On the non-vanishing of theta lifts of Bianchi modular forms to Siegel modular forms

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Submitted for the degree of
Doctor of Philosophy

School of Mathematics and Statistics
University of Sheffield
September 2020

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To my family and Dr Tobias Berger

Abstract

In this thesis we study the theta lifting of a weight 2 Bianchi modular form $\mathcal{F}$ of level $\Gamma_0(\mathfrak{n})$ with $\mathfrak{n}$ square-free to a weight 2 holomorphic Siegel modular form. Motivated by Prasanna's work for the Shintani lifting, we define the local Schwartz function at finite places using a quadratic Hecke character χ of square-free conductor $\mathfrak{f}$ coprime to level $\mathfrak{n}$. Then, at certain 2 by 2 Gram matrices β related to $\mathfrak{f}$, we can express the Fourier coefficient of this theta lifting as a multiple of $L(\mathcal{F}, \chi, 1)$ by a non-zero constant. If the twisted L-value is known to be non-vanishing, we can deduce the non-vanishing of our theta lifting.

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Introduction

Shimura initiated the systematic study of holomorphic modular forms of half-integral weight and provided a correspondence between certain modular forms of even weight and modular forms of half-integral weight. Later, in the other direction, Shintani [Shi75] described a method in terms of weighted periods of holomorphic cusp forms to construct modular forms of half-integral weight. Waldspurger showed in [Wal81] a proportional relation between special values of L-functions attached to an eigenform of even weight and the square of the square-free Fourier coefficients of the Shintani lifting. For the special case of modular forms on the full modular group, Kohnen-Zagier [KZ81] proved a simple version of Waldspurger's theorem with the constant of proportionality given explicitly. Inspired by their work we will analyse the theta lifting of Bianchi modular forms to Siegel modular forms and investigate the relationship between Fourier coefficients of this lifting and special L-values attached to the Bianchi modular forms. This can be used to describe the non-vanishing of the theta lifting which is an open problem in general.

Shintani's result can be recovered as a special case of the Kudla-Milson cohomological theta lifting (see [KM90], [Fun02] and [FM11]) which can be described as an integral of the exterior product of two cohomology classes over arithmetic quotients of the symmetric space attached to orthogonal groups. To construct the Shintani lifting of a weight 2 cusp form f for a congruence subgroup Γ, we can consider a 3-dimensional rational quadratic space V of signature (2,1) associated to which the symmetric space D is isomorphic to the upper half plane $\mathbb{H}_2$. By the Eichler-Shimura isomorphism, the cusp form f can be realised as a cohomology class represented by the differential form η_f on the arithmetic quotient $\Gamma\backslash D$. This is paired against the cohomological theta kernel form θ defined on the product of two locally symmetric spaces attached to $\widetilde{\mathrm{SL}_2} \times \mathrm{SO}(2,1)$ where $\widetilde{\mathrm{SL}_2}$ denotes the metaplectic group of SL_2. Given a cohomological Schwartz form φ on V with the infinity part defined by Kudla-Milson, namely

$\varphi_\infty = \varphi_\infty^{\mathrm{KM}}$, and the usual Weil representation ω, the theta kernel is defined as

$$\theta(g,h,z) := \sum_{\mathbf{x}\in V} \omega(g,h)\varphi(\mathbf{x},z) \quad \text{for } g \in \widetilde{\mathrm{SL}_2}, h \in \mathrm{SO}(2,1), z \in D.$$

Then the theta lifting, which is a weight 3/2 cusp form, is given by

$$\Theta_\varphi(\eta_f)(g) = \int_{\Gamma\backslash D} f(z)dz \wedge \theta(g,h,z).$$

Further results of Kudla-Milson [KM90] and Funke [FM02] imply an interpretation of its Fourier coefficients as period integrals over certain cycles C. More explicitly, the coefficient at $\beta > 0$ is given by

$$\sum_{\mathbf{x}\in\Gamma\backslash\Omega_\beta} \varphi_f^\chi(\mathbf{x}) \int_{C_\mathbf{x}} f(z)dz$$

where $\Omega_\beta := \{\mathbf{x} \in V : -\det(\mathbf{x}) = \beta\}$. For an auxiliary quadratic character χ we here take $\varphi = \varphi_\infty\varphi_f^\chi$ with the finite part φ_f^χ almost the same as that defined in [Pra09]. Then the coefficient of $\Theta_\varphi(\eta_f)$ at certain β (depending on the conductor of χ) can be expressed as the above weighted sum of period integrals over infinite geodesics. In Section 3.3.1, the period integral over the infinite geodesic with one end point at the cusp ∞ can be related to $L(f,\chi,1)$. Those over other infinite geodesics can be transformed by Atkin-Lehner operators to be over infinite geodesics ending in ∞. Eventually the Fourier coefficient at β turns out to be a multiple of $L(f,\chi,1)$ by a non-vanishing constant. If the twisted L-value is known to be non-vanishing then so is this Shintani lifting.

Proposition 0.0.1 (Proposition 3.3.5)**.** *Let f be a weight 2 cusp form of level $\Gamma_0(N)$ with odd square-free N and χ_m a quadratic Dirichlet character of odd square-free conductor m such that $(m,N)=1$. Then, at m^2 the Fourier coefficient of the theta lifting can be expressed as*

$$(*) \cdot L(f,\chi_m,1)$$

where the non-zero factor $(*)$ is given explicitly in (3.7).

To construct the theta lifting of a weight 2 Bianchi modular form $\mathcal{F}$ for level $\Gamma_0(\mathfrak{n})$ with $\mathfrak{n}$ a square-free ideal for an imaginary quadratic field F, following [KM90] and [Ber14] we consider the 4-dimensional rational quadratic space V given by Hermitian matrices with entries in F. Its associated symmetric space D is isomorphic to the upper half space $\mathbb{H}_3$. In our theta integral we use the differential form $\eta_\mathcal{F}$ attached to $\mathcal{F}$ defined on the arithmetic quotient $\Gamma\backslash D$. Different to the Shintani case we choose the Schwartz form $\varphi = \varphi_\infty^{\mathrm{KM}}\varphi_f$ defined on a pair of vectors in V so that the theta kernel is given by

$$\theta(g,h,z) := \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in V^2} \omega(g,h)\varphi(\mathbf{x}_1,\mathbf{x}_2;z) \quad \text{for } g \in \mathrm{Sp}_4 \subset \mathrm{GL}_4, h \in \mathrm{SO}(3,1).$$

Then the theta lifting is constructed as

$$\Theta_\varphi(\eta_{\mathcal{F}})(g) = \int_{\Gamma\backslash D} \eta_{\mathcal{F}}(z) \wedge \theta(g,h,z)$$

which turns out to be a weight 2 holomorphic Siegel modular form. To calculate its Fourier coefficient at a 2×2 symmetric matrix $\beta > 0$, given here by

$$\sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma\backslash\Omega_\beta} \varphi_f(\mathbf{x}_1,\mathbf{x}_2) \int_{C_{U(\mathbf{x}_1,\mathbf{x}_2)}} \eta_{\mathcal{F}}$$

where $\Omega_\beta := \{(\mathbf{x}_1,\mathbf{x}_2) \in V^2 : \frac{1}{2}((\mathbf{x}_i,\mathbf{x}_j)) = \beta\}$ and $U(\mathbf{x}_1,\mathbf{x}_2) := \mathrm{Span}\{\mathbf{x}_1,\mathbf{x}_2\} \subset V$. For an auxiliary quadratic Hecke character χ with its conductor coprime to $\mathfrak{n}$ we define the Schwartz form as $\varphi = \varphi_\infty^{\mathrm{KM}}\varphi_f^\chi$. The choice of the Schwartz function φ_f^χ in Section 4.3 is crucial for us to get the period integral related to some twisted L-values. With this choice we take certain $\beta > 0$ (again depending on the conductor of χ) at which the coefficient of $\Theta_\varphi(\eta_{\mathcal{F}})$ is expressed as the above weighted sum of period integrals over infinite geodesics joining two cusps. By Theorem 1.4.12, the period integral over infinite geodesics ending in ∞ can be related to $L(\mathcal{F},\chi,1)$. We apply Atkin-Lehner operators to transform other infinite geodesics so as to obtain the period integral over geodesic lines through ∞. In Section 4.4 I compute the coefficient at such a β as a multiple of $L(\mathcal{F},\chi,1)$ by a non-vanishing number. By Friedberg-Hoffstein's theorem [FH95, Theorem B], we can deduce that there always exists a character χ such that the twisted L-value is non-vanishing which implies the non-vanishing of the corresponding theta lifting.

Theorem 0.0.2 (Theorem 4.4.17)**.** *Let $F = \mathbb{Q}(\sqrt{d})$ (square-free $d < 0$) be an imaginary quadratic field of class number one and denote by $\mathcal{O}$ its ring of integers. Consider a weight 2 Bianchi cusp form $\mathcal{F}$ of level $\Gamma_0(\mathfrak{n})$ with $\mathfrak{n}$ a square-free ideal away from ramified primes in $F/\mathbb{Q}$. Given a square-free product m of split or inert primes in $F/\mathbb{Q}$ such that $(m,\mathfrak{n}) = 1$, we choose a quadratic Hecke character $\chi_{\mathfrak{m}}$ of conductor $m\sqrt{d}\mathcal{O}$. Then, at certain $\beta > 0$ related to m, the Fourier coefficient of the theta lifting can be computed as*

$$(*)\cdot L(\mathcal{F},\chi_{\mathfrak{m}},1)$$

where the non-zero constant $(*)$ is given explicitly in (4.35).

Chapter 1

Background

§ 1.1 Locally symmetric spaces

Locally symmetric spaces arise in different areas such as differential geometry, number theory, automorphic forms and representation theory. The most important class consists of quotients of symmetric spaces by arithmetic groups, for example, the quotient of the upper half plane $\mathbb{H}_2$ by $\mathrm{SL}_2(\mathbb{Z})$. In this section we review locally symmetric spaces and arithmetic groups based on Ji's lecture notes [Ji].

Let M be a complete Riemannian manifold. For any point $x \in M$, there exists a neighbourhood U such that

(1) every point in U is connected to x by a unique geodesic,
(2) there exists a star-shaped domain $V \subset T_x M$ containing the origin 0 and symmetric with respect to 0 such that the exponential map $\exp : V \to U$ is a diffeomorphism.

On such a neighbourhood U, there is a geodesic symmetry s_x defined by reversing geodesics passing through x, i.e., for any geodesic $\gamma(t)$, $t \in \mathbb{R}$, with $\gamma(0) = x$,

$$s_x(\gamma(t)) = \gamma(-t),$$

when $\gamma(t) \in U$. Since $s_x \neq \mathrm{Id}$ and $s_x^2 = \mathrm{Id}$, s_x is involutive and called the local geodesic symmetry at x.

Definition 1.1.1. (1) A complete Riemannian manifold M is called locally symmetric if for any $x \in M$, the (local) geodesic symmetry s_x is a local isometry.

(2) The manifold M is called a symmetric space if it is locally symmetric and every local isometry s_x extends to a global isometry of M.

If M is symmetric, then for all values of t, we have $s_x(\gamma(t)) = \gamma(-t)$. Clearly, symmetric spaces are also locally symmetric spaces. If M is a locally symmetric space, then its universal covering space $X = \widetilde{M}$ with the lifted Riemannnian metric is symmetric. The fundamental group $\Gamma = \pi_1(M)$ of M acts isometrically and properly, and $M = \Gamma\backslash X$. So locally symmetric spaces are quotients of symmetric spaces.

Denote by $G = \mathrm{Is}_0(X)$ the identity component of the isometry group $\mathrm{Is}(X)$ of the symmetric space X. It is well known that if X is a symmetric space then G is a Lie group and acts transitively on X, see [Ji, Proposition 2.4]. Fix a base point $x_0 \in X$ and the stabilizer of x_0 in G is denoted by

$$K = \{g \in G : g \cdot x_0 = x_0\}.$$

Then K is a compact subgroup of G and we have

$$G/K \simeq X, \quad gK \mapsto gx_0.$$

The fundamental group Γ acts isometrically on X and is a discrete subgroup of G. So any locally symmetric space M is of the form

$$M = \Gamma\backslash G/K.$$

Therefore, each locally symmetric space determines a triple (G, K, Γ).

We can reverse the above process to construct locally symmetric spaces. If G is a connected non-compact semisimple Lie group and $K \subset G$ a maximal compact subgroup, then endowed with a G-invariant metric $X = G/K$ is symmetric space. Any torsion free discrete subgroup Γ of G acts isometrically on X and the quotient $\Gamma\backslash X$ is a locally symmetric space. Such discrete groups Γ are often constructed via algebraic groups, e.g. arithmetic groups which will be defined in the following. More generally Γ can be any discrete subgroup of G, not necessarily torsion free. Since many natural important arithmetic subgroups such as $\mathrm{SL}_2(\mathbb{Z})$ are not torsion free, $\Gamma\backslash X$ is also called a locally symmetric space for a non-torsion free discrete subgroup Γ.

Example 1.1.2. Consider

$$G = \mathrm{SL}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{R}, ad - bc = 1 \right\},$$

$$K = \mathrm{SO}(2) = \left\{ \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} : \theta \in \mathbb{R} \right\}$$

and

$$\Gamma = \mathrm{SL}_2(\mathbb{Z}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, c, d \in \mathbb{Z}, ad - bc = 1 \right\}.$$

The modular group Γ is not torsion free since for example

$$\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq \mathrm{Id} \ \text{ but } \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^4 = \mathrm{Id}\,.$$

Let the upper half plane

$$\mathbb{H}_2 = \{z = x + iy \in \mathbb{C} : x \in \mathbb{R}, y > 0\}$$

with metric $ds^2 = \frac{1}{y^2}(dx^2 + dy^2)$. The group $\mathrm{SL}_2(\mathbb{R})$ acts isometrically and holomorphically on $\mathbb{H}_2$ via fractional linear transformation

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot z = \frac{az + b}{cz + d}.$$

By this action we can show that $\mathbb{H}_2$ is a symmetric space. The geodesic symmetry at the base point $x_0 = i$ is given by

$$s_i(z) = -1/\bar{z} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \cdot z$$

and is an isometry of $\mathbb{H}$. Under the conjugation by elements in $\mathrm{SL}_2(\mathbb{R})$, it follows that for any point $x \in \mathbb{H}_2$ the geodesic symmetry s_x is an isometry. The stabilizer of $x_0 = i$ is $K = \mathrm{SO}(2)$, and hence

$$X = \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2) \simeq \mathbb{H}_2, \quad g\,\mathrm{SO}(2) \mapsto gi$$

and a locally symmetric space is $\Gamma\backslash\mathbb{H}_2$.

A variety $\mathbf{G}$ over a field k is called an algebraic group if it is also a group and the group operations

$$\lambda : \mathbf{G} \times \mathbf{G} \to \mathbf{G}, \quad (g_1, g_2) \mapsto g_1 g_2,$$
$$\mu : \mathbf{G} \to \mathbf{G}, \quad g \mapsto g^{-1}$$

are morphisms of varieties. We are interested in linear algebraic groups. The first example of those is $\mathrm{GL}_n(k)$ which is contained in the affine space of $M_{n\times n}(k) \simeq k^{n^2}$. It can be realised as an affine variety via the embedding

$$\mathrm{GL}_n(k) \to M_{n\times n}(k) \times k = k^{n^2+1}, \quad (g_{ij}) \mapsto ((g_{ij}), (\det(g_{ij}))^{-1}).$$

Let (X_{ij}, Z) be the coordinates of $M_{n\times n}(k) \times k$. Then the image is the affine hypersurface defined by $\det(X_{ij})Z = 1$ which is a polynomial in X_{ij} and Z. One can check that the group actions on $\mathrm{GL}_n(k)$ are given by polynomials in X_{ij} and Z. So $\mathrm{GL}_n(k)$

is an affine algebraic variety.

Linear algebraic groups often occur as the automorphism groups of some structures such as determinant and quadratic forms.

Example 1.1.3.

- Special linear group
$$\mathrm{SL}_n(k) = \{g \in \mathrm{GL}_n(k) : \det(g) = 1\}.$$
- Symplectic group
$$\mathrm{Sp}_{2n}(k) = \{g \in \mathrm{GL}_{2n}(k) : \det(g) = 1; F(gX, gY) = F(X, Y) \text{ for } X, Y \in k^{2n}\}$$
where
$$F(X, Y) = x_1 y_{2n} + x_2 y_{2n-1} + \cdots + x_n y_{n+1} - x_{n+1} y_n - \cdots - x_{2n} y_1$$
is a skew-symmetric form. Note that $\mathrm{Sp}_2 = \mathrm{SL}_2$.

Example 1.1.4. Special orthogonal group

- Let
$$\mathrm{SO}(m, n) = \{g \in \mathrm{SL}_{m+n}(k) : g^t I_{m,n} g = I_{m,n}\}$$
where $I_{m,n}$ stands for a diagonal matrix whose diagonal entries are m +1's followed by n -1's. Denote $\mathrm{SO}(n, 0)$ (or $\mathrm{SO}(0, n)$ which is the same group) by $\mathrm{SO}(n)$.
- A symmetric bilinear form B on k^n is non-degenerate if for all nonzero $v \in k^n$ there exists $w \in k^n$ such that $B(v, w) \neq 0$. Define
$$\mathrm{SO}(B, k) = \{g \in \mathrm{SL}_n(k) : B(gv, gw) = B(v, w) \text{ for all } v, w \in k^n\}.$$
- A quadratic form on k^n is homogeneous polynomial $Q(x_1, \ldots, x_n)$ of degree 2. It is non-degenerate if the corresponding bilinear form is non-degenerate. Define
$$\mathrm{SO}(Q, k) = \{g \in \mathrm{SL}_n(k) : Q(gv) = Q(v) \text{ for all } v, w \in k^n\}.$$

These three approaches give rise to the same groups $\mathrm{SO}(m, n)$.

With the transitive action of $\mathrm{SL}_2(\mathbb{R})$ on the upper half plane $\mathbb{H}_2$ by fractional linear transformation, we have seen that $\mathbb{H}_2 \simeq \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}(2)$. This has a higher-dimensional generalisation, the Siegel upper half space, consisting of symmetric complex $n \times n$ matrices with positive definite imaginary part:
$$\mathcal{H}_n := \{Z = X + iY : X, Y \in M_{n\times n}(\mathbb{R}) : X^t = X, Y^t = Y, Y > 0\}$$
(in particular, $\mathcal{H}_1 = \mathbb{H}_2$ is the upper half plane). We may see this as open subset of $\mathbb{C}^{n(n+1)/2}$ by sending a matrix (z_{ij}) to the point $(z_{ij})_{i\leq j} \in \mathbb{C}^{n(n+1)/2}$, so there is a natural complex structure.

Recall that the symplectic group can also be defined as

$$\mathrm{Sp}_{2n}(\mathbb{R}) := \left\{ g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in M_{2n\times 2n}(\mathbb{R}) : g^t \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} g = \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix} \right\}$$

and it acts transitively on $\mathcal{H}_n$ by

$$g \cdot Z = (AZ+B)(CZ+D)^{-1} \quad \text{for } g \in \mathrm{Sp}_{2n}, Z \in \mathcal{H}_n.$$

We have a subgroup of unitary matrices

$$U(n) = \{Z \in M_{n\times n}(\mathbb{C}) : \bar{Z}^t Z = Z\bar{Z}^t = I_n\}$$

(in particular, $SO(2) \simeq U(1)$). This can can be identified with a subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ via

$$X + iY \mapsto \begin{pmatrix} X & Y \\ -Y & X \end{pmatrix}$$

and is a maximal compact subgroup of $\mathrm{Sp}_{2n}(\mathbb{R})$ stabilising the base point iI_n in $\mathcal{H}_n$. In fact we have as a symmetric space

$$\mathcal{H}_n \simeq \mathrm{Sp}_{2n}(\mathbb{R})/U(n).$$

A linear algebraic group $\mathbf{G}$ is said to be defined over $\mathbb{Q}$ if the polynomials defining $\mathbf{G}$ as a subvariety have coefficients in $\mathbb{Q}$. Let $\mathbf{G} \subset \mathrm{GL}_n(\mathbb{C})$ be a linear algebraic group defined over $\mathbb{Q}$, $\mathbf{G}(\mathbb{Q}) \subset \mathrm{GL}_n(\mathbb{Q})$ the set of its rational elements and $\mathbf{G}(\mathbb{Z}) \subset \mathrm{GL}_n(\mathbb{Z})$ the set of its elements with integral entries. A subgroup $\Gamma \subset \mathbf{G}(\mathbb{Q})$ is called an arithmetic subgroup if it is commensurable to $\mathbf{G}(\mathbb{Z})$, i.e. $\Gamma \cap \mathbf{G}(\mathbb{Z})$ has finite index in both Γ and $\mathbf{G}(\mathbb{Z})$. As an abstract affine algebraic group defined over $\mathbb{Q}$, $\mathbf{G}$ admits different embeddings into $\mathrm{GL}_{n'}(\mathbb{C})$ (n' might different from n). Choosing a different embedding, we will get a different integral subgroup $\mathbf{G}(\mathbb{Z})$. One can show that these different embeddings and different choices of integral structures lead to the same class of arithmetic groups.

For the following discussion, we need a bit more general set-up of arithmetic groups, see [Ji, Section 4]. Let F be a number field and $\mathcal{O}_F$ its ring of integers. Let $\mathbf{G} \subset \mathrm{GL}_n(\mathbb{C})$ be a linear algebraic group defined over F. A subgroup $\Gamma \subset \mathbf{G}(F)$ is called arithmetic if it is commensurable to $\mathbf{G}(\mathcal{O}_F) = \mathbf{G} \cap \mathrm{GL}_n(\mathcal{O}_F)$. In fact, by the functor of restriction of scalars, there is an algebraic group $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ defined over $\mathbb{Q}$ such that $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}(\mathbb{Q}) = \mathbf{G}(F)$ and $\mathbf{G}(\mathcal{O}_F)$ is commensurable to $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}(\mathbb{Z})$ under this identification. If $[F : \mathbb{Q}] = r$, then $\mathrm{Res}_{F/\mathbb{Q}}\mathbf{G}$ is a linear subgroup of $\mathrm{GL}_{nr}(\mathbb{C}) \subset M_{nr\times nr}(\mathbb{C})$.

Example 1.1.5. Consider the upper half space, a model of 3-dimensional hyperbolic space which closely resembles the upper half plane,

$$\mathbb{H}_3 := \{(z, r) : z = x + iy \in \mathbb{C}, r > 0\}.$$

The notation for points in $\mathbb{H}_3$ is

$$P = (z, r) = (x, y, r) = z + rj \quad \text{where } j = (0, 0, 1).$$

We equip $\mathbb{H}_3$ with the hyperbolic metric coming from the line element

$$ds^2 = \frac{dx^2 + dy^2 + dr^2}{r^2}.$$

Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field with square-free $d < 0$. Then $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_2$ is defined over $\mathbb{Q}$ and $\mathrm{Res}_{F/\mathbb{Q}} \mathrm{SL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{C})$. For $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C})$ and $P \in \mathbb{H}_3$, the action of g on $\mathbb{H}_3$ is given by

$$P \mapsto g \cdot P := (aP + b)(cP + d)^{-1}$$

where the inverse is taken in the skew field of quaternions. More explicitly, writing $g \cdot (z + rj) = z' + r'j$, we have

$$z' = \frac{(az + b)(\bar{c}\bar{z}) + \bar{d} + a\bar{c}r^2}{|cz + d|^2 + |c|^2 r^2} \quad \text{and} \quad r' = \frac{r}{|cz + d|^2 + |c|^2 r^2}.$$

The stabilizer of j with respect to this action is $\mathrm{SU}(2) = \{g : g \in \mathrm{U}(2), \det(g) = 1\}$ which is one of maximal compact subgroups of $\mathrm{SL}_2(\mathbb{C})$. Then the symmetric space associated to $\mathrm{SL}_2(\mathbb{C})$ is $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$ which can be realised as $\mathbb{H}_3$. Here we have the map

$$\mathrm{SL}_2(\mathbb{C}) \to \mathbb{H}_3, \quad g \mapsto g \cdot j$$

which gives rise to an $\mathrm{SL}_2(\mathbb{C})$-equivariant bijection between the symmetric space of $\mathrm{SL}_2(\mathbb{C})$ and $\mathbb{H}_3$.

The arithmetic subgroup $\mathrm{SL}_2(\mathcal{O}_F)$ is a discrete subgroup of $\mathrm{SL}_2(\mathbb{C})$ and called the Bianchi group. For $\Gamma \subset \mathrm{SL}_2(\mathcal{O}_F)$, the quotient $\Gamma \backslash \mathbb{H}_3$ is a typical non-compact arithmetic 3-dimensional hyperbolic manifold which is a locally symmetric space.

Let V be a rational vector space of dimension $m = p + q$ with a non-degenerate symmetric bilinear form $(\ ,\ )$ of signature (p, q). Let $\underline{G} = \mathrm{SO}(V)$ and $G(p, q) = \underline{G}_0(\mathbb{R}) \simeq \mathrm{SO}_0(p, q)$ be the connected identity component of the real points of $\underline{G}$. The associated symmetric space is given by $D = G(p, q)/(G(p) \times G(q))$. It can be identified with the space of negative q-planes Z in $V(\mathbb{R})$ on which the bilinear form $(\ ,\ )$ is negative definite:

$$\mathrm{Gr}_q = \{Z \subset V(\mathbb{R}) : \dim Z = q \text{ and } (\ ,\ )|_Z < 0\},$$

see e.g. [KM90, Section 2].

Example 1.1.6. Here we assume the exceptional isomorphism, e.g. $\mathrm{PSL}_2(\mathbb{R}) \simeq G(2,1)$ and $\mathrm{PSL}_2(\mathbb{C}) \simeq G(3,1)$, which we will discuss in details in our next Section 1.2.

- For signature (2,1), over $\mathbb{R}$ we fix an isomorphism
$$V(\mathbb{R}) \simeq \left\{ \mathbf{x} = \begin{pmatrix} b & a \\ c & -b \end{pmatrix} \in M_{2\times 2}(\mathbb{R}) \right\}$$
with quadratic form $q(\mathbf{x}) = -\det(\mathbf{x})$ and corresponding bilinear form $(\mathbf{x}, \mathbf{y}) = \mathrm{tr}(\mathbf{x}, \mathbf{y})$. The group $G(2,1)$ acts isometrically on $V(\mathbb{R})$ by $\mathbf{x} \mapsto g \cdot \mathbf{x} := g\mathbf{x}g^{-1}$ for $g \in G(2,1)$. We fix an orthonormal basis of $V(\mathbb{R})$ given by $e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ and $e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Pick the line Z_0 spanned by e_3, the base point of D with stabilizer $G(2)$. Then we have the isomorphism
$$\mathbb{H}_2 \to D = G(2,1)/G(2) \to \mathrm{Gr}_1$$
with
$$z = x + iy \in \mathbb{H}_2 \mapsto gG(2) \mapsto \mathbb{R} \cdot g \cdot Z_0 =: l(z)$$
where $g \in \mathrm{PSL}_2(\mathbb{R}) \simeq G(2,1)$ such that $g \cdot i = z$, e.g. $g = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{y} & 0 \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix} = \begin{pmatrix} \sqrt{y} & \frac{x}{\sqrt{y}} \\ 0 & \frac{1}{\sqrt{y}} \end{pmatrix}$. We see that $l(z)$ is generated by
$$\mathbf{x}(z) := y^{-1} \begin{pmatrix} -x & x^2 + y^2 \\ -1 & x \end{pmatrix}$$
with $q(\mathbf{x}(z)) = -1$.
- For signature (3,1), over $\mathbb{R}$ we fix an isomorphism
$$V(\mathbb{R}) \simeq \left\{ \mathbf{x} \in M_{2\times 2}(\mathbb{C}) : \mathbf{x}^t = \bar{\mathbf{x}} \right\}$$
with quadratic form $q(\mathbf{x}) = -\det(\mathbf{x})$ and corresponding bilinear form $(\mathbf{x}, \mathbf{y}) = \mathrm{tr}(\mathbf{x}\mathbf{y}^*)$ where $\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. The group $G(3,1)$ acts isometrically on $V(\mathbb{R})$ by $g \cdot \mathbf{x} = g\mathbf{x}\bar{g}^t$ for $g \in G(3,1)$. We fix an orthonomal basis of $V(\mathbb{R})$ given by $e_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $e_3 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$ and $e_4 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. Pick the line Z_0 spanned by e_4, the base point of D with stabilizer $G(3)$. Then we have the isomorphism
$$\mathbb{H}_3 \to D = G(3,1)/G(3) \to \mathrm{Gr}_1$$

with

$$P = z + rj \in \mathbb{H}_3 \mapsto gG(3) \mapsto \mathbb{R}\cdot g \cdot Z_0 =: l(P)$$

where $g \in \mathrm{PSL}_2(\mathbb{C}) \simeq G(3,1)$ such that $g \cdot j = P$, e.g. $g = \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{r} & 0 \\ 0 & \frac{1}{\sqrt{r}} \end{pmatrix} = \begin{pmatrix} \sqrt{r} & \frac{z}{\sqrt{r}} \\ 0 & \frac{1}{\sqrt{r}} \end{pmatrix}$. We can calculate that $l(z)$ is generated by

$$\mathbf{x}(P) := r^{-1}\begin{pmatrix} |z|^2 + r^2 & z \\ \bar{z} & 1 \end{pmatrix}$$

with $q(\mathbf{x}(P)) = -1$.

§ 1.2 The exceptional isomorphism

Given a quadratic space V over a field K of characteristic not 2, we have an exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}(V) \xrightarrow{\Lambda} \mathrm{SO}(V) \to K^\times/(K^\times)^2,$$

see [Hah04, Theorem 7]. Here Spin stands for the spin group associated to this quadratic space which will be defined later. So Λ induces an isomorphism

$$\mathrm{Spin}(V)/\{\pm 1\} \simeq \mathrm{SO}^+(V)$$

where $\mathrm{SO}^+(V) := \mathrm{Im}(\Lambda)$. In the case of $\dim V = 3$, we have $\mathrm{Spin}_3 \simeq \mathrm{SL}_2$ which implies the exceptional isomorphism $\mathrm{PSL}_2 \simeq \mathrm{SO}^+(2,1)$, and furthermore if $K = \mathbb{R}$, we obtain

$$\mathrm{PSL}_2(\mathbb{R}) \simeq \mathrm{SO}_3^+(V(\mathbb{R})) = \mathrm{SO}_0(V(\mathbb{R})). \tag{1.1}$$

In the following, we shall describe the construction of an isomorphism between PSL_2 and $\mathrm{SO}^+(3,1)$ in the case of $\dim V = 4$, see [EGM98, Section 1.3].

We start off with certain facts about Clifford algebras. Let K be a field of characteristic not equal to 2 and V an n-dimensional K-vector space. Suppose that $Q : V \to K$ is a non-degenerate quadratic form with associated symmetric bilinear form $B : V \times V \to K$, that is

$$B(x,y) = Q(x+y) - Q(x) - Q(y),$$
$$Q(x) = \frac{1}{2}B(x,x).$$

Denote by $T(V)$ the tensor algebra of V and by $\mathfrak{a}_Q$ its two-sided ideal generated by the elements $x \otimes y + y \otimes x - B(x,y)$ where x, y run through the elements of V. Define the Clifford algebra of Q to be the quotient $\mathcal{C}(Q) := T(V)/\mathfrak{a}_Q$. The field K and the vector space V inject into $\mathcal{C}(Q)$ with their canonical images. Let $e_1, \ldots, e_n$ be an orthogonal basis of V with respect to Q. Then we have in $\mathcal{C}(Q)$:

$$e_i^2 = Q(e_i) \quad \text{and} \quad e_i \cdot e_j = -e_j \cdot e_i.$$

Let $\mathcal{P}$ be the set of subsets of $\{1, \ldots, n\}$. For $M = \{i_1, \ldots, i_r\} \in \mathcal{P}$ with $i_1 < \cdots < i_r$, we define

$$e_M := e_{i_1} \cdot \cdots \cdot e_{i_r}$$

with the convention $e_\emptyset = 1$. Then the 2^n elements e_M make a vector basis of $\mathcal{C}(Q)$. For $M, N \in \mathcal{P}_n$ the product $e_M \cdot e_N$ can be calculated explicitly as a scalar factor times an appropriate e_L. The Clifford algebra $\mathcal{C}(Q)$ has main anti-involution $*$ and a main involution $'$ commuting with $*$ acting on e_M given by:

$$e_M^* = (-1)^{\frac{r(r-1)}{2}} e_M \cdot e_M \quad \text{and} \quad e_M' = (-1)^r \cdot e_M$$

where r stands for the cardinality of M. The span of the elements e_M with M of even cardinality is a subalgebra called $\mathcal{C}^+(Q)$.

Let us consider a particular example as discussed [EGM98, Section 1.3]. For non-zero $\epsilon \in K$ let $V_\epsilon = K \cdot f_3$ be the one-dimensional K-vector space with basis f_3. The quadratic form Q_ϵ on E_ϵ is given by $Q_\epsilon(f_3) = -\epsilon$. Then the Clifford algebra $\mathcal{C}(Q_\epsilon)$ is two-dimensional and commutative. In case $-\epsilon \in K^{\times 2}$ the K-algebra $\mathcal{C}(Q_\epsilon)$ is isomorphic to $K \times K$, if not then $\mathcal{C}(Q_\epsilon)$ is a quadratic extension of K. Here $K^{\times 2}$ denotes the subgroup of squares in $K^\times$. In case $K = \mathbb{R}$ and $\epsilon = 1$ we call f_3 also i, which makes the identification $\mathcal{C}(Q_1) = \mathbb{C}$.

We now fix a 3-dimensional vector space E_0 with basis f_0, f_1, f_2 and the quadratic form Q_0 on it is given by

$$Q_0(y_0 f_0 + y_1 f_1 + y_2 f_2) = Q_0(y_0, y_1, y_2) = y_0^2 + y_1^2 + y_2^2.$$

It is well known that $\mathcal{C}^+(Q_0) = M_2(K)$, the algebra of 2 by 2 matrices over K. An isomorphism between these two algebras can be constructed as follows. We define the elements of $\mathcal{C}(Q_0)$ depending on λ_i:

$$\tau_0 = \frac{1}{2}(f_0 + f_1), \tau_1 = \frac{1}{2}(f_0 - f_1),$$
$$u = \tau_1 \tau_0 = \frac{1}{2} f_0 f_1, w_1 = \tau_1 f_2 = \frac{1}{2}(f_0 f_2 - f_1 f_2),$$
$$w_0 = \tau_0 f_2 = \frac{1}{2}(f_0 f_2 + f_1 f_2), v = \tau_0 \tau_1 = -\frac{1}{2} f_0 f_1.$$

Then the map

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \mapsto u, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \mapsto w_1, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \mapsto w_0, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \mapsto v$$

extends to an algebra isomorphism

$$\psi : M_2(K) \longrightarrow \mathcal{C}^+(Q_0).$$

We have

$$\psi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)^* = \psi\left(\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}\right).$$

The following construction will be important for our definition of the exceptional isomorphism.

Consider the K-vector space $\widetilde{V}_\epsilon := V_0 \oplus V_\epsilon$ with quadratic form $\widetilde{Q}_\epsilon := Q_0 \perp Q_\epsilon$. We define the map $\bullet : V_\epsilon \to \mathcal{C}^+(\widetilde{Q}_\epsilon)$ with $\dot{x} := f_0 f_1 f_2 \cdot x$ which extends to an injective K-algebra homomorphism $\bullet : \mathcal{C}(Q_\epsilon) \to \mathcal{C}^+(\widetilde{Q}_\epsilon)$. This map commutes with the anti-automorphism *. Then the map $\psi : M_2(\mathcal{C}(Q_\epsilon)) \to \mathcal{C}^+(\widetilde{Q}_\epsilon)$,

$$\psi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) := \dot{\alpha} u + \dot{\beta} w_1 + \dot{\gamma} w_0 + \dot{\delta} v$$

is a K-algebra isomorphism and satisfies

$$\psi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)^* = \psi\left(\begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix}\right).$$

We are now ready to describe an isomorphism between $\mathrm{SL}_2(\mathcal{C}(Q_\epsilon))$ and the spin group of a suitable quadratic form.

Let U be an n-dimensional K vector space with non-degenerate quadratic form q. Then the spin-group of q is defined as

$$\mathrm{Spin}_n(K, q) := \{s \in \mathcal{C}^+(q) : s \cdot U \cdot s^* \subset U, s \cdot s^* = 1\}.$$

The K-algebra isomorphism ψ defined above then restricts to a group homomorphism $\psi : \mathrm{SL}_2(\mathcal{C}(Q_\epsilon)) \to \mathrm{Spin}_4(\widetilde{Q}_\epsilon)$.

The spin-group of a quadratic form has a canonical homomorphism to the corresponding orthogonal group which we will review in the following. Let U be a n-dimensional vector space with non-degenerated quadratic form Q. The space U is identified with a subspace of $\mathcal{C}(Q)$ and for $x \in U$ we have $Q(x) = x \cdot x^*$. Let $\mathrm{O}_n(K, Q) := \{g \in \mathrm{GL}_n(K) : Q \circ g = Q\}$. For $s \in \mathrm{Spin}_n(K, Q)$ we define the linear map $\Lambda(s) \in \mathrm{GL}(U)$ given by $\Lambda(s)(x) := s \cdot x \cdot s^*$. Then the computation

$$sxs^* \cdot (sxs^*)^* = sxs^*sx^*s^* = sxx^*s^* = xx^*$$

shows that $\Lambda(s) \in \mathrm{O}_n(K, Q)$. We have constructed now a homomorphism

$$\Lambda : \mathrm{Spin}_n(K, Q) \longrightarrow \mathrm{O}_n(K, Q).$$

Let $\Omega_n(K, Q)$ be the commutator subgroup of $\mathrm{O}_n(K, Q)$ and $\Gamma(Q)$ the subgroup of $K^\times/K^{\times 2}$ generated by the expressions $Q(x)Q(y)$ with $Q(x) \neq 0 \neq Q(y)$. Let $x \in U$ with $Q(x) \neq 0$. The linear map $\sigma_x : U \to U$,

$$\sigma_x(v) := v - 2\frac{B(v, x)}{Q(x)} \cdot x$$

is called the reflection in the hyperplane perpendicular to x. Every element $g \in \mathrm{O}_n(K, Q)$ can be expressed as a product of reflections $g = \sigma_{x_1} \cdots \sigma_{x_r}$. Associating to g the product $Q(x_1) \cdots Q(x_r)$ in $K^\times/K^{\times 2}$ we get a well-defined homomorphism

$$\Sigma : \mathrm{SO}_n(K, Q) \longrightarrow K^\times/K^{\times 2}$$

which is called the spinorial norm homomorphism. Then we have

Proposition 1.2.1 (Proposition 3.8, [EGM98])**.** *Let U be an n-dimensional K-vector space with non-degenerate quadratic form Q. Then the following hold.*

(1) We have $\Lambda(\mathrm{Spin}_n(K, Q)) \subset \mathrm{SO}_n(K, Q), \Sigma(\mathrm{SO}_n(K, Q)) \subset \Gamma(Q)$ and the resulting exact sequence

$$1 \to \{\pm 1\} \to \mathrm{Spin}_n(K, Q) \xrightarrow{\Lambda} \mathrm{SO}_n(K, Q) \to \Gamma(Q) \to 1.$$

(2) We define $\mathrm{SO}_n^+(K, Q) := Im(\Lambda)$ and get $\Omega_n(K, Q) \subset \mathrm{SO}_n^+(K, Q)$.

For the special quadratic form Q_ϵ defined above we consider $\Psi : \mathrm{SL}_2(\mathcal{C}(Q_\epsilon)) \to \mathrm{O}_4(K, \widetilde{Q}_\epsilon)$ given by $\Psi := \Lambda \circ \psi$. The isomorphisms ψ, Ψ are usually called exceptional isomorphisms. Then we have

Proposition 1.2.2 (Proposition 3.10, [EGM98])**.** *The map $\Psi : \mathrm{SL}_2(\mathcal{C}(Q_\epsilon)) \to \mathrm{O}_4(K, \widetilde{Q}_\epsilon)$ satisfies $\Psi(\mathrm{SL}_2(\mathcal{C}(Q_\epsilon))) = \mathrm{SO}_4^+(K, \widetilde{Q}_\epsilon)$ and the resulting sequence*

$$1 \to \{\pm 1\} \to \mathrm{SL}_2(\mathcal{C}(Q_\epsilon)) \xrightarrow{\Psi} \mathrm{SO}_4(K, \widetilde{Q}_\epsilon) \to K^\times/K^{\times 2} \to 1$$

is exact.

Example 1.2.3. (1) Let $K = \mathbb{R}$ and take the one-dimensional $\mathbb{R}$-vector space $V_1 = \mathbb{R} \cdot f_3$ with quadratic form $Q_1(\lambda f_3) = -\lambda^2$. We have the identification $\mathcal{C}(Q_1) = \mathbb{C}$. Then we have the exact sequence

$$1 \to \{\pm 1\} \to \mathrm{SL}_2(\mathbb{C}) \to \mathrm{SO}(3, 1)(\mathbb{R}) \to \mathbb{R}^\times/\mathbb{R}^{\times 2} \to 1.$$

Note that $\mathrm{SO}^+(3,1)(\mathbb{R})$ is the connected component $\mathrm{SO}_0(3,1)(\mathbb{R})$ of the identity in $\mathrm{SO}(3,1)(\mathbb{R})$, see [EGM98, Proposition 3.12]. It follows that we have the exceptional isomorphism $\mathrm{PSL}_2(\mathbb{C}) \simeq \mathrm{SO}_0(3,1)(\mathbb{R})$ which also implies the symmetric space both for $\mathrm{PSL}_2(\mathbb{C})$ and $\mathrm{SO}_0(3,1)(\mathbb{R})$ is $\mathbb{H}_3$ as in Example 1.1.6.

(2) Let $K = \mathbb{Q}_q$ with odd prime q split in the imaginary quadratic field $\mathbb{Q}(\sqrt{d})$ (square-free $d < 0$). Take the one-dimensional $\mathbb{Q}_q$-vector space $V_{-d} = \mathbb{Q}_q \cdot \sqrt{d}$ with quadratic form $Q_{-d}(\lambda\sqrt{d}) = \lambda^2 d < 0$. We have $\mathcal{C}(Q_{-d}) \simeq \mathbb{Q}_q \times \mathbb{Q}_q$. Then we have the exact sequence

$$1 \to \{\pm 1\} \to \mathrm{SL}_2(\mathbb{Q}_q \times \mathbb{Q}_q) \to \mathrm{SO}(3,1)(\mathbb{Q}_q) \to \mathbb{Q}_q^\times / \mathbb{Q}_q^{\times 2} \to 1$$

from which we can deduce the isomorphism $\mathrm{PSL}_2(\mathbb{Q}_q \times \mathbb{Q}_q) \simeq \mathrm{SO}^+(3,1)(\mathbb{Q}_q)$.

(3) Suppose the assumptions in the second example hold but q is inert or ramified. Then we have $\mathcal{C}(Q_{-d}) \simeq \mathbb{Q}_q(\sqrt{d})$ and the exact sequence

$$1 \to \{\pm 1\} \to \mathrm{SL}_2(\mathbb{Q}_q(\sqrt{d})) \to \mathrm{SO}(3,1)(\mathbb{Q}_q) \to \mathbb{Q}_q^\times / \mathbb{Q}_q^{\times 2} \to 1$$

implying $\mathrm{PSL}_2(\mathbb{Q}_q(\sqrt{d})) \simeq \mathrm{SO}^+(3,1)(\mathbb{Q}_q)$.

§ 1.3 Hecke characters

In this section we review Hecke characters classically and idelically from Shurman's lecture notes (see [Shu]).

Let F be an imaginary quadratic field with the integer ring $\mathcal{O}$. Let $\mathfrak{f}$ be an integral ideal, i.e. an ideal of $\mathcal{O}$. The elements of $F^\times$ that generate fractional ideals coprime to $\mathfrak{f}$ form a subgroup,

$$F(\mathfrak{f}) = \{\alpha \in F^\times : ((\alpha), \mathfrak{f}) = 1\}.$$

Definition 1.3.1 (Multiplicative Congruence)**.** For a pair of nonzero field elements $\alpha, \beta \in F(\mathfrak{f})$, the condition $\alpha \equiv \beta \bmod^\times \mathfrak{f}$ means $\beta - \alpha \in F(\mathfrak{f})\mathfrak{f}$.

Define

$$\begin{aligned} F_{\mathfrak{f}} &= 1 + F(\mathfrak{f})\mathfrak{f} = \{\alpha \in F^\times : \alpha \equiv 1 \bmod^\times \mathfrak{f}\} \subset F(\mathfrak{f})\mathfrak{f}, \\ I(\mathfrak{f}) &= \{\text{fractional ideal of } F \text{ coprime to } \mathfrak{f}\}, \\ P(\mathfrak{f}) &= \{\text{principal fractional ideal } (\alpha) \text{ of } F \text{ coprime to } \mathfrak{f}\}, \\ P_{\mathfrak{f}} &= \{\text{principal fractional ideal } (\alpha) \text{ of } F \text{ where } \alpha \equiv 1 \bmod^\times \mathfrak{f}\}. \end{aligned}$$

We have a map

$$F^\times \longrightarrow \mathbb{C}, \quad \alpha \longmapsto 1 \otimes \alpha,$$

where we identify $\mathbb{R} \otimes F$ (tensoring over $\mathbb{Q}$) with $\mathbb{C}$ in the usual way.

Definition 1.3.2 (Classical Hecke Character)**.** Let $\mathfrak{f}$ be a (nonzero) ideal of $\mathcal{O}$, and let $\chi_\infty : \mathbb{C}^\times \to \mathbb{C}^\times$ be a continuous character. Then the character

$$\chi : I(\mathfrak{f}) \longrightarrow \mathbb{C}^\times$$

is a Hecke character with conductor $\mathfrak{f}$ and infinity-type χ_∞ if χ_∞ determines χ on $P_\mathfrak{f}$ by the rule

$$\chi((\alpha)) = \chi_\infty^{-1}(1 \otimes \alpha) \quad \text{for all } \alpha \in F_\mathfrak{f}.$$

The group of units of the adele ring $\mathbb{A}_F$ is called the group of ideles, denoted $\mathbb{I}_F$. Under the subspace topology inherited from $\mathbb{A}_F$, $\mathbb{I}_F$ is not a topological group since inversion $(-)^{-1} : \mathbb{I}_F \to \mathbb{I}_F$ cannot be continuous. However, $\mathbb{I}_F$ can be endowed with the subspace topology given by the embedding

$$\mathbb{I}_F \to \mathbb{A}_F \times \mathbb{A}_F : x \mapsto (x, x^{-1}).$$

In this way, we get a locally compact topological group. Alternatively, for each finite S containing the set of archimedean places, we have a locally compact group

$$\mathbb{I}_{F,S} = \prod_{v \in S} F_v^\times \times \prod_{v \notin S} \mathcal{O}_v^\times$$

since each unit group $\mathcal{O}_v^\times$ is compact, and the idele group can be described as the colimit over a filtered system of open inclusions

$$\mathbb{I}_F = \operatorname{colim}_S \mathbb{I}_S.$$

Indeed the idele topology coincides with the filtered colimit topology.

Definition 1.3.3 (Idelic Hecke Character)**.** A Hecke character of F is a continuous character of the idele group of F that is trivial on $F^\times$,

$$\chi : \mathbb{I}_F \longrightarrow \mathbb{C}^\times, \quad \chi(F^\times) = 1.$$

A Hecke character $\chi : \mathbb{I}_F \longrightarrow \mathbb{C}^\times$ has a conductor intrinsically built in, a product of local conductors at the finite places, even though its definition makes no direct reference to a conductor. We discuss this in the following.

At any nonarchimedean place v the local character $\chi_v : F_v^\times \to \mathbb{C}^\times$ is determined by its value on the local units $\mathcal{O}_v^\times$ and by its value on a uniformizer ϖ_v. By the nature of the idele topology, the kernel of any continuous group homomorphism $\mathbb{I}_F \to \mathbb{C}^\times$ contains almost all the local unit groups $\mathcal{O}_v^\times$. Therefore χ_v takes the unramified form $\chi_v(x) = |x|_v^s$ (where $s \in \mathbb{C}$) for almost all nonarchimedean v.

If χ_v is unramified then the local conductor of χ is $\mathcal{O}_v$. If χ_v is ramified then the local conductor of χ is $\mathfrak{p}_v^{e_v}$ for the smallest $e_v > 0$ such that χ_v is defined on $\mathcal{O}_v^\times/(1+\mathfrak{p}_v^{e_v}) \simeq (\mathcal{O}_v/\mathfrak{p}_v^{e_v})^\times$.

Given an idelic Hecke character, we show how to produce a corresponding classical Hecke character. Let the idelic Hecke character be

$$\chi = \bigotimes_v \chi_v$$

and let its conductor be

$$\mathfrak{f} = \prod_v \mathfrak{p}_v^{e_v}.$$

Define a character of fractional ideals coprime to $\mathfrak{f}$,

$$\widetilde{\chi} : I(\mathfrak{f}) \longrightarrow \mathbb{C}^\times,$$

by the conditions

$$\widetilde{\chi}(\mathfrak{p}_v) = \chi_v(\mathcal{O}_v^\times \varpi_v), \qquad \text{non-archimedean } v \nmid \mathfrak{f}.$$

The conditions are sensible because the local characters are unramified away from the conductor. In order to get a classical Hecke character $\widetilde{\chi}$, the composition

$$F_{\mathfrak{f}} \longrightarrow I(\mathfrak{f}) \xrightarrow{\widetilde{\chi}} \mathbb{C}^\times$$

needs to take form $a \mapsto \widetilde{\chi}_\infty^{-1}(1 \otimes a)$ for some character $\widetilde{\chi}_\infty$ on $\mathbb{C}^\times$. We compute the composite for any $a \in F_{\mathfrak{f}}$, with $(a) = \prod \mathfrak{q}_v^{\alpha_v}$, using the fact that χ is trivial on $F^\times$ at the last step,

$$a \longrightarrow \prod \widetilde{\chi}(\mathfrak{p}_v)^{\alpha_v} = \prod \chi_v(\mathcal{O}_v^\times \varpi_v)^{\alpha_v} = \chi(a_{\text{fin}}) = \chi^{-1}(a_{\text{inf}}).$$

Here subscripts inf and fin denote the infinite part and the finite part respectively. The natural identification of $\mathbb{I}_\infty$ and $\mathbb{C}^\times$ takes a_{inf} to $1 \otimes a$. Thus, given an idelic Hecke character χ, the corresponding character $\widetilde{\chi}$ of $I(\mathfrak{f})$ is a classical Hecke character whose infinite type matches that of the idelic character, i.e. $\widetilde{\chi}_\infty = \chi_\infty$.

Conversely, given a classical Hecke character χ of F having conductor $\mathfrak{f}$ and infinity-type χ_∞, we have a corresponding idelic Hecke character $\widetilde{\chi}$:

- Since $1 \otimes F_{\mathfrak{f}}$ is dense in $\mathbb{R} \otimes F$, the infinite part $\widetilde{\chi}_\infty$ of $\widetilde{\chi}$ is determined by χ_∞.
- For $v \nmid \mathfrak{f}$, define

$$\widetilde{\chi}_v(\mathcal{O}_v^\times \varpi_v) = \chi(\mathfrak{p}_v).$$

- Any $x \in \prod_{v|\mathfrak{f}} F_v^\times$ is closely approximated by some $a \in F^\times$, and so the desired value $\widetilde{\chi}(x)$ is closely approximated by $\prod_{v\nmid\mathfrak{f}} \widetilde{\chi}_v^{-1}(a_v)$ (including infinite v) as $\widetilde{\chi} = 1$ on $F^\times$.

§ 1.4 Automorphic forms

Throughout this chapter we let G be a reductive group over a number field F. Our goal of this chapter is to review automorphic forms both classically and adelically following [GH19]. We will relate the adelic definition to automorphic forms on locally symmetric spaces, and then to classical modular forms on arithmetic quotients of the upper half plane. In the end, we will discuss two important examples of automorphic forms used in this thesis, Bianchi modular forms and Siegel modular forms.

1.4.1 Classical automorphic forms

Write $\mathbb{A} = \mathbb{A}_F$ and its finite part $\mathbb{A}_f$. Let $K_\infty \subset G(F_\infty)$ be a maximal compact subgroup and $K_f \subset G(\mathbb{A}_f)$ a compact open subgroup. In fact, the quotient

$$G(F)\backslash G(\mathbb{A}_f)/K_f$$

is finite, which is also known as the finiteness of class number. Let h be its size and $t_1, \ldots, t_h$ a set of representatives for this quotient. Then we have a homeomorphism

$$\coprod_{i=1}^{h} \Gamma_i(K_f)\backslash G(F_\infty) \longrightarrow G(F)\backslash G(\mathbb{A})/K_f$$

given on the i-th component by

$$\Gamma_i(K_f) g_\infty \longmapsto G(F) g_\infty t_i \Gamma_i(K_f),$$

where

$$\Gamma_i(K_f) := G(F) \cap t_i \cdot G(F_\infty) K_f \cdot t_i^{-1}.$$

In this subsection we work only at the infinity place and then pass to the adelic setting in the next subsection.

Let

$$\iota' : G \longrightarrow \mathrm{GL}_n$$

be a closed embedding and $\iota : G \to \mathrm{GL}_{2n}$ the embedding defined by

$$\iota(g) := \begin{pmatrix} \iota'(g) & \\ & \iota'^{-t}(g) \end{pmatrix}. \tag{1.2}$$

We then define the norm

$$\|g\| := \|g\|_\iota = \prod_{v|\infty} \sup_{1 \le i,j \le 2n} |g_{ij}|_v.$$

Let $\mathfrak{g}$ be the Lie algebra of $G(F_\infty)$ and $U(\mathfrak{g})$ the universal enveloping algebra of complexion $\mathfrak{g}^{\mathbb{C}}$ with its centre $Z(\mathfrak{g})$.

Definition 1.4.1. A function

$$\Phi_\infty : G(F_\infty) \longrightarrow \mathbb{C}$$

is of moderate growth or slowly increasing if there are constants $c, r \in \mathbb{R}_{>0}$ such that

$$\Phi_\infty(g_\infty)| \leq c\|g_\infty\|^r.$$

In particular, the notion of moderate growth is independent of the choice of ι. For the definition of automorphic forms we will require the notion of K_∞-finite functions and $Z(\mathfrak{g})$-finite functions.

Definition 1.4.2. We say a function $\Phi_\infty : G(F_\infty) \to \mathbb{C}$ is right K_∞-finite if the space spanned by the right translates of Φ_∞ by elements of K_∞ is finite-dimensional.

Recall from [GH19, Section 4.2] that there exists an exponential map

$$\exp : \mathfrak{g} \longrightarrow G(F_\infty)$$

where $\mathfrak{g}$ denotes the Lie algebra of $G(F_\infty)$. In the case where $G = \mathrm{GL}_n$, the Lie algebra $\mathfrak{gl}_n$ is the collection of n by n matrices. In this case the exponential is given by

$$\exp(X) := \sum_{j=0}^{\infty} \frac{X^j}{j!}.$$

In general, the representation $G \to \mathrm{GL}_n$ induces an inclusion $\mathfrak{g} \to \mathfrak{gl}_n$, and the exponential on $\mathfrak{g}$ is obtained by restriction. Let (ρ, V) be a Hilbert representation of $G(F_\infty)$. Given $v \in V$ and $X \in \mathfrak{g}$, we set

$$\begin{aligned}\rho(X)v &= \frac{d}{dt}\rho(\exp(tX))v|_{t=0} \\ &= \left(\lim_{h\to 0} \frac{\rho(\exp(t+h)X)v - \rho(\exp(tX))v}{h}\right)|_{t=0}\end{aligned}$$

if the limit exists. Simply we write Xv for $\rho(X)v$. A vector $v \in V$ is C^1 if for $X \in \mathfrak{g}$, the derivative Xv is defined. We define C^k inductively by stipulating that $v \in V$ is C^k if Xv is C^{k-1} for all $X \in \mathfrak{g}$. A vector $v \in V$ is C^∞, which is called to be smooth, if it is C^k for all $k \geq 1$. The action of $\mathfrak{g}$ defined above is a Lie algebra representation, see e.g. [GH19, Lemma 4.2.2]. We can extend the action of $\mathfrak{g}$ to an action of the complexification $\mathfrak{g}^{\mathbb{C}} := \mathfrak{g} \otimes_{\mathbb{R}} \mathbb{C}$ by setting $(X + iY)v = Xv + iYv$. Let $\mathcal{U}(\mathfrak{g})$ be the universal enveloping algebra of $\mathfrak{g}^{\mathbb{C}}$ and $\mathcal{Z}(\mathfrak{g})$ its centre.

Definition 1.4.3. Let V be a $\mathcal{Z}(\mathfrak{g})$-module. A vector $\Phi_\infty \in V$ if $\mathcal{Z}(\mathfrak{g})$-finite is $\mathcal{Z}(\mathfrak{g})\Phi_\infty$ is finite-dimensional.

We now finally come to the definition of a classical automorphic form:

Definition 1.4.4. Let $\Gamma \subset G(F_\infty)$ be an arithmetic subgroup. A smooth function $\Phi_\infty : G(F_\infty) \to \mathbb{C}$ of moderate growth is an automorphic form on Γ if it is left Γ-invariant, right K_∞-finite, and $\mathcal{Z}(\mathfrak{g})$-finite. We denote by $\mathcal{A}(\Gamma)$ the space of automorphic forms on Γ.

1.4.2 Adelic automorphic forms

Let $\iota : G \to \mathrm{SL}_{2n}$ be the embedding of (1.2). For a place v of F let

$$\|g\|_v := \|g\|_{\iota,v} = \sup_{1\leq i,j\leq 2n} |g_{ij}|_v$$

and for a set of places S of F (finite or infinite) let

$$\|g\|_S := \prod_{v\in S} \|g\|_v.$$

If S is the set of all places of F then we omit it from notation.

As in the archimedean setting we have a definition of an adelic function of moderate growth:

Definition 1.4.5. A function

$$\Phi : G(\mathbb{A}) \longrightarrow \mathbb{C}$$

is of moderate growth or slowly increasing if there are constants $c, r \in \mathbb{R}_{>0}$ such that

$$|\Phi(g)| \leq c\|g\|^r.$$

Let $K_{\max} \subset G(\mathbb{A})$ be a maximal compact subgroup; thus $K_{\max} = K_\infty \times K_{f,\max}$ where $K_\infty \subset G(F_\infty)$ and $K_{f,\max} \subset G(\mathbb{A}_f)$ are maximal compact subgroups. As before, we say that a function $\Phi : G(\mathbb{A}) \to \mathbb{C}$ is right $K_{\max}$-finite if the span of translates

$$\{x \mapsto \Phi(xk) : k \in K_{\max}\}$$

is finite-dimensional. The reason that we do not include the subscript max at the infinite component is that it is rare to consider non-maximal compact subgroups in this setting, although it is very natural to consider non-maximal compact open subgroups at the finite places.

