2.1.26}$$

Recall that

$$\big(\mathrm{ad}_r\,\xi_{j_1}^{-\,(m_1)}\cdots\xi_{j_r}^{-\,(m_r)}\big)\big(\mathrm{ad}_r\,\xi_{j_r}^{+\,(m_r)}\cdots\xi_{j_1}^{+\,(m_1)}\big)k_j^{-1}\xi_j^+ = k_j^{-1}\xi_j^+\,. \tag{2.1.27}$$

This gives

$$\begin{aligned}\kappa_B(B_j^-) &= c_B^{-1}\Big(q^{a_{jj}}k_j^{-1}\xi_j^+k_j - d_j'\big[\big(\mathrm{ad}_r\,\xi_{j_1}^{-\,(m_1)}\cdots\xi_{j_r}^{-\,(m_r)}\big)\xi_{p(j)}^-\big]k_j\Big)\\ &= c_B^{-1}q^{a_{jj}}d_j'(-1)^{m_{(j)}}\big[\big(\mathrm{ad}_r\,\xi_{j_1}^{-\,(m_1)}\cdots\xi_{j_r}^{-\,(m_r)}\big)B_{p(j)}^-\big]\,k_{p(j)}^{-1}k_j\,,\end{aligned} \tag{2.1.28}$$

where $d_j' = d_j c_B^{m_{(j)}+2}q^{\sum_k m_k a_{jj_k} - a_{jj}}$, and we have required $d_j' = q^{a_{jj}}d_{p(j)}^{-1}$. Thus $\kappa_B(B_j^-) \in \mathcal{B}$, and in a similar way one could show that $\kappa_B(B_{p(j)}^-) \in \mathcal{B}$. Finally, the property is manifest for $\mathcal{M}$ and of $\mathcal{T}_\Theta$. This implies that one can replace all generators B_j^- (2.1.22) by an equivalent set of generators B_j^+ that are obtained by interchanging all ξ_i^- and $k_i^{-1}\xi_i^+$ ($i \in \mathbf{I}$) in (2.1.21) and (2.1.22) giving [72]

$$B_j^+ = \xi_j'^+ k_j - d_j\, \tilde{\theta}'(\xi'^+) k_j \quad \text{for all} \quad \alpha_j \in \pi\backslash\pi_\Theta\,, \tag{2.1.29}$$

and suitable $d_j \in \mathbb{C}^\times$; here $\xi_j'^+ = k_j^{-1}\xi_j^+$ and $\tilde{\theta}'$ is defined by

$$\begin{aligned} \tilde{\theta}'(\xi_j'^+) &= (-1)^{m_{(j)}}\big[\big(\mathrm{ad}_r\, \xi_{j_1}^{-\,(m_1)} \cdots \xi_{j_r}^{-\,(m_r)}\big)\xi_{p(j)}^-\big]\,, \\ \tilde{\theta}'(\xi_{p(j)}'^+) &= \big[\big(\mathrm{ad}_r\, \xi_{j_r}^{-\,(m_r)} \cdots \xi_{j_1}^{-\,(m_1)}\big)\xi_j^-\big]\,. \end{aligned} \tag{2.1.30}$$

The coproducts of B_j^+ have the following form

$$\Delta(B_j^+) \in k_j \otimes B_j^+ + \mathcal{U}_q(\mathfrak{g}) \otimes \mathcal{M}^- \mathcal{T}_\Theta\,. \tag{2.1.31}$$

This leads to the following corollaries:

Corollary 2.1.2. There exists an algebra automorphism $\tilde{\theta}'$ of $\mathcal{U}_q(\mathfrak{g})$ such that

$$\begin{aligned} &\tilde{\theta}'(q) = q^{-1}\,, \\ &\tilde{\theta}'(\xi_i^\pm) = \xi_i^\pm \quad \text{for all} \quad \alpha_i \in \pi_\Theta\,, \\ &\tilde{\theta}'(\tau(\lambda)) = \tau(\Theta(-\lambda)) \quad \text{for all} \quad \tau(\lambda) \in \mathcal{T}\,, \end{aligned} \tag{2.1.32}$$

and (2.1.30) holds for all $\alpha_j \in \pi^*$. It is an involution $\tilde{\theta}'^2 = id$ when restricted to $\mathcal{M}$ and to $\mathcal{T}$. In the $q \to 1$ limit $\tilde{\theta}'$ specializes to θ.

Corollary 2.1.3. The subalgebra $\mathcal{B}' \subset \mathcal{U}_q(\mathfrak{g})$ generated by $\mathcal{M}$, $\mathcal{T}_\Theta$ and the elements

$$B_j^+ = \xi_j'^+ k_j - d_j'\, \tilde{\theta}(\xi_j'^+) k_j \quad \text{for all} \quad \alpha_j \in \pi\backslash\pi_\Theta\,, \tag{2.1.33}$$

and suitable $d_j' \in \mathbb{C}^\times$ is a left coideal subalgebra of $\mathcal{U}_q(\mathfrak{g})$.

Corollary 2.1.4. The subalgebra $\mathcal{D}' \subset \mathcal{B}'$ generated by $\mathcal{M}$, $\mathcal{T}_\Theta$ and the elements B_j^+ for any but not all $\alpha_j \in \pi\backslash\pi_\Theta$ is a left coideal subalgebra of $\mathcal{U}_q(\mathfrak{g})$.

Note that in this case the action of $\tilde{\theta}'$ on ξ_j^- is not explicitly defined, but is constrained requiring $\tilde{\theta}'$ to be an automorphism of $\mathcal{U}_q(\mathfrak{g})$.

Quantum affine coideal subalgebras. We will further be interested in two particular extensions of the coideal subalgebras defined above. We will construct coideal subalgebras of the quantum affine algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$ that are associated with the singlet and vector boundaries.

Let $\hat{\mathfrak{g}}$ be the (untwisted) affine Kac–Moody algebra. Let $(\widehat{a}_{ij})_{i,j\in\hat{\mathbf{I}}}$ denote the extended Cartan matrix and $(\widehat{b}_{ij}) = (\widehat{r}_i\, \widehat{a}_{ij})$ be the symmetrized extended Cartan matrix. Here $\hat{\mathbf{I}} = \{0, 1, \ldots, n\}$ denotes the set of Dynkin nodes of $\hat{\mathfrak{g}}$. The set of the simple positive roots is given by $\widehat{\pi} = \alpha_0 \cup \pi$, where α_0 is the affine root. Recall that $\hat{\mathfrak{g}}$ is an one–dimensional central extension of the Lie algebra $\mathcal{L}(\mathfrak{g}) = \mathfrak{g}[z, z^{-1}]$ of Laurent polynomial maps $\mathbb{C}^\times \to \mathfrak{g}$ under point–wise operations, i.e. there exists a well–defined Lie bracket. The triangular decomposition is given by $\hat{\mathfrak{g}} = \hat{\mathbf{n}}^+ \oplus \hat{\mathbf{h}} \oplus \hat{\mathbf{n}}^-$, where

$$\hat{\mathbf{n}}^\pm = z^{\pm 1}\,\mathbb{C}[z^{\pm 1}] \otimes (\mathbf{n}^\mp \oplus \mathbf{h}) \oplus \mathbb{C}[z^{\pm 1}] \otimes \mathbf{n}^\pm\,, \qquad \hat{\mathbf{h}} = (1 \otimes \mathbf{h}) \oplus \mathbb{C}K \oplus \mathbb{C}D\,, \tag{2.1.34}$$

Here K is the central element and D is the derivation of the algebra. The Chevalley generators are given by

$$\begin{aligned} E_i^+ &= 1 \otimes e_i\,, & E_0^+ &= z \otimes e_0 \in z \otimes \mathbf{n}^- \subset \hat{\mathbf{n}}^+\,, \\ E_i^- &= 1 \otimes f_i\,, & E_0^- &= z^{-1} \otimes f_0 \in z^{-1} \otimes \mathbf{n}^+ \subset \hat{\mathbf{n}}^-\,, \\ H_i &= 1 \otimes h_i\,, & H_0 &= [E_0^+, E_0^-] \in [e_0, f_0] + \mathbb{C}K \subset \hat{\mathbf{h}}\,, \end{aligned} \tag{2.1.35}$$

where $e_0 \in \mathfrak{g}_{-\vartheta}$, $f_0 \in \mathfrak{g}_\vartheta$ are such that $\vartheta \in \Phi^+$ is the highest root of $\mathfrak{g}$.

The elements $E_i^\pm$, H_i ($i \in \hat{\mathbf{I}}$) generate a subalgebra $\tilde{\mathfrak{g}} \subset \hat{\mathfrak{g}}$ such that $\hat{\mathfrak{g}} = \tilde{\mathfrak{g}} \oplus \mathbb{C}D$ is a semi–direct product Lie algebra. The derivation $D = z\,d/dz$ of $\mathbb{C}[z, z^{-1}]$ acts on $\tilde{\mathfrak{g}}$ by

$$[D, E_0^\pm] = \pm E_0^\pm \qquad \text{and} \qquad [D, H_0] = [D, H_i] = [D, E_i^\pm] = 0 \quad \text{for all} \quad i \in \mathbf{I}\,. \tag{2.1.36}$$

Set $\delta \in \hat{\mathbf{h}}^*$ such that $\delta(D) = 1$ and $\delta(\mathbf{h} \oplus \mathbb{C}K) = 0$. Then the affine root is given by $\alpha_0 = \delta - \vartheta$.

Consider an involution θ of $\tilde{\mathfrak{g}}$ such that the associated root space involution Θ is given by

$$\Theta(\alpha_0) \in -\alpha_{p(0)} - \mathbb{Z}(\pi \backslash \alpha_{p(0)}) \qquad \text{and} \qquad \Theta(\alpha_i) = \alpha_i \;\text{ for all }\; \alpha_i \in \pi \backslash \alpha_{p(0)}\,, \tag{2.1.37}$$

and satisfying the following constraint,

$$\alpha_0 - \Theta(\alpha_0) = k\delta\,, \qquad \text{where} \qquad \begin{cases} k = 1 \text{ for } p(0) \neq 0\,, \\ k = 2 \text{ for } p(0) = 0\,, \end{cases} \tag{2.1.38}$$

here $p(0) \in \{0, \ldots, n\}$, and $\pi \backslash \alpha_{p(0)} = \pi$ if $p(0) = 0$. Define $\theta(D) = -D$. Then, for the $p(0) = 0$ case, the relations

$$[\theta(D), \theta(E_0^\pm)] = \theta([D, E_0^\pm]) \qquad \text{and} \qquad [\theta(D), \theta(E_i^\pm)] = \theta([D, E_i^\pm]) \quad \text{for all} \quad i \in \mathbf{I} \tag{2.1.39}$$

are satisfied, and thus the involution θ can be naturally lifted to an involution of $\hat{\mathfrak{g}}$. Otherwise, if $p(0) \neq 0$, relations (2.1.39) do not hold and such lift is not possible. Nevertheless, $\theta^2 = id$ on $\hat{\mathfrak{g}}$ for both cases.

Let $\mathcal{U}_q(\widehat{\mathfrak{g}})$ be the universal enveloping algebra of $\hat{\mathfrak{g}}$. The algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$ in the standard Drinfeld-Jimbo realization is generated by the elements $\xi_i^\pm$, $k_i^\pm$ ($i \in \hat{\mathbf{I}}$) satisfying (2.1.16) and (2.1.17) with a_{ij} (resp. b_{ij}) replaced by $\widehat{a}_{ij}$ (resp. $\widehat{B}_{ij}$). The subalgebra of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ generated by $\xi_i^\pm$, $k_i^{\pm 1}$ ($i \in \mathbf{I}$) is a Hopf subalgebra and is isomorphic as a Hopf algebra to $\mathcal{U}_q(\mathfrak{g})$.

In this way, the modules of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ restrict to the modules of $\mathcal{U}_q(\mathfrak{g})$ [90]. The involution θ defines a Hopf subalgebra $\mathcal{M} \subset \mathcal{U}_q(\widehat{\mathfrak{g}})$ generated by $\xi_i^\pm$, $k_i^{\pm 1}$ for all $\alpha_i \in \pi \backslash \alpha_{p(0)}$ and an abelian subgroup $\mathcal{T}_\Theta$ in the sense as described above. Furthermore, the involution θ induces an automorphism of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ in the following way.

Conjecture 2.1.1 (Theorem 2.1.1 for the quantum affine algebras)**.** *Let a root space involution Θ be defined as in* (2.1.37)*. Then there exists a sequence $\{\alpha_{0_1}, \ldots, \alpha_{0_r}\}$, where $\alpha_{0_k} \in \pi\backslash\alpha_{p(0)}$, and a set of positive integers $\{m_1, \ldots, m_r\}$ such that the algebra map $\tilde{\theta}$ defined by*

$$\begin{aligned}
&\tilde{\theta}(q) = q^{-1}\,, \\
&\tilde{\theta}(\xi_i^\pm) = \xi_i^\pm \quad \textit{for all} \quad \alpha_i \in \pi\backslash\alpha_{p(0)}\,, \\
&\tilde{\theta}(\tau(\lambda)) = \tau(\Theta(-\lambda)) \quad \textit{for all} \quad \tau(\lambda) \in \mathcal{T}\,, \\
&\tilde{\theta}(\xi_0^-) = \big[\big(\mathrm{ad}_r\, \xi_{0_1}^{+\,(m_1)} \cdots \xi_{0_r}^{+\,(m_r)}\big) k_{p(0)}^{-1} \xi_{p(0)}^+\big] \\
&\textit{and} \quad \tilde{\theta}(\xi_{p(0)}^-) = (-1)^{m_{(0)}} \big[\big(\mathrm{ad}_r\, \xi_{0_r}^{+\,(m_r)} \cdots \xi_{0_1}^{+\,(m_1)}\big) k_0^{-1} \xi_0^+\big]\,,
\end{aligned} \tag{2.1.40}$$

can be extended to an automorphism of $\mathcal{U}_q(\widehat{\mathfrak{g}})$. Furthermore, it is an involution $\tilde{\theta}^2 = id$ when restricted to $\mathcal{M}$ and to $\mathcal{T}$. In the $q \to 1$ limit $\tilde{\theta}$ specializes to θ.

Note that for $p(0) = 0$ case the last two lines of (2.1.40) are equivalent. The proof of this conjecture would be a lift of the proof of the Theorem 7.1 of [31]. This is because the sequence $\{m_1, \ldots, m_r\}$ does not include the affine root, which makes the whole construction very similar to the non-affine case. However here we will not attempt to give a proof as it goes beyond of the scope of the present work. We will concentrate on the quantum affine coideal subalgebras $\mathcal{B} \subset \mathcal{U}_q(\widehat{\mathfrak{g}})$ compatible with the reflection equation. Set $\hat{\pi}^* = \{\alpha_0, \alpha_{p(0)}\}$ if $p(0) \neq 0$, and $\hat{\pi}^* = \{\alpha_0\}$ otherwise. Then:

Theorem 2.1.3. *The algebra $\mathcal{B}$ generated by $\mathcal{M}$, $\mathcal{T}_\Theta$, and the elements*

$$B_j^- = \xi_j^- k_j - d_j \tilde{\theta}(\xi_j^-) k_j \qquad \textit{for} \qquad \alpha_j \in \hat{\pi}^*\,, \tag{2.1.41}$$

and suitable $d_j \in \mathbb{C}^\times$ is a quantum affine coideal subalgebra of $\mathcal{U}_q(\widehat{\mathfrak{g}})$.

Proof. The proof of this theorem is a direct lift of the proof of the Theorem 2.1.2 (Theorem 7.2 of [31]). We need to check that

$$\Delta(b) \in \mathcal{U}_q(\widehat{\mathfrak{g}}) \otimes \mathcal{B} \qquad \text{for all} \qquad b \in \mathcal{B}\,. \tag{2.1.42}$$

This property is manifest for all generators of $\mathcal{M}$ and of $\mathcal{T}_\Theta$. Next we need to show that it also holds for B_j^-. By the definition $\tilde{\theta}(\xi_j^-)$ is such that

$$\Delta(\tilde{\theta}(\xi_j^-) k_j) \in k_j \otimes \tilde{\theta}(\xi_j^-) k_j + \mathcal{U}_q(\widehat{\mathfrak{g}}) \otimes \mathcal{M}^+ \mathcal{T}_\Theta\,. \tag{2.1.43}$$

Hence

$$\Delta(B_j^-) \in k_j \otimes B_j^- + \mathcal{U}_q(\widehat{\mathfrak{g}}) \otimes \mathcal{M}^+ \mathcal{T}_\Theta \subset \mathcal{U}_q(\widehat{\mathfrak{g}}) \otimes \mathcal{B}\,, \tag{2.1.44}$$

and the theorem follows. □

Analogously to the non-affine case, we could instead introduce an equivalent coideal subalgebra $\mathcal{B}'$ which is obtained by interchanging all ξ_i^- and $k_i^{-1}\xi_i^+$, $\{i \in \hat{\mathbf{I}}\}$. This leads to the following corollaries:

Corollary 2.1.5. There exists an algebra automorphism $\tilde{\theta}'$ of $\mathcal{U}_q(\widehat{\mathfrak{g}})$ such that

$$\begin{aligned}
&\tilde{\theta}'(q) = q^{-1}\,, \\
&\tilde{\theta}'(\xi_i^{\pm}) = \xi_i^{\pm} \quad \text{for all} \quad \alpha_i \in \pi\backslash\alpha_{p(0)}\,, \\
&\tilde{\theta}'(\tau(\lambda)) = \tau(\Theta(-\lambda)) \quad \text{for all} \quad \tau(\lambda) \in \mathcal{T}\,, \\
&\tilde{\theta}'(\xi_0'^{+}) = (-1)^{m(0)}\big[\big(\mathrm{ad}_r\, \xi_{0_1}^{-\,(m_1)} \cdots \xi_{0_r}^{-\,(m_r)}\big)\xi_{p(0)}^-\big] \\
&\text{and} \quad \tilde{\theta}'(\xi_{p(0)}'^{+}) = \big[\big(\mathrm{ad}_r\, \xi_{0_r}^{-\,(m_r)} \cdots \xi_{0_1}^{-\,(m_1)}\big)\xi_0^-\big]\,,
\end{aligned} \tag{2.1.45}$$

Furthermore, it is an involution $\tilde{\theta}'^2 = id$ when restricted to $\mathcal{M}$ and to $\mathcal{T}$. In the $q \to 1$ limit $\tilde{\theta}'$ specializes to the involution θ.

Corollary 2.1.6. The subalgebra $\mathcal{B}' \subset \mathcal{U}_q(\widehat{\mathfrak{g}})$ generated by $\mathcal{M}$, $\mathcal{T}_\Theta$ and the elements

$$B_j^+ = \xi_j'^{+} k_j - d_j'\,\tilde{\theta}(\xi_j'^{+})k_j \qquad \text{for} \qquad \alpha_j \in \hat{\pi}^*\,, \tag{2.1.46}$$

and suitable $d_j' \in \mathbb{C}^\times$ is a left coideal subalgebra of $\mathcal{U}_q(\widehat{\mathfrak{g}})$.

Conjecture 2.1.2. *The coideal subalgebra $\mathcal{B}$ defined above with suitable $d_j \in \mathbb{C}^\times$ is a quantum affine reflection algebra.*

Corollary 2.1.7. The coideal subalgebra $\mathcal{B}'$ defined above with suitable $d_j' \in \mathbb{C}^\times$ is a quantum affine reflection algebra, and is isomorphic to $\mathcal{B}$.

In the cases when it is obvious from the context we will further refer to a quantum affine reflection algebra simply as a reflection algebra. The parameters d_j (resp. d_j') are constrained by solving the intertwining equation (2.1.7) for all generators of $\mathcal{B}$ (resp. $\mathcal{B}'$). In certain cases, in particular for $p(0) \neq 0$, the requirement for d_j (or equivalently for d_j') to be non-zero is too restrictive. Thus in such cases it can be more convenient to deal with a coideal subalgebra defined in the following way.

Conjecture 2.1.3. *The subalgebra $\mathcal{B} \subset \mathcal{U}_q(\widehat{\mathfrak{g}})$ generated by $\mathcal{M}$, $\mathcal{T}_\Theta$ and the elements*

$$B_0^- = \xi_0^- k_0 - d_-\tilde{\theta}(\xi_0^-)k_0 \qquad \textit{and} \qquad B_0^+ = \xi_0'^{+} k_0 - d_+\tilde{\theta}'(\xi_0'^{+})k_0 \tag{2.1.47}$$

with suitable $d_\pm \in \mathbb{C}$ is a quantum affine reflection algebra.

Definition 2.1.3. *We call $B_0^\pm$ the twisted affine generators.*

We will give explicit examples supporting the claims above for both singlet and vector boundaries in the following sections. In Section 2 we will construct coideal subalgebras for $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$, and in Section 4 for $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$.

Note that for the $p(0)=0$ case inclusion of both B_0^+ and B_0^- could potentially lead to an unwanted growth of $\mathcal{B}$. This can be avoided by a suitable choice of $d_\pm$. Let us show this explicitly. Consider the following element,

$$\begin{aligned} &- d_-\big[\big(\mathrm{ad}_r\, \xi_{0_1}^{+\,(m_1)} \cdots \xi_{0_r}^{+\,(m_r)}\big) B_0^+ k_\vartheta\big] \\ &\qquad = \big[- d_-\tilde{\theta}(\xi_0^-) + d_- d_+ \big(\mathrm{ad}_r\, \xi_{0_1}^{+\,(m_1)} \cdots \xi_{0_r}^{+\,(m_r)} \xi_{0_r}^{-\,(m_r)} \cdots \xi_{0_1}^{-\,(m_1)}\big) \xi_0^-\big] k_0 k_\vartheta = B_0^- k_\vartheta \,, \end{aligned} \tag{2.1.48}$$

where $k_\vartheta = k_1 \cdots k_n \in \mathcal{T}_\Theta$. The last equality holds for a suitable choice of $d_\pm$. Finally note that one could equivalently choose $\xi_j'^+ = \xi_j^+ k_j^{-1}$ in (2.1.47). This would introduce a factor of $q^{\pm \widehat{a}_{jj}}$ for $d_\pm$.

2.1.2 Coideal subalgebras for Yangian algebras

Yangian. The Yangian $\mathcal{Y}(\mathfrak{g})$ of a Lie algebra $\mathfrak{g}$ is a deformation of the universal enveloping algebra of the polynomial algebra $\mathfrak{g}[u]$. It is generated by the level-zero $\mathfrak{g}$ generators j^a and the level-one Yangian generators $\widehat{j}^a$. Their commutators have the generic form

$$[j^a, j^b] = f^{ab}{}_c\, j^c, \qquad [j^a, \widehat{j}^b] = f^{ab}{}_c\, \widehat{j}^c, \tag{2.1.49}$$

and are required to obey Jacobi and Serre relations

$$\big[j^{[a}, \big[j^b, j^{c]}\big]\big] = 0\,, \qquad \big[\widehat{j}^{[a}, \big[\widehat{j}^b, j^{c]}\big]\big] = \alpha^2\, a^{abc}{}_{def}\, j^{\{d} j^e j^{f\}}\,, \tag{2.1.50}$$

where ${}^{[a\,b\,c]}$ denotes cyclic permutations, ${}^{\{d\,e\,f\}}$ is the total symmetrization, and $a^{abc}{}_{def} = \frac{1}{24} f^{ag}{}_d\, f^{bh}{}_e\, f^{ck}{}_f\, f_{ghk}$. For $\mathfrak{g} = \mathfrak{sl}(2)$ the second equation in (2.1.50) is trivial and

$$\big[\big[\widehat{j}^a, \widehat{j}^b\big], \big[j^l, \widehat{j}^m\big]\big] + \big[\big[\widehat{j}^l, \widehat{j}^m\big], \big[j^a, \widehat{j}^b\big]\big] = \alpha^2 (a^{abc}{}_{deg} f^{lm}{}_c + a^{lmc}{}_{deg} f^{ab}{}_c)\, \widehat{j}^{\{d} j^e j^{g\}} \tag{2.1.51}$$

needs to be used instead [3,4]. The indices of the structure constants $f^{ab}{}_d$ are lowered by the means of the inverse Killing–Cartan form g_{bd}. Here α is a formal level-one deformation parameter which is used to count the formal level of the algebra elements. In such a way the left and right hand sides of the expressions in (2.1.50) and (2.1.51) are of the same level.

The Hopf algebra structure is then equipped with the following coproduct Δ, antipode S and counit ϵ,

$$\begin{aligned} &\Delta(j^a) = j^a \otimes 1 + 1 \otimes j^a, && S(j^a) = -j^a\,, && \epsilon(j^a) = 0\,, \\ &\Delta(\widehat{j}^a) = \widehat{j}^a \otimes 1 + 1 \otimes \widehat{j}^a + \frac{\alpha}{2} f^a_{bc} j^b \otimes j^c\,, && S(\widehat{j}^a) = -\widehat{j}^a + \frac{c_\mathfrak{g}}{4} j^a\,, && \epsilon(\widehat{j}^a) = 0\,, \end{aligned} \tag{2.1.52}$$

where $c_\mathfrak{g}$ is the eigenvalue of the quadratic Casimir operator in the adjoint representation ($f_a{}^{bc} f_{cbd} = c_\mathfrak{g}\, g_{ad}$) and is required to be non-vanishing.

The finite-dimensional representations of $\mathcal{Y}(\mathfrak{g})$ are realized in one-parameter families, due to the "evaluation automorphism"

$$\tau_u : \mathcal{Y}(\mathfrak{g}) \to \mathcal{Y}(\mathfrak{g}) \qquad j^a \mapsto j^a\,, \qquad \widehat{j}^a \mapsto \widehat{j}^a + u j^a\,, \tag{2.1.53}$$

corresponding to a shift in the polynomial variable. On (the limited set of) finite-dimensional irreducible representations of $\mathfrak{g}$ which may be extended to representations of $\mathcal{Y}(\mathfrak{g})$, these families are explicitly realized via the "evaluation map"

$$\mathrm{ev}_u : \mathcal{Y}(\mathfrak{g}) \to \mathrm{U}(\mathfrak{g}) \qquad j^a \mapsto j^a\,, \qquad \widehat{j}^a \mapsto u j^a\,, \tag{2.1.54}$$

which yields "evaluation modules" and u is the spectral parameter.

The level-two Yangian generators may be obtained by commuting level-one generators as

$$\widehat{\widehat{j}}^a = \frac{1}{c_{\mathfrak{g}}} f^a{}_{bc}\,[\widehat{j}^c, \widehat{j}^b]\,, \qquad \text{and} \qquad [\widehat{j}^a, \widehat{j}^b] = f^{ab}{}_c \widehat{\widehat{j}}^c + X^{ab}, \tag{2.1.55}$$

where the non-zero extra term X^{ab} is constrained by the Serre relations (2.1.50) to satisfy $f^{[ab}{}_d X^{c]d} = Y^{abc}$, here Y^{abc} is the right hand side of the second equation in (2.1.50) (and thus a fixed cubic combination of level-zero generators), and by (2.1.55) to satisfy $f^a{}_{bc} X^{bc} = 0$ [17].

Let $\mathcal{V}$ be a finite dimensional vector space and $T_u : \mathcal{Y}(\mathfrak{g}) \to End(\mathcal{V})$ be an evaluation representation of $\mathcal{Y}(\mathfrak{g})$ on $\mathcal{V}$. Then $(T_u \otimes T_v) : \mathcal{R} \to R(u-v) \in End(\mathcal{V} \otimes \mathcal{V})$ maps the universal R-matrix to a matrix, called the additive R-matrix. In such a way (2.1.1) becomes the intertwining equation,

$$R(u-v)\,(T_u \otimes T_v)[\Delta(a)] = (T_u \otimes T_v)[\Delta^{op}(a)]\,R(u-v) \quad \text{for all} \quad a \in \mathcal{Y}(\mathfrak{g})\,, \tag{2.1.56}$$

and equivalently the Yang-Baxter equation (2.1.2) becomes

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)\,. \tag{2.1.57}$$

Generalized twisted Yangians and the reflection equation. The reflection equation for the Yangian algebra is obtained from (2.1.5) in the same way as (2.1.57) from (2.1.4) giving

$$R_{12}(u-v)K_{13}(u)R_{12}(u+v)K_{23}(v) = K_{23}(v)R_{12}(u+v)K_{13}(v)R_{12}(u-v). \tag{2.1.58}$$

Let $\mathcal{B} \subset \mathcal{Y}(\mathfrak{g})$ be a left coideal subalgebra,

$$\Delta(b) \in \mathcal{Y}(\mathfrak{g}) \otimes \mathcal{B} \quad \text{for all} \quad b \in \mathcal{B}\,. \tag{2.1.59}$$

Let $\bar{T}_s : \mathcal{B} \to End(\mathcal{W})$ denote an evaluation representation of $\mathcal{B}$ on the boundary vector space $\mathcal{W}$. Here we assume $\mathcal{W}$ to be finite dimensional.

Definition 2.1.4. *The coideal subalgebra $\mathcal{B}$ is called a Yangian reflection algebra if the intertwining equation*

$$(T_{-u} \otimes \bar{T}_s)[\Delta(b)]\,K(u) = K(u)\,(T_u \otimes \bar{T}_s)[\Delta(b)] \quad \textit{for all} \quad b \in \mathcal{B}\,, \tag{2.1.60}$$

defines a K-matrix $K(u) \in End(\mathcal{V} \otimes \mathcal{W})$ satisfying the reflection equation (2.1.58).

Let the boundary vector space be one-dimensional, $\mathcal{W} = \mathbb{C}$. Then $\bar{T}_s = \epsilon$ and $K(u) \in End(\mathcal{V} \otimes \mathbb{C})$. In this case the intertwining equation (2.1.60) becomes

$$(T_{-u} \otimes \epsilon)[\Delta(b)]\, K(z) = K(z)\, (T_u \otimes \epsilon)[\Delta(b)] \quad \text{for all} \quad b \in \mathcal{B}\,. \tag{2.1.61}$$

Note that (2.1.60) and (2.1.61) are the Yangian equivalents of the quantum affine intertwining equations (2.1.7) and (2.1.8). Finally, for an irreducible representation T_u (resp. $T_u \otimes \bar{T}_s$) of $\mathcal{B}$ the intertwining equation (2.1.61) (resp. (2.1.60)) defines the K-matrix uniquely up to an overall scalar factor. As in the quantum affine case, the boundary representation $\bar{T}_s$ may be different from T_u. Here we will consider the $\bar{T}_s \cong T_u$ case only.

We will next identify two types of coideal subalgebras of $\mathcal{Y}(\mathfrak{g})$ that are compatible with the reflection equation. These are the so-called generalized twisted Yangians introduced in [40] and [23], and are constructed by defining involutions of $\mathcal{Y}(\mathfrak{g})$ and requiring the coideal property to be satisfied. We will be calling these twisted Yangians to be of type I and type II respectively. These twisted Yangians are formulated using Drinfeld-Jimbo realization of Yangians [2, 3] and hence are conventionally called the generalized twisted Yangians [40, 41] or MacKay twisted Yangians [67] to distinguish them from the Olshanskii twisted Yangians constructed using the RTT realization of Yangians [38].

We give the next two propositions without the proofs as they are straightforward.

Proposition 2.1.1. *Let a subalgebra $\mathfrak{a} \subset \mathfrak{g}$ be such that the splitting $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{b}$ forms a symmetric pair*

$$[\mathfrak{a}, \mathfrak{a}] \subset \mathfrak{a}, \qquad [\mathfrak{a}, \mathfrak{b}] \subset \mathfrak{b}, \qquad [\mathfrak{b}, \mathfrak{b}] \subset \mathfrak{a}\,. \tag{2.1.62}$$

This splitting allows us to introduce an involution θ of $\mathfrak{g}$ such that

$$\theta(j^i) = j^i\,, \qquad \theta(j^p) = -j^p\,, \qquad \textit{where} \qquad j^i \in \mathfrak{a}\,,\ j^p \in \mathfrak{b}\,. \tag{2.1.63}$$

Then θ can be extended to the involution $\bar{\theta}$ of $\mathcal{Y}(\mathfrak{g})$ such that

$$\bar{\theta}(\widehat{j}^i) = -\widehat{j}^i\,, \qquad \theta(\widehat{j}^p) = \widehat{j}^p\,, \qquad \theta(\alpha) = -\alpha\,. \tag{2.1.64}$$

Proposition 2.1.2. *Let θ be the trivial involution of $\mathfrak{g}$,*

$$\theta(j^a) = j^a\,, \qquad j^a \in \mathfrak{g}\,. \tag{2.1.65}$$

Then it can be extended to a non-trivial involution $\bar{\theta}$ of $\mathcal{Y}(\mathfrak{g})$ such that

$$\bar{\theta}(\widehat{j}^a) = -\widehat{j}^a\,, \qquad \theta(\widehat{\widehat{j}}^a) = \widehat{\widehat{j}}^a\,, \qquad \theta(\alpha) = -\alpha\,. \tag{2.1.66}$$

Involution $\bar{\theta}$ endows $\mathcal{Y}(\mathfrak{g})$ with the structure of a filtered algebra which combined with the requirement for the coideal property

$$\Delta\left(\bar{\theta}(\mathcal{Y}(\mathfrak{g}))\right) \subset \mathcal{Y}(\mathfrak{g}) \otimes \bar{\theta}(\mathcal{Y}(\mathfrak{g}))\,, \tag{2.1.67}$$

to be satisfied defines the twisted Yangian $\mathcal{Y}(\mathfrak{g}, \theta(\mathfrak{g}))$. Here $\bar{\theta}(\mathcal{Y}(\mathfrak{g}))$ denotes the $\bar{\theta}$–fixed subalgebra of $\mathcal{Y}(\mathfrak{g})$, and $\theta(\mathfrak{g})$ denotes the θ–fixed subalgebra of $\mathfrak{g}$.

Definition 2.1.5. *Let $\mathfrak{a} = \theta(\mathfrak{g})$ be a non-trivial θ–fixed subalgebra of $\mathfrak{g}$. Then the twisted Yangian $\mathcal{Y}(\mathfrak{g},\mathfrak{a})$ of type I is a left coideal subalgebra of $\mathcal{Y}(\mathfrak{g})$ generated by the level-zero generators j^i and the twisted level-one generators [40, 41].*

$$\widetilde{j}^p = \widehat{j}^p + \alpha\, t\, j^p + \frac{\alpha}{4} f^p{}_{qi} \left(j^q\, j^i + j^i\, j^q \right) , \quad \Delta(\widetilde{j}^p) = \widetilde{j}^p \otimes 1 + 1 \otimes \widetilde{j}^p + \alpha f^p{}_{qi}\, j^q \otimes j^i \, , \tag{2.1.68}$$

where $i(,j,k,...)$ run over the $\mathfrak{a}$-indices and $p,q(,r,...)$ over the $\mathfrak{b}$-indices, and $t \in \mathbb{C}$ is an arbitrary complex number.

Definition 2.1.6. *Let $\theta(\mathfrak{g}) = \mathfrak{g}$ be the trivial involution of $\mathfrak{g}$. Then the twisted Yangian $\mathcal{Y}(\mathfrak{g},\mathfrak{g})$ of the type II is a left coideal subalgebra of $\mathcal{Y}(\mathfrak{g})$ generated by the level-zero generators j^a and the twisted level-two generators [23]*

$$\begin{aligned} \widetilde{\widetilde{j}}^a &= \alpha\, t\, \widehat{j}^a + \frac{1}{c_{\mathfrak{g}}} f^a{}_{bc} \left([\widehat{j}^c, \widehat{j}^b] + \frac{\alpha}{2} f^c{}_{de}\, j^d [\widehat{j}^b, j^e] + \frac{\alpha}{2} f^b{}_{de}\, j^d [j^e, \widehat{j}^c] \right) \\ &= \widehat{\widehat{j}}^a + \alpha\, t\, \widehat{j}^a + \frac{\alpha}{2c_{\mathfrak{g}}} f^a{}_{bc} \left(f^c{}_{de}\, j^d [\widehat{j}^b, j^e] + f^b{}_{de}\, j^d [j^e, \widehat{j}^c] \right) , \end{aligned} \tag{2.1.69}$$

having coproducts of the form

$$\Delta(\widetilde{\widetilde{j}}^a) = \widetilde{\widetilde{j}}^a \otimes 1 + 1 \otimes \widetilde{\widetilde{j}}^a + \frac{\alpha}{c_{\mathfrak{g}}} f^a{}_{bc} \Big(f^c{}_{de} [j^d, \widehat{j}^b] \otimes j^e + f^b{}_{de} [\widehat{j}^c, j^d] \otimes j^e \Big) + \frac{1}{c_{\mathfrak{g}}} \mathcal{O}(\alpha^2) \, , \tag{2.1.70}$$

where $a(,b,c,...)$ run over all indices of $\mathfrak{g}$, and $t \in \mathbb{C}$ is an arbitrary complex number. The order $\mathcal{O}(\alpha^2)$ terms are are cubic in the level-zero generators and thus automatically satisfy the coideal property (2.1.67).

Remark 2.1.1. In the case when $c_{\mathfrak{g}} = 0$ (and g_{ad} is degenerate) the twisted level-two generators can be alternatively defined by

$$\widetilde{\widetilde{j}}^{cb} = [\widehat{j}^c, \widehat{j}^b] + \alpha\, t\, [j^c, \widehat{j}^b] + \frac{\alpha}{2} f^c{}_{de}\, j^d [\widehat{j}^b, j^e] + \frac{\alpha}{2} f^b{}_{de}\, j^d [j^e, \widehat{j}^c] \, . \tag{2.1.71}$$

Theorem 2.1.4. *The twisted Yangian of type I with suitable $t \in \mathbb{C}$ is a Yangian reflection algebra.*

Proof. The proof is given in [67]. □

Conjecture 2.1.4. *The twisted Yangian of type II with suitable $t \in \mathbb{C}$ is a Yangian reflection algebra.*

In the following sections we will give explicit examples of the twisted Yangians of both types and show that they are Yangian reflection algebras. In section 3 we will construct twisted Yangians for $\mathcal{Y}(\mathfrak{sl}(2))$, and in Section 5 for $\mathcal{Y}(\mathfrak{gl}(1|1))$.

Let us give a final conjecture regarding which we will give some explicit details for the coideal subalgebras we will be considering in the remaining parts of this chapter.

Conjecture 2.1.5. *In the rational $q \to 1$ limit the quantum affine coideal subalgebra defined by the conjecture 2.1.2 specializes to the twisted Yangian of type I if $p(0) \neq 0$, and to the type II if $p(0) = 0$.*

2.2 Reflection algebras for $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$

Algebra. The quantum affine Lie algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$ in the Drinfeld-Jimbo realization is generated by the Chevalley generators $\xi_1^{\pm}$, the Cartan generators k_1, k_1^{-1}, the affine Chevalley generators $\xi_0^{\pm}$ and the corresponding Cartan generators k_0, k_0^{-1}. The extended (symmetric) Cartan matrix is given by

$$(\widehat{a}_{ij})_{0\le i,j\le 1} = \begin{pmatrix} 2 & -2 \\ -2 & 2 \end{pmatrix} . \tag{2.2.1}$$

The corresponding root space Φ is generated by $\hat{\pi} = \{\alpha_0,\, \alpha_1\}$. The commutation relations of the algebra are as follows,

$$[k_i, k_j] = 0\,, \qquad k_i\,\xi_j^{\pm} = q^{\widehat{a}_{ij}}\xi_j^{\pm}k_i\,, \qquad [\xi_i^+, \xi_j^-] = \delta_{ij}\frac{k_i - k_i^{-1}}{q - q^{-1}}\,. \tag{2.2.2}$$

Representation. We define the fundamental evaluation representation T_z of $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$ on a two-dimensional vector space $\mathcal{V}$ with basis vectors $\{v_1,\, v_2\}$. Let $e_{j,k}$ be 2×2 matrices satisfying $(e_{j,k})_{j',k'} = \delta_{j,j'}\delta_{k,k'}$ or equivalently $e_{i,j}\,v_k = \delta_{j,k}\,v_i$ (i.e. for any operator A its matrix elements A_{ij} are defined by $A\,v_i = A_{ji}v_j$). Then the representation T_z is defined by

$$\begin{aligned} &T_z(\xi_1^+) = e_{1,2}\,, && T_z(\xi_1^-) = e_{2,1}\,, && T_z(k_1) = q\,e_{1,1} + q^{-1}e_{2,2}\,, \\ &T_z(\xi_0^+) = z\,e_{2,1}\,, && T_z(\xi_0^-) = z^{-1}e_{1,2}\,, && T_z(k_0) = q^{-1}e_{1,1} + q\,e_{2,2}\,, \end{aligned} \tag{2.2.3}$$

We choose the boundary vector space $\mathcal{W}$ to be equivalent to $\mathcal{V}$. The boundary representation T_s is obtained from (2.2.3) by replacing z with s.

The fundamental R-matrix $R_{ij}(z) \in End(\mathcal{V}_i \otimes \mathcal{V}_j)$ satisfying the Yang-Baxter equation (2.1.4)

$$R_{12}(z/w)R_{13}(z)R_{23}(w) = R_{23}(w)R_{13}(z)R_{12}(z/w)\,,$$

is given by

$$R(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 1-q\,r & 0 \\ 0 & 1-r/q & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{where} \qquad r = \frac{z-1}{q\,z - 1/q}\,. \tag{2.2.4}$$

2.2.1 Singlet boundary

Consider the reflection equation (2.1.5) on the space $\mathcal{V}\otimes\mathcal{V}\otimes\mathbb{C}$ with the R-matrix defined by (2.2.4) and the K-matrix being any 2×2 matrix satisfying

$$R_{12}(z/w)K_{13}(z)R_{12}(zw)K_{23}(w) = K_{23}(w)R_{12}(zw)K_{13}(z)R_{12}(z/w)\,.$$

The general solution is [16]

$$K(z)=\begin{pmatrix} 1 & a\,k' \\ b\,k' & k \end{pmatrix}, \qquad \text{where} \qquad k=\frac{c\,z-1}{z(c-z)}\,, \qquad k'=\frac{1-z^2}{z(c-z)}\,, \tag{2.2.5}$$

and $a,\, b,\, c \in \mathbb{C}$ are arbitrary complex numbers.

We are interested in a solution compatible with the underlying Lie algebra. The minimal constraint is to require the reflection matrix to intertwine the Cartan generators,

$$(T_{1/z}\otimes\epsilon)[\Delta(k_i)]\,K(z)-K(z)\,(T_z\otimes\epsilon)[\Delta(k_i)]=0\,. \tag{2.2.6}$$

This constraint restricts the K-matrix (2.2.5) to be of diagonal form ($a=b=0$)[1]. Next, it is easy to see that such a K-matrix does not satisfy the intertwining equation for any of the Chevalley generators,

$$(T_{1/z}\otimes\epsilon)[\Delta(\xi_i^{\pm})]\,K(z)-K(z)\,(T_z\otimes\epsilon)[\Delta(\xi_i^{\pm})]\neq 0\,. \tag{2.2.7}$$

We call Cartan generators k_i the *preserved* generators, while the Chevalley generators $\xi_i^{\pm}$ are the *broken* generators. This setup is consistent with the following quantum affine coideal subalgebra.

Proposition 2.2.1. *Let the involution Θ act on the root space Φ as*

$$\Theta(\alpha_0)=-\alpha_1\,. \tag{2.2.8}$$

Then it defines a quantum affine coideal subalgebra $\mathcal{B}\subset\mathcal{A}=\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$ generated by the Cartan element $k_0k_1^{-1}$ and the twisted affine generators

$$\begin{aligned} B_0^{+} &= \xi_0'^{+}k_0-d_{+}\,\tilde{\theta}(\xi_0'^{+})k_0\,, \qquad & \tilde{\theta}(\xi_0'^{+}) &= \xi_1^{-}\,,\\ B_0^{-} &= \xi_0^{-}k_0-d_{-}\,\tilde{\theta}(\xi_0^{-})k_0\,, \qquad & \tilde{\theta}(\xi_0^{-}) &= \xi_1'^{+}\,, \end{aligned} \tag{2.2.9}$$

where $\xi_0'^{+}=k_0^{-1}\xi_0^{+}$ and $d_{\pm}\in\mathbb{C}$ are arbitrary complex numbers.

Proof. The generators (2.2.9) satisfy the coideal property

$$\begin{aligned} \Delta(B_0^{+}) &= \xi_0'^{+}k_0\otimes 1-d_{+}\,\xi_1^{-}k_0\otimes k_0k_1^{-1}+k_0\otimes B_0^{-}\in\mathcal{A}\otimes\mathcal{B}\,,\\ \Delta(B_0^{-}) &= \xi_0^{-}k_0\otimes 1-d_{-}\,\xi_1'^{+}k_0\otimes k_0k_1^{-1}+k_0\otimes B_0^{+}\in\mathcal{A}\otimes\mathcal{B}\,, \end{aligned} \tag{2.2.10}$$

and the property is obvious for $k_0k_1^{-1}$. □

Proposition 2.2.2. *The quantum affine coideal subalgebra defined above with $d_{+}\,q=d_{-}/q=c$, where $c\in\mathbb{C}$, is a reflection algebra for a singlet boundary.*

[1]This constraint may be alternatively obtained by requiring the unitarity property to hold, $K(z^{-1})K(z)=id$.

Proof. The representation of the generators of $\mathcal{B}$ is given by

$$T_z(k_0k_1^{-1}) = q^{-2}e_{1,1} + q^2e_{2,2}\,, \quad T_z(B_0^+) = q^{-2}(z - q\,d_+)e_{2,1}\,, \quad T_z(B_0^-) = (q\,z^{-1} - d_-)e_{1,2}\,. \tag{2.2.11}$$

Let $K(z)$ be any 2×2 matrix. The intertwining equation for $k_0k_1^{-1}$ restricts $K(z)$ to be of a diagonal form, thus up to an overall scalar factor, $K(z) = e_{1,1} + k\,e_{2,2}$. Next, the intertwining equation for $B^{\pm}$ gives

$$1 + q\,z\,d_+(k-1) - z^2k = 0\,, \qquad d_-(k-1) + q\,(z^{-1} - z\,k) = 0\,, \tag{2.2.12}$$

having a unique solution $d_+\,q = d_-/q = c$ and $k = \dfrac{c\,z-1}{z(c-z)}$, where $c \in \mathbb{C}$ is an arbitrary complex number. This coincides with (2.2.5) provided $a = b = 0$.

□

2.2.2 Vector boundary

Consider the reflection equation (2.1.5) in the tensor space $\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{W}$ with the R-matrix defined by (2.2.4). Then there exists a solution of the reflection equation,

$$K(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-k/q & k & 0 \\ 0 & k & 1-q\,k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{where} \qquad k = \frac{(q-q^{-1})(z^2-1)}{q^{-2} - c\,z + q^2z^2}\,, \tag{2.2.13}$$

and $c \in \mathbb{C}$ is an arbitrary complex number. This K-matrix satisfies the intertwining equation (2.1.7)

$$(T_{1/z}\otimes T_s)[\Delta(b)]\,K(z) - K(z)\,(T_z\otimes T_s)[\Delta(b)] = 0 \quad \text{for all} \quad b \in \mathcal{U}_q(\mathfrak{sl}(2))\,. \tag{2.2.14}$$

We call Cartan generators k_i and Chevalley generators $\xi_1^{\pm}$ the *preserved* generators, while the affine Chevalley generators $\xi_0^{\pm}$ are the *broken* generators. Next, we identify the corresponding quantum affine coideal subalgebra consistent with the reflection matrix (2.2.13).

Proposition 2.2.3. *Let the involution Θ act on the root space Φ as*

$$\Theta(\alpha_0) = -\alpha_0 - 2\alpha_1\,, \qquad \Theta(\alpha_1) = \alpha_1\,. \tag{2.2.15}$$

Then it defines a quantum affine coideal subalgebra $\mathcal{B}\subset\mathcal{A} = \mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$ generated by the Cartan generator k_1, the Chevalley generators $\xi_1^{\pm}$, and the twisted affine generator

$$B_0^- = \xi_0^-k_0 - d_-\,\tilde\theta(\xi_0^-)k_0\,, \qquad \tilde\theta(\xi_0^-) = \big(ad_r\,\xi_1^+\xi_1^+\big)\xi_0'^+\,, \tag{2.2.16}$$

where $\xi_0'^+ = k_0^{-1}\xi_0^+$ and $d_- \in \mathbb{C}^\times$ is an arbitrary non-zero complex number.

Proof. The twisted affine generator (2.2.16) satisfies coideal property

$$\begin{aligned}\Delta(B_0^-) &= \xi_0^- k_0 \otimes 1 - d_-\,\tilde\theta(\xi_0^-)k_0 \otimes k_1^{-2} + k_0 \otimes B_0^- \\ &\quad + d_- q^2(q^2-q^{-2})\big(\xi_0^+ \otimes (\mathrm{ad}_r\,\xi_1^+)\xi_1'^+ - k_0\,(\mathrm{ad}_r\,\xi_1^+)\xi_0'^+ \otimes k_1^{-1}\xi_1'^+\big) \in \mathcal{A}\otimes\mathcal{B}\,. \end{aligned} \tag{2.2.17}$$

The property is satisfied by the definition for the rest of the generators.

□

Remark 2.2.1. This algebra may be alternatively generated by k_1, $\xi_1^\pm$, and the twisted affine generator

$$B_0^+ = \xi_0'^+ k_0 - d_+\,\tilde\theta(\xi_0'^+)k_0\,, \qquad \tilde\theta(\xi_0'^+) = \big(\mathrm{ad}_r\,\xi_1^-\xi_1^-\big)\xi_0^-\,, \tag{2.2.18}$$

having coproduct

$$\begin{aligned}\Delta(B_0^+) &= \xi_0'^+ k_0 \otimes 1 - d_+\,\tilde\theta(\xi_0'^+)k_0 \otimes k_1^{-2} + k_0 \otimes B_0^+ \\ &\quad - d_+(q^2-q^2)\big(\xi_0^- k_0 \otimes (\mathrm{ad}_r\,\xi_1^-)\xi_1^- - q^{-2}\,k_0\,(\mathrm{ad}_r\,\xi_1^-)\xi_0^- \otimes k_1^{-1}\xi_1^-\big) \in \mathcal{A}\otimes\mathcal{B}\,, \end{aligned} \tag{2.2.19}$$

and $d_+ = d_-^{-1}(q^{-1}+q)^{-2}$. Generators $B_0^\pm$ are related by

$$B_0^- = -d_-\big[(\mathrm{ad}_r\,\xi_1^+\xi_1^+)B_0^+ k_1\big]k_1^{-1}\,. \tag{2.2.20}$$

Proposition 2.2.4. *The quantum affine coideal subalgebra defined above with $q^2 d_+ = q^{-2}d_- = (q+q^{-1})^{-1}$ is a reflection algebra for a vector boundary.*

Proof. The representation $(T_z\otimes T_s)$ of the coproducts of the Lie generators of $\mathcal{B}$ is given by

$$\begin{aligned}(T_z\otimes T_s)[\Delta(k_1)] &= q^2 e_{1,1} + e_{2,2} + e_{3,3} + q^{-2}e_{4,4}\,,\\ (T_z\otimes T_s)[\Delta(\xi^+)] &= q\,e_{1,2} + e_{1,3} + e_{2,4} + q^{-1}e_{3,4}\,,\\ (T_z\otimes T_s)[\Delta(\xi^-)] &= e_{2,1} + q^{-1}e_{3,1} + q\,e_{4,2} + e_{4,3}\,, \end{aligned} \tag{2.2.21}$$

and of the twisted affine generators by

$$\begin{aligned}&(T_z\otimes T_s)[\Delta(B_0^+)]\\ &\quad= \big(q^{-3}s + d_+(s^{-1}(q^{-2}+1) - z^{-1}(q^{-4}-1))\big)\,e_{2,1} + q^{-2}\big(z + d_+z^{-1}(q+q^{-1})\big)\,e_{3,1}\\ &\qquad + \big(q^{-2}z + d_+q^2z^{-1}(q+q^{-1})\big)\,e_{4,2} + \big(q^{-1}s + d_+(s^{-1}(q^2+1) - z^{-1}(q^4-1))\big)\,e_{4,3}\,,\\ &(T_z\otimes T_s)[\Delta(B_0^-)]\\ &\quad= \big(s^{-1} + q^{-1}d_-(s(q^{-2}+1) - z(q^{-4}-1))\big)\,e_{1,2} + \big(q\,z^{-1} + d_-z(q^{-4}+q^{-2})\big)\,e_{2,4}\\ &\qquad + \big(q\,z^{-1} + d_-z(q^2+1)\big)\,e_{2,4} + \big(q^2s^{-1} + q^{-1}d_-(s(q^2+1) - z(q^4-1)\big)\,e_{3,4}\,. \end{aligned} \tag{2.2.22}$$

Let $K(z)$ be any 4×4 matrix. Then the intertwining equation for the Lie generators (2.2.21) constrain $K(z)$ to the form given in (2.2.13) up to an unknown function k and an overall scalar factor. Next, the intertwining equation for the twisted affine generators (2.2.22) has a unique solution,

$$q^2 d_+ = q^{-2} d_- = (q+q^{-1})^{-1}\,, \qquad k = \frac{(q-q^{-1})(z^2-1)}{q^{-2}-(s^{-1}+s)z+q^2z^2}\,, \tag{2.2.23}$$

which coincides with k given in (2.2.13) provided $c = s + s^{-1}$. □

Remark 2.2.2. Reflection matrix (2.2.13) satisfies the intertwining equation for all Cartan generators $k_i \in \mathcal{T}$, thus the subalgebra $\mathcal{B} \otimes \mathcal{T} \subset \mathcal{A}$ is also a reflection algebra. The same is true for the reflection algebra of a singlet boundary.

Remark 2.2.3. The coideal subalgebra defined in proposition 2.2.2 is also compatible with a vector boundary. The corresponding reflection matrix is

$$K(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+s(q^2z-c)k & qs(c-s)k & 0 \\ 0 & q(cs-1)k & k'+s(z^{-1}-q^2c)k & 0 \\ 0 & 0 & 0 & k' \end{pmatrix}, \tag{2.2.24}$$

where

$$k = \frac{(q^2-1)(z^2-1)}{(c-z)(q^2z-s)(q^2sz-1)}\,, \qquad k' = \frac{c\,z-1}{z(c-z)}\,, \tag{2.2.25}$$

and $c, s \in \mathbb{C}$ are arbitrary complex numbers. This vector boundary reflection matrix can be obtained using the boundary fusion procedure [15, 91], which in this case is simply $K_V(z) = P\,R(zs)\,(1 \otimes K_S(z))\,P\,R(z/s)$, where $K_V(z)$ is (2.2.24), $K_S(z)$ is (2.2.5) with $a = b = 0$, and $P = R(0)$ with $R(z)$ given by (2.2.4).

Similarly, the coideal subalgebra defined in proposition 2.2.4 is compatible with a singlet boundary. However, the corresponding reflection matrix is trivial,

$$K = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.2.26}$$

This reflection matrix can be obtained by solving the boundary intertwining equation for the Lie algebra generators only, and thus the twisted affine symmetries are redundant in this case. Then the reflection matrix (2.2.13) can be obtained using an equivalent fusion procedure as above. These properties will further reappear in the Yangian case and for the $GL(1|1)$ algebra for both affine and Yangian cases. In these cases we will simply state that the corresponding reflection matrix is trivial and omit repeating the expression for the fusion procedure.

Remark 2.2.4. Let $T_{l,z}$ be a finite-dimensional irreducible representation of the algebra $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$. Let l be an integral or half-integral non-negative number and $\mathcal{V}_l$ be a $(2l+1)$–dimensional complex vector space with a basis $\{v_m \,|\, m = -l, -l+1, \cdots, l\}$. For convenience we set $v_{-l-1} = v_{l+1} = 0$. The operators $T_{l,z}(\xi_i^{\pm}),\ T_{l,z}(k_i)$ act on the space $\mathcal{V}_l$

by

$$T_{l,z}(\xi_1^{\pm})\,v_m = ([l\mp m]_q[l\pm m+1]_q)^{1/2}\,v_{m+1}\,, \qquad T_{l,z}(k_1)\,v_m = q^{2m}v_m\,,$$
$$T_{l,z}(\xi_0^{\pm})\,v_m = z^{\pm 1}\,([l\mp m]_q[l\pm m+1]_q)^{1/2}\,v_{m+1}\,, \qquad T_{l,z}(k_0)\,v_m = q^{-2m}v_m\,, \qquad (2.2.27)$$

Let the boundary vector space $\mathcal{W}_l$ and the boundary representation $T_{l,s}$ be defined in the same way. Then all the constructions of the quantum affine coideal subalgebras presented above apply directly for any finite-dimensional representation $T_{l,z}$ and lead to a unique solution (for fixed l) of the reflection equation (for a singlet boundary this was explicitly shown in [76]).

The coideal subalgebras given in propositions 2.2.2 and 2.2.4 by the construction are closely related to the orthogonal and symplectic twisted q-Yangians $\mathrm{Y}_{\mathrm{q}}^{\mathrm{tw}}(\mathfrak{o}_2)$ and $\mathrm{Y}_{\mathrm{q}}^{\mathrm{tw}}(\mathfrak{sp}_4)$ introduced in [79], however we do not know the exact isomorphism.

2.3 Reflection algebras for $\mathcal{Y}(\mathfrak{sl}(2))$

Algebra. The Yangian $\mathcal{Y}(\mathfrak{sl}(2))$ is generated by the level-zero Chevalley generators $E^{\pm}$, Cartan generator H, and the level-one Yangian generators $\widehat{E}^{\pm}$ and the corresponding level-one Cartan generator $\widehat{H}$. The commutation relations of the algebra are given by

$$[H,E^{\pm}] = \pm 2E^{\pm}\,, \quad [E^+,E^-] = H\,, \quad [H,\widehat{E}^{\pm}] = \pm 2\widehat{E}^{\pm}\,, \quad [E^{\pm},\widehat{E}^{\mp}] = \pm\widehat{H}\,, \quad [H,\widehat{H}] = 0\,. \qquad (2.3.1)$$

The Hopf algebra structure is equipped with the following coproduct,

$$\Delta(H) = H\otimes 1 + 1\otimes H\,, \qquad \Delta(\widehat{H}) = \widehat{H}\otimes 1 + 1\otimes\widehat{H} - \alpha\,(E^+\otimes E^- - E^-\otimes E^+)\,,$$
$$\Delta(E^{\pm}) = E^{\pm}\otimes 1 + 1\otimes E^{\pm}\,, \quad \Delta(\widehat{E}^{\pm}) = \widehat{E}^{\pm}\otimes 1 + 1\otimes\widehat{E}^{\pm} \pm \frac{\alpha}{2}\,(E^{\pm}\otimes H - H\otimes E^{\pm})\,. \qquad (2.3.2)$$

Representation. The fundamental evaluation representation of $\mathcal{Y}(\mathfrak{sl}(2))$ on the two-dimensional vector space $\mathcal{V}$ is defined by

$$T_u(E^+) = e_{1,2}\,, \qquad T_u(E^-) = e_{2,1}\,, \qquad T_u(H) = e_{1,1} - e_{2,2}\,,$$
$$T_u(\widehat{E}^+) = u\,e_{1,2}\,, \qquad T_u(\widehat{E}^-) = u\,e_{2,1}\,, \qquad T_u(\widehat{H}) = u\,(e_{1,1} - e_{2,2})\,. \qquad (2.3.3)$$

We set $T_u(\alpha) = 1$. The boundary representation T_s is obtained by replacing u with s.

The fundamental R-matrix $R_{ij}(u) \in End(\mathcal{V}_i\otimes\mathcal{V}_j)$ satisfying the Yang-Baxter equation (2.1.57)

$$R_{12}(u-v)R_{13}(u)R_{23}(v) = R_{23}(v)R_{13}(u)R_{12}(u-v)\,,$$

is given by

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 1-r & 0 \\ 0 & 1-r & r & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{where} \qquad r = \frac{u}{u-1}\,. \qquad (2.3.4)$$

2.3.1 Singlet boundary

Consider the reflection equation (2.1.58) in the space $\mathcal{V}\otimes\mathcal{V}\otimes\mathbb{C}$ with the R-matrix defined by (2.3.4) and the K-matrix being any 2×2 matrix satisfying

$$R_{12}(u-v)K_{13}(u)R_{12}(u+v)K_{23}(v) = K_{23}(v)R_{12}(u+v)K_{13}(v)R_{12}(u-v)\,.$$

The general solution is [14]

$$K(u) = \begin{pmatrix} 1 & a\,k' \\ b\,k' & k \end{pmatrix}, \qquad \text{where} \qquad k = \frac{c+u}{c-u}\,, \qquad k' = \frac{u}{c-u}\,, \tag{2.3.5}$$

and $a,\, b,\, c \in \mathbb{C}$ are arbitrary complex numbers.

Once again we are interested in a solution compatible with the underlying Lie algebra and thus require the reflection matrix to intertwine the Cartan generator H,

$$(T_u\otimes\epsilon)[\Delta(H)]\,K(u) - K(u)\,(T_u\otimes\epsilon)[\Delta(H)] = 0\,. \tag{2.3.6}$$

This requirement restricts K-matrix (2.2.5) to be of the diagonal form ($a=b=0$). Next, it is easy to check that such a K-matrix does not satisfy the intertwining equation for any other generators of $\mathfrak{sl}(2)$ (and $\mathcal{Y}(\mathfrak{sl}(2))$). Hence we call Cartan generator H the *preserved* generator, while the rest are the *broken* generators. This setup allows us to define the following involution and the twisted Yangian:

Proposition 2.3.1. *Let the involution θ act on Lie algebra $\mathfrak{g}=\mathfrak{sl}(2)$ as*

$$\theta(H) = H\,, \qquad \theta(E^{\pm}) = -E^{\pm}\,, \tag{2.3.7}$$

defining a symmetric pair $(\mathfrak{g},\theta(\mathfrak{g}))$, where $\theta(\mathfrak{g})$ is a θ–fixed subalgebra of $\mathfrak{g}$. Then the involution θ can be extended to the $\bar{\theta}$ involution of $\mathcal{Y}(\mathfrak{g})$ such that

$$\bar{\theta}(\widehat{H}) = -\widehat{H}\,, \qquad \bar{\theta}(\widehat{E}^{\pm}) = \widehat{E}^{\pm}\,, \qquad \bar{\theta}(\alpha) = -\alpha\,. \tag{2.3.8}$$

This involution is obvious and thus we do not give a proof of it; we will follow the same strategy, when appropriate, in further sections.

Proposition 2.3.2. *The twisted Yangian $\mathcal{Y}(\mathfrak{g},\theta(\mathfrak{g}))$ of type I for $\mathfrak{g}=\mathfrak{sl}(2)$ and $\theta(\mathfrak{g})=H$ is the $\bar{\theta}$–fixed coideal subalgebra of $\mathcal{Y}(\mathfrak{g})$ generated by the Cartan generator H and the twisted Yangian generators [38]*

$$\widetilde{E}^{\pm} = \widehat{E}^{\pm} \pm \alpha\, t\, E^{\pm} \pm \frac{\alpha}{4}(H\,E^{\pm} + E^{\pm}H)\,. \tag{2.3.9}$$

Here $t\in\mathbb{C}$ is an arbitrary complex number.

Proof. The twisted generators (2.3.9) are in the positive eigenspace of the involution $\bar{\theta}$ and satisfy the coideal property

$$\Delta(\widetilde{E}^{\pm}) = \widetilde{E}^{\pm}\otimes 1 \pm \alpha\,E^{\pm}\otimes H + 1\otimes\widetilde{E}^{-}_{1} \in \mathcal{Y}(\mathfrak{g})\otimes\mathcal{Y}(\mathfrak{g},\theta(\mathfrak{g}))\,. \tag{2.3.10}$$

The same properties for H follows from the definition. □

Proposition 2.3.3. *The twisted Yangian $\mathcal{Y}(\mathfrak{g},\theta(\mathfrak{g}))$ defined above is a reflection algebra for a singlet boundary.*

Proof. The representation T_u of the generators of $\mathcal{Y}(\mathfrak{g},\theta(\mathfrak{g}))$ is given by

$$T_u(H) = e_{1,1} - e_{2,2}\,, \qquad T_u(\widetilde{E}^{+}) = (u+t)\,e_{1,2}\,, \qquad T_u(\widetilde{E}^{-}) = (u-t)\,e_{1,2}\,. \tag{2.3.11}$$

Let $K(u)$ be any 2×2 matrix. Then the intertwining equation for H restricts $K(u)$ to be of the diagonal form, thus up to an overall scalar factor, $K(u) = e_{1,1} + k\,e_{2,2}$. Next, the intertwining equation for $\widetilde{E}^{\pm}$ has a unique solution $k = \dfrac{t+u}{t-u}$ which coincides with (2.3.5) provided $c = t$ and $a = b = 0$. □

2.3.2 Vector boundary

Consider the reflection equation (2.1.58) in the tensor space $\mathcal{V}\otimes\mathcal{V}\otimes\mathcal{W}$ with the R-matrix defined by (2.3.4). Then there exists a solution of the reflection equation,

$$K(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-k & k & 0 \\ 0 & k & 1-k & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \text{where} \qquad k = \frac{2u}{c^2-(u-1)^2}\,, \tag{2.3.12}$$

and $c\in\mathbb{C}$ is an arbitrary complex number.

This K-matrix satisfies the intertwining equation

$$(T_{-u}\otimes T_s)[\Delta(b)]\,K(u) - K(u)\,(T_u\otimes T_s)[\Delta(b)] = 0 \quad \text{for all} \quad b\in\mathfrak{sl}(2)\,. \tag{2.3.13}$$

Thus we call the level-zero $\mathfrak{sl}(2)$ generators $E^{\pm}$ and H the *preserved* generators, while the level-one generators $\widehat{E}^{\pm}$ and $\widehat{H}$ are the *broken* generators. This setup leads to the following involution and the twisted Yangian.

Proposition 2.3.4. *Let θ be the trivial involution of the Lie algebra $\mathfrak{g} = \mathfrak{sl}(2)$,*

$$\theta(H) = H\,, \qquad \theta(E^{\pm}) = E^{\pm} \qquad \Longrightarrow \qquad \theta(\mathfrak{g}) = \mathfrak{g}\,. \tag{2.3.14}$$

Then it can be extended to a non-trivial involution $\bar\theta$ of $\mathcal{Y}(\mathfrak{sl}(2))$ such that

$$\bar\theta(\widehat{H}) = -\widehat{H}\,, \qquad \bar\theta(\widehat{E}^{\pm}) = -\widehat{E}^{\pm}\,, \qquad \bar\theta(\alpha) = -\alpha\,. \tag{2.3.15}$$

Proposition 2.3.5. *The twisted Yangian $\mathcal{Y}(\mathfrak{g},\mathfrak{g})$ of type II for $\mathfrak{g} = \mathfrak{sl}(2)$ and $\theta(\mathfrak{g}) = \mathfrak{g}$ is the $\bar\theta$–fixed coideal subalgebra of $\mathcal{Y}(\mathfrak{g})$ generated by all level-zero generators and the level-two twisted Yangian generators*

$$\widetilde{\widetilde{E}}^{\pm} = \pm\frac{1}{2}\left([\widehat{H},\,\widehat{E}^{\pm}] - \alpha\left(t\,\widehat{E}^{\pm} + E^{\pm}\widehat{H} - H\,\widehat{E}^{\pm}\right)\right) \qquad \textit{and} \qquad \widetilde{\widetilde{H}} = [\,\widetilde{\widetilde{E}}^{+},\,E^{-}]\,, \tag{2.3.16}$$

where $t\in\mathbb{C}$ is an arbitrary complex number.

Proof. The twisted generators (2.3.16) are in the positive eigenspace of the involution $\bar\theta$ and satisfy the coideal property

$$\begin{aligned}\Delta\big(\,\widetilde{\widetilde{E}}{}^{\pm}\,\big) &= \widetilde{\widetilde{E}}{}^{\pm}\otimes 1 + 1\otimes \widetilde{\widetilde{E}}{}^{\pm} \pm \alpha\,(\widehat{H}\otimes E^{\pm} - \widehat{E}^{\pm}\otimes H) + \mathcal{O}(\alpha^2) \in \mathcal{Y}(\mathfrak{g})\otimes\mathcal{Y}(\mathfrak{g},\mathfrak{g})\,,\\ \Delta\big(\,\widetilde{\widetilde{H}}\,\big) &= \widetilde{\widetilde{H}}\otimes 1 + 1\otimes \widetilde{\widetilde{H}} + 2\alpha\,(\widehat{E}^{-}\otimes E^{+} - \widehat{E}^{+}\otimes E^{-}) + \mathcal{O}(\alpha^2) \in \mathcal{Y}(\mathfrak{g})\otimes\mathcal{Y}(\mathfrak{g},\mathfrak{g})\,.\end{aligned} \tag{2.3.17}$$

The same properties for the level-zero generators follow from the definition. $\square$

Proposition 2.3.6. *The twisted Yangian $\mathcal{Y}(\mathfrak{g},\mathfrak{g})$ defined above with $t=-2$ is a reflection algebra for a vector boundary.*

Proof. The representation $(T_u\otimes T_s)$ of the coproducts of the Lie generators of $\mathcal{Y}(\mathfrak{g},\mathfrak{g})$ is given by

$$\begin{aligned}&(T_u\otimes T_s)[\Delta(E^+)] = e_{1,2}+e_{1,3}+e_{2,4}+e_{3,4}\,, \qquad (T_u\otimes T_s)[\Delta(H)] = 2\,(e_{1,1}+e_{4,4})\,,\\ &(T_u\otimes T_s)[\Delta(E^-)] = e_{2,1}+e_{3,1}+e_{4,2}+e_{4,3}\,,\end{aligned} \tag{2.3.18}$$

and of the twisted Yangian generators by

$$\begin{aligned}(T_u\otimes T_s)[\Delta(\widetilde{\widetilde{E}}{}^{+})] &= \alpha\, e_{1,2}+\beta\, e_{1,3}+\gamma\, e_{2,4}+\delta\, e_{3,4}\,,\\ (T_u\otimes T_s)[\Delta(\widetilde{\widetilde{E}}{}^{-})] &= \delta\, e_{2,1}+\gamma\, e_{3,1}+\beta\, e_{4,2}+\alpha\, e_{4,3}\,,\\ (T_u\otimes T_s)[\Delta(\widetilde{\widetilde{H}})] &= \lambda\, e_{1,1}+\mu\, e_{2,2}-\mu\, e_{3,3}-\lambda\, e_{4,4}+\eta\,(e_{2,3}-e_{3,2})\,,\end{aligned} \tag{2.3.19}$$

where

$$\begin{aligned}&\alpha = \tfrac{1}{16}((4s+t+2)^2-(t+4)^2)-u-\tfrac12\,, && \lambda = \tfrac{1}{16}((4u+t+2)^2+(4c+t+2)^2\\ &\beta = \tfrac14(2u+1)(2u+t+3)\,, && \qquad -2(t+2)^2)-\tfrac12\,,\\ &\gamma = \tfrac14(2u-1)(2u+t+1)\,, && \mu = \tfrac{1}{16}((4u+t+2)^2-(4c+t+2)^2)+1\,,\\ &\delta = \tfrac{1}{16}((4s+t+2)^2-t^2)+u-\tfrac12\,, && \eta = -(2u+t/2+1)\,.\end{aligned} \tag{2.3.20}$$

Let $K(u)$ be any 4×4 matrix. Then the intertwining equation for the Lie generators (2.3.18) constrain $K(u)$ to the form given in (2.3.12) up to an unknown function k and an overall scalar factor. Next, the intertwining equation for the twisted Yangian generators (2.3.19) constrain $t=-2$ and has a unique solution $k=\dfrac{2u}{s^2-(u-1)^2}$ which coincides with (2.3.12) provided $c=s$. $\square$

Remark 2.3.1. Following the same pattern as in the quantum affine case, the twisted Yangian given in proposition 2.3.3 is also compatible with a vector boundary. The corresponding fundamental reflection matrix is

$$K(z) = \begin{pmatrix} 1 & 0 & 0 & 0\\ 0 & 1+(u-1-c)k & (c-s)k & 0\\ 0 & (c+s)k & k'+(1-c-u)k & 0\\ 0 & 0 & 0 & k'\end{pmatrix} \tag{2.3.21}$$

where

$$k = \frac{2u}{(c-u)(1+s-u)(u+s-1)}\,, \qquad k' = \frac{c+u}{c-u}\,, \tag{2.3.22}$$

and $c, s \in \mathbb{C}$ are arbitrary complex numbers.

The twisted Yangian given in proposition 2.3.6 is compatible with a singlet boundary, and the corresponding reflection matrix is trivial.

Remark 2.3.2. Let $T_{l,u}$ be a $(2l+1)$-dimensional representation of $\mathcal{Y}(\mathfrak{sl}(2))$. Let $\mathcal{V}_l$ be a vector space defined in remark 2.2.4. The generators of $\mathcal{Y}(\mathfrak{sl}(2))$ act on the space $\mathcal{V}_l$ by [37]

$$\begin{aligned} T_{l,u}(E^{\pm})\, v_m &= ((l\mp m)(l\pm m+1))^{1/2}\, v_{m+1}\,, & T_{l,u}(H)\, v_m &= 2m\, v_m\,, \\ T_{l,u}(\widehat{E}_0^{\pm})\, v_m &= 2\, l\, u\, ((l\mp m)(l\pm m+1))^{1/2}\, v_{m+1}\,, & T_{l,u}(\widehat{H})\, v_m &= 4m\, l\, u\, v_m\,, \end{aligned} \tag{2.3.23}$$

Let the boundary vector space $\mathcal{W}_l$ and the boundary representation $T_{l,s}$ be defined in the same way. Then all the constructions of twisted Yangians presented above apply directly for any finite-dimensional representation $T_{l,u}$ and lead to unique solution (for fixed l) of the reflection equation.

To finalize this section we want to note that the twisted Yangian of type I given in proposition 2.3.3 is isomorphic to the orthogonal twisted Yangian $\mathcal{Y}^{+}(2)$ of [38,92] and to $B(2,1)$ of [39]. The twisted Yangian of type II given in proposition 2.3.6 is isomorphic to the symplectic twisted Yangian $\mathcal{Y}^{-}(2)$ of [38,92] and to $B(2,0)$ of [39] (see also [66]).

2.3.3 Yangian limit

The algebra $\mathcal{U}_q(\widehat{\mathfrak{g}})$ does not contain any singular elements, and in the $q\to 1$ limit specializes to $\mathcal{U}(\mathfrak{g})$ via the composite map φ such that

$$\mathcal{U}_q(\widehat{\mathfrak{g}}) \underset{q\to 1}{\longrightarrow} \mathcal{U}(\mathcal{L}(g)) \underset{z\to 1}{\longrightarrow} \mathcal{U}(g)\,. \tag{2.3.24}$$

Set $q = \mathrm{e}^{\alpha\hbar}$ and $z = \mathrm{e}^{-2\hbar u}$. Then the $q \to 1$ limit is obtained by setting $\hbar \to 0$, where $\hbar$ is an indeterminate deformation parameter that can be regarded as the Planck's constant when the Yangian is an auxiliary algebra and α is a formal parameter used to track the "level" of the Yangian generators and is usually set to unity [17]. Consider an extended algebra

$$\widetilde{\mathcal{U}_q(\widehat{\mathfrak{g}})} = \mathcal{U}_q(\widehat{\mathfrak{g}}) \otimes_{\mathbb{C}[[\hbar]]} \mathbb{C}((\hbar))\,, \tag{2.3.25}$$

Here $\mathbb{C}[[\hbar]]$ (resp. $\mathbb{C}((\hbar))$) denotes the formal power (resp. Laurent) series in $\hbar$. This algebra contains singular elements those that do not have a properly defined $q \to 1$ limit. Let $\mathcal{A} \subset \widetilde{\mathcal{U}_q(\widehat{\mathfrak{g}})}$ be the subalgebra generated $\mathcal{U}_q(\widehat{\mathfrak{g}})$ and $\hbar^{-1}\mathrm{ker}(\varphi)$. Then the Yangian $\mathcal{Y}(\mathfrak{g})$ as an algebra is isomorphic to the quotient $\mathcal{A}/\hbar\mathcal{A} \cong \mathcal{Y}(\mathfrak{g})$ [4].

In such a way the Yangian $\mathcal{Y}(\mathfrak{sl}(2))$ can be obtained by taking a rational $q \to 1$ limit of certain singular combinations of the generators of $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$. Here we shall be very concise and rely a lot on the evaluation map (for some heuristic arguments see appendix A; for more thorough considerations see e.g. [86–89]). The Yangian generators of $\mathcal{Y}(\mathfrak{sl}(2))$ are obtained by the following prescription,

$$\widehat{E}^{\pm} = \pm\alpha \lim_{q\to 1} \frac{\xi_0^{\mp} - \xi_1^{\pm}}{q - q^{-1}} \,. \tag{2.3.26}$$

The Lie algebra generators are recovered by

$$E^{\pm} = \lim_{q\to 1} \xi_1^{\pm} = \lim_{q\to 1} \xi_0^{\mp} \qquad \text{and} \qquad H = \lim_{q\to 1} h_1 = -\lim_{q\to 1} h_0 \,. \tag{2.3.27}$$

We will next show that the quantum affine reflection algebras considered in sections 2.2.1 and 2.2.2 in the rational $q \to 1$ limit specializes to the Yangian reflection algebras considered in sections 2.3.1 and 2.3.2 respectively.

Proposition 2.3.7. *The quantum affine coideal subalgebra $\mathcal{B} \subset \mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$ defined by the Proposition 2.2.2 in the rational $q \to 1$ limit specializes to the twisted Yangian $\mathcal{Y}(\mathfrak{g}, \mathfrak{a})$ of type I defined by the Proposition 2.3.3.*

Proof. Recall that $\mathcal{B}$ is generated by the twisted affine generators (2.2.9)

$$B_0^{+} = \xi_0'^{+} k_0 - c/q\, \xi_1^{-} k_0 \,, \qquad B_0^{-} = \xi_0^{-} k_0 - c\, q\, \xi_1'^{+} k_0 \,. \tag{2.3.28}$$

and the Cartan element $k_0 k_1^{-1}$. Note that $\xi_0'^{+} k_0 = q^{-2} \xi_0^{+}$. We will be using the following series expansion,

$$k_i \to 1 + (q-1) h_i + \mathcal{O}(\alpha^2) \,, \tag{2.3.29}$$

where $\mathcal{O}(\alpha^2)$ represent the higher order in α terms (here $\alpha \sim \hbar$). Then by substituting $c \to q^{-2t}$ we find

$$\begin{aligned} \lim_{q\to 1} \frac{\alpha\, q^2 B_0^{+}}{q - q^{-1}} &= \lim_{q\to 1} \left[\alpha\, \frac{\xi_0^{+} - \xi_1^{-}}{q - q^{-1}} + \alpha\, \frac{q-1}{q - q^{-1}} \big(2t\, \xi_1^{-} - \xi_1^{-} - \xi_1^{-} h_0 \big) + \mathcal{O}(\alpha^2) \right] \\ &= -\widehat{E}^{-} + \alpha\, t\, E^{-} + \frac{\alpha}{4} \left(E^{-} H + H\, E^{-} \right) = -\widetilde{E}^{-} \,, \end{aligned} \tag{2.3.30}$$

and

$$\begin{aligned} \lim_{q\to 1} \frac{\alpha\, B_0^{-}}{q - q^{-1}} &= \lim_{q\to 1} \left[\alpha\, \frac{\xi_0^{-} - \xi_1^{+}}{q - q^{-1}} + \alpha\, \frac{q-1}{q - q^{-1}} \big(\xi_0^{-} h_0 - (1-2t) \xi_1^{+} - \xi_1^{+} h_0 + h_1 \xi_1^{+} \big) + \mathcal{O}(\alpha^2) \right] \\ &= \widehat{E}^{+} + \alpha\, t\, E^{+} + \frac{\alpha}{4} \left(E^{+} H + H\, E^{+} \right) = \widetilde{E}^{+} \,. \end{aligned} \tag{2.3.31}$$

Finally,

$$\lim_{q\to 1} \frac{1 - k_0 k_1^{-1}}{q - q^{-1}} = H \,. \tag{2.3.32}$$

These coincide with (2.3.9) as required. □

Proposition 2.3.8. *The quantum affine coideal subalgebra $\mathcal{B} \subset \mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$ defined by the Proposition 2.2.4 in the rational $q \to 1$ limit specializes to the twisted Yangian $\mathcal{Y}(\mathfrak{g}, \mathfrak{g})$ of type II defined by the Proposition 2.3.6.*

Proof. We will prove this proposition for the representations only. Let $z = q^{-4\,l\,u}$. Then, upon rescaling $s \to q^{4\,k\,s}$, the twisted affine generators (2.2.16) and (2.2.18) in the rational $q \to 1$ limit specialize to the twisted Yangian generators (2.3.16) by the following prescription,

$$\lim_{q\to 1} \frac{q^{-2}\,T_{l,z}(B_0^-) - 2\,T_{l,z}(\xi_1'^+)}{(q-q^{-1})^2} = T_{l,u}(\widetilde{\widetilde{E}}^+)\,, \qquad \lim_{q\to 1} \frac{q^{2}\,T_{l,z}(B_0^+) - 2\,T_{l,z}(\xi_1^-)}{(q-q^{-1})^2} = T_{l,u}(\widetilde{\widetilde{E}}^-)\,, \tag{2.3.33}$$

and

$$\begin{aligned} \lim_{q\to 1} \frac{q^{-2}\,(T_{l,z}\otimes T_{k,s})[\Delta(B_0^-)] - 2\,(T_{l,z}\otimes T_{k,s})[\Delta(\xi_1'^+)]}{(q-q^{-1})^2} &= (T_{l,u}\otimes T_{k,s})[\Delta(\widetilde{\widetilde{E}}^+)]\,, \\ \lim_{q\to 1} \frac{q^{2}\,(T_{l,z}\otimes T_{k,s})[\Delta(B_0^+)] - 2\,(T_{l,z}\otimes T_{k,s})[\Delta(\xi_1^-)]}{(q-q^{-1})^2} &= (T_{l,u}\otimes T_{k,s})[\Delta(\widetilde{\widetilde{E}}^-)]\,. \end{aligned} \tag{2.3.34}$$

□

2.4 Reflection algebras for $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$

Algebra. The quantum affine Lie superalgebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ in the Drinfeld-Jimbo realization is generated by the fermionic Chevalley generators $\xi_1^\pm$, the Cartan generators k_1, k_2 and their inverses (here $k_2 = q^{h_2}$, and h_2 is the non-supertraceless generator completing the superalgebra $\mathfrak{sl}(1|1)$ to $\mathfrak{gl}(1|1)$), and the affine fermionic Chevalley generators $\xi_0^\pm$ and the corresponding affine Cartan generators k_0, k_0^{-1}. The extended (symmetric) Cartan matrix is given by

$$(\widehat{a}_{ij})_{0\le i,j\le 2} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ -2 & 2 & 0 \end{pmatrix}. \tag{2.4.1}$$

The corresponding root space has a basis of two fermionic roots, $\hat{\pi} = \{\alpha_0, \alpha_1\}$. The commutation relations are as follows, for $0 \le i, j \le 2$ (Chevalley generators corresponding to the Cartan generator k_2 are absent):

$$[k_i, k_j] = 0\,, \qquad [k_i, \xi_j^\pm] = q^{\pm\widehat{a}_{ij}}\xi_j^\pm\,, \qquad \{\xi_i^+, \xi_j^-\} = \delta_{ij}\frac{k_i - k_i^{-1}}{q - q^{-1}}\,. \tag{2.4.2}$$

Here $\{a, b\} = a\,b + b\,a$ denotes the anti-commutator. The graded right adjoint action is defined by

$$\big(\mathrm{ad}_r\,\xi_i^+\big)\,a = (-1)^{[\xi_i^+][a]}k_i^{-1}a\,\xi_i^+ - k_i^{-1}\xi_i^+ a\,, \quad \big(\mathrm{ad}_r\,\xi_i^-\big)\,a = (-1)^{[\xi_i^-][a]}a\,\xi_i^- - \xi_i^- k_i\,a\,k_i^{-1}\,. \tag{2.4.3}$$

where $(-1)^{[\xi_i^{\pm}][a]}$ is the grading factor. Notice that the block $(\widehat{a}_{ij})_{0\leq i,j\leq 1}$ is trivial, and thus the right adjoint action for this block is equivalent to the regular graded commutator. This shall have important consequences for constructing the reflection algebra for a vector boundary. For this reason we shall also be in the need of the level-one Cartan generators $h_2^{\pm}$, which have coproducts defined by

$$\begin{aligned}\Delta h_2^{+} &= h_2^{+}\otimes 1 + 1\otimes h_2^{+} + 2\,\xi_0^{+}k_1\otimes \xi_1^{+}\,,\\ \Delta h_2^{-} &= h_2^{-}\otimes 1 + 1\otimes h_2^{-} + 2\,\xi_0^{-}\otimes k_0^{-1}\xi_1^{-}\,.\end{aligned} \tag{2.4.4}$$

Representation. We define the fundamental evaluation representation T_z of $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ on a graded two-dimensional vector space $\mathcal{V}$. Let $\mathcal{V}=\{v_1,v_2\}$ and $\mathcal{V}'=\{v_1',v_2'\}$, then $v_1v_1'=v_1'v_1$, $v_2v_2'=-v_2'v_2$, and $v_2v_2=v_2'v_2'=0$. Let $e_{j,k}$ be 2×2 matrices satisfying $(e_{j,k})_{j',k'}=\delta_{j,j'}\delta_{k,k'}$. Then the representation T_z is defined by

$$\begin{aligned}&T_z(\xi_1^{+})=e_{1,2}\,, && T_z(\xi_0^{+})=z\,e_{2,1}\,, && T_z(h_2)=e_{1,1}-e_{2,2}\,,\\ &T_z(\xi_1^{-})=e_{2,1}\,, && T_z(\xi_0^{-})=-z^{-1}e_{1,2}\,, && T_z(h_2^{\pm})=\frac{z^{\pm1}q}{q^{-1}-q}(e_{1,1}-e_{2,2})\,,\\ &T_z(k_1)\;=q\,e_{1,1}+q\,e_{2,2}\,, && T_z(k_0)\;=q^{-1}e_{1,1}+q^{-1}e_{2,2}\,, && T_z(k_2)=q\,e_{1,1}+q^{-1}e_{2,2}\,.\end{aligned} \tag{2.4.5}$$

We choose the boundary vector space $\mathcal{W}$ to be equivalent to $\mathcal{V}$. Then the boundary representation T_s on $\mathcal{W}$ is obtained from (2.4.5) by replacing z with s.

The fundamental R-matrix satisfying Yang-Baxter equation (2.1.4) is given by

$$R(z)=\begin{pmatrix}1&0&0&0\\0&r&1-q\,r&0\\0&1-r/q&r&0\\0&0&0&-1+(q+1/q)\,r\end{pmatrix},\qquad\text{where}\qquad r=\frac{z-1}{q\,z-1/q}\,. \tag{2.4.6}$$

2.4.1 Singlet boundary

Consider the reflection equation (2.1.5) in the space $\mathcal{V}\otimes\mathcal{V}\otimes\mathbb{C}$ with the R-matrix defined by (2.4.6). Then the general solution of the reflection equation is [93]

$$K(z)=\begin{pmatrix}1&0\\0&k\end{pmatrix},\qquad\text{where}\qquad k=\frac{c\,z-1}{z(c-z)}\,, \tag{2.4.7}$$

and $c\in\mathbb{C}$ is an arbitrary complex number.

The general solution (2.4.7), in contrast to (2.2.5), is already of the diagonal form and thus intertwines all (level-zero) Cartan generators k_i, but does not satisfy the intertwining equation neither for any of the Chevalley generators nor for the level-one Cartan generators $h_2^{\pm}$. Hence we call Cartan generators k_i the *preserved* generators, while the generators $\xi_i^{\pm}$ and $h_2^{\pm}$ are the *broken* generators. This setup is consistent with the following involution and the quantum affine coideal subalgebra.

Proposition 2.4.1. *Let the involution* Θ *act on the root space* Φ *as*

$$\Theta(\alpha_0) = -\alpha_1 \,. \tag{2.4.8}$$

Then it defines a quantum affine coideal subalgebra $\mathcal{B} \subset \mathcal{A} = \mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ *generated by the Cartan elements* k_2, $k_0 k_1^{-1}$, *and the twisted affine generators*

$$\begin{aligned} B_0^+ &= \xi_0'^+ k_0 - d_+ \,\tilde\theta(\xi_0'^+) k_0 \,, \qquad & \tilde\theta(\xi_0'^+) &= \xi_1^- \,, \\ B_0^- &= \xi_0^- k_0 - d_- \,\tilde\theta(\xi_0^-) k_0 \,, \qquad & \tilde\theta(\xi_0^-) &= \xi_1'^+ \,, \end{aligned} \tag{2.4.9}$$

where $\xi_i'^+ = k_i^{-1}\xi_i^+$ *and* $d_\pm \in \mathbb{C}$ *are arbitrary complex numbers.*

Proof. The coideal property is trivial for the Cartan elements, and for (2.4.9) follows directly from (2.2.10). ☐

Proposition 2.4.2. *The quantum affine coideal subalgebra defined above with* $d_+ = -d_- = q\,c$ *where* $c \in \mathbb{C}$ *is an arbitrary complex number, is a reflection algebra for a singlet boundary.*

Proof. The representation T_z of the generators of $\mathcal{B}$ is given by

$$T_z(k_0 k_1^{-1}) = q^{-2}(e_{1,1} + e_{2,2}) \,, \quad T_z(B_0^+) = (z - q^{-1} d_+)\, e_{2,1} \,, \quad T_z(B_0^-) = q^{-2}(q\, z^{-1} - d_-) e_{1,2} \,. \tag{2.4.10}$$

and $T_z(k_2)$ was given in (2.4.5). Let $K(z)$ be any 2×2 matrix. Then the intertwining equation for $k_0 k_1^{-1}$ and k_2 restricts $K(z)$ to be of the diagonal form. This gives $K(z) = e_{1,1} + k\, e_{2,2}$ up to an overall scalar factor. Next, the intertwining equation for $B^\pm$ gives

$$d_+(k-1)z - q(z^2 k - 1) = 0 \,, \qquad d_-(k-1)z + q(z^2 k - 1) = 0 \,, \tag{2.4.11}$$

having a unique solution $d_+ = -d_- = q\,c$ and $k = \dfrac{c\,z - 1}{z(c-z)}$, where $c \in \mathbb{C}$ is any complex number. This coincides with (2.4.7). ☐

2.4.2 Vector boundary

Consider the reflection equation (2.1.5) in the tensor space $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{W}$ with the R-matrix defined by (2.4.6). Then there exists a solution of the reflection equation

$$K(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 - k/q & k & 0 \\ 0 & k & 1 - q\,k & 0 \\ 0 & 0 & 0 & 1 - (q + q^{-1})\,k \end{pmatrix}, \quad \text{where} \quad k = \frac{(q - q^{-1})(z^2 - 1)}{q^{-2} - c\,z + q^2 z^2} \,, \tag{2.4.12}$$

and $c \in \mathbb{C}$ is an arbitrary complex number.

This K-matrix satisfies the intertwining equation (2.1.7) for all generators of the Lie superalgebra $\mathfrak{gl}(1|1)$. Thus we call Cartan generators k_i and Chevalley generators $\xi_1^\pm$ the *preserved* generators, while the affine Chevalley generators $\xi_0^\pm$ and the level-one Cartan generators $h_2^\pm$ are the *broken* generators. Next, we identify the corresponding quantum affine coideal subalgebra consistent with the reflection matrix (2.4.12).

Proposition 2.4.3. *Let the involution Θ act on the root space Φ as*

$$\Theta(\alpha_0) = -\alpha_0 - 2\alpha_1 \,, \qquad\qquad \Theta(\alpha_1) = \alpha_1 \,. \tag{2.4.13}$$

Then it defines a quantum affine coideal subalgebra $\mathcal{B} \subset \mathcal{A} = \mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ generated by the Cartan generators k_i, the Chevalley generators $\xi_1^{\pm}$, and the twisted affine generator

$$B_0^- = \xi_0^- - d_-\,\tilde{\theta}(\xi_0^-)\,, \qquad\qquad \tilde{\theta}(\xi_0^-) = [h_2^+\,,\,\xi_1'^+]\,, \tag{2.4.14}$$

where $\xi_i'^+ = k_i^{-1}\xi_i^+$ and $d_- \in \mathbb{C}^\times$ is an arbitrary complex number.

Proof. The twisted affine generator (2.4.14) satisfies coideal property,

$$\Delta(B_0^-) = \xi_0^- \otimes k_0^{-1} - d_-\,\tilde{\theta}(\xi_0^-) \otimes k_1^{-1} + 1 \otimes B_0^- + 2d_-\{\xi_0'^+\,,\,\xi_1'^+\} \otimes k_0^{-1}k_1^{-1}\xi_1'^+ \in \mathcal{A} \otimes \mathcal{B}\,. \tag{2.4.15}$$

The property follows by definition for k_i and $\xi_1^{\pm}$. □

Remark 2.4.1. This algebra may alternatively be generated by k_i, $\xi_1^{\pm}$, and the twisted affine generator

$$B_0^+ = \xi_0'^+ - d_+\,\tilde{\theta}(\xi_0'^+)\,, \qquad\qquad \tilde{\theta}(\xi_0'^+) = [h_2^-\,,\,\xi_1^-]\,, \tag{2.4.16}$$

having coproduct

$$\Delta(B_0^+) = \xi_0'^+ \otimes k_0^{-1} - d_x\,\tilde{\theta}(\xi_0'^+) \otimes k_1^{-1} + 1 \otimes B_0^+ + 2d_+\{\xi_0^-\,,\,\xi_1^-\} \otimes k_0^{-1}k_1^{-1}\xi_1^- \in \mathcal{A} \otimes \mathcal{B}\,, \tag{2.4.17}$$

and $d_+ = d_-^{-1}(q^{-1}-q)^2/4$. The generators $B_0^{\pm}$ are related by

$$(\mathrm{ad}_r\,\xi_1^-)B_0^- = \frac{2d_-}{q-q^{-1}}(\mathrm{ad}_r\,\xi_1^+)B_0^+\,. \tag{2.4.18}$$

Remark 2.4.2. This algebra is not of a canonical form (compare (2.1.47), (2.2.16) and (2.4.14)). This is due to the all-zero entries in the block $0 \le i,j \le 1$ of the extended Cartan matrix $\widehat{a}_{ij}$ (2.4.1), and thus the ad_r-action is equivalent to the usual graded commutator. In such a way the level-one generators $h_2^{\pm}$ are employed to ensure the coideal property.

Proposition 2.4.4. *The quantum affine coideal subalgebra defined above with $d_- = d_+ = (q^{-1}-q)/2$ is a reflection algebra for a vector boundary.*

Proof. The representation $(T_z \otimes T_s)$ of the coproducts of the Lie generators of $\mathcal{B}$ is given by

$$\begin{aligned}
(T_z \otimes T_s)[\Delta(\xi^+)] &= q\,e_{1,2} + e_{1,3} + e_{2,4} - q\,e_{3,4}\,,\\
(T_z \otimes T_s)[\Delta(\xi^-)] &= e_{2,1} + q^{-1}e_{3,1} + q^{-1}e_{4,2} - e_{4,3}\,,\\
(T_z \otimes T_s)[\Delta(k_0)] &= q^{-2}(e_{1,1} + e_{2,2} + e_{3,3} + e_{4,4})\,,\\
(T_z \otimes T_s)[\Delta(k_1)] &= q^{2}\,(e_{1,1} + e_{2,2} + e_{3,3} + e_{4,4})\,,\\
(T_z \otimes T_s)[\Delta(k_2)] &= q^{2}e_{1,1} + e_{2,2} + e_{3,3} + q^{-2}e_{4,4}\,,
\end{aligned} \tag{2.4.19}$$

and of the twisted affine generators by

$$(T_z \otimes T_s)[\Delta(B_0^+)] = \alpha\, e_{2,1} + \beta\, e_{3,1} + \beta\, e_{4,2} - \alpha\, e_{4,3}\,,$$
$$(T_z \otimes T_s)[\Delta(B_0^-)] = \gamma\, e_{1,2} + \delta\, e_{1,3} + \gamma\, e_{2,4} - \gamma\, e_{3,4}\,, \tag{2.4.20}$$

where

$$\alpha = q\, s + 2d_+((q^{-2}-1)^{-1}s^{-1} - z^{-1})\,, \quad \beta = q(q\, z - 2d_+(q^2-1)^{-1}z^{-1})\,,$$
$$\gamma = -q^{-1}(q\, s^{-1} + 2d_-((q^{-2}-1)s - z))\, \delta = -(q\, z^{-1} - 2d_-(q^2-1)^{-1}z)\,, \tag{2.4.21}$$

Let $K(z)$ be any 4×4 matrix. Then the intertwining equation for the Lie generators (2.4.19) constrain $K(z)$ to the form given in (2.4.12) up to an unknown function k and an overall scalar factor. Next, the intertwining equation for the twisted affine generators (2.4.20) has a unique solution

$$d_+ = d_- = (q^{-1} - q)/2\,, \qquad k = \frac{(q - q^{-1})(z^2 - 1)}{q^{-2} - (s + s^{-1})z + q^2 z^2}\,, \tag{2.4.22}$$

which coincides with k given in (2.4.12) provided $c = s + s^{-1}$. □

Remark 2.4.3. The coideal subalgebra given in proposition 2.4.2 does not lead to a unique reflection matrix for a vector boundary. The boundary intertwining equation in this case defines the reflection matrix up to an overall scalar factor and one unknown function,

$$K(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 + (q^2 z - c)k' & q(c-s)k' & 0 \\ 0 & q(c - s^{-1})k' & 1 + k & 0 \\ 0 & 0 & 0 & 1 + k + (q^2 z^{-1} - c)k' \end{pmatrix} \tag{2.4.23}$$

where

$$k = \frac{q^2\left(s + z(c-s)(cs-1)k' - s\, z^2\right)}{s\, z\,(q^2 z - c)}\,, \tag{2.4.24}$$

and $c, s \in \mathbb{C}$ are arbitrary complex numbers. Here $k' = k'(z)$ is an unknown function that needs to be obtained by solving the reflection equation (giving i.e. $k' = 0$). In such a way this coideal subalgebra is not a reflection algebra for a vector boundary as it does not lead to a reflection matrix automatically satisfying the reflection equation.

The coideal subalgebra given in proposition 2.4.4 is compatible with a singlet boundary; the corresponding reflection matrix is trivial.

