Geometric models of soliton vortex dynamics

René Israel García Lara

School of Mathematics

University of Leeds

A thesis submitted for the degree of

Doctor of Philosophy

March 2021

To my daughter.

Acknowledgements

I would like to thank my supervisor, Martin Speight, for his valuable guidance and support. His experience and insight were encouraging and motivating, which I truly appreciate.

My gratitude to Derek Harland and Bernd Schroers for their comments on the final version of this thesis.

Many thanks to my family and friends for their kind support both at Leeds and abroad through constant calls and messages, specially to Eyra for her support at home and to Diana who shared with me so many moments even at the distance.

This project would have not been possible without the support of CONACyT through a PhD grant which I kindly acknowledge.

Abstract

In this work we focus on BPS solutions of the gauged $O(3)$ Sigma model, originally due to Schroers, and use these ideas to study the geometry of the moduli space. The model has an asymmetry parameter τ breaking the symmetry of vortices and antivortices on the field equations. It is shown that the moduli space is incomplete both on the Euclidean plane and on a compact surface. On the Euclidean plane, the L^2 metric on the moduli space is approximated for well separated cores and results consistent with similar approximations for the Ginzburg-Landau functional are found. The scattering angle of approaching vortex-antivortex pairs of different effective mass is computed numerically and is shown to be different from the well known scattering of approaching Ginzburg-Landau vortices. The volume of the moduli space for general τ is computed for the case of the round sphere and flat tori.

The model on a compact surface is deformed introducing a neutral field and a Chern-Simons term. A lower bound for the Chern-Simons constant κ such that the extended model admits a solution is shown to exist, and if the total number of vortices and antivortices are different, the existence of an upper bound is also shown. Existence of multiple solutions to the governing elliptic problem is established on a compact surface as well as the existence of two limiting behaviours as $\kappa \to 0$. A localization formula for the deformation is found for both Ginzburg-Landau and the $O(3)$ Sigma model vortices and it is shown that it can be extended to the coalescense set. This rules out the possibility that this is Kim-Lee's term in the case of Ginzburg-Landau vortices, moreover, the deformation term is compared on the plane with the

Ricci form of the surface and it is shown they are different, hence also discarding that this is the term proposed by Collie-Tong to model vortex dynamics with Chern-Simons interaction.

Contents

Chapter 1

Introduction

This work is about the geometry of moduli spaces of vortices and antivortices on a Riemann surface Σ. We are interested mostly in the gauged $O(3)$ Sigma model, where the fields are represented by a connection A and a section ϕ of a fibre bundle with fibres diffeomorphic to $\mathbb{P}^1$, the Riemann sphere. We say ϕ is a Higgs field with target the Riemann sphere. Static solutions of the field equations modulo gauge equivalence form the moduli space of vortices and antivortices, each solution is determined by the cores of the fields: the preimages of the north pole (vortex points) and the south pole (antivortex points). It can be proved the total number of the cores is enumerable and if Σ is compact, it is finite. We will assume without loss of generality this is the case, even though Σ can be the complex plane. The dynamics of slowly varying fields can be described by geodesic motion of curves on the moduli space [32] with a metric called the L^2 metric. This metric is Kähler and well understood for the moduli space of vortices of the Ginzburg-Landau functional, in which case it is known that the moduli space is a complete metric space and if the ambient surface is compact the moduli space is also compact, hence of finite volume.

The $O(3)$ Sigma model we will study is asymmetric, vortices and antivortices have different effective mass, moreover, the existence of two types of cores means vortices and antivortices cannot coalesce, therefore, a natural question is if the moduli space is still complete. Another question we address is how the asymmetry affects the volume of the moduli space. These questions were addressed for the symmetric case in the reference [45]. The techniques used in the reference however do not apply in general, we developed analytical tools to extend the results to the asymmetric case.

Later, we add a Chern-Simons deformation to the model and describe the change in the dynamics of the fields on the moduli space. The deformation is tuned by means of a deformation constant κ which we assume small. It turns out that the dynamics of the theory is described by geodesic motion perturbed with a connection term proportional to κ, i.e. a term dependent on the velocity of the cores. Our model resembles the model of Kim and Lee [26] with the difference that the target is the sphere and there are two types of cores to consider. It is well known for several related models with Chern-Simons deformations that multiple solutions of the field equations occur. We study the problem of existence and multiplicity of solutions to the field equations of the deformed $O(3)$ Sigma model, the main result is that even though multiple solutions of the equations can exist, there is a minimal deformation, such that no matter which configuration of vortices and antivortices on the moduli space we choose, we can find exactly one solution close to the undeformed solution of the $O(3)$ Sigma model.

We conclude with a description of the chapters of the thesis.

In chapter 2 we describe the ideas of localization in abstract terms. Our approach is general and suits equally well Ginzburg-Landau vortices as well as the $O(3)$ Sigma model, with the benefit that it makes clear what we mean by adding a Chern-Simons term. We also present analytical results that are common to other parts of the next chapters.

In chapter 3 we focus on the $O(3)$ Sigma model on the euclidean plane. We study asymmetric vortex-antivortex pairs, supporting our analysis with numerical evidence of the behaviour of colliding vortex-antivortex pairs. We compute the metric on the moduli space of vortex-antivortex pairs numerically and use this computations to study the scattering of approaching cores. The main result is theorem 3.14 which says that the moduli space is incomplete.

In chapter 4 we move to a compact ambient surface. The main results are the incompleteness of the moduli space of vortex-antivortex pairs, theorem 4.15, and the computation of the volume of the moduli space for the round sphere and for flat tori in theorem 4.25, confirming a general conjecture by Romão-Speight [45] in these cases.

Chapter 5 is devoted to the study of Chern-Simons deformations on compact surfaces. We prove the existence of multiple solutions for small deformations of the $O(3)$ Sigma model if the number of vortices and antivortices is different and find bounds for the deformation constant. We also solve the field equations numerically on the sphere for two configurations of vortices and antivortices at antipodal positions. The main result is theorem 5.18, describing the behaviour of the solutions to the field equations. We finalise the chapter applying the localization technique to vortices of the Ginzburg-Landau model and vortices/antivortices of the $O(3)$ Sigma model, both with a Chern-Simons deformation. We found that dynamics is deviated from geodesic motion by a connection term consistent with previous results of Kim-Lee [26] and Collie-Tong [10], and compared our result with theirs.

Chapter 2

Preliminaries

This chapter is for basic definitions and results of field theory that we will use in the successive. To study the geometry of the moduli space of vortices we need several analytical tools, this chapter is intended to be a bridge between field theory and analysis.

In section 2.1 we introduce the $O(3)$ Sigma model, which will play a central role all along the thesis.

In section 2.2 we discuss a localization formula for the $O(3)$ Sigma model, we compute a metric for the moduli space of vortices and antivortices, the L^2 metric, and prove that it is Kähler.

Section 2.3 is about the analytic properties of the Taubes equation, this is the elliptic PDE that guarantees the existence of the moduli space of vortices and antivortices. Several theorems of analysis are introduced in this section to keep them collected in the same place for further reference. In subsection 2.3.1 we prove that the solution to the Taubes equation depends differentiably on the position of the vortices and antivortices.

In section 2.4 we state less known theorems of functional analysis about compact non-linear operators that we will need later.

2.1 Field theory on complex line bundles

In this section we introduce notation and a few facts about $\mathbb{P}^1$ fibre bundles that will be required for most of the work.

Let us start considering a principal $U(1)$ bundle $U(1) \to P \to \mathbb{R} \times \Sigma$, where Σ is a Riemann surface. No further assumption on Σ is needed. Let M be an n-dimensional manifold, such that there exists a homomorphism

$$\rho : U(1) \to \mathrm{Aut}(M), \tag{2.1}$$

from the structure group to the group of automorphisms of M. The word automorphism means that if M has an extra structure, for example, if is a symplectic or Kähler manifold, then ρ should preserve this structure. Let F be the fibre bundle associated to ρ,

$$F = (\mathbb{R} \times \Sigma) \times_\rho M. \tag{2.2}$$

Recall a connection form on P is a $\mathfrak{u}(1)$ valued form ω on P, such that the kernel $\mathrm{Ker}(\omega)$ defines the horizontal sub-bundle of TP. Since $U(1)$ is one dimensional, we can identify ω with a regular form. For any local section $s_a : U_a \subset \mathbb{R} \times \Sigma \to P$, the connection is given by a local form $A_a = s_a^*(\omega)$ such that in any overlap $U_a \cap U_b \neq \emptyset$ there is a transition function $\theta_{ab} : U_a \cap U_b \to \mathbb{R}$ satisfying the condition,

$$A_b = A_a + d\theta_{ab}. \tag{2.3}$$

$U(1)$ is an abelian group, hence the adjoint representation of the structure group is trivial, the group of gauge transformations in this case is

$$\mathcal{G} = C^\infty(\mathbb{R} \times \Sigma, U(1)). \tag{2.4}$$

The space of connections $\mathscr{A}$ is an affine space: for any two connection forms $\omega, \omega' \in \mathscr{A}$, the difference $\omega - \omega'$ determines a unique 1-form $A \in \Omega^1(\mathbb{R} \times \Sigma)$ such that if $s_a : U_a \to P$ is a local trivialisation, then $s_a^*(\omega - \omega')$ is the restriction of A to U_a. Therefore $\mathscr{A}$ is in bijection with $\Omega^1(\mathbb{R} \times \Sigma)$, the space of 1-forms on $\mathbb{R} \times \Sigma$. Let $\mathcal{A} = \Gamma F \times \mathscr{A}$ be the space of pairs of fields (ϕ, A), consisting of a section, $\phi : \mathbb{R} \times \Sigma \to F$, and a connection form $A \in \Omega^1(\mathbb{R} \times \Sigma)$.

The quotient $\mathcal{A}/\mathcal{G}$ is the configuration space $\mathcal{C}$. If $M = \mathbb{S}^2$, then ρ has two antipodal fixed points, the north and south poles. We choose one that we will denote as N and call it the north pole. In this setting F is a $\mathbb{P}^1$ bundle, the fibres are modelled on the complex projective line. The fact that ρ represents the unitary group by rotations of the sphere lets us pull the north pole back into a section $N : \mathbb{R} \times \Sigma \to F$. The south pole can also be pulled back into another section, that we denote by $-N$, however we must emphasise that F lacks any algebraic structure conferring other meaning to the name than a mere notation. We also denote by $X \in \mathfrak{X}(\mathbb{S}^2)$ the Killing field generated by ρ,

$$X_p = \left.\frac{d}{ds}\right|_{s=0} \left(\rho\left(e^{is}\right) \cdot p\right), \qquad p \in \mathbb{S}^2. \tag{2.5}$$

A section $\phi : \mathbb{R} \times \Sigma \to F$ is determined completely by the family of maps $\phi_\alpha : U_\alpha \to \mathbb{S}^2$ defined for each trivialising neighbourhood $U_\alpha \subset \mathbb{R} \times \Sigma$, if $U_{\alpha\beta} = U_\alpha \cap U_\beta \neq \emptyset$, we have

$$\phi_\beta(x) = \rho(\exp(i\theta_{\alpha\beta}(x))) \cdot \phi_\alpha(x), \qquad x \in U_{\alpha\beta} = U_\alpha \cap U_\beta. \tag{2.6}$$

Since ρ acts by isometries, we can define the product $\langle N, \phi\rangle$ using the trivialisations: for $x \in U_\alpha$,

$$\langle N(x), \phi(x)\rangle = \langle N, \phi_\alpha(x)\rangle. \tag{2.7}$$

We also define the covariant derivative of ϕ as the section

$$\mathcal{D}\phi : \mathbb{R} \times \Sigma \to T^*(\mathbb{R} \times \Sigma) \otimes \phi^*(TF) \tag{2.8}$$

determined by the trivialisations $\mathcal{D}\phi_\alpha : U_\alpha \to T^*U_\alpha \otimes T\mathbb{S}^2$ as,

$$\mathcal{D}\phi_\alpha = \mathbf{d}\phi_\alpha - \mathbf{A}_\alpha \otimes X_{\phi_\alpha}, \tag{2.9}$$

where $\mathbf{d}\phi_\alpha : TU_\alpha \to TF$ can be split into its temporal and spatial components,

$$\mathbf{d}\phi_\alpha = dt \otimes \partial_t\phi_\alpha + d\phi_\alpha, \qquad d\phi_\alpha(t, \cdot) \in T^*\Sigma \otimes (\phi_\alpha(t, \cdot))^*(TF). \tag{2.10}$$

Likewise, $\mathbf{A}_\alpha = A^0_\alpha\, dt + A_\alpha$, where $A^0_\alpha \in C^\infty(U_\alpha)$ and $A_\alpha(t, \cdot) \in \Omega^1(U_\alpha)$. If we define

$$\mathrm{D}_t\phi_\alpha = \partial_t\phi_\alpha - A^0_\alpha \otimes X_{\phi_\alpha}, \qquad \mathrm{D}\phi_\alpha = d\phi_\alpha - A_\alpha \otimes X_\phi, \tag{2.11}$$

then,

$$\mathcal{D}\phi_\alpha = dt \otimes \mathrm{D}_t\phi_\alpha + \mathrm{D}\phi_\alpha. \tag{2.12}$$

We introduce a Lorentzian metric as follows. If g denotes a Riemannian metric in Σ then the metric in $\mathbb{R}\times\Sigma$ is the product dt^2-g. This metric induces a metric in $\Omega^2(\mathbb{R}\times\Sigma)$. Recall the curvature form $\omega\in\Omega^2(\mathbb{R}\times\Sigma)$ is given in a local trivialisation by $\omega = d\mathbf{A}_\alpha$ and define the electric and magnetic forms, as the forms $e\in\Omega^1(\mathbb{R}\times\Sigma)$ and $B\in\Omega^2(\mathbb{R}\times\Sigma)$ respectively, such that,

$$\omega = dt\wedge e + B, \tag{2.13}$$

and for fixed t, $e(t,\cdot)\in\Omega^1(\Sigma)$, $B(t,\cdot)\in\Omega^2(\Sigma)$.

Although $|d\phi_\alpha|$ is gauge dependent, at the intersection $U_{\alpha\beta}$ of any two trivialisation neighbourhoods, $|\mathcal{D}\phi_\alpha| = |\mathcal{D}\phi_\beta|$, hence we can define

$$||\mathcal{D}\phi(t,\cdot)||^2 = ||\mathrm{D}_t\phi||^2 - ||\mathrm{D}\phi||^2. \tag{2.14}$$

With all these definitions, we can express the gauged $O(3)$ Lagrangian as,

$$\mathrm{L}_{O(3)} = \frac{1}{2}\left(||\mathrm{D}_t\phi||^2 + ||e||^2 - (||\mathrm{D}\phi||^2 + ||B||^2 + ||\tau - \langle N,\phi\rangle||^2)\right), \tag{2.15}$$

where the asymmetry parameter $\tau\in(-1,1)$ determines the vaccuum manifold and if Σ is non-compact, we must add suitable boundary conditions to ϕ and A to guarantee convergence of the norms. The $O(3)$ Lagrangian admits Bogomolny type static solutions in the temporal gauge, in which $A^0_\alpha = 0$. In this gauge, the total conserved energy of a time independent pair of fields (ϕ, A) is

$$\mathrm{E} = \frac{1}{2}\left(||\mathrm{D}\phi||^2 + ||B||^2 + ||\tau - \langle N,\phi\rangle||^2\right). \tag{2.16}$$

The temporal covariant derivative D_A can be decompose into holomorphic and anti-holomorphic parts,

$$\mathrm{D}_A = \partial_A + \overline{\partial}_A, \tag{2.17}$$

where in a local holomorphic coordinate chart U_α in which ϕ trivialises as $\phi_\alpha : U_\alpha \to \mathbb{S}^2$,

$$\partial_A\phi_\alpha = \frac{1}{2}\left(\mathrm{D}_A\phi_\alpha(\partial_1) - \phi_\alpha\times\mathrm{D}_A\phi_\alpha(\partial_2)\right), \quad \overline{\partial}_A\phi_\alpha = \frac{1}{2}\left(\mathrm{D}_A\phi_\alpha(\partial_1) + \phi_\alpha\times\mathrm{D}_A\phi_\alpha(\partial_2)\right), \tag{2.18}$$

We will consider the sets

$$\mathcal{P} = \phi^{-1}(N), \qquad \mathcal{Q} = \phi^{-1}(-N), \tag{2.19}$$

which we call the set of vortices and antivortices. The term vortex is of wide use for the Abelian Higgs model, where it refers to the zeros of the Higgs field. Both theories, the Abelian Higgs model and the $O(3)$ Sigma model, have similarities, for example the $U(1)$ symmetry of the fields, hence it is natural to refer to vortices of the $O(3)$ Sigma model, on the other hand, the term antivortex, which is also used in the literature, stresses the distinction with vortices, since vortices and antivortices cannot coalesce. We assume that both sets are finite. In proposition 2.1 we define the Bogomolny equations.

Proposition 2.1. *If (ϕ, A) is a solution of the Bogomolny equations,*

$$\overline{\partial}_A \phi = 0, \tag{2.20}$$

$$*B = \langle N, \phi \rangle - \tau, \tag{2.21}$$

then the pair minimises the energy of the $O(3)$ Lagrangian and the minimum energy is,

$$\mathrm{E} = 2\pi(1-\tau)\,k_+ + 2\pi(1+\tau)\,k_-. \tag{2.22}$$

Proposition 2.1 should be attributed to several authors who proved it for the different cases. On the plane it was proved by Schroers [50] for $\tau = 1$ and later for general τ in [49]. On a compact manifold for $\tau = 0$ it was proved by Sibner, Sibner and Yang [51]. Speight and Rõmao [45] give another proof which is suitable for both a compact surface and the euclidean plane, which we adapt.

Proof. We distinguish two cases. Firstly, let us assume that Σ is compact. We can choose an open and dense set $U \subset \Sigma$ holomorphic to the unit disc such that it contains $\mathcal{P} \cup \mathcal{Q}$. Since U is contractible, the restriction $F\,|_U$ can be trivialised. In this trivialisation, ϕ is equivalent to a function $\varphi : U \to \mathbb{S}^2$. Since the action of $U(1)$ in the sphere is Hamiltonian, we can consider the moment map $\mu : \mathbb{S}^2 \to \mathbb{R}$,

$$\mu(p) = \langle N, p \rangle - \tau. \tag{2.23}$$

If ω denotes the symplectic form in the sphere, then $d\mu = \iota_X \omega$. Let us denote by $\mathrm{J} : T\mathbb{S}^2 \to T\mathbb{S}^2$, $\mathrm{J}_x(v) = x \times v$ the almost complex structure on the sphere. Recall the basic identity,

$$\langle \mathrm{J}\, v, w \rangle = \omega(v, w), \qquad v, w \in T_x\mathbb{S}^2. \tag{2.24}$$

We will use the Bogomolny trick,

$$\begin{aligned} 0 &\leq \frac{1}{2}\left(||\mathrm{D}_1\varphi + \mathrm{J}\,\mathrm{D}_2\varphi||^2 + ||{*B} - \mu\circ\varphi||^2\right) \\ &= \mathrm{E} + \langle \mathrm{D}_1\varphi, \mathrm{J}\,\mathrm{D}_2\varphi\rangle - \langle *B, \mu\circ\varphi\rangle \\ &= \mathrm{E} + \langle \partial_1\varphi, \mathrm{J}\,\partial_2\varphi\rangle + \int_U \omega(X_\varphi, A_1\partial_2\varphi - A_2\partial_1\varphi)\,\mathrm{Vol} - \langle *B, \mu\circ\varphi\rangle \\ &= \mathrm{E} + \int_U \left(-\varphi^*\omega + A\wedge d(\mu\circ\varphi) - B\wedge \mu\circ\varphi\right) \\ &= \mathrm{E} - \int_U \left(\varphi^*\omega + d\left(\mu\circ\varphi\cdot A\right)\right). \end{aligned} \tag{2.25}$$

Note that $\varphi^*\omega + d(\mu\circ\varphi\cdot A)$ is gauge invariant and can be extended to all of Σ. Introducing spherical coordinates (ϑ, ϱ) in $\mathbb{S}^2$ with ϱ the azimuthal angle, we define the one form,

$$\varpi = \varphi^*(d\varrho) - A \in \Omega^1(U \setminus \mathcal{P}\cup\mathcal{Q}), \tag{2.26}$$

and note that ϖ is gauge invariant and therefore also extends to $\Sigma\setminus\mathcal{P}\cup\mathcal{Q}$. If we denote by $\mathbb{D}_\epsilon$ a collection of disjoint ϵ-disks, each one centred at one point $x \in \mathcal{P}\cup\mathcal{Q}$, then,

$$\begin{aligned} \int_U \left(\varphi^*\omega + d\left(\mu\circ\varphi\cdot A\right)\right) &= -\int_{\Sigma\setminus\mathcal{P}\cup\mathcal{Q}} d(\langle N, \phi\rangle\varpi) - \tau\int_\Sigma B \\ &= \lim_{\epsilon\to 0}\int_{\partial\mathbb{D}_\epsilon} \langle N, \phi\rangle\varpi - \tau\int_\Sigma B \\ &= 2\pi(k_+ + k_-) - 2\pi\tau(k_+ - k_-) \\ &= 2\pi(1-\tau)\,k_+ + 2\pi(1+\tau)\,k_-. \end{aligned} \tag{2.27}$$

Hence,

$$\mathrm{E} \geq 2\pi(1-\tau)\,k_+ + 2\pi(1+\tau)\,k_-, \tag{2.28}$$

and the energy is minimised if (ϕ, A) is a solution to the Bogomolny equations. If Σ instead is the Euclidean plane, we have to assume that $D\phi$, B and $\mu \circ \phi$ are L^2 sections of their respective bundles. In this case we can take $U = \mathbb{R}^2$, and most of the proof follows verbatim the previous steps, except that to compute the integral (2.27) we must suppose that the fields satisfy the boundary condition,

$$\lim_{|x| \to \infty} (\langle N, \phi \rangle - \tau) = 0. \tag{2.29}$$

□

We started assuming the sets $\mathcal{P}$ and $\mathcal{Q}$ where finite and found that a pair (ϕ, A) of solutions to the Bogomolny equations minimises the static energy. In the compact case, the assumption about the size of the sets is redundant, the proof for $\tau = 0$ found in [51] can be adapted to the asymmetric case.

Proposition 2.2. *If (ϕ, A) is a solution to the Bogomolny equations, then $\mathcal{P}$ and $\mathcal{Q}$ are discrete. In particular, if Σ is compact, these are finite sets. Moreover, if $x \in \mathcal{P} \cup \mathcal{Q}$, then $\phi(x)$ is of finite degree, in the sense that there is a unique positive integer d such that if $x \in \mathcal{P} \cup \mathcal{Q}$ and $\varphi : U \to \mathbb{C}$, $\pi : V \subset \mathbb{S}^2 \to \mathbb{C}$, are holomorphic coordinates about x and $\phi(x)$ with $\varphi(x) = \pi(\phi(x)) = 0$, then there is a smooth function $R : \varphi(U) \to \pi(V)$ such that,*

$$\pi \circ \phi \circ \varphi^{-1}(z) = z^d R(z), \qquad \forall z \in \varphi(U), \tag{2.30}$$

but $R(0) \neq 0$.

Proof. Suppose $x \in \mathcal{P}$ and $\phi_\alpha : U_\alpha \to \mathbb{S}^2$ is a local trivialisation in an holomorphic chart $\varphi_\alpha : U_\alpha \to \mathbb{C}$ with $\varphi_\alpha(x) = 0$. Let $\pi_- : \mathbb{S}^2 \setminus \{-N\} \to \mathbb{C}$ be south pole stereographic projection and let

$$\psi_\alpha = \pi_- \circ \phi_\alpha \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \setminus \mathcal{Q}) \to \mathbb{C}. \tag{2.31}$$

Since π_- is a holomorphic local diffeomorphism, the first Bogomolny equation is equivalent in these charts to,

$$\bar{\partial}\psi_\alpha = \frac{1}{2}(-A_2 + A_1 i)\psi_\alpha. \tag{2.32}$$

If A is smooth, by the $\bar{\partial}$-Poincare lemma, there exists a smooth function $w : \varphi_\alpha(U_\alpha \setminus \mathcal{Q}) \to \mathbb{C}$ such that $\bar{\partial} w = \frac{1}{2}(-A_2 + A_1 i)$, hence the function $e^w \psi_\alpha$ is holomorphic, $\overline{\partial}(e^w\psi_\alpha) = 0$ and the zero set of ψ_α is discrete unless $\psi_\alpha \equiv 0$ which is impossible because it violates the Bogomolny equations. This proves that $\mathcal{P}$ is a discrete set. Since $e^w\psi_\alpha$ is holomorphic, the assertion about the degree follows in these charts and since the degree is an holomorphic invariant, this proves the claim for any other holomorphic chart. Using the north pole stereographic projection proves similar claims for $\mathcal{Q}$. □

We say that $x \in \mathcal{P}$ is the position of a single vortex if the degree is 1 and similarly for $x \in \mathcal{Q}$, if the degree is 1 we say that x is the position of a single antivortex. We will denote the size of the sets $\mathcal{P}$, $\mathcal{Q}$ as $k_\pm$ respectively, where we count each vortex and antivortex with multiplicity.