Now we can give the definition of an adelic automorphic form:

Definition 1.4.6. A function

$$\Phi : G(\mathbb{A}) \longrightarrow \mathbb{C}$$

of moderate growth is an automorphic form on G if it is left $G(F)$-invariant, $K_{\max}$-finite, and $\mathcal{Z}(\mathfrak{g})$-finite. The $\mathbb{C}$-vector space of automorphic forms is denoted by $\mathcal{A}$ of $\mathcal{A}(G)$.

Definition 1.4.7. An automorphic form $\Phi \in \mathcal{A}$ is said to be cuspidal if for every proper parabolic subgroup $P \subset G$ with unipotent radical N one has

$$\int_{[N]} \Phi(ng)dn = 0$$

for all $g \in G(\mathbb{A})$ where the terminology $[\,\cdot\,]$ is described in [GH19, Section 2.6].

1.4.3 From modular forms to automorphic forms

In this subsection we make the connection between classical modular forms and automorphic forms precisely with help from an additional reference [Boo15].

Set $F = \mathbb{Q}$ and $G = \mathrm{GL}_2$. Recall that the strong approximation theorem states that for any compact open subgroup $K_f \subset \mathrm{GL}_2(\mathbb{A}_f)$ such that $\det(K_f) = \mathbb{A}_f^\times$, we have

$$\mathrm{GL}_2(\mathbb{A}_\mathbb{Q}) = \mathrm{GL}_2(\mathbb{Q})\,\mathrm{GL}_2(\mathbb{R})K_f.$$

Note that this relies on the class number of $\mathbb{Q}$ being one. A convenient compact open subgroup to work with will be

$$K_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathbb{Z}}) : c = 0 \mod N \right\}.$$

The connection between the $\mathrm{GL}_2(\mathbb{R})^+$ (or $\mathrm{SL}_2(\mathbb{R})$) and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ is the following proposition:

Proposition 1.4.8. *[Boo15, Proposition 1.2] For any positive integer N, there are natural isomorphisms*

$$\Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{R}) \simeq \mathcal{Z}(\mathbb{A}_\mathbb{Q})\,\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N),$$
$$\Gamma_0(N)\backslash \mathrm{GL}_2(\mathbb{R})^+ \simeq \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/K_0(N).$$

In particular, adding in an archimedean component to $K_0(N)$ such as $\mathrm{SO}(2)$ *would give a more direct comparison with the upper half plane, i.e.*

$$\Gamma_0(N)\backslash\mathbb{H}_2 \simeq \mathcal{Z}(\mathbb{A}_\mathbb{Q})\,\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/\,\mathrm{SO}(2) \times K_0(N),$$
$$\Gamma_0(N)\backslash\mathbb{H}_2 \simeq \mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/\,\mathrm{SO}(2) \times K_0(N).$$

With the identification of spaces in the above proposition, we can set up a correspondence between functions on the upper half plane, $\mathrm{GL}_2(\mathbb{R})^+$ and $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. Instead of working with the quotients $\Gamma_0(N)\backslash\mathbb{H}_2$, we will work with functions on $\mathrm{GL}_2(\mathbb{R})$ and $\mathrm{GL}_2(\mathbb{A}_{\mathbb{Q}})$ which satisfy transformation laws.

We start with the classical modular forms on the upper half plane $\mathbb{H}_2$. Let $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$ be a congruence subgroup. For example, we could set

$$\Gamma = \Gamma_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N|c \right\}.$$

Now we recall the definition of a classical modular form for Γ:

Definition 1.4.9. Let $k \in \mathbb{Z}_{>0}$ and $\mathbb{H}_2$ the complex upper half plane. The space of weight k modular forms for Γ is space $M_k(\Gamma)$ of holomorphic functions $f : \mathbb{H}_2 \to \mathbb{C}$ satisfying the following conditions:

(1) $f\left(\frac{az+b}{cz+d}\right) = (cz+d)^k f(z)$ for all $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \Gamma$ and all $z \in \mathbb{H}_2$,
(2) f extends to holomorphically to the cusps.

If f additionally vanishes at the cusps we say that f is a cusp form. The space of weight k cusp forms is denoted $S_k(\Gamma)$.

We will now discuss how a modular form is an example of an automorphic form on $\mathrm{GL}_2(\mathbb{R})$. The observation that $\mathrm{GL}_2(\mathbb{R})^+$ acts on the upper half plane with stabilizer $K = \mathrm{SO}(2)$ suggests the relationship between modular forms on $\mathbb{H}_2$ and automorphic forms on $\mathrm{GL}_2(\mathbb{R})$. Given a cusp form f, we consider the function defined on $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{R})^+$ by

$$F(g) := (f|_k g)(i) = (ad-bc)^{k/2}(ci+d)^{-k} f\left(\frac{ai+b}{ci+d}\right).$$

One can show that it has many nice transformation properties and then it is the automorphic form for $\mathrm{GL}_2(\mathbb{R})$, see [Boo15, Section 2.2]. Among these properties, we want to highlight the following:

- for $\gamma \in \Gamma_0(N)$, $F(\gamma g) = F(g)$;
- for $\kappa = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \in K = \mathrm{SO}(2)$, we see that
$$F(g\kappa) = e^{-ik\theta} F(g)$$
which implies that F is K-finite;
- for $\gamma \in \mathcal{Z}(\mathbb{R})^+ \subset \mathrm{GL}_2(\mathbb{R})$, we have
$$F(\gamma g) = F(g).$$

It follows that the automorphic form F on $\mathrm{GL}_2(\mathbb{R})$ descends to the function on the locally symmetric space $\Gamma_0(N)\backslash \mathrm{GL}_2(\mathbb{R})^+/\mathcal{Z}(\mathbb{R})^+ \mathrm{SO}(2) \simeq \Gamma_0(N)\backslash\mathbb{H}_2$.

We now describe how to associate to f and F an automorphic form Φ on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$. The idea is similar to that of $\mathrm{GL}_2(\mathbb{R})^+$; that is using Proposition 1.4.8 to see a relation between the spaces, and then a connection between functions on $\mathrm{GL}_2(\mathbb{R})$ and those on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ satisfying certain transformation properties. For $\gamma \in \mathrm{GL}_2(\mathbb{Q})$, $g_\infty \in \mathrm{GL}_2(\mathbb{R})^+$ and $\kappa_f \in K_0(N)$, we define

$$\Phi(\gamma g_\infty \kappa_f) := F(g_\infty) = (f|_k g_\infty)(i). \tag{1.3}$$

One can show that the function Φ is well defined and it is a cusp form, see [Boo15, Proposition 2.3]. For the transformation properties of Φ, see [Boo15, Section 2.3] in details. Again, we point out some properties here:

- Φ is left invariant under $\mathrm{GL}_2(\mathbb{Q})$ by definition;
- for $\kappa = \kappa_\infty \kappa_f \in K = \mathrm{SO}(2)K_0(N)$, we have that
$$\Phi(g\kappa) = e^{-ik\theta}\Phi(g)$$
which implies the K-finiteness;
- for $g \in \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ and $z \in \mathbb{A}_\mathbb{Q}^\times$, we can check that
$$\Phi\left(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix} g\right) = \Phi(g).$$

So, by the above proposition, we can observe that the automorphic form Φ on $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ descends via (1.3) to the function on the locally symmetric space

$$\mathcal{Z}(\mathbb{A}_\mathbb{Q})\,\mathrm{GL}_2(\mathbb{Q})\backslash \mathrm{GL}_2(\mathbb{A}_\mathbb{Q})/\,\mathrm{SO}(2) \times K_0(N) \simeq \Gamma_0(N)\backslash \mathrm{SL}_2(\mathbb{R})/\,\mathrm{SO}(2) \simeq \Gamma_0(N)\backslash\mathbb{H}_2.$$

1.4.4 Bianchi modular forms

In this section we consider cusp forms on the adele group $\mathrm{GL}_2(\mathbb{A})$ for the adele ring $\mathbb{A} = \mathbb{A}_F$ over an imaginary quadratic field F of class number one. For arbitrary class number, see e.g. [Gha99, Section 2]. The class number in this thesis is restricted since we know how to use Atkin-Lehner operators in Section 4.4.2 only in this case of class number one. We will treat larger class numbers in the future.

Let $\widehat{\mathcal{O}} = \prod_{v<\infty} \mathcal{O}_v$. Given an ideal $\mathfrak{N} \subset \mathcal{O}$, we define

$$K_0(\mathfrak{N}) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\widehat{\mathcal{O}}) : c \in \mathfrak{N}\widehat{\mathcal{O}} \right\}$$

which is a compact open subgroup of the finite part of $\mathrm{GL}_2(\mathbb{A})$. Let $V_n(\mathbb{C})$ be the space of homogeneous polynomials of degree n in two variables $\mathbf{s} = \left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)$ with complex coefficients.

In the following we will define the automorphic forms on $\mathrm{GL}_2(\mathbb{A}_F)$ which are eigenforms of the Casimir operator (which is central) in the universal enveloping algebra of the complexification $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$. First we recall the Casimir operator explicitly in this case from [Hid93, Section 1.3]. For

$$\mathfrak{sl}_2(\mathbb{C}) = \{x \in \mathfrak{gl}_2(\mathbb{C}) : \mathrm{tr}(x) = 0\},$$

we have $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$. Here, $\mathfrak{sl}_2(\mathbb{C})$ is embedded into $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ as $x \mapsto x \oplus x^c$ for the complex conjugation c. We write the first (resp. second) projection of $x \in \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{sl}_2(\mathbb{C})$ as x' (resp. x''). Let

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

and then the Casimir operator in $\mathfrak{sl}_2(\mathbb{C})$ is given by

$$C = \frac{1}{8}(ef + fe) + \frac{h^2}{16} - \frac{1}{8}((ie)(if) + (if)(ie)) - \frac{(ih)^2}{16}$$

As an element of the complexification, we have

$$C = \frac{1}{4}(D' \oplus D'')$$

where $D' = e'f' + f'e' + h'^2/2$ and $D'' = e''f'' + f''e'' + h''^2/2$. Following the discussion in Section 1.4.1, the Lie algebra $\mathfrak{sl}_2(\mathbb{C})$ acts on smooth functions Φ on $\mathrm{GL}_2(\mathbb{C})$ by

$$X\Phi(g_\infty) = \frac{d}{dt}\{\Phi(g_\infty \exp(tX))\}\big|_{t=0} \quad \text{for } X \in \mathfrak{sl}_2(\mathbb{C}).$$

Definition 1.4.10. A smooth function $\Phi : \mathrm{GL}_2(\mathbb{A}_F) \longrightarrow V_2(\mathbb{C})$ is said to be a cusp form of weight 2 and level $K_0(\mathfrak{N})$ if it satisfies:

(i) $\Phi(rg, \mathbf{s}) = \Phi(g, \mathbf{s})$ for $r \in \mathrm{GL}_2(F)$;

(ii) $\Phi(zg, \mathbf{s}) = \Phi(g, \mathbf{s})$ for $z \in \mathcal{Z}(\mathrm{GL}_2(\mathbb{C})) \simeq \mathbb{C}^\times$;

(iii) $\Phi(gk, \mathbf{s}) = \Phi\left(g, k_\infty \left(\begin{smallmatrix} S \\ T \end{smallmatrix}\right)\right)$ for $k = k_\infty \cdot k_f \in \mathrm{SU}(2) \times K_0(\mathfrak{N})$;

(iv) Φ is an eigenfunction of the complexification (in the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \otimes_{\mathbb{R}} \mathbb{C}$) of the Casimir operator of $\mathfrak{sl}_2(\mathbb{C})$ with eigenvalue 0, i.e.

$$D'\Phi = 0 \quad \text{and} \quad D''\Phi = 0$$

for D', D'' defined above. Here $\Phi(g_\infty g_f)$ is considered as a function of $g_\infty \in \mathrm{GL}_2(\mathbb{C})$;

(v) Φ satisfies the cuspidal condition

$$\int_{U(F)\backslash U(\mathbb{A}_F)} \Phi(vg)dv = 0$$

for all $g \in \mathrm{GL}_2(\mathbb{A}_F)$, where

$$U(F) = \left\{ v = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in F \right\}, \quad U(\mathbb{A}_F) = \left\{ v = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} : u \in \mathbb{A}_F \right\}$$

and dv is the Lebesgue measure on $\mathbb{A}_F$.

Remark 1.4.11. Condition (iii) implies that Φ is $K_{\max}$-finite as in Definition 1.4.6. Condition (iv) implies that Φ is C-finite for the Casimir operator C. As the centre $\mathcal{Z}(\mathfrak{g})$ of the universal algebra is generated by C and the identity matrix, condition (ii) and (iv) imply that Φ is $\mathcal{Z}(\mathfrak{g})$-finite. It is not obvious to show that Φ has moderate growth. To do so we can follow the treatment in the classical case of $\mathrm{GL}_2(\mathbb{A}_\mathbb{Q})$ (see e.g. [Tro, page 16]) but we omit details here. To conclude, the function Φ defined above on $\mathrm{GL}_2(\mathbb{A}_F)$ is indeed an automorphic form given in Definition 1.4.6.

From now on such a cusp form is called a *Bianchi modular form of weight 2* and we denote the space of these Bianchi modular forms by $S_2(\mathfrak{N})$. The Bianchi modular form Φ has Fourier expansion ([Hid94, Theorem 6.1]):

$$\Phi \begin{pmatrix} y & x \\ 0 & 1 \end{pmatrix} = |y|_F \sum_{\alpha \in F^\times} c(\alpha y \delta, \Phi) W(\alpha y_\infty) e_K(\alpha x) \tag{1.4}$$

where:

(1) $|\cdot|_F$ is the usual idele character of $\mathbb{A}_F^\times$ trivial on $F^\times$;
(2) $\delta = \sqrt{d_F}$ (where d_F is the discriminant of F) is a generator of the different $\mathfrak{D}$ of F, i.e. $\delta\mathcal{O} = \mathfrak{D}$;
(3) the Fourier coefficient $c(\cdot\,, \Phi)$ is a well defined function on the fractional ideals of F such that $c(I, \Phi) = 0$ for I non-integral;
(4) $W : \mathbb{C}^\times \to V_2(\mathbb{C})$ is the Whittaker function

$$W(s) := \sum_{n=0}^{2} \binom{2}{n} \left(\frac{s}{i|s|} \right)^{1-n} \mathbf{K}_{n-1}(4\pi|s|) X^{2-n} Y^n,$$

where $\mathbf{K}_n(x)$ is (a modified Bessel function that is) a solution to differential equation

$$\frac{d^2\mathbf{K}_n}{dx^2} + \frac{1}{x}\frac{d\mathbf{K}_n}{dx} - \left(1 + \frac{n^2}{x^2}\right)\mathbf{K}_n = 0,$$

with asymptotic behaviour $\mathbf{K}_n(x) \sim \sqrt{\frac{\pi}{2x}} e^{-x}$ as $x \to \infty$. Note that $\mathbf{K}_{-n} = \mathbf{K}_n$;

(5) e_K is an additive character of $F\backslash\mathbb{A}_F$ given by

$$e_K = \left(\prod_{\mathfrak{p}} (e_p \circ \mathrm{Tr}_{F_{\mathfrak{p}}/\mathbb{Q}_p})\right) \cdot (e_\infty \circ \mathrm{Tr}_{\mathbb{C}/\mathbb{R}}).$$

Here $e_p(x) = \exp(2\pi i \mathrm{Fr}_p(x)$ for $x \in \mathbb{Q}_p$ where $\mathrm{Fr}_p(x)$ denotes the fractional part of x and $e_\infty(x) = \exp(2\pi i x)$ for $x \in \mathbb{R}$.

The adelic cusp form Φ can descend to a function $\mathfrak{F} : \mathrm{GL}_2(\mathbb{C}) \to V_2(\mathbb{C})$. The strong approximation theorem gives us the decomposition

$$\mathrm{GL}_2(\mathbb{A}_F) = \mathrm{GL}_2(F) \cdot [\mathrm{GL}_2(\mathbb{C}) \times K_0(\mathfrak{N})].$$

We define the discrete subgroup of $\mathrm{SL}_2(F)$ via

$$\Gamma_0(\mathfrak{N}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} : a, b, d \in \mathcal{O}, c \in \mathfrak{N}, ad - bc = 1 \right\}.$$

One can check that $\mathrm{SL}_2(F) \cap [\mathrm{GL}_2(\mathbb{C}) \times U_0(\mathfrak{N})] = \Gamma_0(\mathfrak{N})$.

Define $\mathfrak{F} : \mathrm{GL}_2(\mathbb{C}) \to V_2(\mathbb{C})$ via $\mathfrak{F}(g) = \Phi(g)$. It is a cusp form on $\mathrm{GL}_2(\mathbb{C})$ and determines in turn a cusp form $\mathcal{F}$ on $\mathbb{H}_3$ in the following. We introduce the automorphy factor

$$j(\gamma, (z, r)) = \begin{pmatrix} cz + d & -cr \\ \bar{c}r & \bar{c}\bar{z} + \bar{d} \end{pmatrix} \text{ for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}), (z, r) \in \mathbb{H}_3.$$

Define $\mathcal{F} : \mathbb{H}_3 \to V_2(\mathbb{C})$ by

$$\mathcal{F}((z, r), \mathbf{s}) = \mathfrak{F}(g, j(g, (0, 1))^t \mathbf{s}),$$

where $g \in \mathrm{SL}_2(\mathbb{C})$ is chosen such that $g \cdot (0, 1) = (z, r)$. One can check that $\mathcal{F}$ is well defined and that it satisfies the automorphy condition

$$\mathcal{F}(\gamma \cdot (z, r), \mathbf{s}) = \mathcal{F}((z, r), j(\gamma, (z, r))^t \mathbf{s}) \quad \text{for } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(\mathfrak{N}).$$

Thus $\mathcal{F} \in S_2(\Gamma_0(\mathfrak{N}))$, the space of cusp forms on $\mathbb{H}_3$ satisfying the above condition.

For the associated cusp forms $\mathcal{F}$ on $\mathbb{H}_3$, the Fourier expansion can be worked out to be

$$\mathcal{F}((z, r), \mathbf{s}) = r \sum_{n=0}^{2} \binom{2}{n} \sum_{\alpha \in K^\times} \left[c(\alpha\delta) \left(\frac{\alpha}{i|\alpha|} \right)^{1-n} \times \right.$$
$$\left. \mathbf{K}_{n-1}(4\pi|\alpha|r) e^{2\pi i(\alpha z + \overline{\alpha z})} \right] S^{2-n} T^n.$$

For $n \in \{0, 1, 2\}$, let $\mathcal{F}_n : \mathbb{H}_3 \longrightarrow \mathbb{C}$ be the functions determined by the expression

$$\mathcal{F}((z, r), \mathbf{s}) = r \sum_{n=0}^{2} \mathcal{F}_n(z, r) S^{2-n} T^n. \tag{1.5}$$

More explicitly, we have

$$\mathcal{F}_n(z,r) = \sum_{\alpha\in K^\times} c(\alpha\delta)\left(\frac{\alpha}{i|\alpha|}\right)^{1-n} \mathbf{K}_{n-1}(4\pi|\alpha|r)e^{2\pi i(\alpha z+\overline{\alpha z})}. \tag{1.6}$$

For the cusp form $\mathcal{F}$ corresponding to Φ, we want to define the twist of the L-function by a Hecke character ψ of conductor $\mathfrak{f}$. For each ideal $\mathfrak{m}$ coprime to $\mathfrak{f}$, we have $\psi(\mathfrak{m}) = \prod_{\mathfrak{p}^n||\mathfrak{m}} \psi_{\mathfrak{p}}(\pi_{\mathfrak{p}})^n$. Then define

$$L(\Phi,\psi,s) = L(\mathcal{F},\psi,s) := |\mathcal{O}_F^\times|^{-1} \sum_{\alpha\in K^\times} c((\alpha),\Phi)\psi((\alpha))N((\alpha))^{-s}. \tag{1.7}$$

Theorem 1.4.12 (Theorem 1.8, [Wil17])**.** *Let Φ be a cusp form of weight 2 and of level $K_0(\mathfrak{N})$ corresponding the cusp form $\mathcal{F}$ on $\mathbb{H}_3$. For $n \in \{0,1,2\}$, let $\mathcal{F}_n$ be as defined in* (1.6) *above. Let ψ denote a Hecke character of conductor $\mathfrak{f}$ with infinity type $(-u,-v) = \left(-\frac{1-n}{2}, \frac{1-n}{2}\right)$. Then, for $s \in \mathbb{C}$, we have*

$$L(\Phi,\psi,s) = A(n,\psi,s) \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_F \\ (a\mathfrak{f},\mathfrak{f})=1}} \psi(a\mathfrak{f})^{-1} a^u \bar{a}^v \int_0^\infty r^{2s-2}\mathcal{F}_n(a,r)dr$$

and

$$A(n,\psi,s) = \frac{4\cdot(2\pi)^{2s} i^{1-n}\binom{2}{n}^{-1}}{|\mathcal{O}_F^\times|\cdot|d_F|^s\cdot\Gamma\left(s+\frac{n-1}{2}\right)\Gamma\left(s-\frac{n-1}{2}\right)\tau(\varphi^{-1})}.$$

Here τ denote the Gauss sum defined to be

$$\tau(\psi) := \sum_{\substack{[a]\in\mathfrak{f}^{-1}/\mathcal{O}_F \\ (a\mathfrak{f},\mathfrak{f})=1}} \psi(a\mathfrak{f})\psi_\infty\left(\frac{a}{\delta}\right) e^{2\pi i\,\mathrm{Tr}_{F/\mathbb{Q}}(a/\delta)}.$$

Proof. Williams proved this formula for the cusp form of level $K_1(\mathfrak{N})$ (the adelic analogy of the congruence subgroup $\Gamma_1(\mathfrak{N})$). However the Fourier expansion used by him in the case of $K_1(\mathfrak{N})$ and (k,k) is the same as that for level $K_0(\mathfrak{N})$ and weight (k,k) in [Hid94, Theorem 6.1] and [Gha99, Section 2]. So we can deduce this formula again for the level $K_0(\mathfrak{N})$ following Williams' proof without any change. □

1.4.5 Siegel modular forms

As we will construct the theta lifting of a Bianchi modular form, which is a weight 2 Siegel modular form, the aim of this section is to introduce a brief overview on basic aspects of Siegel modular forms with [AS01] as our main reference.

For considering Siegel modular forms (of degree 2) in the context of automorphic form, we define

$$H := \mathrm{GSp}_4 = \{g \in \mathrm{GL}_4 : \exists\ \lambda(h) \in \mathrm{GL}_1 \text{ s.t. } hJ^th = \lambda(h)J\},$$

where $J = \begin{pmatrix} & I_2 \\ -I_2 & \end{pmatrix}$ for I_2 the 2×2 identity matrix. The function λ is called the multiplier homomorphism. Its kernel is the group Sp_4 and there is an exact sequence

$$1 \longrightarrow \mathrm{Sp}_4 \longrightarrow H \longrightarrow \mathrm{GL}_1 \longrightarrow 1.$$

The centre $\mathcal{Z}$ of G consists of the scalar matrices.

Recall the Siegel upper half plane given by

$$\mathcal{H}_2 = \{Z = X + iY \in M_2(\mathbb{C}) : X, Y \in M_2(\mathbb{R}), X = X^t, Y = Y^t, Y > 0\}.$$

Siegel modular forms of degree 2 are certain holomorphic functions on the Siegel upper half plane $\mathcal{H}_2$. The group $\mathrm{Sp}_4(\mathbb{R})$ acts on the upper half plane by linear fractional transformations, that is

$$h \cdot Z = (AZ + B)(CZ + D)^{-1} \quad \text{for } h = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R}).$$

The full modular group is $\mathrm{Sp}_4(\mathbb{Z})$ and the principal congruence subgroup of level N is given by

$$\Gamma_2(N) = \{\gamma \in \mathrm{Sp}_4(\mathbb{Z}) : \gamma \equiv I_4 \mod N\}.$$

A subgroup Γ of $\mathrm{Sp}_4(\mathbb{Z})$ such that $\Gamma_2(N) \subset \Gamma$ is called a congruence subgroup of level N. We will consider a Siegel modular form of weight 2, degree 2 and character χ with respect to Γ in the sense that $f|_2\gamma = \chi(\det(A))f$ for $\gamma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \Gamma$, where

$$(f|_2\gamma)(Z) := \det(CZ + D)^{-2} f(\gamma \cdot Z) \quad \text{for } \gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{R}), Z \in \mathcal{H}_2.$$

The Siegel modular form has the Fourier expansion

$$f(Z) = \sum_R c_R \exp(2\pi i \operatorname{tr}(RZ)) \quad \text{for } Z \in \mathcal{H}_2,$$

where R runs over semi-integral, positive definite matrices.

We will associate a function $\Phi_f : H(\mathbb{A}) \to \mathbb{C}$ to f on $\mathcal{H}_2$ as follows. One can use strong approximation for Sp_4 to show that

$$H(\mathbb{A}) = H(\mathbb{Q})H(\mathbb{R})^+ \prod_{p<\infty} H(\mathbb{Z}_p),$$

where $H(\mathbb{R})^+$ denotes those elements of $H(\mathbb{R})$ with positive multiplier. Write $h \in H(\mathbb{A})$ as

$$h = h_{\mathbb{Q}} h_\infty k \quad \text{for } h_{\mathbb{Q}} \in H(\mathbb{Q}), h_\infty \in H(\mathbb{R})^+, k \in K_f,$$

where $K_f = \prod_{p<\infty} K_p$ with $K_p = H(\mathbb{Z}_p)$. Then we define

$$\Phi_f(h) = (f|_k h_\infty)(I), \tag{1.8}$$

where $I = \mathrm{diag}(i, i) \in \mathcal{H}_2$. This is well-defined due to the transformation properties of f.

The map $f \mapsto \Phi_f$ injects the space of modular forms of weight 2 into a space of functions Φ_f on $H(\mathbb{A})$ satisfying the following properties

(i) $\Phi_f(\gamma h) = \Phi_f(h)$ for $\gamma \in H(\mathbb{Q})$,
(ii) $\Phi_f(hk_f) = \Phi_f(h)$ for $k_f \in K_f$,
(iii) $\Phi_f(hk_\infty) = \Phi_f(h) j(k_\infty, I)^{-2}$ for $k_\infty \in K_\infty$,
(iv) $\Phi_f(hz) = \Phi_f(h)$ for $z \in \mathcal{Z}(\mathbb{A})$,

where $\mathcal{Z} \simeq \mathrm{GL}_1$ is the centre of GSp_4 and $K_\infty \simeq \mathrm{U}(2)$ is the standard maximal compact subgroup of $\mathrm{Sp}_4(\mathbb{R})$. If f is a cusp form, then the automorphic form Φ_f is cuspidal, i.e.,

$$\int_{N(\mathbb{Q})\backslash N(\mathbb{A})} \Phi_f(nh)dn = 0 \quad \text{for all } h \in H(\mathbb{A})$$

for each unipotent radical N of each proper parabolic subgroup of H.

In the other direction, given the weight 2 adelic form Φ on $H(\mathbb{A})$ of level $K_f \subset \prod_{p<\infty} H(\mathbb{Z}_p)$ satisfying above conditions (i)-(iv), the isomorphism

$$\Gamma\backslash \mathrm{Sp}(\mathbb{R})/U(2) \simeq \mathcal{Z}(\mathbb{A})H(\mathbb{Q})\backslash H(\mathbb{A})/U(2) \times K_f \quad \text{for } \Gamma = \mathrm{Sp}_4(\mathbb{R}) \cap K_f$$

induced by the strong approximation theorem

$$H(\mathbb{A}) = H(\mathbb{Q})H(\mathbb{R})^+ \prod_{p<\infty} H(\mathbb{Z}_p),$$

helps us observe that Φ descends via (1.8) to the classical Siegel modular form on the locally symmetric space $\Gamma\backslash\mathcal{H}_2$.

§ 1.5 Automorphic forms and cohomology

Consider a weight 2 modular (cusp) form $f \in M_2(\Gamma)$ (respectively $S_2(\Gamma)$) for $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. Then

$$\eta_f := f(z) \otimes dz$$

defines a closed holomorphic 1-form on $X = \Gamma\backslash\mathbb{H}_2$ with complex values. It is well known that this assignment induces the Eichler-Shimura isomorphism

$$M_2(\Gamma) \oplus \overline{S_2(\Gamma)} \simeq H^1(X, \mathbb{C})$$

where $\overline{S_2(\Gamma)}$ denotes the space of anti-holomorphic cusp forms in $S_2(\Gamma)$, which is in this case isomorphic to $S_2(\Gamma)$. For arbitrary weight of the modular form, see e.g. [FM11, Section 4]. In the following we will discuss in more details differentials on the upper half space $\mathbb{H}_3$ and the Eichler-Shimura-Harder isomorphism in this case.

1.5.1 Differentials on $\mathbb{H}_3$

In this section we will recall the differential forms on the upper half space as discussed in [Wil16, Section 3.2 and Section 5.2].

Let G be an arbitrary Lie group that has the structure of a real Riemannian manifold. Left translation by an element $g \in G$, denoted L_g, induces a pull-back action L_g^* on differentials. A differential $\omega \in \Omega^r(G, \mathbb{C})$ is said to be left-invariant if $L_g^*\omega = \omega$ for all $g \in G$. We can choose a basis for the space of left-invariant differentials. If a set of complex differentials (β_i) is chosen so that the evaluations $((\beta_i)_0)$ at the identity span the space $(T_0G)^*$, then any left-invariant 1-form ω can written uniquely as

$$\omega = \sum_i \alpha_i \beta_i, \quad \alpha_i \in \mathbb{C}.$$

Now let $G = G_\infty = Z_\infty B_\infty K_\infty$ where

$$\begin{aligned} G_\infty &= \mathrm{GL}_2(\mathbb{C}) \\ Z_\infty &= \left\{ \zeta \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} : \zeta \in \mathbb{C}^\times \right\}, \\ B_\infty &= \left\{ \begin{pmatrix} r & z \\ 0 & 1 \end{pmatrix} : z \in \mathbb{C}, r \in \mathbb{R}_{>0} \right\}, \\ K_\infty &= \mathrm{SU}(2) = \left\{ \begin{pmatrix} u & v \\ -\bar{v} & \bar{u} \end{pmatrix} : u, v \in \mathbb{C}, u\bar{u} + v\bar{v} = 1 \right\}. \end{aligned}$$

In fact B_∞ can be identified with the coset space $G_\infty / Z_\infty K_\infty$ and also with the space $\mathbb{H}_3$ in the obvious manner. We write $\pi : G_\infty \to \mathbb{H}_3$ for the canonical projection of G_∞ onto $\mathbb{H}_3$. The restriction of π to B_∞ is the bijection identifying B_∞ with $\mathbb{H}_3$ so every element of $\mathbb{H}_3$ can be written as $\pi(b)$ for a suitable $b \in B_\infty$.

The action of G_∞ on $\mathbb{H}_3$ is given explicitly by

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot (z,r) = \left(\frac{(\alpha z+\beta)(\bar{\gamma}\bar{z}+\bar{\delta}) + \alpha\bar{\gamma}r^2}{|\gamma z+\delta|^2+|\gamma|^2r^2}, \quad \frac{|\alpha\delta-\beta\gamma|r}{|\gamma z+\delta|^2+|\gamma|^2r^2} \right); \tag{1.9}$$

we write $L_g : \mathbb{H}_3 \to \mathbb{H}_3$ for the map $(z,r) \mapsto g \cdot (z,r)$. We can choose a basis $\beta = (\beta_0, \beta_1, \beta_2)^T$ of left-invariant differentials on $\mathbb{H}_3$ with respect to L_g^* as follows:

$$\beta_0 = -\frac{dz}{r}, \quad \beta_1 = \frac{dr}{r}, \quad \beta_2 = \frac{d\bar{z}}{r}. \tag{1.10}$$

Write $\beta = \beta(z,r)$ as a column vector. For $g \in \mathrm{GL}_2(\mathbb{C})$ and $(z,r) \in \mathbb{H}$, the Jacobian matrix $J(g,(z,r))$ is defined by

$$\beta(g \cdot (z,r)) = J(g;(z,r))\beta(z,r) \tag{1.11}$$

As a function, J satisfies the cocycle relation

$$J(g_1g_2;(z,r) = J(g_1; g_2 \cdot (z,r))J(g_2;(z,r)) \quad \text{for } g_1, g_2 \in \mathrm{GL}_2(\mathbb{C}).$$

Left-invariance under B_∞ gives $J(b;(z,r)) = 1$ for all $b \in B_\infty$, and combined with the cocycle relation we have

$$J(g;(z,r)) = J(\pi(g)^{-1}g;(z,r)),$$

where $\pi(g)^{-1}g \in \mathbb{C}^\times \cdot \mathrm{SU}_2(\mathbb{C})$. Note that $\pi(g) = g \cdot \pi(1)$ where $\pi(1) = (0,1) \in \mathbb{H}_3$.

Let $g = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$. Write $\Delta = ad - bc$, $t = \overline{cz+d}$, and $s = \bar{c}r$. Then we have

$$J(g;(z,r)) = \frac{1}{|\Delta|(t\bar{t}+s\bar{s})} \begin{pmatrix} \Delta & 0 & 0 \\ 0 & |\Delta| & 0 \\ 0 & 0 & \bar{\Delta} \end{pmatrix} \begin{pmatrix} t^2 & -2ts & t^2 \\ t\bar{s} & t\bar{t} - s\bar{s} & -\bar{t}s \\ \bar{s}^2 & 2\bar{t}\bar{s} & \bar{s}^2 \end{pmatrix}. \tag{1.12}$$

We now define the representation ρ on $Z_\infty K_\infty$ to be the restriction of $J(g;(0,1))$ to $g \in Z_\infty K_\infty$. For $\mathcal{F} : \mathbb{H}_3 \to \mathbb{C}^3$ and $g \in G_\infty$ we define $\mathcal{F}|_g$ given explicitly by

$$(\mathcal{F}|_g)(\pi(b)) = \mathcal{F}(g\pi(b))\rho(\pi(gb)^{-1}gb) \quad \text{for } b \in B_\infty. \tag{1.13}$$

In the special case when ρ is given by (1.12), this simplifies to

$$(\mathcal{F}|_g)(z,r) = \mathcal{F}(g \cdot (z,r))J(g;(z,r)) \quad \text{for } (z,r) \in \mathbb{H}_3. \tag{1.14}$$

In the sequel, we attach differential forms to the cusp forms defined in the previous section 1.4.4. Suppose that $\Phi : \mathrm{GL}_2(\mathbb{A}_F) \to V_2(\mathbb{C})$ is a cusp form, giving rise to a cusp form $\mathcal{F}$ on $\mathbb{H}_3$. Let $\mathfrak{F}$ be the corresponding cusp form on $\mathrm{GL}_2(\mathbb{C})$. Restricting to $\mathrm{SL}_2(\mathbb{C})$ and composing with this map, $\mathfrak{F}$ can be considered as $\mathfrak{F} : \mathrm{SL}_2(\mathbb{C}) \to V_2(\mathbb{C})$. Identifying $V_2(\mathbb{C})$ with the space of differentials spanned by the basis $(\beta_0, \beta_1, \beta_2)$ as in (1.10), we can view the map $\mathfrak{F}$ as an element of $\Omega^1(\mathrm{SL}_2(\mathbb{C}), \mathbb{C})$. Then this differential descends to the quotient $\mathbb{H}_3 = \mathrm{SL}_2(\mathbb{C})/\,\mathrm{SU}(2)$.

Proposition 1.5.1. *Let g, b, π be as above. We have*

$$\mathcal{F}(g\pi(b)) \cdot \beta(g \cdot (z,r)) = (\mathcal{F}|_g)(\pi(b)) \cdot \beta(z,r).$$

Proof. Combing (1.11) and (1.13) we observe that

$$\mathcal{F}(g\pi(b)) \cdot \beta(g \cdot (z,r)) = (\mathcal{F}|_g)(\pi(b))\rho(\pi(gb)^{-1}gb)^{-1}J(g;(z,r)) \cdot \beta(z,r).$$

It suffices to show that

$$\rho(\pi(gb)^{-1}gb) = J(g;(z,r)).$$

Write $\pi(b) = (z,r)$ with $\pi(b) = b \cdot \pi(1)$. By the cocycle relation of J, we have

$$J(g;\pi(b)) = J(gb;\pi(1))J(b;\pi(1))^{-1} = J(\pi(gb)^{-1}gb;\pi(1)) = \rho(\pi(gb)^{-1}gb).$$

□

1.5.2 The Eichler-Shimura-Harder isomorphism

In this section we will describe how to realise cusp forms over the imaginary quadratic field as differential forms on arithmetic quotients of the upper half space in an explicit way as outlined in [Gha99, Section 5].

Over F, there are two isomorphisms which are special cases of the isomorphisms for GL_2 over general number fields relating cusp forms to C^∞ harmonic differential forms. We denote these by

$$\sigma_q : S_2(\Gamma_0(\mathfrak{N})) \simeq H^q_{\mathrm{cusp}}(\Gamma_0(\mathfrak{N})\backslash\mathbb{H}_3, \mathbb{C}),$$

with $q = 1, 2$. There is an action of the Hecke algebra on both sides, and the σ_q are Hecke equivariant. In this thesis we only consider the first isomorphism, that is cusp forms over F are realised as differential 1-forms. For simplicity we write σ for σ_1.

Let $\mathcal{F} \in S_2(\Gamma_0(\mathfrak{N}))$ (resp. $\mathfrak{F}$) be the cusp forms defined on $\mathbb{H}_3$ (resp. $\mathrm{GL}_2(\mathbb{C})$). We will describe how to construct $\sigma(\mathcal{F})$ explicitly. Denote the restriction of $\mathfrak{F}$ to $\mathrm{SL}_2(\mathbb{C})$ by $\mathfrak{F}$ for simplicity. The $\mathrm{SL}_2(\mathbb{C})$-action on $V_2(\mathbb{C})$ is given by

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot P\begin{pmatrix} A \\ B \end{pmatrix} = P\left(\begin{pmatrix} \bar{a} & \bar{c} \\ -c & a \end{pmatrix}\begin{pmatrix} A \\ B \end{pmatrix}\right) \text{ for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{C}).$$

Then we define $\sigma(\mathcal{F})(g) = g \cdot \mathfrak{F}(g)$ for $g \in \mathrm{SL}_2(\mathbb{C})$. Here we have replaced $\Omega^1(\mathbb{H}_3)$ with $V_2(\mathbb{C})$ and the pull back action on $\Omega^1(\mathbb{H}_3)$ by the induced action on $V_2(\mathbb{C})$. Thus we must replace (A^2, AB, B^2) by $(-dz, dr, d\bar{z})$. One can show that $\sigma(\mathcal{F})(gu) = \sigma(\mathcal{F})(g)$ for $u \in \mathrm{SU}(2)$ so that $\sigma(\mathcal{F})(gu)$ can be thought of as differential 1-form on $\mathbb{H}_3$.

It is possible to make the above construction completely explicit. Given the auxiliary variables $\mathbf{u} = \binom{U}{V}$, we set

$$\mathbf{Q} = \left(\binom{2}{\alpha} (-1)^{2-\alpha} U^\alpha V^{2-\alpha} \right)_{\alpha=0,1,2} = (V^2, -2UV, U^2).$$

For variable $\mathbf{a} = \binom{A}{B}$, let $\boldsymbol{\psi} = \boldsymbol{\psi}(\mathbf{a}) = (\psi_0, \psi_1, \psi_2)^t$ where the $\psi_i = \psi_i(\mathbf{a})$ is a homogeneous polynomial of degree 2 in $\mathbf{s}$ defined by

$$(AV - BU)^2 = \mathbf{Q} \cdot \boldsymbol{\psi}.$$

It is easy to calculate that $\boldsymbol{\psi}(\mathbf{a}) = (A^2, AB, B^2)$. For $u \in \mathrm{SU}_2(\mathbb{C})$ it has the special property

$$\boldsymbol{\psi}(u\mathbf{a}) = \rho_2(u) \cdot \boldsymbol{\psi}(\mathbf{a}).$$

Here

$$\rho_2 \begin{pmatrix} a & b \\ c & d \end{pmatrix} \cdot \begin{pmatrix} A \\ B \end{pmatrix}^2 = \left(\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} A \\ B \end{pmatrix} \right)^2 \quad \text{for } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{C})$$

where

$$\begin{pmatrix} A \\ B \end{pmatrix}^2 = (A^2, AB, B^2)^t.$$

As $\mathfrak{F}$ takes values in $V_2(\mathbb{C})$ we let $\mathfrak{F}_\alpha$ be the components of $\mathfrak{F}$, namely $\mathfrak{F}(g, \mathbf{a}) = \sum_{\alpha=0}^2 \mathfrak{F}_\alpha(g) A^{2-\alpha} B^\alpha$. Define $\mathfrak{F}' : \mathrm{SL}_2(\mathbb{C}) \to V_2(\mathbb{C})$ by

$$\mathfrak{F}'(g, \mathbf{a}) = (\mathfrak{F}_0(g), \mathfrak{F}_1(g), \mathfrak{F}_2(g)) \cdot \boldsymbol{\psi}(\mathbf{a}).$$

One can prove that $\mathfrak{F}'(gu, \mathbf{a}) = \mathfrak{F}'(g, u\mathbf{a})$ for $u \in \mathrm{SU}_2(\mathbb{C})$. Finally define $\mathfrak{F}'' : \mathrm{SL}_2(\mathbb{C}) \to V_2(\mathbb{C})$ by $\mathfrak{F}''(g, \mathbf{a}) = \mathfrak{F}'(g, g\mathbf{a})$. One can checke that it has the property $\mathfrak{F}''(gu, \mathbf{a}) = \mathfrak{F}''(g, \mathbf{a})$.

Thus, in summary, we have

Definition 1.5.2. [Gha99, Definition 6] $\sigma(\mathcal{F})$ is the $\mathbb{C}$-valued differential form on $\mathbb{H}_3$ obtained from $\mathfrak{F}''$ by replacing (A^2, AB, B^2) by $(-dz, dr, d\bar{z})$. More specifically, if $g \cdot (0,1) = (z, r)$, e.g. $g = \begin{pmatrix} \sqrt{r} & \frac{z}{\sqrt{r}} \\ 0 & \frac{1}{\sqrt{r}} \end{pmatrix}$, then

$$\sigma(\mathcal{F})(z, r) = (\mathfrak{F}_0(g), \mathfrak{F}_1(g), \mathfrak{F}_2(g)) \cdot \frac{1}{r} \boldsymbol{\psi}(\mathbf{a}) = -\mathfrak{F}_0(g) \frac{dz}{r} + \mathfrak{F}_1(g) \frac{dr}{r} + \mathfrak{F}_2(g) \frac{d\bar{z}}{r}.$$

Here $\mathfrak{F}_i(g)$ for $i \in \{0, 1, 2\}$ is in one-to-one correspondence to $\mathcal{F}_i(z, r)$ as in (1.6).

§ 1.6 Automorphic representations

In Section 4.5 we will use results in [FH95] about the non-vanishing of L-values attached to automorphic representations for $\mathrm{GL}_2(\mathbb{A}_F)$ over an imaginary quadratic field F. So in this section we will sketch the passage from weight 2 Bianchi modular forms defined in the previous section 1.4.4 to the corresponding automorphic representations. The material in this section can be found in [GH11] and [Kud03].

For simplicity denote $G = \mathrm{GL}_2$ and $\mathbb{A} = \mathbb{A}_F$ for the imaginary quadratic field of class number 1.

Definition 1.6.1 (Definition 2.1, [Kud03])**.** The space of automorphic forms $\mathcal{A}(G)$ on $G(\mathbb{A})$ with trivial central character is the space of complex valued functions Φ on $G(\mathbb{A})$ as defined in Definition 1.4.6 such that:

(1) $\Phi(z\gamma g) = \Phi(g)$ for $z \in \mathcal{Z}(\mathbb{A}) \simeq \mathbb{A}^\times$ and $\gamma \in G(F)$.
(2) The function $g_\infty \mapsto \Phi(g_\infty g_f)$ is smooth on $G(F_\infty)$.
(3) The space spanned by the right translates of Φ by elements of $K_\infty \simeq \mathrm{U}(2)$ is finite-dimensional, i.e., Φ is right K_∞-finite.
(4) There is a compact open subgroup $K_f \subset G(\mathbb{A}_f)$ such that Φ is invariant under right translation by K_f.
(5) Φ is $\mathcal{Z}(\mathfrak{g})$-finite.
(6) Φ is of moderate growth.

Definition 1.6.2 (Definition 2.2, [Kud03])**.** The space $\mathcal{A}_0(G) \subset \mathcal{A}(G)$ of cuspidal automorphic forms is defined by adding the following cuspidal condition:

(7) For all $g \in G(\mathbb{A})$,
$$\int_{F\backslash\mathbb{A}} \Phi\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} g\right) dx = 0.$$

Recall that $\mathfrak{g}$ denotes the complexification of $\mathfrak{gl}_2(\mathbb{C})$, $\mathcal{U}(\mathfrak{g})$ its enveloping algebra whose elements can be identified with differential operators D_α for $\alpha \in \mathfrak{g}$, and $K_\infty \simeq U(2)$. Next we define two important types of modules playing a major role in the representation theory:

$$(\mathfrak{g}, K_\infty) - \text{module} \quad \text{and} \quad (\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f) - \text{module}.$$

Definition 1.6.3. [Kud03, Definition 2.3(i)] We define a $(\mathfrak{g}, K_\infty)$-module to be a complex vector space V with actions

$$\pi_{\mathfrak{g}} : U(\mathfrak{g}) \to \operatorname{End} V = \{\text{set of all linear maps } V \to V\},$$
$$\pi_{K_\infty} : K_\infty \to \mathrm{GL}(V) = \{\text{set of all invertible linear maps } V \to V\},$$

such that the subspace of V spanned by $\{\pi_{K_\infty}(k) \cdot v : k \in K_\infty\}$ is finite dimensional, and

$$\pi_{\mathfrak{g}}(D_\alpha) \cdot \pi_{\pi_\infty}(k) = \pi_{\pi_\infty}(k) \cdot \pi_{\mathfrak{g}}(D_{k^{-1}\alpha k}) \text{ for all } \alpha \in \mathfrak{g}, k \in K_\infty.$$

Further we require that

$$\pi_{\mathfrak{g}}(D_\alpha) \cdot v = \lim_{t \to 0} \frac{1}{t}(\pi_{K_\infty}(\exp(t\alpha)) \cdot v - v)$$

for all $v \in V$ and α in the Lie algebra of K_∞.

We shall denote the pair of actions $(\pi_{\mathfrak{g}}, \pi_{K_\infty})$ by π and refer to the ordered pair (π, V) as a $(\mathfrak{g}, K_\infty)$-module.

Definition 1.6.4. [Kud03, Definition 2.3(ii)] We define a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module to be a complex vector space with actions

$$\pi_{\mathfrak{g}} : U(\mathfrak{g}) \to \mathrm{End}(V),$$
$$\pi_{K_\infty} : K_\infty \to \mathrm{GL}(V),$$
$$\pi_f : G(\mathbb{A}_f) \to \mathrm{GL}(V),$$

such that $V, \pi_{\mathfrak{g}}$ and π_{K_∞} form a $(\mathfrak{g}, K_\infty)$-module, and in addition the relations

$$\pi_f(a_f) \cdot \pi_{\mathfrak{g}}(D_\alpha) = \pi_{\mathfrak{g}}(D_\alpha) \cdot \pi_f(a_f),$$
$$\pi_f(a_f) \cdot \pi_{K_\infty}(k) = \pi_{K_\infty}(k) \cdot \pi_f(a_f),$$

are satisfied for $\alpha \in \mathfrak{g}$, $D_\alpha \in U(\mathfrak{g})$, $k \in K_\infty$ and $a_f \in G(\mathbb{A}_f)$.

We let $\pi = ((\pi_{\mathfrak{g}}, \pi_{K_\infty}), \pi_f)$ and refer to the ordered pair (π, V) as a $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module. We say the $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module is *smooth* if every vector $v \in V$ is fixed by some open compact subgroup of $G(\mathbb{A}_f)$ under the action π_f. The $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module is said to be *irreducible* if it is non-zero and has no proper non-zero subspace preserved by the actions $\pi_{\mathfrak{g}}, \pi_{K_\infty}, \pi_f$. One main result is that $\mathcal{A}_0(G)$ is a smooth $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$-module, see [Kud03, Section 2]. The representation of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ on $\mathcal{A}_0(G)$ is the so-called *automorphic representation.*

In the previous section 1.4.4 we have defined the weight 2 Bianchi modular form $\mathcal{F}$ on $\mathbb{H}_3$ and the corresponding $\Phi_{\mathcal{F}}$ on $G(\mathbb{A}) = \mathrm{GL}_2(\mathbb{A}_F)$ taking values in $V_2(\mathbb{C})$. To have scalar-valued functions, we can consider any non-zero linear form L on $V_2(\mathbb{C})$, and define

$$\widetilde{\mathcal{F}}(z,r) := L(\mathcal{F}(z,r)) \quad \text{for } (z,r) \in \mathbb{H}_3$$

and

$$\widetilde{\Phi_{\mathcal{F}}}(g) := L(\Phi(g)) \quad \text{for } g \in G(\mathbb{A}).$$

It is straightforward to check that $\widetilde{\Phi_{\mathcal{F}}} \in \mathcal{A}_0(G)$ since $\Phi_{\mathcal{F}} \in \mathcal{A}_0(G)$ as discussed in Remark 1.4.11, e.g.

$$\widetilde{\Phi}(gg_f) = L(\Phi(gg_f)) = L(\Phi(g)) = \widetilde{\Phi}(g) \quad \text{for } g_f \in K_f \subset G(\mathbb{A}_f).$$

The choice of L is irrelevant as we will eventually consider the space $\mathcal{A}_{0,\widetilde{\mathcal{F}}} \subset \mathcal{A}_0$ spanned by all right translates of $\widetilde{\Phi_{\mathcal{F}}}$ under the action of π.

Definition 1.6.5 (Analogue of Definition 5.2, [Tro])**.** Let $\widetilde{\mathcal{F}}$ and $\widetilde{\Phi_{\mathcal{F}}}$ be as above. The automorphic representation $\pi_{\widetilde{\mathcal{F}}}$ attached to $\widetilde{\mathcal{F}}$ is the restriction of the representation of $(\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)$ on the subspace $\mathcal{A}_{0,\widetilde{\mathcal{F}}}(G)$ of $\mathcal{A}_0(G)$ defined by:

$$\mathcal{A}_{0,\widetilde{\mathcal{F}}}(G) := \{\pi(g)\widetilde{\Phi_{\mathcal{F}}} : g \in (\mathfrak{g}, K_\infty) \times G(\mathbb{A}_f)\}.$$

Chapter 2

Weil representation and the Kudla-Millson theory

§ 2.1 Weil representation for symplectic-orthogonal dual pair

In this section we discuss the Weil representation for the dual pair $(\mathrm{Sp}(W), \mathrm{O}(V))$ following Kudla's lecture notes [Kud96].

Let W be a symplectic vector space over a local field F. For a group A with subgroup B, we let

$$\mathrm{Cent}_A(B) = \{a \in A : ab = ba \text{ for all } b \in B\}$$

be the commutant of B in A. A pair of subgroups B and B' of A are said to be mutual commutants if $\mathrm{Cent}_A(B) = B'$ and $\mathrm{Cent}_A(B') = B$. A reductive dual pair (G, G') in $\mathrm{Sp}(W)$ is a pair of subgroups G and G' of $\mathrm{Sp}(W)$ such that G_1 and G_2 are reductive groups and

$$\mathrm{Cent}_{\mathrm{Sp}(W)}(G) = G' \quad \text{and} \quad \mathrm{Cent}_{\mathrm{Sp}(W)}(G') = G.$$

We will simply call such (G, G') as a dual pair. The pair $(\mathrm{Sp}(W), \{\pm 1_W\})$ is the most trivial example of a dual pair.

Dual pairs can be constructed as tensor products. Let W be a finite dimensional left vector space over F with a non-degenerate skew-symmetric bilinear form

$$\langle\ ,\ \rangle : W \times W \longrightarrow F$$

with

$$\langle ax, by\rangle = a\langle x, y\rangle b \quad \text{and} \quad \langle y, x\rangle = -\langle x, y\rangle.$$

Let

$$\mathrm{Sp}(W) = \{g \in \mathrm{GL}(W) : \langle xg, yg\rangle = \langle x, y\rangle \text{ for all } x, y \in W\}$$

be the isometry group of W. Similarly, let V be a finite dimensional right vector space over F with a non-degenerate symmetric bilinear form

$$(\ ,\ ) : V \times V \longrightarrow F$$

with

$$(xa, xb) = a(x, y)b \quad \text{and} \quad (y, x) = \epsilon(x, y).$$

Let

$$\mathrm{O}(V) = \{g \in \mathrm{GL}(V) : (gx, gy) = (x, y) \text{ for all } x, y \in V\}$$

be the isometry group of V.

The rational vector space $\mathbb{W} = W \otimes_F V$ has a non-degenerate bilinear alternating form

$$\langle\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle\rangle = \langle x_1, x_2\rangle(y_1, y_2)$$

and there is a natural map

$$\mathrm{Sp}(W) \times \mathrm{O}(V) \to \mathrm{Sp}(\mathbb{W}), \quad (h, g) \mapsto h \otimes g.$$

Thus we obtain a dual pair $(\mathrm{Sp}(W), \mathrm{O}(V))$ in $\mathrm{Sp}(\mathbb{W})$.

Let $\dim_F W = 2n$ and $\dim_F V = m$. We will describe a Weil representation of $\mathrm{Sp}(W) \times \mathrm{O}(V)$ when m is even, and of $\mathrm{Mp}(W) \times \mathrm{O}(V)$ when m is odd where Mp denotes the metapletic cover of Sp. We can make the identification $\mathrm{Mp}(W) = \mathrm{Sp}(W) \times \mathbb{C}^\times$. In this thesis we are interested in $m = 3$ with signature (2,1) and $m = 4$ with signature (3,1).

Let ψ be an additive character on F and χ_V be the quadratic character of $F^\times$ defined by

$$\chi_V(x) = (x, (-1)^{\frac{m(m-1)}{2}} \det(V))_F$$

where $\det(V)$ is the determinant of the Gram matrix with respect to the bilinear form on V and $(\ ,\ )_F$ denotes the Hilbert symbol. We let, for $x \in F$,

$$\gamma(\psi) = \gamma(\psi \circ x^2)$$

be the Weil index of the character ϕ of second degree on F given by $\phi(x) = \psi(x^2)$.

For $a \in F^\times$, let

$$\gamma(a, \psi) = \gamma(\psi_a)/\gamma(\psi)$$

where $\psi_a(x) = \psi(ax)$. For $z \in \mathbb{C}^\times$, let

$$\chi_V^\psi(x, z) = \chi_V(x) \cdot \begin{cases} z \cdot \gamma(x, \psi)^{-1}, & \text{if } m \text{ is odd,} \\ 1, & \text{if } m \text{ is even.} \end{cases}$$

Fix a direct sum $W = X + Y$ with maximal isotropic subgroups X, Y. Then we have that $\mathbb{W} = X \otimes V + Y \otimes V$ and that

$$X \otimes V \simeq V^n = \{\mathbf{x} = (x_1, \ldots, x_n) : x_i \in V\}.$$

For $\mathbf{x}, \mathbf{y} \in V^n$, we write

$$(\mathbf{x}, \mathbf{y}) = ((x_i, y_j)) \in \mathrm{Sym}_n(F).$$

View elements of W as row vectors (x, y) with $x \in X$ and $y \in Y$. We can write $g \in \mathrm{Sp}(W)$ as

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where $a \in \mathrm{End}(X)$, $b \in \mathrm{Hom}(X, Y)$, $c \in \mathrm{Hom}(Y, X)$ and $d \in \mathrm{End}(Y)$. Let

$$M = \left\{ m(a) = \begin{pmatrix} a & 0 \\ 0 & a^\vee \end{pmatrix} : a \in \mathrm{GL}(X) \right\}$$

and

$$N = \left\{ n(b) = \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} : b \in \mathrm{Hom}(X, Y), \text{ symmetric} \right\}.$$

Here $a^\vee \in \mathrm{GL}(Y)$ is determined by the condition that $\langle xa, ya^\vee \rangle = \langle x, y \rangle$ for all $x \in X$ and $y \in Y$.

The Weil representation of $\mathrm{Mp}(W) = \mathrm{Sp}(W) \times \mathbb{C}^\times$ can be realised on the Schwartz space $S(V^n)$ which is the space of locally constant, compactly supported functions on V^n if F is non-archimedean, and consists of those Schwartz functions of the form $p(\mathbf{x})\varphi_0(\mathbf{x})$ if F is archimedean, where p is a polynomial function on V^n and φ_0 denotes the standard Gaussian later given in (2.8).

Let $\varphi \in S(V^n)$ and the action of $\mathrm{O}(V)$ on $S(V^n)$ is given by

$$\omega_{\psi, W}(h)\varphi(\mathbf{x}) = \varphi(h^{-1}\mathbf{x}) \quad \text{for } h \in \mathrm{O}(V), \mathbf{x} \in V^n.$$

We will describe the action of $\mathrm{Mp}(W)$ on $S(V^n)$ as follows. For $a \in \mathrm{GL}_n(F)$ and $z \in \mathbb{C}^\times$,

$$\omega_{\psi, V}(m(a), z)\varphi(\mathbf{x}) = \chi_V^\psi(\det(a), z) |\det(a)|^{\frac{m}{2}} \varphi(\mathbf{x}a)$$

and for $b \in \mathrm{Sym}_n(F)$,

$$\omega_{\psi,V}(n(b),1)\varphi(x) = \psi\left(\frac{1}{2}\operatorname{tr}((\mathbf{x},\mathbf{x})b)\right)\varphi(\mathbf{x}).$$

For $w = w_n = \begin{pmatrix} 0 & 1_n \\ -1_n & 0 \end{pmatrix}$,

$$\omega_{\psi,V}(w,1)\varphi(\mathbf{x}) = \gamma(\psi\circ V)^{-n}\int_{V^n}\psi(-\operatorname{tr}((\mathbf{x},\mathbf{y})))\varphi(\mathbf{y})d\mathbf{y}$$

where $d\mathbf{y}$ is the measure on V^n which is self dual for this Fourier transform. Here the factor $\gamma(\psi\circ V)$ is the Weil index of the quadratic space associated to the bilinear form on V, which is an eighth root of unity depending only on the isomorphism class of quadratic forms on F^m.