2.5 Reflection algebras for $\mathcal{Y}(\mathfrak{gl}(1|1))$

Algebra. The Yangian $\mathcal{Y}(\mathfrak{gl}(1|1))$ is generated by the level-zero Chevalley generators $E^\pm$, Cartan generators H and the non-supertraceless generator H_2, and the level-one

Yangian generators $\widehat{E}^{\pm}$ and the corresponding level-one Cartan generators $\widehat{H}$ and $\widehat{H}_2$. The commutation relations of the algebra are

$$\begin{aligned}
&\{E^+, E^-\} = H\,, \qquad \{E^{\pm}, \widehat{E}^{\mp}\} = \widehat{H}\,, \qquad [H, E^{\pm}] = [H, \widehat{E}^{\pm}] = [H, \widehat{H}] = 0\,,\\
&[H_2, E^{\pm}] = \pm 2E^{\pm}\,, \quad [H_2, \widehat{E}^{\pm}] = [\widehat{H}_2,\, E^{\pm}] = \pm 2\widehat{E}^{\pm}\,.
\end{aligned} \tag{2.5.1}$$

The Hopf algebra structure is equipped with the following coproduct,

$$\begin{aligned}
\Delta(H) &= H\otimes 1 + 1\otimes H\,, & \Delta(\widehat{H}) &= \widehat{H}\otimes 1 + 1\otimes \widehat{H}\,,\\
\Delta(H_2) &= H_2\otimes 1 + 1\otimes H_2\,, & \Delta(\widehat{H}_2) &= \widehat{H}_2\otimes 1 + 1\otimes \widehat{H}_2 - \alpha\,(E^+\otimes E^- + E^-\otimes E^+)\,,\\
\Delta(E^{\pm}) &= E^{\pm}\otimes 1 + 1\otimes E^{\pm}\,, & \Delta(\widehat{E}^{\pm}) &= \widehat{E}^{\pm}\otimes 1 + 1\otimes \widehat{E}^{\pm} \mp \frac{\alpha}{2}\,(E^{\pm}\otimes H - H\otimes E^{\pm})\,.
\end{aligned} \tag{2.5.2}$$

Representation. The evaluation representation on the graded two-dimensional vector space $\mathcal{V}$ is defined by

$$\begin{aligned}
&T_u(E^+) = e_{1,2}\,, \quad T_u(E^-) = e_{2,1}\,, \quad T_u(H) = e_{1,1}+e_{2,2}\,, \quad T_u(H_2) = e_{1,1}-e_{2,2}\,,\\
&T_u(\widehat{E}^+) = u\,e_{1,2}\,, \quad T_u(\widehat{E}^-) = u\,e_{2,1}\,, \quad T_u(\widehat{H}) = u\,(e_{1,1}+e_{2,2})\,, \quad T_u(\widehat{H}_2) = u\,(e_{1,1}-e_{2,2})\,,
\end{aligned} \tag{2.5.3}$$

and $T_u(\alpha) = 1$. We choose the boundary vector space $\mathcal{W}$ to be equivalent to $\mathcal{V}$. The boundary representation T_s on $\mathcal{W}$ is obtained from (2.5.3) by replacing z with s.

The fundamental R-matrix satisfying Yang-Baxter equation (2.1.57) is

$$R(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & r & 1-r & 0 \\ 0 & 1-r & r & 0 \\ 0 & 0 & 0 & -1+2r \end{pmatrix}, \qquad \text{where} \qquad r = \frac{u}{u+1}\,. \tag{2.5.4}$$

2.5.1 Singlet boundary

Consider the reflection equation (2.1.58) in the tensor space $\mathcal{V}\otimes\mathcal{V}\otimes\mathbb{C}$ with the R-matrix defined by (2.5.4). Then the general solution of the reflection equation is [83]

$$K(u) = \begin{pmatrix} 1 & 0 \\ 0 & k \end{pmatrix}, \qquad \text{where} \qquad k = \frac{c+u}{c-u}\,, \tag{2.5.5}$$

and $c\in\mathbb{C}$ is an arbitrary complex number.

This K-matrix intertwines the Cartan generators H and H_2, but does not satisfy the intertwining equation for any other generators of $\mathfrak{gl}(1|1)$ (and $\mathcal{Y}(\mathfrak{gl}(1|1))$). Hence we call Cartan generators H and H_2 the *preserved* generators, while the rest are the *broken* generators. This setup allows us to define the following twisted Yangian.

Proposition 2.5.1. *Let the involution θ act on the Lie algebra $\mathfrak{g} = \mathfrak{gl}(1|1)$ as*

$$\theta(H) = H\,, \qquad \theta(H_2) = H_2\,, \qquad \theta(E^{\pm}) = -E^{\pm}\,, \tag{2.5.6}$$

defining a symmetric pair $(\mathfrak{g}, \theta(\mathfrak{g}))$. Then involution θ can be extended to the involution $\bar\theta$ of $\mathcal{Y}(\mathfrak{g})$ such that

$$\bar\theta(\widehat{H}) = -\widehat{H}\,, \qquad \bar\theta(\widehat{H}_2) = -\widehat{H}_2\,, \qquad \bar\theta(\widehat{E}^{\pm}) = \widehat{E}^{\pm}\,, \qquad \bar\theta(\alpha) = -\alpha\,. \tag{2.5.7}$$

Proposition 2.5.2. *The twisted Yangian $\mathcal{Y}(\mathfrak{g}, \theta(\mathfrak{g}))$ of type I for $\mathfrak{g} = \mathfrak{gl}(1|1)$ and $\theta(\mathfrak{g}) = \{H, H_2\}$ is $\bar\theta$–fixed coideal subalgebra of $\mathcal{Y}(\mathfrak{g})$ generated by the Cartan generators H and H_2, and the twisted Yangian generators*

$$\widetilde{E}^{\pm} = \widehat{E}^{\pm} \pm \alpha t\, E^{\pm} \mp \frac{\alpha}{2} H E^{\pm}\,, \tag{2.5.8}$$

where $t \in \mathbb{C}$ is an arbitrary complex number.

Proof. The twisted generators (2.5.8) are in the positive eigenspace of the involution $\bar\theta$ (2.5.7) and satisfy the coideal property

$$\Delta(\widetilde{E}^{\pm}) = \widetilde{E}^{\pm} \otimes 1 + 1 \otimes \widetilde{E}^{\pm} \mp \alpha\, E^{\pm} \otimes H \in \mathcal{Y}(\mathfrak{g}) \otimes \mathcal{Y}(\mathfrak{g}, \theta(\mathfrak{g}))\,. \tag{2.5.9}$$

The same properties are obvious for H and H_2. □

Proposition 2.5.3. *The twisted Yangian $\mathcal{Y}(\mathfrak{g}, \theta(\mathfrak{g}))$ defined above is a reflection algebra for a singlet boundary.*

Proof. The representation T_u of the twisted generators of $\mathcal{Y}(\mathfrak{g}, \theta(\mathfrak{g}))$ is given by

$$T_u(\widetilde{E}^{+}) = (u + t - 1/2)\, e_{1,2}\,, \qquad T_u(\widetilde{E}^{-}) = (u - t + 1/2)\, e_{1,2}\,, \tag{2.5.10}$$

For H and H_2 it was given in (2.5.3). Let $K(u)$ be any 2×2 matrix. Then the intertwining equation for H and H_2 restricts $K(u)$ to be of the diagonal form. This gives $K(u) = e_{1,1} + k\, e_{2,2}$ up to an overall scalar factor. Next, the intertwining equation for $\widetilde{E}^{\pm}$ has a unique solution $k = \dfrac{t + u - 1/2}{t - u - 1/2}$ which coincides with (2.5.5) provided $t = c + 1/2$ and $a = b = 0$. □

2.5.2 Vector boundary

Consider the reflection equation (2.1.5) in the tensor space $\mathcal{V} \otimes \mathcal{V} \otimes \mathcal{W}$ with the R-matrix defined by (2.3.4). Then there exists a solution of the reflection equation,

$$K(u) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1-k & k & 0 \\ 0 & k & 1-k & 0 \\ 0 & 0 & 0 & 1-2k \end{pmatrix}, \qquad \text{where} \qquad k = \frac{2u}{(u+1)^2 - c^2}\,, \tag{2.5.11}$$

and $c \in \mathbb{C}$ is an arbitrary complex number.

This K-matrix satisfies the intertwining equation for all generators of $\mathfrak{gl}(1|1)$. We call the level-zero generators $E^{\pm}$, H and H_2 the *preserved* generators, while the level-one generators $\widehat{E}^{\pm}$, $\widehat{H}$ and $\widehat{H}_2$ are the *broken* generators. This setup leads to the following twisted Yangian.

Proposition 2.5.4. *Let θ be the trivial involution of the Lie algebra $\mathfrak{g} = \mathfrak{gl}(1|1)$,*

$$\theta(H) = H\,, \qquad \theta(H_2) = H_2\,, \qquad \theta(E^{\pm}) = E^{\pm} \qquad \Longrightarrow \qquad \theta(\mathfrak{g}) = \mathfrak{g}\,. \tag{2.5.12}$$

Then it can be extended to a non-trivial involution $\tilde{\theta}$ of $\mathcal{Y}(\mathfrak{gl}(1|1))$ such that

$$\bar{\theta}(\widehat{H}) = -\widehat{H}\,, \qquad \bar{\theta}(\widehat{H}_2) = -\widehat{H}_2\,, \qquad \bar{\theta}(\widehat{E}^{\pm}) = -\widehat{E}^{\pm}\,, \qquad \bar{\theta}(\alpha) = -\alpha\,. \tag{2.5.13}$$

Proposition 2.5.5. *The twisted Yangian $\mathcal{Y}(\mathfrak{g},\mathfrak{g})$ of type II for $\mathfrak{g} = \mathfrak{gl}(1|1)$ and $\theta(\mathfrak{g}) = \mathfrak{g}$ is $\bar{\theta}$–fixed coideal subalgebra of $\mathcal{Y}(\mathfrak{g})$ generated by the level-zero generators and the level-two twisted Yangian generators*

$$\widetilde{\widetilde{E}}^{\pm} = \pm\frac{1}{2}\Big([\widehat{H}_2,\,\widehat{E}^{\pm}] + \alpha\,(\pm t\,\widehat{E}^{\pm} + E^{\pm}\widehat{H} - \widehat{E}^{\pm}H)\Big) \qquad \text{and} \qquad \widetilde{\widetilde{H}} = \{E^{\pm},\,\widetilde{\widetilde{E}}^{\mp}\}\,, \tag{2.5.14}$$

where $t \in \mathbb{C}$ is any complex number.

Proof. The twisted generators (2.5.14) are in the positive eigenspace of the involution $\bar{\theta}$ (2.5.13) and satisfy the coideal property

$$\Delta(\widetilde{\widetilde{E}}^{\pm}) = \widetilde{\widetilde{E}}^{\pm}\otimes 1 + 1\otimes\widetilde{\widetilde{E}}^{\pm} \pm \alpha\,(\widehat{H}\otimes E^{\pm} - \widehat{E}^{\pm}\otimes H) + \mathcal{O}(\alpha^2) \in \mathcal{Y}(\mathfrak{g})\otimes\mathcal{Y}(\mathfrak{g},\mathfrak{g})\,, \tag{2.5.15}$$

and for $\widetilde{\widetilde{H}}$ and the level-zero generators these properties follow identically. □

Proposition 2.5.6. *The twisted Yangian $\mathcal{Y}(\mathfrak{g},\mathfrak{g})$ defined above with $t = 0$ is a reflection algebra for a vector boundary.*

Proof. The representation $(T_u \otimes T_s)$ of the coproducts of the Lie generators of $\mathcal{Y}(\mathfrak{g},\mathfrak{g})$ is given by

$$\begin{aligned} &(T_u\otimes T_s)[\Delta(E^+)] = e_{1,2}+e_{1,3}+e_{2,4}-e_{3,4}\,, \qquad (T_u\otimes T_s)[\Delta(H)] \;= 2\textstyle\sum_{i=1}^4 e_{i,i}\,,\\ &(T_u\otimes T_s)[\Delta(E^-)] = e_{2,1}+e_{3,1}+e_{4,2}-e_{4,3}\,, \qquad (T_u\otimes T_s)[\Delta(H_2)] = 2\,(e_{1,1}-e_{4,4})\,, \end{aligned} \tag{2.5.16}$$

and of the twisted Yangian generators by

$$\begin{aligned} &(T_z\otimes T_s)[\Delta(\widetilde{\widetilde{E}}^+)] = \alpha\,e_{1,2}+\beta\,e_{1,3}+\beta\,e_{2,4}-\alpha\,e_{3,4}\,, \quad (T_z\otimes T_s)[\Delta(\widetilde{\widetilde{H}})] = \eta\textstyle\sum_{i=1}^4 e_{i,i}\,,\\ &(T_z\otimes T_s)[\Delta(\widetilde{\widetilde{E}}^-)] = \gamma\,e_{2,1}+\delta\,e_{3,1}+\gamma\,e_{4,2}-\gamma\,e_{4,3}\,, \end{aligned} \tag{2.5.17}$$

where

$$\alpha = 2s(2s+t)+4u+t-3\,, \quad \beta = (2u-1)(2u+t-1), \quad \eta = \tfrac{1}{16}((4s+t)^2 + (4u+t)^2$$
$$\gamma = 2s(2s+t)-4u-t-3\,, \quad \delta = (2u+1)(2u+t+1), \qquad\qquad -2t^2-8). \tag{2.5.18}$$

Let $K(u)$ be any 4×4 matrix. The intertwining equation for the Lie generators (2.5.16) constrain $K(u)$ to the form given in (2.5.11) up to unknown function k and overall scalar factor. Next, the intertwining equation for the twisted Yangian generators (2.5.17) constrain $t=0$ and has a unique solution $k = \dfrac{2u}{(u+1)^2-s^2}$ which coincides with (2.5.11) provided $c=s$. □

Remark 2.5.1. The twisted Yangian given in proposition 2.5.3 does not lead to a unique reflection matrix for a vector boundary. The boundary intertwining equation in this case defines the reflection matrix up to an overall scalar factor and one unknown function,

$$K(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1+(1-c+u)k' & (c-s)k' & 0 \\ 0 & (c+s)k' & 1+k & 0 \\ 0 & 0 & 0 & 1+k+(1-c-u)k' \end{pmatrix} \tag{2.5.19}$$

where

$$k = \frac{(c^2-s^2)k'-2u}{1-c+u}\,, \qquad c = t-1/2\,, \tag{2.5.20}$$

and $t,s\in\mathbb{C}$ are arbitrary complex numbers. Here $k'=k'(u)$ is an unknown function that needs further to be obtained by solving the reflection equation (giving i.e. $k'=0$ or $k'=(2u)/(s^2+c(1+u)^2-u(1+u)^2)$). In such a way this twisted Yangian is not a reflection algebra for a vector boundary as it does not lead to a reflection matrix automatically satisfying the reflection equation. This is in agreement with the remark 2.4.3.

The twisted Yangian given in proposition 2.5.6 is compatible with a singlet boundary; the corresponding reflection matrix is trivial.

2.5.3 Yangian limit

Lie algebras $\mathfrak{sl}(2)$ and $\mathfrak{gl}(1|1)$ are very similar, thus the Yangian limit of coideal subalgebras of $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ is obtained in a very similar way as it was done in section 2.3.3. The Yangian $\mathcal{Y}(\mathfrak{gl}(1|1))$ can be obtained by taking a rational $q\to 1$ limit of the singular combinations of the generators of $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$,

$$\widehat{E}^{\pm} = \alpha \lim_{q\to 1} \frac{\xi_0^{\mp} \pm \xi_1^{\pm}}{q-q^{-1}}\,, \tag{2.5.21}$$

The Lie algebra generators are given by

$$E^{\pm} = \lim_{q\to 1} \xi_1^{\pm} = \mp \lim_{q\to 1} \xi_0^{\mp} \qquad \text{and} \qquad H = \lim_{q\to 1} h_1 = -\lim_{q\to 1} h_0\,. \tag{2.5.22}$$

Next will show that the quantum affine reflection algebras considered in sections 2.4.1 and 2.4.2 in the rational $q \to 1$ limit specializes to the Yangian reflection algebras considered in sections 2.5.1 and 2.5.2 respectively.

Proposition 2.5.7. *The quantum affine coideal subalgebra $\mathcal{B} \subset \mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ defined by the Proposition 2.4.2 in the rational $q \to 1$ limit specializes to the twisted Yangian $\mathcal{Y}(\mathfrak{g},\mathfrak{a})$ of type I defined by the Proposition 2.5.3.*

Proof. Recall that $\mathcal{B}$ is generated by the twisted affine generators (2.4.9)

$$B_0^+ = \xi_0'^+ k_0 - c\, q\, \xi_1^- k_0\,, \qquad B_0^- = \xi_0^- k_0 + c\, q\, \xi_1'^+ k_0\,. \tag{2.5.23}$$

and Cartan element $k_0 k_1^{-1}$. Firstly note that $\xi_0'^+ k_0 = \xi_0^+$. Then by substituting $c \to q^{2(t-1/2)}$ and using the expansion (2.3.29) we find

$$\begin{aligned}\lim_{q\to1} \frac{\alpha\, B_0^+}{q-q^{-1}} &= \lim_{q\to1}\left[\alpha\, \frac{\xi_0^+ - \xi_1^-}{q-q^{-1}} - \alpha\, \frac{q-1}{q-q^{-1}}\big(2\,t\,\xi_1^- + \xi_1^- h_0\big) + \mathcal{O}(\alpha^2)\right] \\ &= \widehat{E}^- - \alpha\, t\, E^- + \frac{\alpha}{2} H E^- = \widetilde{E}^-\,,\end{aligned} \tag{2.5.24}$$

and

$$\begin{aligned}\lim_{q\to1} \frac{\alpha\, B_0^-}{q-q^{-1}} &= \lim_{q\to1}\left[\alpha\, \frac{\xi_0^- + \xi_1^+}{q-q^{-1}} + \alpha\, \frac{q-1}{q-q^{-1}}\big(2\,t\,\xi_1^+ + \xi_0^- h_0 + \xi_1^+(h_0-h_1)\big) + \mathcal{O}(\alpha^2)\right] \\ &= \widehat{E}^+ + \alpha\, t\, E^+ - \frac{\alpha}{2} H E^+ = \widetilde{E}^+\,.\end{aligned} \tag{2.5.25}$$

Finally, $H_2 = \lim\limits_{q\to1} (k_2^2-1)/(q-q^{-1})$, and the Cartan generator H is obtained in an equivalent way as in (2.3.32). These coincide with (2.5.8) as required. □

Proposition 2.5.8. *The quantum affine coideal subalgebra $\mathcal{B} \subset \mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ defined by the Proposition 2.4.4 in the rational $q \to 1$ limit specializes to the twisted Yangian $\mathcal{Y}(\mathfrak{g},\mathfrak{g})$ of type II defined by the Proposition 2.5.6.*

Proof. We will prove this proposition for the representations only. Let $z = q^{2\,u}$. Then, upon rescaling $s \to q^{2\,s}$, the twisted affine generators (2.4.14) and (2.4.16) in the rational $q \to 1$ limit specialize to the twisted Yangian generators (2.5.14) by the following prescription,

$$\lim_{q\to1} \frac{T_z(B_0^+) - 2\,T_z(\xi_1^- k_1)}{(q-q^{-1})^2} = T_u(\widetilde{\widetilde{E}}{}^-)\,, \qquad \lim_{q\to1} \frac{T_z(B_0^-) + 2\,T_z(\xi_1^+)}{(q-q^{-1})^2} = -T_u(\widetilde{\widetilde{E}}{}^+)\,, \tag{2.5.26}$$

and

$$\begin{aligned}\lim_{q\to1} \frac{(T_z\otimes T_s)[\Delta(B_0^+)] - 2\,(T_z\otimes T_s)[\Delta(\xi_1^- k_1)]}{(q-q^{-1})^2} &= (T_u\otimes T_s)[\Delta(\widetilde{\widetilde{E}}{}^-)]\,, \\ \lim_{q\to1} \frac{(T_z\otimes T_s)[\Delta(B_0^-)] + 2\,(T_z\otimes T_s)[\Delta(\xi_1^+)]}{(q-q^{-1})^2} &= -(T_u\otimes T_s)[\Delta(\widetilde{\widetilde{E}}{}^+)]\,.\end{aligned} \tag{2.5.27}$$

□

A The Yangian limit of $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$

Here we give a heuristic derivation of the Yangian limit of $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$. We will recover the level-one generators of $\mathcal{Y}(\mathfrak{sl}(2))$ by calculating the rational $q \to 1$ limit of certain singular combinations of the generators of $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2))$ that otherwise do not have a propierly defined $q \to 1$ limit.

Set $\mathfrak{g} = \mathfrak{sl}(2)$. Recall that $\hat{\mathfrak{sl}}(2) \cong \mathcal{L}(\mathfrak{g})$. Let the Chevalley basis of $\mathcal{U}(\mathfrak{g}[u])$ be given by

$$\begin{aligned} &E^+ = 1 \otimes e\,, && E^- = 1 \otimes f\,, && H = 1 \otimes h\,, \\ &\hat{E}^+ = u \otimes f\,, && \hat{E}^- = -u \otimes e\,, && \hat{H} = u \otimes h\,. \end{aligned} \tag{A.1}$$

Let e_q, f_q and $k_q = q^h$ be the basis of $\mathcal{U}_q(\mathfrak{g})$. Let the basis of $\mathcal{U}_q(\mathcal{L}(g))$ be given by

$$\begin{aligned} &\xi_1^+ = 1 \otimes e_q\,, && \xi_1^- = 1 \otimes f_q\,, && h_1 = 1 \otimes h\,, && k_1 = q^{h_1}\,, \\ &\xi_0^+ = z \otimes f\,, && \xi_0^- = z^{-1} \otimes e\,, && h_0 = -h_1\,, && k_1 = q^{h_1}\,. \end{aligned} \tag{A.2}$$

Set $q = \mathrm{e}^{\alpha\hbar}$ and $z = \mathrm{e}^{-2\hbar u}$. Then the $q \to 1$ limit is obtained by setting $\hbar \to 0$. The expansion in series of the Cartan generators k_i at the point $\hbar = 0$ is given by

$$k_i \to 1 + (q-1)h_i + \mathcal{O}(\alpha^2)\,, \tag{A.3}$$

where $\mathcal{O}(\alpha^2)$ denotes higher order in α terms ($\alpha \sim \hbar$). Note that the (full) $q \to 1$ limit gives

$$E^+ = \lim_{q\to 1} \xi_1^+ = \lim_{q\to 1} \xi_0^-\,, \qquad E^- = \lim_{q\to 1} \xi_1^- = \lim_{q\to 1} \xi_0^+\,. \tag{A.4}$$

Choose

$$\xi_\alpha^+ = \frac{\alpha}{q-q^{-1}}(\xi_0^- - \xi_1^+)\,, \qquad \xi_\alpha^- = -\frac{\alpha}{q-q^{-1}}(\xi_0^+ - \xi_1^-)\,. \tag{A.5}$$

Then

$$\begin{aligned} \lim_{q\to 1} \xi_\alpha^+ &= \lim_{q\to 1} \alpha\,\frac{z^{-1}-1}{q-q^{-1}} \otimes e_q = u \otimes e = \hat{E}_\alpha^+\,, \\ \lim_{q\to 1} \xi_\alpha^- &= \lim_{q\to 1} \alpha\,\frac{1-z}{q-q^{-1}} \otimes f_q = -u \otimes f = \hat{E}_\alpha^-\,, \end{aligned} \tag{A.6}$$

and

$$\begin{aligned} \lim_{q\to 1} \Delta(\xi_\alpha^+) &= \lim_{q\to 1} \frac{\alpha}{q-q^{-1}}(\xi_0^- \otimes k_0^{-1} + 1 \otimes \xi_0^- - \xi_1^+ \otimes 1 - k_1 \otimes \xi_1^+) \\ &= \lim_{q\to 1} \Big[\alpha\,\frac{\xi_0^- - \xi_1^+}{q-q^{-1}} \otimes 1 + 1 \otimes \alpha\,\frac{\xi_0^- - \xi_1^+}{q-q^{-1}} \\ &\qquad - \alpha\,\frac{q-1}{q-q^{-1}}(\xi_0^- \otimes h_0 + h_1 \otimes \xi_1^+) + \mathcal{O}(\alpha^2)\Big] \\ &= \hat{E}_\alpha^+ \otimes 1 + 1 \otimes \hat{E}_\alpha^+ + \frac{\alpha}{2}(E^+ \otimes H - H \otimes E^+) = \Delta(\hat{E}_\alpha^+)\,, \end{aligned} \tag{A.7}$$

$$
\begin{aligned}
\lim_{q\to 1}\Delta(\xi_\alpha^-) &= \lim_{q\to 1}\frac{\alpha}{q-q^{-1}}(-\xi_0^+\otimes 1 - k_0\otimes\xi_0^+ + \xi_1^-\otimes k_1^{-1} + 1\otimes\xi_1^-) \\
&= \lim_{q\to 1}\Big[\alpha\,\frac{\xi_1^- - \xi_0^+}{q-q^{-1}}\otimes 1 + 1\otimes\alpha\,\frac{\xi_1^- - \xi_0^+}{q-q^{-1}} \\
&\qquad\quad - \alpha\,\frac{q-1}{q-q^{-1}}(\xi_1^-\otimes h_1 + h_0\otimes\xi_0^-) + \mathcal{O}(\alpha^2)\Big] \\
&= \hat{E}_\alpha^-\otimes 1 + 1\otimes\hat{E}_\alpha^- - \frac{\alpha}{2}(E^-\otimes H - H\otimes E^+) = \Delta(\hat{E}_\alpha^-)\,,
\end{aligned}
\tag{A.8}
$$

which coincide with (2.3.2).

Chapter 3

Integrable boundaries in AdS/CFT

It has been recognized in recent years that the planar limit of $\mathcal{N} = 4$ super Yang-Mills is integrable, and the worldsheet S-matrix approach allows us to successfully study the spectra of the light-cone superstrings propagating freely in $AdS_5 \times S^5$ spacetime in the framework of the AdS/CFT correspondence conjectured by Maldacena et al. [6] (see also the review [9]). The S-matrix approach [94–96] was first developed in the spin chain framework in the perturbative regime of the gauge theory, where it allows one to conjecture the corresponding (all-loop) Bethe equations describing the asymptotic spectrum of the gauge theory [19, 57, 97]. The integrability allows one to find the exact expressions of the S-matrices by requiring them to respect the underlying symmetries of the model. It is well-known that the S-matrix for the fundamental excitations in the bulk can be determined up to an overall (so-called 'dressing') phase factor from just the centrally extended $\mathfrak{psu}(2|2)_{\mathcal{C}}$ symmetry [18, 19, 98], and the S-matrix obtained in this way respects the Yang-Baxter equation (YBE) and a generalized physical unitarity condition. The overall phase factor is severely constrained by the crossing symmetry [99]. This non-analytic overall phase factor constitutes an important feature of the string S-matrix and has been the subject of intensive research [100–103].

In the limit of infinite light-cone momentum, in addition to the fundamental states, the spectrum of the string sigma model contains an infinite tower of bound states [104–106]. These manifest themselves as poles of the multi-magnon S-matrix built from the fundamental S-matrix S^{AA}, where A is used to denote the fundamental state. The generic bound state S-matrix S^{MN} is then obtained by considering a tensor product of the $4M$- and $4N$-dimensional atypical (short) supersymmetric multiplets of the $\mathfrak{psu}(2|2)_{\mathcal{C}}$ [20, 104, 107]. These multiplets can be obtained from the $(M+N)$-fold tensor product of the fundamental representation by projecting it onto the totally symmetric component.

As was shown in [108], the construction of the bound state S-matrix relies on the observation that an M-magnon bound state representation of the $\mathfrak{psu}(2|2)_{\mathcal{C}}$ algebra may be realized on the space $\mathcal{V}_M$ of homogeneous supersymmetric polynomials of degree M depending on two bosonic and two fermionic variables, ω_a and θ_α respectively. Thus, the representation space is identical to an irreducible short superfield $\Phi^M(\omega, \theta)$. In this realization the algebra generators are represented by differential operators linear in variables

ω_a and θ_α with the scattering coefficients being functions of the parameters describing the representation. The introduction of a space $\mathcal{D}^M$ dual to $\mathcal{V}_M$, which may be realized as the space of differential operators preserving the homogeneous gradation of $\Phi^M(\omega, \theta)$, allows one to define the S-matrix as (an element of)

$$S^{\mathrm{MN}} \in \mathrm{End}(\mathcal{V}_M \otimes \mathcal{V}_N) \approx \mathcal{V}_M \otimes \mathcal{V}_N \otimes \mathcal{D}^M \otimes \mathcal{D}^N \,.$$

Thus the S-matrix S^{MN} may be written as a differential operator of degree $M+N$ acting on the product of two superfields $\Phi^M(\omega, \theta)$ and $\Phi^N(\omega, \theta)$.

Finding the generic bound state S-matrices is very complicated, as the $\mathfrak{psu}(2|2)_{\mathcal{C}}$ symmetry alone is not enough to determine all of the scattering coefficients. Further constraints are required, either from the YBE or the underlying Yangian symmetry [20].

The underlying Yangian symmetry, which goes back to the inception of quantum groups, is at the core of the general strategy for finding the higher-order bound state S-matrices [109]. The Yangian symmetry is essential since the fusion procedure does not work straightforwardly for AdS/CFT S-matrices [108]. These higher-order S-matrices play an important role in understanding the underlying integrability and deriving the transfer matrices, Bethe ansatz equations, writing T- and Y-systems, and other important algebraic objects of the theory. It is worth recalling that Yangians generically have some very nice properties, particularly at the level of representation theory [17, 110]. So the appearance of Yangian symmetry in the string context – for example, via the universal R-matrix [111–113] – is a very welcome feature.

A specific case of worldsheet scattering is the boundary scattering which has attended lots of research interest and development on its own due to a large variety of the boundary conditions that arise when open strings end on D-branes embedded in the $AdS_5 \times S^5$ background (See e.g. [59, 61–63, 114–124]). Boundary conditions depend not only on the type of the D-brane the string is attached to, but also on the type of embedding and the relative orientation of the string and the brane. The emerging integrable configurations have been classified in [65].

Deep in the bulk of an open string the theory is indistinguishable from the pure $\mathcal{N} = 4$ super Yang-Mills, thus the symmetry arguments discussed above remain valid, and the bulk S-matrix may be used without modifications. The task is then to determine the reflection of magnons from the end of the string, where the residual symmetries of the boundary are crucial in determining the structure of the reflection K-matrix. However it is a great challenge to understand the residual boundary symmetries in full generality.

An important feature of quantum integrability is that the presence of suitable boundary conditions may break a bulk Yangian symmetry without spoiling integrability. In such a way one can find a boundary Yangian symmetry which is a coideal subalgebra of the bulk Yangian symmetry, and in many cases appearing in the AdS/CFT this was show to be generalized twisted Yangian algebras discussed in Chapter 2.

This chapter is organized as follows. In Section 3.1 we recap the underlying symmetries and the worldsheet S-matrix of the light-cone superstrings in AdS/CFT. In Section 3.2 we review the spectrum of integrable boundaries and give the necessary preliminaries for the boundary scattering theory. Sections 3.3 and 3.4 contain description of boundary symmetries and boundary scattering theory for giant gravitons and $D7$-branes, that share a lot of common features from the boundary scattering point of view. Section 3.5 considers boundary scattering for the $D5$-brane. This boundary was long thought not to be integrable. Here we show that it is indeed integrable and is of a specific 'achiral' type. This type of boundary symmetry was not considered explicitly in Chapter 2, thus this sections also contains the general considerations for this type of algebra. Appendix B accommodates explicit expressions of selected bound state reflection matrices.

3.1 Worldsheet scattering

3.1.1 Underlying symmetries

The symmetry algebra of excitations in the light-cone superstring theory on the $AdS_5 \times S^5$ background and for the single-trace local operators in the $\mathcal{N}=4$ supersymmetric Yang-Mills gauge theory is given by two copies (left and right) of the centrally-extended Lie superalgebra [18, 125]

$$\mathfrak{psu}(2|2)_{\mathcal{C}} = \mathfrak{psu}(2|2) \ltimes \mathbb{R}^3 \,. \tag{3.1.1}$$

3.1.1.1 Lie algebra

The algebra $\mathfrak{psu}(2|2)_{\mathcal{C}}$ contains two sets of bosonic $\mathfrak{su}(2)$ rotation generators $\mathbb{R}_a^{\ b}$, $\mathbb{L}_\alpha^{\ \beta}$, two sets of fermionic supersymmetry generators $\mathbb{Q}_\alpha^{\ a}$, $\mathbb{G}_a^{\ \alpha}$ and three central elements $\mathbb{H}$, $\mathbb{C}$ and $\mathbb{C}^\dagger$. The non-trivial commutation relations are

$$\begin{aligned}
&[\mathbb{L}_\alpha^{\ \beta}, \mathbb{J}_\gamma] = \delta_\gamma^\beta \,\mathbb{J}_\alpha - \tfrac{1}{2}\delta_\alpha^\beta \,\mathbb{J}_\gamma \,, &\quad& \{\mathbb{Q}_\alpha^{\ a}, \mathbb{Q}_\beta^{\ b}\} = \epsilon^{ab}\epsilon_{\alpha\beta}\,\mathbb{C} \,, \\
&[\mathbb{L}_\alpha^{\ \beta}, \mathbb{J}^\gamma] = -\delta_\alpha^\gamma \,\mathbb{J}^\beta + \tfrac{1}{2}\delta_\alpha^\beta \,\mathbb{J}^\gamma \,, && \{\mathbb{G}_a^{\ \alpha}, \mathbb{G}_b^{\ \beta}\} = \epsilon^{\alpha\beta}\epsilon_{ab}\,\mathbb{C}^\dagger \,, \\
&[\mathbb{R}_a^{\ b}, \mathbb{J}_c] = \delta_c^b \,\mathbb{J}_a - \tfrac{1}{2}\delta_a^b \,\mathbb{J}_c \,, && \{\mathbb{Q}_\alpha^{\ a}, \mathbb{G}_b^{\ \beta}\} = \delta_b^a \,\mathbb{L}_\beta^{\ \alpha} + \delta_\beta^\alpha \,\mathbb{R}_b^{\ a} + \tfrac{1}{2}\delta_b^a\delta_\beta^\alpha \,\mathbb{H} \,, \\
&[\mathbb{R}_a^{\ b}, \mathbb{J}^c] = -\delta_a^c \,\mathbb{J}^b + \tfrac{1}{2}\delta_a^b \,\mathbb{J}^c \,,
\end{aligned} \tag{3.1.2}$$

where $a,\ b, ... = 1, 2$ and $\alpha,\ \beta, ... = 3, 4$, and the symbols $\mathbb{J}_a$, $\mathbb{J}_\alpha$ with lower (or upper) indices represent any generator with the corresponding index structure.

Hopf algebra. This algebra may be equipped with a non-trivial (braided) Hopf algebra structure [126] such that for any $\mathbb{J}^A \in \mathfrak{psu}(2|2)_{\mathcal{C}}$,

$$\Delta(\mathbb{J}^A) = \mathbb{J}^A \otimes 1 + \mathbb{U}^{[[A]]} \otimes \mathbb{J}^A \,, \qquad \Delta^{op}(\mathbb{J}^A) = \mathbb{J}^A \otimes \mathbb{U}^{[[A]]} + 1 \otimes \mathbb{J}^A \,. \tag{3.1.3}$$

Here $\mathbb{U}$ is the so-called braiding factor, and is a group-like element of the algebra,

$$\Delta(\mathbb{U}) = \mathbb{U} \otimes \mathbb{U} \,. \tag{3.1.4}$$

The additive quantum number $[[A]]$ equals 0 for generators in $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ and for $\mathbb{H}$, $\frac{1}{2}$ for $\mathbb{Q}_\alpha^{\ a}$, $-\frac{1}{2}$ for $\mathbb{G}_a^\alpha$, 1 for $\mathbb{C}$ and -1 for $\mathbb{C}^\dagger$. This number is sometimes called the hypercharge. We will come back to it a little bit further.

The Hopf algebra structure becomes complete after defining the antipode map S and the counit map ε,

$$\begin{aligned} &S(1) = 1\,, && S(\mathbb{U}^{\pm1}) = \mathbb{U}^{\mp1}, && S(\mathbb{J}^A) = -\mathbb{U}^{-[[A]]}\mathbb{J}^A\,, \\ &\varepsilon(1) = \varepsilon(\mathbb{U}^{\pm1}) = 1\,, && \varepsilon(\mathbb{J}^A) = 0\,. && \end{aligned} \tag{3.1.5}$$

The braided structure of the algebra imposes additional constraints on the central generators $\mathbb{C}$ and $\mathbb{C}^\dagger$. The co-commutativity requirement gives

$$\begin{aligned} \Delta(\mathbb{C}) &= \Delta^{op}(\mathbb{C}) \quad \Rightarrow \quad \mathbb{C}\otimes\left(1-\mathbb{U}^{+2}\right) = \left(1-\mathbb{U}^{+2}\right)\otimes\mathbb{C} \quad \Rightarrow \quad \mathbb{C}\propto(1-\mathbb{U}^{+2})\,, \\ \Delta(\mathbb{C}^\dagger) &= \Delta^{op}(\mathbb{C}^\dagger) \quad \Rightarrow \quad \mathbb{C}^\dagger\otimes\left(1-\mathbb{U}^{-2}\right) = \left(1-\mathbb{U}^{-2}\right)\otimes\mathbb{C}^\dagger \quad \Rightarrow \quad \mathbb{C}^\dagger\propto(1-\mathbb{U}^{-2})\,. \end{aligned} \tag{3.1.6}$$

One can further introduce universal proportionality coefficients α, $\alpha^\dagger$ and g such that

$$\mathbb{C} = g\,\alpha\,(1-\mathbb{U}^{+2})\,, \qquad \mathbb{C}^\dagger = g\,\alpha^\dagger\,(1-\mathbb{U}^{-2})\,, \qquad \mathbb{C}\,\mathbb{C}^\dagger - g\,\alpha\,\mathbb{C}^\dagger - g\,\alpha^\dagger\,\mathbb{C} = 0\,. \tag{3.1.7}$$

Here g plays the role of the coupling constant of the corresponding string theory. In the $g\to0$ limit central generators $\mathbb{C}$ and $\mathbb{C}^\dagger$ vanish and and the $\mathfrak{psu}(2|2)_\mathcal{C}$ algebra specializes to $\mathfrak{su}(2|2)$.

Automorphism group. The $\mathfrak{psu}(2|2)$ algebra has no matrix representation, but the centrally extended algebra does, and the representations may be traced back from the superalgebra $\mathfrak{gl}(2|2)$ using an $SL(2)$ outer-automorphism group of the algebra [127]. This outer automorphism reveals itself in the $\varepsilon\to0$ limit of the exceptional superalgebra $\mathfrak{d}(2,1;\varepsilon)$ [128]. The $SL(2)$ automorphism transforms the supersymmetry generators of the algebra as

$$\mathbb{Q}'{}_\alpha{}^a = u_1\,\mathbb{Q}_\alpha{}^a - u_2\,\varepsilon^{ab}\,\varepsilon_{\alpha\beta}\,\mathbb{G}_b{}^\beta\,, \qquad \mathbb{G}'{}_a{}^\alpha = v_1\,\mathbb{G}_a{}^\alpha - v_2\,\varepsilon_{ab}\,\varepsilon^{\alpha\beta}\,\mathbb{Q}_\beta{}^b\,, \tag{3.1.8}$$

and the central generators as

$$\begin{aligned} &\mathbb{C}' = u_1^2\,\mathbb{C} + u_2^2\,\mathbb{C}^\dagger + u_1u_2\,\mathbb{H}\,, \qquad \mathbb{C}'^\dagger = v_1^2\,\mathbb{C}^\dagger + v_2^2\,\mathbb{C} + v_1v_2\,\mathbb{H}\,, \\ &\mathbb{H}' = (u_1v_1+u_2v_2)\,\mathbb{H} + 2\,u_1v_2\,\mathbb{C} + 2\,u_2v_1\,\mathbb{C}^\dagger\,. \end{aligned} \tag{3.1.9}$$

The parameters u_i and v_i satisfy the non-degeneracy constraint $u_1v_1-u_2v_2=1$ and may be combined into an $SL(2)$ matrix

$$h^{out} = \begin{pmatrix} u_1 & u_2 \\ v_2 & v_1 \end{pmatrix}. \tag{3.1.10}$$

We shall be interested in the unitary representations of $\mathfrak{psu}(2|2)_\mathcal{C}$. The latter requirement restricts the $SL(2)$ automorphism group to its real form $SU(1,1)$ upon imposing the unitarity constraints $v_1^*=u_1$ and $v_2^*=u_2$.

It is important to note that the outer-automorphism group leaves the combination $\vec{\mathbb{H}}^2 := \mathbb{H}^2 - 4\,\mathbb{C}\,\mathbb{C}^\dagger$ of the central charges invariant, i.e. this combination defines the orbits of the $SL(2)$.

Hypercharge and reflection map. Set $u_2 = v_2 = 0$ and choose $u_1 = \mathrm{e}^{i\phi}$, $v_1 = \mathrm{e}^{-i\phi}$ for some real ϕ. This setup defines a $U(1) \subset SU(1,1)$ subgroup of the outer-automorphism group discussed above given by

$$\mathbb{Q}^a_{\ \alpha} \to \mathrm{e}^{i\phi}\mathbb{Q}^a_{\ \alpha}\,, \qquad \mathbb{G}_a^{\ \alpha} \to \mathrm{e}^{-i\phi}\mathbb{G}_a^{\ \alpha}\,, \qquad \mathbb{C} \to \mathrm{e}^{2i\phi}\mathbb{C}\,, \qquad \mathbb{C}^\dagger \to \mathrm{e}^{-2i\phi}\mathbb{C}^\dagger\,. \tag{3.1.11}$$

This soubgroup has several important and far-reaching properties that we will encounter in the boundary scattering.

The corresponding Lie generator $\mathbb{Y}$ serves as a hypercharge for the generators of $\mathfrak{psu}(2|2)_{\mathcal{C}}$,

$$[\mathbb{Y}, \mathbb{J}^A] = \frac{[[A]]}{2}\,\mathbb{J}^A\,, \tag{3.1.12}$$

more explicitly:

$$\begin{aligned}
&[\mathbb{Y}, \mathbb{Q}^a_\alpha] = +\tfrac{1}{2}\mathbb{Q}^{\ a}_{\alpha}\,, \qquad [\mathbb{Y}, \mathbb{C}] = +\mathbb{C}\,, \qquad [\mathbb{Y}, \mathbb{L}_a^{\ b}] = [\mathbb{Y}, \mathbb{R}^{\ \beta}_\alpha] = [\mathbb{Y}, \mathbb{H}] = 0\,,\\
&[\mathbb{Y}, \mathbb{G}_a^{\ \alpha}] = -\tfrac{1}{2}\mathbb{G}_a^{\ \alpha}\,, \qquad [\mathbb{Y}, \mathbb{C}^\dagger] = -\mathbb{C}^\dagger\,.
\end{aligned} \tag{3.1.13}$$

Set $u_1^2 = -\mathbb{U}^{-2}$, $v_1^2 = -\mathbb{U}^2$. Recall that $\mathbb{U}$ is a group-like element, thus can be represented as $\mathbb{U} \equiv \mathrm{e}^{ip/2}$ for some p. Then the corresponding generator $\kappa \in U(1)$, as we will show in the Section 3.3.2, can be interpreted as the reflection map.

Representations. The $\mathfrak{psu}(2|2)_{\mathcal{C}}$ algebra has several different types of finite-dimensional representations. The most relevant representations for AdS/CFT superstrings are called long (typical) and short (atypical). There are also anomalous (singlet and adjoint) representations. See [18] and [19] for a comprehensive review and details. A tensor product of two short representations generically yields a sum of long multiplets. The long representations are generically irreducible, but become reducible for some specific eigenvalues of the central charges. Next we will briefly review the decomposition of the tensor product of two fundamental representations and the tensor product of two 2-particle bound state representations, as this will be important to us later on.

The fundamental excitations (asymptotic states) of the superstring transform in the 4-d(imensional) short (fundamental) representation $\boxslash$. The tensor product of two fundamental representations gives a 16-d irreducible long multiplet. This is the smallest long representation. At the special points (corresponding to special eigenvalues of the central charges) one may decompose the 16-d long multiplet into two 8-d short representations (totally symmetric and totally antisymmetric), or into two singlets (corresponding to the fundamental singlet state of the spectrum) and a minimal 14-d adjoint, which may further be reduced to $(3 + 2 \times 4 + 3)$-d totally symmetric multiplets. We shall mainly focus on supersymmetric short and singlet representations, where the interesting physical states (magnons and their bound states) of the AdS/CFT superstring live. A multiplet shortening constraint defining a supersymmetric (or equivalently anti-supersymmetric) short representation is given by

$$\mathbb{H}^2 - 4\,\mathbb{C}\,\mathbb{C}^\dagger = 1\,. \tag{3.1.14}$$

Two-particle bound states live in an 8-d supersymmetric short representation ⧄⧄. A tensor product of two such representations decomposes into a sum of two long, 16-d and 48-d, representations. This tensor product is very important in the scattering theory we shall be considering, as it is the simplest representation for which the Lie algebra is not enough to determine all the scattering coefficients and additional constraints are required [108]. Thus this representation serves as the most simple non-trivial test of Yangian symmetry.

The supersymmetric short representation describing an M-particle bound state consists of vectors $|m,n,k,l\rangle \in V(p)$ where $k+l+m+n=M$, and $V(p)$ is the corresponding vector space of the excitations with momentum p. The labels m,n denote fermionic degrees of freedom and k,l denote the bosonic part. The symmetry generators act on the basis vectors as

$$\begin{aligned}
&\mathbb{R}_1{}^1\,|m,n,k,l\rangle = \tfrac{1}{2}(l-k)\,|m,n,k,l\rangle\,, &&\mathbb{L}_3{}^3\,|m,n,k,l\rangle = \tfrac{1}{2}(m-n)\,|m,n,k,l\rangle\,,\\
&\mathbb{R}_2{}^1\,|m,n,k,l\rangle = k\,|m,n,k-1,l+1\rangle\,, &&\mathbb{L}_3{}^4\,|m,n,k,l\rangle = |m+1,n-1,k,l\rangle\,,\\
&\mathbb{R}_1{}^2\,|m,n,k,l\rangle = l\,|m,n,k+1,l-1\rangle\,, &&\mathbb{L}_4{}^3\,|m,n,k,l\rangle = |m-1,n+1,k,l\rangle\,,
\end{aligned}\tag{3.1.15}$$

while the action of the supercharges is defined by

$$\begin{aligned}
&\mathbb{Q}_4{}^2\,|m,n,k,l\rangle = a\,(-1)^m l\,|m,n+1,k,l-1\rangle + b\,|m-1,n,k+1,l\rangle\,,\\
&\mathbb{G}_2{}^4\,|m,n,k,l\rangle = c\,k\,|m+1,n,k-1,l\rangle + d\,(-1)^m\,|m,n-1,k,l+1\rangle\,.
\end{aligned}\tag{3.1.16}$$

The explicit action of the rest of the generators is easily obtained by the defining commutation relations (3.1.2). Finally, the action of central elements is given by

$$\begin{aligned}
&\mathbb{C}\,|m,n,k,l\rangle = M\,a\,b\,|m,n,k,l\rangle\,, &&\mathbb{C}^\dagger\,|m,n,k,l\rangle = M\,c\,d\,|m,n,k,l\rangle\,,\\
&\mathbb{H}\,|m,n,k,l\rangle = M\,(a\,d+b\,c)\,|m,n,k,l\rangle &&\mathbb{U}\,|m,n,k,l\rangle = U\,|m,n,k,l\rangle\,,
\end{aligned}\tag{3.1.17}$$

where the braiding factor in this representation is related to the momentum of the magnon, $U=\mathrm{e}^{ip/2}$. In such a way the multiplet shortening constraint (3.1.14) becomes

$$a\,d - b\,c = 1\,,\tag{3.1.18}$$

while (3.1.7) gives

$$M\,a\,b = g\,\alpha\,(1-U^2)\,,\qquad\qquad M\,c\,d = g\,\alpha^\dagger\,(1-U^2)\,.\tag{3.1.19}$$

A convenient parametrization of the representation labels satisfying (3.1.18) and (3.1.19) is [18, 108]

$$a=\sqrt{\frac{g}{M}}\gamma,\quad b=\sqrt{\frac{g}{M}}\frac{\alpha}{\gamma}\left(1-\frac{x^+}{x^-}\right),\quad c=\sqrt{\frac{g}{M}}\frac{i\,\gamma}{\alpha\,x^+},\quad d=\sqrt{\frac{g}{M}}\frac{i\,x^+}{\gamma}\left(\frac{x^-}{x^+}-1\right),\tag{3.1.20}$$

where M is the bound state number ($M = 1$ corresponds to the fundamental representation), g is the coupling constant, and $x^{\pm}$ are the spectral parameters ($e^{ip} = \frac{x^+}{x^-}$). The multiplet shortening constraint in this parametrization becomes

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{iM}{g}\,. \tag{3.1.21}$$

and is conveniently called the mass-shell constraint. The parameters γ and α are internal parameters of the representation and define the relative normalization between bosons and fermions. The unitarity imposes $\alpha^\dagger = \alpha^{-1}$ and $\gamma = e^{i\varphi}\sqrt{i\,(x^- - x^+)}$, where the arbitrary phase factor $e^{i\varphi}$ reflects the freedom in choosing $x^{\pm}$ and is conveniently set to $\varphi = p/4$. The rapidity of the magnon in the $x^{\pm}$ parametrization is defined to be

$$u = x^+ + \frac{1}{x^+} - \frac{iM}{2\,g}\,. \tag{3.1.22}$$

Finally let us note that this algebra does not have a well-defined quadratic Casimir operator due to the degeneracy of the Killing-Cartan form. Consider a Casimir-like quadratic generator $\mathbb{W}$,

$$\mathbb{W} := \mathbb{R}_a^{\ b}\,\mathbb{R}_b^{\ a} - \mathbb{L}_\alpha^{\ \beta}\,\mathbb{L}_\beta^{\ \alpha} + \mathbb{Q}_\alpha^{\ a}\,\mathbb{G}_a^{\ \alpha} - \mathbb{G}_a^{\ \alpha}\,\mathbb{Q}_\alpha^{\ a}\,. \tag{3.1.23}$$

Enhance the algebra by an extra non-supertraceless generator $\mathbb{B} \in \mathfrak{gl}(2|2)$ such that

$$\mathbb{B}\,|m,n,k,l\rangle = \frac{1}{2(a\,d + b\,c)}(m{+}n{-}k{-}l)\,|m,n,k,l\rangle\,. \tag{3.1.24}$$

Then one can define an extended quadratic generator

$$\mathbb{T} := \mathbb{B}\,\mathbb{H} + \mathbb{W}\,, \qquad \text{such that} \qquad [\mathbb{J}^A,\,\mathbb{T}]\,|m,n,k,l\rangle = 0\,, \tag{3.1.25}$$

for all $\mathbb{J}^A \in \mathfrak{psu}(2|2)_{\mathcal{C}}$. This generator is called the generalized quadratic Casimir operator of $\mathfrak{psu}(2|2)_{\mathcal{C}}$. The generator $\mathbb{B}$ is closely related to the so-called 'secret' symmetry of the AdS/CFT. We will consider this symmetry in more detail in Section 5.

3.1.1.2 Yangian

The Yangian algebra for planar AdS/CFT was constructed in [20] and has been further investigated in [109, 113, 128–132]. The coproducts of the Yangian charges are defined as

$$\Delta(\widehat{\mathbb{J}}^A) = \widehat{\mathbb{J}}^A \otimes 1 + \mathbb{U}^{[[A]]} \otimes \widehat{\mathbb{J}}^A + \tfrac{1}{2} f^A_{\ BC}\,\mathbb{U}^{[[C]]}\mathbb{J}^B \otimes \mathbb{J}^C, \tag{3.1.26}$$

and the opposite coproducts are

$$\Delta^{op}(\widehat{\mathbb{J}}^A) = \widehat{\mathbb{J}}^A \otimes \mathbb{U}^{[[A]]} + 1 \otimes \widehat{\mathbb{J}}^A - \tfrac{1}{2} f^A_{\ BC}\,\mathbb{J}^B \otimes \mathbb{U}^{[[B]]}\,\mathbb{J}^C. \tag{3.1.27}$$

The Cartan-Killing form g_{AD} of $\mathfrak{psu}(2|2)_{\mathcal{C}}$ is degenerate and thus the symbols $f^A_{\ BC}$ can not be obtained directly from the structure constants $f^{AB}_{\quad C}$ of the Lie algebra (3.1.2). However there is a number of ways to overcome this obstacle, for example by employing the exceptional superalgebra $\mathfrak{d}(2,1;\epsilon)$ [128]. In such a way the explicit expressions of the coproducts are given by

$$\begin{aligned}
\Delta(\widehat{\mathbb{R}}_a{}^b) &= \widehat{\mathbb{R}}_a{}^b \otimes 1 + 1 \otimes \widehat{\mathbb{R}}_a{}^b + \tfrac{1}{2}\,\mathbb{R}_a{}^c \otimes \mathbb{R}_c{}^b - \tfrac{1}{2}\,\mathbb{R}_c{}^b \otimes \mathbb{R}_a{}^c \\
&\quad - \tfrac{1}{2}\,\mathbb{U}^{+1}\,\mathbb{G}_a{}^\gamma \otimes \mathbb{Q}_\gamma{}^b - \tfrac{1}{2}\,\mathbb{U}^{-1}\,\mathbb{Q}_\gamma{}^b \otimes \mathbb{G}_a{}^\gamma \\
&\quad + \tfrac{1}{4}\,\delta_a^b\,\mathbb{U}^{+1}\,\mathbb{G}_c{}^\gamma \otimes \mathbb{Q}_\gamma{}^c + \tfrac{1}{4}\,\delta_a^b\,\mathbb{U}^{-1}\,\mathbb{Q}_\gamma{}^c \otimes \mathbb{G}_c{}^\gamma\,, \\
\Delta(\widehat{\mathbb{L}}_\alpha{}^\beta) &= \widehat{\mathbb{L}}_\alpha{}^\beta \otimes 1 + 1 \otimes \widehat{\mathbb{L}}_\alpha{}^\beta - \tfrac{1}{2}\,\mathbb{L}_\alpha{}^\gamma \otimes \mathbb{L}_\gamma{}^\beta + \tfrac{1}{2}\,\mathbb{L}_\gamma{}^\beta \otimes \mathbb{L}_\alpha{}^\gamma \\
&\quad + \tfrac{1}{2}\,\mathbb{U}^{+1}\,\mathbb{G}_c{}^\beta \otimes \mathbb{Q}_\alpha{}^c + \tfrac{1}{2}\,\mathbb{U}^{-1}\,\mathbb{Q}_\alpha{}^c \otimes \mathbb{G}_c{}^\beta \\
&\quad - \tfrac{1}{4}\,\delta_\alpha^\beta\,\mathbb{U}^{+1}\,\mathbb{G}_c{}^\gamma \otimes \mathbb{Q}_\gamma{}^c - \tfrac{1}{4}\,\delta_\alpha^\beta\,\mathbb{U}^{-1}\,\mathbb{Q}_\gamma{}^c \otimes \mathbb{G}_c{}^\gamma\,, \\
\Delta(\widehat{\mathbb{Q}}_\alpha{}^a) &= \widehat{\mathbb{Q}}_\alpha{}^a \otimes 1 + \mathbb{U}^{+1} \otimes \widehat{\mathbb{Q}}_\alpha{}^a + \tfrac{1}{2}\,\mathbb{Q}_\alpha{}^c \otimes \mathbb{R}_c{}^a - \tfrac{1}{2}\,\mathbb{U}^{+1}\,\mathbb{R}_c{}^a \otimes \mathbb{Q}_\alpha{}^c \\
&\quad + \tfrac{1}{2}\,\mathbb{Q}_\gamma{}^a \otimes \mathbb{L}_\alpha{}^\gamma - \tfrac{1}{2}\,\mathbb{U}^{+1}\,\mathbb{L}_\alpha{}^\gamma \otimes \mathbb{Q}_\gamma{}^a + \tfrac{1}{4}\,\mathbb{Q}_\alpha{}^a \otimes \mathbb{H} - \tfrac{1}{4}\,\mathbb{U}^{+1}\,\mathbb{H} \otimes \mathbb{Q}_\alpha{}^a \\
&\quad + \tfrac{1}{2}\,\varepsilon_{\alpha\gamma}\varepsilon^{ac}\,\mathbb{C}\,\mathbb{U}^{-1} \otimes \mathbb{G}_c{}^\gamma - \tfrac{1}{2}\,\varepsilon_{\alpha\gamma}\varepsilon^{ac}\,\mathbb{G}_c{}^\gamma \otimes \mathbb{C}\,, \\
\Delta(\widehat{\mathbb{G}}_a{}^\alpha) &= \widehat{\mathbb{G}}_a{}^\alpha \otimes 1 + \mathbb{U}^{-1} \otimes \widehat{\mathbb{G}}_a{}^\alpha - \tfrac{1}{2}\,\mathbb{G}_c{}^\alpha \otimes \mathbb{R}_a{}^c + \tfrac{1}{2}\,\mathbb{U}^{-1}\,\mathbb{R}_a{}^c \otimes \mathbb{G}_c{}^\alpha \\
&\quad - \tfrac{1}{2}\,\mathbb{G}_a{}^\gamma \otimes \mathbb{L}_\gamma{}^\alpha + \tfrac{1}{2}\,\mathbb{U}^{-1}\,\mathbb{L}_\gamma{}^\alpha \otimes \mathbb{G}_a{}^\gamma - \tfrac{1}{4}\,\mathbb{G}_a{}^\alpha \otimes \mathbb{H} + \tfrac{1}{4}\,\mathbb{U}^{-1}\,\mathbb{H} \otimes \mathbb{G}_a{}^\alpha \\
&\quad - \tfrac{1}{2}\,\varepsilon_{ac}\varepsilon^{\alpha\gamma}\,\mathbb{C}^\dagger\,\mathbb{U}^{+1} \otimes \mathbb{Q}_\gamma{}^c + \tfrac{1}{2}\,\varepsilon_{ac}\varepsilon^{\alpha\gamma}\,\mathbb{U}^{-2}\,\mathbb{Q}_\gamma{}^c \otimes \mathbb{C}^\dagger\,, \\
\Delta(\widehat{\mathbb{H}}) &= \widehat{\mathbb{H}} \otimes 1 + 1 \otimes \widehat{\mathbb{H}} + \mathbb{U}^{-2}\,\mathbb{C} \otimes \mathbb{C}^\dagger - \mathbb{U}^{+2}\,\mathbb{C}^\dagger \otimes \mathbb{C}\,, \\
\Delta(\widehat{\mathbb{C}}) &= \widehat{\mathbb{C}} \otimes 1 + \mathbb{U}^{+2} \otimes \widehat{\mathbb{C}} - \tfrac{1}{2}\,\mathbb{U}^{+2}\,\mathbb{H} \otimes \mathbb{C} + \tfrac{1}{2}\,\mathbb{C} \otimes \mathbb{H}\,, \\
\Delta(\widehat{\mathbb{C}}^\dagger) &= \widehat{\mathbb{C}}^\dagger \otimes 1 + \mathbb{U}^{-2} \otimes \widehat{\mathbb{C}}^\dagger + \tfrac{1}{2}\,\mathbb{U}^{-2}\,\mathbb{H} \otimes \mathbb{C}^\dagger - \tfrac{1}{2}\,\mathbb{C}^\dagger \otimes \mathbb{H}\,.
\end{aligned} \tag{3.1.28}$$

Quasi-commutativity. In order to have a quasi-commutative Hopf algebra the centre of the algebra is required to be co-commutative. It was shown in [20] that the coproducts of the central charges may be chosen to be co-commutative not only at the algebra level, but also at the Yangian level. For this purpose one needs to define the following combinations of Yangian generators,

$$\widehat{\mathbb{H}}' := \widehat{\mathbb{H}} + \alpha^\dagger\mathbb{C} - \alpha\,\mathbb{C}^\dagger\,, \quad \widehat{\mathbb{C}}' := \widehat{\mathbb{C}} + \tfrac{1}{2}\,\mathbb{H}\,(\mathbb{C} - 2\alpha)\,, \quad \widehat{\mathbb{C}}^{\dagger\prime} := \widehat{\mathbb{C}}^\dagger - \tfrac{1}{2}\,\mathbb{H}\,(\mathbb{C}^\dagger - 2\alpha^\dagger)\,, \tag{3.1.29}$$

which we can call the 'deformed central charges'. These new deformed charges have almost–trivial coproducts

$$\Delta(\widehat{\mathbb{H}}') = \widehat{\mathbb{H}}' \otimes 1 + 1 \otimes \widehat{\mathbb{H}}'\,, \quad \Delta(\widehat{\mathbb{C}}') = \widehat{\mathbb{C}}' \otimes 1 + \mathbb{U}^{+2} \otimes \widehat{\mathbb{C}}'\,, \quad \Delta(\widehat{\mathbb{C}}^{\dagger\prime}) = \widehat{\mathbb{C}}^{\dagger\prime} \otimes 1 + \mathbb{U}^{-2} \otimes \widehat{\mathbb{C}}^{\dagger\prime}\,. \tag{3.1.30}$$

The coproduct of $\widehat{\mathbb{H}}'$ is already co-commutative, while the co-commutativity of $\widehat{\mathbb{C}}'$ and $\widehat{\mathbb{C}}^{\dagger\prime}$ can be ensured by imposing additional constraints

$$\widehat{\mathbb{C}}' = \beta\, v_C\,\mathbb{C}\,, \qquad \widehat{\mathbb{C}}^{\dagger\prime} = \beta\, v_{C^\dagger}\,\mathbb{C}^\dagger\,, \tag{3.1.31}$$

with some universal parameters v_C, $v_{C^\dagger}$ and β. In such a way the generators $\widehat{\mathbb{H}}$, $\widehat{\mathbb{C}}$, $\widehat{\mathbb{C}}^\dagger$ are also required to be co-commutative as they differ from the deformed central charges by the central elements of the algebra only. We can also introduce a similar ansatz

$$\widehat{\mathbb{H}}' = \beta\, v_H\, \mathbb{H}, \tag{3.1.32}$$

to have a complete set of expressions of the deformed central charges with v_H being some universal parameter as well. We have not introduced or assumed any relations between the parameters v_C, $v_{C^\dagger}$ and v_H so far, we have merely required them to be universal[1]. We will arrive at a set of constraints by constructing the evaluation representation. However it is easy to see, that even at the representation level (on-shell) v_H will remain unconstrained. This is because $\mathbb{H}$ and $\widehat{\mathbb{H}}'$ are not only co-commutative but also commutative charges.

Evaluation representation. The evaluation representation is constructed with the help of the evaluation map ansatz

$$\widehat{\mathbb{J}}\,|v\rangle = \gamma\,(v + v_0)\,\mathbb{J}\,|v\rangle\,. \tag{3.1.33}$$

In such a way the deformed central charges (3.1.29) give

$$\begin{aligned}
\widehat{\mathbb{C}}'\ |v\rangle &= \left[\gamma\,(v+v_0)\,\mathbb{C} + \tfrac{1}{2}\,\mathbb{H}\,(\mathbb{C}-2\alpha)\right]|v\rangle = \beta\, v_C\,\mathbb{C}\,|v\rangle\,,\\
\widehat{\mathbb{C}}^{\dagger\prime}\,|v\rangle &= \left[\gamma\,(v+v_0)\,\mathbb{C}^\dagger - \tfrac{1}{2}\,\mathbb{H}\,(\mathbb{C}^\dagger-2\alpha^\dagger)\right]|v\rangle = \beta\, v_{C^\dagger}\,\mathbb{C}^\dagger\,|v\rangle\ ,
\end{aligned} \tag{3.1.34}$$

where at the final step we have imposed the constraints (3.1.31) and (3.1.32). Assuming that all universal constants are equal $v_C = v_C^\dagger = v_0$, these equations have a family of solutions

$$v = \frac{i\,g}{\gamma}u + \frac{\beta}{\gamma}v_0 - v_0\,, \tag{3.1.35}$$

where u is the rapidity (3.1.22) of the magnon and the parameters β and γ remain unconstrained. A natural choice is

$$\beta = \gamma = i\,g\,, \tag{3.1.36}$$

giving a simple, sensible solution $v = u$. The parameter v_0 remains unconstrained, but is conventionally set to $v_0 = 0$.

3.1.2 S-matrix

Superspace representation. The worldsheet S-matrix is conveniently described on a $4M$–dimensional graded vector space $\mathcal{V}_M$ of monomials of degree M of two bosonic ω_a ($a = 1, 2$), and two fermionic variables θ_α ($\alpha = 3, 4$) [108]. Any homogeneous supersymmetric polynomial of degree M can be expressed as

$$\begin{aligned}
\Phi_M\,(\omega,\theta) = \phi^{a_1\ldots a_M}\omega_{a_1}\cdots\omega_{a_M} &+ \phi^{a_1\ldots a_{M-1}\alpha}\omega_{a_1}\cdots\omega_{a_{M-1}}\theta_\alpha\\
&+ \phi^{a_1\ldots a_{M-2}\alpha_1\alpha_2}\omega_{a_1}\cdots\omega_{a_{M-2}}\theta_{\alpha_1}\theta_{\alpha_2}\,.
\end{aligned} \tag{3.1.37}$$

[1]One the other hand, the universality condition is quite strong on its own.

In such a way $\mathcal{V}_M = \mathrm{Span}(\Phi_M(\omega, \theta))$.

Recall that M-magnon bound states form a supersymmetric short $4M$–dimensional representation of $\mathfrak{psu}(2|2)_{\mathcal{C}}$ spanned by $|m, n, k, l\rangle \in V_M(p)$. Set $\big\langle m, n, k, l\big| := |m, n, k, l\big\rangle^\dagger$ and require the basis of $V_M(p)$ to be orthonormal,

$$\big\langle a, b, c, d\big|m, n, k, l\big\rangle = \delta_{am}\delta_{bn}\delta_{ck}\delta_{dl}\,. \tag{3.1.38}$$

Then there exists a canonical isomorphism φ between the vectors spaces $V_M(p)$ and $\mathcal{V}_M$ given by

$$\varphi : |m, n, k, l\rangle \mapsto N_{mnkl}\, \omega_1^k\, \omega_2^l\, \theta_3^m\, \theta_4^n\,, \qquad N_{mnkl} = \left(\frac{1}{k!\, l!}\right)^{1/2}. \tag{3.1.39}$$

Choose the hermitian conjugate operators to be

$$(\omega_a)^\dagger = \frac{\partial}{\partial\omega_a}\,, \qquad (\theta_\alpha)^\dagger = \frac{\partial}{\partial\theta_\alpha}\,, \tag{3.1.40}$$

and consider them to be real. Then the dual to vector space $\mathcal{D}^M \simeq \mathcal{V}_M^\dagger$ is realized as the space of polynomials of degree M of the differential operators $\frac{\partial}{\partial\omega_a}$ and $\frac{\partial}{\partial\theta_\alpha}$ with a natural pairing between $\mathcal{D}^M$ and $\mathcal{V}_M$ induced by the relations

$$\frac{\partial}{\partial\omega_a}\omega_b = \delta^a_b\,, \qquad \frac{\partial}{\partial\theta_\alpha}\theta_\beta = \delta^\alpha_\beta\,. \tag{3.1.41}$$

In such a way a superspace representation T_M of $\mathfrak{psu}(2|2)_{\mathcal{C}}$ can be defined via the following differentian operators:

$$\begin{aligned}
\mathrm{T}_M(\mathbb{L}_a{}^b) &= \omega_a\frac{\partial}{\partial\omega_b} - \frac{1}{2}\delta_a^b\,\omega_c\frac{\partial}{\partial\omega_c}\,, \qquad \mathrm{T}_M(\mathbb{Q}_\alpha{}^a) = a\,\theta_\alpha\frac{\partial}{\partial\omega_a} + b\,\epsilon^{ab}\epsilon_{\alpha\beta}\,\omega_b\frac{\partial}{\partial\theta_\beta}\,,\\
\mathrm{T}_M(\mathbb{R}_\alpha{}^\beta) &= \theta_\alpha\frac{\partial}{\partial\theta_\beta} - \frac{1}{2}\delta_\alpha^\beta\,\theta_\gamma\frac{\partial}{\partial\theta_\gamma}\,, \qquad \mathrm{T}_M(\mathbb{G}_a{}^\alpha) = c\,\epsilon_{ab}\epsilon^{\alpha\beta}\,\theta_\beta\frac{\partial}{\partial\omega_b} + d\,\omega_a\frac{\partial}{\partial\theta_\alpha}\,,
\end{aligned} \tag{3.1.42}$$

while the representation of the central generators is given by

$$\mathrm{T}_M(\mathbb{C}) \;= a\,b\,N\,, \qquad \mathrm{T}_M(\mathbb{C}^\dagger) = c\,d\,N\,, \qquad \mathrm{T}_M(\mathbb{H}) = (a\,d + b\,c)\,N\,, \tag{3.1.43}$$

where N is a number operator

$$N = \omega_a\frac{\partial}{\partial\omega_a} + \theta_\alpha\frac{\partial}{\partial\theta_\alpha}\,. \tag{3.1.44}$$

The labels a, b, c, d are defined by (3.1.20). Finally, $\mathrm{T}_M(\mathbb{U}) = U$.

S-matrix. The S-matrix one the superspace can be viewed as an element of

$$S^{\mathrm{MN}}(p_1,p_2)\in\mathrm{End}(\mathcal{V}_M\otimes\mathcal{V}_N)\approx\mathcal{V}_M\otimes\mathcal{V}_N\otimes\mathcal{D}^M\otimes\mathcal{D}^N\,, \tag{3.1.45}$$

and is explicitly expressed as a differential operator

$$S^{\mathrm{MN}}(p_1,p_2)=\sum_{i=1}a_i\,(p_1,p_2)\,\Lambda_i\,, \tag{3.1.46}$$

where $\Lambda_i\in\mathcal{V}_M\otimes\mathcal{V}_N\otimes\mathcal{D}^M\otimes\mathcal{D}^N$ span a complete basis of the differential operators invariant under the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ algebra, and $a_i(p_1,p_2)$ are the scattering coefficients.

The fundamental S-matrix is obtained by setting $M=N=1$ and is conveniently denoted by S^{AA}. The tensor product of two fundamental vector spaces $\mathcal{V}_1\otimes\mathcal{V}_1=\mathcal{W}_2$ is isomorphic to a 16-dimensional fundamental long representation of $\mathfrak{psu}(2|2)_{\mathcal{C}}$. Hence the fundamental S-matrix is described as the second order differential operator

$$S^{\mathrm{AA}}(p_1,p_2)=\sum_{i=1}^{10}a_i(p_1,p_2)\,\Lambda_i\,, \tag{3.1.47}$$

where Λ_i are given by

$$\begin{aligned}
\Lambda_1&=\frac{1}{2}\left(\omega^1_a\omega^2_b+\omega^1_b\omega^2_a\right)\frac{\partial^2}{\partial\omega^2_b\partial\omega^1_a}\,, &\qquad \Lambda_6&=\omega^2_a\theta^1_\alpha\frac{\partial^2}{\partial\omega^2_a\partial\theta^1_\alpha}\,,\\
\Lambda_2&=\frac{1}{2}\left(\omega^1_a\omega^2_b-\omega^1_b\omega^2_a\right)\frac{\partial^2}{\partial\omega^2_b\partial\omega^1_a}\,, & \Lambda_7&=\epsilon^{ab}\omega^1_a\omega^2_b\epsilon_{\alpha\beta}\frac{\partial^2}{\partial\theta^2_\beta\partial\theta^1_\alpha}\,,\\
\Lambda_3&=\frac{1}{2}\left(\theta^1_\alpha\theta^2_\beta+\theta^1_\beta\theta^2_\alpha\right)\frac{\partial^2}{\partial\theta^2_\beta\partial\theta^1_\alpha}\,, & \Lambda_8&=\frac{1}{2}\epsilon^{\alpha\beta}\theta^1_\alpha\theta^2_\beta\epsilon_{ab}\frac{\partial^2}{\partial\omega^2_b\partial\omega^1_a}\,,\\
\Lambda_4&=\frac{1}{2}\left(\theta^1_\alpha\theta^2_\beta-\theta^1_\beta\theta^2_\alpha\right)\frac{\partial^2}{\partial\theta^2_\beta\partial\theta^1_\alpha}\,, & \Lambda_9&=\omega^1_a\theta^2_\alpha\frac{\partial^2}{\partial\omega^2_a\partial\theta^1_\alpha}\,,\\
\Lambda_5&=\omega^1_a\theta^2_\alpha\frac{\partial^2}{\partial\omega^1_a\partial\theta^2_\alpha}\,, & \Lambda_{10}&=\omega^2_a\theta^1_\alpha\frac{\partial^2}{\partial\omega^1_a\partial\theta^2_\alpha}\,.
\end{aligned} \tag{3.1.48}$$

The coefficients $a_i(p_1,p_2)$ are obtained by solving the intertwining equation

$$\Big((\mathrm{T}_M\otimes\mathrm{T}_N)[\Delta^{op}(\mathbb{J}^A)]\,S^{\mathrm{MN}}(p_1,p_2)-S^{\mathrm{MN}}(p_1,p_2)\,(\mathrm{T}_M\otimes\mathrm{T}_N)[\Delta(\mathbb{J}^A)]\Big)\,\mathcal{V}_M\otimes\mathcal{V}_N=0\,. \tag{3.1.49}$$

with $M=N=1$ for all $\mathbb{J}^A\in\mathfrak{psu}(2|2)_{\mathcal{C}}$. The equation above is sufficient to constrain all coefficients $a_i(p_1,p_2)$ of $S^{\mathrm{AA}}(p_1,p_2)$ up to an overall scalar factor, the so-called dressing phase. This can be done using `Wolfram Mathematica` and involves a heavy usage of the mass-shell constraint (3.1.21) giving [18]:

$$
\begin{aligned}
a_1 &= 1\,, & a_6 &= \frac{x_1^- - x_2^-}{x_1^- - x_2^+}\, U_2\,,\\
a_2 &= -1 + 2\,\frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\,\frac{x_2^- - \frac{1}{x_1^+}}{x_2^- - \frac{1}{x_1^-}}\,, & a_7 &= \frac{\left(x_1^- - x_1^+\right)\left(x_1^+ - x_2^+\right)\left(x_2^+ - x_2^-\right)}{\alpha^{-1}\left(x_1^- - x_2^+\right)\left(1 - x_1^- x_2^-\right) U_1\gamma_1\gamma_2}\,,\\
a_3 &= \frac{x_1^+ - x_2^-}{x_1^- - x_2^+}\,\frac{U_2}{U_1}\,, & a_8 &= -\frac{\alpha^{-1}\left(x_1^+ - x_2^+\right)}{\left(x_1^- - x_2^+\right)\left(1 - x_1^- x_2^-\right)}\,\frac{\gamma_1\gamma_2}{U_1^2 U_2}\,,\\
a_4 &= \left[\frac{x_2^- - x_1^+}{x_1^- - x_2^+} + 2\,\frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\,\frac{x_1^- - \frac{1}{x_2^+}}{x_1^- - \frac{1}{x_2^-}}\right]\frac{U_2}{U_1}\,, & a_9 &= \frac{x_1^- - x_1^+}{x_1^- - x_2^+}\,\frac{\gamma_2}{\gamma_1}\,,\\
a_5 &= \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}\,\frac{1}{U_1}\,, & a_{10} &= \frac{x_2^- - x_2^+}{x_1^- - x_2^+}\,\frac{U_2}{U_1}\,\frac{\gamma_1}{\gamma_2}\,.
\end{aligned}
\tag{3.1.50}
$$

In the same way, by solving the intertwining equation (3.1.49) for all Lie algebra generators, the bound state S-matrices S^{M1} and S^{1M} for any $M \geq 2$ can be obtained. This is because the tensor space $\mathcal{V}_M \otimes \mathcal{V}_1 \cong \mathcal{V}_1 \otimes \mathcal{V}_M = \mathcal{W}_{M+1}$ is isomorphic to an irreducible $16M$–dimensional long representation of $\mathfrak{psu}(2|2)_{\mathcal{C}}$.

However, in the case when $M, N \geq 2$, the Lie algebra alone is not enough to fix all coefficients of the bound state S-matrix S^{MN}. This is because the tensor product of higher order supersymmetric short representations generically yields a sum of long representations. To remedy this, one either needs to invoke the Yang-Baxter equation or use the Yangian symmetry [108, 129].

The case $M = N = 2$ is exceptional. The tensor space $\mathcal{V}_2 \otimes \mathcal{V}_2 = \mathcal{W}_{16} \oplus \mathcal{W}_{48}$ is isomorphic to a sum of two 16– and 48–dimensional irreducible long representations of $\mathfrak{psu}(2|2)_{\mathcal{C}}$. The corresponding S-matrix S^{BB} has 48 scattering coefficients. Choose $a_1(p_1, p_2) = 1$. Then the Lie algebra constrains all but one of the remaining 47 coefficients. The last coefficient can be found by solving the Yang-Baxter equation [108] or by employing the Yangian symmetry, i.e. by solving the intertwining equation (3.1.49) for all $\widehat{\mathbb{J}}^A \in \mathcal{Y}(\mathfrak{psu}(2|2)_{\mathcal{C}})$ [20, 109]. In such a way S^{BB} serves as the simplest non-trivial test for the Yangian symmetry.

Interestingly, there is an alternative way to obtain all 48 coefficients (up to an overall dressing phase) of S^{BB}. This S-matrix is specific because only 44 scattering coefficients are non-zero. Thus by setting $a_1(p_1, p_2) = 1$ and $a_j(p_1, p_2) = 0$ for $j = 45 \ldots 48$ (in terms of the notation used in [108]) the remaining 43 coefficients are then uniquely defined by the Lie algebra only [133]. However one needs to know that $a_{45\ldots48}$ are trivial in advance.

The non-local representation. In the last part of this section we will present a non-local representation of $\mathfrak{psu}(2|2)_{\mathcal{C}}$. This representation has a very nice interpretation as we will show a little bit further and will be heavily employed in Section 3.5 where we will discuss boundary scattering for the $D5$-brane.

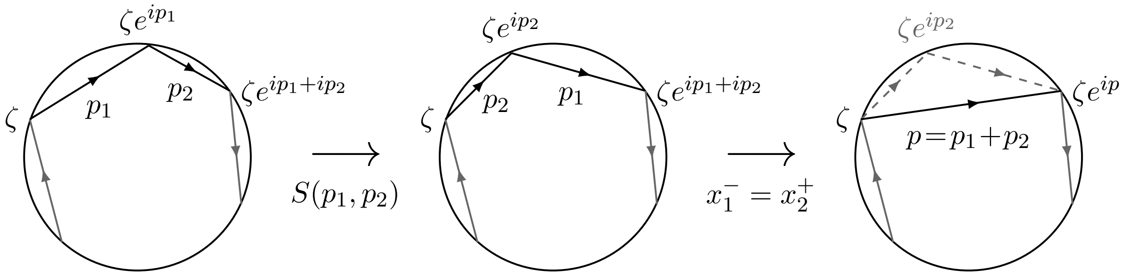

Figure 3.1: *Step 1.* Scattering of two well-separated magnons in an LLM-type coordinates [64,134]. Here the circle represents $S^2 \subset S^5$ of $AdS_5 \times S^5$ where the string configuration is extended (the standard choice being the Z_{56}-plane) and the poly-line represents a segment of a very long (closed) string with straight lines representing excitations. The length of each straight line is proportional to the momentum p_i of the excitation and thus the total momentum of the excitations of string must be $\sum_i p_i = 2\pi n,\ n \in \mathbb{N}$. The direction of the string defines the increment of the phase. Here two excitations with momenta p_1 and p_2 (ζ being the phase reference point) are participating in the scattering. The diagram shows that the scattering of two states does not affect the rest of the string. *Step 2.* Creation of a two-magnon bound state at the pole $x_1^- = x_2^+$ of the S-matrix. The resulting bound state has momentum $e^{ip} = e^{ip_1+ip_2} = \frac{x_1^+}{x_1^-}\frac{x_2^+}{x_2^-} = \frac{x_1^+}{x_2^-} \equiv \frac{x^+}{x^-}$.

The non-local representation is obtained by absorbing the non-trivial braiding of the Hopf algebra into the definition of the representation labels a, b, c, d:

$$a = \sqrt{\frac{g}{2M}}\eta, \quad b = \sqrt{\frac{g}{2M}}\frac{i\zeta}{\eta}\left(\frac{x^+}{x^-} - 1\right), \quad c = -\sqrt{\frac{g}{2M}}\frac{\eta}{\zeta x^+}, \quad d = -\sqrt{\frac{g}{2M}}\frac{x^+}{i\eta}\left(\frac{x^-}{x^+} - 1\right). \tag{3.1.51}$$

These labels can be obtained from (3.1.20) by a simple map, $\alpha \to -i\zeta$, $g \to g/2$ and $\gamma \to \eta$. Here $\zeta = e^{2i\xi}$ is the magnon phase, and the unitarity constrains, $\eta = \mathrm{e}^{i\xi}\,\mathrm{e}^{i\varphi}\sqrt{i\,(x^- - x^+)}$. This representation is sometimes also called the *'Utrecht'* basis, while the one in (3.1.20) is called the *'Potsdam'* basis.[2]

The magnon phase ζ is the non-local element in this representation and carries the information about the total momentum of magnons standing left with respect to the magnon under the consideration. In such a way the S-matrix in this representation has a nice interpretation as shown in the Step 1 of figure 3.1. This representation is also very transparent in showing the creation of bound states at the pole of the S-matrix, see Step 2 of figure 3.1.

The S-matrix on the superspace in the non-local basis is represented by

$$S^{\mathrm{MN}} : \mathcal{V}_M(p_1, \zeta) \otimes \mathcal{V}_N(p_2, \zeta\mathrm{e}^{ip_1}) \to \mathcal{V}_M(p_1, \zeta\mathrm{e}^{ip_2}) \otimes \mathcal{V}_N(p_2, \zeta)\,, \tag{3.1.52}$$

[2]Such naming arose by the different groups working on the same topic at Utrecht and Potsdam but using different notation. Furthermore, 'Potsdam' basis usually follows by Gothic scipt used to denote the algebra elements of $\mathfrak{psu}(2|2)_{\mathcal{C}}$.

where each vector space $\mathcal{V}_M(p_i,\zeta_i)$ is now carrying an extra phase factor. The scattering coefficient of S^{AA} in this basis can be obtained from the ones in (3.1.50) by the following prescription:

$$\begin{aligned} &a_{1,2}\to a_{1,2}\,, && a_5\to U_1\frac{\widetilde{\eta}_2}{\eta_2}\,a_5\,, && a_7\to\frac{\zeta U_1}{i\alpha}\frac{\gamma_1\gamma_2}{\eta_1\eta_2}\,a_7\,, && a_9\to\frac{\gamma_1\widetilde{\eta}_2}{\gamma_2\eta_1}\,a_9\,,\\ &a_{3,4}\to\frac{U_1}{U_2}\frac{\widetilde{\eta}_1\widetilde{\eta}_2}{\eta_1\eta_2}a_{3,4}\,, && a_6\to\frac{1}{U_2}\frac{\widetilde{\eta}_1}{\eta_1}\,a_6\,, && a_8\to\frac{i\alpha U_2}{\zeta U_1^2}\frac{\widetilde{\eta}_1\widetilde{\eta}_2}{\gamma_1\gamma_2}\,a_8\,, && a_{10}\to\frac{U_1}{U_2}\frac{\gamma_2\widetilde{\eta}_1}{\gamma_1\eta_2}\,a_{10}\,, \end{aligned} \tag{3.1.53}$$

where

$$\eta_1=\eta(p_1,\zeta)\,,\qquad \widetilde{\eta}_1=\eta(p_1,\zeta\mathrm{e}^{ip_1})\,,\qquad \eta_2=\eta(p_2,\zeta\mathrm{e}^{ip_1})\,,\qquad \widetilde{\eta}_2=\eta(p_2,\zeta)\,. \tag{3.1.54}$$

3.2 Boundary scattering

3.2.1 The spectrum of boundaries

The best known and most studied boundaries are the so-called $Y=0$ and $Z=0$ giant gravitons that are $D3$-branes occupying the maximal $S^3\subset S^5$ of the $AdS_5\times S^5$ spacetime [21,22,62,63,135–141]. In the gauge theory side of the duality these branes correspond to determinant operators of scalar fields. We will consider the boundary Yangian symmetry and scattering theory for these branes in detail in Section 3.3.

The second type of boundaries we will focus on is the so-called $Y=0$ and $Z=0$ $D7$-branes spanning $AdS_5\times S^3$ [23,59,61,133,142,143]. The explicit description of these branes in the gauge theory side of the duality is very different that of the $D3$-branes, however the boundary scattering theory is almost the same: the reflection matrices for the $Y=0$ giant graviton and for the $Y=0$ $D7$-brane are equivalent, while the one for the $Z=0$ $D7$-brane factorizes in to two non-equivalent factors, left and right. The scattering in the right factor is identical to that of the $Z=0$ giant graviton, while the left factor does not respect any supersymmetries and is somewhat similar to the $Y=0$ case. We will give the details on these branes in Section 3.4.

The last set of boundaries we will encounter are the 'horizontal' and 'vertical' $D5$-branes wrapping a defect hypersurface $AdS_4\subset AdS_5$ and a maximal $S^2\subset S^5$ [59,61,133,142–144]. These boundaries are very distinct as they are of the 'achiral' type and were long thought not to be integrable [143,145,146]. The boundary scattering in this case does not factorize into independent left and right factors as it does for the giant gravitons and $D7$-branes, but becomes entangled. An incoming magnon living in the left factor of theory emerges in the right factor after the reflection, and vice-versa for an incoming right magnon [144]. Such boundary scattering has an underlying boundary Yangian symmetry of an 'achiral' type which can be understood as a particular specialization of the twisted Yangian of type I [24]. We will present this algebra and corresponding scattering theory in Section 3.5.

The main properties characterizing the boundary scattering theory for the branes described above are as follows (here $\mathfrak{psu}(2|2)_+$ is the diagonally embedded subalgebra):

	Lie algebra	representation
bulk magnons	$\mathfrak{psu}(2\|2) \times \mathfrak{psu}(2\|2) \times \mathbb{R}^3$	$vector \times vector$
$Y\!=\!0$ graviton	$\mathfrak{su}(2\|1) \times \mathfrak{su}(2\|1)$	$singlet \times singlet$
$Z\!=\!0$ graviton	$\mathfrak{psu}(2\|2) \times \mathfrak{psu}(2\|2) \times \mathbb{R}^3$	$vector \times vector$
$Y\!=\!0$ $D7$-brane	$\mathfrak{su}(2\|1) \times \mathfrak{su}(2\|1)$	$singlet \times singlet$
$Z\!=\!0$ $D7$-brane	$\mathfrak{su}(2) \times \mathfrak{su}(2) \times \mathfrak{psu}(2\|2) \times \mathbb{R}^3$	$singlet \times vector$
horizontal $D5$-brane	$\mathfrak{psu}(2\|2)_+ \times \mathbb{R}^3$	$singlet$
vertical $D5$-brane	$\mathfrak{psu}(2\|2)_+ \times \mathbb{R}^3$	$vector$

3.2.2 Reflected algebra

Reflection autmorphism. The braided structure of the algebra allows us to define an automorphism κ which acts by inverting the braiding factor,

$$\kappa : \mathfrak{psu}(2|2)_{\mathcal{C}} \to \mathfrak{psu}(2|2)_{\underline{\mathcal{C}}} \qquad \mathbb{J}^A \mapsto \underline{\mathbb{J}}^A\,, \quad \mathbb{U} \mapsto \underline{\mathbb{U}} = \mathbb{U}^{-1}\,, \tag{3.2.1}$$

where the under-barred generators $\underline{\mathbb{J}}^A \in \mathfrak{psu}(2|2)_{\underline{\mathcal{C}}}$. It is an involution, $\kappa^2 = id$, and plays the role of the reflection map of the algebra as we will show explicitly a bit further.

The automorphism κ is compatible with the Hopf algebra in the sense that one can introduce a 'reflected' coproduct,

$$\Delta^{ref} := (\kappa \otimes id) \circ \Delta\,, \qquad\qquad \Delta^{op.ref} := (id \otimes \kappa) \circ \Delta^{op}\,, \tag{3.2.2}$$

giving

$$\Delta^{ref}(\mathbb{J}^A) = \underline{\mathbb{J}}^A \otimes 1 + \mathbb{U}^{-[[A]]} \otimes \mathbb{J}^A\,, \qquad \Delta^{op.ref}(\mathbb{J}^A) = \mathbb{J}^A \otimes \mathbb{U}^{-[[A]]} + 1 \otimes \underline{\mathbb{J}}^A\,. \tag{3.2.3}$$

The $\mathbb{U}^{[[A]]}$-braiding structure of the Hopf algebra implies that κ acts trivially on the generators that are not charged under $\mathbb{U}$ (i.e. $\underline{\mathbb{J}}^A = \mathbb{J}^A$ if $[[A]] = 0$), thus

$$\underline{\mathbb{R}}_a{}^b = \mathbb{R}_a{}^b\,, \qquad \underline{\mathbb{L}}_\alpha{}^\beta = \mathbb{L}_\alpha{}^\beta\,, \qquad \underline{\mathbb{H}} = \mathbb{H}\,, \tag{3.2.4}$$

while (3.1.7) gives

$$\underline{\mathbb{C}} = g\,\alpha\,(1 - \mathbb{U}^{-2})\,, \qquad \underline{\mathbb{C}}^\dagger = g\,\alpha^\dagger(1 - \mathbb{U}^{+2})\,. \tag{3.2.5}$$

We can further require κ to be in the outer–automorphism group, $\kappa \in SL(2)$. Then

$$\begin{aligned}
&\underline{\mathbb{Q}}_\alpha{}^a = u_1\,\mathbb{Q}_\alpha{}^a - u_2\,\varepsilon^{ab}\,\varepsilon_{\alpha\beta}\,\mathbb{G}_b{}^\beta\,, && \underline{\mathbb{C}}\; = u_1^2\,\mathbb{C} + u_2^2\,\mathbb{C}^\dagger + u_1 u_2\,\mathbb{H}\,,\\
&\underline{\mathbb{G}}_a{}^\alpha = v_1\,\mathbb{G}_a{}^\alpha - v_2\,\varepsilon_{ab}\,\varepsilon^{\alpha\beta}\,\mathbb{Q}_\beta{}^b\,, && \underline{\mathbb{C}}^\dagger = v_1^2\,\mathbb{C}^\dagger + v_2^2\,\mathbb{C} + v_1 v_2\,\mathbb{H}\,,\\
&\underline{\mathbb{H}} = (u_1 v_1 + u_2 v_2)\,\mathbb{H} + 2\,u_1 v_2\,\mathbb{C} + 2\,u_2 v_1\,\mathbb{C}^\dagger\,.
\end{aligned} \tag{3.2.6}$$

These relations are satisfied provided $u_1^2 = -\mathbb{U}^2$, $v_1^2 = -\mathbb{U}^{-2}$ and $u_2 = v_2 = 0$. Set $\lambda = i\,\mathbb{U}$, then κ is explicitly given by

$$\kappa : \left\{ \begin{array}{l} \mathbb{R}_a^{\ b},\ \mathbb{L}_\alpha^{\ \beta},\ \mathbb{H} \\ \mathbb{Q}_\alpha^{\ a},\ \mathbb{G}_a^{\ \alpha} \\ \mathbb{C},\ \mathbb{C}^\dagger \end{array} \right\} \to \left\{ \begin{array}{l} \mathbb{R}_a^{\ b},\ \mathbb{L}_\alpha^{\ \beta},\ \mathbb{H} \\ \lambda^{-1}\,\mathbb{Q}_\alpha^{\ a},\ \lambda\,\mathbb{G}_a^{\ \alpha} \\ \lambda^{-2}\,\mathbb{C},\ \lambda^2\,\mathbb{C}^\dagger \end{array} \right\}. \tag{3.2.7}$$

Reflected representation. The representation defined in section 3.1.1.1 describes magnons with momentum p. The algebra automorphism κ, as can be easily deduced from (3.2.1), acts on the representation by inverting momentum,

$$\kappa : \mathbb{U} \mapsto \mathbb{U}^{-1} \qquad \Longrightarrow \qquad \kappa : p \mapsto -p\,, \tag{3.2.8}$$

In such a way the map κ represents the reflection automorphism of the algebra, i.e. maps magnons with momentum p to $-p$ and vice versa. Let us now explicitly define the reflected representation.

The representation constraints are easily deduced to be

$$\underline{a}\,\underline{d} - \underline{b}\,\underline{c} = 1\,, \qquad M\,\underline{a}\,\underline{b} = g\,\alpha\,(1 - U^{-2})\,, \qquad M\,\underline{c}\,\underline{d} = g\,\alpha^{-1}\,(1 - U^2)\,. \tag{3.2.9}$$

These can be solved in terms of the previously defined labels a, b, c, d. Thus (3.2.9) together with (3.1.18) and (3.1.19) leads to

$$\underline{a} = \frac{\underline{\gamma}}{\gamma}a\,, \qquad \underline{b} = \frac{\alpha^2\gamma}{\underline{\gamma}}\frac{c\,d}{a}\,, \qquad \underline{c} = \frac{\underline{\gamma}}{\alpha^2\gamma}\frac{a\,b}{d}\,, \qquad \underline{d} = \frac{\gamma}{\underline{\gamma}}d\,, \tag{3.2.10}$$

giving

$$\underline{a} = \sqrt{\frac{g}{M}}\,\underline{\gamma}, \quad \underline{b} = \sqrt{\frac{g}{M}}\frac{\alpha}{\underline{\gamma}}\left(1 - \frac{x^-}{x^+}\right), \quad \underline{c} = -\sqrt{\frac{g}{M}}\frac{i\underline{\gamma}}{\alpha x^-}, \quad \underline{d} = \sqrt{\frac{g}{M}}\frac{ix^+}{\underline{\gamma}}\left(\frac{x^-}{x^+} - 1\right). \tag{3.2.11}$$

Here we have chosen $\underline{a} = a\,\underline{\gamma}/\gamma$ as an initial constraint. Then, by comparing (3.2.11) with (3.1.20) we find that $\kappa : x^\pm \mapsto -x^\mp$ and $\kappa : \gamma \mapsto \underline{\gamma}$. We will give an explicit form of $\underline{\gamma}$ a little bit further.

The relation between the representation labels (3.1.20) and (3.2.11) can be represented by a compact matrix relation,

$$\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} D = T \begin{pmatrix} a & b \\ c & d \end{pmatrix} T^{-1} \quad \text{where} \quad D = \begin{pmatrix} \gamma/\underline{\gamma} & 0 \\ 0 & \underline{\gamma}/\gamma \end{pmatrix}, \quad T = \begin{pmatrix} U^{-2} & 0 \\ 0 & -1 \end{pmatrix}, \tag{3.2.12}$$

Recall that $\kappa \in SL(2)$ is in the outer–automorphism group and is given by

$$\kappa = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}, \qquad \text{where} \qquad \lambda = i\,\mathbb{U}\,. \tag{3.2.13}$$

Therefore

$$\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} = \begin{pmatrix} -i\,U^{-1} & 0 \\ 0 & i\,U \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}. \tag{3.2.14}$$

Then upon requiring (3.2.12) and (3.2.14) to be in consistency we find $\underline{\gamma} = -i\,U^{-1}\gamma$. Note that the solution of (3.2.6) is $\lambda^2 = -\mathbb{U}^2$, thus we could have equivalently chosen $\lambda = -i\,\mathbb{U}$. This choice leads to $\underline{\gamma} = i\,U^{-1}\gamma$. Thus we will further always retain γ and $\underline{\gamma}$ without going into explicit form in order to avoid any disambiguities.

3.3 Giant gravitons

The maximal giant graviton is a $D3$-brane in $AdS_5 \times S^5$ wrapping a topologically-trivial cycle enclosing maximal $S^3 \subset S^5$, and is prevented from collapsing by coupling to the background supergravity fields [114]. The usual parametrization of S^5 is expressed in terms of the complex coordinates $X = \Phi_1 + i\Phi_2$, $Y = \Phi_3 + i\Phi_4$, $Z = \Phi_5 + i\Phi_6$ respecting $|X|^2 + |Y|^2 + |Z|^2 = 1$, where the radius of S^5 has been set to unity, $R = 1$. In this parametrization the maximal giant graviton is obtained by setting any two Φ_i's to zero. However, any two such configurations are related to each other by an $SO(6)$ rotation. This symmetry can be broken by attaching an open string to the brane and giving it a charge J corresponding to the preferred $SO(2) \subset SO(6)$ rotation.

The parametrization in complex coordinates makes it easy to translate this setup to the gauge theory side. The triplet X, Y, Z can be thought of as representing the three complex scalar fields of the $\mathcal{N} = 4$ super Yang-Mills. Then the field theory description of the string in the large J limit carries a large number of insertions, called the Bethe vacuum state, of the field corresponding to the preferred rotation, and a relatively small number of other fields, called excitations (or simply magnons). The explicit description of the string in the gauge theory depends on the choice of the particular generator J and the relevant orientation of the giant graviton inside S^5. The two relevant cases are obtained by choosing $J = J_{56}$ and the giant graviton to be the maximal three sphere given by $Y = 0$ or $Z = 0$ with the standard Bethe vacuum on the string being $Z = X_5 + iX_6$ [135].

Let us take a closer look at the string theory description of this setup. The $Y = 0$ giant graviton covers the whole disk in the Z plane and wraps an S^1 inside the S^3 attached to each point of the plane. In such a way an open string attached to the $Y = 0$ giant graviton in the large J limit is just a string stretched between two points on the rim. Note that there is an additional S^3 at each point of the Z-plane, thus the string does not necessary need to be contained on the D-brane, see figure 3.2 (left).

In this picture, the $Z = 0$ giant graviton is simply a point at $Z = 0$ and wraps entire S^3 attached to the $Z = 0$ point. Then the open string attached to the $Z = 0$ giant graviton and carrying a large charge J starts at the centre of the circle, extends to the rim, and ends at centre again, see figure 3.2 (right). This configuration suggest that the string is carrying boundary degrees freedom. This is indeed true as we will see explicitly at gauge theory side of the duality.

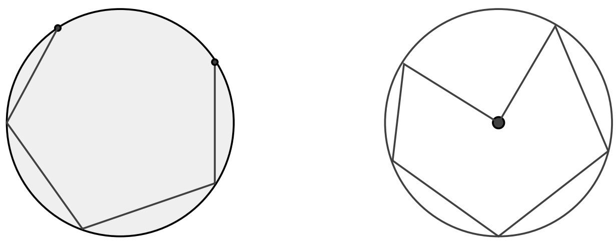

Figure 3.2: *Left:* A large J string attached to the $Y = 0$ giant graviton in the Z plane. The graviton is filling the whole plane. The black dots represent the points where the string is attached to the graviton. Straight segments of the string represent the worldsheet excitations propagating along the string. *Right:* A large J string attached to the $Z = 0$ giant graviton in the Z plane. The graviton is the thick black dot at the centre of the Z plane. The boundary degrees of freedom correspond to the segments of the string connecting the centre to the rim of the circle.

The $Y = 0$ giant graviton in the gauge theory is described as a single trace gauge invariant baryon-like operator det(Y) with $J = N$ [116]. In the large-N limit this operator has no dynamics of its own and serves as 'an infinitely heavy boundary'. Then by attaching an open string one obtains

$$\mathcal{O}_Y = \epsilon^{j_1 j_2 \ldots j_{N-1} A}_{i_1 i_2 \ldots i_{N-1} B} Y^{i_1}_{j_1} Y^{i_2}_{j_2} \ldots Y^{i_{N-1}}_{j_{N-1}} (ZZ \ldots ZZ)^B_A \,. \tag{3.3.1}$$

Here the string is in the 'ground state' and thus is composed entirely of the Bethe vacuum states Z. One can further introduce some impurities χ to propagate along the string,

$$\mathcal{O}_Y = \epsilon^{j_1 j_2 \ldots j_{N-1} A}_{i_1 i_2 \ldots i_{N-1} B} Y^{i_1}_{j_1} Y^{i_2}_{j_2} \ldots Y^{i_{N-1}}_{j_{N-1}} (Z \ldots Z\chi_1 Z \ldots Z\chi_2 Z \ldots Z)^B_A \,. \tag{3.3.2}$$

We assume that the number of impurities χ is much smaller than the number of Z's and all impurities are separated from each other by a large number of Z's. In such a way the scattering of two impurities along the chain of Z's is described by the worldsheet S-matrix considered in Section 3.2. In a similar way, the reflection of the impurities from the ends of the chain is described by the worldsheet reflection K-matrix. We will construct this K-matrix in the following section.