For any solution (ϕ, A) to the Bogomolny equations, we define the function $h : \Sigma \setminus \mathcal{P} \cup \mathcal{Q} \to \mathbb{R}$,

$$h = \log\left(\frac{1 - \langle N, \phi\rangle}{1 + \langle N, \phi\rangle}\right). \tag{2.33}$$

If we define the map $\psi_\alpha : \pi_- \circ \phi_\alpha : U_\alpha \to \mathbb{C}$ as in the proof of proposition 2.2, where $\phi_\alpha : U_\alpha \to \mathbb{S}^2$ represents ϕ in a local trivialisation $U_\alpha \subset \Sigma \setminus \mathcal{P} \cup \mathcal{Q}$, and $\pi_- : \mathbb{S}^2 \setminus \{-N\} \to \mathbb{C}$ is south pole's stereographic projection, then $\exp(h) = |\psi_\alpha|^2$ and $\log\psi_\alpha = \frac{h}{2} + \chi_\alpha i$, where the argument function $\chi_\alpha : U_\alpha \to \mathbb{R}$ is gauge dependent. By equation (2.32),

$$-\frac{1}{4}\Delta(\log\psi_\alpha) = \frac{1}{2}\partial(-A_2 + A_1 i) = \frac{1}{4}\left(-(\partial_1 A_2 - \partial_2 A_1) + (\partial_1 A_1 + \partial_2 A_2)i\right), \tag{2.34}$$

where Δ is geometer's laplacian, which in the holomorphic coordinates we are considering is of the form $\Delta = -e^{-\Lambda}(\partial_1^2 + \partial_2^2)$, where e^Λ is the conformal factor of the metric. Taking the real part of the previous equation, we find,

$$-\Delta h = -2 * B = 2\left(\frac{e^h - 1}{e^h + 1} + \tau\right). \tag{2.35}$$

If $x \in \mathcal{P} \cup \mathcal{Q}$ has degree d_x, we can extend the definition of h to the core set $\mathcal{P} \cup \mathcal{Q}$ by requiring it to be a solution to [49],

$$-\Delta h = 2\left(\frac{e^h - 1}{e^h + 1} + \tau\right) + 4\pi\sum_{x\in\mathcal{P}} d_x\delta_x - 4\pi\sum_{x\in\mathcal{Q}} d_x\delta_x, \tag{2.36}$$

where δ_x is Dirac's measure concentrated at x. For any test function $\varphi \in C_0^\infty(\Sigma)$,

$$\int_\Sigma \varphi\, \delta_x = \varphi(x). \tag{2.37}$$

Notice δ_x includes the measure on Σ. We will call equation (2.36) *the Taubes equation*, as is analogous to the equation studied by Taubes for the Ginzburg-Landau functional [56]. The Taubes equation as given by (2.36) was also obtained by Schroers in [49].

2.2 Localization

The idea of a localization formula originates in the work of Strachan [54]. It was later generalised by Samols [47] and is based on ideas about geodesic approximation originating in [32]. From his work, Strachan and Samols developed approximations to the dynamics of the Abelian Higgs model in the moduli space of static solutions of the field equations, later, Stuart proved in [55] that the moduli space approximation is correct. The results of Stuart also extended to other field theories, for example in [12] Demoulini-Stuart proved a moduli space approximation to the dynamics of the Chern-Simons-Schrödinger model proposed by Manton in [33]. On the other hand, for some field theories it is possible to find an explicit formula for a metric on the moduli space governing the dynamics, such that it only depends of local data, i.e., the position of the cores of the field ϕ. Over the time, the localization formula has been refined and extended to other field theories, e.g. Chern-Simons vortices [10, 25] or Ginzburg-Landau vortices with electric and magnetic impurities [58]. We can describe in an unified way the idea behind localization if we introduce the L^2 metric in the space of fields modulo gauge transformations. By this we mean the space of sections of a given $U(1)$ fibre bundle as described on section 2.1. There are several situations in which this space is finite dimensional, for example for BPS solitons of the Ginzburg-Landau functional. In this case, there are rigorous proofs of this fact [56, 60]. We make no assumption on finite dimensionality though, since the theory can be written in full generality. We restrict the previous field theoretic setup to the static case and think of $\mathcal{A} \to \mathcal{C}$ as an infinite dimensional principal

$\mathscr{G}$-bundle [43]. A curve $(\phi_s, A_s) : I \to \mathcal{A}$ is said to be differentiable, if for any $x \in \Sigma$, the curves $s \mapsto \phi_s(x)$, $s \mapsto A_s(x)$, where,

$$\phi_s(x) : I \to F, \qquad A_s(x) : I \to T_x\Sigma, \tag{2.38}$$

are differentiable. For a differentiable curve in field space, the variation is the pair $(\delta\phi, \delta A)$,

$$\delta\phi : \Sigma \to \phi^* TF, \qquad \delta A \in \Omega^1(\Sigma), \tag{2.39}$$

of pointwise derivatives:

$$\delta\phi(x) = \frac{d}{ds}\bigg|_{s=0} \phi_s(x), \qquad \delta A(x) = \frac{d}{ds}\bigg|_{s=0} A_s(x). \tag{2.40}$$

We will think of the space of variations as the tangent space of $\mathcal{A}$ and denote it as $\mathcal{T}$. If $\alpha \in \mathcal{G}$ is a gauge transformation, the fields transform as,

$$e^{i\alpha} \cdot \phi, \qquad A + d\alpha, \tag{2.41}$$

where the product $e^{i\alpha} \cdot \phi$ is to be understood as the action of $e^{i\alpha}$ in ϕ via the representation ρ. Equation (2.41) defines an action $\alpha * (\phi, A)$, of the gauge group in the space of fields. This action extends naturally to tangent space. By an abuse in notation, let us denote by X the vector field induced in target space by this action, then $\mathcal{G}$ acts in $\mathcal{T}$ as,

$$\alpha * (\delta\phi, \delta A) = (\delta\phi + \alpha X_\phi, \delta A + d\alpha). \tag{2.42}$$

Moreover, the vertical space,

$$\mathcal{V}_{(\phi,A)} = \{(\alpha X_\phi, d\alpha) \mid \alpha \in \mathcal{G}\}, \tag{2.43}$$

determines a sub-bundle of $\mathcal{T}$ whose fibre is in bijection with the Lie algebra $\mathcal{G}$ of gauge transformations [43].

The Riemannian metrics in Σ and M extend to metrics in the cotangent bundle $T^*\Sigma$ and F respectively, which on the other hand, extend to a metric in the space of fields: if $(\phi, A) \in \mathcal{A}$ and $(\delta\phi, \delta A) \in \mathcal{T}_{(\phi,A)}$, the L^2-metric is the product of metrics induced by the Riemannian structure in the domain and the target space,

$$||(\delta\phi, \delta A)||^2_{\mathcal{A}} = ||\delta\phi||^2_{\mathrm{L}^2(\Sigma,F)} + ||\delta A||^2_{\mathrm{L}^2(\Sigma)}. \tag{2.44}$$

$\mathcal{C}$ is the relevant space for applications, as two field configurations differing by a gauge transformation are regarded as physically the same. In analogy to a finite dimensional vector bundle, the L^2-metric can be used to split $\mathcal{T}$ in a direct sum of the vertical space $\mathcal{V}$ and its orthogonal complement. If the quotient $\mathcal{C}$ has a finite dimensional differentiable structure, this complement can be identified with its tangent space. This is not necessarily the case, however we can consider that the orthogonal complement describes tangent vectors to $\mathcal{C}$, whether this space is finite dimensional or not. Hence, the orthogonal complement describes the dynamics of curves $[(\phi_s, A_s)] : I \to \mathcal{C}$ with a lift to field space, even if the quotient lacks regularity.

Given $(\delta\phi, \delta A) \in \mathcal{T}_{(\phi,A)}$, let $\beta \in \mathcal{G}$ be the projection onto $\mathcal{V}_{(\phi,A)}$ with respect to the L^2 product. If $\alpha \in \mathcal{G}$ represents another arbitrary vertical vector at (ϕ, A), then

$$\langle(\delta\phi - \beta X_\phi, \delta A - d\beta), (\alpha X_\phi, d\alpha)\rangle = 0. \tag{2.45}$$

Since α is arbitrary, the perpendicularity condition is equivalent to the equation,

$$\left(\Delta + |X_\phi|^2\right)\beta = \langle X_\phi, \delta\phi\rangle + d^*\delta A, \tag{2.46}$$

where $d^* : \Omega^1(\Sigma) \to \Omega^0(\Sigma)$ is the codifferential, $d^* = - * d*$. What is interesting about the perpendicularity condition is that it is independent of the theory because no Lagrangian or functional for the fields was necessary to deduce it. At the same time, we can talk of kinetic energy in configuration space, at least for curves $[\phi_s, A_s]$ admitting a lift to $\mathcal{A}$. For such a curve, we could define its instant energy as,

$$\mathrm{E}[\delta\phi, \delta A] = \frac{1}{2}||(\delta\phi^\perp, \delta A^\perp)||^2_{\mathcal{A}}. \tag{2.47}$$

We think of the kinetic energy of a dynamic pair (ϕ, A) of solutions to the field equations slowly varying in time, as approximated by the energy of the variation of a static pair of solutions. In this way, we reduce the full theory in spacetime to variations of the static solutions to the Bogomolny equations. With this point of view, the components of the gauge potential are curves defined in some interval $I \subset \mathbb{R}$,

$$A_0 : I \subset \mathbb{R} \to C^\infty(\Sigma), \qquad\qquad A : I \subset \mathbb{R} \to \Omega^1(\Sigma). \tag{2.48}$$

Likewise for the electric and magnetic fields. Let us denote by $\mathcal{A}'$ the subset of $\mathcal{A}$ of solutions of the Bogomolny equations, and by $\mathcal{M}$ the quotient space $\mathcal{A}'/\mathcal{G}$. There is a bundle inclusion,

$$\begin{array}{ccc} \mathcal{A}' & \hookrightarrow & \mathcal{A} \\ \downarrow & & \downarrow \\ \mathcal{M} & \hookrightarrow & \mathcal{C} \end{array} \tag{2.49}$$

and since the Bogomolny equations are gauge invariant, both bundles share the same vertical space. Therefore, the orthogonal projection onto $\mathcal{V}$ is the same.

Example. (*Localization of Ginzburg-Landau vortices*). A an example we consider the Ginzburg-Landau functional. In this case the target space is $\mathbb{C}$ and we can think of sections ϕ as complex valued functions $\mathbb{R} \times \Sigma \to \mathbb{C}$. As described above, static fields are the same as pairs (ϕ, A) of a function $\phi : \Sigma \to \mathbb{C}$ and a connection A on a principal bundle $U(1) \to P \to \Sigma$. As it turns out [31], static configurations in the radiation gauge minimise the energy

$$\mathrm{E} = \frac{1}{2}\left(||\mathrm{D}\phi||^2 + ||B||^2 + \frac{1}{4}\,||1 - |\phi|^2||^2\right) \tag{2.50}$$

and satisfy the following Bogomolny equations,

$$\bar{\partial}_A \phi = 0, \tag{2.51}$$

$$*B = \frac{1}{2}(1 - |\phi|^2). \tag{2.52}$$

The action of $U(1)$ on the target manifold gives rise to the vector field $X_\phi = i\phi$. As is well known from the work of Taubes, solutions to the field equations modulo gauge equivalence are determined by the zeros of ϕ, $(p_1, \ldots, p_n)$. If we let the zeros vary with respect to a parameter, $p_k(s)$, $k = 1, \ldots, n$, identified as the *time* parameter, then the perpendicularity condition is equivalent to,

$$(\Delta + |\phi|^2)\beta = -\frac{i}{2}\left(\phi\dot{\phi}^\dagger - \dot{\phi}\phi^\dagger\right) + d^*\dot{A}, \tag{2.53}$$

and the projection of the variation on the horizontal subspace of $\mathcal{T}$ is,

$$\dot{\phi}^{\perp} = \dot{\phi} - i\phi\beta, \qquad \dot{A}^{\perp} = \dot{A} - d\beta. \tag{2.54}$$

Since the variation is determined by variations of the zeros, if each p_k is in the same open and dense holomorphic neighbourhood U,

$$\dot{\phi} = \dot{p}_k \frac{\partial \phi}{\partial p_k}, \qquad \dot{A} = \dot{p}_k \frac{\partial A}{\partial p_k}. \tag{2.55}$$

In the sequel we make the convention that repeated indices represent sums. If β_k is the projection onto vertical space corresponding to the variation $(\partial_{p_k}\phi, \partial_{p_k}A)$, then $\beta = \dot{p}_k\,\beta_k$. If we denote the pair (ϕ, A) by Φ, the instant energy of a trajectory in the moduli space is therefore,

$$\begin{aligned} \mathrm{E}[\dot{\Phi}] &= \frac{1}{2}||\dot{\Phi}^{\perp}||^2_{\mathcal{A}} \\ &= \frac{1}{2}\dot{p}_k\,\dot{p}_r \left\langle (\partial_{p_k}\Phi)^{\perp}, (\partial_{p_r}\Phi)^{\perp} \right\rangle_{\mathcal{A}} \\ &= \frac{1}{2}\,\dot{p}_k\dot{p}_r\, g_{p_k p_r}. \end{aligned} \tag{2.56}$$

The coefficients $g_{p_k p_r}$ determine a metric in the moduli space. Manton proposed an interpretation of this metric in [32]. In our language, the static energy in the sub-bundle $\mathcal{A}'$ must be preserved by solutions of the Bogomonly equations, because they are energy minimisers. Thence, $\mathrm{E}[\dot{\Phi}]$ approximates the energy of slow moving solutions of the full field equations. Equation (2.56) opens the possibility to study the dynamics of the full field equations as geodesic motion in a finite dimensional manifold. It was Samols who proved that this metric depends only in the first derivatives of ϕ at the zeros [47] of the Higgs field ϕ, obtaining the formula bearing his name on $\mathbb{R}^2$,

$$ds^2 = \pi \sum_{rs} (\delta_{rs} + 2\,\partial_r b_s)\; dp_r\, \overline{dp_s}, \tag{2.57}$$

where the coefficients depend on the position of the zeros of ϕ, in fact, if $h = \log|\phi|^2$, then,

$$b_s = 2\partial_z|_{z=p_s}(h - \log|z - p_s|^2), \tag{2.58}$$

which explains why (2.57) is called a localization formula, in the sense that the data needed to compute the metric is only local to the position of the zeros of ϕ.

□

2.2.1 Localization of BPS solitons of the gauged O(3) Sigma model

Having discussed localization of Ginzburg-Landau vortices as example, we turn attention to the gauged $O(3)$ Sigma model and apply the same technique in detail. Let $\varphi_\alpha : U_\alpha \to \mathbb{C}$ be a holomorphic chart admitting a trivialisation on U_α such that ϕ is equivalent to a function $\phi_\alpha : U_\alpha \to \mathbb{S}^2$. As before, let us define the stereographic projection of ϕ_α as $\psi_\alpha = \pi_- \circ \phi_\alpha \circ \varphi_\alpha^{-1}$. In this chart, $\psi_\alpha = \exp(h/2 + i\chi_\alpha)$ where the function h can be extended to a well defined gauge invariant function on $\Sigma \setminus \mathcal{P} \cup \mathcal{Q}$; however, χ_α is only defined on $U_\alpha \setminus \mathcal{P} \cup \mathcal{Q}$ modulo 2π. If U_β is another holomorphic chart, we can also define a related function χ_β in U_β, if the domains overlap, then for all x in the intersection $U_{\alpha\beta}$,

$$\chi_\beta = \chi_\alpha + \theta_{\alpha\beta} + 2\pi n, \qquad n \in \mathbb{Z}, \tag{2.59}$$

where $\theta_{\alpha\beta} : U_{\alpha\beta} \to \mathbb{R}$ are transition functions. Therefore $d\chi_\beta = d\chi_\alpha + d\theta_{\alpha\beta}$ and the arguments of the family ψ_α define a connection on $\Sigma \setminus \mathcal{P} \cup \mathcal{Q}$ which we call $d\chi$. Let $(\phi, A) : I \subset \mathbb{R} \to \mathcal{C}$ be a curve on the space of solutions to the Bogomolny equations, for each $t \in I$, we denote the core positions of $(\phi(t, \cdot), A(t, \cdot))$ as $p_j(t) \in \mathcal{P} \cup \mathcal{Q}$, $j = 1, \ldots, k_+ + k_-$. We assume the cores are not intersecting and each curve $p_j(t)$ is differentiable. Given $t \in I$, we choose a gauge such that $(\dot\phi, \dot A)$ is perpendicular to the gauge orbit, by (2.46) choosing this gauge is equivalent to

$$\langle \dot\phi, X_\phi \rangle + d^* \dot A = 0. \tag{2.60}$$

By (2.59), (ϕ, A) defines a function

$$\dot\chi : \Sigma_I \to \mathbb{R}, \tag{2.61}$$

where

$$\Sigma_I = (I \times \Sigma) \setminus \{(t, p_j(t)) \mid t \in I,\ j = 1, \ldots k_+ + k_-\}. \tag{2.62}$$

Let $\eta = \frac{\dot h}{2} + \dot\chi i : \Sigma_I \to \mathbb{C}$, then in any holomorphic trivialisation, $\dot\psi_\alpha = \eta\, \psi_\alpha$, moreover, by (2.60) and the Taubes equation, on each time slice

$$\Sigma_t = \Sigma \setminus \{p_j(t) \mid j = 1, \ldots, k_+ + k_-\}, \tag{2.63}$$

η is a solution to

$$-\Delta\eta = \frac{4e^h}{(1+e^h)^2}\,\eta. \tag{2.64}$$

Now we will extend (2.64) to an equation valid on all of $I \times \Sigma$, not just Σ_I. Let us assume U_α is dense and $p_j(t) \in U_\alpha$ for all $t \in I$ and $j \in \{1, \ldots, k_+ + k_-\}$. For any given $t \in I$, let $z_j(t) = \varphi_\alpha(p_j(t)) \in \mathbb{C}$ and to simplify notation, let us write $z_j(t)$ as z_j since time will play no role in the following. We define the signature $s_j \in \{\pm 1\}$ as,

$$s_j = \begin{cases} 1, & p_j \in \mathcal{P}, \\ -1, & p_j \in \mathcal{Q}. \end{cases} \tag{2.65}$$

By proposition 2.2, there is a smooth function $R_\alpha : \mathbb{C} \to \mathbb{C}$ such that,

$$\psi_\alpha(z) = \prod_{j=1}^{k_+ + k_-} (z - z_j)^{s_j} R_\alpha(z), \qquad z \in \mathbb{C} \setminus \varphi_\alpha(\mathcal{P} \cup \mathcal{Q}), \tag{2.66}$$

where the remainder also satisfies $R_\alpha(z_j) \neq 0$. Whence,

$$h(\varphi_\alpha^{-1}(z)) = \sum_{j=1}^{k_+ + k_-} s_j \log|z - z_j|^2 + h_\alpha(z), \tag{2.67}$$

where $h_\alpha : \mathbb{C} \setminus \varphi_\alpha(\mathcal{P} \cup \mathcal{Q}) \to \mathbb{R}$ is smooth. Since the chart is holomorphic, the metric can be written as $e^{\Lambda(z)}|dz|^2$ and the Laplacian as $\Delta = -4e^{-\Lambda}\partial_z\bar{\partial}_z$. Therefore, as distributions,

$$\Delta \log|z - z_j|^2 = -4\pi\, e^{-\Lambda}\delta_{z_j}. \tag{2.68}$$

Recall the volume form of the surface in holomorphic coordinates is $\mathrm{Vol} = i/2\, e^\Lambda\, dz \wedge d\bar{z}$, equation (2.68) means that for any test function $\varphi : \mathbb{C} \to \mathbb{R}$,

$$\int_{\mathbb{C}} \log|z - z_j|^2\, \Delta\varphi\, \mathrm{Vol} = -4\pi\, \varphi(z_j). \tag{2.69}$$

Let $z_j = z_j^1 + z_j^2\, i$, we denote by $D_\epsilon(z_j)$ the holomorphic disk $|z - z_j| < \epsilon$ and by (r_j, θ_j) polar coordinates centred at z_j. Now, we let t vary and compute the following time derivative,

$$\begin{aligned}\partial_t \int_{\mathbb{C}} \log|z - z_j(t)|^2 \, \Delta\varphi \, \mathrm{Vol} &= \int_{\mathbb{C}} -2 \left(\frac{\dot{z}_j^1 \cos(\theta_j) + \dot{z}_j^2 \sin(\theta_j)}{r_j} \right) \Delta\varphi \, \mathrm{Vol} \\ &= \lim_{\epsilon \to 0} \int_{\mathbb{C} \setminus D_\epsilon(z_j)} -2 \left(\frac{\dot{z}_j^1 \cos(\theta_j) + \dot{z}_j^2 \sin(\theta_j)}{r_j} \right) \Delta\varphi \, \mathrm{Vol} \\ &= -4\pi \left(\dot{z}_j^1 \, \partial_1 \varphi(z_j) + \dot{z}_j^2 \, \partial_2 \varphi(z_j) \right) \\ &= -8\pi \Re(\dot{z}_j \, \partial_z \varphi(z_j)). \end{aligned} \tag{2.70}$$

where we applied the divergence theorem to compute the limit, hence, in the sense of distributions,

$$\Delta(\partial_t \log|z - z_j|^2) = 8\pi \Re\left(\dot{z}_j \, \partial_z \delta_{p_j}\right) = -8\pi \Re\left(\dot{z}_j \, \partial_{z_j} \delta_{p_j}\right). \tag{2.71}$$

For a given trajectory of the cores, the right side of this equation defines a distribution on Σ, on the other hand, on the left side is the time derivative of the singular part of $h \circ \varphi^{-1}$, from this observation we state formally (i.e. without considering details about convergence in function spaces) that $\dot{h}$ must be a solution to the equation,

$$-\Delta \dot{h} = \frac{4e^h \dot{h}}{(1 + e^h)^2} + 8\pi \sum_j s_j \Re(\dot{z}_j \, \partial_{z_j} \delta_{p_j}) \tag{2.72}$$

Similarly, for any $z_j \in \varphi_\alpha(\mathcal{P} \cup \mathcal{Q})$ there is a small neighbourhood $D \subset \mathbb{C}$ such that,

$$\dot{\chi} = \Im\left(\frac{\dot{\psi}_\alpha}{\psi_\alpha}\right) = -s_j \left(\frac{-\dot{z}_j^1 \, \sin(\theta_j) + \dot{z}_j^2 \, \cos(\theta_j)}{r_j} \right) + \tilde{\chi}_\alpha, \tag{2.73}$$

for some smooth function $\tilde{\chi}_\alpha : D \to \mathbb{R}$. For the singular part of this equation we have,

$$\begin{aligned}\Delta \left(\frac{-\dot{z}_j^1 \, \sin(\theta_j) + \dot{z}_j^2 \, \cos(\theta_j)}{r_j} \right) &= -2\pi \left(-\dot{z}_j^1 \partial_2 \delta_{p_j} + \dot{z}_j^2 \partial_1 \delta_{p_j} \right) \\ &= -4\pi \, \Im(\dot{z}_j \partial_z \delta_{p_j}) \\ &= 4\pi \, \Im(\dot{z}_j \partial_{z_j} \delta_{p_j}). \end{aligned} \tag{2.74}$$

Hence, $\dot{\chi}$ is a solution to the equation,

$$-\Delta\dot{\chi} = \frac{4e^h}{(1+e^h)^2}\,\dot{\chi} + 4\pi\sum_j s_j \Im(\dot{z}_j\partial_{z_j}\delta_{p_j}). \tag{2.75}$$

We conclude that $\eta = \dot{h}/2 + \dot{\chi}\,i$ is a solution to the equation,

$$-\Delta\eta = \frac{4e^h}{(1+e^h)^2}\,\eta + 4\pi\sum_j s_j\dot{z}_j\,\partial_{z_j}\delta_{p_j}. \tag{2.76}$$

Equation (2.76) is formal, in order to make sense of it, we have to supplement it with analytical properties of the solution h to the Taubes equation and in the case of the plane with proper limiting behaviour at infinity. In the successive chapters we will address these issues. We assume however the existence of exactly one solution to (2.76). Under this assumption, the solution is given by the function

$$\eta = \sum_j \dot{z}_j\,\partial_{z_j}h. \tag{2.77}$$

Note that although each core position z_j is defined up to holomorphic coordinates, the right hand side is well defined independently of the chart chosen, provided the cores are contained in it. With this initial setup, we compute the localization formula (2.85).