Using the local Weil representation, we can define a global representation of $\mathrm{Sp}(\mathbb{W})(\mathbb{A})$ for the adele ring $\mathbb{A}$ of a number field, see [Ral84, Section VIII]. There is a projective representation of $\mathrm{Sp}(\mathbb{W})(\mathbb{A})$ on $S(V(\mathbb{A})^n)$ by taking the tensor product of local representations.

We end this section via showing the Weil representation in two cases of interest in this thesis:

Example 2.1.1. Fix an additive character ψ_q of $\mathbb{Q}_q$ with kernel $q\mathbb{Z}_q$. Note that its kernel is crucial for us to determine the level of the corresponding theta lift.

(1) As given in Example 1.1.6, let V be a 3-dimensional quadratic space of signature (2,1) with quadratic form Q. In this case we consider the metaplectic cover $\widetilde{\mathrm{SL}_2} = \mathrm{SL}_2\times\{\pm 1\}$ paired against the orthogonal group $\mathrm{SO}(2,1)$. For $x\in\mathbb{Q}_q$ and $\alpha\in\mathbb{Q}_q^\times$, let

$$\overline{\mathbf{n}}(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix},\ \underline{\mathbf{n}}(x) = \begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix},\ \mathbf{d}(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \alpha^{-1} \end{pmatrix},\ w = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

and notice that

$$\underline{\mathbf{n}}(x) = \mathbf{d}(-1)\cdot w\cdot\overline{\mathbf{n}}(-x)\cdot w.$$

The local Weil representation ω_{ψ_q} of $\mathrm{SL}_2(\mathbb{Q}_q)\times\{\pm1\}$ on $S(V(\mathbb{Q}_q))$, which can also be found in [Pra09, Section 2.1.3], is characterised by

$$\begin{aligned}
\omega_{\psi_q}(\overline{\mathbf{n}}(y))\varphi_q(\mathbf{x}) &= \psi(yQ(\mathbf{x}))\varphi_q(\mathbf{x}),\\
\omega_{\psi_q}(\mathbf{d}(\alpha))\varphi_q(\mathbf{x}) &= (1/q,\alpha)_q^3(\alpha,-1)_q|\alpha|^{3/2}\varphi_q(\alpha\mathbf{x}),\\
\omega_{\psi_q}(w)\varphi_q(\mathbf{x}) &= \gamma_{\psi_q}\hat{\varphi}_q(\mathbf{x}),\\
\omega_{\psi_q}(1,\epsilon)\varphi_q(\mathbf{x}) &= \epsilon\varphi_q(\mathbf{x}),
\end{aligned}$$

where $\epsilon \in \{\pm 1\}$ and $\hat{\varphi}_q(\mathbf{x})$ denotes the Fourier transform with respect to the pairing $(\mathbf{x}_1, \mathbf{x}_2) \mapsto \psi_q((\mathbf{x}_1, \mathbf{x}_2))$ and γ_{ψ_q} is a certain complex number of absolute value 1.

(2) Let V be a 4-dimensional quadratic space of signature (3,1) with quadratic form Q as in Example 1.1.6. In this case we consider the pair $\mathrm{Sp}_4 \times \mathrm{SO}(3,1)$, of which the Weil representation on $S(V(\mathbb{Q}_q)^2)$ (see [Ber14, Section 3.1]) is given by, for $\mathbf{X} \in V(\mathbb{Q}_q)^2$,

$$\omega(1,h)\varphi_q(\mathbf{X}) = \varphi_q(h^{-1}\mathbf{X}), \tag{2.1}$$

$$\omega\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}, 1\right)\varphi_q(\mathbf{X}) = \psi_q\left(\frac{1}{2}\operatorname{tr}(b(\mathbf{X},\mathbf{X}))\right)\varphi_q(\mathbf{X}), \tag{2.2}$$

$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1} \end{pmatrix}, 1\right)\varphi_q(\mathbf{X}) = \chi_{V,q}(\det(a))|\det(a)|_q^2\varphi_q(\mathbf{X}a), \tag{2.3}$$

$$\omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, 1\right)\varphi_q(\mathbf{X}) = \gamma\hat{\varphi}_q(\mathbf{X}). \tag{2.4}$$

Here the Fourier transform is defined by

$$\hat{\varphi}_q(\mathbf{X}) = \int_{V(\mathbb{Q}_q)^2} \varphi_q(\mathbf{Y})\psi_q(\operatorname{tr}(\mathbf{X},\mathbf{Y}))d\mathbf{Y}$$

and γ is a certain complex number of absolute value 1.

§ 2.2 Theta series

We first recall the classical theta series from Funke's notes [Fun08].

Let V be a rational vector space of dimension $m = p + q$ with a non-degenerate positive definite symmetric bilinear form $(\ ,\ )$. Assume the dimension m is even so that we do not need to consider the metaplectic cover. Let L be an even lattice of level N; that is $Q(x) := (x,x) \in 2\mathbb{Z}$ for $x \in L$ and $Q(L^\sharp)\mathbb{Z} = \frac{1}{N}\mathbb{Z}$. Here $L^\sharp$ is the dual lattice. Furthermore, we fix a vector $h \in L^\sharp/L$ once and for all and write $\mathcal{L} = h + L$. It is well known that for $\tau \in \mathbb{H}_2 = \mathcal{H}_1$, the upper half plane, the associated theta series

$$\theta(\tau, \mathcal{L}) = \sum_{x \in \mathcal{L}} e^{\pi i (x,x) z} \in M_{\frac{m}{2}}(\Gamma(N)) \tag{2.5}$$

is a modular form for the principal congruence subgroup $\Gamma(N) \subset \mathrm{SL}_2(\mathbb{Z})$ of weight $\frac{m}{2}$.

We have a more representation-theoretic approach to describe this classical theta series. Let $S(V(\mathbb{R}))$ be the space of Schwartz functions $\varphi(\mathbb{R})$ on $V_\mathbb{R}$. We write $G' = \mathrm{SL}_2(\mathbb{R})$ and let $K' = \mathrm{SO}(2)$ be its standard maximal compact subgroup. Let $G =$

$\mathrm{O}(V(\mathbb{R}))$ be the orthogonal group of $V(\mathbb{R})$. Then $G' \times G$ acts on $S(V(\mathbb{R}))$ via the Weil representation $\omega_{\mathbb{R}}$ for the additive character $t \mapsto e^{2\pi i t}$. For $x \in V(\mathbb{R})$ and $g \in G$, G acts naturally on $S(V(\mathbb{R}))$ by

$$\omega_{\mathbb{R}}(g)\varphi_{\mathbb{R}}(x) = \varphi_{\mathbb{R}}(g^{-1}x).$$

Following the discussion in the previous subsection 2.1, the action of G' is given as follows:

$$\omega_{\mathbb{R}}\left(\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}\right)\varphi_{\mathbb{R}}(x) = a^{m/2}\varphi_{\mathbb{R}}(xa),$$
$$\omega_{\mathbb{R}}\left(\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}\right)\varphi_{\mathbb{R}}(x) = e^{\pi i b(x,x)}\varphi_{\mathbb{R}}(x),$$
$$\omega_{\mathbb{R}}\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right)\varphi_{\mathbb{R}}(x) = i^{m/2}\hat{\varphi}_{\mathbb{R}}(x),$$

where $a > 0$ and $\hat{\varphi}_{\mathbb{R}}(y) = \int_{V_{\mathbb{R}}} \varphi_{\mathbb{R}}(x)e^{-2\pi i(x,y)}dx$ is the Fourier transform.

Let $\varphi_{\mathbb{R}} \in S(V(\mathbb{R}))$ be an eigenfunction under $\mathrm{SO}(2)$ of weight r; that is $\omega_{\mathbb{R}}(k')\varphi_{\mathbb{R}} = \chi_r(k')\varphi_{\mathbb{R}}$ for $k' \in \mathrm{SO}(2)$, where χ_r is the standard one-dimensional character of $\mathrm{SO}(2) \simeq U(1)$ given by $z \mapsto \chi_r(z) = z^r$. Then we can define

$$\varphi_{\mathbb{R}}(x,\tau) = j(g'_\tau, i)^r\omega(g'_\tau)\varphi_{\mathbb{R}}(x) = v^{-r/2+m/4}\varphi_{\mathbb{R}}(\sqrt{v}x)e^{\pi i(x,x)u},$$

where $g'_\tau \in \mathrm{SL}_2(\mathbb{R})$ is any element which moves the base point $i \in \mathbb{H}_2$ to $\tau = u+iv \in \mathbb{H}_2$ and $j(g'_\tau, i) = v^{-1/2}$ denotes the usual automorphy factor. Then the associated theta series is defined as

$$\theta(\tau, \varphi_{\mathbb{R}}, \mathcal{L}) := \sum_{x\in\mathcal{L}} \varphi_{\mathbb{R}}(x,\tau), \tag{2.6}$$

which is in general non-holomorphic modular form of level N and weight r, see e.g. [FM02, Theorem 4.5]. For the above classical theta series (2.5), we have $\theta(\tau,\mathcal{L}) = \theta(\tau,\varphi_0,\mathcal{L})$ with the Gaussian $\varphi_0(x) := e^{-\pi(x,x)}$.

In the following we review a family of Schwartz forms in $S(V(\mathbb{R})^n) \otimes \Omega^n(D)$ taking values in the space of differential forms on the symmetric space D, constructed by Kudla and Millson, see [FM02, Section 4] and [FM06, Section 5].

Let V be a real quadratic space of dimension $m = p+1$ and signature $(p,1)$. Denote by $S(V^n)$ the space of complex-valued Schwartz functions on V^n. Let $G' = \mathrm{Mp}_n(\mathbb{R})$ be the metaplectic cover of the symplectic group $\mathrm{Sp}_n(\mathbb{R})$ and K' the inverse image of the standard maximal compact subgroup $\mathrm{U}(n) \subset \mathrm{Sp}_n(\mathbb{R})$ under the covering map $\mathrm{Mp}_n(\mathbb{R}) \to \mathrm{Sp}_n(\mathbb{R})$. The embedding of $\mathrm{U}(n)$ into $\mathrm{Sp}_n(\mathbb{R})$ is given by $A+iB \mapsto$

$\begin{pmatrix} A & B \\ -B & A \end{pmatrix}$. Let $\omega = \omega_V$ be the Weil representation of $G' \times \mathrm{O}(V)$ acting on $S(V^n)$ associated to the additive character $t \mapsto e^{2\pi i t}$.

Let $\mathcal{H}_n = \{\tau = u + iv \in \mathrm{Sym}_n(\mathbb{C}) : v > 0\} \simeq \mathrm{Sp}_n(\mathbb{R})/\,\mathrm{U}(n)$ be the Siegel upper half space of genus n. Write $\mathfrak{g}'$ and $\mathfrak{k}'$ for the complexified Lie algebra of $\mathrm{Sp}_n(\mathbb{R})$ and $\mathrm{U}(n)$ respectively. We have the Cartan decomposition as $\mathfrak{g}' = \mathfrak{k}' \oplus \mathfrak{p}'$ and write $\mathfrak{p}' = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ for the decomposition of the tangent space of the base point $i1_n$ into the holomorphic and anti-holomorphic tangent spaces.

Let $G = \mathrm{SO}_0(V(\mathbb{R}))$ be the identity component of $\mathrm{SO}(V(\mathbb{R}))$ and K the maximal compact subgroup of G stablising the base point Z_0 of D. We have seen in Section 1.1 that the symmetric space $D \simeq G/K$ can be realised as the set of negative 1-planes in V and we have demonstrated this in Example 1.1.6 for signature (2,1) and (3,1).

We pick an orthonomal basis $\{e_i\}$ of V such that $(e_\alpha, e_\alpha) = 1$ for $\alpha = 1, \dots, p$ and $(e_\mu, e_\mu) = -1$ for $\mu = p+1$. Let $\mathfrak{g}$ be the Lie algebra of G and $\mathfrak{k}$ of K. We write the Cartan decomposition as $\mathfrak{g} = \mathfrak{p} + \mathfrak{k}$. Then $\mathfrak{p} \simeq \mathfrak{g}/\mathfrak{k}$ is isomorphic to the tangent space at the base point of D. The elements $X_{\alpha\mu}$ of the standard basis of $\mathfrak{p}$ is induced by the basis $\{e_i\}$ of V, i.e.,

$$X_{\alpha\mu}(e_i) = \begin{cases} e_\mu, & \text{if } i = \alpha, \\ e_\alpha, & \text{if } i = \mu, \\ 0, & \text{otherwise.} \end{cases} \tag{2.7}$$

We let $\omega_{\alpha\mu} \in \mathfrak{p}^*$ be the elements of the associated dual basis, and $\Omega^k(D)$ the space of complex-valued differential k-forms on D.

The main result of [KM90] is the construction of a certain differential n-form of D with values in the Schwartz space $S(V(\mathbb{R})^n)$.

Theorem 2.2.1 (Theorem 4.1, [FM02])**.** *For each n with $0 \le n \le p$, there is a non-zero Schwartz form*

$$\varphi_n \in [S(V(\mathbb{R})^n) \otimes \Omega^n(D)]^G \simeq \left[S(V(\mathbb{R})^n) \otimes \bigwedge^n (\mathfrak{p}^*) \right]^K,$$

such that

(1)

$$d\varphi_n = 0,$$

i.e., for each $X \in V(\mathbb{R})^n$, φ_n *is a closed* n*-form on* D *which is* G_X*-invariant:*

$$g^*\varphi_n(X) = \varphi_n(X)$$

for $g \in G_X$, *the stabilizer of* X *in* G.

(2) *The forms are compatible with the wedge product:*

$$\varphi_{n_1} \wedge \varphi_{n_2} = \varphi_{n_1+n_2},$$

where $\varphi_n = 0$ *for* $n > p$.

Fundamental for its relationship to modular forms is that

$$\omega(k')\varphi_n = \det(k')^{m/2}\varphi_n$$

for $k' \subset K'$ the maximal compact subgroup of $\mathrm{Mp}_n(\mathbb{R})$ where ω is the Weil representation.

We now give some explicit formulae for the form $\varphi_n \in [S(V(\mathbb{R})^n) \otimes \bigwedge^n(\mathfrak{p}^*)]^K$, see [FM02, Section 4]. At the end of this subsection, we will show two examples used in this thesis, φ_1 for signature (2,1) and φ_2 for (3,1).

Consider the standard Gaussian,

$$\varphi_0(X) = e^{-\pi \operatorname{tr}(X,X)_{Z_0}} \in S(V^n) \quad \text{for } X = (x_1, \dots, x_n) \in V^n, \tag{2.8}$$

where the majorant $(\ ,\ )_z$ is given by

$$(X,X)_Z = \begin{cases} (X,X), & \text{if } X \in Z^\perp, \\ -(X,X), & \text{if } X \in \mathbb{R}Z. \end{cases}$$

Following (2.7) we can determine the basis of $\mathfrak{p}$ as $\{e_1, \dots, e_p\}$ so $\mathfrak{p}$ can be identified with $\mathbb{R}^p$. Then ω_i becomes the functional on $\mathfrak{p}$ which picks out the i-th coordinate. For $X = (x_1, \dots, x_n) \in V(\mathbb{R})^n \simeq M_{m,n}(\mathbb{R})$ w.r.t. the basis $\{e_1, \dots, e_{p+1}\}$, and for $1 \le s \le n$, we define the 1-form

$$\omega(s,X) = \sum_{i=1}^{p} x_{is}\omega_i.$$

We set

$$\begin{aligned} 2^{-n/2}\varphi_n(X) &= \left(\bigwedge_{s=1}^{n} \omega(s,X)\right) \cdot \varphi_0(X) \\ &= \varphi_1(x_1) \wedge \dots \wedge \varphi_1(x_n). \end{aligned}$$

It can be seen that

$$\varphi_n(X) = 2^{n/2} \sum_{1 \le j_1 < \dots < j_n \le p} P_{j_1,\dots,j_n}(X) \exp(-\pi \operatorname{tr}(X,X)_{Z_0}) \otimes \omega_{j_1} \wedge \dots \wedge \omega_{j_n},$$

where $P_{j_1,\ldots,j_n}(X)$ is the determinant of the n by n matrix obtained from X by removing all rows except the $j_1,\ldots,j_n$. Then we have

$$\varphi_n(X)(W) = 2^{n/2}\det(X,W)\exp(-\pi\,\mathrm{tr}(X,X)_{Z_0})$$

for $W \in T_{Z_0}(D)^n \simeq \mathfrak{p}^n \simeq (Z_0^\perp)^n$. We demonstrate this in two cases of interest in this thesis:

Example 2.2.2. (1) Let $p = 2$ and $n = 1$. For $X = x_1 = x_{11}e_1 + x_{21}e_2 + x_{31}e_3 \in V(\mathbb{R})$, we have the 1-form

$$\omega(1,X) = x_{11}\omega_1 + x_{21}\omega_2.$$

Set

$$2^{-1/2}\varphi_1(X) = (x_{11}\omega_1 + x_{21}\omega_2)\varphi_0(X).$$

For $W = \nu_1 e_1 + \nu_2 e_2 \in Z_0^\perp$, we have

$$\begin{aligned}\varphi_1(X)(W) &= 2^{1/2}(x_{11}\omega_1 + x_{21}\omega_2)(W)\varphi_0(X) = 2^{1/2}(x_{11}\nu_1 + x_{21}\nu_2)\varphi_0(X)\\ &= 2^{1/2}(X,W)\varphi_0(X)\end{aligned}$$

where the last equality is the consequence of

$$\begin{aligned}(X,W) &= (x_{11}e_1 + x_{21}e_2 + x_{31}e_3, \nu_1 e_1 + \nu_2 e_2)\\ &= x_{11}\nu_1(e_1,e_1) + x_{21}\nu_2(e_2,e_2) = x_{11}\nu_1 + x_{21}\nu_2.\end{aligned}$$

(2) Let $p = 3$ and $n = 2$. For

$$V(\mathbb{R})^2 \ni X = (x_1,x_2) \simeq \begin{pmatrix} x_{11} & x_{12}\\ x_{21} & x_{22}\\ x_{31} & x_{32}\\ x_{41} & x_{42}\end{pmatrix},$$

we define the 1-form for $s \in \{1,2\}$

$$\omega(s,X) = x_{1s}\omega_1 + x_{2s}\omega_2 + x_{3s}\omega_3,$$

and then

$$\begin{aligned}2^{-1}\varphi_2(X) &= (\omega(1,X)\wedge\omega(2,X))\cdot\varphi_0(X)\\ &= ((x_{11}x_{22} - x_{12}x_{21})\omega_1\wedge\omega_2 + (x_{11}x_{32} - x_{12}x_{31})\omega_1\wedge\omega_3\\ &\quad + (x_{21}x_{32} - x_{31}x_{22})\omega_2\wedge\omega_3)\cdot\varphi_0(X).\end{aligned}$$

For $W = (W_1,W_2) \in (Z_0^\perp)^2$ with $W_j = \nu_{1j}e_1 + \nu_{2j}e_2 + \nu_{3j}e_3$, $j = 1,2$, we can show

$$\begin{aligned}\varphi_2(X)(W) =& 2\left((x_{11}x_{22} - x_{12}x_{21})\nu_{11}\nu_{22} + (x_{11}x_{32} - x_{12}x_{31})\nu_{11}\nu_{32}\right.\\ &\left.+(x_{21}x_{32} - x_{31}x_{22})\nu_{22}\nu_{32}\right) \cdot \varphi_0(X)\\ =& 2\det(X, W)\varphi_0(X),\end{aligned}$$

where the last identity is deduced via expanding

$$\det(X, W) = (x_1, W_1)(x_2, W_2) - (x_1, W_2)^2.$$

Now we can form the theta series for φ_n, as in (2.6), which can be seen to be a Γ-invariant differential form on D for a subgroup Γ of finite index of the stabilizer of $\mathcal{L}$ in G. Hence it descends to a form on $X = \Gamma\backslash D$. Thus

$$\theta(\tau, \varphi_n, \mathcal{L}) \in \mathrm{NHolM}_{m/2}(\Gamma(N)) \otimes \Omega^n(X)$$

is a non-holomorphic modular form of weight $m/2$ with values in the differential n-forms of X, see [FM02, Theorem 4.5].

§ 2.3 Special cycles

In this section we recall special cycles and their basic properties from [KM90, Section 2].

Let V be a rational vector space of dimension m and L a $\mathbb{Z}$-lattice. Let $(\ ,\ )$ be a non-degenerate quadratic form on V, which is integral ($\mathbb{Z}$-valued) on L and has signature $(p, 1)$ with $p + 1 = m$. Denote by $G = \mathrm{SO}_0(p, 1)$ the identity component in $\mathrm{SO}(V(\mathbb{R}))$. Let Γ be a torsion-free congruence subgroup of $\mathrm{GL}(V)$ preserving L and $(\ ,\ )$. Let $U \subset V$ be an oriented subspace such that $(\ ,\ )|_U$ is non-degenerate. Then we will construct special cycles $C_U \subset \Gamma\backslash D$, where D is the symmetric space associated to G. Recall from Section 1.1 that D can be viewed as the open subset of the Grassmannian $\mathrm{Gr}_1(V)$ consisting of those lines Z such that $(\ ,\ )|_Z$ is negative definite. Since $(\ ,\ )|_U$ is non-degenerate, we have a direct sum decomposition $V = U + U^\perp$.

Suppose that $(\ ,\ )|_U$ is positive definite and we define a subset $D_U \subset D$ by

$$D_U = \{Z \in D : Z = Z \cap U^\perp\}.$$

We let G_U denote the stabilizer of U in G and put $\Gamma_U = \Gamma \cap G_U$. We let $C_U = \Gamma_U\backslash D_U$ and note that C_U is an orientable manifold.

We now explain how an orientation of U gives rise to an orientation of D_U. We choose a base point $Z_0 \in D$ and choose an orientation of Z_0 and an orientation of V once and for all. Propagate the orientation of Z_0 continuously to orient all other $Z \in D$. Orienting D is equivalent to giving an orientation of $\mathrm{Hom}(Z, Z^\perp)$ which depends continuously on Z. Since V is oriented we obtain an induced orientation of $Z^\perp$ such that the orientation of $Z^\perp$ followed by that of Z is the orientation of V. Then $T_Z(D) \simeq \mathrm{Hom}(Z, Z^\perp)$ is oriented. When $(\ ,\ )|_U$ is positive definite, there are canonical isomorphims of the tangent space $T_Z(D_U)$ and the normal space $\nu_Z(D_U)$

$$T_Z(D_U) \simeq \mathrm{Hom}(Z, Z^\perp \cap U^\perp) \text{ and } \nu_Z(D_U) \simeq \mathrm{Hom}(Z, U).$$

Then $T_Z(D_U)$ will receive an orientation by the rule that the orientation of $T_Z(D_U)$ followed by the orientation of $\nu_z(D_U)$ is the orientation of $T_Z(D)$.

We now define $\Omega_\beta \subset V^n$ for β an n by n symmetric matrix by

$$\Omega_\beta = \{X \in V(\mathbb{Q})^n : \frac{1}{2}(X, X) = \beta\}.$$

If $X \in \Omega_\beta$ then the G-orbit $\mathcal{O} := GX \subset \Omega_\beta$. In case β is positive definite, then G acts transitively on $\mathcal{L}_\beta$ and $\mathcal{O} = \Omega_\beta$. We will write C_β instead of $C_\mathcal{O}$ for the cycle corresponding to $\mathcal{O} = \Omega_\beta$. In this case C_β is a locally finite cycle such that each irreducible component has real dimension $p - n$. Indeed pairs of frames $X = (x_1, x_2, \dots, x_n)$ and $X' = (-x_1, x_2, \dots, x_n)$ would occur in Ω_β if β were diagonal. To avoid such cases where C_β would be trivially zero we can introduce a congruence condition. Let $h \in L^n$ and $\mathfrak{a} \subset \mathcal{O}$ an ideal. Then we replace $\Omega_\beta \cap L^n$ by $\Omega_\beta \cap (h + \mathfrak{a}L^n)$. We assume that $\gamma \in \Gamma$ implies that $\gamma \equiv 1 \mod \mathfrak{a}$ so that Γ acts on this intersection. This congruence condition will be used to construct the theta lift in the following section.

§ 2.4 The work of Kudla-Millson and Fourier coefficients

We first recall the classical theta lift as a function on the Siegel upper half plane from [KM90] and [FM02], and then discuss that constructed on $\mathrm{Sp}_{2n}(\mathbb{A})$, see [KM90] and [Ber14].

2.4.1 Classical theta lift

Let $V(\mathbb{Q})$ be a rational vector space of dimension $m = p+1$ with a symmetric bilinear form $(\ ,\ )$ of signature $(p, 1)$ and put $\underline{G}(\mathbb{Q}) = \mathrm{SO}(V(\mathbb{Q}))$. Let $G = \underline{G}_0(\mathbb{R}) \simeq \mathrm{SO}_0(p, 1)$ be the connected component of the identity of the real points of $\underline{G}$. Let L be an even lattice of level N with the dual lattice $L^\sharp$. Fix a vector $h \in L^\sharp/L$ and write $\mathcal{L} = h + L$. Let Γ be a torsion free subgroup of finite index of the stabilizer of $\mathcal{L}$ in G. We denote by D the symmetric space associated to G.

Let $\varphi_n(X, Z)$ be the Schwartz form as given in Section 2.2 for $X \in V(\mathbb{R})^n$ and $Z \in D$. For $\tau = u + iv \in \mathcal{H}_n$, the Siegel space of genus n, following [FM02, Section 4] we define

$$\varphi_n(\tau, X, Z) = \det(v)^{-m/4}\omega(g'_\tau)\varphi_n(X, Z)$$

where

$$g'_\tau = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} = \begin{pmatrix} v^{1/2} & v^{-1/2}u \\ 0 & v^{-1/2} \end{pmatrix}$$

moves the base point $iI_n \in \mathcal{H}_n$ to τ. One obtains (see [FM02, Section 4])

$$\varphi(\tau, X, Z)(W) = 2^{n/2}\det(v)^{1/2}\det(X, W)\exp(\pi i \operatorname{tr}(X, X)_{\tau,Z})$$

for $W \in (T_Z(D))^n \simeq (Z^\perp)^n$ and with $(X, X)_{\tau,Z} = u(X, X) + iv(X, X)_Z$.

For a congruence condition $h \in (L^\sharp)^n$, we define the theta series $\theta(\tau)$ with values in the differential n-forms on D by

$$\theta(\tau, Z) = \sum_{X \in h + L^n} \varphi_n(\tau, X, Z).$$

One can show that it is a non-holomorphic Siegel modular forms of weight $m/2$ with values in the Γ-invariant differential forms of D for some suitable subgroup of $\mathrm{Sp}_{2n}(\mathbb{Z})$, see [FM02, Theorem 4.5].

For a rapidly decreasing closed differential $(p-n)$-form η in $\Gamma\backslash D$, Kudla and Millson defined the transform

$$\Theta(\eta)(\tau) = \int_{\Gamma\backslash D} \eta \wedge \theta(\tau, Z). \tag{2.9}$$

and showed that it is a holomorphic Siegel modular form of weight $m/2$, see [KM90, Theorem 1]. Moreover, the Fourier coefficients are given as periods of η over certain special cycles C_β in $\Gamma\backslash D$ attached to positive definite $\beta \in \mathrm{Sym}_n(\mathbb{Q})$, i.e.,

$$\Theta(\eta)(\tau) = \sum_{\beta > 0} a_\beta(\eta) e^{2\pi i \operatorname{tr}(\beta\tau)}$$

with

$$a_\beta(\eta) = \int_{\Gamma\backslash D} \eta \wedge \theta_\beta(\tau). \tag{2.10}$$

The derivation of the Fourier coefficient will be discussed with more details in the following subsection. Recall from Section 2.3 that

$$\Omega_\beta = \left\{ X \in V(\mathbb{Q})^n : \frac{1}{2}(X, X) = \beta \right\}.$$

The main point of Kudla-Millson's work is that

$$\theta_\beta = \sum_{X \in \Omega_\beta \cap (h + L^n)} \varphi_n(iv, Z, X) e^{-2\pi \operatorname{tr}(\beta v)}$$

is a Poincaré dual form for the composite cycle C_β, i.e.,

$$a_\beta(\eta) = \int_{\Gamma \backslash D} \eta \wedge \theta_\beta = \int_{C_\beta} \eta$$

for all rapidly decreasing closed $(p-n)$-forms in $\Gamma \backslash D$.

Example 2.4.1. Let V be the rational vector space of signature (2,1) as in the Example 1.1.6. Recall from Example 2.2.2 (1) that, for $\mathbf{x} = x_1 e_1 + x_2 e_2 + x_3 e_3 \in V(\mathbb{R})$ and $W = \nu_1 e_1 + \nu_2 e_2 \in \mathfrak{p} \simeq T_{Z_0}(D)$, we have

$$\varphi_1(\mathbf{x})(W) = 2^{1/2}(x_1 \omega_1 + x_2 \omega_2)(W) \varphi_0(\mathbf{x}).$$

Recall the theta series associated to φ_1, for $z = x + iy \in D$ and $\tau = u + iy \in \mathbb{H}_2$,

$$\theta_{\mathcal{L}}(\tau, z, \varphi_1) = \sum_{\mathbf{x} \in \mathcal{L}} \varphi_0(\sqrt{v}\mathbf{x}) e^{\pi i (\mathbf{x}, \mathbf{x}) u}.$$

As discussed in [FM11, Remark 7.2], Shintani defines a scalar-valued theta kernel $\theta(\tau, z, \varphi_S)$ which is integrated against a holomorphic cusp form f. His kernel function at the base point $Z_0 = i$ is given by

$$\varphi_S(\mathbf{x}) = (x_1 + i x_2) \varphi_0(\mathbf{x}).$$

For such input, the kernels are closely related, namely one has

$$\eta_f \wedge \theta(\tau, z, \varphi_1) = 2^{1/2} \theta(\tau, z, \varphi_S) f(z) \frac{dx \wedge dy}{y}.$$

This can be seen by a direct calculation of

$$\begin{aligned} dz \wedge \varphi_1(\mathbf{x}) &= (dx + i dy) \wedge 2^{1/2} \varphi_0(\mathbf{x}) \left(x_1 \frac{dy}{y} - x_2 \frac{dx}{y} \right) \\ &= 2^{1/2} \varphi_0(\mathbf{x})(x_1 + i x_2) \frac{dx \wedge dy}{y} \\ &= 2^{1/2} \varphi_S(\mathbf{x}) \frac{dx \wedge dy}{y}. \end{aligned}$$

2.4.2 Adelic theta lift

In the following we discuss the adelic theta lift on $\mathrm{Sp}_{2n}(\mathbb{A})$.

Let $G = \mathrm{SO}(V)$ and recall the symmetric space D associated to $G_0(\mathbb{R}) :=$

$\mathrm{SO}_0(V(\mathbb{R}))$. We have seen the Schwartz form $\varphi_n \in [S(V(\mathbb{R}))^n \otimes \Omega^n(D)]^{G_0(\mathbb{R})}$ in Section 2.2. For a finite Schwartz function $\varphi_f \in S(V(\mathbb{A}_f)^n)$, let

$$\varphi = \varphi_n \otimes \varphi_f \in [S(V(\mathbb{A})^n) \otimes \Omega^n(D)]^{G_0(\mathbb{R})}.$$

Let $G' = \mathrm{Res}_{\mathbb{R}/\mathbb{Q}} \mathrm{Sp}_{2n}$ and let $\widetilde{G}'(\mathbb{A})$ be the metaplectic cover of $G'(\mathbb{A})$. For convenience, we can take this group to be the extension of $G'(\mathbb{A})$ by $\mathbb{C}^1$. Recall from Section 2.1 that $\widetilde{G}'(\mathbb{A})$ acts on $S(V(\mathbb{A})^n)$ via the global Weil representation associated to an additive character ψ of $\mathbb{A}_{\mathbb{Q}}$.

Let $\Gamma \subset G(\mathbb{Q}) \cap \mathrm{SO}_0(V(\mathbb{R}))$. For $g' \in \widetilde{G}'(\mathbb{A})$, if φ_f is Γ-invariant, the theta series

$$\theta(g', \varphi_f, z) = \sum_{X \in V(\mathbb{Q})^n} \omega(g')\varphi(X, z) \tag{2.11}$$

defines a closed n-form on $\Gamma\backslash D$. Now we can define an adelic theta lift, for a rapidly decreasing $(p-n)$-form η in $\Gamma\backslash D$, to be

$$\Theta(\eta)(g') = \int_{\Gamma\backslash D} \eta(z) \wedge \theta(g', \varphi_f, z). \tag{2.12}$$

By Theorem 1 of [KM90], it is an adelic Siegel modular form on $\widetilde{G}'(\mathbb{A})$ of weight $m/2$.

Remark 2.4.2. For any prime q and any lattice $L \subset V \otimes \mathbb{Q}_q$ define its dual lattice by $L^\sharp = \{X \in V \otimes \mathbb{Q}_q : 2(X, Y) \in \mathbb{Z}_q \ \forall\ Y \in L\}$. Now let L be an integral lattice on V and put $L_q = L \otimes_{\mathbb{Z}} \mathbb{Z}_q$. Fix a $h \in (L^\sharp)^n/L^n$. If we take the finite Schwartz function as the product of the characteristic function of the lattice $h_q + L_q^n$ at each prime q, i.e.,

$$\varphi_f := \prod_q \mathbb{1}_{\{h_q + L_q^n\}}$$

then the above theta series (2.11) can be rewritten as

$$\theta(g', \varphi_f) = \sum_{X \in h + L^n} \omega(g')\varphi_n(X).$$

In the case of sign (2,1) and $n = 1$, it recovers the theta series on the upper half plane in (2.6). Invariance properties of $\theta(g', \varphi_f)$ under subgroups of $\widetilde{G}'(\mathbb{A})$ allow us to use the strong approximation theorem. As discussed in Section 1.3.5, the adelic Siegel modular form can be realised as the classical one defined on the Siegel upper half plane. It follows that with φ_q being the characteristic function the theta lift as define above in (2.12) descends to the classical one on the Siegel upper half plane as given in (2.9).

Write $\tau = u + iv \in g'(i \cdot 1_n) \in \mathcal{H}_n$, the Siegel upper half plane of genus n. By the Iwasawa decomposition, we have

$$g'_\infty = \begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v^{1/2} & 0 \\ 0 & v^{-1/2} \end{pmatrix} k'$$

where $k' \subset K'$, the inverse image of $\mathrm{U}(n)$ under the covering map $\mathrm{Mp}_n(\mathbb{R}) \to \mathrm{Sp}_n(\mathbb{R})$. Then the Whittaker function is given by

$$W_\beta(g'_\infty) = \det(v)^{m/4} \exp(\mathrm{tr}\, \beta\tau) \det(k')^{m/2} \quad \text{for } \beta \in \mathrm{Sym}_n(\mathbb{R}).$$

Returning to our space V of signature $(p, 1)$. Let $U \subset V$ be a $\mathbb{Q}$-subspace with $\dim_{\mathbb{Q}} U = n$ such that $(\ ,\ )|_U$ is positive definite. Following Section 2.3 we have $D_U = \{Z \in D : Z \perp U\}$ and let G_U^0 be the stabilizer of U in $G_0(\mathbb{R})$. Set $\Gamma_U = \Gamma \cap G_U^0$ and then we have the cycle $C_U = \Gamma_U \backslash D_U$ as defined in Section 2.3.

For a positive definite symmetric matrix $\beta \in M_n(\mathbb{Q})$, consider the corresponding hyperboloid

$$\Omega_\beta = \left\{ X \in V(\mathbb{Q})^n : \frac{1}{2}(X, X) = \beta \right\}.$$

Let $S(V(\mathbb{A}_f)^n)_{\mathbb{Z}}$ be space of locally constant $\mathbb{Z}$-valued functions on $S(V(\mathbb{A}_f)^n)$ of compact support. Given any commutative ring R, let

$$S(V(\mathbb{A}_f)^n)_R = S(V(\mathbb{A}_f)^n)_{\mathbb{Z}} \otimes_{\mathbb{Z}} R.$$

We now make the following definition, motivated by [Kud97, Proposition 5.4],

Definition 2.4.3 (Definition 5, [Ber14])**.** For a Γ-invariant Schwartz function $\varphi_f \in S(V(\mathbb{A}_f)^n)_R$, let

$$Z(\beta, \varphi_f, \Gamma) = \sum_{X \in \Gamma \backslash \Omega_\beta} \varphi_f(X) \cdot C_{U(X)}$$

where $U(X)$ is the $\mathbb{Q}$-subspace of V spanned by the components of X.

The following result, called Thom Lemma, is stated in [KM90, Theorem 9.1], where the results of [KM86] for $\Gamma_U \backslash D$ compact and [KM87] for $\Gamma_U \backslash D$ finite volume are recorded. In fact, these results do not cover the case of an infinite geodesic, which can rise for signature $(p, 1)$. It is proved in the case of signature (2,1) and $n = 1$ in [FM02], and signature (3,1) and $n = 2$ in [Ber14, Section 4.3].

Lemma 2.4.4. *Let $\beta > 0$ and $X \in \Omega_\beta$. Put $U = U(X)$. Let Γ_U be a discrete subgroup of G_U^0. For any closed and bounded $(p-n)$-form η on $\Gamma_U \backslash D$,*

$$\int_{\Gamma_U \backslash D} (\omega(g'_\infty)\varphi_n)(X) \wedge \eta = W_\beta(g'_\infty) \int_{\Gamma_U \backslash D_U} \eta.$$

This lemma is just about the archimedean situation so it can be used to prove the result in (2.10). For $p = 2$ and $n = 1$, and $p = 3$ and $n = 2$, applying the above lemma we can prove:

Theorem 2.4.5 (Theorem 8.1, [Kud97] and Theorem 9, [Ber14])**.** *Let* $[\eta] \in H_c^1(\Gamma\backslash D, \mathbb{C})$ *and* $\varphi_f \in S(V(\mathbb{A}_f)^n)^\Gamma$*. For* $g' \in G'(\mathbb{A})$*, let*

$$\varphi' = \omega(g'_f)\varphi_f \in S(V(\mathbb{A}_f)^n).$$

Then we have

$$\Theta_{\varphi_f}(\eta)(g') = c_{K_f} * \sum_{\beta>0} W_\beta(g'_\infty) \cdot \int_{Z(\beta,\varphi',\Gamma)} \eta,$$

where $c_{K_f} = \begin{cases} 2, & \textit{if } -1 \in \Gamma; \\ 1, & \textit{else} . \end{cases}$

Proof. Put $U = U(X)$. Choose a Γ such that φ_f is Γ-invariant and such that, if p is even, then -1 is not in the image of Γ in $\mathrm{SO}(V)$. Let η be a closed 1-form on $\Gamma\backslash D$. Then

$$\int_{\Gamma\backslash D} \theta(g', \varphi') \wedge \eta = \sum_{\beta \in \mathrm{Sym}_n(\mathbb{Q})} \sum_{X \in \Gamma\backslash\Omega_\beta} \int_{\Gamma_U\backslash D} \omega(g')\varphi'(X) \wedge \eta. \tag{2.13}$$

One main result of [KM90] is that the terms in (2.13) where β is not positive definite vanish. Thus, using the Thom Lemma, we obtain

$$\begin{aligned} &\sum_{\beta>0} \sum_{X \in \Gamma\backslash\Omega_\beta} \varphi'(X) \cdot \int_{\Gamma_U\backslash D} \omega(g')\varphi_n(X) \wedge \eta \\ =&\sum_{\beta>0} \sum_{X \in \Gamma\backslash\Omega_\beta} \varphi'(X) \cdot W_\beta(g'_\infty) \cdot \int_{\Gamma_U\backslash D_U} \eta. \end{aligned} \tag{2.14}$$

If p is even and -1 is in the image of Γ in $\mathrm{SO}(V)$, then all terms in (2.14) must be multiplied by a factor of 2. □

Chapter 3

Shintani lift and Fourier coefficient

In this chapter, we explain the adelic theta lifting of a weight 2 cusp form f. For an finite Schwartz function related to an auxiliary quadratic character χ, we express certain Fourier coefficients of this lifting in terms of the twisted L-value $L(f,\chi,1)$.

§ 3.1 Orthogonal group of sign $(2,1)$ and cycles

In this section we recall some basic aspects on orthogonal groups of signature (2,1) and cycles in this case from [FM02] and [FM11].

Let V be a rational vector space of dimension 3 with a non-degenerate symmetric bilinear form $(\ ,\ )$ of signature $(2,1)$. We write $q(\mathbf{x}) = \frac{1}{2}(\mathbf{x},\mathbf{x})$ for the associated quadratic form. We denote the discriminant of the quadratic space by a square-free negative integer d. Throughout we assume that V is isotropic, and in fact we can pick the isomorphism

$$V(\mathbb{Q}) \simeq \left\{ \begin{pmatrix} \sqrt{-d}x_1 & x_2 \\ x_3 & -\sqrt{-d}x_1 \end{pmatrix} : x_i \in \mathbb{Q} \right\} =: B_0(-d;\mathbb{Q}).$$

Then $q(\mathbf{x}) = -\det(\mathbf{x})$ and $(\mathbf{x},\mathbf{y}) = \mathrm{tr}(\mathbf{xy})$. For simplicity we assume that the discriminant is -1. In this model, we define the action of GL_2 on V given by $g\cdot\mathbf{x} := g\mathbf{x}g^{-1}$. Noth that this action preserves the quadratic form, i.e. $q(g\cdot\mathbf{x}) = q(\mathbf{x})$ and the bilinear form as well since the computation

$$(g\cdot\mathbf{x}, g\cdot\mathbf{y}) = \mathrm{tr}(g\mathbf{x}g^{-1}g\mathbf{y}g^{-1}) = \mathrm{tr}(\mathbf{x}\mathbf{y}) = (\mathbf{x},\mathbf{y}).$$

We pick an orthogonal basis e_1, e_2, e_3 of $V(\mathbb{R})$ such that $(e_1, e_1) = (e_2, e_2) = 1$ and $(e_3, e_3) = -1$. This also gives rise to an orientation of V. Explicitly, we set

$$e_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, e_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, e_3 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

We let $K \simeq \mathrm{SO}(2)$ be the stabilizer of e_3 in $G = \mathrm{SL}_2(\mathbb{R})$, and recall from Section 1.1 that the symmetric space $D = G/K \simeq \mathbb{H}_2$ can be identified with the hyperboloid

$$D \simeq \{\mathbf{x} \in V(\mathbb{R}) : (\mathbf{x},\mathbf{x}) = -1, (\mathbf{x}, e_3) < 0\}.$$

Hence e_3 represents the base point z_0 of D. The tangent space $T_{z_0}(D)$ at the base point is canonically isomorphic to $e_3^{\perp}$. We orient D by stipulating that e_1, e_2 is an oriented basis of $T_{z_0}(D)$ and propagate this orientation continuously around D.

Recall the complex upper half plane $\mathbb{H}_2 = \{z = x + iy \in \mathbb{C} : y > 0\}$ and the action of $\mathrm{GL}_2^+(\mathbb{R})$ on it is given by linear fractional transformations as follows

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot z = \frac{\alpha z + \beta}{\gamma z + \delta} \quad \text{for} \quad \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{GL}_2^+(\mathbb{R}), z \in \mathbb{H}_2$$

The isomorphism $\mathbb{H}_2 \simeq D$ as in Example 1.1.6 is given explicitly by

$$\mu : z = x + iy \longmapsto \frac{1}{y}\begin{pmatrix} -x & z\bar{z} \\ -1 & x \end{pmatrix}. \tag{3.1}$$

Proposition 3.1.1. *The above map μ in* (3.1) *intertwines the action of $\mathrm{GL}_2^+(\mathbb{R})$ on V and $\mathbb{H}_2$; that is $\mu(g\cdot z) = g\cdot\mu(z)$ for $z \in \mathbb{H}_2$ and $g \in \mathrm{GL}_2^+(\mathbb{R})$.*

Proof. Let $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2^+(\mathbb{R})$ and $z = x + iy \in \mathbb{H}_2$. Then we compute

$$\begin{aligned} z' = g\cdot z = \frac{\alpha z + \beta}{\gamma z + \delta} &= \frac{(\alpha z + \beta)(\gamma\bar{z} + \delta)}{(\gamma z + \delta)(\gamma\bar{z} + \delta)} \\ &= \frac{\alpha\gamma z\bar{z} + (\alpha\delta + \beta\gamma)x + \beta\delta + (\alpha\delta - \beta\gamma)yi}{|\gamma z + \delta|^2} \end{aligned}$$

and

$$z'\bar{z}' = \frac{\alpha z + \beta}{\gamma z + \delta}\cdot\frac{\alpha\bar{z} + \beta}{\gamma\bar{z} + \delta} = \frac{\alpha^2 z\bar{z} + 2\alpha\beta x + \beta^2}{|\gamma z + \delta|^2}.$$

It follows that

$$\mu(z') = \frac{1}{(\alpha\delta - \beta\gamma)y}\begin{pmatrix} -\alpha\gamma z\bar{z} - (\alpha\delta + \beta\gamma)x - \beta\delta & \alpha^2 z\bar{z} + 2\alpha\beta x + \beta^2 \\ -|\gamma z + \delta|^2 & \alpha\gamma z\bar{z} + (\alpha\delta + \beta\gamma)x + \beta\delta \end{pmatrix}.$$

In the other direction we compute

$$\begin{aligned} g\cdot\mu(z) &= \frac{1}{(\alpha\delta-\beta\gamma)y}\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}-x & z\bar{z}\\ -1 & \delta\end{pmatrix}\begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}\\ &= \frac{1}{(\alpha\delta-\beta\gamma)y}\begin{pmatrix}-\alpha x-\beta & \alpha z\bar{z}+\beta x\\ -\gamma x-\delta & \gamma z\bar{z}+\delta x\end{pmatrix}\begin{pmatrix}\delta & -\beta\\ -\gamma & \alpha\end{pmatrix}\\ &= \frac{1}{(\alpha\delta-\beta\gamma)y}\begin{pmatrix}-\alpha\delta x-\beta\delta-\alpha\gamma z\bar{z}-\alpha\beta x & \alpha\beta x+\beta^2+\alpha^2 z\bar{z}+\alpha\beta x\\ -\gamma\delta x-\delta^2-\gamma^2 z\bar{z}-\gamma\delta x & \beta\gamma x+\beta\delta+\alpha\gamma z\bar{z}+\alpha\delta x\end{pmatrix}.\end{aligned}$$

Comparing the entries we can deduce that $\mu(g\cdot z)=g\cdot\mu(z)$. □

Let $L\subset V(\mathbb{Q})$ be an integral lattice of full rank, i.e. $L\subset L^\sharp$, the dual lattice of L which is given by $\{\mathbf{x}\in V:(\mathbf{x},\mathbf{y})\in\mathbb{Z}\ \forall\mathbf{y}\in L\}$. We let Γ be a torsion-free congruence subgroup of $\mathrm{SL}_2(\mathbb{Z})$ preserving L. We let $X=X_\Gamma=\Gamma\backslash D$ be the associated arithmetic quotient which is a modular curve due to the identification $D\simeq\mathbb{H}_2$.

The set of cusps of $\mathbb{H}_2$ is denoted $\mathbb{P}^1(\mathbb{Q})=\mathbb{Q}\cup\infty$. Here we use $[a:b]$ to denote homogeneous coordinates for a point in $\mathbb{P}^1(\mathbb{Q})$ and $[1:0]$ for ∞. We define the action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{P}^1(\mathbb{Q})$ by

$$\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\cdot[a:b]=[\alpha a+\beta b:\gamma a+\delta b]\quad\text{for}\quad\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\in\mathrm{GL}_2(\mathbb{Q}).$$

Note that the action of $\mathrm{GL}_2(\mathbb{Q})$ on $\mathbb{P}^1(\mathbb{Q})$ extends the action on $\overline{\mathbb{H}}_2=\mathbb{H}_2\cup\mathbb{P}_1(\mathbb{Q})$ if we treat a cusp $a/b+0\cdot i\in\mathbb{H}$. The set of all isotropic lines in V, i.e. $\mathrm{Iso}(V)=\{\mathbf{x}\in V:q(\mathbf{x})=0\}$, can be identified with $\mathbb{P}^1(\mathbb{Q})$ by means of the map

$$\nu:[a:b]\longmapsto\operatorname{span}\begin{pmatrix}-ab & a^2\\ -b^2 & ab\end{pmatrix}\in\mathrm{Iso}(V). \tag{3.2}$$

Proposition 3.1.2. *The above map ν in (3.2) commutes with the $\mathrm{GL}_2(\mathbb{Q})$-action; that is $\nu(g\cdot[a:b])=g\cdot\nu([a:b])$ for $g\in\mathrm{GL}_2(\mathbb{Q})$ and $[a:b]\in\mathbb{P}^1(\mathbb{Q})$.*

Proof. Let $g=\left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right)\in\mathrm{GL}_2(\mathbb{Q})$ and $[a:b]\in\mathbb{P}^1(\mathbb{Q})$. Then we compute

$$\nu(g\cdot[a:b])=\nu\left([\alpha a+\beta b:\gamma a+\delta b]\right)=\begin{pmatrix}-(\alpha a+\beta b)(\gamma a+\delta b) & (\alpha a+\beta b)^2\\ -(\gamma a+\delta b)^2 & (\alpha a+\beta b)(\gamma a+\delta b)\end{pmatrix}.$$

On the other hand we have

$$\begin{aligned} g \cdot \nu([a:b]) &= \frac{1}{\det(g)} \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} -ab & a^2 \\ -b^2 & ab \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \\ &= \frac{1}{\det(g)} \begin{pmatrix} -\alpha ab - \beta b^2 & \alpha a^2 + \beta ab \\ -\gamma ab - \delta b^2 & \gamma a^2 + \delta ab \end{pmatrix} \begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix} \\ &= \frac{1}{\det(g)} \begin{pmatrix} -\alpha\delta ab - \beta\delta b^2 - \alpha\gamma a^2 - \beta\gamma ab & \alpha\beta ab + \beta^2 b^2 + \alpha^2 a^2 + \alpha\beta ab \\ -\gamma\delta ab - \delta^2 b^2 - \gamma^2 a^2 - \gamma\delta ab & \beta\gamma ab + \beta\delta b^2 + \alpha\gamma a^2 + \alpha\delta ab \end{pmatrix}. \end{aligned}$$

Thus we can deduce that $\nu(g \cdot [a:b]) = g \cdot \nu([a:b])$ in $\mathrm{Iso}(V)$. □

The cusp ∞ corresponds to the isotropic line l_∞ spanned by $\mathbf{x}_\infty = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. For $\mathbf{y} \in \mathrm{Iso}(V)$, pick $g \in \mathrm{SL}_2(\mathbb{Q})$ such that $g \cdot \mathbf{y} = \beta \mathbf{x}_\infty$ with $\beta \in \mathbb{Q}^\times$. Put $\Gamma' = g\Gamma g^{-1}$. Hence, $\Gamma'_{\mathbf{x}_\infty} = g\Gamma_{\mathbf{y}} g^{-1}$ is equal to $\left\{\pm \left(\begin{smallmatrix} 1 & k\alpha \\ 0 & 1 \end{smallmatrix}\right) : k \in \mathbb{Z}\right\}$ (if $-I \in \Gamma$) for some $\alpha \in \mathbb{Q}_+$. We call such α the width of the cusp κ corresponding to $\mathbf{y}$. However this is not well defined, since it depends on the choice of $g \in \mathrm{SL}_2(\mathbb{Q})$ and hence on β. Instead we define the width of the cusp κ as $\epsilon(\mathbf{y}, \Gamma) = \alpha/|\beta|$ which only depends on $\mathbf{y}$ and Γ.

Following Section 2.3, a vector $\mathbf{x} \in V(\mathbb{R})$ of positive length defines a geodesic $D_{\mathbf{x}}$ in D via

$$D_{\mathbf{x}} = \{z \in D : z \perp \mathbf{x}\}$$

where being orthogonal is in the sense of the bilinear form. In the upper half plane model, the cycle $D_{\mathbf{x}}$ is given for $\mathbf{x} = \left(\begin{smallmatrix} b & a \\ -c & -b \end{smallmatrix}\right)$ by

$$D_{\mathbf{x}} = \{z \in \mathbb{H}_2 : \mu(z) \perp \mathbf{x}\} = \{z \in \mathbb{H}_2 : c|z|^2 + 2b\mathrm{Re}(z) + a = 0\}.$$

We orient $D_{\mathbf{x}}$ by requiring that a tangent vector $v \in T_z(D_{\mathbf{x}}) \simeq z^\perp \cap \mathbf{x}^\perp$ followed by $z^\perp \cap \mathbf{x}$ gives a properly oriented basis of $T_z(D) \simeq z^\perp$. Then $\langle z^\perp \cap \mathbf{x}^\perp, z^\perp \cap \mathbf{x}, z\rangle$ has the same orientation as $\langle e_1, e_2, e_3\rangle$, i.e. the determinant of the base change is positive. We let $\Gamma_{\mathbf{x}}$ be the stabilizer of $\mathbf{x}$ in $\Gamma \cap \mathrm{SO}_0(2,1)(\mathbb{R})$. We denote the image of the quotient $\Gamma_{\mathbf{x}} \backslash D_{\mathbf{x}}$ in X by $C_{\mathbf{x}}$.

A space with quadratic form is said to split if there is a subspace which is equal to its own orthogonal complement. The stabilizer $\Gamma_{\mathbf{x}}$ is either trivial or infinite cyclic which can be classified by the following lemma. If $\Gamma_{\mathbf{x}}$ is infinite, then $C_{\mathbf{x}}$ is a closed geodesic in X, while $C_{\mathbf{x}}$ is infinite if $\Gamma_{\mathbf{x}}$ is trivial. In the latter case $C_{\mathbf{x}}$ is exactly a classical modular symbol.

Lemma 3.1.3. *Let $q(\mathbf{x}) > 0$ for $\mathbf{x} \in V(\mathbb{Q})$, so $\mathbf{x}^\perp$ has signature (1,1). Then $\Gamma_{\mathbf{x}}$ is trivial if $\mathbf{x}^\perp$ splits over $\mathbb{Q}$. Conversely, if $\mathbf{x}^\perp$ is non-split, i.e., anisotropic over $\mathbb{Q}$,* *then* $\Gamma_{\mathbf{x}}$ *is infinite cyclic.*

Proof. See [Fun02, Lemma 4.2]. □

Proposition 3.1.4. *For* $\mathbf{x} \in V(\mathbb{Q}) \simeq B_0(1;\mathbb{Q})$ *with* $q(\mathbf{x}) > 0$ *the following statements are equivalent:*

(1) $\mathbf{x}^{\perp}$ *is split over* $\mathbb{Q}$*,*
(2) $q(\mathbf{x}) \in (\mathbb{Q}^{\times})^2$*.*

Proof. See [Fun02, Lemma 3.6] for an arbitrary discriminant d. □

If $q(\mathbf{x}) = m^2$ for $m \in \mathbb{Q}^{\times}$, then $\mathbf{x}$ is orthogonal to two cusps κ_1 and κ_2 corresponding to isotropic lines l_{κ_1} and l_{κ_2} with generators u_{κ_1} and u_{κ_2} respectively. We distinguish l_{κ_1} and l_{κ_2} by requiring that $u_{\kappa_1}, \mathbf{x}, u_{\kappa_2}$ gives a properly oriented basis of V which also gives a different way of characterizing the orientation of $D_{\mathbf{x}}$. Note that $\langle u_{\kappa_1}, \mathbf{x}, u_{\kappa_2}\rangle$ and $\langle u_{\kappa_2}, -\mathbf{x}, u_{\kappa_1}\rangle$ share the same orientation as the base change has a positive determinant. Consider

$$L_{m^2} = \{\mathbf{x} \in L : q(\mathbf{x}) = m^2\}$$

For a fixed cusp κ_i, we write

$$L_{m^2,\kappa_i,+} = \{\mathbf{x} \in L_{m^2} : \mathbf{x} \perp \kappa_i, \mathbf{x} \text{ pos orient}\}$$

and note that the stabilizer $\Gamma_{\kappa_i} \subset \Gamma$ of the cusp κ_i acts on this set.