In case of an open string attached to the $Z = 0$ giant graviton the corresponding gauge invariant operator is given by

$$\mathcal{O}_Z = \epsilon^{j_1 j_2 \ldots j_{N-1} A}_{i_1 i_2 \ldots i_{N-1} B} Z^{i_1}_{j_1} Z^{i_2}_{j_2} \ldots Z^{i_{N-1}}_{j_{N-1}} (\chi' Z \ldots Z\chi_1 Z \ldots Z\chi_2 Z \ldots Z\chi'')^B_A \,, \tag{3.3.3}$$

where the impurities χ' and χ'' are attached to the ends of the Z-chain and are called the boundary states. In the absence of the boundary states the operator $\mathcal{O}_Z$ would factorize into a determinant and an independent single trace operator and thus would describe a non-interacting system of a giant graviton and a closed string [63].

In the large J limit the string worldsheet is a very long segment. Consequently, the left and right boundaries are well separated and can be treated independently; thus the boundary scattering becomes equivalent to scattering on a semi-infinite line. In AdS/CFT this translates into the description of a magnon incoming from infinity, reflecting from the boundary, and returning back to infinity. Hence the asymptotic states are interpolating between the usual vacuum of BMN states [58] and the boundary. This treatment allows us to employ the usual S-matrix technique to study the boundary scattering.

3.3.1 Y=0 giant graviton

We will further consider boundary symmetries and the scattering theory for the $Y = 0$ giant graviton. For complete details on the setup of this boundary we refer to [135].

3.3.1.1 Boundary symmetries

The symmetry algebra in the bulk is given by two copies, left and right, of $\mathfrak{psu}(2|2)_{\mathcal{C}}$. The $Y = 0$ giant graviton preserves the subgroup which is also preserved by the field Y. This restricts the symmetry algebra on the boundary to be two copies $\mathfrak{h} = \mathfrak{su}(2|1)$ and has no degrees of freedom attached to the end of the spin chain, as we have discussed above. This allows us to consider the left and right factors of the boundary algebra independently.

The commutation relations of $\mathfrak{su}\,(2|1)$ are acquired from (3.1.2) by dropping the generators with bosonic indices $a,\ b,\ c,\ ... = 1$ (or equivalently with 2). It is straightforward to check that the subalgebra $\mathfrak{h}$ and the subset of 'broken' generators $\mathfrak{m} = \mathfrak{psu}(2|2)_{\mathcal{C}}\backslash\mathfrak{h}$ given by

$$\mathfrak{h} = \{\mathbb{L}_{\alpha}{}^{\beta},\, \mathbb{R}_{2}{}^{2},\, \mathbb{Q}_{\alpha}{}^{2},\, \mathbb{G}_{2}{}^{\alpha},\, \mathbb{H}\}\,, \qquad \mathfrak{m} = \{\mathbb{R}_{1}{}^{2},\, \mathbb{R}_{2}{}^{1},\, \mathbb{Q}_{\gamma}{}^{1},\, \mathbb{G}_{1}{}^{\gamma},\, \mathbb{C},\, \mathbb{C}^{\dagger}\}\,, \tag{3.3.4}$$

satisfy

$$[\mathfrak{h},\mathfrak{h}] \subset \mathfrak{h}\,, \qquad [\mathfrak{h},\mathfrak{m}] \subset \mathfrak{m}\,, \qquad [\mathfrak{m},\mathfrak{m}] \subset \mathfrak{h}\,, \tag{3.3.5}$$

thus form a symmetric pair. In such a way the theory should have an underlying Yangian symmetry, which corresponds to a generalized twisted Yangian of type I.

Twisted Yangian. The twisted Yangian of the $Y = 0$ giant graviton is of type I and is generated by the level-0 generators and the twisted level-1 Yangian generators

$$\widetilde{\mathbb{J}}^{P} := \widehat{\mathbb{J}}^{P} + \tfrac{1}{4}\, f^{P}{}_{QI}\left(\mathbb{J}^{Q}\,\mathbb{J}^{I} + (-1)^{[Q][I]}\,\mathbb{J}^{I}\,\mathbb{J}^{Q}\right) = \widehat{\mathbb{J}}^{P} - \tfrac{1}{4}\,[\,\mathbb{T}^{\mathfrak{h}},\, \mathbb{J}^{P}\,]\,, \tag{3.3.6}$$

where $\mathbb{J}^{P}, \mathbb{J}^{Q} \in \mathfrak{h}$ and $\mathbb{J}^{I} \in \mathfrak{m}$ and $(-1)^{[Q][I]}$ is the fermionic grading factor; $\mathbb{T}^{\mathfrak{h}}$ is the generalized quadratic Casimir operator (3.1.25) restricted to the subalgebra $\mathfrak{h}$,

$$\mathbb{T}^{\mathfrak{h}} = -2\,\mathbb{H}\,\mathbb{Y} + 2\,\mathbb{R}_{1}^{\ 1}\,\mathbb{R}_{1}^{\ 1} - \mathbb{L}_{\gamma}^{\ \delta}\,\mathbb{L}_{\delta}^{\ \gamma} + \mathbb{Q}_{\gamma}^{\ 2}\,\mathbb{G}_{2}^{\ \gamma} - \mathbb{G}_{2}^{\ \gamma}\,\mathbb{Q}_{\gamma}^{\ 2}\,. \tag{3.3.7}$$

In such a way we obtain [3]

$$
\begin{aligned}
&\widetilde{\mathbb{Q}}_\alpha^{\ 1} = \widehat{\mathbb{Q}}_\alpha^{\ 1} - \tfrac{1}{4}[\mathbb{T}^\flat, \mathbb{Q}_\alpha^{\ 1}]\,, \qquad \widetilde{\mathbb{R}}_1^{\ 2} = \widehat{\mathbb{R}}_1^{\ 2} - \tfrac{1}{4}[\mathbb{T}^\flat, \mathbb{R}_1^{\ 2}]\,, \qquad \widetilde{\mathbb{C}} = \widehat{\mathbb{C}} - \tfrac{1}{4}[\mathbb{T}^\flat, \mathbb{C}]\,,\\
&\widetilde{\mathbb{G}}_1^{\ \alpha} = \widehat{\mathbb{G}}_1^{\ \alpha} - \tfrac{1}{4}[\mathbb{T}^\flat, \mathbb{G}_1^{\ \alpha}]\,, \qquad \widetilde{\mathbb{R}}_2^{\ 1} = \widehat{\mathbb{R}}_2^{\ 1} - \tfrac{1}{4}[\mathbb{T}^\flat, \mathbb{R}_2^{\ 1}]\,, \qquad \widetilde{\mathbb{C}}^\dagger = \widehat{\mathbb{C}}^\dagger - \tfrac{1}{4}[\mathbb{T}^\flat, \mathbb{C}^\dagger]\,.
\end{aligned}
\tag{3.3.9}
$$

These twisted Yangian generators satisfy the usual Lie algebra commutation relations; their coproducts are given by

$$
\begin{aligned}
\Delta(\widetilde{\mathbb{Q}}_\alpha^{\ 1}) &= \widetilde{\mathbb{Q}}_\alpha^{\ 2} \otimes 1 + \mathbb{U} \otimes \widetilde{\mathbb{Q}}_\alpha^{\ 2} + \mathbb{Q}_\alpha^{\ 2} \otimes \mathbb{H}' - \mathbb{U}\,\mathbb{R}_1^{\ 2} \otimes \mathbb{Q}_\alpha^{\ 1} + \mathbb{Q}_\gamma^{\ 2} \otimes \mathbb{L}_\alpha^{\ \gamma} - \varepsilon_{\alpha\gamma}\,\mathbb{U}^{-1}\,\mathbb{C} \otimes \mathbb{G}_1^{\ \gamma}\,,\\
\Delta(\widetilde{\mathbb{G}}_1^{\ \alpha}) &= \widetilde{\mathbb{G}}_1^{\ \alpha} \otimes 1 + \mathbb{U}^{-1} \otimes \widetilde{\mathbb{G}}_1^{\ \alpha} - \mathbb{G}_1^{\ \alpha} \otimes \mathbb{H}' + \mathbb{U}^{-1}\mathbb{R}_1^{\ 2} \otimes \mathbb{G}_2^{\ \alpha} - \mathbb{G}_1^{\ \gamma} \otimes \mathbb{L}_\gamma^{\ \alpha} - \varepsilon^{\alpha\gamma}\mathbb{U}\,\mathbb{C}^\dagger \otimes \mathbb{Q}_\gamma^{\ 1},\\
\Delta(\widetilde{\mathbb{R}}_1^{\ 2}) &= \widetilde{\mathbb{R}}_1^{\ 2} \otimes 1 + 1 \otimes \widetilde{\mathbb{R}}_1^{\ 2} - 2\,\mathbb{R}_1^{\ 2} \otimes \mathbb{R}_1^{\ 1} - \mathbb{U}\,\mathbb{G}_1^{\ \gamma} \otimes \mathbb{Q}_\gamma^{\ 2}\,,\\
\Delta(\widetilde{\mathbb{R}}_2^{\ 1}) &= \widetilde{\mathbb{R}}_2^{\ 1} \otimes 1 + 1 \otimes \widetilde{\mathbb{R}}_2^{\ 1} + 2\,\mathbb{R}_2^{\ 1} \otimes \mathbb{R}_1^{\ 1} - \mathbb{U}^{-1}\,\mathbb{Q}_\gamma^{\ 1} \otimes \mathbb{G}_2^{\ \gamma}\,,\\
\Delta(\widetilde{\mathbb{C}}) &= \widetilde{\mathbb{C}} \otimes 1 + \mathbb{U}^2 \otimes \widetilde{\mathbb{C}} + \mathbb{C} \otimes \mathbb{H}\,,\\
\Delta(\widetilde{\mathbb{C}}^\dagger) &= \widetilde{\mathbb{C}}^\dagger \otimes 1 + \mathbb{U}^{-2} \otimes \widetilde{\mathbb{C}}^\dagger - \mathbb{C}^\dagger \otimes \mathbb{H}\,,
\end{aligned}
\tag{3.3.10}
$$

and thus the coideal property is satisfied. Here $\mathbb{H}' = \mathbb{R}_1^{\ 1} + \frac{1}{2}\mathbb{H}$, and we have used the relation $\mathbb{R}_1^{\ 1} = -\mathbb{R}_2^{\ 2}$. Note that the structure constants $f^P_{\ QI}$ in (3.3.6) are obtained from the structure of Yangian $\mathcal{Y}(\mathfrak{psu}(2|2)_\mathcal{C})$, but not from the Lie algebra itself. This is because Lie algebra has a degenerate Cartan-Killing form and thus otherwise the terms with central charges would absent from the twist. This is a specific feature of this algebra and we will encounter it again when constructing the twisted Yangian algebras for the other boundaries in the following sections. Finally we note that the twisted generator $\widetilde{\mathbb{R}}_2^{\ 1}$ is equivalent to the conserved charge $\widetilde{\mathbb{Q}}$ introduced in [21].

3.3.1.2 Boundary scattering

We will construct the boundary scattering for a single factor only. The complete reflection matrix then follows straightforwardly.

Consider a fundamental magnon which transforms irreducibly in the fundamental representation $\square$ of $\mathfrak{psu}(2|2)_\mathcal{C}$. In terms of the boundary Lie algebra, the fundamental magnon transforms in the supersymmetric representation $\square\square$ of $\mathfrak{su}(2|1)$. Thus the boundary Lie algebra defines the fundamental reflection matrix up to an overall dressing phase. The dressing phase can obtained by considering the crossing equations. In such a way the reflection matrix and the dressing phase found in [135].

[3] The explicit form of the twisted Yangian generators is

$$
\begin{aligned}
&\widetilde{\mathbb{Q}}_\alpha^{\ 1} = \widehat{\mathbb{Q}}_\alpha^{\ 1} - \tfrac{1}{4}\{\mathbb{Q}_\alpha^{\ 2}, \mathbb{R}_2^{\ 1}\} + \tfrac{1}{4}\{\mathbb{R}_1^{\ 1}, \mathbb{Q}_\alpha^{\ 1}\} + \tfrac{1}{4}\{\mathbb{Q}_\gamma^{\ 1}, \mathbb{L}_\alpha^{\ \gamma}\} + \tfrac{1}{4}\,\mathbb{Q}_\alpha^{\ 1}\,\mathbb{H} + \tfrac{1}{2}\,\varepsilon_{\alpha\gamma}\,\mathbb{C}\,\mathbb{G}_2^{\ \gamma}\,,\\
&\widetilde{\mathbb{G}}_1^{\ \alpha} = \widehat{\mathbb{G}}_1^{\ \alpha} + \tfrac{1}{4}\{\mathbb{G}_2^{\ \alpha}, \mathbb{R}_1^{\ 2}\} - \tfrac{1}{4}\{\mathbb{R}_1^{\ 1}, \mathbb{G}_1^{\ \alpha}\} - \tfrac{1}{4}\{\mathbb{G}_1^{\ \gamma}, \mathbb{L}_\gamma^{\ \alpha}\} - \tfrac{1}{4}\,\mathbb{G}_1^{\ \alpha}\,\mathbb{H} - \tfrac{1}{2}\,\varepsilon^{\alpha\gamma}\,\mathbb{C}^\dagger\,\mathbb{Q}_\gamma^{\ 2}\,,\\
&\widetilde{\mathbb{R}}_1^{\ 2} = \widehat{\mathbb{R}}_1^{\ 2} - \tfrac{1}{2}\{\mathbb{R}_1^{\ 2}, \mathbb{R}_1^{\ 1}\} + \tfrac{1}{4}[\mathbb{Q}_\gamma^{\ 2}, \mathbb{G}_1^{\ \gamma}]\,, \qquad \widetilde{\mathbb{C}} = \widehat{\mathbb{C}} + \tfrac{1}{2}\,\mathbb{C}\,\mathbb{H}\,,\\
&\widetilde{\mathbb{R}}_2^{\ 1} = \widehat{\mathbb{R}}_2^{\ 1} + \tfrac{1}{2}\{\mathbb{R}_2^{\ 1}, \mathbb{R}_1^{\ 1}\} + \tfrac{1}{4}[\mathbb{G}_2^{\ \gamma}, \mathbb{Q}_\gamma^{\ 1}]\,, \qquad \tilde{\mathbb{C}}^\dagger = \widehat{\mathbb{C}}^\dagger - \tfrac{1}{2}\,\mathbb{C}^\dagger\,\mathbb{H}\,.
\end{aligned}
\tag{3.3.8}
$$

A two-magnon bound state transforms irreducibly in the supersymmetric representation $\square\square$ of $\mathfrak{psu}(2|2)_C$, but in a *reducible* representation $\square\square\square\square \oplus \begin{smallmatrix}\square\square\\ \square\square\end{smallmatrix}$ of $\mathfrak{su}(2|1)$. Therefore one further needs either the boundary Yang-Baxter equation or boundary Yangian symmetry to fix the ratio between the representations of $\mathfrak{su}(2|1)$. A conserved Yangian charge and the two-magnon bound state reflection matrix was found in [21]. The complete boundary Yangian was then constructed in [22].

Consequently, the number of reducible components grows linearly with bound state number M and the Yangian symmetry becomes crucial. The generic M-magnon bound state reflection matrix was constructed in [139].

Our goal is to show the role of the boundary Yangian, thus we will not worry about the overall dressing phase. We will use an orthogonal, but not orthonormal basis. This is to avoid unpleasant numeric factors appearing in the expressions. The reflection matrix in the orthonormal basis can be found in [139].

Reflection matrix. The boundary we are considering forms a trivial (singlet) representation of the boundary algebra, and is conveniently represented by a boundary vacuum state $|0\rangle_B \in V(0)$ which is annihilated by generators of the boundary algebra [16]. Here $V(0)$ is an one-dimensional boundary vector space and will not play any role in the boundary scattering we will be considering further in this section.

Let us define the boundary reflection matrix to be the intertwining matrix

$$K(p)\,|m,n,k,l\rangle \otimes |0\rangle_B = K^{(a,b,c,d)}_{(m,n,k,l)}(p)\,|a,b,c,d\rangle \otimes |0\rangle_B\,, \tag{3.3.11}$$

where $|m,n,k,l\rangle \in V(p)$. The vector space $V(p)$ is $4M$-dimensional and can be decomposed into four $4M = (M+1)+(M-1)+M+M$ subspaces that have an orthogonal basis

$$\begin{aligned}
|k\rangle^1 &= |0,0,k,M{-}k\rangle\,, && k = 0\ldots M\,,\\
|k\rangle^2 &= |1,1,k{-}1,M{-}k{-}1\rangle\,, && k = 1\ldots M-1\,,\\
|k\rangle^3 &= |1,0,k,M{-}k{-}1\rangle\,, && k = 0\ldots M-1\,,\\
|k\rangle^4 &= |0,1,k,M{-}k{-}1\rangle\,, && k = 0\ldots M-1\,.
\end{aligned} \tag{3.3.12}$$

Symmetry constraints. The reflection matrix (3.3.11) is required to satisfy the following intertwining equation

$$\big(K\,\Delta(\mathbb{J}^A) - \Delta^{ref}(\mathbb{J}^A)\,K\big)\,|m,n,k,l\rangle \otimes |0\rangle_B = 0\,, \tag{3.3.13}$$

for all $\mathbb{J}^A$ in the boundary algebra. The invariance under the bosonic generators $\mathbb{R}_1^{\;1}$ and $\mathbb{L}_\alpha^{\;\beta}$ constrain the reflection matrix up to five independent sets of reflection coefficients

$$\begin{aligned}
K\,|k\rangle^1 &= A_k\,|k\rangle^1 + D_k\,|k\rangle^2,\\
K\,|k\rangle^2 &= B_k\,|k\rangle^2 + E_k\,|k\rangle^1,\\
K\,|k\rangle^\alpha &= C_k\,|k\rangle^\alpha,
\end{aligned} \tag{3.3.14}$$

where we have dropped the boudary vacuum state, and the basis (3.3.12) was chosen is such a way that the reflection matrix would act diagonally on the quantum number k.

For computational purposes it is convenient represent the reflection matrix on the superspace as the following differential operator

$$K^{\mathrm{M}}(p) = \sum_i r_i(p)\,\Lambda_i\,, \tag{3.3.15}$$

where $\Lambda_i \in \mathcal{V}_M \otimes \mathcal{D}^M$ span a complete basis of the differential operators invariant under the boundary algebra, and $r_i(p)$ are the reflection coefficients. Set

$$|k\rangle^1 = \omega_1^k\,\omega_2^{M-k}\,, \qquad |k\rangle^2 = \omega_1^{k-1}\,\omega_2^{M-k-1}\,\theta_3\,\theta_4\,, \qquad |k\rangle^\alpha = \omega_1^k\,\omega_2^{M-k-1}\,\theta_\alpha\,. \tag{3.3.16}$$

Then (3.3.11) on the superspace is given by

$$K^{\mathrm{M}} = \sum_{k=0}^{M} A_k\,\Lambda_1^k + \sum_{k=1}^{M-1} B_k\,\Lambda_2^k + \sum_{k=0}^{M-1} C_k\,\Lambda_3^k + \sum_{k=0}^{M-1} D_k\,\Lambda_4^k + \sum_{k=0}^{M-1} E_k\,\Lambda_5^k\,, \tag{3.3.17}$$

where

$$\begin{aligned}
\Lambda_1^k &= \frac{\omega_1^k\,\omega_2^{M-k}}{k!\,(M-k)!}\,\frac{\partial^2}{\partial\omega_1^k\,\partial\omega_2^{M-k}}\,,\\
\Lambda_2^k &= \frac{\omega_1^{k-1}\,\omega_2^{M-k-1}\,\theta_3\theta_4}{(k-1)!\,(M-k-1)!}\,\frac{\partial^{M-2}}{\partial\omega_1^{k-1}\,\partial\omega_2^{M-k-1}}\,\frac{\partial^2}{\partial\theta_4\,\partial\theta_3}\,,\\
\Lambda_3^k &= \frac{\omega_1^k\,\omega_1^{M-k-1}\,\theta_\alpha}{k!\,(M-k-1)!}\,\frac{\partial^{M-1}}{\partial\omega_1^k\,\partial\omega_1^{M-k-1}}\,\frac{\partial}{\partial\theta_\alpha}\,,\\
\Lambda_4^k &= \frac{\omega_1^{k-1}\,\omega_2^{M-k-1}\,\theta_3\theta 4}{k!\,(M-k)!}\,\frac{\partial^2}{\partial\omega_1^k\,\partial\omega_2^{M-k}}\,,\\
\Lambda_5^k &= \frac{\omega_1^k\,\omega_2^{M-k}}{(k-1)!\,(M-k-1)!}\,\frac{\partial^{M-2}}{\partial\omega_1^{k-1}\,\partial\omega_2^{M-k-1}}\,\frac{\partial^2}{\partial\theta_4\,\partial\theta_3}\,.
\end{aligned} \tag{3.3.18}$$

Finally, the intertwining equation (3.3.13) on the superspace becomes

$$\big(\,\mathrm{T}_M(\,\underline{\mathbb{J}}^A)\,K^{\mathrm{M}}(p) - K^{\mathrm{M}}(p)\,\mathrm{T}_M(\,\mathbb{J}^A)\,\big)\,\mathcal{V}_M = 0\,. \tag{3.3.19}$$

Let us start by determining the 'corner' relations - the constraints for the coefficients A_0, D_0, C_0 and A_M, D_M, C_M. Consider the scattering of the 'lowest' state $|0\rangle^1 = \omega_2^M$,

$$K^{\mathrm{M}}\,|0\rangle^1 = A_0\,|0\rangle^1, \qquad \text{thus} \qquad D_0 = 0\,. \tag{3.3.20}$$

The invariance equation for the generator $\mathbb{Q}_4^2$

$$\big(K^{\mathrm{M}}\,\mathrm{T}_M(\mathbb{Q}_4^2) - \mathrm{T}_M(\underline{\mathbb{Q}}_4^2)\,K^{\mathrm{M}}\big)\,|0\rangle^1 = 0 \qquad \text{gives} \qquad C_0 = \frac{\underline{a}}{a}A_0 = \frac{\underline{\gamma}}{\gamma}A_0\,. \tag{3.3.21}$$

We choose the overall normalization to be $A_0 = 1$. Then similar considerations for the 'highest' state $|M\rangle^1 = \omega_1^M$ give

$$D_M = 0 \qquad \text{and} \qquad A_M = \frac{c}{\underline{c}} C_{M-1} = -U^{-2}\frac{\gamma}{\underline{\gamma}} C_{M-1}\,. \tag{3.3.22}$$

The next step is to consider states $|k\rangle^\alpha$. The twisted Yangian generator $\widetilde{\mathbb{R}}_1^{\;2}$ acts on these states as a raising operator

$$\mathrm{T}_M(\widetilde{\mathbb{R}}_1^{\;2})\,|k\rangle^\alpha = f_k(u)|k+1\rangle^\alpha\,, \qquad \mathrm{T}_M(\widetilde{\mathbb{R}}_1^{\;2})\,|k\rangle^\alpha = f_k(-u)|k+1\rangle^\alpha,$$
$$\text{where} \qquad f_k(u) = \tfrac{1}{2}(M-k-1)(2igu + M - 2k - 2)\,. \tag{3.3.23}$$

Notice that these states scatter diagonally. Thus the interwtining equation gives

$$C_{k+1}\, f_k(u) - f_k(-u)\, C_k = 0, \tag{3.3.24}$$

leading to an iterative relation

$$C_k = \frac{f_{k-1}(-u)}{f_{k-1}(u)}\, C_{k-1} = \frac{2igu - M + 2k}{-2igu - M + 2k}\, C_{k-1}\,. \tag{3.3.25}$$

This relation is then simply solved by

$$C_k = C_0 \prod_{n=1}^{k} \frac{2igu - M + 2n}{-2igu - M + 2n}\,. \tag{3.3.26}$$

The coefficients C_k are (anti)symmetric under the interchange $k \to M-k-1$ for M being (even)odd,

$$\begin{aligned} &C_k = -C_{M-k-1} && \text{for} \quad M = \mathit{even} && \text{and} \quad k = 0,\,\ldots,\, M/2-1, \\ &C_k = C_{M-k-1} && \text{for} \quad M = \mathit{odd} && \text{and} \quad k = 0,\,\ldots,\, (M-1)/2-1. \end{aligned} \tag{3.3.27}$$

This symmetry comes by requiring the reflection to be symmetric under the renaming of bosonic indices $1 \leftrightarrow 2$ as the reflection is of a diagonal type for the states $|k\rangle^\alpha$. However this is not the case for the states $|k\rangle^{1,2}$, thus there is no such symmetry for the rest of the reflection coefficients.

The remaining reflection coefficients, as we shall show, will be expressed in terms of C_k and C_{k-1}. Thus by solving the intertwining equation for the generators $\mathbb{Q}_4^{\;2}$ and $\mathbb{G}_2^{\;4}$ and the states $|k\rangle^{1,2}$, we obtain the following set of separable equations

$$\begin{aligned} &D_k\,\underline{b} - (M-k)\,(C_k\, a - A_k\,\underline{a}) = 0\,, && C_k\, b - (M-k)E_k\,\underline{a} - B_k\,\underline{b} = 0\,, \\ &D_k\,\underline{d} + k\,(C_{k-1}\, c - A_k\,\underline{c}) = 0\,, && C_{k-1}\, d + kE_k\,\underline{c} - B_k\,\underline{d} = 0\,, \end{aligned} \tag{3.3.28}$$

having a unique solution

$$\begin{aligned} &A_k = (k\, C_{k-1}\,\underline{b}\, c + (M-k)C_k\, a\,\underline{d})\,/N\,, && D_k = k(M-k)\,(C_k\, a\,\underline{c} - C_{k-1}\,\underline{a}\, c)\,/N\,, \\ &B_k = (k\, C_k\, b\,\underline{c} + (M-k)C_{k-1}\,\underline{a}\, d)\,/N\,, && E_k = (C_k\, b\, - C_{k-1}\,\underline{b}\, d)\,/N\,, \end{aligned} \tag{3.3.29}$$

where the normalization factor is $N = M\,\underline{a}\,\underline{d} - k$. In terms of the $x^{\pm}$ parametrization these read as

$$A_k = \big((M-k)C_k(x^+)^2 - k\,C_{k-1}\big)\,\frac{x^-}{x^+N'}\frac{\gamma}{\underline{\gamma}}\,, \quad D_k = \frac{\gamma\underline{\gamma}}{\alpha}\frac{k(M-k)\,(C_kx^+ + C_{k-1}x^-)}{N'(x^+-x^-)}\,,$$
$$B_k = \big((M-k)C_{k-1}(x^-)^2 - k\,C_k\big)\,\frac{x^+}{x^-N'}\frac{\underline{\gamma}}{\gamma}\,, \quad E_k = \big(C_kx^+ + C_{k-1}x^-\big)\,\frac{x^- - x^+}{N'}\frac{\alpha}{\gamma\underline{\gamma}}\,, \tag{3.3.30}$$

where $N' = k + (M-k)x^-x^+$.

Finally, a straightforward check shows that the unitary property holds,

$$K^M(-p)\,K^M(p) = 1\,. \tag{3.3.31}$$

Fundamental representation. The fundamental reflection matrix $K^A(p)$ is obtained by setting $M = 1$. The states $|k\rangle^2$ are absent in this case and the reflection matrix is of diagonal form. In such a way $K^A(p)$ may be conveniently represented on the superspace as

$$K^A(p) = A_1(p)\,\omega_1\frac{\partial}{\partial\omega_1} + A_0(p)\,\omega_2\frac{\partial}{\partial\omega_2} + C_0(p)\,\theta_\alpha\frac{\partial}{\partial\theta_\alpha}\,. \tag{3.3.32}$$

In this case the boundary Lie algebra is enough to constrain all of the reflection coefficients up to an overall scalar factor,

$$A_0 = \frac{a}{\underline{a}}C_0 = \frac{\gamma}{\underline{\gamma}}C_0\,, \qquad\qquad A_1 = \frac{c}{\underline{c}}C_0 = -\frac{\gamma}{\underline{\gamma}}\frac{x^-}{x^+}C_0\,. \tag{3.3.33}$$

Then, by choosing the normalization to be $A_0 = 1$, this is in agreement with [135].

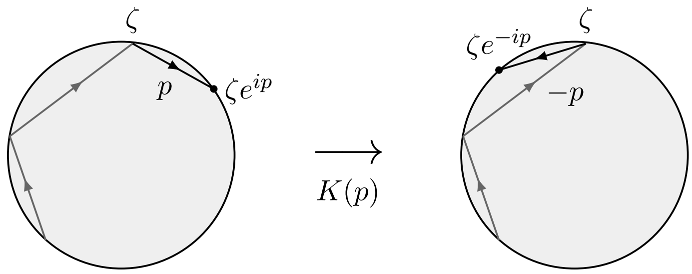

Figure 3.3: Reflection from the right boundary of a magnon living on a semi-infinite string with ζ being the reference point.

Non-local representation. To end up this section we want to give the non-local superspace representation of the reflection from the $Y = 0$ giant graviton. The reflection matrix in this representation is given by the following map,

$$K^{\mathrm{M}} : \mathcal{V}_M(p,\zeta)\otimes 1 \to \mathcal{V}_M(-p,\zeta)\otimes 1\,, \tag{3.3.34}$$

which can be nicely represented on a LLM-type diagram, see figure 3.3.

3.3.1.3 Mirror model

A mirror model of the $Y = 0$ graviton corresponds to a spin chain ending on a boundary which has no degrees of freedom, and preserves only a $\tilde{\mathfrak{h}} = \mathfrak{su}(2|1)$ subalgebra of the bulk symmetry algebra, which is obtained from the $\mathfrak{psu}(2|2)_{\mathcal{C}}$ by dropping generators with fermionic indices $\alpha, \beta, \gamma, \ldots = 3$ (or equivalently with 4). In such a way the boundary algebra structure is

$$\tilde{\mathfrak{h}} = \{\mathbb{R}_a{}^b, \mathbb{L}_3^3, \mathbb{Q}_3^a, \mathbb{G}_a{}^3\}, \qquad \tilde{\mathfrak{m}} = \{\mathbb{L}_3^4, \mathbb{L}_4^3, \mathbb{Q}_4^a, \mathbb{G}_a{}^4, \mathbb{C}, \mathbb{C}^\dagger\}, \tag{3.3.35}$$

where $\tilde{\mathfrak{m}} = \mathfrak{psu}(2|2)_{\mathcal{C}} \backslash \tilde{\mathfrak{h}}$, and $\tilde{\mathfrak{h}}$ and $\tilde{\mathfrak{m}}$ is a symmetric pair. Therefore this boundary has an underlying generalized twisted Yangian algebra of type I. However the Yangian algebra in this case is redundant from the boundary scattering point of view. We will show this in the following paragraphs.

Twisted Yangian. The twisted Yangian for the mirror model is obtained in the same way as for the regular model, and is generated by the level-0 generators $\mathbb{J}^A \in \tilde{\mathfrak{h}}$ and level-1 twisted Yangian generators

$$\begin{aligned}
&\widetilde{\mathbb{Q}}_4^a = \widehat{\mathbb{Q}}_4^a - \tfrac{1}{4}[\mathbb{T}^{\tilde{\mathfrak{h}}}, \mathbb{Q}_4^a], \qquad \widetilde{\mathbb{L}}_3^4 = \widehat{\mathbb{L}}_3^4 - \tfrac{1}{4}[\mathbb{T}^{\tilde{\mathfrak{h}}}, \mathbb{L}_3^4], \qquad \widetilde{\mathbb{C}} = \widehat{\mathbb{C}} - \tfrac{1}{4}[\mathbb{T}^{\tilde{\mathfrak{h}}}, \mathbb{C}], \\
&\widetilde{\mathbb{G}}_a{}^4 = \widehat{\mathbb{G}}_a{}^4 - \tfrac{1}{4}[\mathbb{T}^{\tilde{\mathfrak{h}}}, \mathbb{G}_a^4], \qquad \widetilde{\mathbb{L}}_4^3 = \widehat{\mathbb{L}}_4^3 - \tfrac{1}{4}[\mathbb{T}^{\tilde{\mathfrak{h}}}, \mathbb{L}_4^3], \qquad \widetilde{\mathbb{C}}^\dagger = \widehat{\mathbb{C}}^\dagger - \tfrac{1}{4}[\mathbb{T}^{\tilde{\mathfrak{h}}}, \mathbb{C}^\dagger].
\end{aligned} \tag{3.3.36}$$

where

$$\mathbb{T}^{\tilde{\mathfrak{h}}} = -2\,\mathbb{H}\,\mathbb{Y} + \mathbb{R}_a{}^b\,\mathbb{R}_b{}^a - 2\,\mathbb{L}_3^3\,\mathbb{L}_3^3 + \mathbb{Q}_3^a\,\mathbb{G}_a{}^3 - \mathbb{G}_a{}^3\,\mathbb{Q}_3^a\,. \tag{3.3.37}$$

is the generalized quadratic Casimir operator restricted to $\tilde{\mathfrak{h}}$. The twisted Yangian generators satisfy the usual Lie algebra commutation relations; their coproducts are given by

$$\begin{aligned}
\Delta(\widetilde{\mathbb{Q}}_4^a) &= \widetilde{\mathbb{Q}}_4^a \otimes 1 + \mathbb{U} \otimes \widetilde{\mathbb{Q}}_4^a + \mathbb{Q}_4^c \otimes \mathbb{R}_c{}^a + \mathbb{Q}_4^a \otimes \mathbb{H}' - \mathbb{U}\,\mathbb{L}_4^3 \otimes \mathbb{Q}_3^a - \varepsilon^{ad}\,\mathbb{U}^{-1}\,\mathbb{C} \otimes \mathbb{G}_d{}^3\,, \\
\Delta(\widetilde{\mathbb{G}}_a{}^4) &= \widetilde{\mathbb{G}}_a{}^4 \otimes 1 + \mathbb{U}^{-1} \otimes \widetilde{\mathbb{G}}_a{}^4 - \mathbb{G}_c{}^4 \otimes \mathbb{R}_a{}^c - \mathbb{G}_a{}^4 \otimes \mathbb{H}' + \mathbb{U}^{-1}\,\mathbb{L}_3^4 \otimes \mathbb{G}_a{}^3 + \varepsilon_{ac}\,\mathbb{U}\,\mathbb{C}^\dagger \otimes \mathbb{Q}_3^c, \\
\Delta(\widetilde{\mathbb{L}}_3^4) &= \widetilde{\mathbb{L}}_3^4 \otimes 1 + 1 \otimes \widetilde{\mathbb{L}}_3^4 + 2\,\mathbb{L}_3^4 \otimes \mathbb{L}_3^3 + \mathbb{U}\,\mathbb{G}_c{}^4 \otimes \mathbb{Q}_3^c\,, \\
\Delta(\widetilde{\mathbb{L}}_4^3) &= \widetilde{\mathbb{L}}_4^3 \otimes 1 + 1 \otimes \widetilde{\mathbb{L}}_4^3 + 2\,\mathbb{L}_4^3 \otimes \mathbb{L}_4^4 + \mathbb{U}^{-1}\,\mathbb{Q}_4^c \otimes \mathbb{G}_c{}^3\,, \\
\Delta(\widetilde{\mathbb{C}}) &= \widetilde{\mathbb{C}} \otimes 1 + 1 \otimes \widetilde{\mathbb{C}} + \mathbb{C} \otimes \mathbb{H}\,, \\
\Delta(\widetilde{\mathbb{C}}^\dagger) &= \widetilde{\mathbb{C}}^\dagger \otimes 1 + 1 \otimes \widetilde{\mathbb{C}}^\dagger - \mathbb{C}^\dagger \otimes \mathbb{H}\,,
\end{aligned} \tag{3.3.38}$$

where $\mathbb{H}' = -\mathbb{L}_3^3 + \frac{1}{2}\mathbb{H}$, and we have used $\mathbb{L}_3^3 = -\mathbb{L}_4^4$.

Boundary scattering. The boundary scattering for the mirror model is very similar to the $Y=0$ giant graviton. The boundary is a singlet and thus the reflection matrix can be represented on the superspace as the following differential operator

$$K^{\mathrm{M}}(p)=\sum_i r_i(p)\,\Lambda_i\,, \tag{3.3.39}$$

where $\Lambda_i\in\mathcal{V}_M\otimes\mathcal{D}^M$ span a complete basis of the differential operators invariant under the boundary algebra. However, in this case, the matrix $K^{\mathrm{M}}(p)$ is diagonal for any bound state number M, and the boundary Lie algebra is enough to find all the reflection coefficients up to an overall scalar factor. We will show this by considering reflection matrix for fundamental and two-magnon bound states, and then we will generalize the obtained results for arbitrary bound states.

The fundamental reflection matrix is given by

$$K^{\mathrm{A}}(p)=k_1(p)\,\omega_a\frac{\partial}{\partial\omega_a}+k_2(p)\,\theta_3\frac{\partial}{\partial\theta_3}+k_3(p)\,\theta_4\frac{\partial}{\partial\theta_4}. \tag{3.3.40}$$

Then solving the intertwining equation for generators $\mathbb{Q}_a^{\;3}$ and $\mathbb{G}_3^{\;a}$ leads to

$$k_1(p)=1\,,\qquad k_2(p)=\frac{\underline{a}}{a}=\frac{\underline{\gamma}}{\gamma}\,,\qquad k_3(p)=\frac{\underline{b}}{b}=-U^2\frac{\underline{\gamma}}{\gamma}\,, \tag{3.3.41}$$

where we have chosen the overall normalization to be $k_1(p)=1$.

In the case of the reflection of the two-magnon bound states, the most general structure of the reflection matrix K^B one may write is

$$K^{\mathrm{B}}(p)=\sum_{i=1}^{6}k_i(p)\,\Lambda_i\,, \tag{3.3.42}$$

where Λ_i with $i=1,\dots,4$ are diagonal and Λ_5, Λ_6 are off-diagonal differential operators

$$\begin{aligned}
&\Lambda_1=\frac{1}{2}\omega_b\omega_a\frac{\partial^2}{\partial\omega_b\partial\omega_a}\,, &&\Lambda_2=\omega_a\theta_3\frac{\partial^2}{\partial\omega_a\partial\theta_3}\,, &&\Lambda_3=\omega_a\theta_4\frac{\partial^2}{\partial\omega_a\partial\theta_4}\,,\\
&\Lambda_4=\theta_3\theta_4\frac{\partial^2}{\partial\theta_4\partial\theta_3}\,, &&\Lambda_5=\theta_3\theta_4\frac{\partial^2}{\partial\omega_2\partial\omega_1}\,, &&\Lambda_6=\omega_1\omega_2\frac{\partial^2}{\partial\theta_4\partial\theta_3}\,.
\end{aligned} \tag{3.3.43}$$

However, the off-diagonal reflection channels are forbidden by the boundary symmetry. It is easy to see this by solving the intertwining equation for $\mathbb{R}_2^{\;1}$ and $M=2$, giving

$$\mathrm{T}_M(\mathbb{R}_2^{\;1})\,K^{\mathrm{B}}\,\omega_1\omega_1=2\,k_1\,\omega_1\omega_2\,,\qquad K^{\mathrm{B}}\,\mathrm{T}_M(\mathbb{R}_2^{\;1})\,\omega_1\omega_1=2\,k_1\,\omega_1\omega_2+2\,k_5\,\theta_3\theta_4\,, \tag{3.3.44}$$

leading to $k_5=0$, and similarly

$$\mathrm{T}_M(\mathbb{R}_2^{\;1})\,K^{\mathrm{B}}\,\theta_3\theta_4=k_6\,\omega_2\omega_2\,,\qquad K^{\mathrm{B}}\,\mathrm{T}_M(\mathbb{R}_2^{\;1})\,\theta_3\theta_4=0\,, \tag{3.3.45}$$

leading to $k_6 = 0$. This is a general feature for the reflection of any M-magnon bound states. Hence the generic M-magnon reflection matrix

$$K^{\mathrm{M}}(p) = \sum_{i=1}^{4} k_i(p)\,\Lambda_i\,, \tag{3.3.46}$$

is a diagonal matrix with

$$\begin{aligned}
\Lambda_1 &= \frac{1}{M!}\,\omega_a^M \frac{\partial^M}{\partial^M \omega_a}\,, & \Lambda_2 &= \frac{1}{(M-1)!}\,\omega_a^{M-1}\theta_3 \frac{\partial^M}{\partial\omega_a^{M-1}\partial\theta_3}\,,\\
\Lambda_3 &= \frac{1}{(M-1)!}\,\omega_a^{M-1}\theta_4 \frac{\partial^M}{\partial\omega_a^{M-1}\partial\theta_4}\,, & \Lambda_4 &= \frac{1}{(M-2)!}\,\omega_a^{M-2}\theta_3\theta_4 \frac{\partial^M}{\partial\omega_a^{M-2}\partial\theta_4\partial\theta_3}\,.
\end{aligned} \tag{3.3.47}$$

The boundary symmetry algebra constrains the reflection coefficients up to an overall factor without need of the boundary Yangian symmetry. They are

$$k_1(p) = 1\,, \qquad k_2(p) = \frac{\underline{\gamma}}{\gamma}\,, \qquad k_3(p) = -U^2\frac{\underline{\gamma}}{\gamma}\,, \qquad k_4(p) = -U^2\frac{\underline{\gamma}^2}{\gamma^2}\,, \tag{3.3.48}$$

Finally we want to give a remark noted in [139]. While giving a somewhat trivial boundary scattering theory for the supersymmetric short representations, this model has a non-trivial scattering theory for the anti-supersymmetric short representations. This is the so-called mirror channel and is obtained by a double Wick rotation, i.e. by interchanging bosonic and fermionic indices. In such a way the role of the boundary Yangian becomes crucial for the mirror model and leads to a reflection matrix equivalent to the one of the $Y = 0$ giant graviton considered above. Consequently, the boundary scattering for the $Y = 0$ giant graviton in the mirror channel becomes 'trivial' and is equivalent to the one described in here.

3.3.2 Z=0 giant graviton

We will further consider boundary symmetries and the scattering theory for the $Z = 0$ giant graviton. Once again, for complete details on the setup of this boundary we refer to [135].

3.3.2.1 Boundary symmetries

Boundary algebra. The $Z = 0$ giant graviton preserves the same supersymmetries as the field Z. The boundary Lie algebra consists of two copies, left and right, of the non-braided subalgebra of $\mathfrak{psu}(2|2)_{\mathcal{C}}$. We will denote this algebra as $\mathfrak{psu}(2|2)_{\mathcal{C}_B}$. Furthermore, it is a coideal subalgebra,

$$\Delta(b) \in \mathfrak{psu}(2|2)_{\mathcal{C}} \otimes \mathfrak{psu}(2|2)_{\mathcal{C}_B} \quad \text{for all} \quad b \in \mathfrak{psu}(2|2)_{\mathcal{C}_B}\,. \tag{3.3.49}$$

Let us show this explicitly.

The central generators of the boundary algebra are required to be coreflective,

$$\Delta(\mathbb{H}) = \Delta^{ref}(\mathbb{H})\,, \qquad \Delta(\mathbb{C}) = \Delta^{ref}(\mathbb{C})\,, \qquad \Delta(\mathbb{C}^\dagger) = \Delta^{ref}(\mathbb{C}^\dagger)\,. \tag{3.3.50}$$

This property is clearly satisfied for $\mathbb{H}$ as coproduct is not braided $\mathbb{U}$ (the hypercharge is $[[\mathbb{H}]] = 0$), while for $\mathbb{C}$ and $\mathbb{C}^\dagger$ this property introduces additional constraints on the boundary algebra. Recall that

$$\begin{aligned}
\Delta(\mathbb{C}) &= \mathbb{C}\otimes 1 + \mathbb{U}^{+2}\otimes\mathbb{C}\,, &\qquad \Delta^{ref}(\mathbb{C}) &= \underline{\mathbb{C}}\otimes 1 + \mathbb{U}^{-2}\otimes\mathbb{C}\,,\\
\Delta(\mathbb{C}^\dagger) &= \mathbb{C}^\dagger\otimes 1 + \mathbb{U}^{-2}\otimes\mathbb{C}^\dagger\,, &\qquad \Delta^{ref}(\mathbb{C}^\dagger) &= \underline{\mathbb{C}}^\dagger\otimes 1 + \mathbb{U}^{+2}\otimes\mathbb{C}^\dagger\,.
\end{aligned} \tag{3.3.51}$$

Thus (3.3.50) gives

$$(\mathbb{C}-\underline{\mathbb{C}})\otimes 1 = (\mathbb{U}^{-2}-\mathbb{U}^{+2})\otimes\mathbb{C}, \qquad (\mathbb{C}^\dagger-\underline{\mathbb{C}}^\dagger)\otimes 1 = (\mathbb{U}^{+2}-\mathbb{U}^{-2})\otimes\mathbb{C}^\dagger, \tag{3.3.52}$$

which together with (3.1.7) lead to the following relations

$$\begin{aligned}
\underline{\mathbb{C}}\otimes 1 &= g\,\alpha\left(1-\mathbb{U}^{-2}\right)\otimes 1\,, &\qquad 1\otimes\mathbb{C} &= 1\otimes g\,\alpha\,,\\
\underline{\mathbb{C}}^\dagger\otimes 1 &= g\,\alpha^\dagger\left(1-\mathbb{U}^{+2}\right)\otimes 1\,, &\qquad 1\otimes\mathbb{C}^\dagger &= 1\otimes g\,\alpha^\dagger\,.
\end{aligned} \tag{3.3.53}$$

Note that the tensor space structure above is $\mathfrak{psu}(2|2)_{\mathcal{C}}\otimes\mathfrak{psu}(2|2)_{\mathcal{C}B}$. In such a way the element $\mathbb{U}$ never appears in the second factor, and thus is not in the boundary algebra $\mathfrak{psu}(2|2)_{\mathcal{C}B}$.

Boundary representation. The boundary forms a vector representation of $\mathfrak{psu}(2|2)_{\mathcal{C}B}$. The boundary representation constraints are easily deduced to be

$$M\,a_B\,b_B = g\,\alpha\,, \qquad M\,c_B\,d_B = g\,\alpha^\dagger\,, \qquad a_B\,d_B - b_B\,c_B = 1\,, \tag{3.3.54}$$

where the last relation is the boundary mass-shell constraint. Here we have added the subscript $_B$ to discriminate boundary representation labels from the bulk labels a, b, c, d. A convenient parametrization satisfying these constraints is given by [135]

$$a_B = \sqrt{\frac{g}{M}}\gamma_B\,, \qquad b_B = \sqrt{\frac{g}{M}}\frac{\alpha}{\gamma_B}\,, \qquad c_B = \sqrt{\frac{g}{M}}\frac{i\gamma_B}{\alpha x_B}\,, \qquad d_B = \sqrt{\frac{g}{M}}\frac{x_B}{i\gamma_B}\,. \tag{3.3.55}$$

The boundary mass-shell constraint in this parametrization becomes

$$x_B + \frac{1}{x_B} = \frac{i\,M}{g}\,. \tag{3.3.56}$$

The unitarity requirement imposes an additional constraint, $\gamma_B = e^{i\varphi_B}\sqrt{-ix_B}$. In such a way this representation is just an M-particle bound state representation with different labels. Interestingly, boundary labels can be obtained from the bulk ones in (3.1.20) by a simple bulk-to-boundary map $x^\pm \mapsto \pm x_B$ together with the rescaling of the coupling constant $g \to g/2$. This rescaling is introduced to cancel the factor of $\sqrt{2}$ appearing due to the bulk-to-boundary map of γ, i.e. $\gamma \mapsto \sqrt{2}\,\gamma_B$. In such a way the M-magnon boundary bound state can be interpreted as a bulk $2M$-magnon bound state with a maximal momentum, $p=\pi$, i.e. it is the state at the end of the Brillouin zone.

Twisted Yangian. The twisted Yangian of the $Z = 0$ giant graviton is of type II and is generated by the level-0 generators $\mathbb{J}^A$ of $\mathfrak{psu}(2|2)_\mathcal{C}$ and the twisted level-2 charges $\widetilde{\widetilde{\mathbb{J}}}^{CB}$ [23]

$$\widetilde{\widetilde{\mathbb{J}}}^{CB} = [\,\widehat{\mathbb{J}}^C, \widehat{\mathbb{J}}^B\} + \tfrac{1}{2}(-1)^{|D||B|} f^C_{\ DE}\, \mathbb{J}^D [\,\widehat{\mathbb{J}}^B, \mathbb{J}^E\} + \tfrac{1}{2}(-1)^{|D||C|} f^B_{\ DE}\, \mathbb{J}^D [\,\mathbb{J}^E, \widehat{\mathbb{J}}^C\}\,, \quad (3.3.57)$$

where $[\,,\}$ represents a graded commutator and $(-1)^{|D||B|}$ with $(-1)^{|D||C|}$ are grade factors. The explicit form of these twisted level-2 charges is rather bulky and not very illuminating, thus we do not write them all explicitly. As an example we give an explicit form of a single twisted level-2 supercharge:

$$\widetilde{\widetilde{\mathbb{G}}}{}^{\beta,e}_{a,d} := \left[\,\widehat{\mathbb{G}}{}_a^{\ \beta}, \widehat{\mathbb{R}}{}_d^{\ e}\,\right] + \tfrac{1}{2}\,\widetilde{\mathbb{R}}{}^{\beta,e}_{a,d} - \tfrac{1}{2}\,\widetilde{\mathbb{G}}{}^{\beta,e}_{a,d}\,, \quad (3.3.58)$$

where

$$\begin{aligned}
\widetilde{\mathbb{R}}{}^{\beta,e}_{a,d} &= \left[\mathbb{R}_a^{\ c},\, \widehat{\mathbb{R}}{}_d^{\ e},\, \mathbb{G}_c^{\ \beta}\right] + \left[\mathbb{L}_\gamma^{\ \beta},\, \widehat{\mathbb{R}}{}_d^{\ e},\, \mathbb{G}_a^{\ \gamma}\right] + \epsilon_{ac}\epsilon^{\beta\rho}\left[\,\mathbb{Q}_\rho^{\ c},\, \widehat{\mathbb{R}}{}_d^{\ e}\,\right]\mathbb{C}^\dagger + \tfrac{1}{2}\left[\,\widehat{\mathbb{R}}{}_d^{\ e},\, \mathbb{G}_a^{\ \beta}\,\right]\mathbb{H}\,, \\
\widetilde{\mathbb{G}}{}^{\delta,b}_{d,a} &= \left[\mathbb{R}_a^{\ c},\, \widehat{\mathbb{G}}{}_d^{\ \delta},\, \mathbb{R}_c^{\ b}\right] + \left\{\mathbb{G}_a^{\ \gamma},\, \widehat{\mathbb{G}}{}_d^{\ \delta},\, \mathbb{Q}_\gamma^{\ b}\right\} - \tfrac{1}{2}\,\delta_a^b\left\{\mathbb{Q}_\rho^{\ c},\, \widehat{\mathbb{G}}{}_d^{\ \delta},\, \mathbb{G}_c^{\ \rho}\right\}.
\end{aligned} \quad (3.3.59)$$

Here $[a, b, c] = abc - cba$ and $\{a, b, c\} = abc + cba$. Note that the term with parameter t is not present in (3.3.57). This is because the intertwining equation gives the additional constraint $t = 0$.

Finally, for finding the expressions of the reflected coproducts one has to use (3.2.2) together with

$$\Delta^{ref}(\widehat{\mathbb{J}}^A) = \underline{\widehat{\mathbb{J}}}^A \otimes 1 + \mathbb{U}^{-[[A]]} \otimes \widehat{\mathbb{J}}^A + f^A_{\ BC}\, \mathbb{U}^{-[[C]]}\, \underline{\mathbb{J}}^B \otimes \mathbb{J}^C. \quad (3.3.60)$$

The prescription (3.3.57) has a down side. In general, it gives a linear combination of the level-2 and level-0 charges, and thus it is hard to identify the central elements of the algebra. Knowing these is very important as they are required to be coreflective, and thus commute with the reflection matrix. This allows us to obtain the evaluation map for the boundary states. For example, the level-2 charges defined by

$$\widehat{\widehat{\mathbb{C}}}{}' = \epsilon^{\alpha\beta}\epsilon_{ab}\,\{\widehat{\mathbb{Q}}{}_\alpha^{\ a}, \widehat{\mathbb{Q}}{}_\beta^{\ b}\}\,, \qquad \widehat{\widehat{\mathbb{C}}}{}^{\dagger\prime} = \epsilon^{ab}\epsilon_{\alpha\beta}\,\{\widehat{\mathbb{G}}{}_a^{\ \alpha}, \widehat{\mathbb{G}}{}_b^{\ \beta}\}\,, \quad (3.3.61)$$

in contrast to $\widehat{\mathbb{C}}$, $\widehat{\mathbb{C}}^\dagger$ and $\mathbb{C}$, $\mathbb{C}^\dagger$ are not central, but are shifted from the center by some combination of the level-0 generators. Thus, it is not clear that the twisted charges $\widetilde{\widetilde{\mathbb{C}}}{}'$ and $\widetilde{\widetilde{\mathbb{C}}}{}^{\dagger\prime}$ obtained using prescription (3.3.57) are coreflective. In fact, it is readily checked that they are not. This problem can be resolved by switching to the Drinfeld second realization of $\mathcal{Y}(\mathfrak{psu}(2|2)_\mathcal{C})$ [5]. As was shown in [131], the level-2 charges[4]

$$\widehat{\widehat{\mathbb{C}}} = \{i\widehat{\mathbb{Q}}{}_4^{\ 1} - w_2\,, i\widehat{\mathbb{Q}}{}_3^{\ 2} - w_3\}\,, \qquad \widehat{\widehat{\mathbb{C}}}{}^\dagger = \{i\widehat{\mathbb{G}}{}_1^{\ 4} - z_2\,, i\widehat{\mathbb{G}}{}_2^{\ 3} - z_3\}\,, \quad (3.3.62)$$

[4]We use the same notation as in [113].

are central. Here

$$
\begin{aligned}
w_2 &= -\tfrac{1}{4}\,\{i\mathbb{Q}_4^{\;1},\kappa_{2,0}\} + \tfrac{3i}{4}\,\mathbb{Q}_3^{\;1}\mathbb{L}_4^{\;3} - \tfrac{i}{4}\,\mathbb{R}_2^{\;1}\mathbb{Q}_4^{\;2} - \tfrac{i}{4}\,\mathbb{Q}_4^{\;2}\mathbb{R}_2^{\;1} - \tfrac{i}{4}\,\mathbb{L}_4^{\;3}\mathbb{Q}_3^{\;1} - \tfrac{i}{2}\,\mathbb{G}_2^{\;3}\mathbb{C}\,,\\
w_3 &= -\tfrac{1}{4}\,\{i\mathbb{Q}_3^{\;2},\kappa_{3,0}\} - \tfrac{i}{4}\,\mathbb{Q}_3^{\;1}\mathbb{R}_1^{\;2} + \tfrac{3i}{4}\,\mathbb{R}_1^{\;2}\mathbb{Q}_3^{\;1} - \tfrac{i}{4}\,\mathbb{Q}_4^{\;2}\mathbb{L}_3^{\;4} - \tfrac{i}{4}\,\mathbb{L}_3^{\;4}\mathbb{Q}_4^{\;2} - \tfrac{i}{2}\,\mathbb{G}_1^{\;4}\mathbb{C}\,,\\
z_2 &= -\tfrac{1}{4}\,\{i\mathbb{G}_1^{\;4},\kappa_{2,0}\} - \tfrac{i}{4}\,\mathbb{G}_1^{\;3}\mathbb{L}_3^{\;4} + \tfrac{3i}{4}\,\mathbb{L}_3^{\;4}\mathbb{G}_1^{\;3} - \tfrac{i}{4}\,\mathbb{G}_2^{\;4}\mathbb{R}_1^{\;2} - \tfrac{i}{4}\,\mathbb{R}_1^{\;2}\mathbb{G}_2^{\;4} - \tfrac{i}{2}\,\mathbb{Q}_3^{\;2}\mathbb{C}^\dagger\,,\\
z_3 &= -\tfrac{1}{4}\,\{i\mathbb{G}_2^{\;3},\kappa_{3,0}\} - \tfrac{i}{4}\,\mathbb{G}_2^{\;4}\mathbb{L}_4^{\;3} - \tfrac{i}{4}\,\mathbb{L}_4^{\;3}\mathbb{G}_2^{\;4} + \tfrac{3i}{4}\,\mathbb{G}_1^{\;3}\mathbb{R}_2^{\;1} - \tfrac{i}{4}\,\mathbb{R}_2^{\;1}\mathbb{G}_1^{\;3} - \tfrac{i}{2}\,\mathbb{Q}_4^{\;1}\mathbb{C}^\dagger\,,
\end{aligned}
\tag{3.3.63}
$$

and

$$
\kappa_{2,0} = -\mathbb{R}_1^{\;1} + \mathbb{L}_3^{\;3} - \tfrac{1}{2}\,\mathbb{H}\,, \qquad \kappa_{3,0} = \mathbb{R}_1^{\;1} - \mathbb{L}_3^{\;3} - \tfrac{1}{2}\,\mathbb{H}\,. \tag{3.3.64}
$$

In such a way, applying (3.3.57) to (3.3.62) we find the twisted level-2 central charges

$$
\tilde{\tilde{\mathbb{C}}} = \tilde{\tilde{\mathbb{C}}}{}' + \{w_2, w_3\}\,, \qquad \tilde{\tilde{\mathbb{C}}}{}^\dagger = \tilde{\tilde{\mathbb{C}}}{}^{\dagger\prime} + \{z_2, z_3\}\,. \tag{3.3.65}
$$

Evaluation representation. The evaluation representation of the twisted Yangian generators is obtained by considering the coreflectivity property of the twisted central charges (3.3.65). In such a way we find

$$
\Delta(\,\tilde{\tilde{\mathbb{C}}}\,) = \Delta^{ref}(\,\tilde{\tilde{\mathbb{C}}}\,)\,, \quad \Delta(\,\tilde{\tilde{\mathbb{C}}}{}^\dagger) = \Delta^{ref}(\,\tilde{\tilde{\mathbb{C}}}{}^\dagger) \quad \Longrightarrow \quad \mathrm{ev}_w : \widehat{\mathbb{J}}^A \mapsto igw\,\mathbb{J}^A\,, \tag{3.3.66}
$$

where $w = \frac{iM}{2g}$ is the boundary spectral parameter. The same result may be obtained heuristically by applying the bulk-to-boundary map $x^\pm \mapsto \pm x_B$ to the bulk rapidity (3.1.22) and using the boundary mass-shell constraint (3.3.56),

$$
u = x^+ + \frac{1}{x^+} - \frac{iM}{2g} \;\longmapsto\; x_B + \frac{1}{x_B} - \frac{iM}{2g} = \frac{iM}{2g} = w\,. \tag{3.3.67}
$$

3.3.2.2 Boundary scattering

The boundary scattering for the $Z = 0$ giant graviton was presented in [135]. The left and right sectors are equivalent and thus only one sector needs to be considered. Here we shall give the description of the reflection matrix in terms of the superspace formalism, which was introduced in [133].

The boundary forms a vector representation of the boundary algebra, thus the reflection matrix has essentially the same matrix structure as the worldsheet S-matrix. In such a way the reflection matrix can be conveniently represented on the superspace exactly in the same way as the corresponding S-matrix (see (3.1.45) and (3.1.46)),

$$
K^{\mathrm{MN}}(p,s) = \sum_{i=1} k_i\,(p,s)\,\Lambda_i\,, \tag{3.3.68}
$$

where $\Lambda_i \in \mathcal{V}_M \otimes \mathcal{V}_N \otimes \mathcal{D}^M \otimes \mathcal{D}^N$ span a complete basis of the differential operators invariant under the $\mathfrak{su}(2)\oplus\mathfrak{su}(2)$ algebra, and $k_i(p,s)$ are the reflection coefficients; here p is the momentum of the incoming magnon and s represents the parameters of the boundary magnon. The reflection coefficients can be obtained by solving the boundary intertwining equation

$$\Big((\mathrm{T}_M\otimes\mathrm{T}^B_N)[\Delta^{ref}(\mathbb{J}^A)]\,K^{\mathrm{MN}}(p_1,p_2)-K^{\mathrm{MN}}(p_1,p_2)\,(\mathrm{T}_M\otimes\mathrm{T}^B_N)[\Delta(\mathbb{J}^A)]\Big)\,\mathcal{V}_M\otimes\mathcal{V}_N=0\,. \tag{3.3.69}$$

for all $\mathbb{J}^A$ in the boundary algebra; here T^B_N is the boundary N-magnon bound state representation given in the paragraph above.

The fundamental reflection matrix is obtained by setting $M=N=1$ and is conveniently denoted by K^{Aa},

$$K^{\mathrm{Aa}}(p,s)=\sum_{i=1}^{10}k_i(p,s)\,\Lambda_i\,. \tag{3.3.70}$$

Here Λ_i are the same as in (3.1.48). By solving the boundary intertwining equation for the boundary Lie algebra one finds [135]

$$\begin{aligned}
k_1&=1\,, & k_6&=-\frac{(x^-)^2+x^+x_B}{(x_B-x^-)\,x^-}\frac{\underline{\gamma}}{\gamma}\,,\\
k_2&=1+2\,\frac{(x_B+x^-)\left((x^-)^2-(x^+)^2\right)}{(x_B-x^-)\,x^-x^+}\,, & k_7&=\frac{\alpha\,x_B\,(x_B+x^--x^+)\left((x^-)^2-(x^+)^2\right)}{U\,(x_B-x^-)\,x^-\,\gamma\gamma_B}\,,\\
k_3&=-\frac{(x_B+x^+)}{(x_B-x^-)}\frac{\underline{\gamma}}{\gamma}\,, & k_8&=\frac{(x_B+x^--x^+)\,(x^-+x^+)\,\underline{\gamma}\gamma_B}{\alpha\,U\,(x_B-x^-)\,x^-}\,,\\
k_4&=\left[1-\frac{(2x^+-x^-)(x_B+x^--x^+)(x^-+x^+)}{(x_B-x^-)\,(x^-)^2}\right]\frac{\underline{\gamma}}{U^2\gamma}\,, & k_9&=\frac{(x^+)^2-(x^-)^2}{U\,(x_B-x^-)\,x^-}\frac{\gamma_B}{\gamma}\,,\\
k_5&=\frac{x^-x_B-(x^+)^2}{(x_B-x^-)\,x^+}\,, & k_{10}&=\frac{x_B\,(x^-+x^+)}{U\,(x_B-x^-)\,x^-}\frac{\underline{\gamma}}{\gamma_B}\,,
\end{aligned} \tag{3.3.71}$$

where the overall normalization is set to $k_1=1$. This reflection matrix was first obtained in [135] and the corresponding dressing phase was considered in [136].

The boundary Lie algebra also defines the bound state reflection matrices K^{M1} and K^{1M} for any $M\geq 2$ uniquely up to an overall dressing phase. The most simple cases, with $M=2$, were reported in [133]. The generic bound state reflection matrix can be found by employing the twisted Yangian algebra and using the same approach as it was done for the bound state S-matrix in [109]. However, due to the highly complicated structure of the twisted Yangian generators (see e.g. (3.3.59)) it is extremely challenging to find reasonable and compact expressions of the generic bound state reflection coefficients. Even the most simple case, with $M=N=2$, is already of a very complicated form. Some higher order bound state reflection matrices were calculated numerically and were checked to satisfy the boundary Yang-Baxter equation, thus proving the validity of the proposed Yangian [23]. The most simple and elegant bound state reflection matrices are spelled out in Appendix B.

Non-local representation. To end up the discussion on the giant gravitons we want to give the non-local superspace representation of the reflection from the $Z=0$ giant

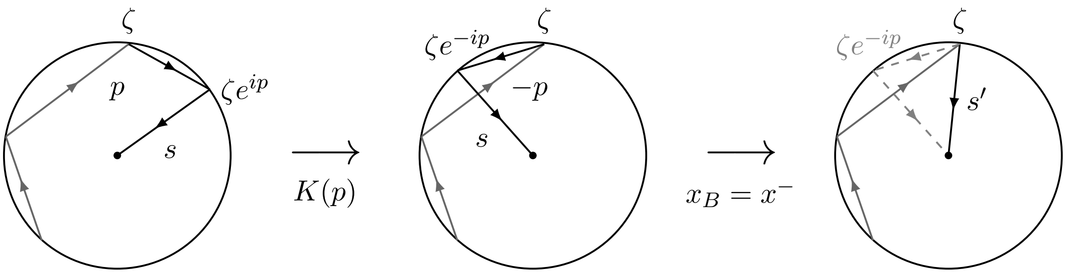

Figure 3.4: *Step 1*. Reflection from the right boundary of a magnon living on a semi-infinite string with ζ being the reference point. Here the dot in the center of the circle corresponds to the $Z = 0$ giant graviton in the Z plane. The boundary degree of freedom corresponds to the straight line connecting the rim of the circle with the dot (brane) in the center. The boundary magnon has zero momentum but is allowed to have a continuous parameter s which represents the energy of the state. *Step 2*. Construction of a two-magnon boundary bound state appearing at the pole $x^- = x_B$ of the K-matrix. The spectral parameter of the emerging boundary bound state is $x'_B = x_B e^{ip} = x_B \frac{x^+}{x^-} = x^+$.

graviton. The non-local boundary labels are

$$a_B = \sqrt{\frac{g}{2}}\eta_B, \quad b_B = -\sqrt{\frac{g}{2}}\frac{i\zeta}{\eta_B}, \quad c_B = -\sqrt{\frac{g}{2}}\frac{\eta_B}{\zeta x_B}, \quad d_B = \sqrt{\frac{g}{2}}\frac{x_B}{i\eta_B}\,. \tag{3.3.72}$$

These can be obtained from (3.3.55) by substituting $\alpha \to -i\zeta$, $g \to g/2$ and $\gamma_B \to \eta_B$ where $\eta_B = \mathrm{e}^{i\xi}\sqrt{-ix_B}$. The reflection matrix in this representation is given by the following map,

$$K^{\mathrm{MN}} : \mathcal{V}_M(p,\zeta) \otimes \mathcal{V}_N(s,\zeta \mathrm{e}^{ip}) \to \mathcal{V}_M(-p,\zeta) \otimes \mathcal{V}_N(s,\zeta \mathrm{e}^{-ip})\,. \tag{3.3.73}$$

The LLM-type diagram for this boundary scattering is given step 1 of figure 3.4. The pole $1/(x_B - x^-)$ in (3.3.71) signals the appearance of the boundary bound states. In such a way an incoming magnon with an appropriate momentum gets adsorbed to the boundary, as it is shown in step 2 of figure 3.4).

Finally, the reflection coefficients of K^{Aa} in this basis can be obtained from the ones in (3.3.71) by the following prescription:

$$\begin{aligned}
&k_{1,2} \to k_{1,2}\,, && k_5 \to \sqrt{U}\frac{\widetilde{\eta}_B}{\eta_B}\,k_5\,, && k_7 \to \frac{\zeta U}{i\alpha}\frac{\gamma\,\gamma_B}{\eta\,\eta_B}\,k_7\,, && k_9 \to U\frac{\gamma\,\widetilde{\eta}_B}{\gamma_B\,\eta}\,k_9\,,\\
&k_{3,4} \to U^2\frac{\gamma\,\widetilde{\eta}\,\widetilde{\eta}_B}{\underline{\gamma}\,\eta\,\eta_B}k_{3,4}\,, && k_6 \to \frac{\gamma\,\widetilde{\eta}}{\underline{\gamma}\,\eta}\,k_6\,, && k_8 \to \frac{i\alpha U}{\zeta}\frac{\widetilde{\eta}\,\widetilde{\eta}_B}{\gamma\,\gamma_B}\,k_8\,, && k_{10} \to U\frac{\gamma_B\,\widetilde{\eta}}{\underline{\gamma}\,\eta_B}\,k_{10}\,,
\end{aligned} \tag{3.3.74}$$

where

$$\eta = \eta(p,\zeta)\,, \qquad \widetilde{\eta} = \eta(-p,\zeta)\,, \qquad \eta_B = \eta_B(\zeta\mathrm{e}^{ip})\,, \qquad \widetilde{\eta}_B = \eta_B(\zeta\mathrm{e}^{-ip})\,. \tag{3.3.75}$$

3.4 D3-D7 brane system

The second system we will consider is the so-called '$D3$-$D7$-brane' system, where the stack of the $D3$-branes leads to the $AdS_5 \times S^5$ background, and the $D7$-brane is wrapping the entire AdS_5 and a maximal $S^3 \subset S^5$. This setup is conformal only in the strict large-N limit where the backreaction of the $D7$-brane can be ignored.

The presence of the $D7$-brane breaks exactly half of the background supersymmetries. In such a way the dual description is a $\mathcal{N} = 2$ super-Yang Mills gauge theory with a single chiral hypermultiplet of fundamental matter [142]. The addition of fundamental matter provides a new way to form local gauge-invariant operators. In addition to the usual closed chains of $\mathcal{N} = 4$ fields, e.g.

$$\mathrm{Tr}\{Z \dots Z\chi_1 Z \dots Z\chi_2 Z \dots Z\} \tag{3.4.1}$$

constructed by taking the trace over the $SU(N)$ colour indices, there are also operators of the form

$$\bar{q} Z \dots Z\chi_1 Z \dots Z\chi_2 Z \dots Z q \tag{3.4.2}$$

where q, $\bar{q}$ are fields in, respectively, the fundamental and anti-fundamental of $SU(N)$. Such operators in the planar limit $N \to \infty$ can be thought of as open spin chains, with the dilatation operator D playing the role of the Hamiltonian.

The boundary scattering depends crucially on the relative orientation of S^3 and the bulk Bethe vacuum field Z, and was presented in [143]. There are two important cases of such configurations that we will consider in this section. They are the so-called $Y = 0$ and the $Z = 0$ $D7$-branes.

3.4.1 Z=0 D7-brane

The $Z = 0$ $D7$-brane is obtained by setting $X_5 = X_6 = 0$ in the parametrization of S^5. This choice breaks the $SO(6)$ symmetry down to $SO(4)_{1234} \times SO(2)_{56}$. There are 16 supercharges invariant under the combination $D - J_{56}$. However only half of them are symmetries of the $Z = 0$ $D7$-brane. Thus this choice of embedding breaks half of the residual supercharges – the left copy of $\mathfrak{psu}(2|2)_{\mathcal{C}}$. This leaves the boundary algebra to be

$$\mathfrak{su}(2) \times \mathfrak{su}(2) \times \widetilde{\mathfrak{psu}}(2|2) \ltimes \mathbb{R}^3 \,. \tag{3.4.3}$$

The field content of the $\mathcal{N} = 2$ fundamental hypermultiplet consists of a doublet of complex scalars ϕ and two Weyl fermions ψ_+, ψ_-. They are charged under the residual symmetries as follows:

	J_{12}	J_{34}	J_{56}	$\mathfrak{so}(1,3)$	D
ϕ	0	$1/2$	0	$[\,0\,,\,0\,]$	1
ψ_+	0	0	$+1/2$	$[0,1/2]$	$3/2$
ψ_-	0	0	$-1/2$	$[1/2,0]$	$3/2$

(3.4.4)

Here $\mathfrak{so}(1,3) \subset \mathfrak{so}(2,4)$ of the AdS_5. The fundamental matter fields listed above form the basis of the rightmost site of the underlying spin chain. Furthermore, they are required to fall into the representations of the residual symmetry algebra. The leading role in the boundary scattering theory is played by the states with the lowest value of $\Delta = D - J_{56}$, namely the doublet ϕ and ψ_+, for which $\Delta = 1$. These states form the fundamental representation $\boxslash$ of $\widetilde{\mathfrak{psu}}(2|2)$, and correspond to the boundary degrees of freedom in the scattering theory of the unexcited boundary. Similarly, the leftmost site of the spin chain has an equivalent configurations furnished by the conjugate fields.

In such a way the fundamental matter fields transform in a $(1, \boxslash)$ representation of the bulk symmetry algebra $\mathfrak{psu}(2|2) \times \widetilde{\mathfrak{psu}}(2|2)$. This setup leads to the factorization $K \otimes \widetilde{K}$ of the complete reflection matrix, and thus two independent reflection processes need to be considered, the reflection in the left and the reflection in the right factor of the brane.

The reflection in the right factor

$$\widetilde{K} : V(p) \otimes V(s) \to V(-p) \otimes V(s) \,, \tag{3.4.5}$$

is equivalent to the reflection from the $Z = 0$ giant graviton discussed in Section 3.3.2. The reflection in the left factor

$$K : V(p) \otimes 1 \to V(-p) \otimes 1 \,, \tag{3.4.6}$$

is a reflection from a non-supersymmetric singlet boundary. The fundamental reflection matrix was found in [143], the bound state one was found in [133]; the boundary Yangian algebra was revealed in [26]. We will next show the latter how the Yangian symmetry was constructed.

3.4.1.1 Boundary symmetries

The boundary Lie algebra for the left factor of the $Z = 0$ $D7$-brane can be formally decomposed as $\mathfrak{h} = \mathfrak{g}/(\mathfrak{m} + \mathfrak{c})$, where

$$\mathfrak{h} = \{\mathbb{R}_b^{\ a},\, \mathbb{L}_\beta^{\ \alpha},\, \mathbb{H}\} \,, \qquad \mathfrak{m} = \{\mathbb{Q}_b^{\ \alpha},\, \mathbb{G}_\beta^{\ a}\} \,, \qquad \mathfrak{c} = \{\mathbb{C},\ \mathbb{C}^\dagger\} \,, \tag{3.4.7}$$

and $\mathfrak{g} = \mathfrak{psu}(2|2)_{\mathcal{C}}$. This setup almost resembles the structure of a symmetric pair. In the latter case the boundary scattering would be governed by a twisted Yangian $\mathrm{Y}(\mathfrak{g},\mathfrak{h})$ of type I in a similar way as for the $Y = 0$ giant graviton. Unfortunately, in the present case the symmetric pair structure breaks down due to the following relations

$$\{\mathbb{Q}_\alpha^{\ a}, \mathbb{Q}_\beta^{\ b}\} = \epsilon^{ab}\epsilon_{\alpha\beta}\, \mathbb{C} \,, \qquad\qquad \{\mathbb{G}_a^{\ \alpha}, \mathbb{G}_b^{\ \beta}\} = \epsilon^{\alpha\beta}\epsilon_{ab}\, \mathbb{C}^\dagger \,. \tag{3.4.8}$$

In other words, the presence of the central charges prevents us from applying the generic formalism discussed earlier. However, the algebra $\mathfrak{psu}(2|2)_{\mathcal{C}}$ has an $SL(2)$ outer automorphism, which is realized as a mixing of the supercharges. This automorphism can be used to rotate the central charges to a trivial point, $\mathbb{C} \equiv \mathbb{C}^\dagger \equiv 0$, in such a way the commutation relations (3.4.8) in the rotated realization of the algebra are absent. We will use an analogue of this automorphism on the level of the twisted charges to construct the twisted Yangian.

Modified twisted Yangian $\mathcal{Y}(\mathfrak{g},\mathfrak{h})$. Let us first ignore the fact that the central charges $\mathbb{C}$ and $\mathbb{C}^\dagger$ are not symmetries of the boundary, and suppose they are in the boundary algebra $\mathfrak{h}$. Then following the general twisted Yangian prescription and using the structural constants obtained from the Yangian $\mathrm{Y}(\mathfrak{psu}(2|2)_\mathcal{C})$ as was explained in the previous section, we obtain

$$\begin{aligned}\widetilde{\mathbb{Q}}'^{\,a}_{\alpha} &= \widehat{\mathbb{Q}}^{\,a}_{\alpha} + t_{Q}\mathbb{Q}^{\,a}_{\alpha} + \tfrac{1}{4}\big(\mathbb{Q}^{\,c}_{\alpha}\,\mathbb{R}_{c}^{\;a} + \mathbb{R}_{c}^{\;a}\,\mathbb{Q}^{\,c}_{\alpha} + \mathbb{Q}^{\,a}_{\gamma}\,\mathbb{L}_{\alpha}^{\;\gamma} + \mathbb{L}_{\alpha}^{\;\gamma}\,\mathbb{Q}^{\,a}_{\gamma} + \mathbb{H}\,\mathbb{Q}^{\,a}_{\alpha} - 2\,\varepsilon_{\alpha\gamma}\varepsilon^{ac}\,\mathbb{C}\,\mathbb{G}_{c}^{\;\gamma}\big),\\ \widetilde{\mathbb{G}}'^{\;\alpha}_{a} &= \widehat{\mathbb{G}}_{a}^{\;\alpha} - t_{G}\,\mathbb{G}_{a}^{\;\alpha} - \tfrac{1}{4}\big(\mathbb{G}_{c}^{\;\alpha}\,\mathbb{R}_{a}^{\;c} + \mathbb{R}_{a}^{\;c}\,\mathbb{G}_{c}^{\;\alpha} + \mathbb{G}_{a}^{\;\gamma}\,\mathbb{L}_{\gamma}^{\;\alpha} + \mathbb{L}_{\gamma}^{\;\alpha}\,\mathbb{G}_{a}^{\;\gamma} + \mathbb{H}\,\mathbb{G}_{a}^{\;\alpha} - 2\,\varepsilon_{ac}\varepsilon^{\alpha\gamma}\,\mathbb{C}^\dagger\mathbb{Q}^{\,c}_{\gamma}\big).\end{aligned} \tag{3.4.9}$$

The coproducts of these charges are

$$\begin{aligned}\Delta(\widetilde{\mathbb{Q}}'^{\,a}_{\alpha}) &= \widetilde{\mathbb{Q}}'^{\,a}_{\alpha}\otimes 1 + \mathcal{U}^{+1}\otimes\widetilde{\mathbb{Q}}'^{\,a}_{\alpha} + \mathbb{Q}^{\,c}_{\alpha}\otimes\mathbb{R}_{c}^{\;a} + \mathbb{Q}^{\,a}_{\gamma}\otimes\mathbb{L}_{\alpha}^{\;\gamma} + \tfrac{1}{2}\mathbb{Q}^{\,a}_{\alpha}\otimes\mathbb{H} - \varepsilon_{\alpha\gamma}\varepsilon^{ac}\mathbb{G}_{c}^{\;\gamma}\,\mathcal{U}^{+2}\otimes\mathbb{C},\\ \Delta(\widetilde{\mathbb{G}}'^{\;\alpha}_{a}) &= \widetilde{\mathbb{G}}'^{\;\alpha}_{a}\otimes 1 + \mathcal{U}^{-1}\otimes\widetilde{\mathbb{G}}'^{\;\alpha}_{a} - \mathbb{G}_{c}^{\;\alpha}\otimes\mathbb{R}_{a}^{\;c} - \mathbb{G}_{a}^{\;\gamma}\otimes\mathbb{L}_{\gamma}^{\;\alpha} - \tfrac{1}{2}\mathbb{G}_{a}^{\;\alpha}\otimes\mathbb{H} + \varepsilon_{ac}\varepsilon^{\alpha\gamma}\mathbb{Q}^{\,c}_{\gamma}\,\mathcal{U}^{-2}\otimes\mathbb{C}^\dagger.\end{aligned} \tag{3.4.10}$$

As expected, we see that these charges violate the coideal property due to central charges acting on the boundary. We can overcome this problem by adding a twist resembling the $SL(2)$ automorphism,

$$\begin{aligned}\widetilde{\mathbb{Q}}^{\,a}_{\alpha} &= \widetilde{\mathbb{Q}}'^{\,a}_{\alpha} + \varepsilon_{\alpha\gamma}\varepsilon^{ac}\,(\mathbb{C} - g\alpha)\,\mathbb{G}_{c}^{\;\gamma}\,,\\ \widetilde{\mathbb{G}}_{a}^{\;\alpha} &= \widetilde{\mathbb{G}}'^{\;\alpha}_{a} - \varepsilon_{ac}\varepsilon^{\alpha\gamma}\,(\mathbb{C}^\dagger - g\alpha^{-1})\mathbb{Q}^{\,c}_{\gamma}\,.\end{aligned} \tag{3.4.11}$$

The coproducts of the new charges are then readily found to be

$$\begin{aligned}\Delta(\widetilde{\mathbb{Q}}^{\,a}_{\alpha}) &= \widetilde{\mathbb{Q}}^{\,\alpha}_{b}\otimes 1 + \mathcal{U}^{+1}\otimes\widetilde{\mathbb{Q}}^{\,\alpha}_{b} + \mathbb{Q}^{\,c}_{\alpha}\otimes\mathbb{R}_{c}^{\;a} + \mathbb{Q}^{\,a}_{\gamma}\otimes\mathbb{L}_{\alpha}^{\;\gamma} + \tfrac{1}{2}\,\mathbb{Q}^{\,a}_{\alpha}\otimes\mathbb{H}\,,\\ \Delta(\widetilde{\mathbb{G}}_{a}^{\;\alpha}) &= \widetilde{\mathbb{G}}_{a}^{\;\alpha}\otimes 1 + \mathcal{U}^{-1}\otimes\widetilde{\mathbb{G}}_{a}^{\;\alpha} - \mathbb{G}_{c}^{\;\alpha}\otimes\mathbb{R}_{a}^{\;c} - \mathbb{G}_{a}^{\;\gamma}\otimes\mathbb{L}_{\gamma}^{\;\alpha} - \tfrac{1}{2}\,\mathbb{G}_{a}^{\;\alpha}\otimes\mathbb{H}\,,\end{aligned} \tag{3.4.12}$$

and thus the coideal property is satisfied.

The parameters t_Q and t_G in the twist (3.4.9) are constrained by requiring the twisted central charges

$$\widetilde{\widetilde{\mathbb{C}}} = \epsilon_{ab}\epsilon^{\alpha\beta}\{\widetilde{\mathbb{Q}}^{\,a}_{\alpha},\,\widetilde{\mathbb{Q}}^{\,b}_{\beta}\}\,, \qquad\qquad \widetilde{\widetilde{\mathbb{C}}}^\dagger = \epsilon_{\alpha\beta}\epsilon^{ab}\{\widetilde{\mathbb{G}}_{a}^{\;\alpha},\,\widetilde{\mathbb{G}}_{b}^{\;\beta}\}\,, \tag{3.4.13}$$

to be coreflective. This gives a constraint $t_Q = t_G = \sqrt{g^2+1/4}$. The square root may be eliminated by using the fundamental mass-shell condition $x_B + 1/x_B = i/g$ (3.3.56). In such a way we obtain a very elegant expression, $t_Q = t_G = ig/x_B + 1/2$.

3.4.1.2 Boundary scattering

The boundary we are considering is a singlet with respect to the boundary algebra, thus it may be represented by the boundary vacuum state $|0\rangle_B$ which is annihilated by all generators of the boundary algebra [16]. We define the reflection matrix to be an intertwining matrix

$$K\,|m,n,k,l\rangle\otimes|0\rangle_B = K^{(a,b,c,d)}_{(m,n,k,l)}\,|a,b,c,d\rangle\otimes|0\rangle_B\,. \tag{3.4.14}$$

The space of states $|m,n,k,l\rangle$ is $4M$-dimensional and can be decomposed into four $4M = (M+1)+(M-1)+M+M$ subspaces that have the orthogonal basis

$$\begin{aligned}
|k\rangle^1 &= |0,0,k,M-k\rangle\,, && k=0\dots M\,,\\
|k\rangle^2 &= |1,1,k-1,M-k-1\rangle\,, && k=1\dots M-1\,,\\
|k\rangle^3 &= |1,0,k,M-k-1\rangle\,, && k=0\dots M-1\,,\\
|k\rangle^4 &= |0,1,k,M-k-1\rangle\,, && k=0\dots M-1\,.
\end{aligned} \tag{3.4.15}$$

The setup so far is of the same type as for the $Y=0$ giant graviton considered in section 3.3.1.2, but the reflection matrix will be of a very simple form in this case.

The boundary Lie algebra in the left factor is generated by the bosonic generators $\mathbb{R}_a^{\ b}$ and $\mathbb{L}_\alpha^{\ \beta}$, and the central charge $\mathbb{H}$ only. It constrains the reflection matrix K to be diagonal for any k and M,

$$K\,|k\rangle^1 = A\,|k\rangle^1\,, \qquad K\,|k\rangle^2 = B\,|k\rangle^2\,, \qquad K\,|k\rangle^\alpha = C\,|k\rangle^\alpha\,, \tag{3.4.16}$$

where $\alpha = 3,\,4$ and we have dropped the boundary vacuum state. The standard normalization is $A=1$. This leaves the coefficients B and C undetermined. However, due to the simple form of the reflection matrix, these can readily be found by solving the boundary Yang-Baxter equation. It factorizes in this case, and thus can be solved by the method of separating variables. Consequently one finds

$$B = \frac{x_B+x^+}{x_B-x^-}\,\frac{\underline{\gamma}}{\gamma}\,, \qquad C = \frac{(x_B+x^+)(1-x_Bx^+)}{(x_B-x^-)(1+x_Bx^-)}\,\frac{\underline{\gamma}^2}{\gamma^2}\,, \tag{3.4.17}$$

where the parameter x_B satisfies the fundamental mass-shell constraint $x_B+1/x_B = i/g$. This constraint is obtained by considering the 'supersymmetric' matrix elements of the boundary Yang-Baxter equation, e.g. ${}^3\langle k_i|\otimes{}^4\langle k_j|\,\mathrm{BYBE}\,|k_m\rangle^1\otimes|k_n\rangle^1$ leading to

$$\big(A_2\,\underline{\gamma}_2 - B_2\,\gamma_2\big)\big(B_1\,\gamma_1\,x_1^- + A_1\,\underline{\gamma}_1\,x_1^+\big) - \big(A_1\,\underline{\gamma}_1 - B_1\,\gamma_1\big)\big(B_2\,\gamma_2\,x_2^- + A_2\,\underline{\gamma}_2\,x_2^+\big) = 0\,, \tag{3.4.18}$$

where $A_i = A(p_i)$ and $B_i = B(p_i)$. This equation can be solved by separating variables and setting

$$\frac{B_i\,\gamma_i\,x_i^- + A_i\,\underline{\gamma}_i\,x_i^+}{A_i\,\underline{\gamma}_i - B_i\,\gamma_i} = x_B\,. \tag{3.4.19}$$

The inspection of the rest of the matrix elements of BYBE constrains x_B to satisfy the fundamental mass-shell constraint. The explicit expression of C_i then follows after lengthy but quite straightforward calculations.

In the next paragraph we will show how to obtain the same results in a much easier way, by employing the twisted Yangian generators (3.4.11).

Symmetry constraints. The complete reflection matrix K (3.4.16) follows from simple Yangian symmetry arguments. Indeed, solving

$$\big(K\,\widetilde{\mathbb{Q}}_3^1 - \underline{\widetilde{\mathbb{Q}}}_3^1\,K\big)\,|k\rangle^1 = 0 \qquad \text{and} \qquad \big(K\,\widetilde{\mathbb{Q}}_3^1 - \underline{\widetilde{\mathbb{Q}}}_3^1\,K\big)\,|k\rangle^2 = 0 \tag{3.4.20}$$

leads to the reflection coefficients that coincide with (3.4.17).

3.4.2 Y=0 D7-brane

The $Y = 0$ $D7$-brane is obtained by setting $X_3 = X_4 = 0$ in the parametrization of S^5. This orientation breaks the $SO(6)$ symmetry down to $SO(4)_{1256} \times SO(2)_{34}$. Then the charge J_{56} further brakes the $SO(4)_{1256}$, and the residual boundary symmetry is

$$\mathfrak{su}(2|1) \times \widetilde{\mathfrak{su}}(2|1) . \tag{3.4.21}$$

The field content at the gauge theory side of this setup is equivalent to the $Z = 0$ case given in (3.4.4), but with J_{34} and J_{56} interchanged. In such a way there is a unique complex scalar field at the rightmost site for which $\Delta = 1/2$. The same considerations apply to the leftmost site, and thus there is a unique unexcited spin chain with an antiquark at the left end and a quark at the right end which from the scattering point of view is identical to the $Y = 0$ giant graviton, and thus the reflection matrix is the same.

3.5 D3-D5 brane system

The third system we will consider is the so-called '$D3$-$D5$-brane' system, where the stack of the $D3$-branes leads to the $AdS_5 \times S^5$ geometry, and the $D5$-brane spans an $AdS_4 \times S^2$. In the same way as in Section 3.4, we will consider the strict large-N limit where the backreaction of the $D5$-brane can be ignored.

Our goal is to build the boundary scattering theory for this system. This system, as we will show, has an underlying Yangian algebra of the so-called 'achiral' type that was not considered in Chapter 2. For this reason we will start by considering achiral boundary conditions in the bosonic Principal Chiral Model. We will then build the boundary scattering theory for the $D5$-brane using the non-local representation of $\mathfrak{psu}(2|2)_{\mathcal{C}}$.

3.5.1 Achiral boundary conditions in the bosonic Principal Chiral Model

Consider a $1+1$-dimensional bosonic principal chiral field $g(t,x) \in G$ on a half-line $x \leq 0$ for a compact, simple Lie group G, with Lagrangian density

$$\mathcal{L} = \frac{1}{2} \operatorname{Tr} \left(\partial_+ g^{-1} \partial_- g \right) . \tag{3.5.1}$$

The model has Lie algebra $\mathfrak{g}$-valued conserved currents

$$j^L_\mu = \partial_\mu g g^{-1} \qquad \text{and} \qquad j^R_\mu = -g^{-1} \partial_\mu g \,, \tag{3.5.2}$$

which generate $\mathfrak{g}_L$ and $\mathfrak{g}_R$ and thereby the model's $G \times G$ symmetry. On these currents at $x = 0$, the boundary conditions are either *chiral*,

$$j^L_\pm = \alpha(j^L_\mp) \,, \quad j^R_\pm = \alpha(j^R_\mp) \quad \Rightarrow \quad j_0 = \alpha(j_0) \,, \quad j_1 = -\alpha(j_1) \quad (\text{both } L \text{ and } R) \,, \tag{3.5.3}$$

or *achiral*,

$$j^L_\pm = \alpha(j^R_\mp) \quad \Rightarrow \quad j^L_0 = \alpha(j^R_0) \,, \quad j^L_1 = -\alpha(j^R_1) \,, \tag{3.5.4}$$

where α is an involutive automorphism of $\mathfrak{g}$.

For the chiral conditions the residual Lie symmetry is $H \times H \subset G \times G$, where H is the subgroup fixed by α. In fact (3.5.3) may be generalized by independent conjugation of the currents, and the general boundary condition on the fields g is that

$$g(t,0) \in k_L H k_R^{-1}\,, \tag{3.5.5}$$

so that at $x = 0$

$$k_L^{-1} j_+^L k_L = \alpha(k_L^{-1} j_-^L k_L)\,, \qquad k_R^{-1} j_+^R k_R = \alpha(k_R^{-1} j_-^R k_R)\,. \tag{3.5.6}$$

The constant group elements k_L and k_R parametrize left- and right-cosets of H in G and may be taken to lie in the Cartan immersion, $G/H = \{\alpha(k)k^{-1} \,|\, k \in G\}$, of G/H in G.[5] Then the Yangian symmetry is two (L and R) copies of a generalization of the twisted Yangian of type I, $\mathcal{Y}(\mathfrak{g},\mathfrak{h}) \subset \mathcal{Y}(\mathfrak{g})$ [40, 41]. For different choices of $\mathfrak{h}$ this encompasses twisted Yangians [38] and reflection algebras [39], and the 'soliton-preserving' and '-non-preserving' boundary conditions of [148].

The second, achiral class of boundary condition was not fully investigated in [27]. Its Lie symmetry was under-identified there as the diagonal $H \subset G \times G$, but the full symmetry is a diagonal $G \subset G \times G$, due to the conservation (where $\mathbb{J} := \int_{-\infty}^{0} j_0$)

$$\frac{d}{dt}\left(\mathbb{J}_L + \alpha(\mathbb{J}_R)\right) = \int_{-\infty}^{0} \partial_0 j_0^L + \partial_0 \alpha(j_0^R) = j_1^L(0) + \alpha(j_1^R(0)) = 0. \tag{3.5.7}$$

Again conjugation is allowed, and

$$g_L^{-1} j_+^L g_L = \alpha(g_R^{-1} j_-^R g_R) \tag{3.5.8}$$

at $x = 0$ follows from

$$g(t,0) \in g_L\{\alpha(k)k^{-1} \,|\, k \in G\} g_R^{-1} = g_L\, G/H\, g_R^{-1}. \tag{3.5.9}$$

Henceforth we set $g_L = g_R = 1$ (the identity in G) for simplicity.

What is the remnant of the Yangian symmetry $\mathcal{Y}(\mathfrak{g} \times \mathfrak{g})$? One might at first think that it is simply $\Delta\mathcal{Y}(\mathfrak{g})$, but it is not. Rather it is again associated with a symmetric space structure, this time $G \times G/G$, and is the co-ideal subalgebra $\mathcal{Y}(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g})$. This is expected from [149], which analysed boundary conditions for symmetric-space sigma models. We write the symmetric pair structure as $\mathfrak{g}_L \oplus \mathfrak{g}_R = \mathfrak{g}_+ \oplus \mathfrak{g}_-$, where $\mathfrak{g}_+$ is the α-twisted diagonal subalgebra and $\mathfrak{g}_-$ its complement (which is *not* a Lie algebra). Note especially that in this case different choices of α merely give different α-twisted embeddings of $\mathfrak{g}$ in $\mathfrak{g} \times \mathfrak{g}$, rather than the different proper subalgebras $\mathfrak{a}$ of $\mathfrak{g}$ we saw in the chiral case. Thus by a change of basis for $\mathfrak{g}_R$ we can set $\alpha = \mathrm{id}$ (the identity map), and indeed we shall need such a change in the next section. For the moment we retain α, but the reader may like to bear in mind that $\alpha = \mathrm{id}$ captures the essence of the construction.

[5] Here = means 'is locally diffeomorphic to'; there may be global ambiguities [147].

The subalgebra $\mathfrak{g}_+$ and its complement $\mathfrak{g}_-$ are spanned by $\mathbb{J}^a_\pm = \mathbb{J}^a_L \pm \alpha(\mathbb{J}^a_R)$, which have eigenvalues ± 1 under the involution $\sigma(\alpha \times \alpha)$. The boundary Yangian symmetry $Y(\mathfrak{g} \times \mathfrak{g}, \mathfrak{g}) = Y(\mathfrak{g}_L \times \mathfrak{g}_R, \mathfrak{g}_+)$ then has Lie subalgebra $\mathfrak{g}_+$, generated by the $\mathbb{J}^a_+$. At level 1 its generators $\tilde{\mathbb{J}}^a_-$ are constructed from the level-1 $Y(\mathfrak{g}_L \times \mathfrak{g}_R)$ generators $\hat{\mathbb{J}}^a_L$, $\hat{\mathbb{J}}^a_R$ as

$$\begin{aligned}\tilde{\mathbb{J}}^a_- &:= \hat{\mathbb{J}}^a_- + \tfrac{1}{8} f^a{}_{cb} \left(\mathbb{J}^c_- \mathbb{J}^b_+ + \mathbb{J}^b_+ \mathbb{J}^c_- \right) \\ &= \hat{\mathbb{J}}^a_- + \tfrac{1}{2} f^a{}_{bc} \mathbb{J}^b_L \, \alpha(\mathbb{J}^c_R),\end{aligned} \tag{3.5.10}$$

where again $\hat{\mathbb{J}}^a_\pm = \hat{\mathbb{J}}^a_L \pm \alpha(\hat{\mathbb{J}}^a_R)$. Notice the factor of two in (3.5.10) relative to (2.1.68), due to the normalization of $\mathbb{J}^a_\pm$. It is easy to see that this is a specific case of the twisted Yangian of type I, which we call *the 'achiral twisted Yangian'* to emphasize its achiral properties.

It is an easy calculation to check that these charges are classically conserved by the achiral boundary condition (3.5.4). Further, the co-product of the level-1 charges is

$$\begin{aligned}\Delta(\tilde{\mathbb{J}}^a_-) &= \Delta(\hat{\mathbb{J}}^a_-) + \tfrac{1}{8} f^a{}_{cb} \big(\Delta(\mathbb{J}^b_+) \, \Delta(\mathbb{J}^c_-) + \Delta(\mathbb{J}^c_-) \, \Delta(\mathbb{J}^b_+) \big) \\ &= \hat{\mathbb{J}}^a_- \otimes 1 + 1 \otimes \hat{\mathbb{J}}^a_- + \tfrac{1}{8} f^a{}_{cb} \big(\mathbb{J}^b_+ \mathbb{J}^c_- + \mathbb{J}^c_- \mathbb{J}^b_+ \big) \otimes 1 + \tfrac{1}{8} f^a{}_{cb} \, 1 \otimes \big(\mathbb{J}^b_+ \mathbb{J}^c_- + \mathbb{J}^c_- \mathbb{J}^b_+ \big) \\ &\quad + \tfrac{1}{4} f^a{}_{bc} \big(\mathbb{J}^b_- \otimes \mathbb{J}^c_+ + \mathbb{J}^b_+ \otimes \mathbb{J}^c_- \big) + \tfrac{1}{4} f^a{}_{cb} \big(\mathbb{J}^b_+ \otimes \mathbb{J}^c_- - \mathbb{J}^b_- \otimes \mathbb{J}^c_+ \big) \\ &= \tilde{\mathbb{J}}^a_- \otimes 1 + 1 \otimes \tilde{\mathbb{J}}^a_- + \tfrac{1}{2} f^a{}_{bc} \mathbb{J}^b_- \otimes \mathbb{J}^c_+ \\ &\in \mathcal{Y}(\mathfrak{g}_L \times \mathfrak{g}_R) \otimes \mathcal{Y}(\mathfrak{g}_L \times \mathfrak{g}_R, \mathfrak{g}_+) \ ,\end{aligned} \tag{3.5.11}$$

and thus satisfies the coideal property.

We commented earlier that the structure of $\mathcal{Y}(\mathfrak{g}_L \times \mathfrak{g}_R, \mathfrak{g}_+)$ is subtly different from a simple diagonal embedding $\Delta\mathcal{Y}(\mathfrak{g}_+) \subset \mathcal{Y}(\mathfrak{g}_L \times \mathfrak{g}_R)$. This difference can be made precise, and is seen in the algebra isomorphism

$$\mathcal{Y}(\mathfrak{g}_L \times \mathfrak{g}_R, \mathfrak{g}_+) \cong \tilde{\Delta}\mathcal{Y}(\mathfrak{g}). \tag{3.5.12}$$

Here, for any Yangian charge $\mathbb{J}$, we define $\tilde{\Delta}(\mathbb{J}) := (1 \circ (-1)^l \alpha)\Delta(\mathbb{J})$, where l is the level of $\mathbb{J}$, and $\Delta(\mathbb{J})$ acts on the tensor product of L and R components, written using a circle $\circ$. Thus the difference lies entirely in the twisting by $1 \circ (-1)^l \alpha$, and at its simplest, with $\alpha =$id, solely in some relative signs. At level 0 the isomorphism relates $\mathbb{J}^a_\pm$ on the left of (3.5.12) to $\mathbb{J}^a \circ 1 \pm 1 \circ \alpha(\mathbb{J}^a)$ on the right, and at level 1 it relates $\tilde{\mathbb{J}}^a_-$ to $\hat{\mathbb{J}}^a \circ 1 - 1 \circ \alpha(\hat{\mathbb{J}}^a)$.

At the co-algebra level this isomorphism takes the form

$$\Delta\mathcal{Y}(\mathfrak{g}_L \times \mathfrak{g}_R, \mathfrak{g}_+) \cong \Sigma \cdot \big(\Delta \circ \Delta' \big(\tilde{\Delta}\mathcal{Y}(\mathfrak{g}) \big) \big) \, . \tag{3.5.13}$$

This equation is valued in the fourfold product of $\mathcal{Y}(\mathfrak{g})$ which acts on the $L \circ R$ components of a $bulk \otimes boundary$ state; Δ and $\Delta' := \sigma \cdot \Delta$ are the usual and flipped coproducts. The role of the operator Σ is simply to re-arrange the order of factors, $x_1 \otimes x_2 \circ x_3 \otimes x_4 \mapsto x_1 \circ x_4 \otimes x_2 \circ x_3$. The relations (3.5.12, 3.5.13) then capture the 'folding' of a bulk into a boundary scattering process [144] which we shall meet shortly.