Lemma 2.3. *Let $\varphi : U \subset \Sigma \to \mathbb{C}$ be a holomorphic chart, U open and dense, such that $\mathcal{P}\cup\mathcal{Q} \subset U$. Assume the cores are simple, for each $p_j \in \mathcal{P}\cup\mathcal{Q}$ define,*

$$b_j = 2\,\bar{\partial}\big|_{z=z_j}\left(s_j h(\varphi^{-1}(z)) - \log|z-z_j|^2\right), \tag{2.78}$$

where $z = \varphi(x)$, $z_j = \varphi(p_j)$. Then the coefficients b_j have the symmetries,

$$\partial_{z_i}b_j = \bar{\partial}_{z_j}\bar{b}_i, \qquad\qquad \partial_{z_i}\bar{b}_j = \partial_{z_j}\bar{b}_i. \tag{2.79}$$

Proof. For the proof we generalise the argument of Manton and Sutcliffe [31, pg. 209] given for vortices of the Ginzburg-Landau functional on the Euclidean plane. Let $\mathrm{K} = -(\Delta + 4e^h(e^h+1)^{-2})$, by the Taubes equation, $s_i\partial_{z_i}h$ (no summation) is a fundamental solution of K,

$$\mathrm{K}(s_i\partial_{z_i}h) = -4\pi\partial\delta_{p_i}. \tag{2.80}$$

If $i \neq j$, $\partial_{z_i} h$ and $\partial_{z_j} h$ have different singularities and we can integrate by parts to obtain,

$$\int_\Sigma \left(s_j \partial_{z_j} h \, \mathrm{K}(s_i \partial_{z_i} h) - s_i \partial_{z_i} h \, \mathrm{K}(s_j \partial_{z_j} h) \right) \mathrm{Vol} = 0, \tag{2.81}$$

where the integration by parts involves computing a limit at each singularity, we omit the details for clarity of the argument.

On the other hand,

$$\begin{aligned} \int_\Sigma \left(s_j \partial_{z_j} h \, (-4\pi \partial \delta_{p_i}) - s_i \partial_{z_i} h \, (-4\pi \partial \delta_{p_j}) \right) \mathrm{Vol} &= 4\pi \left(s_j \partial(\partial_{z_j} h)(p_i) - s_i \partial(\partial_{z_i} h)(p_j) \right) \\ &= 2\pi s_j s_i \left(\partial_{z_j} \overline{b}_i - \partial_{z_i} \overline{b}_j \right). \end{aligned} \tag{2.82}$$

Therefore, $\partial_{z_i} \overline{b}_j = \partial_{z_j} \overline{b}_i$. Since K is a real operator,

$$\mathrm{K}(s_i \overline{\partial} h_{z_i}) = -4\pi \overline{\partial} \delta_{z_i}, \tag{2.83}$$

hence,

$$\begin{aligned} \int_\Sigma \left(s_j \overline{\partial}_{z_j} h \, \mathrm{K}(s_i \partial_{z_i} h) - s_i \partial_{z_i} h \, \mathrm{K}(s_j \overline{\partial}_{z_j} h) \right) \mathrm{Vol} &= -4\pi \int_\Sigma \left(s_j \overline{\partial}_{z_j} h \, \partial \delta_{p_i} - s_i \partial_{z_i} h \, \overline{\partial} \delta_{p_j} \right) \mathrm{Vol} \\ &= 2\pi s_j s_i \, (\overline{\partial}_{z_j} \overline{b}_i - \partial_{z_i} b_j). \end{aligned} \tag{2.84}$$

As in the previous case, we can apply integration by parts to prove that the first integral is zero. Therefore $\partial_{z_i} b_j = \overline{\partial}_{z_j} \overline{b}_i$.

□

We denote by $\mathcal{M}^{k_+,k_-}$ the moduli space of solutions to the Bogomolny equations with k_+ vortices and k_- antivortices.

Theorem 2.4. *If $\varphi : U \subset \Sigma \to \mathbb{C}$ is an open and dense holomorphic chart, containing the cores of a time varying trajectory $(\phi, A) : I \subset \mathbb{R} \to \mathcal{M}^{k_+,k_-}$, such that the variation $(\dot{\phi}, \dot{A})$ satisfies Gauss's equation and each core $p_i \in \mathcal{P} \cup \mathcal{Q}$ is simple, then the kinetic energy of the trajectory can be computed as,*

$$\mathrm{E} = \pi \sum_{i,j=1}^{k_+ + k_-} \left(e^{\Lambda(z_i)} (1 - s_i \tau) \delta_{ij} + \partial_{z_i} b_j \right) \dot{z}_i \, \overline{\dot{z}_j}, \tag{2.85}$$

where $z_j = \varphi(p_j)$*. Moreover, the quadratic form,*

$$\mathrm{K} = 2\pi \sum_{i,j=1}^{k_+ + k_-} \left(e^{\Lambda(z_i)}(1 - s_i\tau)\delta_{ij} + \partial_{z_i} b_j\right) \, dz_i \, d\overline{z}_j, \tag{2.86}$$

determines a Kähler metric in the open and dense set of non intersecting vortices and antivortices.

Theorem 2.4 was proved for $\tau = 0$ in [45]. We follow the authors ideas and extend them to the remaining cases.

Proof. Let D_ϵ be a collection of disjoint holomorphic ϵ-disks, each one centred at one of the cores in $\varphi(\mathcal{P} \cup \mathcal{Q})$ and let $U_\epsilon = U \setminus D_\epsilon$. We will make a calculation similar to the one done in [31] for the Ginzburg-Landau functional. The energy of the trajectory can be computed as,

$$\begin{aligned} \mathrm{E} &= \frac{1}{2}\left(||\dot{\psi}||^2 + ||\dot{A}||^2\right) \\ &= \lim_{\epsilon \to 0} \frac{1}{2} \int_{U_\epsilon} \left(\frac{4e^h \left(\frac{1}{4}\dot{h}^2 + \dot{\chi}^2\right)}{(1+e^h)^2} + |\dot{A}|^2 \right) \mathrm{Vol}, \end{aligned} \tag{2.87}$$

by the first of the Bogomolny equation, $\overline{\partial}\psi = \frac{1}{2}(-A_2 + A_1 i)\psi$, on the other hand, $\overline{\partial}\psi = \psi \, \overline{\partial}(\frac{1}{2}h + \chi i)$, hence,

$$A = d\chi - \frac{1}{2} * dh, \tag{2.88}$$

which implies,

$$|\dot{A}|^2 = |d\dot{\chi}|^2 - \langle d\dot{\chi}, *d\dot{h}\rangle + \frac{1}{4}|*d\dot{h}|^2. \tag{2.89}$$

Integrating by parts,

$$\begin{aligned} \int_{U_\epsilon} |d\dot{\chi}|^2 \, \mathrm{Vol} &= \int_{U_\epsilon} d\dot{\chi} \wedge *d\dot{\chi} \\ &= \int_{\partial U_\epsilon} \dot{\chi} * d\dot{\chi} + \int_{U_\epsilon} \dot{\chi} \, \Delta \dot{\chi} \, \mathrm{Vol} \\ &= -\int_{\partial D_\epsilon} \dot{\chi} * d\dot{\chi} - \int_{U_\epsilon} \frac{4e^h \dot{\chi}^2}{(1+e^h)^2} \mathrm{Vol}. \end{aligned} \tag{2.90}$$

Proceeding in a similar way, we obtain a second pair of equations,

$$\int_{U_\epsilon} \langle d\dot{\chi}, *d\dot{h}\rangle \,\mathrm{Vol} = \int_{\partial D_\epsilon} \dot{\chi}\, d\dot{h}, \tag{2.91}$$

$$\int_{U_\epsilon} |*d\dot{h}|^2 \,\mathrm{Vol} = -\int_{\partial D_\epsilon} \dot{h} * d\dot{h} - \int_{U_\epsilon} \frac{4e^h \dot{h}^2}{(1+e^h)^2}\mathrm{Vol}. \tag{2.92}$$

Substituting into the equation for the energy, we obtain,

$$\begin{aligned} \mathrm{E} &= -\frac{1}{2}\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\left(\dot{\chi} * d\dot{\chi} + \dot{\chi}\, d\dot{h} + \frac{1}{4}\dot{h} * d\dot{h}\right) \\ &= -\frac{1}{2}\lim_{\epsilon\to 0}\int_{\partial D_\epsilon}\left(\dot{\chi} * \dot{A} - \frac{1}{2}\,\dot{h}\,\dot{A}\right), \end{aligned} \tag{2.93}$$

where we have used the time derivative of equation (2.88) to simplify the energy. Since $\epsilon \to 0$, the only terms that contribute to the energy are the singular terms. We will compute each of these terms at the respective core. For any $z_j \in \varphi(\mathcal{P}\cup\mathcal{Q})$, let $D_\epsilon(z_j)$ be the ϵ holomorphic disk centred at this point. If ϵ is small, for $z \in \mathbb{D}_\epsilon(z_j)$ we have the approximations,

$$\dot{h} = -2s_j\left(\frac{\dot{z}_j^1\cos(\theta_j) + \dot{z}_j^2\sin(\theta_j)}{r_j}\right) + R_h(z), \tag{2.94}$$

$$\dot{\chi} = -s_j\left(\frac{-\dot{z}_j^1\sin(\theta_j) + \dot{z}_j^2\cos(\theta_j)}{r_j}\right) + R_\chi(z), \tag{2.95}$$

for some residual smooth functions R_h and R_χ. We also expand $\dot{A}$ in polar coordinates centred at z_j,

$$\dot{A} = \dot{A}_r\, dr_j + \dot{A}_\theta\, r_j\, d\theta_j, \tag{2.96}$$

where,

$$\dot{A}_r = \dot{A}_1\,\cos(\theta_j) + \dot{A}_2\,\sin(\theta_j), \qquad \dot{A}_\theta = -\dot{A}_1\,\sin(\theta_j) + \dot{A}_2\,\cos(\theta_j). \tag{2.97}$$

The singular terms in the energy integral contribute as,

$$\begin{aligned} \lim_{\epsilon\to 0}\int_{\partial D_\epsilon(z_j)} \dot{\chi} * \dot{A} &= -s_j\lim_{\epsilon\to 0}\int_{D_\epsilon(z_j)}\left(\frac{-\dot{z}_j^1\sin(\theta_j) + \dot{z}_j^2\cos(\theta_j)}{\epsilon}\right)\dot{A}_r\,\epsilon d\theta_j \\ &= \pi s_j(\dot{z}_j^1\,\dot{A}_2(z_j) - \dot{z}_j^2\,\dot{A}_1(z_j)) \\ &= -\pi s_j \Im\left(\dot{z}_j\,\left(\dot{A}_1(z_j) - \dot{A}_2(z_j)\, i\right)\right) \end{aligned} \tag{2.98}$$

and

$$\lim_{\epsilon\to 0}\int_{\partial D_\epsilon(z_j)} \dot{h}\,\dot{A} = -2s_j \lim_{\epsilon\to 0}\int_{\partial D_\epsilon(z_j)} \left(\frac{\dot{z}_j^1\cos(\theta_j)+\dot{z}_j^2\sin(\theta_j)}{\epsilon}\right)\dot{A}_\theta\,\epsilon\, d\theta_j$$
$$= -2\pi s_j \left(\dot{z}_j^1\,\dot{A}_2(z_j) - \dot{z}_j^2\,\dot{A}_1(z_j)\right)$$
$$= 2\pi s_j\,\Im\left(\dot{z}_j\,\left(\dot{A}_1(z_j)-\dot{A}_2(z_j)\,i\right)\right), \tag{2.99}$$

Therefore, the energy of a moving pair is,

$$\mathrm{E} = \pi\sum_{j=1}^{k_++k_-} s_j\Im\left(\dot{z}_j\,\left(\dot{A}_1(z_j)-\dot{A}_2(z_j)\,i\right)\right). \tag{2.100}$$

By equations (2.88) and (2.77),

$$\dot{A}_1-\dot{A}_2\,i = \left(\partial_1\dot{\chi}+\frac{1}{2}\,\partial_2\dot{h}\right)-\left(\partial_2\dot{\chi}-\frac{1}{2}\partial_1\dot{h}\right)i$$
$$= \partial_1\left(\frac{1}{2}\dot{h}-\dot{\chi}\,i\right)\,i+\partial_2\left(\frac{1}{2}\dot{h}-\dot{\chi}\,i\right)$$
$$= 2i\,\partial_z\overline{\eta}$$
$$= 2i\sum_j \dot{\bar{z}}_j\,\partial_z\overline{\partial}_{z_j}h. \tag{2.101}$$

In a small neighbourhood of any z_j, we have the asymptotic expansion,

$$s_jh(\varphi^{-1}(z)) = \log|z-z_j|^2 + a_j + \frac{1}{2}\overline{b}_j\,(z-z_j)+\frac{1}{2}b_j\,(\overline{z}-\overline{z}_j) + \overline{c}_j\,(z-z_j)^2+d_j\,|z-z_j|^2+c_j\,(\overline{z}-\overline{z}_j)^2 + \mathcal{O}(|z_j|^3). \tag{2.102}$$

Hence,

$$d_j = \lim_{z\to z_j}\partial_z\overline{\partial}_z(s_jh(\varphi^{-1}(z))-\log|z-z_j|^2)$$
$$= \frac{1}{4}\lim_{z\to z_j}(-e^{\Lambda(z)}\Delta)(s_jh(\varphi^{-1}(z))-\log|z-z_j|^2)$$
$$= \frac{1}{2}s_j\,e^{\Lambda(z_j)}\lim_{z\to z_j}\left(\frac{e^h-1}{e^h+1}+\tau\right)$$
$$= -\frac{1}{2}e^{\Lambda(z_j)}\,(1-s_j\tau) \tag{2.103}$$

and since $\partial_z \overline{\partial}_{z_k} \log |z - z_j|^2 = 0$ for any $z \neq z_j$,

$$\partial_z(\overline{\partial}_{z_k} h)(z_j) = s_j \left(\frac{1}{2} \overline{\partial}_{z_k} \overline{b}_j - d_j \, \delta_{jk} \right). \tag{2.104}$$

Hence,

$$\begin{aligned}
\mathrm{E} &= \pi \sum_j s_j \Im \left(\dot{p}_j \cdot 2i \sum_k \overline{\dot{z}_k} \, \partial_z(\overline{\partial}_{z_k} h)\big|_{z=z_j} \right) \\
&= 2\pi \sum_{j,k} \Re \left(\dot{z}_j \, \overline{\dot{z}_k} \, \left(\frac{1}{2} \overline{\partial}_{z_k} \overline{b}_j - d_j \delta_{jk} \right) \right) \\
&= \pi \sum_{j,k} \Re \left(\left(e^{\Lambda(z_j)}(1 - s_j \tau) \delta_{jk} + \overline{\partial}_{z_k} \overline{b}_j \right) \, \dot{z}_j \, \overline{\dot{z}_k} \right) \\
&= \pi \sum_{j,k} \Re \left(\left(e^{\Lambda(z_j)}(1 - s_j \tau) \delta_{jk} + \partial_{z_j} b_k \right) \, \dot{z}_j \, \overline{\dot{z}_k} \right).
\end{aligned} \tag{2.105}$$

The last equation is a consequence of the symmetry $\partial_{z_j} b_k = \overline{\partial}_{z_k} \overline{b}_j$. Also by this symmetry, (2.85) is a real quantity and therefore coincides with (2.105) as expected, since E represents the kinetic energy of a trajectory on the moduli space. To prove that the metric is Kähler we must prove that the induced form,

$$\omega = \pi i \sum_{j,k} \left(e^{\Lambda(z_j)}(1 - s_j \tau) \delta_{jk} + \partial_{z_j} b_k \right) \, dz_j \wedge d\overline{z}_k, \tag{2.106}$$

is closed. For the following computation, we employ lemma 2.3 and the fact that each term $e^{\Lambda(z_j)}(1 - s_j \tau) \, dz_j \wedge d\overline{z}_k$ is closed,

$$\begin{aligned}
d\omega &= \pi i \sum_{r,s,t} \left(\partial_{z_t} \partial_{z_r} b_s \, dz_t \wedge dz_r \wedge d\overline{z}_s + \overline{\partial}_{z_t} \partial_{z_r} z_s \, d\overline{z}_t \wedge dz_r \wedge d\overline{z}_s \right) \\
&= \pi i \sum_{r,s,t} \left(\partial_{z_t} \partial_{z_r} b_s \, dz_t \wedge dz_r \wedge d\overline{z}_s - \overline{\partial}_{z_t} \partial_{z_r} b_s \, d\overline{z}_t \wedge d\overline{z}_s \wedge dz_r \right) \\
&= \pi i \sum_{r,s,t} \left(\partial_{z_t} \partial_{z_r} b_s \, dz_t \wedge dz_r \wedge d\overline{z}_s - \overline{\partial}_{z_t} \partial_{z_s} b_r \, d\overline{z}_t \wedge d\overline{z}_r \wedge dz_s \right) \\
&= \pi i \sum_{r,s,t} \left(\partial_{z_t} \partial_{z_r} b_s \, dz_t \wedge dz_r \wedge d\overline{z}_s - \overline{\partial}_{z_t} \overline{\partial}_{z_r} \overline{b}_s \, d\overline{z}_t \wedge d\overline{z}_r \wedge dz_s \right) \\
&= -2\pi \Im \left(\sum_{r,s,t} \partial_{z_t} \partial_{z_r} b_s \, dz_t \wedge dz_r \wedge d\overline{z}_s \right) \\
&= 0,
\end{aligned} \tag{2.107}$$

where the last sum is zero by the commutativity of the mixed derivatives. □

2.3 The governing elliptic problem

Equation (2.36) is the governing elliptic problem. Once h is determined, the Bogomolny equations determine B and then A and ϕ up to gauge equivalence. We let,

$$\begin{aligned} &F : \mathbb{R} \to \mathbb{R}, && V : \mathbb{R} \to \mathbb{R}^+, \\ &F(t) = 2\left(\frac{e^t - 1}{e^t + 1} + \tau\right), && V(t) = \frac{4e^t}{(1+e^t)^2}. \end{aligned} \tag{2.108}$$

Note that $V = F'$ and that F and all of its derivatives are bounded functions, moreover, if $\mu = \log\left((1-\tau)(1+\tau)^{-1}\right)$, then F satisfies the following properties,

$$F(\mu) = 0, \tag{2.109}$$

$$F'(\mu) > 0, \tag{2.110}$$

$$F(t) < 0, \qquad t < \mu, \tag{2.111}$$

$$F(t) > 0, \qquad t > \mu, \tag{2.112}$$

and,

$$||F||_{\mathrm{L}^\infty(\mathbb{R})} + ||(1+e^{-t})V||_{\mathrm{L}^\infty(\mathbb{R})} + ||e^{-t}(e^t-1)^{-1}V'||_{\mathrm{L}^\infty(\mathbb{R})} < \infty. \tag{2.113}$$

If $\mathcal{P}$ or $\mathcal{Q}$ is non-empty, there exists exactly one function $h \in C^\infty(\mathbb{R}^2 \setminus \mathcal{P} \cup \mathcal{Q})$ [17], such that,

$$-\Delta h = F(h) + 4\pi \sum_{p\in\mathcal{P}} \delta(x-p) - 4\pi \sum_{q\in\mathcal{Q}} \delta(x-q), \qquad \lim_{|x|\to\infty} h = \mu, \tag{2.114}$$

moreover, for any $\epsilon \in (0,1)$, there exist positive constants $C = C(\epsilon)$ and $R = R(\epsilon)$ such that

$$|h(x) - \mu| \leq C \exp\left(-\frac{1}{2}\sqrt{(1-\tau^2)(1-\epsilon)}|x|\right), \qquad |x| \geq R. \tag{2.115}$$

Therefore, in the euclidean plane, there exists a unique solution to the Taubes equation. For a compact surface, existence of a solution to the Taubes equation was proved for $\tau = 0$ in [51]. We will prove that this is also the case for $\tau \neq 0$ in chapter 4.

In this section we prove that solutions to the Taubes equation depend smoothly on vortex positions. Recall Sobolev's space $\mathrm{W}^{k,p}$ is the completion of the space of C_0^∞ functions compactly supported with respect to Sobolev's norm,

$$||\varphi||_{\mathrm{W}^{k,p}} = \left(\sum_{j=0}^{k} ||\nabla^k \varphi||_{\mathrm{L}^p}^p \right)^{1/p}, \tag{2.116}$$

where $\nabla^j \varphi \in (T^*\Sigma)^{\otimes j}$ is the jth exterior covariant derivative. We denote the space $\mathrm{W}^{k,2}$ as H^k. This is a Hilbert space with the product,

$$\langle \varphi, \psi \rangle_{\mathrm{H}^k} = \sum_{j=0}^{k} \langle \nabla^j \varphi, \nabla^j \psi \rangle_{\mathrm{L}^2}. \tag{2.117}$$

For the inner product in L^2 we omit the subindex if is clear from the context that we refer to L^2 functions.

In the sequel, we will use some results of analysis that we quote here for further reference. The proofs are standard and can be found in the literature, for example in [15] and [13].

Theorem 2.5 (Banach-Alaoglu)**.** *Let* X *be a Banach space, then the closed unit ball of the dual* X^* *is compact with respect to the weak-* topology.*

Theorem 2.6 (Rellich-Kondrachov)**.** *If* Ω *is a an open bounded Lipschitz domain of* $\mathbb{R}^n$*,* $1 \leq p < n$*,* $p^* = \frac{np}{n-p}$*, then* $\mathrm{W}^{1,p}(\Omega)$ *is continuously embedded in* $\mathrm{L}^{p^*}(\Omega)$ *and compactly embedded in* $\mathrm{L}^q(\Omega)$ *for any* $1 \leq q < p^*$*.*

If Ω *is a compact manifold of dimension* n*,* $k > l$*,* $k - n/p > l - n/q$*, then the embedding* $\mathrm{W}^{k,p} \subset \mathrm{W}^{l,q}$ *is completely continuous.*

That the embedding $\mathrm{W}^{k,p} \subset \mathrm{W}^{l,q}$ is completely continuous is equivalent to claiming that any bounded sequence of functions in $\mathrm{W}^{k,p}$ has a subsequence converging in $\mathrm{W}^{l,q}$. In practice, we will use the Rellich-Kondrachov theorem to guarantee that given a bounded sequence of $\mathrm{W}^{1,p}$ functions either on a compact surface or on an open bounded subset of $\mathbb{R}^2$, we can find a subsequence convergent in L^p.

Theorem 2.7 (Sobolev's embedding). *If Ω is either $\mathbb{R}^n$ or a bounded domain of with Lipschitz boundary of a compact Riemannian manifold of dimension n, and if $k > l$, $1 \leq p < q < \infty$ and $\alpha \in (0, 1]$ are such that,*

$$\frac{1}{p} - \frac{k}{n} = -\frac{r + \alpha}{n}, \tag{2.118}$$

then we have the continuous embedding $\mathrm{W}^{k,p}(\Omega) \subset C^{r,\alpha}(\Omega)$.

Theorem 2.8 (Lax-Milgram). *If $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ is a continuous bilinear form in a Hilbert space $\mathcal{H}$ and there is a positive constant α such that,*

$$|B(u, u)| \geq \alpha ||u||^2, \tag{2.119}$$

then, for any $u \in \mathcal{H}$ there is a unique $v \in \mathcal{H}$, such that,

$$B(v, x) = \langle u, x \rangle \qquad \forall x \in \mathcal{H}. \tag{2.120}$$

Moreover,

$$||v|| \leq \frac{1}{\alpha} ||u||. \tag{2.121}$$

The proof can be found in [15, p. 83].