Proposition 3.1.5. *We have*

$$\#\Gamma\backslash L_{m^2} = \sum_i \#\Gamma_{\kappa_i}\backslash L_{m^2,\kappa_i,+}$$

where the sum is over all the non-equivalent cusps and

$$\#\Gamma_{\kappa_i}\backslash L_{m^2,\kappa_i,+} = 2m\epsilon(\mathbf{x}_i, \Gamma)$$

where $\mathbf{x}_i$ *is the isotropic line corresponding to the cusp* κ_i*.*

Proof. See [Fun02, Lemma 3.7] in a bit more general setting. □

Lemma 3.1.6. *Let* $\mathbf{x} \in \left\{\left(\begin{smallmatrix} b & 2a \\ 0 & -b \end{smallmatrix}\right) : a \in \mathbb{Q}, b \in \mathbb{Q}^{\times}\right\} \subset V$ *and the associated cycle is given by*

$$D_{\mathbf{x}} = \{z \in \mathbb{H}_2 : bRe(z) + a = 0\}.$$

Then the sign of b *determines the orientation of* $T_z(D_{\mathbf{x}})$*.*

Proof. Given an element $z = -\frac{a}{b} + iy \in D_{\mathbf{x}}$, we have

$$z = -\frac{a}{b} + iy \longmapsto \frac{1}{y}\begin{pmatrix} \frac{a}{b} & \frac{a^2}{b^2} + y^2 \\ -1 & -\frac{a}{b} \end{pmatrix} \in V(\mathbb{R}).$$

Let $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in z^{\perp}$ and we compute

$$\frac{1}{y}\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\begin{pmatrix} \frac{a}{b} & \frac{a^2}{b^2} + y^2 \\ -1 & -\frac{a}{b} \end{pmatrix} = \begin{pmatrix} \frac{a}{b}\alpha - \beta & * \\ * & \frac{a^2}{b^2}\gamma + y^2\gamma + \frac{a}{b}\alpha \end{pmatrix}.$$

It follows that $\beta = \frac{a^2}{b^2}\gamma + y^2\gamma + \frac{2a}{b}\alpha$ and thus we have

$$z^{\perp} = \left\{ \begin{pmatrix} \alpha & \frac{a^2}{b^2}\gamma + y^2\gamma + \frac{2a}{b}\alpha \\ \gamma & -\alpha \end{pmatrix} : \alpha, \gamma \in \mathbb{Q} \right\}.$$

Suppose $\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix} \in \mathbf{x}^{\perp}$ and we compute

$$\begin{pmatrix} \alpha & \beta \\ \gamma & -\alpha \end{pmatrix}\begin{pmatrix} b & 2a \\ 0 & -b \end{pmatrix} = \begin{pmatrix} b\alpha & * \\ * & 2a\gamma + b\alpha \end{pmatrix}.$$

It follows that $b\alpha + a\gamma = 0$ and thus we have

$$\mathbf{x}^{\perp} = \left\{ \begin{pmatrix} -\frac{a}{b}\gamma & \beta \\ \gamma & \frac{a}{b}\gamma \end{pmatrix} : \beta, \gamma \in \mathbb{Q} \right\}.$$

Then we can calculate

$$\begin{aligned}
z^{\perp} \cap \mathbf{x}^{\perp} &= \left\{ \begin{pmatrix} -\frac{a}{b}\gamma & -\frac{a^2}{b^2}\gamma + y^2\gamma \\ \gamma & \frac{a}{b}\gamma \end{pmatrix} : \gamma \in \mathbb{Q} \right\} \\
&= \left\langle \varepsilon \begin{pmatrix} -\frac{a}{b} & -\frac{a^2}{b^2} + y^2 \\ 1 & \frac{a}{b} \end{pmatrix} \right\rangle \quad (\varepsilon = \pm 1 \text{ describes the orientation of } T_z(D_{\mathbf{x}})) \\
&= \left\langle \varepsilon \left(-\frac{a}{b}e_2 + \frac{1}{2}(e_1 + e_3)\left(-\frac{a^2}{b^2} + y^2 \right) + \frac{1}{2}(e_1 - e_3) \right) \right\rangle \\
&= \left\langle \varepsilon \left(\frac{1}{2}\left(-\frac{a^2}{b^2} + y^2 + 1 \right) e_1 - \frac{a}{b}e_2 + \frac{1}{2}\left(-\frac{a^2}{b^2} + y^2 - 1 \right) e_3 \right) \right\rangle,
\end{aligned}$$

$$z^{\perp} \cap \mathbf{x} = \left\langle \begin{pmatrix} b & 2a \\ 0 & -b \end{pmatrix} \right\rangle = \langle ae_1 + be_2 + ae_3 \rangle,$$

and

$$z \simeq \frac{1}{y}\begin{pmatrix} \frac{a}{b} & \frac{a^2}{b^2}+y^2 \\ -1 & -\frac{a}{b} \end{pmatrix} = \frac{1}{y}\left(\frac{a}{b}e_2 + \frac{1}{2}(e_1+e_3)\left(\frac{a^2}{b^2}+y^2\right) - \frac{1}{2}(e_1-e_3)\right)$$
$$= \frac{1}{y}\left(\frac{1}{2}\left(\frac{a^2}{b^2}+y^2-1\right)e_1 + \frac{a}{b}e_2 + \frac{1}{2}\left(\frac{a^2}{b^2}+y^2+1\right)e_3\right).$$

To conclude we obtain that

$$\begin{pmatrix} z^\perp \cap \mathbf{x}^\perp \\ z^\perp \cap \mathbf{x} \\ z \end{pmatrix} = \begin{pmatrix} \frac{\varepsilon}{2}\left(-\frac{a^2}{b^2}+y^2+1\right) & -\frac{\varepsilon a}{b} & \frac{\varepsilon}{2}\left(-\frac{a^2}{b^2}+y^2-1\right) \\ a & b & a \\ \frac{1}{2y}\left(\frac{a^2}{b^2}+y^2-1\right) & \frac{a}{yb} & \frac{1}{2y}\left(\frac{a^2}{b^2}+y^2+1\right) \end{pmatrix}\begin{pmatrix} e_1 \\ e_2 \\ e_3 \end{pmatrix}$$

where the base change has the determinant

$$\det = \frac{\varepsilon b}{4y}\begin{vmatrix} -\frac{a^2}{b^2}+y^2+1 & -\frac{2a}{b} & -\frac{a^2}{b^2}+y^2-1 \\ \frac{a}{b} & 1 & \frac{a}{b} \\ \frac{a^2}{b^2}+y^2-1 & \frac{2a}{b} & \frac{a^2}{b^2}+y^2+1 \end{vmatrix}$$
$$= \frac{\varepsilon b}{4y}\begin{vmatrix} 2y^2 & 0 & 2y^2 \\ \frac{a}{b} & 1 & \frac{a}{b} \\ \frac{a^2}{b^2}+y^2-1 & \frac{2a}{b} & \frac{a^2}{b^2}+y^2+1 \end{vmatrix}$$
$$= \frac{\varepsilon by}{2}\left(\left(-\frac{a^2}{b^2}+y^2+1\right)+\left(\frac{a^2}{b^2}-y^2+1\right)\right) = \varepsilon by > 0.$$

Thus we can deduce that the sign of b determines the orientation ε of $T_z(D_{\mathbf{x}})$. □

Corollary 3.1.7. *Consider the lattice* $L = \left\{\left(\begin{smallmatrix} b & 2a \\ 2c & -b \end{smallmatrix}\right) : a,b,c \in \mathbb{Z}\right\}$ *such that the cusp* ∞ *has width* $1/2$*. Assume that* m *is a positive integer. We can choose a set of representatives in* $\Gamma_\infty \backslash L_{m^2,\infty,+}$ *given by*

$$\left\{\begin{pmatrix} m & 0 \\ 0 & -m \end{pmatrix}, \begin{pmatrix} m & 2 \\ 0 & -m \end{pmatrix}, \dots, \begin{pmatrix} m & 2(m-1) \\ 0 & -m \end{pmatrix}\right\}.$$

Proof. Given an element $\mathbf{x} \in L_{m^2,\infty,+}$ of form $\left(\begin{smallmatrix} b & 2a \\ 0 & -b \end{smallmatrix}\right)$, Lemma 3.1.6 allows us to orient $\mathbf{x}$ positively by requiring that the top left entry of $\mathbf{x}$ is positive, e.g. $b = m > 0$. From the proof of Proposition 3.1.5, we know that the Γ_∞-action on $\mathbf{x}$ only makes the top right entry of $\mathbf{x}$ vary. Thus we can choose such representatives in $\Gamma_\infty \backslash L_{m^2,\infty,+}$ as stated above. □

§ 3.2 Dirichlet character and Schwartz function

Following Section 1.3, we first recall from [GH11, Section 2.1] the idelic lift of a Dirichlet character of conductor p^f where p^f is a fixed prime power in the following. We define the idelic lift of $\chi : (\mathbb{Z}/p^f)^\times \to \mathbb{C}^\times$ to be a Hecke character $\widetilde{\chi} : \mathbb{Q}^\times/\mathbb{A}_\mathbb{Q}^\times \to \mathbb{C}^\times$ defined as

$$\widetilde{\chi}(a) = \widetilde{\chi}_2(a_\infty) \cdot \widetilde{\chi}(a_2) \cdots, \qquad \text{for } a = (a_\infty, a_2, \dots) \in \mathbb{A}_\mathbb{Q}^\times,$$

where

$$\widetilde{\chi}_\infty(a_\infty) = \begin{cases} 1, & \text{if } \chi(-1) = 1, \\ 1, & \text{if } \chi(-1) = -1 \text{ and } a_\infty > 0, \\ -1, & \text{if } \chi(-1) = -1 \text{ and } a_\infty < 0, \end{cases}$$

and where

$$\widetilde{\chi}_v(a_v) = \begin{cases} \chi(v)^m, & \text{if } a_v \in v^m \mathbb{Z}_v^\times \text{ and } v \neq p, \\ \chi(j)^{-1}, & \text{if } a_v \in p^k(j + p^f\mathbb{Z}_p) \text{ with } j, k \in \mathbb{Z}, (j,p) = 1, \text{ and } v = p. \end{cases}$$

One can verify that this actually defines a Hecke character as given in Section 1.3.

More generally, every Dirichlet character $\chi \pmod{m}$, with $m = \prod_{i=1}^r p_i^{f_i}$, where $p_1, p_2, \dots, p_r$ are distinct primes and $f_1, f_2, \cdots, f_r \geq 1$ can be factored as

$$\chi = \prod_{i=1}^r \chi^{(i)}$$

where $\chi^{(i)}$ is a Dirichlet character of conductor $p_i^{f_i}$. It follows that χ can be lifted to a Hecke character $\widetilde{\chi}$ on $\mathbb{A}_\mathbb{Q}^\times$ as being

$$\widetilde{\chi} = \prod_{i=1}^r \widetilde{\chi}^{(i)},$$

where

$$\widetilde{\chi}_v(a_v) = \begin{cases} \prod_{i=1}^r \chi^{(i)}(p_v)^{\mathrm{ord}_v(a_v)}, & \text{if } p_v \nmid m \\ \chi^{(v)}(j)^{-1} \cdot \prod_{i=1, p_v \neq p_i}^r \chi^{(i)}(p_v)^{\mathrm{ord}_v(a_v)}, & \text{if } a_v \in p_v^k(j + p_v^{f_v}\mathbb{Z}_{p_v}) \text{ with } j, k \in \mathbb{Z}, \\ & (j, p_v) = 1, \text{ and } p_v | m. \end{cases}$$

Remark 3.2.1. To avoid the trivial vanishing of the theta lifting, one approach is to introduce a congruence condition and take the finite Schwartz function as the product of characteristic functions of $h_q + L_q$ as discussed in Remark 2.4.2. Instead of this approach which is adopted in [Shi75] and [FM02], we follow Prasanna's treatment [Pra09, Section 3.2], that is carefully choosing the finite Schwartz function φ_f on $S(V_f)$ related to a quadratic Dirichlet character χ_m of square-free conductor m. Then we construct $\varphi := \varphi_1 \varphi_f$ where φ_1 denotes the Schwartz form at the archimedean place as in Example 2.2.2 (1).

In this chapter we need to consider the pair $\widetilde{\mathrm{SL}_2} \times \mathrm{SO}(2,1)$ to construct the theta lift where $\widetilde{\mathrm{SL}_2} = \mathrm{SL}_2 \times \{\pm 1\}$. We have seen the Weil representation of $\widetilde{\mathrm{SL}_2}$ in Example 2.1.1(1).

Let N be an odd square-free integer coprime to m, which will be the level of modular form considered in next section. Now, adapting Prasanna's choice (see [Pra09, Section 3.2]), we define the local Schwartz function φ_q at each finite place q away from N in the following:

(1) If q is odd and $q \nmid mN$, $\varphi_q = \mathbb{1}_{\{L \otimes \mathbb{Z}_q\}}$ where $L := \left\{ \begin{pmatrix} b & a \\ c & -b \end{pmatrix} : a, b, c \in \mathbb{Z} \right\}$,

(2) If $q | m$ and $q \nmid N$, $\varphi_q \begin{pmatrix} b & a \\ c & -b \end{pmatrix} = 0$, unless $a, b, c \in \mathbb{Z}_p$, $-b^2 - ac \in q\mathbb{Z}_q$, in which case

$$\varphi_q \begin{pmatrix} b & a \\ c & -b \end{pmatrix} = \begin{cases} \widetilde{\chi}_{m,q}(-a) \text{ (resp. } \widetilde{\chi}_{m,q}(c)), & \text{if } \mathrm{ord}_q(a) = 0 \text{ (resp.} \, \mathrm{ord}_q(c) = 0), \\ 0, & \text{if both } \mathrm{ord}_q(a) \neq 0 \text{ and} \\ & \mathrm{ord}_q(c) \neq 0, \end{cases}$$

(3) If $q = 2$, $\varphi_2 \begin{pmatrix} b & a \\ c & -b \end{pmatrix} = \mathbb{1}_{\mathbb{Z}_2}(b) \mathbb{1}_{2\mathbb{Z}_2}(a) \mathbb{1}_{2\mathbb{Z}_2}(c)$.

Definition 3.2.2. At each place $q | N$, we define the local Schwartz function φ_q^N to be the characteristic function of

$$\left\{ \begin{pmatrix} b & a \\ c & -b \end{pmatrix} \in V(\mathbb{Q}_q) : a, b, c \in \mathbb{Z}_q, b \equiv m \mod q \right\}.$$

Remark 3.2.3. We cannot take the local Schwartz function φ_q^N simply as the characteristic function of integral lattices otherwise the theta lifting constructed in our next section would be vanishing for some trivial reason. The vanishing will be seen clearly after the whole treatment of next section, and at the end of next section (see Remark 3.3.4 (i)), we will give some concrete examples to show why we don't take the characteristic function as our local Schwartz function.

With the above φ_q defined at each finite place, one can check that the finite

Schwartz function φ_f is invariant under

$$\overline{\Gamma}(2mN) := \left\{ \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}) : \pm \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \pm \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 2mN \right\}$$
$$\subset \mathrm{PSL}_2(\mathbb{R}) \simeq \mathrm{SO}_0(2,1)(\mathbb{R}) \quad \text{(by the exceptionl isomorphism in (1.1))}$$

via observing the expansion

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \cdot \begin{pmatrix} b & a \\ c & -b \end{pmatrix} = \begin{pmatrix} \alpha\delta b + \beta\delta c - \alpha\gamma a + \beta\gamma b & -\alpha\beta b - \beta^2 c + \alpha^2 a - \alpha\beta b \\ \gamma\delta b + \delta^2 c - \gamma^2 a + \gamma\delta b & -\beta\gamma b - \beta\delta c + \alpha\gamma a - \alpha\delta b \end{pmatrix}.$$

Here $\overline{\Gamma}$ denotes the image of $\Gamma \subset \mathrm{SL}_2$ in PSL_2.

We want to construct a $\overline{\Gamma}_0(2mN)$-invariant finite Schwartz function φ_f^{new} related to φ_f defined above. For $\gamma \in \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z})$, there is an homomorphism

$$\lambda_N : \begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \quad \bmod N$$

with $\ker(\lambda_N) = \overline{\Gamma}(N)$. As

$$\lambda_N(\Gamma_0(N)) = \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z}) : a \in (\mathbb{Z}/N\mathbb{Z})^\times, b \in \mathbb{Z}/N\mathbb{Z} \right\},$$

we have a complete set of representatives of $\overline{\Gamma}_0(N)/\overline{\Gamma}(N)$ consisting of the elements

$$\gamma \in \left\{ \begin{pmatrix} a & b \\ 0 & a^{-1} \end{pmatrix} \in \mathrm{PSL}_2(\mathbb{Z}/N\mathbb{Z}) : a \in (\mathbb{Z}/N\mathbb{Z})^\times, b \in \mathbb{Z}/N\mathbb{Z} \right\}.$$

Then, at each finite place $q|2mN$, φ_q^{new} is defined to be

$$\varphi_q^{\mathrm{new}}(\mathbf{x}) := \sum_{[\gamma] \in \overline{\Gamma}_0(q)/\overline{\Gamma}(q)} \omega_{\psi_q}(\gamma)\varphi_q(\mathbf{x}) = \sum_{[\gamma] \in \overline{\Gamma}_0(q)/\overline{\Gamma}(q)} \varphi_q(\gamma^{-1} \cdot \mathbf{x}).$$

At all other finite places, we do not make any changes to the local Schwartz function. Hence, the corresponding φ_f^{new} is $\overline{\Gamma}_0(2mN)$-invariant.

On the symplectic side, the following invariance properties under subgroups of $\mathrm{SL}_2(\mathbb{Z}_q) \times \{\pm 1\}$ will help us determine the level of our theta lifting in our next section. This procedure will be repeated in a bit more complicated setting in our next chapter on the theta liftings of Bianchi modular forms, see Section 4.3.

Proposition 3.2.4. *(1) For* $\epsilon = 1$, $\omega_{\psi_q}(1,\epsilon)\varphi_q(\mathbf{x}) = \varphi_q(\mathbf{x})$.

(2) For $q \nmid 4N$ *and* $\sigma \in \mathrm{SL}_2(\mathbb{Z}_q)$, $\omega_{\psi_q}(\sigma)\varphi_q = \varphi_q$,

(3) For $q = 2$ *and* $\sigma \in \Gamma_0(4) \subset \mathrm{SL}_2(\mathbb{Z}_2)$, $\omega_{\psi_2}(\sigma)\varphi_2 = \varphi_2$,

(4) For $q|N$ *and* $\sigma \in \mathrm{SL}_2(\mathbb{Z}_q)$ *such that* $h \equiv \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix} \bmod q$, $\omega_{\psi_q}(\sigma)\varphi_q^N = \varphi_q^N$.

Proof. It is clear for the first part, and for (2) and (3) see the proof of [Pra09, Proposition 3.4]. We will prove (4) in details inspired by his proof. For simplicity, we set $\varphi_q = \varphi_q^N$. Setting $\mathbf{x} = \begin{pmatrix} b & a \\ c & -b \end{pmatrix}$ and $\mathbf{y} = \begin{pmatrix} \beta & \alpha \\ \gamma & -\beta \end{pmatrix}$, we write the Fourier transform

$$\begin{aligned} \hat{\varphi}_q(\mathbf{x}) &= \int (\mathbf{x}, \mathbf{y}) \varphi_q(\mathbf{y}) d\mathbf{y} \\ &= \int \frac{1}{2}(a\gamma + c\alpha + 2b\beta) \varphi_q(\mathbf{y}) d\mathbf{y}. \end{aligned}$$

By definition of φ_q above, we know that $\varphi_q(\mathbf{y})$ is invariant under $\beta \mapsto \beta + q$, $\alpha \mapsto \alpha + \mathbb{Z}_q$ and $\gamma \mapsto \gamma + \mathbb{Z}_q$. For the non-vanishing of $\hat{\varphi}_q(\mathbf{x})$, we need $a \in q\mathbb{Z}_q$, $b \in \mathbb{Z}_q$ and $c \in q\mathbb{Z}_q$. It follows that

$$\omega_{\psi_q}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q = \omega_{\psi_q}\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q \quad \text{for } x \in q\mathbb{Z}_q$$

which implies

$$\omega_{\psi_q}\left(\begin{pmatrix} 1 & 0 \\ x & 1 \end{pmatrix}\right)\varphi_q = \varphi_q \quad \text{for } x \in q\mathbb{Z}_q.$$

Also it is clear that

$$\omega_{\psi_q}\left(\begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}\right)\varphi_q = \varphi_q \quad \text{for } x \in q\mathbb{Z}_q.$$

Then the assertion follows. □

§ 3.3 Shintani lift and Fourier coefficient

We have seen in Example 2.2.2 (1) the Schwartz form at the archimedean place:

$$\varphi_1(\mathbf{x}_{\mathbb{R}}, z) \in S(V(\mathbb{R})) \otimes \Omega(D) \quad \text{for } \mathbf{x}_{\mathbb{R}} \in V(\mathbb{R}), z \in D.$$

Given the finite Schwartz function φ_f^{new} on $S(V(\mathbb{A}_f))$ defined in the previous section, we can construct

$$\varphi(\mathbf{x}, z) := \varphi_1 \otimes \varphi_f^{\text{new}} \in S(V(\mathbb{A})) \otimes \Omega(D) \quad \text{for } \mathbf{x} \in V(\mathbb{A}), z \in D. \tag{3.3}$$

Following (2.11), the theta series in this case is given by

$$\theta(g', \varphi_f^{\text{new}}, z) = \sum_{\mathbf{x} \in V(\mathbb{Q})} \omega(g')\varphi(\mathbf{x}, z) \quad \text{for } g' \in \mathrm{SL}_2(\mathbb{A}) \times \{\pm 1\}$$

which defines a closed differential 1-form on $\overline{\Gamma}_0(2mN)\backslash D$. For the non-vanishing of φ_f^{new}, this theta series descends to the sum over the integral lattice

$$L := \left\{ \begin{pmatrix} b & 2a \\ 2c & -b \end{pmatrix} : a, b, c \in \mathbb{Z}, -b^2 - 4ac \equiv 0 \mod m \right\}, \tag{3.4}$$

i.e. $\theta(g', \varphi_f^{\text{new}}, z) = \sum_{\mathbf{x}\in L} \omega(g')\varphi(\mathbf{x}, z)$.

Remark 3.3.1. In [Fun02, (3.17)], the theta series is given as the sum over the shift $h+L$ of L with the finite Schwartz function being the characteristic function. But we take the non-trivial φ_f^{new} as our finite Schwartz function and then the corresponding theta series becomes a sum over the integral lattice in (3.4).

Let $f \in S_2(\Gamma_0(N))$ be a weight 2 cusp form corresponding to a differential form η_f on $\Gamma_0(N)\backslash D$ and $\Gamma(L) \subset \mathrm{SO}_0(2,1)(\mathbb{R})$ the stabilizer group of L. Following (2.12), we have the theta lifting of f, which is cusp form of weight 3/2, given by

$$\Theta_\varphi(\eta_f)(g') = \int_{\Gamma\backslash D} f(z)dz \wedge \theta(g', \varphi_f^{\text{new}}, z)$$

where $\Gamma := \Gamma_0(N) \cap \Gamma(L) \cap \overline{\Gamma}_0(2mN) = \overline{\Gamma}_0(2mN)$. With the Schwartz function defined in the previous section, Proposition 3.2.4 implies that this theta lifting has level

$$\mathcal{L} := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) : N|b, 4N|c \right\}.$$

Set $\beta = m^2$. By Theorem 2.4.5, the Fourier coefficient at m^2 is given by

$$I = \sum_{\mathbf{x}\in\Gamma\backslash L_{m^2}} \varphi_f^{\text{new}}(\mathbf{x}) \int_{C_\mathbf{x}} f(z)dz \tag{3.5}$$

where $L_{m^2} = \{\mathbf{x} \in L : q(\mathbf{x}) = m^2\}$. By Proposition 3.1.5 we have $\Gamma\backslash L_{m^2} = \sum_i \Gamma_{\kappa_i}\backslash L_{m^2,\kappa_i,+}$. It follows that the Fourier coefficient at m^2 can be decomposed as

$$I = \sum_{\mathbf{x}\in\Gamma\backslash L_{m^2}} \varphi_f^{\text{new}}(\mathbf{x}) \int_{C_\mathbf{x}} f(z)dz = \sum_i \sum_{\mathbf{x}\in\Gamma_{\kappa_i}\backslash L_{m^2,\kappa_i,+}} \varphi_f^{\text{new}}(\mathbf{x}) \int_{C_\mathbf{x}} f(z)dz. \tag{3.6}$$

In this case $\mathbf{x}^\perp$ is split over $\mathbb{Q}$ due to Proposition 3.1.4, and then by Lemma 3.1.3 the stabilizer $\Gamma_\mathbf{x}$ is trivial. So the cycle $C_\mathbf{x} = \Gamma_\mathbf{x}\backslash D_\mathbf{x}$ is an infinite geodesic joining two cusps.

Remark 3.3.2. To express our coefficient I at m^2 in terms of twisted L-values, we were inspired by Kohnen's computations in [Koh85, Corollary 1]. Over cycles through the cusp ∞, the above period integral can be related to the special L-value $L(f,\chi,1)$ in subsection 3.3.1. Treating other cycles not through ∞ in subsection 3.3.2, we need Atkin-Lehner operators. In [Koh85, Theorem 3], Kohnen calculated the product $c(m)\overline{c(n)}$ of Fourier coefficients in terms of period integrals. With the condition that $-n$ is a fundamental discriminant, one can derive the formula for the square of the Fourier coefficient at a square-free integer in [Koh85, Corollary 1]. If $-n$ is not a fundamental discriminant, he also gave a bit more complicated version of $c(m)\overline{c(n)}$ in a Remark right after [Koh85, Theorem 3]. This might also lead to a relationship between the square of the Fourier coefficient at a square integer and the twisted L-value, but we have not pursued this.

3.3.1 On cycles through ∞

The goal of this subsection is to calculate

$$I_\infty := \sum_{\mathbf{x}\in\Gamma_\infty\backslash L_{m^2,\infty,+}} \varphi_f^{\text{new}}(\mathbf{x}) \int_{C_{\mathbf{x}}} f(z)dz.$$

We have seen a set of representatives of $\Gamma_\infty\backslash L_{m^2,\infty,+}$ in Corollary 3.1.7. Write $\mathbf{x} = \left(\begin{smallmatrix} m & 2a \\ 0 & -m \end{smallmatrix}\right) \in \Gamma_\infty\backslash L_{m^2,\infty,+}$ with a positive. By Definition 3.2.2, for the non-vanishing of φ_q^N we can exclude $-\mathbf{x}$ from $L_{m^2,\infty,+}$ and only need to count $\left(\begin{smallmatrix} m & 2a \\ 0 & -m \end{smallmatrix}\right) \in \Gamma_\infty\backslash L_{m^2,\infty,+}$. By Corollary 3.1.7, we observe that a ranges over $\mathbb{Z}/m\mathbb{Z}$.

Lemma 3.3.3. *For above m and a, we have*

$$\varphi_f^{\text{new}}\begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix} = 2\prod_{q|mN} [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \cdot \varphi_f\begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix}.$$

Proof. Set $q|2mN$. Write the representative in $\overline{\Gamma}_0(q)/\overline{\Gamma}(q)$

$$\gamma = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \qquad \text{with } x \in (\overline{\Gamma}_0(q)/\overline{\Gamma}(q))^\times, y \in \overline{\Gamma}_0(q)/\overline{\Gamma}(q).$$

Then we compute

$$\begin{aligned} \varphi_q^{\text{new}}\begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix} &= \sum_{[\gamma]\in\overline{\Gamma}_0(q)/\overline{\Gamma}(q)} \varphi_q\left(\gamma^{-1}\cdot\begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix}\right) \\ &= \sum_{[\gamma]\in\overline{\Gamma}_0(q)/\overline{\Gamma}(q)} \varphi_q\begin{pmatrix} m & 2x^{-1}ym+2x^{-2}a \\ 0 & -m \end{pmatrix}. \end{aligned}$$

At $q|m$, we see

$$\varphi_q\begin{pmatrix} m & 2x^{-1}ym+2x^{-2}a \\ 0 & -m \end{pmatrix} = \widetilde{\chi}_{m,q}(-2a) = \varphi_q\begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix}.$$

Similarly, at $q|2N$, we observe that

$$\varphi_q\begin{pmatrix} m & 2x^{-1}ym+2x^{-2}a \\ 0 & -m \end{pmatrix} = \varphi_q\begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix}.$$

So, we have

$$\varphi_q^{\mathrm{new}}\begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix} = [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \cdot \varphi_q \begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix},$$

and deduce that

$$\varphi_f^{\mathrm{new}}\begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix} = 2\prod_{q|mN} [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \cdot \varphi_f \begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix}.$$

□

Now following the above lemma we are ready to calculate

$$\begin{aligned} I_\infty &= \sum_{\mathbf{x}\in\Gamma_\infty\backslash L_{m^2,\infty,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}) \int_{C_{\mathbf{x}}} f(z)dz \\ &= 2\prod_{q|mN} [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \cdot \sum_{a\in\mathbb{Z}/m\mathbb{Z}} \varphi_f \begin{pmatrix} m & 2a \\ 0 & -m \end{pmatrix} \int_{-\frac{a}{m}}^{i\infty} f(z)dz \\ &= 2\prod_{q|mN} [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \cdot \sum_{a\in(\mathbb{Z}/m\mathbb{Z})^\times} \prod_{q|m} \widetilde{\chi}_{m,q}((-2a)_q) \int_{-\frac{a}{m}}^{i\infty} f(z)dz \\ &= 2\prod_{q|mN} [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \cdot \sum_{a\in(\mathbb{Z}/m\mathbb{Z})^\times} \chi_m^{-1}(-2a) \int_0^\infty f(-\frac{a}{m}+it)dt \\ &= 2\prod_{q|mN} [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \cdot \frac{1}{2\pi}\chi_m^{-1}(2)\tau(\chi_m^{-1})L(f,\chi_m,1). \end{aligned} \tag{3.7}$$

The last equality is the consequence of Birch's lemma, which can be derived from the computation on [DFK04, page 4] repeated in the following:

Let χ be a Dirichlet character of conductor m with Gauss sum

$$\tau(\chi) = \sum_{a \bmod m} \chi(a)e^{2\pi ia/m}. \tag{3.8}$$

The twisted L-function $L(f,\chi,s)$ can be expressed as

$$\begin{aligned} L(f,\chi,s) &= \sum_{n\geq 1}\chi(n)c(n)n^{-s} \\ &= \sum_{n\geq 1}\chi(n)c(n)(2\pi)^s\Gamma(s)^{-1}\int_0^\infty t^{s-1}e^{-2\pi nt}dt \\ &= (2\pi)^s\Gamma(s)^{-1}\int_0^\infty t^{s-1}\sum_{n\geq 1}\chi(n)c(n)e^{-2\pi nt}dt. \end{aligned}$$

From the identity

$$\sum_{n\geq 1}\chi(n)c(n)e^{2\pi inz} = \frac{1}{\tau(\chi^{-1})}\sum_{a \bmod m}\chi^{-1}(a)f(z+a/m),$$

we have

$$\begin{aligned} L(f,\chi,1) &= 2\pi \int_0^\infty \frac{1}{\tau(\chi^{-1})} \sum_{a \bmod m} \chi^{-1}(a) f(it + a/m) dt \\ &= \frac{2\pi}{\tau(\chi^{-1})} \sum_{a \bmod m} \chi^{-1}(a) \int_0^\infty f(it + a/m) dt. \end{aligned}$$

Remark 3.3.4. [Koj97] discusses Fourier coefficients of the Shintani lifting of a weight 2 cusp form f. Theorem 1 tells us that the integral appearing in 3.7 can be expressed in terms of twisted L-values, i.e.,

$$\int_0^\infty f(-\frac{a}{m} + it) dt = |(\mathbb{Z}/m\mathbb{Z})^\times|^{-1} \sum_\psi \psi^{-1}(a) H_0(f,\psi) \tag{3.9}$$

where the sum $\sum_\psi$ is taken over all Dirichlet characters ψ modulo m and

$$H_0(f,\psi) = i(2\pi)^{-1} K_0(\psi)(c(p_1),\dots,c(p_r)) L(f,\psi^{-1},1).$$

Here $p_1,\dots,p_r$ are all prime divisors of m, c stands for Fourier coefficients of f and $K_0(\psi)$ is a rational function of r variables given in [Koj97, Lemma 2.2]. Following the identity in (3.9), our I_∞ turns out to be

$$I_\infty = 2 \prod_{q|mN} [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \cdot \sum_{a \in (\mathbb{Z}/m\mathbb{Z})^\times} |(\mathbb{Z}/m\mathbb{Z})^\times|^{-1} \chi_m^{-1}(-2a) \sum_\psi \psi^{-1}(a) H_0(f,\psi).$$

The difference comes from having the particular weighted sum and being able to apply Birch's lemma in (3.7).

3.3.2 On other cycles

In this subsection we aim to calculate, for $\kappa_i \neq \infty$,

$$I_{\kappa_i} := \sum_{\mathbf{x} \in \Gamma_{\kappa_i} \backslash L_{m^2,\kappa_i,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}) \int_{C_\mathbf{x}} f(z) dz$$

with the help of Atkin-Lehner operators.

We recall Asai's treatment of cusps [Asa76, Section 1.1] in the following. Each cusp can be expressed as a reduced fraction with positive numerator except $0 = 0/1$. It is known that equivalence classes of cusps are in one-to-one correspondence with ordered decompositions $N = M_0 M$ of two positive divisors. We say a cusp $\kappa_i = \kappa_{i,2}/\kappa_{i,1}$ belongs to M_i-class if g.c.d.$(\kappa_{i,1}, N) = M_i$. For each decomposition $N = M_{0,i} M_i$ and any cusp $\kappa_i = \kappa_{i,2}/\kappa_{i,1}$ of M_i-class, we can take a typical matrix which transforms κ_i

to $\infty = 1/0$:

$$\omega_{\kappa_i} = \begin{pmatrix} 1 & 0 \\ 0 & M_{0,i} \end{pmatrix} \alpha_{\kappa_i} \text{ with } \alpha_{\kappa_i} = \begin{pmatrix} M_{0,i}\lambda_1 & \lambda_2 \\ -\kappa_{i,1} & \kappa_{i,2} \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z}) \text{ and } \lambda_i \in \mathbb{Z}. \tag{3.10}$$

As $(\kappa_{i,1}, \kappa_{i,2}) = 1$, there exists an integer b such that $b\kappa_{i,2} \equiv 1 (\mathrm{mod}\ \kappa_{i,1})$ and as $(\kappa_{i,1}, M_{0,i}) = 1$ there exists an integer c such that $cM_{0,i} \equiv 1 (\mathrm{mod}\ \kappa_{i,1})$. Taking $\lambda_1 = bc$ we observe that $\lambda_2 = \frac{1-M_{0,i}\lambda_1\kappa_{i,2}}{\kappa_{i,1}}$ is an integer. So such a ω_{κ_i} always exists but is not unique.

Proposition 3.3.5. *Let f be a weight 2 cusp form of level $\Gamma_0(N)$ with N square-free and m a square-free positive integer. Assume that $(2, m, N) = 1$. Choose the Schwartz form as in* (3.3). *Then the Fourier coefficient of the theta lifting of f as in* (3.6) *at m^2 is $I = I_\infty$ which is calculated in* (3.7).

Proof. Inspired by Kohnen's work [Koh85, Corollary 1] we calculate the Fourier coefficient in the following.

It is well known that the fractional linear transformation on the extended upper half plane is the composition of an even number of inversions which means it preserves the orientation ([Ber05, Section 2.3]). Note that ω_{κ_i} acts on integral $\mathbf{x}$ via $\omega_{\kappa_i} \cdot \mathbf{x} = \omega_{\kappa_i}\mathbf{x}\omega_{\kappa_i}^{-1}$. Thus, by Proposition 3.1.2, for $\mathbf{x} \in L_{m^2,\kappa_i,+}$ we have $\omega_{\kappa_i} \cdot \mathbf{x} \in L_{m^2,\infty,+}$. Note that the integrality of lattice is preserved under the action of ω_{κ_i}. Then we have

$$\begin{aligned} I_{\kappa_i} &= \sum_{\mathbf{x} \in \Gamma_{\kappa_i} \backslash L_{m^2,\kappa_i,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}) \int_{C_{\mathbf{x}}} f(z)dz \\ &= \sum_{\mathbf{x} \in \Gamma_{\kappa_i} \backslash L_{m^2,\kappa_i,+}} \varphi_f^{\mathrm{new}}(\omega_{\kappa_i}^{-1} \cdot \omega_{\kappa_i} \cdot \mathbf{x}) \int_{C_{\mathbf{x}}} f(z)dz \\ &= \sum_{\mathbf{x} \in \Gamma_\infty \backslash L_{m^2,\infty,+}} \varphi_f^{\mathrm{new}}(\omega_{\kappa_i}^{-1} \cdot \mathbf{x}) \int_{C_{\omega_{\kappa_i}^{-1} \cdot \mathbf{x}}} f(z)dz. \end{aligned}$$

We will describe $\varphi_f^{\mathrm{new}}(\omega_{\kappa_i}^{-1} \cdot \mathbf{x})$ in the following.

Write

$$\omega_{\kappa_i} = \begin{pmatrix} M_{0,i}\lambda_1 & \lambda_2 \\ -M_{0,i}\kappa_{i,1} & M_{0,i}\kappa_{i,2} \end{pmatrix} \quad \text{with } \det(\omega_{\kappa_i}) = M_{0,i}$$

and

$$\omega_{\kappa_i}^{-1} = M_{0,i}^{-1} \begin{pmatrix} M_{0,i}\kappa_{i,2} & -\lambda_2 \\ M_{0,i}\kappa_{i,1} & M_{0,i}\lambda_1 \end{pmatrix} \quad \text{with } \det(\omega_{\kappa_i}^{-1}) = M_{0,i}^{-1}.$$

For $\mathbf{x} = \begin{pmatrix} b & 2a \\ 0 & -b \end{pmatrix} \in \Gamma_\infty \backslash L_{m^2,\infty,+}$ with $b = m$, we compute

$$\begin{aligned}
&\omega_{\kappa_i}^{-1} \cdot \begin{pmatrix} b & 2a \\ 0 & -b \end{pmatrix} = \omega_{\kappa_i}^{-1} \begin{pmatrix} b & 2a \\ 0 & -b \end{pmatrix} \omega_{\kappa_i} \\
=&M_{0,i}^{-1} \begin{pmatrix} M_{0,i}\kappa_{i,2} & -\lambda_2 \\ M_{0,i}\kappa_{i,1} & M_{0,i}\lambda_1 \end{pmatrix} \begin{pmatrix} b & 2a \\ 0 & -b \end{pmatrix} \begin{pmatrix} M_{0,i}\lambda_1 & \lambda_2 \\ -M_{0,i}\kappa_{i,1} & M_{0,i}\kappa_{i,2} \end{pmatrix} \\
=&M_{0,i}^{-1} \begin{pmatrix} M_{0,i}\kappa_{i,2}b & 2M_{0,i}\kappa_{i,2}a + \lambda_2 b \\ M_{0,i}\kappa_{i,1}b & 2M_{0,i}\kappa_{i,1}a - M_{0,i}\lambda_1 b \end{pmatrix} \begin{pmatrix} M_{0,i}\lambda_1 & \lambda_2 \\ -M_{0,i}\kappa_{i,1} & M_{0,i}\kappa_{i,2} \end{pmatrix} \\
=&\begin{pmatrix} M_{0,i}\kappa_{i,2}\lambda_1 b - 2M_{0,i}\kappa_{i,2}\kappa_{i,1}a - \kappa_{i,1}\lambda_2 b & 2\kappa_{i,2}\lambda_2 b + 2M_{0,i}\kappa_{i,2}^2 a \\ 2M_{0,i}\kappa_{i,1}\lambda_1 b - 2M_{0,i}\kappa_{i,1}^2 a & \kappa_{i,1}\lambda_2 b + 2M_{0,i}\kappa_{i,1}\kappa_{i,2}a - M_{0,i}\kappa_{i,2}\lambda_1 b \end{pmatrix} \\
=:&\begin{pmatrix} b' & 2a' \\ 2c' & -b' \end{pmatrix}
\end{aligned}$$

Note that $M_{0,i}\lambda_1\kappa_{i,2} + \lambda_2\kappa_{i,1} = 1$ as $\det(\omega_{\kappa_i}) = M_{0,i}$.

To analyse $\varphi_q^{\mathrm{new}}(\omega_{\kappa_i}^{-1} \cdot \mathbf{x})$ at $q | 2mN$, for $\gamma = \begin{pmatrix} x & y \\ 0 & x^{-1} \end{pmatrix} \in \Gamma_0(q)/\Gamma(q)$ we compute

$$\gamma^{-1} \cdot \begin{pmatrix} b' & 2a' \\ 2c' & -b' \end{pmatrix} = \begin{pmatrix} b' - 2xyc' & 2x^{-1}yb' - 2y^2c' + 2x^{-2}a' \\ 2x^2c' & -b' + 2xyc' \end{pmatrix}.$$

Return to the decomposition $N = M_{0,i}M_i$. At $q | M_{0,i} | N$, we observe that

$$\begin{aligned}
&\begin{pmatrix} b' - 2xyc' & 2x^{-1}yb' - 2y^2c' + 2x^{-2}a' \\ 2x^2c' & -b' + 2xyc' \end{pmatrix} \\
&\qquad \equiv \begin{pmatrix} -\kappa_{i,1}\lambda_2 b & * \\ * & \kappa_{i,1}\lambda_2 b \end{pmatrix} \equiv \begin{pmatrix} -b & * \\ * & b \end{pmatrix} \not\equiv \begin{pmatrix} m & * \\ * & -m \end{pmatrix} \mod q.
\end{aligned}$$

So $\varphi_q^N \left(\gamma^{-1} \cdot \begin{pmatrix} b' & 2a' \\ 2c' & -b' \end{pmatrix} \right)$ is vanishing which implies φ_q^{new} at $q | M_{0,i} | N$ is vanishing on $\omega_{\kappa_i}^{-1} \cdot \mathbf{x}$.

Therefore, we can deduce that φ_f^{new} is vanishing on $\omega_{\kappa_i}^{-1} \cdot \mathbf{x}$ for $\mathbf{x} \in L_{m^2,\infty,+}$ which implies $I_{\kappa_i} = 0$ for $\kappa_i \neq \infty$. Then the assertion follows. □

Remark 3.3.6. (i) Let φ_q^N be the characteristic function of integral lattice and we will explain why the theta lifting would be vanishing in the following. Assume $\chi_m(-1) = 1$. Then $\sum_{\mathbf{x} \in \Gamma \backslash L_{m^2}} \varphi_f^{\mathrm{new}}(\mathbf{x}) \int_{C_{\mathbf{x}}} f(z)dz$ in (3.6) would be vanishing since both $\mathbf{x}$ and $-\mathbf{x}$ (giving rise to different orientation) lie in $\Gamma \backslash L_{m^2}$. Also, on the last part of (3.6) we would calculate

$$I = \left(1 + \sum_{\text{non-trivial } M_{0,i}|N} \varepsilon_{M_{0,i}} \chi_m(M_{0,i})\right) I_\infty$$

where $\varepsilon_{M_{0,i}}$ is the Atkin-Lehner eigenvalue corresponding to ω_{κ_i}. Recall from [Miy06, Theorem 4.3.12] that the sign of functional equation attached to $L(f, \chi_m, s)$ is $-\varepsilon_N \chi_m(N)$ in our case. For example, let $N = q_1 q_2$. To achieve the non-vanishing of I_∞, we need $L(f, \chi_m, 1) \neq 0$, i.e. we need to have

$$\varepsilon_N \chi_m(N) = \varepsilon_{q_1} \chi_m(q_1) \varepsilon_{q_2} \chi_m(q_2) = -1$$

which implies

$$1 + \varepsilon_{q_1} \chi_m(q_1) + \varepsilon_{q_2} \chi_m(q_2) + \varepsilon_N \chi_m(N) = 0.$$

Then we see that the Fourier coefficient I would be vanishing.

(ii) Another approach to avoid the trivial vanishing of the theta lifting is to take the local Schwartz function at the finite place as the characteristic of a shift of the integral lattice by a suitable rational vector, see e.g. [FM11]. Following this, the finite Schwartz function is invariant under some principal congruence subgroups. Then, a modular forms whose level is a principal congruence subgroup is required to be paired against this theta series while we want to construct theta liftings of modular forms for Γ_0-subgroups. So, in preparation for our work in the case of Bianchi modular forms, we instead incorporated an auxiliary quadratic Hecke character which also gives us flexibility in applying the result of Friedberg and Hoffstein about the non-vanishing of the twisted L-values by Hecke characters in Section 4.5.

The work of Bump, Friedberg and Hoffstein [BFH90, Theorem] guarantees the existence of infinitely many quadratic characters χ such that the twisted L-value $L(f, \chi, 1)$ is non-vanishing. So if $L(f, \chi_m, 1)$ is known to be non-vanishing, we can deduce that the Fourier coefficient at m^2 of the theta lifting defined using χ_m is non-vanishing which implies the non-vanishing of our theta lift.

Chapter 4

Theta lift of Bianchi modular form

§ 4.1 Binary Hermitian forms

In this section we recall some basics from linear algebra about Hermitian matrices and Hermitian binary forms from [EGM98, Chapter 9]. For a complex matrix A, the matrix $\bar{A}$ is obtained from A by applying complex conjugation to all entries and the matrix A^t is the transpose of A. An $n \times n$ matrix A with complex entries is called *Hermitian* if $\bar{A}^t = A$. By the definition we see that an Hermitian matrix is unchanged by taking its conjugate transpose. Note that any Hermitian matrix must have real diagonal entries.

Let R be a subring of $\mathbb{C}$ with $R = \bar{R}$. We write $\mathcal{H}(R)$ for the set of Hermitian 2×2 matrices with entries in R, i.e.

$$\mathcal{H}(R) = \{A \in M_2(R) : \bar{A}^t = A\}.$$

Every $f \in \mathcal{H}(R)$ defines a binary Hermitian form with coefficients in R. If $f = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix}$ then the associated binary Hermitian form is the semi quadratic map $f : \mathbb{C} \times \mathbb{C} \to \mathbb{R}$ defined by

$$f(u,v) = (u,v) \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} (\bar{u}, \bar{v})^t = au\bar{u} + bu\bar{v} + \bar{b}\bar{u}v + dv\bar{v}.$$

We shall often call an element $f \in \mathcal{H}(R)$ a binary hermitian form with coefficients in R. The discriminant $\Delta(f)$ of $f \in \mathcal{H}(R)$ is defined as $\Delta(f) = \det(f)$. Set $|a| = (a\bar{a})^{1/2}$

for $a \in \mathbb{C}$ where $^-$ denotes the complex conjugation. We define the $\mathrm{GL}_2(R)$-action on $\mathcal{H}(R)$ given by the formula

$$\sigma \cdot f = (|\det(\sigma)|^{-1/2}\sigma) f (|\det(\bar{\sigma}^t)|^{-1/2}\bar{\sigma}^t) = |\det(\sigma)|^{-1}\sigma f \bar{\sigma}^t \tag{4.1}$$

for $\sigma \in \mathrm{GL}_2(R)$ and $f \in \mathcal{H}(R)$. If $\sigma = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(R)$ we have

$$\sigma \cdot f = |\det(\sigma)|^{-1} \begin{pmatrix} (\alpha,\beta)f(\bar{\alpha},\bar{\beta})^t & (\alpha,\beta)f(\gamma,\delta)^t \\ (\gamma,\delta)f(\bar{\alpha},\bar{\beta})^t & (\gamma,\delta)f(\bar{\gamma},\bar{\delta})^t \end{pmatrix}.$$

Note that $\Delta(\sigma \cdot f) = \Delta(f)$ for every $\sigma \in \mathrm{GL}_2(R)$ and $f \in \mathcal{H}(R)$. Two elements $f, g \in \mathcal{H}(R)$ are called $\mathrm{GL}_2(R)$-equivalent if $g = \sigma \cdot f$ for some $\sigma \in \mathrm{GL}_2(R)$; $\mathrm{SL}_2(R)$-equivalence is defined analogously.

A binary Hermitian form $f \in \mathcal{H}(R)$ is positive definite if $f(u,v) > 0$ for all $(u,v) \in \mathbb{C} \times \mathbb{C} \setminus \{(0,0)\}$. If $-f$ is positive definite f is called negative definite. If $\Delta(f) < 0$ then f is called indefinite.

We define

$$\mathcal{H}^+(R) = \{f \in \mathcal{H}(R) : f \text{ is positive definite}\}$$
$$\mathcal{H}^-(R) = \{f \in \mathcal{H}(R) : f \text{ is indefinite}\}.$$

Clearly the group $\mathrm{GL}_2(R)$ leaves the $\mathcal{H}^\pm$ invariant. It is easy to see that $f \in \mathcal{H}^+(R)$ if and only if $a > 0$ and $\Delta(f) > 0$. The group $\mathbb{R}_{>0}$ acts on $\mathcal{H}^+(\mathbb{C})$ by scalar multiplication. Similarly $\mathbb{R}^\times$ acts on $\mathcal{H}^-(\mathbb{C})$. We define

$$\tilde{\mathcal{H}}^+(\mathbb{C}) := \mathcal{H}^+(\mathbb{C})/\mathbb{R}_{>0}, \quad \tilde{\mathcal{H}}^-(\mathbb{C}) := \mathcal{H}^-(\mathbb{C})/\mathbb{R}^\times.$$

For $f \in \mathcal{H}^\pm(\mathbb{C}), [f]$ stands for the class of f in $\tilde{\mathcal{H}}^\pm(\mathbb{C})$. The action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathcal{H}^\pm(\mathbb{C})$ clearly induces an action of $\mathrm{GL}_2(\mathbb{C})$ on $\tilde{\mathcal{H}}^\pm(\mathbb{C})$. The centre of $\mathrm{SL}_2(\mathbb{C})$ acts trivially on $\mathcal{H}(\mathbb{C})$, so we get an induced action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathcal{H}(\mathbb{C})$ and $\tilde{\mathcal{H}}^\pm(\mathbb{C})$.

Recall the upper half space $\mathbb{H}_3 = \mathbb{C} \times \mathbb{R}_{>0}$, elements of which can be written as (z, r) with $z = x + iy$ for $x, y \in \mathbb{R}, r \in \mathbb{R}_{>0}$.

Definition 4.1.1. The map $\phi : \mathcal{H}^+(\mathbb{C}) \to \mathbb{H}_3$ is defined as

$$\phi : f = \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \to \frac{b}{d} + \frac{\sqrt{\Delta(f)}}{d} \cdot j$$

In fact ϕ induces a map $\phi : \tilde{\mathcal{H}}^+(\mathbb{C}) \to \mathbb{H}_3$.

This map is a bijection since for a point $(z, r) \in \mathbb{H}_3$ there exists $f = \left(\begin{smallmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{smallmatrix}\right)$ such that $\phi(f) = z + rj \in \mathbb{H}_3$. Therefore, this map gives a one to one correspondence between equivalence classes of positive definite Hermitian forms and points in the upper half space. Note that ϕ is the analogue of identification of the set of equivalence classes of binary positive definite quadratic forms with points of the upper half plane as in (3.1).

Proposition 4.1.2. *The map* $\phi : \tilde{\mathcal{H}}^+(\mathbb{C}) \to \mathbb{H}_3$ *is a* $\mathrm{PSL}_2(\mathbb{C})$*-equivariant bijection; that is* $\phi(\sigma \cdot f) = \sigma \cdot \phi(f)$ *for every* $\sigma \in \mathrm{PSL}_2(\mathbb{C})$ *and* $f \in \tilde{\mathcal{H}}^+$*.*

Proof. See [EGM98, Proposition 9.1.2, Chapter 9]. □

Definition 4.1.3. For a binary Hermitian form $f = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \in \mathcal{H}^-(\mathbb{C})$ we define

$$\psi(f) = \{z + rj \in \mathbb{H}_3 : a - \bar{b}z - b\bar{z} + dz\bar{z} + r^2 d = 0\}$$

and $\mathbf{G} = \{\psi(f) \mid f \in \mathcal{H}^-(\mathbb{C})\}$ which is a set of geodesic planes in $\mathbb{H}_3$.

Remark 4.1.4. This map ψ is slightly different to the map in [EGM98, Definition 1.3, Chapter 9] which is given by

$$f \mapsto \{z + rj \in \mathbb{H}_3 : a + \bar{b}z + b\bar{z} + dz\bar{z} + r^2 d = 0\}.$$

The above map ψ is chosen for us to prove Proposition 4.2.4. In addition we will consider the cycle D_U as in Section 2.3 for positive definite U generated by f with $f \in \mathcal{H}^-(\mathbb{C})$.

If $d \neq 0$ then $\psi(f)$ is the following geodesic hemisphere

$$\psi(f) = \{z + rj \in \mathbb{H}_3 : |dz - b|^2 + |d|^2 r^2 = -\Delta(f)\}.$$

If $d = 0$ then $\psi(f)$ is a vertical plane. The group $\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathbf{G}$ by its induced action on subsets of $\mathbb{H}_3$. Clearly ψ induces a map $\psi : \tilde{\mathcal{H}}^-(\mathbb{C}) \to \mathbf{G}$.

Proposition 4.1.5. *The map* $\psi : \tilde{\mathcal{H}}^-(\mathbb{C}) \to \mathbf{G}$ *is a* $\mathrm{PSL}_2(\mathbb{C})$*-equivariant bijection; that is* $\psi(\sigma \cdot f)) = \sigma \cdot \psi(f)$ *for every* $\sigma \in \mathrm{PSL}_2(\mathbb{C})$ *and* $f \in \tilde{\mathcal{H}}^-(\mathbb{C})$*.*

Proof. We will prove the equivariance property only for the generators of $\mathrm{PSL}_2(\mathbb{C})$.

Let $\sigma = \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$ where $\beta \in \mathbb{C}$. Then

$$\sigma \cdot f = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \begin{pmatrix} 1 & 0 \\ \bar{\beta} & 1 \end{pmatrix} = \begin{pmatrix} a + \beta\bar{b} + \bar{\beta}b + \beta\bar{\beta}d & b + \beta d \\ \bar{b} + \bar{\beta}d & d \end{pmatrix}.$$

It follows that

$$\psi(\sigma \cdot f) = \{z + rj \in \mathbb{H}_3 \mid a + \beta\bar{b} + \bar{\beta}b + \beta\bar{\beta}d - (\bar{b} + \bar{\beta}d)z - (b + \beta d)\bar{z} + dz\bar{z} + r^2 d = 0\}.$$

On the other hand, for $z + rj \in \psi(f)$, we have $\sigma \cdot (z + rj) = (z + \beta) + rj \in \mathbb{H}_3$. Setting $z' = z + \beta$ and $r' = r$, we observe that

$$a - \bar{b}(z' - \beta) - b(\bar{z}' - \bar{\beta}) + d(z' - \beta)(\bar{z}' - \bar{\beta}) + r'^2 d = 0.$$

Then it is not hard to see that $\psi(\sigma \cdot f)) = \sigma \cdot \psi(f)$ for $\sigma = \left(\begin{smallmatrix} 1 & \beta \\ 0 & 1 \end{smallmatrix}\right)$.

In the same way we prove this property for $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. We have

$$\sigma \cdot f = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} d & -\bar{b} \\ -b & a \end{pmatrix}.$$

It follows that

$$\psi(\sigma \cdot f) = \{z + rj \in \mathbb{H}_3 : d + bz + \bar{b}\bar{z} + az\bar{z} + r^2 a = 0\}.$$

For $z + rj \in \psi(f)$, we have $z' + r'j := \sigma \cdot (z + rj) = -\frac{\bar{z}}{|z|^2+r^2} + \frac{r}{|z|^2+r^2} j$. Then $|z'|^2 + r'^2 = \frac{1}{|z|^2+r^2}$. It follows that $z = -\frac{\bar{z}'}{|z'|^2+r'^2}$ and $r = \frac{r'}{|z'|^2+r'^2}$. Hence the following identity holds

$$a + \bar{b}\frac{\bar{z}'}{|z'|^2 + r'^2} + b\frac{z}{|z'|^2 + r'^2} + d\frac{z'\bar{z}'}{(|z'|^2 + r'^2)^2} + d\frac{r'^2}{(|z'|^2 + r'^2)^2} = 0.$$

Then we can see that $\psi(\sigma \cdot f)) = \sigma \cdot \psi(f)$ for $\sigma = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. □

§ 4.2 Orthogonal group of sign $(3, 1)$ and cycles

In this section we recall some basic aspects on orthogonal groups of signature (3,1) and cycles in this case from [Ber14, Section 4].

Let $F = \mathbb{Q}(\sqrt{d})$ $(d < 0)$ be an imaginary quadratic field of class number 1 with discriminant $d_F < 0$. Denote by $\mathcal{O}$ by its ring of integers. For an ideal $\mathfrak{n} \subset \mathcal{O}$ put

$$\Gamma_0(\mathfrak{n}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}) : c \in \mathfrak{n} \right\}.$$

Assume that the four-dimensional space V over $\mathbb{Q}$ is given by the hermitian matrices

$$V = \{\mathbf{x} \in M_2(F) : \mathbf{x}^t = \overline{\mathbf{x}}\},$$

with quadratic form

$$\mathbf{x} \longmapsto q(\mathbf{x}) = \frac{1}{2}(\mathbf{x}, \mathbf{x}) = -\det(\mathbf{x})$$

and corresponding bilinear form

$$(\mathbf{x}, \mathbf{y}) \longmapsto -\operatorname{tr}(\mathbf{x}\mathbf{y}^*),$$

where

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^* = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}. \tag{4.2}$$

Note that this bilinear form is preserved under the action of $\mathrm{GL}_2(\mathbb{C})$ where its action is given in (4.1); that is, for $g \in \mathrm{GL}_2(\mathbb{C})$,

$$\begin{aligned}(g\cdot\mathbf{x}, g\cdot\mathbf{y}) &= (|\det(g)|^{-1} g\mathbf{x}\bar{g}^t, |\det(g)|^{-1} g\mathbf{y}\bar{g}^t) \\ &= -\frac{1}{2}\operatorname{tr}(|\det(g)|^{-1} g\mathbf{x}\bar{g}^t |\det(g)| (\bar{g}^t)^{-1}\mathbf{y}^{-1} g^{-1} \det(|\det(g)|^{-1} g\mathbf{y}\bar{g}^t)) \\ &= -\frac{1}{2}\operatorname{tr}(g\mathbf{x}\mathbf{y}^{-1}\det(\mathbf{y}) g^{-1}) = (\mathbf{x},\mathbf{y}). \end{aligned} \tag{4.3}$$

We fix an orthogonal basis of $V(\mathbb{Q})$ given by $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$, $e_3 = \left(\begin{smallmatrix} 0 & \sqrt{d} \\ -\sqrt{d} & 0 \end{smallmatrix}\right)$ and $e_4 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) = Z_0$ such that the discriminant of V is d. The basis of $Z_0^{\perp}$ can be identified with $\{e_1, e_2, e_3\}$.