Let us demonstrate (3.5.13) explicitly. First,

$$\begin{aligned}\Delta\circ\Delta'\big(\tilde{\Delta}(\hat{\mathbb{J}}^a)\big) &= \Delta\circ\Delta'\big(\hat{\mathbb{J}}^a\circ 1 - 1\circ\alpha(\hat{\mathbb{J}}^a) + \tfrac{1}{2}f^a_{\ bc}\,\mathbb{J}^b\circ\alpha(\mathbb{J}^c)\big)\\ &= \big(\hat{\mathbb{J}}^a\otimes 1 + 1\otimes\hat{\mathbb{J}}^a + \tfrac{1}{2}f^a_{\ bc}\,\mathbb{J}^b\otimes\mathbb{J}^c\big)\circ 1\otimes 1\\ &\quad -\ 1\otimes 1\circ\big(\alpha(\hat{\mathbb{J}}^a)\otimes 1 + 1\otimes\alpha(\hat{\mathbb{J}}^a) - \tfrac{1}{2}f^a_{\ bc}\,\alpha(\mathbb{J}^b)\otimes\alpha(\mathbb{J}^c)\big)\\ &\quad +\ \tfrac{1}{2}f^a_{\ bc}\big(\mathbb{J}^b\otimes 1 + 1\otimes\mathbb{J}^b\big)\circ\big(\alpha(\mathbb{J}^c)\otimes 1 + 1\otimes\alpha(\mathbb{J}^c)\big).\end{aligned} \tag{3.5.14}$$

Then, acting with Σ,

$$\begin{aligned}\Sigma\cdot\big(\Delta\circ\Delta'\big(\tilde{\Delta}(\hat{\mathbb{J}}^a)\big)\big) &= \big(\hat{\mathbb{J}}^a\circ 1 - 1\circ\alpha(\hat{\mathbb{J}}^a) + \tfrac{1}{2}f^a_{\ bc}\,\mathbb{J}^b\circ\alpha(\mathbb{J}^c)\big)\otimes 1\circ 1\\ &\quad +\ 1\circ 1\otimes\big(\hat{\mathbb{J}}^a\circ 1 - 1\circ\alpha(\hat{\mathbb{J}}^a) + \tfrac{1}{2}f^a_{\ bc}\,\mathbb{J}^b\circ\alpha(\mathbb{J}^c)\big)\\ &\quad +\ \tfrac{1}{2}f^a_{\ bc}\big(\mathbb{J}^b\circ 1 - 1\circ\alpha(\mathbb{J}^b)\big)\otimes\big(\mathbb{J}^c\circ 1 + 1\circ\alpha(\mathbb{J}^c)\big)\,,\end{aligned} \tag{3.5.15}$$

which, applying (3.5.10), corresponds to (3.5.11).

Now consider the implications for the scattering theory. Recall that the bulk multiplets in the bosonic PCM form representations $V\circ V$ of $\mathcal{Y}(\mathfrak{g}_L\times\mathfrak{g}_R)$, where V is a fundamental representation of $\mathcal{Y}(\mathfrak{g})$.[6] The bulk multiplet carries a rapidity u, corresponding to the application of the shift automorphism $L_u\circ L_u$ to $Y(\mathfrak{g}_L\times\mathfrak{g}_R)$. The bulk scattering of $U\circ U$ from $V\circ V$ is then constructed as a product of minimal factors $S_L\circ S_R$ (each factor acting on $U\otimes V$), multiplied by an overall scalar factor [150].

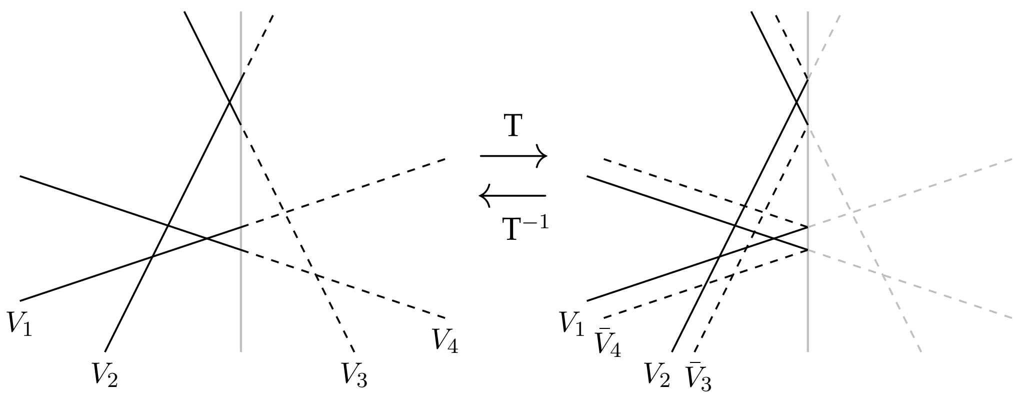

Figure 3.5: The action of the conjugation operator T on 4-particle scattering, with solid L and dashed R lines.

The state $V\circ V$ scatters off the boundary into $\bar{V}\circ\bar{V}$, with $u\mapsto -u$. Thus, on the states, in the isomorphism (3.5.12) we write the action of Σ as conjugation by an operator T whose effect is to re-order and conjugate multiplets,

$$\mathrm{T}:\mathrm{V}_1\otimes\mathrm{V}_2\circ\mathrm{V}_3\otimes\mathrm{V}_4\to\mathrm{V}_1\circ\bar{\mathrm{V}}_4\otimes\mathrm{V}_2\circ\bar{\mathrm{V}}_3\,. \tag{3.5.16}$$

[6] This is typically a reducible rep of $\mathfrak{g}$ with the corresponding fundamental $\mathfrak{g}$-rep as a component, although for $\mathfrak{g}=\mathfrak{su}(n)$ they are identical.

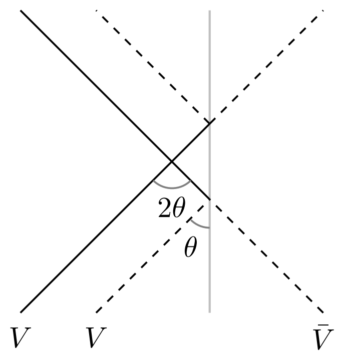

Figure 3.6: Achiral reflection process.

The meaning of the map T (T^{-1}) is revealed, as the folding (unfolding) of a bulk to a boundary scattering process, in figure 3.5. Similar unfolding processes relate boundary unitarity and crossing-unitarity [16] to bulk unitarity and crossing relations. The reversing of rapidity by the boundary can be seen in how the shift automorphism acts on (3.5.12), where, crucially, $L_u \circ L_u(1 \circ (-1)^l \alpha) = (1 \circ (-1)^l \alpha) L_u \circ L_{-u}$ on $\Delta\mathcal{Y}(\mathfrak{g})$.

In the simplest case, where the boundary is in a singlet state, the boundary scattering matrix $K_{V \circ V}(u)$ is conjugate to $(1 \circ \alpha) S_{V\bar{V}}(2u)$ (see figure 3.6). Its direct construction via conservation of the $\mathcal{Y}(\mathfrak{g}_L \times \mathfrak{g}_R, \mathfrak{g}_+)$ charges is isomorphic, via (3.5.12), to that of the bulk S-matrix (using, for example, the Tensor Product Graph method [151]), in which the doubling of u is traced back to the extra factor of two in (3.5.10). We therefore expect a spectrum of boundary bound states in non-trivial multiplets whose mass ratios are those of the bulk states, inherited through the pole structure of $S_{V\bar{V}}(2u)$.

The folding construction of the boundary scattering process straightforwardly accommodates such non-trivial boundary multiplets, and this will play an important role in understanding reflection from the $D5$-brane in the next section. The boundary scattering matrix, the relevant solution of the boundary Yang-Baxter equation (BYBE) [13, 15], then becomes a product of three non-trivial factors, analogous to the boundary fusion procedure [91, 152]. These are a bulk S-matrix and two (what we shall call) 'achiral reflection matrices', which are trivial in the case of the singlet boundary and which participate in the reflection process as on the right of figure 3.5. This threefold process inherits, via (3.5.12), a Yang-Baxter property: the order of the factorization does not matter, and our apparent placing of the bulk S-matrix to the left of the boundary in figures 3.5 and 3.6 is merely an artefact (figure 3.7).

3.5.2 The $D5$-brane: general considerations

The $D5$-brane considered in [143, 144] wraps an $AdS_4 \subset AdS_5$ and a maximal $S^2 \subset S^5$. Such a configuration defines a $2+1$ dimensional defect hypersurface of the $3+1$ dimensional conformal boundary of AdS_5. The fundamental matter living on this hypersurface is a 3d hypermultiplet [60]. The presence of the $D5$-brane breaks the $\mathfrak{so}(6)$ symmetry of

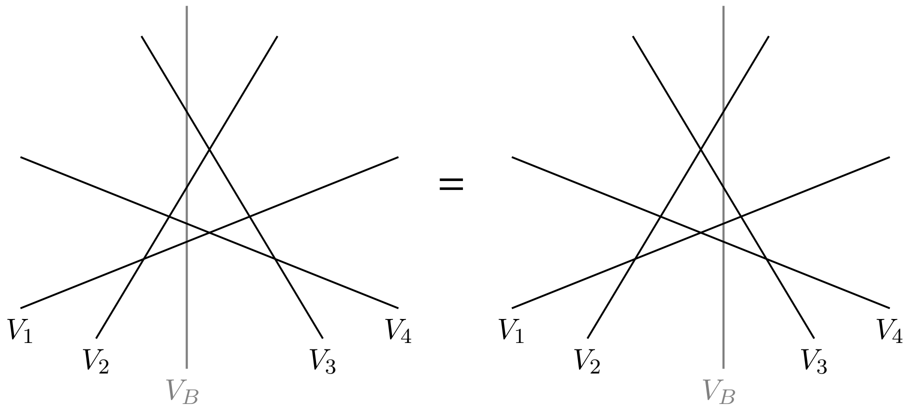

Figure 3.7: The unfolded BYBE as a 5-particle bulk process. The vertical line corresponding to the boundary may be shifted left or right by employing the bulk YBE.

S^5 down to $\mathfrak{so}(3)_H \times \mathfrak{so}(3)_V$. As is the convention, we fix the bulk vacuum state to be $Z = X^5 + iX^6$ and then consider two different orientations of the maximal S^2 inside S^5:

- the maximal S^2 specified by $X^1 = X^2 = X^3 = 0$, – this orientation in [144] was termed 'horizontal' $D5$-brane[7] and, from the scattering theory point of view, corresponds to a singlet boundary;

- the maximal S^2 specified by $X^4 = X^5 = X^6 = 0$, – this orientation is perpendicular to the previous and is termed the 'vertical' $D5$-brane; now the boundary carries a field multiplet transforming in the vector representation of the boundary algebra.

The Lie algebra. The $D5$-brane preserves a diagonal subalgebra $\mathfrak{psu}(2|2)_+ \ltimes \mathbb{R}^3$ of the bulk algebra $\mathfrak{psu}(2|2)_L \times \mathfrak{psu}(2|2)_R \ltimes \mathbb{R}^3$ generated by

$$\begin{aligned} \mathbb{L}_{\check{\alpha}}^{\ \check{\beta}} &= \mathbb{L}_{\alpha}^{\ \beta} + \mathbb{L}_{\bar{\dot{\alpha}}}^{\ \bar{\dot{\beta}}}, & \mathbb{Q}_{\check{\alpha}}^{\ \check{a}} &= \mathbb{Q}_{\alpha}^{\ a} + \tau\,\mathbb{Q}_{\bar{\dot{\alpha}}}^{\ \dot{a}}, \\ \mathbb{R}_{\check{a}}^{\ \check{b}} &= \mathbb{R}_{a}^{\ b} + \mathbb{R}_{\dot{a}}^{\ \dot{b}}, & \mathbb{G}_{\check{a}}^{\ \check{\alpha}} &= \mathbb{G}_{a}^{\ \alpha} + \tau^{-1}\mathbb{G}_{\dot{a}}^{\ \bar{\dot{\alpha}}}, \end{aligned} \tag{3.5.17}$$

where $\tau = -1$ for the horizontal case and $\tau = -i$ for the vertical one.[8] The notation for the dotted and checked indices is the same as for undotted ones, $\dot{a}$, $\check{a}$, $\dot{b}$, $\check{b} = 1, 2$ and $\dot{\alpha}$, $\check{\alpha}$, $\dot{\beta}$, $\check{\beta} = 3, 4$. The generators with the undotted indices generate $\mathfrak{psu}(2|2)_L$ and the generators with the dotted indices generate $\mathfrak{psu}(2|2)_R$. Rather than make the involution α explicit, it is easier to absorb it into the combination of the scale τ and a change of basis, denoted by a bar, which acts on the dotted greek indices acts as $\bar{\dot{3}} = \dot{4}$ and $\bar{\dot{4}} = \dot{3}$. We also

[7] This configuration corresponds to horizontal vacuum orientation in the scattering theory.

[8] The supersymmetries preserved by the $D5$-brane were worked out in Appendix B of [143] and lead to two scattering theories with $\tau^2 = \pm 1$, representing the horizontal and vertical cases.

define the complementary charges

$$
\begin{aligned}
&\overline{\mathbb{L}}_{\check{\alpha}}{}^{\check{\beta}} = \mathbb{L}_{\alpha}{}^{\beta} - \mathbb{L}_{\bar{\dot{\alpha}}}{}^{\bar{\dot{\beta}}}, && \overline{\mathbb{Q}}_{\check{\alpha}}{}^{\check{a}} = \mathbb{Q}_{\alpha}{}^{a} - \tau\, \mathbb{Q}_{\bar{\dot{\alpha}}}{}^{\dot{a}}, \\
&\overline{\mathbb{R}}_{\check{a}}{}^{\check{b}} = \mathbb{R}_{a}{}^{b} - \mathbb{R}_{\dot{a}}{}^{\dot{b}}, && \overline{\mathbb{G}}_{\check{a}}{}^{\check{\alpha}} = \mathbb{G}_{a}{}^{\alpha} - \tau^{-1}\mathbb{G}_{\dot{a}}{}^{\bar{\dot{\alpha}}}\,,
\end{aligned}
\tag{3.5.18}
$$

which do not in themselves form a Lie algebra (and are not preserved by the boundary), but together with (3.5.17) and the central charges $\mathbb{C}$, $\mathbb{C}^{\dagger}$ and $\mathbb{H}$ generate bulk algebra $\mathfrak{psu}(2|2)_L \times \mathfrak{psu}(2|2)_R \ltimes \mathbb{R}^3$.

The representation. We will consider the fundamental representation $\boxslash$ of $\mathfrak{psu}(2|2)_{\mathcal{C}}$ only. Let us denote the basis of this representation as

$$
\begin{aligned}
&|\phi_1\rangle := |0,0,1,0\rangle\,, \qquad && |\psi_3\rangle := |1,0,0,0\rangle\,, \\
&|\phi_2\rangle := |0,0,0,1\rangle\,, \qquad && |\psi_4\rangle := |0,1,0,0\rangle\,,
\end{aligned}
\tag{3.5.19}
$$

and the same for the dotted indices. The bulk magnon transforms in the $\boxslash_{(a,b,c,d)}$ of the left and the $\widetilde{\boxslash}_{(a,b,c,d)}$ of the right representation of the bulk symmetry algebra and they both carry the same set (a,b,c,d) of representation labels. Our goal is to build the canonical representation of bulk states with respect to the boundary algebra (3.5.17). It is easy to see that the left representation transforms canonically with respect to the boundary algebra. However, the right representation does not, and one thus has to choose a different basis for it in order to obtain the algebra action in the canonical form. It was shown in [144] that by choosing the basis

$$
(\tilde{\phi}_{\check{1}}, \tilde{\phi}_{\check{2}} | \tilde{\psi}_{\check{3}}, \tilde{\psi}_{\check{4}}) := (\tilde{\phi}_{\dot{1}}, \tilde{\phi}_{\dot{2}} | \lambda\tilde{\psi}_{\dot{4}}, \lambda\tilde{\psi}_{\dot{3}})
\tag{3.5.20}
$$

to be the new basis of $\widetilde{\boxslash}$, with some arbitrary constant λ representing the rescaling of the new base with the respect to the old, one acquires the canonical action of the boundary algebra (3.5.17) on the right rep:

$$
\begin{aligned}
&\mathbb{Q}_{\check{\alpha}}{}^{\check{a}}\, |\tilde{\phi}_{\check{b}}\rangle = \tilde{a}\, \delta_{\check{b}}^{\check{a}} |\tilde{\psi}_{\check{\alpha}}\rangle, && \mathbb{G}_{\check{a}}{}^{\check{\alpha}}\, |\tilde{\phi}_{\check{b}}\rangle = \tilde{c}\, \varepsilon^{\check{\alpha}\check{\beta}}\, \varepsilon_{\check{a}\check{b}}\, |\tilde{\psi}_{\check{\beta}}\rangle, \\
&\mathbb{Q}_{\check{\alpha}}{}^{\check{a}}\, |\tilde{\psi}_{\check{\beta}}\rangle = \tilde{b}\, \varepsilon_{\check{\alpha}\check{\beta}}\, \varepsilon^{\check{a}\check{b}}\, |\tilde{\phi}_{\check{b}}\rangle, && \mathbb{G}_{\check{a}}{}^{\check{\alpha}}\, |\lambda\tilde{\psi}_{\check{\beta}}\rangle = \tilde{d}\, \delta_{\check{\beta}}^{\check{\alpha}}\, |\tilde{\phi}_{\check{a}}\rangle,
\end{aligned}
\tag{3.5.21}
$$

where $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ are the representation labels in the new basis. They are related to the old basis by

$$
\tilde{a} = \frac{\tau}{\lambda} a, \quad \tilde{b} = -\tau\lambda\, b, \quad \tilde{c} = -\frac{1}{\tau\lambda} c, \quad \tilde{d} = \frac{\lambda}{\tau} d,
\tag{3.5.22}
$$

where the minus sign comes from the relation $\varepsilon^{\bar{\dot{\alpha}}\bar{\dot{\beta}}} = \overline{\varepsilon^{\dot{\alpha}\dot{\beta}}} = \overline{\varepsilon^{\check{\alpha}\check{\beta}}} = -\varepsilon^{\check{\alpha}\check{\beta}}$ and similarly for $\varepsilon_{\bar{\dot{\alpha}}\bar{\dot{\beta}}}$. By choosing $\tilde{a} = a$ one fixes the rescaling constant to be $\lambda = \tau$ and arrives at the relation between new and old representation labels

$$
(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}) = (a, -\tau^2 b, -\tau^{-2} c, d).
\tag{3.5.23}
$$

Thus the canonical representation of the bulk magnon with respect to the boundary algebra and the corresponding labels are

$$\boxslash_{(a,b,c,d)} \circ \widetilde{\boxslash}_{(a,-\tau^2 b,-\tau^{-2}c,d)}. \tag{3.5.24}$$

As in the previous section, we use $\circ$ to denote the tensor product of L and R representations of bulk magnon, reserving the usual $\otimes$ for the tensor product of the bulk and boundary reps. The action of the complementary charges (3.5.18) on the right representation is almost of canonical form, except for an extra minus sign

$$\begin{aligned} \overline{\mathbb{Q}}_{\check{\alpha}}^{\ \check{a}} \, |\tilde{\phi}_{\check{b}}\rangle &= -\tilde{a}\, \delta_{\check{b}}^{\check{a}} \, |\tilde{\psi}_{\check{\alpha}}\rangle, & \overline{\mathbb{G}}_{\check{a}}^{\ \check{\alpha}} \, |\tilde{\phi}_{\check{b}}\rangle &= -\tilde{c}\, \varepsilon^{\check{\alpha}\check{\beta}}\, \varepsilon_{\check{a}\check{b}} \, |\tilde{\psi}_{\check{\beta}}\rangle, \\ \overline{\mathbb{Q}}_{\check{\alpha}}^{\ \check{a}} \, |\tilde{\psi}_{\check{\beta}}\rangle &= -\tilde{b}\, \varepsilon_{\check{\alpha}\check{\beta}}\, \varepsilon^{\check{a}\check{b}} \, |\tilde{\phi}_{\check{b}}\rangle, & \overline{\mathbb{G}}_{\check{a}}^{\ \check{\alpha}} \, |\lambda\tilde{\psi}_{\check{\beta}}\rangle &= -\tilde{d}\, \delta_{\check{\beta}}^{\check{\alpha}} \, |\tilde{\phi}_{\check{a}}\rangle. \end{aligned} \tag{3.5.25}$$

The total eigenvalues of the central charges on the bulk representations (3.5.24) in the new basis become

$$\begin{aligned} \mathbb{C} \ &:= \mathbb{C} \circ 1 + 1 \circ \mathbb{C} \equiv ab + \tilde{a}\tilde{b} = (1-\tau^2)ab\,, \\ \mathbb{C}^\dagger &:= \mathbb{C}^\dagger \circ 1 + 1 \circ \mathbb{C}^\dagger \equiv cd + \tilde{c}\tilde{d} = (1-\tau^{-2})ab\,, \\ \mathbb{H} \ &:= \mathbb{H} \circ 1 + 1 \circ \mathbb{H} \equiv ad + bc + \tilde{a}\tilde{d} + \tilde{b}\tilde{c} = 2(ad+bc)\,, \end{aligned} \tag{3.5.26}$$

hence the bulk magnon lives in the following tensor product of fundamental representations:

$$\langle 0,0;H,C,C^\dagger\rangle \circ \langle 0,0;H,-\tau^2 C,-\tau^{-2}C^\dagger\rangle = \{0,0,2H,(1-\tau^2)C,(1-\tau^{-2})C^\dagger\}\,, \tag{3.5.27}$$

which depends on the value of τ. Let us explain this result in more detail.

The symmetry algebra in the bulk is $\mathfrak{g}_L \oplus \mathfrak{g}_R \oplus \mathfrak{m}$, where $\mathfrak{m}$ is the central extension, which is invariant under α; thus the central charges' eigenvalues should not depend on τ either. What has happened is that, by using a different basis for $\mathfrak{g}_R$, which allowed us to write the action of the $\mathfrak{g}_+$ charges in untwisted diagonal form, we have introduced τ-dependence into the $L \circ R$ basis of the central charges. That is, the price we have to pay for simplifying the action of α is that the central charges $\mathbb{C}$ and $\mathbb{C}^\dagger$ become dependent on τ. We resolve this by introducing complementary central charges

$$\begin{aligned} \overline{\mathbb{C}} \ &:= \mathbb{C} \circ 1 - 1 \circ \mathbb{C} \equiv ab - \tilde{a}\tilde{b} = (1+\tau^2)ab, \\ \overline{\mathbb{C}}^\dagger &:= \mathbb{C}^\dagger \circ 1 - 1 \circ \mathbb{C}^\dagger \equiv cd - \tilde{c}\tilde{d} = (1+\tau^{-2})ab, \\ \overline{\mathbb{H}} \ &:= \mathbb{H} \circ 1 - 1 \circ \mathbb{H} \equiv ad + bc - \tilde{a}\tilde{d} - \tilde{b}\tilde{c} = 0. \end{aligned} \tag{3.5.28}$$

This formal enlargement of the algebra does not – cannot – add any new constraints to the system. The charge $\overline{\mathbb{H}}$ has zero eigenvalues on the bulk and boundary representations independently of τ and we have introduced it merely to enable us to write the Yangian charges in a nicely symmetric form. The new charges $\overline{\mathbb{C}}$ and $\overline{\mathbb{C}}^\dagger$ will have non-zero eigenvalues on the bulk representations when $\tau^2 = +1$, while the charges $\mathbb{C}$ and $\mathbb{C}^\dagger$ then vanish; and vice versa for the $\tau^2 = -1$ case. Thus there are always exactly three non-trivial central charges in the system.

The Yangian algebra. We now have enough ingredients to write down the general form of the boundary Yangian. Following the discussion in Section 3.5.1 we define the achiral twisted boundary Yangian $\mathcal{Y}(\mathfrak{g}_L \times \mathfrak{g}_R, \mathfrak{g}_+)$ to be generated by the Lie algebra generators (3.5.17) and (3.5.18), central charges and the twisted Yangian generators (3.5.29).

$$
\begin{aligned}
\widetilde{\mathbb{R}}_{\hat{a}}{}^{\hat{b}} &= \hat{\overline{\mathbb{R}}}_{\hat{a}}{}^{\hat{b}} - \tfrac{1}{4}\,\mathbb{R}_{\check{a}}{}^{\check{c}}\,\overline{\mathbb{R}}_{\check{c}}{}^{\check{b}} + \tfrac{1}{4}\,\mathbb{R}_{\check{c}}{}^{\check{b}}\,\overline{\mathbb{R}}_{\check{a}}{}^{\check{c}} + \tfrac{1}{4}\,\mathbb{G}_{\check{a}}{}^{\check{\gamma}}\,\overline{\mathbb{Q}}_{\check{\gamma}}{}^{\check{b}} + \tfrac{1}{4}\,\mathbb{Q}_{\check{\gamma}}{}^{\check{b}}\,\overline{\mathbb{G}}_{\check{a}}{}^{\check{\gamma}} \\
&\qquad - \tfrac{1}{8}\,\delta_{\check{a}}^{\check{b}}\,\mathbb{G}_{\check{c}}{}^{\check{\gamma}}\,\overline{\mathbb{Q}}_{\check{\gamma}}{}^{\check{c}} - \tfrac{1}{8}\,\delta_{\check{a}}^{\check{b}}\,\mathbb{Q}_{\check{\gamma}}{}^{\check{c}}\,\overline{\mathbb{G}}_{\check{c}}{}^{\check{\gamma}}\,, \\
\widetilde{\mathbb{L}}_{\hat{\alpha}}{}^{\hat{\beta}} &= \hat{\overline{\mathbb{L}}}_{\hat{\alpha}}{}^{\hat{\beta}} + \tfrac{1}{4}\,\mathbb{L}_{\check{\alpha}}{}^{\check{\gamma}}\,\overline{\mathbb{L}}_{\check{\gamma}}{}^{\check{\beta}} - \tfrac{1}{4}\,\mathbb{L}_{\check{\gamma}}{}^{\check{\beta}}\,\overline{\mathbb{L}}_{\check{\alpha}}{}^{\check{\gamma}} - \tfrac{1}{4}\,\mathbb{G}_{\check{c}}{}^{\check{\beta}}\,\overline{\mathbb{Q}}_{\check{\alpha}}{}^{\check{c}} - \tfrac{1}{4}\mathbb{Q}_{\check{\alpha}}{}^{\check{c}}\,\overline{\mathbb{G}}_{\check{c}}{}^{\check{\beta}} \\
&\qquad + \tfrac{1}{8}\,\delta_{\check{\alpha}}^{\check{\beta}}\,\mathbb{G}_{\check{c}}{}^{\check{\gamma}}\,\overline{\mathbb{Q}}_{\check{\gamma}}{}^{\check{c}} + \tfrac{1}{8}\,\delta_{\check{\alpha}}^{\check{\beta}}\,\mathbb{Q}_{\check{\gamma}}{}^{\check{c}}\,\overline{\mathbb{G}}_{\check{c}}{}^{\check{\gamma}}\,, \\
\widetilde{\mathbb{Q}}_{\hat{\alpha}}{}^{\hat{a}} &= \hat{\overline{\mathbb{Q}}}_{\hat{\alpha}}{}^{\hat{a}} - \tfrac{1}{4}\,\mathbb{Q}_{\check{\alpha}}{}^{\check{c}}\,\overline{\mathbb{R}}_{\check{c}}{}^{\check{a}} + \tfrac{1}{4}\,\mathbb{R}_{\check{c}}{}^{\check{a}}\,\overline{\mathbb{Q}}_{\check{\alpha}}{}^{\check{c}} - \tfrac{1}{4}\,\mathbb{Q}_{\check{\gamma}}{}^{\check{a}}\,\overline{\mathbb{L}}_{\check{\alpha}}{}^{\check{\gamma}} + \tfrac{1}{4}\,\mathbb{L}_{\check{\alpha}}{}^{\check{\gamma}}\,\overline{\mathbb{Q}}_{\check{\gamma}}{}^{\check{a}} \\
&\qquad + \tfrac{1}{8}\,\mathbb{Q}_{\check{\alpha}}{}^{\check{a}}\,\overline{\mathbb{H}} + \tfrac{1}{8}\,\mathbb{H}\,\overline{\mathbb{Q}}_{\check{\alpha}}{}^{\check{a}} - \tfrac{1}{4}\,\varepsilon_{\check{\alpha}\check{\gamma}}\,\varepsilon^{\check{a}\check{d}}\,\mathbb{C}\,\overline{\mathbb{G}}_{\check{d}}{}^{\check{\gamma}} + \tfrac{1}{4}\,\varepsilon_{\check{\alpha}\check{\gamma}}\,\varepsilon^{\check{a}\check{d}}\,\mathbb{G}_{\check{d}}{}^{\check{\gamma}}\,\overline{\mathbb{C}}\,, \\
\widetilde{\mathbb{G}}_{\hat{a}}{}^{\hat{\alpha}} &= \hat{\overline{\mathbb{G}}}_{\hat{a}}{}^{\hat{\alpha}} + \tfrac{1}{4}\,\mathbb{G}_{\check{c}}{}^{\check{\alpha}}\,\overline{\mathbb{R}}_{\check{a}}{}^{\check{c}} - \tfrac{1}{4}\,\mathbb{R}_{\check{a}}{}^{\check{c}}\,\overline{\mathbb{G}}_{\check{c}}{}^{\check{\alpha}} + \tfrac{1}{4}\,\mathbb{G}_{\check{a}}{}^{\check{\gamma}}\,\overline{\mathbb{L}}_{\check{\gamma}}{}^{\check{\alpha}} - \tfrac{1}{4}\,\mathbb{L}_{\check{\gamma}}{}^{\check{\alpha}}\,\overline{\mathbb{G}}_{\check{a}}{}^{\check{\gamma}} \\
&\qquad + \tfrac{1}{8}\,\mathbb{G}_{\check{a}}{}^{\check{\alpha}}\,\overline{\mathbb{H}} - \tfrac{1}{8}\,\mathbb{H}\,\overline{\mathbb{G}}_{\check{a}}{}^{\check{\alpha}} + \tfrac{1}{4}\,\varepsilon_{\check{a}\check{c}}\,\varepsilon^{\check{\alpha}\check{\gamma}}\,\mathbb{C}^{\dagger}\,\overline{\mathbb{Q}}_{\check{\gamma}}{}^{\check{c}} - \tfrac{1}{4}\,\varepsilon_{\check{a}\check{c}}\,\varepsilon^{\check{\alpha}\check{\gamma}}\,\mathbb{Q}_{\check{\gamma}}{}^{\check{c}}\,\overline{\mathbb{C}}^{\dagger}\,, \\
\widetilde{\mathbb{C}} &= \hat{\overline{\mathbb{C}}} + \tfrac{1}{4}\,\mathbb{H}\,\overline{\mathbb{C}} - \tfrac{1}{4}\,\mathbb{C}\,\overline{\mathbb{H}}\,, \\
\widetilde{\mathbb{C}}^{\dagger} &= \hat{\overline{\mathbb{C}}}^{\dagger} - \tfrac{1}{4}\,\mathbb{H}\,\overline{\mathbb{C}}^{\dagger} + \tfrac{1}{4}\,\mathbb{C}^{\dagger}\,\overline{\mathbb{H}}\,.
\end{aligned}
\tag{3.5.29}
$$

Here the hat-bar operators $\hat{\overline{\mathbb{J}}}$ are the grade-1 partners of the complementary charges (3.5.18) and (3.5.28). This is the general form of the twisted boundary Yangian for the reflection from the $D5$-brane and represents the explicit realization of (3.5.10). The co-products of the twisted charges (3.5.29) have the canonical form

$$
\begin{aligned}
\Delta\big(\widetilde{\mathbb{R}}_{\check{a}}{}^{\check{b}}\big) &= \widetilde{\mathbb{R}}_{\check{a}}{}^{\check{b}} \otimes 1 + 1 \otimes \widetilde{\mathbb{R}}_{\check{a}}{}^{\check{b}} + \tfrac{1}{2}\overline{\mathbb{R}}_{\check{a}}{}^{\check{c}} \otimes \mathbb{R}_{\check{c}}{}^{\check{b}} - \tfrac{1}{2}\overline{\mathbb{R}}_{\check{c}}{}^{\check{b}} \otimes \mathbb{R}_{\check{a}}{}^{\check{c}} + \tfrac{1}{2}\overline{\mathbb{G}}_{\check{a}}{}^{\check{\gamma}} \otimes \mathbb{Q}_{\check{\gamma}}{}^{\check{b}} + \tfrac{1}{4}\overline{\mathbb{Q}}_{\check{\gamma}}{}^{\check{b}} \otimes \mathbb{G}_{\check{a}}{}^{\check{\gamma}} \\
&\qquad - \tfrac{1}{4}\delta_{\check{a}}^{\check{b}}\,\overline{\mathbb{G}}_{\check{c}}{}^{\check{\gamma}} \otimes \mathbb{Q}_{\check{\gamma}}{}^{\check{c}} - \tfrac{1}{4}\delta_{\check{a}}^{\check{b}}\,\overline{\mathbb{Q}}_{\check{\gamma}}{}^{\check{c}} \otimes \mathbb{G}_{\check{c}}{}^{\check{\gamma}}\,, \\
\Delta\big(\widetilde{\mathbb{L}}_{\check{\alpha}}{}^{\check{\beta}}\big) &= \widetilde{\mathbb{L}}_{\check{\alpha}}{}^{\check{\beta}} \otimes 1 + 1 \otimes \widetilde{\mathbb{L}}_{\check{\alpha}}{}^{\check{\beta}} + \tfrac{1}{2}\overline{\mathbb{L}}_{\check{\alpha}}{}^{\check{\gamma}} \otimes \mathbb{L}_{\check{\gamma}}{}^{\check{\beta}} - \tfrac{1}{2}\overline{\mathbb{L}}_{\check{\gamma}}{}^{\check{\beta}} \otimes \mathbb{L}_{\check{\alpha}}{}^{\check{\gamma}} - \tfrac{1}{2}\overline{\mathbb{G}}_{\check{c}}{}^{\check{\beta}} \otimes \mathbb{Q}_{\check{\alpha}}{}^{\check{c}} - \tfrac{1}{2}\overline{\mathbb{Q}}_{\check{\alpha}}{}^{\check{c}} \otimes \mathbb{G}_{\check{c}}{}^{\check{\beta}} \\
&\qquad + \tfrac{1}{4}\delta_{\check{\alpha}}^{\check{\beta}}\,\overline{\mathbb{G}}_{\check{c}}{}^{\check{\gamma}} \otimes \mathbb{Q}_{\check{\gamma}}{}^{\check{c}} + \tfrac{1}{4}\delta_{\check{\alpha}}^{\check{\beta}}\,\overline{\mathbb{Q}}_{\check{\gamma}}{}^{\check{c}} \otimes \mathbb{G}_{\check{c}}{}^{\check{\gamma}}\,, \\
\Delta\big(\widetilde{\mathbb{Q}}_{\check{\alpha}}{}^{\check{a}}\big) &= \widetilde{\mathbb{Q}}_{\check{\alpha}}{}^{\check{a}} \otimes 1 + 1 \otimes \widetilde{\mathbb{Q}}_{\check{\alpha}}{}^{\check{a}} - \tfrac{1}{2}\overline{\mathbb{Q}}_{\check{\alpha}}{}^{\check{c}} \otimes \mathbb{R}_{\check{c}}{}^{\check{a}} + \tfrac{1}{2}\overline{\mathbb{R}}_{\check{c}}{}^{\check{a}} \otimes \mathbb{Q}_{\check{\alpha}}{}^{\check{c}} - \tfrac{1}{2}\overline{\mathbb{Q}}_{\check{\gamma}}{}^{\check{a}} \otimes \mathbb{L}_{\check{\alpha}}{}^{\check{\gamma}} + \tfrac{1}{2}\overline{\mathbb{L}}_{\check{\alpha}}{}^{\check{\gamma}} \otimes \mathbb{Q}_{\check{\gamma}}{}^{\check{a}} \\
&\qquad + \tfrac{1}{4}\overline{\mathbb{H}} \otimes \mathbb{Q}_{\check{\alpha}}{}^{\check{a}} - \tfrac{1}{4}\overline{\mathbb{Q}}_{\check{\alpha}}{}^{\check{a}} \otimes \mathbb{H} - \tfrac{1}{2}\varepsilon_{\check{\alpha}\check{\gamma}}\,\varepsilon^{\check{a}\check{d}}\,\overline{\mathbb{C}} \otimes \mathbb{G}_{\check{d}}{}^{\check{\gamma}} + \tfrac{1}{2}\varepsilon_{\check{\alpha}\check{\gamma}}\,\varepsilon^{\check{a}\check{d}}\,\overline{\mathbb{G}}_{\check{d}}{}^{\check{\gamma}} \otimes \mathbb{C}\,, \\
\Delta\big(\widetilde{\mathbb{G}}_{\check{a}}{}^{\check{\alpha}}\big) &= \widetilde{\mathbb{G}}_{\check{a}}{}^{\check{\alpha}} \otimes 1 + 1 \otimes \widetilde{\mathbb{G}}_{\check{a}}{}^{\check{\alpha}} + \tfrac{1}{2}\overline{\mathbb{G}}_{\check{c}}{}^{\check{\alpha}} \otimes \mathbb{R}_{\check{a}}{}^{\check{c}} - \tfrac{1}{2}\overline{\mathbb{R}}_{\check{a}}{}^{\check{c}} \otimes \mathbb{G}_{\check{c}}{}^{\check{\alpha}} + \tfrac{1}{2}\overline{\mathbb{G}}_{\check{a}}{}^{\check{\gamma}} \otimes \mathbb{L}_{\check{\gamma}}{}^{\check{\alpha}} - \tfrac{1}{2}\overline{\mathbb{L}}_{\check{\gamma}}{}^{\check{\alpha}} \otimes \mathbb{G}_{\check{a}}{}^{\check{\gamma}} \\
&\qquad - \tfrac{1}{4}\overline{\mathbb{H}} \otimes \mathbb{G}_{\check{a}}{}^{\check{\alpha}} + \tfrac{1}{4}\overline{\mathbb{G}}_{\check{a}}{}^{\check{\alpha}} \otimes \mathbb{H} + \tfrac{1}{2}\varepsilon_{\check{a}\check{c}}\,\varepsilon^{\check{\alpha}\check{\gamma}}\,\overline{\mathbb{C}}^{\dagger} \otimes \mathbb{Q}_{\check{\gamma}}{}^{\check{c}} - \tfrac{1}{2}\varepsilon_{\check{a}\check{c}}\,\varepsilon^{\check{\alpha}\check{\gamma}}\,\overline{\mathbb{Q}}_{\check{\gamma}}{}^{\check{c}} \otimes \mathbb{C}^{\dagger}\,, \\
\Delta\big(\widetilde{\mathbb{C}}\,\big) &= \widetilde{\mathbb{C}} \otimes 1 + 1 \otimes \widetilde{\mathbb{C}} + \tfrac{1}{2}\overline{\mathbb{C}} \otimes \mathbb{H} - \tfrac{1}{2}\overline{\mathbb{H}} \otimes \mathbb{C}\,, \\
\Delta\big(\widetilde{\mathbb{C}}^{\dagger}\big) &= \widetilde{\mathbb{C}}^{\dagger} \otimes 1 + 1 \otimes \widetilde{\mathbb{C}}^{\dagger} - \tfrac{1}{2}\overline{\mathbb{C}}^{\dagger} \otimes \mathbb{H} + \tfrac{1}{2}\overline{\mathbb{H}} \otimes \mathbb{C}^{\dagger},
\end{aligned}
\tag{3.5.30}
$$

as expected from (3.5.11). Note that the terms of the form $1 \otimes \widetilde{\mathbb{J}}$ annihilate the boundary and give no contribution to the explicit calculations. Also note that the expressions above may be reduced to a more compact and transparent form using the Lie algebra relations (3.1.2). We will do this by considering the reflection from the vertical and horizontal $D5$-branes separately.

3.5.3 The horizontal $D5$-brane

The boundary algebra of the horizontal $D5$-brane is acquired from (3.5.17) by setting $\tau = -1$. As was shown in [143], the boundary carries no degrees of freedom in the scattering theory, and thus is a singlet of $\mathfrak{psl}(2|2)_+$. The total central charges $\mathbb{C}$ and $\mathbb{C}^\dagger$ vanish with respect to the boundary symmetry,

$$\langle 0, 0; H, C, C^\dagger \rangle \circ \langle 0, 0; H, -C, -C^\dagger \rangle = \{0, 0, 2H, 0, 0\}\,, \tag{3.5.31}$$

and the bulk magnon transforms in the tensor representation

$$\boxslash_{(a,b,c,d)} \circ \ \widetilde{\boxslash}_{(a,-b,-c,d)}, \tag{3.5.32}$$

with respect to the boundary algebra. The reflection matrix is simply a map

$$\mathcal{K}^h : \boxslash \circ \widetilde{\boxslash} \otimes 1 \to \boxslash \circ \widetilde{\boxslash} \otimes 1, \tag{3.5.33}$$

and may be neatly represented on superspace as an operator

$$\mathcal{K}^h : \quad \mathcal{V}\,(p, \zeta) \circ \mathcal{V}(-p, \zeta e^{ip}) \to \mathcal{V}(-p, \zeta) \circ \mathcal{V}(p, \zeta e^{-ip}), \tag{3.5.34}$$

where $\mathcal{V}(p, \zeta)$ is the corresponding vector space. Thus $\mathcal{K}^h$ differs from the bulk S-matrix $\mathcal{S}(p, -p)$ by an overall phase at most.

Boundary scattering. The boundary is achiral in the sense that the incoming L state becomes a R state after the reflection and vice versa. This feature of the achiral boundary may be neatly displayed graphically (see figure 3.8, left side). The picture of the reflection nicely accommodates the fact that $\mathcal{K}^h$ is equivalent to $\mathcal{S}$ as discussed above and suggests that it should be related as

$$\mathcal{K}^h = \mathcal{K} \cdot \mathcal{S}(p, -p) \cdot \mathcal{K}\,, \tag{3.5.35}$$

with $\mathcal{K}$ here[9] being the achiral map $\mathcal{K} : 1 \circ \widetilde{\boxslash} \otimes 1 \to \boxslash \circ 1 \otimes 1$ for an incoming right state and $\mathcal{K} : \boxslash \circ 1 \otimes 1 \to 1 \circ \widetilde{\boxslash} \otimes 1$ for an incoming left state. This relation implies that the structure of the boundary Yangian for the horizontal $D5$-brane should be related to the bulk Yangian by (3.5.12). Let us show this explicitly.

The boundary is a singlet; thus the only surviving terms in (3.5.30) are of the form $\widetilde{\mathbb{J}} \otimes 1$, since all non-local two-site operators of the form $\mathbb{J} \otimes \mathbb{J}$ annihilate the boundary and give no contribution.[10] Then using (3.5.29, 3.5.17, 3.5.18, 3.5.28) and performing some Lie

[9] This enlarged $\mathcal{K}$ denoting the achiral map should not be confused with the reflection map κ.

[10] See e.g. [16] for the formulation of the scattering theory on the half-line.

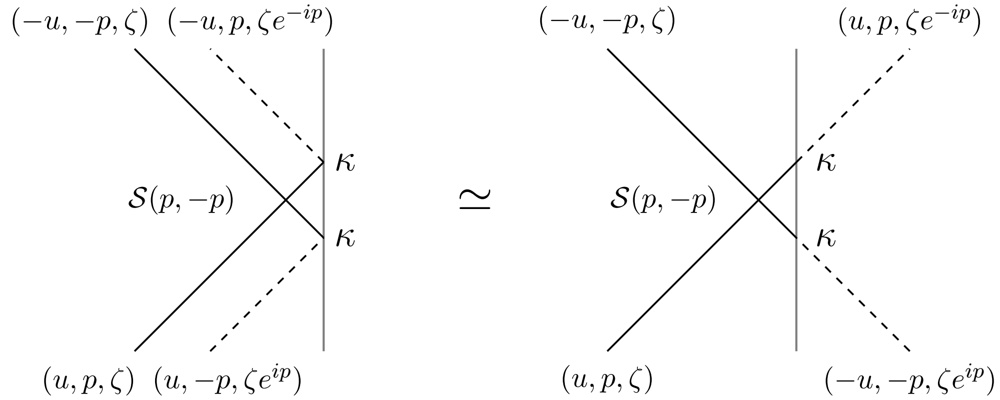

Figure 3.8: Unfolding of the reflection from the horizontal $D5$-brane. Solid lines correspond to the left representations while the dotted lines correspond to right reps. The vertical gray lines correspond to the singlet boundary which acts merely as an achiral map κ mapping right (left) representations into left (right) representations (and conjugates multiplets by maping $u \mapsto -u$ in the unfolded picture). The left and right sides of the figure are related through the conjugation map T.

algebra manipulations, (3.5.30) becomes

$$\begin{aligned}
\Delta\big(\widetilde{\mathbb{R}}_{\check a}^{\;\check b}\big) &= \big(\widehat{\mathbb{R}}_{\check a}^{\;\check b}\circ 1 - 1\circ \widehat{\mathbb{R}}_{\check a}^{\;\check b} + \tfrac12\,\mathbb{R}_{\check a}^{\;\check c}\circ \mathbb{R}_{\check c}^{\;\check b} - \tfrac12\,\mathbb{R}_{\check c}^{\;\check b}\circ \mathbb{R}_{\check a}^{\;\check c} - \tfrac12\,\mathbb{G}_{\check a}^{\;\check\gamma}\circ \mathbb{Q}_{\check\gamma}^{\;\check b} - \tfrac12\,\mathbb{Q}_{\check\gamma}^{\;\check b}\circ \mathbb{G}_{\check a}^{\;\check\gamma}\\
&\qquad + \tfrac14\,\delta_{\check a}^{\check b}\,\mathbb{G}_{\check c}^{\;\check\gamma}\circ \mathbb{Q}_{\check\gamma}^{\;\check c} + \tfrac14\,\delta_{\check a}^{\check b}\,\mathbb{Q}_{\check\gamma}^{\;\check c}\circ \mathbb{G}_{\check c}^{\;\check\gamma}\big)\otimes 1\,,\\
\Delta\big(\widetilde{\mathbb{L}}_{\check\alpha}^{\;\check\beta}\big) &= \big(\widehat{\mathbb{L}}_{\check\alpha}^{\;\check\beta}\circ 1 - 1\circ \widehat{\mathbb{L}}_{\check\alpha}^{\;\check\beta} - \tfrac12\,\mathbb{L}_{\check\alpha}^{\;\check\gamma}\circ \mathbb{L}_{\check\gamma}^{\;\check\beta} + \tfrac12\,\mathbb{L}_{\check\gamma}^{\;\check\beta}\circ \mathbb{L}_{\check\alpha}^{\;\check\gamma} + \tfrac12\,\mathbb{G}_{\check c}^{\;\check\beta}\circ \mathbb{Q}_{\check\alpha}^{\;\check c} + \tfrac12\,\mathbb{Q}_{\check\alpha}^{\;\check c}\circ \mathbb{G}_{\check c}^{\;\check\beta}\\
&\qquad - \tfrac14\,\delta_{\check\alpha}^{\check\beta}\,\mathbb{G}_{\check c}^{\;\check\gamma}\circ \mathbb{Q}_{\check\gamma}^{\;\check c} - \tfrac14\,\delta_{\check\alpha}^{\check\beta}\,\mathbb{Q}_{\check\gamma}^{\;\check c}\circ \mathbb{G}_{\check c}^{\;\check\gamma}\big)\otimes 1\,,\\
\Delta\big(\widetilde{\mathbb{Q}}_{\check\alpha}^{\;\check a}\big) &= \big(\widehat{\mathbb{Q}}_{\check\alpha}^{\;\check a}\circ 1 - 1\circ \widehat{\mathbb{Q}}_{\check\alpha}^{\;\check a} + \tfrac12\,\mathbb{Q}_{\check\alpha}^{\;\check c}\circ \mathbb{R}_{\check c}^{\;\check a} - \tfrac12\,\mathbb{R}_{\check c}^{\;\check a}\circ \mathbb{Q}_{\check\alpha}^{\;\check c} + \tfrac12\,\mathbb{Q}_{\check\gamma}^{\;\check a}\circ \mathbb{L}_{\check\alpha}^{\;\check\gamma} - \tfrac12\,\mathbb{L}_{\check\alpha}^{\;\check\gamma}\circ \mathbb{Q}_{\check\gamma}^{\;\check a}\\
&\qquad + \tfrac14\,\mathbb{Q}_{\check\alpha}^{\;\check a}\circ \mathbb{H} - \tfrac14\,\mathbb{H}\circ \mathbb{Q}_{\check\alpha}^{\;\check a} + \tfrac12\varepsilon_{\check\alpha\check\gamma}\,\varepsilon^{ad}\,\mathbb{C}\circ \mathbb{G}_{\check d}^{\;\check\gamma} - \tfrac12\varepsilon_{\check\alpha\check\gamma}\,\varepsilon^{ad}\,\mathbb{G}_{\check d}^{\;\check\gamma}\circ \mathbb{C}\big)\otimes 1\,,\\
\Delta\big(\widetilde{\mathbb{G}}_{\check a}^{\;\check\alpha}\big) &= \big(\widehat{\mathbb{G}}_{\check a}^{\;\check\alpha}\circ 1 - 1\circ \widehat{\mathbb{G}}_{\check a}^{\;\check\alpha} - \tfrac12\,\mathbb{G}_{\check c}^{\;\check\alpha}\circ \mathbb{R}_{\check a}^{\;\check c} + \tfrac12\,\mathbb{R}_{\check a}^{\;\check c}\circ \mathbb{G}_{\check c}^{\;\check\alpha} - \tfrac12\,\mathbb{G}_{\check a}^{\;\check\gamma}\circ \mathbb{L}_{\check\gamma}^{\;\check\alpha} + \tfrac12\,\mathbb{L}_{\check\gamma}^{\;\check\alpha}\circ \mathbb{G}_{\check a}^{\;\check\gamma}\\
&\qquad - \tfrac14\,\mathbb{G}_{\check a}^{\;\check\alpha}\circ \mathbb{H} + \tfrac14\,\mathbb{H}\circ \mathbb{G}_{\check a}^{\;\check\alpha} - \tfrac12\varepsilon_{\check a\check c}\,\varepsilon^{\check\alpha\check\gamma}\,\mathbb{C}^\dagger\circ \mathbb{Q}_{\check\gamma}^{\;\check c} + \tfrac12\varepsilon_{\check a\check c}\,\varepsilon^{\check\alpha\check\gamma}\,\mathbb{Q}_{\check\gamma}^{\;\check c}\circ \mathbb{C}^\dagger\big)\otimes 1\,,\\
\Delta\big(\widetilde{\mathbb{C}}\big)\;\; &= \big(\widehat{\mathbb{C}}\circ 1 - 1\circ\widehat{\mathbb{C}} - \tfrac12\,\mathbb{H}\circ\mathbb{C} + \tfrac12\,\mathbb{C}\circ\mathbb{H}\big)\otimes 1\,,\\
\Delta\big(\widetilde{\mathbb{C}}^\dagger\big) &= \big(\widehat{\mathbb{C}}^\dagger\circ 1 - 1\circ\widehat{\mathbb{C}}^\dagger + \tfrac12\,\mathbb{H}\circ\mathbb{C}^\dagger - \tfrac12\,\mathbb{C}^\dagger\circ\mathbb{H}\big)\otimes 1\,.
\end{aligned}\tag{3.5.36}$$

We have checked that these co-products commute with the reflection matrix $\mathcal{K}^h$ calculated in [144] and also with the two-magnon bound state reflection matrix which is constructed from $S^{BB}(p_1,p_2)$ by setting $p_2 := -p_1$.

It is easy to observe that these co-products have almost the same form as (3.1.28), as we expected. The crucial difference is the negative sign of terms of the form $1\circ\widehat{\mathbb{J}}$ in (3.5.36), in contrast to (3.1.28). This is the outcome of the graded map $1\circ(-1)^l$ relating $\tilde\Delta$ to the usual Δ in (3.5.12). In this particular case it has a lucid physical interpretation. Consider the scattering in the bulk of two magnons with momenta p and $-p$. The residual symmetry of such scattering is described by (3.1.28). The rapidities of the states in the bulk are u and $-u$ and are facing the same direction as their momenta. But in the case of a single bulk magnon reflecting from the horizontal $D5$-brane the rapidity of the right rep, which has the effective momentum $-p$ with respect to the boundary algebra, is u and faces the physical direction, but not the effective one i.e. is not $-u$). Thus this minus sign difference is explicitly seen in the co-products (3.5.36).

Interestingly, in the unfolded picture of the reflection (the right side of figure 3.8), which is related to the left side by the map T (3.5.16), the rapidity of the right representation is facing the same direction as the momentum. This is because the map T not only re-orders the states, but also sends $V_R \mapsto \bar{V}_R$, $u \mapsto -u$. Thus the reflection from the boundary in the unfolded picture,

$$\mathcal{K}^h : \boxslash \otimes 1 \otimes \widetilde{\boxslash} \to \boxslash \otimes 1 \otimes \widetilde{\boxslash} \,, \tag{3.5.37}$$

may be regarded as a 'scattering through the boundary' and is governed by the Yangian

$$\Delta\big(\widetilde{\mathbb{J}}^A\big) = \widehat{\mathbb{J}}^A \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \widehat{\mathbb{J}}^A + \tfrac{1}{2} f^A_{\ BC}\, \mathbb{J}^B \otimes 1 \otimes \mathbb{J}^C \,, \tag{3.5.38}$$

which is equivalent to (3.1.28) (by removing the middle singlet in (3.5.37) and (3.5.38) as it effectively plays no role).

Boundary Yang-Baxter equation. In order to check that the boundary is integrable one has to consider the boundary Yang-Baxter equation (BYBE), which computes the difference between the two possible ways of factorizing the scattering of two incoming magnons off a boundary.

The BYBE represents two incoming bulk magnons reflecting from the boundary:

$$\begin{aligned} \text{BYBE} : \mathcal{V}_L(p_1,\zeta) \otimes \mathcal{V}_R(-p_1,\zeta e^{ip_1}) \otimes \mathcal{V}_L(p_2,\zeta e^{ip_1}) \otimes \mathcal{V}_R(-p_2,\zeta e^{i(p_1+p_2)}) \to \\ \mathcal{V}_L(-p_1,\zeta) \otimes \mathcal{V}_R(p_1,\zeta e^{-ip_1}) \otimes \mathcal{V}_L(-p_2,\zeta e^{-ip_1}) \otimes \mathcal{V}_R(p_2,\zeta e^{-i(p_1+p_2)}). \end{aligned} \tag{3.5.39}$$

Here $\mathcal{V}_L$ ($\mathcal{V}_R$) are representations of the boundary algebra originating as left (respectively, right) factors of bulk magnons. We must not lose track of this information, because it affects how the representations scatter, as follows.

For the bulk scattering, left (right) states scatter with left (respectively, right) states only. When scattering two left representations we use the standard S-matrix, but when scattering two right representations we must allow for the change of basis, (3.5.23), which produces additional signs in the ζ-dependent components:

$$\begin{aligned} \langle\psi^{\check{3}}\psi^{\check{4}}|\, \mathcal{S}\, |\phi^{\check{1}}\phi^{\check{2}}\rangle &= -\langle\psi^{\dot{3}}\psi^{\dot{4}}|\, \mathcal{S}\, |\phi^{\dot{1}}\phi^{\dot{2}}\rangle = -a_7 \,, \\ \langle\phi^{\check{1}}\phi^{\check{2}}|\, \mathcal{S}\, |\psi^{\check{3}}\psi^{\check{4}}\rangle &= -\langle\phi^{\dot{1}}\phi^{\dot{2}}|\, \mathcal{S}\, |\psi^{\dot{3}}\psi^{\dot{4}}\rangle = -a_8 \,. \end{aligned} \tag{3.5.40}$$

Given that a_7 and a_8 depend linearly on the phase, this sign change is just $\zeta \mapsto -\zeta$. Next, to exchange a left state with a right state in the tensor product one must use a graded permutation, which also produces certain minus signs.[11]

[11] This graded permutation was overlooked in the calculations of [143] thus obscuring the integrability of the $D5$-brane boundary conditions.

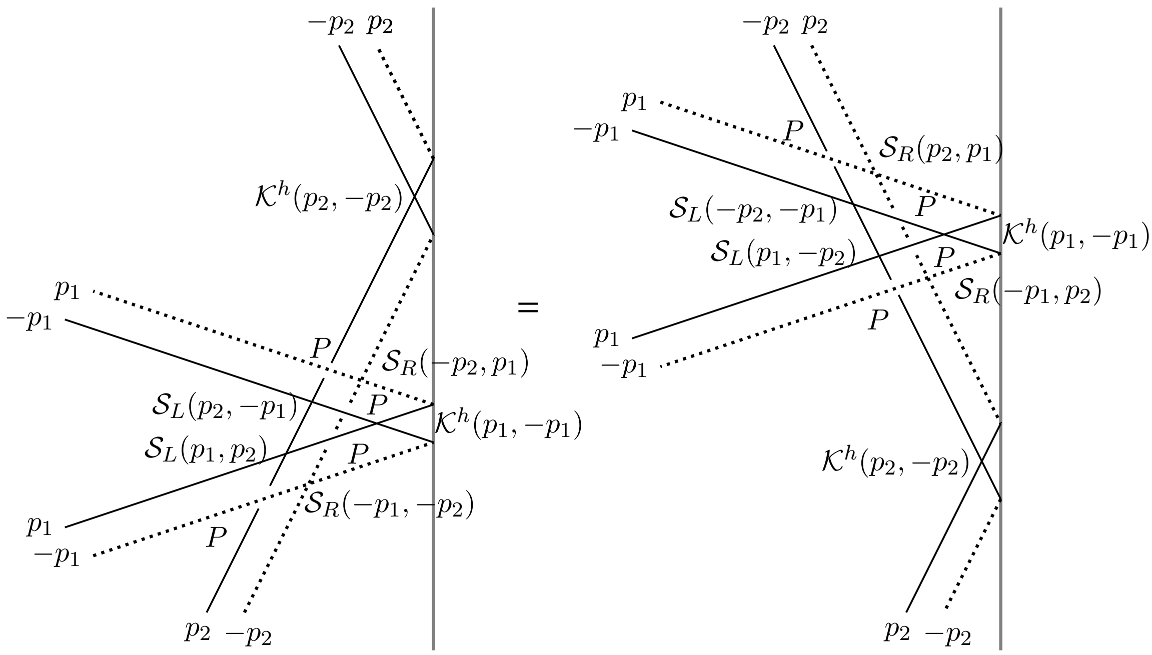

Figure 3.9: BYBE for the reflection in the horizontal case. Solid lines correspond to the left representations while the dotted lines correspond to right reps.

The pictorial version of the BYBE is presented in figure 3.9 and the equation itself is

$$\begin{aligned}
&\mathcal{K}_{34}(p_2,\zeta e^{-ip_1};-p_2,\zeta e^{-i(p_1-p_2)})\,P_{23}\,\mathcal{S}_{34}(-p_2,-\zeta e^{ip_2};p_1,-\zeta e^{i(p_2-p_1)})\\
&\quad\times\mathcal{S}_{12}(p_2,\zeta;-p_1,\zeta e^{ip_2})\,P_{23}\,\mathcal{K}_{34}(p_1,\zeta e^{ip_2};-p_1,\zeta e^{i(p_1+p_2)})\\
&\qquad\times P_{23}\,\mathcal{S}_{12}(p_1,\zeta;p_2,\zeta e^{ip_1})\,\mathcal{S}_{34}(-p_1,-\zeta e^{ip_1};-p_2,-\zeta e^{i(p_1+p_2)})\,P_{23}\\
&-P_{23}\,\mathcal{S}_{34}(p_2,-\zeta e^{-ip_2};p_1,-\zeta e^{-i(p_2+p_1)})\,\mathcal{S}_{12}(-p_2,\zeta;-p_1,\zeta e^{-ip_2})\,P_{23}\\
&\quad\times\mathcal{K}_{34}(p_1,\zeta e^{-p_2};-p_1,\zeta e^{-i(p_2-p_1)})\,P_{23}\,\mathcal{S}_{12}(p_1,\zeta;-p_2,\zeta e^{ip_1})\\
&\qquad\times\mathcal{S}_{34}(-p_1,-\zeta e^{ip_1};p_2,-\zeta e^{i(p_1-p_2)})\,P_{23}\,\mathcal{K}_{34}(p_2,\zeta e^{ip_1};-p_2,\zeta e^{i(p_1+p_2)})=0,
\end{aligned}\tag{3.5.41}$$

where the subscripts 12, 23, 34 indicate the tensor factors on which the operators act, P_{ij} is the graded permutation operator permuting left-right states, $\mathcal{S}^L_{ij}$ and $\mathcal{S}^R_{ij}$ are the left and right bulk S-matrices[12] and $\mathcal{K}_{34}$ is the reflection matrix. We have checked directly that this boundary YBE is satisfied.

Another way to verify that the boundary YBE is satisfied is to note that it may be mapped to a standard *bulk* YBE, as follows. One can verify that whenever a phase-dependent component appears, the extra sign in the right S-matrix is canceled with a

[12]Here $\mathcal{S}_{ij} = P_{ij}S_{ij}$, thus $\mathcal{S} : V(p_1)\otimes V(p_2) \to V(p_2)\otimes V(p_1)$.

minus sign from a graded permutation. Then, using also the relation between $\mathcal{K}$ and the bulk S-matrix, the above equation is equivalent to

$$\begin{aligned} &\mathcal{S}_{23}(p_2,\zeta e^{-ip_1};-p_2,\zeta e^{ip_2-ip_1})\,\mathcal{S}_{34}(p_1,\zeta e^{ip_2-ip_1};-p_2,\zeta e^{ip_2})\,\mathcal{S}_{12}(p_2,\zeta;-p_1,\zeta e^{ip_2}) \\ &\qquad\times\mathcal{S}_{23}(p_1,\zeta e^{ip_2};-p_1,\zeta e^{ip_1+ip_2})\,\mathcal{S}_{34}(-p_2,\zeta e^{ip_1+ip_2};-p_1,\zeta e^{ip_1})\,\mathcal{S}_{12}(p_1,\zeta;p_2,\zeta e^{ip_1}) \\ &-\mathcal{S}_{34}(p_1,\zeta^{-ip_1-ip_2};p_2,\zeta e^{-ip_2})\,\mathcal{S}_{12}(-p_2,\zeta;-p_1,\zeta e^{-ip_2})\,\mathcal{S}_{23}(p_1,\zeta e^{-ip_2};-p_1,\zeta e^{ip_1-ip_2}) \\ &\qquad\times\mathcal{S}_{34}(p_2,\zeta e^{ip_1-ip_2};-p_1,\zeta e^{ip_1})\,\mathcal{S}_{12}(p_1,\zeta;-p_2,\zeta e^{ip_1})\,\mathcal{S}_{23}(p_2,\zeta e^{ip_1};-p_2,\zeta e^{ip_1+ip_2})=0. \end{aligned} \tag{3.5.42}$$

In this way we have 'unfolded' the BYBE into a succession of bulk scattering processes. Consequently, the boundary YBE follows from a particular case of the bulk YBE. The meaning of (3.5.42) is represented in figure 3.10.

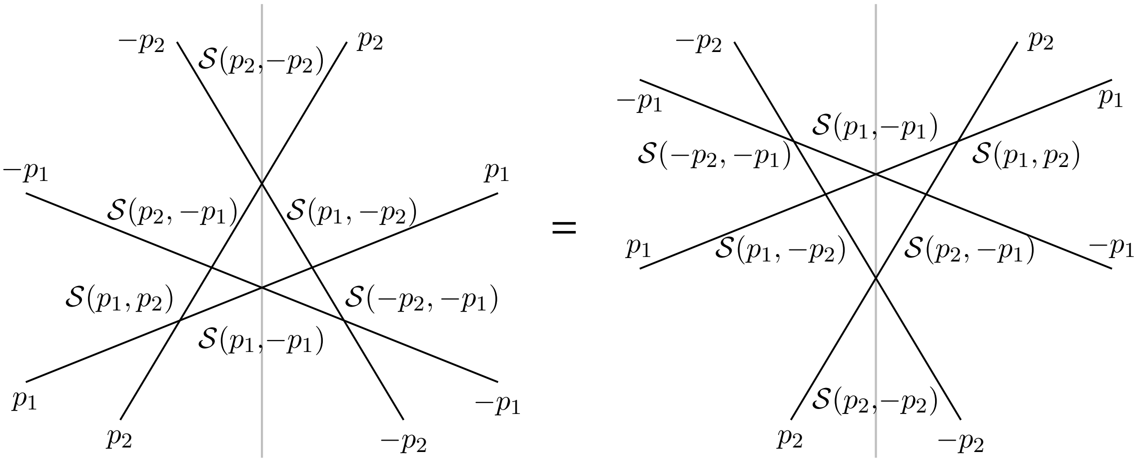

Figure 3.10: The unfolded BYBE as a 4-particle scattering in the bulk. The vertical line plays no role in the unfolded picture, but is drawn as a reminder that $S(p,-p)$ represents boundary reflections.

From this second, 'unfolded', point of view, the boundary is seen to be 'achiral', meaning that an incoming left state becomes a right one after the reflection and a right one becomes a left.

3.5.4 The vertical $D5$-brane

The boundary algebra of the vertical $D5$-brane is acquired from (3.5.17) by setting $\tau=-i$. We will consider reflection from the right boundary, which carries a $\check{\boxslash}$ spanned by the fields ϕ^a and $\psi^{\check{\alpha}\dot{1}}$ [143, 144]. The scattering problem under consideration is described by the triple tensor product

$$\boxslash_{(a,b,c,d)}\circ\widetilde{\boxslash}_{(a,b,c,d)}\otimes\check{\boxslash}_{(a_B,b_B,c_B,d_B)}, \tag{3.5.43}$$

where once again $\circ$ describes $L \circ R$ representations of the bulk magnon while $\otimes$ describes the usual tensor product of $bulk \otimes boundary$ reps.

The eigenvalues of the total central charges of the bulk magnon with the respect to the boundary algebra are

$$\mathbb{C} \equiv 2C, \quad \mathbb{C}^\dagger \equiv 2C^\dagger, \quad \mathbb{H} \equiv 2H, \tag{3.5.44}$$

and satisfy the multiplet shortening constraint according to which

$$\{0, 0; 2H, 2C, 2C^\dagger\} = \langle 1, 0; 2H, 2C, 2C^\dagger\rangle \oplus \langle 0, 1; 2H, 2C, 2C^\dagger\rangle = \square\!\square \oplus \begin{smallmatrix}\square\\ \square\end{smallmatrix}\,. \tag{3.5.45}$$

The representation labels specifying the boundary are[13]

$$a_B = \sqrt{g}\eta_B, \quad b_B = -\sqrt{g}\frac{i\zeta}{\eta_B}, \quad c_B = -\sqrt{g}\frac{\eta_B}{\zeta x_B}, \quad d_B = \sqrt{g}\frac{x_B}{i\eta_B}. \tag{3.5.46}$$

This representation is related to a radial line segment in the LLM-disc picture [64, 134, 135]. The unitarity and mass-shell conditions give

$$|\eta_B|^2 = -ix_B\,, \qquad x_B \equiv \frac{i(1+\sqrt{1+4g^2})}{2g}\,. \tag{3.5.47}$$

Thus, the exact energy of the boundary excitation is

$$\check{H} = D - J_{56} = \tfrac{1}{2}\sqrt{1+4g^2}\,. \tag{3.5.48}$$

For the boundary degree of freedom, the representation labels (3.5.46) and the mass-shell condition (3.5.47) are those of the boundary fundamental degree of freedom in the $Z = 0$ giant graviton case (3.3.72), but with a coupling constant g twice bigger. This doubling of the coupling constant is crucial for integrability to hold and for the exact boundary energy (3.5.48) to consistently reproduce 1-loop anomalous dimensions.

As we saw, the elementary bulk magnons transform, under the boundary symmetry algebra, in direct sum of two $M = 2$ bound state representations (symmetric and anti-symmetric). Therefore we have the following two scattering processes:

$$\mathcal{K}^{Ba} : \square\!\square \otimes \check{\square} \to \square\!\square \otimes \check{\square}, \tag{3.5.49}$$

$$\mathcal{K}^{\overline{Ba}} : \begin{smallmatrix}\square\\ \square\end{smallmatrix} \otimes \check{\square} \to \begin{smallmatrix}\square\\ \square\end{smallmatrix} \otimes \check{\square}, \tag{3.5.50}$$

As in [133, 143], following [108], the reflection matrices in the symmetric and antisymmetric channels (3.5.49) are

$$\mathcal{K}^{Ba} = \sum_{i=1}^{19} k_i^{(\mathrm{S})}\,\Lambda_i, \qquad \mathcal{K}^{\overline{Ba}} = \sum_{i=1}^{19} k_i^{(\mathrm{A})}\bar{\Lambda}_i, \tag{3.5.51}$$

[13] Note that here we use a different parametrization from the one used in [143].

where Λ_i are certain differential operators (see appendix C for details), $\bar{\Lambda}_i$ are obtained from Λ_i by exchanging indices $\check{1} \leftrightarrow \check{3}$ and $\check{2} \leftrightarrow \check{4}$, and where $k_i^{(\mathrm{S,A})}$ are the reflection coefficients. In both cases, the symmetry algebra alone fixes all reflection coefficients up to an overall phase. Interestingly, the two channels are related by $k_i^{(\mathrm{A})}(p, x_B) = k_i^{(\mathrm{S})}(-p, x_B)$ [143]. Note that the reflection coefficients do not explicitly depend on g, thus they coincide with the ones for $Z = 0$ giant graviton found in [133].

It is easy to check that symmetric and antisymmetric reflection matrices $\mathcal{K}^{Ba}$ and $\mathcal{K}^{\overline{Ba}}$ do satisfy BYBE on their own. The BYBE invariance of $\mathcal{K}^{Ba}$ was checked in [133], while for checking the BYBE invariance of $\mathcal{K}^{\overline{Ba}}$ we had to construct an antisymmetric bound state S-matrix $\mathcal{S}^{\overline{BB}}$ which is the mirror-model partner of the ordinary bound state S-matrix $\mathcal{S}^{BB}$.

For the vertical vacuum case the complete reflection matrix must be some linear combination:

$$\mathcal{K}^v = k_0\, \mathcal{K}^{Ba} + \mathcal{K}^{\overline{Ba}}, \tag{3.5.52}$$

with k_0 being a function of bulk and boundary representation parameters. The important question is whether there exists any choice of this function, such that the system is integrable, i.e. such that the complete reflection matrix obeys the boundary Yang-Baxter equation. For this purpose one needs to consider the complete bulk 16×16-dim. S-matrix $\mathcal{S}^{A\dot{A}A\dot{A}}$ which may be constructed as a tensor product of two fundamental S-matrices $\mathcal{S}^{AA}$ and $\mathcal{S}^{\dot{A}\dot{A}}$. It is convenient to compute $\mathcal{S}^{A\dot{A}A\dot{A}}$ in the basis of (graded) symmetric and antisymmetric states i.e. on the superspace and the mirror-superspace. The complete S-matrix is not block-diagonal in this basis; rather it mixes symmetric and antisymmetric states during the scattering. But it is important to note that it is invariant under the symmetries preserved by the boundary (this is natural as the boundary algebra is a subalgebra of the bulk algebra).

The BYBE for the reflection in this vertical case reads as

$$\begin{aligned}\mathrm{BYBE}: &\mathcal{V}_L(p_1,\zeta)\otimes\mathcal{V}_R(p_1,\zeta)\otimes\mathcal{V}_L(p_2,\zeta e^{ip_1})\otimes\mathcal{V}_R(p_2,\zeta e^{ip_1})\otimes\mathcal{V}_B(x_B,\zeta e^{i(p_1+p_2)})\to\\ &\mathcal{V}_L(-p_1,\zeta)\otimes\mathcal{V}_R(-p_1,\zeta)\otimes\mathcal{V}_L(-p_2,\zeta e^{-ip_1})\otimes\mathcal{V}_R(-p_2,\zeta e^{-ip_1})\otimes\mathcal{V}_B(x_B,\zeta e^{-i(p_1+p_2)}),\end{aligned} \tag{3.5.53}$$

where once again the scattering in the bulk is between left-left and right-right states only, while the permutation of left-right and right-left states produces a graded minus sign. In the contrast to the horizontal case, the right S-matrix is equivalent to the left S-matrix, i.e. it does not acquire an extra minus sign in the ζ-dependent components, since now $-\tau^2 = +1$. Also, all phases in (3.5.53) are increasing from left to right. The graphical interpretation of BYBE is almost the same as for the horizontal case. The difference is that the boundary in this case does not act diagonally but mixes bulk and boundary flavours.

A general matrix element of the BYBE (3.5.53) has a complicated structure. We found the particular matrix element

$$\big\langle \phi_1^{\{\check{3}\check{4}\}} \otimes \phi_2^{\{\check{1}\check{1}\}} \otimes \phi_B^{\check{1}} \big|\, \mathrm{BYBE}\, \big| \psi_1^{\{\check{1}\check{3}\}} \otimes \phi_2^{\{\check{1}\check{1}\}} \otimes \psi_B^{\check{4}} \big\rangle \tag{3.5.54}$$

to be quite tractable and by treating minus signs coming from permuting left and right representations carefully (i.e. $\mathcal{S}^{A_1\dot{A}_2A_3\dot{A}_4} = (-1)^{[\dot{A}_2][A_3]}\mathcal{S}^{A_1A_3}\otimes\mathcal{S}^{\dot{A}_2\dot{A}_4}$) we find the required ratio has to be

$$k_0 = -\frac{x^-(x_B - x^-)^2}{x^+(x_B + x^+)^2}\frac{\eta^2\eta_B}{\tilde{\eta}^2\tilde{\eta}_B}. \tag{3.5.55}$$

for (3.5.54) to vanish. We have then checked that, using this ratio, the reflection matrix K^v (3.5.52) satisfies all matrix elements of BYBE (3.5.53). Thus we conclude that the reflection in the vertical case is indeed integrable. We also claim that it is an achiral boundary in the same sense as in the horizontal case: at this stage the 'unfolded" picture of the reflection is not obvious, but becomes very transparent when we considering the nested Bethe ansatz [144].

Factorized approach. As was shown using the Bethe ansatz technique in [144], reflection from the vertical $D5$-brane is achiral. Hence it may be represented by a diagram (figure 3.11) very similar to the one describing the reflection from the horizontal $D5$-brane (figure 3.8).

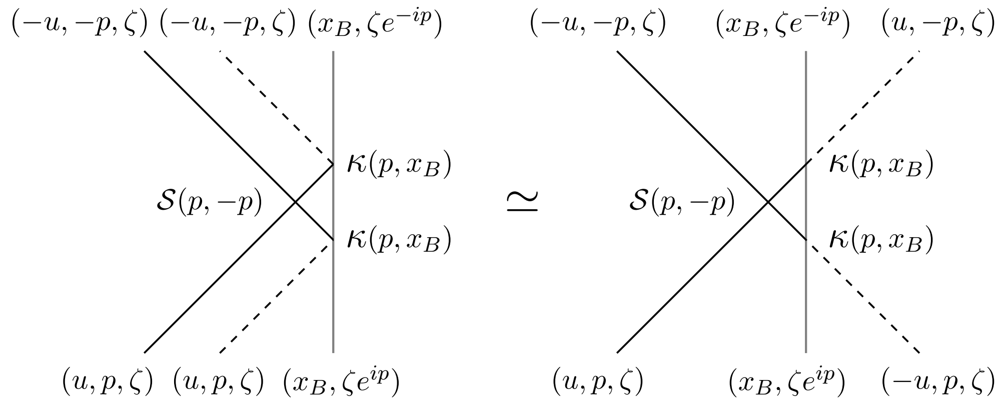

Figure 3.11: Unfolding of the reflection from the vertical $D5$-brane. Solid lines correspond to the left representations while the dotted lines correspond to right reps. The vertical gray lines correspond to the boundary rep. In the contrast to the horizontal case, the achiral reflection not only maps left (right) representations to right (left) representations but is also an intertwining matrix mapping momentum $p \mapsto -p$.

Thus, as one can see from figure 3.11, the reflection factorizes as a composition of a bulk S-matrix and two achiral reflection matrices $\mathscr{K}$, with

$$\mathcal{K}^v(p,x_B) = \mathscr{K}(p,x_B)\,\mathcal{S}(p,-p)\,\mathscr{K}(p,x_B). \tag{3.5.56}$$

The achiral reflection matrix $\mathscr{K}$ in the folded picture maps incoming right states into outgoing left states as

$$\mathscr{K} : 1 \circ \widetilde{\boxslash} \otimes \check{\boxslash} \mapsto \boxslash \circ 1 \otimes \check{\boxslash}, \tag{3.5.57}$$

and incoming left states into outgoing right ones as

$$\mathcal{K} : \boxslash \circ 1 \otimes \check{\boxslash} \mapsto 1 \circ \widetilde{\boxslash} \otimes \check{\boxslash}. \tag{3.5.58}$$

These two expressions may be combined into one using vector space notation

$$\mathcal{K} : \mathcal{V}_{L(R)}(p,\zeta) \otimes \mathcal{V}_B(x_B, \zeta e^{ip}) \mapsto \mathcal{V}_{R(L)}(-p, \zeta e^{ip}) \otimes \mathcal{V}_B(x_B,\zeta), \tag{3.5.59}$$

and thereby may be defined on superspace in the usual way

$$\mathcal{K}(p, x_B) = \sum_{i=1}^{10} k_i(p, x_B)\, \Lambda_i, \tag{3.5.60}$$

where Λ_i are the $\mathfrak{su}(2) \times \mathfrak{su}(2)$ invariant differential operators (see [108,144]). Invariance under the boundary algebra (3.5.17) fixes k_i up to an overall phase to be [24]

$$\begin{aligned}
k_1 &= -\frac{x_B - x^-}{x_B + x^+}\frac{\eta\eta_B}{\tilde{\eta}\tilde{\eta}_B}, & k_2 &= \frac{5(x^+ - x_B)x^- - 3((x^-)^2 - x_B x^+)}{(x_B + x^+)(x^- + x^+)}\frac{\eta\eta_B}{\tilde{\eta}\tilde{\eta}_B}, \\
k_3 &= 1, & k_4 &= \frac{5(x^- + x_B)x^+ - 3((x^+)^2 + x_B x^-)}{(x_B + x^+)(x^- + x^+)}, \\
k_5 &= -\frac{x_B - x^+}{x_B + x^+}\frac{\eta}{\tilde{\eta}}, & k_6 &= \frac{x_B + x^-}{x_B + x^+}\frac{\eta_B}{\tilde{\eta}_B}, \\
k_7 &= \frac{i\sqrt{2}\,\zeta\, x_B(x_B - x^+)(x^- - x^+)}{(x_B + x^+)(1 + x_B x^-)\tilde{\eta}\tilde{\eta}_B}, & k_8 &= \frac{i\sqrt{2}(x_B + x^-)\eta\eta_B}{\zeta(x_B + x^+)(1 - x_B x^-)}, \\
k_9 &= \sqrt{2}\,\frac{x^- - x^+}{x_B + x^+}\frac{\eta_B}{\tilde{\eta}}, & k_{10} &= -\sqrt{2}\,\frac{x_B}{x_B + x^+}\frac{\eta}{\tilde{\eta}_B}.
\end{aligned} \tag{3.5.61}$$

We have checked explicitly that the factorization (3.5.56) is correct. It obeys the Yang-Baxter relation and the reflection coefficients coincide with the ones found in [144]. For example, the reflection of the bulk state $\phi_1 \circ \tilde{\phi}_1$ from the boundary state $\check{\phi}_1$ gives a relation

$$k_0\, k_1^{(\mathrm{S})}(p_1, \zeta; x_B) = k_1(p, \zeta e^{-ip}; x_B)\, a_1(p, -p, \zeta)\, k_1(p, \zeta; x_B), \tag{3.5.62}$$

where a_1 and $k_1^{(\mathrm{S})}$ are the coefficients of the fundamental S-matrix and reflection matrix $\mathcal{K}^{Ba}$ respectively; all of them are spelled out in the appendices of [144].[14]

The achiral reflection in the unfolded picture can be interpreted as a scattering through the achiral boundary and the choice of phases in (3.5.59) can then be easily read from the LLM-type diagram (figure 3.12). The maps (3.5.57) and (3.5.58) in the unfolded picture become

$$\mathcal{K}^{\mathrm{unf}} : 1 \otimes \check{\boxslash} \otimes \widetilde{\boxslash} \mapsto \boxslash \otimes \check{\boxslash} \otimes 1, \tag{3.5.63}$$

and

$$\mathcal{K}^{\mathrm{unf}} : \boxslash \otimes \check{\boxslash} \otimes 1 \mapsto 1 \otimes \check{\boxslash} \otimes \widetilde{\boxslash}, \tag{3.5.64}$$

[14] The achiral reflection matrix is equivalent to the S-matrix by identifying $x^{\pm} = \pm x_B$ up to a graded permutation and an extra factor of $-i$ in $k_{3,4,6,8,10}$ due to the change of the basis (3.5.20) for the right rep. This map identifies the boundary magnon with a bulk magnon of momentum $p = \pi$.

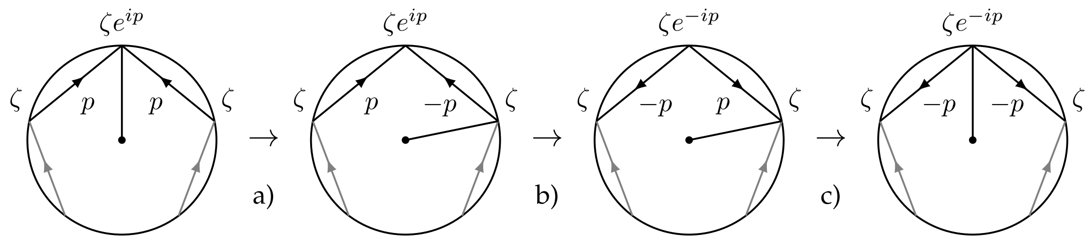

Figure 3.12: LLM-type diagram for the scattering through the right boundary in the unfolded picture. The vertical $D5$-brane corresponds to the dot in the center of the circle. The line adjoining the center and the circle corresponds to the boundary rep. The line segments to the left from the boundary line correspond to the left reps, while the line segments to the right from the boundary correspond to the right reps. The phase is increasing towards the boundary for left and right reps. Here a) is the scattering of the right representation through the boundary, b) is the scattering of two left states in the bulk and c) is is the scattering of the left representation through the boundary. The gray line segments do not participate in the scattering.

respectively; thus folded and unfolded achiral reflection matrices are related to each other as $\mathcal{K} = \mathrm{T} \cdot \mathcal{K}^{\mathrm{unf}}$, where T is the specialization of the folding map (3.5.16) in which the boundary carries an irreducible $\mathfrak{g}_+$ representation (as opposed, more generally, to a $\mathfrak{g}_+$-reducible $\mathfrak{g}_L \times \mathfrak{g}_R$ rep). Note that the reflection coefficients k_3, k_4 and k_8 acquire an extra minus sign in the unfolded picture (because of graded permutation of two fermionic states).

Yangian approach. Now we are ready to consider the explicit realization of the Yangian (3.5.30) for the vertical $D5$-brane. The boundary $\check{\boxslash}$ in this case is an evaluation irrep of the achiral twisted Yangian with rapidity zero. We consider the folded picture first. All complementary central charges (3.5.28) have zero eigenvalues on all (bulk and boundary) reps, hence are trivial in this case and do not need to be considered. Thus only the coproducts of non-central charges in (3.5.30) contribute. Writing their action on the tensor product $\boxslash \circ \widetilde{\boxslash} \otimes \check{\boxslash}$, we have, for example,

$$
\begin{aligned}
\Delta\big(\widetilde{\mathbb{R}}_{\check{a}}^{\,\check{b}}\big) = {} & \check{\mathbb{R}}_{\check{a}}^{\,\check{c}} \circ 1 \otimes 1 - 1 \circ \check{\mathbb{R}}_{\check{a}}^{\,\check{c}} \otimes 1 \\
& + \tfrac{1}{2}\, \mathbb{R}_{\check{a}}^{\,\check{c}} \circ \mathbb{R}_{\check{c}}^{\,\check{b}} \otimes 1 - \tfrac{1}{2} \mathbb{R}_{\check{c}}^{\,\check{b}} \circ \mathbb{R}_{\check{a}}^{\,\check{c}} \otimes 1 - \tfrac{1}{2} \mathbb{G}_{\check{a}}^{\,\check{\gamma}} \circ \mathbb{Q}_{\check{\gamma}}^{\,\check{b}} \otimes 1 - \tfrac{1}{2} \mathbb{Q}_{\check{\gamma}}^{\,\check{b}} \circ \mathbb{G}_{\check{a}}^{\,\check{\gamma}} \otimes 1 \\
& + \tfrac{1}{4} \delta_{\check{a}}^{\check{b}}\, \mathbb{G}_{\check{c}}^{\,\check{\gamma}} \circ \mathbb{Q}_{\check{\gamma}}^{\,\check{c}} \otimes 1 + \tfrac{1}{4} \delta_{\check{a}}^{\check{b}}\, \mathbb{Q}_{\check{\gamma}}^{\,\check{c}} \circ \mathbb{G}_{\check{c}}^{\,\check{\gamma}} \otimes 1 \\
& + \tfrac{1}{2}\, \mathbb{R}_{\check{a}}^{\,\check{c}} \circ 1 \otimes \mathbb{R}_{\check{c}}^{\,\check{b}} - \tfrac{1}{2} \mathbb{R}_{\check{c}}^{\,\check{b}} \circ 1 \otimes \mathbb{R}_{\check{a}}^{\,\check{c}} + \tfrac{1}{2} \mathbb{G}_{\check{a}}^{\,\check{\gamma}} \circ 1 \otimes \mathbb{Q}_{\check{\gamma}}^{\,\check{b}} + \tfrac{1}{4} \mathbb{Q}_{\check{\gamma}}^{\,\check{b}} \circ 1 \otimes \mathbb{G}_{\check{a}}^{\,\check{\gamma}} \\
& - \tfrac{1}{4} \delta_{\check{a}}^{\check{b}}\, \mathbb{G}_{\check{c}}^{\,\check{\gamma}} \circ 1 \otimes \mathbb{Q}_{\check{\gamma}}^{\,\check{c}} - \tfrac{1}{4} \delta_{\check{a}}^{\check{b}}\, \mathbb{Q}_{\check{\gamma}}^{\,\check{c}} \circ 1 \otimes \mathbb{G}_{\check{c}}^{\,\check{\gamma}} \\
& - \tfrac{1}{2}\, 1 \circ \mathbb{R}_{\check{a}}^{\,\check{c}} \otimes \mathbb{R}_{\check{c}}^{\,\check{b}} + \tfrac{1}{2}\, 1 \circ \mathbb{R}_{\check{c}}^{\,\check{b}} \otimes \mathbb{R}_{\check{a}}^{\,\check{c}} - \tfrac{1}{2}\, 1 \circ \mathbb{G}_{\check{a}}^{\,\check{\gamma}} \otimes \mathbb{Q}_{\check{\gamma}}^{\,\check{b}} - \tfrac{1}{4}\, 1 \circ \mathbb{Q}_{\check{\gamma}}^{\,\check{b}} \otimes \mathbb{G}_{\check{a}}^{\,\check{\gamma}} \\
& + \tfrac{1}{4} \delta_{\check{a}}^{\check{b}}\, 1 \circ \mathbb{G}_{\check{c}}^{\,\check{\gamma}} \otimes \mathbb{Q}_{\check{\gamma}}^{\,\check{c}} + \tfrac{1}{4} \delta_{\check{a}}^{\check{b}}\, 1 \circ \mathbb{Q}_{\check{\gamma}}^{\,\check{c}} \otimes \mathbb{G}_{\check{c}}^{\,\check{\gamma}}\,,
\end{aligned}
\tag{3.5.65}
$$

and similarly for other charges. Thus the co-products in (3.5.30) may be cast in the form

$$\Delta\big(\widetilde{\mathbb{J}}^{\check{A}}\big) = \big(\widehat{\mathbb{J}}^{\check{A}} \circ 1 - 1 \circ \widehat{\mathbb{J}}^{\check{A}} + \tfrac{1}{2} f^{\check{A}}_{\check{B}\check{C}} (\mathbb{J}^{\check{B}} \circ \mathbb{J}^{\check{C}})\big) \otimes 1 + \tfrac{1}{2} f^{\check{A}}_{\check{B}\check{C}} \big(\mathbb{J}^{\check{B}} \circ 1 - 1 \circ \mathbb{J}^{\check{B}}\big) \otimes \mathbb{J}^{\check{C}}\,, \tag{3.5.66}$$

revealing the factorization (3.5.56) explicitly. Here the first line corresponds to the Yangian of the S-matrix in (3.5.56)

$$\Delta(\widetilde{\mathbb{J}}^{\check{A}})\Big|_{\mathcal{S}} = \big(\widehat{\mathbb{J}}^{\check{A}} \circ 1 - 1 \circ \widehat{\mathbb{J}}^{\check{A}} + \tfrac{1}{2} f^{\check{A}}_{\check{B}\check{C}} (\mathbb{J}^{\check{B}} \circ \mathbb{J}^{\check{C}})\big) \otimes 1\,, \tag{3.5.67}$$

and was explicitly spelled out in (3.5.36), while the terms in the second line originate from the achiral reflection matrices. Therefore the part of the Yangian charges governing the achiral $\mathcal{K}$-matrices is

$$\Delta(\widetilde{\mathbb{J}}^{\check{A}})\Big|_{\mathcal{K}} = \big(\widehat{\mathbb{J}}^{\check{A}} \circ 1 - 1 \circ \widehat{\mathbb{J}}^{\check{A}}\big) \otimes 1 + \tfrac{1}{2} f^{\check{A}}_{\check{B}\check{C}} \big(\mathbb{J}^{\check{B}} \circ 1 - 1 \circ \mathbb{J}^{\check{B}}\big) \otimes \mathbb{J}^{\check{C}}. \tag{3.5.68}$$

Once again we meet minus signs that need to be understood. The origin of the minus sign in front of the level-1 charge in (3.5.67) was discussed in section 4.1, and its physical interpretation is almost the same as for the horizontal reflection. The difference is that now both momenta and rapidities of left and right representations are facing in the *same* direction (towards the boundary) in the initial configuration i.e. for the incoming state; see left side of figure 3.11) but the scattering always follows after the reflection as seen from (3.5.56). Thus the S-matrix in (3.5.56) acts on the state with momentum and rapidity *reversed* with respect to the initial configuration.

In order to understand the origin of the minus sign in front of the level-1 charge in (3.5.68) it is better to consider the unfolded picture of the achiral reflection (the right side of figure 3.11) first. As for the horizontal case, the reflection in the unfolded picture may be thought of as achiral scattering through the boundary state; this was nicely shown in figure 3.12. The Yangian charges (3.5.68) in the unfolded picture become

$$\Delta(\widetilde{\mathbb{J}}^{\check{A}})\Big|^{\text{unf}}_{\mathcal{K}} = \widehat{\mathbb{J}}^{\check{A}} \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \widehat{\mathbb{J}}^{\check{A}} + \tfrac{1}{2} f^{\check{A}}_{\check{B}\check{C}} \big(\mathbb{J}^{\check{B}} \otimes \mathbb{J}^{\check{C}} \otimes 1 + 1 \otimes \mathbb{J}^{\check{B}} \otimes \mathbb{J}^{\check{C}}\big)\,. \tag{3.5.69}$$

The minus sign in front of level-1 charge was absorbed by the unfolding map T^{-1}, while the minus sign in front of the two-site term was absorbed into $f^{\check{A}}_{\check{B}\check{C}}$ using the antisymmetry under exchange of $\check{B}$ and $\check{C}$. Thus the achiral scattering through the boundary is governed by a Yangian symmetry equivalent to the bulk Yangian (3.1.28) up to the different underlying tensor space structures. But here lies the most important feature of the achiral scattering. In contrast to the horizontal case, the right representation in the unfolded picture has momentum and rapidity pointing in *opposite* directions, as may be seen in the right side of figure 3.11. Thus the minus sign in front of the level-1 charge in (3.5.68) effectively reverses the rapidity of the right representation in the folded picture, where it is pointing the same direction as the momentum.

We have checked that the Yangian charges (3.5.66) commute with the reflection matrix $\mathcal{K}^v$ found in [144]. Hence reflection from the vertical $D5$-brane may be viewed in two ways: as a single reflection matrix $\mathcal{K}^v$ which is governed by the Yangian charges (3.5.66), or as a factorized reflection (3.5.56) which is governed by the Yangian charges (3.5.67) and (3.5.68). Either way the result is the same, as required.

B Reflection matrices

This appendix contains selected bound state reflection matrices for vector boundary and are relevant to $Z = 0$ giant graviton and $D7$-branes and to 'vertical' $D5$-brane. The reflection matrices are given in the non-local representation. The local one can be obtained by the following substitution

$$\eta \to \gamma\,, \qquad \widetilde{\eta} \to \gamma\,, \qquad \eta_B \to U\gamma_B\,, \qquad \widetilde{\eta}_B \to U^{-1}\gamma_B\,, \qquad \zeta \to i\alpha\,. \tag{B.1}$$

Reflection matrices $\mathcal{K}^{Ba}$ and $\mathcal{K}^{\overline{Ba}}$. The supersymmetric reflection K-matrix $\mathcal{K}^{Ba}$ describing the reflection of the two-magnon bound states in the bulk from the fundamental states on the boundary may be defined as a differential operator

$$\mathcal{K}^{Ba}(p, x_B) = \sum_i k_i^{(S)}(p, x_B)\,\Lambda_i\;, \tag{B.2}$$

acting on the superspace, where Λ_i are

$$\begin{aligned}
\Lambda_1 &= \frac{1}{6}\left(\omega_a^1\omega_b^1\omega_c^2 + \omega_c^1\omega_b^1\omega_a^2 + \omega_a^1\omega_c^1\omega_b^2\right)\frac{\partial^3}{\partial\omega_c^2\partial\omega_b^1\partial\omega_a^1}, &
\Lambda_{10} &= \frac{1}{2}\epsilon^{kl}\omega_c^1\omega_k^1\omega_l^2\epsilon_{\alpha\beta}\frac{\partial^3}{\partial\omega_c^2\partial\theta_\beta^1\partial\theta_\alpha^1},\\
\Lambda_2 &= \frac{1}{6}\left(\epsilon_{bc}\omega_a^1 + \epsilon_{ac}\omega_b^1\right)\epsilon^{kl}\omega_k^1\omega_l^2\frac{\partial^3}{\partial\omega_c^2\partial\omega_b^1\partial\omega_a^1}, &
\Lambda_{11} &= \frac{1}{2}\epsilon^{\alpha\beta}\omega_b^2\theta_\alpha^1\theta_\beta^1\epsilon_{ac}\frac{\partial^3}{\partial\omega_c^2\partial\omega_b^1\partial\omega_a^1},\\
\Lambda_3 &= \frac{1}{2}\theta_\beta^1\left(\omega_a^1\omega_c^2 + \omega_c^1\omega_a^2\right)\frac{\partial^3}{\partial\omega_c^2\partial\omega_a^1\partial\theta_\beta^1}, &
\Lambda_{12} &= \epsilon^{\alpha\beta}\omega_c^2\theta_\alpha^1\theta_\beta^1\epsilon_{\gamma\delta}\frac{\partial^3}{\partial\omega_a^1\partial\theta_\delta^2\partial\theta_\gamma^1},\\
\Lambda_4 &= \frac{1}{2}\theta_\beta^1\left(\omega_a^1\omega_c^2 - \omega_c^1\omega_a^2\right)\frac{\partial^3}{\partial\omega_c^2\partial\omega_a^1\partial\theta_\beta^1}, &
\Lambda_{13} &= \epsilon^{\alpha\beta}\omega_b^1\theta_\alpha^1\theta_\beta^2\epsilon_{ac}\frac{\partial^3}{\partial\omega_c^2\partial\omega_b^1\partial\omega_a^1},\\
\Lambda_5 &= \frac{1}{2}\omega_a^1\omega_b^1\theta_\gamma^2\frac{\partial^3}{\partial\omega_b^1\partial\omega_a^1\partial\theta_\gamma^2}, &
\Lambda_{14} &= \frac{1}{2}\epsilon^{\alpha\beta}\omega_a^2\theta_\alpha^1\theta_\beta^1\epsilon_{\gamma\delta}\frac{\partial^3}{\partial\omega_a^1\partial\theta_\beta^2\partial\theta_\gamma^1},\\
\Lambda_6 &= \frac{1}{2}\omega_c^2\theta_\alpha^1\theta_\beta^1\frac{\partial^3}{\partial\omega_c^2\partial\theta_\beta^1\partial\theta_\alpha^1}, &
\Lambda_{15} &= \frac{1}{2}\epsilon^{\alpha\beta}\omega_c^1\theta_\alpha^1\theta_\beta^2\epsilon_{\gamma\delta}\frac{\partial^3}{\partial\omega_c^2\partial\theta_\delta^1\partial\theta_\gamma^1},\\
\Lambda_7 &= \frac{1}{2}\omega_a^1\left(\theta_\beta^1\theta_\gamma^2 + \theta_\gamma^1\theta_\beta^2\right)\frac{\partial^3}{\partial\omega_a^1\partial\theta_\gamma^2\partial\theta_\beta^1}, &
\Lambda_{16} &= \epsilon^{\alpha\gamma}\theta_\alpha^1\theta_\beta^1\theta_\gamma^2\epsilon_{ac}\frac{\partial^3}{\partial\omega_c^2\partial\omega_a^1\partial\theta_\beta^1},\\
\Lambda_8 &= \frac{1}{2}\omega_a^1\left(\theta_\beta^1\theta_\gamma^2 - \theta_\gamma^1\theta_\beta^2\right)\frac{\partial^3}{\partial\omega_a^1\partial\theta_\gamma^2\partial\theta_\beta^1}, &
\Lambda_{17} &= \epsilon^{kl}\omega_k^1\omega_l^2\theta_\beta^1\epsilon_{\alpha\gamma}\frac{\partial^3}{\partial\theta_\gamma^2\partial\theta_\beta^1\partial\theta_\alpha^1},\\
\Lambda_9 &= \theta_\alpha^1\theta_\beta^1\theta_\gamma^2\frac{\partial^3}{\partial\theta_\gamma^2\partial\theta_\beta^1\partial\theta_\alpha^1}, &
\Lambda_{18} &= \omega_b^1\omega_a^2\theta_\gamma^1\frac{\partial^3}{\partial\omega_b^1\partial\omega_a^1\partial\theta_\gamma^2},\\
& & \Lambda_{19} &= \omega_a^1\omega_c^1\theta_\beta^2\frac{\partial^3}{\partial\omega_c^2\partial\omega_a^1\partial\theta_\beta^1}.
\end{aligned} \tag{B.3}$$

The coefficients of the symmetric reflection matrix $\mathcal{K}^{Ba}$ are:

$$\begin{aligned}
k_1^{(S)} &= 1\\
k_2^{(S)} &= \frac{3x_B(x^-)^2 - x_B(x^+)^2(2+3(x^+)^2) + x^-x^+(x_B - 4x^+ + x_B(x^+)^2)}{2(x_B + (-1+x_B^2)x^- - x_B(x^-)^2)(x^+)^2},\\
k_3^{(S)} &= -\frac{((x^-)^2 + x_Bx^+)}{(x_B - x^-)x^-}\frac{\tilde{\eta}}{\eta},\\
k_4^{(S)} &= -\frac{(x_B + x^+)(x^- + x_B(x^+)^2)}{(x_B + (-1+x_B^2)x^- - x_B(x^-)^2)x^+}\frac{\tilde{\eta}}{\eta},\\
k_5^{(S)} &= \frac{(x_Bx^- - (x^+)^2)}{(x_B - x^-)x^-}\frac{\tilde{\eta}_B}{\eta_B},\\
k_6^{(S)} &= \frac{(-x_B(x^-)^4 + x_B(x^+)^2 + x^-x^+(x_B + x_B(x^-)^2 + x^-(4 - 2x_Bx^+)))}{2(x^-)^2(x_B + (-1+x_B^2)x^- - x_B(x^-)^2)}\frac{\tilde{\eta}^2}{\eta^2},\\
k_7^{(S)} &= -\frac{x^+(x_B + x^+)}{(x_B - x^-)x^-}\frac{\tilde{\eta}\tilde{\eta}_B}{\eta\eta_B},\\
k_8^{(S)} &= \frac{(2x_B^2(x^-)^3 + x_B(x^-)^2(x_B - x^+)x^+ + 2(x^+)^3 + x^-x^+(-x_B + x^+))}{(x^-)^2(x_B + (-1+x_B^2)x^- - x_B(x^-)^2)}\frac{\tilde{\eta}\tilde{\eta}_B}{\eta\eta_B},\\
k_9^{(S)} &= \frac{x^+(x_B + x^+)(-x_B(x^-)^2 + x^+)}{(x^-)^2(x_B + (-1+x_B^2)x^- - x_B(x^-)^2)}\frac{\tilde{\eta}^2\tilde{\eta}_B}{\eta^2\eta_B},\\
k_{10}^{(S)} &= \frac{i\zeta((x^-)^2 - (x^+)^2)^2(x_Bx^+ + x_B(x^-)^2x^+ + x^-(x_B + 2x^+ - x_B(x^+)^2))}{4(x^-)^2(x_B + (-1+x_B^2)x^- - x_B(x^-)^2)x^+(-1 + x_Bx^+)}\frac{1}{\eta^2},\\
k_{11}^{(S)} &= -\frac{ix_B(x^- + x^+)^2}{2\zeta(x_B - x^-)x^-(1 + x_Bx^-)x^+}\tilde{\eta}^2,\\
k_{12}^{(S)} &= -\frac{i\zeta x_B(x_Bx^- - (x^+)^2)((x^-)^2 - (x^+)^2)}{\sqrt{2}x^-(x_B + (-1+x_B^2)x^- - x_B(x^-)^2)x^+}\frac{1}{\eta\eta_B},\\
k_{13}^{(S)} &= \frac{i(x^- + x^+)(x_Bx^- - (x^+)^2)}{\sqrt{2}\zeta(x_B - x^-)x^-(1 + x_Bx^-)x^+}\tilde{\eta}\tilde{\eta}_B,\\
k_{14}^{(S)} &= \frac{x_B(x_B(x^-)^2 - x^+)(x^- + x^+)}{\sqrt{2}(x^-)^2(-x_B + x^-)(1 + x_Bx^-)}\frac{\tilde{\eta}^2}{\eta\eta_B},\\
k_{15}^{(S)} &= \frac{(x_B(x^-)^2 - x^+)((x^-)^2 - (x^+)^2)}{\sqrt{2}(x^-)^2(x_B + (-1+x_B^2)x^- - x_B(x^-)^2)}\frac{\tilde{\eta}\tilde{\eta}_B}{\eta^2},\\
k_{16}^{(S)} &= -\frac{i(x_B + x^+)(x^- + x^+)}{\sqrt{2}\zeta x^-(-x_B + x^-)(1 + x_Bx^-)}\frac{\tilde{\eta}^2\tilde{\eta}_B}{\eta},\\
k_{17}^{(S)} &= \frac{i\zeta x_B(x_B + x^+)(-(x^-)^2 + (x^+)^2)}{\sqrt{2}x^-(x_B + (-1+x_B^2)x^- - x_B(x^-)^2)}\frac{\tilde{\eta}}{\eta^2\eta_B},\\
k_{18}^{(S)} &= \frac{x_B(x^- + x^+)}{\sqrt{2}(x_B - x^-)x^-}\frac{\tilde{\eta}}{\eta_B},\\
k_{19}^{(S)} &= \frac{((x^-)^2 - (x^+)^2)}{\sqrt{2}x^-(-x_B + x^-)}\frac{\tilde{\eta}_B}{\eta}.
\end{aligned} \tag{B.4}$$

The anti-supersymmetric reflection K-matrix $\mathcal{K}^{\overline{Ba}}$ describing the reflection of the two-magnon bound states in the mirror bulk theory from the fundamental states on the boundary may be defined as a differential operator

$$\mathcal{K}^{\overline{Ba}}(p, x_B) = \sum_i k_i^{(A)}(p, x_B)\, \overline{\Lambda}_i \,, \tag{B.5}$$

acting on the mirror superspace, where $\overline{\Lambda}_i$ are the differential operators acting on the mirror superspace. They may be acquired from (B.3) by interchange of bosonic and fermionic indices, $(a,\ b) \leftrightarrow (\alpha,\ \beta)$. The reflection coefficients $k_i^{(A)}$ may be obtained from B.4 using the relation $k_i^{(A)}(p, x_B) = k_i^{(S)}(-p, x_B)$.