Theorem 2.9 (Schauder's estimates). *If $\Omega' \Subset \Omega$ are open sets of any manifold M, $f \in \mathrm{H}^k(\Omega)$ and $u \in \mathrm{H}^1(\Omega)$ is a weak solution to the equation*

$$\Delta_M u = f, \tag{2.122}$$

then $u \in \mathrm{H}^{k+2}(\Omega')$ and

$$||u||_{\mathrm{H}^{k+2}(\Omega')} \leq C \left(||f||_{\mathrm{H}^k(\Omega)} + ||u||_{\mathrm{L}^2(\Omega)} \right), \tag{2.123}$$

for some constant $C = C(k, \Omega, \Omega')$. In a compact manifold M, we also have the estimate,

$$||u - \overline{u}||_{\mathrm{H}^{k+2}} \leq C \, ||f||_{\mathrm{H}^k}, \tag{2.124}$$

for some constant $C = C(k, \Omega, \Omega')$, where $\overline{u} = \frac{1}{|M|} \int_M u \,\mathrm{Vol}$ is the trace of u.

Given a pair of open sets Ω', Ω on a topological space, the notation $\Omega' \Subset \Omega$ means $\overline{\Omega'} \subset \Omega$.

2.3.1 Smooth parametric dependence of h

The moduli space can be identified with $\left(\Sigma^{k_+} \times \Sigma^{k_-} \setminus \Delta_{k_+,k_-}\right)/\mathrm{S}_{k_+} \times \mathrm{S}_{k_-}$, where Δ_{k_+,k_-} is the big fat diagonal of intersecting vortices and antivortices and the product of symmetric groups $\mathrm{S}_{k_+} \times \mathrm{S}_{k_-}$ act permuting the components of $\Sigma^{k_+} \times \Sigma^{k_-}$. Let us focus in the open and dense subset of non-overlapping cores. We can identify this space with $\Sigma^{k_+} \times \Sigma^{k_-} \setminus \Delta_{k_+,k_-}$. We aim to prove that in this subspace, h depends smoothly on the positions of the cores.

Lemma 2.10. *Let Σ be either the plane or a compact surface. If $V \in C^\infty(\Sigma)$, is a non-negative smooth function with only finite zeros, such that if Σ is the plane, $\lim_{|x|\to\infty} V(x) \in (0,1]$, and all the derivatives $\nabla^k V$ are bounded, then for any $r \geq 0$, Schrodinger's operator,*

$$\Delta + V : \mathrm{H}^{r+2}(\Sigma) \to \mathrm{H}^r(\Sigma), \tag{2.125}$$

is a Hilbert space isomorphism.

Proof. By the hypothesis on the potential function V, the operator $\Delta + V$ is continuous. Let us define the bilinear form $B : \mathrm{H}^1 \times \mathrm{H}^1 \to \mathbb{R}$ and the linear functional $A : \mathrm{H}^1 \to \mathbb{R}$ such that,

$$\begin{aligned} B(u,v) &= \langle \nabla u, \nabla v\rangle + \langle Vu, v\rangle, \\ A(u) &= \langle b, u\rangle, \end{aligned} \tag{2.126}$$

where $b \in \mathrm{H}^r$. By the Cauchy-Schwarz inequality, A and B are continuous.

Firstly, we claim B is coercive, i.e., there is a positive constant α such that,

$$\alpha\,||u||^2_{\mathrm{H}^1} \leq B(u,u). \tag{2.127}$$

Let Ω be either the compact surface Σ or an open disk $\mathbb{D}_R(0) \subset \mathbb{R}^2$ such that there is a constant $a \in (0,1]$ for which $V(x) \geq a > 0$ if $x \in \mathbb{R}^2 \setminus \Omega$. In the latter case,

$$||\nabla u||^2_{\mathrm{L}^2(\mathbb{R}^2\setminus\Omega)} + ||V^{1/2}u||^2_{\mathrm{L}^2(\mathbb{R}^2\setminus\Omega)} \geq a||u||^2_{\mathrm{H}^1(\mathbb{R}^2\setminus\Omega)}. \tag{2.128}$$

Assume towards a contradiction the existence of a sequence $\{u_n\} \subset \mathrm{H}^1(\Omega)$, such that

$$||u_n||_{\mathrm{H}^1(\Omega)} = 1, \qquad\qquad B(u_n,u_n) \leq \frac{1}{n}. \tag{2.129}$$

By the Banach-Aloglu theorem, we can assume $u_n \rightharpoonup u_*$ in $\mathrm{H}^1(\Omega)$, and by the Rellich theorem, we can assume the strong convergence $u_n \to u_*$ in $\mathrm{L}^2(\Omega)$. Since $B(u_n, u_n) \to 0$,

$$||\nabla u_n||_{\mathrm{L}^2(\Omega)} \to 0, \tag{2.130}$$

hence u_* is constant almost everywhere, because by the strong convergence in L^2 and the convergence,

$$\langle u_n, u_* \rangle_{\mathrm{H}^1(\Omega)} \to ||u_*||^2_{\mathrm{H}^1(\Omega)}, \tag{2.131}$$

we deduce $\langle \nabla u_n, \nabla u_* \rangle_{\mathrm{L}^2(\Omega)} \to ||\nabla u_*||^2_{\mathrm{L}^2(\Omega)}$, but $\langle \nabla u_n, \nabla u_* \rangle_{\mathrm{L}^2(\Omega)} \to 0$, hence $\nabla u_* = 0$ almost everywhere. On the other hand,

$$||V^{1/2} u_n||_{\mathrm{L}^2(\Omega)} \to 0, \tag{2.132}$$

and V is positive except for a finite set, thence $u_* = 0$. We conclude $u_n \to 0$ in $\mathrm{H}^1(\Omega)$, but this is a contradiction because each u_n has unit norm. Therefore, there is a positive constant a' such that if $u \in \mathrm{H}^1(\Omega)$, then $B(u,u) \geq a' ||u||^2_{\mathrm{H}^1(\Omega)}$. If Σ is compact we conclude B is coercive. If Σ is the plane, let us take $\alpha = \min(a, a')$. If $u \in \mathrm{H}^1(\mathbb{R}^2)$,

$$B(u,u) \geq a||u||^2_{\mathrm{H}^1(\mathbb{R}^2 \setminus \Omega)} + a'||u||^2_{\mathrm{H}^1(\Omega)} \geq \alpha ||u||^2_{\mathrm{H}^1(\mathbb{R}^2)}. \tag{2.133}$$

Secondly, we prove the basic inequality,

$$||u||_{\mathrm{H}^{r+2}} \leq C \, ||(\Delta + V)\, u||_{\mathrm{H}^r}, \tag{2.134}$$

where $u \in \mathrm{H}^{r+2}$ is arbitrary. If Σ is compact this is by Schauder's estimates and coercivity of B. If Σ is the plane, we first prove the inequality for $\varphi \in C_0^\infty$. Assume $r = 0$, by coercivity,

$$||\varphi||_{\mathrm{H}^1} \leq C \, ||(\Delta + V)\varphi||_{\mathrm{L}^2}. \tag{2.135}$$

We know in this case $||\nabla^2 \varphi||_{\mathrm{L}^2} = ||\Delta \varphi||_{\mathrm{L}^2}$ [15, Thm. 9.9], hence,

$$\begin{aligned} ||\varphi||_{\mathrm{H}^2} &\leq C \, (||(\Delta + V)\varphi||_{\mathrm{L}^2} + ||\nabla^2 \varphi||_{\mathrm{L}^2}) \\ &= C \, (||(\Delta + V)\varphi||_{\mathrm{L}^2} + ||\Delta \varphi||_{\mathrm{L}^2}) \\ &\leq C \, (||(\Delta + V)\varphi||_{\mathrm{L}^2} + ||\varphi||_{\mathrm{L}^2}) \\ &\leq C \, ||(\Delta + V)\varphi||_{\mathrm{L}^2}. \end{aligned} \tag{2.136}$$

Let $\psi = (\Delta + V)\,\varphi \in C_0^\infty$. Given the test function φ, $\partial_j\varphi$ is a solution to the problem,

$$(\Delta + V)\,\partial_j\varphi = \partial_j\psi - \partial_j V\,\varphi. \tag{2.137}$$

By hypothesis, the derivatives of the potential are bounded. Applying the previous bound to $\nabla\varphi$,

$$\begin{aligned}
||\nabla\varphi||_{\mathrm{H}^2} &\leq C\,||(\Delta + V)\,\nabla\varphi||_{\mathrm{L}^2} \\
&\leq C\,(||\nabla\psi||_{\mathrm{L}^2} + ||\varphi\,\nabla V||_{\mathrm{L}^2}) \\
&\leq C\,(||\psi||_{\mathrm{H}^1} + ||\varphi||_{\mathrm{L}^2}) \\
&\leq C\,(||(\Delta + V)\,\varphi||_{\mathrm{H}^1}).
\end{aligned} \tag{2.138}$$

We apply this argument recursively. Having found bounds for φ and $\nabla\varphi$ up to some r,

$$\begin{aligned}
||\varphi||_{\mathrm{H}^{r+3}} &\leq ||\varphi||_{\mathrm{H}^{r+2}} + ||\nabla\varphi||_{\mathrm{H}^{r+2}} \\
&\leq C(||(\Delta + V)\,\varphi||_{\mathrm{H}^r} + ||(\Delta + V)\,\varphi||_{\mathrm{H}^{r+1}}) \\
&\leq C\,||(\Delta + V)\,\varphi||_{\mathrm{H}^{r+1}}.
\end{aligned} \tag{2.139}$$

Thus, for all $r \geq 0$ there is a constant C such that for any $\varphi \in C_0^\infty$,

$$||\varphi||_{\mathrm{H}^{r+2}} \leq C\,||(\Delta + V)\,\varphi||_{\mathrm{H}^r}. \tag{2.140}$$

Since C_0^∞ is dense in H^r and $(\Delta + V)$ is continuous, we conclude (2.134) is also valid on the plane.

Thirdly, we prove $(\Delta + V)$ is surjective. By the Lax-Milgram theorem, for any $b \in \mathrm{H}^r$ there is a unique $u \in \mathrm{H}^1$, such that $B(u, v) = A(v)$ for all $v \in \mathrm{H}^1$. This function is a weak solution of the equation,

$$(\Delta + V)\,u = b. \tag{2.141}$$

If Σ is compact, elliptic regularity implies u is a strong solution in $\mathrm{H}^{r+2}(\Sigma)$. We prove this is also the case on the plane. Let $\psi \in C_0^\infty$ and denote by φ the weak solution to the equation,

$$(\Delta + V)\,\varphi = \psi. \tag{2.142}$$

Elliptic regularity and Sobolev's embedding imply φ is a strong solution in C^∞. Notice $\varphi \in \mathrm{H}^{r+2}$ $\forall r \geq 0$ because our previous argument can still be applied to show (2.140) holds. Let $\{\psi_n\} \subset C_0^\infty$ be a sequence of test functions converging to b in H^r. For each ψ_n let $\varphi_n \in C^\infty$ be a strong solution of the elliptic problem. By (2.140) $\{\varphi_n\}$ is a Cauchy sequence in H^{r+2}, thus there is $u \in \mathrm{H}^{r+2}$ such that $\varphi_n \to u$. By continuity of $\Delta + V$, $u \in \mathrm{H}^{r+2}$ is a strong solution of (2.141).

Finally, (2.134) implies $\Delta + V$ is injective. By the open mapping theorem, the inverse is also continuous and the operator is an isomorphism. □

For a compact manifold in general, we can estimate the norm of solutions to linear problems,

Proposition 2.11. *If M is a compact Riemannian manifold of dimension n, $-\Delta$ is the Laplace-Beltrami operator of the metric, $a, b \in \mathrm{L}^2(M)$ are functions such that a is non-negative and bounded with positive integral $\int_M a \,\mathrm{Vol}$, then the problem,*

$$-\Delta u = au + b, \tag{2.143}$$

has exactly one solution $u \in \mathrm{H}^2(M)$. Moreover, there is a positive constant $C(a)$ such that,

$$||u||_{\mathrm{H}^2} \leq K\,||b||_{\mathrm{L}^2}, \tag{2.144}$$

where the constant $K(a)$ depends on the bound for a and $\int_M a \,\mathrm{Vol}$.

If $n = 2, 3$, by Sobolev's embedding, $u \in C^0(M)$, in general, we only have $u \in \mathrm{H}^2(M)$ unless we know a and b have more regularity. This problem has been studied for different conditions on the coefficients in the references [41, 42].

Proof. We will prove the existence of solutions to the linear problem and continuity on the datum as an application of the Lax-Milgram theorem.

Let $\mathcal{X}$ be the subspace of $\mathrm{H}^1(M)$ of functions of zero average,

$$\mathcal{X} = \left\{ u \in \mathrm{H}^1(M) \;\middle|\; \int_M u \,\mathrm{Vol} = 0 \right\}. \tag{2.145}$$

$\mathrm{H}^1(M)$ can be decomposed as

$$\mathrm{H}^1(M) = \mathcal{X} \oplus \mathbb{R}. \tag{2.146}$$

Finding a solution to equation (2.143) is equivalent to find $(u_0, c) \in \mathcal{X} \oplus \mathbb{R}$, such that

$$-\Delta u_0 = a u_0 + a\, c + b. \tag{2.147}$$

By the divergence theorem, the constant is

$$c = \frac{-\int_M (a u_0 + b)\,\mathrm{Vol}}{\int_M a\,\mathrm{Vol}}. \tag{2.148}$$

(2.143) is equivalent to finding $u_0 \in \mathcal{X}$ such that

$$-\Delta\, u_0 = a u_0 + b - \frac{a \cdot \int_M (a u_0 + b)\,\mathrm{Vol}}{\int_M a\,\mathrm{Vol}}. \tag{2.149}$$

Let us define the operators $\mathrm{A} : \mathcal{X} \times \mathcal{X} \to \mathbb{R}$, $\mathrm{B} : \mathcal{X} \to \mathbb{R}$, as

$$\mathrm{A}(u, v) = \langle du, dv \rangle + \langle au, v \rangle - \frac{1}{\int_M a\,\mathrm{Vol}} \int_M au\,\mathrm{Vol} \cdot \int_M av\,\mathrm{Vol}, \tag{2.150}$$

$$\mathrm{B}(v) = \frac{1}{\int_M a\,\mathrm{Vol}} \int_M \left(a \cdot \int_M b\,\mathrm{Vol} - b \cdot \int_M a\,\mathrm{Vol} \right) \cdot v\,\mathrm{Vol}. \tag{2.151}$$

Equation (2.149) can be rewritten in variational form as the problem of finding $u_0 \in \mathcal{X}$, such that for any $v \in \mathcal{X}$,

$$\mathrm{A}(u_0, v) = \mathrm{B}(v). \tag{2.152}$$

B is bounded and A continuous because $a, b \in \mathrm{L}^2(M)$. By Cauchy-Schwartz's inequality,

$$\left| \int_M au\,\mathrm{Vol} \right| \leq ||\sqrt{a}||_{\mathrm{L}^2}\, ||\sqrt{a}u||_{\mathrm{L}^2}, \tag{2.153}$$

hence,

$$\begin{aligned}
\mathrm{A}(u, u) &= ||du||^2_{\mathrm{L}^2} + \frac{1}{\int_M a\,\mathrm{Vol}} \left(\int_M au^2\,\mathrm{Vol} \cdot \int_M a\,\mathrm{Vol} - \left(\int_M au\,\mathrm{Vol} \right)^2 \right) \\
&= ||du||^2_{\mathrm{L}^2} + \frac{1}{\int_M a\,\mathrm{Vol}} \left(||\sqrt{a}u||^2_{\mathrm{L}^2}\, ||\sqrt{a}||^2_{\mathrm{L}^2} - \langle a, u \rangle^2_{\mathrm{L}^2} \right) \\
&\geq ||du||^2_{\mathrm{L}^2}.
\end{aligned} \tag{2.154}$$

By Poincaré's inequality, there is a positive constant α, such that

$$\alpha\, ||u||^2_{\mathrm{H}^1} \leq ||du||^2_{\mathrm{L}^2} \leq \mathrm{A}(u, u). \tag{2.155}$$

Therefore, there exists a unique solution $u \in \mathrm{H}^1(M)$ to (2.143). By standard elliptic regularity estimates, $u \in \mathrm{H}^2(M)$. By the Lax-Milgram theorem,

$$||u_0||_{\mathrm{H}^1} \leq \frac{1}{\alpha \int_M a \,\mathrm{Vol}} \left|\left| a \cdot \int_M b \,\mathrm{Vol} - b \cdot \int_M a \,\mathrm{Vol} \right|\right|_{\mathrm{L}^2}. \tag{2.156}$$

By equation (2.148),

$$|c| \leq \frac{1}{\int_M a \,\mathrm{Vol}} \left(||a||_{\mathrm{L}^2} \cdot ||u_0||_{\mathrm{L}^2} + \left| \int_M b \,\mathrm{Vol} \right| \right). \tag{2.157}$$

Therefore, u is bounded in $\mathrm{H}^1(M)$ by,

$$||u||_{\mathrm{H}^1} \leq \frac{C}{\int_M a \,\mathrm{Vol}} \left(\frac{||a||_{\mathrm{L}^2}^2}{\int_M a \,\mathrm{Vol}} + ||a||_{\mathrm{L}^2} + 1 \right) ||b||_{\mathrm{L}^2}, \tag{2.158}$$

for some suitable constant C. By the elliptic estimate we conclude $u \in \mathrm{H}^2(M)$ and since a is bounded,

$$\begin{aligned} ||u||_{\mathrm{H}^2} &\leq K \left(||\Delta u||_{\mathrm{L}^2} + ||u||_{\mathrm{L}^2} \right) \\ &\leq K \left(||a u_0||_{\mathrm{L}^2} + ||b||_{\mathrm{L}^2} + ||u_0||_{\mathrm{L}^2} \right) \\ &\leq K \left(||u_0||_{\mathrm{L}^2} + ||b||_{\mathrm{L}^2} \right) \\ &\leq K \, ||b||_{\mathrm{L}^2}. \end{aligned} \tag{2.159}$$

where the constant K was renamed from one inequality to the following. □

We prove smooth dependence on parameters by the implicit function theorem. If $\Sigma = \mathbb{R}^2$, we define,

$$v_c = -\log\left(1 + \frac{1}{|x-c|^2}\right), \qquad g_c = -\frac{4}{(1+|x-c|^2)^2}, \tag{2.160}$$

then

$$-\Delta v_c = g_c + 4\pi\delta(x-c). \tag{2.161}$$

If Σ is compact, we rely on the existence of Green's function [2]. This is a smooth symmetric function $G : \Sigma \times \Sigma \setminus \Delta \to \mathbb{R}$, such that,

$$-\Delta_x G(x,y) = \delta_y - \frac{1}{|\Sigma|}, \qquad \int_\Sigma G(x,y)\mathrm{Vol}_x = 0. \tag{2.162}$$

Notice that we have chosen the oposite sign for $G(x,y)$ with respect to [2]. In this case, we define,

$$v_c = 4\pi\, G(x,c). \tag{2.163}$$

Given tuples $\mathbf{p} = (p_1, \dots, p_{k_+})$, $\mathbf{q} = (q_1, \dots, q_{k_-})$ of non intersecting vortices and antivortices, let

$$\mathbf{c} = (p_1, \dots, p_{k+}, q_1, \dots, q_{k_-}) \in \Sigma^{k_+ + k_-}, \tag{2.164}$$

$$v = \sum_j s_j v_{c_j}. \tag{2.165}$$

$$g = \begin{cases} \sum_j s_j g_{c_j}, & \Sigma = \mathbb{R}^2, \\ -\frac{4\pi}{|\Sigma|}(k_+ - k_-), & \Sigma \text{ compact.} \end{cases} \tag{2.166}$$

Let $\tilde{h} = h - v - \mu$, then the Taubes equation is equivalent to its regularised counterpart,

$$-\Delta \tilde{h} = F(\tilde{h} + v + \mu) - g. \tag{2.167}$$

If Σ is the euclidean plane, we add the boundary condition,

$$\lim_{|x| \to \infty} \tilde{h} = 0. \tag{2.168}$$

Theorem 2.12. *Let $\mathbf{p} = (p_1, \dots, p_{k_+})$, $\mathbf{q} = (q_1, \dots, q_{k_-})$ be sequences of non-intersecting simple cores in Σ, either a compact surface or the Euclidean plane. Let us denote by $h(x; \mathbf{p}, \mathbf{q})$ the solution to the Taubes equation for this configuration. For any families $U_r \subset \Sigma$, $r = 1, \dots, k_+$, $V_s \subset \Sigma$, $s = 1, \dots, k_-$, of open neighbourhoods on Σ such that $U_r \cap V_s = \emptyset$, let $W = (\cup_r U_r) \bigcup (\cup_s V_s)$, then the restriction*

$$h : \left(\Sigma \setminus \overline{W}\right) \times_r U_r \times_s V_s \to \mathbb{R}, \tag{2.169}$$

is smooth.

Proof. Consider the function

$$f(r) = \frac{1}{(1 + r^2)^2}, \qquad r \in \mathbb{R}, \tag{2.170}$$

this function has the property that it and all of its derivatives are dominated by r^{-4} as $r \to \infty$. This guarantees that,

$$g \in \mathrm{H}^r(\mathbb{R}^2), \qquad \forall r \geq 0. \tag{2.171}$$

and that as a function $\mathbb{R}^{2n} \to \mathrm{H}^r(\mathbb{R}^2)$, g varies smoothly. We note that in the plane, the function

$$e^{v_c} = \frac{|x-c|^2}{1+|x-c|^2}, \tag{2.172}$$

and all of its derivatives are bounded, and that $e^{v_\mathbf{p}}$ and $e^{v_\mathbf{q}}$ have no common zeros if $\mathbf{p}$ and $\mathbf{q}$ have no common elements. In the compact case, it is known that for fixed y, $G(x,y)$ has a singularity at y, however, locally in any open disk $\mathbb{D}_r(y)$ of smaller radius than the injectivity radius, $G(x,y)$ has the asymptotic expansion,

$$G(x,y) = \frac{1}{2\pi} \log\left(d(x,y)\right) + \tilde{G}(x,y), \qquad \forall x \in \mathbb{D}_r(y), \tag{2.173}$$

where $d(x,y)$ is the Riemannian distance and $\tilde{G}(x,y)$ is a smooth function defined on the disk. Hence e^{v_c} is also smooth and well defined on Σ.