We have seen in Example 1.1.6 that the symmetric space $D \simeq \mathbb{H}_3$ is isomorphic to the Grassmannian Gr_1 via the map

$$\mu : z + rj \in \mathbb{H}_3 \longmapsto \frac{1}{r}\begin{pmatrix} |z|^2 + r^2 & z \\ \bar{z} & 1 \end{pmatrix}. \tag{4.4}$$

The GL_2-action on the Hermitian form defined as in (4.1) induces that on $\mathbb{H}_3$ in the following. For $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$ we have $g \cdot \frac{1}{r}\left(\begin{smallmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{smallmatrix}\right) = \frac{1}{r'}\left(\begin{smallmatrix} |z'|^2+r'^2 & z' \\ \bar{z}' & 1 \end{smallmatrix}\right)$; expand the LHS,

$$\begin{aligned} g\cdot\frac{1}{r}\left(\begin{smallmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{smallmatrix}\right) &= r^{-1}|\det(g)|^{-1}\left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)\left(\begin{smallmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{smallmatrix}\right)\left(\begin{smallmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{smallmatrix}\right) \\ &= r^{-1}|\det(g)|^{-1}\left(\begin{smallmatrix} \alpha|z|^2+\alpha r^2+\beta\bar{z} & \alpha z+\beta \\ \gamma|z|^2+\gamma r^2+\delta\bar{z} & \gamma z+\delta \end{smallmatrix}\right)\left(\begin{smallmatrix} \bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta} \end{smallmatrix}\right) \\ &= r^{-1}|\det(g)|^{-1}\left(\begin{smallmatrix} \alpha\bar{\alpha}|z|^2+\alpha\bar{\alpha}r^2+\bar{\alpha}\beta\bar{z}+\alpha\bar{\beta}z+\beta\bar{\beta} & \alpha\bar{\gamma}|z|^2+\alpha\bar{\gamma}r^2+\beta\bar{\gamma}\bar{z}+\alpha\bar{\delta}z+\beta\bar{\delta} \\ \bar{\alpha}\gamma|z|^2+\bar{\alpha}\gamma r^2+\bar{\alpha}\delta\bar{z}+\bar{\beta}\gamma z+\bar{\beta}\delta & \gamma\bar{\gamma}|z|^2+\gamma\bar{\gamma}r^2+\bar{\gamma}\delta\bar{z}+\gamma\bar{\delta}z+\delta\bar{\delta} \end{smallmatrix}\right), \end{aligned}$$

and then

$$z' = \frac{(\alpha z+\beta)(\bar{\gamma}\bar{z}+\bar{\delta})+\alpha\bar{\gamma}r^2}{|\gamma z+\delta|^2+|\gamma|^2r^2}, \quad r' = \frac{|\alpha\delta-\beta\gamma|r}{|\gamma z+\delta|^2+|\gamma|^2r^2}.$$

By (4.4), we can deduce the action of $\mathrm{GL}_2(\mathbb{C})$ on $\mathbb{H}_3$ (as in (1.9)) to be as

$$\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\cdot(z,r) = \left(\frac{(\alpha z+\beta)(\bar{\gamma}\bar{z}+\bar{\delta})+\alpha\bar{\gamma}r^2}{|\gamma z+\delta|^2+|\gamma|^2r^2}, \quad \frac{|\alpha\delta-\beta\gamma|r}{|\gamma z+\delta|^2+|\gamma|^2r^2}\right). \tag{4.5}$$

Proposition 4.2.1 (Analogue of Proposition 3.1.1)**.** *The above map* μ *as in* (4.4) *intertwines the* $\mathrm{GL}_2(\mathbb{C})$*-action on* $V(\mathbb{R})$ *and* $\mathbb{H}_3$*; that is* $\mu(g\cdot(z,r)) = g\cdot\mu(z,r)$ *for* $g \in \mathrm{GL}_2(\mathbb{C})$*.*

Proof. It suffices to calculate $\mu(g\cdot(z,r))$ for $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right)$. Writing $(z',r') = g\cdot(z,r)$, by the formula (4.5) we have

$$\mu(g\cdot(z,r)) = \frac{1}{r'}\begin{pmatrix} |z'|^2+r'^2 & z' \\ \bar{z}' & 1 \end{pmatrix}.$$

It is not difficult to observe that the entries of $\mu(g\cdot(z,r))$ are the same as the counterparts of $g \cdot \mu(z,r)$ except for the top left one

$$\frac{|z'|^2 + r'^2}{r'} = \frac{|(\alpha z + \beta)(\bar\gamma \bar z + \bar\delta) + \alpha\bar\gamma r^2|^2 + |\alpha\delta - \beta\gamma|^2 r^2}{r|\det g| \cdot (|\gamma z + \delta|^2 + |\gamma|^2 r^2)}. \tag{4.6}$$

We want to show that

$$\text{the numerator in (4.6)} = (|\alpha z + \beta|^2 + |\alpha|^2 r^2) \times (|\gamma z + \delta|^2 + |\gamma|^2 r^2).$$

Expanding both sides, we have

$$\begin{aligned}\text{LHS} =& |\alpha z + \beta|^2 |\gamma z + \delta|^2 + \alpha\bar\alpha\gamma\bar\gamma r^4 + (\alpha\bar\gamma z\bar z + \alpha\bar\delta z + \beta\bar\gamma\bar z + \beta\bar\delta)\bar\alpha\gamma r^2 \\ &+ (\bar\alpha\gamma z\bar z + \bar\beta\gamma z + \bar\alpha\delta\bar z + \bar\beta\delta)\alpha\bar\gamma r^2 + (\alpha\bar\alpha\delta\bar\delta - \alpha\bar\beta\bar\gamma\delta - \bar\alpha\beta\gamma\bar\delta + \beta\bar\beta\gamma\bar\gamma) r^2\end{aligned}$$

and

$$\begin{aligned}\text{RHS} =& |\alpha z + \beta|^2 |\gamma z + \delta|^2 + \alpha\bar\alpha\gamma\bar\gamma r^4 \\ &+ (\alpha\bar\alpha z\bar z + \alpha\bar\beta z + \bar\alpha\beta\bar z + \beta\bar\beta)\gamma\bar\gamma r^2 + (\gamma\bar\gamma z\bar z + \gamma\bar\delta z + \bar\gamma\delta\bar z + \delta\bar\delta)\alpha\bar\alpha r^2.\end{aligned}$$

As LHS=RHS, we have that

$$\frac{|z'|^2 + r'^2}{r'} = \frac{1}{r|\det g|}(\alpha\bar\alpha|z|^2 + \alpha\bar\alpha r^2 + \bar\alpha\beta\bar z + \alpha\bar\beta z + \beta\bar\beta)$$

which is equal to the top left entry of $g \cdot \mu(z,r)$. Hence the $\mathrm{GL}_2(\mathbb{C})$-equivariance property of μ has been proven. □

The set $\mathrm{Iso}(V)$ of all isotropic lines (1-dimensional $\mathbf{x} \in V$ such that $q(\mathbf{x}) = 0$) in $V(\mathbb{Q})$ can be identified with $\mathbb{P}^1(F) = F \cup \infty$ ($\infty = [1:0]$). Assume that the cusp ∞ corresponds to the isotropic line spanned by $u_\infty = \left(\begin{smallmatrix}1 & 0\\ 0 & 0\end{smallmatrix}\right)$. Given an element $g = \left(\begin{smallmatrix}\alpha & \beta\\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{SL}_2(F)$ transforming the cusp ∞ to another cusp $\kappa = [\alpha : \gamma]$, we can see that

$$g \cdot u_\infty = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix}\begin{pmatrix}\bar\alpha & \bar\gamma\\ \bar\beta & \bar\delta\end{pmatrix} = \begin{pmatrix}\alpha\bar\alpha & \alpha\bar\gamma\\ \bar\alpha\gamma & \gamma\bar\gamma\end{pmatrix}.$$

Hence we can identify the cusp with the isotropic line by means of the map

$$\nu : [a:b] \longmapsto \operatorname{span}\begin{pmatrix}a\bar a & a\bar b\\ \bar a b & b\bar b\end{pmatrix} \in \mathrm{Iso}(V). \tag{4.7}$$

Proposition 4.2.2 (Analogue of Proposition 3.1.2)**.** *The above map ν satisfies*

$$\nu(g \cdot [a:b]) = g \cdot \nu([a:b])$$

for $g \in \mathrm{GL}_2(F)$ and $[a:b] \in \mathbb{P}^1(F)$.

Proof. We compute

$$\nu(g \cdot [a:b]) = \operatorname{span}\begin{pmatrix}(\alpha a + \beta b)(\bar\alpha\bar a + \bar\beta\bar b) & (\alpha a + \beta b)(\bar\gamma\bar a + \bar\delta\bar b)\\ (\bar\alpha\bar a + \bar\beta\bar b)(\gamma a + \delta b) & (\gamma a + \delta b)(\bar\gamma\bar a + \bar\delta\bar b)\end{pmatrix}$$

and for $g = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(F)$ with $|\det g| \in \mathbb{Q}$,

$$\begin{aligned}
&g \cdot \nu([a:b]) \\
=&\mathrm{span}\frac{1}{|\det g|}\begin{pmatrix}\alpha & \beta \\ \gamma & \delta\end{pmatrix}\begin{pmatrix}a\bar{a} & a\bar{b} \\ \bar{a}b & b\bar{b}\end{pmatrix}\begin{pmatrix}\bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta}\end{pmatrix} = \mathrm{span}\begin{pmatrix}\alpha a\bar{a} + \beta\bar{a}b & \alpha a\bar{b} + \beta b\bar{b} \\ \gamma a\bar{a} + \delta\bar{a}b & \gamma a\bar{b} + \delta b\bar{b}\end{pmatrix}\begin{pmatrix}\bar{\alpha} & \bar{\gamma} \\ \bar{\beta} & \bar{\delta}\end{pmatrix} \\
=&\mathrm{span}\begin{pmatrix}\alpha\bar{\alpha}a\bar{a} + \bar{\alpha}\beta\bar{a}b + \alpha\bar{\beta}a\bar{b} + \beta\bar{\beta}b\bar{b} & \alpha\bar{\gamma}a\bar{a} + \beta\bar{\gamma}\bar{a}b + \alpha\bar{\delta}a\bar{b} + \beta\bar{\delta}b\bar{b} \\ \bar{\alpha}\gamma a\bar{a} + \bar{\alpha}\delta\bar{a}b + \bar{\beta}\gamma a\bar{b} + \bar{\beta}\delta b\bar{b} & \gamma\bar{\gamma}a\bar{a} + \bar{\gamma}\delta\bar{a}b + \gamma\bar{\delta}a\bar{b} + \delta\bar{\delta}b\bar{b}\end{pmatrix}.
\end{aligned}$$

Comparing the entries we have proven this property. □

Let $U \subset V$ be a $\mathbb{Q}$-subspace with $\dim_{\mathbb{Q}} U = 2$ such that $(\ ,\ )|_U$ is positive definite. Then its orthogonal complement $U^\perp$ has signature (1,1). As in Section 2.3, we have the special cycle

$$D_U = \{Z \in D : Z \perp U\}$$

and let Γ_U be the stabilizer of U in $\mathrm{SO}_0(3,1)(V(\mathbb{R}))$. We denote the image of the quotient $\Gamma_U \backslash D_U$ in $\Gamma\backslash D$ by C_U. The stabilizer Γ_U is either trivial (if the orthogonal complement $U^\perp \subset V$ is split over $\mathbb{Q}$) or infinite cyclic (if $U^\perp$ is non-split over $\mathbb{Q}$) (see [Fun02, Lemma 4.2]). If Γ_U is infinite, then C_U is a closed geodesic in $\Gamma\backslash D$, while C_U is infinite if Γ_U is trivial (see [Ber14, Section 4.3]).

Lemma 4.2.3 (Analogue of Proposition 3.1.4)**.** *For above U, the following two statements are equivalent:*

(1) $U^\perp$ *is split over* $\mathbb{Q}$*,*
(2) $\mathrm{disc}(U) \in -d(\mathbb{Q}^\times)^2$.

Proof. For an arbitrary subspace U of a non-degenerate quadratic space V we have $\dim(V) = \dim(U) + \dim(U^\perp)$. Thus $U^\perp$ is also 2-dimensional. By assumption $U^\perp$ is a hyperbolic plane. By Witt's Theorem (a 2-dimensional quadratic space over a field F is a hyperbolic plane if and only if its discriminant lies in $-(F^\times)^2$), we have $\mathrm{disc}(U^\perp) \in -(\mathbb{Q}^\times)^2$. Thus $\mathrm{disc}(U) \in -d(\mathbb{Q}^\times)^2$ as $\mathrm{disc}(V) = \mathrm{disc}(U)\mathrm{disc}(U^\perp) \in d(\mathbb{Q}^\times)^2$.

Conversely suppose $\mathrm{disc}(U) \in -d(\mathbb{Q}^\times)^2$. Again by $\mathrm{disc}(V) = \mathrm{disc}(U)\mathrm{disc}(U^\perp)$, we have $\mathrm{disc}(U^\perp) \in -(\mathbb{Q}^\times)^2$ which implies that $U^\perp$ is split over $\mathbb{Q}$. □

We orient D_U by requiring that a tangent vector $v \in T_Z(D_U) \simeq Z^\perp \cap U^\perp$ followed by $Z^\perp \cap U$ gives a properly oriented basis of $T_Z(D) \simeq Z^\perp$. Then $\langle Z^\perp \cap U^\perp, Z^\perp \cap U, Z\rangle$

has the same orientation as $\langle e_1, e_2, e_3, e_4 \rangle$, i.e. the determinant of the base change is positive.

For $\beta = \beta^t \in M_2(\mathbb{Q})$ a positive definite symmetric matrix, let

$$\Omega_\beta = \left\{ (\mathbf{x}_1, \mathbf{x}_2) \in V^2(\mathbb{Q}) : \frac{1}{2} \begin{pmatrix} (\mathbf{x}_1, \mathbf{x}_1) & (\mathbf{x}_1, \mathbf{x}_2) \\ (\mathbf{x}_1, \mathbf{x}_2) & (\mathbf{x}_2, \mathbf{x}_2) \end{pmatrix} = \beta \right\}.$$

Consider the subspace $U(\mathbf{x}_1, \mathbf{x}_2) := \operatorname{Span}\{\mathbf{x}_1, \mathbf{x}_2\} \subset V$. For a fixed cusp κ_i corresponding to the isotropic line l_{κ_i}, we write

$$\Omega_{\beta,\kappa_i} = \{(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_\beta : U(\mathbf{x}_1, \mathbf{x}_2) \perp l_{\kappa_i}\}.$$

From now on, fix a β such that $\det \beta \in -d(\mathbb{Q}^\times)^2$, i.e. $\operatorname{disc}(U(\mathbf{x}_1, \mathbf{x}_2)) \in -d(\mathbb{Q}^\times)^2$ for $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_\beta$. Let $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_\beta$ and $U = U(\mathbf{x}_1, \mathbf{x}_2)$. Given a vector $\mathbf{x} \in U$, by Lemma 4.2.3, it is orthogonal to two isotropic lines l_{κ_1} and l_{κ_2} generated by u_{κ_1} and u_{κ_2} respectively associated to two cusps κ_1 and κ_2. Again, if these two cusps are not equivalent with respect to Γ, we can give a positive orientation to U to distinguish the cusps in the sense that the new base $\langle u_{\kappa_1}, \mathbf{x}_1, \mathbf{x}_2, u_{\kappa_2} \rangle$ preserves the orientation of $\langle e_1, e_2, e_3, e_4 \rangle$. For a fixed cusp κ_i corresponding to the isotropic line l_{κ_i}, we write

$$\begin{aligned} \Omega_{\beta,\kappa_i,+} = \{(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_{\beta,\kappa_i} :& U(\mathbf{x}_1, \mathbf{x}_2) \perp \langle u_{\kappa_i}, u_{\kappa_j} \rangle, \\ & \langle u_{\kappa_i}, \mathbf{x}_1, \mathbf{x}_2, u_{\kappa_j} \rangle \text{ has a positive orientation}\}. \end{aligned}$$

It should be mentioned here that $\langle u_{\kappa_i}, \mathbf{x}_1, \mathbf{x}_2, u_{\kappa_j} \rangle$ and $\langle u_{\kappa_i}, -\mathbf{x}_1, -\mathbf{x}_2, u_{\kappa_j} \rangle$ have the same orientation which means that we need to count $(\mathbf{x}_1, \mathbf{x}_2)$ and $(-\mathbf{x}_1, -\mathbf{x}_2)$ simultaneously in $\Omega_{\beta,\kappa_i,+}$. Alternatively, the following Lemma 4.2.7 describes the orientations associated to two pairs $(\mathbf{x}_1, \mathbf{x}_2)$ and $(-\mathbf{x}_1, -\mathbf{x}_2)$ in $\Omega_{\beta,\infty,+}$. Note that the stabilizer $\Gamma_{\kappa_i} \subset \Gamma$ of the cusp κ_i acts on $\Omega_{\beta,\kappa_i,+}$ as $\mathrm{GL}_2(\mathbb{C})$ preserves bilinear forms and the orientation.

Proposition 4.2.4 (Analogue of Proposition 3.1.5). *For* $\det(\beta) \in -d(\mathbb{Q}^\times)^2$*, we have*

$$\Gamma \backslash \Omega_\beta = \sum_{\kappa_i \in \Gamma \backslash \mathbb{P}^1(F)} \Gamma_{\kappa_i} \backslash \Omega_{\beta,\kappa_i,+}.$$

Proof. Given a representative $[(\mathbf{x}_1, \mathbf{x}_2)]$ in $\Gamma \backslash \Omega_\beta$ such that $U(\mathbf{x}_1, \mathbf{x}_2) \perp u_{\kappa_i}$, we consider its Γ-orbit $\Gamma \cdot (\mathbf{x}_1, \mathbf{x}_2)$. The corresponding D_U for $U = \langle \Gamma \cdot (\mathbf{x}_1, \mathbf{x}_2) \rangle$ has the image $C_{(\mathbf{x}_1, \mathbf{x}_2)}$ in $\Gamma \backslash \mathbb{H}_3$ under the natural projection $\mathbb{H}_3 \to \Gamma \backslash \mathbb{H}_3$. For a $\gamma \in \Gamma$ we have $U_{\Gamma \cdot (\mathbf{x}_1, \mathbf{x}_2)} \perp \gamma \cdot u_{\kappa_i}$. By Proposition 4.2.1, we know that $\gamma \cdot u_{\kappa_i} = u_{\gamma \cdot \kappa_i}$. It follows that $\gamma \cdot (\mathbf{x}_1, \mathbf{x}_2)$ lies in $\Omega_{\beta, \gamma \cdot \kappa_i, +}$. Thus, modulo the Γ-action, we have a well-defined map:

$$\iota : \Gamma \backslash \Omega_\beta \longrightarrow \coprod_{\kappa_i \in \Gamma \backslash \mathbb{P}^1(F)} \Gamma_{\kappa_i} \backslash \Omega_{\beta,\kappa_i,+}.$$

If two pairs $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2)$ are not Γ-equivalent then they are not Γ_{κ_i}-equivalent since $\Gamma_{\kappa_i} \subset \Gamma$. Hence this map is injective.

We will show that the inverse map ι^{-1} is injective in the following. For $\mathbf{x} = \left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right)$, we calculate its orthogonal complement in $\mathbb{H}_3$ using the isomorphism (4.4),

$$\mathbf{x}^{\perp} \cap \mathbb{H}_3 = \{z + rj \in \mathbb{H}_3 : d(|z|^2 + r^2) - b\bar{z} - \bar{b}z + a = 0\} = \psi(\mathbf{x})$$

where ψ is defined as in Definition 4.1.3. Observe that $\mathbf{x}_1^{\perp} \cap \mathbf{x}_2^{\perp} \cap \mathbb{H}_3 = \psi(\mathbf{x}_1) \cap \psi(\mathbf{x}_2)$ of which one boundary point on the extended complex plane is κ_i. Suppose that two pairs $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2)$ are not Γ_{κ_i}-equivalent in $\Gamma_{\kappa_i} \backslash \Omega_{\beta, \kappa_i, +}$. Note that $\psi(\mathbf{x}_1) \cap \psi(\mathbf{x}_2)$ and $\psi(\mathbf{y}_1) \cap \psi(\mathbf{y}_2)$ have a boundary point in common, the cusp κ_i. Assume that there exists an element $\gamma \in \Gamma$ such that $\gamma \cdot (\mathbf{x}_1, \mathbf{x}_2) = (\mathbf{y}_1, \mathbf{y}_2)$. Then, by Proposition 4.1.5, we have $\gamma \cdot \psi(\mathbf{x}_1) = \psi(\mathbf{y}_1)$ and $\gamma \cdot \psi(\mathbf{x}_2) = \psi(\mathbf{y}_2)$. It is easy to observe that

$$\gamma \cdot (\psi(\mathbf{x}_1) \cap \psi(\mathbf{x}_2)) = \gamma \cdot \psi(\mathbf{x}_1) \cap \gamma \cdot \psi(\mathbf{x}_2) = \psi(\mathbf{y}_1) \cap \psi(\mathbf{y}_2).$$

It follows that γ must be in Γ_{κ_i}, which is a contradiction to that $(\mathbf{x}_1, \mathbf{x}_2)$ and $(\mathbf{y}_1, \mathbf{y}_2)$ are not Γ_{κ_i}-equivalent. So such a γ does not exist. We have proven the injectivity of ι^{-1}. □

Set $\det(\beta) \in -d(\mathbb{Q}^{\times})^2$. It is easy to observe that, for the cusp ∞, we have

$$\Omega_{\beta,\infty} = \left\{ \left(\begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ \bar{b}_2 & 0 \end{pmatrix} \right) \in \Omega_{\beta} : a_1, a_2 \in \mathbb{Q}, b_1, b_2 \in F \right\}.$$

For $(\mathbf{x}_1, \mathbf{x}_2) = \left(\left(\begin{smallmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ \bar{b}_2 & 0 \end{smallmatrix}\right) \right) \in \Omega_{\beta,\infty}$, we have

$$\beta = \frac{1}{2} \begin{pmatrix} (\mathbf{x}_1, \mathbf{x}_1) & (\mathbf{x}_1, \mathbf{x}_2) \\ (\mathbf{x}_1, \mathbf{x}_2) & (\mathbf{x}_2, \mathbf{x}_2) \end{pmatrix} = \begin{pmatrix} b_1\bar{b}_1 & \frac{1}{2}(b_1\bar{b}_2 + \bar{b}_1 b_2) \\ \frac{1}{2}(b_1\bar{b}_2 + \bar{b}_1 b_2) & b_2\bar{b}_2 \end{pmatrix}.$$

of which the determinant is

$$\det(\beta) = \operatorname{disc}(U(\mathbf{x}_1, \mathbf{x}_2)) = -\frac{1}{4}(b_1\bar{b}_2 - \bar{b}_1 b_2)^2.$$

We are not interested in the case when $b_1\bar{b}_2 \in \mathbb{Q}$ since then $\det(\beta) = 0$.

Let $U = U(\mathbf{x}_1, \mathbf{x}_2)$ for $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_{\beta,\infty}$. We will calculate its corresponding special cycle D_U in the following. Given a point $z + rj \in \mathbb{H}_3$ identified with $\frac{1}{r}\left(\begin{smallmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{smallmatrix}\right)$, we compute

$$\frac{1}{r}\begin{pmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{pmatrix}\begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{pmatrix}^* = \frac{1}{r}\begin{pmatrix} |z|^2+r^2 & z \\ \bar{z} & 1 \end{pmatrix}\begin{pmatrix} 0 & -b_1 \\ -\bar{b}_1 & a_1 \end{pmatrix} = \frac{1}{r}\begin{pmatrix} -\bar{b}_1 z & * \\ * & -b_1\bar{z}+a_1 \end{pmatrix}.$$

Thus we have

$$\mathbf{x}_1^{\perp} = \{z + rj \in \mathbb{H}_3 : a_1 - b_1\bar{z} - \bar{b}_1 z = 0\},$$

and similarly,

$$\mathbf{x}_2^{\perp} = \{z + rj \in \mathbb{H}_3 : a_2 - b_2\bar{z} - \bar{b}_2 z = 0\}.$$

Then, solving above equations, we can deduce that the special cycle D_U consists of the infinite geodesic line joining two cusps ∞ and

$$z_U = \frac{a_2 b_1 - a_1 b_2}{b_1\bar{b}_2 - \bar{b}_1 b_2}. \tag{4.8}$$

Lemma 4.2.5. *Suppose that $\mathcal{O} = \mathbb{Z}[\omega]$ with ω is either $\sqrt{d}$ or $\frac{1+\sqrt{d}}{2}$ and denote the stabilizer of the cusp ∞ by $\Gamma_\infty = \{\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right) : \alpha \in \mathcal{O}\}$. Denote*

$$L_{\infty,\dagger} = \left\{ \left(\begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ \bar{b}_2 & 0 \end{pmatrix} \right) : a_1, a_2 \in \mathbb{Z}, b_1, b_2 \in \mathcal{O}, \text{ the condition } \dagger \text{ holds} \right\}$$

where the condition † is given by

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix} \tag{4.9}$$

with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, $\alpha\delta - \beta\gamma = \pm 1$ and $m \in F$. Then the cusp $z_{U(\mathbf{x}_1,\mathbf{x}_2)}$ associated to the pair $(\mathbf{x}_1, \mathbf{x}_2)$ in $\Gamma_\infty \backslash L_{\infty,\dagger}$ runs through all the representatives in $(\overline{m}\sqrt{d_F}|)^{-1}\mathcal{O}/\mathcal{O}$.

Proof. Write $U = U(\mathbf{x}_1, \mathbf{x}_2)$. The Γ_∞-action on $(\mathbf{x}_1, \mathbf{x}_2) \in L_{\infty,\dagger}$ is given explicitly by

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \cdot (\mathbf{x}_1, \mathbf{x}_2) = \left(\begin{pmatrix} a_1 + \alpha\bar{b}_1 + \bar{\alpha} b_1 & b_1 \\ \bar{b}_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 + \alpha\bar{b}_2 + \bar{\alpha} b_2 & b_2 \\ \bar{b}_2 & 0 \end{pmatrix} \right).$$

Under the Γ_∞-action, the cusp z_U becomes z'_U; that is

$$\begin{aligned} z'_U =& \frac{(a_2 + \alpha\bar{b}_2 + \bar{\alpha}b_2)b_1 - (a_1 + \alpha\bar{b}_1 + \bar{\alpha}b_1)b_2}{b_1\bar{b}_2 - \bar{b}_1 b_2} \\ =& \frac{a_2 b_1 + \alpha\bar{b}_2 b_1 - a_1 b_2 - \alpha\bar{b}_1 b_2}{b_1\bar{b}_2 - \bar{b}_1 b_2} = z_U + \alpha. \end{aligned}$$

By our assumption, the cusp z_U can be rewritten as

$$\begin{aligned} z_U &= \frac{m a_2(\alpha + \beta\omega) - m a_1(\gamma + \delta\omega)}{-m\overline{m}(\alpha + \beta\omega)(\gamma + \delta\bar{\omega}) + m\overline{m}(\alpha + \beta\bar{\omega})(\gamma + \delta\omega)} \\ &= \frac{a_2(\alpha + \beta\omega) - a_1(\gamma + \delta\omega)}{-\overline{m}(\alpha\delta - \beta\gamma)(\bar{\omega} - \omega)} = \frac{a_2(\alpha + \beta\omega) - a_1(\gamma + \delta\omega)}{\overline{m}\sqrt{d_F}} \end{aligned}$$

of which the numerator ranges over the whole $\mathcal{O}$.

Thus, modulo the Γ_∞-action, the corresponding cusp z_U runs through all the representatives in $(\overline{m}\sqrt{d_F})^{-1}\mathcal{O}/\mathcal{O}$. □

Remark 4.2.6. Let m be a square-free product of split or inert primes.

(1) Let $d \equiv 1 \bmod 4$ and then $d_F = d$. The above z_U ranges over $(m\sqrt{d})^{-1}\mathcal{O}$. Writing $\mathfrak{f} = (m\sqrt{d})\mathcal{O}$, we have
$$z_U \mathfrak{f} = \left(\frac{a_2b_1 - a_1b_2}{b_1\bar{b}_2 - \bar{b}_1 b_2}\right)\mathfrak{f} = \left(\frac{a_2b_1 - a_1b_2}{m}\right)\mathcal{O}.$$

(2) Let $d \equiv 2, 3 \bmod 4$ and then $d_F = 4d$. Note that in this case prime 2 is ramified in $F = \mathbb{Q}(\sqrt{d})$. Rewrite (4.9) above as
$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{2}m \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$
with $\alpha, \beta, \gamma, \delta \in \mathbb{Z}$, $\alpha\delta - \beta\gamma = \pm 1$. Then the above z_U ranges over $(m\sqrt{d})^{-1}\mathcal{O}$. Writing $\mathfrak{f} = (m\sqrt{d})\mathcal{O}$, we have
$$z_U \mathfrak{f} = \left(\frac{a_2b_1 - a_1b_2}{b_1\bar{b}_2 - \bar{b}_1 b_2}\right)\mathfrak{f} = \left(\frac{2(a_2b_1 - a_1b_2)}{m}\right)\mathcal{O}.$$

In Section 4.3 we will define the Schwartz function evaluated at $\frac{a_2b_1 - a_1b_2}{m}$ or $\frac{2(a_2b_1 - a_1b_2)}{m}$ as above depending on d.

Let $U = \langle \mathbf{x}_1, \mathbf{x}_2 \rangle = \left\langle \begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ \bar{b}_2 & 0 \end{pmatrix} \right\rangle$ where $a_1, a_2 \in \mathbb{Q}$ and $b_1, b_2 \in F^\times$. We have seen that D_U consists of the infinite geodesic line joining the cusps ∞ and z_U as in (4.8). Choose a point $Z = z_U + rj$ on D_U and then the orientation of $T_Z(D_U)$ depends on the sign of $\mathrm{Im}(b_1\bar{b}_2)$ (assuming $\mathrm{Im}(b_1\bar{b}_2) \neq 0$) by the following lemma.

Lemma 4.2.7 (Analogue of Lemma 3.1.6)**.** *Let U, D_U, Z be as above. Then the sign of $Im(b_1\bar{b}_2)$ (assuming $Im(b_1\bar{b}_2) \neq 0$) determines the orientation of $T_Z(D_U)$.*

Proof. Let $Z = z_U + rj$ be a point on D_U which can be identified with $\frac{1}{r}\begin{pmatrix} z_U\bar{z}_U + r^2 & z_U \\ \bar{z}_U & 1 \end{pmatrix}$. Suppose that $\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \delta \end{pmatrix} \in Z^\perp$ and we compute, recalling $*$ action in (4.2)
$$\begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \delta \end{pmatrix} \begin{pmatrix} z_U\bar{z}_U + r^2 & z_U \\ \bar{z}_U & 1 \end{pmatrix}^* = \begin{pmatrix} \alpha - \beta\bar{z}_U & * \\ * & -\bar{\beta}z_U + \delta(z_U\bar{z}_U + r^2) \end{pmatrix}.$$
It follows that
$$Z^\perp = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & \delta \end{pmatrix} : \alpha - \beta\bar{z}_U - \bar{\beta}z_U + \delta(z_U\bar{z}_U + r^2) = 0 \right\}.$$

We describe the subspace $Z^\perp \cap U$ as
$$Z^\perp \cap U = \left\{ \begin{pmatrix} \alpha & \beta \\ \bar{\beta} & 0 \end{pmatrix} : \alpha - \beta\bar{z}_U - \bar{\beta}z_U = 0, \alpha \in \mathbb{R}, \beta \in \mathbb{C} \right\} = \left\{ \begin{pmatrix} \beta\bar{z}_U + \bar{\beta}z_U & \beta \\ \bar{\beta} & 0 \end{pmatrix} \right\}$$
where $\beta = \mathrm{Span}\{b_1, b_2\}$. Set $\beta = \beta_1 + \beta_2 i$, $z_U = z_{U,1} + z_{U,2}i$ and $b_j = b_{j1} + b_{j2}i$ ($j = 1, 2$). Suppose that $\beta = ub_1 + vb_2$ for $u, v \in \mathbb{Q}$ and then we observe that

$\beta_1 = ub_{11} + vb_{21}$ and $\beta_2 = ub_{12} + vb_{22}$. Consider

$$\begin{aligned}
&\begin{pmatrix} \beta\bar{z}_U+\bar{\beta}z_U & \beta \\ \bar{\beta} & 0 \end{pmatrix}\\
=&\begin{pmatrix} 2(\beta_1 z_{U,1}+\beta_2 z_{U,2}) & \beta_1+\beta_2 i \\ \beta_1-\beta_2 i & 0 \end{pmatrix}\\
=&\beta_1 \begin{pmatrix} 2z_{U,1} & 1 \\ 1 & 0 \end{pmatrix} + \beta_2 \begin{pmatrix} 2z_{U,2} & i \\ -i & 0 \end{pmatrix}\\
=&(ub_{11}+vb_{21}) \begin{pmatrix} 2z_{U,1} & 1 \\ 1 & 0 \end{pmatrix} + (ub_{12}+vb_{22}) \begin{pmatrix} 2z_{U,2} & i \\ -i & 0 \end{pmatrix}\\
=&u\left(b_{11} \begin{pmatrix} 2z_{U,1} & 1 \\ 1 & 0 \end{pmatrix} + b_{12} \begin{pmatrix} 2z_{U,2} & i \\ -i & 0 \end{pmatrix}\right) + v\left(b_{21} \begin{pmatrix} 2z_{U,1} & 1 \\ 1 & 0 \end{pmatrix} + b_{22} \begin{pmatrix} 2z_{U,2} & i \\ -i & 0 \end{pmatrix}\right)\\
=&u\left(\frac{1}{2}(b_1+\bar{b}_1) \begin{pmatrix} 2z_{U,1} & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2}i(b_1-\bar{b}_1) \begin{pmatrix} 2z_{U,2} & i \\ -i & 0 \end{pmatrix}\right)\\
&+v\left(\frac{1}{2}(b_2+\bar{b}_2) \begin{pmatrix} 2z_{U,1} & 1 \\ 1 & 0 \end{pmatrix} - \frac{1}{2}i(b_2-\bar{b}_2) \begin{pmatrix} 2z_{U,2} & i \\ -i & 0 \end{pmatrix}\right)\\
=&u\left(\frac{1}{4}(b_1+\bar{b}_1)(z_U+\bar{z}_U)(e_1+e_4) + \frac{1}{2}(b_1+\bar{b}_1)e_2\right.\\
&\left.-\frac{1}{4}(b_1-\bar{b}_1)(z_U-\bar{z}_U)(e_1+e_4) - \frac{1}{2}i(b_1-\bar{b}_1)e_3\right)\\
&+v\left(\frac{1}{4}(b_2+\bar{b}_2)(z_U+\bar{z}_U)(e_1+e_4) + \frac{1}{2}(b_2+\bar{b}_2)e_2\right.\\
&\left.-\frac{1}{4}(b_2-\bar{b}_2)(z_U-\bar{z}_U)(e_1+e_4) - \frac{1}{2}i(b_2-\bar{b}_2)e_3\right)\\
=&u\left(\frac{1}{2}(b_1\bar{z}_U+\bar{b}_1 z_U)(e_1+e_4) + \frac{1}{2}(b_1+\bar{b}_1)e_2 - \frac{1}{2}i(b_1-\bar{b}_1)e_3\right)\\
&+v\left(\frac{1}{2}(b_2\bar{z}_U+\bar{b}_2 z_U)(e_1+e_4) + \frac{1}{2}(b_2+\bar{b}_2)e_2 - \frac{1}{2}i(b_2-\bar{b}_2)e_3\right).
\end{aligned}$$

Observe that

$$\begin{aligned}
b_1\bar{z}_U+\bar{b}_1 z_U =&b_1\frac{a_2\bar{b}_1-a_1\bar{b}_2}{\bar{b}_1 b_2-b_1\bar{b}_2} + \bar{b}_1\frac{a_2 b_1-a_1 b_2}{b_1\bar{b}_2-\bar{b}_1 b_2}\\
=&\frac{-a_2 b_1\bar{b}_1+a_1 b_1\bar{b}_2+a_2 b_1\bar{b}_1-a_1\bar{b}_1 b_2}{b_1\bar{b}_2-\bar{b}_1 b_2} = a_1
\end{aligned}$$

and similarly that $b_2\bar{z}_U+\bar{b}_2 z_U = a_2$. Then we deduce that

$$\begin{aligned}
Z^\perp\cap U = \Big\langle &\frac{1}{2}a_1(e_1+e_4) + \frac{1}{2}(b_1+\bar{b}_1)e_2 - \frac{1}{2}i(b_1-\bar{b}_1)e_3,\\
&\frac{1}{2}a_2(e_1+e_4) + \frac{1}{2}(b_2+\bar{b}_2)e_2 - \frac{1}{2}i(b_2-\bar{b}_2)e_3\Big\rangle
\end{aligned}$$

which coincides with $U = \left\langle \begin{pmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ \bar{b}_2 & 0 \end{pmatrix} \right\rangle$.

To describe the subspace $U^\perp$, we consider $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in U^\perp$ which satisfies that, due to the orthogonality,

$$\begin{cases} \delta a_1 = \beta \bar{b}_1 + \bar{\beta} b_1 \\ \delta a_2 = \beta \bar{b}_2 + \bar{\beta} b_2 \end{cases}.$$

Then we have by solving equations

$$\beta = \frac{(a_2 b_1 - a_1 b_2)\delta}{b_1 \bar{b}_2 - \bar{b}_1 b_2} = z_U \delta \quad \text{and} \quad \bar{\beta} = \frac{(a_2 \bar{b}_1 - a_1 \bar{b}_2)\delta}{\bar{b}_1 b_2 - b_1 \bar{b}_2} = \bar{z}_U \delta.$$

It follows that

$$\begin{aligned} Z^{\perp} \cap U^{\perp} &= \left\{ \begin{pmatrix} (z_U \bar{z}_U - r^2)\delta & z_U \delta \\ \bar{z}_U \delta & \delta \end{pmatrix} : \delta \in \mathbb{Q} \right\} = \left\langle \varepsilon \begin{pmatrix} z_U \bar{z}_U - r^2 & z_U \\ \bar{z}_U & 1 \end{pmatrix} \right\rangle \\ &= \left\langle \frac{\varepsilon}{2}(z_U \bar{z}_U - r^2 - 1)e_1 + \frac{1}{2}(z_U + \bar{z}_U)e_2 - \frac{1}{2}i(z_U - \bar{z}_U) + e_3 \frac{\varepsilon}{2}(z_U \bar{z}_U - r^2 + 1)e_4 \right\rangle \end{aligned}$$

where $\varepsilon = \pm 1$ describes the orientation of $T_Z(D_U) \simeq Z^{\perp} \cap U^{\perp}$. If the cycle D_U is directed from z_U to ∞ then we take $\varepsilon = -1$. For a different direction of D_U we take $\varepsilon = 1$.

For the point Z on D_U we consider its corresponding vector in V:

$$\begin{aligned} &\frac{1}{r} \begin{pmatrix} z_U \bar{z}_U + r^2 & z_U \\ \bar{z}_U & 1 \end{pmatrix} \\ =&\frac{1}{r} \left(\frac{1}{2}(z_U \bar{z}_U + r^2)(e_1 + e_4) + \frac{1}{2}(z_U + \bar{z}_U)e_2 - \frac{1}{2}i(z_U - \bar{z}_U)e_3 - \frac{1}{2}(e_1 - e_4) \right) \\ =&\frac{1}{r} \left(\frac{1}{2}(z_U \bar{z}_U + r^2 - 1)e_1 + \frac{1}{2}(z_U + \bar{z}_U)e_2 - \frac{1}{2}i(z_U - \bar{z}_U)e_3 + \frac{1}{2}(z_U \bar{z}_U + r^2 + 1)e_4 \right). \end{aligned}$$

Then we consider the base change, which describes the orientation related to introducing $\Omega_{\beta,\kappa_i,+}$, given by

$$\begin{pmatrix} z^{\perp} \cap U^{\perp} \\ z^{\perp} \cap U \\ z \end{pmatrix} = M \begin{pmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \end{pmatrix}$$

where

$$M = \begin{pmatrix} \frac{\varepsilon}{2}(z_U \bar{z}_U - r^2 - 1) & \frac{1}{2}(z_U + \bar{z}_U) & -\frac{1}{2}i(z_U - \bar{z}_U) & \frac{\varepsilon}{2}(z_U \bar{z}_U - r^2 + 1) \\ \frac{1}{2}a_1 & \frac{1}{2}(b_1 + \bar{b}_1) & -\frac{1}{2}i(b_1 - \bar{b}_1) & \frac{1}{2}a_1 \\ \frac{1}{2}a_2 & \frac{1}{2}(b_2 + \bar{b}_2) & -\frac{1}{2}i(b_2 - \bar{b}_2) & \frac{1}{2}a_2 \\ \frac{1}{2r}(z_U \bar{z}_U + r^2 - 1) & \frac{1}{2r}(z_U + \bar{z}_U) & -\frac{1}{2r}i(z_U - \bar{z}_U) & \frac{1}{2r}(z_U \bar{z}_U + r^2 + 1) \end{pmatrix}.$$

We calculate its determinant in the following:

$$\begin{aligned}
\det M =& \frac{\varepsilon}{r}\begin{vmatrix} -r^2 & 0 & 0 & -r^2 \\ \frac{1}{2}a_1 & \frac{1}{2}(b_1+\bar{b}_1) & -\frac{1}{2}i(b_1-\bar{b}_1) & \frac{1}{2}a_1 \\ \frac{1}{2}a_2 & \frac{1}{2}(b_2+\bar{b}_2) & -\frac{1}{2}i(b_2-\bar{b}_2) & \frac{1}{2}a_2 \\ \frac{1}{2}(z_U\bar{z}_U+r^2-1) & \frac{1}{2}(z_U+\bar{z}_U) & -\frac{1}{2}i(z_U-\bar{z}_U) & \frac{1}{2}(z_U\bar{z}_U+r^2+1) \end{vmatrix} \\
=& -\varepsilon r \begin{vmatrix} 1 & 0 & 0 & 0 \\ \frac{1}{2}a_1 & \frac{1}{2}(b_1+\bar{b}_1) & -\frac{1}{2}i(b_1-\bar{b}_1) & 0 \\ \frac{1}{2}a_2 & \frac{1}{2}(b_2+\bar{b}_2) & -\frac{1}{2}i(b_2-\bar{b}_2) & 0 \\ \frac{1}{2}(z_U\bar{z}_U+r^2-1) & \frac{1}{2}(z_U+\bar{z}_U) & -\frac{1}{2}i(z_U-\bar{z}_U) & 1 \end{vmatrix} \\
=& -\varepsilon r \begin{vmatrix} \frac{1}{2}(b_1+\bar{b}_1) & -\frac{1}{2}i(b_1-\bar{b}_1) \\ \frac{1}{2}(b_2+\bar{b}_2) & -\frac{1}{2}i(b_2-\bar{b}_2) \end{vmatrix} \\
=& -\varepsilon r\frac{1}{2}i(b_1\bar{b}_2-\bar{b}_1 b_2) = \frac{1}{2}\varepsilon r \operatorname{Im}(b_1\bar{b}_2) > 0
\end{aligned}$$

which implies that the sign of $\operatorname{Im}(b_1\bar{b}_2)$ determines the orientation ε of $T_Z(D_U)$. □

§ 4.3 Schwartz function

Let $F=\mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field and denote by $\mathcal{O}$ its ring of integers. Choose $m\in\mathbb{Z}$ as a square-free product of inert or split primes and put $\mathfrak{m}=m\mathcal{O}$. Let $\chi_{\mathfrak{m}}$ be a quadratic Hecke character of conductor $\mathfrak{f}=\sqrt{d}\mathfrak{m}$. Denote by $\widetilde{\chi}_{\mathfrak{m}}$ the induced idelic one as in Section 1.3 and by $\widetilde{\chi}_{\mathfrak{m},v}$ its local component. In this section we will define a finite Schwartz function related to this character $\chi_{\mathfrak{m}}$. At the archimedean place we take the Schwartz form $\varphi_2\in S(V(\mathbb{R})^2)\otimes\Omega^2(D)$ as in Example 2.2.2 (2).

We first describe how to localise the quadratic space in the following proposition. In Section 3.2 we have chosen the rational quadratic space of dimension 4 such that $(V(\mathbb{Q}),q)\simeq(\mathcal{H}(F),-\det)$. Thus, to extend it to a 4-dimensional quadratic space over p-adic numbers $\mathbb{Q}_p$, we can consider $\mathcal{H}(F)\otimes\mathbb{Q}_p$. Following [Rob01, p.273], there are two four dimensional quadratic spaces over $\mathbb{Q}_p$ with discriminant $\mathfrak{d}\in\mathbb{Q}_p^\times/(\mathbb{Q}_p^\times)^2$ up to isometry. If $\mathfrak{d}=1$, it is isometric to $M_{2\times 2}(\mathbb{Q}_p)$ equipped with the determinant; if $\mathfrak{d}\neq 1$, it is isometric to

$$V_1(\mathbb{Q}_p)=\left\{\begin{pmatrix} e & f\sqrt{\mathfrak{d}} \\ g\sqrt{\mathfrak{d}} & \bar{e}\end{pmatrix} : f,g\in\mathbb{Q}_p, e\in\mathbb{Q}_p(\sqrt{\mathfrak{d}})\right\}\subset M_2\left(\mathbb{Q}_p(\sqrt{\mathfrak{d}})\right)$$

equipped with the determinant.

Proposition 4.3.1. *For a prime p, the four dimensional quadratic space over $\mathbb{Q}_p$ is* *isometric to either* $(V_1(\mathbb{Q}_p), \det)$ *when* p *is inert or ramified in* $F/\mathbb{Q}$, *or* $(M_2(\mathbb{Q}_p), \det)$ *when* p *splits in* $F/\mathbb{Q}$.

Proof. Given a diagonal quadratic form $Q = \sum_{i=1}^{4} a_i x_i^2$ with $a_i \in \mathbb{Q}_p^\times$, we define the Hasse invariant as $c_p(Q) = c(Q) = \prod_{i<j}(a_i, a_j)_p = \pm 1$ where $(\ ,\ )$ denotes the Hilbert symbol. The non-degenerate quadratic spaces over $\mathbb{Q}_p$ $(p < \infty)$ are in 1-1 correspondence with the triples $(4, \mathfrak{d}, c)$, where $\mathfrak{d}$ is the discriminant and c is the Hasse invariant, see [Cas78, Theorem 1.1, Chapter 4].

Let p be inert in $F/\mathbb{Q}$ which implies that $\sqrt{d} \notin \mathbb{Q}_p$ and that $F \otimes \mathbb{Q}_p = \mathbb{Q}_p(\sqrt{d})$. Then we have

$$\mathcal{H}(F) \otimes \mathbb{Q}_p = \left\{ \begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} : a, d \in \mathbb{Q}_p, b \in \mathbb{Q}_p(\sqrt{d}) \right\} = \mathcal{H}(F \otimes \mathbb{Q}_p),$$

where $^-$ denotes the non-trivial action in $\mathrm{Gal}(\mathbb{Q}_p(\sqrt{d})/\mathbb{Q}_p)$. Equipping $\mathcal{H}(F) \otimes \mathbb{Q}_p$ with the quadratic form being $-\det$ and choosing an orthogonal basis $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $e_4 = \left(\begin{smallmatrix} 0 & \sqrt{d} \\ -\sqrt{d} & 0 \end{smallmatrix}\right)$, we have an associated diagonal form $Q = -x_1^2 + x_2^2 + x_3^2 - dx_4^2$. It follows that $\mathfrak{d} = d$ and $c = (-1, -d)_p = 1$ since $p \nmid d$. Similarly, for $V_1(\mathbb{Q}_p)$ with the discriminant $\mathfrak{d} = d$, choosing an orthogonal basis $e_1' = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e_2' = \left(\begin{smallmatrix} \sqrt{d} & 0 \\ 0 & -\sqrt{d} \end{smallmatrix}\right)$, $e_3' = \left(\begin{smallmatrix} 0 & \sqrt{d} \\ \sqrt{d} & 0 \end{smallmatrix}\right)$ and $e_4' = \left(\begin{smallmatrix} 0 & \sqrt{d} \\ -\sqrt{d} & 0 \end{smallmatrix}\right)$ in $V_1(\mathbb{Q}_p)$ above, we have a diagonal form $Q' = x_1^2 - dx_2^2 - dx_3^2 + dx_4^2$. Then $\mathfrak{d}' = d^3$ and $c' = (1, -d)_p^2 (1, d)_p (-d, -d)_p (-d, d)_p^2 = 1$ since $p \nmid d$. Thus, we can deduce that $(\mathcal{H}(F) \otimes \mathbb{Q}_p, -\det) \simeq (V_1(\mathbb{Q}_p), \det)$ if p is inert in $F/\mathbb{Q}$.

Let p split in $F/\mathbb{Q}$ such that $(p) = \mathfrak{p}\bar{\mathfrak{p}}$. Then d has a square root α in the ring $\mathbb{Z}_p$ of p-adic integers by Hensel's lemma. It is known that $F \otimes \mathbb{Q}_p = F_{\mathfrak{p}} \times F_{\bar{\mathfrak{p}}}$ where $F_{\mathfrak{p}}, F_{\bar{\mathfrak{p}}}$ are both isomorphic to $\mathbb{Q}_p$. Consider the map $\mathcal{H}(F) \otimes \mathbb{Q}_p \to M_2(\mathbb{Q}_p)$ via $\left(\begin{smallmatrix} a & b \\ \bar{b} & d \end{smallmatrix}\right) \otimes x \mapsto \left(\begin{smallmatrix} a_p x & b_{\mathfrak{p}} x \\ b_{\bar{\mathfrak{p}}} x & d_p x \end{smallmatrix}\right)$ where the subscripts $p, \mathfrak{p}, \bar{\mathfrak{p}}$ denote images under $\mathbb{Q} \hookrightarrow \mathbb{Q}_p$, $F \hookrightarrow F_{\mathfrak{p}}$ and $F \hookrightarrow F_{\bar{\mathfrak{p}}}$ respectively. Note that $\bar{b}_{\mathfrak{p}}, b_{\bar{\mathfrak{p}}}$ have the same image in $\mathbb{Q}_p$. It is not hard to see the map $\mathcal{H}(F) \otimes \mathbb{Q}_p \to M_2(\mathbb{Q}_p)$ is surjective: for any element $\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \cdot \lambda$ with $\lambda \in \mathbb{Q}_p$ we can find its preimage $\left(\begin{smallmatrix} 0 & \sqrt{d} \\ -\sqrt{d} & 0 \end{smallmatrix}\right) \otimes \alpha^{-1}\lambda$ in $\mathcal{H}(F) \otimes \mathbb{Q}_p$; for $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \cdot \lambda$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \cdot \lambda$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \cdot \lambda$, we can find their preimages $\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right) \otimes \lambda$, $\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right) \otimes \lambda$ and $\left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right) \otimes \lambda$ respectively in $\mathcal{H}(F) \otimes \mathbb{Q}_p$. Then $\mathcal{H}(F) \otimes \mathbb{Q}_p \simeq M_2(\mathbb{Q}_p)$ as they are both 4-dimensional over $\mathbb{Q}_p$. In fact $M_2(\mathbb{Q}_p)$ equipped with the determinant is one isometric class of four dimensional quadratic spaces of discriminant 1 [Rob01, p.273]. Again, equipping $\mathcal{H}(F) \otimes \mathbb{Q}_p$ with minus determinant and choosing an orthogonal basis $e_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $e_4 = \left(\begin{smallmatrix} 0 & \sqrt{d} \\ -\sqrt{d} & 0 \end{smallmatrix}\right)$, we have an associated diagonal form $Q = -x_1^2 + x_2^2 + x_3^2 - dx_4^2$. It follows that $\mathfrak{d} = d$ (square in $\mathbb{Z}_p$) and

$c = (-1, -d)_p = 1$. Choosing an orthogonal basis $e'_1 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$, $e'_2 = \left(\begin{smallmatrix} 1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$, $e'_3 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix}\right)$ and $e'_4 = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$ in $M_2(\mathbb{Q}_p)$ we have $\mathfrak{d}' = 1$ and $c' = 1$. Thus we can deduce that $(\mathcal{H}(F) \otimes \mathbb{Q}_p, -\det) \simeq (M_2(\mathbb{Q}_p), \det)$ if p splits in $F/\mathbb{Q}$.

Suppose that p is ramified in $F/\mathbb{Q}_p$ and then we have $F \otimes \mathbb{Q}_p = \mathbb{Q}_p(\sqrt{d})$. As in the inert case, $\mathcal{H}(F) \otimes \mathbb{Q}_p = \mathcal{H}(F \otimes \mathbb{Q}_p)$. Corresponding to $(\mathcal{H}(F) \otimes \mathbb{Q}_p, -\det)$ the Hasse invariant $c = (-1, -d)_p = (-1)^{\frac{p-1}{2}}$. For $(V_1(\mathbb{Q}_p), \det)$ we calculate

$$c' = (1, d)_p(-d, -d)_p = (1, -d)_p(1, -1)_p(1, -d)_p(-d, -d)_p = (-d, -d)_p = (-1)^{\frac{p-1}{2}}$$

where the last equality holds as d is square-free and divisible by q. Thus, if p is ramified we have $(\mathcal{H}(F \otimes \mathbb{Q}_p), -\det) \simeq (V_1(\mathbb{Q}_p), \det)$. □

In this chapter we need to consider the pair $\mathrm{Sp}_4 \times \mathrm{SO}(3,1)$ to construct the theta liftings of weight 2 Bianchi modular forms. We have seen its Weil representation in Example 2.1.1 (2). In the following subsections 4.3.1, 4.3.2 and 4.3.3, we define local Schwartz functions at split primes dividing m, inert primes dividing m and ramified primes away from 2 respectively. In Section 4.4 we will construct the theta lifting of a weight 2 Bianchi modular form of $\Gamma_0(\mathfrak{n})$ with square-free $\mathfrak{n}$ coprime to $(m|d_F|)$. To avoid the vanishing of our theta lifting as discussed in Remark 3.3.6 (i), in subsection 4.3.4 we define the local Schwartz function at each place dividing $N(\mathfrak{n})$ (norm of $\mathfrak{n}$) and ramified prime 2, to be different to the characteristic function of integral lattice . In subsection 4.3.5 we consider all other finite places.

4.3.1 At split prime dividing m

Let $q|m$ be a split prime such that $(q) = \mathfrak{q}\bar{\mathfrak{q}}$. According to Proposition 4.3.1, there is an isomorphism $(\mathcal{H}(F) \otimes \mathbb{Q}_q, -\det) \simeq (V_1(\mathbb{Q}_q), \det)$ for q split in $F/\mathbb{Q}$ given by

$$\begin{pmatrix} a & b \\ \bar{b} & d \end{pmatrix} \longmapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

where c is the image of $\bar{b} \in F$ under $F \hookrightarrow F_{\mathfrak{q}} \simeq \mathbb{Q}_q$.

Definition 4.3.2. (1) Suppose that $d \equiv 1 \bmod 4$. The local Schwartz function $\varphi_q^{\chi_{\mathfrak{m}}}$ at q is vanishing unless

$$a_i \in \mathbb{Z}_q, b_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, c_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, d_i \in \mathbb{Z}_q, a_2 d_1 - a_1 d_2 \in q\mathbb{Z}_q,$$

in which case

$$\varphi_q^{\chi_{\mathfrak{m}}}\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right)$$

$$= \begin{cases} (\widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}\widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}})\left(\frac{a_2b_1-a_1b_2}{m} + \frac{c_2d_1-c_1d_2}{m}\right), & \text{if } \frac{a_2b_1-a_1b_2}{m} + \frac{c_2d_1-c_1d_2}{m} \in \mathcal{O}_{\mathfrak{q}}^{\times} \times \mathcal{O}_{\bar{\mathfrak{q}}}^{\times}, \\ 0, & \text{if } \frac{a_2b_1-a_1b_2}{m} + \frac{c_2d_1-c_1d_2}{m} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}} \text{ or } \bar{\mathfrak{q}}\mathcal{O}_{\bar{\mathfrak{q}}}. \end{cases}$$

where $(\widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}\widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}})(-) = \widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}(-)\widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}}(-)$. Note that $b_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, c_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ is equivalent to $b_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}} \times \bar{\mathfrak{q}}\mathcal{O}_{\bar{\mathfrak{q}}}$.

(2) Suppose that $d \equiv 2, 3 \bmod 4$. Replace above $\frac{a_2b_1-a_1b_2}{m}$ by $\frac{2(a_2b_1-a_1b_2)}{m}$ and $\frac{c_2d_1-c_1d_2}{m}$ by $\frac{2(c_2d_1-c_1d_2)}{m}$ as discussed in Remark 4.2.6

In the following we will check the invariance properties of this local Schwartz function under some congruence subgroups of Sp_4 and $\mathrm{SO}(3,1)$ in details in case of $d \equiv 1 \bmod 4$ and the other case can be treated similarly. We need to calculate the transformation properties (2.1), (2.2), (2.3) and (2.4). For simplicity we write $\varphi_q = \varphi_q^{\chi_{\mathfrak{m}}}$ and $\chi = \chi_{\mathfrak{m}}$ in the following computation.

Set $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2) = \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix}\right)\right)$. For $a_i \in \mathbb{Z}_q$, $b_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, $c_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, $d_i \in \mathbb{Z}_q$, it is not difficult to observe that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\varphi_q(\mathbf{X}) = \varphi_q(\mathbf{x}_1, \mathbf{x}_2) \quad \text{for } u \in M_2(q\mathbb{Z}_q) \tag{4.10}$$

as $\psi_q\left(\frac{1}{2}\operatorname{tr}(u(\mathbf{X}, \mathbf{X}))\right)$ is trivial for such $(\mathbf{x}_1, \mathbf{x}_2)$ and u.

Set $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2) = \left(\left(\begin{smallmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{smallmatrix}\right), \left(\begin{smallmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{smallmatrix}\right)\right)$. Inspired by Prasanna's computations in the proof of [Pra09, Proposition 3.4], we will calculate the Fourier transform

$$\hat{\varphi}_q(\mathbf{Y}) = \int_{\substack{a_i \in \mathbb{Z}_q, b_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}} \\ c_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, d_i \in \mathbb{Z}_q}} \psi_q(\operatorname{tr}(\mathbf{X}, \mathbf{Y}))\varphi_q(\mathbf{X})d\mathbf{X} \quad \text{for } i = 1, 2,$$

where

$$\operatorname{tr}(\mathbf{X}, \mathbf{Y}) = -(a_1\delta_1 - b_1\gamma_1 - c_1\beta_1 + d_1\alpha_1 + a_2\delta_2 - b_2\gamma_2 - c_2\beta_2 + d_2\alpha_2).$$

Denote by $\lambda_{\sqrt{d}}$ the image of $\sqrt{d}$ in $\mathcal{O}_{\mathfrak{q}}$. By the above definition, φ_q is invariant under the transformations $a_i \mapsto a_i + q$, $b_i \mapsto b_i + q^2$, $b_i \mapsto b_i + q^2\lambda_{\sqrt{d}}$ (or $b_i \mapsto b_i + q^2\frac{1+\lambda_{\sqrt{d}}}{2}$) and $d_i \mapsto d_i + q$. Sending $a_i \mapsto a_i + q$, we will have $\psi_q(-q\delta_i)$, factored out of the above integral, which has to be trivial for the non-vanishing of $\hat{\varphi}_q$. So for $\hat{\varphi}_q(\mathbf{Y})$ non-vanishing we need $\delta_i \in \mathbb{Z}_q$. Sending $b_1 \mapsto b_1 + q^2$, we get $\psi_q(q^2(\gamma_1 + \beta_1))$. For $b_1 \mapsto b_1 + q^2\lambda_{\sqrt{d}}$ and $b_i \mapsto b_i + q^2\frac{1+\lambda_{\sqrt{d}}}{2}$, we get $\psi_q(q^2(\gamma_1 - \beta_1)\lambda_{\sqrt{d}})$ and $\psi_q(\frac{1}{2}q^2(\gamma_1 + \beta_1) + \frac{1}{2}q^2(\gamma_1 - \beta_1)\lambda_{\sqrt{d}})$ respectively. For $\hat{\varphi}_q(\mathbf{Y})$ non-vanishing we need

$\gamma_1-\beta_1,\gamma_1-\beta_1 \in q^{-1}\mathbb{Z}_q$ which implies $\beta_1,\gamma_1 \in q^{-1}\mathbb{Z}_q$. Repeating the same argument we can deduce that for the non-vanishing of $\hat{\varphi}_q$ the following conditions must be satisfied,

$$\alpha_i \in \mathbb{Z}_q, \beta_i \in q^{-1}\mathbb{Z}_q, \gamma_i \in q^{-1}\mathbb{Z}_q, \delta_i \in \mathbb{Z}_q.$$

It follows that $\omega\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)\right)\varphi_q(\mathbf{Y})$ is vanishing unless $\alpha_i \in \mathbb{Z}_q$, $\beta_i \in q^{-1}\mathbb{Z}_q$, $\gamma_i \in q^{-1}\mathbb{Z}_q$ and $\delta_i \in \mathbb{Z}_q$.