Reflection matrix $\mathcal{K}^{Ab}$. The supersymmetric reflection K-matrix $\mathcal{K}^{Ab}$ describing the reflection of the fundamental states in the bulk from the two-magnon bound states on the boundary may be defined as a differential operator

$$\mathcal{K}^{Ab}(p, x_B) = \sum_i k_i(p, x_B)\, \Lambda_i \,, \tag{B.6}$$

acting on the superspace, where Λ_i are

$$\Lambda_1 = \frac{1}{6}(w_a^1 w_b^2 w_c^2 + w_a^2 w_b^1 w_c^2 + w_a^2 w_b^2 w_c^1)\frac{\partial^3}{\partial w_c^2 \partial w_b^2 \partial w_a^1}, \qquad \Lambda_{10} = \frac{1}{2}\epsilon^{kl} w_k^1 w_l^2 w_a^2 \epsilon_{\alpha\beta}\, \frac{\partial^3}{\partial\theta_\beta^2 \partial\theta_\alpha^2 \partial w_a^1},$$

$$\Lambda_2 = \frac{1}{6}(\epsilon_{ab} w_c^2 + \epsilon_{ac} w_b^2)\epsilon^{kl} w_k^1 w_l^2\, \frac{\partial^3}{\partial w_c^2 \partial w_b^2 \partial w_a^1}, \qquad \Lambda_{11} = \frac{1}{2}\epsilon^{\alpha\beta}\theta_\alpha^2\theta_\beta^2 \epsilon_{ab} w_c^1\, \frac{\partial^3}{\partial w_c^2 \partial w_b^2 \partial w_a^1},$$

$$\Lambda_3 = \frac{1}{2}(w_a^1 w_b^2 + w_a^2 w_b^1)\theta_\alpha^2\, \frac{\partial^3}{\partial\theta_\alpha^2 \partial w_b^2 \partial w_a^1}, \qquad \Lambda_{12} = \epsilon^{kl} w_k^1 w_l^2 \epsilon_{\alpha\beta} w_a^2\, \frac{\partial^3}{\partial w_a^2 \partial\theta_\beta^2 \partial\theta_\alpha^1},$$

$$\Lambda_4 = \frac{1}{2}(w_a^1 w_b^2 - w_a^2 w_b^1)\theta_\alpha^2\, \frac{\partial^3}{\partial\theta_\alpha^2 \partial w_b^2 \partial w_a^1}, \qquad \Lambda_{13} = \epsilon^{\alpha\beta}\theta_\alpha^1\theta_\beta^2 \epsilon_{ab} w_c^2\, \frac{\partial^3}{\partial w_c^2 \partial w_b^2 \partial w_a^1},$$

$$\Lambda_5 = \frac{1}{2} w_a^2 w_b^2 \theta_\alpha^1\, \frac{\partial^3}{\partial w_b^2 \partial w_a^2 \partial\theta_\alpha^1}, \qquad \Lambda_{14} = \frac{1}{2}\epsilon^{\alpha\beta}\theta_\alpha^2\theta_\beta^2 \epsilon_{\gamma\delta} w_a^1\, \frac{\partial^3}{\partial w_a^2 \partial\theta_\delta^2 \partial\theta_\gamma^1},$$

$$\Lambda_6 = \frac{1}{2} w_a^1 \theta_\alpha^2 \theta_\beta^2\, \frac{\partial^3}{\partial\theta_\beta^2 \partial\theta_\alpha^2 \partial w_a^1}, \qquad \Lambda_{15} = \frac{1}{2}\epsilon^{\alpha\beta}\theta_\alpha^1\theta_\beta^2 \epsilon_{\gamma\delta} w_a^2\, \frac{\partial^3}{\partial\theta_\delta^2 \partial\theta_\gamma^2 \partial w_a^1},$$

$$\Lambda_7 = \frac{1}{2} w_a^2 (\theta_\alpha^1\theta_\beta^2 + \theta_\beta^1\theta_\alpha^2)\, \frac{\partial^3}{\partial w_a^2 \partial\theta_\beta^2 \partial\theta_\alpha^1}, \qquad \Lambda_{16} = \epsilon_{ab}\epsilon^{\alpha\beta}\theta_\alpha^1\theta_\beta^2\theta_\gamma^2\, \frac{\partial^3}{\partial\theta_\gamma^2 \partial w_b^2 \partial w_a^1},$$

$$\Lambda_8 = \frac{1}{2} w_a^2 (\theta_\alpha^1\theta_\beta^2 - \theta_\beta^1\theta_\alpha^2)\, \frac{\partial^3}{\partial w_a^2 \partial\theta_\beta^2 \partial\theta_\alpha^1}, \qquad \Lambda_{17} = \epsilon^{kl} w_k^1 w_l^2 \epsilon_{\alpha\beta}\theta_\gamma^2\, \frac{\partial^3}{\partial\theta_\gamma^2 \partial\theta_\beta^2 \partial\theta_\alpha^1},$$

$$\Lambda_9 = \frac{1}{2}\theta_\alpha^1\theta_\beta^2\theta_\gamma^2\, \frac{\partial^3}{\partial\theta_\gamma^2 \partial\theta_\beta^2 \partial\theta_\alpha^1}, \qquad \Lambda_{18} = w_a^1 w_b^2 \theta_\alpha^2\, \frac{\partial^3}{\partial w_b^2 \partial w_a^2 \partial\theta_\alpha^1},$$

$$\Lambda_{19} = w_a^2 w_b^2 \theta_\alpha^1\, \frac{\partial^3}{\partial\theta_\alpha^2 \partial w_b^2 \partial w_a^1} \tag{B.7}$$

The reflection coefficients k_i are as follows:

$$
\begin{aligned}
k_1 &= 1\,,\\
k_2 &= 1-\frac{3}{2}\frac{(x^-+x^+)((x^-)^2+x_B^2(x^+)^2)}{(x_B-x^-)x^-x^+(-1+x_Bx^+)}\,,\\
k_3 &= \frac{(x_Bx^- -(x^+)^2)}{(x_B-x^-)x^-}\frac{\tilde\eta_B}{\eta_B}\,,\\
k_4 &= -\frac{(x_B+x^+)(x^-+x_B(x^+)^2)}{(x_B-x^-)x^-(-1+x_Bx^+)}\frac{\tilde\eta_B}{\eta_B}\,,\\
k_5 &= -\frac{((x^-)^2+x_Bx^+)}{(x_B-x^-)x^-}\frac{\tilde\eta}{\eta}\,,\\
k_6 &= \frac{x^+(x_B^2(x^-)^3-2x_Bx^-x^+-x_B^2(x^-)^2x^+-x^-(x^+)^2-2x_B(x^-)^2(x^+)^2+(x^+)^3)\tilde\eta_B^2}{2(x_B-x^-)(x^-)^3(-1+x_Bx^+)\eta_B^2}\,,\\
k_7 &= -\frac{x^+(x_B+x^+)}{(x_B-x^-)x^-}\frac{\tilde\eta\,\tilde\eta_B}{\eta\,\eta_B}\,,\\
k_8 &= \frac{x^+((x^-)^2x^+-x_B(x^-)^2-2x_B(x^-)^4-x_B^2(x^-)^2x^++2x_B(x^+)^2+x_B(x^-)^2(x^+)^2)\tilde\eta\,\tilde\eta_B}{(x_B-x^-)(x^-)^3(-1+x_Bx^+)\eta\,\eta_B}\,,\\
k_9 &= \frac{(x^+)^2(x_B+x^+)(-x_B(x^-)^2+x^+)}{(x_B-x^-)(x^-)^3(-1+x_Bx^+)}\frac{\tilde\eta\,\tilde\eta_B^2}{\eta\,\eta_B^2}\,,\\
k_{10} &= -\frac{i\zeta x_B^2(x^- - x^+)(x^-+x^+)^2}{2(x_B-x^-)(x^-)^2(-1+x_Bx^+)\eta_B^2}\,,\\
k_{11} &= -\frac{i(x^- - x^+)(x^-+x^+)^2\tilde\eta_B^2}{2\zeta(x_B-x^-)(x^-)^2(-1+x_Bx^+)}\,,\\
k_{12} &= \frac{i\zeta x_B((x^-)^2+x_Bx^+)((x^-)^2-(x^+)^2)}{\sqrt{2}(x_B-x^-)(x^-)^2(-1+x_Bx^+)\eta\,\eta_B}\,,\\
k_{13} &= \frac{i(x^-+x^+)((x^-)^2+x_Bx^+)\tilde\eta\,\tilde\eta_B}{\sqrt{2}\zeta(x^-)^2(-x_B+x^-)(-1+x_Bx^+)}\,,\\
k_{14} &= \frac{(x_B(x^-)^2-x^+)x^+((x^-)^2-(x^+)^2)}{\sqrt{2}(x^-)^3(-x_B+x^-)(-1+x_Bx^+)}\frac{\tilde\eta_B^2}{\eta\,\eta_B}\,,\\
k_{15} &= \frac{x_B(x_B(x^-)^2-x^+)x^+(x^-+x^+)}{\sqrt{2}(x_B-x^-)(x^-)^3(-1+x_Bx^+)}\frac{\tilde\eta\,\tilde\eta_B}{\eta_B^2}\,,\\
k_{16} &= \frac{ix^+(x_B+x^+)(x^-+x^+)}{\sqrt{2}\zeta(x^-)^2(-x_B+x^-)(-1+x_Bx^+)}\frac{\tilde\eta\,\tilde\eta_B^2}{\eta_B}\,,\\
k_{17} &= \frac{i\zeta x_Bx^+(x_B+x^+)((x^-)^2-(x^+)^2)}{\sqrt{2}(x_B-x^-)(x^-)^2(-1+x_Bx^+)}\frac{\tilde\eta_B}{\eta\,\eta_B^2}\,,\\
k_{18} &= \frac{((x^-)^2-(x^+)^2)}{\sqrt{2}x^-(-x_B+x^-)}\frac{\tilde\eta_B}{\eta}\,,\\
k_{19} &= \frac{x_B(x^-+x^+)}{\sqrt{2}(x_B-x^-)x^-}\frac{\tilde\eta}{\eta_B}\,.
\end{aligned}
\tag{B.8}
$$

Chapter 4

q-deformed scattering in AdS/CFT

The quantum deformed (q-deformed) scattering in AdS/CFT is also known as a deformed Hubbard chain, or, more precisely, the one-dimensional double-deformed Hubbard chain. These two systems are equivalent as they have the same underlying symmetry, the quantum affine algebra $\widehat{\mathcal{Q}}$ [30]. The physical interpretation of the q-deformations is not known at the present time, nevertheless the existence of such symmetries is very welcome. Quantum affine algebras generally are of more symmetric and elegant form than their Yangian avatars. This is particularly important for the boundary scattering in AdS/CFT. In the previous chapter we have encountered a wide variety of twisted Yangians that look very different from each other. Here we will show that q-deformed approach leads to a very elegant and uniform boundary scattering theory. But first of all let us briefly recall some details about the Hubbard model.

The Hubbard model, which was named after John Hubbard, is the simplest model of interacting particles on a lattice. It has only two terms in the Hamiltonian: the hopping term (kinetic energy) and the Coulomb potential [153]. The model describes an ensemble of particles in a periodic potential at sufficiently low temperatures such that all the particles may be considered to be in the lowest Bloch band. Moreover, any long-range interactions between the particles are considered to be weak enough and are consequently ignored. It is based on the tight-binding approximation of superconducting systems and the motion of electrons between the atoms of a crystalline solid. Despite its apparent simplicity, there are different applications and generalizations describing a plethora of interesting phenomena. In the case when interactions between particles on different sites of the lattice can not be neglected and are taken into account, the model is often referred to as the Extended Hubbard model. The particles can either be fermions, as in Hubbard's original work, or bosons, and the model is then referred as either the Bose-Hubbard model or the boson Hubbard model. The latter can be used to study systems such as bosonic atoms on an optical lattice (for a decent overview of various generalizations see reprint volumes [154–156] and also a more recent book [157]).

A very specific class of models that share features with the one-dimensional Hubbard model and the supersymmetric t-J model [158] is the so-called Alcaraz and Bariev model [159]. It contains an extra spin-spin interaction term in the Hamiltonian and it shows some characteristics of superconductivity. This model can be viewed as a quantum deformation of the Hubbard model in much the same way as the Heisenberg XXZ model is a quantum deformation of the XXX model. This model has a specific R-matrix which can not be written as a function of the difference of two associated spectral parameters. This paradigm is related to the very interesting but at the same time complicated algebraic properties of the model.

The exact integrability of the one-dimensional Hubbard model was established by B. Shastry [160]. It was also shown that the model exhibits $\mathcal{Y}(\mathfrak{su}(2)) \oplus \mathcal{Y}(\mathfrak{su}(2))$ Yangian symmetry [161]. However this symmetry is insufficient to constrain Shastry's S-matrix completely. Similarly, the worldsheet S-matrix for the $AdS_5 \times S^5$ superstring also turns out to have Yangian symmetry [20]. However the Yangian in this model is based on a larger Lie algebra, the centrally extended $\mathfrak{su}(2|2)$ Lie superalgebra, which has been considered in Chapter 3.

An interesting approach to the Hubbard model was put forward in [28]. Here the model was based on the quantum deformation $\mathcal{Q}$ of the centrally extended $\mathfrak{psu}(2|2)_{\mathcal{C}}$ algebra. This q-deformed algebra has a number of interesting features such as a rather symmetric realization of the different central elements. This model describes spectrum of deformed supersymmetric one-dimensional Hubbard models [28, 162]. The undeformed Hubbard model is revealed by taking a specific limit of deformed model [163]. Moreover, by sending the quantum deformation parameter $q \to 1$, the S-matrix of this model specializes to the AdS/CFT worldsheet S-matrix which we have discussed in the previous chapter. As such, this matrix encompasses both different varieties of Hubbard models as well as the AdS/CFT worldsheet S-matrix and seems to provide a unifying algebraic framework for describing this class of models.

The fundamental q-deformed S-matrix is constrained up to an overall phase by requiring invariance under $\mathcal{Q}$ itself. However, in the light that both the AdS/CFT and the Hubbard model S-matrices are actually invariant under an infinite dimensional symmetry algebra, it should not be surprising that such a structure is also present here. Indeed, the larger algebraic structure underlying this S-matrix is the quantum affine algebra $\widehat{\mathcal{Q}}$ [30]. This infinite dimensional algebra is obtained by adding an additional fermionic node to the Dynkin diagram of $\mathcal{Q}$. In the $q \to 1$ limit one can retrieve the Yangian generators of $\mathfrak{psu}(2|2)_{\mathcal{C}}$ by considering the appropriate combinations of generators of $\widehat{\mathcal{Q}}$. This fuels the idea that $\widehat{\mathcal{Q}}$ plays a similar role as the Yangian in the undeformed case. More specifically, it is expected that the S-matrix in the higher representations is uniquely defined up to an overall phase by the underlying quantum affine algebra $\widehat{\mathcal{Q}}$. This indeed turns out to be the case as we will show in this chapter.

The type of representations we will be considering in here are the supersymmetric short representations. In order to construct these representations, we employ the formalism of quantum oscillators. It is a quantum version of the well-known harmonic oscillator algebra and is defined by

$$[N, \mathsf{a}] = -\mathsf{a}\,, \qquad [N, \mathsf{a}^\dagger] = \mathsf{a}^\dagger\,, \qquad \mathsf{a}\,\mathsf{a}^\dagger - q\,\mathsf{a}^\dagger \mathsf{a} = q^{-N}\,.$$

The use of quantum oscillators in the context of quantum groups was investigated earlier in [164–166]. By employing Fock space type modules, q-oscillators naturally give rise to the bound state representations of quantum groups. This approach was first formulated for the quantum deformed algebra $\mathcal{U}_q(\mathfrak{sl}(2))$ and later extended to simple Lie (super)algebras of a more general type, see e.g. [167]. Since then quantum oscillators have become an important part of the theory of quantum deformed algebras.

The q-oscillator approach to bound states is not only an interesting mathematical playground for studying the quantum affine algebra $\widehat{\mathcal{Q}}$ and its S-matrix; there is also a more elaborate motivation for considering these representations and the corresponding S-matrix. Firstly, there might be some possible applications in the context of the deformed Hubbard model. Secondly, the study of bound states is necessary to understand some fundamental properties of the q-deformed AdS/CFT. For example, bound states usually play a crucial role in the thermodynamics of the model. In the case of the non-deformed AdS/CFT, the thermodynamic Bethe ansatz (TBA) formalism is key in describing the complete spectrum of the theory [168–171]. The bound state S-matrix then governs the large volume solutions of both the TBA equations and the Y-system. Thus this is one of the first steps towards the TBA and Y-system formalism for the q-deformed model. And, consequently, it might give some useful insights for the general structures of the AdS/CFT superstring. For example, there might be an interesting link to the recently constructed q-deformed Pohlmeyer reduced version of the superstrings in the $AdS_5 \times S^5$ background [135,172] which seems to be closely related to the q-deformed model constructed in [28].

Open boundary conditions for the deformed Hubbard model have received less attention than their closed chain counterpart, nevertheless they exhibit a rich variety of structures (see e.g. [173–175]). Reflection matrices for open boundary conditions for the deformed Hubbard model of [28] have been first studied in [29]. Here the q-deformed models of the giant gravitons were considered and the corresponding fundamental reflection matrices were obtained. In the light of the algebra $\widehat{\mathcal{Q}}$ these models have been further developed to incorporate the underlying affine symmetries. Three types of coideal subalgebras of $\widehat{\mathcal{Q}}$ and reflection matrices were formulated in [25,26] that govern the boundary scattering for the $Y = 0$ and $Z = 0$ giant gravitons and the left factor of the $Z = 0$ $D7$-brane. These boundary algebras follow the pattern of the quantum affine coideal subalgebras discussed in Chapter 2.

This Chapter is organized as follows. In Section 4.1 we introduce the quantum affine algebra $\widehat{\mathcal{Q}}$ of the deformed Hubbard chain, and its bound state representations. In Section 4.2 we construct the bound state q-deformed S-matrix. Section 4.3 gives the necessary preliminaries for the q-deformed boundary scattering theory. Then in Sections 4.4, 4.4 and 4.5 we construct the q-deformed models of the $Y = 0$ and $Z = 0$ giant gravitons and the left factor of the $Z = 0$ $D7$-brane, which we have previously considered in Chapter 3. We then construct the corresponding boundary algebras and build the the q-deformed boundary scattering theory. The majority of the q-deformed S-matrix coefficients and results of the intermediate steps of calculations are spelled out in Appendices C, D and E. Appendix F contains selected bound state reflection matrices for the q-deformed $Z = 0$ giant graviton.

4.1 Quantum affine algebra $\widehat{\mathcal{Q}}$

In this section we review the quantum deformation of the centrally extended $\mathfrak{psu}(2|2)_{\mathcal{C}}$ algebra [28], its affine extension [30] and the bound state representation [176].

4.1.1 Quantum deformation of $\mathfrak{psu}(2|2)_{\mathcal{C}}$

The quantum deformed $\mathfrak{psu}(2|2)_{\mathcal{C}}$ algebra $\mathcal{Q}$ was introduced in [28]. This algebra is generated by the three sets of Chevalley-Serre generators $\{E_j, K_j, F_j\}$ ($j = 1, 2, 3$) where E_j and F_j are raising and lowering generators respectively and $K_j = q^{H_j}$ are the Cartan generators. We will consider the case when E_2 and F_2 are fermionic generators and the rest are bosonic. This corresponds to the $\mathfrak{su}(2|2)$ Dynkin diagram in Figure 4.1. In addition, this algebra has two central charges U and $V = q^C$ and two parameters: the deformation parameter q and the coupling constant g. There is also a third parameter α, which describes the relative scaling of E_2 and F_2. Even though it is possible absorb this parameter into these generators by a suitable redefinition, we will keep it unspecified.

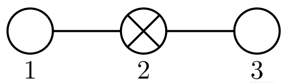

Figure 4.1: Dynkin diagram for the $\mathfrak{su}(2|2)$ algebra.

Algebra. The commutation relations which include the mixed Chevalley-Serre generators are ($j, k = 1, 2, 3$)

$$K_j E_k = q^{+DA_{jk}} E_k K_j, \qquad K_j F_k = q^{-DA_{jk}} F_k K_j, \qquad [E_j, F_k\} = D_{jj}\delta_{jk}\frac{K_j - K_j^{-1}}{q - q^{-1}}, \tag{4.1.1}$$

where the associated Cartan matrix A and normalization matrix D are given by

$$DA = \begin{pmatrix} +2 & -1 & 0 \\ -1 & 0 & +1 \\ 0 & +1 & -2 \end{pmatrix}, \qquad D = \mathrm{diag}(+1, -1, -1)\ . \tag{4.1.2}$$

There are also the unmixed commutation relations, called the Serre relations ($j = 1, 3$),

$$\begin{aligned} &[E_1, E_3] = \{E_2, E_2\} = \big[E_j, [E_j, E_2]\big] - (q - 2 + q^{-1})E_j E_2 E_j = 0, \\ &[F_1, F_3] = \{F_2, F_2\} = \big[F_j, [F_j, F_2]\big] - (q - 2 + q^{-1})F_j F_2 F_j = 0. \end{aligned} \tag{4.1.3}$$

In addition, this algebra satisfies the extended Serre relations that give rise to two central elements U and V as follows,

$$\begin{aligned} g\alpha(1 - U^2V^2) &= \big\{[E_2, E_1], [E_2, E_3]\big\} - (q - 2 + q^{-1})E_2E_1E_3E_2, \\ g\alpha^{-1}(V^{-2} - U^{-2}) &= \big\{[F_2, F_1], [F_2, F_3]\big\} - (q - 2 + q^{-1})F_2F_1F_3F_2. \end{aligned} \tag{4.1.4}$$

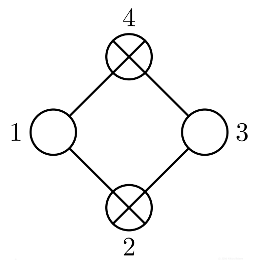

Figure 4.2: Dynkin diagram for the affine $\widehat{\mathfrak{su}}(2|2)$ algebra.

The central element V is also related to the Cartan generators through

$$V^{-2} = K_1 K_2^2 K_3 \,. \tag{4.1.5}$$

The conventional $\mathcal{U}_q(\mathfrak{su}(2|2))$ algebra is obtained in the limit $g \to 0$.

Coalgebra. The defining relations of $\mathcal{Q}$ are compatible with the following coalgebra structure. The coproduct of the group like elements $X \in \{U, V, K\}$ is $\Delta(X) = X \otimes X$ and the coproducts of the Chevalley-Serre generators E_j and F_j ($j = 1, 3$) take the standard forms. However the coproducts of the fermionic generators E_2 and F_2 involve an additional braiding factor U, which is one of the central charges of the algebra alluded to in the previous paragraph,

$$\Delta(E_j) = E_j \otimes 1 + K_j^{-1} U^{+\delta_{j,2}} \otimes E_j \,, \qquad \Delta(F_j) = F_j \otimes K_j + U^{-\delta_{j,2}} \otimes F_j \,. \tag{4.1.6}$$

The coalgebra can be extended to a Hopf algebra. We will give the relevant definitions of the antipode and counit later on.

4.1.2 Affine Extension

The infinite dimensional quantum affine algebra $\widehat{\mathcal{Q}}$ is the affine extension of $\mathcal{Q}$ introduced in [30]. The affine extension is obtained by adding an additional node to the Dynkin diagram as depicted in Figure 4.2. The remarkable property of this diagram is that the additional fermionic node is a copy of the second node. Therefore, we introduce the affine Chevalley-Serre generators $\{E_4, F_4, K_4\}$ as copies of $\{E_2, F_2, K_2\}$ and assume that they satisfy the same commutation relations as are given in (4.1.1), (4.1.3) and (4.1.4) and also have the same coalgebra structure (4.1.6). Thus, we introduce an additional set of the parameters g, α and central charges U, V. We distinguish these two sets by adhering subscripts to them arising from the generators to which they are associated,

$$g \to g_k, \qquad \alpha \to \alpha_k, \qquad U \to U_k, \qquad V \to V_k, \qquad \text{with} \qquad k = 2, 4. \tag{4.1.7}$$

Next, we need to determine the commutation relations $\{E_2, F_4\}$ and $\{E_4, F_2\}$ in such way that they would be compatible with the coalgebra structure,

$$\Delta(\{E_2, F_4\}) = \{\Delta(E_2), \Delta(F_4)\} \qquad \text{and} \qquad \Delta(\{E_4, F_2\}) = \{\Delta(E_4), \Delta(F_2)\} \,. \tag{4.1.8}$$

Algebra. As a result, we obtain the quantum affine algebra $\widehat{\mathcal{Q}}$ [30]. The mixed commutation relations of it are given by ($i, j = 1, 3$)

$$\begin{aligned} K_i E_j &= q^{+DA_{ij}} E_j K_i, & K_i F_j &= q^{-DA_{ij}} F_j K_i, \\ \{E_2, F_4\} &= -\tilde{g}\tilde{\alpha}^{-1}(K_4 - U_2 U_4^{-1} K_2^{-1}), & \{E_4, F_2\} &= \tilde{g}\tilde{\alpha}(K_2 - U_4 U_2^{-1} K_4^{-1}), \\ [E_j, F_j\} &= D_{jj}\frac{K_j - K_j^{-1}}{q - q^{-1}} & [E_i, F_j\} &= 0, \quad \text{for } i \neq j,\ i + j \neq 6\ . \end{aligned} \tag{4.1.9}$$

with the two new constants $\tilde{g}$ and $\tilde{\alpha}$ and the associated supersymmetric Cartan matrix A and normalization matrix D given by

$$DA = \begin{pmatrix} +2 & -1 & 0 & -1 \\ -1 & 0 & +1 & 0 \\ 0 & +1 & -2 & +1 \\ -1 & 0 & +1 & 0 \end{pmatrix}, \qquad D = \mathrm{diag}(1, -1, -1, -1). \tag{4.1.10}$$

These are supplemented by the following Serre relations ($j = 1, 3$ and $k = 2, 4$)

$$\begin{aligned} &[E_1, E_3] = E_2 E_2 = E_4 E_4 = \{E_2, E_4\} = 0\ , \\ &[F_1, F_3] = F_2 F_2 = F_4 F_4 = \{F_2, F_4\} = 0\ , \\ &[E_j, [E_j, E_k]] - (q - 2 + q^{-1}) E_j E_k E_j = 0\ , \\ &[F_j, [F_j, F_k]] - (q - 2 + q^{-1}) F_j F_k F_j = 0\ . \end{aligned} \tag{4.1.11}$$

The central elements are related to the quartic Serre relations as ($k = 2, 4$)

$$\begin{aligned} g_k \alpha_k (1 - U_k^2 V_k^2) &= \big\{[E_k, E_1], [E_k, E_3]\big\} - (q - 2 + q^{-1}) E_k E_1 E_3 E_k\ , \\ g_k \alpha_k^{-1} (V_k^{-2} - U_k^{-2}) &= \big\{[F_k, F_1], [F_k, F_3]\big\} - (q - 2 + q^{-1}) F_k F_1 F_3 F_k\ . \end{aligned} \tag{4.1.12}$$

In such a way this algebra has three regular central charges,

$$C_1 = K_1 K_2^2 K_3\ , \qquad C_2 = g_2 \alpha_2 (1 - U_2^2 V_2^2)\ , \qquad C_3 = g_2 \alpha_2^{-1} (V_2^{-2} - U_2^{-2})\ , \tag{4.1.13}$$

and three affine ones,

$$\hat{C}_1 = K_1 K_4^2 K_3\ , \qquad \hat{C}_2 = g_4 \alpha_4 (1 - U_4^2 V_4^2)\ , \qquad \hat{C}_3 = g_4 \alpha_4^{-1} (V_4^{-2} - U_4^{-2})\ . \tag{4.1.14}$$

The central elements V_k are constrained by the relation $K_1^{-1} K_k^{-2} K_3^{-1} = V_k^2$.

Coalgebra. The group-like elements $X \in \{1, K_j, U_k, V_k\}$ ($j = 1, 2, 3, 4$ and $k = 2, 4$) have the coproduct Δ, the antipode S and the counit ε defined in the usual way,

$$\Delta(X) = X \otimes X, \qquad S(X) = X^{-1}, \qquad \varepsilon(X) = 1, \tag{4.1.15}$$

while the coproducts of the Chevalley-Serre generators are deformed by the central elements U_k as follows ($j = 1, 2, 3, 4$),

$$\begin{aligned}
&\Delta(E_j) = E_j \otimes 1 + K_j^{-1} U_2^{+\delta_{j,2}} U_4^{+\delta_{j,4}} \otimes E_j, \quad \mathrm{S}(E_j) = -U_2^{-\delta_{j,2}} U_4^{-\delta_{j,4}} K_j E_j, \quad \varepsilon(E_j) = 0, \\
&\Delta(F_j) = F_j \otimes K_j + U_2^{-\delta_{j,2}} U_4^{-\delta_{j,4}} \otimes F_j, \qquad \mathrm{S}(F_j) = -U_2^{+\delta_{j,2}} U_4^{+\delta_{j,4}} F_j K_j^{-1}, \quad \varepsilon(F_j) = 0.
\end{aligned} \tag{4.1.16}$$

It is important to note that the above coproducts are compatible with all the defining relations, including the commutators $\{E_2, F_4\}$ and $\{E_4, F_2\}$ in (4.1.9). The opposite coproduct is defined as $\Delta^{op} = \mathcal{P} \circ \Delta \circ \mathcal{P}$ with $\mathcal{P}$ being the graded permutation operator.

Parameter constraints. In general, the quantum affine algebra $\widehat{\mathcal{Q}}$ has seven parameters $g_k, \alpha_k, \tilde{\alpha}, \tilde{g}, q$ ($k = 2, 4$). A suitable choice of them which lead to an interesting fundamental representation was performed in [30]:

$$g_2 = g_4 = g, \qquad \alpha_2 = \alpha_4\, \tilde{\alpha}^{-2} = \alpha, \qquad \tilde{g}^2 = \frac{g^2}{1 - g^2(q - q^{-1})^2}\,. \tag{4.1.17}$$

This choice of parameters is also compatible with the bound state representations. Thus in this paper we only consider the quantum affine algebra $\widehat{\mathcal{Q}}$, parametrized by four independent parameters $g, \alpha, \tilde{\alpha}, q$ given in the relations above.

4.1.3 Quantum oscillators and representations

In this section we will provide all the necessary background for constructing the bound state S-matrix for the q-deformed Hubbard model. We will build the bound state representation by introducing q-oscillator formalism linking it to the aforementioned quantum affine algebra.

4.1.3.1 q-Oscillators

We first introduce the notion of q-oscillators and discuss how to obtain the representations of the quantum deformed algebras using q-oscillators. A concise overview of the q-oscillators and their relation to such representations may be found in [167, 177].

Definitions. The q-oscillator (q-Heisenberg-Weyl algebra) $\mathcal{U}_q(\mathfrak{h}_4)$ is the associative unital algebra consisting of the generators $\{\mathsf{a}^\dagger, \mathsf{a}, w, w^{-1}\}$ that satisfy the following relations,

$$\begin{aligned}
&w\, \mathsf{a}^\dagger = q\, \mathsf{a}^\dagger w, && qw\, \mathsf{a} = \mathsf{a}\, w, \\
&ww^{-1} = w^{-1}w = 1, && \mathsf{a}\, \mathsf{a}^\dagger - q\, \mathsf{a}^\dagger \mathsf{a} = w^{-1}.
\end{aligned} \tag{4.1.18}$$

From the defining relations one can see that the element $w^{-1}(\mathsf{a}^\dagger \mathsf{a} - \frac{w - w^{-1}}{q - q^{-1}})$ is central. As such, we will set it to zero in the remainder. Then one easily obtains

$$\mathsf{a}^\dagger \mathsf{a} = \frac{w - w^{-1}}{q - q^{-1}}, \qquad \mathsf{a}\, \mathsf{a}^\dagger = \frac{qw - q^{-1}w^{-1}}{q - q^{-1}}. \tag{4.1.19}$$

We will also need to consider the fermionic version of the q-oscillator. The above notion is extended to include fermionic operators by adjusting the defining relations in the following way (we keep the same notation for bosonic and fermionic a, $\mathsf{a}^\dagger$ for now)

$$w\,\mathsf{a}^\dagger = q\,\mathsf{a}^\dagger w, \qquad qw\,\mathsf{a} = \mathsf{a}\,w, \qquad (4.1.20)$$
$$ww^{-1} = w^{-1}w = 1, \qquad \mathsf{a}\,\mathsf{a}^\dagger + q\,\mathsf{a}^\dagger\mathsf{a} = w.$$

In this case, the central element is $w(\mathsf{a}^\dagger\mathsf{a} - \frac{w-w^{-1}}{q-q^{-1}})$. Again we set this element to zero, resulting in the following identities

$$\mathsf{a}^\dagger\mathsf{a} = \frac{w-w^{-1}}{q-q^{-1}}, \qquad \mathsf{a}\,\mathsf{a}^\dagger = \frac{qw^{-1}-q^{-1}w}{q-q^{-1}}. \qquad (4.1.21)$$

Of course in the fermionic case the operators $\mathsf{a}, \mathsf{a}^\dagger$ square to zero. Equation (4.1.21) implies that this only is consistent if $w^2 = 1, q^2$. Below we will identify $w \equiv q^N$, where $N = 0, 1$ is the number of fermions making it indeed compatible.

Fock space. The q-oscillator algebra can be used to define representations of $\mathcal{U}_q(\mathfrak{sl}(2))$ in a very simple way. Let us first build the Fock representation of $\mathcal{U}_q(\mathfrak{h}_4)$. For this purpose consider a vacuum state $|0\rangle$ such that

$$\mathsf{a}|0\rangle = 0, \qquad (4.1.22)$$

then the Fock vector space $\mathcal{F}$ generated by the states of the form

$$|n\rangle = (\mathsf{a}^\dagger)^n|0\rangle\,, \qquad (4.1.23)$$

is an irreducible module of $\mathcal{U}_q(\mathfrak{h}_4)$. Let us first consider the bosonic q-oscillators. With the help of the defining relations (4.1.18) and (4.1.19) one finds that the action of the oscillator algebra generators on this module is

$$\mathsf{a}^\dagger|n\rangle = |n+1\rangle, \qquad \mathsf{a}|n\rangle = [n]_q|n-1\rangle, \qquad w|n\rangle = q^n|n\rangle. \qquad (4.1.24)$$

This makes it natural to identify $w \equiv q^N$, where N is understood as a number operator. Analogously, fermionic generators are found to act as

$$\mathsf{a}^\dagger|n\rangle = |n+1\rangle, \qquad \mathsf{a}|n\rangle = [2-n]_q|n-1\rangle, \qquad w|n\rangle = q^n|n\rangle. \qquad (4.1.25)$$

However, due to the fermionic nature, n can only take the values 0 and 1 and thus the identity $[2-n]_q = [n]_q$ holds.

Next consider two copies of bosonic q-oscillators $\mathsf{a}_i, \mathsf{a}_i^\dagger, w_i = q^{N_i}$ which mutually commute. Then the Fock space is naturally spanned by vectors of the form

$$|m,n\rangle = (\mathsf{a}_1^\dagger)^m(\mathsf{a}_2^\dagger)^n|0\rangle. \qquad (4.1.26)$$

It is easy to see that under the identification

$$E = \mathsf{a}_2^\dagger\mathsf{a}_1, \qquad F = \mathsf{a}_1^\dagger\mathsf{a}_2, \qquad H = N_2 - N_1, \qquad (4.1.27)$$

the Fock space forms an infinite dimensional $\mathcal{U}_q(\mathfrak{sl}(2))$-representation. Moreover, the subspace $\mathcal{F}_M = \mathrm{span}\{\, |m, M-m\rangle \mid m = 0, \dots, M\,\}$ is an irreducible $\mathcal{U}_q(\mathfrak{sl}(2))$-representation of dimension $M+1$. This can be straightforwardly generalized to $\mathfrak{sl}(n)$ and more generally, by including fermionic oscillators, this space is extended to the representations of $\mathfrak{sl}(n|m)$ [167].

Representations of $\mathcal{U}_q(\mathfrak{psu}(2|2)_\mathcal{C})$. We will now construct the bound state representation of $\mathcal{U}_q(\mathfrak{psu}(2|2)_\mathcal{C})$ in the q-oscillator language. We need to consider two copies of $\mathfrak{sl}(2)$, a bosonic and a fermionic one. Thus we need four sets of q-oscillators $\mathsf{a}_i, \mathsf{a}_i^\dagger, w_i = q^{N_i}$, where the index $i = 1,2$ denotes bosonic oscillators and $i = 3,4$ – fermionic ones. Using these we write

$$E_1 = \mathsf{a}_2^\dagger \mathsf{a}_1, \qquad F_1 = \mathsf{a}_1^\dagger \mathsf{a}_2, \qquad H_1 = N_2 - N_1, \tag{4.1.28}$$

$$E_2 = a\, \mathsf{a}_4^\dagger \mathsf{a}_2 + b\, \mathsf{a}_1^\dagger \mathsf{a}_3 \quad F_2 = c\, \mathsf{a}_3^\dagger \mathsf{a}_1 + d\, \mathsf{a}_2^\dagger \mathsf{a}_4, \quad H_2 = -C + \frac{N_1 + N_3 - N_2 - N_4}{2}, \tag{4.1.29}$$

$$E_3 = \mathsf{a}_3^\dagger \mathsf{a}_4, \qquad F_3 = \mathsf{a}_4^\dagger \mathsf{a}_3, \qquad H_3 = N_4 - N_3, \tag{4.1.30}$$

where C is central. It is then straightforward to check that this set of generators forms a representation of $\mathcal{U}_q(\mathfrak{su}(2|2))$ on the Fock space when restricting to the subspace of total particle number M upon setting

$$ad = \frac{[C + \frac{M}{2}]_q}{[M]_q}, \qquad bc = \frac{[C - \frac{M}{2}]_q}{[M]_q}, \qquad ab = \frac{\mathfrak{P}}{[M]_q}, \qquad cd = \frac{\mathfrak{K}}{[M]_q}. \tag{4.1.31}$$

In the above $\mathfrak{K}, \mathfrak{P}$ correspond to the right hand side of the Serre relations (4.1.12) following [28]. As a consequence, the central charges satisfy the shortening condition

$$[C]_q^2 - \mathfrak{P}\mathfrak{K} = \left[\tfrac{M}{2}\right]_q^2. \tag{4.1.32}$$

Here the q-numbers are defined as

$$[k]_q = \frac{q^k - q^{-k}}{q - q^{-1}}. \tag{4.1.33}$$

This way of constructing representations of the centrally extended algebra reminds us of the procedure used in, e.g. [113], where long representations were be obtained by twisting $\mathfrak{sl}(n|m)$ in a similar way.

In the $q \to 1$ limit the q-oscillators get reduced to regular oscillators and their representations coincide with the superspace formalism introduced in [108]. The identification is as follows

$$\mathsf{a}_{1,2} \leftrightarrow \frac{\partial}{\partial w_{1,2}}, \qquad \mathsf{a}_{1,2}^\dagger \leftrightarrow w_{1,2}, \qquad \mathsf{a}_{3,4} \leftrightarrow \frac{\partial}{\partial \theta_{3,4}}, \qquad \mathsf{a}_{3,4}^\dagger \leftrightarrow \theta_{3,4}. \tag{4.1.34}$$

Parametrization and central elements. Introducing $V = q^C$ and U as in [30], we rewrite (4.1.31) as

$$ad = \frac{q^{\frac{M}{2}}V - q^{-\frac{M}{2}}V^{-1}}{q^M - q^{-M}}, \qquad bc = \frac{q^{-\frac{M}{2}}V - q^{\frac{M}{2}}V^{-1}}{q^M - q^{-M}},$$
$$ab = \frac{g\alpha}{[M]_q}(1 - U^2V^2), \qquad cd = \frac{g\alpha^{-1}}{[M]_q}(V^{-2} - U^{-2}). \tag{4.1.35}$$

which altogether leads to a constraint for U and V,

$$\frac{g^2}{[M]_q^2}(V^{-2} - U^{-2})(1 - U^2V^2) = \frac{(V - q^M V^{-1})(V - q^{-M}V^{-1})}{(q^M - q^{-M})^2} \; . \tag{4.1.36}$$

This constraint agrees with the one in [30] by identifying $q \to q^M, \; g \to g/[M]_q$. The explicit parametrization of the labels a, b, c, d shall be given a bit further.

4.1.3.2 Affine extension

Next we want to consider the affine extension introduced in [30]. Here we will show that our representation allows an affine extension. Analogously to [30] we make the ansatz that the affine charges act as copies of E_2, F_2, H_2. In other words, we set

$$E_4 = a_4\, \mathsf{a}_4^\dagger \mathsf{a}_2 + b_4\, \mathsf{a}_1^\dagger \mathsf{a}_3, \quad F_4 = c_4\, \mathsf{a}_3^\dagger \mathsf{a}_1 + d_4\, \mathsf{a}_2^\dagger \mathsf{a}_4, \quad H_4 = -C_4 + \frac{N_1 + N_3 - N_2 - N_4}{2}. \tag{4.1.37}$$

Checking all of the commutation relations is straightforward. Also, due to the defining relations (4.1.35), the equivalent expressions for the affine representation parameters are obtained

$$a_4d_4 = \frac{q^{\frac{M}{2}}V_4 - q^{-\frac{M}{2}}V_4^{-1}}{q^M - q^{-M}}, \qquad b_4c_4 = \frac{q^{-\frac{M}{2}}V_4 - q^{\frac{M}{2}}V_4^{-1}}{q^M - q^{-M}},$$
$$a_4b_4 = \frac{g_4\alpha_4}{[M]_q}(1 - U_4^2V_4^2), \qquad c_4d_4 = \frac{g_4\alpha_4^{-1}}{[M]_q}(V_4^{-2} - U_4^{-2}). \tag{4.1.38}$$

However the commutators between the generators E_2 and E_4 and also between F_2 and F_4 induce relations between a_2, a_4, etc. These are found to be

$$a_2d_4 = \frac{\tilde{g}\tilde{\alpha}^{-1}}{[M]_q}(q^{\frac{M}{2}}U_2U_4^{-1}V_2 - q^{-\frac{M}{2}}V_4^{-1}), \qquad b_2c_4 = \frac{\tilde{g}\tilde{\alpha}^{-1}}{[M]_q}(q^{-\frac{M}{2}}U_2U_4^{-1}V_2 - q^{\frac{M}{2}}V_4^{-1}),$$
$$c_2b_4 = \frac{\tilde{g}\tilde{\alpha}}{[M]_q}(q^{\frac{M}{2}}V_2^{-1} - q^{-\frac{M}{2}}U_2^{-1}U_4V_2), \qquad d_2a_4 = \frac{\tilde{g}\tilde{\alpha}}{[M]_q}(q^{-\frac{M}{2}}V_2^{-1} - q^{\frac{M}{2}}U_2^{-1}U_4V_2), \tag{4.1.39}$$

and agree with [30] upon sending $q \to q^M$, $\tilde{g} \to \frac{\tilde{g}}{[M]_q}$, as in the non-affine case. The tilded $\tilde{g}, \; \tilde{\alpha}$ are not independent but constrained parameters; thus there are 12 constraints for 12 parameters $\{a_k, b_k, c_k, d_k, U_k, V_k\}$.

Hopf algebra and variables. The Hopf algebra structure is just as previously discussed in Chapter 3. Here we will introduce Zhukowksy variables that will parameterize the representation labels $\{a_k, b_k, c_k, d_k\}$ and central elements U_k, V_k for the bound state representation. Following [30] we choose

$$g_2 = g_4 = g, \qquad \alpha_2 = \alpha_4\,\tilde{\alpha}^{-2} = \alpha, \qquad \tilde{g}^2 = \frac{g^2}{1 - g^2(q - q^{-1})^2}. \tag{4.1.40}$$

Note that the powers of q in the expressions above are 1 and not M because $g^2(q - q^{-1})^2$ is invariant under the bound state map $(g,\, q) \mapsto (g/[M]_q,\, q^M)$, thus these equations are identical to the ones for the fundamental representation.

Also, there is a relation between the central elements of the algebra,

$$U_4 = \pm U_2^{-1}, \qquad V_4 = \pm V_2^{-1}, \tag{4.1.41}$$

that are called the two-parameter family of the representation [30]. We shall be using the *plus* relation in our calculations.

The mass-shell constraint (multiplet shortening condition) obtained from the expressions (4.1.35) and (4.1.38) reads as

$$(a_k d_k - q^M b_k c_k)(a_k d_k - q^{-M} b_k c_k) = 1, \tag{4.1.42}$$

and holds independently for $k = 2, 4$. In terms of the conventional $x^\pm$ parametrization it becomes

$$\frac{1}{q^M}\big(x^+ + \frac{1}{x^+}\big) - q^M\big(x^- + \frac{1}{x^-}\big) = \big(q^M - \frac{1}{q^M}\big)\big(\xi + \frac{1}{\xi}\big), \tag{4.1.43}$$

where $\xi = -i\tilde{g}(q - q^{-1})$. One can further introduce a function $\zeta(x)$

$$\zeta(x) = -\frac{x + 1/x + \xi + 1/\xi}{\xi - 1/\xi}, \tag{4.1.44}$$

in terms of which (4.1.43) becomes $q^{-M}\zeta(x^+) = q^M\zeta(x^-)$. This parametrization leads to the following expressions of the labels a_k, b_k, c_k, d_k of a 'canonical form':

$$\begin{aligned}
a_k &= \sqrt{\frac{g}{[M]_q}}\gamma_k, & b_k &= \sqrt{\frac{g}{[M]_q}}\frac{\alpha_k}{\gamma_k}\frac{x_k^- - x_k^+}{x_k^-}, \\
c_k &= \sqrt{\frac{g}{[M]_q}}\frac{\gamma_k}{V_k\,\alpha_k}\frac{i\,q^{\frac{M}{2}}\,\tilde{g}}{g(x_k^+ + \xi)}, & d_k &= \sqrt{\frac{g}{[M]_q}}\frac{V_k\,\tilde{g}\,q^{\frac{M}{2}}}{i\,g\,\gamma_k}\frac{x_k^+ - x_k^-}{\xi x_k^+ + 1},
\end{aligned} \tag{4.1.45}$$

where the representation of the central elements is

$$U_k^2 = \frac{1}{q^M}\frac{x_k^+ + \xi}{x_k^- + \xi} = q^M\frac{x_k^+}{x_k^-}\frac{\xi x_k^- + 1}{\xi x_k^+ + 1}, \qquad V_k^2 = \frac{1}{q^M}\frac{\xi x_k^+ + 1}{\xi x_k^- + 1} = q^M\frac{x_k^+}{x_k^-}\frac{x_k^- + \xi}{x_k^+ + \xi}, \tag{4.1.46}$$

and the relations between $x_2^\pm,\ \gamma_2$ and $x_4^\pm,\ \gamma_4$ are constrained by (4.1.39) to be

$$x_2^\pm = x^\pm, \qquad x_4^\pm = \frac{1}{x^\pm}, \qquad \gamma_2 = \gamma, \qquad \gamma_4 = \frac{i\tilde{\alpha}\gamma}{x^+}. \tag{4.1.47}$$

The relation between normalization coefficients α_2 and α_4 was given in (4.1.40). Finally, the convenient multiplicative evaluation parameter z for the bound state representation is

$$z = \frac{1 - U^2V^2}{V^2 - U^2} = q^{-M}\zeta(x^+) = q^M\,\zeta(x^-)\,. \tag{4.1.48}$$

4.1.3.3 Summary

For the convenience of the reader we want to summarize all expressions that will be used in the subsequent calculations of the bound state S-matrix. We will slightly change the notation for parameters related to the fermionic nodes. We rename the representation parameters and the central elements of the algebra as

$$\begin{aligned}(a_2, b_2, c_2, d_2, U_2, V_2) &\to (a, b, c, d, U, V),\\ (a_4, b_4, c_4, d_4, U_4, V_4) &\to (\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}, \widetilde{U}, \widetilde{V}),\end{aligned} \tag{4.1.49}$$

in order to reserve the subscript position for discriminating states living in different tensor spaces. We will also give some relations that we found to be very useful.

Explicit representation. The bound state representation is defined as

$$|m,n,k,l\rangle = (\mathsf{a}_3^\dagger)^m(\mathsf{a}_4^\dagger)^n(\mathsf{a}_1^\dagger)^k(\mathsf{a}_2^\dagger)^l\,|0\rangle. \tag{4.1.50}$$

The total number of excitations is $k + l + m + n = M$. The triple corresponding to the bosonic $\mathfrak{sl}(2)$ is given by

$$\begin{aligned}&H_1|m,n,k,l\rangle = (l-k)|m,n,k,l\rangle,\\ &E_1|m,n,k,l\rangle = [k]_q\,|m,n,k-1,l+1\rangle, \quad F_1|m,n,k,l\rangle = [l]_q\,|m,n,k+1,l-1\rangle.\end{aligned} \tag{4.1.51}$$

The fermionic part is

$$\begin{aligned}&H_3|m,n,k,l\rangle = (n-m)|m,n,k,l\rangle,\\ &E_3|m,n,k,l\rangle = |m+1,n-1,k,l\rangle, \qquad F_3|m,n,k,l\rangle = |m-1,n+1,k,l\rangle.\end{aligned} \tag{4.1.52}$$

The action of the supercharges is given by

$$\begin{aligned}H_2|m,n,k,l\rangle &= -\{C - \tfrac{1}{2}(k-l+m-n)\}|m,n,k,l\rangle,\\ E_2|m,n,k,l\rangle &= a\,(-1)^m[l]_q\,|m,n+1,k,l-1\rangle + b\,|m-1,n,k+1,l\rangle,\\ F_2|m,n,k,l\rangle &= c\,[k]_q\,|m+1,n,k-1,l\rangle + d\,(-1)^m\,|m,n-1,k,l+1\rangle.\end{aligned} \tag{4.1.53}$$

The parameters a, b, c, d are related to the central charges via (4.1.31). The affine charges are defined exactly in the same way,

$$\begin{aligned}H_4|m,n,k,l\rangle &= -\{\widetilde{C} - \tfrac{1}{2}(k-l+m-n)\}|m,n,k,l\rangle,\\ E_4|m,n,k,l\rangle &= \tilde{a}\,(-1)^m[l]_q|m,n+1,k,l-1\rangle + \tilde{b}\,|m-1,n,k+1,l\rangle,\\ F_4|m,n,k,l\rangle &= \tilde{c}\,[k]_q\,|m+1,n,k-1,l\rangle + \tilde{d}\,(-1)^m\,|m,n-1,k,l+1\rangle.\end{aligned} \tag{4.1.54}$$

The representation labels a, b, c, d are given by

$$a = \sqrt{\frac{g}{[M]_q}}\gamma, \qquad b = \sqrt{\frac{g}{[M]_q}}\frac{\alpha}{\gamma}\frac{x^- - x^+}{x^-},$$
$$c = \sqrt{\frac{g}{[M]_q}}\frac{\gamma}{\alpha V}\frac{i\, q^{\frac{M}{2}}\tilde{g}}{g(x^+ + \xi)}, \qquad d = \sqrt{\frac{g}{[M]_q}}\frac{\tilde{g}\, q^{\frac{M}{2}} V}{i\, g\, \gamma}\frac{x^+ - x^-}{\xi x^+ + 1}, \tag{4.1.55}$$

and the affine parameters $\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d}$ are acquired by replacing $V \to \widetilde{V} = V^{-1}$, $\gamma \to \frac{i\tilde{\alpha}\gamma}{x^+}$, $\alpha \to \alpha\,\tilde{\alpha}^2$ and $x^\pm \to \frac{1}{x^\pm}$; the corresponding central elements are given by $V = q^C$, $\widetilde{V} = q^{\widetilde{C}}$.

Useful relations. The evaluation parameter z may be expressed explicitly in terms of $x^\pm$ parametrization as

$$z\,(q - q^{-1})(\xi - \xi^{-1}) = -\frac{1}{[M]_q}\Big(x^+ - x^- + \frac{1}{x^+} - \frac{1}{x^-}\Big). \tag{4.1.56}$$

Then using the identity

$$\xi - \xi^{-1} = \frac{\tilde{g}}{i(q - q^{-1})g^2}, \tag{4.1.57}$$

one can further show that it is related to the representation labels (4.1.55) and their affine partners in a very nice way,

$$z = \frac{g}{\tilde{g}\,\alpha\,\tilde{\alpha}}(a\tilde{b} - b\tilde{a}), \qquad \frac{1}{z} = \frac{g\,\alpha\,\tilde{\alpha}}{\tilde{g}}(c\tilde{d} - d\tilde{c}), \tag{4.1.58}$$

while the consistency conditions (4.1.39) give

$$z = \frac{1 - U^2V^2}{V^2 - U^2} = \frac{1 - \widetilde{U}^2\widetilde{V}^2}{\widetilde{V}^2 - \widetilde{U}^2}\,. \tag{4.1.59}$$

Rational limit. The rational limit is usually obtained by substituting $q = 1 + h$ and then finding the $h \to 0$ limit. Thus by defining the evaluation parameter (4.1.48) as $z = q^{-2u}$ we can expand it in series of h as [30]

$$z = 1 - 2hu + \mathcal{O}(h^2), \qquad \text{where} \qquad u = \frac{ig}{2}(x^+ + x^-)(1 + 1/x^+x^-). \tag{4.1.60}$$

It is noted that the $x^\pm$ parameters in (4.1.60) satisfies the leading order of the following relation which is stemming from the mass-shell constraint (4.1.43) in the $h \to 0$ limit,

$$x^+ + \frac{1}{x^+} - x^- - \frac{1}{x^-} = \frac{iM}{g} + 2hMu + \mathcal{O}(h^2). \tag{4.1.61}$$

In fact, this is consistent with the rational constraint for $x^\pm$ parameters [109]. Finally, it would be important to see how the representation parameters reduce in the rational limit.

The representation labels (4.1.55) in the $q \to 1$ limit reduce to the usual (undeformed) labels (a, b, c, d) of [109]. On the other hand, the affine parameters are related to the non-affine ones $(\tilde{a}, \tilde{b}, \tilde{c}, \tilde{d})$ through [30]

$$M\widetilde{T} = \begin{pmatrix} z^{-1} & 0 \\ 0 & 1 \end{pmatrix} T \begin{pmatrix} w^{-1} & 0 \\ 0 & wz \end{pmatrix} \quad \text{with} \quad M = \begin{pmatrix} 0 & \alpha\tilde{\alpha} \\ -\alpha^{-1}\tilde{\alpha}^{-1} & 0 \end{pmatrix}, \quad T = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix} \tag{4.1.62}$$

where z is the evaluation parameter given in (4.1.48), (4.1.59) and w is defined by

$$w = \frac{\tilde{g}V}{gq^{1/2}} \frac{qU^2 - 1}{V^2U^2 - 1} = \frac{gq^{1/2}}{\tilde{g}V} \frac{U^2 - V^2}{U^2 - q}. \tag{4.1.63}$$

Since the central elements specialize to $(U, V) \to (\sqrt{\frac{x^+}{x^-}}, 1)$ in the limit $q \to 1$, it is easy to see that the matrix relation (4.1.62) reduces the following simple form,

$$M\widetilde{T} = T. \tag{4.1.64}$$

4.2 q-deformed S-matrix

In this section we will construct the bound state S-matrix which is an intertwining matrix of the tensor space furnished by the vectors

$$|m_1, n_1, k_1, l_1\rangle \otimes |m_2, n_2, k_2, l_2\rangle \ \in V_{M_1} \otimes V_{M_2}. \tag{4.2.1}$$

Here $0 \le m_1, n_1, m_2, n_2 \le 1$ and $k_1, l_1, k_2, l_2 \ge 0$ denote the numbers of fermionic and bosonic excitations respectively with the bound state number M_i being the total number of excitations, $M_i = m_i + n_i + k_i + l_i$. Thus the S-matrix is the automorphism of the quantum deformed tensor space and is required to intertwine the coproduct and the opposite coproduct of the affine algebra $\widehat{\mathcal{Q}}$,

$$\left[S\,\Delta(J) - \Delta^{op}(J)\,S\right] V_{M_1} \otimes V_{M_2}\,, \qquad \text{for all} \qquad J \in \widehat{\mathcal{Q}}\,. \tag{4.2.2}$$

We normalize the S-matrix in such a way that the state $|0, 0, 0, M_1\rangle \otimes |0, 0, 0, M_2\rangle$ is invariant under the scattering. Therefore we will denote the state

$$|0\rangle = |0, 0, 0, M_1\rangle \otimes |0, 0, 0, M_2\rangle, \tag{4.2.3}$$

as the vacuum state.

The invariance under bosonic symmetries $\Delta(H_1)$ and $\Delta(H_3)$ requires the total number of fermions and the total number of fermions of one type[1]

$$\begin{aligned} N_f &= m_1 + m_2 + n_1 + n_2 + 2l_1 + 2l_2, \\ N_{f_3} &= m_1 + m_2 + l_1 + l_2. \end{aligned} \tag{4.2.4}$$

to be conserved. This conservation divides the space (4.2.1) into five types of invariant subspaces of the S-matrix:

[1] Note that a bosonic excitation may be interpreted as a combined excitation of two fermions of different type.

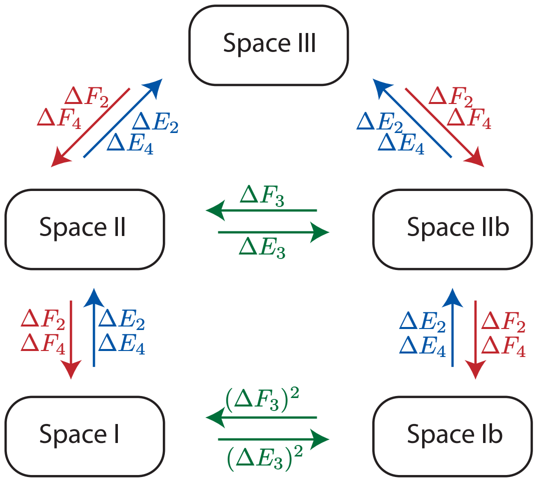

Figure 4.3: The invariant subspaces of the S-matrix and the algebraic relations between them.

I $|0,1,k_1,l_1\rangle \otimes |0,1,k_2,l_2\rangle$,

Ib $|1,0,k_1,l_1\rangle \otimes |1,0,k_2,l_2\rangle$,

II $\{|0,0,k_1,l_1\rangle \otimes |0,1,k_2,l_2\rangle, |1,1,k_1,l_1\rangle \otimes |0,1,k_2,l_2\rangle,$
$|0,1,k_1,l_1\rangle \otimes |0,0,k_2,l_2\rangle, |0,1,k_1,l_1\rangle \otimes |1,1,k_2,l_2\rangle\}$,

IIb $\{|0,0,k_1,l_1\rangle \otimes |1,0,k_2,l_2\rangle, |1,1,k_1,l_1\rangle \otimes |1,0,k_2,l_2\rangle,$
$|1,0,k_1,l_1\rangle \otimes |0,0,k_2,l_2\rangle, |1,0,k_1,l_1\rangle \otimes |1,1,k_2,l_2\rangle\}$,

III $\{|0,0,k_1,l_1\rangle \otimes |0,0,k_2,l_2\rangle, |0,0,k_1,l_1\rangle \otimes |1,1,k_2,l_2\rangle, |1,1,k_1,l_1\rangle \otimes |0,0,k_2,l_2\rangle,$
$|1,1,k_1,l_1\rangle \otimes |1,1,k_2,l_2\rangle, |0,1,k_1,l_1\rangle \otimes |1,0,k_2,l_2\rangle, |1,0,k_1,l_1\rangle \otimes |0,1,k_2,l_2\rangle\}$.

Subspaces I, Ib and II, IIb are isomorphic, hence we need to find the S-matrix for one of the isomorphic subspaces only. In the following we will consider the scattering in the subspaces I, II and III only.

The invariant subspaces differ by the numbers N_{f,f_3}. By considering the action of the algebra charges it is easy to see that the different subspaces are related to each other in the way shown in figure 4.3.

Finally we want to give a remark on our choice of the basis. The q-oscillator basis we are considering is orthogonal, but not orthonormal,

$$\langle m',n',k',l'|m,n,k,l\rangle = \frac{1}{[k]!\,[l]!}\,\delta_{m,m'}\delta_{n,n'}\delta_{k,k'}\delta_{l,l'}, \tag{4.2.5}$$

where $[n]! = [n]_q[n-1]_q \cdots [1]_q$ is the quantum factorial. We shall choose the normalization for the *bra* vectors to be

$$\langle m,n,k,l| := \frac{1}{[k]!\,[l]!}\,|m,n,k,l\rangle^\dagger. \tag{4.2.6}$$

which helps us to normalize the scalar product to unity and avoid the appearance of unpleasant numerical factors of the form $\big([k]![l]!\big)^{-1/2}$ in the derivations. The price we have to pay for this choice of the basis is that, for real q, the inverse of the S-matrix is related to the Hermitian conjugate only up to a basis transformation. For complex q this property is not valid even for the fundamental representation [28].
For further convenience we introduce these shorthands

$$\begin{aligned} &M = M_1 + M_2, \qquad \delta M = M_1 - M_2, \qquad K = k_1 + k_2, \qquad \delta K = k_1 - k_2,\\ &\bar{k}_i = M_i - k_i - 1, \quad \delta k_i = \bar{k}_i - k_i = M_i - 2k_i - 1, \quad z_{12} = z_1/z_2, \quad \delta u = u_1 - u_2. \end{aligned} \tag{4.2.7}$$

4.2.1 Scattering in subspace I

The conserved fermionic numbers (4.2.4) for the subspace I are $N_f = 2K + 2$ and $N_{f_3} = K + 2$. Thus for the fixed K ($0 \leq K \leq M_1 + M_2 - 2$) the dimension of the space is $K + 1$ and the states in this space are defined as

$$|k_1, k_2\rangle^{\mathrm{I}} = |0, 1, k_1, M_1 - k_1 - 1\rangle \otimes |0, 1, k_2, M_2 - k_2 - 1\rangle. \tag{4.2.8}$$

We start by considering the highest weight state (the state with $k_1 = k_2 = 0$). The invariance under $\Delta(H_1$ and $\Delta(H_3$ requires it to be an eigenstate of the S-matrix,

$$S\,|0,0\rangle^{\mathrm{I}} = \mathscr{D}\,|0,0\rangle^{\mathrm{I}}. \tag{4.2.9}$$

Let us compute $\mathscr{D}$. First, we construct the highest weight state by acting with the combination $\Delta(E_2)\Delta(E_4)$ on the vacuum state (4.2.3) (we use the notation $a_i \equiv a(p_i)$ etc.)

$$\Delta(E_2)\Delta(E_4)\,|0\rangle = q^{\frac{M_1}{2}}[M_1]_q[M_2]_q\,(a_1\tilde{a}_2\,\widetilde{U}_1\widetilde{V}_1 - a_2\tilde{a}_1\,U_1V_1)\,|0,0\rangle^{\mathrm{I}}. \tag{4.2.10}$$

This construction let us to rewrite (4.2.9) as

$$\begin{aligned} S\,|0,0\rangle^{\mathrm{I}} &= \frac{S\,\Delta(E_2)\Delta(E_4)}{q^{\frac{M_1}{2}}[M_1]_q[M_2]_q\,(a_1\tilde{a}_2\,\widetilde{U}_1\widetilde{V}_1 - a_2\tilde{a}_1\,V_1U_1)}\,|0\rangle\\ &= \frac{\Delta^{op}(E_2)\Delta^{op}(E_4)\,S}{q^{\frac{M_1}{2}}[M_1]_q[M_2]_q\,(a_1\tilde{a}_2\,\widetilde{U}_1\widetilde{V}_1 - a_2\tilde{a}_1\,V_1U_1)}\,|0\rangle\\ &= -q^{\frac{M_2-M_1}{2}}\,\frac{a_2\tilde{a}_1\,\widetilde{U}_2\widetilde{V}_2 - a_1\tilde{a}_2\,V_2U_2}{a_1\tilde{a}_2\,\widetilde{U}_1\widetilde{V}_1 - a_2\tilde{a}_1\,V_1U_1}\,|0,0\rangle^{\mathrm{I}}, \end{aligned} \tag{4.2.11}$$

where we have used the invariance condition (4.2.2) when going from the first to the second line. Comparing (4.2.11) with (4.2.9) we find $\mathscr{D}$ to be

$$\mathscr{D} = -q^{\frac{M_2-M_1}{2}}\,\frac{a_2\tilde{a}_1\,\widetilde{U}_2\widetilde{V}_2 - a_1\tilde{a}_2\,V_2U_2}{a_1\tilde{a}_2\,\widetilde{U}_1\widetilde{V}_1 - a_2\tilde{a}_1\,V_1U_1} = q^{-\delta M/2}\frac{U_2V_2}{U_1V_1}\frac{x_1^+ - x_2^-}{x_1^- - x_2^+}. \tag{4.2.12}$$

In the $q \to 1$ limit this is the inverse of the result found in [109] due to the interchange of Δ and Δ^{op} with respect to the ones in [109].

Next we define the action of the S-matrix on the subspace I to be

$$S\,|k_1,k_2\rangle^{\mathrm{I}} = \sum_{n=0}^{K} \mathscr{X}_n^{k_1,k_2} |n, K-n\rangle^{\mathrm{I}}. \tag{4.2.13}$$

The strategy for finding coefficients $\mathscr{X}_n^{k_1,k_2}$ will be based on building the generic state $|k_1,k_2\rangle^{\mathrm{I}}$ by starting from the highest weight state $|0,0\rangle^{\mathrm{I}}$. This allows us to relate $\mathscr{X}_n^{k_1,k_2}$ with any k_1, k_2 and n to the already known coefficient $\mathscr{D}$. Thus we need to construct k_1- and k_2-raising operators. We start from inspecting the action of the coproduct of the bosonic charge F_1 giving

$$\Delta(F_1)|k_1,k_2\rangle^{\mathrm{I}} = [\bar{k}_1]_q\, q^{\delta k_2}|k_1+1,k_2\rangle^{\mathrm{I}} + [\bar{k}_2]_q\,|k_1,k_2+1\rangle^{\mathrm{I}}, \tag{4.2.14}$$

and

$$\Delta^{op}(F_1)|k_1,k_2\rangle^{\mathrm{I}} = [\bar{k}_1]_q\,|k_1+1,k_2\rangle^{\mathrm{I}} + [\bar{k}_2]_q\, q^{\delta k_1}|k_1,k_2+1\rangle^{\mathrm{I}}. \tag{4.2.15}$$

These coproducts do not have the desired properties we want, but are very close. However, with the help of E_2, E_3 and E_4 we can construct a new charge with a similar action,

$$\hat{F}_1 = \frac{g}{\tilde{g}\,\alpha\,\tilde{\alpha}}\,\{E_2,[E_4,E_3]\}. \tag{4.2.16}$$

We call this new charge 'the affine partner' of the raising charge F_1. The action of $\hat{F}_1$ on the state of the form $|0,1,k,l\rangle$ is

$$\hat{F}_1\,|0,1,k,l\rangle = z\,[l]_q\,|0,1,k+1,l-1\rangle, \tag{4.2.17}$$

where we have used (4.1.56) implicitly[2]. Then it is straightforward to see that the new affine raising charge acts on generic states in subspace I as

$$\Delta(\hat{F}_1)\,|k_1,k_2\rangle^{\mathrm{I}} = z_1\,[\bar{k}_1]_q\,|k_1+1,k_2\rangle^{\mathrm{I}} + z_2\,q^{\delta k_1}\,[\bar{k}_2]_q\,|k_1,k_2+1\rangle^{\mathrm{I}}. \tag{4.2.19}$$

And the action of $\Delta^{op}(\hat{F}_1)$ is

$$\Delta^{op}(\hat{F}_1)\,|k_1,k_2\rangle^{\mathrm{I}} = z_1\,q^{\delta k_2}\,[\bar{k}_1]_q\,|k_1+1,k_2\rangle^{\mathrm{I}} + z_2\,[\bar{k}_2]_q\,|k_1,k_2+1\rangle^{\mathrm{I}}. \tag{4.2.20}$$

By combining $\Delta(\hat{F}_1)$ with $\Delta(F_1)$ we obtain composite operators having the action of the desired form – raising k_1 and k_2 separately:

$$|k_1+1,k_2\rangle^{\mathrm{I}} = \frac{1}{[\bar{k}_1]_q}\,\frac{\Delta(\hat{F}_1) - z_2\,q^{\delta k_1}\Delta(F_1)}{z_1 - z_2\,q^{\delta k_1+\delta k_2}}\,|k_1,k_2\rangle^{\mathrm{I}}, \tag{4.2.21}$$

$$|k_1,k_2+1\rangle^{\mathrm{I}} = \frac{1}{[\bar{k}_2]_q}\,\frac{z_1\,\Delta(F_1) - q^{\delta k_2}\Delta(\hat{F}_1)}{z_1 - z_2\,q^{\delta k_1+\delta k_2}}\,|k_1,k_2\rangle^{\mathrm{I}}. \tag{4.2.22}$$

[2]For the consistency of the algebra we also give a definition of the 'affine lowering charge' $\hat{E}_1$:

$$\hat{E}_1 = \frac{g\,\alpha\,\tilde{\alpha}}{\tilde{g}}\,\{F_2,[F_4,F_3]\}, \qquad \hat{E}_1|0,1,k,l\rangle = \frac{[k]_q}{z}\,|0,1,k-1,l+1\rangle. \tag{4.2.18}$$

Then by induction we find that the generic state $|k_1, k_2\rangle^{\mathrm{I}}$ may be constructed as

$$|k_1, k_2\rangle^{\mathrm{I}} = \frac{\prod_{j_2=0}^{k_2-1}(z_1\, \Delta(F_1) - q^{\delta j_2}\Delta(\hat{F}_1)) \prod_{i_1=0}^{k_1-1}(\Delta(\hat{F}_1) - z_2\, q^{\delta i_1}\Delta(F_1))}{\prod_{i=1}^{k_1}[M_1 - i]_q \prod_{j=1}^{k_2}[M_2 - j]_q \prod_{j=1}^{k_1+k_2}(z_1 - z_2 q^{M-2j})} |0, 0\rangle^{\mathrm{I}}. \quad (4.2.23)$$

Finding $\mathscr{X}_n^{k_1,k_2}$ is then straightforward. We only need to act with the S-matrix on the expression above and sandwich with a *bra*-vector as

$$\mathscr{X}_n^{k_1,k_2} = {}^{\mathrm{I}}\langle n, K - n|\, S\, |k_1, k_2\rangle^{\mathrm{I}}. \quad (4.2.24)$$

Performing similar steps as we did in (4.2.11) and employing the relations

$$\begin{aligned}
&(\Delta^{op}(\hat{F}_1) - z_2\, q^{\delta k_1}\Delta^{op}(F_1))\, |n_1, n_2\rangle^{\mathrm{I}} \\
&\quad = [\bar{n}_2]_q\, z_2\, (1 - q^{\delta k_1 + \delta n_1})\, |n_1, n_2 + 1\rangle^{\mathrm{I}} + [\bar{n}_1]_q\, (z_1\, q^{\delta n_2} - z_2\, q^{\delta k_1})\, |n_1 + 1, n_2\rangle^{\mathrm{I}}, \quad (4.2.25)\\
&(z_1\, \Delta^{op}(F_1) - q^{\delta k_2}\Delta^{op}(\hat{F}_1))\, |n_1, n_2\rangle^{\mathrm{I}} \\
&\quad = [\bar{n}_1]_q\, z_1\, (1 - q^{\delta n_2 + \delta k_2})\, |n_1 + 1, n_2\rangle^{\mathrm{I}} + [\bar{n}_2]_q\, (z_1 q^{\delta n_1} - z_2 q^{\delta k_2})|n_1, n_2 + 1\rangle^{\mathrm{I}}, \quad (4.2.26)
\end{aligned}$$

we find the coefficients of the S-matrix in the subspace I to be

$$\begin{aligned}
\mathscr{X}_n^{k_1,k_2} = \mathscr{D}\, &\frac{\prod_{i=1}^{n}[M_1 - i]_q \prod_{j=1}^{K-n}[M_2 - j]_q}{\prod_{i=1}^{k_1}[M_1 - i]_q \prod_{j=1}^{k_2}[M_2 - j]_q} \frac{1}{\prod_{l=1}^{K}(z_{12} - q^{M-2l})} \\
\times &\sum_{m=0}^{k_1} \Bigg(z_{12}^{n-m} q^{k_2(n-m) - k_1 m - k_2^2} \left[\begin{array}{c} k_1 \\ m \end{array}\right]_q \left[\begin{array}{c} k_2 \\ n - m \end{array}\right]_q \\
&\times \prod_{p=0}^{m-1}(z_{12}\, q^{M_2+2p} - q^{M_1}) \prod_{p=1+m}^{k_1}(1 - q^{2(M_1-p)}) \\
&\times \prod_{p=1}^{n-m}(1 - q^{2(M_2-K+n-p)}) \prod_{p=-m}^{k_2-n-1}(z_{12}\, q^{M_1+2p} - q^{M_2}) \Bigg), \quad (4.2.27)
\end{aligned}$$

where $z_{12} = \frac{z_1}{z_2}$ and the q-binomials are defined as

$$\left[\begin{array}{c} a \\ b \end{array}\right]_q \equiv \frac{[a]_q!}{[b]_q![a - b]_q!}. \quad (4.2.28)$$

Apart from the prefactor $\mathscr{D}$, this expression only depends on the quotient z_{12} and on simple q-factors. The expression above has exactly the form that one would expect to obtain by an educated guess relying on the one given in [109].

Quantum $6j$-Symbol. The coefficients $\mathscr{X}_n^{k_1,k_2}$ of the bound state S-matrix may be regarded as the coefficients which arise in the fusion rule of the irreducible representations of $\mathcal{U}_q(su(2))$, thus it is expected that the expression (4.2.27) is related to the quantum $6j$-symbol, which is the q-deformation of $6j$-symbol and was first introduced in [178].

In order to see the relation with the quantum $6j$-symbol, we first rewrite (4.2.27) in terms of quantum factorials. This can be done by introducing the notation $z_{12} = q^{-2\delta u}$ and using the following identity several times,

$$\frac{q^A - q^B}{q - q^{-1}} = q^{\frac{A+B}{2}} \left[\frac{A-B}{2}\right]_q . \tag{4.2.29}$$

Secondary, we shift the index of summation m to $M_1 - 2 - m$. After some computation, we obtain the following form,

$$\begin{aligned}\mathscr{X}_n^{k_1,k_2} &= \mathscr{D}\, q^{(k_1-n)(k_2-n+\delta u+\frac{\delta M}{2})} \frac{[M_2-k_2-1]!}{[M_1-n-1]!} \frac{[\delta u + \frac{M}{2} - 1 - K]!}{[\delta u + \frac{M}{2} - 1]!} \\ &\times [k_1]![k_2]![\delta u + \tfrac{\delta M}{2}]![\delta u - \tfrac{\delta M}{2} - k_2 + n + 1]! \\ &\times \sum_{m\geq 0} [m+1]!\, ([m - M_1 + 2 + k_1]!\, [m - M_1 + 2 + n]!\, [k_2 - n + M_1 - 2 - m]! \\ &\quad \times [m + \delta u - \tfrac{M}{2} + 2]!\, [\delta u + \tfrac{M}{2} - 1 - m]!\, [M_1 - 2 - m]!\, [M - K - 3 - m]!)^{-1} .\end{aligned} \tag{4.2.30}$$

where the summation index m runs over the non-negative integers such that all arguments of the quantum factorials, which do not include δu, are non-negative. Finally, replacing the six variables $(M_1, M_2, k_1, k_2, n, \delta u)$ by the appropriate combinations of the set $(j_1, j_2, j_3, j_4, j_5, j_6)$ as (see also [109]),

$$\begin{aligned} j_1 &= \tfrac{1}{2}(K - n + \tfrac{\delta M}{2} + \delta u), & j_4 &= \tfrac{1}{2}(\tfrac{\delta M}{2} - 1 + k_2 - \delta u), \\ j_2 &= \tfrac{1}{2}(\tfrac{M}{2} - 2 - k_2 - \delta u), & j_5 &= \tfrac{1}{2}(\tfrac{M}{2} - 1 - K + n + \delta u), \\ j_3 &= \tfrac{1}{2}(M_1 - 2 - k_1 - n), & j_6 &= \tfrac{1}{2}(M_2 - 1), \end{aligned} \tag{4.2.31}$$

we have found that the expression (4.2.27) obtains a quite elegant form

$$\begin{aligned}\mathscr{X}_n^{k_1,k_2} &= \mathscr{D}\, (-1)^{j_1-j_3-j_4+2j_5+j_6} q^{(j_1-j_2+j_3)(j_1+j_2-j_4-j_5)} \frac{[j_1+j_2-j_3]!}{[1+j_1+j_2+j_3]!} \frac{[j_1+j_5-j_6]!}{[j_1+j_5+j_6]!} \\ &\times [j_3 - j_4 + j_5]!\, [j_3 + j_4 - j_5]!\, [j_2 - j_4 + j_6]!\, [-j_2 + j_4 + j_6]! \begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix},\end{aligned} \tag{4.2.32}$$

where we have defined the rescaled quantum $6j$-symbol by

$$\begin{aligned}\begin{vmatrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{vmatrix} &= \sum_{m\geq 0} (-1)^m [m+1]! \big([j_{1245} - m]!\, [j_{1346} - m]!\, [j_{2356} - m]! \\ &\quad \times [m - j_{123}]!\, [m - j_{345}]!\, [m - j_{246}]!\, [m - j_{156}]!\big)^{-1}.\end{aligned} \tag{4.2.33}$$

Here we have used bookkeeping notations $j_{abc} = j_a + j_b + j_c$ and $j_{abcd} = j_a + j_b + j_c + j_d$. The above expression is related with the quantum $6j$-symbol introduced in [178] as

$$\left\{ \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right\} = \sqrt{2j_3 - 1}\sqrt{2j_6 - 1}\,(-1)^{-j_1 - j_2 + 2j_3 + j_4 + j_5}$$
$$\times \Delta(j_1, j_2, j_3)\Delta(j_1, j_5, j_6)\Delta(j_2, j_4, j_6)\Delta(j_3, j_4, j_5) \left| \begin{matrix} j_1 & j_2 & j_3 \\ j_4 & j_5 & j_6 \end{matrix} \right|, \quad (4.2.34)$$

where the triangle coefficient $\Delta(a, b, c)$ is defined to be

$$\Delta(a, b, c) = \left[\frac{[a + b - c]!\,[b + c - a]!\,[c + a - b]!}{[1 + a + b + c]!} \right]^{1/2}. \quad (4.2.35)$$

Rational Limit. In order to find the rational limit of the matrix $\mathscr{X}$ (4.2.27) we first use the expansion (4.1.60) for the spectral parameter z. This leads to

$$\begin{aligned} \mathscr{X}_n^{k_1,k_2} = \mathscr{D}\, &\frac{\prod_{i=1}^{n}[M_1 - i]_q \prod_{j=1}^{K-n}[M_2 - j]_q}{\prod_{i=1}^{k_1}[M_1 - i]_q \prod_{j=1}^{k_2}[M_2 - j]_q} \frac{1}{\prod_{l=1}^{K}(z_{12}^{1/2}[\delta u]_q + q^{M/2 - l}[\frac{M}{2} - l]_q)} \\ &\times \sum_{m=0}^{k_1} \left(z_{12}^{n-m} q^{k_2(n-m) - k_1 m - k_2^2} \left[\begin{matrix} k_1 \\ m \end{matrix} \right]_q \left[\begin{matrix} k_2 \\ n - m \end{matrix} \right]_q \right. \\ &\qquad \times \prod_{p=0}^{m-1} \left(z_{12}^{1/2}\, q^{M_2/2 + p} \left[\delta u - \frac{M_2}{2} - p \right]_q + q^{M_1/2} \left[\frac{M_1}{2} \right]_q \right) \\ &\qquad \times \prod_{p=-m}^{k_2 - n - 1} \left(z_{12}^{1/2}\, q^{M_1/2 + p} \left[\delta u - \frac{M_1}{2} - p \right]_q + q^{M_2/2} \left[\frac{M_2}{2} \right]_q \right) \\ &\qquad \left. \times \prod_{p=1+m}^{k_1} q^{M_1 - p}[M_1 - p]_q \prod_{p=1}^{n-m} q^{M_2 - K + n - p}[M_2 - K + n - p]_q \right), \end{aligned} \quad (4.2.36)$$

where $\delta u = u_1 - u_2$. Now we are ready to find $q \to 1$ limit. The q-numbers $[x]_q$ coalesce to x, thus (4.2.36) becomes

$$\begin{aligned} \mathscr{X}_n^{k_1,k_2} = \mathscr{D}\, &\frac{\prod_{i=1}^{n}(M_1 - i) \prod_{j=1}^{K-n}(M_2 - j)}{\prod_{i=1}^{k_1}(M_1 - i) \prod_{j=1}^{k_2}(M_2 - j)} \frac{1}{\prod_{l=1}^{K}(\delta u + \frac{M}{2} - l)} \\ &\times \sum_{m=0}^{k_1} \left(\left(\begin{matrix} k_1 \\ m \end{matrix} \right) \left(\begin{matrix} k_2 \\ n - m \end{matrix} \right) \prod_{p=0}^{m-1} \left(\delta u + \frac{\delta M}{2} - p \right) \prod_{p=-m}^{k_2 - n - 1} \left(\delta u - \frac{\delta M}{2} - p \right) \right. \\ &\qquad \left. \times \prod_{p=1+m}^{k_1} (M_1 - p) \prod_{p=1}^{n-m} (M_2 - K + n - p) \right). \end{aligned} \quad (4.2.37)$$

This result coincides exactly with the expression obtained in [109][3]

Classical Limit. It is also important to find the classical limit $g \to \infty$ of (4.2.27). This limit corresponds to the case 'T(h)' in the analysis of the classical algebra [163], where the deformation parameter q is expanded as

$$q = 1 + \frac{h}{2g} + \mathcal{O}(g^{-2}), \tag{4.2.38}$$

and the $x^{\pm}$ parameters become

$$x^{\pm} = x\left[1 \pm \frac{hM}{2g}\frac{(x+\tilde{h})(1+1/x\tilde{h})}{x-x^{-1}} + \mathcal{O}(g^{-2})\right], \quad \text{where} \quad \tilde{h} = -\frac{ih}{\sqrt{1-h^2}}\,. \tag{4.2.39}$$

The above expressions are compatible with the constraint (4.1.43) up to a given order. Since $\xi \to \tilde{h}$ and $x^{\pm} \to x$ in the classical limit, it is easy to see that the evaluation parameter z reduces to[4]

$$z = -\frac{(x+\tilde{h})(1+1/x\tilde{h})}{\tilde{h}-\tilde{h}^{-1}} = -\frac{C+D}{C-D}\,, \tag{4.2.40}$$

where elements C and D are the classical limits of $U = q^D$ and $V = q^C$ respectively, and are given by

$$D = \tfrac{1}{2}(z+1)\,\tilde{q}\,, \qquad C = \tfrac{1}{2}(z-1)\,\tilde{q}\,, \qquad \text{where} \qquad \tilde{q} = -M\frac{\tilde{h}-\tilde{h}^{-1}}{x-x^{-1}}\,. \tag{4.2.41}$$

With these preliminaries, we find the classical limit of (4.2.27) to be

$$\begin{aligned}
\mathscr{X}^{k_1,k_2}_n \sim{}& (1+\mathscr{D}_{cl})\,\frac{\prod_{i=1}^{n}(M_1-i)\prod_{j=1}^{K-n}(M_2-j)}{\prod_{i=1}^{k_1}(M_1-i)\prod_{j=1}^{k_2}(M_2-j)}\left(1+\frac{h}{g}\sum_{l=1}^{k_1+k_2}\frac{\frac{M}{2}-l}{z_{12}-1}\right)\\
&\times\sum_{m=0}^{k_1}\Bigg[\left(-\frac{h}{g}\frac{1}{z_{12}-1}\right)^{k_1+n-2m} z_{12}^{n-m}\binom{k_1}{m}\binom{k_2}{n-m}\\
&\times\Bigg(1+\frac{h}{g}\sum_{p=0}^{m-1}\frac{z_{12}\left(\frac{M_2}{2}+p\right)-\frac{M_1}{2}}{z_{12}-1}+\frac{h}{g}\sum_{p=-m}^{k_2-n-1}\frac{z_{12}\left(\frac{M_1}{2}+p\right)-\frac{M_2}{2}}{z_{12}-1}\\
&\qquad+\frac{h}{2g}\big(k_2(n-m)-k_1m-k_2^2\big)\Bigg)\prod_{p=1+m}^{k_1}(M_1-p)\prod_{p=1}^{n-m}(M_2-K+n-p)\Bigg],
\end{aligned} \tag{4.2.42}$$

where $\mathscr{D}_{cl}$ is $\mathcal{O}(g^{-1})$ term of $\mathscr{D}$ in (4.2.27). Since the binomial coefficients force the index m to be $m \le \min\{k_1, n\}$, we will discuss the two possible cases separately. They are the

[3]The normalization of the evaluation parameter is slightly different in here, $u_{\text{here}} = -2u$ [109].

[4]The classical evaluation parameter given in [163] is related with ours as $z^{cl}_{[34]} = (z^{cl}_{\text{here}})^{-1}$ and the classical parameter is $x_{[34]} = -ih\tilde{h}^{-1}(x_{\text{here}} + \tilde{h})$.

$n \neq k_1$ case (off-diagonal sector) and the $n = k_1$ case (diagonal sector).

Off-diagonal sector. In the case when n is different from k_1, it is further classified by two more cases – if n is bigger or smaller than k_1. Firstly, in the $n > k_1$ case, the leading order of (4.2.42) is $\mathcal{O}(g^{-(n-k_1)})$ with $m = k_1$. Therefore the $\mathcal{O}(g^{-1})$ term, which contributes to the classical r-matrix, is obtained by setting $n = k_1 + 1$. In this situation, the classical limit of (4.2.42) turns out to be of a simple form,

$$\mathscr{X}^{k_1,k_2}_{k_1+1} \sim -\frac{h}{g}\frac{z_1}{z_1 - z_2}k_2(M_1 - k_1 - 1)\ . \tag{4.2.43}$$

Secondary, in the $n < k_1$ case, the leading order is $\mathcal{O}(g^{-(k_1-n)})$ with $m = n$. Therefore, the $\mathcal{O}(g^{-1})$ contribution is given by $n = k_1 - 1$. In this case the amplitude becomes

$$\mathscr{X}^{k_1,k_2}_{k_1-1} \sim -\frac{h}{g}\frac{z_2}{z_1 - z_2}k_1(M_2 - k_2 - 1)\ . \tag{4.2.44}$$

The other matrix elements do not contribute to the classical r-matrix.

Diagonal sector. This is the $n = k_1$ case and it needs a more elaborate treatment in comparison with the off-diagonal sector. In this case the leading order in (4.2.42) is $\mathcal{O}(1)$ with $m = k_1 = n$. Thus the classical limit turns out to be

$$\begin{aligned}\mathscr{X}^{k_1,k_2}_{k_1} \sim 1 + \mathscr{D}_{cl} - \frac{h}{2g}(k_1^2 + k_2^2) + \frac{h}{g}\frac{1}{z_1 - z_2}&\left[\sum_{l=1}^{k_1+k_2} z_2\left(\frac{M}{2} - l\right)\right.\\ &\left. + \sum_{p=0}^{k_1-1}\left(\frac{z_1M_2 - z_2M_1}{2} + z_1 p\right) + \sum_{p=-k_1}^{k_2-k_1-1}\left(\frac{z_1M_1 - z_2M_2}{2} + z_1 p\right)\right]\ .\end{aligned} \tag{4.2.45}$$

Full Rational Limit. It is noted that the classical limit still depends on the deformation parameter h. This allows us to take $h \to 0$ limit further, which corresponds to the case "R(full)" in the analysis of [163]. In this limit, the classical evaluation parameter (4.2.40) reads,

$$z \sim 1 - \frac{h}{g}u + \mathcal{O}(h^2), \qquad \text{with} \qquad u = x + \frac{1}{x}. \tag{4.2.46}$$

Then the off-diagonal elements of the classical r-matrix (4.2.43) and (4.2.44) turns out to be

$$\mathscr{X}^{k_1,k_2}_{k_1+1} \sim \frac{1}{\delta u}k_2(M_1 - k_1 - 1)\ , \qquad \mathscr{X}^{k_1,k_2}_{k_1-1} \sim \frac{1}{\delta u}k_1(M_2 - k_2 - 1)\ . \tag{4.2.47}$$

On the other hand, the diagonal elements (4.2.45) reduce to

$$\mathscr{X}^{k_1,k_2}_{k_1} \sim 1 + \mathscr{D}_{cl} - \frac{1}{\delta u}\left[\sum_{l=1}^{k_1+k_2}\left(\tfrac{M}{2} - l\right) + \sum_{p=0}^{k_1-1}\left(-\tfrac{\delta M}{2} + p\right) + \sum_{p=-k_1}^{k_2-k_1-1}\left(\tfrac{\delta M}{2} + p\right)\right]. \tag{4.2.48}$$

The above expressions (4.2.47) and (4.2.48) agree with the classical limits of rational case [109].

4.2.2 Scattering in subspace II

The S-matrix in the subspace II is defined to be

$$S\,|k_1,k_2\rangle_i^{\mathrm{II}} = \sum_{n=0}^{K}\sum_{j=1}^{4} |n,K-n\rangle_j^{\mathrm{II}} \left(\mathscr{Y}_n^{k_1,k_2}\right)_i^j\,, \tag{4.2.49}$$

and the standard $4N+2$–dimensional basis is

$$\begin{aligned}
|k_1,k_2\rangle_1^{\mathrm{II}} &= |0,1,k_1,M_1-k_1-1\rangle \otimes |0,0,k_2,M_2-k_2\rangle,\\
|k_1,k_2\rangle_2^{\mathrm{II}} &= |0,0,k_1,M_1-k_1\rangle \otimes |0,1,k_2,M_2-k_2-1\rangle,\\
|k_1,k_2\rangle_3^{\mathrm{II}} &= |0,1,k_1,M_1-k_1-1\rangle \otimes |1,1,k_2-1,M_2-k_2-1\rangle,\\
|k_1,k_2\rangle_4^{\mathrm{II}} &= |1,1,k_1-1,M_1-k_1-1\rangle \otimes |0,1,k_2,M_2-k_2-1\rangle.
\end{aligned} \tag{4.2.50}$$

We shall express the coefficients $(\mathscr{Y}_n^{k_1,k_2})_i^j$ in terms of already known $\mathscr{X}_n^{k_1,k_2}$ with the help of the charges $\Delta(E_2)$ and $\Delta(E_4)$ that relate the states in the subspace II to the states in subspace I:

$$\Delta(E_2)\,|k_1,k_2\rangle_j^{\mathrm{II}} = Q_j(k_1,k_2)\,|k_1,k_2\rangle^{\mathrm{I}}, \qquad \Delta(E_4)\,|k_1,k_2\rangle_j^{\mathrm{II}} = \widetilde{Q}_j(k_1,k_2)\,|k_1,k_2\rangle^{\mathrm{I}}. \tag{4.2.51}$$

The coefficients $Q_j(k_1,k_2)$, $\widetilde{Q}_j(k_1,k_2)$ and their partners for $\Delta^{op}(E_2)$ and $\Delta^{op}(E_4)$ are spelled out in the Appendix C.