In any case, $F(u+v+\mu) \in \mathrm{H}^r(\Sigma)$ for any $u \in \mathrm{H}^r(\Sigma)$. Let $\Delta\Sigma = \Sigma^{k_+} \times \Sigma^{k_-} \setminus \Delta_{k_+,k_-}$, then the function,

$$\Delta\Sigma \times \mathrm{H}^r(\Sigma) \to \mathrm{H}^r(\Sigma), \qquad (\mathbf{p},\mathbf{q},u) \mapsto F(u+v+\mu) - g, \tag{2.174}$$

is smooth. Therefore, the operator

$$\mathrm{T}\,u = \Delta u + F(u+v+\mu) - g, \qquad u \in \mathrm{H}^{r+2}(\Sigma), \tag{2.175}$$

is a well defined, smooth operator $\Delta\Sigma \times \mathrm{H}^{r+2}(\Sigma) \to \mathrm{H}^r(\Sigma)$. If $\tilde{h}$ is a solution to the regularised Taubes equation, then $\partial_{\tilde{h}}\mathrm{T} : \mathrm{H}^{r+2} \to \mathrm{H}^r$ is the operator,

$$(\partial_{\tilde{h}}\mathrm{T})\,\delta u = (\Delta + V(\tilde{h}+v+\mu))\,\delta u, \tag{2.176}$$

where as a function of Σ,

$$V(x) = V\left(\tilde{h}+v+\mu\right) \in C^r \tag{2.177}$$

is a positive function whose zero set is $\mathcal{P} \cup \mathcal{Q}$ and if Σ is $\mathbb{R}^2$, has the property that $\lim_{|x|\to\infty} V(x) = (1-\tau^2)$. By lemma 2.10, $\partial_{\tilde{h}}\mathrm{T}$ is an isomorphism, by the implicit function theorem, the mapping $(\mathbf{p},\mathbf{q}) \mapsto \tilde{h}$ is smooth as a map $\Delta\Sigma \to \mathrm{H}^r(\Sigma)$. By Sobolev's embedding, it is also smooth as a map $\Delta\Sigma \to C^{r-2}(\Sigma)$ for all $r \geq 2$. Hence, it depends smoothly on $(\mathbf{p},\mathbf{q})$. Finally, since the solution to the Taubes equation is $u = \tilde{h} + v + \mu$, we have that for any neighbourhood W of $\mathcal{P} \cup \mathcal{Q}$, the restriction $u : \Sigma \setminus \overline{W} \to \mathbb{R}$ depends smoothly on the cores. □

Corollary 2.13. *Let U be either an open and dense subset of the compact surface Σ or the euclidean plane. In any holomorphic chart $\varphi : U \to \mathbb{C}$ containing the cores, the localization formula (2.85) can be extended continuously to the coincidence set.*

Proof. For any given core $p_j \in U$, let $z_j = \varphi(p_j) \in \mathbb{C}$. We assume each p_j is simple and that all the z_j are contained in a bounded domain $D \subset \mathbb{C}$. This assumption is superfluous for the Euclidean plane but for a compact surface is necessary for the existence of a smooth function $H : D \times D \to \mathbb{R}$ such that for any $z, w \in D$,

$$G(\varphi^{-1}(z), \varphi^{-1}(w)) = \frac{1}{2\pi} \log|z - w| + H(z, w). \tag{2.178}$$

Assume without loss of generality $s_1 = s_2$, to prove the result it is enough to show that for any pair of indices $i, j \in \{1, \ldots, k_+ + k_-\}$, $\lim_{z_1 \to z_2} \partial_{z_j} b_i(\mathbf{z})$ exists, where $\mathbf{z} = (z_1, \ldots, z_{k_+ + k_-})$. In the following computation, we denote by $h_\varphi(z) = h(\varphi^{-1}(z))$, $G_\varphi(z, w) = G(\varphi^{-1}(z), \varphi^{-1}(w))$, $\tilde{h}_\varphi(z) = \tilde{h}(\varphi^{-1}(z))$ the local representation of the functions,

$$\begin{aligned} b_i &= 2\,\overline{\partial}_{z=z_i}\left(s_i\, h_\varphi(z) - \log|z - z_i|^2\right) \\ &= 2\,\overline{\partial}_{z=z_i}\left(4\pi\, G_\varphi(z, z_i) - \log|z - z_i|^2\right) + 8\pi \sum_{k \neq i} s_i s_k \overline{\partial}_z G_\varphi(z_i, z_k) + 2 s_i\, \overline{\partial}_z \tilde{h}(z_i, \mathbf{z}) \\ &= 8\pi\, \overline{\partial}_z H_\varphi(z_i, z_i) + 8\pi \sum_{k \neq i} s_i s_k \overline{\partial}_z G_\varphi(z_i, z_k) + 2 s_i\, \overline{\partial}_z \tilde{h}_\varphi(z_i, \mathbf{z}), \end{aligned} \tag{2.179}$$

where $\overline{\partial}_z$ refers to the derivative with respect to the first variable in each term. Hence,

$$\partial_{z_j} b_i = 8\pi\, \partial_{z_j} \overline{\partial}_z H_\varphi(z_i, z_i) + 8\pi \sum_{k \neq i} s_i s_k \partial_{z_j} \overline{\partial}_z G_\varphi(z_i, z_k) + 2 s_i\, \partial_{z_j} \overline{\partial}_z \tilde{h}_\varphi(z_i, \mathbf{z}). \tag{2.180}$$

The functions $\partial_{z_j}\overline{\partial}_z H_\varphi(z_i, z_i)$ and $\partial_{z_j}\overline{\partial}_z \tilde{h}_\varphi(z_i, \mathbf{z})$ vary continuously with $(z_i, \mathbf{z})$, whence, the limits,

$$\lim_{z_1\to z_2} \partial_{z_j}\overline{\partial}_z H_\varphi(z_i, z_i), \qquad \lim_{z_1\to z_2} \partial_{z_j}\tilde{h}_\varphi(z_i, \mathbf{z}), \tag{2.181}$$

both exist. In the above sum, if $i \neq 1$ and $k \neq 1$, or if either $i = 1$ or $k = 1$ and the other is not index 2, the limit

$$\lim_{z_1\to z_2} \partial_{z_j}\overline{\partial}_z G_\varphi(z_i, z_k) \tag{2.182}$$

exists because G is smooth away of the diagonal set of $\Sigma \times \Sigma$. Finally, if $\{i, k\} = \{1, 2\}$, we can assume without loss of generality $i = 1$, $k = 2$, to compute,

$$\begin{aligned}
\lim_{z_1\to z_2} \partial_{z_j}\overline{\partial}_z G_\varphi(z_1, z_2) &= \lim_{z_1\to z_2} \partial_{z_j}\overline{\partial}_{z=z_1} \left(\frac{1}{2\pi}\log|z - z_2| + H_\varphi(z, z_2)\right) \\
&= \lim_{z_1\to z_2} \partial_{z_j}\overline{\partial}_z H_\varphi(z_1, z_2) \\
&= \partial_{z_j}\overline{\partial}_z H_\varphi(z_2, z_2).
\end{aligned} \tag{2.183}$$

Therefore, $\lim_{z_1\to z_2} \partial_{z_j} b_i(\mathbf{z})$ exists, implying the localization formula can be extended to the coincidence set. □

In later applications we will also focus on vortices of the Ginzburg-Landau functional, in this case, the governing elliptic problem is the orginal Taubes equation,

$$-\Delta h = e^h - 1 + 4\pi \sum_i \delta_{p_i}. \tag{2.184}$$

If Σ is the euclidean plane, we add the condition $\lim_{|x|\to\infty} h = 0$. In both cases, we know that there exists a solution h to the Taubes equation for any configuration $\mathbf{p}$ of points. On the plane this is proved in [56] whereas in a compact surface Σ the proof can be found in [61]. As for the $O(3)$ Sigma model, given a configuration $\mathbf{p} = (p_1, \ldots, p_n)$ of cores, if we define $\tilde{h}$ such that $h = \tilde{h} + v_\mathbf{p}$, then $\tilde{h}$ is the unique solution of the regularized Taubes equation for the Ginzburg-Landau functional,

$$-\Delta\tilde{h} = e^{\tilde{h}+v_\mathbf{p}} - 1 - g_\mathbf{p}, \tag{2.185}$$

where the functions $v_{\mathbf{p}}$, $g_{\mathbf{p}}$ are defined either as in equation (2.160) if Σ is the euclidean plane or $v_{\mathbf{p}}$ is defined as in equation (2.160) and $g_{\mathbf{p}}$ is the constant function $4\pi n|\Sigma|^{-1}$ if Σ is compact.

Mimicking the proof of theorem 2.12, we prove the following proposition.

Proposition 2.14. *Let $\mathbf{p} \in \Sigma^n$ be a sequence of points on a Riemann surface Σ either compact or the euclidean plane. If Σ is compact assume $n \in \mathbb{Z}^+$ satisfies Bradlow's bound for vortices of the Ginzburg-Landau functional,*

$$4\pi\, n \leq |\Sigma|. \tag{2.186}$$

Let $\tilde{h} : \Sigma \times \Sigma^n \to \mathbb{R}$ be such that $\tilde{h}(x;\mathbf{p})$ is the unique solution to equation (2.185) with data $\mathbf{p}$, then $\tilde{h}$ is a smooth function of x and the data.

Proof. As in the proof of theorem 2.12, we define an operator $\mathrm{T} : \Sigma^n \times \mathrm{H}^{r+2} \to \mathrm{H}^r$, such that,

$$\mathrm{T}\,(\mathbf{p}, u) = \Delta u + e^{u+v_{\mathbf{p}}} - 1 - g_{\mathbf{p}}, \tag{2.187}$$

and observe that as in the proof of the theorem, this operator is smooth. Moreover, the derivative $\partial_u \mathrm{T} : \mathrm{H}^{r+2} \to \mathrm{H}^r$ at $(\mathbf{p}, \tilde{h})$ is,

$$\partial_u \mathrm{T}(\delta u) = (\Delta + e^{\tilde{h}+v_{\mathbf{p}}})\, \delta u. \tag{2.188}$$

We notice the potential $V(x) = e^{\tilde{h}+v_{\mathbf{p}}}$ and all the derivatives are bounded functions. If Σ is compact, this is because V is smooth and if the surface is $\mathbb{R}^2$, this is becuase $e^{v_{\mathbf{p}}}$ has this property, as shown in the proof of theorem 2.12 and because h, the solution to the Taubes equation, and all of their derivatives decay exponentially as $|x| \to \infty$. Hence, $\tilde{h} = h - v_{\mathbf{p}}$ and the derivatives are continuous bounded functions. By lemma 2.10, $\partial_u \mathrm{T}$ is an isomorphism. By the implicit function theorem, for any $\mathbf{p} \in \Sigma^n$ and any $r \geq 2$, there is a neighbourhood $U \subset \Sigma^n$ of $\mathbf{p}$, such that the map $\mathbf{p} \mapsto \tilde{h}(x;\mathbf{p})$ is smooth as a function $U \to \mathrm{H}^{r+2}$. By Sobolev's embedding, $\tilde{h}(x)$ is of class C^r as a function $\Sigma \times U \to \mathbb{R}$. Since differentiability is a local property, this implies $\tilde{h}$ is of class C^r on $\Sigma \times \Sigma^n$ for any $r \geq 0$. Therefore $\tilde{h}$ is smooth. ☐

2.4 Topological methods

We finalize this chapter with a brief exposition of some results of analysis that we will use to prove existence of solutions to the elliptic problem on compact manifolds in chapters 4 and 5. Both methods are attributable to Leray and Schauder. Our exposition will be short and will focus on the results we need. Details can be found in the books [8] and [11]. Recall a subset of a topological space is precompact if the closure is compact.

Definition 2.15. *Let X, Y be Banach spaces and $\Omega \subset X$. A continuous map $T : \Omega \to Y$ is compact if it maps bounded subsets of Ω to precompact subsets of Y.*

As a caveat, in [11] compact operators are called completely continuous.

Theorem/Definition 2.16. *Let $\Omega \subset X$ be an open and bounded subset of a real Banach space, $T : \Omega \to X$ compact and $y \notin (I - T)(\partial\Omega)$. For each admisible triple (T, Ω, y), there is a unique integer $\deg(I - T, \Omega, y) \in \mathbb{Z}$, with the following properties:*

1. $\deg(I, \Omega, y) = 1$ *for* $y \in \Omega$.

2. $\deg(I - T, \Omega, y) = \deg(I - T, \Omega_1, y) + \deg(I - T, \Omega_2, y)$ *whenever* Ω_1, Ω_2 *are disjoint open subsets of* Ω *such that* $y \notin (I - T)(\overline{\Omega} \setminus (\Omega_1 \cup \Omega_2))$.

3. *Homotopy invariance:* $\deg(I - H(t, \cdot), \Omega, y(t))$ *is independent of* $t \in [0, 1]$ *whenever* $H : [0, 1] \times \overline{\Omega} \to X$ *is compact,* $y : [0, 1] \to X$ *is continuous and* $y(t) \notin (I - H(t, \cdot))(\partial\Omega)$ *on* $[0, 1]$.

4. *General homotopy invariance: Let* $\Theta \subset [0, 1] \times X$ *be bounded and open in* $[0, 1] \times X$ *with* $\Theta_t = \{x \in X : (t, x) \in \Theta\}$. *If* $H : \overline{\Theta} \to X$ *is compact and* $y : [0, 1] \to X$ *is continuous with* $y(t) \notin (I - H(t, \cdot))(\partial\Theta_t)$ *for all* $t \in [0, 1]$, *then* $\deg(I - H(t, \cdot), \Theta_t, y(t))$ *is independent of* t.

$\deg(I - T, \Omega, y)$ is Leray-Schauder's degree. It can be proved [11, Thm. 8.2] that $\deg(I - T, \Omega, y) \neq 0$ implies $(I - T)^{-1}(y) \neq \emptyset$. As an application of this concept, there is the following result of Schäfer [48],

Theorem 2.17. *Let $T : X \to X$ be compact. Then the following alternative holds:*

1. $x - \lambda T(x) = 0$ *has a solution for every* $\lambda \in [0, 1]$*, or*
2. $S = \{x \in X : \exists\lambda \in [0, 1]\, s.t.\, x - \lambda T(x) = 0\}$ *is unbounded.*

For a linear operator, alternative 1 always holds by choosing the solution $x = 0$, however, for non-linear operators this is not always the case. A proof of the theorem can be found in [8, Cor. I.1.18]. In general, computing the degree is a difficult task. Suppose $x_0 \in (I - T)^{-1}(y)$ isolated, then x_0 is the only solution of the equation $x - T(x) = y$ in some disk $\mathbb{D}_{\epsilon_0}(x_0)$. By homotopy invariance, $\deg(I - T, \mathbb{D}_\epsilon(x_0), y)$ is independent of ϵ for $0 < \epsilon < \epsilon_0$.

Definition 2.18. *With the previous assumptions, the index of an isolated solution x_0 to the equation $x - T(x) = y$ is*

$$\operatorname{ind}(I - T, x_0, y) = \deg(I - T, \mathbb{D}_\epsilon(x_0), y), \tag{2.189}$$

where $\epsilon > 0$ is any sufficiently small radius.

If T is compact and differentiable at x_0, then $T'(x_0)$ is a compact linear operator. We state the following theorems,

Theorem 2.19 (Leray-Schauder)**.** *If $T : \Omega \subset X \to X$ is compact and differentiable at x_0 and if $I - T'(x_0)$ is injective, then $\operatorname{ind}(I - T, x_0, y) = \pm 1$. More precisely,*

$$\begin{aligned} \operatorname{ind}(I - T, x_0, y) &= \operatorname{ind}(I - T'(x_0), x_0, y) \\ &= (-1)^\beta, \qquad \beta = \sum_{\lambda > 1} m(\lambda). \end{aligned} \tag{2.190}$$

The sum is taken over all eigenvalues $\lambda > 1$ of $T'(x_0)$ and $m(\lambda)$ is the algebraic multiplicity of λ.

Definition 2.20. *An operator $H(\lambda, x)$, $H : \mathbb{R} \times X \to X$, is continuous in λ uniformly with respect to x in balls in X if for any given ball $B \subset X$ and for any $\epsilon > 0$, there is a $\delta > 0$ such that $|\lambda_2 - \lambda_1| < \delta$ implies $|H(\lambda_2, x) - H(\lambda_1, x)| < \epsilon$ for all $x \in B$.*

The following theorem can be found in [8, Thm. I.3.3].

Theorem 2.21. *Let* $H : \mathbb{R} \times X \to X$ *be such that for all* $\lambda \in \mathbb{R}$ *the map* $H(\lambda, \cdot) : X \to X$ *is compact and* $H(\lambda, x)$ *is continuous in* λ *uniformly with respect to* x *in balls in* X *(definition 2.20). Let* (λ_0, x_0) *be a solution of the equation*

$$x - H(\lambda, x) = 0. \tag{2.191}$$

Suppose $\mathcal{U} \subset X$ *is an open, bounded set such that* $x_0 \in \mathcal{U}$ *and,*

1. *for fixed* λ_0 *there is no other solution in* $\overline{\mathcal{U}}$*,*
2. $\deg(I - H(\lambda_0, \cdot), \mathcal{U}, 0) \neq 0$*.*

Then there exist two connected and closed sets (=continua) $\mathcal{C}^+ \subset [\lambda_0, \infty) \times X$ *and* $\mathcal{C}^- \subset (-\infty, \lambda_0] \times X$ *of solutions of* (2.191) *with* $(\lambda_0, x_0) \in \mathcal{C}^+ \cap \mathcal{C}^-$*. For* $\mathcal{C}^+$ *one of the following two alternatives hold:*

1. $\mathcal{C}^+$ *is unbounded or,*
2. $\mathcal{C}^+ \cap (\{\lambda_0\} \times (X \setminus \overline{\mathcal{U}})) \neq \emptyset$*.*

The same alternatives hold for $\mathcal{C}^-$*.*

The hypotesis on H implies the restriction to bounded subsets of $\mathbb{R} \times X$ is compact. The definition of compact operator on the reference is slightly different, however, it is not difficult to go through the proof and adapt it to our current definition.

We conclude the section mentioning that several results related to theorem 2.21 can be found in the literature. A good survey of related applications is [40].

Chapter 3

Asymmetric vortex-antivortex systems in the euclidean plane

In this chapter we study the moduli space of vortex-antivortex pairs on the euclidean plane in detail. Our approach will be analytical and numerical. To understand the geometry of the moduli space, we need to analyse the properties of the Taubes equation in the critical case when a vortex and an antivortex collide.

In section 3.1, we study the space of vortex-antivortex pairs, the main result is that it is incomplete. To prove this theorem, we find bounds for h_ϵ, the solution to the Taubes equation in several lemmas in subsection 3.1.1.

In section 3.2, we develop an asymptotic approximation for the L^2 metric of vortex-antivortex pairs in the centre of mass frame, and complement it with the point source formalism in subsection 3.2.1, in which we approximate the fields linearising the field equations. The main result is the Lagrangian (3.214) which confirms the asymptotic formula obtained previously. In subsection 3.2.2, we find another asymptotic aproximation for the metric, this time for small ϵ, the main result is equation (3.269).

In section 3.3 we approximate the L^2 metric numerically, using the data found by numerical methods to study the scattering of vortex-antivortex pairs in subsection 3.3.1 and in this way testing our approximations of the previous section.

Finally, in section 3.4 we study Ricci magnetic geodesics. These curves are of mathematical interest, there are a few results about the relation between extensibility of them and completeness of the underlying space, as the moduli space of vortex-antivortex pairs is incomplete, the question of whether or not it is complete in the Ricci magnetic sense is interesting in its own.

In order to start, let us note that on $\mathbb{R}^2$ any fibre bundle is trivial, therefore we can consider sections on the target manifold as pairs (ϕ, A) of a function $\phi : \mathbb{R}^2 \to \mathbb{S}^2$ and a 1-form $A \in \Omega^1(\mathbb{R}^2)$. Since the Lagrangian is isometrically invariant, by Noether's theorem there will be conserved currents. In the Euclidean case, the conserved quantities are the total energy, E, the linear and angular momenta. We already know how to compute the energy. For the remaining constants of motion note that the Laplacian is invariant under the action of the group of isometries of the plane, $\mathbb{E}(2) \cong \mathbb{R}^2 \rtimes \mathbb{O}(2)$, which is a Lie group of dimension three. If $h(x; \mathbf{p}, \mathbf{q}) : \mathbb{R}^2 \setminus (P \cup Q) \to \mathbb{R}$ is the solution to the Taubes equation, and $\gamma \in \mathbb{E}(2)$ acts in $\mathbf{p}$ and $\mathbf{q}$ component-wise, this implies,

$$h(x; \gamma\mathbf{p}, \gamma\mathbf{q}) = h(\gamma^{-1}x; \mathbf{p}, \mathbf{q}). \tag{3.1}$$

Lemma 3.1. *Let $\mathbf{c} = (c_1, \ldots, c_{k_+ + k_-})$ be a sequence of cores, ordered such that the first k_+ are the vortices. Let b_j be the coefficients defined in lemma 2.3. If $\gamma x = \alpha x + \beta$, $\alpha, x, \beta \in \mathbb{R}^2$, $|\alpha| = 1$, is an orientation preserving isometry, then*

$$b_j(\gamma\mathbf{c}) = \alpha\, b_j(\mathbf{c}). \tag{3.2}$$

If γ is the orientation reversing generator, $\gamma x = \overline{x}$, we have,

$$b_j(\overline{\mathbf{c}}) = \overline{b}_j(\mathbf{c}). \tag{3.3}$$

Proof. After some algebraic manipulation and the chain rule,

$$\begin{aligned} b_j(\gamma\mathbf{c}) &= 2\,\overline{\partial}_{x=\gamma c_j} \left(s_j h(x;\gamma\mathbf{c}) - \log|x - \gamma c_j|^2\right) \\ &= 2\,\overline{\partial}_{x=\gamma c_j} \left(s_j h(\gamma^{-1}x;\mathbf{c}) - \log|\gamma^{-1}x - c_j|^2\right) \\ &= 2\,\partial_{x=c_j} \left(s_j h(x;\mathbf{c}) - \log|x - c_j|^2\right) \overline{\partial}_{x=\gamma c}(\gamma^{-1}x) \\ &\quad + 2\,\overline{\partial}_{x=c_j} \left(s_j h(x;\mathbf{c}) - \log|x - c_j|^2\right) \overline{\partial}_{x=\gamma c}\overline{(\gamma^{-1}x)} \\ &= 2\,\overline{\partial}_{x=c_j} \left(s_j h(x;\mathbf{c}) - \log|x - c_j|^2\right) \frac{1}{\overline{\alpha}} \\ &= \alpha\, b_j(\mathbf{c}), \end{aligned} \tag{3.4}$$

where we used the fact that $\gamma^{-1}x$ is holomorphic and α unitary to simplify the result of the chain rule. For the second identity we proceed analogously,

$$\begin{aligned} b_j(\overline{\mathbf{c}}) &= 2\,\overline{\partial}_{x=\overline{c}_j}\left(s_j h(x;\overline{\mathbf{c}}) - \log|x-\overline{c}_j|^2\right) \\ &= 2\,\overline{\partial}_{x=\overline{c}_j}\left(s_j h(\overline{x};\mathbf{c}) - \log|\overline{x}-c_j|^2\right) \\ &= 2\,\partial_{x=c_j}\left(s_j h(x;\mathbf{c}) - \log|x-c_j|^2\right) \\ &= \overline{b}_j(\mathbf{c}). \end{aligned} \tag{3.5}$$

□

As a consequence of the lemma, the coefficients b_j are translation invariants: if X is the Killing field generated by a one parameter family of isometries γ_s, then

$$\mathcal{L}_X b_j(\mathbf{c}) = \dot{a}_0\, b_j(\mathbf{c}), \tag{3.6}$$

where $\mathcal{L}_X$ denotes the Lie derivative. If $\gamma_s x = x + sb$ is a one parameter family of translations, then $\dot{a}_0 = 0$, on the other hand,

$$X = \sum_k \left(b\,\partial_{c_k} + \overline{b}\,\overline{\partial}_{c_k}\right), \tag{3.7}$$

Letting $b = 1$ and $b = i$, we obtain,

$$\sum_k \left(\partial_{c_k} + \overline{\partial}_{c_k}\right) b_j(\mathbf{c}) = 0, \qquad \sum_k \left(\partial_{c_k} - \overline{\partial}_{c_k}\right) b_j(\mathbf{c}) = 0. \tag{3.8}$$

Hence, $\sum_k \partial_{c_k} b_j = \sum_k \overline{\partial}_{c_k} b_j = 0$. Applying the symmetries of the coefficients,

$$\overline{\partial}_{c_j} \sum_k b_k = \sum_k \overline{\partial}_{c_j} b_k = \sum_k \overline{\partial}_{c_k} b_j = 0, \tag{3.9}$$

$$\partial_{c_j} \sum_k b_k = \sum_k \partial_{c_j} b_k = \sum_k \overline{\partial}_{c_k} \overline{b}_j = \overline{\sum_k \partial_{c_k} b_j} = 0. \tag{3.10}$$

Therefore, $\sum_k b_k$ is constant. Repeating this argument with the one-parameter family of rotations $\gamma_s\, x = e^{s\,i}\, x$, we find that $X = i \sum_k \left(c_k\, \partial_{c_k} - \overline{c}_k\, \overline{\partial}_{c_k}\right)$, thence,

$$\sum_k \left(c_k\, \partial_{c_k} - \overline{c}_k\, \overline{\partial}_{c_k}\right)\, b_j = b_j. \tag{3.11}$$

Summing over j, we find,

$$\sum_j b_j = \sum_j \sum_k \left(c_k\, \partial_{c_k} - \bar{c}_k\, \overline{\partial}_{c_k}\right)\, b_j = \sum_k \left(c_k\, \partial_{c_k} - \bar{c}_k\, \overline{\partial}_{c_k}\right) \cdot \sum_j b_j = 0, \tag{3.12}$$

since $\sum_j b_j$ is constant. This result is analogous to the similar result obtained by Samols for vortices of the Ginzburg-Landau functional in [47]. As a consequence of this symmetry we have the following proposition about conservation of momentum.