Recall from (2.2) that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\hat{\varphi}_q(\mathbf{Y}) = \psi_q\left(\frac{1}{2}\operatorname{tr}(u(\mathbf{Y},\mathbf{Y}))\right)\hat{\varphi}_q(\mathbf{Y}).$$

For $\alpha_i \in \mathbb{Z}_q, \beta_i \in q^{-1}\mathbb{Z}_q, \gamma_i \in q^{-1}\mathbb{Z}_q, \delta_i \in \mathbb{Z}_q$ and $u \in M_2(q^3\mathbb{Z}_q)$, $\psi_q\left(\frac{1}{2}\operatorname{tr}(u(\mathbf{Y},\mathbf{Y}))\right)$ is trivial. Thus we can deduce that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q(\mathbf{Y}) = \omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q(\mathbf{Y}) \quad \text{for } u \in M_2(q^3\mathbb{Z}_q),\ u = u^t$$

which implies

$$\omega\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right)\varphi_q(\mathbf{Y}) = \varphi_q(\mathbf{Y}) \quad \text{for } u \in M_2(q^3\mathbb{Z}_q),\ u = u^t. \tag{4.11}$$

For $a = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_q)$, we compute

$$\mathbf{X}a = (\mathbf{x}_1,\mathbf{x}_2)\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} = \left(\begin{pmatrix} \alpha a_1+\gamma a_2 & \alpha b_1+\gamma b_2 \\ \alpha c_1+\gamma c_2 & \alpha d_1+\gamma d_2 \end{pmatrix}, \begin{pmatrix} \beta a_1+\delta a_2 & \beta b_1+\delta b_2 \\ \beta c_1+\delta c_2 & \beta d_1+\delta d_2 \end{pmatrix}\right)$$

due to which we obtain that

$$\begin{aligned} \frac{a_2'b_1'-a_1'b_2'}{m} &= \frac{(\beta a_1+\delta a_2)(\alpha b_1+\gamma b_2)-(\alpha a_1+\gamma a_2)(\beta b_1+\delta b_2)}{m} \\ &= \frac{(\alpha\delta-\beta\gamma)a_2b_1-(\alpha\delta-\beta\gamma)a_1b_2}{m} = \det(a)\cdot\frac{a_2b_1-a_1b_2}{m} \end{aligned}$$

and similarly that $\frac{c_2'd_1'-c_1'd_2'}{m} = \det(a)\cdot\frac{c_2d_1-c_1d_2}{m}$. Then from (2.3) we see that if $\det(a) \in \mathbb{Z}_q^\times$ then

$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^ta^{-1} \end{pmatrix}\right)\varphi_q(\mathbf{X}) = \chi_{V,q}(\det(a))|\det(a)|_q^2(\widetilde{\chi}_{\mathfrak{q}}\widetilde{\chi}_{\bar{\mathfrak{q}}})(\det(a))\varphi_q(\mathbf{X}). \tag{4.12}$$

Combining (4.10), (4.11) and (4.12), we have proved the following lemma:

Lemma 4.3.3. *We have*

$$\omega(k_1)\varphi_q = \chi_{V,q}(\det(A))|\det(A)|_q^2(\widetilde{\chi}_{\mathfrak{q}}\widetilde{\chi}_{\bar{\mathfrak{q}}})(\det(A))\varphi_q$$

for

$$k_1 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q) : B \in M_2(q\mathbb{Z}_q), C \in M_2(q^3\mathbb{Z}_q)\right\}.$$

Proof. The assertion follows from the Iwahori decomposition of Sp_4. □

We next discuss the action of $\mathrm{SO}(3,1)(V(\mathbb{Q}_q))$ on φ_q characterised by

$$\omega(1,h)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(h^{-1}\mathbf{x}_1,h^{-1}\mathbf{x}_2) \quad \text{for } h\in\mathrm{SO}(3,1)(V(\mathbb{Q}_q)).$$

We have seen the exceptional isomorphism $\mathrm{PSL}_2\simeq\mathrm{SO}^+(3,1)$ in Section 1.2, and in this case of split q as in Example 1.2.3 (2), we want to check congruence subgroups of $\mathrm{PSL}_2(\mathbb{Z}_q\times\mathbb{Z}_q)$ under which the Schwartz function φ_q is invariant. Recall from [Rob01, Section 2] that $h^{-1}\mathbf{x}_i := h_1^{-1}\mathbf{x}_i{}^t(h_2^{-1})^*$ for $h=(h_1,h_2)\in\mathrm{PSL}_2(\mathbb{Z}_q)\times\mathrm{PSL}_2(\mathbb{Z}_q)=\mathrm{PSL}_2(\mathbb{Z}_q\times\mathbb{Z}_q)$.

Lemma 4.3.4. *For $h=(h_1,h_2)\in\mathrm{PSL}_2(\mathbb{Z}_q)\times\mathrm{PSL}_2(\mathbb{Z}_q)$ satisfying*

$$h_i\in\overline{\Gamma}(q)=\left\{\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\in\mathrm{PSL}_2(\mathbb{Z}_q):\pm\begin{pmatrix}\alpha&\beta\\\gamma&\delta\end{pmatrix}\equiv\pm\begin{pmatrix}1&0\\0&1\end{pmatrix}\bmod q\right\},$$

we have that

$$\omega(1,h)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2). \tag{4.13}$$

Proof. Set

$$h_j^{-1}=\begin{pmatrix}\alpha_j&\beta_j\\\gamma_j&\delta_j\end{pmatrix} \text{ and } \mathbf{x}_i=\begin{pmatrix}a_i&b_i\\c_i&d_i\end{pmatrix} \quad \text{for } i,j\in\{1,2\}$$

with $\alpha_j,\delta_j\equiv 1 \bmod q$ and $\beta_j,\gamma_j\equiv 0\bmod q$.

First we assume that $a_i,d_i\in\mathbb{Z}_q$ and $b_i,c_i\in q\mathbb{Z}_q$ so that φ_q is non-vanishing on $(\mathbf{x}_1,\mathbf{x}_2)$.

We compute

$$\begin{aligned}\begin{pmatrix}a_i'&b_i'\\c_i'&d_i'\end{pmatrix} :=& h_1^{-1}\mathbf{x}_i{}^t(h_2^{-1})^*\\ =&\begin{pmatrix}\delta_2(\alpha_1a_i+\beta_1c_i)-\beta_2(\alpha_1b_i+\beta_1d_i) & -\gamma_2(\alpha_1a_i+\beta_1c_i)+\alpha_2(\alpha_1b_i+\beta_1d_i)\\ \delta_2(\gamma_1a_i+\delta_1c_i)-\beta_2(\gamma_1b_i+\delta_1d_i) & -\gamma_2(\gamma_1a_i+\delta_1c_i)+\alpha_2(\gamma_1b_i+\delta_1d_i)\end{pmatrix}.\end{aligned}$$

It is not hard to observe that $b_i',c_i'\in q\mathbb{Z}_q$ as $b_i,c_i,\beta_j,\gamma_j\in q\mathbb{Z}_q$, and

$$a_2'd_1'-a_1'd_2'\equiv\alpha_1\alpha_2\delta_1\delta_2(a_2d_1-a_1d_2)\equiv a_2d_1-a_1d_2 \mod q.$$

Modulo q^2, we have

$$\begin{aligned}a_2'b_1'-a_1'b_2' \equiv&\delta_2\alpha_1a_2(-\gamma_2\alpha_1a_1+\alpha_2\alpha_1b_1+\alpha_2\beta_1d_1)-\delta_2\alpha_1a_1(-\gamma_2\alpha_1a_2+\alpha_2\alpha_1b_2+\alpha_2\beta_1d_2)\\ \equiv&\alpha_1^2\alpha_2\delta_2(a_2b_1-a_1b_2)+\alpha_1\alpha_2\beta_1\delta_2(a_2d_1-a_1d_2).\end{aligned}$$

It follows that
$$\frac{a_2'b_1' - a_1'b_2'}{m} \equiv \frac{a_2b_1 - a_1b_2}{m} \mod q.$$

Similarly, we obtain that modulo q^2
$$\begin{aligned} c_2'd_1' - c_1'd_2' \equiv & \alpha_2\delta_1 d_1(\delta_2\gamma_1 a_2 + \delta_2\delta_1 c_2 - \beta_2\delta_1 d_2) - \alpha_2\delta_1 d_2(\delta_2\gamma_1 a_1 + \delta_2\delta_1 c_1 - \beta_2\delta_1 d_1) \\ \equiv & \alpha_2\delta_1^2\delta_2(c_2d_1 - c_1d_2) + \alpha_2\gamma_1\delta_1\delta_2(a_2d_1 - a_1d_2). \end{aligned}$$
and
$$\frac{c_2'd_1' - c_1'd_2'}{m} \equiv \frac{c_2d_1 - c_1d_2}{m} \mod q.$$

Therefore, when φ_q is non-vanishing, we can deduce that
$$\omega(1,h)\varphi_q((\mathbf{x}_1,\mathbf{x}_2)) = \varphi_q((\mathbf{x}_1,\mathbf{x}_2)) \quad \text{for } h = (h_1,h_2) \in \overline{\Gamma}(q) \times \overline{\Gamma}(q).$$
When φ_q is vanishing, we consider that $b_1 \in \mathbb{Z}_q^\times$ and other cases that b_2, d_1 or d_2 in $\mathbb{Z}_q^\times$ can be treated similarly . For $h^{-1} = \left(\begin{pmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{pmatrix}, \begin{pmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{pmatrix}\right) \in \overline{\Gamma}(q)$, it is observed that
$$b_1' = -\gamma_2(\alpha_1 a_1 + \beta_1 c_1) + \alpha_2(\alpha_1 b_1 + \beta_1 d_1) \in \mathbb{Z}_q^\times$$
which makes φ_q vanish on $(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2)$. Now we have proven this lemma. □

4.3.2 At inert prime dividing m

Let q be an inert prime dividing m such that $(q) = \mathfrak{q}$. According to Proposition 4.3.1, there is an isomorphism $(\mathcal{H}(F) \otimes \mathbb{Q}_q, -\det) \simeq (V_1(\mathbb{Q}_q\sqrt{d}), \det)$ for q inert in $F/\mathbb{Q}$ given by
$$\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \longmapsto \begin{pmatrix} b & a\sqrt{d} \\ c\sqrt{d} & \bar{b} \end{pmatrix} \qquad \text{for } a, c \in \mathbb{Q}_q, b \in \mathbb{Q}_q(\sqrt{d})$$
where $^-$ on the right hand side denotes the non-trivial action in $\mathrm{Gal}(\mathbb{Q}_q(\sqrt{d})/\mathbb{Q}_q)$.

Definition 4.3.5. (1) Suppose that $d \equiv 1 \bmod 4$. The local Schwartz function $\varphi_q^{\chi_\mathfrak{m}}$ at q is vanishing unless, for $i = 1, 2$,
$$a_i \in \mathbb{Z}_q, b_i \in \mathfrak{q}\mathcal{O}_\mathfrak{q}, c_i \in \mathbb{Z}_q \text{ and } a_2c_1 - a_1c_2 \in q\mathbb{Z}_q,$$

in which case

$$\varphi_q^{\chi_{\mathfrak{m}}}\left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix}\right)$$
$$= \begin{cases} \widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}\left(\frac{a_2b_1-a_1b_2}{m}+\frac{\bar{b}_2c_1-\bar{b}_1c_2}{m}\right), & \text{if } \frac{a_2b_1-a_1b_2}{m}+\frac{\bar{b}_2c_1-\bar{b}_1c_2}{m}\in\mathcal{O}_{\mathfrak{q}}^{\times}, \\ 0, & \text{if } \frac{a_2b_1-a_1b_2}{m}+\frac{\bar{b}_2c_1-\bar{b}_1c_2}{m}\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}. \end{cases}$$

(2) Suppose that $d\equiv 2,3 \bmod 4$. We replace above $\frac{a_2b_1-a_1b_2}{m}$ by $\frac{2(a_2b_1-a_1b_2)}{m}$ and $\frac{\bar{b}_2c_1-\bar{b}_1c_2}{m}$ by $\frac{2(\bar{b}_2c_1-\bar{b}_1c_2)}{m}$ as discussed in Remark 4.2.6.

In the following we will check the invariance properties of this local Schwartz function with respect to $\mathrm{Sp}_4\times\mathrm{SO}(3,1)$ in detail in case of $d\equiv 1 \bmod 4$ and the other case can be treated similarly. For simplicity we write $\varphi_q=\varphi_q^{\chi_{\mathfrak{m}}}$ and $\chi=\chi_{\mathfrak{m}}$.

Set $\mathbf{X}=(\mathbf{x}_1,\mathbf{x}_2)=\left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix}\right)$. For $a_i\in\mathbb{Z}_q, b_i\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ and $c_i\in\mathbb{Z}_q$, it is easy to observe that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2) \quad \text{for } u\in M_2(q\mathbb{Z}_q). \tag{4.14}$$

Set $\mathbf{Y}=(\mathbf{y}_1,\mathbf{y}_2)=\left(\begin{pmatrix} \beta_1 & \alpha_1\sqrt{d} \\ \gamma_1\sqrt{d} & \bar{\beta}_1 \end{pmatrix}, \begin{pmatrix} \beta_2 & \alpha_2\sqrt{d} \\ \gamma_2\sqrt{d} & \bar{\beta}_2 \end{pmatrix}\right)$. Consider the Fourier transform

$$\hat{\varphi}_q(\mathbf{Y})=\int\psi_q(\mathrm{tr}(\mathbf{X},\mathbf{Y}))\varphi_q(\mathbf{X})d\mathbf{X}$$

where

$$\mathrm{tr}(\mathbf{X},\mathbf{Y})=-(b_1\bar{\beta}_1+\bar{b}_1\beta_1-a_1\gamma_1 d-\alpha_1c_1d+b_2\bar{\beta}_2+\bar{b}_2\beta_2-a_2\gamma_2 d-\alpha_2c_2d).$$

By the definition, φ_q is invariant under the transformations $a_i\mapsto a_i+q$, $b_i\mapsto b_i+q^2$, $b_i\mapsto b_i+q^2\sqrt{d}$ (or $b_i\mapsto b_i+q^2\cdot\frac{1+\sqrt{d}}{2}$) and $c_i\mapsto c_i+q$. Repeating arguments in the previous subsection, we can observe that the Fourier transform $\hat{\varphi}(\mathbf{y}_1,\mathbf{y}_2)$ is vanishing unless, for $i=1,2$,

$$\alpha_i\in\mathbb{Z}_q;\beta,\bar{\beta}\in\mathfrak{q}^{-1}\mathcal{O}_{\mathfrak{q}}(\text{as } \beta_i+\bar{\beta}_i\in q^{-1}\mathbb{Z}_q,\beta_i-\bar{\beta}_i\in q^{-1}\sqrt{d}\mathbb{Z}_q);\gamma_i\in\mathbb{Z}_q.$$

It follows that, for $u\in M_2(q^3\mathbb{Z}_q)$ such that $u=u^t$,

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q(\mathbf{y}_1,\mathbf{y}_2)=\omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\right)\varphi_q(\mathbf{y}_1,\mathbf{y}_2)$$

which implies

$$\omega\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix}\right)\varphi_q(\mathbf{y}_1,\mathbf{y}_2)=\varphi_q(\mathbf{y}_1,\mathbf{y}_2) \quad \text{for } u\in M_2(q^3\mathbb{Z}_q),\ u=u^t. \tag{4.15}$$

For $a \in \mathrm{GL}_2(\mathbb{Z}_q)$, write $\left(\begin{pmatrix} b'_1 & a'_1\sqrt{d} \\ c'_1\sqrt{d} & \bar{b}'_1 \end{pmatrix}, \begin{pmatrix} b'_2 & a'_2\sqrt{d} \\ c'_2\sqrt{d} & \bar{b}'_2 \end{pmatrix}\right) = \mathbf{X}a$. We have $a'_2b'_1 - a'_1b'_2 = \det(a)(a_2b_1 - a_1b_2)$, $\bar{b}'_2c'_1 - \bar{b}'_1c'_2 = \det(a)(\bar{b}_2c_1 - \bar{b}_1c_2)$ and $a'_2c'_1 - a'_1c'_2 = \det(a)(a_2c_1 - a_1c_2)$. So for $\det(a) \in \mathbb{Z}_q^\times$, we obtain

$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^ta^{-1} \end{pmatrix}, 1\right)\varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \chi_{V,q}(\det(a))|\det(a)|_q^2\widetilde{\chi}_{\mathfrak{q}}(\det(a))\varphi_q(\mathbf{x}_1, \mathbf{x}_2). \quad (4.16)$$

Combining (4.14),(4.15) and (4.16), we can deduce the following lemma:

Lemma 4.3.6. *We have*

$$\omega(k_2)\varphi_q = \chi_{V,q}(\det(A))|\det(A)|_q^2\widetilde{\chi}_{\mathfrak{q}}(\det(A)\varphi_q$$

for

$$k_2 = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \left\{\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q) : B \in M_2(q\mathbb{Z}_q), C \in M_2(q^3\mathbb{Z}_q)\right\}.$$

We next discuss the action of $\mathrm{SO}(3,1)(V(\mathbb{Q}_q))$ on φ_q characterised by

$$\omega(1,h)\varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \varphi_q(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2) \quad \text{for } h \in \mathrm{SO}(3,1)(V(\mathbb{Q}_q)).$$

Due to the exceptional isomorphism $\mathrm{PSL}_2(\mathbb{Q}_q(\sqrt{d})) \simeq \mathrm{SO}^+(3,1)(V(\mathbb{Q}_q))$ as in Example 1.2.3 (3), in this case we check the invariance property under some congruence subgroups of $\mathrm{PSL}_2(\mathcal{O}_{\mathfrak{q}})$. Here we have that $h^{-1}\mathbf{x}_i := h^{-1}\mathbf{x}_i(\bar{h}^{-1})^*$ for $i = 1, 2$ where $^-$ denotes the non-trivial action in $\mathrm{Gal}(\mathbb{Q}_q(\sqrt{d})/\mathbb{Q}_q)$ (see [Rob01, Section 2]).

Lemma 4.3.7. *For*

$$h \in \overline{\Gamma}(q) = \left\{\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_{\mathfrak{q}}) : \pm\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \equiv \pm\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod q\right\},$$

we have that

$$\omega(1,h)\varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \varphi_q(\mathbf{x}_1, \mathbf{x}_2). \quad (4.17)$$

Proof. Set

$$h^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ and } \mathbf{x}_i = \begin{pmatrix} b_i & a_i\sqrt{d} \\ c_i\sqrt{d} & \bar{b}_i \end{pmatrix}.$$

with $\alpha, \delta \equiv 1 \bmod q$ and $\beta, \gamma \equiv 0 \bmod q$.

First we assume $a_i \in \mathbb{Z}_q$, $\bar{b}_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}, c_i \in \mathbb{Z}_q$ so that φ_q is non-vanishing on

$(\mathbf{x}_1, \mathbf{x}_2)$. Writing $h^{-1}\mathbf{x}_i = \begin{pmatrix} b'_i & a'_i\sqrt{d} \\ c'_i\sqrt{d} & \bar{b}'_i \end{pmatrix}$, we compute

$$\begin{aligned}\begin{pmatrix} b'_i & a'_i\sqrt{d} \\ c'_i\sqrt{d} & \bar{b}'_i \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} b_i & a_i\sqrt{d} \\ c_i\sqrt{d} & \bar{b}_i \end{pmatrix} \begin{pmatrix} \bar{\delta} & -\bar{\beta} \\ -\bar{\gamma} & \bar{\alpha} \end{pmatrix} \\ &= \begin{pmatrix} \alpha b_i + \beta c_i\sqrt{d} & \alpha a_i\sqrt{d} + \beta\bar{b}_i \\ \gamma b_i + \delta c_i\sqrt{d} & \gamma a_i\sqrt{d} + \delta\bar{b}_i \end{pmatrix} \begin{pmatrix} \bar{\delta} & -\bar{\beta} \\ -\bar{\gamma} & \bar{\alpha} \end{pmatrix} \\ &= \begin{pmatrix} \bar{\delta}(\alpha b_i + \beta c_i\sqrt{d}) - \bar{\gamma}(\alpha a_i\sqrt{d} + \beta\bar{b}_i) & -\bar{\beta}(\alpha b_i + \beta c_i\sqrt{d}) + \bar{\alpha}(\alpha a_i\sqrt{d} + \beta\bar{b}_i) \\ \bar{\delta}(\gamma b_i + \delta c_i\sqrt{d}) - \bar{\gamma}(\gamma a_i\sqrt{d} + \delta\bar{b}_i) & -\bar{\beta}(\gamma b_i + \delta c_i\sqrt{d}) + \bar{\alpha}(\gamma a_i\sqrt{d} + \delta\bar{b}_i) \end{pmatrix}.\end{aligned}$$

It is not hard to observe that $a'_i \in \mathbb{Z}_q, b'_i \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, $c'_i \in \mathbb{Z}_q$ and that $a'_2c'_1 - a'_1c'_2 \equiv a_2c_1 - a_1c_2 \bmod q$. Then we expand

$$\begin{aligned}&a'_2b'_1 - a'_1b'_2 \\ =&\left(\frac{\bar{\alpha}\beta\bar{b}_2 - \bar{\beta}\alpha b_2}{\sqrt{d}} - \bar{\beta}\beta c_2 + \bar{\alpha}\alpha a_2\right)(\bar{\delta}(\alpha b_1 + \beta c_1\sqrt{d}) - \bar{\gamma}(\alpha a_1\sqrt{d} + \beta\bar{b}_1)) \\ &- \left(\frac{\bar{\alpha}\beta\bar{b}_1 - \bar{\beta}\alpha b_1}{\sqrt{d}} - \bar{\beta}\beta c_1 + \bar{\alpha}\alpha a_1\right)(\bar{\delta}(\alpha b_2 + \beta c_2\sqrt{d}) - \bar{\gamma}(\alpha a_2\sqrt{d} + \beta\bar{b}_2)),\end{aligned}$$

and, modulo q^2, we get

$$a'_2b'_1 - a'_1b'_2 \equiv \alpha^2\bar{\alpha}\bar{\delta}(a_2b_1 - a_1b_2) + \alpha\bar{\alpha}\beta\bar{\delta}\sqrt{d}(a_2c_1 - a_1c_2).$$

Similarly, we have, modulo q^2,

$$\begin{aligned}&\bar{b}'_2c'_1 - \bar{b}'_1c'_2 \\ \equiv&\delta\bar{\delta}c_1(-\bar{\beta}\delta c_2\sqrt{d} + \bar{\alpha}(\gamma a_2\sqrt{d} + \delta\bar{b}_2)) - \delta\bar{\delta}c_2(-\bar{\beta}\delta c_1\sqrt{d} + \bar{\alpha}(\gamma a_1\sqrt{d} + \delta\bar{b}_1)) \\ \equiv&\delta^2\bar{\alpha}\bar{\delta}(\bar{b}_2c_1 - \bar{b}_1c_2) + \bar{\alpha}\gamma\delta\bar{\delta}\sqrt{d}(a_2c_1 - a_1c_2).\end{aligned}$$

Then we can deduce that

$$\frac{a'_2b'_1 - a'_1b'_2}{m} + \frac{\bar{b}'_2c'_1 - \bar{b}'_1c'_2}{m} \equiv \frac{a_2b_1 - a_1b_2}{m} + \frac{\bar{b}_2c_1 - \bar{b}_1c_2}{m} \mod q$$

which implies that

$$\varphi_q\left(\frac{a'_2b'_1 - a'_1b'_2}{m} + \frac{\bar{b}'_2c'_1 - \bar{b}'_1c'_2}{m}\right) = \varphi_q\left(\frac{a_2b_1 - a_1b_2}{m} + \frac{\bar{b}_2c_1 - \bar{b}_1c_2}{m}\right).$$

Next we assume $b_1 \in \mathcal{O}_{\mathfrak{q}}^\times$ so that φ_q is vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$. It follows that $\bar{\delta}\alpha b_1 \in \mathcal{O}_{\mathfrak{q}}^\times$ and then $b'_1 \in \mathcal{O}_{\mathfrak{q}}^\times$ which makes φ_q is vanishing on $(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2)$. Other cases that $\bar{b}'_1, b'_2, \bar{b}'_2 \in \mathcal{O}_{\mathfrak{q}}^\times$ can be treated in the same way and recall that $a'_2c'_1 - a'_1c'_2 \equiv a_2c_1 - a_1c_2 \bmod q$. Hence, if φ_q is vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$, so is that on $(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2)$.

Now we have proven this lemma. □

4.3.3 At ramified prime away from 2

Let q be a ramified prime away from 2 such that $(q)=\mathfrak{q}^2$. According to Proposition 4.3.1, there is an isomorphism $(\mathcal{H}(F)\otimes\mathbb{Q}_q,-\det)\simeq(V_1(\mathbb{Q}_q),\det)$ for q ramified in $F/\mathbb{Q}$ given by

$$\begin{pmatrix} a & b \\ \bar{b} & c \end{pmatrix} \longmapsto \begin{pmatrix} b & a\sqrt{d} \\ c\sqrt{d} & \bar{b} \end{pmatrix} \qquad \text{for } a,c\in\mathbb{Q}_q, b\in\mathbb{Q}_q(\sqrt{d})$$

where $^-$ on the right hand side denotes the non-trivial action in $\mathrm{Gal}(\mathbb{Q}_q(\sqrt{d})/\mathbb{Q}_q)$. Note that when $d\equiv 2,3 \pmod 4$ the prime 2 is ramified and at the ramified 2 the local Schwartz function is defined in the next subsection.

Definition 4.3.8. (1) Suppose that $d\equiv 1 \bmod 4$. The local Schwartz function $\varphi_q^{\chi_\mathfrak{m}}$ at q is vanishing unless, for $i=1,2$,

$$a_i\in\mathbb{Z}_q, c_i\in\mathbb{Z}_q, b_i\in\mathcal{O}_\mathfrak{q}, b_1\bar{b}_2-\bar{b}_1b_2\in\mathfrak{q}\mathcal{O}_\mathfrak{q},$$

in which case

$$\varphi_q^{\chi_\mathfrak{m}}\left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix},\begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix}\right)$$
$$=\begin{cases} \widetilde{\chi}_{\mathfrak{m},\mathfrak{q}}\left(\frac{a_2b_1-a_1b_2}{m}+\frac{\bar{b}_2c_1-\bar{b}_1c_2}{m}\right), & \text{if } \frac{a_2b_1-a_1b_2}{m}+\frac{\bar{b}_2c_1-\bar{b}_1c_2}{m}\in\mathcal{O}_\mathfrak{q}^\times, \\ 0, & \text{if } \frac{a_2b_1-a_1b_2}{m}+\frac{\bar{b}_2c_1-\bar{b}_1c_2}{m}\in\mathfrak{q}\mathcal{O}_\mathfrak{q}. \end{cases}$$

(2) Suppose that $d\equiv 2,3 \bmod 4$. We replace above $\frac{a_2b_1-a_1b_2}{m}$ by $\frac{2(a_2b_1-a_1b_2)}{m}$ and $\frac{\bar{b}_2c_1-\bar{b}_1c_2}{m}$ by $\frac{2(\bar{b}_2c_1-\bar{b}_1c_2)}{m}$ as discussed in Remark 4.2.6.

In the following we will check the invariance properties of this local Schwartz function with respect to $\mathrm{Sp}_4\times\mathrm{SO}(3,1)$ in case of $d\equiv 1 \bmod 4$ and the other case can be treated similarly. For simplicity we write $\varphi_q=\varphi_q^{\chi_\mathfrak{m}}$ and $\chi=\chi_\mathfrak{m}$.

Set $\mathbf{X}=(\mathbf{x}_1,\mathbf{x}_2)=\left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix},\begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix}\right)$. For $a_i,c_i\in\mathbb{Z}_q$ and $b_i,\bar{b}_i\in\mathcal{O}_\mathfrak{q}$, it is easy to observe that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix}\right)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2) \quad \text{for } u\in M_2(q\mathbb{Z}_q). \tag{4.18}$$

Set $\mathbf{Y}=(\mathbf{y}_1,\mathbf{y}_2)=\left(\begin{pmatrix} \beta_1 & \alpha_1\sqrt{d} \\ \gamma_1\sqrt{d} & \bar{\beta}_1 \end{pmatrix},\begin{pmatrix} \beta_2 & \alpha_2\sqrt{d} \\ \gamma_2\sqrt{d} & \bar{\beta}_2 \end{pmatrix}\right)$. Consider the Fourier transform

$$\hat{\varphi}_q(\mathbf{Y})=\int\psi_q(\mathrm{tr}(\mathbf{X},\mathbf{Y}))\varphi_q(\mathbf{X})d\mathbf{X}$$

where

$$\operatorname{tr}(\mathbf{X},\mathbf{Y}) = -(b_1\bar{\beta}_1 + \bar{b}_1\beta_1 - a_1\gamma_1 d - \alpha_1 c_1 d + b_2\bar{\beta}_2 + \bar{b}_2\beta_2 - a_2\gamma_2 d - \alpha_2 c_2 d).$$

By the definition, φ_q is invariant under the transformations $a_i \mapsto a_i + q$, $b_i \mapsto b_i + q$, $b_i \mapsto b_i + \sqrt{d}$ and $c_i \mapsto c_i + q$. Repeating arguments in the previous subsection, we can observe that the Fourier transform $\hat{\varphi}(\mathbf{y}_1,\mathbf{y}_2)$ is vanishing unless, for $i = 1, 2$,

$$\alpha_i, \gamma_i \in q^{-1}\mathbb{Z}_q \text{ and } \beta_i, \bar{\beta}_i \in \mathcal{O}_{\mathfrak{q}}(\text{as } \beta_i + \bar{\beta}_i \in \mathbb{Z}_q, \beta_i - \bar{\beta}_i \in \mathbb{Z}_q\sqrt{d}).$$

It follows that, for $u \in M_2(q^2\mathbb{Z}_q)$ such that $u = u^t$,

$$\omega\left(\begin{pmatrix}1 & u\\ 0 & 1\end{pmatrix}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\varphi_q(\mathbf{y}_1,\mathbf{y}_2) = \omega\left(\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right)\varphi_q(\mathbf{y}_1,\mathbf{y}_2)$$

which implies

$$\omega\left(\begin{pmatrix}1 & 0\\ u & 1\end{pmatrix}\right)\varphi_q(\mathbf{y}_1,\mathbf{y}_2) = \varphi_q(\mathbf{y}_1,\mathbf{y}_2) \quad \text{for } u \in M_2(q^2\mathbb{Z}_q),\ u = u^t. \tag{4.19}$$

For $a \in \mathrm{GL}_2(\mathbb{Z}_q)$ with $\det(a) \in \mathbb{Z}_q^\times$, we also have

$$\omega\left(\begin{pmatrix}a & 0\\ 0 & {}^t a^{-1}\end{pmatrix}, 1\right)\varphi_q(\mathbf{x}_1,\mathbf{x}_2) = \chi_{V,q}(\det(a))|\det(a)|_q^2\widetilde{\chi}_{\mathfrak{q}}(\det(a))\varphi_q(\mathbf{x}_1,\mathbf{x}_2). \tag{4.20}$$

Again, combining (4.18),(4.19) and (4.20), we can deduce the following lemma:

Lemma 4.3.9. *We have*

$$\omega(k_3)\varphi_q = \chi_{V,q}(\det(A))|\det(A)|_q^2\widetilde{\chi}_{\mathfrak{q}}(\det(A))\varphi_q$$

for

$$k_3 = \begin{pmatrix}A & B\\ C & D\end{pmatrix} \in \left\{\begin{pmatrix}A & B\\ C & D\end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q) : B \in M_2(q\mathbb{Z}_q), C \in M_2(q^2\mathbb{Z}_q)\right\}.$$

We next discuss the action of $\mathrm{SO}(3,1)(V(\mathbb{Q}_q))$ on φ_q characterised by

$$\omega(1,h)\varphi_q(\mathbf{x}_1,\mathbf{x}_2) = \varphi_q(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2) \quad \text{for } h \in \mathrm{SO}(3,1)(V(\mathbb{Q}_q)).$$

In this case, we check the invariance property under congruence subgroups of $\mathrm{PSL}_2(\mathcal{O}_{\mathfrak{q}})$ and have that $h^{-1}\mathbf{x}_i := h^{-1}\mathbf{x}_i(\bar{h}^{-1})^*$ for $i = 1, 2$ where $^-$ denotes the non-trivial action in $\mathrm{Gal}(\mathbb{Q}_q(\sqrt{d})/\mathbb{Q}_q)$ (see [Rob01, Section 2]).

Lemma 4.3.10. *For*

$$h \in \overline{\Gamma}(\mathfrak{q}) = \left\{\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \in \mathrm{PSL}_2(\mathcal{O}_{\mathfrak{q}}) : \pm\begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix} \equiv \pm\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} \bmod \mathfrak{q}\right\},$$

we have that

$$\omega(1,h)\varphi_q(\mathbf{x}_1,\mathbf{x}_2) = \varphi_q(\mathbf{x}_1,\mathbf{x}_2). \tag{4.21}$$

Proof. Set
$$h^{-1} = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \text{ and } \mathbf{x}_i = \begin{pmatrix} b_i & a_i\sqrt{d} \\ c_i\sqrt{d} & \bar{b}_i \end{pmatrix}.$$
with $\alpha, \delta \equiv 1 \bmod \mathfrak{q}$ and $\beta, \gamma \equiv 0 \bmod \mathfrak{q}$.

First we assume $a_i, c_i \in \mathbb{Z}_q$ and $b_i, \bar{b}_i \in \mathcal{O}_\mathfrak{q}$ so that φ_q is non-vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$. Writing $h^{-1}\mathbf{x}_i = \begin{pmatrix} b_i' & a_i'\sqrt{d} \\ c_i'\sqrt{d} & \bar{b}_i' \end{pmatrix}$, we compute
$$\begin{aligned}
\begin{pmatrix} b_i' & a_i'\sqrt{d} \\ c_i'\sqrt{d} & \bar{b}_i' \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} b_i & a_i\sqrt{d} \\ c_i\sqrt{d} & \bar{b}_i \end{pmatrix} \begin{pmatrix} \bar{\delta} & -\bar{\beta} \\ -\bar{\gamma} & \bar{\alpha} \end{pmatrix} \\
&= \begin{pmatrix} \alpha b_i + \beta c_i\sqrt{d} & \alpha a_i\sqrt{d} + \beta\bar{b}_i \\ \gamma b_i + \delta c_i\sqrt{d} & \gamma a_i\sqrt{d} + \delta\bar{b}_i \end{pmatrix} \begin{pmatrix} \bar{\delta} & -\bar{\beta} \\ -\bar{\gamma} & \bar{\alpha} \end{pmatrix} \\
&= \begin{pmatrix} \bar{\delta}(\alpha b_i + \beta c_i\sqrt{d}) - \bar{\gamma}(\alpha a_i\sqrt{d} + \beta\bar{b}_i) & -\bar{\beta}(\alpha b_i + \beta c_i\sqrt{d}) + \bar{\alpha}(\alpha a_i\sqrt{d} + \beta\bar{b}_i) \\ \bar{\delta}(\gamma b_i + \delta c_i\sqrt{d}) - \bar{\gamma}(\gamma a_i\sqrt{d} + \delta\bar{b}_i) & -\bar{\beta}(\gamma b_i + \delta c_i\sqrt{d}) + \bar{\alpha}(\gamma a_i\sqrt{d} + \delta\bar{b}_i) \end{pmatrix}.
\end{aligned}$$
It is not hard to observe that $a_i', c_i' \in \mathbb{Z}_q, b_i', \bar{b}_i' \in \mathcal{O}_\mathfrak{q}$ and $b_1'\bar{b}_2' - \bar{b}_1'b_2' \equiv b_1\bar{b}_2 - \bar{b}_1 b_2 \bmod \mathfrak{q}$. Modulo $\mathfrak{q}$, we have
$$\begin{aligned}
&a_2'b_1' - a_1'b_2' \\
\equiv&\bar{\delta}\alpha b_1 \left(\frac{\bar{\alpha}\beta\bar{b}_2 - \bar{\beta}\alpha b_2}{\sqrt{d}} + \bar{\alpha}\alpha a_2 \right) - \bar{\delta}\alpha b_2 \left(\frac{\bar{\alpha}\beta\bar{b}_1 - \bar{\beta}\alpha b_1}{\sqrt{d}} + \bar{\alpha}\alpha a_1 \right) \\
\equiv&\alpha^2\bar{\alpha}\bar{\delta}(a_2b_1 - a_1b_2) - \alpha\bar{\alpha}\beta\bar{\delta}/\sqrt{d}(b_1\bar{b}_2 - \bar{b}_1b_2),
\end{aligned}$$
and
$$\begin{aligned}
&c_2'\bar{b}_1' - c_1'\bar{b}_2' \\
\equiv&\bar{\alpha}\delta\bar{b}_1 \left(\frac{\bar{\delta}\gamma b_2 - \delta\bar{\gamma}\bar{b}_2}{\sqrt{d}} + \bar{\delta}\delta c_2 \right) - \bar{\alpha}\delta\bar{b}_2 \left(\frac{\bar{\delta}\gamma b_1 - \delta\bar{\gamma}\bar{b}_1}{\sqrt{d}} + \bar{\delta}\delta c_1 \right) \\
\equiv&\delta^2\bar{\alpha}\bar{\delta}(c_2\bar{b}_1 - c_1\bar{b}_2) + \bar{\alpha}\gamma\delta\bar{\delta}/\sqrt{d}(b_1\bar{b}_2 - \bar{b}_1b_2).
\end{aligned}$$
So, modulo $\mathfrak{q}$, we get
$$\frac{a_2'b_1' - a_1'b_2'}{m} \equiv \alpha^2\bar{\alpha}\bar{\delta}\frac{a_2b_1 - a_1b_2}{m} \quad \text{and} \quad \frac{c_2'\bar{b}_1' - c_1'\bar{b}_2'}{m} \equiv \delta^2\bar{\alpha}\bar{\delta}\frac{c_2\bar{b}_1 - c_1\bar{b}_2}{m}.$$
It follows that
$$\varphi_q\left(\frac{a_2'b_1' - a_1'b_2'}{m} + \frac{c_2'\bar{b}_1' - c_1'\bar{b}_2'}{m} \right) = \varphi_q\left(\frac{a_2b_1 - a_1b_2}{m} + \frac{c_2\bar{b}_1 - c_1\bar{b}_2}{m} \right).$$

Next we assume $b_1 \in \mathfrak{q}^{-1}\mathcal{O}_\mathfrak{q}^\times$ so that φ_q is vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$. It follows that $\bar{\delta}\alpha b_1 \in \mathfrak{q}^{-1}\mathcal{O}_\mathfrak{q}^\times$ and then $b_1' \in \mathfrak{q}^{-1}\mathcal{O}_\mathfrak{q}^\times$ which makes φ_q be vanishing on $(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2)$. Other cases for $a_i, c_i, \bar{b}_1, b_2, \bar{b}_2$ can be treated in the same way. Hence, if φ_q is vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$, so is that on $(h^{-1}\mathbf{x}_1, h^{-1}\mathbf{x}_2)$.

Now we have proven this lemma. □

4.3.4 At places dividing $N(\mathfrak{n})$ and ramified 2

In this subsection we consider the local Schwartz function at finite places dividing $N(\mathfrak{n})$ and at ramified prime 2 (when $d \equiv 2,3 \bmod 4$). For a place q and an integral lattice X on V, we put $X_q = X \otimes_{\mathbb{Z}} \mathbb{Z}_q$.

Definition 4.3.11. (1) Let $q|N(\mathfrak{n})$ be split with $(q) = \mathfrak{q}\bar{\mathfrak{q}}$.

- Suppose that $(\mathfrak{n},(q)) = \mathfrak{q}$. Define the local Schwartz function φ_q to be the characteristic function of
$$\left\{\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \in X_q^2 : b_1c_2 + c_1b_2 \in \mathcal{O}_{\mathfrak{q}}^{\times}, d_i \in q\mathbb{Z}_q\right\}.$$
- Suppose that $(\mathfrak{n},(q)) = \bar{\mathfrak{q}}$. Define φ_q to be the characteristic function of
$$\left\{\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \in X_q^2 : b_1c_2 + c_1b_2 \in \mathcal{O}_{\bar{\mathfrak{q}}}^{\times}, d_i \in q\mathbb{Z}_q\right\}.$$
- Suppose that $(\mathfrak{n},(q)) = (q)$. Define φ_q to be the characteristic function of
$$\left\{\left(\begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix}, \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix}\right) \in X_q^2 : b_1c_2 + c_1b_2 \in \mathcal{O}_{\mathfrak{q}}^{\times} \times \mathcal{O}_{\bar{\mathfrak{q}}}^{\times}, d_i \in q\mathbb{Z}_q\right\}.$$

(2) At inert place $q|\mathfrak{n}$ with $(q) = \mathfrak{q}$, we define φ_q to be the characteristic function of
$$\left\{\left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix}\right) \in X_q^2 : b_1\bar{b}_2 + \bar{b}_1b_2 \in \mathcal{O}_{\mathfrak{q}}^{\times}, c_i \in q\mathbb{Z}_q\right\}$$

(3) If 2 is ramified with $(2) = \mathfrak{q}_2^2$, we define φ_2 to be the characteristic function of
$$\left\{\left(\begin{pmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1 \end{pmatrix}, \begin{pmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2 \end{pmatrix}\right) : a_i, c_i \in \mathbb{Z}_2, b_i \in \frac{1}{2}\mathcal{O}_{\mathfrak{q}_2}, b_1\bar{b}_2 + \bar{b}_1b_2 \in \frac{1}{2}\mathcal{O}_{\mathfrak{q}_2}^{\times}\right\}.$$

Note that if we take the local Schwartz function at finite places dividing $N(\mathfrak{n})$ and ramified 2 as the characteristic function of integral lattice, the resulting theta lifting would be vanishing as discussed in Remark 3.3.6 (i).

In the following we will check the invariance properties of this local Schwartz function with respect to $\mathrm{Sp}_4 \times \mathrm{SO}(3,1)$.

Lemma 4.3.12. *(1) For φ_q as in above Definition 4.3.11 (1) and (2), We have*
$$\omega(k_4)\varphi_q = \varphi_q$$

for

$$k_4 \in \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q) : A \in \begin{pmatrix} \mathbb{Z}_q & q\mathbb{Z}_q \\ q\mathbb{Z}_q & \mathbb{Z}_q \end{pmatrix}, B \in M_2(q\mathbb{Z}_q), C \in M_2(q\mathbb{Z}_q) \right\}.$$

(2) At ramified 2, we have

$$\omega(k_5)\varphi_2 = \varphi_2$$

for

$$k_5 \in \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q) : A \in \begin{pmatrix} \mathbb{Z}_2 & 2\mathbb{Z}_2 \\ 2\mathbb{Z}_2 & \mathbb{Z}_2 \end{pmatrix}, B \in M_2(2^4\mathbb{Z}_2) \right\}.$$

Proof. (1) We prove this lemma in details only for split q with $(\mathfrak{n}, (q)) = \mathfrak{q}$ and other cases can be treated similarly.

Set $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2) = \left(\left(\begin{smallmatrix} a_1 & b_1 \\ c_1 & d_1 \end{smallmatrix} \right), \left(\begin{smallmatrix} a_2 & b_2 \\ c_2 & d_2 \end{smallmatrix} \right) \right) \in X_q^2$. It is not difficult to observe that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \right) \varphi_q(\mathbf{X}) = \varphi_q(\mathbf{x}_1, \mathbf{x}_2) \quad \text{for } u \in M_2(q\mathbb{Z}_q).$$

Set $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2) = \left(\left(\begin{smallmatrix} \alpha_1 & \beta_1 \\ \gamma_1 & \delta_1 \end{smallmatrix} \right), \left(\begin{smallmatrix} \alpha_2 & \beta_2 \\ \gamma_2 & \delta_2 \end{smallmatrix} \right) \right)$. Consider the Fourier transform

$$\hat{\varphi}_q(\mathbf{Y}) = \int_{X_{q,v}^2} \psi_q(\mathrm{tr}(\mathbf{X}, \mathbf{Y}))\varphi_q(\mathbf{X}) d\mathbf{X} \quad \text{for } i = 1, 2,$$

where

$$\mathrm{tr}(\mathbf{X}, \mathbf{Y}) = -(a_1\delta_1 - b_1\gamma_1 - c_1\beta_1 + d_1\alpha_1 + a_2\delta_2 - b_2\gamma_2 - c_2\beta_2 + d_2\alpha_2).$$

By the above definition, φ_q is invariant under the transformations $a_i \mapsto a_i + \mathbb{Z}_q$, $b_i \mapsto b_i + q$, $b_i \mapsto b_i + q\lambda_{\sqrt{d}}$ (or $b_i \mapsto b_i + q\frac{1+\lambda_{\sqrt{d}}}{2}$) and $d_i \mapsto d_i + q$. Note that $b_1c_2 + c_1b_2 \in \mathcal{O}_{\mathfrak{q}}^{\times}$ is not preserved under $b_i \mapsto b_i + \mathfrak{q}$ or $b_i \mapsto b_i + \bar{\mathfrak{q}}$. Then we can deduce that $\omega\left(\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix} \right) \right) \varphi_q(\mathbf{Y})$ is vanishing unless $\alpha_i \in \mathbb{Z}_q$, $\beta_i \in \mathbb{Z}_q$, $\gamma_i \in \mathbb{Z}_q$ and $\delta_i \in q\mathbb{Z}_q$. It follows that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \varphi_q(\mathbf{Y}) = \omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \right) \varphi_q(\mathbf{Y}) \quad \text{for } u \in M_2(q\mathbb{Z}_q),\ u = u^t$$

which implies

$$\omega\left(\begin{pmatrix} 1 & 0 \\ u & 1 \end{pmatrix} \right) \varphi_q(\mathbf{Y}) = \varphi_q(\mathbf{Y}) \quad \text{for } u \in M_2(q\mathbb{Z}_q),\ u = u^t.$$

For $a = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta \end{smallmatrix} \right) \in \mathrm{GL}_2(\mathbb{Z}_q)$, set $\left(\left(\begin{smallmatrix} a_1' & b_1' \\ c_1' & d_1' \end{smallmatrix} \right), \left(\begin{smallmatrix} a_2' & b_2' \\ c_2' & d_2' \end{smallmatrix} \right) \right) := (\mathbf{x}_1, \mathbf{x}_2)a$. It is clear that

$$d_1' = \alpha d_1 + \beta d_2 \text{ and } d_2' = \gamma d_1 + \delta d_2$$

lie in $q\mathbb{Z}_q$. Also we have

$$b_1'c_2' + c_1'b_2' = (\alpha b_1 + \gamma b_2)(\beta c_1 + \delta c_2) + (\beta b_1 + \delta b_2)(\alpha c_1 + \gamma c_2).$$

If $\beta, \gamma \equiv 0 \bmod q$ and $\det(a) \in \mathbb{Z}_q^\times$, then $b_1'c_2' + c_1'b_2' \in \mathcal{O}_{\mathfrak{q}}^\times$. We can deduce that

$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1}\end{pmatrix}\right)\varphi_q(\mathbf{X}) = \varphi_q(\mathbf{X}) \quad \text{for } a \in \begin{pmatrix} \mathbb{Z}_q & q\mathbb{Z}_q \\ q\mathbb{Z}_q & \mathbb{Z}_q\end{pmatrix} \text{ and } \det(a) \in \mathbb{Z}_q^\times.$$

(2) Set $\mathbf{X} = (\mathbf{x}_1, \mathbf{x}_2) = \left(\left(\begin{smallmatrix} b_1 & a_1\sqrt{d} \\ c_1\sqrt{d} & \bar{b}_1\end{smallmatrix}\right), \left(\begin{smallmatrix} b_2 & a_2\sqrt{d} \\ c_2\sqrt{d} & \bar{b}_2\end{smallmatrix}\right)\right)$. For $a_i, c_i \in \mathbb{Z}_2$ and $b_i \in \frac{1}{2}\mathcal{O}_{\mathfrak{q}_2}$, it is easy to observe that

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\right)\varphi_q(\mathbf{x}_1, \mathbf{x}_2) = \varphi_q(\mathbf{x}_1, \mathbf{x}_2) \quad \text{for } u \in M_2(2^4\mathbb{Z}_2).$$

Set $\mathbf{Y} = (\mathbf{y}_1, \mathbf{y}_2) = \left(\left(\begin{smallmatrix} \beta_1 & \alpha_1\sqrt{d} \\ \gamma_1\sqrt{d} & \bar{\beta}_1\end{smallmatrix}\right), \left(\begin{smallmatrix} \beta_2 & \alpha_2\sqrt{d} \\ \gamma_2\sqrt{d} & \bar{\beta}_2\end{smallmatrix}\right)\right)$. Consider the Fourier transform

$$\hat{\varphi}_q(\mathbf{Y}) = \int \psi_q(\operatorname{tr}(\mathbf{X}, \mathbf{Y}))\varphi_q(\mathbf{X})d\mathbf{X}$$

where

$$\operatorname{tr}(\mathbf{X}, \mathbf{Y}) = -(b_1\bar{\beta}_1 + \bar{b}_1\beta_1 - a_1\gamma_1 d - \alpha_1 c_1 d + b_2\bar{\beta}_2 + \bar{b}_2\beta_2 - a_2\gamma_2 d - \alpha_2 c_2 d).$$

By the definition, φ_q is invariant under the transformations $a_i \mapsto a_i + \mathbb{Z}_2$, $b_i \mapsto b_i + \mathbb{Z}_2$, $b_i \mapsto b_i + \mathbb{Z}_2\sqrt{d}$ and $c_i \mapsto c_i + \mathbb{Z}_2$. Repeating arguments in the previous subsection, we can observe that the Fourier transform $\hat{\varphi}(\mathbf{y}_1, \mathbf{y}_2)$ is vanishing unless, for $i = 1, 2$,

$$\alpha_i, \gamma_i \in 2\mathbb{Z}_2 \text{ and } \beta_i \in 2\mathcal{O}_{\mathfrak{q}_2} (\text{as } \beta_i + \bar{\beta}_i \in 2\mathbb{Z}_q, \beta_i - \bar{\beta}_i \in 2\mathbb{Z}_q\sqrt{d}).$$

It follows that, for $u \in M_2(\mathbb{Z}_2)$ such that $u = u^t$,

$$\omega\left(\begin{pmatrix} 1 & u \\ 0 & 1\end{pmatrix}\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right)\varphi_q(\mathbf{y}_1, \mathbf{y}_2) = \omega\left(\begin{pmatrix} 0 & 1 \\ -1 & 0\end{pmatrix}\right)\varphi_q(\mathbf{y}_1, \mathbf{y}_2)$$

which implies

$$\omega\left(\begin{pmatrix} 1 & 0 \\ u & 1\end{pmatrix}\right)\varphi_q(\mathbf{y}_1, \mathbf{y}_2) = \varphi_q(\mathbf{y}_1, \mathbf{y}_2) \quad \text{for } u \in M_2(\mathbb{Z}_2),\ u = u^t.$$

For $a = \left(\begin{smallmatrix} \alpha & \beta \\ \gamma & \delta\end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}_q)$, set $\left(\left(\begin{smallmatrix} b_1' & a_1'\sqrt{d} \\ c_1'\sqrt{d} & \bar{b}_1'\end{smallmatrix}\right), \left(\begin{smallmatrix} b_2' & a_2'\sqrt{d} \\ c_2'\sqrt{d} & \bar{b}_2'\end{smallmatrix}\right)\right) := (\mathbf{x}_1, \mathbf{x}_2)a$. We have

$$b_1' = \alpha b_1 + \gamma b_2 \text{ and } b_2' = \beta b_1 + \delta b_2$$

and then

$$b_1'\bar{b}_2' + \bar{b}_1'b_2' = (\alpha b_1 + \gamma b_2)(\beta\bar{b}_1 + \delta\bar{b}_2) + (\alpha\bar{b}_1 + \gamma\bar{b}_2)(\beta b_1 + \delta b_2).$$

If $\beta, \gamma \in 2\mathbb{Z}_2$ and $\alpha\gamma \in \mathbb{Z}_2^\times$, we have

$$\omega\left(\begin{pmatrix} a & 0 \\ 0 & {}^t a^{-1}\end{pmatrix}, 1\right)\varphi_2(\mathbf{x}_1, \mathbf{x}_2) = \varphi_2(\mathbf{x}_1, \mathbf{x}_2).$$

□

Lemma 4.3.13. *(1) Let $q|N(\mathfrak{n})$ split with $(q)=\mathfrak{q}\bar{\mathfrak{q}}$.*

- *Suppose that $(\mathfrak{n},(q))=\mathfrak{q}$. We have*
$$\omega(1,h_1)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)$$
for $h_1=h_{1,1}\times h_{1,2}$ with
$$h_{1,i}\in\overline{\Gamma}(\mathfrak{q})=\left\{\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\in\mathrm{PSL}_2(\mathcal{O}_{\mathfrak{q}}):\pm\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}\equiv\pm\begin{pmatrix}1&0\\ 0&1\end{pmatrix}\bmod\mathfrak{q}\right\}.$$
- *If $(\mathfrak{n},(q))=\bar{\mathfrak{q}}$, we have*
$$\omega(1,h_2)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)\quad \textit{for } h_2\in\overline{\Gamma}(\bar{\mathfrak{q}}).$$
- *If $(\mathfrak{n},(q))=(q)$, we have*
$$\omega(1,h_3)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)\quad \textit{for } h_3\in\overline{\Gamma}(q).$$

(2) For inert $q|N(\mathfrak{n})$ with $(q)=\mathfrak{q}$, we have
$$\omega(1,h_4)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)\quad \textit{for } h_4\in\overline{\Gamma}(\mathfrak{q}).$$

(3) For 2 ramified with $(2)=\mathfrak{q}_2^2$, we have
$$\omega(1,h_5)\varphi_q(\mathbf{x}_1,\mathbf{x}_2)=\varphi_q(\mathbf{x}_1,\mathbf{x}_2)\quad \textit{for } h_5\in\overline{\Gamma}(\mathfrak{q}_2).$$

Proof. In part (1), we prove the first statement and other cases can be treated similarly. Let $(q)=\mathfrak{q}\bar{\mathfrak{q}}$. Set $h_j^{-1}=\begin{pmatrix}\alpha_j&\beta_j\\ \gamma_j&\delta_j\end{pmatrix}$ and $\mathbf{x}_i=\begin{pmatrix}a_i&b_i\\ c_i&d_i\end{pmatrix}$ with $\alpha_j,\delta_j\equiv 1\bmod\mathfrak{q}$ and $\beta_j,\gamma_j\equiv 0\bmod\mathfrak{q}$. We compute
$$\begin{aligned}\begin{pmatrix}a_i'&b_i'\\ c_i'&d_i'\end{pmatrix}:=&h_1^{-1}\mathbf{x}_i{}^t(h_2^{-1})^*\\ =&\begin{pmatrix}\delta_2(\alpha_1a_i+\beta_1c_i)-\beta_2(\alpha_1b_i+\beta_1d_i)&-\gamma_2(\alpha_1a_i+\beta_1c_i)+\alpha_2(\alpha_1b_i+\beta_1d_i)\\ \delta_2(\gamma_1a_i+\delta_1c_i)-\beta_2(\gamma_1b_i+\delta_1d_i)&-\gamma_2(\gamma_1a_i+\delta_1c_i)+\alpha_2(\gamma_1b_i+\delta_1d_i)\end{pmatrix}\\ \equiv&\begin{pmatrix}*&b_i\\ c_i&d_i\end{pmatrix}\quad\bmod\mathfrak{q}.\end{aligned}$$
So the conditions on $b_1c_2+b_2c_1$ and d_i' for φ_q non-vanishing are preserved.

Let q be an inert prime. Set $h^{-1}=\begin{pmatrix}\alpha&\beta\\ \gamma&\delta\end{pmatrix}$ and $\mathbf{x}_i=\begin{pmatrix}b_i&a_i\sqrt{d}\\ c_i\sqrt{d}&\bar{b}_i\end{pmatrix}$ with

$\alpha, \delta \equiv 1 \bmod q$ and $\beta, \gamma \equiv 0 \bmod q$. It suffices to show that

$$\begin{aligned}\begin{pmatrix} b_i' & a_i'\sqrt{d} \\ c_i'\sqrt{d} & \bar{b}_i' \end{pmatrix} &= \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \begin{pmatrix} b_i & a_i\sqrt{d} \\ c_i\sqrt{d} & \bar{b}_i \end{pmatrix} \begin{pmatrix} \bar{\delta} & -\bar{\beta} \\ -\bar{\gamma} & \bar{\alpha} \end{pmatrix} \\ &= \begin{pmatrix} \bar{\delta}(\alpha b_i + \beta c_i\sqrt{d}) - \bar{\gamma}(\alpha a_i\sqrt{d} + \beta\bar{b}_i) & * \\ \bar{\delta}(\gamma b_i + \delta c_i\sqrt{d}) - \bar{\gamma}(\gamma a_i\sqrt{d} + \delta\bar{b}_i) & -\bar{\beta}(\gamma b_i + \delta c_i\sqrt{d}) + \bar{\alpha}(\gamma a_i\sqrt{d} + \delta\bar{b}_i) \end{pmatrix} \\ &\equiv \begin{pmatrix} b_i & * \\ c_i\sqrt{d} & \bar{b}_i \end{pmatrix} \quad \bmod q.\end{aligned}$$

It is clear that if φ_q is vanishing on $(\mathbf{x}_1, \mathbf{x}_2)$, then so is $\omega(1, h_j)\varphi_q$ on $(\mathbf{x}_1, \mathbf{x}_2)$ in the same way as dicussed in previous subsections. Similarly at ramified 2 we obtain the same result. □

4.3.5 At other finite places

We consider non-archimedean places away from $m|d_F|N(\mathfrak{n})$. For such a place q and an integral lattice X on V, we put $X_q = X \otimes_{\mathbb{Z}} \mathbb{Z}_q$. Define its dual lattice

$$X_q^\sharp = \{\mathbf{x} \in V \otimes \mathbb{Q}_q : (\mathbf{x}, \mathbf{y}) \in \mathbb{Z}_q \ \forall \mathbf{y} \in X_q\}$$

and let (q^{-l_q}) be the $\mathbb{Z}_q$-module generated by $\{(\mathbf{x}, \mathbf{x}) : \mathbf{x} \in X_q^\sharp\}$. In [Ber14, Lemma 27], it is shown that $l_q = 0$ at these places. At each place q, we define the local Schwartz function φ_q to be the characteristic function of X_q^2. Note that φ_q is invariant under $\mathrm{PSL}_2(\mathbb{Z}_q) \times \mathrm{PSL}_2(\mathbb{Z}_q)$ for split q and $\mathrm{PSL}_2(\mathcal{O}_\mathfrak{q})$ for inert or ramified q due to $l_q = 0$ (see [Ber14, Section 5.2]).