The strategy of finding $\mathscr{X}_n^{k_1,k_2}$ is the following. We start by considering the matrix element

$$\begin{aligned}
{}^{\mathrm{I}}\langle n,K-n|\,\Delta^{op}(E_2)\,S\,|k_1,k_2\rangle_i^{\mathrm{II}} &= \sum_{j=1}^{4}\sum_{m=0}^{K} {}^{\mathrm{I}}\langle n,K-n|\,\Delta^{op}(E_2)\,|m,K-m\rangle_j^{\mathrm{II}} \left(\mathscr{Y}_m^{k_1,k_2}\right)_i^j\\
&= \sum_{j=1}^{4}\sum_{m=0}^{K} {}^{\mathrm{I}}\langle n,K-n|m,K-m\rangle^{\mathrm{I}}\, Q_j^{op}(m,K-m) \left(\mathscr{Y}_m^{k_1,k_2}\right)_i^j\\
&= \sum_{j=1}^{4} Q_j^{op}(n,K-n) \left(\mathscr{Y}_n^{k_1,k_2}\right)_i^j\,.
\end{aligned} \tag{4.2.52}$$

Next, using the invariance of the S-matrix $\Delta^{op}(E_2)\,S = S\,\Delta(E_2)$, we rewrite (4.2.52) as

$$\begin{aligned}
{}^{\mathrm{I}}\langle n,K-n|\,S\,\Delta(E_2)\,|k_1,k_2\rangle_i^{\mathrm{II}} &= {}^{\mathrm{I}}\langle n,K-n|\,S\,|k_1,k_2\rangle^{\mathrm{I}}\, Q_i(k_1,k_2)\\
&= \sum_{m=0}^{N} {}^{\mathrm{I}}\langle n,K-n|m,K-m\rangle^{\mathrm{I}}\, \mathscr{X}_m^{k_1,k_2}\, Q_i(k_1,k_2)\\
&= \mathscr{X}_n^{k_1,k_2}\, Q_i(k_1,k_2).
\end{aligned} \tag{4.2.53}$$

Likewise we get a similar set of relations by considering the charge E_4. These relations can be conveniently summarized in terms of a matrix equation

$$\begin{pmatrix} Q_1^{op}(n,K-n) & Q_2^{op}(n,K-n) & Q_3^{op}(n,K-n) & Q_4^{op}(n,K-n) \\ \tilde{Q}_1^{op}(n,K-n) & \tilde{Q}_2^{op}(n,K-n) & \tilde{Q}_3^{op}(n,K-n) & \tilde{Q}_4^{op}(n,K-n) \end{pmatrix} \mathscr{Y}_n^{k_1,k_2} = \\ = \mathscr{X}_n^{k_1,k_2} \begin{pmatrix} Q_1(k_1,k_2) & Q_2(k_1,k_2) & Q_3(k_1,k_2) & Q_4(k_1,k_2) \\ \tilde{Q}_1(k_1,k_2) & \tilde{Q}_2(k_1,k_2) & \tilde{Q}_3(k_1,k_2) & \tilde{Q}_4(k_1,k_2) \end{pmatrix}, \tag{4.2.54}$$

giving a total number of 8 constraints. However, there is a further need of 8 more constraints. These can be obtained by considering a composite operator

$$\check{E}_2 = e_0\Big(e_1\,\hat{F}_1F_3F_2 + e_2\,F_1F_3F_2 + e_3\,F_3F_2F_1\Big), \tag{4.2.55}$$

where

$$e_0 = q^{1+K+\frac{M_1}{2}}(q^M z_1 - q^{2K+2}z_2)^{-1}, \qquad e_1 = (q-q^{-1}), \\ e_2 = q^{M_2+2n}(q^{-2-2K}z_1 - q^{2-M}z_2), \qquad e_3 = -q^{M_2+2n}(q^{-1-2K}z_1 - q^{1-M}z_2), \tag{4.2.56}$$

and its affine partner $\check{E}_4$. These operators act on the states in the subspace II as

$$\Delta(\check{E}_2)|k_1,k_2\rangle_i^{\mathrm{II}} = Z_i(k_1,k_2)|k_1,k_2\rangle^{\mathrm{I}} + Z_i^+(k_1,k_2)|k_1+1,k_2-1\rangle^{\mathrm{I}} \\ + Z_i^-(k_1,k_2)|k_1-1,k_2+1\rangle^{\mathrm{I}}, \tag{4.2.57}$$

giving

$$^{\mathrm{I}}\langle n,K-n|\;\Delta^{op}(\check{E}_2)\;S\;|k_1,k_2\rangle_i^{\mathrm{II}} = \sum_{j=1}^{4}\sum_{m=0}^{K}\,{}^{\mathrm{I}}\langle n,K-n|\;\Delta^{op}(\check{E}_2)\;|m,K-m\rangle_j^{\mathrm{II}}\;\big(\mathscr{Y}_m^{k_1,k_2}\big)_i^j \\ = \sum_{j=1}^{4}\bigg(Z_j^{op}(n,K-n)\;\big(\mathscr{Y}_n^{k_1,k_2}\big)_i^j + Z_j^{+,op}(n-1,K-n+1)\;\big(\mathscr{Y}_{n-1}^{k_1,k_2}\big)_i^j \\ + Z_j^{-,op}(n+1,K-n-1)\;\big(\mathscr{Y}_{n+1}^{k_1,k_2}\big)_i^j\bigg). \tag{4.2.58}$$

The coefficients (4.2.56) are chosen in a such way that the 'non-diagonal' part of this relation is vanishing, $Z_j^{+,op}(n-1,K-n+1) = Z_j^{-,op}(n+1,K-n-1) = 0$. Therefore the only surviving part of (4.2.58) is

$$^{\mathrm{I}}\langle n,K-n|\;\Delta^{op}(\check{E}_2)\;S\;|k_1,k_2\rangle_i^{\mathrm{II}} = \sum_{j=1}^{4} Z_j^{op}(n,K-n)\;\big(\mathscr{Y}_n^{k_1,k_2}\big)_i^j\,. \tag{4.2.59}$$

This results in the following matrix equation for $Z_j^{op}(n,K-n)$:

$$\begin{aligned}&\Big(Z_1^{op}(n,K-n) \quad Z_2^{op}(n,K-n) \quad Z_3^{op}(n,K-n) \quad Z_4^{op}(n,K-n)\Big)\,\mathscr{Y}_n^{k_1,k_2}\\ &\quad = \Big(Z_1(k_1,k_2) \quad Z_2(k_1,k_2) \quad Z_3(k_1,k_2) \quad Z_4(k_1,k_2)\Big)\,\mathscr{X}_n^{k_1,k_2}\\ &\qquad + \Big(Z_1^+(k_1,k_2) \quad 0 \quad Z_3^-(k_1,k_2) \quad 0\Big)\,\mathscr{X}_n^{k_1+1,k_2-1}\\ &\qquad + \Big(0 \quad Z_2^-(k_1,k_2) \quad 0 \quad Z_4^-(k_1,k_2)\Big)\,\mathscr{X}_n^{k_1-1,k_2+1},\end{aligned} \tag{4.2.60}$$

plus a similar set of equations arising from the affine charge $\check{E}_4$. Both sets can further be united into a compact matrix form

$$A\,\mathscr{Y}_n^{k_1,k_2} = B\,\mathscr{X}_n^{k_1,k_2} + B^+\,\mathscr{X}_n^{k_1+1,k_2-1} + B^-\,\mathscr{X}_n^{k_1-1,k_2+1}, \tag{4.2.61}$$

which multiplied from the left by A^{-1} defines all coefficients of $\mathscr{Y}_n^{k_1,k_2}$ in terms of already known $\mathscr{X}_n^{k_1,k_2}$, $\mathscr{X}_n^{k_1\pm1,k_2\mp1}$. The explicit expressions of matrices A, A^{-1}, B, $B^{\pm}$, their $q\to1$ limit and the coefficients $Z_i(k_1,k_2)$, $Z_j^{op}(n,K-n)$ and their affine partners are spelled out the Appendix C.1.

To finalize we want to note that not all of the constraints in (4.2.60) are linearly independent. The set of independent constraints is chosen in such way that the inverse matrix A^{-1} would exist.

4.2.3 Scattering in subspace III

We will compute the S-matrix components in the subspace III in a very similar way as we did in the previous section for the scattering in subspace II. We start by defining the S-matrix for the subspace III as

$$S\,|k_1,k_2\rangle_i^{\rm III} = \sum_{n=0}^{K}\sum_{j=1}^{6} |n,K-n\rangle_j^{\rm III}\big(\mathscr{Z}_n^{k_1,k_2}\big)_i^{\,j}\,, \tag{4.2.62}$$

where the standard basis for the $6N$-dimensional vector space is

$$\begin{aligned}|k_1,k_2\rangle_1^{\rm III} &= |0,0,k_1,M_1-k_1\rangle\otimes|0,0,k_2,M_2-k_2\rangle,\\ |k_1,k_2\rangle_2^{\rm III} &= |0,0,k_1,M_1-k_1\rangle\otimes|1,1,k_2-1,M_2-k_2-1\rangle,\\ |k_1,k_2\rangle_3^{\rm III} &= |1,1,k_1-1,M_1-k_1-1\rangle\otimes|0,0,k_2,M_2-k_2\rangle,\\ |k_1,k_2\rangle_4^{\rm III} &= |1,1,k_1-1,M_1-k_1-1\rangle\otimes|1,1,k_2-1,M_2-k_2-1\rangle,\\ |k_1,k_2\rangle_5^{\rm III} &= |1,0,k_1-1,M_1-k_1\rangle\otimes|0,1,k_2,M_2-k_2-1\rangle,\\ |k_1,k_2\rangle_6^{\rm III} &= |0,1,k_1,M_1-k_1-1\rangle\otimes|1,0,k_2-1,M_2-k_2\rangle.\end{aligned} \tag{4.2.63}$$

Next we shall employ the same strategy as before. We perform the same steps as in

(4.2.52) and (4.2.53) only with $\Delta^{op}(E_2)$, giving

$$
\begin{aligned}
{}_{i}^{\mathrm{II}}\langle n, K-n|\, \Delta^{op}(E_2)\, S\, |k_1,k_2\rangle_j^{\mathrm{III}} &= \sum_{l=1}^{6} \big(G^{op}(n,K-n)\big)_l^i \big(\mathscr{X}_n^{k_1,k_2}\big)_j^l,\\
{}_{i}^{\mathrm{II}}\langle n, K-n|\, S\, \Delta(E_2)\, |k_1,k_2\rangle_j^{\mathrm{III}} &= \sum_{m=1}^{4} \big(\mathscr{Y}_n^{k_1,k_2}\big)_m^i \big(G(k_1,k_2)\big)_j^m,
\end{aligned}
\tag{4.2.64}
$$

where $G^{(op)}$ are the matrix representations of the charges $\Delta^{(op)}E_2$. Once again these equations (together with the affine ones coming from E_4) do not provide enough constraints to define the matrix $\mathscr{X}_n^{k_1,k_2}$ uniquely, and we need additional constraints. They are obtained with the help of $\Delta^{(op)}(F_3F_2)$, namely

$$
\begin{aligned}
{}_{i}^{\mathrm{II}}\langle n-\theta_i, K-n+\theta_i-1|\, \Delta^{op}(F_3F_2)\, S\, |k_1,k_2\rangle_j^{\mathrm{III}} &= \sum_{l=1}^{6} \big(H^{op}(n,n-K)\big)_l^i \big(\mathscr{X}_n^{k_1,k_2}\big)_j^l\,,\\
{}_{i}^{\mathrm{II}}\langle n-\theta_i, K-n+\theta_i-1|\, S\, \Delta(F_3F_2)\, |k_1,k_2\rangle_j^{\mathrm{III}} &= \sum_{m=1}^{4} \big(\overline{\mathscr{Y}}_n^{k_1,k_2}\big)_m^i \big(H(k_1,k_2)\big)_j^m\,,
\end{aligned}
\tag{4.2.65}
$$

where θ_i is defined by $\theta_i = (1-(-1)^i)/2$ and $H^{(op)}$ is the matrix representation of $\Delta^{(op)}(F_3F_2)$. Here we have also introduced $\overline{\mathscr{Y}}_n^{k_1,k_2}$ as

$$
\big(\overline{\mathscr{Y}}_n^{k_1,k_2}\big)_j^i = \big(\mathscr{Y}_{n-\theta_i}^{k_1-\theta_j,k_2+\theta_j-1}\big)_j^i\,. \tag{4.2.66}
$$

These equations may be written in a compact way using matrix notation

$$
\begin{aligned}
G^{op}(n,K-n)\, \mathscr{X}_n^{k_1,k_2} &= \mathscr{Y}_n^{k_1,k_2}\, G(k_1,k_2),\\
H^{op}(n,K-n)\, \mathscr{X}_n^{k_1,k_2} &= \overline{\mathscr{Y}}_n^{k_1,k_2}\, H(k_1,k_2).
\end{aligned}
\tag{4.2.67}
$$

The explicit realization of the matrices in the expressions above are spelled out in the Appendix C.2.

Similarly as in the previous case, not all rows and columns of $G^{(op)}$ and $H^{(op)}$ are linearly independent, thus we have to select the independent ones only. Therefore by taking the following linear combinations,

$$
\overline{G}^{(op)} = q^{K-n-\frac{M_2}{2}}\big(\tilde{a}_2 G^{(op)} - a_2 \widetilde{G}^{(op)}\big) \quad \text{and} \quad \overline{H}^{(op)} = \tilde{c}_2 V_1 H^{(op)} - c_2 V_1^{-1} \widetilde{H}^{(op)}\,, \tag{4.2.68}
$$

where the tilded matrices are the affine counterparts and selecting the first three rows of each, we are able to combine them into the non-singular quadratic matrix A (6×6) and the rectangular matrix B (8×6) as follows ($j = 1,\cdots,6$),

$$
(A)_j^i = \begin{cases} (\overline{G}^{op})_j^i\,, & i = 1,2,3\,,\\ (\overline{H}^{op})_j^{i-3}\,, & i = 4,5,6\,, \end{cases} \quad \text{and} \quad (B)_j^i = \begin{cases} (\overline{G})_j^i\,, & i = 1,2,3,4\,,\\ (\overline{H})_j^{i-4}\,, & i = 5,6,7,8\,. \end{cases} \tag{4.2.69}
$$

This approach let us to rewrite the constraints (4.2.67) in terms of a single matrix relation

$$A\,\mathscr{Z}_n^{k_1,k_2} = \check{\mathscr{Y}}_n^{k_1,k_2} B \qquad \text{giving} \qquad \mathscr{Z}_n^{k_1,k_2} = A^{-1}\,\check{\mathscr{Y}}_n^{k_1,k_2} B\,. \tag{4.2.70}$$

This relation let us to obtain any matrix element $(\mathscr{Z}_n^{k_1,k_2})^i_j$ of the scattering in the subspace III. Here we have also introduced the block diagonal matrix $\check{\mathscr{Y}}_n^{k,l}$ (6×8) as

$$\left(\check{\mathscr{Y}}_n^{k,l}\right)^i_j, = \begin{cases} \left(\mathscr{Y}_n^{k,l}\right)^i_j, & i=1,2,3\,, \quad \text{and} \quad j=1,2,3,4\,, \\ \left(\overline{\mathscr{Y}}_n^{k,l}\right)^{i-3}_{j-4}, & i=4,5,6\,, \quad \text{and} \quad j=5,6,7,8\,, \\ 0\,, & \text{the rest}\,. \end{cases} \tag{4.2.71}$$

The explicit form of matrices A, A^{-1}, B and their $q\to 1$ limit are given in Appendix C.2.

4.2.4 Special cases of the S-matrix

In this section we consider the reduction of the S-matrix in the case when one or both factors of the tensor space (4.2.1) are transforming in the fundamental representation.

4.2.4.1 Fundamental S-matrix

As a most simple case of the derivations presented in section 4.2, we want to compute the fundamental S-matrix found in [28]. The fundamental representation is defined by setting $M_1 = M_2 = 1$ and the corresponding S-matrix is 16×16 – dimensional. In order to make the comparison with [28] more explicit, let us denote

$$\mathsf{a}^\dagger_{1,2} = \phi^{1,2}, \qquad\qquad \text{and} \qquad\qquad \mathsf{a}^\dagger_{3,4} = \psi^{1,2}. \tag{4.2.72}$$

Then, starting with the subspaces I and Ib, we find

$$S\,|\psi^\alpha\psi^\alpha\rangle = \mathscr{D}\,|\psi^\alpha\psi^\alpha\rangle, \tag{4.2.73}$$

where $\mathscr{D}$ is given by (4.2.12). Further, due to our normalization

$$S\,|\phi^a\phi^a\rangle = |\phi^a\phi^a\rangle. \tag{4.2.74}$$

Here we would like to remark that our normalization differs from [28] where the S-matrix is normalized such that $S\,|\psi^\alpha\psi^\alpha\rangle = -|\psi^\alpha\psi^\alpha\rangle$. In other words, the quantities given here need to be divided by an additional factor of $\mathscr{D}$.

Next we proceed to the subspaces II and IIb. For the subspace II (and analogously for IIb) the parameters k_1, k_2, n indexing the matrix $\mathscr{Y}$ can take the values 0 and 1, but fortunately, we find that $\mathscr{Y}$ is the same for both of these values. Next it is easy to observe that the matrices A (C.4) and B (C.5) get reduced to the upper left 2×2 blocks

$$A = \begin{pmatrix} -a_2 & q^{1/2}U_2V_2a_1 \\ -\tilde{a}_2 & q^{1/2}\widetilde{U}_2\widetilde{V}_2\tilde{a}_1 \end{pmatrix}, \qquad\qquad B = \begin{pmatrix} -a_2\sqrt{q}U_1V_1 & a_1 \\ -\tilde{a}_2\sqrt{q}\widetilde{U}_1\widetilde{V}_1 & \tilde{a}_1 \end{pmatrix}, \tag{4.2.75}$$

while the matrices B^+ and B^- do not contribute at all. This gives the following solution of (4.2.61)

$$
\begin{aligned}
\mathscr{Y}_0^{0,0} &= \mathscr{D} \begin{pmatrix} \frac{\sqrt{q}(a_2\tilde{a}_1 U_1^2 V_1^2 - a_1\tilde{a}_2 U_2^2 V_2^2)}{U_1 V_1 (a_2\tilde{a}_1 - a_1\tilde{a}_2 U_2^2 V_2^2)} & \frac{a_1\tilde{a}_1(1-U_2^2V_2^2)}{a_1\tilde{a}_2 U_2^2 V_2^2 - a_2\tilde{a}_1} \\ \frac{a_2\tilde{a}_2 U_2 (U_1^2V_1^2-1)V_2}{U_1 V_1 (a_2\tilde{a}_1 - a_1\tilde{a}_2 U_2^2 V_2^2)} & \frac{(a_2\tilde{a}_1 - a_1\tilde{a}_2)U_2V_2}{\sqrt{q}(a_2\tilde{a}_1 - a_1\tilde{a}_2 U_2^2 V_2^2)} \end{pmatrix} \\
&= \begin{pmatrix} q^{1/2} U_2 V_2 \frac{x_2^- - x_1^-}{x_2^+ - x_1^-} & \frac{\gamma_1}{\gamma_2}\frac{U_2V_2}{U_1V_1}\frac{x_2^+ - x_2^-}{x_2^+ - x_1^-} \\ \frac{\gamma_2}{\gamma_1}\frac{x_1^+ - x_1^-}{x_2^+ - x_1^-} & \frac{1}{q^{1/2}U_1V_1}\frac{x_2^+ - x_1^+}{x_2^+ - x_1^-} \end{pmatrix} .
\end{aligned}
\tag{4.2.76}
$$

Then the corresponding explicit form of the fundamental S-matrix acting on the inequivalent states is

$$
\begin{aligned}
S|\psi^\alpha \phi^b\rangle &= q^{1/2} U_2 V_2 \frac{x_2^- - x_1^-}{x_2^+ - x_1^-} |\psi^\alpha\phi^b\rangle + \frac{\gamma_2}{\gamma_1}\frac{x_1^+ - x_1^-}{x_2^+ - x_1^-}|\phi^b\psi^\alpha\rangle \,, \\
S|\phi^a \psi^\beta\rangle &= \frac{\gamma_1}{\gamma_2}\frac{U_2V_2}{U_1V_1}\frac{x_2^+ - x_2^-}{x_2^+ - x_1^-}|\psi^\beta\phi^a\rangle + \frac{1}{q^{1/2}U_1V_1}\frac{x_2^+ - x_1^+}{x_2^+ - x_1^-}|\phi^a\psi^\beta\rangle \,.
\end{aligned}
\tag{4.2.77}
$$

Finally we turn to the subspace III which is four dimensional in this case. Analogously to our strategy presented section 4.2.2, we inspect the action of $\Delta(E_2)$ and $\Delta(E_4)$ obtaining

$$
\begin{aligned}
&\Delta(E_2)|1,0\rangle_1^{\mathrm{III}} = \frac{U_1V_1}{\sqrt{q}} a_2 |1,0\rangle_2^{\mathrm{II}}, && \Delta(E_2)|1,0\rangle_5^{\mathrm{III}} = b_1|1,0\rangle_2^{\mathrm{II}}, \\
&\Delta(E_2)|0,1\rangle_1^{\mathrm{III}} = a_1|0,0\rangle_1^{\mathrm{II}}, && \Delta(E_2)|0,1\rangle_6^{\mathrm{III}} = -U_1V_1\sqrt{q}\, b_2|0,0\rangle_1^{\mathrm{II}},
\end{aligned}
\tag{4.2.78}
$$

plus similar expressions for E_4. For completeness, let us spell out the opposite coproduct as well

$$
\begin{aligned}
&\Delta^{op}(E_2)|1,0\rangle_1^{\mathrm{III}} = a_2|1,0\rangle_2^{\mathrm{II}}, && \Delta^{op}(E_2)|1,0\rangle_5^{\mathrm{III}} = b_1 U_2 V_2 \sqrt{q}|1,0\rangle_2^{\mathrm{II}}, \\
&\Delta^{op}(E_2)|0,1\rangle_1^{\mathrm{III}} = a_1 \frac{U_2V_2}{\sqrt{q}}|0,0\rangle_1^{\mathrm{II}}, && \Delta^{op}(E_2)|0,1\rangle_6^{\mathrm{III}} = -b_2|0,0\rangle_1^{\mathrm{II}}.
\end{aligned}
\tag{4.2.79}
$$

The equation (4.2.70) in this case becomes

$$
\begin{pmatrix} a_2 & b_1\sqrt{q}U_2V_2 \\ \tilde{a}_2 & \tilde{b}_1\sqrt{q}\widetilde{U}_2\widetilde{V}_2 \end{pmatrix} \begin{pmatrix} (\mathscr{Z}_1^{1,0})_1^1 & (\mathscr{Z}_1^{1,0})_1^5 \\ (\mathscr{Z}_1^{1,0})_1^5 & (\mathscr{Z}_1^{1,0})_5^5 \end{pmatrix} = \begin{pmatrix} \frac{U_1V_1}{\sqrt{q}} a_2 & b_1 \\ \frac{\widetilde{U}_1\widetilde{V}_1}{\sqrt{q}}\tilde{a}_2 & \tilde{b}_1 \end{pmatrix} (\mathscr{Y}_1^{1,0})_2^2 \,,
\tag{4.2.80}
$$

the explicit solution of which is

$$
\begin{pmatrix} (\mathscr{Z}_1^{1,0})_1^1 & (\mathscr{Z}_1^{1,0})_1^5 \\ (\mathscr{Z}_1^{1,0})_1^5 & (\mathscr{Z}_1^{1,0})_5^5 \end{pmatrix} = \begin{pmatrix} \frac{(1-x_2^- x_1^+)(x_1^+ - x_2^+)}{(1-x_1^- x_2^-)(x_1^- - x_2^+)}\frac{x_1^-}{q x_1^+} & \frac{\alpha(x_1^- - x_1^+)(x_2^- - x_2^+)(x_1^+ - x_2^+)}{\sqrt{q}U_1V_1\gamma_1\gamma_2(x_1^- x_2^- - 1)(x_1^- - x_2^+)} \\ \frac{\gamma_1\gamma_2(x_1^+ - x_2^+)}{U_2V_2\alpha(1-x_1^- x_2^-)(x_2^+ - x_1^-)}\frac{x_1^-}{q^{3/2}x_1^+} & \frac{(1-x_1^- x_2^+)(x_1^+ - x_2^+)}{(1-x_1^- x_2^-)(x_1^- - x_2^+)}\frac{U_2V_2}{U_1V_1}\frac{x_2^-}{q x_2^+} \end{pmatrix} .
\tag{4.2.81}
$$

The remaining matrix elements are then easily deduced from similar derivations. These results are in agreement with [28]. For a complete list of all the scattering elements we refer to the Appendix D.1.

4.2.4.2 The S-matrix $\mathbf{S}_{Q1}$

In this section we will derive the S-matrix describing the scattering of an arbitrary bound state with a fundamental one, S_{Q1}. Once again, we will follow the derivations performed in section 4.2 step by step. First, by setting $M_2 = 1$, we find that the states in subspaces I and Ib scatter almost trivially

$$S\,|k,0\rangle^{\mathrm{I}} = \mathscr{D}\,|k,0\rangle^{\mathrm{I}}. \tag{4.2.82}$$

However the scattering in the subspace II does not get simplified that much. Nevertheless, for fixed $k_1 + k_2$, the corresponding vector space gets restricted to

$$\{|k_1,0\rangle_1^{\mathrm{II}},\, |k_1-1,1\rangle_1^{\mathrm{II}},\, |k_1,0\rangle_2^{\mathrm{II}},\, |k_1,0\rangle_4^{\mathrm{II}}\}. \tag{4.2.83}$$

This is because the states $|k_1,k_2\rangle_3^{\mathrm{II}}$ have $M_2 \geq 2$ and thus they are not present. By reducing our general expressions to accommodate these 4 states, we are lead to 16 inequivalent scattering elements, however we found 2 of them to be vanishing. The rest may be casted in quite compact form as

$$\begin{aligned}
S\,|k,0\rangle_1^{\mathrm{II}} &= (\mathscr{Y}_0^{k,0})_1^1|k,0\rangle_1^{\mathrm{II}} + (\mathscr{Y}_1^{k,0})_1^1|k-1,1\rangle_1^{\mathrm{II}} + (\mathscr{Y}_0^{k,0})_1^2|k,0\rangle_2^{\mathrm{II}} + (\mathscr{Y}_0^{k,0})_1^4|k,0\rangle_4^{\mathrm{II}}\,,\\
S\,|k-1,1\rangle_1^{\mathrm{II}} &= (\mathscr{Y}_0^{k-1,1})_1^1|k,0\rangle_1^{\mathrm{II}} + (\mathscr{Y}_1^{k-1,1})_1^1|k-1,1\rangle_1^{\mathrm{II}} + (\mathscr{Y}_0^{k-1,1})_1^2|k,0\rangle_2^{\mathrm{II}} + (\mathscr{Y}_0^{k-1,1})_1^4|k,0\rangle_4^{\mathrm{II}}\,,\\
S\,|k,0\rangle_2^{\mathrm{II}} &= (\mathscr{Y}_0^{k,0})_2^1|k,0\rangle_1^{\mathrm{II}} + (\mathscr{Y}_1^{k,0})_2^1|k-1,1\rangle_1^{\mathrm{II}} + (\mathscr{Y}_0^{k,0})_2^2|k,0\rangle_2^{\mathrm{II}}\,,\\
S\,|k,0\rangle_4^{\mathrm{II}} &= (\mathscr{Y}_0^{k,0})_4^1|k,0\rangle_1^{\mathrm{II}} + (\mathscr{Y}_1^{k,0})_4^1|k-1,1\rangle_1^{\mathrm{II}} + (\mathscr{Y}_0^{k,0})_4^4|k,0\rangle_4^{\mathrm{II}}\,.
\end{aligned} \tag{4.2.84}$$

The explicit expressions of the coefficients above are given in Appendix D.2. Upon setting $M_1 = 1$ the coefficients with indices 1 and 2 reduce to the ones of the fundamental S-matrix (4.2.77) derived previously.

The scattering in the subspace III simplifies considerably. It is easy to see, that the states $|k_1,k_2\rangle_{2,4}^{\mathrm{III}}$ need not to be considered. Thus we are led to the reduced case of our general expressions for subspace III that involve the states (4.2.83) and

$$\{|k,0\rangle_1^{\mathrm{III}},\, |k,0\rangle_3^{\mathrm{III}},\, |k,0\rangle_5^{\mathrm{III}},\, |k-1,1\rangle_1^{\mathrm{III}},\, |k-1,1\rangle_3^{\mathrm{III}},\, |k-1,1\rangle_6^{\mathrm{III}}\} \tag{4.2.85}$$

only. However, there is a more straightforward way to obtain the S-matrix in this particular case.

There are 36 scattering coefficients in subspace III that need to be determined, but not all of them are independent. Firstly we can relate the half of them to the other half by considering the identity

$$\Delta(E_3)|k-1,0\rangle^{\mathrm{I}} = |k,0\rangle_5^{\mathrm{III}} + q^{-1}|k-1,1\rangle_6^{\mathrm{III}}, \tag{4.2.86}$$

giving

$$S\,|k-1,1\rangle_6^{\mathrm{III}} = \mathscr{D}\left(|k,0\rangle_5^{\mathrm{III}} + q\,|k-1,1\rangle_6^{\mathrm{III}}\right) - q\,S\,|k,0\rangle_5^{\mathrm{III}}\,. \tag{4.2.87}$$

Subsequently we can express the states $|k-1,1\rangle_1^{\rm III}$, $|k-1,1\rangle_3^{\rm III}$ as follows

$$\begin{aligned}
\frac{\Delta(F_1E_1)-q[k]_q[M-k+1]_q}{[k]_q}\,|k,0\rangle_1^{\rm III} &= |k-1,1\rangle_1^{\rm III},\\
\frac{\Delta(F_1E_1)-q[k-1]_q[M-k]_q}{[k-1]_q}\,|k,0\rangle_3^{\rm III} &= |k-1,1\rangle_3^{\rm III}.
\end{aligned} \tag{4.2.88}$$

The explicit constraints that follow from these identities are listed in the Appendix D.2.

Then instead of reducing the general expression of the matrix $\mathscr{X}$, we follow its derivation path. By considering the action of the charges F_2 and F_4 on the subspace II states we are able to find simple expressions that relate subspaces III to subspace II as

$$\begin{aligned}
|k,0\rangle_1^{\rm III} &= \frac{\tilde{c}_1V_2\Delta(F_2)-c_1\widetilde{V}_2\Delta(F_4)}{\tilde{c}_1d_2\widetilde{U}_1V_2-c_1\tilde{d}_2U_1\widetilde{V}_2}\,|k,0\rangle_2^{\rm II}\,,\\
|k,0\rangle_3^{\rm III} &= \frac{\tilde{d}_1V_2\Delta(F_2)-d_1\widetilde{V}_2\Delta(F_4)}{\tilde{d}_1d_2\widetilde{U}_1V_2-\tilde{d}_2d_1U_1\widetilde{V}_2}\,|k,0\rangle_4^{\rm II}\,,\\
|k,0\rangle_5^{\rm III} &= \frac{\sqrt{q}}{[k]_q}\,\frac{\tilde{d}_2U_1\Delta(F_2)-d_2\widetilde{U}_1\Delta(F_4)}{c_1\tilde{d}_2U_1\widetilde{V}_2-\tilde{c}_1d_2\widetilde{U}_1V_2}\,|k,0\rangle_2^{\rm II}\,.
\end{aligned} \tag{4.2.89}$$

This approach let us to find the expressions of the matrix elements of $\mathscr{X}$ in terms of the matrix elements of $\mathscr{Y}$ for this particular case in quite an easy way. The explicit expressions are once again given in the Appendix D.2.

4.3 q-deformed boundary scattering

In the following sections we will consider the q-deformed models of the boundary scattering for the $Z=0$ and $Y=0$ giant gravitons and the left factor of the $Z=0$ $D7$-brane considered in Chapter 3. We will start by briefly recalling the construction of the quantum affine coideal subalgebras [179] presented in Chapter 2, and the necessary preliminaries for the boundary scattering theory. Then in the subsequent sections we will construct the corresponding boundary algebras using the same approach as we did in Chapter 2.

4.3.1 Quantum affine coideal subalgebras

Let the quantum deformed universal enveloping algebra $\mathcal{U}_q(\mathfrak{g})$ of a semisimple complex Lie algebra $\mathfrak{g}$ of rank n be generated by the elements $E_i,\ F_i,\ K_i^{\pm1}$ ($K_i=q^{H_i}$, $i=1,\dots,n$), that correspond to the standard Drinfeld-Jimbo realization. The Hopf algebra structure of $\mathcal{U}_q(\mathfrak{g})$ is given by

$$\begin{aligned}
&\Delta(K_i)=K_i\otimes K_i\,, && S(K_i^{-1})=K_i\,, && \epsilon(K_i)=1\,,\\
&\Delta(E_i)=E_i\otimes 1+K_i^{-1}\otimes E_i\,, && S(E_i)=-K_iE_i\,, && \epsilon(E_i)=0\,,\\
&\Delta(F_i)=F_i\otimes K_i+1\otimes F_i\,, && S(F_i)=-F_iK_i^{-1}\,, && \epsilon(F_i)=0\,.
\end{aligned} \tag{4.3.1}$$

Being a Hopf algebra, $\mathcal{U}_q(\mathfrak{g})$ admits a right adjoint actions that makes $\mathcal{U}_q(\mathfrak{g})$ into a right module. The right adjoint action (in Sweedler notation $(\mathrm{ad}_r\, b)a = \sum_i S\big(b^{(i)}_{(1)}\big)\, a\, b^{(i)}_{(2)}\,)$ is given by

$$
\begin{aligned}
&(\mathrm{ad}_r\, E_i)A = (-1)^{[A][E_i]} K_i A E_i - K_i E_i A\,, \\
&(\mathrm{ad}_r\, F_i)A = (-1)^{[A][F_i]} A F_i - F_i K_i^{-1} A K_i\,, \qquad \big(\mathrm{ad}_r\, K_i^{-1}\big)A = K_i A K_i^{-1}\,,
\end{aligned} \tag{4.3.2}
$$

where $(-1)^{[A][E_i]}$ and $(-1)^{[A][F_i]}$ are the fermionic grade factors. We shall also be using a short–hand notation $\big(\mathrm{ad}_r\, E_i \cdots E_j\big)A = \big(\mathrm{ad}_r\, E_i \cdots \mathrm{ad}_r\, E_j\big)A$ and similarly for F_i.

Let $\mathcal{U}_q(\widehat{\mathfrak{g}})$ be the universal enveloping algebra of the Kac–Moody algebra $\hat{\mathfrak{g}}$, the affine extension of $\mathfrak{g}$. Let $\pi = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ be the set of simple positive roots of $\mathfrak{g}$, and let $\widehat{\pi} = \alpha_0 \cup \pi$, where α_0 denotes the affine root. Let $E_0,\, F_0,\, K_0^{\pm 1}$ be the affine generators of $\mathcal{U}_q(\widehat{\mathfrak{g}})$, and let $\mathcal{T}$ denote the abelian subgroup $\mathcal{T} \subset \mathcal{U}_q(\widehat{\mathfrak{g}})$ generated by all $K_i^{\pm 1}$ and $K_0^{\pm 1}$.

Consider an involution θ of $\hat{\mathfrak{g}}$ such that the associated root space automorphism Θ can be represented by

$$
\Theta(\alpha_0) \in -\alpha_{p(0)} - \mathbb{Z}(\pi \backslash \alpha_{p(0)}) \quad \text{and} \quad \Theta(\alpha_i) = \alpha_i \quad \text{for all} \quad \alpha_i \in \pi_\Theta = \pi \backslash \alpha_{p(0)}\,, \tag{4.3.3}
$$

where $p(0) \in \{0, 1, \ldots, n\}$, and satisfying

$$
\alpha_0 - \Theta(\alpha_0) = k\delta\,, \qquad \text{where} \qquad \begin{cases} k = 1 \text{ for } p(0) \neq 0\,, \\ k = 2 \text{ for } p(0) = 0\,, \end{cases} \tag{4.3.4}
$$

where δ is the imaginary root. Then Θ induces a subalgebra $\mathcal{M} \subset \mathcal{U}_q(\widehat{\mathfrak{g}})$ generated by E_i, F_i and $K_i^{\pm}$ for all $\alpha_i \in \pi_\Theta$ and a Θ–fixed subgroup $\mathcal{T}_\Theta$. Furthermore, there exists a sequence $\{\alpha_{i_1}, \ldots, \alpha_{i_r}\}$, $\alpha_{i_k} \in \pi_\Theta$, and a set of positive integers $\{m_1, \ldots, m_r\}$ such that the algebra elements defined by

$$
\begin{aligned}
&\widetilde{E}_0 = F_0 K_0^{-1} - d_y\, \tilde{\theta}(F_0) K_0^{-1}\,, \qquad \tilde{\theta}(F_0) = \big(\mathrm{ad}_r\, E_{i_1}{}^{(m_1)} \cdots E_{i_r}{}^{(m_r)}\big) E'_{p(0)}\,, \\
&\widetilde{F}_0 = E'_0 K_0^{-1} - d_x\, \tilde{\theta}(E'_0) K_0^{-1}\,, \qquad \tilde{\theta}(E'_0) = \big(\mathrm{ad}_r\, F_{i_1}{}^{(m_1)} \cdots F_{i_r}{}^{(m_r)}\big) F_{p(0)}\,,
\end{aligned} \tag{4.3.5}
$$

where $E'_i = E_i K_i$, together with $\mathcal{T}_\Theta$, $\mathcal{M}$ and suitable $d_x,\, d_y \in \mathbb{C}$ generate a quantum affine coideal subalgebra $\widehat{\mathcal{B}} \subset \mathcal{U}_q(\widehat{\mathfrak{g}})$ which is compatible with the reflection equation. Note that quite often boundary algebra includes all of the Cartan subgroup $\mathcal{T}$. In such cases the factor of K_0^{-1} in (4.3.5) can be omitted. The boundary algebras we will be considering in the next sections will be exactly of this type. We will show that the q-deformed model of the $Z = 0$ giant graviton is described by a coideal subalgebra which corresponds to the $p(0) = 0$ case, while the q-deformed models of the $Y = 0$ giant graviton and the left factor of the $Z = 0$ $D7$-brane will be described by coideal subalgebras which correspond to the $p(0) \neq 0$ case.

4.3.2 Reflected algebra

Reflection autmorphism. The representation defined in section 4.1.3 describes incoming states carrying momentum p. The representation corresponding to the reflected states with momentum $-p$ will have the deformation parameter equal to $e^{-ip} = U^{-2}$. The conservation of the total number of bosons and fermions together with the energy conservation constrains central element V and Cartan generators K_i to be invariant under the reflection. These arguments imply that there exists a reflection map $\kappa : \widehat{\mathcal{Q}} \to \widehat{\mathcal{Q}}^{ref}$ defined by

$$\kappa : (V, U) \mapsto (\underline{V}, \underline{U}) \qquad \text{and} \qquad \kappa : (E_j, F_j, K_j) \mapsto (\underline{E}_j, \underline{F}_j, \underline{K}_j)\,, \tag{4.3.6}$$

where the underlined elements generate the reflected algebra $\widehat{\mathcal{Q}}^{ref}$. Furthermore, the constraints

$$\underline{U} = U^{-1}\,, \qquad \underline{V} = V\,, \qquad \underline{K}_i = K_i\,, \tag{4.3.7}$$

define the representation of the reflected algebra. Let us show the explicit form of the reflection map. The U-braiding structure of the Hopf algebra implies that κ acts trivially on the generators that are not braided, thus

$$\underline{E}_1 = E_1\,, \quad \underline{E}_3 = E_3\,, \quad \underline{F}_1 = F_1\,, \quad \underline{F}_3 = F_3\,, \tag{4.3.8}$$

while (4.1.13) and (4.1.14) give

$$\begin{aligned} &\underline{C}_2 = g\alpha(1 - U^{-2}V^2) = \lambda^{-2}\, C_2\,, && \underline{C}_3 = g\alpha^{-1}(V^{-2} - U^2) = \lambda^2\, C_3\,, \\ &\underline{\widehat{C}}_2 = g\alpha\tilde{\alpha}^2(1 - U^2V^{-2}) = \hat{\lambda}^{-2}\, \hat{C}_2\,, && \underline{\widehat{C}}_3 = g\alpha^{-1}\tilde{\alpha}^{-2}(V^2 - U^{-2}) = \hat{\lambda}^2\, \hat{C}_2\,, \end{aligned} \tag{4.3.9}$$

where $\lambda^2 = -z\, U^2$, $\hat{\lambda}^2 = -z\, U^{-2}$. Then the quartic Serre relations (4.1.12) for reflected algebra $\widehat{\mathcal{Q}}^{ref}$ give the remaining constraints,

$$\underline{E}_2 = \lambda^{-1} E_2\,, \qquad \underline{F}_2 = \lambda\, F_2\,, \qquad \underline{E}_4 = \hat{\lambda}\, E_4\,, \qquad \underline{F}_4 = \hat{\lambda}\, F_4\,. \tag{4.3.10}$$

In such a way the map κ becomes an automorphism of $\widehat{\mathcal{Q}}$ given by

$$\kappa : \left\{ \begin{array}{l} E_1,\, F_1,\, E_3,\, F_3, \\ E_2,\, F_2,\, E_4,\, F_4, \\ K_i,\, V,\, U \end{array} \right\} \to \left\{ \begin{array}{l} E_1,\, F_1,\, E_3,\, F_3, \\ \lambda^{-1}E_2,\, \lambda F_2,\, \hat{\lambda}^{-1}E_4,\, \hat{\lambda}F_4, \\ K_i,\, V,\, U^{-1} \end{array} \right\}. \tag{4.3.11}$$

Finally, we introduce the reflected coproducts of E_i and F_i ,

$$\Delta^{ref}(E_j) = \underline{E}_j \otimes 1 + K_j^{-1} U^{-\delta_{j,2}+\delta_{j,4}} \otimes E_j\,, \quad \Delta^{ref}(F_j) = \underline{F}_j \otimes K_j + U^{+\delta_{j,2}-\delta_{j,4}} \otimes F_j\,, \tag{4.3.12}$$

where $\Delta^{ref} := (\kappa \otimes id) \circ \Delta$ and we have used (4.3.7) implicitly. These shall play an important role in finding the explicit form of the reflection matrix.

Reflected representation. The representation labels $\underline{a}$, $\underline{b}$, $\underline{c}$, $\underline{d}$ associated to the generators $\underline{E}_j, \underline{F}_j$ can be obtained by replacing $U \mapsto U^{-1}$ in (4.1.45) and similarly for the affine ones. Then the labels of the reflected charges are related to the initial ones as

$$\underline{a} = \frac{\underline{\gamma}}{\gamma} a, \qquad \underline{b} = \frac{\gamma\alpha^2}{\underline{\gamma}} \frac{cd}{a} V^2, \qquad \underline{c} = \frac{\underline{\gamma}}{\gamma\alpha^2} \frac{ab}{d} V^{-2}, \qquad \underline{d} = \frac{\gamma}{\underline{\gamma}} d, \tag{4.3.13}$$

giving

$$\underline{a} = \sqrt{\frac{g}{[M]_q}}\underline{\gamma}, \qquad \underline{b} = \sqrt{\frac{g}{[M]_q}}\frac{\alpha}{\underline{\gamma}}\frac{\tilde{g}^2(x^+ - x^-)}{g^2(1+\xi x^-)(\xi + x^+)},$$
$$\underline{c} = \sqrt{\frac{g}{[M]_q}}\frac{\underline{\gamma}}{\alpha V}\frac{g q^{\frac{M}{2}}(\xi x^- + 1)}{i\tilde{g}\, x^-}, \qquad \underline{d} = \sqrt{\frac{g}{[M]_q}}\frac{\tilde{g}\, q^{\frac{M}{2}} V}{i\, g\, \underline{\gamma}}\frac{x^+ - x^-}{\xi x^+ + 1}, \tag{4.3.14}$$

The extension to the affine case is straightforward. Here we have chosen $\underline{a} = \frac{\underline{\gamma}}{\gamma} a$ as an initial constraint with $\underline{\gamma}$ being the reflected version of γ, i.e. $\kappa(\gamma) = \underline{\gamma}$. The reflection map for the $x^\pm$ parametrization is found by comparing (4.3.14) with (4.1.55), giving

$$\kappa : x^\pm \mapsto -\frac{x^\mp + \xi}{\xi x^\mp + 1}\,. \tag{4.3.15}$$

It is involutive, $\kappa^2 = id$, and is in agreement with the one conjectured in [29][5]. In the $q \to 1$ limit this maps specializes to the usual reflection map, $\kappa : x^\pm \mapsto -x^\mp$, as required. Finally, the spectral parameter z is required to transform as $\kappa : z \mapsto z^{-1}$ under the reflection map. This is indeed true and follows straightforwardly when applying map κ to (4.1.48).

The expressions in (4.3.13) may be casted in a matrix form

$$\begin{pmatrix} \underline{a} & \underline{b} \\ \underline{c} & \underline{d} \end{pmatrix} D = T \begin{pmatrix} a & b \\ c & d \end{pmatrix} T^{-1} \quad \text{with} \quad D = \begin{pmatrix} \gamma/\underline{\gamma} & 0 \\ 0 & \underline{\gamma}/\gamma \end{pmatrix}, \quad T = \begin{pmatrix} U^{-2} & 0 \\ 0 & -z \end{pmatrix}, \tag{4.3.16}$$

revealing the explicit relation between two isomorphic representations of $\widehat{\mathcal{Q}}$. Here γ and $\underline{\gamma}$ are unconstrained parameters defining the representations of incoming and reflected states. In the $q \to 1$ limit (4.3.16) specializes to (3.2.12) as required.

Finally, by requiring (4.3.11) and (4.3.16) to be in consistency we find $\underline{\gamma}^2 = -z^{-1}U^{-2}\gamma^2$. This relation is in agreement with the one obtained in Section 3.3.2.

4.4 q-deformed Y=0 giant graviton

Having all the required algebraic structures presented we are ready to construct the quantum affine coideal subalgebra and the boundary scattering theory for the q-deformed model of the $Y = 0$ giant graviton considered in Section 3.3.1. We will denote the coideal subalgebra for this boundary by $\widehat{\mathcal{B}}_Y$.

Let us start from inspecting the charges of $\widehat{\mathcal{Q}}$. It has eight regular supercharges, namely

$$F_2,\ F_{21},\ F_{32},\ F_{321} \qquad \text{and} \qquad E_2,\ E_{21},\ E_{32},\ E_{321}\,, \tag{4.4.1}$$

where we have used a shorthand notation $F_{ijk} = [F_i, [F_j, F_k]]$ and the same for E_{ijk}. By replacing $F_2 \to F_4$ and $E_2 \to E_4$ eight affine supercharges are obtained,

$$F_4,\ F_{41},\ F_{34},\ F_{341} \qquad \text{and} \qquad E_4,\ E_{41},\ E_{34},\ E_{341}\,. \tag{4.4.2}$$

[5] The authors of [29] are using the $x^\pm$ parametrization of [28], while we use the one of [30]. The map between these two is $x^\pm_{[28]} = g\tilde{g}^{-1}(x^\pm_{[30]} + \xi)$.

The replacement of the same type applied to (4.1.13) produces affine partners of the central charges, $\hat{C}_1$, $\hat{C}_2$ and $\hat{C}_3$ (4.1.14). And finally, the affine partners of F_1, E_1 and F_3, E_3 are

$$\hat{F}_1 = E_{432}\,,\ \hat{E}_1 = F_{432} \qquad \text{and} \qquad \hat{F}_3 = E_{421}\,,\ \hat{E}_3 = F_{421}\,. \tag{4.4.3}$$

The boundary we are considering does not respect bosonic symmetries E_1, F_1, central charges C_2, C_3 and affine charges E_4, F_4 (let us name these charges as the *broken*, while the rest will be named as *preserved*), thus it breakes exactly half of the supercharges (4.4.1) and (4.4.2) with the broken regular supercharges are

$$E_{21}\,,\ E_{321} \qquad \text{and} \qquad F_{21}\,,\ F_{321}\,. \tag{4.4.4}$$

The construction presented in section 4.3.1 and the relation to the algebraic structures of the $Y=0$ giant graviton implies that for each broken regular charge, the algebra $\widehat{\mathcal{B}}_Y$ must possess a corresponding twisted affine charge satisfying coideal property. We shall denote these charges by

$$B = \{\widetilde{F}_1,\ \widetilde{F}_{21},\ \widetilde{F}_{321},\ \widetilde{E}_1,\ \widetilde{E}_{21},\ \widetilde{E}_{321},\ \widetilde{C}_2,\ \widetilde{C}_3\}\,. \tag{4.4.5}$$

4.4.1 Coideal subalgebra

The set of positive simple roots of $\widehat{\mathcal{Q}}$ is $\pi = \{\alpha_1,\ \alpha_2,\ \alpha_3,\ \alpha_4\}$. The boundary conditions imply that the corresponding root space automorphism Θ_Y (4.3.3) acts on the simple roots as

$$\begin{aligned} \Theta_Y(\alpha_2) &= \alpha_2\,, & \Theta_Y(\alpha_1) &= -\alpha_2-\alpha_3-\alpha_4\,, \\ \Theta_Y(\alpha_3) &= \alpha_3\,, & \Theta_Y(\alpha_4) &= -\alpha_1-\alpha_2-\alpha_3\,. \end{aligned} \tag{4.4.6}$$

Thus $\pi_\Theta = \{\alpha_2,\ \alpha_3\}$ and it gives rise to a subalgebra $\mathcal{M}_Y$ of $\widehat{\mathcal{Q}}$. Note that $\alpha_4 = \delta - \bar{\theta}$ is the affine root where $\bar{\theta} = \alpha_1+\alpha_2+\alpha_3$ is the highest root of the non-affine algebra $\mathcal{Q}$. However we are interested in the finite dimensional representations which are constructed by dropping all imaginary roots; thus giving the constraint $K_1K_2K_3K_4 = 1$ [30,176].

We shall build $\widehat{\mathcal{B}}_Y$ based on the affine extension. Thus by composing (4.3.5) with (4.4.6) we obtain the following twisted affine generators, [6]

$$\widetilde{E}_{321} = F_4 + d_y\,\tilde{\theta}(F_4)\,, \qquad \tilde{\theta}(F_4) = (\mathrm{ad}_r E_3E_2)\,E_1'\,, \tag{4.4.7}$$

$$\widetilde{F}_{321} = E_4' + d_x\,\tilde{\theta}(E_4')\,, \qquad \tilde{\theta}(E_4') = (\mathrm{ad}_r F_3F_2)\,F_1\,, \tag{4.4.8}$$

with suitable d_x and d_y. Then with the help of the right adjoint action $\mathrm{ad}_r\mathcal{M}$ we construct the rest of the twisted affine generators,

$$\widetilde{E}_{21} = (\mathrm{ad}_r F_3)\,\widetilde{E}_{321}\,, \qquad \widetilde{F}_{21} = (\mathrm{ad}_r E_3)\,\widetilde{F}_{321}\,, \tag{4.4.9}$$

$$\widetilde{E}_1 = (\mathrm{ad}_r F_2F_3)\,\widetilde{E}_{321}\,, \qquad \widetilde{F}_1 = (\mathrm{ad}_r E_2E_3)\,\widetilde{F}_{321}\,, \tag{4.4.10}$$

$$\widetilde{C}_2 = (\mathrm{ad}_r E_2)\,\widetilde{E}_{321}\,, \qquad \widetilde{C}_3 = (\mathrm{ad}_r F_2)\,\widetilde{F}_{321}\,. \tag{4.4.11}$$

[6] Alternitavely, one could choose the root α_1 as the starting point giving $\widetilde{E}_1 = E_1'K_1^{-1} + d_x\,\tilde{\theta}(E_1')K_1^{-1}$ and $\widetilde{F}_1 = F_1K_1^{-1} + d_y\,\tilde{\theta}(F_1)K_1^{-1}$ where $\tilde{\theta}(E_1') = (\mathrm{ad}_r F_2\,\mathrm{ad}_r F_3)\,F_4$ and $\tilde{\theta}(F_1) = (\mathrm{ad}_r E_2\,\mathrm{ad}_r E_3)\,E_4'$. However, these generators are completely equivalent to the ones given above.

Let us show the coideal property for the these charges explicitly. However it is enough to show this property for the charges (4.4.7) and (4.4.8) only,

$$\begin{aligned}\Delta(\widetilde{E}_{321}) &= F_4 \otimes K_4 + U \otimes \widetilde{E}_{321} + d_y\,\tilde{\theta}(F_4) \otimes K_{123} \\ &\quad + d_y(q-q^{-1})\big((\mathrm{ad}_r E_2)E'_1 \otimes K_{12}E'_3 - UE'_1 \otimes K_1(\mathrm{ad}_r E_3)E'_2\big) \\ &\in \widehat{\mathcal{Q}} \otimes \widehat{\mathcal{B}}_Y\,,\end{aligned} \tag{4.4.12}$$

and

$$\begin{aligned}\Delta(\widetilde{F}_{321}) &= E'_4 \otimes K_4 + U^{-1} \otimes \widetilde{F}_{321} + d_x\,\tilde{\theta}(E'_4) \otimes K_{123} \\ &\quad - d_x(q-q^{-1})\big((\mathrm{ad}_r F_2)F_1 \otimes F_3K_{12} - U^{-1}F_1 \otimes [(\mathrm{ad}_r F_3)F_2]K_1\big) \\ &\in \widehat{\mathcal{Q}} \otimes \widehat{\mathcal{B}}_Y\,.\end{aligned} \tag{4.4.13}$$

Here $K_{12} = K_1K_2$ and $K_{123} = K_1K_2K_3$. The coideal property for the rest of the charges, (4.4.9), (4.4.10) and (4.4.11), is obvious since $\widehat{\mathcal{B}}_Y$ is invariant under the $\mathrm{ad}_r\mathcal{M}_Y$ action.

Finally, we want to perform some checks of our constructions. The twisted affine central charges $\widetilde{C}_2$ and $\widetilde{C}_3$ (4.6.9) must be conserved under the reflection. Thus requiring $\underline{\widetilde{C}}_2 = \widetilde{C}_2$ and $\underline{\widetilde{C}}_3 = \widetilde{C}_3$ we find

$$d_y = \frac{\tilde{g}}{g\alpha\tilde{\alpha}} \qquad \text{and} \qquad d_x = -\alpha\tilde{\alpha}\frac{\tilde{g}}{g}\,. \tag{4.4.14}$$

Then, requiring $\tilde{g}/g$ to be real, we find $(\alpha\tilde{\alpha})^2 = -1$ having a solution $\tilde{\alpha} = 1$ and $\alpha = i$ which corresponds to the usual setting of the unitary representations.

Yangian limit. The algebra $\widehat{\mathcal{Q}}$ in the $q \to 1$ limit has no singular elements and the naive $q \to 1$ limit leads to the undeformed universal enveloping algebra. The relation to the associated Yangian algebra was explicitly shown in [30] by considering the the specific combinations of charges of $\widehat{\mathcal{Q}}$ that are singular in the $q \to 1$ limit. The construction presented in [30] is very closely related to the so-called Drinfeldian [180]. However the twisted affine generators (4.6.2 - 4.6.9) are already of the required form. Thus the algebra $\widehat{\mathcal{B}}_Y$ in the rational $q \to 1$ limit is isomorphic to the associated twisted Yangian considered in Section 3.3.1 proposed by [21, 22]. The explicit relations between the quantum affine and Yangian charges are

$$\begin{aligned}
&\lim_{q\to1}\frac{\alpha\tilde{\alpha}\widetilde{E}_{321}}{2(q-1)} = -\widetilde{\mathbb{Q}}_3^{\ 1} - g\alpha\,\mathbb{G}_2^{\ 4}\,, &&\lim_{q\to1}\frac{\widetilde{F}_{321}}{2\alpha\tilde{\alpha}(q-1)} = -\widetilde{\mathbb{G}}_1^{\ 3} - \frac{g}{\alpha}\mathbb{Q}_4^{\ 2}\,,\,, \\
&\lim_{q\to1}\frac{\alpha\tilde{\alpha}\widetilde{E}_{21}}{2(q-1)} = \widetilde{\mathbb{Q}}_4^{\ 1} + g\alpha\,\mathbb{G}_2^{\ 3}, &&\lim_{q\to1}\frac{\widetilde{F}_{21}}{2\alpha\tilde{\alpha}(q-1)} = -\widetilde{\mathbb{G}}_1^{\ 4} + \frac{g}{\alpha}\mathbb{Q}_3^{\ 2}, \\
&\lim_{q\to1}\frac{\alpha\tilde{\alpha}\widetilde{E}_{1}}{2(q-1)} = \widetilde{\mathbb{R}}_2^{\ 1}, &&\lim_{q\to1}\frac{\widetilde{F}_{1}}{2\alpha\tilde{\alpha}(q-1)} = -\widetilde{\mathbb{R}}_1^{\ 2}, \\
&\lim_{q\to1}\frac{\alpha\tilde{\alpha}\widetilde{C}_{2}}{2(q-1)} = -\widetilde{\mathbb{C}} - g\alpha(\mathbb{R}_1^{\ 1} - \mathbb{L}_3^{\ 3} - \tfrac{1}{2}\mathbb{H}), &&\lim_{q\to1}\frac{\widetilde{C}_{3}}{2\alpha\tilde{\alpha}(q-1)} = -\widetilde{\mathbb{C}}^{\dagger} + \frac{g}{\alpha}(\mathbb{R}_1^{\ 1} - \mathbb{L}_3^{\ 3} - \tfrac{1}{2}\mathbb{H}).
\end{aligned} \tag{4.4.15}$$

We have checked that these relations for bound state representations at both algebra and coalgebra level.

4.4.2 Boundary scattering

In this section we consider the boundary scattering theory for the q-deformed $Y = 0$ giant graviton and find the explicit form of the bound state reflection matrix. Moreover, we explicitly solve the reflection equation and show that the reflection matrix K is indeed invariant under the coideal subalgebra $\widehat{\mathcal{B}}_Y$.

Reflection matrix. The boundary we are considering is a singlet with respect to the boundary algebra $\widehat{\mathcal{B}}_Y$, thus it may be represented via the boundary vacuum state $|0\rangle_B$. It is annihilated by all charges of it, with the exception of the generators K_i, which keep the boundary invariant,

$$K_i\,|0\rangle_B = q^{H_i}\,|0\rangle_B = |0\rangle_B\,. \tag{4.4.16}$$

We define the reflection matrix to be the intertwining matrix

$$K\,|m,n,k,l\rangle \otimes |0\rangle_B = K^{(a,b,c,d)}_{(m,n,k,l)}\,|a,b,c,d\rangle \otimes |0\rangle_B\,. \tag{4.4.17}$$

The space of states $|m,n,k,l\rangle$ is $4M$-dimensional and can be decomposed into four $4M = (M+1)+(M-1)+M+M$ subspaces that have the orthogonal basis

$$\begin{aligned}
|k\rangle^1 &= |0,0,k,M{-}k\rangle, && k = 0\ldots M,\\
|k\rangle^2 &= |1,1,k{-}1,M{-}k{-}1\rangle, && k = 1\ldots M-1,\\
|k\rangle^3 &= |1,0,k,M{-}k{-}1\rangle, && k = 0\ldots M-1,\\
|k\rangle^4 &= |0,1,k,M{-}k{-}1\rangle, && k = 0\ldots M-1.
\end{aligned} \tag{4.4.18}$$

Symmetry constraints. The reflection matrix (4.4.17) is required to intertwine the coproducts of the boundary algebra,

$$\big(K\,\Delta(J) - \Delta^{ref}(J)\,K\big)\,|k\rangle^i \otimes |0\rangle_B = 0\,, \quad \text{for all} \quad J \in \widehat{\mathcal{B}} \quad \text{and} \quad i = 1\ldots 4. \tag{4.4.19}$$

The form of reflection matrix is constrained by the bosonic generators E_3 and F_3 into five independent sets of coefficients

$$\begin{aligned}
K\,|k\rangle^1 &= A_k\,|k\rangle^1 + D_k\,|k\rangle^2,\\
K\,|k\rangle^2 &= B_k\,|k\rangle^2 + E_k\,|k\rangle^1,\\
K\,|k\rangle^\alpha &= C_k\,|k\rangle^\alpha,
\end{aligned} \tag{4.4.20}$$

where $\alpha = 3, 4$ and we have dropped the boundary vacuum state. We note that the basis (4.4.18) was chosen is such a way that the reflection matrix would act diagonally on the quantum number k. Also we are working in an orthogonal, but not orthonormal basis in order to avoid having normalization factors appearing in explicit expressions.

However switching to the orthonormal basis is rather easy and requires only extra factors of $([k]![M-k]!)^{\frac{1}{2}}$ and $([k]![M-k]!)^{-\frac{1}{2}}$ to be added to D_k and E_k respectively.

We start by determining the corner relations - the constraints for reflection coefficients A_0, D_0, C_0 and A_M, D_M, C_M. This can be achieved by considering reflection of the lowest state $|0\rangle^1$:

$$K\,|0\rangle^1 = A_0\,|0\rangle^1, \qquad \text{thus} \qquad D_0 = 0. \tag{4.4.21}$$

Then the invariance condition (4.4.19) for the charge E_2,

$$(K\,E_2 - \underline{E}_2\,K)\,|0\rangle^1 = 0, \qquad \text{gives} \qquad C_0 = \frac{\underline{a}}{a}A_0 = \frac{\underline{\gamma}}{\gamma}A_0. \tag{4.4.22}$$

We choose the overall normalization to be $A_0 = 1$. The same constraint may be found by considering the reflection of states $|0\rangle^\alpha$ and the charge F_2. Similar considerations for the highest state $|M\rangle^1$ give

$$D_M = 0 \qquad \text{and} \qquad A_M = \frac{c}{\underline{c}}C_{M-1} = -\frac{\gamma}{z\,U^2\,\underline{\gamma}}C_{M-1}. \tag{4.4.23}$$

Next we turn to the states $|k\rangle^\alpha$ as they scatter from the boundary diagonally. The twisted affine generator $\widetilde{F}_1$ acts on these states as a raising operator

$$\begin{aligned} &\widetilde{F}_1|k\rangle^\alpha = f_k(z)|k+1\rangle^\alpha, \qquad \underline{\widetilde{F}}_1|k\rangle^\alpha = f_k(1/z)|k+1\rangle^\alpha, \\ \text{with} \qquad &f_k(z) \equiv d_x[M-k-1]_q q^{-M/2-k-1}\left(q^M - q^{2k+2}z\right)V^{-1}. \end{aligned} \tag{4.4.24}$$

The invariance condition then straightforwardly gives

$$C_{k+1}\,f_k(z) - f_k(1/z)\,C_k = 0, \tag{4.4.25}$$

leading to an iterative relation

$$C_k = \frac{f_{k-1}(1/z)}{f_{k-1}(z)}C_{k-1} = \frac{q^M - q^{2k}/z}{q^M - q^{2k}z}C_{k-1}. \tag{4.4.26}$$

This relation is then simply solved by

$$C_k = C_0\prod_{n=1}^{k}\frac{q^M - q^{2n}/z}{q^M - q^{2n}z}. \tag{4.4.27}$$

The coefficients C_k are (anti)symmetric up to a factor of z under the interchange $k \to M-k-1$ for M being (even)odd,

$$\begin{aligned} z^kC_k &= -z^{M-k-1}C_{M-k-1} \quad \text{for} \quad M = even \quad \text{and} \quad k = 0, \ldots, M/2-1, \\ z^kC_k &= z^{M-k-1}C_{M-k-1} \quad \text{for} \quad M = odd \quad \text{and} \quad k = 0, \ldots, (M-1)/2-1. \end{aligned} \tag{4.4.28}$$

This symmetry comes from the requirement that the reflection is covariant under the renaming of bosonic indices $1 \leftrightarrow 2$ as the reflection is of a diagonal type for the states

$|k\rangle^\alpha$. However this is not the case for the states $|k\rangle^{1,2}$, thus there is no such symmetry for the rest of the reflection coefficients. The factors of z in (4.4.28) arise due to the non-commutative nature of the model. In the $q \to 1$ limit this (anti)covariance specializes to (anti)symmetry for M being (even)odd, as observed in [139].

The remaining reflection coefficients, as we shall show, will be expressed in terms of C_k and C_{k-1}. Requiring the reflection matrix to be invariant under the generators E_2 and F_2 on the bosonic states $|k\rangle^{1,2}$, we obtain the following set of separable equations

$$\begin{aligned} &D_k\, \underline{b} - [M-k]_q\,(C_k\, a - A_k\, \underline{a}) = 0, && C_k\, b - [M-k]_q E_k\, \underline{a} - B_k\, \underline{b} = 0,\\ &D_k\, \underline{d} + [k]_q\,(C_{k-1}\, c - A_k\, \underline{c}) = 0, && C_{k-1}\, d + [k]_q E_k\, \underline{c} - B_k\, \underline{d} = 0, \end{aligned} \tag{4.4.29}$$

with an unique solution

$$\begin{aligned} &A_k = (C_{k-1}[k]_q \underline{b} c + C_k[M-k]_q a \underline{d})\,/N, && D_k = [k]_q[M-k]_q\,(C_k a \underline{c} - C_{k-1}\underline{a} c)\,/N,\\ &B_k = (C_k[k]_q b \underline{c} + C_{k-1}[M-k]_q \underline{a} d)\,/N, && E_k = (C_k\, b \underline{d} - C_{k-1}\underline{b} d)\,/N, \end{aligned} \tag{4.4.30}$$

where the normalization factor N is

$$N = [k]_q\, \underline{b}\, \underline{c} + [M-k]_q\, \underline{a}\, \underline{d} = \frac{V q^{M/2-k} - V^{-1} q^{-M/2+k}}{q - q^{-1}}\,. \tag{4.4.31}$$

Writing the coefficients explicitly in terms of the $x^\pm$ parametrization we finally obtain

$$\begin{aligned} A_k &= \frac{\gamma\, \tilde{g}\, q^{\frac{M}{2}}\,(x^- - x^+)\left(\tilde{g}^2 q^M [k]_q C_{k-1} - g^2 [M-k]_q C_k\,(\xi + x^+)^2\right) V}{i\underline{\gamma} g^2 [M]_q\,(\xi + x^+)^2\,(1 + \xi x^+)\, N},\\ B_k &= \frac{i\underline{\gamma}\, q^{-\frac{M}{2}}\,(x^- - x^+)\left(\tilde{g}^2 [M-k]_q C_{k-1}\,(x^-)^2 - g^2 q^M [k]_q C_k\,(1 + \xi x^-)^2\right)}{\gamma \tilde{g} [M]_q\,(x^-)^2\,(1 + \xi x^-)\, V\, N},\\ D_k &= \frac{\gamma\underline{\gamma}\, q^{\frac{M}{2}}\,[k]_q[M-k]_q\,\left(\tilde{g}^2 C_{k-1} x^- + g^2 C_k\,(1 + \xi x^-)\,(\xi + x^+)\right)}{i\alpha \tilde{g} [M]_q\, x^-\,(\xi + x^+)\, V\, N},\\ E_k &= \frac{i\alpha\, \tilde{g}\, q^{\frac{M}{2}}\,(x^- - x^+)^2\,\left(\tilde{g}^2 C_{k-1} x^- + g^2 C_k\,(1 + \xi x^-)\,(\xi + x^+)\right) V}{\gamma\underline{\gamma}\, g^2 [M]_q\, x^-\,(1 + \xi x^-)\,(\xi + x^+)\,(1 + \xi x^+)\, N}. \end{aligned} \tag{4.4.32}$$

Finally, it is easy to check that the reflection matrix satisfies the unitarity constraint

$$K(p)\, K(-p) = 1\,. \tag{4.4.33}$$

Rational limit. In the $q \to 1$ limit the reflection coefficients specialize to

$$\begin{aligned} &A_k = \frac{\gamma}{\underline{\gamma}} \frac{x^-}{x^+ N}\left((M-k) C_k (x^+)^2 - k C_{k-1}\right), && B_k = \frac{\underline{\gamma}}{\gamma} \frac{x^+}{x^- N}\left((M-k) C_{k-1} (x^-)^2 - k C_k\right),\\ &D_k = \frac{\gamma\underline{\gamma}}{\alpha} \frac{k(M-k)\,(C_k x^+ + C_{k-1} x^-)}{N(x^+ - x^-)}, && E_k = \frac{\alpha}{\gamma\underline{\gamma}} \frac{x^- - x^+}{N}\left(C_k x^+ + C_{k-1} x^-\right), \end{aligned} \tag{4.4.34}$$

and the coefficients C_k and the normalization N are given by [7]

$$C_k = \frac{2igu - M + 2k}{-2igu - M + 2k} C_{k-1}, \qquad N = k + (M-k)x^- x^+. \tag{4.4.35}$$

These exactly reproduce the ones found in Section 3.3.1 and are in agreement with the ones found in [139].[8]

Fundamental representation. In this case $M = 1$ and the state $|k\rangle^2$ is absent, thus the reflection matrix is purely diagonal. The charges E_2 and F_2 constrain the reflection coefficients to be

$$A_0 = \frac{a}{\underline{a}} C_0 = \frac{\gamma}{\underline{\gamma}} C_0, \tag{4.4.36}$$

$$A_1 = \frac{c}{\underline{c}} C_0 = -\frac{\gamma}{\underline{\gamma}} \frac{C_0}{z\, U^2} \underset{q\to 1}{\longrightarrow} -\frac{\gamma}{\underline{\gamma}} \frac{x^-}{x^+} C_0. \tag{4.4.37}$$

Again, choosing the normalization to be $A_0 = 1$ this is in agreement with [29] and with [135] in the rational limit.

Reflection Equation. In order to show the integrability of the model, we have to show that the reflection matrix is a solution of the reflection equation (boundary Yang-Baxter equation). In fact, we shall explicitly derive the coefficient C_k by solving the reflection equation. The unique solution we find agrees perfectly with the coefficients that are derived from the symmetry considerations. This explicitly proves that the reflection matrix respects the boundary algebra $\widehat{\mathcal{B}}_Y$.

Consider two states with bound state numbers $M_1,\ M_2$ and spectral parameters $z_1,\ z_2$. Let us denote $K_i = K_{M_i}(z_i)$ and $S_{ij} = S_{M_i M_j}(z_i, z_j)$ and also let the underscored index indicate that the corresponding representation is reflected. Then the reflection equation is then given by

$$K_2 S_{2\underline{1}} K_1 S_{12} = S_{\underline{2}\underline{1}} K_1 S_{1\underline{2}} K_2, \tag{4.4.38}$$

which explicitly written out in components reads as

$$K_\gamma^\delta(z_2)\, S_{\beta,c}^{\gamma,d}(z_2, z_1^{-1})\, K_b^c(z_1)\, S_{a,\alpha}^{b,\beta}(z_1, z_2) = S_{\gamma,c}^{\delta,d}(z_2^{-1}, z_1^{-1})\, K_b^c(z_1)\, S_{a,\beta}^{b,\gamma}(z_1, z_2^{-1})\, K_\alpha^\beta(z_2), \tag{4.4.39}$$

where we have used Roman and Greek letter to distinguish indices of the fist and second states respectively.

Let us first consider states of the form $|k_1\rangle^\alpha \otimes |k_2\rangle^\alpha$, because the reflection matrix acts diagonally on these states. This corresponds to the *subspace I* case in terms of the analysis

[7] Here we have rescaled the normalization factor as $N_{(4.4.31)} \to N_{(4.4.35)}/(x^+x^- - 1)$ in $q \to 1$ limit.

[8] Up to some factors due to different choice of the basis (4.4.18) with respect to the one in [139].

performed in Section 4.2.1. The reflection equation in this subspace becomes

$$\sum_{n=0}^{k_1+k_2} C_m(z_1)\mathscr{X}_m^{K-n,n}(z_2,z_1^{-1})C_n(z_2)\mathscr{X}_n^{k_1,k_2}(z_1,z_2) = \sum_{n=0}^{k_1+k_2} \mathscr{X}_m^{n,K-n}(z_2^{-1},z_1^{-1})C_n(z_1)\mathscr{X}_n^{k_1,k_2}(z_1,z_2^{-1})C_{k_2}(z_2). \tag{4.4.40}$$

We will now proceed with the derivation of C_k. For $k_1 = k_2 = 0$ we easily find that the reflection equation is satisfied. Next we consider the state where $k_1 = 1, k_2 = 0$. In this case the reflection equation is satisfied provided that C_1 satisfies the following relation

$$C_1(z_1) = C_0(z_1)\left[1 - \frac{z_1^2-1}{z_1\left(\frac{q^{M_1-M_2}\left(z_2^2C_1(z_2)-C_0(z_2)\right)}{z_2(C_1(z_2)-C_0(z_2))} + z_1\right)}\right]. \tag{4.4.41}$$

The right hand side is allowed to depend solely on z_1 thus there are two solutions, a trivial one $C_1 = C_0$ and

$$C_1 = \frac{z^{-1} - A\,q^M}{z - A\,q^M}C_0. \tag{4.4.42}$$

The latter solution has an undetermined constant A. This coefficient may be determined by either considering the rational limit, or by studying the reflection equation involving states from the *subspace II* (see Section 4.2). Both arguments lead to $A = q^{-2}$. Finally, by studying a state with $k_1 = 2,\ k_2 = 0$ we can solve for C_2 and so on. This leads to the following solution

$$C_k = C_0\prod_{i=1}^{k}\frac{q^M - q^{2i}/z}{q^M - q^{2i}z}, \tag{4.4.43}$$

which perfectly agrees with (4.4.27). As expected, the trivial solution does not solve the reflection equation in general case.

Subsequently we have numerically checked the reflection equation for generic values of k_1, k_2, n, M_1, M_2 for all different states from *subspace II* and *subspace III* and found it to be satisfied.

4.5 q-deformed Z=0 giant graviton

The $Z = 0$ giant graviton and the boundary algebra associated to it was considered in Section 3.3.2. Here we will construct a quantum affine coideal subalgebra governing the boundary scattering from the q-deformed model of the $Z = 0$ giant graviton.

The $Z = 0$ giant graviton preserves all of the bulk Lie algebra. Therefore the corresponding q-deformed model of this boundary preserves all regular charges and all of the Cartan subalgebra $\mathcal{T}$ of $\widehat{\mathcal{Q}}$. The affine generators E_4 and F_4 are not preserved by the boundary itself, but give rise to the twisted affine generators of the quantum affine coideal subalgebra $\widehat{\mathcal{B}}_Z \subset \widehat{\mathcal{Q}}$.

We will essentially follow the same way as we did for the q-deformed model of $Y = 0$ giant graviton in the section above. A new element in this case will be the construction of boundary representation of $\widehat{\mathcal{B}}_Z$.

4.5.1 Coideal subalgebra

The boundary conditions define the root space automorphism Θ_Z associated to this boundary to act on the simple roots as

$$\Theta_Z(\alpha_i) = \alpha_i \quad \text{for} \quad i = 1, 2, 3, \qquad \text{and} \qquad \Theta_Z(\alpha_4) = -\alpha_4 - 2\alpha_3 - 2\alpha_2 - 2\alpha_1 \,. \tag{4.5.1}$$

Thus $\pi_{\Theta_Z} = \{\alpha_1,\, \alpha_2,\, \alpha_3\}$ and it gives rise to the subalgebra $\mathcal{M}_Z$ of $\widehat{\mathcal{Q}}$. The affine part of the boundary algebra $\widehat{\mathcal{B}}_Z$ is generated by the twisted affine generators

$$\widetilde{E}_{312} = F_4 - d_y\, \tilde{\theta}(F_4)\,, \qquad \tilde{\theta}(F_4) = (\mathrm{ad}_r E_1 E_3 E_2 E_3 E_2 E_1)\, E_4'\,, \tag{4.5.2}$$

$$\widetilde{F}_{312} = E_4' - d_x\, \tilde{\theta}(E_4')\,, \qquad \tilde{\theta}(E_4') = (\mathrm{ad}_r F_1 F_3 F_2 F_3 F_2 F_1)\, F_4\,, \tag{4.5.3}$$

with suitable d_x and d_y; the action of $\tilde{\theta}$ is induced by (4.5.1). Any other non-trivial ordering of the generators in the adjoint action above is equivalent up to a sign. Here by non-trivial we assume the obtained operator is non-zero. The rest of $\widehat{\mathcal{B}}_Z$ can be furnished with the help of the right adjoint action of $\mathrm{ad}_r \mathcal{M}_Z$,

$$\widetilde{E}_{12} = (\mathrm{ad}_r F_3)\, \widetilde{E}_{312}\,, \qquad \widetilde{F}_{12} = (\mathrm{ad}_r E_3)\, \widetilde{F}_{312}\,, \tag{4.5.4}$$

$$\widetilde{E}_{32} = (\mathrm{ad}_r F_1)\, \widetilde{E}_{312}\,, \qquad \widetilde{F}_{32} = (\mathrm{ad}_r E_1)\, \widetilde{F}_{312}\,, \tag{4.5.5}$$

$$\widetilde{E}_{2} = (\mathrm{ad}_r F_1\, \mathrm{ad}_r F_3)\, \widetilde{E}_{312}\,, \qquad \widetilde{F}_{2} = (\mathrm{ad}_r E_1\, \mathrm{ad}_r E_3)\, \widetilde{F}_{312}\,, \tag{4.5.6}$$

$$\widetilde{C}_{2} = (\mathrm{ad}_r E_2)\, \widetilde{E}_{312}\,, \qquad \widetilde{C}_{3} = (\mathrm{ad}_r F_2)\, \widetilde{F}_{312}\,. \tag{4.5.7}$$

Let us show the coideal property for the these generators explicitly. It is enough to show the coideal property for a pair of twisted affine generators only. For simplicity reasons we choose (4.5.6),

$$\begin{aligned}
\Delta(\widetilde{E}_2) = (\mathrm{ad}_r\, F_1F_3)F_4K_4^{-1}\otimes K_{13} - d_y(\mathrm{ad}_r\, E_2E_3E_2E_1)E_4'K_4^{-1}\otimes K_{2321} + UK_4^{-1}\otimes \widetilde{E}_2 \\
+ (q-q^{-1})\Big[q^{-1}F_4K_4^{-1}\otimes K_4\,[F_1,F_3]_{q^2} \\
- q(\mathrm{ad}_r\, F_1)F_4\otimes K_{14}F_3 + q^{-1}(\mathrm{ad}_r\, F_3)F_4\otimes K_{34}F_1\Big] \\
- d_y(q-q^{-1})(U\otimes 1)\Big[q^{-2}UE_4'\otimes K_4\,\big\{E_2',\big[E_3',\big[E_1',E_2'\big]_q\big]_{q^3}\big\} \\
- U(\mathrm{ad}_r\, E_1)E_4'\otimes K_{14}(\mathrm{ad}_r\, E_2E_3)E_2' - U(\mathrm{ad}_r\, E_3)E_4'\otimes K_{34}(\mathrm{ad}_r\, E_2E_1)E_2' \\
+ (\mathrm{ad}_r\, E_2E_3)E_4'\otimes K_{234}(\mathrm{ad}_r\, E_2)E_1' + (\mathrm{ad}_r\, E_2E_1)E_4'\otimes K_{214}(\mathrm{ad}_r\, E_2)E_3' \\
+ (\mathrm{ad}_r\, E_2E_1E_3)E_4'\otimes K_{2134}E_2'\Big] \\
\in \widehat{\mathcal{Q}}\otimes\widehat{\mathcal{B}}_Z\,,
\end{aligned} \tag{4.5.8}$$

and

$$\begin{aligned}
\Delta(\widetilde{F}_2) = (\mathrm{ad}_r\, E_1E_3)E_4'\otimes K_{134} - d_x(\mathrm{ad}_r\, F_2F_3F_2F_1)F_4\otimes K_{23214} + U^{-1}\otimes \widetilde{F}_2 \\
+ (q-q^{-1})\Big[q^{-1}E_4'\otimes K_4\,[E_1,E_3]_{q^2} \\
+ (\mathrm{ad}_r\, E_1)E_4'\otimes K_{14}E_3' - (\mathrm{ad}_r\, E_3)E_4'\otimes K_{34}E_1'\Big] \\
- d_x(q-q^{-1})(U^{-1}\otimes 1)\Big[-q^2\,U^{-1}F_4\otimes K_4\,\big\{F_2,\big[F_3,[F_1,F_2]_{q^{-1}}\big]_{q^{-3}}\big\} \\
+ q^{-1}U^{-1}(\mathrm{ad}_r\, F_1)F_4\otimes K_{14}(\mathrm{ad}_r\, F_2F_3)F_2 + q\,U^{-1}(\mathrm{ad}_r\, F_3)F_4\otimes K_{34}(\mathrm{ad}_r\, F_2F_1)F_2 \\
- q^{-1}(\mathrm{ad}_r\, F_2F_3)F_4\otimes K_{234}(\mathrm{ad}_r\, F_2)F_1 - q(\mathrm{ad}_r\, F_2F_1)F_4\otimes K_{214}(\mathrm{ad}_r\, F_2)F_3 \\
- (\mathrm{ad}_r\, F_2F_1F_3)F_4\otimes K_{2134}F_2\Big] \\
\in \widehat{\mathcal{Q}}\otimes\widehat{\mathcal{B}}_Z\,,
\end{aligned} \tag{4.5.9}$$

where $\{a,b\} = ab + ba$ denotes the anti-commutator, $[a,b]_x = ab - q^x\, ba$ is a q-deformed commutator, and the short-hand notation $K_{i\ldots j} = K_i\cdots K_j$ has been employed. The coideal property for the rest of the twisted affine generators follows straightforwardly from the $\mathrm{ad}_r\mathcal{M}_Z$-invariance of $\widehat{\mathcal{B}}_Z$.

Boundary representation. The next step is to construct the boundary bound state representation of the coideal subalgebra $\widehat{\mathcal{B}}_Z$. The constraints defining the representation are the commutation relations in the third line of (4.1.9), and the coreflectivity of the regular central charges C_2, C_3 (4.1.13) and the twisted affine central charges $\widetilde{C}_2$, $\widetilde{C}_3$ (4.5.7). We will start by constructing the boundary representation of the regular supercharges E_2 and F_2 and the central element V. We will denote the latter as V_B in order to distinguish it from the bulk one, V. Note that the deformation parameter U is *not* in the boundary algebra and thus does not have a boundary representation. In such a way the algebra constraints (4.1.13) get modified for the boundary algebra.

The algebra constraints for C_2 and C_3 for incoming and reflected states in the bulk are given by

$$\begin{aligned} C_2 \otimes 1 &= g\alpha(1-U^2V^2)\otimes 1\,, & C_3 \otimes 1 &= g\alpha^{-1}(V^{-2}-U^{-2})\otimes 1\,, \\ \underline{C}_2 \otimes 1 &= g\alpha(1-U^{-2}V^2)\otimes 1\,, & \underline{C}_3 \otimes 1 &= g\alpha^{-1}(V^{-2}-U^{2})\otimes 1\,. \end{aligned} \tag{4.5.10}$$

Here we have used (4.3.7) implicitly and the tensor space structure is $bulk \otimes boundary$. Then requiring their coproducts

$$\begin{aligned} \Delta(C_2) &= C_2\otimes 1 + V^2U^2\otimes C_2\,, & \Delta(C_3) &= C_3\otimes V_B^{-2} + U^{-2}\otimes C_3\,, \\ \Delta^{ref}(C_2) &= \underline{C}_2\otimes 1 + V^2U^{-2}\otimes C_2\,, & \Delta^{ref}(C_3) &= \underline{C}_3\otimes V_B^{-2} + U^{2}\otimes C_3\,, \end{aligned} \tag{4.5.11}$$

to be coreflective, $\Delta(C_i) = \Delta^{ref}(C_i)$, we find the boundary algebra constraints for the regular central charges to be

$$1\otimes C_2 = 1\otimes g\alpha\,, \qquad\qquad 1\otimes C_3 = 1\otimes g\alpha^{-1}V_B^{-2}\,. \tag{4.5.12}$$

Therefore the representation constraints for the boundary algebra are

$$\begin{aligned} a_Bd_B &= \frac{q^{\frac{M}{2}}V_B - q^{-\frac{M}{2}}V_B^{-1}}{q^M-q^{-M}}\,, & b_Bc_B &= \frac{q^{-\frac{M}{2}}V_B - q^{\frac{M}{2}}V_B^{-1}}{q^M-q^{-M}}\,, \\ a_Bb_B &= \frac{g\alpha}{[M]_q}\,, & c_Bd_B &= \frac{g\alpha^{-1}}{[M]_q}V_B^{-2}\,. \end{aligned} \tag{4.5.13}$$

These relations force the boundary labels to be

$$\begin{aligned} a_B &= \sqrt{\tfrac{g}{[M]_q}}\gamma_B\,, & b_B &= \sqrt{\tfrac{g}{[M]_q}}\frac{\alpha}{\gamma_B}, \\ c_B &= \sqrt{\tfrac{g}{[M]_q}}\frac{\gamma_B}{\alpha}\frac{i\tilde{g}}{g\xi}\frac{q^{M/2}\left(1-q^{-M}V_B^2\right)}{V_B}\,, & d_B &= \sqrt{\tfrac{g}{[M]_q}}\frac{\tilde{g}}{\gamma_B}\frac{q^{M/2}\left(V_B^2-q^{-M}\right)}{ig\xi\, V_B}\,, \end{aligned} \tag{4.5.14}$$

where V_B is required to satisfy

$$\left(V_B^2-q^{-M}\right)\left(V_B^2-q^{M}\right) = \frac{\xi^2}{\xi^2-1}\,. \tag{4.5.15}$$

A convenient parametrization satisfying this constraint is

$$V_B^2 = q^M\frac{x_B}{x_B+\xi} = q^{-M}\frac{1+\xi x_B}{1-\xi^2}\,. \tag{4.5.16}$$

In this way the boundary labels become

$$\begin{aligned} a_B &= \sqrt{\tfrac{g}{[M]_q}}\gamma_B\,, & b_B &= \sqrt{\tfrac{g}{[M]_q}}\frac{\alpha}{\gamma_B}\,, \\ c_B &= \sqrt{\tfrac{g}{[M]_q}}\frac{\gamma_B}{\alpha}\frac{i\tilde{g}}{g}\frac{q^{M/2}}{V_B(x_B+\xi)}\,, & d_B &= \sqrt{\tfrac{g}{[M]_q}}\frac{\tilde{g}}{ig\gamma_B}\frac{V_Bq^{M/2}\,(x_B+\xi)}{\xi x_B+1}\,. \end{aligned} \tag{4.5.17}$$

Consequently, the mass-shell constraint

$$\left(a_B d_B - q^M b_B c_B\right)\left(a_B d_B - q^{-M} b_B c_B\right) = 1\,, \tag{4.5.18}$$

in this parametrization becomes

$$\frac{q^{-2M} g^2 \left(1 + x_B^2 + 2 x_B \xi\right)^2}{[M]_q^2 \, (\xi^2 - 1)\, x_B^2} = 1\,. \tag{4.5.19}$$

In the $q \to 1$ limit it gives the usual (non-deformed) mass-shell constraint (3.3.56)

$$-\frac{g^2}{M^2}\left(x_B + \frac{1}{x_B}\right)^2 = 1 \qquad \Longrightarrow \qquad x_B + \frac{1}{x_B} = \frac{iM}{g}\,. \tag{4.5.20}$$

Furthermore, the $q \to 1$ limit gives $V_B \to 1$, and labels (4.5.17) reproduce the usual non-deformed boundary labels (3.3.55), as required.

Let us turn now to the construction of the boundary representation labels of the affine generators E_4 and F_4. We will construct the affine representation in a similar way as we did for the regular ones above, except we will not give the explicit details of the coreflectivity of the twisted affine central charges as we did for the regular ones. This is because the explicit form of the coproducts of $\widetilde{C}_2$ and $\widetilde{C}_3$ is very large and thus we will only state the final constraints we have obtained.

The representation constraints that follow from the commutation relations (4.1.9) are

$$\tilde a_B \tilde d_B = \frac{q^{\frac{M}{2}} \widetilde{V}_B - q^{-\frac{M}{2}} \widetilde{V}_B^{-1}}{q^M - q^{-M}}\,, \qquad \tilde b_B \tilde c_B = \frac{q^{-\frac{M}{2}} \widetilde{V}_B - q^{\frac{M}{2}} \widetilde{V}_B^{-1}}{q^M - q^{-M}}\,. \tag{4.5.21}$$

Bearing on the analogy to the affine bulk labels we choose the following ansatz for the affine boundary labels,

$$\begin{aligned}
\tilde a_B &= \sqrt{\tfrac{g}{[M]_q}}\,\frac{\gamma_B \tilde\alpha}{A_B}\,, & \tilde b_B &= \sqrt{\tfrac{g}{[M]_q}}\,\frac{\alpha\tilde\alpha}{\gamma_B} B_B\,,\\
\tilde c_B &= \frac{q^{-\frac{M}{2}} \widetilde{V}_B - q^{\frac{M}{2}} \widetilde{V}_B^{-1}}{(q^M - q^{-M})\,\tilde b_B}\,, & \tilde d_B &= \frac{q^{\frac{M}{2}} \widetilde{V}_B - q^{-\frac{M}{2}} \widetilde{V}_B^{-1}}{(q^M - q^{-M})\,\tilde a_B}\,,
\end{aligned} \tag{4.5.22}$$

where A_B and B_B are undetermined parameters. Then using this ansatz and requiring $\widetilde{C}_2$ and $\widetilde{C}_3$ to be coreflective we find additional constraints that solve this requirement,

$$A_B = -i\,x_B\,, \qquad B_B = -i(x_B + 2\xi)\,, \qquad V_B^2 \widetilde{V}_B^2 = 1 + \frac{\xi^2}{\xi^2 - 1}\,. \tag{4.5.23}$$

These define the affine boundary labels to be

$$\begin{aligned}
\tilde a_B &= \sqrt{\tfrac{g}{[M]_q}}\,\frac{i\gamma_B \tilde\alpha}{x_B}\,, & \tilde b_B &= \sqrt{\tfrac{g}{[M]_q}}\,\frac{\alpha\tilde\alpha}{i\gamma_B}(x_B + 2\xi)\,,\\
\tilde c_B &= -\sqrt{\tfrac{g}{[M]_q}}\,\frac{\tilde g\, q^{\frac{M}{2}} \gamma_B}{g\alpha\tilde\alpha(1 + \xi x_B)\widetilde{V}_B}\,, & \tilde d_B &= \sqrt{\tfrac{g}{[M]_q}}\,\frac{\tilde g\, q^{-\frac{M}{2}}}{g\tilde\alpha\gamma_B \widetilde{V}_B}\,\frac{1 - \xi(x_B + 2\xi)}{\xi^2 - 1}\,.
\end{aligned} \tag{4.5.24}$$

The coreflectivity property also constrains the parameters d_y and d_x to be

$$d_y = (\alpha\tilde{\alpha})^{-2}\,, \qquad\qquad d_x = -(\alpha\tilde{\alpha})^{2}\,, \tag{4.5.25}$$

thus revealing the last undetermined elements of $\widehat{\mathcal{B}}_Z$. The reflection map acts trivially on the boundary spectral parameter, $\kappa : x_B \mapsto x_B$ and the boundary labels, both regular and affine, are invariant under the reflection as required.

Finally we want to give two useful relations of the boundary representation that are closely linked to those of the bulk representation. Recall that the evaluation parameter z may be expressed in terms of the bulk representation labels as (4.1.58)

$$z = \frac{g}{\tilde{g}\,\alpha\,\tilde{\alpha}}(a\tilde{b} - b\tilde{a})\,, \qquad z^{-1} = \frac{g\,\alpha\,\tilde{\alpha}}{\tilde{g}}(c\tilde{d} - d\tilde{c})\,. \tag{4.5.26}$$

In a similar way, for the boundary representation, we obtain

$$q^{M} = \frac{g}{\tilde{g}\,\alpha\,\tilde{\alpha}}(a_B\tilde{b}_B - b_B\tilde{a}_B)\,, \qquad q^{-M} = V_B\widetilde{V}_B\frac{g\,\alpha\,\tilde{\alpha}}{\tilde{g}}(c_B\tilde{d}_B - d_B\tilde{c}_B)\,. \tag{4.5.27}$$

4.5.2 Boundary scattering

The boundary algebra $\widehat{\mathcal{B}}_Z$ allows us to find any bound state reflection matrix up to the overall dressing phase. This can be done in a similar way as in [176], where the bound state S-matrix for the algebra $\widehat{\mathcal{Q}}$ was found. However these calculations are rather complicated and thus we will reduce our goal to finding the analytic expressions of the reflection matrices with the total bound state number $M \leq 3$. These are the fundamental reflection matrix K_q^{Aa} and the bound state reflection matrices K_q^{Ba} and K_q^{Ab}. Here indices ^{A} and ^{B} denote the fundamental and $M = 2$ bound states in the bulk, and in the same way ^{a} and ^{b} denote the boundary states. These matrices are given in the Appendix F. We have checked that they are unitary and satisfy the reflection equation. Also we have calculated some higher order bound state reflection matrices numerically, and checked that they satisfy the reflection equation.

4.6 q-deformed Z=0 $D7$-brane

The $Z = 0$ $D7$-brane was considered in Section 3.3.2. This boundary from the scattering theory point of view factorizes into two inequivalent factors, left and right.

The right factor respects all of the bulk Lie algebra and thus the boundary algebra is equivalent to that of the $Z = 0$ giant graviton. The same story follows for the q-deformed approach, and thus the corresponding boundary algebra was presented in the section above. The reflection matrices are also the same.

The left factor of the $Z = 0$ $D7$-brane does not respect any of the Lie supercharges $\mathbb{Q}_a^{\;\alpha}$, $\mathbb{G}_\alpha^{\;a}$ or central charges $\mathbb{C}$, $\mathbb{C}^\dagger$. Hence the corresponding q-deformed model of this boundary in addition to the affine supercharges E_4 and F_4 does not respect regular supercharges F_2 and E_2 (and central elements C_2, C_3). These generators combined together will give rise to the twisted affine generators of the quantum affine coideal subalgebra $\widehat{\mathcal{B}}_X \subset \widehat{\mathcal{Q}}$. The boundary is a singlet with respect to the boundary algebra, thus the boundary algebra, as will be shown in this section, is of the same type as for the q-deformed model of the $Y=0$ giant graviton considered in Section 4.4.

4.6.1 Coideal subalgebra

The boundary conditions define the root space automorphism Θ_X associated to the left factor of the $D7$-brane to act on the simple roots as

$$\Theta_X(\alpha_1)=\alpha_1, \qquad \Theta_X(\alpha_2)=-\alpha_4-\alpha_1-\alpha_3,$$
$$\Theta_X(\alpha_3)=\alpha_3, \qquad \Theta_X(\alpha_4)=-\alpha_2-\alpha_1-\alpha_3. \tag{4.6.1}$$

Thus $\pi_{\Theta_X}=\{\alpha_1,\,\alpha_3\}$ and it gives rise to the subalgebra $\mathcal{M}_X$ of $\widehat{\mathcal{Q}}$.

As in the previous cases, we build $\widehat{\mathcal{B}}_X$ based on the affine extension. This setup fixes the twisted affine generators to be

$$\widetilde{E}_{312}=F_4-d_y\,\widetilde{\Theta}(F_4)\,, \qquad \widetilde{\Theta}(F_4)=(\mathrm{ad}_r E_3E_1)\,E'_2\,, \tag{4.6.2}$$
$$\widetilde{F}_{312}=E'_4-d_x\,\widetilde{\Theta}(E'_4)\,, \qquad \widetilde{\Theta}(E'_4)=(\mathrm{ad}_r F_3F_1)\,F_2\,, \tag{4.6.3}$$

with suitable d_x and d_y. Let us show the coideal property for the these twisted generators explicitly,

$$\begin{aligned}\Delta(\widetilde{E}_{312}) &= F_4\otimes K_4-d_y(\mathrm{ad}_r\,E_3E_1)E'_2\otimes K_{312}-U\otimes\widetilde{E}_{312}\\ &\quad -d_y(q-q^{-1})\Big((\mathrm{ad}_r\,E_1)E'_2\otimes K_{12}E'_3\\ &\qquad\qquad -(\mathrm{ad}_r\,E_3)E'_2\otimes K_{32}E'_1+q^{-1}E'_2\otimes K_2\left[E'_1,E'_3\right]_{q^2}\Big)\\ &\in\widehat{\mathcal{Q}}\otimes\widehat{\mathcal{B}}_X\ ,\end{aligned}\tag{4.6.4}$$

and

$$\begin{aligned}\Delta(\widetilde{F}_{312}) &= E_4\otimes K_4-d_x(\mathrm{ad}_r\,F_3F_1)F_2\otimes K_{312}-U^{-1}\otimes\widetilde{F}_{312}\\ &\quad -d_x\,q^{-1}(q-q^{-1})\Big((\mathrm{ad}_r\,F_3)F_2\otimes K_{32}F_1\\ &\qquad\qquad -q^2(\mathrm{ad}_r\,F_1)F_2\otimes K_{12}F_3+F_2\otimes K_2\,[F_1,F_3]_{q^2}\Big)\\ &\in\widehat{\mathcal{Q}}\otimes\widehat{\mathcal{B}}_X\ .\end{aligned}\tag{4.6.5}$$

The rest of $\widehat{\mathcal{B}}_X$ can be furnished with the help of the right adjoint action, $\mathrm{ad}_r\mathcal{M}_X$,

$$\widetilde{E}_{12}=(\mathrm{ad}_r F_3)\,\widetilde{E}_{312}, \qquad \widetilde{F}_{12}=(\mathrm{ad}_r E_3)\,\widetilde{F}_{312}, \tag{4.6.6}$$
$$\widetilde{E}_{32}=(\mathrm{ad}_r F_1)\,\widetilde{E}_{312}, \qquad \widetilde{F}_{32}=(\mathrm{ad}_r E_1)\,\widetilde{F}_{312}, \tag{4.6.7}$$
$$\widetilde{E}_{2}=(\mathrm{ad}_r F_1F_3)\,\widetilde{E}_{312}, \qquad \widetilde{F}_{2}=(\mathrm{ad}_r E_1E_3)\,\widetilde{F}_{312}. \tag{4.6.8}$$

The coideal property for these charges follows identically since $\widehat{\mathcal{B}}_X$ is invariant under the adjoint action of $\mathcal{M}_X$.

The final ingredients of $\widehat{\mathcal{B}}_X$ are the twisted affine central charges $\tilde{\tilde{C}}_2$ and $\tilde{\tilde{C}}_3$ that can be obtained by anticommuting two twisted affine generators, e.g.

$$\tilde{\tilde{C}}_2 = \{\widetilde{E}_{12}\,, \widetilde{E}_{32}\}\,, \qquad \tilde{\tilde{C}}_3 = \{\widetilde{F}_{12}\,, \widetilde{F}_{32}\}. \tag{4.6.9}$$

These twisted affine central charges must be coreflective. And because the boundary is a singlet we require $\underline{\tilde{\tilde{C}}}_2 = \tilde{\tilde{C}}_2$ and $\underline{\tilde{\tilde{C}}}_3 = \tilde{\tilde{C}}_3$. This requirement gives us the following constraints,

$$1 + d_x\chi(q+q^{-1}) - \frac{d_x^2\chi^2}{\xi^2-1} = 0\,, \qquad \frac{1}{\xi^2-1} + \frac{d_y}{\chi}(q+q^{-1}) - \frac{d_y^2}{\chi^2} = 0\,, \tag{4.6.10}$$

where $\chi = \dfrac{\tilde{g}}{g\alpha\tilde{\alpha}}$. These constraints can be solved by introducing a simple ansatz,

$$d_y = \frac{\tilde{g}}{g\,\alpha\tilde{\alpha}}\,V_B' \qquad \text{and} \qquad d_x = -\frac{g\,\alpha\tilde{\alpha}}{\tilde{g}}\,V_B'(1-\xi^2)\,, \tag{4.6.11}$$

where

$$V_B' = q\frac{1-\xi x_B'}{1-\xi^2} = q^{-1}\frac{x_B'}{x_B'-\xi}\,. \tag{4.6.12}$$

Note that V_B' is related to V_B in (4.5.16) by setting $M = 1$ and inverting the deformation parameter, $q \to q^{-1}$, giving $\xi \to -\xi$. Thus x_B' may be understood as the spectral parameter of the *oppositely* deformed fundamental boundary.[9]

4.6.2 Boundary scattering

The structure of the q-deformed reflection matrix is equivalent to the non-deformed case (3.4.14) and the corresponding vector space is the same (3.4.15). The bosonic generators E_1, F_1 and E_3, F_3 constrain the reflection matrix to be diagonal,

$$K\,|k\rangle^1 = A_q\,|k\rangle^1, \qquad K\,|k\rangle^2 = B_q\,|k\rangle^2, \qquad K\,|k\rangle^\alpha = C_q\,|k\rangle^\alpha, \tag{4.6.13}$$

and we have added the subscript $_q$ to distinguish the q-deformed reflection coefficients from the ones in (3.4.16). Next we choose the normalization for the reflection of the state $|k\rangle^1$ to be $A_q = 1$. Then the intertwining equation for $\widetilde{E}_2$ gives

$$\big(K\,\widetilde{E}_2 - \underline{\widetilde{E}}_2\,K\big)|k\rangle^1 = 0 \qquad \Longrightarrow \qquad B_q = \frac{x_B' + x^+}{x_B' + \kappa(x^+)}\frac{\underline{\gamma}}{\gamma}. \tag{4.6.14}$$

[9] It is possible to choose a parametrization of d_x and d_y that it would agree with the one used for the $Z = 0$ giant graviton, i.e. in terms of x_B, not x_B'. However this would make expressions of the reflection matrices much more complicated and the pole structure would not be transparent.

Equivalently, the same constraint may be found by considering the reflection of states $|0\rangle^{\alpha}$ and employing the generator $\widetilde{F}_2$. Next we consider the reflection of the $|k\rangle^2$ state. The intertwining equation in this case leads to

$$\big(K\,\widetilde{E}_2 - \widetilde{\underline{E}}_2\,K\big)|k\rangle^2 = 0 \quad\Longrightarrow\quad C_q = \frac{(1+\xi x^-)(1+\xi x^+)}{1-\xi^2}\,\frac{(1+x'_B\kappa(x^-))(x'_B+x^+)}{(1+x'_B x^-)(x'_B+\kappa(x^+))}\,\frac{\underline{\gamma}^2}{\gamma^2}. \tag{4.6.15}$$

Let us perform some consistency checks. It is straightforward to check that this reflection matrix satisfies the unitarity condition $K(p)K(-p) = 1$. In the $q \to 1$ limit the q-deformed reflection coefficients A_q, B_q and C_q specialize to the non-deformed ones given in (3.4.17) as required. Finally we have verified that it satisfies the reflection equation when the total bound state number $M \leq 5$. This is sufficient to claim that reflection equation should be satisfiend for any bound state numbers.