Proposition 3.2. *The total conserved momentum of a vortex-antivortex system with cores at position* $\mathbf{c}$ *is,*

$$P_1 + P_2\, i = 2\pi \sum_j (1 - s_j \tau)\, \dot{c}_j, \tag{3.13}$$

where $s_j = s_{c_j}$ *is the sign function determining the type of the core.*

Proof. By lemma 3.1 the translation group acts isometrically on the moduli space. Hence for any $b \in \mathbb{C}$ the fields, $X = \sum_k (b\, \partial_{c_k} + \bar{b}\, \overline{\partial}_{c_k})$ are Killing fields and the product

$$P_b = \langle \dot{\mathbf{c}}, X \rangle \tag{3.14}$$

is constant along geodesic trajectories and corresponds to the projection of momentum on the b direction. If K denotes the Kähler metric, equation (2.86),

$$\begin{aligned} P_b &= \Re\left(\mathrm{K}(\dot{\mathbf{c}}, X)\right) \\ &= \frac{1}{2}\left(\mathrm{K}(\dot{\mathbf{c}}, X) + \overline{\mathrm{K}(\dot{\mathbf{c}}, X)}\right) \\ &= \frac{1}{2}\left(\mathrm{K}(\dot{\mathbf{c}}, X) + \mathrm{K}(X, \dot{\mathbf{c}})\right) \\ &= \frac{1}{2} \sum_{i,j} \mathrm{K}_{i\bar{j}}\, \left(\dot{c}_i \overline{X}_j + X_i \overline{\dot{c}_j}\right). \end{aligned} \tag{3.15}$$

On the other hand, $\mathrm{K}_{i\bar{j}} = 2\pi\left((1 - s_i\tau)\delta_{ij} + \partial_{c_i} b_j\right)$. Note that by the invariance of the coefficients b_j, we have,

$$\sum_{i,j} \partial_{c_i} b_j\, \left(\dot{c}_i \overline{X}_j + X_i \overline{\dot{c}_j}\right) = \bar{b} \sum_i \dot{c}_i \partial_{c_i} \left(\sum_j b_j\right) + b \sum_j \left(\dot{c}_j \sum_i \partial_{c_i} b_j\right) = 0. \tag{3.16}$$

Hence,

$$P_b = \pi \sum_i (1 - s_i\tau)(\dot{c}_i\,\overline{b} + b\,\overline{\dot{c}_i}). \tag{3.17}$$

If we let $b = 1$ and $b = i$ we get the momentum in the direction of the real an imaginary axes are the real and imaginary parts of the vector,

$$2\pi \sum_j (1 - s_j\tau)\,\dot{c}_j. \tag{3.18}$$

□

Proposition 3.3. *The angular momentum of a vortex-antivortex system is,*

$$\ell = \omega(\mathbf{c}, \dot{\mathbf{c}}), \tag{3.19}$$

where $\omega \in \Omega^2(\mathbb{R}^2)$ is the Kähler form of the metric.

Proof. Conservation of angular momentum corresponds to the action of $SO(2)$ on the moduli space. Let $X = i\sum_k \left(c_k\partial_{c_k} - \overline{c}_k\,\overline{\partial}_{c_k}\right)$ be the Killing field generating the action of $SO(2)$, the conserved angular momentum is,

$$\begin{aligned}
\ell &= \langle \dot{\mathbf{c}}, X\rangle \\
&= \frac{1}{2}\left(\mathrm{K}(\dot{\mathbf{c}}, X) + \mathrm{K}(X, \dot{\mathbf{c}})\right) \\
&= \pi i \sum_{j,k} \left(\left((1 - s_j)\,\delta_{jk} + \partial_{c_j} b_k\right)\left(-\dot{c}_j\,\overline{c}_k + c_j\,\overline{\dot{c}_k}\right)\right) \\
&= \omega(\mathbf{c}, \dot{\mathbf{c}}).
\end{aligned} \tag{3.20}$$

□

It is convenient to express the dynamics of vortex-antivortex systems in the centre of mass frame. Let us define,

$$C = \frac{1}{(1-\tau)k^+ + (1+\tau)k^-} \sum_j (1 - s_j\tau)c_j, \tag{3.21}$$

$$M = 2\pi\,(1-\tau)k^+ + 2\pi\,(1+\tau)k^-, \tag{3.22}$$

M is the total mass and C the centre of mass of the system as determined by conservation of momentum and energy. Let us define the variables $\xi_j \in \mathbb{R}^2$ such that,

$$c_j = C + \xi_j. \tag{3.23}$$

Let $m_j = 2\pi(1 - s_j \tau)$ be the effective mass of a core, then $\sum_j m_j \, \xi_j = 0$. Note that this linear combination is invariant under the action of $S_{k_+} \times S_{k_-}$ on the moduli space, where S_n is the symmetric group of order n, hence it determines a well defined subspace $\mathcal{M}_0^{k_+,k_-} \subset \mathcal{M}^{k_+,k_-}$ where $C = 0$.

Proposition 3.4. *Let* K_0 *be the restriction of the Kähler metric to* $\mathcal{M}_0^{k_+,k_-}$*, then,*

$$\mathrm{K} = M|dC|^2 + \mathrm{K}_0.$$

Proof. This is a consequence of translation invariance, let $m_i = 2\pi\,(1 - s_i\,\tau)$ be the mass of the core at c_i,

$$\mathrm{K} = \sum_i m_i \, |dC + d\xi_i|^2 + 2\pi \sum_{i,j} \partial_{c_i} b_j \, (dC + d\xi_i) \, \overline{(dC + d\xi_j)}. \tag{3.24}$$

The first terms can be split into

$$M|dC|^2 + 2\Re\left(\overline{dC} \sum_i m_i d\xi_i\right) + \sum_i m_i |d\xi_i|^2 = M|dC|^2 + \sum_i m_i |d\xi_i|^2, \tag{3.25}$$

and the second terms can be split as,

$$\begin{aligned}2\pi\left(\sum_j (|dC|^2 + \overline{d\xi}_j dC) \cdot \sum_i \partial_{c_i} b_j + \overline{dC} \sum_i d\xi_i \cdot \sum_j \partial_{c_i} b_j + \sum_{i,j} \partial_{c_i} b_j \overline{d\xi}_j d\xi_{c_i}\right) \\ = 2\pi \sum_{i,j} \partial_{c_i} b_j \, \overline{d\xi}_j d\xi_i,\end{aligned} \tag{3.26}$$

where the first two terms cancelled because the coefficients b_j are translation invariant. Substituting back into the formula for K we conclude the claim of the proposition. ☐

As a consequence, the moduli space decomposes in a product of Kähler manifolds,

$$\mathcal{M}^{k_+,k_-} \cong \mathbb{R}^2 \times \mathcal{M}_0^{k_+,k_-}, \tag{3.27}$$

such that the metric splits in a trivial flat metric in $\mathbb{R}^2$ and the nontrivial restriction to $\mathcal{M}_0^{k_+,k_-}$. This splitting was first observed by Samols [47] for vortices in the Abelian Higgs model. $\mathcal{M}_0^{k_+,k_-}$ is the space of vortices and antivortices with fixed centre of mass. Given the decomposition of the metric in the moduli space, the energy and angular momentum in the centre of mass frame are,

$$E = \frac{1}{2}\mathrm{K}(\dot{\xi}, \dot{\xi}), \tag{3.28}$$

$$\ell = \omega(\xi, \dot{\xi}), \tag{3.29}$$

where K and ω are the Kähler metric and Kähler form of ambient space at $\xi = (\xi_1, \ldots, \xi_{k_+ + k_-})$.

3.1 The moduli space of vortex-antivortex pairs

In this section we focus on the moduli space of vortex-antivortex pairs on Euclidean space and extend the analysis done by Romão-Speight in [45] for $\tau = 0$. We focus on the non trivial part of the metric in the submanifold $\mathcal{M}_0^{1,1} \cong \mathbb{R}^2 \setminus \{0\}$, of pairs with centre of mass at the origin. Let

$$b(x) = b_1(x, -x), \qquad x \in \mathbb{R}^+. \tag{3.30}$$

By the invariance of the coefficient b_1 with respect to conjugation, b is a real function. Let us assume c_1 is the vortex position, introducing (ϵ, θ) coordinates such that $c_1 - c_2 = 2\epsilon e^{i\theta}$, we have,

$$b_1(c_1, c_2) = e^{i\theta} b(\epsilon). \tag{3.31}$$

Recall $b_1 + b_2 = 0$ and $\partial_1 b_1 + \partial_2 b_1 = 0$, then the restriction of the metric to $\mathcal{M}_0^{1,1}$ is,

$$g_0 = \Omega(\epsilon)\,(d\epsilon^2 + \epsilon^2 d\theta^2), \tag{3.32}$$

where the conformal factor is,

$$\Omega(\epsilon) = 2\pi \left(2(1 - \tau^2) + \frac{1}{\epsilon}\frac{d}{d\epsilon}\left(\epsilon b(\epsilon)\right) \right). \tag{3.33}$$

3.1.1 The singularity at $\epsilon = 0$

In this section we study the limiting behaviour of solutions to the Taubes equation for vortex-antivortex pairs as $\epsilon \to 0$. We aim to prove bounds for h_ϵ in order to estimate the length of radial geodesics and finalize proving that the moduli space of vortex-antivortex pairs is incomplete. We start defining the following constant and functions,

$$\mu = \frac{1-\tau}{1+\tau}, \qquad F_\mu(t) = 2\,\frac{e^t - 1}{\mu\, e^t + 1}, \qquad V_\mu(t) = \frac{2(\mu+1)\, e^t}{(\mu\, e^t + 1)^2}. \tag{3.34}$$

If $h_T(x, \epsilon, -\epsilon)$ is the solution to the Taubes equation with a vortex at position $(\epsilon, 0)$ and an antivortex at $(-\epsilon, 0)$, let us define the function h_ϵ such that $h_T = h_\epsilon + \mu$. To express the Taubes equation in a convenient way, we make the change of variable,

$$x' = (1-\tau^2)^{-1/2}\, x, \tag{3.35}$$

under this change of variable, the position of a vortex or antivortex is $(\pm\epsilon', 0) = (\pm(1-\tau^2)^{-1/2}\,\epsilon, 0)$. By an abuse of notation, we still denote by x coordinates in the rescaled Euclidean plane and by $(\pm\epsilon, 0)$ the positions of the cores. With these definitions, the Taubes equation is equivalent to,

$$-\Delta h_\epsilon = F_\mu(h_\epsilon) + 4\pi\delta_\epsilon - 4\pi\delta_{-\epsilon}, \tag{3.36}$$

together with the constraint,

$$\lim_{|x|\to\infty} h_\epsilon = 0. \tag{3.37}$$

Let u be the solution to the Taubes equation for the Ginzburg-Landau functional [56],

$$-\Delta u = e^u - 1 + 4\pi\delta_0, \tag{3.38}$$

Yang proves in [60] that $u < 0$. For the following results, we will assume $\tau \in [0, 1)$, the case $\tau < 0$ being similar. Repeating the argument of Yang, the function $u_\epsilon(x) = u(x - \epsilon)$ is a sub-solution of h_ϵ, i.e. $u_\epsilon < 0$ and

$$-\Delta u_\epsilon \geq F_\mu(u_\epsilon), \qquad x \in \mathbb{R}^2 \setminus \{\epsilon, -\epsilon\}\,. \tag{3.39}$$

On the other hand, the function $-u_{-\epsilon} = -u(x+\epsilon)$ is a super-solution: it is positive and

$$-\Delta(-u_{-\epsilon}) \leq F_\mu(-u_{-\epsilon}), \qquad x \in \mathbb{R}^2 \setminus \{\epsilon, -\epsilon\}\,. \tag{3.40}$$

By the maximum principle,

$$u_\epsilon(x) < h_\epsilon(x) < -u_{-\epsilon}(x), \qquad x \in \mathbb{R}^2 \setminus \{\epsilon, -\epsilon\}\,. \tag{3.41}$$

Lemma 3.5. *For any $\delta \in (0,1)$ there exist two constants $C(\delta)$ and $R(\delta)$ such that*

$$|u(x)| \leq C\, e^{-(1-\delta)\,|x|}, \qquad |\nabla u(x)| \leq C\, e^{-(1-\delta)\,|x|}, \qquad |x| > R. \tag{3.42}$$

In particular, $||u||_{\mathrm{L}^p} < \infty$ for any $p > 0$.

Proof. That u and its derivatives decay exponentially fast at infinity can be found in the literature, for example in [23, 56], here we adapt a proof of Yang for solutions of the elliptic problem of the $O(3)$ Sigma model in the symmetric case [59, Lemma 8.3]. Since $\lim_{|x|\to\infty} u = 0$, we linearise (3.38) about $u = 0$ in a neighbourhood of infinity to obtain,

$$-\Delta u = f(x)\, u, \qquad |x| \geq R. \tag{3.43}$$

where $f(x)$ is a function such that $f(x) \to 1$ as $|x| \to \infty$. Let us introduce the comparison function

$$w(x) = C\, e^{-(1-\delta)\,|x|}, \qquad |x| \geq R, \tag{3.44}$$

where $C(\delta)$ and $R(\delta)$ are positive constants yet to be determined. The Laplacian of this function is $-\Delta w = (1-\delta)\,(1-\delta-|x|^{-1})\,w$. Choosing R sufficiently large, we can guarantee that

$$f(x) > (1-\delta)\left(1-\delta-\frac{1}{|x|}\right), \tag{3.45}$$

hence,

$$-\Delta(u-w) > f(x)(u-w), \tag{3.46}$$

for $|x| > R$. Let us choose C big enough for the continuous function $u - w$ to be negative at the boundary $|x| = R$. Since $u(x) - w(x) \to 0$ as $|x| \to \infty$, by the maximum principle $u(x) < w(x)$ for all $|x| \geq R$. Since (3.43) is linear, we can apply the same argument to $-u$. Choosing the bigger of each pair of constants (C, R) the decay rate of u is proved.

For the decay rate of ∇u, we know that $u \in \mathrm{H}^r$ for all $r \geq 2$ [56], in particular, $\nabla u \to 0$ as $|x| \to \infty$. Linearising in a neighbourhood of infinity, ∇u is a solution to the equation,

$$-\Delta\,(\nabla u) = f(x)\,\nabla u, \tag{3.47}$$

for some function $f(x)$ such that $f(x) \to 1$ as $|x| \to 0$. We can apply the same argument as before to obtain the exponential decay estimate of ∇u. To prove the assertion about the L^p norm of u, note that $|u|^p$ also decays exponentially fast at infinity for any $p > 0$ and since the singularity at $x = 0$ is logarithmic and $\lim_{|x|\to 0} |x|\,(\log|x|)^p = 0$, the integral

$$\int_{\mathbb{R}^2} |u|^p\, dx = \int_0^{2\pi} \int_0^{\infty} |u|^p\, r\, dr\, d\theta \tag{3.48}$$

is convergent. □

For any $R > 0$ and $\epsilon_0 > 0$, if $|x| > R$ and $\epsilon < \epsilon_0$, by the triangle inequality $|x \pm \epsilon| > R - \epsilon_0$. As a consequence of this observation and Lemma 3.5, we have the following corollary,

Corollary 3.6. *For any $\delta \in (0,1)$ and $\epsilon_0 > 0$, there exists constants $C(\delta)$, and $R(\delta, \epsilon_0)$, such that if $\epsilon < \epsilon_0$ and $|x| > R$,*

$$|u_{\pm\epsilon}(x)| \leq C\, e^{-(1-\delta)|x|}, \qquad |\nabla u_{\pm\epsilon}(x)| \leq C\, e^{-(1-\delta)|x|}. \tag{3.49}$$

We also have the following uniform bounds, valid for any $p > 0$,

$$\begin{aligned} ||h_\epsilon||_{\mathrm{L}^p} &\leq ||{-u_{-\epsilon}} - u_\epsilon||_{\mathrm{L}^p} \\ &\leq ||u_{-\epsilon}||_{\mathrm{L}^p} + ||u_\epsilon||_{\mathrm{L}^p} \qquad (3.50) \\ &= 2\,||u||_{\mathrm{L}^p}. \qquad (3.51) \end{aligned}$$

Let us introduce the functions,

$$v(t) = -\log\left(1 + t^{-2}\right), \qquad g(t) = \frac{4}{(1+t^2)^2}, \qquad t > 0. \tag{3.52}$$

and let $v_\epsilon = v(|x-\epsilon|) - v(|x+\epsilon|)$, $g_\epsilon = g(|x-\epsilon|) - g(|x+\epsilon|)$. We have the norm estimates,

$$||g_\epsilon||_{\mathrm{L}^p} \leq 2\,||g(|x|)||_{\mathrm{L}^p}, \tag{3.53}$$

$$||v_\epsilon||_{\mathrm{L}^p} \leq 2\,||v(|x|)||_{\mathrm{L}^p}. \tag{3.54}$$

Each of the functions $|v_\epsilon|^p$, $|g_\epsilon|^p$ is pointwise convergent to zero. Therefore,

$$\lim_{\epsilon\to 0} ||g_\epsilon||_{\mathrm{L}^p} = 0, \qquad p > \frac{1}{2}, \tag{3.55}$$

$$\lim_{\epsilon\to 0} ||v_\epsilon||_{\mathrm{L}^p} = 0, \qquad p > 1. \tag{3.56}$$

Let us define $\tilde{h}_\epsilon = h_\epsilon - v_\epsilon$. Then $\tilde{h}_\epsilon$ is a solution to the regularised Taubes equation,

$$-\Delta\tilde{h}_\epsilon = F_\mu(v_\epsilon + \tilde{h}_\epsilon) - g_\epsilon. \tag{3.57}$$

From now onwards, we will use the same variable C to denote a positive constant, independent of ϵ, that can change from one inequality to the following. By our estimates for the p norm of h_ϵ and v_ϵ, $\tilde{h}_\epsilon$ is uniformly bounded in L^p for $p > 1$.

Lemma 3.7. *Let $\epsilon_0 > 0$ be an arbitrary positive constant, $||\tilde{h}_\epsilon||_{\mathrm{H}^1} \leq C$ for $\epsilon < \epsilon_0$.*

Proof. Since $\tilde{h}_\epsilon$ is uniformly bounded on L^2, we aim to show that $||\nabla\tilde{h}_\epsilon||_{\mathrm{L}^2}$ is also bounded if $\epsilon < \epsilon_0$. We have,

$$\begin{aligned} ||\nabla\tilde{h}_\epsilon||^2_{\mathrm{L}^2} &= -\langle F_\mu(v_\epsilon + \tilde{h}_\epsilon), \tilde{h}_\epsilon\rangle + \langle g_\epsilon, \tilde{h}_\epsilon\rangle, \\ &= -\langle F_\mu(h_\epsilon), \tilde{h}_\epsilon\rangle + \langle g_\epsilon, \tilde{h}_\epsilon\rangle, \\ &\leq |\langle F_\mu(h_\epsilon), \tilde{h}_\epsilon\rangle| + ||g_\epsilon||_{\mathrm{L}^2}\,||\tilde{h}_\epsilon||_{\mathrm{L}^2} \\ &\leq |\langle F_\mu(h_\epsilon), \tilde{h}_\epsilon\rangle| + C, \end{aligned} \tag{3.58}$$

where $\langle \cdot, \cdot \rangle$ is the L^2 product. It remains to show $\langle F_\mu(h_\epsilon), \tilde{h}_\epsilon \rangle$ is uniformly bounded. Let $\delta \in (0,1)$ be any given number, by corollary 3.6, there are positive constants R, C such that if $|x| > R$ and $\epsilon < \epsilon_0$,

$$|h_\epsilon(x)| \leq |u(x-\epsilon) - u(x+\epsilon)| \tag{3.59}$$

$$\leq C\, e^{-(1-\delta)|x|}. \tag{3.60}$$

Hence, there is another constant, such that,

$$F_\mu(h_\epsilon) \leq C\, e^{-(1-\delta)|x|}, \qquad |x| \geq R. \tag{3.61}$$

Let U be the exterior of the disk $\mathbb{D}_R(0)$, by the previous bound,

$$|\langle F_\mu(h_\epsilon), \tilde{h}_\epsilon \rangle|_{\mathrm{L}^2(U)} \leq ||F_\mu(h_\epsilon)||_{\mathrm{L}^2(U)}\, ||\tilde{h}_\epsilon||_{\mathrm{L}^2(U)} \leq C\, ||e^{-(1-\delta)|x|}||_{\mathrm{L}^2(U)}, \tag{3.62}$$

since $\tilde{h}_\epsilon$ is uniformly bounded on L^2. On the other hand, F_μ is a bounded function, hence,

$$|\langle F_\mu(h_\epsilon), \tilde{h}_\epsilon \rangle|_{\mathrm{L}^2(\mathbb{D}_R(0))} \leq ||F_\mu(h_\epsilon)||_{\mathrm{L}^2(\mathbb{D}_R(0))}\, ||\tilde{h}_\epsilon||_{\mathrm{L}^2(\mathbb{D}_R(0))} \leq C. \tag{3.63}$$

This concludes the proof that $\langle F_\mu(h_\epsilon), \tilde{h}_\epsilon \rangle$ is bounded on L^2. □

Proposition 3.8. $\lim_{\epsilon \to 0} ||\tilde{h}_\epsilon||_{\mathrm{L}^2} = 0$.

Proof. Let $\tilde{h}_n = \tilde{h}_{\epsilon_n}$ be any sequence such that $\epsilon_n \to 0$. By lemma 3.7 $\left\{\tilde{h}_n\right\}$ is bounded on H^1, hence, by the Banach-Alaoglu theorem, after passing to a subsequence if necessary, there is a function $\tilde{h}_* \in \mathrm{H}^1$ such that $\tilde{h}_n \rightharpoonup \tilde{h}_*$ weakly on H^1 and by the Rellich-Kondrashov theorem, after passing to another subsequence if necessary, we can assume that for any bounded domain $\mathcal{D}$, $\tilde{h}_n \to \tilde{h}_*$ strongly on $\mathrm{L}^2(\mathcal{D})$. We will assume without further notice that domains are bounded and their boundaries have at least Lipschitz regularity. Let $\varphi \in C_0^1(\mathbb{R}^2)$ and let $\mathcal{D}$ be a domain containing the support of φ,

$$\begin{aligned}
\langle \varphi, \tilde{h}_* \rangle_{\mathrm{H}^1} &= \lim \langle \varphi, \tilde{h}_n \rangle_{\mathrm{H}^1} \\
&= \lim \langle \nabla\varphi, \nabla\tilde{h}_n \rangle_{\mathrm{L}^2} + \lim \langle \varphi, \tilde{h}_n \rangle_{\mathrm{L}^2} \\
&= \lim \langle \varphi, \Delta\tilde{h}_n \rangle_{\mathrm{L}^2} + \lim \langle \varphi, \tilde{h}_n \rangle_{\mathrm{L}^2} \\
&= -\lim \langle \varphi, F_\mu(v_n + \tilde{h}_n) - g_n \rangle_{\mathrm{L}^2} + \langle \varphi, \tilde{h}_* \rangle_{\mathrm{L}^2}.
\end{aligned} \tag{3.64}$$

The last equation because the convergence $\tilde{h}_n \to \tilde{h}_*$ is strong on bounded domains and φ is compactly supported. Consequently,

$$\langle \nabla\varphi, \nabla\tilde{h}_* \rangle_{L^2} = -\lim \langle \varphi, F_\mu(v_n + \tilde{h}_n) - g_n \rangle_{L^2}. \tag{3.65}$$

By the mean value theorem, we have the estimate,

$$||F_\mu(v_n + \tilde{h}_n) - F_\mu(\tilde{h}_*)||_{L^2(\mathcal{D})} \leq C \left(||v_n||_{L^2(\mathcal{D})} + ||\tilde{h}_n - \tilde{h}_*||_{L^2(\mathcal{D})} \right). \tag{3.66}$$

Therefore, $F_\mu(v_n + \tilde{h}_n) \to F_\mu(\tilde{h}_*)$ and $g_n \to 0$ strongly on $L^2(\mathcal{D})$, thence $\tilde{h}_*$ is a weak solution of the equation,

$$-\Delta\tilde{h}_* = F_\mu(\tilde{h}_*). \tag{3.67}$$

By elliptic regularity $\tilde{h}_*$ is a strong solution and by the maximum principle $\tilde{h}_* = 0$.