Lemma 4.3.14. *([Yos84, Lemma 2.1]) At non-archimedean $q \nmid m|d_F|N(\mathfrak{n})$ we have*

$$\omega(\sigma)\varphi_q = \chi_{V,q}(\det A)\varphi_q \quad \text{for } \sigma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}_q).$$

§ 4.4 Theta lift and Fourier coefficient

Let $F = \mathbb{Q}(\sqrt{d})$ be an imaginary quadratic field of class number 1. Denote its ring of integers by $\mathcal{O} = \mathbb{Z}[\omega]$ and the discriminant by d_F. Let m be a product of distinct inert or split primes as introduced in Section 4.3 and choose a quadratic Hecke character $\chi_\mathfrak{m}$ ($\mathfrak{m} = m\mathcal{O}$) of conductor $\mathfrak{f} = \sqrt{d}\mathfrak{m}$. Let $\mathfrak{n}$ be square-free and coprime to $(m|d_F|)$. Suppose that $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2) : \mathbb{H}_3 \to \mathbb{C}^3$ is a weight 2 cusp form for $\Gamma_0(\mathfrak{n})$ with the corresponding $\Gamma_0(\mathfrak{n})$-invariant differential $\eta_\mathcal{F}$ on $\mathbb{H}_3$ of the form $-\mathcal{F}_0\frac{dz}{r} + \mathcal{F}_1\frac{dr}{r} + \mathcal{F}_2\frac{d\bar{z}}{r}$ for $(z, r) \in \mathbb{H}_3$, see Definition 1.5.2.

Remark 4.4.1. (1) Suppose that $d \equiv 1 \bmod 4$ with $d_F = d$. We choose the local Schwartz function $\varphi_q^{\chi_\mathfrak{m}}$ at each place q dividing $m|d_F|$ as defined in Definition 4.3.2, 4.3.5 and 4.3.8. At each place q dividing $N(\mathfrak{n})$, the local Schwartz function $\varphi_q^{\mathfrak{n}}$ is chosen to be as in Definition 4.3.11. For all other finite places we take the local Schwartz function as in Section 4.3.5.

(2) Suppose that $d \equiv 2, 3 \bmod 4$ with $d_F = 4d$. We choose the local Schwartz function $\varphi_q^{\chi_\mathfrak{m}}$ at each place q dividing m as defined in Definition 4.3.2 and 4.3.5, and that at ramified place away from 2 as in Definition 4.3.8. At each place q dividing $2N(\mathfrak{n})$, the local Schwartz functions $\varphi_q^{\mathfrak{n}}$ and φ_2 are chosen to be as in Definition 4.3.11. For all other places we take the local Schwartz function as in Section 4.3.5.

It has been shown in Lemma 4.3.4, Lemma 4.3.7, Lemma 4.3.10 and Lemma 4.3.13 that the local Schwartz function φ_v at each place v dividing $m|d_F|N(\mathfrak{n})$ is invariant under the action of the principal congruence subgroup $\overline{\Gamma}(\mathfrak{q}_v) \subset \mathrm{SO}^+(3,1)(V(\mathbb{Q}_v))$. As what we have done on the Shintani lifting in Chapter 3, we now consider a $\overline{\Gamma}_0(\mathfrak{q}_v)$-invariant local Schwartz function φ_v^{new} at these places defined by

$$\varphi_v^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) = \sum_{[\gamma] \in \overline{\Gamma}_0(\mathfrak{q}_v)/\overline{\Gamma}(\mathfrak{q}_v)} \omega(1, \gamma)\varphi_v(\mathbf{x}_1, \mathbf{x}_2)$$

where the sum is taken over all the representatives of $\overline{\Gamma}_0(\mathfrak{q}_v)/\overline{\Gamma}(\mathfrak{q}_v)$. With this new local Schwartz function we know that φ_f^{new} is invariant under $\overline{\Gamma}_0(\mathfrak{f}\mathfrak{n})$ when $d \equiv 1 \bmod 4$ or $\overline{\Gamma}_0(\mathfrak{f}\mathfrak{n}\mathfrak{q}_2)$ when $d \equiv 2, 3 \bmod 4$.

Give the Schwartz form $\varphi_2 \in S(V(\mathbb{R})^2) \otimes \Omega^2(D)$ as in Example 2.2.2 (2) and the above finite Schwartz function φ_f^{new} on $V(\mathbb{A}_f)^2$, we now consider a Schwartz form

$$\varphi(\mathbf{X}, z) := \varphi_2 \otimes \varphi_f^{\mathrm{new}} \in S(V(\mathbb{A})^2) \otimes \Omega^2(D) \quad \text{for } \mathbf{X} \in V(\mathbb{A})^2, z \in D.$$

Following (2.11), the theta series in this case is given by

$$\theta(g', \varphi_f^{\mathrm{new}}, z) = \sum_{\mathbf{X} \in V(\mathbb{Q})^2} \omega(g')\varphi(\mathbf{X}, z) \quad \text{for } g' \in \mathrm{Sp}_4(\mathbb{A})$$

which defines a closed differential 2-form on $\overline{\Gamma}_0(2mN)\backslash D$.

Following (2.12), the theta lifting of $\mathcal{F}$, which is a holomophic Siegel modular form of weight 2, is given by

$$\Theta_\varphi(\eta_\mathcal{F})(g') = \int_{\Gamma\backslash D} \eta_\mathcal{F}(z) \wedge \theta(g', \varphi_f^{\mathrm{new}}, z)$$

where $\Gamma = \overline{\Gamma}_0(\mathfrak{f}\mathfrak{n}) \cap \Gamma_0(\mathfrak{n}) = \overline{\Gamma}_0(\mathfrak{f}\mathfrak{n})$ when $d \equiv 1 \bmod 4$ or $\Gamma = \overline{\Gamma}_0(\mathfrak{f}\mathfrak{n}\mathfrak{q}_2) \cap \Gamma_0(\mathfrak{n}) = \overline{\Gamma}_0(\mathfrak{f}\mathfrak{n}\mathfrak{q}_2)$ when $d \equiv 2, 3 \bmod 4$. By Lemma 4.3.3, Lemma 4.3.6, Lemma 4.3.9 and

Lemma 4.3.14, we can determine that it has level

$$\mathcal{L}_{m,\mathfrak{n}} = \left\{ \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathrm{Sp}_4(\mathbb{Z}) : A \in \begin{pmatrix} \mathbb{Z} & n_1 N(\mathfrak{n})\mathbb{Z} \\ n_1 N(\mathfrak{n})\mathbb{Z} & \mathbb{Z} \end{pmatrix}, \right.$$
$$\left. B \in M_2(n_2 m|d|N(\mathfrak{n})\mathbb{Z}), C \in M_2(m^3 d^2 N(\mathfrak{n})\mathbb{Z}) \right\}$$

with

$$\begin{cases} n_1 = n_2 = 1, & \text{if } d \equiv 1 \bmod 4 \\ n_1 = 2, n_2 = 2^4, & \text{if } d \equiv 2,3 \bmod 4. \end{cases}$$

Recall

$$\Omega_\beta = \left\{ (\mathbf{x}_1, \mathbf{x}_2) \in V(\mathbb{Q})^2 : \frac{1}{2} \begin{pmatrix} (\mathbf{x}_1, \mathbf{x}_1) & (\mathbf{x}_1, \mathbf{x}_2) \\ (\mathbf{x}_1, \mathbf{x}_2) & (\mathbf{x}_2, \mathbf{x}_2) \end{pmatrix} = \beta \right\}.$$

By Theorem 2.4.5, the Fourier coefficient of the theta lifting $\Theta_\varphi(\eta_\mathcal{F})$ at $\beta > 0$ is given by

$$\begin{aligned} C_{\Theta_\varphi(\eta_\mathcal{F}),\beta} &= \sum_{(\mathbf{x}_1,\mathbf{x}_2) \in \Gamma \backslash \Omega_\beta} \varphi_f^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) \int_{C_{U(\mathbf{x}_1,\mathbf{x}_2)}} \eta_\mathcal{F} \\ &= \sum_{[\kappa_i] \in \Gamma \backslash \mathbb{P}^1(F)} \sum_{(\mathbf{x}_1,\mathbf{x}_2) \in \Gamma_{\kappa_i} \backslash \Omega_{\beta,\kappa_i,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) \int_{C_{U(\mathbf{x}_1,\mathbf{x}_2)}} \eta_\mathcal{F} \end{aligned} \tag{4.22}$$

where the second equality is the consequence of Proposition 4.2.4. For simplicity we will denote $C_{\Theta_{\varphi^{\mathrm{new}}}(\eta_\mathcal{F}),\beta} := I = \sum_{[\kappa_i] \in \Gamma \backslash \mathbb{P}^1(F)} I_{\kappa_i}$ where

$$I_{\kappa_i} = \sum_{(\mathbf{x}_1,\mathbf{x}_2) \in \Gamma_{\kappa_i} \backslash \Omega_{\beta,\kappa_i,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) \int_{C_{U(\mathbf{x}_1,\mathbf{x}_2)}} \eta_\mathcal{F}. \tag{4.23}$$

Similar to what we have done in Section 3.3, we will first express I_∞ in terms of the twisted L-value $L(\mathcal{F}, \chi_\mathfrak{m}, 1)$ in subsection 4.4.1 and then use Atkin-Lehner operators to calculate I_{κ_i} for $\kappa_i \neq \infty$ in subsection 4.4.2.

Remark 4.4.2. We will describe how to choose the Gram matrix β for which we will show that $C_{\Theta_\varphi(\eta_\mathcal{F}),\beta}$ is non-vanishing.

- Let $\det \beta \in -d(\mathbb{Q}^\times)^2$. Then for $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_\beta$, $U(\mathbf{x}_1, \mathbf{x}_2)^\perp$ is split over $\mathbb{Q}$ due to Proposition 4.2.3 and $U(\mathbf{x}_1, \mathbf{x}_2)^\perp$ has signature (1,1). The same arguments as in the proof of Lemma 3.1.3 show that the stabilizer $\Gamma_U \subset \Gamma$ of $U = U(\mathbf{x}_1, \mathbf{x}_2)$ is trivial if $U^\perp$ is split over $\mathbb{Q}$.
- For $(\mathbf{x}_1, \mathbf{x}_2) = \left(\left(\begin{smallmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{smallmatrix} \right), \left(\begin{smallmatrix} a_2 & b_2 \\ \bar{b}_2 & 0 \end{smallmatrix} \right) \right) \in \Omega_\beta, \infty, +$, we have

$$\beta = \begin{pmatrix} \beta_{11} & \beta_{12} \\ \beta_{21} & \beta_{22} \end{pmatrix} = \begin{pmatrix} b_1 \bar{b}_1 & \frac{1}{2}(b_1 \bar{b}_2 + \bar{b}_1 b_2) \\ \frac{1}{2}(b_1 \bar{b}_2 + \bar{b}_1 b_2) & b_2 \bar{b}_2 \end{pmatrix}.$$

We want this pair to satisfy the condition $\dagger$ as in Remark 4.2.6. This will allow us to apply Lemma 4.2.5 to deduce that the corresponding cusp $z_{U(\mathbf{x}_1,\mathbf{x}_2)}$ runs through all the representatives in $\mathfrak{f}^{-1}/\mathcal{O}$. For the non-vanishing of $\varphi_q^{\chi_{\mathfrak{m}}}$ at split or inert q dividing $\mathfrak{f}$, we only count $(\mathbf{x}_1,\mathbf{x}_2)$ such that b_i $(i=1,2)$ is divisible by m. Via imposing conditions on β itself, we can achieve that for any pair $(\mathbf{x}_1,\mathbf{x}_2)$ in $\Omega_{\beta,\infty,+}$ the assumption $\dagger$ as in Remark 4.2.6 holds. Explicit examples of β will be given at the end of the subsection 4.4.1, see from Example 4.4.6 to Example 4.4.11.

Assume that β is given as in above Remark 4.4.2. For $(\mathbf{x}_1,\mathbf{x}_2)$ in $\Omega_{\beta,\infty,+}$, we have

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix} \text{ if } d \equiv 1 \tag{4.24}$$

or

$$\begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{2} m \begin{pmatrix} x & y \\ z & w \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix} \text{ if } d \equiv 2,3 \tag{4.25}$$

with $x,y,z,w\in\mathbb{Z}$ and $xw-yz=\pm 1$. We want to find out if there is another pair $(\mathbf{y}_1,\mathbf{y}_2)$ in $\Omega_{\beta,\infty,+}$ such that it gives rise to the same cycle D_U as that generated by $(\mathbf{x}_1,\mathbf{x}_2)$.

Assume that such a pair $(\mathbf{y}_1,\mathbf{y}_2)$ exists in $\Omega_{\beta,\infty,+}$. For $U(\mathbf{x}_1,\mathbf{x}_2)=U(\mathbf{y}_1,\mathbf{y}_2)$ we consider an element $\sigma=\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)\in \mathrm{GL}_2(\mathbb{Q})$ such that

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{pmatrix} = \begin{pmatrix} \mathbf{y}_1 \\ \mathbf{y}_2 \end{pmatrix}. \tag{4.26}$$

To make $\langle u_\infty,\mathbf{x}_1,\mathbf{x}_2,u_{\kappa_j}\rangle$ and $\langle u_\infty,\mathbf{y}_1,\mathbf{y}_2,u_{\kappa_j}\rangle$ represent the same orientation, we need $\sigma\in\mathrm{GL}_2^+(\mathbb{Q})$. Additionally the Gram matrix β's corresponding to $(\mathbf{x}_1,\mathbf{x}_2)$ and $(\mathbf{y}_1,\mathbf{y}_2)$ must be identical.

Expressing $(\mathbf{y}_i,\mathbf{y}_j)$ in terms of $(\mathbf{x}_i,\mathbf{x}_j)$ and using bilinearity, we have

$$\begin{aligned}
(\mathbf{y}_1,\mathbf{y}_1) &= a^2(\mathbf{x}_1,\mathbf{x}_1)+2ab(\mathbf{x}_1,\mathbf{x}_2)+b^2(\mathbf{x}_2,\mathbf{x}_2),\\
(\mathbf{y}_2,\mathbf{y}_2) &= c^2(\mathbf{x}_1,\mathbf{x}_1)+2cd(\mathbf{x}_1,\mathbf{x}_2)+d^2(\mathbf{x}_2,\mathbf{x}_2),\\
(\mathbf{y}_1,\mathbf{y}_2) &= ac(\mathbf{x}_1,\mathbf{x}_1)+(ad+bc)(\mathbf{x}_1,\mathbf{x}_2)+bd(\mathbf{x}_2,\mathbf{x}_2).
\end{aligned}$$

Consider that $\det\beta$ is preserved; more explicitly,

$$\begin{aligned}\det\beta =& \det((\mathbf{y}_i,\mathbf{y}_j)) = (\mathbf{y}_1,\mathbf{y}_1)(\mathbf{y}_2,\mathbf{y}_2) - (\mathbf{y}_1,\mathbf{y}_2)^2\\ =& a^2c^2(\mathbf{x}_1,\mathbf{x}_1)^2 + b^2d^2(\mathbf{x}_2,\mathbf{x}_2)^2 + 4abcd(\mathbf{x}_1,\mathbf{x}_2)^2 + (a^2d^2+b^2c^2)(\mathbf{x}_1,\mathbf{x}_1)(\mathbf{x}_2,\mathbf{x}_2)\\ &+ 2ac(ad+bc)(\mathbf{x}_1,\mathbf{x}_1)(\mathbf{x}_1,\mathbf{x}_2) + 2bd(ad+bc)(\mathbf{x}_1,\mathbf{x}_2)(\mathbf{x}_2,\mathbf{x}_2)\\ &- a^2c^2(\mathbf{x}_1,\mathbf{x}_1)^2 - b^2d^2(\mathbf{x}_2,\mathbf{x}_2)^2 - (ad+bc)^2(\mathbf{x}_1,\mathbf{x}_2)^2 - 2abcd(\mathbf{x}_1,\mathbf{x}_1)(\mathbf{x}_2,\mathbf{x}_2)\\ &- 2ac(ad+bc)(\mathbf{x}_1,\mathbf{x}_1)(\mathbf{x}_1,\mathbf{x}_2) - 2bd(ad+bc)(\mathbf{x}_1,\mathbf{x}_2)(\mathbf{x}_2,\mathbf{x}_2)\\ =& (\det\sigma)^2\det((\mathbf{x}_i,\mathbf{x}_j)) = \det((\mathbf{x}_i,\mathbf{x}_j)).\end{aligned}$$

Since σ has positive determinant we know that $\sigma\in\mathrm{SL}_2(\mathbb{Q})$.

Moreover, to preserve β the following identities must hold:

$$(\mathbf{x}_1,\mathbf{x}_1) = a^2(\mathbf{x}_1,\mathbf{x}_1) + 2ab(\mathbf{x}_1,\mathbf{x}_2) + b^2(\mathbf{x}_2,\mathbf{x}_2), \tag{4.27}$$

$$(\mathbf{x}_2,\mathbf{x}_2) = c^2(\mathbf{x}_1,\mathbf{x}_1) + 2cd(\mathbf{x}_1,\mathbf{x}_2) + d^2(\mathbf{x}_2,\mathbf{x}_2), \tag{4.28}$$

$$(\mathbf{x}_1,\mathbf{x}_2) = ac(\mathbf{x}_1,\mathbf{x}_1) + (ad+bc)(\mathbf{x}_1,\mathbf{x}_2) + bd(\mathbf{x}_2,\mathbf{x}_2). \tag{4.29}$$

As $\det\sigma = ad-bc=1$, we can rewrite (4.29) as

$$ac(\mathbf{x}_1,\mathbf{x}_1) + 2bc(\mathbf{x}_1,\mathbf{x}_2) + bd(\mathbf{x}_2,\mathbf{x}_2) = 0. \tag{4.30}$$

We will describe σ in different cases in the following.

(I) Let b be 0. From (4.27) we know that $a^2=1$, and from (4.29) we have that $(\mathbf{x}_1,\mathbf{x}_2) = ac(\mathbf{x}_1,\mathbf{x}_1)+(\mathbf{x}_1,\mathbf{x}_2)$ which implies $c=0$. So $\sigma=\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. In the same way, if $c=0$ then $\sigma=\pm\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$.

(II) Assume that $bc\neq 0$. Substituting $(\mathbf{x}_1,\mathbf{x}_2)$ in (4.27), we have

$$(\mathbf{x}_1,\mathbf{x}_1) = a^2(\mathbf{x}_1,\mathbf{x}_1) + b^2(\mathbf{x}_2,\mathbf{x}_2) - \frac{a}{c}(ac(\mathbf{x}_1,\mathbf{x}_1)+bd(\mathbf{x}_2,\mathbf{x}_2))$$

which is simplified to be

$$c(\mathbf{x}_1,\mathbf{x}_1) + b(\mathbf{x}_2,\mathbf{x}_2) = 0. \tag{4.31}$$

Combining (4.30) and (4.31) we have

$$ac(\mathbf{x}_1,\mathbf{x}_1) + 2bc(\mathbf{x}_1,\mathbf{x}_2) - cd(\mathbf{x}_1,\mathbf{x}_1) = 0$$

and then $d = a + 2b\frac{(\mathbf{x}_1,\mathbf{x}_2)}{(\mathbf{x}_1,\mathbf{x}_1)}$. As $ad-bd=1$, we have

$$a^2 + 2ab\frac{(\mathbf{x}_1,\mathbf{x}_2)}{(\mathbf{x}_1,\mathbf{x}_1)} + b^2\frac{(\mathbf{x}_2,\mathbf{x}_2)}{(\mathbf{x}_1,\mathbf{x}_1)} = 1. \tag{4.32}$$

Set $\mathbf{x}_1 = \begin{pmatrix} a_1 & b_1\\ \bar{b}_1 & d_1\end{pmatrix}$, $\mathbf{x}_2 = \begin{pmatrix} a_2 & b_2\\ \bar{b}_2 & d_2\end{pmatrix}$, $\mathbf{y}_1 = \begin{pmatrix} a_1' & b_1'\\ \bar{b}_1' & d_1'\end{pmatrix}$ and $\mathbf{y}_2 = \begin{pmatrix} a_2' & b_2'\\ \bar{b}_2' & d_2'\end{pmatrix}$. Combining (4.24) and (4.26), we can rewrite

$$b_1' = ab_1 + bb_2 = m((ax+bz)+(ay+bw)\omega)$$

or combining (4.25) and (4.26)

$$b_1' = ab_1 + bb_2 = \frac{1}{2}((ax+bz) + (ay+bw)\omega).$$

Then we need $ax+bz = \mu \in \mathbb{Z}$ and $ay+bw = \nu \in \mathbb{Z}$. Solving these two equations we get

$$a = \frac{\mu\delta - \nu\gamma}{\alpha\delta - \beta\gamma} \in \mathbb{Z} \quad \text{and} \quad b = \frac{\nu\alpha - \mu\beta}{\alpha\delta - \beta\gamma} \in \mathbb{Z}.$$

Similarly, we can deduce that $c, d \in \mathbb{Z}$ when treating b_2'. Therefore, the linear transform $\sigma \in \mathrm{SL}_2(\mathbb{Q})$ on $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_{\beta,\kappa_i,+}$ generates the same cycle, but the Schwartz function on $(\mathbf{y}_1, \mathbf{y}_2)$ vanishes if $\sigma \notin \mathrm{SL}_2(\mathbb{Z})$.

For particular choices of β we get limited possibilities of above σ. We can rewrite (4.32) as

$$\left(a + b\frac{(\mathbf{x}_1, \mathbf{x}_2)}{(\mathbf{x}_1, \mathbf{x}_1)}\right)^2 + b^2 \frac{(\mathbf{x}_1, \mathbf{x}_1)(\mathbf{x}_2, \mathbf{x}_2) - (\mathbf{x}_1, \mathbf{x}_2)^2}{(\mathbf{x}_1, \mathbf{x}_1)^2} = 1. \tag{4.33}$$

(II.1) If $(\mathbf{x}_1, \mathbf{x}_2) = 0$ and $(\mathbf{x}_1, \mathbf{x}_1) = (\mathbf{x}_2, \mathbf{x}_2)$, then $a^2 = 0$ as $bc \neq 0$ by our assumption. So in this case $\sigma = \pm \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$.

(II.2) If $(\mathbf{x}_1, \mathbf{x}_2) = 0$ and $(\mathbf{x}_1, \mathbf{x}_1) < (\mathbf{x}_2, \mathbf{x}_2)$, i.e. $\frac{\det \beta}{(\mathbf{x}_1, \mathbf{x}_1)^2} > 1$, then there is no such a σ that $bc \neq 0$.

(II.3) If $(\mathbf{x}_1, \mathbf{x}_2) \neq 0$ and $\frac{(\mathbf{x}_1, \mathbf{x}_1)(\mathbf{x}_2, \mathbf{x}_2) - (\mathbf{x}_1, \mathbf{x}_2)^2}{(\mathbf{x}_1, \mathbf{x}_1)^2} > 1$, then b has to be 0 which is a contradiction to $bc \neq 0$.

Remark 4.4.3. The possibilities of σ in (4.26) will determine the constant μ_β in Proposition 4.4.5. After the whole treatment of this section 4.4, we will see that this μ_β does not effect the non-vanishing of our theta liftings since it appears in the Fourier coefficient as a non-zero multiplier. In Example from 4.4.6 to 4.4.11, we will show how to get the exact values of μ_β.

4.4.1 On cycles through ∞

We will first calculate the part I_∞ corresponding to the cusp ∞ as in (4.23). We pick a fundamental domain for $\Gamma_\infty \backslash D_U$ and integrate with respect to the cycle. Since we are integrating along a vertical path with z-coordinate constant, we can ignore dz and $d\bar{z}$. We obtain

$$I_\infty = \sum_{(\mathbf{x}_1, \mathbf{x}_2) \in \Gamma_\infty \backslash \Omega_{\beta,\infty,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) \int_{(z,r) \in C_{U(\mathbf{x}_1, \mathbf{x}_2)}} \frac{1}{2} \mathcal{F}_1(z, r) dr. \tag{4.34}$$

Lemma 4.4.4. *For $(\mathbf{x}_1, \mathbf{x}_2) \in \Gamma_\infty \backslash \Omega_{\beta,\infty,+}$, we have*

$$\varphi_f^{\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) = \lambda_{\mathfrak{m},\mathfrak{n}} \varphi_f(\mathbf{x}_1, \mathbf{x}_2)$$

where

$$\lambda_{\mathfrak{m},\mathfrak{n}} = \prod_{q_1|m|d|} [\overline{\Gamma}_0(q_1) : \overline{\Gamma}(q_1)] \prod_{\mathfrak{q}_2 \text{ above ramified } 2} [\overline{\Gamma}_0(\mathfrak{q}_2) : \overline{\Gamma}(\mathfrak{q}_2)] \prod_{\mathfrak{q}_3|\mathfrak{n}} [\overline{\Gamma}_0(\mathfrak{q}_3) : \overline{\Gamma}(\mathfrak{q}_3)].$$

Proof. Note that any pair $(\mathbf{x}_1, \mathbf{x}_2)$ in $\Omega_{\beta,\infty}$ is of form $\left(\left(\begin{smallmatrix} a_1 & b_1 \\ \bar{b}_1 & 0 \end{smallmatrix}\right), \left(\begin{smallmatrix} a_2 & b_2 \\ \bar{b}_2 & 0 \end{smallmatrix}\right)\right)$. Recall from Section 1.3, for $a \in \mathcal{O}^\times$ satisfying $((a), \mathfrak{f}) = 1$ we have

$$\prod_{v|\mathfrak{f}} \widetilde{\chi}_v(a_v) = \prod_{v\nmid\mathfrak{f}} \widetilde{\chi}_v^{-1}(a_v) = \chi^{-1}((a)).$$

Then, for our choice of finite Schwartz function φ_f, we have

$$\begin{aligned} \varphi_f(\mathbf{x}_1, \mathbf{x}_2) &= \prod_{v|\mathfrak{f}} \widetilde{\chi}_{\mathfrak{m},v}\left(\frac{a_2 b_1 - a_1 b_2}{m}\right) \\ &= \prod_{v\nmid\mathfrak{f}} \widetilde{\chi}_{\mathfrak{m},v}^{-1}\left(\frac{a_2 b_1 - a_1 b_2}{m}\right) = \chi^{-1}\left(\left(\frac{a_2 b_1 - a_1 b_2}{m}\right)\right) \end{aligned}$$

or

$$\varphi_f(\mathbf{x}_1, \mathbf{x}_2) = \chi^{-1}\left(\left(\frac{2(a_2 b_1 - a_1 b_2)}{m}\right)\right).$$

Let q be the split prime dividing m. Consider the representative

$$\gamma = (\gamma_1, \gamma_2) = \left(\begin{pmatrix} x_{\gamma,1} & y_{\gamma,1} \\ 0 & x_{\gamma,1}^{-1} \end{pmatrix}, \begin{pmatrix} x_{\gamma,2} & y_{\gamma,2} \\ 0 & x_{\gamma,2}^{-1} \end{pmatrix}\right)$$

for $\overline{\Gamma}_0(q)/\overline{\Gamma}(q)$ with $[x_{\gamma,j}] \in (\mathcal{O}/(q))^\times$ and $[y_{\gamma,j}] \in \mathcal{O}/(q)$. By the computation in the proof of Lemma 4.3.4, we can observe that

$$\begin{aligned} \varphi_q^{\chi_\mathfrak{m},\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) &= \sum_{[\gamma]\in\overline{\Gamma}_0(q)/\overline{\Gamma}(q)} (\widetilde{\chi}_{\mathfrak{m},\mathfrak{q}} \widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}})(x_{\gamma,1}^{-2})(\widetilde{\chi}_{\mathfrak{m},\mathfrak{q}} \widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}})\left(\frac{a_2 b_1 - a_1 b_2}{m}\right) \\ &= \sum_{[\gamma]\in\overline{\Gamma}_0(q)/\overline{\Gamma}(q)} (\widetilde{\chi}_{\mathfrak{m},\mathfrak{q}} \widetilde{\chi}_{\mathfrak{m},\bar{\mathfrak{q}}})\left(\frac{a_2 b_1 - a_1 b_2}{m}\right) \\ &= [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \varphi_q^{\chi_\mathfrak{m}}(\mathbf{x}_1, \mathbf{x}_2) \end{aligned}$$

Similarly we have for inert $q|m$

$$\varphi_q^{\chi_\mathfrak{m},\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) = [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \varphi_q^{\chi_\mathfrak{m}}(\mathbf{x}_1, \mathbf{x}_2),$$

and for ramified prime q with $(q) = \mathfrak{q}^2$

$$\varphi_q^{\chi_\mathfrak{m},\mathrm{new}}(\mathbf{x}_1, \mathbf{x}_2) = [\overline{\Gamma}_0(\mathfrak{q}) : \overline{\Gamma}(\mathfrak{q})] \varphi_q^{\chi_\mathfrak{m}}(\mathbf{x}_1, \mathbf{x}_2).$$

At the place $q|N(\mathfrak{n})$ we will show that

$$\varphi_q^{\mathfrak{n},\mathrm{new}} = [\overline{\Gamma}_0(q) : \overline{\Gamma}(q)] \varphi_q^{\mathfrak{n}}.$$

For q split, set $\mathbf{x}_i = \begin{pmatrix} a_i & b_i \\ c_i & 0 \end{pmatrix}$ and compute

$$w(1,\gamma)\cdot(\mathbf{x}_1,\mathbf{x}_2) = \left(\begin{pmatrix} * & x_{\gamma_1}^{-1}x_{\gamma_2}^{-1}b_1 \\ x_{\gamma_1}x_{\gamma_2}c_1 & 0 \end{pmatrix}, \begin{pmatrix} * & x_{\gamma_1}^{-1}x_{\gamma_2}^{-1}b_2 \\ x_{\gamma_1}x_{\gamma_2}c_2 & 0 \end{pmatrix} \right).$$

It is not hard to observe that the condition on $b_1c_2+b_2c_1$ is preserved as $x_{\gamma,j}\in\mathbb{Z}_q^\times$. It follow that

$$w(1,\gamma)\varphi_q(\mathbf{x}_1,\mathbf{x}_2) = \varphi_q(\mathbf{x}_1,\mathbf{x}_2)$$

which implies the assertion. For q inert, set $\mathbf{x}_i = \begin{pmatrix} b_i & a_i\sqrt{d} \\ 0 & \bar{b}_i \end{pmatrix}$ and compute

$$\gamma^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2) = \left(\begin{pmatrix} x_\gamma^{-1}\bar{x}_\gamma b_1 & * \\ 0 & x_\gamma\bar{x}_\gamma^{-1}\bar{b}_1 \end{pmatrix}, \begin{pmatrix} x_\gamma^{-1}\bar{x}_\gamma b_2 & * \\ 0 & x_\gamma\bar{x}_\gamma^{-1}\bar{b}_2 \end{pmatrix} \right).$$

Again the condition on $b_1\bar{b}_2+\bar{b}_1b_2$ is preserved and so the assertion follows. The case at ramified 2 can be treated similarly.

Now we have proven the lemma. □

Proposition 4.4.5. *Assume that the Gram matrix β is chosen so that the condition † in Lemma 4.2.5 is satisfied. Then we can calculate*

$$I_\infty = \frac{\mu_\beta\lambda_{\mathfrak{m},\mathfrak{n}}L(\mathcal{F},\chi_\mathfrak{m},1)}{2A(1,\chi_\mathfrak{m},1)} \tag{4.35}$$

where μ_β is a non-zero integer depending on β as stated in Remark 4.4.3 and $A(1,\chi_\mathfrak{m},1)$ is given explicitly in Theorem 1.4.12.

Proof. By the above lemma, we can express

$$\begin{aligned} I_\infty =& \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}} \varphi_f^{\text{new}}(\mathbf{x}_1,\mathbf{x}_2)\int_{(z,r)\in C_{U(\mathbf{x}_1,\mathbf{x}_2)}}\frac{1}{2}\mathcal{F}_1(z,r)dr \\ =&\lambda_{\mathfrak{m},\mathfrak{n}}\sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}} \varphi_f(\mathbf{x}_1,\mathbf{x}_2)\int_{(z,r)\in C_{U(\mathbf{x}_1,\mathbf{x}_2)}}\frac{1}{2}\mathcal{F}_1(z,r)dr. \end{aligned}$$

Under our assumption on β, by Lemma 4.2.5 we have

$$I_\infty = \mu_\beta\lambda_{\mathfrak{m},\mathfrak{n}}\sum_{[z_U]\in\mathfrak{f}^{-1}/\mathcal{O},(z_U\mathfrak{f},\mathfrak{f})=1}\chi_\mathfrak{m}^{-1}(z_U\mathfrak{f})\int_0^\infty\frac{1}{2}\mathcal{F}_1(z,r)dr$$

where μ_β is a non-zero integer depending on the possibilities of σ as discussed in Remark 4.4.3. At last, by Theorem 1.4.12 with $n=1$, we can compute

$$I_\infty = \mu_\beta\lambda_{\mathfrak{m},\mathfrak{n}}\sum_{[z_U]\in\mathfrak{f}^{-1}/\mathcal{O},(z_U\mathfrak{f},\mathfrak{f})=1}\chi_\mathfrak{m}^{-1}(z_U\mathfrak{f})\int_0^\infty\frac{1}{2}\mathcal{F}_1(z,r)dr = \frac{\mu_\beta\lambda_{\mathfrak{m},\mathfrak{n}}L(\mathcal{F},\chi_\mathfrak{m},1)}{2A(1,\chi_\mathfrak{m},1)}.$$

□

For a diagonal Gram matrix β, the pair $(\mathbf{x}_1, \mathbf{x}_2) \in \Omega_{\beta,\infty,+}$ has $b_1\bar{b}_2 + \bar{b}_1 b_2 = 0$. It follows that $\varphi_q^{\mathfrak{n}}$ is vanishing on such a pair $(\mathbf{x}_1, \mathbf{x}_2)$. So, for the non-vanishing of I_∞, the Gram matrix β being diagonal is ruled out of our consideration. In the following we give some examples of β satisfying the condition † (as promised in Remark 4.4.2) for which I_∞ can be expressed in terms of $L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)$.

Example 4.4.6. Let $F = \mathbb{Q}(\sqrt{-3})$ with $d_F = -3$ and $\mathcal{O} = \mathbb{Z}[\omega]$ with $\omega = \frac{1+\sqrt{-3}}{2}$. Suppose that

$$\beta = \begin{pmatrix} b_1\bar{b}_1 & \frac{1}{2}(b_1\bar{b}_2 + \bar{b}_1 b_2) \\ \frac{1}{2}(b_1\bar{b}_2 + \bar{b}_1 b_2) & b_2\bar{b}_2 \end{pmatrix} = \begin{pmatrix} m^2 & \frac{1}{2}m^2 \\ \frac{1}{2}m^2 & m^2 \end{pmatrix}.$$

We have

$$I_\infty = \frac{4\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, \chi_{\mathfrak{m}}, 1)}.$$

Proof. For the non-vanishing of $\varphi_q^{\chi_{\mathfrak{m}}}$, we need $m|b_i$. Solving $b_i\bar{b}_i = m^2$, we must take $b_i = \pm m$, $\pm m\omega$ or $\pm m\bar{\omega}$. Observing

$$(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2}(b_1\bar{b}_2 + \bar{b}_1 b_2) = \frac{1}{2}m^2,$$

we can determine b_i with the condition † as in Proposition 4.2.5 satisfied:

$$\begin{cases} b_1 = m \\ b_2 = m\omega \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

$$\begin{cases} b_1 = m \\ b_2 = m\bar{\omega} \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} 1 & 0 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

$$\begin{cases} b_1 = -m \\ b_2 = -m\omega \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

$$\begin{cases} b_1 = -m \\ b_2 = -m\bar{\omega} \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} -1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

$$\begin{cases} b_1 = m\omega \text{ or } m\bar{\omega} \\ b_2 = m, \end{cases} \quad \begin{cases} b_1 = -m\omega \text{ or } -m\bar{\omega} \\ b_2 = -m. \end{cases}$$

We have seen in Lemma 4.2.7 that the sign of $\mathrm{Im}(b_1\bar{b}_2)$ determines the orientation ε of $T_Z(D_U)$ via $\varepsilon\mathrm{Im}(b_1\bar{b}_2) > 0$. If the cycle D_U integrated over is directed from the cusp on the complex plane to the cusp ∞, we need $\varepsilon < 0$ which implies $\mathrm{Im}(b_1\bar{b}_2) < 0$. We will list all pairs in $\Gamma_\infty \backslash \Omega_{\beta,\infty,+}$ with $\mathrm{Im}(b_1\bar{b}_2) < 0$.

First we consider one pair

$$(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}) = \left(\begin{pmatrix} a_{1,1} & m \\ m & 0 \end{pmatrix}, \begin{pmatrix} a_{1,2} & m\omega \\ m\bar{\omega} & 0 \end{pmatrix} \right)$$

with $a_{1,1}, a_{1,2} \in \mathbb{Z}$ which gives rise to the cycle $D_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})}$ directed from the cusp $z_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} = \frac{a_{1,1}\omega - a_{1,2}}{m\sqrt{-3}} \in F$ to the cusp ∞. Rewriting (4.32) as $a^2 - ab + b^2 = 1$, we have either $a^2 = 1, b^2 = 0$ or $a^2 = 0, b^2 = 1$ and then $\sigma = \pm\left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$ or $\pm\left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. So the following four pairs give rise to the same cycle $D_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})}$:

$$(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}), (-\mathbf{x}_{1,1}, -\mathbf{x}_{1,2}), (\mathbf{x}_{1,2}, -\mathbf{x}_{1,1}), (-\mathbf{x}_{1,2}, \mathbf{x}_{1,1}). \tag{4.36}$$

Lemma 4.2.5 tells us that for $(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}) \in \Gamma_\infty \backslash \Omega_{\beta,\infty,+}$, $z_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})}$ ranges over $\mathfrak{f}^{-1}/\mathcal{O}$ with $\mathfrak{f} = \sqrt{-3}\mathfrak{m}$.

Suppose that

$$(\mathbf{x}_{2,1}, \mathbf{x}_{2,2}) = \left(\begin{pmatrix} a_{2,1} & -m \\ -m & 0 \end{pmatrix}, \begin{pmatrix} a_{2,2} & -m\bar{\omega} \\ -m\omega & 0 \end{pmatrix} \right)$$

with $a_{2,1}, a_{2,2} \in \mathbb{Z}$ is another pair in $\Omega_{\beta,\infty,+}$ which gives rise to the cycle $D_{U(\mathbf{x}_{2,1},\mathbf{x}_{2,2})}$ directed from the cusp $z_{U(\mathbf{x}_{2,1},\mathbf{x}_{2,2})} = \frac{a_{2,1}\bar{\omega} - a_{2,2}}{m\sqrt{-3}} \in F$ to the cusp ∞. Similarly we have following pairs

$$(\mathbf{x}_{2,1}, \mathbf{x}_{2,2}), (-\mathbf{x}_{2,1}, -\mathbf{x}_{2,2}), (\mathbf{x}_{2,2}, -\mathbf{x}_{2,1}), (-\mathbf{x}_{2,2}, \mathbf{x}_{2,1}) \tag{4.37}$$

giving rise to the same cycle $D_{U(\mathbf{x}_{2,1},\mathbf{x}_{2,2})}$. Also for $(\mathbf{x}_{2,1}, \mathbf{x}_{2,2}) \in \Gamma_\infty \backslash \Omega_{\beta,\infty,+}$ we have $z_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})}$ running through $\mathfrak{f}^{-1}/\mathcal{O}$ with $\mathfrak{f} = \sqrt{-3}\mathfrak{m}$.

It is obvious that the eight pairs in (4.36) and (4.37) are not Γ_∞-equivalent since the Γ_∞-action on the pair preserves off-diagonal entries of each component of the pair. Then we can split I_∞ as

$$\begin{aligned} I_\infty = & I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} + I_{(-\mathbf{x}_{1,1},-\mathbf{x}_{1,2})} + I_{(\mathbf{x}_{1,2},-\mathbf{x}_{1,1})} + I_{(-\mathbf{x}_{1,2},\mathbf{x}_{1,1})} \\ & + I_{(\mathbf{x}_{2,1},\mathbf{x}_{2,2})} + I_{(-\mathbf{x}_{2,1},-\mathbf{x}_{2,2})} + I_{(\mathbf{x}_{2,2},-\mathbf{x}_{2,1})} + I_{(-\mathbf{x}_{2,2},\mathbf{x}_{2,1})}, \end{aligned}$$

where the subscript $(-,-)$ indicates the sum as in (4.34) over $[z_{U(-,-)}] \in \mathfrak{f}^{-1}/\mathcal{O}$. By Theorem 1.4.12 with $n = 1$, we can calculate

$$I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} = I_{(\mathbf{x}_{2,1},\mathbf{x}_{2,2})} = \frac{\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_\mathfrak{m}, 1)}{2A(1, \chi_\mathfrak{m}, 1)}.$$

So, in this case we have $\mu_\beta = 8$ and then we can deduce that

$$I_\infty = 8 \cdot I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} = \frac{4\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_\mathfrak{m}, 1)}{A(1, \chi_\mathfrak{m}, 1)}.$$

□

Example 4.4.7. Let $F = \mathbb{Q}(\sqrt{d})$ with $d \equiv 1 \pmod 4$ and $d \neq -3$ in which case $d_F = d$ and $\mathcal{O} = \mathbb{Z}[\omega]$ with $\omega = \frac{1+\sqrt{d}}{2}$. Suppose that

$$\beta = \begin{pmatrix} m^2 & \frac{1}{2}m^2 \\ \frac{1}{2}m^2 & \frac{1-d}{4}m^2 \end{pmatrix}.$$

We have

$$I_\infty = \frac{2\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_\mathfrak{m}, 1)}{A(1, \chi_\mathfrak{m}, 1)}.$$

Proof. For the non-vanishing of $\varphi_q^{\chi_\mathfrak{m}}$ and by solving $(\mathbf{x}_i, \mathbf{x}_j)$, we can determine that

$$\begin{cases} b_1 = m \\ b_2 = m\omega \text{ or } m\bar{\omega}, \end{cases} \begin{cases} b_1 = -m \\ b_2 = -m\omega \text{ or } -m\bar{\omega} \end{cases}$$

with the condition † holding.

Again in this case we need $\mathrm{Im}(b_1\bar{b}_2) < 0$. Suppose that

$$(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}) = \left(\begin{pmatrix} a_{1,1} & m \\ m & 0 \end{pmatrix}, \begin{pmatrix} a_{1,2} & m\omega \\ m\bar{\omega} & 0 \end{pmatrix} \right)$$

gives rise to the cycle $D_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})}$ directed from $z_{U(\mathbf{x}_{1,1},\mathbf{x}_{1,2})}$ to the cusp ∞ and that

$$(\mathbf{x}_{2,1}, \mathbf{x}_{2,2}) = \left(\begin{pmatrix} a_{2,1} & -m \\ -m & 0 \end{pmatrix}, \begin{pmatrix} a_{2,2} & -m\omega \\ -m\bar{\omega} & 0 \end{pmatrix} \right)$$

to the cycle $D_{U(\mathbf{x}_{2,1},\mathbf{x}_{2,2})}$ directed from $z_{U(\mathbf{x}_{2,1},\mathbf{x}_{2,2})}$ to the cusp ∞. Combing (I) and (II.3), we have $\sigma = \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. So, in this case we have $\mu_\beta = 4$ and then we can calculate

$$\begin{aligned} I_\infty =& I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} + I_{(-\mathbf{x}_{1,1},-\mathbf{x}_{1,2})} + I_{(\mathbf{x}_{2,1},\mathbf{x}_{2,2})} + I_{(-\mathbf{x}_{2,1},-\mathbf{x}_{2,2})} \\ =& 4I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} = \frac{2\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_\mathfrak{m}, 1)}{A(1, \chi_\mathfrak{m}, 1)}. \end{aligned}$$

□

Example 4.4.8. Let $F = \mathbb{Q}(\sqrt{d})$ with $d \equiv 2, 3 \pmod 4$ and $d \neq -1$ in which case $d_F = 4d$ and $\mathcal{O} = \mathbb{Z}[d]$. Let

$$\beta = \begin{pmatrix} \frac{1}{4}m^2 & \frac{1}{4}nm^2 \\ \frac{1}{4}nm^2 & \frac{1}{4}(n^2 - d)m^2 \end{pmatrix}$$

with $n \in \mathbb{Z}$ coprime to $2\mathfrak{n}$ (for the non-vanishing of $\varphi_q^\mathfrak{n}$ and φ_2). We have

$$I_\infty = \frac{2\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_\mathfrak{m}, 1)}{A(1, \chi_\mathfrak{m}, 1)}.$$

Proof. For the non-vanishing of φ_f, via solving

$$\begin{cases} (\mathbf{x}_1, \mathbf{x}_1) = \frac{1}{4} \\ m^2(\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2}(b_1\bar{b}_2 + \bar{b}_1 b_2) = \frac{1}{4}nm^2 \\ (\mathbf{x}_2, \mathbf{x}_2) = b_2\bar{b}_2 = \frac{1}{4}(n^2 - d)m^2 \end{cases}$$

we can determine that

$$\begin{cases} b_1 = \frac{1}{2}m \\ b_2 = \frac{1}{2}(n+\sqrt{d})m \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{2}m \begin{pmatrix} 1 & 0 \\ n & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{d} \end{pmatrix}$$

$$\begin{cases} b_1 = \frac{1}{2}m \\ b_2 = \frac{1}{2}(n-\sqrt{d})m \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{2}m \begin{pmatrix} 1 & 0 \\ n & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{d} \end{pmatrix}$$

$$\begin{cases} b_1 = -\frac{1}{2}m \\ b_2 = -\frac{1}{2}(n+\sqrt{d})m \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{2}m \begin{pmatrix} -1 & 0 \\ -n & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{d} \end{pmatrix}$$

$$\begin{cases} b_1 = -\frac{1}{2}m \\ b_2 = -\frac{1}{2}(n-\sqrt{d})m \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = \frac{1}{2}m \begin{pmatrix} -1 & 0 \\ -n & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \sqrt{d} \end{pmatrix}$$

with the condition † holding.

For $\mathrm{Im}(b_1\bar{b}_2) < 0$, we consider

$$(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}) = \left(\begin{pmatrix} a_{1,1} & \frac{1}{2}m \\ \frac{1}{2}m & 0 \end{pmatrix}, \begin{pmatrix} a_{1,2} & \frac{1}{2}(n+\sqrt{d})m \\ (n-\sqrt{d})\frac{1}{2}m & 0 \end{pmatrix} \right)$$

and

$$(\mathbf{x}_{2,1}, \mathbf{x}_{2,2}) = \left(\begin{pmatrix} a_{2,1} & -\frac{1}{2}m \\ -\frac{1}{2}m & 0 \end{pmatrix}, \begin{pmatrix} a_{2,2} & \frac{1}{2}(n-\sqrt{d})m \\ \frac{1}{2}(n+\sqrt{d})m & 0 \end{pmatrix} \right)$$

By (I) and (II.3), we have $\sigma = \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. In this case we have $\mu_\beta = 4$ and then we get

$$\begin{aligned} I_\infty =& I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} + I_{(-\mathbf{x}_{1,1},-\mathbf{x}_{1,2})} + I_{(\mathbf{x}_{2,1},\mathbf{x}_{2,2})} + I_{(-\mathbf{x}_{2,1},-\mathbf{x}_{2,2})} \\ =& 4I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} = \frac{2\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, \chi_{\mathfrak{m}}, 1)}. \end{aligned}$$

□

Example 4.4.9. Let $F = \mathbb{Q}(i)$ with $d_F = -4$ and $\mathcal{O} = \mathbb{Z}[i]$. Set

$$\beta = \begin{pmatrix} \frac{1}{4}m^2 & \frac{1}{4}nm^2 \\ \frac{1}{4}nm^2 & \frac{1}{4}(n^2+1)m^2 \end{pmatrix}$$

with $1 < n \in \mathbb{Z}$ coprime to $2\mathfrak{n}$ (for the non-vanishing of $\varphi_q^{\mathfrak{n}}$ and φ_2 at ramified 2). We have

$$I_\infty = \frac{4\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, \chi_{\mathfrak{m}}, 1)}.$$

Proof. For the non-vanishing of $\varphi_q^{\chi_{\mathfrak{m}}}$, via solving

$$\begin{cases} (\mathbf{x}_1, \mathbf{x}_1) = \frac{1}{4}m^2 \\ (\mathbf{x}_1, \mathbf{x}_2) = \frac{1}{2}(b_1\bar{b}_2 + \bar{b}_1 b_2) = \frac{1}{4}nm^2 \\ (\mathbf{x}_2, \mathbf{x}_2) = b_2\bar{b}_2 = \frac{1}{4}(n^2+1)m^2 \end{cases}$$

we can determine that

$$\begin{cases} b_1 = \frac{1}{2}m \\ b_2 = \frac{1}{2}(n \pm i)m, \end{cases} \begin{cases} b_1 = \frac{1}{2}mi \\ b_2 = \frac{1}{2}(ni \pm 1)m, \end{cases} \begin{cases} b_1 = -\frac{1}{2}m \\ b_2 = -\frac{1}{2}(n \pm i)m, \end{cases} \begin{cases} b_1 = -\frac{1}{2}mi \\ b_2 = -\frac{1}{2}(ni \pm 1)m. \end{cases}$$

For $\mathrm{Im}(b_1\bar{b}_2) < 0$, we write

$$\begin{aligned} (\mathbf{x}_{1,1}, \mathbf{x}_{1,2}) &= \left(\begin{pmatrix} a_{1,1} & \frac{1}{2}m \\ \frac{1}{2}m & 0 \end{pmatrix}, \begin{pmatrix} a_{1,2} & \frac{1}{2}(n+i)m \\ \frac{1}{2}(n-i)m & 0 \end{pmatrix} \right), \\ (\mathbf{x}_{2,1}, \mathbf{x}_{2,2}) &= \left(\begin{pmatrix} a_{2,1} & -\frac{1}{2}m \\ -\frac{1}{2}m & 0 \end{pmatrix}, \begin{pmatrix} a_{2,2} & \frac{1}{2}(n-i)m \\ \frac{1}{2}(n+i)m & 0 \end{pmatrix} \right), \\ (\mathbf{x}_{3,1}, \mathbf{x}_{3,2}) &= \left(\begin{pmatrix} a_{3,1} & \frac{1}{2}mi \\ -\frac{1}{2}mi & 0 \end{pmatrix}, \begin{pmatrix} a_{3,2} & \frac{1}{2}(ni-1)m \\ \frac{1}{2}(-ni-1)m & 0 \end{pmatrix} \right), \\ (\mathbf{x}_{4,1}, \mathbf{x}_{4,2}) &= \left(\begin{pmatrix} a_{4,1} & -\frac{1}{2}mi \\ \frac{1}{2}mi & 0 \end{pmatrix}, \begin{pmatrix} a_{4,2} & -\frac{1}{2}(ni-1)m \\ \frac{1}{2}(ni+1)m & 0 \end{pmatrix} \right). \end{aligned}$$

By (I) and (II.3), we have $\sigma = \pm \left(\begin{smallmatrix} 1 & 0 \\ 0 & 1 \end{smallmatrix}\right)$. So we have $\mu_\beta = 8$ and then

$$\begin{aligned} I_\infty =& I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} + I_{(-\mathbf{x}_{1,1},-\mathbf{x}_{1,2})} + I_{(\mathbf{x}_{2,1},\mathbf{x}_{2,2})} + I_{(-\mathbf{x}_{2,1},-\mathbf{x}_{2,2})} \\ & I_{(\mathbf{x}_{3,1},\mathbf{x}_{3,2})} + I_{(-\mathbf{x}_{3,1},-\mathbf{x}_{3,2})} + I_{(\mathbf{x}_{4,1},\mathbf{x}_{4,2})} + I_{(-\mathbf{x}_{4,1},-\mathbf{x}_{4,2})} \\ =& 8I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} = \frac{4\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, \chi_{\mathfrak{m}}, 1)}. \end{aligned}$$

□

Example 4.4.10. Let $F = \mathbb{Q}(\sqrt{d})$ with $d \equiv 1 \pmod 4$ and $d \neq -3$ in which case $d_F = d$ and $\mathcal{O} = \mathbb{Z}[\omega]$ with $\omega = \frac{1+\sqrt{d}}{2}$. Set

$$\beta = \begin{pmatrix} m^2 & \frac{nm^2}{2} \\ \frac{nm^2}{2} & \frac{n^2-d}{4}m^2 \end{pmatrix}$$

with $1 < n \in \mathbb{Z}$ coprime to $q_{\mathfrak{n}}$. We have

$$I_\infty = \frac{2\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, \chi_{\mathfrak{m}}, 1)}.$$

Proof. We can determine that

$$\begin{cases} b_1 = m \\ b_2 = \frac{n \pm \sqrt{d}}{2}m \end{cases} \quad \text{or} \quad \begin{cases} b_1 = -m \\ b_2 = -\frac{n \pm \sqrt{d}}{2}m. \end{cases}$$

It is not hard to check that the condition † is satisfied. For $\mathrm{Im}(b_1\bar{b}_2) < 0$, we write

$$(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}) = \left(\begin{pmatrix} a_{1,1} & m \\ m & 0 \end{pmatrix}, \begin{pmatrix} a_{1,2} & \frac{n+\sqrt{d}}{2}m \\ \frac{n-\sqrt{d}}{2}m & 0 \end{pmatrix} \right)$$

and

$$(\mathbf{x}_{2,1}, \mathbf{x}_{2,2}) = \left(\begin{pmatrix} a_{2,1} & -m \\ -m & 0 \end{pmatrix}, \begin{pmatrix} a_{2,2} & -\frac{n+\sqrt{d}}{2}m \\ -\frac{n-\sqrt{d}}{2}m & 0 \end{pmatrix} \right).$$

Similarly we get $\mu_\beta = 4$ and

$$\begin{aligned} I_\infty =& I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} + I_{(-\mathbf{x}_{1,1},-\mathbf{x}_{1,2})} + I_{(\mathbf{x}_{2,1},\mathbf{x}_{2,2})} + I_{(-\mathbf{x}_{2,1},-\mathbf{x}_{2,2})} \\ =& 4I_{(\mathbf{x}_{1,1},\mathbf{x}_{1,2})} = \frac{2\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, \chi_{\mathfrak{m}}, 1)}. \end{aligned}$$

□

Example 4.4.11. Let $F = \mathbb{Q}(\sqrt{-3})$ with $d_F = d = -3$ and $\mathcal{O} = \mathbb{Z}[\omega]$ with $\omega = \frac{1+\sqrt{-3}}{2}$. Set

$$\beta = \begin{pmatrix} m^2 & \frac{nm^2}{2} \\ \frac{nm^2}{2} & \frac{n^2-d}{4}m^2 \end{pmatrix}$$

with odd $n \in \mathbb{Z}$ greater than 1 and coprime to $\mathfrak{n}$. We have

$$I_\infty = \frac{3\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, \chi_{\mathfrak{m}}, 1)}.$$

Proof. We can determine b_i:

$$\begin{cases} b_1 = m \\ b_2 = \frac{n+\sqrt{d}}{2}m \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} 1 & 0 \\ \frac{n-1}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

$$\begin{cases} b_1 = m \\ b_2 = \frac{n-\sqrt{d}}{2}m \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} 1 & 0 \\ \frac{n+1}{2} & -1 \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

$$\begin{cases} b_1 = m\omega \\ b_2 = \frac{n+\sqrt{d}}{2}m\omega \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} 0 & 1 \\ \frac{d-1}{4} & \frac{n+1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

$$\begin{cases} b_1 = m\omega \\ b_2 = \frac{n-\sqrt{d}}{2}m\omega \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} 0 & 1 \\ \frac{1-d}{4} & \frac{n-1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

$$\begin{cases} b_1 = m\bar{\omega} \\ b_2 = \frac{n+\sqrt{d}}{2}m\bar{\omega} \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} 1 & -1 \\ \frac{2n-d-1}{4} & \frac{1-n}{2} \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

$$\begin{cases} b_1 = m\bar{\omega} \\ b_2 = \frac{n-\sqrt{d}}{2}m\bar{\omega} \end{cases} \sim \begin{pmatrix} b_1 \\ b_2 \end{pmatrix} = m \begin{pmatrix} 1 & -1 \\ \frac{2n+d+1}{4} & -\frac{n+1}{2} \end{pmatrix} \begin{pmatrix} 1 \\ \omega \end{pmatrix}$$

and

$$\begin{cases} b_1 = -m \\ b_2 = -\frac{n\pm\sqrt{d}}{2}m \end{cases} \begin{cases} b_1 = -m\omega \\ b_2 = -\frac{n\pm\sqrt{d}}{2}m\omega \end{cases} \begin{cases} b_1 = -m\bar{\omega} \\ b_2 = -\frac{n\pm\sqrt{d}}{2}m\bar{\omega}. \end{cases}$$

For $\mathrm{Im}(b_1\bar{b}_2) < 0$, we consider

$$(\mathbf{x}_{1,1}, \mathbf{x}_{1,2}) = \left(\begin{pmatrix} a_{1,1} & m \\ m & 0 \end{pmatrix}, \begin{pmatrix} a_{1,2} & \frac{n+\sqrt{d}}{2}m \\ \frac{n-\sqrt{d}}{2}m & 0 \end{pmatrix} \right),$$

$$(\mathbf{x}_{2,1}, \mathbf{x}_{2,2}) = \left(\begin{pmatrix} a_{2,1} & m\omega \\ m\bar{\omega} & 0 \end{pmatrix}, \begin{pmatrix} a_{2,2} & \frac{n+\sqrt{d}}{2}\omega \\ \frac{n-\sqrt{d}}{2}m\bar{\omega} & 0 \end{pmatrix} \right),$$

$$(\mathbf{x}_{3,1}, \mathbf{x}_{3,2}) = \left(\begin{pmatrix} a_{3,1} & m\bar{\omega} \\ m\omega & 0 \end{pmatrix}, \begin{pmatrix} a_{3,2} & \frac{n-\sqrt{d}}{2}\bar{\omega} \\ \frac{n+\sqrt{d}}{2}m\omega & 0 \end{pmatrix} \right).$$

and $(-\mathbf{x}_{1,1}, -\mathbf{x}_{1,2}), (-\mathbf{x}_{2,1}, -\mathbf{x}_{2,2}), (-\mathbf{x}_{3,1}, -\mathbf{x}_{3,2})$. In this case we have $\mu_\beta = 6$ and

$$I_\infty = 6I_{(\mathbf{x}_{1,1}, \mathbf{x}_{1,2})} = \frac{3\lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{A(1, \chi_{\mathfrak{m}}, 1)}.$$

□

Remark. We can swap the diagonal entries of each β in Example from 4.4.6 to 4.4.11 and obtain same results.