C Elements of the S-matrix

In this Appendix we have spelled out various coefficients and matrices that have been heavily used in the intermediate steps in deriving the final expressions of the S-matrix for the subspaces II and III.

C.1 Subspace II

The coefficients for the charge $\Delta(E_2)$ in (4.2.51) are

$$\begin{aligned} Q_1(k_1,k_2) &= -q^{M_1/2-k_1}\,a_2\,U_1V_1\,[\bar{k}_2+1]_q, & Q_2(k_1,k_2) &= a_1\,[\bar{k}_1+1]_q,\\ Q_3(k_1,k_2) &= -q^{M_1/2-k_1}\,b_2\,U_1V_1, & Q_4(k_1,k_2) &= b_1. \end{aligned} \tag{C.1}$$

Similarly, the coefficients for the charge $\Delta^{op}(E_2)$ are

$$\begin{aligned} Q_1^{op}(k_1,k_2) &= -a_2\,[\bar{k}_2+1]_q, & Q_2^{op}(k_1,k_2) &= q^{M_2/2-k_2}\,a_1\,U_2V_2\,[\bar{k}_1+1]_q,\\ Q_3^{op}(k_1,k_2) &= -b_2, & Q_4^{op}(k_1,k_2) &= q^{M_2/2-k_2}\,b_1\,U_2V_2. \end{aligned} \tag{C.2}$$

By replacing $a,b \to \tilde{a},\tilde{b}$ and $U,V \to \widetilde{U},\widetilde{V}$, one obtains $\widetilde{Q}_i(k_1,k_2)$ and $\widetilde{Q}_i^{op}(k_1,k_2)$ related to the affine charge E_4.

The coefficients in (4.2.60) are

$$\begin{aligned} Z_1^{op}(n,K-n) &= c_2\,\widetilde{V}_1\,[M_2-K+n]_q, & Z_2^{op}(n,K-n) &= c_1\,\widetilde{U}_2\,[n-M_1]_q\,q^{n-K-\frac{M_1}{2}},\\ Z_3^{op}(n,K-n) &= d_2\,\widetilde{V}_1\,q^{-M_2}, & Z_4^{op}(n,K-n) &= -d_1\,\widetilde{U}_2\,q^{n-K+\frac{M_1}{2}}. \end{aligned} \tag{C.3}$$

and

$$Z_1(k_1,k_2) = \frac{c_2\widetilde{U}_1\,[\bar{k}_2+1]_q}{q^M z_{12}-q^{2(K+1)}} q^{M_1/2-k_1+M_2}\Big(q^{2n}z_{12}-q^{\delta M}\big(q^{2(n-\bar{k}_1)}-1\big)-q^{2k_2+\delta M}\Big),$$
$$Z_2(k_1,k_2) = \frac{z_{12}\,c_1\widetilde{V}_2\,[\bar{k}_1+1]_q}{q^M z_{12}-q^{2(K+1)}} q^{-\delta M/2+2}\Big(q^{2n}z_{21}-q^{\delta M}\big(q^{2(n+\bar{k}_2)}-q^{2K}\big)-q^{2k_2+\delta M}\Big),$$
$$Z_3(k_1,k_2) = \frac{d_2\widetilde{U}_1}{q^M z_{12}-q^{2(K+1)}} q^{M_1/2-k_1}\Big(q^{2n}z_{12}-q^{M}\big(q^{2(n-\bar{k}_1)}-1\big)-q^{2k_2+\delta M}\Big),$$
$$Z_4(k_1,k_2) = \frac{z_{12}\,d_1\widetilde{V}_2}{q^M z_{12}-q^{2(K+1)}} q^{M/2+2}\Big(q^{2n}z_{21}-q^{-M}\big(q^{2(n+\bar{k}_2)}-q^{2K}\big)-q^{2k_2+\delta M}\Big).$$

The matrices in (4.2.61) are defined as

$$A = \begin{pmatrix} Q_1^{op}(n,K-n) & Q_2^{op}(n,K-n) & Q_3^{op}(n,K-n) & Q_4^{op}(n,K-n) \\ \tilde{Q}_1^{op}(n,K-n) & \tilde{Q}_2^{op}(n,K-n) & \tilde{Q}_3^{op}(n,K-n) & \tilde{Q}_4^{op}(n,K-n) \\ Z_1^{op}(n,K-n) & Z_2^{op}(n,K-n) & Z_3^{op}(n,K-n) & Z_4^{op}(n,K-n) \\ \tilde{Z}_1^{op}(n,K-n) & \tilde{Z}_2^{op}(n,K-n) & \tilde{Z}_3^{op}(n,K-n) & \tilde{Z}_4^{op}(n,K-n) \end{pmatrix}, \tag{C.4}$$

$$B = \begin{pmatrix} Q_1(k_1,k_2) & Q_2(k_1,k_2) & Q_3(k_1,k_2) & Q_4(k_1,k_2) \\ \tilde{Q}_1(k_1,k_2) & \tilde{Q}_2(k_1,k_2) & \tilde{Q}_3(k_1,k_2) & \tilde{Q}_4(k_1,k_2) \\ Z_1(k_1,k_2) & Z_2(k_1,k_2) & Z_3(k_1,k_2) & Z_4(k_1,k_2) \\ \tilde{Z}_1(k_1,k_2) & \tilde{Z}_2(k_1,k_2) & \tilde{Z}_3(k_1,k_2) & \tilde{Z}_4(k_1,k_2) \end{pmatrix}, \tag{C.5}$$

and

$$B^+ = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ Z_1^+(k_1,k_2) & 0 & Z_3^+(k_1,k_2) & 0 \\ \tilde{Z}_1^+(k_1,k_2) & 0 & \tilde{Z}_3^+(k_1,k_2) & 0 \end{pmatrix}, \quad B^- = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & Z_2^-(k_1,k_2) & 0 & Z_4^-(k_1,k_2) \\ 0 & \tilde{Z}_2^-(k_1,k_2) & 0 & \tilde{Z}_4^-(k_1,k_2) \end{pmatrix}. \tag{C.6}$$

The latter two have a quite compact explicit form

$$B^+ = [\bar{k}_1]_q \frac{q^{1+k_1-k_2-\frac{M_1}{2}}}{(q-q^{-1})^{-1}} \frac{q^{M_1+2k_2}z_{12}-q^{M_2+2(n+1)}}{q^M z_{12}-q^{2(K+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -c_2\widetilde{U}_1[k_2]_q & 0 & d_2\widetilde{U}_1 & 0 \\ -\tilde{c}_2 U_1[k_2]_q & 0 & \tilde{d}_2 U_1 & 0 \end{pmatrix}, \tag{C.7}$$

$$B^- = [\bar{k}_2]_q \frac{q^{1-k_1+\frac{\delta M}{2}}}{(q-q^{-1})^{-1}} \frac{q^{M_2+2n}z_{12}-q^{M_1+2(k_2+1)}}{q^M z_{12}-q^{2(K+1)}} \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -c_1\widetilde{V}_2[k_1]_q & 0 & d_1\widetilde{V}_2 \\ 0 & -\tilde{c}_1 V_2[k_1]_q & 0 & \tilde{d}_1 V_2 \end{pmatrix}, \tag{C.8}$$

The inverse of A has a very complex form, however it can be decomposed intro three quite compact matrices as $A^{-1} = CVD$, where

$$C = \begin{pmatrix} \frac{z_{12}\tilde{b}_2}{[M_2-K+n]_q} & 0 & \frac{z_{12}\tilde{\alpha}b_2}{[M_2-K+n]_q} & 0 \\ 0 & \frac{q^{K-\frac{M_2}{2}-n}\tilde{\alpha}b_1U_2V_2}{[n-M_1]_q} & 0 & \frac{q^{K-\frac{M_2}{2}-n}\tilde{b}_1}{[M_1-n]_qU_2V_2} \\ -z_{12}\tilde{a}_2 & 0 & -z_{12}\tilde{\alpha}a_2 & 0 \\ 0 & q^{K-\frac{M_2}{2}-n}\tilde{\alpha}a_1U_2V_2 & 0 & -\frac{q^{K-\frac{M_2}{2}-n}\tilde{a}_1}{U_2V_2} \end{pmatrix}, \tag{C.9}$$

$$D = \mathrm{diag}\left(\frac{ig\xi}{\tilde{g}\alpha\tilde{\alpha}z_2}, \frac{ig\xi}{\tilde{g}\alpha\tilde{\alpha}^2z_2}, \frac{q^{\frac{M_2}{2}}}{\widetilde{V}_1\widetilde{V}_2\tilde{\alpha}}, \frac{q^{\frac{M_2}{2}}}{V_1V_2}\right), \tag{C.10}$$

$$V = \frac{1}{W}\begin{pmatrix} \frac{1}{i\xi}\left[U_z\xi^2-V_z+\frac{\widetilde{V}_zV_z-\widetilde{U}_zU_z\xi^2}{z_{12}}\right] & V_z-U_z & i\xi U_z & -V_z \\ \widetilde{U}_z-\widetilde{V}_z & \frac{i}{\xi}\left(\widetilde{V}_z-\widetilde{U}_z\xi^2\right) & \widetilde{V}_z & i\widetilde{U}_z\xi \\ \widetilde{V}_z-\widetilde{U}_z & \frac{i}{\xi}\left[\widetilde{U}_z\xi^2-\widetilde{V}_z+\frac{\widetilde{V}_zV_z-\widetilde{U}_zU_z\xi^2}{z_{12}}\right] & -\widetilde{V}_z & -i\widetilde{U}_z\xi \\ \frac{i}{\xi}\left(V_z-U_z\xi^2\right) & V_z-U_z & iU_z\xi & -V_z \end{pmatrix}, \tag{C.11}$$

here

$$W = \widetilde{V}_zV_z - \widetilde{U}_zU_z\xi^2, \qquad U_z = z_{12} - U_1^2U_2^2, \qquad \widetilde{U}_z = z_{12} - \widetilde{U}_1^2\widetilde{U}_2^2, \tag{C.12}$$

plus similar expressions for V_z.

Rational limit. The matrices B^+ (C.7) and B^- (C.8) in the $q \to 1+h$ ($h \to 0$) limit become

$$B^+ = 2h\,\bar{k}_1\frac{\delta u - \frac{\delta M}{2} - k_2 + n + 1}{\delta u - \frac{M}{2} + K + 1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -k_2c_2/U_1 & 0 & d_2/U_1 & 0 \\ k_2a_2U_1/\alpha\tilde{\alpha} & 0 & -b_2U_1/\alpha\tilde{\alpha} & 0 \end{pmatrix}, \tag{C.13}$$

$$B^- = 2h\,\bar{k}_2\frac{\delta u + \frac{\delta M}{2} + k_2 - n + 1}{\delta u - \frac{M}{2} + K + 1}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -k_1c_1 & 0 & d_1 \\ 0 & k_1a_1/\alpha\tilde{\alpha} & 0 & -b_1/\alpha\tilde{\alpha} \end{pmatrix}. \tag{C.14}$$

The matrices A (C.4) and B (C.5) in the $q \to 1$ limit become

$$A = \begin{pmatrix} -(M_2-K+n)g_2\gamma_2 & (M_1-n)g_1U_2\gamma_1 & -\frac{\alpha g_2(x_2^- - x_2^+)}{\gamma_2 x_2^-} & \frac{\alpha g_1 U_2(x_1^- - x_1^+)}{\gamma_1 x_1^-} \\ -\frac{i(M_2-K+n)\tilde{\alpha} g_2\gamma_2}{x_2^+} & \frac{i(M_1-n)\tilde{\alpha} g_1\gamma_1}{U_2 x_1^+} & -\frac{i\alpha\tilde{\alpha} g_2(x_2^- - x_2^+)}{\gamma_2} & \frac{i\alpha\tilde{\alpha} g_1(x_1^- - x_1^+)}{U_2\gamma_1} \\ \frac{i(M_2-K+n)g_2\gamma_2}{\alpha x_2^+} & -\frac{i(M_1-n)g_1\gamma_1}{\alpha U_2 x_1^+} & \frac{ig_2(x_2^- - x_2^+)}{\gamma_2} & -\frac{ig_1(x_1^- - x_1^+)}{U_2\gamma_1} \\ -\frac{(M_2-K+n)g_2\gamma_2}{\alpha\tilde{\alpha}} & \frac{(M_1-n)g_1U_2\gamma_1}{\alpha\tilde{\alpha}} & -\frac{g_2(x_2^- - x_2^+)}{\tilde{\alpha}\gamma_2 x_2^-} & \frac{g_1U_2(x_1^- - x_1^+)}{\tilde{\alpha}\gamma_1 x_1^-} \end{pmatrix}, \tag{C.15}$$

$$B = \begin{pmatrix} -(M_2-k_2)g_2U_1\gamma_2 & (M_1-k_1)g_1\gamma_1 & -\frac{\alpha g_2 U_1(x_2^- - x_2^+)}{\gamma_2 x_2^-} & \frac{\alpha g_1(x_1^- - x_1^+)}{\gamma_1 x_1^-} \\ -\frac{i(M_2-k_2)\tilde{\alpha} g_2\gamma_2}{U_1 x_2^+} & \frac{i(M_1-k_1)\tilde{\alpha} g_1\gamma_1}{x_1^+} & -\frac{i\alpha\tilde{\alpha} g_2(x_2^- - x_2^+)}{U_1\gamma_2} & \frac{i\alpha\tilde{\alpha} g_1(x_1^- - x_1^+)}{\gamma_1} \\ \frac{i(M_2-k_2)g_2\gamma_2}{\alpha U_1 x_2^+} & -\frac{i(M_1-k_1)g_1\gamma_1}{\alpha x_1^+} & \frac{ig_2(x_2^- - x_2^+)}{U_1\gamma_2} & -\frac{ig_1(x_1^- - x_1^+)}{\gamma_1} \\ -\frac{(M_2-k_2)g_2U_1\gamma_2}{\alpha\tilde{\alpha}} & \frac{(M_1-k_1)g_1\gamma_1}{\alpha\tilde{\alpha}} & -\frac{g_2U_1(x_2^- - x_2^+)}{\tilde{\alpha}\gamma_2 x_2^-} & \frac{g_1(x_1^- - x_1^+)}{\tilde{\alpha}\gamma_1 x_1^-} \end{pmatrix}. \tag{C.16}$$

The notation used in here is $g_i = \sqrt{\frac{g}{M_i}}$ and $U_i = \sqrt{\frac{x_i^+}{x_i^-}}$.

It might seem that the matrices B^+ and B^- do not contribute in the $q \to 1$ limit as they are of order $\mathcal{O}(h)$, however the combinations $A^{-1}B^+$ and $A^{-1}B^-$ in (4.2.61) are of order $\mathcal{O}(1)$, thus are defined correctly. We do not spell out the explicit expression of A^{-1} in the $q \to 1$ limit as it is quite sizy and also not much illuminative.

C.2 Subspace III

The coefficients' matrices in the expressions (4.2.67)

$$G^{op}(n, K-n)\,\mathscr{Z}_n^{k_1,k_2} = \mathscr{Y}_n^{k_1,k_2}\,G(k_1,k_2),$$
$$H^{op}(n, K-n)\,\mathscr{Z}_n^{k_1,k_2} = \overline{\mathscr{Y}}_n^{k_1,k_2}\,H(k_1,k_2),$$

are

$$G^{op} = \begin{pmatrix} \frac{q^{\frac{M_2}{2}-K+n}[M_1-n]_q\,a_1}{\widetilde{U}_2\widetilde{V}_2} & 0 & \frac{q^{\frac{M_2}{2}-K+n}b_1}{\widetilde{U}_2\widetilde{V}_2} & 0 & 0 & -b_2 \\ [M_2-K+n]_q a_2 & b_2 & 0 & 0 & \frac{q^{\frac{M_2}{2}-K+n}b_1}{\widetilde{U}_2\widetilde{V}_2} & 0 \\ 0 & \frac{q^{\frac{M_2}{2}-K+n}[M_1-n]_q a_1}{\widetilde{U}_2\widetilde{V}_2} & 0 & \frac{q^{\frac{M_2}{2}-K+n}b_1}{\widetilde{U}_2\widetilde{V}_2} & 0 & [M_2-K+n]_q a_2 \\ 0 & 0 & [M_2-K+n]_q a_2 & b_2 & \frac{q^{\frac{M_2}{2}-K+n}[M_1-n]_q a_1}{-\widetilde{U}_2\widetilde{V}_2} & 0 \end{pmatrix}, \tag{C.17}$$

$$G = \begin{pmatrix} [M_1-k_1]_q a_1 & 0 & b_1 & 0 & 0 & \frac{q^{\frac{M_1}{2}-k_1}b_2}{-\widetilde{U}_1\widetilde{V}_1} \\ \frac{q^{\frac{M_1}{2}-k_1}[M_2-k_2]_q a_2}{\widetilde{U}_1\widetilde{V}_1} & \frac{q^{\frac{M_1}{2}-k_1}b_2}{\widetilde{U}_1\widetilde{V}_1} & 0 & 0 & b_1 & 0 \\ 0 & [M_1-k_1]_q a_1 & 0 & b_1 & 0 & \frac{q^{\frac{M_1}{2}-k_1}[M_2-k_2]a_2}{\widetilde{U}_1\widetilde{V}_1} \\ 0 & 0 & \frac{q^{\frac{M_1}{2}-k_1}[M_2-k_2]_q a_2}{\widetilde{U}_1\widetilde{V}_1} & \frac{q^{\frac{M_1}{2}-k_1}b_2}{\widetilde{U}_1\widetilde{V}_1} & -[M_1-k_1]_q a_1 & 0 \end{pmatrix}, \tag{C.18}$$

and

$$H^{op} = \begin{pmatrix} \frac{[n]_q c_1}{U_2} & 0 & -\frac{d_1}{U_2} & 0 & -\frac{q^{n-\frac{M_1}{2}} d_2}{V_1} & 0 \\ \frac{q^{n-\frac{M_1}{2}}[K-n]_q c_2}{V_1} & \frac{q^{n-\frac{M_1}{2}} d_2}{-V_1} & 0 & 0 & 0 & \frac{d_1}{U_2} \\ 0 & \frac{[n]_q c_1}{U_2} & 0 & -\frac{d_1}{U_2} & -\frac{q^{n-\frac{M_1}{2}}[K-n]_q c_2}{V_1} & 0 \\ 0 & 0 & \frac{q^{n-\frac{M_1}{2}}[K-n]_q c_2}{V_1} & \frac{q^{n-\frac{M_1}{2}} d_2}{-V_1} & 0 & \frac{[n]_q c_1}{U_2} \end{pmatrix}, \tag{C.19}$$

$$H = \begin{pmatrix} \frac{q^{k_2-\frac{M_2}{2}}[k_1]_q c_1}{V_2} & 0 & \frac{q^{k_2-\frac{M_2}{2}} d_1}{-V_2} & 0 & -d_2\widetilde{U}_1 & 0 \\ [k_2]_q c_2\widetilde{U}_1 & -d_2\widetilde{U}_1 & 0 & 0 & 0 & \frac{q^{k_2-\frac{M_2}{2}} d_1}{V_2} \\ 0 & \frac{q^{k_2-\frac{M_2}{2}}[k_1]_q c_1}{V_2} & 0 & \frac{q^{k_2-\frac{M_2}{2}} d_1}{-V_2} & -[k_2]_q c_2\widetilde{U}_1 & 0 \\ 0 & 0 & [k_2]_q c_2\widetilde{U}_1 & -d_2\widetilde{U}_1 & 0 & \frac{q^{k_2-\frac{M_2}{2}}[k_1]_q c_1}{V_2} \end{pmatrix}. \tag{C.20}$$

Their affine counterparts $\widetilde{G}$, $\widetilde{G}^{op}$ and $\widetilde{H}$, $\widetilde{H}^{op}$ are obtained by the replacing non-affine (or affine) parameter to affine (or non-affine) ones. The matrix $\overline{\mathscr{Y}}_n^{k_1,k_2}$ is a slightly modified version of $\mathscr{Y}_n^{k_1,k_2}$,

$$\overline{\mathscr{Y}}_n^{k_1,k_2} \equiv \begin{pmatrix} (\mathscr{Y}_{n-1}^{k_1-1,k_2})_1^1 & (\mathscr{Y}_{n-1}^{k_1,k_2-1})_2^1 & (\mathscr{Y}_{n-1}^{k_1-1,k_2})_3^1 & (\mathscr{Y}_{n-1}^{k_1,k_2-1})_4^1 \\ (\mathscr{Y}_{n}^{k_1-1,k_2})_1^2 & (\mathscr{Y}_{n}^{k_1,k_2-1})_2^2 & (\mathscr{Y}_{n}^{k_1-1,k_2})_3^2 & (\mathscr{Y}_{n}^{k_1,k_2-1})_4^2 \\ (\mathscr{Y}_{n-1}^{k_1-1,k_2})_1^3 & (\mathscr{Y}_{n-1}^{k_1,k_2-1})_2^3 & (\mathscr{Y}_{n-1}^{k_1-1,k_2})_3^3 & (\mathscr{Y}_{n-1}^{k_1,k_2-1})_4^3 \\ (\mathscr{Y}_{n}^{k_1-1,k_2})_1^4 & (\mathscr{Y}_{n}^{k_1,k_2-1})_2^4 & (\mathscr{Y}_{n}^{k_1-1,k_2})_3^4 & (\mathscr{Y}_{n}^{k_1,k_2-1})_4^4 \end{pmatrix}. \tag{C.21}$$

The coefficient matrices in (4.2.70), $A\,\mathscr{X}_n^{k_1,k_2} = \check{\mathscr{Y}}_n^{k,l}B$, are

$$A = \begin{pmatrix} -\frac{[M_1-n]\mathcal{A}_3}{U_2V_2} & 0 & \frac{\mathcal{A}_1}{U_2V_2} & 0 & 0 & q_2\tilde{z}_2 \\ 0 & -q_2\tilde{z}_2 & 0 & 0 & \frac{\mathcal{A}_1}{U_2V_2} & 0 \\ 0 & -\frac{[M_1-n]\mathcal{A}_3}{U_2V_2} & 0 & \frac{\mathcal{A}_1}{U_2V_2} & 0 & 0 \\ -\frac{[n]_q\mathcal{A}_2}{U_2V_1} & 0 & -\frac{\mathcal{A}_4}{U_2V_1} & 0 & \frac{\tilde{g}^2 q_1}{g^2\tilde{z}_2} & 0 \\ 0 & \frac{\tilde{g}^2 q_1}{g^2\tilde{z}_2} & 0 & 0 & 0 & \frac{\mathcal{A}_4}{U_2V_1} \\ 0 & -\frac{[n]_q\mathcal{A}_2}{U_2V_1} & 0 & -\frac{\mathcal{A}_4}{U_2V_1} & 0 & 0 \end{pmatrix}, \tag{C.22}$$

$$A^{-1} = \begin{pmatrix} -\frac{U_2V_2}{\mathcal{A}_0\mathcal{A}_4^{-1}} & \frac{\tilde{g}^2 q_1 U_2^2 V_1 V_2}{g^2\mathcal{A}_0\tilde{z}_2} & 0 & -\frac{U_2V_1}{\mathcal{A}_0\mathcal{A}_1^{-1}} & \frac{q_2U_2^2V_1V_2\tilde{z}_2}{\mathcal{A}_0} & 0 \\ 0 & 0 & -\frac{U_2V_2}{\mathcal{A}_0\mathcal{A}_4^{-1}} & 0 & 0 & -\frac{U_2V_1}{\mathcal{A}_0\mathcal{A}_1^{-1}} \\ \frac{[n]_qU_2V_2}{\mathcal{A}_0\mathcal{A}_2^{-1}} & \frac{\tilde{g}^2[M_1-n]q_1U_2^2V_1V_2}{g^2\mathcal{A}_0\mathcal{A}_1\mathcal{A}_3^{-1}\tilde{z}_2} & \frac{\tilde{g}^2q_1q_2U_2^3V_1V_2^2}{-g^2\mathcal{A}_0\mathcal{A}_1} & \frac{[M_1-n]U_2V_1}{-\mathcal{A}_0\mathcal{A}_3^{-1}} & \frac{[n]_qq_2U_2^2V_1V_2\tilde{z}_2}{-\mathcal{A}_0\mathcal{A}_2^{-1}\mathcal{A}_4} & \frac{\tilde{g}^2q_1q_2U_2^3V_1^2V_2}{-g^2\mathcal{A}_0\mathcal{A}_4} \\ 0 & 0 & \frac{[n]_qU_2V_2}{\mathcal{A}_0\mathcal{A}_2^{-1}} & 0 & 0 & -\frac{[M_1-n]U_2V_1}{\mathcal{A}_0\mathcal{A}_3^{-1}} \\ 0 & \frac{U_2V_2}{\mathcal{A}_1} & -\frac{q_2U_2^2V_2^2\tilde{z}_2}{\mathcal{A}_0\mathcal{A}_1\mathcal{A}_4^{-1}} & 0 & 0 & -\frac{q_2U_2^2V_1V_2\tilde{z}_2}{\mathcal{A}_0} \\ 0 & 0 & \frac{\tilde{g}^2q_1U_2^2V_1V_2}{g^2\mathcal{A}_0\tilde{z}_2} & 0 & \frac{U_2V_1}{\mathcal{A}_4} & \frac{\tilde{g}^2q_1U_2^2V_1^2}{g^2\mathcal{A}_0\mathcal{A}_1^{-1}\mathcal{A}_4\tilde{z}_2} \end{pmatrix}, \tag{C.23}$$

here we have defined $\tilde{z}_i = \frac{\tilde{g}\alpha\tilde{\alpha}}{g} z_i$ and $\mathcal{A}_0 = [n]_q\, \mathcal{A}_1\mathcal{A}_2 + [M_1 - n]_q\, \mathcal{A}_3\mathcal{A}_4$ where

$$\begin{aligned}
\mathcal{A}_1 &= b_1\tilde{a}_2U_2^2V_2^2 - a_2\tilde{b}_1, & \mathcal{A}_2 &= c_2\tilde{c}_1U_2^2 - c_1V_1^2\tilde{c}_2, \\
\mathcal{A}_3 &= a_2\tilde{a}_1 - a_1\tilde{a}_2U_2^2V_2^2, & \mathcal{A}_4 &= d_1\tilde{c}_2V_1^2 - c_2\tilde{d}_1U_2^2.
\end{aligned} \tag{C.24}$$

$$B = \begin{pmatrix}
-[M_1-k_1]_q q_2\mathcal{B}_3 & 0 & q_2\mathcal{B}_2 & 0 & 0 & -\frac{q_3\mathcal{B}_1}{q_1U_1V_1} \\
-\frac{[M_2-k_2]_q q_3\mathcal{B}_7}{q_1U_1V_1} & \frac{q_3\mathcal{B}_1}{q_1U_1V_1} & 0 & 0 & q_2\mathcal{B}_2 & 0 \\
0 & -[M_1-k_1]_q q_2\mathcal{B}_3 & 0 & q_2\mathcal{B}_2 & 0 & -\frac{[M_2-k_2]_q q_3\mathcal{B}_7}{q_1U_1V_1} \\
0 & 0 & -\frac{[M_2-k_2]_q q_3\mathcal{B}_7}{q_1U_1V_1} & \frac{q_3\mathcal{B}_1}{q_1U_1V_1} & [M_1-k_1]_q q_2\mathcal{B}_3 & 0 \\
-\frac{[k_1]_q q_3\mathcal{B}_4}{V_1V_2} & 0 & -\frac{q_3\mathcal{B}_5}{V_1V_2} & 0 & -\frac{\mathcal{B}_6}{U_1V_1} & 0 \\
-\frac{[k_2]_q\mathcal{B}_8}{U_1V_1} & -\frac{\mathcal{B}_6}{U_1V_1} & 0 & 0 & 0 & \frac{q_3\mathcal{B}_5}{V_1V_2} \\
0 & -\frac{[k_1]_q q_3\mathcal{B}_4}{V_1V_2} & 0 & -\frac{q_3\mathcal{B}_5}{V_1V_2} & \frac{[k_2]_q\mathcal{B}_8}{U_1V_1} & 0 \\
0 & 0 & -\frac{[k_2]_q\mathcal{B}_8}{U_1V_1} & -\frac{\mathcal{B}_6}{U_1V_1} & 0 & -\frac{[k_1]_q q_3\mathcal{B}_4}{V_1V_2}
\end{pmatrix}, \tag{C.25}$$

and we are using the shorthand notation $q_1 = q^{n-\frac{M_1}{2}}$, $q_2 = q^{K-n-\frac{M_2}{2}}$, $q_3 = q^{k_2-\frac{M_2}{2}}$ and

$$\begin{aligned}
\mathcal{B}_1 &= b_2\tilde{a}_2U_1^2V_1^2 - a_2\tilde{b}_2, & \mathcal{B}_2 &= b_1\tilde{a}_2 - a_2\tilde{b}_1, \\
\mathcal{B}_3 &= a_2\tilde{a}_1 - a_1\tilde{a}_2, & \mathcal{B}_4 &= c_2\tilde{c}_1V_2^2 - c_1\tilde{c}_2V_1^2, \\
\mathcal{B}_5 &= d_1\tilde{c}_2V_1^2 - c_2\tilde{d}_1V_2^2, & \mathcal{B}_6 &= d_2\tilde{c}_2V_1^2 - c_2\tilde{d}_2U_1^2, \\
\mathcal{B}_7 &= a_2\tilde{a}_2(1 - U_1^2V_1^2), & \mathcal{B}_8 &= c_2\tilde{c}_2(U_1^2 - V_1^2).
\end{aligned} \tag{C.26}$$

The matrix $\check{\mathscr{Y}}_n^{k_1,k_2}$ is defined as

$$\check{\mathscr{Y}}_n^{k_1,k_2} = \begin{pmatrix} \mathscr{Y}_n^{k_1,k_2} & 0 \\ 0 & \overline{\mathscr{Y}}_n^{k_1,k_2} \end{pmatrix}, \tag{C.27}$$

where only first three rows of both $\mathscr{Y}_n^{k_1,k_2}$ and $\overline{\mathscr{Y}}_n^{k_1,k_2}$ are taken.

Rational limit. In the rational limit $q \to 1$ the coefficients (C.24) and (C.26) acquire quite compact expressions

$$\begin{aligned}
\frac{\mathcal{A}_1}{\alpha\,\tilde{\alpha}} &= \alpha\,\tilde{\alpha}\,\mathcal{A}_4 = i\sqrt{\frac{g}{M_1}}\sqrt{\frac{g}{M_2}}\frac{\left(x_1^- - x_1^+\right)\left(1 - x_1^-x_2^-\right)\gamma_2}{x_1^-x_2^-\gamma_1}, \\
\frac{\mathcal{A}_3}{\tilde{\alpha}} &= \tilde{\alpha}\,\mathcal{A}_2 = i\sqrt{\frac{g}{M_1}}\sqrt{\frac{g}{M_2}}\frac{\left(x_2^- - x_1^+\right)\gamma_1\gamma_2}{x_2^-x_1^+},
\end{aligned} \tag{C.28}$$

giving

$$\mathcal{A}_0 = -\frac{g^2}{\alpha M_2}\frac{\left(1 - x_1^-x_2^-\right)\left(x_1^- - x_1^+\right)\left(x_2^- - x_1^+\right)\gamma_2^2}{x_1^-(x_2^-)^2x_1^+}, \tag{C.29}$$

and also

$$\begin{aligned}
\frac{\mathcal{B}_1}{\alpha\,\tilde{\alpha}} &= \alpha\,\tilde{\alpha}\,\mathcal{B}_6 = i\frac{g}{M_2}\frac{\left(x_2^- - x_2^+\right)\left(x_1^+ - x_1^- x_2^- x_2^+\right)}{x_1^- x_2^- x_2^+}, \\
\frac{\mathcal{B}_2}{\alpha\,\tilde{\alpha}} &= \alpha\,\tilde{\alpha}\,\mathcal{B}_5 = i\sqrt{\frac{g}{M_1}}\sqrt{\frac{g}{M_2}}\frac{\left(x_1^- - x_1^+\right)\left(1 - x_1^- x_2^+\right)\gamma_2}{x_1^- x_2^+\gamma_1}, \\
\frac{\mathcal{B}_3}{\tilde{\alpha}} &= \alpha^2\tilde{\alpha}\,\mathcal{B}_4 = -i\sqrt{\frac{g}{M_1}}\sqrt{\frac{g}{M_2}}\frac{\left(x_1^+ - x_2^+\right)\gamma_1\gamma_2}{x_1^+ x_2^+}, \\
\frac{\mathcal{B}_7}{\tilde{\alpha}} &= \alpha^2\tilde{\alpha}\,\mathcal{B}_8 = i\frac{g}{M_2}\frac{\left(x_1^- - x_1^+\right)\gamma_2^2}{x_1^- x_2^+}.
\end{aligned} \tag{C.30}$$

D Elements of the special cases of the S-matrix

D.1 Elements of the fundamental S-matrix

The fundamental S-matrix for the space III acquires the following form,

$$\begin{aligned}
S\,|\phi^1\phi^2\rangle &= (\mathscr{X}_1^{1,0})_1^1|\phi^1\phi^2\rangle + (\mathscr{X}_0^{1,0})_1^1|\phi^2\phi^1\rangle + (\mathscr{X}_1^{1,0})_1^5|\psi^1\psi^2\rangle + (\mathscr{X}_0^{1,0})_1^6|\psi^2\psi^1\rangle, \\
S\,|\phi^2\phi^1\rangle &= (\mathscr{X}_1^{0,1})_1^1|\phi^1\phi^2\rangle + (\mathscr{X}_0^{0,1})_1^1|\phi^2\phi^1\rangle + (\mathscr{X}_1^{0,1})_1^5|\psi^1\psi^2\rangle + (\mathscr{X}_0^{0,1})_1^6|\psi^2\psi^1\rangle, \\
S\,|\psi^1\psi^2\rangle &= (\mathscr{X}_1^{1,0})_5^1|\phi^1\phi^2\rangle + (\mathscr{X}_0^{1,0})_5^1|\phi^2\phi^1\rangle + (\mathscr{X}_1^{1,0})_5^5|\psi^1\psi^2\rangle + (\mathscr{X}_0^{1,0})_5^6|\psi^2\psi^1\rangle, \\
S\,|\psi^2\psi^1\rangle &= (\mathscr{X}_1^{0,1})_6^1|\phi^1\phi^2\rangle + (\mathscr{X}_0^{0,1})_6^1|\phi^2\phi^1\rangle + (\mathscr{X}_1^{0,1})_6^5|\psi^1\psi^2\rangle + (\mathscr{X}_0^{0,1})_6^6|\psi^2\psi^1\rangle.
\end{aligned} \tag{D.1}$$

In order to find these coefficients $\mathscr{X}$ it is sufficient to consider the first relation of (4.2.67) and its affine counterpart only. In fact, the constraints read as follows,

$$\begin{aligned}
&\begin{pmatrix} (G^{op})_1^2 & (G^{op})_5^2 \\ (\widetilde{G}^{op})_1^2 & (\widetilde{G}^{op})_5^2 \end{pmatrix}(1,0)\begin{pmatrix} (\mathscr{X}_1^{1,0})_1^1 & (\mathscr{X}_1^{1,0})_5^1 \\ (\mathscr{X}_1^{1,0})_1^5 & (\mathscr{X}_1^{1,0})_5^5 \end{pmatrix} = (\mathscr{Y}_1^{1,0})_2^2\begin{pmatrix} (G)_1^2 & (G)_5^2 \\ (\widetilde{G})_1^2 & (\widetilde{G})_5^2 \end{pmatrix}(1,0)\,, \\
&\begin{pmatrix} (G^{op})_1^2 & (G^{op})_5^2 \\ (\widetilde{G}^{op})_1^2 & (\widetilde{G}^{op})_5^2 \end{pmatrix}(1,0)\begin{pmatrix} (\mathscr{X}_1^{0,1})_1^1 & (\mathscr{X}_1^{0,1})_6^1 \\ (\mathscr{X}_1^{0,1})_1^5 & (\mathscr{X}_1^{0,1})_6^5 \end{pmatrix} = (\mathscr{Y}_1^{0,1})_1^2\begin{pmatrix} (G)_1^1 & (G)_6^1 \\ (\widetilde{G})_1^1 & (\widetilde{G})_6^1 \end{pmatrix}(0,1)\,, \\
&\begin{pmatrix} (G^{op})_1^1 & (G^{op})_6^1 \\ (\widetilde{G}^{op})_1^1 & (\widetilde{G}^{op})_6^1 \end{pmatrix}(0,1)\begin{pmatrix} (\mathscr{X}_0^{1,0})_1^1 & (\mathscr{X}_0^{1,0})_5^1 \\ (\mathscr{X}_0^{1,0})_1^6 & (\mathscr{X}_0^{1,0})_5^6 \end{pmatrix} = (\mathscr{Y}_0^{1,0})_2^1\begin{pmatrix} (G)_1^2 & (G)_5^2 \\ (\widetilde{G})_1^2 & (\widetilde{G})_5^2 \end{pmatrix}(1,0)\,, \\
&\begin{pmatrix} (G^{op})_1^1 & (G^{op})_6^1 \\ (\widetilde{G}^{op})_1^1 & (\widetilde{G}^{op})_6^1 \end{pmatrix}(0,1)\begin{pmatrix} (\mathscr{X}_0^{0,1})_1^1 & (\mathscr{X}_0^{0,1})_6^1 \\ (\mathscr{X}_0^{0,1})_1^6 & (\mathscr{X}_0^{0,1})_6^6 \end{pmatrix} = itself(\mathscr{Y}_0^{0,1})_1^1\begin{pmatrix} (G)_1^1 & (G)_6^1 \\ (\widetilde{G})_1^1 & (\widetilde{G})_6^1 \end{pmatrix}(0,1)\,.
\end{aligned} \tag{D.2}$$

It is easy to solve these relations for $\mathscr{X}$ and we find them to agree with [28]. For the completeness, we have listed the relations of our elements $\mathscr{X}$ to those of [28][10]

$$\begin{aligned}
\begin{pmatrix} (\mathscr{X}_1^{1,0})_1^1 & (\mathscr{X}_1^{1,0})_5^1 \\ (\mathscr{X}_1^{1,0})_1^5 & (\mathscr{X}_1^{1,0})_5^5 \end{pmatrix} = \begin{pmatrix} (\mathscr{X}_0^{0,1})_1^1 & (\mathscr{X}_0^{0,1})_6^1 \\ (\mathscr{X}_0^{0,1})_1^6 & (\mathscr{X}_0^{0,1})_6^6 \end{pmatrix} &= \frac{1}{A_{12}} \begin{pmatrix} \frac{A_{12}-B_{12}}{q+q^{-1}} & -\frac{F_{12}}{q+q^{-1}} \\ \frac{C_{12}}{q+q^{-1}} & -\frac{D_{12}-E_{12}}{q+q^{-1}} \end{pmatrix}, \\
\begin{pmatrix} (\mathscr{X}_1^{0,1})_1^1 & (\mathscr{X}_1^{0,1})_6^1 \\ (\mathscr{X}_1^{0,1})_1^5 & (\mathscr{X}_1^{0,1})_6^5 \end{pmatrix} &= \frac{1}{A_{12}} \begin{pmatrix} \frac{q^{-1}A_{12}+qB_{12}}{q+q^{-1}} & \frac{qF_{12}}{q+q^{-1}} \\ -\frac{qC_{12}}{q+q^{-1}} & -\frac{q^{-1}D_{12}+qE_{12}}{q+q^{-1}} \end{pmatrix}, \\
\begin{pmatrix} (\mathscr{X}_0^{0,1})_1^1 & (\mathscr{X}_0^{0,1})_6^1 \\ (\mathscr{X}_0^{0,1})_1^6 & (\mathscr{X}_0^{0,1})_6^6 \end{pmatrix} &= \frac{1}{A_{12}} \begin{pmatrix} \frac{qA_{12}+q^{-1}B_{12}}{q+q^{-1}} & \frac{q^{-1}F_{12}}{q+q^{-1}} \\ -\frac{q^{-1}C_{12}}{q+q^{-1}} & -\frac{qD_{12}+q^{-1}E_{12}}{q+q^{-1}} \end{pmatrix}.
\end{aligned} \tag{D.3}$$

D.2 Elements of the S-matrix $\mathbf{S}_{Q1}$

Here we list the explicit forms of the coefficients of the matrix S_{Q1}.

Subspace II. First we give the coefficients of the matrix $\mathscr{Y}$ in the case of a bound state scattering with a fundamental particle. There are four different combinations of the parameters k_1, k_2, n that contribute. Thus we have to consider the case where $k_2 = 0$ and $k_1 = n = k$ leading to

$$\begin{aligned}
&(\mathscr{Y}_k^{k,0})_1^1 = q^{\frac{1}{2}+k} U_2 V_2 \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{z_{12} - q^{Q-2k-1}}{z_{12} - q^{Q-1}}, && (\mathscr{Y}_k^{k,0})_2^2 = \frac{1}{q^{\frac{Q}{2}} U_1 V_1} \frac{x_1^+ - x_2^+}{x_1^- - x_2^+}, \\
&(\mathscr{Y}_k^{k,0})_2^1 = q^{\frac{1-Q}{2}} \frac{[Q-k]_q}{\sqrt{[Q]_q}} \frac{x_2^- - x_2^+}{x_1^- - x_2^+} \frac{U_2 V_2}{U_1 V_1} \frac{\gamma_1}{\gamma_2}, && (\mathscr{Y}_k^{k,0})_1^2 = \frac{1}{\sqrt{[Q]_q}} \frac{x_1^- - x_1^+}{x_1^- - x_2^+} \frac{\gamma_2}{\gamma_1}, \\
&(\mathscr{Y}_k^{k,0})_4^1 = \frac{q^{\frac{1-Q}{2}} \alpha}{\sqrt{[Q]_q}} \frac{U_2 V_2}{U_1 V_1} \frac{[x_1^- - x_1^+][x_2^- - x_2^+][x_2^- - x_1^+]}{(x_1^- - x_2^+)(x_1^- x_2^- - 1)\gamma_1 \gamma_2}, && (\mathscr{Y}_k^{k,0})_2^4 = (\mathscr{Y}_{k-1}^{k,0})_4^2 = 0, \\
&(\mathscr{Y}_k^{k,0})_1^4 = \frac{q^{-Q}[k]_q}{\sqrt{[Q]_q}} \frac{x_1^+ - x_2^-}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)} \frac{x_1^-}{x_1^+} \frac{\gamma_1 \gamma_2}{\alpha}, && (\mathscr{Y}_k^{k,0})_4^4 = \frac{q^{-\frac{Q}{2}}}{U_1 V_1} \frac{x_1^+ - x_1^-}{x_1^- - x_2^+} \frac{1 - x_1^- x_2^+}{1 - x_1^- x_2^-}.
\end{aligned} \tag{D.4}$$

Next we have three elements corresponding to $k_2 = 1$ and $k_1 + 1 = n = k$ giving

$$\begin{aligned}
&(\mathscr{Y}_k^{k-1,1})_1^1 = q^{\frac{1}{2}-Q} U_2 V_2 \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{(q^{2(k+1)} - q^{2Q}) z_{12}}{z_{12} - q^{Q-1}}, && (\mathscr{Y}_k^{k-1,1})_1^2 = \frac{q^{1+k-Q}}{\sqrt{[Q]_q}} \frac{x_1^- - x_1^+}{x_1^- - x_2^+} \frac{\gamma_2}{\gamma_1}, \\
&(\mathscr{Y}_k^{k-1,1})_1^4 = \frac{[Q-k-1]_q}{q^{Q-k-1}\sqrt{[Q]_q}} \frac{x_2^- - x_1^+}{(x_1^- - x_2^+)(1 - x_1^- x_2^-)} \frac{x_1^-}{x_1^+} \frac{\gamma_1 \gamma_2}{\alpha}.
\end{aligned} \tag{D.5}$$

[10]We remind that our $x^\pm$ parameterization is based on the one of [30] which are related to those of [28] by $x^\pm_{[35]} = g\tilde{g}^{-1}(x^\pm_{[10]} + \xi)$. This point must be taken into account when performing the concrete comparison.

Then we have another three scattering entries for $k_2 = 0$ and $k_1 = n + 1 = k$ contributing

$$(\mathscr{Y}^{k,0}_{k-1})^1_1 = q^{\frac{1}{2}+Q} U_2 V_2 \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{1 - q^{-2k}}{q^Q - q z_{12}}, \quad (\mathscr{Y}^{k,0}_{k-1})^1_2 = q^{\frac{1+Q-2k}{2}} \frac{[k]_q}{\sqrt{[Q]_q}} \frac{x_2^- - x_2^+}{x_1^- - x_2^+} \frac{U_2 V_2}{U_1 V_1} \frac{\gamma_1}{\gamma_2},$$
$$(\mathscr{Y}^{k,0}_{k-1})^1_4 = -q^{-k} (\mathscr{Y}^{k,0}_{k})^1_4. \tag{D.6}$$

Finally, there is one element with $k_2 = 1$ and $k_1 = n = k - 1$ providing the last element

$$(\mathscr{Y}^{k-1,1}_{k-1})^1_1 = q^{\frac{1}{2}-k} U_2 V_2 \frac{x_1^- - x_2^-}{x_1^- - x_2^+} \frac{q^{2k} - q^{1+Q} z_{12}}{q^Q - q z_{12}}. \tag{D.7}$$

Subspace III. There are 36 elements of the matrix $\mathscr{Z}$ that need be determined. As mentioned in Section 4.2.4, it follows that (4.2.87) becomes

$$S\,|k-1,1\rangle^{\rm III}_6 = \mathscr{D}\left(|k,0\rangle^{\rm III}_5 + q\,|k-1,1\rangle^{\rm III}_6\right) - q\,S\,|k,0\rangle^{\rm III}_5. \tag{D.8}$$

Acting with the S-matrix on both sides of the equations (4.2.88) and using its invariance property allows us to express the elements of the S-matrix of the left hand side to the ones on the right hand side. Explicitly we find

$$\begin{aligned}
(\mathscr{Z}^{k-1,1}_{k})^1_1 =& (\mathscr{Z}^{k,0}_{k})^1_1 [Q-k+1]_q (q^{2k-Q-2} - q) + (\mathscr{Z}^{k,0}_{k-1})^1_1 \frac{[Q-k+1]_q}{[k]_q}\,,\\
(\mathscr{Z}^{k-1,1}_{k-1})^1_1 =& (\mathscr{Z}^{k,0}_{k})^1_1 + \frac{[k-1]_q [Q-k+2]_q (q^{2k-Q-4} - q) - [Q-2k+1]_q}{[k]_q} (\mathscr{Z}^{k,0}_{k-1})^1_1\,,\\
(\mathscr{Z}^{k-1,1}_{k})^3_1 =& (\mathscr{Z}^{k,0}_{k})^3_1 \frac{[k-1]_q [Q-k]_q q^{2k-Q-2} - q[k]_q [Q-k+1]_q}{[k]_q} + (\mathscr{Z}^{k,0}_{k-1})^3_1 \frac{[Q-k]_q}{[k]_q}\,,\\
(\mathscr{Z}^{k-1,1}_{k-1})^3_1 =& (\mathscr{Z}^{k,0}_{k-1})^3_1 \frac{[k-2]_q [Q-k+1]_q q^{2k-Q-4} + q[k-1]_q [k-Q-2]_q + [2k-Q-1]_q}{[k]_q}\\
&+ (\mathscr{Z}^{k,0}_{k})^3_1 \frac{[k-1]_q)}{[k]_q}\,,\\
(\mathscr{Z}^{k-1,1}_{k})^5_1 =& (\mathscr{Z}^{k,0}_{k})^5_1 \frac{([k-1]_q q^{2k-3-Q} - q[k]_q)[Q-k+1]_q}{[k]_q}\,,\\
(\mathscr{Z}^{k-1,1}_{k-1})^6_1 =& (\mathscr{Z}^{k,0}_{k-1})^6_1 \frac{([k-1]_q q^{2k-3-Q} - q[k]_q)[Q-k+1]_q}{[k]_q}\,,\\
(\mathscr{Z}^{k-1,1}_{k})^1_3 =& (\mathscr{Z}^{k,0}_{k})^1_3 \left[\frac{[k]_q [Q-k+1]_q q^{2k-Q-2}}{[k-1]_q} - q[Q-k]_q\right] + (\mathscr{Z}^{k,0}_{k-1})^1_3 \frac{[Q-k+1]_q}{[k-1]_q}\,,\\
(\mathscr{Z}^{k-1,1}_{k-1})^1_3 =& (\mathscr{Z}^{k,0}_{k-1})^1_3 \left[\frac{[Q-k+2]_q}{q^{Q+4-2k}} + [k-Q+1]_q - \frac{[k-2]_q q^{Q-k+1}}{[k-1]_q}\right] + (\mathscr{Z}^{k,0}_{k})^1_3 \frac{[k]_q}{[k-1]_q}\,,\\
(\mathscr{Z}^{k-1,1}_{k})^3_3 =& (\mathscr{Z}^{k,0}_{k})^3_3 [Q-k]_q (q^{2k-Q-2} - q) + (\mathscr{Z}^{k,0}_{k-1})^3_3 \frac{[Q-k]_q}{[k-1]_q}\,,
\end{aligned}$$

$$
\begin{aligned}
(\mathscr{Z}^{k-1,1}_{k-1})^3_3 =& (\mathscr{Z}^{k,0}_{k-1})^3_3 \left[\frac{[k-2]_q[Q-k+1]_q}{q^{Q-2k+4}[k-1]_q} + \frac{q[k]_q[k-Q+1]_q - q^2[2k-Q-1]_q}{[k-1]_q}\right] + (\mathscr{Z}^{k,0}_{k})^3_3\,,\\
(\mathscr{Z}^{k-1,1}_{k})^5_3 =& (\mathscr{Z}^{k,0}_{k})^5_1([Q-k+1]_q q^{2k-Q-3} - q[Q-k])\,,\\
(\mathscr{Z}^{k-1,1}_{k-1})^6_3 =& (\mathscr{Z}^{k,0}_{k-1})^6_1([Q-k+1]_q q^{2k-Q-3} - q[Q-k])\,. \qquad\qquad \text{(D.9)}
\end{aligned}
$$

Finally, the remaining elements are

$$
\begin{aligned}
(\mathscr{Z}^{k,0}_{k})^1_5 =& \frac{\alpha}{U_1V_1}\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)\left[\frac{(\xi x_1^+ +1)[Q-k]_q(q(\xi+x_2^-)-x_2^+-\xi)}{(\xi^2-1)q^Q} - [k]_q(x_1^+ - x_2^+)\right]}{\gamma_1\gamma_2[k]_q\sqrt{[Q]}(1-x_1^-x_2^-)(x_1^- - x_2^+)q^{\frac{Q}{2}}}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^1_5 =& \frac{\alpha}{U_1V_1}\frac{(x_1^- - x_1^+)(x_2^- - x_2^+)[q(\xi+x_2^-)(\xi x_1^+ +1) - (\xi + x_1^+)(\xi x_2^+ +1)]}{\gamma_1\gamma_2(\xi^2-1)\sqrt{[Q]}(1-x_1^-x_2^-)(x_1^- - x_2^+)q^{k+\frac{Q}{2}}}\,,\\
(\mathscr{Z}^{k,0}_{k})^3_5 =& \frac{\gamma_1}{\gamma_2 q^{\frac{Q}{2}}}\frac{[Q-k]_q}{\sqrt{[Q]_q}}\frac{(x_2^- - x_2^+)[q(\xi+x_1^-)(\xi+x_2^-) - (\xi x_1^- +1)(\xi x_2^+ +1)]}{(\xi^2-1)(1-x_1^-x_2^-)U_1V_1(x_1^- - x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^3_5 =& \frac{\gamma_1}{\gamma_2}\frac{[k-1]_q}{\sqrt{[Q]_q}}\frac{(x_2^- - x_2^+)[q(\xi+x_1^-)(\xi+x_2^-) - (\xi x_1^- +1)(\xi x_2^+ +1)]}{(\xi^2-1)(1-x_1^-x_2^-)U_1V_1(x_1^- - x_2^+)q^{k-\frac{Q}{2}}}\,,\\
(\mathscr{Z}^{k,0}_{k})^5_5 =& \frac{(x_1^+ - x_2^+)[(\xi x_1^- +1)(\xi x_2^+ +1) - q(\xi+x_1^-)(\xi+x_2^-)]}{(\xi^2-1)(x_1^-x_2^- - 1)(x_1^- - x_2^+)U_1V_1U_2V_2 q^{\frac{Q+1}{2}}}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^6_5 =& \frac{z_{12}(x_2^- - x_2^+)(x_1^+(\xi x_2^- +1)(\xi x_2^+ +1) - V_1^4 x_1^-(\xi+x_2^-)(\xi+x_2^+))}{(\xi^2-1)V_1^2 x_2^+(x_1^-x_2^- - 1)(x_1^- - x_2^+)q^{\frac{1+Q}{2}}}\frac{U_2V_2}{U_1V_1}\,,\\
(\mathscr{Z}^{k,0}_{k})^1_1 =& \frac{x_2^-(x_1^- - x_1^+)[Q-k]_Q[(\xi x_1^- +1)(\xi x_1^+ +1) - V_2^2(\xi+x_1^-)(\xi+x_1^+)]}{(\xi^2-1)x_1^+ z_{12}[Q]_q(x_1^-x_2^- - 1)(x_1^- - x_2^+)q^Q} +\\
&+ \frac{x_1^-(x_2^-x_1^+ - 1)(x_1^+ - x_2^+)q^{k-2Q}}{x_1^+(x_1^-x_2^- - 1)(x_1^- - x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^1_1 =& \frac{x_2^-[k]_q q^{-k}(x_1^- - x_1^+)[(\xi x_1^- +1)(\xi x_1^+ +1) - V_2^2(\xi+x_1^-)(\xi+x_1^+)]}{(1-\xi^2)x_1^+ z_{12}[Q]_q(1-x_1^-x_2^-)(x_1^- - x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k})^3_1 =& \frac{\gamma_1^2 x_1^-[k]_q q^{-Q-1}(\xi x_2^+ +1)[Q-k]_q[q x_2^+(\xi+x_2^-) - x_2^-(\xi+x_2^+)]}{\alpha(\xi^2-1)x_1^+x_2^+[Q]_q(1-x_1^-x_2^-)(x_1^- - x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^3_1 =& \frac{\gamma_1^2 x_1^-[k-1]_q[k]q^{-k-1}(\xi x_2^+ +1)[q x_2^+(\xi+x_2^-) - x_2^-(\xi+x_2^+)]}{\alpha(\xi^2-1)x_1^+x_2^+[Q]_q(1-x_1^-x_2^-)(x_1^- - x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k})^5_1 =& \frac{\gamma_1\gamma_2(V_2^2-1)x_1^-x_2^-[k]_q q^{-Q-\frac{3}{2}}(x_1^+ - x_2^+)(\xi+x_2^+)(\xi x_2^+ +1)}{\alpha(\xi^2-1)x_1^+x_2^+\sqrt{[Q]_q}(1-x_1^-x_2^-)(x_1^- - x_2^+)(x_2^- - x_2^+)U_2V_2}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^6_1 =& \frac{\gamma_1\gamma_2 x_1^-[k]_q q^{-Q-\frac{3}{2}}(x_1^+ - x_2^+)}{\alpha x_1^+\sqrt{[Q]_q}(x_1^-x_2^- - 1)(x_1^- - x_2^+)U_2V_2}\,,\\
(\mathscr{Z}^{k,0}_{k})^1_3 =& \frac{\alpha q(1-V_2^2)(x_1^- - x_1^+)^2(\xi+x_2^-)(\xi x_2^- +1)}{\gamma_1^2(\xi^2-1)[Q]_q(x_1^-x_2^- - 1)(x_1^- - x_2^+)}\,,
\end{aligned}
$$

$$
\begin{aligned}
(\mathscr{Z}^{k,0}_{k-1})^1_3 =& \frac{\alpha(1-V_2^2)q^{k-3}(x_1^- - x_1^+)^2(\xi + x_2^-)(\xi x_2^- + 1)}{\gamma_1^2(\xi^2-1)[Q]_q(1-x_1^- x_2^-)(x_1^- - x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k})^3_3 =& \frac{q^k(x_1^- - x_2^-)(x_1^- x_2^+ - 1)}{(x_1^- x_2^- - 1)(x_1^- - x_2^+)} + \\
&- \frac{x_2^-[k]_q(x_1^- - x_1^+)[V_2^2(\xi x_1^- + 1)(\xi x_1^+ + 1) - (\xi + x_1^-)(\xi + x_1^+)]}{(\xi^2-1)x_1^+ z_{12}[Q]_q(x_1^- x_2^- - 1)(x_1^- - x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^3_3 =& \frac{x_2^-[k-1]_q q^{-k}(x_1^- - x_1^+)[V_2^2(\xi x_1^- + 1)(\xi x_1^+ + 1) - (\xi + x_1^-)(\xi + x_1^+)]}{(\xi^2-1)x_1^+ z_{12}[Q]_q(x_1^- x_2^- - 1)(x_1^- - x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k})^5_3 =& \sqrt{\frac{q}{[Q]_q}}\frac{(V_2^2-1)(x_1^- - x_1^+)(1 - x_1^- x_2^+)(\xi + x_2^-)(\xi x_2^- + 1)}{(\xi^2-1)(x_1^- x_2^- - 1)(x_1^- - x_2^+)(x_2^- - x_2^+)}\,,\\
(\mathscr{Z}^{k,0}_{k-1})^6_3 =& \frac{\gamma_2 q^{-Q-\frac{1}{2}}(x_1^- - x_1^+)(x_1^- x_2^+ - 1)}{\gamma_1\sqrt{[Q]_q}(x_1^- x_2^- - 1)(x_1^- - x_2^+)U_2 V_2}.
\end{aligned}
\tag{D.10}
$$

E Yang-Baxter equation

In this section we briefly summarize some details on the checks of the Yang-Baxter equation (YBE) that we have preformed for the bound state S-matrix.

Let us first focus on subspace I. The block $\mathscr{X}$ governing the scattering in this subspace is required to satisfy YBE on its own right. Thus we need to consider the following scattering sequences,

$$
|0,1,k_1,\bar{k}_1\rangle \otimes |0,1,k_2,\bar{k}_2\rangle \otimes |0,1,k_3,\bar{k}_3\rangle \xrightarrow{\text{YBE}} |0,1,m_1,\bar{m}_1\rangle \otimes |0,1,m_2,\bar{m}_2\rangle \otimes |0,1,m_3,\bar{m}_3\rangle
\tag{E.1}
$$

which give the explicit form of the YBE in subspace I,

$$
\begin{aligned}
&\sum_{n=0}^{k_1+k_2} \mathscr{X}^{k_1,k_2}_n(z_1,z_2)\mathscr{X}^{n,k_3}_{m_2}(z_1,z_3)\mathscr{X}^{k_1+k_2-n,k_3+n-m_2}_{m_1-m_2}(z_2,z_3) = \\
&\qquad \sum_{n=0}^{k_2+k_3} \mathscr{X}^{m_1-n,n}_{m_2}(z_1,z_2)\mathscr{X}^{k_1,k_2+k_3-n}_{m_1-n}(z_1,z_3)\mathscr{X}^{k_2,k_3}_n(z_2,z_3)\,.
\end{aligned}
\tag{E.2}
$$

We did not attempt to prove this identity in full generality, but we did check it for a large set of different values of the parameters k_i, m_i and bound state numbers and found it to be perfectly satisfied.

In a similar way, this approach for checking YBE may be extended to include the subspaces II and III. For example, acting with $S_{12}S_{13}S_{23} - S_{23}S_{13}S_{12}$ on the states of the following form,

$$
|0,0,k_1,l_1\rangle \otimes |0,0,k_2,l_2\rangle \otimes |0,1,k_3,\bar{k}_3\rangle,
\tag{E.3}
$$

will result in a plethora of different types of states and the coefficients will depend on all three scattering blocks $\mathscr{X}$, $\mathscr{Y}$, $\mathscr{Z}$. Due to the large size of these expressions we have not spelled them out explicitly here. Nevertheless we have explicitly computed, for different values of the parameters, various matrix elements of the YBE that include states from all three subspaces that in general may be written as

$$\langle \text{ out-state} \mid \text{YBE} \mid \text{in-state} \rangle . \tag{E.4}$$

We have performed the checks for a wide range of numerical values of the representation parameters and in each case it proved to be compatible with the YBE.

F q-deformed reflection matrices

In this Appendix we present the explicit forms of the q-deformed reflection matrices for the $Z = 0$ giant graviton. We enumerate the basis for fundamental particles as

$$e_1 = |0,0,1,0\rangle , \qquad e_2 = |0,0,0,1\rangle, \qquad e_3 = |1,0,0,0\rangle, \qquad e_4 = |0,1,0,0\rangle . \tag{F.1}$$

and two-particle bound states as

$$\begin{aligned} &\hat{e}_1 = |0,0,2,0\rangle , && \hat{e}_2 = |0,0,1,1\rangle , && \hat{e}_3 = |0,0,0,2\rangle && \hat{e}_4 = |1,0,1,0\rangle \\ &\hat{e}_5 = |1,0,0,1\rangle , && \hat{e}_6 = |0,1,1,0\rangle , && \hat{e}_7 = |0,1,0,1\rangle && \hat{e}_8 = |1,1,0,0\rangle . \end{aligned} \tag{F.2}$$

We will use the symbol " $\circ$ " to denote the tensor product of states to keep the expressions as compact as possible. Our normalization is such that $K\, e_1 \circ e_1 = e_1 \circ e_1$ and equivalently for the bound states. We have checked that these reflection matrices satisfy the reflection equation and unitarity requirement, $K(-p)K(p) = 1$.

Reflection matrix K_q^{Aa}

$$\begin{aligned} &K\, e_a \circ e_a = e_a \circ e_a , \\ &K\, e_a \circ e_\alpha = k_3\, e_\alpha \circ e_a + k_2\, e_a \circ e_\alpha , \\ &K\, e_\alpha \circ e_a = k_8\, e_\alpha \circ e_a + k_4\, e_a \circ e_\alpha , \\ &K\, e_\alpha \circ e_\alpha = k_9\, e_\alpha \circ e_\alpha , \\ &K\, e_1 \circ e_2 = k_1\, e_2 \circ e_1 + (1 - q^{-1} k_1)\, e_1 \circ e_2 - q^{-1} k_6\, e_4 \circ e_3 + q^{-2} k_6\, e_3 \circ e_4 , \\ &K\, e_2 \circ e_1 = (1 - q k_1)\, e_2 \circ e_1 + k_1\, e_1 \circ e_2 + k_6\, e_4 \circ e_3 - q^{-1} k_6\, e_3 \circ e_4 , \\ &K\, e_3 \circ e_4 = -q k_5\, e_2 \circ e_1 + k_5\, e_1 \circ e_2 + k_7\, e_4 \circ e_3 + (-q^{-1} k_7 + k_9)\, e_3 \circ e_4 , \\ &K\, e_4 \circ e_3 = q^2 k_5\, e_2 \circ e_1 - q k_5\, e_1 \circ e_2 + (-q k_7 + k_9)\, e_4 \circ e_3 + k_7\, e_3 \circ e_4 , \end{aligned} \tag{F.3}$$

here $a = 1, 2$ and $\alpha = 3, 4$.

The coefficients k_i above are given by

$$
\begin{aligned}
k_1 &= \left[\frac{U^2(\xi+x^+)-q(\xi+x^-)}{x_B-x^-} - U^2(1-U^2V^2)\frac{x_B+\xi}{x_B-x^-}\frac{x^--\kappa(x^-)}{\xi+x^+}\right]\frac{V^2}{U^2}\,,\\
k_2 &= \frac{q(\xi+x_B)-U^2(\xi+x^+)}{qU^2(x_B-x^-)}\,,\\
k_3 &= q^{\frac{1}{2}}(1-U^2V^2)\frac{x^--\kappa(x^-)}{x^--x_B}\frac{V}{U}\frac{\gamma_B}{\gamma}\,,\\
k_4 &= zq^{-\frac{1}{2}}\frac{(x^--\kappa(x^-))(x_B+\xi)}{(x_B-x^-)(\xi+x^-)}\frac{V}{U}\frac{\underline{\gamma}}{\gamma_B}\,,\\
k_5 &= \frac{q^{-\frac{3}{2}}}{\alpha}\left[\frac{q\,(\xi+x^-)-U^2\,(\xi+x^+)}{(x_B-x^-)} + \frac{z\,(x^+-\kappa\,(x^+))\,(x_B+\xi)}{q^2\,(x_B-x^-)\,(\xi+x^-)}\right]\frac{V}{U}\,\underline{\gamma}\gamma_B\,,\\
k_6 &= \alpha q^{\frac{1}{2}}\frac{U^4-1}{U^2}\left[qV^2\frac{\xi+x^+}{x_B-x^-} + \frac{(1-U^2V^2)(x_B+\xi)}{x^--x_B}\right]\frac{V}{U}\frac{1}{\gamma\gamma_B}\,,\\
k_7 &= \left[\frac{zV^2}{q}\frac{U^4-1}{U^2}\frac{x_B+\xi}{x_B-x^-} + \frac{(1-U^2V^2)(x^+-\kappa(x^+))}{x_B-x^-}\right]\frac{\underline{\gamma}}{\gamma}\,,\\
k_8 &= \frac{zU^2(x_B+\xi)+\xi+x^-}{x^--x_B}\frac{\underline{\gamma}}{\gamma}\,,\\
k_9 &= z\frac{x_B-\kappa(x^-)}{x^--x_B}\frac{\underline{\gamma}}{\gamma}\,.
\end{aligned}
\tag{F.4}
$$

Reflection matrix K_q^{Ba}

$$
\begin{aligned}
&K\,\hat{e}_1{\circ}e_1 = \,\hat{e}_1{\circ}e_1\,,\\
&K\,\hat{e}_1{\circ}e_2 = qk_3\,\hat{e}_2{\circ}e_1 - \tfrac{k_5}{q}\,\hat{e}_8{\circ}e_1 + \big(1-k_3-\tfrac{k_3}{q^2}\big)\,\hat{e}_1{\circ}e_2 - \tfrac{k_4}{q}\,\hat{e}_6{\circ}e_3 + \tfrac{k_4}{q^2}\,\hat{e}_4{\circ}e_4\,,\\
&K\,\hat{e}_1{\circ}e_3 = k_1\,\hat{e}_4{\circ}e_1 + k_2\,\hat{e}_1{\circ}e_3\,,\\
&K\,\hat{e}_1{\circ}e_4 = k_1\,\hat{e}_6{\circ}e_1 + k_2\,\hat{e}_1{\circ}e_4\,,\\
&K\,\hat{e}_2{\circ}e_1 = (1-q^2k_3)\,\hat{e}_2{\circ}e_1 + k_5\,\hat{e}_8{\circ}e_1 + \big(\tfrac{1}{q}+q\big)k_3\,\hat{e}_1{\circ}e_2 + k_4\,\hat{e}_6{\circ}e_3 - \tfrac{k_4}{q}\,\hat{e}_4{\circ}e_4\,,\\
&K\,\hat{e}_2{\circ}e_2 = \big(1+\tfrac{1}{q^2}\big)k_3\,\hat{e}_3{\circ}e_1 + \big(1-\tfrac{k_3}{q^2}\big)\,\hat{e}_2{\circ}e_2 - \tfrac{k_5}{q^2}\,\hat{e}_8{\circ}e_2 - \tfrac{k_4}{q^2}\,\hat{e}_7{\circ}e_3 + \tfrac{k_4}{q^3}\,\hat{e}_5{\circ}e_4\,,\\
&K\,\hat{e}_2{\circ}e_3 = \tfrac{k_1}{q}\,\hat{e}_5{\circ}e_1 + k_1\,\hat{e}_4{\circ}e_2 + k_2\,\hat{e}_2{\circ}e_3\,,\\
&K\,\hat{e}_2{\circ}e_4 = \tfrac{k_1}{q}\,\hat{e}_7{\circ}e_1 + k_1\,\hat{e}_6{\circ}e_2 + k_2\,\hat{e}_2{\circ}e_4\,,\\
&K\,\hat{e}_3{\circ}e_1 = (1-(1+q^2)k_3)\,\hat{e}_3{\circ}e_1 + k_3\,\hat{e}_2{\circ}e_2 + k_5\,\hat{e}_8{\circ}e_2 + k_4\,\hat{e}_7{\circ}e_3 - \tfrac{k_4}{q}\,\hat{e}_5{\circ}e_4\,,\\
&K\,\hat{e}_3{\circ}e_2 = \,\hat{e}_3{\circ}e_2\,,\\
&K\,\hat{e}_3{\circ}e_3 = k_1\,\hat{e}_5{\circ}e_2 + k_2\,\hat{e}_3{\circ}e_3\,,\\
&K\,\hat{e}_3{\circ}e_4 = k_1\,\hat{e}_7{\circ}e_2 + k_2\,\hat{e}_3{\circ}e_4\,,\\
&K\,\hat{e}_4{\circ}e_1 = k_6\,\hat{e}_4{\circ}e_1 + \big(\tfrac{1}{q}+q\big)k_{11}\,\hat{e}_1{\circ}e_3\,,\\
&K\,\hat{e}_4{\circ}e_2 = k_{12}\,\hat{e}_5{\circ}e_1 + \big(k_6-\tfrac{k_{12}}{q}\big)\,\hat{e}_4{\circ}e_2 + qk_{11}\,\hat{e}_2{\circ}e_3 - \tfrac{k_{13}}{q}\,\hat{e}_8{\circ}e_3\,,\\
&K\,\hat{e}_4{\circ}e_3 = k_7\,\hat{e}_4{\circ}e_3\,,\\
&K\,\hat{e}_4{\circ}e_4 = k_8\,\hat{e}_2{\circ}e_1 + k_{10}\,\hat{e}_8{\circ}e_1 - \tfrac{1+q^2}{q^3}k_8\,\hat{e}_1{\circ}e_2 + k_9\,\hat{e}_6{\circ}e_3 + \big(k_7-\tfrac{k_9}{q}\big)\,\hat{e}_4{\circ}e_4\,,\\
&K\,\hat{e}_5{\circ}e_1 = (k_6-qk_{12})\,\hat{e}_5{\circ}e_1 + k_{12}\,\hat{e}_4{\circ}e_2 + k_{11}\,\hat{e}_2{\circ}e_3 + k_{13}\,\hat{e}_8{\circ}e_3\,,\\
&K\,\hat{e}_5{\circ}e_2 = k_6\,\hat{e}_5{\circ}e_2 + \big(\tfrac{1}{q}+q\big)k_{11}\,\hat{e}_3{\circ}e_3\,,\\
&K\,\hat{e}_5{\circ}e_3 = k_7\,\hat{e}_5{\circ}e_3\,,\\
&K\,\hat{e}_5{\circ}e_4 = \big(1+\tfrac{1}{q^2}\big)k_8\,\hat{e}_3{\circ}e_1 - \tfrac{k_8}{q^2}\,\hat{e}_2{\circ}e_2 + k_{10}\,\hat{e}_8{\circ}e_2 + k_9\,\hat{e}_7{\circ}e_3 + \big(k_7-\tfrac{k_9}{q}\big)\,\hat{e}_5{\circ}e_4\,,\\
&K\,\hat{e}_6{\circ}e_1 = k_6\,\hat{e}_6{\circ}e_1 + \big(\tfrac{1}{q}+q\big)k_{11}\,\hat{e}_1{\circ}e_4\,,\\
&K\,\hat{e}_6{\circ}e_2 = k_{12}\,\hat{e}_7{\circ}e_1 + \big(k_6-\tfrac{k_{12}}{q}\big)\,\hat{e}_6{\circ}e_2 + qk_{11}\,\hat{e}_2{\circ}e_4 - \tfrac{k_{13}}{q}\,\hat{e}_8{\circ}e_4\,,\\
&K\,\hat{e}_6{\circ}e_3 = -qk_8\,\hat{e}_2{\circ}e_1 - qk_{10}\,\hat{e}_8{\circ}e_1 + \big(1+\tfrac{1}{q^2}\big)k_8\,\hat{e}_1{\circ}e_2 + (k_7-qk_9)\,\hat{e}_6{\circ}e_3 + k_9\,\hat{e}_4{\circ}e_4\,,\\
&K\,\hat{e}_6{\circ}e_4 = k_7\,\hat{e}_6{\circ}e_4\,,\\
&K\,\hat{e}_7{\circ}e_1 = (k_6-qk_{12})\,\hat{e}_7{\circ}e_1 + k_{12}\,\hat{e}_6{\circ}e_2 + k_{11}\,\hat{e}_2{\circ}e_4 + k_{13}\,\hat{e}_8{\circ}e_4\,,\\
&K\,\hat{e}_7{\circ}e_2 = k_6\,\hat{e}_7{\circ}e_2 + \big(\tfrac{1}{q}+q\big)k_{11}\,\hat{e}_3{\circ}e_4\,,\\
&K\,\hat{e}_7{\circ}e_3 = -(\tfrac{1}{q}+q)k_8\,\hat{e}_3{\circ}e_1 + \tfrac{k_8}{q}\,\hat{e}_2{\circ}e_2 - qk_{10}\,\hat{e}_8{\circ}e_2 + (k_7-qk_9)\,\hat{e}_7{\circ}e_3 + k_9\,\hat{e}_5{\circ}e_4\,,\\
&K\,\hat{e}_7{\circ}e_4 = k_7\,\hat{e}_7{\circ}e_4\,,\\
&K\,\hat{e}_8{\circ}e_1 = k_{14}\,\hat{e}_2{\circ}e_1 + k_{16}\,\hat{e}_8{\circ}e_1 - \tfrac{1+q^2}{q^3}k_{14}\,\hat{e}_1{\circ}e_2 + k_{15}\,\hat{e}_6{\circ}e_3 - \tfrac{k_{15}}{q}\,\hat{e}_4{\circ}e_4\,,\\
&K\,\hat{e}_8{\circ}e_2 = \big(1+\tfrac{1}{q^2}\big)k_{14}\,\hat{e}_3{\circ}e_1 - \tfrac{k_{14}}{q^2}\,\hat{e}_2{\circ}e_2 + k_{16}\,\hat{e}_8{\circ}e_2 + k_{15}\,\hat{e}_7{\circ}e_3 - \tfrac{k_{15}}{q}\,\hat{e}_5{\circ}e_4\,,\\
&K\,\hat{e}_8{\circ}e_3 = -qk_{17}\,\hat{e}_5{\circ}e_1 + k_{17}\,\hat{e}_4{\circ}e_2 + k_{18}\,\hat{e}_8{\circ}e_3\,,\\
&K\,\hat{e}_8{\circ}e_4 = -qk_{17}\,\hat{e}_7{\circ}e_1 + k_{17}\,\hat{e}_6{\circ}e_2 + k_{18}\,\hat{e}_8{\circ}e_4\,.
\end{aligned}
\tag{F.5}
$$

The reflection coefficients of K^{Ba} are

$$
\begin{aligned}
k_1 &= \sqrt{\tfrac{q}{1+q^2}}\frac{q(U^4-1)(\xi+x^-)}{x_B-x^-}\frac{V}{U}\frac{\gamma_B}{\gamma}\,,\\
k_2 &= \frac{q^2(x_B+\xi)-U^2(\xi+x^+)}{q^2U^2\,(x_B-x^-)}\,,\\
k_3 &= \frac{U^4-1}{1+q^2}\frac{\xi+x^-}{x_B-x^-}\left[q^2+\frac{1}{U^2z}\frac{x_B+\xi-U^2(\xi+x^+)}{(x_B+\xi)x^+}\frac{1+\xi x^+}{\chi+x^-}\right],\\
k_4 &= \frac{k_{13}}{1+\chi x^+}\left[\left[\frac{1}{x_B}+\xi-\frac{\xi}{U^2z}\right]x^+-\frac{1}{U^2z}\right]\frac{\gamma}{\underline{\gamma}}\,,\\
k_5 &= \alpha\frac{U^4-1}{1+q^{-2}}\frac{x^+-\kappa(x^+)-z^{-1}(q^2-1)(\xi+x^-)}{\chi+x^-}\frac{\xi+x^-}{x_B-x^-}\frac{V^2-U^{-2}}{\gamma^2}\,,\\
k_6 &= \left[z\,U^2\frac{x_B+\xi}{x^--x_B}+\frac{\xi+x^-}{x^--x_B}\right]\frac{\underline{\gamma}}{\gamma}\,,\\
k_7 &= \frac{U^2(\xi+x^-)+(x_B+\xi)z}{x^--x_B}\frac{\underline{\gamma}}{\gamma}\,,\\
k_8 &= \sqrt{\tfrac{q}{1+q^2}}\frac{V}{zU}\frac{x_B+\xi-U^2(\xi+x^+)}{\kappa(x^-)-x_B}\frac{U^2(\xi+x^-)+z(x_B+\xi)}{x^--x_B}\frac{1+\xi x^+}{x_B+\xi}\frac{\kappa(x^-)-x^-}{x^+\,(\chi+x^-)}\frac{\underline{\gamma}\gamma_B}{q\alpha}\,,\\
k_9 &= qz\frac{x^+}{x^-}\frac{x^--\kappa(x^-)}{x_B-x^-}\left[\frac{\left(\frac{1}{x_B}+\xi\right)x^+-\frac{1+\xi x^+}{U^2z}}{q^2U^2(1+\chi x^+)}\frac{U^2(\xi+x^-)+z(x_B+\xi)}{\chi+x^-}-1\right]\frac{\underline{\gamma}}{\gamma}\,,\\
k_{10} &= -\frac{k_1(\xi+x^+)\left(q^2U^2\,(1+\xi x^+)-(x_B+\xi)V^2x^-\right)\underline{\gamma}}{q^4\,(1+x_B\xi+(x_B+\xi)x^-)\,x^+\gamma}\,,\\
k_{11} &= \sqrt{\tfrac{q}{1+q^2}}\frac{U}{q\xi V}\frac{x_B+\xi}{x^--x_B}\left[\frac{z}{\kappa(x^-)}+\frac{V^2}{x^+}\right]\frac{\underline{\gamma}}{\gamma_B}\,,\\
k_{12} &= \sqrt{\tfrac{q}{1+q^2}}\frac{1}{UV}\left[\frac{qk_{11}(x^--x^+)\gamma_B}{x^-\gamma}+\frac{k_{13}\gamma\gamma_B}{q\alpha}\right],\\
k_{13} &= \sqrt{\tfrac{q}{1+q^2}}\frac{\alpha(x^--x^+)\,(x^+-\kappa(x^+))}{q\,UV\,(\chi+x^-)\,x^-}\frac{U^2(\xi+x^-)+(x_B+\xi)z}{x^--x_B}\frac{\underline{\gamma}}{\gamma^2\gamma_B}\,,\\
k_{14} &= \frac{q^2\,(x^++\xi)-(\kappa(x^+)+\xi)}{(1+q^2)(x_B-\kappa(x^-))}\frac{U^2(\xi+x^-)+(x_B+\xi)z}{V^2(\chi+x^-)(x^--x_B)}\left[\frac{1}{x^-}-\frac{1}{\kappa(x^-)}\right]\frac{\underline{\gamma}^2}{\alpha}\,,\\
k_{15} &= -\sqrt{\tfrac{q}{1+q^2}}\frac{U}{q\xi V}\frac{x_B+\xi}{x^--x_B}\left[\frac{z}{\kappa(x^-)}+\frac{V^2}{x^+}\right]\left[\frac{\xi+x^-}{\chi+x^-}-\frac{U^2z}{x_B}\frac{1+x_B\xi}{\chi+x^-}\right]\frac{\underline{\gamma}^2}{\gamma\gamma_B}\,,\\
k_{16} &= \alpha k_{14}V^2\frac{(\xi+x^+)(x^+-x^-)}{(1+\xi x^+)\,\gamma^2}+k_{15}\sqrt{\tfrac{1+q^2}{q^3}}\frac{V}{U}\frac{\xi+x^+}{x_B+\xi}\frac{\gamma_B}{\gamma}+k_6\frac{\xi+x^+}{\xi+\kappa(x^+)}\frac{\underline{\gamma}}{\gamma}\,,\\
k_{17} &= \sqrt{\tfrac{q}{1+q^2}}\frac{q^{-2}(\xi+x^+)+(x_B+\xi)z}{UVx^-(x^--x_B)}\frac{1+\xi x^+}{x_B+\xi}\frac{\kappa(x^-)-x^-}{\chi+x^-}\frac{\gamma_B\underline{\gamma}^2}{q\alpha\gamma}\,,\\
k_{18} &= \frac{q^{-2}(\xi+x^+)+(x_B+\xi)z}{x^--x_B}\left[\frac{\xi+x^-}{\chi+x^-}-\frac{U^2z}{q^2x_B}\frac{1+x_B\xi}{\chi+x^-}\right]\frac{\underline{\gamma}^2}{\gamma^2}\,,
\end{aligned}
\tag{F.6}
$$

here $\chi=\frac{1+\xi x_B}{x_B+\xi}$.

Reflection matrix K_q^{Ab}

$$
\begin{aligned}
K\, e_1\circ\hat e_1 &= e_1\circ\hat e_1\,,\\
K\, e_1\circ\hat e_2 &= \big(1+\tfrac{1}{q^2}\big)k_5\, e_2\circ\hat e_1 + \big(1-\tfrac{k_5}{q^2}\big)\, e_1\circ\hat e_2 - \tfrac{k_7}{q^2}\, e_4\circ\hat e_4 + \tfrac{k_7}{q^3}\, e_3\circ\hat e_6 - \tfrac{k_6}{q^2}\, e_1\circ\hat e_8\,,\\
K\, e_1\circ\hat e_3 &= qk_5\, e_2\circ\hat e_2 + \big(1-k_5+\tfrac{k_5}{q^2}\big)\, e_1\circ\hat e_3 - \tfrac{k_7}{q}\, e_4\circ\hat e_5 + \tfrac{k_7}{q^2}\, e_3\circ\hat e_7 - \tfrac{k_6}{q}\, e_2\circ\hat e_8\,,\\
K\, e_1\circ\hat e_4 &= \big(\tfrac{1}{q}+q\big)k_9\, e_3\circ\hat e_1 + k_1\, e_1\circ\hat e_4\,,\\
K\, e_1\circ\hat e_5 &= qk_9\, e_3\circ\hat e_2 + k_8\, e_2\circ\hat e_4 + \big(k_1-\tfrac{k_8}{q}\big)\, e_1\circ\hat e_5 - \tfrac{k_{10}}{q}\, e_3\circ\hat e_8\,,\\
K\, e_1\circ\hat e_6 &= \big(\tfrac{1}{q}+q\big)k_9\, e_4\circ\hat e_1 + k_1\, e_1\circ\hat e_6\,,\\
K\, e_1\circ\hat e_7 &= qk_9\, e_4\circ\hat e_2 + k_8\, e_2\circ\hat e_6 + \big(k_1-\tfrac{k_8}{q}\big)\, e_1\circ\hat e_7 - \tfrac{k_{10}}{q}\, e_4\circ\hat e_8\,,\\
K\, e_1\circ\hat e_8 &= -(1+q^2)k_2\, e_2\circ\hat e_1 + k_2\, e_1\circ\hat e_2 + k_4\, e_4\circ\hat e_4 - \tfrac{k_4}{q}\, e_3\circ\hat e_6 + k_3\, e_1\circ\hat e_8\,,\\
K\, e_2\circ\hat e_1 &= (1-(1+q^2)k_5)\, e_2\circ\hat e_1 + k_5\, e_1\circ\hat e_2 + k_7\, e_4\circ\hat e_4 - \tfrac{k_7}{q}\, e_3\circ\hat e_6 + k_6\, e_1\circ\hat e_8\,,\\
K\, e_2\circ\hat e_2 &= (1-q^2k_5)\, e_2\circ\hat e_2 + \big(\tfrac{1}{q}+q\big)k_5\, e_1\circ\hat e_3 + k_7\, e_4\circ\hat e_5 - \tfrac{k_7}{q}\, e_3\circ\hat e_7 + k_6\, e_2\circ\hat e_8\,,\\
K\, e_2\circ\hat e_3 &= e_2\circ\hat e_3\,,\\
K\, e_2\circ\hat e_4 &= k_9\, e_3\circ\hat e_2 + (k_1-qk_8)\, e_2\circ\hat e_4 + k_8\, e_1\circ\hat e_5 + k_{10}\, e_3\circ\hat e_8\,,\\
K\, e_2\circ\hat e_5 &= \big(\tfrac{1}{q}+q\big)k_9\, e_3\circ\hat e_3 + k_1\, e_2\circ\hat e_5\,,\\
K\, e_2\circ\hat e_6 &= k_9\, e_4\circ\hat e_2 + (k_1-qk_8)\, e_2\circ\hat e_6 + k_8\, e_1\circ\hat e_7 + k_{10}\, e_4\circ\hat e_8\,,\\
K\, e_2\circ\hat e_7 &= \big(\tfrac{1}{q}+q\big)k_9\, e_4\circ\hat e_3 + k_1\, e_2\circ\hat e_7\,,\\
K\, e_2\circ\hat e_8 &= -q^2k_2\, e_2\circ\hat e_2 + \big(\tfrac{1}{q}+q\big)k_2\, e_1\circ\hat e_3 + k_4\, e_4\circ\hat e_5 - \tfrac{k_4}{q}\, e_3\circ\hat e_7 + k_3\, e_2\circ\hat e_8\,,\\
K\, e_3\circ\hat e_1 &= k_{12}\, e_3\circ\hat e_1 + k_{11}\, e_1\circ\hat e_4\,,\\
K\, e_3\circ\hat e_2 &= k_{12}\, e_3\circ\hat e_2 + \tfrac{k_{11}}{q}\, e_2\circ\hat e_4 + k_{11}\, e_1\circ\hat e_5\,,\\
K\, e_3\circ\hat e_3 &= k_{12}\, e_3\circ\hat e_3 + k_{11}\, e_2\circ\hat e_5\,,\\
K\, e_3\circ\hat e_4 &= k_{13}\, e_3\circ\hat e_4\,,\\
K\, e_3\circ\hat e_5 &= k_{13}\, e_3\circ\hat e_5\,,\\
K\, e_3\circ\hat e_6 &= -(1+q^2)k_{14}\, e_2\circ\hat e_1 + k_{14}\, e_1\circ\hat e_2 + k_{16}\, e_4\circ\hat e_4 + \big(k_{13}-\tfrac{k_{16}}{q}\big)\, e_3\circ\hat e_6 + k_{15}\, e_1\circ\hat e_8\,,\\
K\, e_3\circ\hat e_7 &= -q^2k_{14}\, e_2\circ\hat e_2 + \big(\tfrac{1}{q}+q\big)k_{14}\, e_1\circ\hat e_3 + k_{16}\, e_4\circ\hat e_5 + \big(k_{13}-\tfrac{k_{16}}{q}\big)\, e_3\circ\hat e_7 + k_{15}\, e_2\circ\hat e_8\,,\\
K\, e_3\circ\hat e_8 &= -qk_{17}\, e_2\circ\hat e_4 + k_{17}\, e_1\circ\hat e_5 + k_{18}\, e_3\circ\hat e_8\,,\\
K\, e_4\circ\hat e_1 &= k_{12}\, e_4\circ\hat e_1 + k_{11}\, e_1\circ\hat e_6\,,\\
K\, e_4\circ\hat e_2 &= k_{12}\, e_4\circ\hat e_2 + \tfrac{k_{11}}{q}\, e_2\circ\hat e_6 + k_{11}\, e_1\circ\hat e_7\,,\\
K\, e_4\circ\hat e_3 &= k_{12}\, e_4\circ\hat e_3 + k_{11}\, e_2\circ\hat e_7\,,\\
K\, e_4\circ\hat e_4 &= (q+q^3)k_{14}\, e_2\circ\hat e_1 - qk_{14}\, e_1\circ\hat e_2 + (k_{13}-qk_{16})\, e_4\circ\hat e_4 + k_{16}\, e_3\circ\hat e_6 - qk_{15}\, e_1\circ\hat e_8\,,\\
K\, e_4\circ\hat e_5 &= q^3k_{14}\, e_2\circ\hat e_2 - (1+q^2)k_{14}\, e_1\circ\hat e_3 + (k_{13}-qk_{16})\, e_4\circ\hat e_5 + k_{16}\, e_3\circ\hat e_7 - qk_{15}\, e_2\circ\hat e_8\,,\\
K\, e_4\circ\hat e_6 &= k_{13}\, e_4\circ\hat e_6\,,\\
K\, e_4\circ\hat e_7 &= k_{13}\, e_4\circ\hat e_7\,,\\
K\, e_4\circ\hat e_8 &= -qk_{17}\, e_2\circ\hat e_6 + k_{17}\, e_1\circ\hat e_7 + k_{18}\, e_4\circ\hat e_8\,.
\end{aligned}
\tag{F.7}
$$

The reflection coefficients of K^{Ab} are

$$
\begin{aligned}
k_1 &= \frac{(x_B+\xi)-U^4(\xi+x^-)}{U^2(x_B-x^-)}\,,\\
k_2 &= \frac{q^{-1}k_4+qU^{-2}k_9}{\alpha\sqrt{1+q^2}}\frac{UV\gamma\gamma_B}{U^2V^2-1}\,,\\
k_3 &= \frac{k_{10}-\alpha U^2k_4}{\alpha q\sqrt{1+q^2}\,(1-U^2V^2)}\frac{V}{U}-k_{18}\frac{\gamma}{zU^2}\,,\\
k_4 &= \frac{\alpha q^2k_{17}}{\gamma\underline{\gamma}}\frac{\kappa(x^+)x^-}{(U^2-V^2)^{-1}}-k_{16}\frac{q^2UV}{z\sqrt{1+q^2}}\frac{x^-+\xi}{x_B+\xi}\frac{\gamma_B}{\underline{\gamma}}\,,\\
k_5 &= \frac{U^2-U^{-2}}{1+q^2}\left[\frac{V^2U^2}{\xi q^2}\frac{1+x_B\xi}{x_B-x^-}\frac{q^2-1}{1-x_Bx^+}\left[\frac{1+\xi x^+}{U^2}-\frac{x^+(x_B+\xi)}{U^2-V^2}\right]\right.\\
&\qquad\left.+q\frac{\xi+x^+}{x_B-x^-}\left[1-\frac{V^2-U^2}{q\kappa(x^-)x^+}+\frac{qV^2(\xi^2-1)}{(x_B+\xi)(\xi+x^+)}\right]\right],\\
k_7 &= -q\alpha\sqrt{1+q^2}k_5\frac{1-U^2V^2}{UV\gamma\gamma_B}+\frac{q^3\alpha}{\sqrt{1+q^2}}\frac{1-U^2V^2}{UV\gamma\gamma_B}-\frac{q^3\alpha k_{12}}{\sqrt{1+q^2}}\frac{1-U^{-2}V^2}{UV\underline{\gamma}\gamma_B}\,,\\
k_6 &= \frac{1}{\sqrt{1+q^2}}\frac{UV}{U^2-V^2}\frac{x_B+\xi}{\xi+x^+}\frac{\gamma}{\gamma_B}(k_{10}U^2-k_7)\,,\\
k_8 &= \frac{q}{q^2+1}\frac{(1+x_B^2+2x_B\xi)x^+}{(x_B-x^-)(1-x_Bx^+)}\frac{U^4-1}{V^2-U^2}\frac{V^2}{U^2}\,,\\
k_9 &= \frac{U^2-U^{-2}}{\sqrt{1+q^2}}\frac{V}{U}\frac{\xi+x^+}{x_B-x^-}\,,\\
k_{10} &= \alpha\frac{U^2-U^{-2}}{\sqrt{q^2+1}}\frac{V}{U}\frac{\xi+x^+}{x_B-x^-}\left[1+\frac{(1-q^4)x_B(1+\xi x^+)}{\xi(1-x_Bx^+)}\right],\\
k_{11} &= \sqrt{\frac{1}{1+q^2}}\frac{U^4-1}{U^2-V^2}\frac{V}{U}\frac{x_B+\xi}{x_B-x^-}\,,\\
k_{12} &= -\left[\frac{\xi+x^-}{x_B-x^-}+zU^2\frac{x_B+\xi}{x_B-x^-}\right],\\
k_{13} &= -\frac{\xi+x^+}{q(x_B-x^-)}-z\frac{x_B+\xi}{x_B-x^-}\,,\\
k_{14} &= \frac{U^2-U^{-2}}{\alpha q^4\sqrt{1+q^2}}\frac{V}{U}\frac{1+x_B\xi}{x_B-x^-}\left[\frac{U^2V^2}{U^2-V^2}\left[\frac{(q^2-1)(1+\xi x^+)}{\xi(1-x_Bx^+)}-\frac{1}{x_B}\right]-\frac{z}{q^2x_B}\frac{x_B+\xi}{\xi+x^-}\right],\\
k_{15} &= \frac{q^{-1}}{U^2V^2-1}\left[k_{16}\frac{UV}{\sqrt{1+q^2}}+\alpha^{-1}k_{10}\right],\\
k_{16} &= \frac{U^4-1}{x_B-x^-}\frac{V^2}{U^2}\left[z(x_B+\xi)\frac{q^2(x^++\xi)-(1+x_B\xi)x^+}{q^3\xi(1-x_Bx^+)}-(x^++\xi)\right],\\
k_{17} &= \frac{1}{\alpha q\sqrt{1+q^2}}\frac{V}{U}\frac{qz(x_B+\xi)+(\xi+x^+)}{\kappa(x^+)(1-x_Bx^+)}\frac{x^+-\kappa(x^+)}{x_B-x^-}\,,\\
k_{18} &= \frac{(x_B+\xi)x^-+(\xi+x^+)\frac{1}{\kappa(x^+)}}{x_B-x^-}\frac{qz(x_B+\xi)+(\xi+x^+)}{1-x_Bx^+}\frac{V^2}{q}\,.
\end{aligned}
\tag{F.8}
$$

Chapter 5

Secret symmetries of the AdS/CFT scattering matrices

The Hopf superalgebra relevant to AdS/CFT, which we have considered in Chapter 3, is quite unconventional, and, as of today, its properties are only partially understood. It is infinite dimensional, with a structure similar to Yangians [17,37,181–183]. It admits a *level zero* given by the centrally extended $\mathfrak{psu}(2|2)_{\mathcal{C}}$ Lie superalgebra, and *level one* generators giving rise to an infinite dimensional tower. Nevertheless, the actual algebra sits rather outside the standard theory of Yangians, in that it displays an additional symmetry at level one, which is absent at level zero. Were this symmetry present at level zero, it would extend the Yangian to that of $\mathfrak{gl}(2|2)$. However, this is not compatible with the central extension. Moreover, if one starts commuting the new generator with the old ones, one obtains a growth in the algebra which is not completely clear how to control.

At the time it was discovered [130], it was unclear whether the secret symmetry was an accidental feature of the choice of vacuum for the spin-chain, or the choice of gauge for the string sigma model. This is because the centrally extended $\mathfrak{psu}(2|2)_{\mathcal{C}}$ algebra is intimately linked to those specific choices. More recently, however, there have been observations of the very same mechanism in several sectors of AdS/CFT. For instance, the secret symmetry reveals itself as symmetries of the boundary scattering matrices [184] and also appear as a so-called 'bonus' Yangian symmetry in [185, 186]. Thus, the secret symmetry should perhaps be regarded as an integral part of the symmetries of the model.

The need for such an extension seems to respond to a consistency issue of the underlying quantum group description of the integrable structure, following a general prescription by Khoroshkin and Tolstoy [187]. According to this argument, in the case of superalgebras with a degenerate Cartan matrix (as the present one is), one may adopt the S-matrix of the smallest non-degenerate algebra containing the original one. The S-matrix found in such a way intertwines *a fortiori* the coproducts of the original algebra. This leads to the natural question of whether a similar symmetry is also hidden in the S-matrices of the deformed Hubbard model.

A first hint that this is the case is found in the so-called classical limit. For the rational case the secret symmetry plays there a crucial role, where it is needed to achieve factorization of the classical r-matrix in the form of a quantum-double [129, 188–190]. Similarly, the secret symmetry generator is also appearing in the factorized expression of the q-deformed classical r-matrix [163].

Another natural limit to investigate is the 'conventional' affine limit of the algebra $\widehat{\mathcal{Q}}$ considered in the previous chapter. This limit is obtained by sending one of the (complex) parameters of the relevant representation (namely, the coupling constant g) to zero, followed by a suitable transformation that removes the twist factors of [126, 191]. In this limit, two of the three central charges of $\widehat{\mathcal{Q}}$ vanish; thus, the algebra becomes isomorphic to the conventional quantum affine superalgebra $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2|2))$. By adjoining the non-supertraceless Cartan generators $h_{4,0}$ and $h_{4,\pm1}$, one may extend $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2|2))$ to $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$. The representations of $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$ can be obtained from [70]. In such a way the secret symmetry of $\widehat{\mathcal{Q}}$ can be revealed by the intuition inspired by the corresponding $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$ generator.

The full $\widehat{\mathcal{Q}}$ S-matrix in this conventional limit is naturally found to have $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$ symmetry. In other words, we automatically find an extended symmetry in this limit, corresponding to the operators $h_{4,i}$. However, at non-zero g, we see the appearance of the same phenomenon as in the rational case: the level one non-supertraceless generator is once again a symmetry, while the level zero is not. We find two secret symmetries which we call B_E and B_F, and which extends to all the bound state $\widehat{\mathcal{Q}}$ S-matrices. More precisely, while these symmetries are an analog of the Cartan generators $h_{4,\pm1}$ of $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$, they get promoted to full $\widehat{\mathcal{Q}}$ symmetries only in specific linear combinations. In the rational $q \to 1$ limit they exactly reproduce the secret symmetry of the worldsheet S-matrix [130].

The facts listed above show that we are not dealing with an accidental problem. On the other hand, even in the light of these new observations, the fundamental nature of the secret symmetry remains unclear, and it is still not known how to consistently embed it into a satisfactory mathematical framework. After all, we might simply have in front of us a new type of quantum group[1].

This chapter is organized as follows. In Section 5.1 we review the secret symmetry of the worldsheet S-matrix and some of its properties. In Section 5.2 we present secret symmetries of the reflection matrices for the $Y = 0$ giant graviton and $D5$-branes. In Section 5.2 we investigate the quantum affine origins of the secret symmetry.

5.1 Secret symmetries of the S-matrix

The $\mathfrak{su}(2|2)$ algebra has a $\mathfrak{u}(1)$ outer automorphism extending the algebra to $\mathfrak{u}(2|2)$. However, the additional $\mathfrak{u}(1)$ charge, which acts as a boson–fermion discriminator,

$$\mathbb{B}\,|\phi_a\rangle = +I\,|\phi_a\rangle\,, \qquad \mathbb{B}\,|\psi_\alpha\rangle = -I\,|\psi_\alpha\rangle\,, \tag{5.1.1}$$

[1] P. Etingof, A. Torrielli, private communication.

is not a symmetry of the S-matrix and the eigenvalue I is not constrained to any particular value.[2] Strikingly, this charge has a Yangian partner $\widehat{\mathbb{B}}$ which is known as a 'secret symmetry' of the S-matrix and is the same for left and right factors. It was shown in [130] and confirmed in [190] that the additional charge

$$\Delta(\widehat{\mathbb{B}}) = \widehat{\mathbb{B}} \otimes 1 + 1 \otimes \widehat{\mathbb{B}} - \tfrac{1}{2}(\mathbb{U}^{-1}\mathbb{Q}_\alpha^{\ a} \otimes \mathbb{G}_a^{\ \alpha} + \mathbb{U}\,\mathbb{G}_a^{\ \alpha} \otimes \mathbb{Q}_\alpha^{\ a})\,, \tag{5.1.2}$$

where $\widehat{\mathbb{B}}$ is the level-1 partner of (5.1.1) with the eigenvalue

$$\widehat{I} = \frac{ig}{8}\big(x^+ - \frac{1}{x^+} + x^- - \frac{1}{x^-}\big), \tag{5.1.3}$$

is a symmetry of the S-matrix. Interestingly, the non-trivial part of the co-product of $\widehat{\mathbb{B}}$ appears to be the same as the ε-correction of $\Delta(\widehat{\mathbb{H}})$ in the limit $\varepsilon \to 0$ of the exceptional superalgebra $\mathfrak{d}(2,1;\varepsilon)$ [128]. Furthermore, this novel symmetry generates several new symmetries of the S-matrix that do not have a Lie algebra analog. They were originally found by computing the commutators $[\Delta(\widehat{\mathbb{B}}),\Delta(\mathbb{Q}_a^{\ \alpha})]$ and $[\Delta(\widehat{\mathbb{B}}),\Delta(\mathbb{G}_\alpha^{\ a})]$ and taking linear combinations with the Yangian charges $\Delta(\widehat{\mathbb{Q}}_a^{\ \alpha})$ and $\Delta(\widehat{\mathbb{G}}_\alpha^{\ a})$ (see [130] for the details).