Let $\mathcal{D}$ be any other domain, our previous argument shows that any sequence $\tilde{h}_n$ has a convergent subsequence $\tilde{h}_{n_j} \to 0$ on $L^2(\mathcal{D})$. Therefore for any domain $\mathcal{D}$, $\lim_{\epsilon\to 0}||\tilde{h}_\epsilon||_{L^2(\mathcal{D})} = 0$. Now we prove that $\lim_{\epsilon\to 0}||\tilde{h}||_{L^2} = 0$, to this end, let $\rho > 0$ and let us take $R > 0$ such that

$$||u||_{L^2(\mathbb{R}^2\setminus\mathbb{D}_R(0))} < \frac{\rho}{2}. \tag{3.68}$$

Let ϵ_0 be small enough such that $|x \pm \epsilon| > R$ for all $\epsilon < \epsilon_0$ and $|x| > 2R$. In this situation we have,

$$\begin{aligned} ||\tilde{h}_\epsilon||_{L^2(\mathbb{R}^2\setminus\mathbb{D}_{2R}(0))} &< ||u_\epsilon||_{L^2(\mathbb{R}^2\setminus\mathbb{D}_{2R}(0))} + ||u_{-\epsilon}||_{L^2(\mathbb{R}^2\setminus\mathbb{D}_{2R}(0))} \\ &\leq 2||u||_{L^2(\mathbb{R}^2\setminus\mathbb{D}_R(0))} \\ &< \rho. \end{aligned} \tag{3.69}$$

On the other hand, there exists ϵ_1 such that if $\epsilon < \epsilon_1$, then,

$$||\tilde{h}_\epsilon||_{L^2(\mathbb{D}_{2R}(0))} < \rho, \tag{3.70}$$

taking $\epsilon' = \min(\epsilon_0, \epsilon_1)$, we conclude that

$$||\tilde{h}_\epsilon||_{L^2} < 2\rho, \qquad \forall\epsilon < \epsilon' \tag{3.71}$$

and the limit $\lim_{\epsilon\to 0}||\tilde{h}_\epsilon||_{L^2} = 0$ holds. □

Since $\tilde{h}_\epsilon \to 0$ strongly as $\epsilon \to 0$, by the mean value theorem as in the proof of the proposition, we have,

$$\lim_{\epsilon \to 0} ||F_\mu(v_\epsilon + \tilde{h}_\epsilon)||_{\mathrm{L}^2} = 0. \tag{3.72}$$

Moreover,

$$||\Delta \tilde{h}_\epsilon||_{\mathrm{L}^2} \leq ||F_\mu(v_\epsilon + \tilde{h}_\epsilon)||_{\mathrm{L}^2} + ||g_\epsilon||_{\mathrm{L}^2}, \tag{3.73}$$

since both terms on the right side of the inequality converge to 0, we have the limit

$$\lim_{\epsilon \to 0} ||\Delta \tilde{h}_\epsilon||_{\mathrm{L}^2} = 0. \tag{3.74}$$

Lemma 3.9. *Let $\mathcal{D}$ be any domain on the plane, the restrictions $\tilde{h}_\epsilon|_{\mathcal{D}}$ and $\nabla \tilde{h}_\epsilon|_{\mathcal{D}}$ converge uniformly to 0.*

Proof. If we take any pair of domains $\mathcal{D} \Subset \mathcal{D}'$, by Schauder's estimates,

$$||\tilde{h}_\epsilon||_{\mathrm{H}^2(\mathcal{D})} \leq C(||\Delta \tilde{h}_\epsilon||_{\mathrm{L}^2(\mathcal{D}')} + ||\tilde{h}_\epsilon||_{\mathrm{L}^2(\mathcal{D}')}), \tag{3.75}$$

which implies $\tilde{h}_\epsilon \to 0$ in $\mathrm{H}^2(\mathcal{D})$ as $\epsilon \to 0$. By Sobolev's embedding, we have that for any domain, $\lim_{\epsilon \to 0} \tilde{h}_\epsilon = 0$ uniformly. Let $p > 2$ be any real number and let $\tilde{h}_n$ be any sequence of functions such that $\epsilon_n \to 0$. Since the convergence is uniform on $\mathcal{D}$, we can apply the dominated convergence theorem to obtain,

$$||\tilde{h}_n||_{\mathrm{L}^p(\mathcal{D})} \to 0, \qquad ||F(v_n + \tilde{h}_n)||_{\mathrm{L}^p(\mathcal{D})} \to 0 \tag{3.76}$$

and since the sequence is arbitrary, we conclude the limits,

$$\lim_{\epsilon \to 0} ||\tilde{h}_\epsilon||_{\mathrm{L}^p(\mathcal{D})} = 0, \qquad \lim_{\epsilon \to 0} ||F(v_\epsilon + \tilde{h}_\epsilon)||_{\mathrm{L}^p(\mathcal{D})} = 0, \tag{3.77}$$

are valid for any domain. In particular, both limits are valid for the domain $\mathcal{D}'$ of equation (3.75). By Schauder's estimates $||\tilde{h}_\epsilon||_{\mathrm{W}^{2,p}(\mathcal{D})} \to 0$ as $\epsilon \to 0$ and by Sobolev's embedding,

$$\lim_{\epsilon \to 0} ||\tilde{h}_\epsilon||_{C^1(\mathcal{D})} = 0. \tag{3.78}$$

□

Proposition 3.10. *The convergence $\tilde{h}_\epsilon \to 0$ is uniform on $\mathbb{R}^2$.*

Proof. Recall

$$|\tilde{h}_\epsilon| \leq |h_\epsilon| + |v_\epsilon| \leq |u(x-\epsilon) - u(x+\epsilon)| + |v(|x-\epsilon|) - v(|x+\epsilon|)| \,. \tag{3.79}$$

Let $R > 0$ be any large positive constant, such that the estimates of lemma 3.5 hold for $\delta = \frac{1}{2}$. If $|x| > 2R$ and $\epsilon < R$, then $|x \pm \epsilon| > R$. We can apply the mean value theorem to obtain the estimate

$$\begin{aligned} |v(|x-\epsilon|) - v(|x+\epsilon|)| &= \left|\log\left(1 + |x-\epsilon|^{-2}\right) - \log\left(1+|x+\epsilon|^{-2}\right)\right| \\ &\leq \frac{1}{R^2}\left||x-\epsilon|^{-2} - |x+\epsilon|^{-2}\right| \\ &= \frac{4\epsilon\,|x_1|}{R^2\,|x-\epsilon|^2|x+\epsilon|^2} \\ &= \left|\frac{4\epsilon\,(x_1+\epsilon)}{R^2\,|x-\epsilon|^2|x+\epsilon|^2} - \frac{4\epsilon^2}{R^2\,|x-\epsilon|^2|x+\epsilon|^2}\right| \\ &\leq \frac{4\epsilon}{R^5} + \frac{4\epsilon^2}{R^6}. \end{aligned} \tag{3.80}$$

Likewise, there is some ξ in the linear segment joining $x-\epsilon$ to $x+\epsilon$ such that,

$$|u(x-\epsilon) - u(x+\epsilon)| = 2\,|\partial_1 u(\xi)\,\epsilon| \leq 2\,Ce^{-\frac{1}{2}|\xi|}\,\epsilon \leq 2\,C\epsilon, \tag{3.81}$$

where we have used lemma 3.5. We conclude that $\tilde{h}_\epsilon \to 0$ uniformly on $\mathbb{R}^2 \backslash \mathbb{D}_R(0)$, but by lemma 3.9, $\tilde{h}_\epsilon$ also converges uniformly on $\mathbb{D}_R(0)$. □

Recall Poincare's constant of a domain $\mathcal{D}$ is the best constant $C_p(\mathcal{D})$ such that for any zero average function $u : \mathcal{D} \to \mathbb{R}$,

$$||u||_{\mathrm{L}^2(\mathcal{D})} \leq C_p\,||\nabla u||_{\mathrm{L}^2(\mathcal{D})}. \tag{3.82}$$

Lemma 3.11. *Let $a : \mathbb{R}^2 \to [0, M)$ be a continuous function, such that*

1. *For some convex domain $\mathcal{D}$ with diameter $d < \pi/M$, $\int_{\mathcal{D}} a\,\mathrm{Vol} > 0$,*
2. *a is positive on $\Omega = \mathbb{R}^2 \setminus \mathcal{D}$.*

If $m = \inf_\Omega a > 0$, the bilinear form

$$B : \mathrm{H}^1 \times \mathrm{H}^1 \to \mathbb{R}, \qquad B(u,v) = \langle \nabla u, \nabla v\rangle_{\mathrm{L}^2} + \langle u,\, av\rangle_{\mathrm{L}^2}, \tag{3.83}$$

is coercive with coercivity constant,

$$0 < \alpha < \min\left(m, 1, \frac{\int_{\mathcal{D}} a \operatorname{Vol}}{\operatorname{Vol}(\mathcal{D})}, \frac{1 - \frac{M\,d}{\pi}}{1 + Cp(\mathcal{D})}, \frac{1}{\operatorname{Vol}(\mathcal{D})} \int_{\mathcal{D}} a \left(1 - \frac{a}{M}\right) \operatorname{Vol} \right). \quad (3.84)$$

Proof. We aim to prove the existence of a positive constant α such that for any $u \in \mathrm{H}^1$,

$$||u||^2_{\mathrm{H}^1}\, \alpha \leq B(u,u), \quad (3.85)$$

Let $\alpha_1 = \min(m, 1)$, in the exterior Ω of the given domain,

$$||u||^2_{\mathrm{H}^1(\Omega)}\, \alpha_1 \leq ||\nabla u||^2_{\mathrm{L}^2(\Omega)} + \langle u,\, au\rangle_{\mathrm{L}^2(\Omega)}. \quad (3.86)$$

On the other hand, any $u \in \mathrm{H}^1(\mathcal{D})$ can be decomposed as $u_0 + \overline{u}$, where u_0 is of zero average on $\mathcal{D}$ and $\overline{u} \in \mathbb{R}$, hence, u_0 is orthogonal to $\overline{u}$ in $\mathrm{H}^1(\mathcal{D})$. Coercivity in $\mathcal{D}$ is equivalent to find a positive constant α_2 such that,

$$\begin{aligned}\left(||u_0||^2_{\mathrm{H}^1(\mathcal{D})} + \overline{u}^2 \operatorname{Vol}(\mathcal{D})\right) \alpha_2 \leq{}& ||\nabla u_0||^2_{\mathrm{L}^2(\mathcal{D})} + \langle a,\, u_0^2\rangle_{\mathrm{L}^2(\mathcal{D})} \\ &+ 2\,\overline{u}\,\langle a,\, u_0\rangle_{\mathrm{L}^2(\mathcal{D})} + \overline{u}^2\,\langle a, 1\rangle_{\mathrm{L}^2(\mathcal{D})},\end{aligned} \quad (3.87)$$

or equivalently,

$$\begin{aligned}\left(\langle a, 1\rangle_{\mathrm{L}^2(\mathcal{D})} - \alpha_2 \operatorname{Vol}(\mathcal{D})\right) \overline{u}^2 + 2\,\langle a, u_0\rangle_{\mathrm{L}^2(\mathcal{D})}\,\overline{u} + (1 - \alpha_2)\,||\nabla u_0||^2_{\mathrm{L}^2(\mathcal{D})} \\ + \langle a,\, u_0^2\rangle_{\mathrm{L}^2(\mathcal{D})} - \alpha_2 ||u_0||^2_{\mathrm{L}^2(\mathcal{D})} \geq 0.\end{aligned} \quad (3.88)$$

For this is quadratic inequality on $\overline{u}$ to hold regardless of $\overline{u}$, the leading coefficient with respect to $\overline{u}$ must be positive and the discriminant of the quadratic must be non-positive, from these two conditions we deduce the following restrictions:

$$\alpha_2 < \frac{\langle a, 1\rangle_{\mathrm{L}^2(\mathcal{D})}}{\operatorname{Vol}(\mathcal{D})} = \frac{\int_{\mathcal{D}} a \operatorname{Vol}}{\operatorname{Vol}(\mathcal{D})}. \quad (3.89)$$

$$\begin{aligned}\langle a, u_0\rangle^2_{\mathrm{L}^2(\mathcal{D})} \leq{}& \left(\langle a, 1\rangle_{\mathrm{L}^2(\mathcal{D})} - \alpha_2 \operatorname{Vol}(\mathcal{D})\right) \\ &\left((1 - \alpha_2)\,||\nabla u_0||^2_{\mathrm{L}^2(\mathcal{D})} + \langle a,\, u_0^2\rangle_{\mathrm{L}^2(\mathcal{D})} - \alpha_2 ||u_0||^2_{\mathrm{L}^2(\mathcal{D})}\right).\end{aligned} \quad (3.90)$$

We claim the existence of a positive constant α_2 such that the second restriction is independent of u_0. To this end, let us divide this inequality by M,

$$\begin{aligned}\left\langle \frac{a}{M}, u_0 \right\rangle^2_{\mathrm{L}^2(\mathcal{D})} \le & \left(\left\langle \frac{a}{M}, 1 \right\rangle_{\mathrm{L}^2(\mathcal{D})} - \frac{\alpha_2}{M} \operatorname{Vol}(\mathcal{D}) \right) \\ & \left(\frac{1-\alpha_2}{M} \, ||\nabla u_0||^2_{\mathrm{L}^2(\mathcal{D})} + \left\langle \frac{a}{M},\, u_0^2 \right\rangle_{\mathrm{L}^2(\mathcal{D})} - \frac{\alpha_2}{M} ||u_0||^2_{\mathrm{L}^2(\mathcal{D})} \right). \end{aligned} \tag{3.91}$$

By Cauchy-Schwarz,

$$\left\langle \frac{a}{M},\, u_0 \right\rangle^2_{\mathrm{L}^2(\mathcal{D})} \le \left|\left| \frac{a}{M} \right|\right|^2_{\mathrm{L}^2(\mathcal{D})} ||u_0||^2_{\mathrm{L}^2(\mathcal{D})} . \tag{3.92}$$

Notice that,

$$\left|\left| \frac{a}{M} \right|\right|^2_{\mathrm{L}^2(\mathcal{D})} \le \left\langle \frac{a}{M}, 1 \right\rangle_{\mathrm{L}^2(\mathcal{D})} - \frac{\alpha_2}{M} \operatorname{Vol}(\mathcal{D}), \tag{3.93}$$

if and only if

$$\alpha_2 \le \frac{1}{\operatorname{Vol}(\mathcal{D})} \int_{\mathcal{D}} a \left(1 - \frac{a}{M} \right) \operatorname{Vol}. \tag{3.94}$$

On the other hand, the inequality

$$||u_0||^2_{\mathrm{L}^2(\mathcal{D})} \le \left(\frac{1-\alpha_2}{M} \, ||\nabla u_0||^2_{\mathrm{L}^2(\mathcal{D})} + \left\langle \frac{a}{M},\, u_0^2 \right\rangle_{\mathrm{L}^2(\mathcal{D})} - \frac{\alpha_2}{M} \, ||u_0||^2_{\mathrm{L}^2(\mathcal{D})} \right) \tag{3.95}$$

is equivalent to,

$$||u_0||^2_{\mathrm{L}^2(\mathcal{D})} - \left\langle \frac{a}{M},\, u_0^2 \right\rangle_{\mathrm{L}^2(\mathcal{D})} + \frac{\alpha_2}{M} \, ||u_0||^2_{\mathrm{L}^2(\mathcal{D})} \le \frac{1-\alpha_2}{M} \, ||\nabla u_0||^2_{\mathrm{L}^2(\mathcal{D})}. \tag{3.96}$$

By Poincare's inequality and the bound $0 \le a/M < 1$,

$$\begin{aligned} ||u_0||^2_{\mathrm{L}^2(\mathcal{D})} - \left\langle \frac{a}{M},\, u_0^2 \right\rangle_{\mathrm{L}^2(\mathcal{D})} + \frac{\alpha_2}{M} \, ||u_0||^2_{\mathrm{L}^2(\mathcal{D})} &\le \left(1 + \frac{\alpha_2}{M} \right) ||u_0||^2_{\mathrm{L}^2(\mathcal{D})} \\ &\le \left(1 + \frac{\alpha_2}{M} \right) C_p \, ||\nabla u_0||^2_{\mathrm{L}^2(\mathcal{D})}. \end{aligned} \tag{3.97}$$

For the right side of this inequality to be lesser than $(1-\alpha_2)/M$, we require,

$$\alpha_2 < \frac{1 - M\, C_p}{1 + C_p}. \tag{3.98}$$

Since $\mathcal{D}$ is convex, we know by a result of Payne and Weinberger [44] that $C_p \leq d/\pi$. Since $d/\pi < 1/M$ implies $MC_p < 1$ and

$$\frac{1 - \frac{Md}{\pi}}{1 + C_p} \leq \frac{1 - M\, C_p}{1 + C_p}, \tag{3.99}$$

it is enough to require $\alpha_2 < \left(1 - \frac{Md}{\pi}\right)(1 + C_p)^{-1}$ to obtain the final bound. Defining $\alpha \leq \min(\alpha_1, \alpha_2)$, we prove coercivity with a constant as stated in the lemma. □

Lemma 3.12. *For any $\epsilon_0 > 0$, there is a positive constants $C(\epsilon_0)$, such that for all $\epsilon \leq \epsilon_0$,*

$$||g_\epsilon||_{\mathrm{L}^p} \leq C\epsilon, \qquad p > \frac{2}{5}, \tag{3.100}$$

$$||v_\epsilon||_{\mathrm{L}^2} \leq C\epsilon\, |\log \epsilon|, \tag{3.101}$$

$$||v_\epsilon||_{\mathrm{L}^p} \leq C\epsilon^{2/p}, \qquad p > 1,\, p \neq 2, \tag{3.102}$$

Proof. Let us rewrite g_ϵ,

$$\begin{aligned} g_\epsilon(x) &= \frac{4}{(1 + |x - \epsilon|^2)^2} - \frac{4}{(1 + |x + \epsilon|^2)^2} \\ &= \frac{4\,(|x+\epsilon|^2 - |x-\epsilon|^2)\,(2 + |x+\epsilon|^2 + |x-\epsilon|^2)}{(1 + |x+\epsilon|^2)^2\,(1 + |x-\epsilon|^2)^2} \\ &= \frac{16\,\epsilon x_1\,(2 + |x+\epsilon|^2 + |x-\epsilon|^2)}{(1 + |x+\epsilon|^2)^2\,(1 + |x-\epsilon|^2)^2} \end{aligned} \tag{3.103}$$

and let us take $R > \epsilon_0$. If $\Omega = \mathbb{R}^2 \setminus \mathbb{D}_R(0)$,

$$||g_\epsilon||_{\mathrm{L}^p(\Omega)} \leq 16\,\epsilon \left|\left| \frac{x_1\,(2 + 2(|x| + R)^2)}{(1 + (|x| - R)^2)^4} \right|\right|_{\mathrm{L}^p(\Omega)}. \tag{3.104}$$

The norm on the right decay as $|x|^{-5}$ as $|x| \to \infty$, hence is convergent for $p > 2/5$. On the other hand, we have,

$$||g_\epsilon||_{\mathrm{L}^p(\mathbb{D}_R(0))} \leq 16\,\epsilon \left|\left| x_1 \left(2 + 2(|x| + R)^2\right) \right|\right|_{\mathrm{L}^p(\mathbb{D}_R(0))}. \tag{3.105}$$

Thence $||g_\epsilon||_{\mathrm{L}^p} \leq C\epsilon$ if $\epsilon < R$. For v_ϵ we follow several steps, dividing the plane in subregions where we can have control of the logarithmic singularities.

We start with an algebraic rearrangement,

$$\begin{aligned} v_\epsilon(x) &= \log\left(\frac{1+|x+\epsilon|^{-2}}{1+|x-\epsilon|^{-2}}\right) \\ &= \log\left(1+\frac{|x+\epsilon|^{-2}-|x-\epsilon|^{-2}}{1+|x-\epsilon|^{-2}}\right) \\ &= \log\left(1-\frac{4\epsilon x_1}{|x+\epsilon|^2(1+|x-\epsilon|^2)}\right). \end{aligned} \tag{3.106}$$

Let $R > 2\epsilon_0$ be a large positive constant such that if $|x| \geq R$ and $\epsilon < R/2$, we have the approximation,

$$\begin{aligned} |v_\epsilon(x)| &= \frac{4\epsilon|x_1|}{|x+\epsilon|^2(1+|x-\epsilon|^2)} + \mathcal{O}(\epsilon^2) \\ &\leq \frac{4\epsilon|x_1|}{\left(|x|-\frac{R}{2}\right)^2\left(1+(|x|-\frac{R}{2})^2\right)}. \end{aligned} \tag{3.107}$$

v_ϵ is bounded in $\Omega = \mathbb{R}^2 \setminus \mathbb{D}_R(0)$ by a function of order $|x|^{-3}$, hence,

$$||v_\epsilon||_{\mathrm{L}^p(\Omega)} \leq 4\epsilon \left|\left|\frac{x_1}{\left(|x|-\frac{R}{2}\right)^2\left(1+(|x|-\frac{R}{2})^2\right)}\right|\right|_{\mathrm{L}^p(\Omega)}, \tag{3.108}$$

for any $p > 1$. On the other hand,

$$\begin{aligned} ||v(|x-\epsilon|)-v(|x+\epsilon|)||_{\mathrm{L}^p(\mathbb{D}_R(0))} \leq{}& \left|\left|\log(1+|x+\epsilon|^2)-\log(1+|x-\epsilon|^2)\right|\right|_{\mathrm{L}^p(\mathbb{D}_R(0))} \\ &+ \left|\left|\log(|x-\epsilon|^2)-\log(|x+\epsilon|^2)\right|\right|_{\mathrm{L}^p(\mathbb{D}_R(0))}. \end{aligned} \tag{3.109}$$

For the first term, the difference can be bounded as,

$$\begin{aligned} \left|\left|\log(1+|x+\epsilon|^2)-\log(1+|x-\epsilon|^2)\right|\right|_{\mathrm{L}^p(\mathbb{D}_R(0))} &\leq \left|\left|\,|x+\epsilon|^2-|x-\epsilon|^2\right|\right|_{\mathrm{L}^p(\mathbb{D}_R(0))} \\ &\leq 4\epsilon\,||x_1||_{\mathrm{L}^p(\mathbb{D}_R(0))} \\ &\leq 4\epsilon R\left(\pi R^2\right)^{1/p}. \end{aligned} \tag{3.110}$$

For the second term, we proceed in two steps. Firstly, let us consider the annulus $2\epsilon \leq |x| \leq R$ and note that,

$$\log|x-\epsilon|^2 - \log|x+\epsilon|^2 = \log\left|1-\frac{\epsilon}{x}\right|^2 - \log\left|1+\frac{\epsilon}{x}\right|^2. \tag{3.111}$$

Let $A(R,2\epsilon) = \mathbb{D}_R(0) \setminus \mathbb{D}_{2\epsilon}(0)$ be the given annulus, with $A(1/(2\epsilon),1/R)$ defined accordingly. We make the change of variables $x' = 1/x$ and compute,

$$\begin{aligned}
\left|\left|\log|x-\epsilon|^2 - \log|x+\epsilon|^2\right|\right|_{\mathrm{L}^p(A(R,2\epsilon))} &= \left|\left|\left(\log|1-\epsilon x'|^2 - \log|1+\epsilon x'|^2\right)|x'|^{-2}\right|\right|_{\mathrm{L}^p(A(1/(2\epsilon),1/R))} \\
&\leq 2\left|\left|\left(|1-\epsilon x'|^2 - |1+\epsilon x'|^2\right)|x'|^{-2}\right|\right|_{\mathrm{L}^p(A(1/(2\epsilon),1/R))} \\
&\leq 8\epsilon\left|\left|\,|x'|^{-1}\right|\right|_{\mathrm{L}^p(A(1/(2\epsilon),1/R))}. \qquad (3.112)
\end{aligned}$$

The last norm can be computed exactly, we found that,

$$\left|\left|\,|x'|^{-1}\right|\right|_{\mathrm{L}^p(A(1/(2\epsilon),1/R))} = \begin{cases} \sqrt{2\pi}\left(\log\left(\frac{R}{2\epsilon}\right)\right)^{1/2}, & p=2, \\ \frac{4}{|p-2|^{1/p}}\left|\frac{\epsilon^2}{2^{p-2}} - \frac{\epsilon^p}{R^{p-2}}\right|^{1/p}, & p\neq 2. \end{cases} \qquad (3.113)$$

Secondly, we use the inequality $|x| \leq |x \pm \epsilon| + \epsilon$, which can be obtained by an application of the triangle inequality. With this inequality at hand,

$$\begin{aligned}
\left|\left|\log(|x-\epsilon|^2) - \log(|x+\epsilon|^2)\right|\right|_{\mathrm{L}^p(\mathbb{D}_{2\epsilon}(0))} &\leq \left|\left|\log(|x-\epsilon|^2)\right|\right|_{\mathrm{L}^p(\mathbb{D}_{2\epsilon}(0))} + \left|\left|\log(|x+\epsilon|^2)\right|\right|_{\mathrm{L}^p(\mathbb{D}_{2\epsilon}(0))} \\
&\leq 2\left|\left|\log|x|^2\right|\right|_{\mathrm{L}^p(\mathbb{D}_{3\epsilon}(0))}. \qquad (3.114)
\end{aligned}$$

The last norm can also be computed,

$$\left|\left|\log|x|^2\right|\right|_{\mathrm{L}^p(\mathbb{D}_{3\epsilon}(0))} = \begin{cases} 6\sqrt{\pi}\,\epsilon\,\left(\log^2(3\epsilon) - \log(3\epsilon) + \frac{1}{2}\right)^{1/2}, & p=2, \\ \frac{\pi^{1/p}}{2}\left(\int_{-2\log(3\epsilon)}^{\infty} u^p\, e^{-u}\, du\right)^{1/p}, & p\neq 2. \end{cases} \qquad (3.115)$$

In the last integral, e^{-u} dominates u^p, hence

$$\left|\left|\log|x|^2\right|\right|_{\mathrm{L}^p(\mathbb{D}_{3\epsilon}(0))} \leq C\begin{cases} \epsilon\,|\log\epsilon| & p=2, \\ \epsilon^{2/p} & p\neq 2, \end{cases} \qquad (3.116)$$

where the constant is independent of ϵ.