4.4.2 On other cycles

We introduce the Atkin-Lehner operator as defined in M. Lingham's thesis [Lin05, Section 5.3]. Lingham developed this for all odd class numbers while we shall only use results for class number 1 since then we can follow Asai's treatment of cusps (see [Asa76, Section 1.1]) in the case of principal ideal domain. For $\mathfrak{m}$ in $\mathcal{O}$ dividing $\mathfrak{n}$ such that $\mathfrak{m}$ and $\frac{\mathfrak{n}}{\mathfrak{m}}$ are coprime, take

$$W_{\mathfrak{m}} = \begin{pmatrix} x & y \\ z & w \end{pmatrix} \tag{4.38}$$

where $x \in \mathfrak{m}$, $y \in \mathcal{O}$, $z \in \mathfrak{n}, w \in \mathfrak{m}$ and $\langle xw - yz \rangle = \mathfrak{m}$.

Proposition 4.4.12. *(1) For any ideal $\mathfrak{m}$ dividing $\mathfrak{n}$ such that $\mathfrak{m}$ and $\frac{\mathfrak{n}}{\mathfrak{m}}$ are coprime, we can find a matrix of the form* (4.38)*.*

(2) $W_{\mathfrak{m}}$ is an involution (i.e. $W_{\mathfrak{m}}^2$ (modulo scalars) lies in $\Gamma_0(\mathfrak{n})$), normalizes $\Gamma_0(\mathfrak{n})$ and is independent of the particular choice of x, y, z, w.

Proof. See [Lin05, Lemma 5.3.1 and Lemma 5.3.2]. □

In particular if we take $\mathfrak{m} = \mathcal{O}$ we get an element of $\Gamma_0(\mathfrak{n})$ and if we take $\mathfrak{m} = \mathfrak{n}$ we get the analogue of the classical Fricke involution. One can check that the Fricke involution can be formed as a product of Atkin-Lehner involutions, where $\mathfrak{m}$ runs over prime power divisors of $\mathfrak{n}$.

Lemma 4.4.13. *Let $\alpha_1 = \frac{p_1}{q_1}$, $\alpha_2 = \frac{p_2}{q_2}$ be two cusps such that $\langle p_1, q_1 \rangle = \langle p_2, q_2 \rangle = \mathcal{O}$. Then the following are equivalent*

(1) $\alpha_2 = M\alpha_1$ for some $M \in \Gamma_0(\mathfrak{n})$;
(2) $q_2 s_1 - q_1 s_2 \in q_1 q_2 \mathcal{O} + \mathfrak{n}$, where s_i satisfies $p_i s_i \equiv 1 \pmod{q_i}$.

Proof. See [Lin05, Lemma 1.5.1] for a more general version holding over any number field. □

It follows that two cusps are equivalent relative to $\Gamma_0(\mathfrak{n})$ if and only if the ideals generated by the denominators have the same 'greatest common divisor' with $\mathfrak{n}$, so each equivalence class of cusps is in one-to-one correspondence with each ordered decomposition $\mathfrak{n} = \mathfrak{M}\mathfrak{L}$. Following Asai's treatment (see [Asa76, Section 1.1]) again as in section 3.3.2, we say a cusp κ_2/κ_1 belongs to $\mathfrak{L}$-class if $\gcd(\kappa_1\mathcal{O}, \mathfrak{n}) = \mathfrak{L}$. For each decomposition $\mathfrak{n} = \mathfrak{M}\mathfrak{L}$ with $\mathfrak{M} = M\mathcal{O}$ and any cusp $\kappa = \kappa_2/\kappa_1$ of $\mathfrak{L}$-class, we can take a typical matrix W_κ which transforms κ to ∞:

$$W_\kappa = \begin{pmatrix} 1 & 0 \\ 0 & M \end{pmatrix} \alpha_\kappa \quad \text{with} \quad \alpha_\kappa = \begin{pmatrix} M\lambda_1 & \lambda_2 \\ -\kappa_1 & \kappa_2 \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}). \tag{4.39}$$

As $\langle \kappa_1, \kappa_2 \rangle = \langle \kappa_1, M \rangle = \mathcal{O}$ there exist $b, c \in \mathcal{O}$ such that $b\kappa_2 \equiv 1 \pmod{\kappa_1}$ and $cM \equiv 1 \pmod{\kappa_1}$. Taking $\lambda_1 = bc \in \mathcal{O}$ we observe that $\lambda_2 = \frac{1 - M\lambda_1\kappa_2}{\kappa_1}$ belongs to $\mathcal{O}$. So W_κ is well-defined. It is not difficult to see that W_κ is of type of Atkin-Lehner operator as defined in (4.38).

Fix a representative $\kappa_i = \kappa_{i,2}/\kappa_{i,1} \in \mathbb{P}^1(F)/\Gamma$ of each equivalence class of cusps corresponding to the ordered decomposition $m\sqrt{d_F}\mathfrak{n} = \mathfrak{M}_i\mathfrak{L}_i$ with $\mathfrak{M}_i$ generated by M_i and $\mathfrak{L}_i$ by L_i. Write as defined in (4.39)

$$W_{\kappa_i} = \begin{pmatrix} 1 & 0 \\ 0 & M_i \end{pmatrix} \begin{pmatrix} M_i\lambda_1 & \lambda_2 \\ -\kappa_{i,1} & \kappa_{i,2} \end{pmatrix}$$

which transforms κ_i to ∞.

It is well known that the fractional linear transformation on the extended upper half space is composition of an even number of inversions (see e.g. [Ber05, Section 2.3]). So the action of $\mathrm{GL}_2(\mathbb{C})$ on the subspace U preserves the orientation. By Proposition 4.2.2 we know that if $U(\mathbf{x}_1, \mathbf{x}_2) \perp \nu(\infty)$ then $U(W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)) \perp \nu(W_{\kappa_i}^{-1} \cdot \infty)$. We have proven that the bilinear form on a pair of vectors is preserved under the action of

$\mathrm{GL}_2(\mathbb{C})$ in (4.3) and hence so is the Gram matrix β. Thus for $(\mathbf{x}_1,\mathbf{x}_2)\in\Omega_{\beta,\infty,+}$ we have $|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)\in\Omega_{\beta,\kappa_i,+}$. Then we obtain

$$\begin{aligned} I_{\kappa_i} &= \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_{\kappa_i}\backslash\Omega_{\beta,\kappa_i,+}} \varphi_f^{\mathrm{new}}(\mathbf{x}_1,\mathbf{x}_2)\int_{C_{U(\mathbf{x}_1,\mathbf{x}_2)}}\eta_{\mathcal{F}}\\ &= \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_{\infty}\backslash\Omega_{\beta,\infty,+}} \varphi_f^{\mathrm{new}}(|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))\int_{C_{U(|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))}}\eta_{\mathcal{F}}\\ &= \sum_{(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_{\infty}\backslash\Omega_{\beta,\infty,+}} \varphi_f^{\mathrm{new}}(|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))\int_{C_{U(W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))}}\eta_{\mathcal{F}} \end{aligned}$$

where the last equality is the consequence of $U(|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))=U(W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))$.

Remark 4.4.14. Slightly different to what we have done in the Shintani case, we introduce the factor $|\det(W_{\kappa_i})|$ to make sure that for $(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_{\kappa_i}\backslash\Omega_{\beta,\kappa_i,+}$, $|\det(W_{\kappa_i})|^{-1}\cdot W_{\kappa_i}\cdot(\mathbf{x}_1,\mathbf{x}_2)$ lies in $V(\mathbb{Q})^2$ (so in $\Gamma_\infty\backslash\Omega_{\beta,\infty,+}$) but not just in $V(\mathbb{R})^2$.

Next we will analyze $\varphi_f^{\mathrm{new}}(|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2))$ for $(\mathbf{x}_1,\mathbf{x}_2)\in\Gamma_\infty\backslash\Omega_{\beta,\infty,+}$. For simplicity we write $\chi=\chi_{\mathfrak{m}}$.

We begin the calculation in a slightly more general setting. Given $g=\begin{pmatrix}x&y\\z&w\end{pmatrix}$ and $(\mathbf{x}_1,\mathbf{x}_2)=\left(\begin{pmatrix}a_1&b_1\\\bar b_1&0\end{pmatrix},\begin{pmatrix}a_2&b_2\\\bar b_2&0\end{pmatrix}\right)$, we compute

$$\begin{aligned}\left(\begin{pmatrix}a_1'&b_1'\\\bar b_1'&d_1'\end{pmatrix},\begin{pmatrix}a_2'&b_2'\\\bar b_2'&d_2'\end{pmatrix}\right) &:= |\det g|^{-1}\cdot g\cdot\left(\begin{pmatrix}a_1&b_1\\\bar b_1&0\end{pmatrix},\begin{pmatrix}a_2&b_2\\\bar b_2&0\end{pmatrix}\right)\\ =&|\det(g)|^{-2}\left(\begin{pmatrix}a_1x\bar x+\bar b_1\bar xy+b_1x\bar y & a_1x\bar z+\bar b_1y\bar z+b_1x\bar w\\ a_1\bar xz+\bar b_1\bar xw+b_1\bar yz & a_1z\bar z+\bar b_1\bar zw+b_1z\bar w\end{pmatrix},\begin{pmatrix}a_2x\bar x+\bar b_2\bar xy+b_2x\bar y & a_2x\bar z+\bar b_2y\bar z+b_2x\bar w\\ a_2\bar xz+\bar b_2\bar xw+b_2\bar yz & a_2z\bar z+\bar b_2\bar zw+b_2z\bar w\end{pmatrix}\right)\end{aligned} \tag{4.40}$$

and then

$$\begin{aligned}(a_2'b_1'-a_1'b_2') =&|\det(g)|^{-4}[(a_2x\bar x+\bar b_2\bar xy+b_2x\bar y)(a_1x\bar z+\bar b_1y\bar z+b_1x\bar w)\\ &-(a_1x\bar x+\bar b_1\bar xy+b_1x\bar y)(a_2x\bar z+\bar b_2y\bar z+b_2x\bar w)]\\ =&|\det(g)|^{-4}[a_2b_1(x\bar xx\bar w-x\bar yx\bar z)-a_1b_2(x\bar xx\bar w-x\bar yx\bar z)\\ &+b_1\bar b_2(x\bar xy\bar w-x\bar yx\bar z)+\bar b_1b_2(x\bar yy\bar z-\bar xyx\bar w)]\\ =&\det(g)^{-2}\det(\bar g)^{-1}[(a_2b_1-a_1b_2)x^2+(b_1\bar b_2-\bar b_1b_2)xy],\end{aligned} \tag{4.41}$$

$$\begin{aligned}(\bar{b}_2'd_1' - \bar{b}_1'd_2') =& |\det(g)|^{-4}[(a_2\bar{x}z + \bar{b}_2\bar{x}w + b_2\bar{y}z)(a_1z\bar{z} + \bar{b}_1\bar{z}w + b_1z\bar{w}) \\ &- (a_1\bar{x}z + \bar{b}_1\bar{x}w + b_1\bar{y}z)(a_2z\bar{z} + \bar{b}_2\bar{z}w + b_2z\bar{w})] \\ =& |\det(g)|^{-4}[a_2b_1(\bar{x}zz\bar{w} - \bar{y}zz\bar{z}) - a_1b_2(\bar{x}zz\bar{w} - \bar{y}zz\bar{z})] \\ &+ b_1\bar{b}_2(\bar{x}wz\bar{w} - \bar{y}z\bar{z}w) - \bar{b}_1b_2(\bar{x}wz\bar{w} - \bar{y}z\bar{z}w) \\ =& \det(g)^{-2}\det(\bar{g})^{-1}[(a_2b_1 - a_1b_2)z^2 + (b_1\bar{b}_2 - \bar{b}_1b_2)wz]. \qquad (4.42)\end{aligned}$$

Remark 4.4.15. With our choice of β, the pair $(\mathbf{x}_1, \mathbf{x}_2) \in \Gamma_\infty\backslash\Omega_{\beta,\infty,+}$ satisfies the condition † as in Remark 4.2.6. It means that $b_i \in \mathfrak{q}\mathcal{O}_\mathfrak{q} \times \bar{\mathfrak{q}}\mathcal{O}_{\bar{\mathfrak{q}}}$ (i.e. $q|b_i$) for split $q|m$ with $(q) = \mathfrak{q}\bar{\mathfrak{q}}$, and $b_i \in \mathfrak{q}\mathcal{O}_\mathfrak{q}$ for inert $q|m$ with $(q) = \mathfrak{q}$. So $b_1\bar{b}_2 - \bar{b}_1b_2$ appearing in (4.41) and (4.42) turns out to be divisible by q^2 for each prime $q|m$.

Recall the ordered decomposition $\mathfrak{f}\mathfrak{n} = \mathfrak{M}_i\mathfrak{L}_i$ (or $\mathfrak{f}\mathfrak{n}\mathfrak{q}_2 = \mathfrak{M}_i\mathfrak{L}_i$ with $\mathfrak{q}_2$ above 2 when $d \equiv 2, 3 \bmod 4$) and its corresponding representative $\kappa_i = \frac{\kappa_{i,2}}{\kappa_{i,1}}$ of equivalence class of cusps with $\kappa_{i,1}$ and $\kappa_{i,2}$ coprime.

Lemma 4.4.16 (Analogue of Lemma 3.3.3). *For $(\mathbf{x}_1, \mathbf{x}_2) \in \Gamma_\infty\backslash\Omega_{\beta,\infty,+}$ and non-trivial $\mathfrak{M}_i$ dividing $\mathfrak{f}\mathfrak{n}$ when $d \equiv 1 \bmod 4$ (or $\mathfrak{f}\mathfrak{n}\mathfrak{q}_2$ with $\mathfrak{q}_2$ above 2 when $d \equiv 2, 3 \bmod 4$), we have that φ_f^{new} is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$.*

Proof. Write

$$W_{\kappa_i}^{-1} = \begin{pmatrix}\kappa_{i,2} & -\frac{\lambda_2}{M_i} \\ \kappa_{i,1} & \lambda_1\end{pmatrix} =: \begin{pmatrix}x & y \\ z & w\end{pmatrix} \qquad \text{with } \det(W_{\kappa_i}^{-1}) = \frac{1}{M_i}$$

and for $(\mathbf{x}_1, \mathbf{x}_2) = \left(\begin{pmatrix}a_1 & b_1 \\ \bar{b}_1 & 0\end{pmatrix}, \begin{pmatrix}a_2 & b_2 \\ \bar{b}_2 & 0\end{pmatrix}\right) \in \Gamma_\infty\backslash\Omega_{\beta,\infty,+}$, set

$$(\mathbf{x}_1', \mathbf{x}_2') = \left(\begin{pmatrix}a_1' & b_1' \\ \bar{b}_1' & d_1'\end{pmatrix}, \begin{pmatrix}a_2' & b_2' \\ \bar{b}_2' & d_2'\end{pmatrix}\right) := |\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot \left(\begin{pmatrix}a_1 & b_1 \\ \bar{b}_1 & 0\end{pmatrix}, \begin{pmatrix}a_1 & b_1 \\ \bar{b}_1 & 0\end{pmatrix}\right).$$

By (4.40), we have for $j = 1, 2$

$$\begin{aligned}a_j' =& |M_i|^2(a_jx\bar{x} + \bar{b}_j\bar{x}y + b_jx\bar{y}) = |M_i|^2a_j\kappa_{i,2}\bar{\kappa}_{i,2} - \overline{M}_i\bar{b}_j\bar{\kappa}_{i,2}\lambda_2 - M_ib_j\kappa_{i,2}\bar{\lambda}_2, \\ b_j' =& |M_i|^2(a_jx\bar{z} + \bar{b}_jy\bar{z} + b_jx\bar{w}) = |M_i|^2a_j\kappa_{i,2}\bar{\kappa}_{i,1} - \overline{M}_i\bar{b}_j\lambda_2\bar{\kappa}_{i,1} + |M_i|^2b_j\kappa_{i,2}\bar{\lambda}_1, \\ \bar{b}_j' =& |M_i|^2(a_j\bar{x}z + b_j\bar{y}z + \bar{b}_j\bar{x}w) = |M_i|^2a_j\bar{\kappa}_{i,2}\kappa_{i,1} - M_ib_j\bar{\lambda}_2\kappa_{i,1} + |M_i|^2\bar{b}_j\bar{\kappa}_{i,2}\lambda_1, \\ d_j' =& |M_i|^2(a_jz\bar{z} + \bar{b}_j\bar{z}w + b_jz\bar{w}) = |M_i|^2(a_j\kappa_{i,1}\bar{\kappa}_{i,1} + \bar{b}_j\bar{\kappa}_{i,1}\lambda_1 + b_j\kappa_{i,1}\bar{\lambda}_1).\end{aligned}$$

(I) Let q a prime dividing $m|d_F|$ which is split, inert or ramified. We will only treat in details the case when q is split with $(q) = \mathfrak{q}\bar{\mathfrak{q}}$ and other cases can be treated similarly. We want to show that if

$$(\mathfrak{M}_i, (m)) = \mathfrak{q}, (\mathfrak{M}_i, (m)) = \bar{\mathfrak{q}} \text{ or } (\mathfrak{M}_i, (m)) = (q)$$

then $\omega(1,\gamma)\varphi_q^{\chi}$ is vanishing on $|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)$ for $[\gamma]\in\overline{\Gamma}_0(q)/\overline{\Gamma}(q)$.

(I.1) Let $(\mathfrak{M}_i,(m))=\mathfrak{q}$ and then we have

$$\mathfrak{q}|\mathfrak{M}_i,\ \bar{\mathfrak{q}}|\overline{\mathfrak{M}}_i,\ \bar{\mathfrak{q}}|(\kappa_{i,1}),\ \mathfrak{q}|(\bar{\kappa}_{i,1}),\ \bar{\mathfrak{q}}\nmid(\kappa_{i,2}),\ \mathfrak{q}\nmid(\bar{\kappa}_{i,2})$$

By Remark 4.4.15, there is no need to discuss the integrality of b_j but we care for that of a_j.

Suppose that $a_j\in\mathbb{Z}_q$. It is easy to observe that $a'_j,b'_j,\bar{b}'_j,d'_j\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$. Set

$$\gamma=(\gamma_1,\gamma_2)=\left(\begin{pmatrix}u_1 & v_1\\ 0 & u_1^{-1}\end{pmatrix},\begin{pmatrix}u_2 & v_2\\ 0 & u_2^{-1}\end{pmatrix}\right)$$

with $[u_1],[u_2]\in(\mathcal{O}/(q))^\times$ and $[v_1],[v_2]\in\mathcal{O}/(q)$. We write

$$(\mathbf{x}''_1,\mathbf{x}''_2)=\left(\begin{pmatrix}a''_1 & b''_1\\ c''_1 & d''_1\end{pmatrix},\begin{pmatrix}a''_2 & b''_2\\ c''_2 & d''_2\end{pmatrix}\right):=(\gamma_1^{-1}\mathbf{x}'_1{}^t(\gamma_2^{-1})^*,\gamma_1^{-1}\mathbf{x}'_2{}^t(\gamma_2^{-1})^*)$$

and compute

$$\begin{aligned}\begin{pmatrix}a''_j & b''_j\\ c''_j & d''_j\end{pmatrix}&=\begin{pmatrix}u_1^{-1} & -v_1\\ 0 & u_1\end{pmatrix}\begin{pmatrix}a'_j & b'_j\\ \bar{b}'_j & d'_j\end{pmatrix}\begin{pmatrix}u_2 & 0\\ v_2 & u_2^{-1}\end{pmatrix}\\ &=\begin{pmatrix}u_1^{-1}u_2a'_j-v_1u_2\bar{b}'_j+u_1^{-1}v_2b'_j-v_1v_2d'_j & u_1^{-1}u_2^{-1}b'_j-v_1u_2^{-1}d'_j\\ u_1u_2\bar{b}'_j+u_1v_2d'_j & u_1u_2^{-1}d'_j\end{pmatrix}.\end{aligned}$$

Then, as $a'_j,b'_j,\bar{b}'_j,d'_j\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, we can observe that $a''_j,b''_j,c''_j,d''_j\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ as well which implies that

$$\frac{a''_2b''_1-a''_1b''_2}{m}+\frac{c''_2d'_1-c''_1d'_2}{m}\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}.$$

It immediately follows that $\omega(1,\gamma)\varphi_q^{\chi,\mathrm{new}}$ is vanishing on $|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)$.

Suppose that $a_j\notin\mathbb{Z}_q$ and set $l_q=\min\{\mathrm{ord}_q(a_j)\}\le -1$. Assume that $\omega(1,\gamma)\varphi_q^{\chi,\mathrm{new}}$ is non-vanishing on $|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)$ which requires that $a''_j,d''_j\in\mathbb{Z}_q$ and $b''_j,c''_j\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$.

- We first consider $v_2\in(\mathcal{O}/(q))^\times$. Observing

$$c''_j=u_1u_2\bar{b}'_j+u_1v_2d'_j\quad\text{and}\quad d''_j=u_1u_2^{-1}d'_j,$$

we know that for $c''_j\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ and $d''_j\in\mathbb{Z}_q$ we need $d'_j\in\mathbb{Z}_q$ and at least $\bar{b}'_j\in\mathcal{O}_{\mathfrak{q}}$. As

$$\bar{b}'_j=|M_i|^2a_j\bar{\kappa}_{i,2}\kappa_{i,1}-M_ib_j\bar{\lambda}_2\kappa_{i,1}+|M_i|^2\bar{b}_j\bar{\kappa}_{i,2}\lambda_1$$

with $\bar{\kappa}_{i,2},\kappa_{i,1}\in\mathcal{O}_{\mathfrak{q}}^\times$, we then need $M_i\in\mathfrak{q}^{-l_q}$ which makes

$$d'_j=|M_i|^2(a_j\kappa_{i,1}\bar{\kappa}_{i,1}+\bar{b}_j\bar{\kappa}_{i,1}\lambda_1+b_j\kappa_{i,1}\bar{\lambda}_1)\quad(q|\kappa_{i,1}\bar{\kappa}_{i,1})$$

lie in $q\mathbb{Z}_q$. Looking back to $c''_j=u_1u_2\bar{b}'_j+u_1v_2d'_j\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, that $d'_j\in q\mathbb{Z}_q$ makes $\bar{b}'_j\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$. It follows that we need $M_i\in\mathfrak{q}^{-l_q+1}\mathcal{O}_{\mathfrak{q}}$, a contradiction to that $\mathfrak{M}_i$ is square-free.

- Let $v_2 = 0$. Then we have
$$\begin{pmatrix} a_j'' & b_j'' \\ c_j'' & d_j'' \end{pmatrix} = \begin{pmatrix} * & * \\ u_1 u_2 \bar{b}_j' & u_1 u_2^{-1} d_j' \end{pmatrix}.$$
For $c_j'' \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, we need
$$\bar{b}_j' = |M_i|^2 a_j \bar{\kappa}_{i,2}\kappa_{i,1} - M_i b_j \bar{\lambda}_2 \kappa_{i,1} + |M_i|^2 \bar{b}_j \bar{\kappa}_{i,2}\lambda_1 \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}} \quad (\bar{\kappa}_{i,2}, \kappa_{i,1} \in \mathcal{O}_{\mathfrak{q}}^{\times})$$
which requires $M_i \in \mathfrak{q}^{-l_q+1}\mathcal{O}_{\mathfrak{q}}$ contradicting to that $\mathfrak{M}_i$ is square-free.

Therefore, when $(\mathfrak{M}_i, (m)) = \mathfrak{q}$, we can deduce that $\omega(1,\gamma)\varphi_q^{\chi}$ is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$.

(I.2) When $(\mathfrak{M}_i, (m)) = \bar{\mathfrak{q}}$, we can prove it in the same way.

(I.3) Let $(\mathfrak{M}_i, (m)) = q$ and then we have $q | M_i$ and $\kappa_{i,1}, \bar{\kappa}_{i,1} \in \mathcal{O}_{\mathfrak{q}}^{\times}$. It is clear for $a_j \in \mathbb{Z}_q$.

Suppose that $a_j \notin \mathbb{Z}_q$ and set $l_q = \min\{\mathrm{ord}_q(a_j)\} \leq -1$. First assume v_1 is a unit. Let $l_q = -1$. Then we have that all $a_j', d_j', b_j', \bar{b}_j'$ are divisible by q and so are $a_j'', d_j'', b_j'', c_j''$. Let $l_q = -2$. It is clear that $d_j' \in \mathbb{Z}_q$. Then, for $b_j'', \bar{b}_j'' \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}$, we need $b_j', \bar{b}_j' \in \mathcal{O}_{\mathfrak{q}}$. Expand
$$\begin{aligned} a_2'' b_1'' - a_1'' b_2'' =& u_1^{-2}(a_2' b_1' - a_1' b_2') + u_1^{-1} v_1 (a_2' d_1' - a_1' d_2') \\ &+ u_1^{-1} v_1 (b_1' \bar{b}_2' - \bar{b}_1' b_2') + v_1^2 (\bar{b}_2' d_1' - \bar{b}_1' d_2') \end{aligned}$$
and
$$c_2'' d_1'' - c_1'' d_2'' = u_1^2 (\bar{b}_2' d_1' - \bar{b}_1' d_2').$$
It is not hard to observe that $a_2' d_1' - a_1' d_2' \in q^2 \mathbb{Z}_q$ and $b_1' \bar{b}_2' - \bar{b}_1' b_2' \in \mathfrak{q}^2 \mathcal{O}_{\mathfrak{q}}$. By (4.41) and (4.42), we have
$$a_2' d_1' - a_1' d_2' = M_i |M_i|^2 ((a_2 b_1 - a_1 b_2)\kappa_{i,2}^2 + (b_1' \bar{b}_2' - \bar{b}_1' b_2')\kappa_{i,2}^2 (-\lambda / M_i))$$
and
$$\bar{b}_2' d_1' - \bar{b}_1' d_2' = M_i |M_i|^2 ((a_2 b_1 - a_1 b_2)\kappa_{i,1}^2 + (b_1' \bar{b}_2' - \bar{b}_1' b_2')\kappa_{i,1}\lambda_1),$$
both of which lie in $\mathfrak{q}^2 \mathcal{O}_{\mathfrak{q}}$. It follows again that
$$\frac{a_2'' b_1'' - a_1'' b_2''}{m} + \frac{c_2'' d_1' - c_1'' d_2'}{m} \in \mathfrak{q}\mathcal{O}_{\mathfrak{q}}.$$
For $l_q < -2$ there is no chance for $d_j'' \in \mathbb{Z}_q$ as $M_{i,1}$ is square-free. It is clear for $v_1 = 0$. Therefore $\omega(1,\gamma)\varphi_q^{\chi}$ is vanishing on $|\det(W_{\kappa_i})| \cdot W_{\kappa_i}^{-1} \cdot (\mathbf{x}_1, \mathbf{x}_2)$ in this case.

(II) Next we consider $\varphi_q^{\mathfrak{n}}$ for split $q | N(\mathfrak{n})$ and omit details for inert q. As discussed in subsection 4.3.4, we can have $(\mathfrak{n}, (q)) = \mathfrak{q}$, $(\mathfrak{n}, (q)) = \bar{\mathfrak{q}}$ or $(\mathfrak{n}, (q)) = (q)$. Again we want to show that if $(\mathfrak{n}, (q)) | \mathfrak{M}_i$ then $\omega(1,\gamma)\varphi_q^{\mathfrak{n}}$ is vanishing on $|\det(W_{\kappa_i})| \cdot$

$W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)$ for $[\gamma]\in\overline{\Gamma}_0(\mathfrak{q})/\overline{\Gamma}(\mathfrak{q})$, $\overline{\Gamma}_0(\bar{\mathfrak{q}})/\overline{\Gamma}(\bar{\mathfrak{q}})$ or $\overline{\Gamma}_0(q)/\overline{\Gamma}(q)$ respectively.
Let $a_j\in\mathbb{Z}_q$. Assume that $(\mathfrak{n},(q))=\mathfrak{q}$ and $\mathfrak{q}|\mathfrak{M}_i$. Then it is clear that $\bar{b}'_j\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ and $d'_j\in q\mathbb{Z}_q$. Expanding

$$\begin{aligned}b''_1c''_2+b''_2c''_1=&(u_1^{-1}u_2^{-1}b'_1-v_1u_2^{-1}d'_1)(u_1u_2\bar{b}'_2+u_1v_2d'_2)\\&+(u_1^{-1}u_2^{-1}b'_2-v_1u_2^{-1}d'_2)(u_1u_2\bar{b}'_1+u_1v_2d'_1),\end{aligned}$$

we see it is in $\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$. So $\omega(1,\gamma)\varphi_q^{\mathfrak{n}}$ is vanishing on $|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)$. Also it is clear for $\bar{\mathfrak{q}}|\mathfrak{M}_i$ or $(q)|\mathfrak{M}_i$.
Let $a_j\notin\mathbb{Z}_q$ and set $l_q=\min\{\mathrm{ord}_q(a_j)\}\le -1$. Assume that $(\mathfrak{n},(q))=\mathfrak{q}$ and $\mathfrak{q}|\mathfrak{M}_i$. Then we have $q\nmid\kappa_{i,1}$. Look at

$$d'_j=|M_i|^2(a_j\kappa_{i,1}\bar{\kappa}_{i,1}+\bar{b}_j\bar{\kappa}_{i,1}\lambda_1+b_j\kappa_{i,1}\bar{\lambda}_1).$$

Then there is no chance for $d''_j=d'_j$ to be in $q\mathbb{Z}_q$ as M_i is square-free. So $\omega(1,\gamma)\varphi_q^{\mathfrak{n}}$ is vanishing on $|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)$. This also occurs in the case that $(\mathfrak{n},(q))=\bar{\mathfrak{q}}$ and $\bar{\mathfrak{q}}|\mathfrak{M}_i$. Now assume that $(\mathfrak{n},(q))=(q)$ and $(q)|\mathfrak{M}_i$. If $l_q=-1$, then we have $b'_j,\bar{b}'_j\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$ which implies that $b''_1c''_2+b''_2c''_1\in\mathfrak{q}\mathcal{O}_{\mathfrak{q}}$. If $l_q\le -2$, we can observe that there is no room for $d''_j=u_1u_2^{-1}d'_j\in q\mathbb{Z}$ for square-free M_i.
Let q' be another prime dividing $N(\mathfrak{n})$ with $q'=\mathfrak{q}'\bar{\mathfrak{q}}'$. Similarly, if $\mathfrak{q}'|\mathfrak{M}_i$, $\bar{\mathfrak{q}}'|\mathfrak{M}_i$ or $(q)|\mathfrak{M}_i$, we can show that $\omega(1,\gamma)\varphi_{q'}^{\mathfrak{n}}$ is vanishing on $|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)$.

(III) To finish our proof we consider φ_2 if 2 is ramified with $(2)=\mathfrak{q}_2^2$. Set $\gamma=\begin{pmatrix}u&v\\0&u^{-1}\end{pmatrix}$ with $[u]\in(\mathcal{O}/\mathfrak{q}_2)^\times$ and $[v]\in\mathcal{O}/\mathfrak{q}_2$. We write

$$(\mathbf{x}''_1,\mathbf{x}''_2)=\left(\begin{pmatrix}\bar{b}''_1&a''_1\sqrt{d}\\c''_1\sqrt{d}&b''_1\end{pmatrix},\begin{pmatrix}\bar{b}''_2&a''_2\sqrt{d}\\c''_2\sqrt{d}&\bar{b}''_2\end{pmatrix}\right):=(\gamma\mathbf{x}'_1\bar{\gamma}^*,\gamma\mathbf{x}'_2\bar{\gamma}^*).$$

Suppose that $\mathfrak{q}_2|\mathfrak{M}_i$. If $l_q=\min\{\mathrm{ord}_q(a_j)\}\ge -1$, then we have $b'_j\in\frac{1}{\mathfrak{q}_2}\mathcal{O}_{\mathfrak{q}_2}$ and then $b''_j\in\frac{1}{\mathfrak{q}_2}\mathcal{O}_{\mathfrak{q}_2}$ as well. So $b''_1\bar{b}''_2\in\frac{1}{2}\mathcal{O}_{\mathfrak{q}_2}$ and then $b''_1\bar{b}''_2+\bar{b}''_1b''_2=2\mathrm{Re}(b''_1\bar{b}''_2)\in\mathbb{Z}_2$ which makes φ_2 vanish on $|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)$. If $l_q=\min\{\mathrm{ord}_q(a_j)\}\le -2$, then there is no chance for $d'_j\in\mathbb{Z}_2$ as $\mathfrak{M}_i$ is square-free, and so for c''_j. So again we have $\omega(1,\gamma)\varphi_2$ vanishing on $|\det(W_{\kappa_i})|\cdot W_{\kappa_i}^{-1}\cdot(\mathbf{x}_1,\mathbf{x}_2)$.

□

If follows that I_{κ_i} is vanishing for $\kappa_i\ne\infty$. So we have proven our main theorem:

Theorem 4.4.17. *Suppose that $F=\mathbb{Q}(\sqrt{d})$ is an imaginary quadratic field of class number 1 with the discriminant d_F and denote its ring of integers by $\mathcal{O}$. Let m be a square-free product of inert or split primes, and put $\mathfrak{m}=m\mathcal{O}$ and $\mathfrak{f}=\sqrt{d}\mathfrak{m}$. Choose a* *quadratic Hecke character $\chi_{\mathfrak{m}}$ of conductor $\mathfrak{f}$. Given a square-free ideal $\mathfrak{n}$ coprime to $(m|d_F|)$, let $\mathcal{F}$ be a weight 2 Bianchi cusp form of level $\Gamma_0(\mathfrak{n})$. Choose the Schwartz function as in Remark 4.4.1 (1) and (2), and β as in Remark 4.4.2. Then the Fourier coefficient of the theta lift at β as in* (4.22) *is*

$$I_\infty = \frac{\mu_\beta \lambda_{\mathfrak{m},\mathfrak{n}} L(\mathcal{F}, \chi_{\mathfrak{m}}, 1)}{2A(1,1,\chi_{\mathfrak{m}},1)}$$

as in Proposition 4.4.5.

So, if $L(\mathcal{F}, \chi_{\mathfrak{m}}, 1) \neq 0$, we can deduce the non-vanishing of our theta lifting as above.

§ 4.5 Non-vanishing of theta lifting

Recall from [CW94] that a new form in $S_2(\Gamma_0(\mathfrak{n}))$ is an eigenform for all the Hecke operators $T_{\mathfrak{p}}$ for $\mathfrak{p}$ not dividing $\mathfrak{n}$, which is not induced from in $S_2(\Gamma_0(\mathfrak{m}))$ for any level $\mathfrak{m}$ properly dividing $\mathfrak{n}$. There is an involution J induced by the action on $\mathbb{H}_3$ of the matrix $\left(\begin{smallmatrix} \epsilon & 0 \\ 0 & 1 \end{smallmatrix}\right)$, where ϵ generates the unit group of $\mathcal{O}$. The effect of J on Fourier coefficients is $c(\alpha) \to c(\epsilon\alpha)$; the involution commutes with the Hecke operators, and splits $S_2(\Gamma_0(\mathfrak{n}))$ into two eigenspaces,

$$S_2(\Gamma_0(\mathfrak{n})) = S_2^+(\Gamma_0(\mathfrak{n})) \oplus S_2^-(\Gamma_0(\mathfrak{n})).$$

Newforms in $S_2^+(\Gamma_0(\mathfrak{n}))$ were called plusforms, and their Fourier coefficients satisfy the additional condition $c(\epsilon\alpha) = c(\alpha)$ for all $\alpha \in \mathcal{O}$. Denote by $S_2^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ the space of newforms in $S_2(\Gamma_0(\mathfrak{n}))$ and by $S_2^{\mathrm{new},+}(\Gamma_0(\mathfrak{n}))$ the space of plusforms in $S_2^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$. More discussions on newforms and plusforms of weight 2 Bianchi modular forms can be found in [CW94].

As discussed in Section 1.5,

$$\beta = \left(-\frac{dz}{r}, \frac{dr}{r}, \frac{d\bar{z}}{r}\right) \qquad \text{for } (z,r) \in \mathbb{H}_3$$

is a basis for the left-invariant differential forms on $\mathbb{H}_3$. Let $\mathcal{F} \in S_2^{\mathrm{new}}(\Gamma_0(\mathfrak{n}))$ and recall its Mellin transform from [CW94, Section 2.5]:

$$\Lambda(\mathcal{F}, s) = \frac{(4\pi)^2}{|d_F|} \cdot \int_0^\infty t^{2s-2} \mathcal{F} \cdot \beta \tag{4.43}$$

for $\mathcal{F} = (\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_2)$ as given in (1.5) and (1.6).

Proposition 4.5.1. *[CW94, Proposition 2.1] Let $\mathcal{F} \in S_2^{\mathrm{new},+}(\Gamma_0(\mathfrak{n}))$. Then*

(1) *For $Re(s) > 3/2$ we have*

$$\Lambda(\mathcal{F}, s) = (2\pi)^{2-2s}|d_F|^{s-1}\Gamma(s)^2 L(\mathcal{F}, s) \tag{4.44}$$

for $L(\mathcal{F}, s)$ given in (1.7) *with trivial character.*

(2) *Assume that $\mathcal{F}$ is an eigenform for the Fricke involution $\omega_{\mathfrak{n}} = \begin{pmatrix} 0 & -1 \\ \mathfrak{n} & 0 \end{pmatrix}$, i.e., $\mathcal{F}|\omega_{\mathfrak{n}} = \varepsilon_{\mathfrak{n}}\mathcal{F}$ with $\varepsilon_{\mathfrak{n}} = \pm 1$. Then $\Lambda(\mathcal{F}, s)$ satisfies the functional equation*

$$\Lambda(\mathcal{F}, s) = -\varepsilon_{\mathfrak{n}} N(\mathfrak{n})^{1-s}\Lambda(\mathcal{F}, 2-s). \tag{4.45}$$

Put $\alpha(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}$ and $\omega_N = \omega(N) = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}$. Let ψ be a character of $(\mathcal{O}/\mathfrak{m}_\psi)^\times$ with conductor $\mathfrak{m}_\psi$. Similar to the twisted Hilbert modular forms [SW93, Section 5], the twist of $\mathcal{F}$ by ψ can be defined as, for $m \in \mathfrak{m}_\psi$,

$$\mathcal{F}_\psi = G(\psi^{-1}, 1/m)^{-1} \sum_{u \in (\mathcal{O}/\mathfrak{m}_\psi)^\times} \psi^{-1}(u)\mathcal{F}|_2\alpha(u/m)$$

where $G(\psi^{-1}, 1/m)^{-1}$ is the Gauss sum of ψ^{-1}.

Lemma 4.5.2. *Let $\mathcal{F} \in S_2(\Gamma_0(\mathfrak{n}))$, ψ a character of $(\mathcal{O}/\mathfrak{m}_\psi)^\times$, and $\mathfrak{M}$ the least common multiple of $\mathfrak{n}$, $\mathfrak{m}_\psi^2$, and $\mathfrak{m}_\psi$. Then $\mathcal{F}_\psi \in S_2(\Gamma_0(\mathfrak{M}), \psi^2)$.*

Proof. We will apply Miyake's treatment in [Miy06, Lemma 4.3.10] to our case without any new techniques.

Let $\gamma = \begin{pmatrix} a & b \\ cM & d \end{pmatrix} \in \Gamma_0(\mathfrak{M})$ where $M \in \mathfrak{M}$ and put

$$\gamma' = \alpha(u/m)\gamma\alpha(d^2u/m)^{-1},$$

then $\gamma' \in \Gamma_0(\mathfrak{M}) \subset \Gamma_0(\mathfrak{n})$. Writing $\gamma' = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$, we have

$$\mathcal{F}|_2\alpha(u/m)\gamma = \mathcal{F}|_2\gamma'\alpha(d^2u/m) = \mathcal{F}|_2\alpha(d^2u/m).$$

Therefore

$$\begin{aligned}
\mathcal{F}_\psi|_2\gamma &= G(\psi^{-1}, 1/m)^{-1} \sum_{u \in (\mathcal{O}/\mathfrak{m}_\psi)^*} \psi^{-1}(u)\mathcal{F}|_2\alpha(u/m)\gamma \\
&= G(\psi^{-1}, 1/m)^{-1} \sum_{u \in (\mathcal{O}/\mathfrak{m}_\psi)^*} \psi^{-1}(u)\mathcal{F}|_2\alpha(d^2u/m) \\
&= \psi(d^2)G(\psi^{-1}, 1/m)^{-1} \sum_{u \in (\mathcal{O}/\mathfrak{m}_\psi)^*} \psi^{-1}(d^2u)\mathcal{F}|_2\alpha(d^2u/m) \\
&= \psi^2(d)\mathcal{F}_\psi
\end{aligned}$$

which implies that $f_\psi \in S_2(\Gamma_0(\mathfrak{M}), \psi^2)$. □

Lemma 4.5.3. *Let $\mathcal{F} \in S_2(\Gamma_0(\mathfrak{n}))$ and ψ a character of $(\mathcal{O}/\mathfrak{m}_\psi)^\times$. If $(\mathfrak{n}, \mathfrak{m}_\psi) = 1$, then*

$$\mathcal{F}_\psi|_2\omega(\mathfrak{n}\mathfrak{m}_\psi^2) = C_\psi \mathcal{G}_{\psi^{-1}}$$

where $\mathcal{G} = \mathcal{F}|_2\omega_\mathfrak{n}$ and

$$C_\psi = C_{\psi,\mathfrak{n}} = \psi(\mathfrak{n})G(\psi)/G(\psi^{-1}).$$

Proof. We will apply Miyake's treatment in [Miy06, Lemma 4.3.11] to our case without any new techniques.

For $u \in \mathcal{O}$ prime to $m \in \mathfrak{m}_\psi$, take $n, v \in \mathcal{O}$ and $N \in \mathfrak{n}$ so that $nm - Nuv = 1$. Then

$$\alpha(u/m)\omega(Nm^2) = m \cdot \omega(N) \begin{pmatrix} m & -v \\ -uN & n \end{pmatrix} \alpha(v/m). \tag{4.46}$$

Since $\mathcal{G} = \mathcal{F}|_2\omega_\mathfrak{n}$ belongs to $S_2(\Gamma_0(\mathfrak{n}))$, (4.46) implies

$$\mathcal{F}|_2\alpha(u/m)\omega(Nm^2) = \mathcal{G}|_2\alpha(v/m),$$

so that

$$\begin{aligned} G(\psi^{-1})\mathcal{F}_\psi|_2\omega(Nm^2) &= \sum_{u\in(\mathcal{O}/\mathfrak{m}_\psi)^\times} \psi^{-1}(u)\mathcal{F}|_2\alpha(u/m)\omega(Nm^2) \\ &= \sum_{v\in(\mathcal{O}/\mathfrak{m}_\psi)^\times} \psi(-Nv)\mathcal{G}|_2\alpha(v/m) \\ &= \psi(-N) \sum_{v\in(\mathcal{O}/\mathfrak{m}_\psi)^\times} \psi(v)\mathcal{G}|_2\alpha(v/m). \end{aligned}$$

Then the assertion follows immediately. □

Combining Lemma 4.5.3 and Proposition 4.5.1, for the central value at $s = 1$ we obtain:

Proposition 4.5.4. *For $\mathcal{F} \in S_2^{\mathrm{new},+}(\Gamma_0(\mathfrak{n}))$ and ψ a quadratic Hecke character, we have*

$$L(\mathcal{F}_\psi, 1) = -\varepsilon_\mathfrak{n}\psi(\mathfrak{n})L(\mathcal{F}_\psi, 1).$$

Let $\mathfrak{n}$, $\chi_\mathfrak{m}$ and $\mathfrak{m}$ be as in Theorem 4.4.17. For $\mathcal{F} \in S_2^{\mathrm{new},+}(\Gamma_0(\mathfrak{n}))$, it follows that for the non-vanishing of $L(\mathcal{F}, \chi_\mathfrak{m}, 1) = L(\mathcal{F}_{\chi_\mathfrak{m}}, 1)$, we need at least $\varepsilon_\mathfrak{n}\chi_\mathfrak{m}(\mathfrak{n}) = -1$.

Lemma 4.5.5. *Given a Bianchi modular form $\mathcal{F} \in S_2^{\mathrm{new},+}(\Gamma_0(\mathfrak{n}))$, there always exists a quadratic Hecke character $\chi_\mathfrak{m}$ of conductor $\mathfrak{m}$ such that $\varepsilon_\mathfrak{n}\chi_\mathfrak{m}(\mathfrak{n}) = -1$.*

Proof. Assume that

$$\varepsilon_{\mathfrak{n}}\chi_{\mathfrak{m}}(\mathfrak{n}) = \prod_{\text{prime } \mathfrak{q}_i|\mathfrak{n}} \varepsilon_{\mathfrak{q}_i}\chi_{\mathfrak{m}}(\mathfrak{q}_i) = -1.$$

We denote, for each prime $\mathfrak{q}_i$ dividing $\mathfrak{n}$,

$$\lambda_{\mathfrak{q}_i} := \chi_{\mathfrak{m}}(\mathfrak{q}_i)\varepsilon_{\mathfrak{q}_i} \in \{\pm 1\}. \tag{4.47}$$

Recall the Chinese Remainder Theorem in the following. Let $N = \prod_i n_i$ with the n_i being pairwise coprime. Given any integer a_i there exists an integer x such that $x \equiv a_i \pmod{n_i}$ for every i. To solve the system of congruences consider $N_i = N/n_i$ and then there exists integers M_i such that $N_i M_i \equiv 1 \pmod{n_i}$. A solution of the system of congruences is $x = \sum_i a_i N_i M_i$. The way for computing the solution can also be applied into principal ideal domains.

Recall the quadratic residue symbol from [Neu99, Chapter V]. The quadratic residue symbol for $\mathcal{O}$ is defined by, for a prime ideal $\mathfrak{p} \subset \mathcal{O}$,

$$\left(\frac{\alpha}{\mathfrak{p}}\right) = \alpha^{\frac{N\mathfrak{p}-1}{2}} \bmod \mathfrak{p}.$$

It has properties completely analogous to those of classical Legendre symbol

$$\left(\frac{\alpha}{\mathfrak{p}}\right) = \begin{cases} 0, & \alpha \in \mathfrak{p}, \\ 1, & \alpha \notin \mathfrak{p} \text{ and } \exists \eta \in \mathcal{O} : \alpha \equiv \eta^2 \bmod \mathfrak{p}, \\ -1, & \alpha \notin \mathfrak{p} \text{ and there is no such } \eta. \end{cases}$$

The quadratic residue symbol can be extended to take non-prime ideals or non-zero elements as its denominator, in the same way that the Jacobi symbol extends the Legendre symbol. For $0 \neq \beta \in \mathcal{O}$ then we define $\left(\frac{\alpha}{\beta}\right) := \left(\frac{\alpha}{(\beta)}\right)$ where (β) is the principal ideal generated by β. Analogous to the Jacobi symbol, this symbol is multiplicative in the top and bottom parameters.

We are interested in the quadratic reciprocity law in the case of the imaginary quadratic field $F = \mathbb{Q}(\sqrt{d})$ with class number one (see [Hec81, Chapter VIII]. For any $\alpha \in \mathcal{O}$ with odd norm we define elements $t_\alpha, t'_\alpha \in \mathbb{Z}/2\mathbb{Z}$ by

$$\alpha \equiv \sqrt{d}^{t_\alpha}(1+2\sqrt{d})^{t'_\alpha}\xi^2 \bmod 4 \quad \text{for } \xi \in \mathcal{O}.$$

Then the quadratic reciprocity law for coprime elements of odd norm is given by

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = (-1)^T$$

where

$$T \equiv \begin{cases} t_\alpha t'_\beta + t'_\alpha t_\beta + t_\alpha t_\beta \bmod 2, & \text{if } d \equiv 1, 2 \bmod 4 \\ t_\alpha t'_\beta + t'_\alpha t_\beta \bmod 2, & \text{if } d \equiv 3 \bmod 4. \end{cases}$$

In particular, if $\alpha \equiv 1$ mod 4, we can observe that $t_\alpha = t'_\alpha = 0$ which implies that $T \equiv 0$ mod 2. It follows that

$$\left(\frac{\alpha}{\beta}\right)\left(\frac{\beta}{\alpha}\right) = 1 \quad \text{for } \alpha \equiv 1 \text{ mod } 4. \tag{4.48}$$

We want to find a quadratic character defined by the quadratic residue symbol, $\chi_{\mathfrak{m}} = \left(\frac{\cdot}{m\sqrt{d}}\right)$, such that $\chi_{\mathfrak{m}}(\mathfrak{q}_i) = \varepsilon_{\mathfrak{q}_i}\lambda_{\mathfrak{q}_i}$ for $\lambda_{\mathfrak{q}_i}$ given in (4.47). By our assumption m is the product of inert or split primes. We can impose that $m \equiv 1$ mod 4 to get $\left(\frac{\cdot}{m}\right) = \left(\frac{m}{\cdot}\right)$ by the above quadratic reciprocity law (4.48). To achieve $\chi_{\mathfrak{m}}(\mathfrak{q}_i) = \left(\frac{m}{\mathfrak{q}_i}\right)\left(\frac{\mathfrak{q}_i}{\sqrt{d}}\right) = \varepsilon_{\mathfrak{q}_i}\lambda_{\mathfrak{q}_i}$, we need $\left(\frac{m}{\mathfrak{q}_i}\right) = \left(\frac{\mathfrak{q}_i}{\sqrt{d}}\right)\varepsilon_{\mathfrak{q}_i}\lambda_{\mathfrak{q}_i}$ which can be done via imposing congruence conditions $(*)$ on m modulo $\mathfrak{q}_i$. Therefore, by the Chinese remainder theorem, there exists a m satisfying

$$\begin{cases} m \equiv 1 \text{ mod } 4 \\ \text{congruence conditions } (*) \text{ on } m \text{ mod } \mathfrak{q}_i \text{ for each prime } \mathfrak{q}_i | \mathfrak{n}. \end{cases} \tag{4.49}$$

Now we have proven this lemma. □

Write $S := \{\text{place } v : v \mid 2|d|\mathfrak{n}\}$. Let ξ be a quadratic idelic Hecke character of conductor $M_\xi\mathcal{O}$ such that $M_\xi \equiv 1 \bmod 4$, $M_\xi \equiv m \bmod \mathfrak{q}_i$ for each $\mathfrak{q}_i|\mathfrak{n}$ and at v dividing $\sqrt{d}\mathcal{O}$ the local component ξ_v is ramified with square-free conductor. Note that its conductor is coprime to $2\mathfrak{n}$ and divisible by $\sqrt{d}\mathcal{O}$, and so is its induced character χ_ξ of $(\mathcal{O}/M_\xi\mathcal{O})^\times$. Also we can observe that M_ξ satisfies the conditions in (4.49). So, by the preceding lemma there exists a χ_ξ attached to ξ such that $\varepsilon_{\mathfrak{n}}\chi_\xi(\mathfrak{n}) = -1$. Let $\Psi(S;\xi)$ denote the set of quadratic characters χ_ξ such that $\widetilde{\chi}_{\xi,v} = \xi_v$ for all $v \in S$. Recall from [FH95, Theorem B(1)]

Proposition 4.5.6. *Suppose π is a cuspidal automorphic representation of $\mathrm{GL}_2(\mathbb{A})$ which is self-contragredient. Suppose that for some quadratic character $\chi \in \Psi(S;\xi)$ one has root number $\epsilon(\pi\otimes\chi) = 1$. Then there exist infinitely many quadratic characters $\chi' \in \Psi(S;\xi)$ such that $L(\pi\otimes\chi', 1) \neq 0$.*

In Section 1.6 we have discussed the automorphic representation π on the space of weight 2 Bianchi modular forms. Also, we have shown that there exists a $\chi_\xi \in \Psi(S;\xi)$ such that $\varepsilon_{\mathfrak{n}}\chi_\xi(\mathfrak{n}) = -1$, i.e., $\epsilon(\pi\otimes\chi_\xi) = 1$. So we can apply the above proposition to deduce that, for $\mathcal{F} \in S_2^{\text{new},+}$, there are infinitely many quadratic characters $\chi \in \Psi(S;\xi)$ such that $L(\mathcal{F},\chi,1)$ is non-vanishing.

We will explain that these infinitely many quadratic characters always include a quadratic character with square-free conductor. This is necessary since the quadratic character $\chi_{\mathfrak{m}}$ as in Theorem 4.4.17 has the square-free conductor $\mathfrak{m}$. Suppose that $\Psi(S;\xi) \ni \chi_{\mathfrak{M}} : (\mathcal{O}/\mathfrak{M})^\times \to \mathbb{C}^\times$ is a quadratic Hecke character. Set $\mathfrak{M} = \prod_{\text{prime } \mathfrak{p}_i|\mathfrak{M}} \mathfrak{p}_i^{r_i}$ with $r_i \geq 1$. By the Chinese Remainder Theorem, we have $(\mathcal{O}/\mathfrak{M})^\times \simeq \prod_{\mathfrak{p}_i|\mathfrak{M}}(\mathcal{O}/\mathfrak{p}_i^{r_i})^\times$ and then can write $\chi_{\mathfrak{M}} = \prod \chi_{\mathfrak{M},\mathfrak{p}_i}$ with $\chi_{\mathfrak{M},\mathfrak{p}_i}$ defined on $(\mathcal{O}/\mathfrak{p}_i^{r_i})^\times$. It is known that $(\mathcal{O}/\mathfrak{p}^r)^\times$ has cyclic order of either $p^r(p-1)$ for $\mathfrak{p}$ above split prime p or $p^{2(r-1)}(p^2-1)$ for $\mathfrak{p}$ above inert prime p. So $\chi_{\mathfrak{M},\mathfrak{p}_i}$ is induced from a character defined on $(\mathcal{O}/\mathfrak{p})^\times$ which implies that $\chi_{\mathfrak{M}}$ is induced from a primitive character $\chi_{\mathfrak{m}_0}$ of square-free conductor $\mathfrak{m}_0$.

We will show that the non-vanishing of $L(\mathcal{F}, \chi_{\mathfrak{M}}, 1)$ is equivalent to that of $L(\mathcal{F}, \chi_{\mathfrak{m}_0}, 1)$. Write $\mathfrak{M} = \mathfrak{m}_0\mathfrak{n}_0^2$. It is a fact that

$$L(\mathcal{F}, \chi_{\mathfrak{M}}, s) = L(\mathcal{F}, \chi_{\mathfrak{m}_0}, s) \prod_{v|\mathfrak{n}_0}(1 - a_{\mathcal{F}}(\mathfrak{p}_v)\chi_{\mathfrak{m}_0}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-s} + N(\mathfrak{p}_v)^{1-2s})$$

where $a_{\mathcal{F}}$ denotes the Fourier coefficient of $\mathcal{F}$. It suffices to show the non-vanishing of

$$1 - a_{\mathcal{F}}(\mathfrak{p}_v)\chi_{\mathfrak{m}_0}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-s} + N(\mathfrak{p}_v)^{1-2s} \quad \text{at } s = 1$$

which can be rewritten as the Hecke polynomial

$$(1 - \alpha_{\mathcal{F}}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-s})(1 - \beta_{\mathcal{F}}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-s}).$$

As $|\alpha_{\mathcal{F}}(\mathfrak{p}_v)| < N(\mathfrak{p}_v)$ and $|\beta_{\mathcal{F}}(\mathfrak{p}_v)| < N(\mathfrak{p}_v)$, we can deduce the non-vanishing of

$$(1 - \alpha_{\mathcal{F}}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-1})(1 - \beta_{\mathcal{F}}(\mathfrak{p}_v)N(\mathfrak{p}_v)^{-1}).$$

As $\mathfrak{m}_0$ is square-free, divisible by $\sqrt{d}$ and coprime to $\mathfrak{n}$ such that $L(\mathcal{F}, \chi_{\mathfrak{m}_0}, 1)$ is non-vanishing, following Theorem 4.4.17 we can deduce that

Theorem 4.5.7. *Given a Bianchi modular form $\mathcal{F} \in S_2^{\text{new},+}$ with $\mathfrak{n}$ coprime to $d_F\mathcal{O}$, there always exists a quadratic Hecke character such that the theta lifting as in Theorem 4.4.17 is non-vanishing.*

Example 4.5.8. Let $F = \mathbb{Q}(\sqrt{-3})$ with $\mathcal{O} = \mathbb{Z}[\omega]$ and $d_F = d = -3$. Consider the weight 2 Bianchi modular form $\mathcal{F}$ of level $\Gamma_0(\mathfrak{p}_{283.1})$ with $\mathfrak{p}_{283.1} = (19\omega - 13)$ above split prime 283 (LMFDB label: 2.0.3.1-283.1-a). It has root number -1 which implies that $L(\mathcal{F}, 1)$ is vanishing. The Atkin-Lehner eigenvalue is $\varepsilon_{\mathfrak{p}_{283.1}} = 1$. Using Magma calculator, we can find a quadratic Hecke character χ of conductor $\mathfrak{f} = (7\sqrt{d})$ such that $\chi(\mathfrak{p}_{283.1})\varepsilon_{\mathfrak{p}_{283.1}} = -1$:

```
K:=QuadraticField(-3);
OK<w>:=Integers(K);
I:=7*OK*(2*w-1);
H:= HeckeCharacterGroup(I);
```

```
H;
chi:= H.1;
Order(chi);
p283_1:=Factorization(283*OK)[1,1];
p283_1;
chi(p283_1);

Abelian Group isomorphic to Z/2 + Z/6 given as Z/2 + Z/6
Group of Hecke characters H of modulus of norm 147
over Quadratic Field with defining polynomial $.1^2 + 3
over the Rational Field mapping to Cyclotomic Field
of order 6 and degree 2
2
Prime Ideal of OK
Two element generators:
    283
    2*w + 88
-1
```

Then, by Theorem 4.4.17, the Fourier coefficient at β (as in Example 4.4.6 or 4.4.11) equals to I_∞ which is non-vanishing if $L(\mathcal{F}, \chi, 1) \neq 0$.

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