These new symmetries generated by (5.1.2) are

$$\begin{aligned}
\Delta(\mathbb{Q}_{\alpha,+1}^{\ a}) &= \mathbb{Q}_{\alpha,+1}^{\ a} \otimes 1 + \mathbb{U} \otimes \mathbb{Q}_{a,+1}^{\ \alpha} - \tfrac{1}{2}\,\mathbb{U}\,\mathbb{L}_\alpha^{\ \gamma} \otimes \mathbb{Q}_\gamma^{\ a} + \tfrac{1}{2}\,\mathbb{Q}_\gamma^{\ a} \otimes \mathbb{L}_\alpha^{\ \gamma} \\
&\quad - \tfrac{1}{2}\,\mathbb{U}\,\mathbb{R}_c^{\ a} \otimes \mathbb{Q}_\alpha^{\ c} + \tfrac{1}{2}\,\mathbb{Q}_\alpha^{\ c} \otimes \mathbb{R}_c^{\ a} - \tfrac{1}{4}\,\mathbb{U}\,\mathbb{H} \otimes \mathbb{Q}_\alpha^{\ a} + \tfrac{1}{4}\mathbb{Q}_\alpha^{\ a} \otimes \mathbb{H}\,, \\
\Delta(\mathbb{Q}_{\alpha,-1}^{\ a}) &= \mathbb{Q}_{\alpha,-1}^{\ a} \otimes 1 + \mathbb{U} \otimes \mathbb{Q}_{a,-1}^{\ \alpha} - \tfrac{1}{2}\varepsilon_{\alpha\gamma}\,\varepsilon^{ac}\,\mathbb{U}^2\,\mathbb{G}_c^{\ \gamma} \otimes \mathbb{C} + \tfrac{1}{2}\varepsilon_{\alpha\gamma}\,\varepsilon^{ac}\,\mathbb{U}^{-1}\,\mathbb{C} \otimes \mathbb{G}_c^{\ \gamma}\,, \\
\Delta(\mathbb{G}_{a,+1}^{\ \alpha}) &= \mathbb{G}_{a,+1}^{\ \alpha} \otimes 1 + \mathbb{U}^{-1} \otimes \mathbb{G}_{a,+1}^{\ \alpha} + \tfrac{1}{2}\,\mathbb{U}^{-1}\,\mathbb{L}_\gamma^{\ \alpha} \otimes \mathbb{G}_a^{\ \gamma} - \tfrac{1}{2}\,\mathbb{G}_a^{\ \gamma} \otimes \mathbb{L}_\gamma^{\ \alpha} \\
&\quad + \tfrac{1}{2}\,\mathbb{U}^{-1}\,\mathbb{R}_a^{\ c} \otimes \mathbb{G}_c^{\ \alpha} - \tfrac{1}{2}\,\mathbb{G}_c^{\ \alpha} \otimes \mathbb{R}_a^{\ c} + \tfrac{1}{4}\,\mathbb{U}^{-1}\,\mathbb{H} \otimes \mathbb{G}_a^{\ \alpha} - \tfrac{1}{4}\mathbb{G}_a^{\ \alpha} \otimes \mathbb{H}\,, \\
\Delta(\mathbb{G}_{a,-1}^{\ \alpha}) &= \mathbb{G}_{a,-1}^{\ \alpha} \otimes 1 + \mathbb{U}^{-1} \otimes \mathbb{G}_{a,-1}^{\ \alpha} + \tfrac{1}{2}\varepsilon_{ac}\,\varepsilon^{\alpha\gamma}\,\mathbb{U}^{-2}\,\mathbb{Q}_\gamma^{\ c} \otimes \mathbb{C}^\dagger - \tfrac{1}{2}\varepsilon_{ac}\,\varepsilon^{\alpha\gamma}\,\mathbb{U}\,\mathbb{C}^\dagger \otimes \mathbb{Q}_\gamma^{\ c},
\end{aligned} \tag{5.1.4}$$

where the eigenvalues of the new charges $\mathbb{Q}_{\alpha,\pm1}^{\ a}$ and $\mathbb{G}_{a,\pm1}^{\ \alpha}$ are given by

$$\begin{aligned}
\mathbb{Q}_{\alpha,+1}^{\ a} &= i\frac{g}{2}\,\mathbb{Q}_\alpha^{\ a}\,(u\Pi_b + v\Pi_f), & \mathbb{Q}_{\alpha,-1}^{\ a} &= i\frac{g}{2}\,\mathbb{Q}_\alpha^{\ a}\,(v\Pi_b + u\Pi_f)\,, \\
\mathbb{G}_{a,+1}^{\ \alpha} &= i\frac{g}{2}\,\mathbb{G}_a^{\ \alpha}\,(v\Pi_b + u\Pi_f), & \mathbb{G}_{a,-1}^{\ \alpha} &= i\frac{g}{2}\,\mathbb{G}_a^{\ \alpha}\,(u\Pi_b + v\Pi_f),
\end{aligned} \tag{5.1.5}$$

with Π_b and Π_f being the projectors onto bosons and fermions respectively and

$$u = \frac{1}{2}(x^+ + x^-), \qquad v = \frac{1}{2}\left(\frac{1}{x^+} + \frac{1}{x^-}\right). \tag{5.1.6}$$

The charges of such form were first considered in constructing the classical S-matrix of AdS/CFT [189]. It is easy to convince ourselves that these charges do not have Lie algebra analog, because the naive Lie algebra limit (i.e. $u \to 1$, $v \to 1$) leads to the usual Lie algebra supercharges $\mathbb{Q}_\alpha^{\ a}$ and $\mathbb{G}_a^{\ \alpha}$.

[2] The charge $\mathbb{B}$ can be related to $\mathfrak{psu}(2|2)_{\mathcal{C}}$ by rather specific non-linear commutation relations, see [192].

5.2 Secret symmetries of the K-matrices

An very natural question is if secret symmetries (5.1.2) and (5.1.4) manifest themselves in the twisted Yangians associated to the integrable boundaries considered in Chapter 3. This question was raised in [184] and a positive answer was found. It was shown that the $Y = 0$ giant graviton and $D5$-branes inherit additional symmetries that originate from the secret symmetry of the S-matrix.

An open question is the secret symmetry of the reflection matrix for the $Z = 0$ giant graviton. The search for such symmetry would require the knowledge of the level-2 partner of $\widehat{\mathbb{B}}$ which is not know at the present time.

We will not present the explicit calculations of the invariance conditions for the secret symmetries we will construct as they are quite straightforward and not very illuminating, but at the same time involve very large computer algebra calculations that we have performed with `Mathematica`.[3]

5.2.1 $Y = 0$ giant graviton

The $Y = 0$ giant graviton preserves a $\mathfrak{h} = \mathfrak{su}(2|1)_L = \{\mathbb{L}_\alpha^{\ \beta},\ \mathbb{R}_1^{\ 1},\ \mathbb{R}_2^{\ 2},\ \mathbb{Q}_\alpha^{1},\ \mathbb{G}_1^{\ \alpha},\ \mathbb{H}\}$ subalgebra of the bulk $\mathfrak{psu}(2|2)_C$ algebra. The boundary Yangian symmetry is generated by the twisted charges

$$\widetilde{\mathbb{J}}^p := \widehat{\mathbb{J}}^p + \tfrac{1}{4} f^p_{\ qi} \left(\mathbb{J}^q\, \mathbb{J}^i + \mathbb{J}^i\, \mathbb{J}^q\right) , \tag{5.2.1}$$

the co-products of which are of the form

$$\Delta(\widetilde{\mathbb{J}}^p) = \widetilde{\mathbb{J}}^p \otimes 1 + 1 \otimes \widetilde{\mathbb{J}}^p + f^p_{\ qi}\, \mathbb{J}^q \otimes \mathbb{J}^i \ , \tag{5.2.2}$$

where $\mathbb{J}^i \in \mathfrak{h}$ and $\mathbb{J}^{p\,(q)} \in \mathfrak{m} = \{\mathbb{R}_1^{\ 2},\ \mathbb{R}_2^{\ 1},\ \mathbb{Q}_\gamma^{2},\ \mathbb{G}_2^{\ \gamma},\ \mathbb{C},\ \mathbb{C}^\dagger\}$ are the generators of the subset $\mathfrak{m} = \mathfrak{g}\backslash\mathfrak{h}$. The boundary is a singlet in the scattering theory, thus only terms of the form $\widetilde{\mathbb{J}}^p \otimes 1$ in (5.2.2) need to be considered. See Section 3.3.1 for complete details on the setup.

The fundamental reflection matrix, describing the scattering of fundamental magnons from the boundary, is diagonal and the generator $\mathbb{B}$ (5.1.1) is a symmetry of it, but the twisted (5.2.1) partner of the secret Yangian charge (5.1.2) is not. Higher order reflection matrices are of non-diagonal form and do not respect either additional symmetry $\mathbb{B}$, or the twisted partner of $\widehat{\mathbb{B}}$. To be more precise, due to the form of the tail in (5.1.2) the twist (5.2.1) does not ensure coideal property, and we do not see any other way to ensure coideal property for this charge.

The next step is to check if the twisted (5.2.1) partners of the additional secret charges (5.1.4) are symmetries of the reflection matrix. By performing the twist (5.2.1) we found

[3] For the explicit calculations we are using the superspace formalism, in which the secret charge (5.1.2) is defined as $\mathrm{T}_M(\widehat{\mathbb{B}}) = \widehat{I}\left(w_a \frac{\partial}{\partial w_a} - \theta_\alpha \frac{\partial}{\partial \theta_\alpha}\right)$ and is equivalent (up to a prefactor) to the charge Σ introduced in [129].

the new additional twisted secret charges to be

$$\begin{aligned}
\widetilde{\mathbb{Q}}_{\alpha,+1}^{\;2} &:= \mathbb{Q}_{\alpha,+1}^{\;2} + \tfrac{1}{2}\mathbb{Q}_{\alpha}^{\;2}\,\mathbb{R}_{2}^{\;2} - \tfrac{1}{2}\mathbb{R}_{1}^{\;2}\,\mathbb{Q}_{\alpha}^{\;1} + \tfrac{1}{2}\mathbb{Q}_{\gamma}^{\;2}\,\mathbb{L}_{\alpha}^{\;\gamma} + \tfrac{1}{4}\mathbb{Q}_{\alpha}^{\;2}\,\mathbb{H}\,,\\
\widetilde{\mathbb{Q}}_{\alpha,-1}^{\;2} &:= \mathbb{Q}_{\alpha,-1}^{\;2} - \tfrac{1}{2}\varepsilon_{\alpha\gamma}\,\mathbb{C}\,\mathbb{G}_{1}^{\;\gamma}\,,\\
\widetilde{\mathbb{G}}_{2,+1}^{\;\alpha} &:= \mathbb{G}_{2,+1}^{\;\alpha} - \tfrac{1}{2}\mathbb{G}_{2}^{\;\alpha}\,\mathbb{R}_{2}^{\;2} + \tfrac{1}{2}\mathbb{R}_{2}^{\;1}\,\mathbb{G}_{1}^{\;\alpha} - \tfrac{1}{2}\mathbb{G}_{2}^{\;\gamma}\,\mathbb{L}_{\gamma}^{\;\alpha} - \tfrac{1}{4}\mathbb{G}_{2}^{\;\alpha}\,\mathbb{H}\,,\\
\widetilde{\mathbb{G}}_{2,-1}^{\;\alpha} &:= \mathbb{G}_{2,-1}^{\;\alpha} + \tfrac{1}{2}\varepsilon^{\alpha\gamma}\,\mathbb{C}^{\dagger}\,\mathbb{Q}_{\gamma}^{\;1}\,,
\end{aligned}\tag{5.2.3}$$

and checked that they intertwine both fundamental and bound-state reflection matrices. Hence they are the symmetries of the reflection matrix.

The mirror model of the $Y = 0$ maximal giant graviton preserves the subalgebra $\mathfrak{h} = \mathfrak{su}(2|1)_R = \{\mathbb{R}_a^{\;b},\ \mathbb{L}_3^{\;3},\ \mathbb{L}_4^{\;4},\ \mathbb{Q}_3^{\;a},\ \mathbb{G}_a^{\;3},\ \mathbb{H}\}$ and the complementary subset is $\mathfrak{m} = \{\mathbb{L}_3^{\;4},\ \mathbb{L}_4^{\;3},\ \mathbb{Q}_4^{\;a},\ \mathbb{G}_a^{\;4},\ \mathbb{C},\ \mathbb{C}^{\dagger}\}$. The boundary is a singlet and the reflection matrices are diagonal at all orders of the bound-state number (see Section 3.3.1.3 for details); thus $\mathbb{B}$ (5.1.1) is a symmetry at all orders. Similarly to the previous case we have checked that the twisted partners

$$\begin{aligned}
\widetilde{\mathbb{Q}}_{4,+1}^{\;a} &:= \mathbb{Q}_{4,+1}^{\;a} + \tfrac{1}{2}\mathbb{Q}_{4}^{\;c}\,\mathbb{R}_{c}^{\;a} + \tfrac{1}{2}\mathbb{Q}_{4}^{\;a}\,\mathbb{L}_{4}^{\;4} - \tfrac{1}{2}\mathbb{L}_{4}^{\;3}\,\mathbb{Q}_{3}^{\;a} + \tfrac{1}{4}\mathbb{Q}_{4}^{\;a}\,\mathbb{H}\,,\\
\widetilde{\mathbb{Q}}_{4,-1}^{\;a} &:= \mathbb{Q}_{4,-1}^{\;a} - \tfrac{1}{2}\varepsilon^{ad}\,\mathbb{C}\,\mathbb{G}_{d}^{\;3}\,,\\
\widetilde{\mathbb{G}}_{a,+1}^{\;4} &:= \mathbb{G}_{a,+1}^{\;4} - \tfrac{1}{2}\mathbb{G}_{c}^{\;4}\,\mathbb{R}_{a}^{\;c} - \tfrac{1}{2}\mathbb{G}_{a}^{\;4}\,\mathbb{L}_{4}^{\;4} + \tfrac{1}{2}\mathbb{L}_{3}^{\;4}\,\mathbb{G}_{a}^{\;3} - \tfrac{1}{4}\mathbb{G}_{a}^{\;4},\mathbb{H}\,,\\
\widetilde{\mathbb{G}}_{a,-1}^{\;4} &:= \mathbb{G}_{a,-1}^{\;4} + \tfrac{1}{2}\varepsilon_{ac}\,\mathbb{C}^{\dagger}\,\mathbb{Q}_{3}^{\;c}\,,
\end{aligned}\tag{5.2.4}$$

of the secret charges (5.1.4) are symmetries of the reflection matrix.

5.2.2 $D5$-brane

The $D5$-brane preserves a diagonal subalgebra $\mathfrak{psu}(2|2)_+ \ltimes \mathbb{R}^3$ of the complete bulk algebra $\mathfrak{psu}(2|2) \times \widetilde{\mathfrak{psu}}(2|2) \ltimes \mathbb{R}^3$ generated by [143,144]

$$\begin{aligned}
\mathbb{L}_{\check\alpha}^{\;\check\beta} &= \mathbb{L}_{\alpha}^{\;\beta} + \mathbb{L}_{\bar{\dot\alpha}}^{\;\bar{\dot\beta}}\,, &\qquad \mathbb{Q}_{\check\alpha}^{\;\check a} &= \mathbb{Q}_{\alpha}^{\;a} + \kappa\,\mathbb{Q}_{\bar{\dot\alpha}}^{\;\dot a}\,,\\
\mathbb{R}_{\check a}^{\;\check b} &= \mathbb{R}_{a}^{\;b} + \mathbb{R}_{\dot a}^{\;\dot b}\,, &\qquad \mathbb{G}_{\check a}^{\;\check\alpha} &= \mathbb{G}_{a}^{\;\alpha} + \kappa^{-1}\mathbb{G}_{\dot a}^{\;\bar{\dot\alpha}}\,.
\end{aligned}\tag{5.2.5}$$

where $\kappa^2 = \pm 1$; the notation for the dotted and checked indices is the same as for undotted ones, $\dot a$, $\check a$, $\dot b$, $\check b = \dot 1$, $\dot 2$ and $\dot\alpha$, $\check\alpha$, $\dot\beta$, $\check\beta = \dot 3$, $\dot 4$; the bar above the dotted indices acts as $\bar{\dot 3} = \dot 4$ and $\bar{\dot 4} = \dot 3$. The generators with the undotted indices generate 'left' $\mathfrak{psu}(2|2)$ and the generators with the dotted indices generate 'right' $\widetilde{\mathfrak{psu}}(2|2)$. The complementary charges are defined as

$$\begin{aligned}
\overline{\mathbb{L}}_{\check\alpha}^{\;\check\beta} &= \mathbb{L}_{\alpha}^{\;\beta} - \mathbb{L}_{\bar{\dot\alpha}}^{\;\bar{\dot\beta}}\,, &\qquad \overline{\mathbb{Q}}_{\check\alpha}^{\;\check a} &= \mathbb{Q}_{\alpha}^{\;a} - \kappa\,\mathbb{Q}_{\bar{\dot\alpha}}^{\;\dot a}\,,\\
\overline{\mathbb{R}}_{\check a}^{\;\check b} &= \mathbb{R}_{a}^{\;b} - \mathbb{R}_{\dot a}^{\;\dot b}\,, &\qquad \overline{\mathbb{G}}_{\check a}^{\;\check\alpha} &= \mathbb{G}_{a}^{\;\alpha} - \kappa^{-1}\mathbb{G}_{\dot a}^{\;\bar{\dot\alpha}}\,,
\end{aligned}\tag{5.2.6}$$

and in the contrast to (5.2.5) annihilate the boundary by definition. See Section 3.5.2 for complete details on the setup.

The Yangian symmetry of the $D5$-brane is generated by the twisted charges

$$\widetilde{\mathbb{J}}^{\check{A}} := \widehat{\mathbb{J}}^{\check{A}} + \tfrac{1}{8} f^{\check{A}}_{\check{B}\check{C}} \big(\mathbb{J}^{\check{B}}\, \mathbb{J}^{\check{C}} + \mathbb{J}^{\check{C}}\, \mathbb{J}^{\check{B}} \big) \,, \tag{5.2.7}$$

where the indices $\check{A},\ \check{B},\ \check{C}$, run through all possible charges. The co-products of (5.2.7) acquire the form

$$\Delta(\widetilde{\mathbb{J}}^{A}) = \widetilde{\mathbb{J}}^{\check{A}} \otimes 1 + 1 \otimes \widetilde{\mathbb{J}}^{\check{A}} + \tfrac{1}{2} f^{A}_{BC}\, \mathbb{J}^{\check{B}} \otimes \mathbb{J}^{\check{C}} \,. \tag{5.2.8}$$

Based on this construction it is easy to see that the twisted (5.2.7) partner for the $D5$-brane of the secret charge (5.1.2) is

$$\Delta(\widetilde{\mathbb{B}}) = \widetilde{\mathbb{B}} \otimes 1 - 1 \otimes \widetilde{\mathbb{B}} - \tfrac{1}{2} \big(\overline{\mathbb{Q}}{}^{\check{a}}_{\check{\alpha}} \otimes \mathbb{G}^{\check{\alpha}}_{\check{a}} + \overline{\mathbb{G}}{}^{\check{\alpha}}_{\check{a}} \otimes \mathbb{Q}^{\check{a}}_{\check{\alpha}} \big) \,, \tag{5.2.9}$$

while the twisted partners of (5.1.4) are

$$\begin{aligned}
\Delta(\widetilde{\mathbb{Q}}{}^{\check{a}}_{\check{\alpha},+1}) &= \widetilde{\mathbb{Q}}{}^{\check{a}}_{\check{\alpha},+1} \otimes 1 + 1 \otimes \widetilde{\mathbb{Q}}{}^{\check{a}}_{\check{\alpha},+1} - \tfrac{1}{2}\, \overline{\mathbb{Q}}{}^{\check{c}}_{\check{\alpha}} \otimes \mathbb{R}^{\check{b}}_{\check{c}} + \tfrac{1}{2}\, \overline{\mathbb{R}}{}^{\check{a}}_{\check{c}} \otimes \mathbb{Q}^{\check{c}}_{\check{\alpha}} \\
&\quad - \tfrac{1}{2}\, \overline{\mathbb{Q}}{}^{\check{a}}_{\check{\gamma}} \otimes \mathbb{L}^{\check{\gamma}}_{\check{\alpha}} + \tfrac{1}{2}\, \overline{\mathbb{L}}{}^{\check{\gamma}}_{\check{\alpha}} \otimes \mathbb{Q}^{\check{a}}_{\check{\gamma}} + \tfrac{1}{4}\, \overline{\mathbb{H}} \otimes \mathbb{Q}^{\check{a}}_{\check{\alpha}} - \tfrac{1}{4}\, \overline{\mathbb{Q}}{}^{\check{a}}_{\check{\alpha}} \otimes \mathbb{H} \,, \\
\Delta(\widetilde{\mathbb{Q}}{}^{\check{a}}_{\check{\alpha},-1}) &= \widetilde{\mathbb{Q}}{}^{\check{a}}_{\check{\alpha},-1} \otimes 1 + \otimes \widetilde{\mathbb{Q}}{}^{\check{a}}_{\check{\alpha},-1} - \tfrac{1}{2}\, \varepsilon_{\check{\alpha}\check{\gamma}}\, \varepsilon^{\check{a}\check{d}\,-1}\, \overline{\mathbb{C}} \otimes \mathbb{G}^{\check{\gamma}}_{\check{d}} + \tfrac{1}{2}\, \varepsilon_{\check{\alpha}\check{\gamma}}\, \varepsilon^{\check{a}\check{d}}\, \overline{\mathbb{G}}{}^{\check{\gamma}}_{\check{d}} \otimes \mathbb{C} \,, \\
\Delta(\widetilde{\mathbb{G}}{}^{\check{\alpha}}_{\check{a},+1}) &= \widetilde{\mathbb{G}}{}^{\check{\alpha}}_{\check{a},+1} \otimes 1 + 1 \otimes \widetilde{\mathbb{G}}{}^{\check{\alpha}}_{\check{a},+1} + \tfrac{1}{2}\, \overline{\mathbb{G}}{}^{\check{\alpha}}_{\check{c}} \otimes \mathbb{R}^{\check{c}}_{\check{a}} - \tfrac{1}{2}\, \overline{\mathbb{R}}{}^{\check{c}}_{\check{a}} \otimes \mathbb{G}^{\check{\alpha}}_{\check{c}} \\
&\quad + \tfrac{1}{2}\, \overline{\mathbb{G}}{}^{\check{\gamma}}_{\check{a}} \otimes \mathbb{L}^{\check{\alpha}}_{\check{\gamma}} - \tfrac{1}{2}\, \overline{\mathbb{L}}{}^{\check{\alpha}}_{\check{\gamma}} \otimes \mathbb{G}^{\check{\gamma}}_{\check{a}} - \tfrac{1}{4}\, \overline{\mathbb{H}} \otimes \mathbb{G}^{\check{\alpha}}_{\check{a}} + \tfrac{1}{4}\, \overline{\mathbb{G}}{}^{\check{\alpha}}_{\check{a}} \otimes \mathbb{H} \,, \\
\Delta(\widetilde{\mathbb{G}}{}^{\check{\alpha}}_{\check{a},-1}) &= \widetilde{\mathbb{G}}{}^{\check{\alpha}}_{\check{a},-1} \otimes 1 + 1 \otimes \widetilde{\mathbb{G}}{}^{\check{\alpha}}_{\check{a},-1} + \tfrac{1}{2}\, \varepsilon_{\check{a}\check{c}}\, \varepsilon^{\check{\alpha}\check{\gamma}}\, \overline{\mathbb{C}}{}^{\dagger} \otimes \mathbb{Q}^{\check{c}}_{\check{\gamma}} - \tfrac{1}{2}\, \varepsilon_{\check{a}\check{c}}\, \varepsilon^{\check{\alpha}\check{\gamma}}\, \overline{\mathbb{Q}}{}^{\check{c}}_{\check{\gamma}} \otimes \mathbb{C}^{\dagger}.
\end{aligned} \tag{5.2.10}$$

Note that were considering the algebra in its non-local realization to be in consistency with Section 3.5. This is the general structure of the secret symmetries for the reflection from $D5$-brane. The definitions of $\overline{\mathbb{C}}$, $\overline{\mathbb{C}}^{\dagger}$ and $\overline{\mathbb{H}}$ need to be developed a little further (see Section 3.5.2 for complete details). Two inequivalent orientations of the $D5$-brane, horizontal and vertical, that look rather different in the scattering theory are known. Thus we will consider the explicit realization of the secret symmetries (5.2.9) and (5.2.10) for both orientations separately.

Horizontal $D5$-brane. In the case of reflection from the horizontal $D5$-brane ($\kappa = -1$), the boundary is a singlet; thus neglecting the irrelevant terms in (5.2.9) and (5.2.10) and with the help of the Lie algebra the remaining parts may be simplified to

$$\Delta(\widetilde{\mathbb{B}}) = \big(\widehat{\mathbb{B}} \circ 1 - 1 \circ \widehat{\mathbb{B}} - \tfrac{1}{2} \big(\mathbb{Q}^{a}_{\alpha} \circ \mathbb{G}^{\ \alpha}_{a} + \mathbb{G}^{\ \alpha}_{a} \circ \mathbb{Q}^{a}_{\alpha} \big) \big) \otimes 1 \,, \tag{5.2.11}$$

and

$$\begin{aligned}
\Delta(\widetilde{\mathbb{Q}}{}^{a}_{\alpha,+1}) &= \big(\mathbb{Q}^{a}_{\alpha,+1} \circ 1 - 1 \circ \mathbb{Q}^{\alpha}_{a,+1} - \tfrac{1}{2} \mathbb{L}^{\ \gamma}_{\alpha} \circ \mathbb{Q}^{a}_{\gamma} + \tfrac{1}{2} \mathbb{Q}^{a}_{\gamma} \circ \mathbb{L}^{\ \gamma}_{\alpha} \\
&\quad - \tfrac{1}{2} \mathbb{R}^{\ a}_{c} \circ \mathbb{Q}^{c}_{\alpha} + \tfrac{1}{2} \mathbb{Q}^{c}_{\alpha} \circ \mathbb{R}^{\ a}_{c} - \tfrac{1}{4} \mathbb{H} \circ \mathbb{Q}^{a}_{\alpha} + \tfrac{1}{4} \mathbb{Q}^{a}_{\alpha} \circ \mathbb{H} \big) \otimes 1 \,, \\
\Delta(\widetilde{\mathbb{Q}}{}^{a}_{\alpha,-1}) &= \big(\mathbb{Q}^{a}_{\alpha,-1} \circ 1 - 1 \circ \mathbb{Q}^{\alpha}_{a,-1} - \tfrac{1}{2} \varepsilon_{\alpha\gamma}\, \varepsilon^{ac}\, \mathbb{G}^{\ \gamma}_{c} \circ \mathbb{C} + \tfrac{1}{2} \varepsilon_{\alpha\gamma}\, \varepsilon^{ac}\, \mathbb{C} \circ \mathbb{G}^{\ \gamma}_{c} \big) \otimes 1 \,, \\
\Delta(\widetilde{\mathbb{G}}{}^{\ \alpha}_{a,+1}) &= \big(\mathbb{G}^{\ \alpha}_{a,+1} \circ 1 - 1 \circ \mathbb{G}^{\ \alpha}_{a,+1} + \tfrac{1}{2} \mathbb{L}^{\ \alpha}_{\gamma} \circ \mathbb{G}^{\ \gamma}_{a} - \tfrac{1}{2} \mathbb{G}^{\ \gamma}_{a} \circ \mathbb{L}^{\ \alpha}_{\gamma} \\
&\quad + \tfrac{1}{2} \mathbb{R}^{\ c}_{a} \circ \mathbb{G}^{\ \alpha}_{c} - \tfrac{1}{2} \mathbb{G}^{\ \alpha}_{c} \circ \mathbb{R}^{\ c}_{a} + \tfrac{1}{4} \mathbb{H} \circ \mathbb{G}^{\ \alpha}_{a} - \tfrac{1}{4} \mathbb{G}^{\ \alpha}_{a} \circ \mathbb{H} \big) \otimes 1, \\
\Delta(\widetilde{\mathbb{G}}{}^{\ \alpha}_{a,-1}) &= \big(\mathbb{G}^{\ \alpha}_{a,-1} \circ 1 - 1 \circ \mathbb{G}^{\ \alpha}_{a,-1} + \tfrac{1}{2} \varepsilon_{ac}\, \varepsilon^{\alpha\gamma}\, \mathbb{Q}^{c}_{\gamma} \circ \mathbb{C}^{\dagger} - \tfrac{1}{2} \varepsilon_{ac}\, \varepsilon^{\alpha\gamma}\, \mathbb{C}^{\dagger} \circ \mathbb{Q}^{c}_{\gamma} \big) \otimes 1,
\end{aligned} \tag{5.2.12}$$

here '$\circ$' describes the tensor product of 'left' and 'right' representations (hereafter 'reps') of the bulk magnon and the usual tensor product '$\otimes$' separates the bulk and boundary reps. The central charges in this picture act on the bulk states as $\mathbb{C} := \mathbb{C} \circ 1 + 1 \circ \mathbb{C}$, $\overline{\mathbb{C}} := \mathbb{C} \circ 1 - 1 \circ \mathbb{C}$ and analogously for $\mathbb{C}^\dagger$, $\mathbb{H}$. Note that the secret charges (5.2.11) and (5.2.12) effectively differ from (5.1.2) and (5.1.4) by a minus sign only (we refer to Section 3.5.3 for the details on this similarity). We have checked that these charges commute with the reflection matrix $\mathcal{K}^h$ (3.5.35) [144], and thus are secret symmetries of the horizontal $D5$-brane.

Vertical $D5$-brane. In the case of reflection from the vertical $D5$-brane ($\kappa = -i$), the boundary carries a field multiplet transforming in the vector representation of the boundary algebra thus the non-local terms in (5.2.9) and (5.2.10) may no longer be neglected. Nevertheless the general expressions may be casted in a quite transparent form, as

$$\begin{aligned} \Delta(\widetilde{\mathbb{B}}) = \big(&\widehat{\mathbb{B}} \circ 1 - 1 \circ \widehat{\mathbb{B}} - \tfrac{1}{2}\big(\mathbb{Q}_\alpha^{\ a} \circ \mathbb{G}_a^{\ \alpha} + \mathbb{G}_a^{\ \alpha} \circ \mathbb{Q}_\alpha^{\ a}\big)\big) \otimes 1 \\ &- \tfrac{1}{2}(\mathbb{Q}_\alpha^{\ a} \circ 1 - 1 \circ \mathbb{Q}_\alpha^{\ a}) \otimes \mathbb{G}_a^{\ \alpha} - \tfrac{1}{2}(\mathbb{G}_a^{\ \alpha} \circ 1 - 1 \circ \mathbb{G}_a^{\ \alpha}) \otimes \mathbb{Q}_\alpha^{\ a} , \end{aligned} \tag{5.2.13}$$

and

$$\begin{aligned} \Delta(\widetilde{\mathbb{Q}}_{\alpha,+1}^{\ a}) = \big(&\mathbb{Q}_{\alpha,+1}^{\ a} \circ 1 - 1 \circ \mathbb{Q}_{a,+1}^{\ \alpha} - \tfrac{1}{2}\mathbb{L}_\alpha^{\ \gamma} \circ \mathbb{Q}_\gamma^{\ a} + \tfrac{1}{2}\mathbb{Q}_\gamma^{\ a} \circ \mathbb{L}_\alpha^{\ \gamma} \\ &- \tfrac{1}{2}\mathbb{R}_c^{\ a} \circ \mathbb{Q}_\alpha^{\ c} + \tfrac{1}{2}\mathbb{Q}_\alpha^{\ c} \circ \mathbb{R}_c^{\ a} - \tfrac{1}{4}\mathbb{H} \circ \mathbb{Q}_\alpha^{\ a} + \tfrac{1}{4}\mathbb{Q}_\alpha^{\ a} \circ \mathbb{H}\big) \otimes 1 \, , \\ &- \tfrac{1}{2}(\mathbb{L}_\alpha^{\ \gamma} \circ 1 - 1 \circ \mathbb{L}_\alpha^{\ \gamma}) \otimes \mathbb{Q}_\gamma^{\ a} + \tfrac{1}{2}(\mathbb{Q}_\gamma^{\ a} \circ 1 - 1 \circ \mathbb{Q}_\gamma^{\ a}) \otimes \mathbb{L}_\alpha^{\ \gamma} \\ &- \tfrac{1}{2}(\mathbb{R}_c^{\ a} \circ 1 - 1 \circ \mathbb{R}_c^{\ a}) \otimes \mathbb{Q}_\alpha^{\ c} + \tfrac{1}{2}(\mathbb{Q}_\alpha^{\ c} \circ 1 - 1 \circ \mathbb{Q}_\alpha^{\ c}) \otimes \mathbb{R}_c^{\ a} \\ &- \tfrac{1}{4}(\mathbb{H} \circ 1 - 1 \circ \mathbb{H}) \otimes \mathbb{Q}_\alpha^{\ a} + \tfrac{1}{4}(\mathbb{Q}_\alpha^{\ a} \circ 1 - 1 \circ \mathbb{Q}_\alpha^{\ a}) \otimes \mathbb{H} \, , \\ \Delta(\widetilde{\mathbb{Q}}_{\alpha,-1}^{\ a}) = \big(&\mathbb{Q}_{\alpha,-1}^{\ a} \circ 1 - 1 \circ \mathbb{Q}_{a,-1}^{\ \alpha} - \tfrac{1}{2}\varepsilon_{\alpha\gamma}\,\varepsilon^{ac}\,\mathbb{G}_c^{\ \gamma} \circ \mathbb{C} + \tfrac{1}{2}\varepsilon_{\alpha\gamma}\,\varepsilon^{ac}\,\mathbb{C} \circ \mathbb{G}_c^{\ \gamma}\big) \otimes 1 \\ &- \tfrac{1}{2}\varepsilon_{\alpha\gamma}\,\varepsilon^{ac}(\mathbb{G}_c^{\ \gamma} \circ 1 - 1 \circ \mathbb{G}_c^{\ \gamma}) \otimes \mathbb{C} + \tfrac{1}{2}\varepsilon_{\alpha\gamma}\,\varepsilon^{ac}(\mathbb{C} \circ 1 - 1 \circ \mathbb{C}) \otimes \mathbb{G}_c^{\ \gamma} \, , \\ \Delta(\widetilde{\mathbb{G}}_{a,+1}^{\ \alpha}) = \big(&\mathbb{G}_{a,+1}^{\ \alpha} \circ 1 - 1 \circ \mathbb{G}_{a,+1}^{\ \alpha} + \tfrac{1}{2}\mathbb{L}_\gamma^{\ \alpha} \circ \mathbb{G}_a^{\ \gamma} - \tfrac{1}{2}\mathbb{G}_a^{\ \gamma} \circ \mathbb{L}_\gamma^{\ \alpha} \\ &+ \tfrac{1}{2}\mathbb{R}_a^{\ c} \circ \mathbb{G}_c^{\ \alpha} - \tfrac{1}{2}\mathbb{G}_c^{\ \alpha} \circ \mathbb{R}_a^{\ c} + \tfrac{1}{4}\mathbb{H} \circ \mathbb{G}_a^{\ \alpha} - \tfrac{1}{4}\mathbb{G}_a^{\ \alpha} \circ \mathbb{H}\big) \otimes 1 \\ &+ \tfrac{1}{2}(\mathbb{L}_\gamma^{\ \alpha} \circ 1 - 1 \circ \mathbb{L}_\gamma^{\ \alpha}) \otimes \mathbb{G}_a^{\ \gamma} - \tfrac{1}{2}(\mathbb{G}_a^{\ \gamma} \circ 1 - 1 \circ \mathbb{G}_a^{\ \gamma}) \otimes \mathbb{L}_\gamma^{\ \alpha} \\ &+ \tfrac{1}{2}(\mathbb{R}_a^{\ c} \circ 1 - 1 \circ \mathbb{R}_a^{\ c}) \otimes \mathbb{G}_c^{\ \alpha} - \tfrac{1}{2}(\mathbb{G}_c^{\ \alpha} \circ 1 - 1 \circ \mathbb{G}_c^{\ \alpha}) \otimes \mathbb{R}_a^{\ c} \\ &+ \tfrac{1}{4}(\mathbb{H} \circ 1 - 1 \circ \mathbb{H}) \otimes \mathbb{G}_a^{\ \alpha} - \tfrac{1}{4}(\mathbb{G}_a^{\ \alpha} \circ 1 - 1 \circ \mathbb{G}_a^{\ \alpha}) \otimes \mathbb{H} \, , \\ \Delta(\widetilde{\mathbb{G}}_{a,-1}^{\ \alpha}) = \big(&\mathbb{G}_{a,-1}^{\ \alpha} \circ 1 - 1 \circ \mathbb{G}_{a,-1}^{\ \alpha} + \tfrac{1}{2}\varepsilon_{ac}\,\varepsilon^{\alpha\gamma}\,\mathbb{Q}_\gamma^{\ c} \circ \mathbb{C}^\dagger - \tfrac{1}{2}\varepsilon_{ac}\,\varepsilon^{\alpha\gamma}\,\mathbb{C}^\dagger \circ \mathbb{Q}_\gamma^{\ c}\big) \otimes 1 \\ &+ \tfrac{1}{2}\varepsilon_{ac}\,\varepsilon^{\alpha\gamma}(\mathbb{Q}_\gamma^{\ c} \circ 1 - 1 \circ \mathbb{Q}_\gamma^{\ c}) \otimes \mathbb{C}^\dagger - \tfrac{1}{2}\varepsilon_{ac}\,\varepsilon^{\alpha\gamma}(\mathbb{C}^\dagger \circ 1 - 1 \circ \mathbb{C}^\dagger) \otimes \mathbb{Q}_\gamma^{\ c}. \end{aligned} \tag{5.2.14}$$

Once again we have checked that these new secret charges commute with the complete reflection matrix $\mathcal{K}^v$ (3.5.52) [144] and the achiral reflection matrix (3.5.59) [24], thus are the secret symmetries of the reflection from the vertical $D5$-brane.

5.3 Secret symmetries of the q-deformed S-matrix

In this final section we will show the quantum affine origin of the secret symmetry (5.1.2). We start by recalling the Chevalley-Serre and Drinfeld's second realization of the quantum affine superalgebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$. We consider its fundamental representation and give the explicit realization of the corresponding S-matrix and the non-supertraceless charges $h_{2,0}$ and $h_{2,\pm 1}$. Thus Section 5.3.1 can be considered both as a warm-up exercise, and as a treatment relevant to a wealth of subsectors of the full algebra and corresponding S-matrix, later discussed in Section 5.3.3. In Section 5.3.2 we review the superalgebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$ and its fundamental representation, and give the necessary background for building the secret symmetry of $\widehat{\mathcal{Q}}$. In Section 5.3.3, bearing on the construction presented in Section 5.3.2, we build the secret symmetry of the bound state S-matrices of $\widehat{\mathcal{Q}}$ in both the conventional limit ($g \to 0$) and the full case of $\widehat{\mathcal{Q}}$.

5.3.1 The quantum affine superalgebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$

In this section we provide both the Chevalley-Serre realization and the so called Drinfeld's second realization [5] of the quantum affine superalgebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$, in the conventions of [193] (see also [194–199]). We choose a complex number $q \neq 0$ and not a root of unity, and define

$$[y]_q = \frac{q^y - q^{-y}}{q - q^{-1}} \,. \tag{5.3.1}$$

We will also set the central charge c of the quantum affine algebra to zero for the rest of this section, and generically indicate with [,] the graded (or super-)commutator. We instead reserve the symbol { , } for the anti-commutator.

5.3.1.1 Chevalley-Serre realization

In the Chevalley-Serre realization, the Lie superalgebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ is generated by fermionic Chevalley generators $\xi_1^\pm$, Cartan generators h_1, h_2, with h_2 the non-supertraceless element completing the superalgebra $\mathfrak{sl}(1|1)$ to $\mathfrak{gl}(1|1)$, and the affine fermionic Chevalley generators $\xi_0^\pm$ and corresponding Cartan generator h_0 .

The generalized symmetric Cartan matrix is given by

$$(a_{ij})_{0 \leqq i,j \leqq 2} = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 2 \\ -2 & 2 & 0 \end{pmatrix} . \tag{5.3.2}$$

Notice that this matrix is degenerate, but the Lie superalgebra block $1 \leqq i,j \leqq 2$ is not. The defining relations are as follows, for $0 \leqq i,j \leqq 2$ (Chevalley generators corresponding to the Cartan generator h_2 are absent):

$$[h_i, h_j] = 0\,, \qquad [h_i, \xi_j^\pm] = \pm a_{ij} x_j^\pm\,, \qquad \{\xi_i^+, \xi_j^-\} = \delta_{ij} \frac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\,, \tag{5.3.3}$$

supplemented by a suitable set of Serre relations. We refer to [70] for the explicit form of the Serre relations, as we will instead spell out the complete set of relations in Drinfeld's second realization, see (5.3.6).

One can define a Hopf algebra structure with the following coproduct, antipode and counit:

$$\begin{aligned}
&\Delta(h_i) = h_i \otimes 1 + 1 \otimes h_i\,, && S(h_i) = -h_i\,,\\
&\Delta(\xi_i^+) = \xi_i^+ \otimes 1 + q^{h_i} \otimes \xi_i^+\,, && S(\xi_i^\pm) = -q^{\mp h_i}\xi_i^\pm\,,\\
&\Delta(\xi_i^-) = \xi_i^- \otimes q^{-h_i} + 1 \otimes \xi_i^-\,, && \epsilon(h_i) = \epsilon(\xi_i^\pm) = 0\,.
\end{aligned} \tag{5.3.4}$$

5.3.1.2 Drinfeld's second realization

The same algebra is also generated by an infinite set of Drinfeld's generators, which in some sense make explicit the infinite set of 'levels' of the quantum affine algebra obtained, in the Chevalley-Serre realization, by subsequent commutations with the affine generators $\xi_0^\pm$. Drinfeld's generators are

$$\xi_{1,m}^\pm\,,\ h_{i,n}\,, \qquad \text{with} \qquad i = 1,2\,, \quad m,n \in \mathbb{Z}\,. \tag{5.3.5}$$

The defining relations are as follows:

$$\begin{aligned}
&[h_{i,m}\,,\, h_{j,n}] = 0\,, && \{\xi_{1,n}^+\,,\, \xi_{1,m}^-\} = \frac{1}{q-q^{-1}}\left(\psi_{1,n+m}^+ - \psi_{1,n+m}^-\right),\\
&[h_{i,0}\,,\, \xi_{1,m}^\pm] = \pm a_{i1}\,\xi_{1,m}^\pm\,, && \{\xi_{1,n}^\pm\,,\, \xi_{1,m}^\pm\} = 0\,,\\
&[h_{i,n}\,,\, \xi_{1,m}^\pm] = \pm\frac{[a_{i1}n]_q}{n}\,\xi_{1,n+m}^\pm\,, && \text{for} \quad n \neq 0\,.
\end{aligned} \tag{5.3.6}$$

We have used the definition

$$\psi_1^\pm(z) = q^{\pm h_{1,0}} \exp\left(\pm(q-q^{-1})\sum_{m>0} h_{1,\pm m} z^{\mp m}\right) = \sum_{n\in\mathbb{Z}} \psi_{1,n}^\pm z^{-n}\,. \tag{5.3.7}$$

The above expression (5.3.7) should be understood as defining a generating function for the individual $\psi_{1,n}^\pm$'s, which in turn can be obtained by Laurent expanding both sides of the equation and matching the powers of the parameter z.

We call 'level' the index n of Drinfeld's generators. One typically introduces a 'derivation' operator d that counts the level, in the following way:

$$[d, \tau_n] = n\,\tau_n\,, \tag{5.3.8}$$

for any generator τ_n at level n.

The map between the Chevalley-Serre and Drinfeld's second realization, which constitutes a Hopf algebra isomorphism, is given by the following assignment, for $i = 1,2$:

$$\begin{aligned}
&h_i = h_{i,0}\,, && \xi_1^\pm = \xi_{1,0}^\pm\,,\\
&h_0 = -h_{1,0}\,, && \xi_0^\pm = \pm\xi_{1,\pm 1}^\mp\, q^{\mp h_{1,0}}\,,
\end{aligned} \tag{5.3.9}$$

where we have used the fact that $a_{11} = 0$. As one can see, the Chevalley generator associated to the positive (respectively, negative) affine root generates the positive (respectively, negative) tower of levels in Drinfeld's second realization.

The coalgebra structure in Drinfeld's second realization satisfies the following triangular decomposition, for $n \in \mathbb{Z}, n \neq 0$ (for $n = 0$ the coproduct can be obtained directly from (5.3.9), (5.3.4)):

$$\begin{aligned}
\Delta(h_{1,n}) &= h_{1,n} \otimes 1 + 1 \otimes h_{1,n} \ mod\ N_- \otimes N_+, \\
\Delta(\xi^+_{1,n}) &= \xi^+_{1,n} \otimes 1 + q^{sign(n) h_{1,0}} \otimes \xi^+_{1,n} \\
&\quad + \sum_{k=\frac{1}{2}(1-sign(n))}^{|n|-1} \psi^{sign(n)}_{1,sign(n)(|n|-k)} \otimes \xi^+_{1,sign(n)k} \ mod\ N_- \otimes N_+^2, \\
\Delta(\xi^-_{1,n}) &= \xi^-_{1,n} \otimes q^{sign(n) h_{1,0}} + 1 \otimes \xi^-_{1,n} \\
&\quad + \sum_{k=\frac{1}{2}(1+sign(n))}^{|n|-1} \xi^-_{1,sign(n)k} \otimes \psi^{sign(n)}_{1,sign(n)(|n|-k)} \ mod\ N_-^2 \otimes N_+,
\end{aligned} \tag{5.3.10}$$

with $N_\pm$ (respectively, $N_\pm^2$) the left ideals generated by $\xi^\pm_{1,m}$ (respectively, $\xi^\pm_{1,m}\xi^\pm_{1,m'}$), with $m, m' \in \mathbb{Z}$.

The coproduct for the generators $h_{2,n}$ is obtained by imposing that Δ is an algebra homomorphism, namely, that it respects the defining relations (5.3.6). Making use of (5.3.10), we obtain for instance

$$\begin{aligned}
\Delta(h_{2,+1}) &= h_{2,+1} \otimes 1 + 1 \otimes h_{2,+1} + (q^{-2} - q^2) \xi^-_{1,+1} \otimes \xi^+_{1,0}\,, \\
\Delta(h_{2,-1}) &= h_{2,-1} \otimes 1 + 1 \otimes h_{2,-1} - (q^{-2} - q^2) \xi^-_{1,0} \otimes \xi^+_{1,-1}\,.
\end{aligned} \tag{5.3.11}$$

5.3.1.3 Fundamental representation

We provide here what we will call the 'fundamental evaluation' representation in Drinfeld's second realization, as obtained from [70] by specializing to a particular case. By the terminology 'fundamental evaluation' representation we mean a representation which coincides with the fundamental representation at level zero, while the level one generators of the quantum affine algebra are obtained by multiplying the entries of the level zero generators by appropriate linear polynomials in a certain (sometimes called 'evaluation' or 'spectral') parameter z. To obtain the corresponding representation in the Chevalley-Serre realization, one can make use of Drinfeld's map (5.3.9). For v_1 and v_2 a bosonic and fermionic state, respectively, η_{ij} the matrix with 1 in position (i, j) and zero elsewhere, and z a spectral parameter counting the level, we have for instance

$$\begin{aligned}
&\xi^+_{1,0} = \eta_{12}\,, && \xi^-_{1,0} = \eta_{21}\,, && h_{1,0} = \eta_{11} + \eta_{22}\,, \\
&h_{2,0} = \eta_{11} - \eta_{22}\,, && h_{2,\pm 1} = \tfrac{1}{2}(z\,q)^{\pm 1} [2]_q\,(\eta_{11} - \eta_{22})\,, && \\
&\xi^+_{1,\pm 1} = (z\,q)^{\pm 1}\,\eta_{12}\,, && \xi^-_{1,\pm 1} = (z\,q)^{\pm 1}\,\eta_{21}\,. &&
\end{aligned} \tag{5.3.12}$$

The derivation (5.3.8) in this representation is given by $d = z\frac{d}{dz}$. The S-matrix R is defined by the requirement that it satisfies the intertwining property

$$\Delta^{op}(\tau)\, R \,= R\,\Delta(\tau)\,, \tag{5.3.13}$$

with $\Delta^{op}(\tau)$ defined as $\Delta(\tau)$ followed by a graded permutation, and τ any generator of the algebra. In this specific instance, such an S-matrix is given (up to an overall factor) by (see also [200])

$$\begin{aligned} R = \eta_{11}\otimes\eta_{11} + \frac{\frac{z}{w}-1}{q\frac{z}{w}-q^{-1}}(\eta_{11}\otimes\eta_{22}+\eta_{22}\otimes\eta_{11}) \\ + \frac{\frac{z}{w}(q-q^{-1})}{q\frac{z}{w}-q^{-1}}(\eta_{21}\otimes\eta_{12}-\frac{w}{z}\eta_{12}\otimes\eta_{21}) + \frac{q^{-1}\frac{z}{w}-q}{q\frac{z}{w}-q^{-1}}\eta_{22}\otimes\eta_{22}\,, \end{aligned} \tag{5.3.14}$$

where z, w are the spectral parameters corresponding to the first and second copy of the algebra respectively.

Finally, we want to translate the expressions (5.3.11) into the Chevalley-Serre realization, as this shall be important to us later on. This can be done with the help of (5.3.9). However, the charges $h_{2,\pm1}$ have no canonical image under Drinfeld's map. For this reason, let us introduce new charges

$$\mathcal{B}_{\pm} = \frac{(z\,q)^{\pm1}}{q^{-1}-q}\,h_{2,0}\,. \tag{5.3.15}$$

In the Chevalley-Serre realization, (5.3.11) then reads as

$$\begin{aligned} \Delta(\mathcal{B}_+) &= \mathcal{B}_+\otimes 1 + 1\otimes\mathcal{B}_+ + 2\,\xi_0^+ k_1\otimes\xi_1^+\,, \\ \Delta(\mathcal{B}_-) &= \mathcal{B}_-\otimes 1 + 1\otimes\mathcal{B}_- + 2\,\xi_1^-\otimes k_1^{-1}\xi_0^-\,. \end{aligned} \tag{5.3.16}$$

5.3.2 The quantum affine superalgebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$

We will now specialize the presentation of [70] to the case of $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$. While the previous section is strictly related to certain subsectors of the q-deformed AdS/CFT algebra (which we will treat in the second part of the paper), this section is related to the full algebra and corresponding S-matrix. We will directly focus on Drinfeld's second realization for simplicity, referring to [70] for further details (see also [201]).

5.3.2.1 Drinfeld's second realization

The algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$ (for an all-fermionic Dynkin diagram) is generated by an infinite set of Drinfeld's generators

$$\xi^{\pm}_{i,m}\,,\ h_{j,n}\,,\qquad \text{with}\qquad i=1,2,3,\quad j=1,2,3,4,\quad m,n\in\mathbb{Z}. \tag{5.3.17}$$

The defining relations are as follows:

$$
\begin{aligned}
&[h_{j,m}\,,\,h_{j',n}]=0\,,\\
&[h_{j,0}\,,\,\xi^{\pm}_{i,m}]=\pm a_{ji}\,\xi^{\pm}_{i,m}\,,\\
&[h_{j,n}\,,\,\xi^{\pm}_{i,m}]=\pm\frac{[a_{ji}\,n]_q}{n}\,\xi^{\pm}_{i,n+m}\,,\qquad n\neq 0\,,\\
&\{\xi^{+}_{i,n}\,,\,\xi^{-}_{i',m}\}=\frac{\delta_{i,i'}}{q-q^{-1}}\left(\psi^{+}_{i,n+m}-\psi^{-}_{i,n+m}\right),
\end{aligned}
\tag{5.3.18}
$$

combined with a suitable set of Serre relations [70] which read

$$
\begin{aligned}
&\{\xi^{\pm}_{i,m}\,,\,\xi^{\pm}_{i',n}\}=0,\quad \text{if}\quad a_{ii'}=0\,,\\
&\{\xi^{\pm}_{i,m+1}\,,\,\xi^{\pm}_{i',n}\}_{q^{\pm a_{ii'}}}=\{\xi^{\pm}_{i',n+1},\xi^{\pm}_{i,m}\}_{q^{\pm a_{ii'}}}\,,\\
&[\{\xi^{\pm}_{2,m}\,,\,\xi^{\pm}_{1,n}\}_q,\{\xi^{\pm}_{2,p}\,,\,\xi^{\pm}_{3,r}\}_{q^{-1}}]=[\{\xi^{\pm}_{2,p}\,,\,\xi^{\pm}_{1,n}\}_q,\{\xi^{\pm}_{2,m}\,,\,\xi^{\pm}_{3,r}\}_{q^{-1}}]\,.
\end{aligned}
\tag{5.3.19}
$$

The symmetric Cartan matrix reads

$$
(a_{ij})_{1\leqq i,j\leqq 4}=\begin{pmatrix} 0 & 1 & 0 & 2\\ 1 & 0 & -1 & -2\\ 0 & -1 & 0 & 2\\ 2 & -2 & 2 & 0\end{pmatrix}.
\tag{5.3.20}
$$

We have once again used the definition

$$
\psi^{\pm}_i(z)=q^{\pm h_{i,0}}\exp\left(\pm(q-q^{-1})\sum_{m>0}h_{i,\pm m}z^{\mp m}\right)=\sum_{n\in\mathbb{Z}}\psi^{\pm}_{i,n}z^{-n}\,.
\tag{5.3.21}
$$

The 'derivation' operator d counting the level is once again introduced in the following way:

$$
[d,\tau_n]=n\,\tau_n,
\tag{5.3.22}
$$

for any generator τ_n at level n.

Let us comment on the Serre relations (5.3.19). The first line expresses the fermionic nature of the generators associated to the simple roots, while the second one ensures that a good filtration is preserved. This means that one is free to combine levels in different ways to obtain one and the same 'sum' level as a result. The third line, taken at level 0 (namely, for $m=n=p=r=0$), tells us that there are only three generators associated to the non-simple roots, two obtained as $\{\xi^{\pm}_{2,0},\xi^{\pm}_{1,0}\}_q$ and $\{\xi^{\pm}_{2,0},\xi^{\pm}_{3,0}\}_{q^{-1}}$, and one obtained by commuting, for instance, the very first of these generators with $\xi^{\pm}_{3,0}$. In fact, the third Serre relation implies that commuting the two generators associated to the non-simple roots with each other returns zero, which truncates any further growth in the number of generators.

The coproduct has the natural structure (we define $sign(0) \equiv +1$)

$$
\begin{aligned}
\Delta(h_{i,n}) &= h_{i,n} \otimes 1 + 1 \otimes h_{i,n} \ mod\ N_- \otimes N_+ \,, \\
\Delta(\xi^+_{i,n}) &= \xi^+_{i,n} \otimes 1 + q^{sign(n) h_{i,0}} \otimes \xi^+_{i,n} \\
&\quad + \sum_{k=\frac{1}{2}(1-sign(n))}^{|n|-1} \psi^{sign(n)}_{i,sign(n)(|n|-k)} \otimes \xi^+_{i,sign(n)k} \ mod\ N_- \otimes N_+^2 \,, \\
\Delta(\xi^-_{i,n}) &= \xi^-_{i,n} \otimes q^{sign(n) h_{i,0}} + 1 \otimes \xi^-_{i,n} \\
&\quad + \sum_{k=\frac{1}{2}(1+sign(n))}^{|n|-1} \xi^-_{i,sign(n)k} \otimes \psi^{sign(n)}_{i,sign(n)(|n|-k)} \ mod\ N_-^2 \otimes N_+ \,,
\end{aligned} \tag{5.3.23}
$$

with $N_\pm$ (respectively, $N^2_\pm$) the left ideals generated by $\xi^\pm_{i,m}$ (respectively, $\xi^\pm_{i,m}\xi^\pm_{i,m'}$), with $m, m' \in \mathbb{Z}$ and $i = 1, 2, 3$.

The coproduct for the generators $h_{4,n}$ is obtained by imposing that Δ respects the defining relations (5.3.18). With respect to the case of $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$, the 'tail' of the coproduct (*i.e.*, the quadratic part that comes after the trivial comultiplication rule for the generator itself) now contains generators associated to non-simple roots (which before where simply absent). By carefully taking into account (5.3.23), we find

$$
\begin{aligned}
\Delta(h_{4,+1}) &= h_{4,+1} \otimes 1 + 1 \otimes h_{4,+1} \\
&\qquad + (q^{-1} - q) \sum_{i=1}^{3} [a_{4i}]_q \, \xi^-_{i,+1} \otimes \xi^+_{i,0} \ + \text{non-simple roots} \,, \\
\Delta(h_{4,-1}) &= h_{4,-1} \otimes 1 + 1 \otimes h_{4,-1} \\
&\qquad - (q^{-1} - q) \sum_{i=1}^{3} [a_{4i}]_q \, \xi^-_{i,0} \otimes \xi^+_{i,-1} \ + \text{non-simple roots} \,.
\end{aligned} \tag{5.3.24}
$$

We will specify the non-simple part of the tail of the coproduct in the fundamental representation in the following section.

5.3.2.2 Fundamental representation

The fundamental evaluation representation in Drinfeld's second realization can be obtained from [70] in a particular case. For v_1, v_2 and v_3, v_4 two bosonic and two fermionic states, respectively, η_{ij} the matrix with 1 in position (i, j) and zero elsewhere, and z a spectral parameter counting the level, we have this time

$$
\begin{aligned}
&\xi^+_{1,0} = \eta_{13}\,, && \xi^+_{2,0} = \eta_{32}\,, && \xi^+_{3,0} = \eta_{24}\,,\\
&\xi^-_{1,0} = \eta_{31}\,, && \xi^-_{2,0} = -\eta_{23}\,, && \xi^-_{3,0} = \eta_{42}\,,\\
&h_{1,0} = (\eta_{11}+\eta_{33}), && h_{2,0} = -(\eta_{33}+\eta_{22}), && h_{3,0} = (\eta_{22}+\eta_{44})\\
&h_{4,0} = \sum_{k=1}^{4}(-)^{[k]}\eta_{kk}\,, && && \\
&\xi^+_{1,\pm1} = (z\,q)^{\pm1}\,\eta_{13}\,, && \xi^+_{2,\pm1} = z^{\pm1}\,\eta_{32}\,, && \xi^+_{3,\pm1} = (z\,q)^{\pm1}\,\eta_{24}\,,\\
&\xi^-_{1,\pm1} = (z\,q)^{\pm1}\,\eta_{31}\,, && \xi^-_{2,\pm1} = -z^{\pm1}\,\eta_{23}\,, && \xi^-_{3,\pm1} = (z\,q)^{\pm1}\,\eta_{42}\,,\\
&h_{1,\pm1} = z^{\pm1}(\eta_{11}+\eta_{33})\,, && h_{2,\pm1} = -(z\,q)^{\pm1}(\eta_{22}+\eta_{33})\,, && h_{3,\pm1} = z^{\pm1}(\eta_{22}+\eta_{44})\,,\\
&h_{4,\pm1} = z^{\pm1}\,[2]_q\Big(y^{\pm}\eta_{11}+(y^{\pm}+1-q^{\pm1})\eta_{22}+(y^{\pm}-q^{\pm1})\eta_{33}+(y^{\pm}+1-2q^{\pm1})\eta_{44}\Big)\,, && &&
\end{aligned}
\tag{5.3.25}
$$

with $[k]$ the grading of the state v_k. The derivation (5.3.22) in the fundamental evaluation representation (5.3.25) is given by $d = z\,\frac{d}{dz}$. The algebra $\mathfrak{gl}(n|n)$ is non-semisimple ($\mathfrak{sl}(n|n)$ being a non-trivial ideal strictly contained in it). Hence, one can always add a constant times the identity to the non-supertraceless generator who lives outside the ideal (and, therefore, never appears on the right-hand-side of any commutation relations). The generator $h_{4,1}$ of the quantum-affine version also does not appear on the r.h.s. of any commutation relations, and one can use the freedom we just mentioned to redefine this generator by adding a multiple of the identity. This is reflected in the choice of $y^{\pm}$ (which we tacitly fixed to a convenient value in the previous section). The term multiplying $y^{\pm}$ is a multiple of the identity matrix, and its coproduct is trivial hence it drops out of the defining relation for the S-matrix (5.3.13).

Let us spell out the coproduct (5.3.24) in this representation (z and w once again refer to the first and, respectively, the second factor in the tensor product):

$$
\begin{aligned}
\Delta(h_{4,+1}) &= h_{4,+1}\otimes 1 + 1\otimes h_{4,+1} + (q^{-2}-q^2)\,z\,\Big(q\eta_{31}\otimes\eta_{13} + (q-1)\eta_{21}\otimes\eta_{12}\\
&\qquad + (2q-1)\eta_{41}\otimes\eta_{14} + \eta_{23}\otimes\eta_{32} + (1-q)\eta_{43}\otimes\eta_{34} + q\eta_{42}\otimes\eta_{24}\Big).\\
\Delta(h_{4,-1}) &= h_{4,-1}\otimes 1 + 1\otimes h_{4,-1} - (q^{-2}-q^2)\,w^{-1}\,\Big(q^{-1}\eta_{31}\otimes\eta_{13} + (q^{-1}-1)\eta_{21}\otimes\eta_{12}\\
&\qquad + (2q^{-1}-1)\eta_{41}\otimes\eta_{14} + \eta_{23}\otimes\eta_{32} + (1-q^{-1})\eta_{43}\otimes\eta_{34} + q^{-1}\eta_{42}\otimes\eta_{24}\Big).
\end{aligned}
\tag{5.3.26}
$$

Notice that the bosonic part of the tail is higher order in the $q\to 1$ limit, and therefore it disappears in the Yangian limit. The parameter y does not appear in the coefficients of the tail, according to the above discussion. We can once again fix the constant y to a convenient value, for instance

$$
y^{\pm} = q^{\pm1} - \tfrac{1}{2}\,, \tag{5.3.27}
$$

which produces the following representation:

$$h_{4,\pm 1} = z^{\pm 1}\,[2]_q \left((q^{\pm 1} - \tfrac{1}{2})\eta_{11} + \tfrac{1}{2}\eta_{22} - \tfrac{1}{2}\eta_{33} - (q^{\pm 1} - \tfrac{1}{2})\eta_{44} \right), \tag{5.3.28}$$

The S-matrix satisfying the interwining property (5.3.13) is given (up to an overall factor) by (see also [202])

$$\begin{aligned} R = {} & \eta_{11} \otimes \eta_{11} + \eta_{22} \otimes \eta_{22} + \frac{q^2 - \frac{z}{w}}{1 - q^2 \frac{z}{w}} (\eta_{33} \otimes \eta_{33} + \eta_{44} \otimes \eta_{44}) \\ & + \frac{q\left(1 - \frac{z}{w}\right)}{1 - q^2 \frac{z}{w}} \sum_{i \neq j} \eta_{ii} \otimes \eta_{jj} - \frac{q^2 - 1}{q^2 \frac{z}{w} - 1} \left(\sum_{(i,j) \in A} \eta_{ij} \otimes \eta_{ji} - \eta_{12} \otimes \eta_{21} - \eta_{32} \otimes \eta_{23} \right) \\ & + \frac{q^2 - 1}{q^2 - \frac{w}{z}} \left(\sum_{(i,j) \in B} \eta_{ij} \otimes \eta_{ji} - \eta_{23} \otimes \eta_{32} - \eta_{43} \otimes \eta_{34} \right), \end{aligned} \tag{5.3.29}$$

As a consistency check, one can notice that in the scaling limit $q = e^h$ and $z/w = e^{2\,\delta u\, h}$ with $h \to 0$, the above S-matrix reduces to the Yangian S-matrix

$$R_Y = \frac{\delta u}{\delta u + 1} \left(1 + \frac{P}{\delta u} \right), \tag{5.3.30}$$

with P being the graded permutation operator $P = \sum_{i,j=1}^{4} (-)^j\, \eta_{ij} \otimes \eta_{ji}$.

One can show that the combination

$$\mathcal{B}_{\pm} = \frac{q^{\pm 1}}{q^{-1} - q} \left(\frac{2}{q^{\pm 1} [2]_q} h_{4,\pm 1} \;+\; (q^{\mp 1} - 1)(h_{1,\pm 1} - h_{3,\pm 1}) \right), \tag{5.3.31}$$

is such that, in the representation (5.3.25), one obtains an analog of (5.3.15),

$$\mathcal{B}_{\pm} = \frac{(z\,q)^{\pm 1}}{q^{-1} - q} \sum_{i=1}^{4} (-)^{[i]}\, \eta_{ii}\,. \tag{5.3.32}$$

Then, using (5.3.26) and

$$\begin{aligned} \Delta(h_{1,+1}) &= h_{1,+1} \otimes 1 + 1 \otimes h_{1,+1} + (q^{-1} - q)\, z\, (\eta \otimes \eta)_h\,, \\ \Delta(h_{1,-1}) &= h_{1,-1} \otimes 1 + 1 \otimes h_{1,-1} - (q^{-1} - q)\, w^{-1}\, (\eta \otimes \eta)_h\,, \\ \Delta(h_{3,+1}) &= h_{3,+1} \otimes 1 + 1 \otimes h_{3,+1} - (q^{-1} - q)\, z\, (\eta \otimes \eta)_h\,, \\ \Delta(h_{3,-1}) &= h_{3,-1} \otimes 1 + 1 \otimes h_{3,-1} + (q^{-1} - q)\, w^{-1}\, (\eta \otimes \eta)_h\,, \end{aligned} \tag{5.3.33}$$

where

$$(\eta \otimes \eta)_h = \eta_{21} \otimes \eta_{12} - \eta_{23} \otimes \eta_{32} + \eta_{41} \otimes \eta_{14} + \eta_{43} \otimes \eta_{34}\,, \tag{5.3.34}$$

we find

$$\Delta(\mathcal{B}_+) = \mathcal{B}_+ \otimes 1 + 1 \otimes \mathcal{B}_+ + 2\, z\, q\, (\eta_{31} \otimes \eta_{13} \, + \eta_{23} \otimes \eta_{32} \, + \eta_{41} \otimes \eta_{14} \, + \eta_{42} \otimes \eta_{24})\,,$$
$$\Delta(\mathcal{B}_-) = \mathcal{B}_- \otimes 1 + 1 \otimes \mathcal{B}_- + 2\, (w\, q)^{-1}\, (\eta_{31} \otimes \eta_{13} \, + \eta_{23} \otimes \eta_{32} \, + \eta_{41} \otimes \eta_{14} \, + \eta_{42} \otimes \eta_{24})\,. \tag{5.3.35}$$

As in the previous section, we translate these expressions into the Chevalley-Serre realization. The map between the Chevalley-Serre and Drinfeld's second realization, in the fundamental representation which is relevant to the present discussion, is given by the following assignment:

$$h_i = h_{i,0}\,, \qquad \xi_i^{\pm} = \xi_{i,0}^{\pm}\,,$$
$$h_0 = -h_{1,0} - h_{2,0} - h_{3,0}\,, \qquad \xi_0^{\pm} = \pm (q\, z)^{\pm 1} [[\xi_{1,0}^{\mp}, \xi_{2,0}^{\mp}], \xi_{3,0}^{\mp}]\, q^{\mp (h_{1,0} + h_{2,0} + h_{3,0})}\,. \tag{5.3.36}$$

Thus with the help of (5.3.25) we find

$$\begin{aligned}\Delta(\mathcal{B}_+) = \mathcal{B}_+ \otimes 1 + 1 \otimes \mathcal{B}_+ + 2\, \big(&\xi_0^+ k_{123} \otimes \xi_{123}^+ + \xi_{012}^+ k_3 \otimes \xi_3^+ \\ &- q^2 \xi_{013}^+ k_2 \otimes \xi_2^+ + \xi_{230}^+ k_1 \otimes \xi_1^+\big)\,,\\ \Delta(\mathcal{B}_-) = \mathcal{B}_- \otimes 1 + 1 \otimes \mathcal{B}_- + 2\, \big(&\xi_{123}^- \otimes k_{123}^{-1} \xi_0^- + \xi_3^- \otimes k_3^{-1} \xi_{012}^- \\ &- q^{-2} \xi_2^- \otimes k_2^{-1} \xi_{013}^- + \xi_1^- \otimes k_1^{-1} \xi_{230}^-\big)\,,\end{aligned} \tag{5.3.37}$$

where we have used the short-hand notation $k_{ijk} = k_i k_j k_k$ and $\xi_{ijk} = [[\xi_i, \xi_j], \xi_k]$. One can observe that these expressions can formally be written as

$$\begin{aligned}\Delta(\mathcal{B}_+) &= \mathcal{B}_+ \otimes 1 + 1 \otimes \mathcal{B}_+ + 2 \sum_{\alpha \in \Phi_0} c_\alpha\, \xi_{\delta - \alpha} k_\alpha \otimes \xi_\alpha\,,\\ \Delta(\mathcal{B}_-) &= \mathcal{B}_- \otimes 1 + 1 \otimes \mathcal{B}_- + 2 \sum_{\alpha \in \Phi_0} c_\alpha\, \xi_{-\alpha} \otimes k_\alpha^{-1} \xi_{\alpha - \delta}\,,\end{aligned} \tag{5.3.38}$$

where Φ_0 is the set of all positive non-affine roots, δ is the affine root and c_α's are complex parameters.

Let us make a final remark concerning the symmetry we have just obtained. We derived the coproduct (5.3.35) starting from an all-fermionic Dynkin diagram, and the pattern of simple and non-simple roots which appear in the tail of the coproduct respects the original choice of Dynkin diagram. For later purposes, it will turn out to be convenient to work with a so-called *distinguished* Dynkin diagram. This is associated to a basis with only one fermionic root. The assignment of simple roots will be different and this will reflect on the appearence of the generators associated to non-simple roots in the tail. In order to be able to match with the expressions we will later find, it is useful to perform a twist of the coalgebra structure (and of the corresponding S-matrix) in the spirit of [203] (see also [70]), where it is explained that such twists may involve factors of the universal

S-matrix itself. One can check that the following transformation

$$\Psi = \mathrm{Id} - (q - q^{-1})(\eta_{23} \otimes \eta_{32} + \eta_{32} \otimes \eta_{23}) - \frac{w}{z}\,\eta_{22} \otimes \eta_{33} - \frac{z}{w}\,\eta_{33} \otimes \eta_{22}$$
$$- \eta_{34} \otimes \eta_{43} - \eta_{43} \otimes \eta_{34} - \eta_{33} \otimes \eta_{44} - \eta_{44} \otimes \eta_{33}\,, \tag{5.3.39}$$

is such that

$$\Delta' = \Psi\,\Delta\,\Psi^{-1} \qquad \text{and} \qquad R' = \Psi^{op}\,R\,\Psi^{-1}. \tag{5.3.40}$$

gives

$$\Delta'(\mathcal{B}_+) = \mathcal{B}_+ \otimes 1 + 1 \otimes \mathcal{B}_+ + 2\,z\,q\,(\eta_{31} \otimes \eta_{13} + \eta_{32} \otimes \eta_{23} + \eta_{41} \otimes \eta_{14} + \eta_{42} \otimes \eta_{24})\,,$$
$$\Delta'(\mathcal{B}_-) = \mathcal{B}_- \otimes 1 + 1 \otimes \mathcal{B}_- + \frac{2}{w\,q}\,(\eta_{31} \otimes \eta_{13} + \eta_{32} \otimes \eta_{23} + \eta_{41} \otimes \eta_{14} + \eta_{42} \otimes \eta_{24})\,, \tag{5.3.41}$$

which is an analog of (5.3.35) for the case of the distinguished Dynkin diagram. The inverse of (5.3.39) can be explicitly calculated, and it reads

$$\Psi^{-1} = \mathrm{Id} + \tau_1\,\eta_{22} \otimes \eta_{33} + \tau_2\,\eta_{33} \otimes \eta_{22} + \tau_3\,\eta_{23} \otimes \eta_{32} + \tau_4\,\eta_{32} \otimes \eta_{23}\,, \tag{5.3.42}$$

with

$$\tau_1 = -\big((1 - w/z) + (q^{-1} - q)^2\big)\,\omega^{-1}\,, \qquad \tau_3 = \tau_4 = (q - q^{-1})\,\omega^{-1}\,,$$
$$\tau_2 = -\big((1 - z/w) + (q^{-1} - q)^2\big)\,\omega^{-1}\,, \tag{5.3.43}$$

and

$$\omega = (1 - z/w)(1 - w/z) + (q^{-1} - q)^2\,. \tag{5.3.44}$$

The non-supertraceless generator we have been focusing our attention on is what will be promoted to the secret symmetry of the full q-deformed AdS/CFT model in the next section. While, in the conventional case we have just been treating, this generator literally extends the superalgebra $\mathfrak{su}(2|2)$ to $\mathfrak{gl}(2|2)$, it will instead only appear at the first quantum-affine level in the subsequent treatment, in parallel to the rational case. The need for such an extension is however the same as in the conventional situation. Its presence corresponds to a consistency issue of the underlying quantum group description of the integrable structure, according to the prescription of Khoroshkin and Tolstoy [187]. In their analysis, an additional Cartan generator is needed to invert the otherwise degenerate Cartan matrix. In turn, the invertibility of the Cartan matrix allows one to write down the universal S-matrix, which appears to be in exponential form with precisely the inverse Cartan matrix appearing at the exponent (see also [204]).

5.3.3 Deformed quantum affine algebra $\widehat{\mathcal{Q}}$

Having explored the fundamental evaluation representations of the algebras $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ and $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$, we are now ready to turn to the quantum affine algebra $\widehat{\mathcal{Q}}$ constructed in [30] and presented in Section 4.1. Thus bearing on the construction presented in the previous sections, we will build the secret symmetry of the bound state S-matrix of Section 4.2. Finally we show that this new symmetry is a quantum analog of the secret symmetry discovered in [130].

5.3.3.1 Conventional affine limit

Before moving to the analysis of the secret symmetry of $\widehat{\mathcal{Q}}$ we would like to first consider the conventional affine limit obtained by setting $g \to 0$ [30]. It is going to be a warm-up exercise and also shall serve as a bridge between the secret symmetry of $\widehat{\mathcal{Q}}$ and the symmetries of $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$ considered in the previous section. In fact, we will prepare all formulas in such a way that it will be easy for the reader to appreciate the cross-over to the full q-deformed case. Note that the 'braiding' by the element U is preserved in the $g \to 0$ limit, while the Serre relations (4.1.12) are restored to their usual form. A suitable twist could remove the U-deformation, however we choose to keep it to facilitate once again the transition to the AdS/CFT case later on. Thus we obtain what we will call a 'U-deformed' $\mathcal{U}_q(\widehat{\mathfrak{sl}}(2|2))$.

Parametrization. To find the explicit relation with $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$ we need to parametrize the conventional affine limit of $\widehat{\mathcal{Q}}$ in terms of the spectral parameter z. This may be achieved by expanding parameters $x^\pm$ in series of g,

$$x^\pm = \frac{i}{g}\frac{q^{\pm M}z-1}{(q-q^{-1})} + \mathcal{O}(g). \tag{5.3.45}$$

Upon rescaling $\gamma \to \bar\gamma\,(g/[M]_q)^{-1/2}$, we find the representation labels to be

$$\begin{aligned}
&a = \bar\gamma\,, && b = 0\,, && c = 0\,, && d = \frac{1}{\bar\gamma}\,,\\
&\tilde a = 0\,, && \tilde b = \frac{\alpha\tilde\alpha z}{\bar\gamma}\,, && \tilde c = -\frac{\bar\gamma}{\alpha\tilde\alpha z}\,, && \tilde d = 0\,.
\end{aligned} \tag{5.3.46}$$

The central elements of the algebra become

$$U^2 = U_2^2 = U_4^{-2} = \frac{1-q^M z}{q^M-1}\,, \qquad V^2 = V_2^2 = V_4^{-2} = q^M. \tag{5.3.47}$$

Fundamental representation. The algebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$ is larger than the one obtained from $\widehat{\mathcal{Q}}$ in the conventional limit due to the presence of the non-supertraceless operators. Let us denote these additional generators originating from $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$ as

$$B_F = \frac{z^{-1}\,q}{q^{-1}-q}\,B_0\,, \qquad B_E = \frac{z\,q^{-1}}{q^{-1}-q}\,B_0 \qquad \text{and} \qquad B_0 = \mathrm{diag}(1,1,-1,-1)\,. \tag{5.3.48}$$

They are equivalent to (5.3.32) up to the redefinition $z \mapsto z^{-1}$. The charge B_0 has a trivial coproduct, while the coproducts of the charges $B_{E/F}$ are defined to have the following form:

$$\Delta(B_F) = B_F \otimes 1 + 1 \otimes B_F - 2\alpha\tilde{\alpha}\Big(U^{-1}F_4 \otimes K_4F_{123} + U^{-1}F_{43} \otimes K_{43}F_{21} \\ + U^{-1}F_{14} \otimes K_{14}F_{32} + U^{-1}F_{341} \otimes K_{143}F_2\Big),$$

$$\Delta(B_E) = B_E \otimes 1 + 1 \otimes B_E - \frac{2}{\alpha\tilde{\alpha}}\Big(U^{-1}E_2K_{341}^{-1} \otimes E_{143} + U^{-1}E_{23}K_{41}^{-1} \otimes E_{41} \\ + U^{-1}E_{12}K_{34}^{-1} \otimes E_{34} + U^{-1}E_{321}K_4^{-1} \otimes E_4\Big). \quad (5.3.49)$$

Here $K_{ij} = K_iK_j$, $K_{ijk} = K_iK_jK_k$, $E_{ij} = [E_i, E_j]$, $E_{ijk} = [[E_i, E_j], E_k]$ and similar expressions hold for the F's. The explicit matrix representation is

$$\begin{aligned} &E_1 = \eta_{21}\,, && E_2 = \bar{\gamma}\,\eta_{42}\,, && E_3 = \eta_{34}\,, && E_4 = \alpha\tilde{\alpha}\,z\,\eta_{13}\,, \\ &F_1 = \eta_{12}\,, && F_2 = \bar{\gamma}^{-1}\,\eta_{24}\,, && F_3 = \eta_{43}\,, && F_4 = -(\alpha\tilde{\alpha}\,z)^{-1}\,\eta_{31}\,, \end{aligned} \quad (5.3.50)$$

and

$$\begin{aligned} &K_1 = \mathrm{diag}(q^{-1}, q, 1, 1)\,, && K_2 = \mathrm{diag}(1, q^{-1}, 1, q^{-1})\,, \\ &K_3 = \mathrm{diag}(1, 1, q^{-1}, q)\,, && K_4 = \mathrm{diag}(q, 1, q, 1)\,. \end{aligned} \quad (5.3.51)$$

All three charges B_0, $B_{E/F}$ are symmetries of the ($g \to 0$) fundamental S-matrix of $\widehat{\mathcal{Q}}$. This is because in this limit the central charges C_2, C_3 vanish and the S-matrix becomes equivalent to (5.3.29) up to the U-deformation and similarity transformation (5.3.39).

The coproducts in (5.3.49) are of the generic form (5.3.38) and are equivalent to (5.3.37). Let us be more precise on this equivalence. By removing the U-deformation, setting the representation parameters to $\alpha = \tilde{\alpha} = 1$ and mapping the spectral parameter as $z \mapsto z^{-1}$, the above expressions (5.3.49) exactly coincide with (5.3.41).

The algebra $\widehat{\mathcal{Q}}$ has an outer automorphism which flips the nodes 2 and 4 of its Dynkin diagram [30]. This automorphism leads to the 'doubling' of the charges (5.3.48),[4]

$$B_F \to B_F^{\pm} = \frac{z^{-1}q^{\pm1}}{q^{-1}-q}\,B_0 \qquad \text{and} \qquad B_E \to B_E^{\pm} = \frac{z\,q^{\pm1}}{q^{-1}-q}\,B_0\,. \quad (5.3.52)$$

The coproducts of B_E^- and B_F^+ are given by (5.3.49), while the coproducts of B_E^+ and B_F^- are obtained by interchanging indices $2 \leftrightarrow 4$ and inverting the U-deformation $U^{-1} \to U$. These new charges shall be important in obtaining a correct Yangian limit. In the following sections we shall concentrate on the charges B_F^+ and B_E^+, or in a shorthand notation $B_{E/F}^+$.

Bound state representation. Let us lift the definitions presented in the previous paragraph to the case of generic bound state representations. For this purpose we redefine the charges in (5.3.52) as

$$B_F^{\pm} = \frac{z^{-1}q^{\pm M}}{q^{-1}-q}\,B_0\,, \qquad B_E^{\pm} = \frac{z\,q^{\pm M}}{q^{-1}-q}\,B_0 \qquad \text{and} \qquad B_0 = N_1 + N_2 - N_3 - N_4\,, \quad (5.3.53)$$

[4]In terms of (5.3.38) this automorphism corresponds to the shifting of the affine root δ from the left to the right factor of the tensor product, and *vice versa*.

where M is the bound state number and N_i are the number operators (see Section 4.1.3 for their realization in terms of quantum oscillators). The charge B_0 has a trivial coproduct. In order to define the explicit realization of the coproducts of $B^{\pm}_{E/F}$ for arbitrary bound states we need to introduce the notion of right adjoint action,

$$\begin{aligned}
(\mathrm{ad}_r\, E_i)A &= (-1)^{[i][A]} K_i A E_i - K_i E_i A\,,\\
(\mathrm{ad}_r\, F_i)A &= (-1)^{[i][A]} A F_i - F_i K_i^{-1} A K_i\,,\\
(\mathrm{ad}_r\, K_i)A &= K_i A K_i^{-1}\,,
\end{aligned} \tag{5.3.54}$$

for any $A \in \widehat{\mathcal{Q}}$. Here $(-1)^{[i][A]}$ represents the grading factor of the supercharges. We shall also be using the shorthand notation $\mathrm{ad}_r\, A_{i_1}\cdots A_{i_l} = \mathrm{ad}_r\, A_{i_1}\cdots \mathrm{ad}_r\, A_{i_l}$ and $E'_i = K_i E_i$. The right adjoint action is used to define the bound state representation of generators corresponding to non-simple roots in the coproducts of the charges (5.3.53). In such a way we obtain expressions of the generic form (5.3.38),

$$\begin{aligned}
\Delta(B^+_F) &= B^+_F \otimes 1 + 1 \otimes B^+_F\\
&\quad - 2\,\alpha\tilde{\alpha}\Big(U^{-1}F_4 \otimes ((\mathrm{ad}_r F_3 F_2)F_1)K_4 + U^{-1}(\mathrm{ad}_r F_1 F_4)F_3 \otimes F_2 K_2^{-1}\\
&\qquad + U^{-1}(\mathrm{ad}_r F_1)F_4 \otimes ((\mathrm{ad}_r F_3)F_2)K_{14} + U^{-1}(\mathrm{ad}_r F_4)F_3 \otimes K_{43}(\mathrm{ad}_r F_2)F_1\\
&\qquad + F_3 \otimes ((\mathrm{ad}_r F_2 F_1)F_4)K_3 + (\mathrm{ad}_r F_2 F_1)F_4 \otimes F_3 K_3^{-1}\Big)\,,\\
\Delta(B^+_E) &= B^+_E \otimes 1 + 1 \otimes B^+_E\\
&\quad - \frac{2}{\alpha\tilde{\alpha}}\Big(U E'_4 \otimes K_4(\mathrm{ad}_r E_3 E_2)E'_1 + U(\mathrm{ad}_r E_1 E_4)E'_3 \otimes E_2\\
&\qquad + U(\mathrm{ad}_r E_1)E'_4 \otimes K_{14}(\mathrm{ad}_r E_3)E'_2 + U(\mathrm{ad}_r E_4)E'_3 \otimes K_{43}(\mathrm{ad}_r E_2)E'_1\\
&\qquad + E'_3 \otimes K_3(\mathrm{ad}_r E_2 E_1)E'_4 + (\mathrm{ad}_r E_2 E_1)E'_4 \otimes E_3\Big)\,.
\end{aligned} \tag{5.3.55}$$

The coproducts of $B^-_{E/F}$ are obtained from the ones of $B^+_{E/F}$ above in the same fashion as for the fundamental representation, *i.e.* by interchanging indices $2 \leftrightarrow 4$ and $U \leftrightarrow U^{-1}$. Notice the extra two 'bosonic' terms in (5.3.55) in contrast to (5.3.49). These terms ensure that $\Delta(B^{\pm}_{E/F})$ are symmetries of the bound state S-matrix[5].

We would like to point out that the extra terms in the tail display a quite surprising discrepancy between the two $\mathcal{U}_q(\mathfrak{sl}(2))$ subalgebras generated by E_1, F_1 and E_3, F_3. We do not fully understand the algebraic reason for this fact. The natural explanation would be that the bound state representations manifestly break the symmetry between bosons and fermions and hence between the two $\mathcal{U}_q(\mathfrak{sl}(2))$'s. This means that in the case of the S-matrix of the anti-bound states (for anti-supersymmetric representations) we might expect the tail to be modified by interchanging indices $1 \leftrightarrow 3$ for the last two terms. For the case of a generic S-matrix all four extra terms (the ones in (5.3.55) plus the ones

[5]Notice that, in the case of the fundamental representation, these symmetries differ from (5.3.35) for the addition of precisely the above mentioned bosonic terms. However, these terms are by themselves a symmetry of the S-matrix in the fundamental representation, and can therefore always be added to the coproduct.

with indices $1 \leftrightarrow 3$ interchanged) would then possibly be included, and the different representations would only see a part of them survive. Alternatively, we would also like to point the reader to the asymmetry between the indices $1, 2$ (corresponding to bosons) and $3, 4$ (corresponding to fermions) in (5.3.39), meaning that these bosonic terms could also be an artifact of the choice of Dynkin diagram. It would be interesting to gain a better understanding of the origin of this discrepancy.

Finally we note that $\Delta(B^{\pm}_F)$ is related to $\Delta(B^{\pm}_E)$ by renaming $E'_i \mapsto F_i$ and transposing the ordering $K_i A \mapsto A K_i$, where A represents any ad_r-type operator, thus $E_i \mapsto F_i K_i^{-1}$.

Restriction to the $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ subsectors. The bound state representations of $\widehat{\mathcal{Q}}$ furnished by the vectors

$$|m, n, k, l\rangle = (\mathsf{a}_3^\dagger)^m (\mathsf{a}_4^\dagger)^n (\mathsf{a}_1^\dagger)^k (\mathsf{a}_2^\dagger)^l \, |0\rangle \,, \tag{5.3.56}$$

have four $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$-invariant subsectors. These subsectors are spanned by the vectors

$$|m, 0, k, 0\rangle_{\mathrm{I}}, \quad |0, n, 0, l\rangle_{\mathrm{II}}, \quad |0, n, k, 0\rangle_{\mathrm{III}}, \quad |m, 0, 0, l\rangle_{\mathrm{IV}}, \tag{5.3.57}$$

where Roman subscripts enumerate the different subsectors. Each of these subsectors is isomorphic to the bound state representations of the superalgebra $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$ considered in section 5.3.1. They lead to four independent copies of the corresponding bound state $\mathrm{S}_{1|1}$-matrix embedded into the (complete) bound state S-matrix. Thus one can introduce a formal restriction of the coproducts (5.3.55) onto the $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$-invariant subsectors,

$$\begin{aligned}
\Delta(B^+_F)\Big|_{\mathrm{A}} &= B^+_F \otimes 1 + 1 \otimes B^+_F \\
&\quad - 2\,\alpha\tilde{\alpha}\Big(\delta_{\mathrm{A,I}}\, U^{-1} F_4 \otimes (\mathrm{ad}_r F_3 F_2) F_1 K_4 + \delta_{\mathrm{A,II}}\, U^{-1} (\mathrm{ad}_r F_1 F_4) F_3 \otimes F_2 K_2^{-1} \\
&\quad + \delta_{\mathrm{A,III}}\, U^{-1} (\mathrm{ad}_r F_1) F_4 \otimes (\mathrm{ad}_r F_3) F_2 K_{14} + \delta_{\mathrm{A,IV}}\, U^{-1} (\mathrm{ad}_r F_4) F_3 \otimes K_{43} (\mathrm{ad}_r F_2) F_1\Big), \\
\Delta(B^+_E)\Big|_{\mathrm{A}} &= B^+_E \otimes 1 + 1 \otimes B^+_E \\
&\quad - \frac{2}{\alpha\tilde{\alpha}}\Big(\delta_{\mathrm{A,I}}\, U E'_4 \otimes K_4 (\mathrm{ad}_r E_3 E_2) E'_1 + \delta_{\mathrm{A,II}}\, U (\mathrm{ad}_r E_1 E_4) E'_3 \otimes E_2 \\
&\quad + \delta_{\mathrm{A,III}}\, U (\mathrm{ad}_r E_1) E'_4 \otimes K_{14} (\mathrm{ad}_r E_3) E'_2 + \delta_{\mathrm{A,IV}}\, U (\mathrm{ad}_r E_4) E'_3 \otimes K_{43} (\mathrm{ad}_r E_2) E'_1\Big).
\end{aligned} \tag{5.3.58}$$

In this fashion, for each subsector we obtain charges equivalent to (5.3.16). The last two terms in the tails of (5.3.55) do not play any role in this case, as they vanish on these subsectors.

5.3.3.2 q-deformed AdS/CFT: the Secret symmetry

Having prepared all the suitable formulas, we can now come back to the full q-deformed AdS/CFT case. In the previous section we have explored the symmetries of the conventional affine limit of $\widehat{\mathcal{Q}}$ whose S-matrix is effectively isomorphic to the one of $\mathcal{U}_q(\widehat{\mathfrak{gl}}(2|2))$, thus the charges B_0 and $B^{\pm}_{E/F}$ are proper symmetries. The question we want to answer is whether any of these charges are symmetries of the bound state representations of $\widehat{\mathcal{Q}}$. Naturally, B_0 is not a symmetry. However we find that the charges $B^{\pm}_{E/F}$ are symmetries of $\widehat{\mathcal{Q}}$, upon a redefinition

$$\begin{aligned} B^+_F &= \frac{\tilde{g}g^{-1}[M]_q}{U^2 - V^{-2}}\, B_0\,, & B^+_E &= \frac{\tilde{g}g^{-1}[M]_q}{U^{-2} - V^{-2}}\, B_0\,, \\ B^-_F &= \frac{\tilde{g}g^{-1}[M]_q}{V^2 - U^{-2}}\, B_0\,, & B^-_E &= \frac{\tilde{g}g^{-1}[M]_q}{V^2 - U^2}\, B_0\,, \end{aligned} \tag{5.3.59}$$

while keeping the form of coproducts as in (5.3.55). We have checked numerically the intertwining property for these new symmetries for the bound states representations with the total bound state number up to $M_1 + M_2 \leq 5$. It is important to notice that in the conventional limit these charges exactly reduce to (5.3.53), and so they correspond to the natural lift of the conventional affine limit case to the generic representations of $\widehat{\mathcal{Q}}$.

This striking similarity between $B^{\pm}_{E/F}$ is not accidental. The charges B^+_E and B^+_F (and equivalently B^-_E and B^-_F) are related to each other by the map $U \mapsto U^{-1}$ and $E'_i \mapsto F_i$ (as described above) as this is the automorphism of the coalgebra which interchanges lowering and raising Chevalley generators. The relation between B^+_E and B^-_E (and equivalently B^+_F and B^-_F) corresponds to the algebra automorphism of flipping the nodes 2 and 4 of the Dynkin diagram and represents the symmetry between states (particles) and anti-states (anti-particles), i.e. the corresponding representations are self-adjoint. Thus $B^+_{E/F}$ and $B^-_{E/F}$ are not independent, rather two isomorphic representations of charges $B_{E/F}$.

An important difference between $\widehat{\mathcal{Q}}$ and its conventional affine limit is that the previously mentioned extra two 'bosonic' terms in (5.3.55) are not a symmetry of the fundamental S-matrix by themselves anymore and thus (5.3.55) is unique for all bound state representations. Another important difference is that the $\mathcal{U}_q(\widehat{\mathfrak{gl}}(1|1))$-invariant subsectors I and II and subsectors III and IV become entangled from the algebra point of view. This is because the generators $E_{2/4}$ and $F_{2/4}$ act non-trivially on two subsectors simultaneously, while in the conventional affine limit this was not the case (as it can easily be seen from (4.1.53), (4.1.54) and (5.3.46)). Therefore, the formal restriction in (5.3.58) needs to be modified by identifying the delta functions with indices I and II, and with indices III and IV.

Yangian limit. Finally, we can consider the rational limit of the symmetry we have just found. Accordingly, we write $q \sim 1 + h$ with $h \to 0$. In this limit the secret charges we have constructed become[6]

$$B^+_F = -B^-_E = \frac{M\, x^-}{x^+ - x^-}\, B_0 + \mathcal{O}(h)\,, \qquad B^+_E = -B^-_F = \frac{M\, x^+}{x^- - x^+}\, B_0 + \mathcal{O}(h)\,. \tag{5.3.60}$$

[6]The rational factor $(q^{-1} - q)^{-1}$ is already included in the definition of the charges, as one can easily trace back using (5.3.53) and (5.3.59).

Thus

$$\underset{q\to 1}{lim}\ \tfrac{1}{4}(B_E^+ - B_F^+) = \underset{q\to 1}{lim}\ \tfrac{1}{4}(B_E^- - B_F^-) = ig\, u_s\, B_0\,, \tag{5.3.61}$$

where $u_s = \frac{1}{4}(x^+ - \frac{1}{x^+} + x^- - \frac{1}{x^-})$ is the rapidity found for the secret symmetry [130]. Subsequently, at the coalgebra level we find

$$\underset{q\to 1}{lim}\ \tfrac{1}{4}\big(\Delta(B_E^+) - \Delta(B_F^+)\big) = \underset{q\to 1}{lim}\ \tfrac{1}{4}\big(\Delta(B_E^-) - \Delta(B_F^-)\big) = \Delta(\widehat{\mathbb{B}})\,, \tag{5.3.62}$$

where precisely coincides with the secret symmetry (5.1.2). We note that all the checks are done for the bound state representations only.

We remark that the outer-automorphism flipping roots 2 and 4, which leads to the doubling of the charges $B_{E/F} \to B^{\pm}_{E/F}$, turns out to be crucial in obtaining the secret Yangian charge $\widehat{\mathbb{B}}$. This is because the rational limit of the linear combinations $B_E^{\pm} - B_F^{\mp}$ corresponds instead to a bilinear combination of Lie algebra charges plus a central element.

Conclusions and Outlook

This manuscript has presented the research performed during the authors PhD studies, which were devoted to the exploration of quantum groups and integrable boundaries in AdS/CFT. The results obtained are threefold. First, there has been a solid contribution to the theory of quantum groups, in particular to the theory of reflection algebras. A new type of (generalized) twisted Yangians for boundaries preserving all of the bulk Lie algebra were constructed, and a new 'achiral' form of the (generalized) twisted Yangian was uncovered. The author has also generalized the theory of quantum symmetric pairs for quantum affine algebras which led to coideal subalgebras that can be thought of as quantum affine analogues of the aforementioned twisted Yangians.

Secondly, all of the above mentioned algebras were shown to play an important role in the worldsheet scattering in the AdS/CFT duality, which was shown to be very rich in integrable boundaries. The best known ones are the $D3$-, $D5$- and $D7$-branes. The corresponding boundary conditions depend crucially on the type of embedding of the D-brane into the $AdS_5 \times S^5$ background and the relative orientation of the brane and the open string attached to it. This leads to five different boundary conditions from the boundary scattering theory point of view. Each of these boundaries were shown to be integrable and the corresponding twisted Yangian algebras that govern (bound state) boundary scattering were constructed. A particularly important results are related to the $D5$-brane which was long thought not to be integrable [143,145,146]. Here it was shown that this boundary is of a specific 'achiral' type, and is indeed integrable. These results were later shown to play an important role in calculating the quark-antiquark potential (generalized cusp anomalous dimension) in AdS/CFT [205,206].

Thirdly, a quantum deformed approach to the AdS/CFT worldsheet scattering has been developed. A generic bound state representation of the quantum affine algebra of the Deformed Hubbard Chain [30] was constructed and the corresponding S-matrix was obtained. This S-matrix was shown to be a quantum deformed analogue of the AdS/CFT worldsheet S-matrix. Furthermore, the quantum deformed models of the $D3$- and $D7$-branes were considered and the corresponding boundary scattering theories were constructed. These were shown to obey coideal quantum affine subalgebras of the afore mentioned type. These coideal subalgebras, in contrast to their Yangian avatars, are of a very elegant and compact form.

The quantum deformed approach was also employed in search of the origins of the so-called 'secret' symmetry of the AdS/CFT, which appears as a level-one (Yangian) generator in the symmetry algebra without the level-zero (Lie algebra) analogue. It was shown that this symmetry in the quantum deformed model has two relatives, level-one and level-minus-one symmetries. This is in agreement with what was expected from the theory of quantum groups. However there are still quite a few mysteries related to this peculiar symmetry, – for example, if there are higher-level relatives of this symmetry. These higher-level symmetries are expected to play a role in the boundary scattering [184].

An important aspect of the quantum deformations is that they offer a quite different understanding of the system than the conventional approach. For example, the spectrum of bound states is limited from above when the deformation parameter q is a (higher-order) root of unity [207]. This leads to a finite number of mirror TBA equations [208]. Hence the quantum deformed approach offers a new and elegant approach to complex integrable systems.

To finalize we want to note that there are still quite a lot of open questions in the gauge/gravity dualities that are closely related to the results presented in this manuscript. It would be very interesting to explore boundary TBA and Y-systems for various boundary conditions along the lines of [141]. There has been very little work done in exploring boundary scattering in other backgrounds and dualities, e.g. in the AdS_3/CFT_2 and the ABJM models. Here a question of particular importance is the spectrum of the $\mathfrak{d}(2,1;\alpha)$ spin-chain. This algebra, in contrast to the $\mathfrak{psu}(2|2)_{\mathcal{C}}$, has a non-degenerate Cartan-Killing form and thus is better behaved. However the corresponding spin-chain has a much more complex structure [209], and the boundary scattering in this context has not been explored at all. As we have mentioned earlier, boundary scattering is a necessary component in calculating the quark-antiquark potential, which is currently emerging as a new mainstream topic in the exploration of dualities, and thus will require a good understanding of the boundary scattering. The methods presented in this manuscript offer a solid background for enhancing the exploration of the boundary effects in other backgrounds and gauge/gravity dualities.

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