Taking into account all the regions in which we divided the plane, we find that the dominant term is $\epsilon\,|\log\epsilon|$ for $p=2$ and $\epsilon^{2/p}$ in other case. This concludes the proof of the lemma. □

Proposition 3.13. *For any domain neighbourhood $\mathcal{D}$ of the origin, there is an $\epsilon_0 > 0$ such that,*

$$\max_{\mathcal{D}} |\partial_1 \tilde{h}_\epsilon(x)| \leq C\epsilon^{2/p}, \qquad (3.117)$$

for all $\epsilon < \epsilon_0$.

Proof. We start defining a family of potentials a_ϵ which approximate $V_\mu(v_\epsilon + \tilde{h}_\epsilon)$ as $\epsilon \to 0$. If $x \neq \pm\epsilon$, there is a $\xi_\epsilon(x)$ such that $|\xi_\epsilon(x)| \leq \left|v_\epsilon(x) + \tilde{h}_\epsilon(x)\right|$ and,

$$F_\mu(v_\epsilon(x) + \tilde{h}_\epsilon(x)) = V_\mu(\xi_\epsilon(x))(v_\epsilon(x) + \tilde{h}_\epsilon(x)). \tag{3.118}$$

Let $a_\epsilon = V_\mu(\xi_\epsilon)$, this is a positive function such that if $v_\epsilon(x) + \tilde{h}_\epsilon(x) \neq 0$,

$$a_\epsilon(x) = \frac{F_\mu(v_\epsilon(x) + \tilde{h}_\epsilon(x))}{v_\epsilon(x) + \tilde{h}_\epsilon(x)}, \tag{3.119}$$

hence a_ϵ is continuous in the complement of the zeros of $v_\epsilon + \tilde{h}_\epsilon$. Moreover, if x_0 is in the set of zeros of $v_\epsilon + \tilde{h}_\epsilon$,

$$\lim_{x \to x_0} a_\epsilon(x) = \lim_{x \to x_0} \frac{F_\mu(v_\epsilon(x) + \tilde{h}_\epsilon(x))}{v_\epsilon(x) + \tilde{h}_\epsilon(x)} = V_\mu(0) = a_\epsilon(x_0), \tag{3.120}$$

since $\xi_\epsilon(x_0) = 0$ because ξ_ϵ is bounded by $|v_\epsilon + \tilde{h}_\epsilon|$ and $v_\epsilon + \tilde{h}_\epsilon \to 0$ as $x \to x_0$.

Hence, a_ϵ is a continuous function on $\mathbb{R}^2 \setminus \{\pm\epsilon\}$ which we can extend continuously to $\pm\epsilon$, because F_μ and $\tilde{h}_\epsilon$ are bounded functions and v_ϵ diverges to $\pm\infty$ at the poles $\pm\epsilon$, hence, $\lim_{x \to \pm\epsilon} a_\epsilon(x) = 0$. Redefining a_ϵ as this extension, notice that it determines a family of bounded non-negative, continuous functions, each of them with only two zeros at the vortex-antivortex positions. Let $\mathcal{D}'$ be a convex domain neighbourhood of the origin, with diameter $d < \pi/M$ for some strict upper bound M of V_μ. Pointwise, each $\xi_\epsilon(x) \to 0$ as $\epsilon \to 0$, hence we also have the convergence $a_\epsilon(x) \to 2(\mu+1)^{-1}$ as $\epsilon \to 0$. By the dominated convergence theorem,

$$\int_{\mathcal{D}'} a_\epsilon \mathrm{Vol} \to \frac{2}{1+\mu} |\mathcal{D}'|, \tag{3.121}$$

$$\int_{\mathcal{D}'} a_\epsilon \left(1 - \frac{a_\epsilon}{M}\right) \mathrm{Vol} \to \frac{2}{1+\mu} \left(1 - \frac{2}{M(1+\mu)}\right) |\mathcal{D}'|. \tag{3.122}$$

Let $\Omega = \mathbb{R}^2 \setminus \mathcal{D}'$, $m_\epsilon = \inf_\Omega a_\epsilon$ and let us assume ϵ_0 is small enough for $\pm\epsilon \in \mathcal{D}'$ provided $\epsilon \leq \epsilon_0$. We know that $v_\epsilon + \tilde{h}_\epsilon \to 0$ uniformly in Ω, hence,

$$\lim_{\epsilon \to 0} m_\epsilon = \lim_{\epsilon \to 0} \inf_\Omega V_\mu(v_\epsilon + \tilde{h}_\epsilon) = \frac{2}{\mu+1}. \tag{3.123}$$

By lemma 3.11, the potentials a_ϵ define coercive continuous bilinear functions $\mathrm{H}^1 \times \mathrm{H}^1 \to \mathbb{R}$, such that

$$C_\epsilon \, ||\tilde{h}_\epsilon||^2_{\mathrm{H}^1} \leq ||\nabla \tilde{h}_\epsilon||^2_{\mathrm{L}^2} + \langle a_\epsilon \, \tilde{h}_\epsilon, \tilde{h}_\epsilon \rangle. \tag{3.124}$$

Let,

$$m = \frac{1}{\mu+1}, \qquad \alpha_1 = \frac{1}{\mu+1}, \qquad \alpha_2 = \frac{1}{\mu+1}\left(1 - \frac{2}{M(\mu+1)}\right), \tag{3.125}$$

if we select a positive constant $C > 0$ such that,

$$C < \min\left(m, 1, \alpha_1, \frac{1 - \frac{M\,d}{\pi}}{1 + Cp(\mathcal{D}')}, \alpha_2\right), \tag{3.126}$$

then according to lemma 3.11 we can use C as a common coercivity constant for all the potential functions a_ϵ with $\epsilon \leq \epsilon_0$. Therefore,

$$||\nabla\tilde{h}_\epsilon||^2_{\mathrm{L}^2} = -\langle F_\mu(v_\epsilon + \tilde{h}_\epsilon), \tilde{h}_\epsilon\rangle + \langle g_\epsilon, \tilde{h}_\epsilon\rangle = -\langle a_\epsilon \cdot (v_\epsilon + \tilde{h}_\epsilon), \tilde{h}_\epsilon\rangle + \langle g_\epsilon, \tilde{h}_\epsilon\rangle. \tag{3.127}$$

If we apply the uniform coercivity constant, we obtain the bound,

$$\begin{aligned} C\,||\tilde{h}_\epsilon||^2_{\mathrm{H}^1} &\leq ||\nabla\tilde{h}_\epsilon||^2_{\mathrm{L}^2} + \langle a_\epsilon\tilde{h}_\epsilon, \tilde{h}_\epsilon\rangle \\ &= -\langle a_\epsilon v_\epsilon, \tilde{h}_\epsilon\rangle + \langle g_\epsilon, \tilde{h}_\epsilon\rangle \\ &\leq C_2\left(||v_\epsilon||_{\mathrm{L}^2} + ||g_\epsilon||_{\mathrm{L}^2}\right)||\tilde{h}_\epsilon||_{\mathrm{L}^2}, \end{aligned} \tag{3.128}$$

where we have used Cauchy-Schwarz and the fact that the set $\{a_\epsilon : \epsilon \leq \epsilon_0\}$ is uniformly bounded. From this inequality, we deduce the existence of a positive constant C, such that,

$$\max\left(||\tilde{h}_\epsilon||_{\mathrm{L}^2}, ||\nabla\tilde{h}_\epsilon||_{\mathrm{L}^2}\right) \leq C\left(||v_\epsilon||_{\mathrm{L}^2} + ||g_\epsilon||_{\mathrm{L}^2}\right) \tag{3.129}$$

Applying lemma 3.12 we infer the existence of another constant, such that,

$$||v_\epsilon||_{\mathrm{L}^2} + ||g_\epsilon||_{\mathrm{L}^2} \leq C\,\epsilon\,|\log\epsilon|, \tag{3.130}$$

for $\epsilon \leq \epsilon_0$. By the elliptic estimates and Sobolev's embedding,

$$||\tilde{h}_\epsilon||_{C^0(\mathcal{D})} \leq C_1||\tilde{h}_\epsilon||_{\mathrm{H}^2(\mathcal{D})} \leq C_2\left(||\Delta\tilde{h}_\epsilon||_{\mathrm{L}^2(\mathcal{D})} + ||\tilde{h}_\epsilon||_{\mathrm{L}^2(\mathcal{D})}\right). \tag{3.131}$$

Since,

$$||\Delta\tilde{h}_\epsilon||_{\mathrm{L}^2(\mathcal{D})} = ||a_\epsilon\,(v_\epsilon + \tilde{h}_\epsilon)||_{\mathrm{L}^2(\mathcal{D})} \leq C\left(||v_\epsilon||_{\mathrm{L}^2(\mathcal{D})} + ||\tilde{h}_\epsilon||_{\mathrm{L}^2(\mathcal{D})}\right), \tag{3.132}$$

we apply lemma 3.12 again and the estimate for the L^2 norm of $\tilde{h}_\epsilon$ we have obtained to deduce that,

$$||\tilde{h}_\epsilon||_{C^0(\mathcal{D})} \leq C\epsilon|\log(\epsilon)|, \qquad \epsilon \leq \epsilon_0. \tag{3.133}$$

We use this estimate and Sobolev's embedding again to estimate the supremum of $\partial_1 \tilde{h}_\epsilon$ at $\mathcal{D}$. If $p > 2$, we have,

$$||\tilde{h}_\epsilon||_{C^1(\mathcal{D})} \leq C_1||\tilde{h}_\epsilon||_{\mathrm{W}^{2,p}(\mathcal{D})} \leq C_2\left(||\Delta\tilde{h}_\epsilon||_{\mathrm{L}^p(\mathcal{D})} + ||\tilde{h}_\epsilon||_{\mathrm{L}^p(\mathcal{D})}\right). \tag{3.134}$$

Again by lemma 3.12 and the previous estimate on the C^0 norm of $\tilde{h}_\epsilon$,

$$\begin{aligned}
||\Delta\tilde{h}_\epsilon||_{\mathrm{L}^p(\mathcal{D})} + ||\tilde{h}_\epsilon||_{\mathrm{L}^p(\mathcal{D})} &\leq ||a_\epsilon\,(v_\epsilon + \tilde{h}_\epsilon)||_{\mathrm{L}^p(\mathcal{D})} + ||g_\epsilon||_{\mathrm{L}^p(\mathcal{D})} + ||\tilde{h}_\epsilon||_{\mathrm{L}^p(\mathcal{D})}\\
&\leq C\left(||v_\epsilon||_{\mathrm{L}^p(\mathcal{D})} + ||\tilde{h}_\epsilon||_{\mathrm{L}^p(\mathcal{D})} + ||g_\epsilon||_{\mathrm{L}^p(\mathcal{D})}\right)\\
&\leq C\left(\epsilon^{2/p} + \epsilon\,|\log\epsilon|\cdot|\mathcal{D}|^{1/p} + \epsilon\right)\\
&\leq C\,\epsilon^{2/p}.
\end{aligned} \tag{3.135}$$

Since asymptotically $\epsilon\,|\log\epsilon| \leq \epsilon^{2/p}$ as $\epsilon \to 0$. Therefore, $||\partial_1\tilde{h}_\epsilon||_{C^0(\mathcal{D})} \leq C\,\epsilon^{2/p}$ if ϵ is small. □

Going back to the original, undilated coordinates $x \in \mathbb{R}^2$, we can state the following theorem,

Theorem 3.14. *The moduli space $\mathcal{M}_0^{1,1}$ is an incomplete metric space, such that geodesic discs centred at the singular point $\epsilon = 0$ have finite area.*

In comparison, the moduli space of vortices for the Ginzburg-Landau functional is complete, as can be seen in the results of Strachan [54] who studied geodesic motion on hyperbolic space or Samols [47] on the euclidean plane. Incompleteness of the moduli space was expected by previous results of Romão-Speight, who conjectured an asymptotic logarithmic approximation to $\Omega(\epsilon)$ for small ϵ at $\tau = 0$ [45].

Proof. We will prove that $\mathcal{M}_0^{1,1}$ is incomplete exhibiting a curve of finite length reaching the singularity at $\epsilon = 0$. Let us take any radial geodesic parametrized as

$$\gamma_\theta : (0, \epsilon_0] \to \mathcal{M}_0^{1,1}, \qquad \gamma_\theta(\epsilon) = \epsilon\, e^{i\theta}. \tag{3.136}$$

By Cauchy-Schwarz, the length of this curve is bounded since,

$$\ell = \int_0^{\epsilon_0} \Omega(\epsilon)^{1/2}\, d\epsilon \leq \epsilon_0^{1/2} \left(\int_0^{\epsilon_0} \Omega(\epsilon)\, d\epsilon \right)^{1/2}. \tag{3.137}$$

We will prove that the energy, and therefore the length, is finite. Recall the interaction coefficient is given by

$$\begin{aligned} b(\epsilon) &= 2\, \partial_1|_{x=\epsilon} \left(h_\epsilon(x) - \log|x-\epsilon|^2 \right) \\ &= 2\, \partial_1|_{x=\epsilon} \left(\tilde{h}_\epsilon(x) + v(|x+\epsilon|) - \log\left(1 + |x-\epsilon|^2\right) + \mu \right) \\ &= 2\, \partial_1 \tilde{h}_\epsilon(\epsilon) + \frac{8\epsilon}{1+4\epsilon^2} - \frac{2}{\epsilon}. \end{aligned} \tag{3.138}$$

Let $\tilde{b}(\epsilon) = 2\, \partial_1 \tilde{h}_\epsilon(\epsilon) + 8\, \epsilon(1+4\epsilon^2)^{-1}$, we have,

$$\begin{aligned} \int_0^{\epsilon_0} \Omega(\epsilon)\, d\epsilon &= 2\pi \int_0^{\epsilon_0} 2(1-\tau^2) + \frac{1}{\epsilon}\frac{d}{d\epsilon}\left(\epsilon b(\epsilon)\right)\, d\epsilon \\ &= 4\pi(1-\tau^2)\, \epsilon_0 + 2\pi \int_0^{\epsilon_0} \frac{1}{\epsilon}\frac{d}{d\epsilon}\left(\epsilon \tilde{b}(\epsilon)\right)\, d\epsilon \\ &= 4\pi(1-\tau^2)\, \epsilon_0 + 2\pi \left(\tilde{b}(\epsilon_0) - \lim_{\epsilon\to 0} \tilde{b}(\epsilon) + \int_0^{\epsilon_0} \frac{\tilde{b}(\epsilon)}{\epsilon}\, d\epsilon \right), \end{aligned} \tag{3.139}$$

where we have used integration by parts in the last equation. Let us assume ϵ_0 is so small we can use the estimate in proposition 3.13,

$$\begin{aligned} \int_0^{\epsilon_0} \Omega(\epsilon)\, d\epsilon &= 4\pi(1-\tau^2)\, \epsilon_0 + 2\pi\tilde{b}(\epsilon_0) + 8\pi \tan^{-1}(2\epsilon_0) + 4\pi \int_0^{\epsilon_0} \frac{\partial_1 \tilde{h}_\epsilon(\epsilon)}{\epsilon}\, d\epsilon \\ &\leq 4\pi(1-\tau^2)\, \epsilon_0 + 8\pi \tan^{-1}(2\epsilon_0) + \frac{16\pi\epsilon_0}{1+4\epsilon_0^2} + C\left(\epsilon_0^{2/p} + \int_0^{\epsilon_0} \epsilon^{\frac{2}{p}-1}\, d\epsilon \right) \\ &\leq 4\pi(1-\tau^2)\, \epsilon_0 + 8\pi \tan^{-1}(2\epsilon_0) + \frac{16\pi\epsilon_0}{1+4\epsilon_0^2} + C\epsilon_0^{2/p}. \end{aligned} \tag{3.140}$$

Therefore, the energy is finite, hence, the length of the geodesic is also finite, moreover, the length is bounded by,

$$\ell \leq 2\pi \left(5 - \tau^2\right)^{1/2} \epsilon_0 + C\, \epsilon_0^{1/p}. \tag{3.141}$$

For the area of a disk, we have a similar calculation,

$$\begin{aligned}
\mathrm{Vol}(\mathbb{D}_R(0)) &= 2\pi \int_0^R \Omega(\epsilon)\,\epsilon\,d\epsilon \\
&= 4\pi^2\,(1-\tau^2)\,R^2 + 4\pi^2 R\,\tilde{b}(R) \\
&\leq 4\pi^2\,(1-\tau^2)\,R^2 + \frac{32\pi^2 R^2}{1+4R^2} + 8\pi^2 R\,\partial_1 \tilde{h}_\epsilon(\epsilon) \\
&\leq 4\pi^2\,(1-\tau^2)\,R^2 + \frac{32\pi^2 R^2}{1+4R^2} + C\,R^{1+\frac{2}{p}}.
\end{aligned} \tag{3.142}$$

□

Samols compared the area of small disks on the moduli space for the Ginzburg-Landau functional with the area of a cone with deficit angle π. Recall each vortex/antivortex has effective mass $2\pi(1 \mp \tau)$ respectively, in the centre of mass coordinates, the reduced mass of the vortex-antivortex system is $\pi(1-\tau^2)$, hence, if we normalize (3.142) dividing by the reduced mass, we find that the first term in the upper bound is $4\pi R^2$, the area of a right circular cone of radius R and deficit angle $3\pi/2$. The second and third terms in the upper bound are far from optimal, because they do not depend on τ and the third term is of order smaller than 2, however, the conjectured asymptotics of the conformal factor for small ϵ (3.269) leads us to also conjecture that the first term in the upper bound is the first term of an approximation to $\mathrm{Vol}(\mathbb{D}_R(0))$ for small R.

3.2 Asymptotic approximation at large separation

If the cores are separated by a large distance, it is plausible to assume that the interactions are so weak, that in the neighbourhood of any of them, they can be described by the solution corresponding to one vortex plus a small perturbation term due to the interactions. We use this idea to approximate dynamics in the moduli space for well separated vortices. For Ginzburg-Landau vortices this was done by Speight in [52] and Manton-Speight in [39]. We start finding Hedgehog solutions to the Bogomolny equations. Let us assume that there are exactly N

vortices at the origin. We will use the Ansatz,

$$\begin{aligned} \phi &= (\sin(f)\cos(N\theta),\, \sin(f)\sin(N\theta),\, \cos(f)),\\ A &= N a(r)\, d\theta, \end{aligned} \tag{3.143}$$

which assumes circular symmetry of the field equations. This Ansatz was used before by Schroers to study solutions of the $U(1)$-gauged $O(3)$ Sigma model for $\tau = 1$ [50]. The energy density of this static configuration is,

$$\mathcal{E} = \left(\frac{N(a-1)\sin(f)}{2r}\right)^2 + (\tau - \cos(f))^2 . \tag{3.144}$$

For these fields to represent N vortices at the origin with finite energy, we add the boundary conditions,

$$f(0) = 0, \qquad a(0) = 0, \qquad \lim_{r\to\infty} f = \cos^{-1}\tau, \qquad \lim_{r\to\infty} a = 1. \tag{3.145}$$

With this Ansatz, the Bogomolny equations reduce to the system of ODEs,

$$f' = \frac{N}{r}(a-1)\sin(f), \qquad a' = \frac{r}{N}(\cos(f) - \tau). \tag{3.146}$$

Unfortunately, we cannot extend these equations to the origin, instead, we select a small initial value δ and perturb the Bogomolny equations to lowest order in δ. We found that to lowest order,

$$f(\delta) = \alpha\, \delta^N, \qquad a(\delta) = \frac{1-\tau}{2N}\, \delta^2, \tag{3.147}$$

then we used α as a shooting parameter. In practice, we chose $\delta = 10^{-8}$ and for the boundary condition at infinity, we selected $r_\infty = 10$ except for the last τ, for which $r_\infty = 20$. We took r_∞ as infinity and shot until $(f(r_\infty), a(r_\infty))$ satisfied the boundary condition, as in the paper of Speight [52]. We used the solver *solve_ivp* of the scientific library *SciPy* with default parameters. Internally, it uses the Runge-Kutta method of order 5(4), which controls the error using a local extrapolation and uses a quartic interpolation polynomial to compute the solution at the preconfigured set of points shown in Figure 3.1.

If (ϕ, a) is the solution to the Bogomolny equations with N vortices at the origin and parameter τ and we invert the orientation of the sphere, selecting

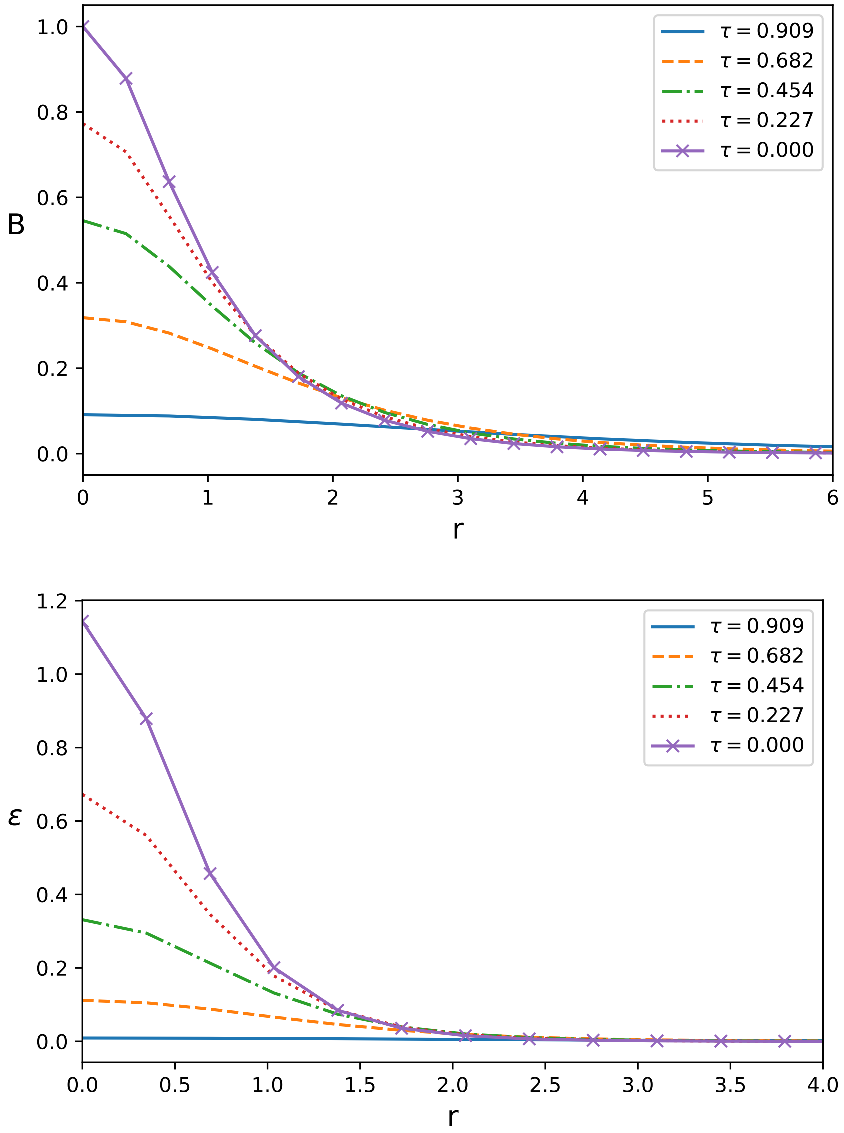


Figure 3.1: Magnetic field and energy density of hedgehog solutions for positive values of τ. The graphs show how as τ grows, the energy and magnetic field weaken.

$-n$ as the north pole, it is not difficult to see that $(\phi, -a)$ is also a solution to the Bogomolny equations, this time with parameter $-\tau$ and N antivortices at the origin, hence the qualitative properties of an antivortex hedgehog are the same, except that to a τ-vortex corresponds a $-\tau$ antivortex and to a B (vortex) magnetic field corresponds a $-B$ (antivortex) magnetic field.

Assuming there is only one core at the origin, the solution to the Taubes equation, h, is also radial, and away of the origin, is a solution to the equation,

$$\frac{d^2h}{dr^2} + \frac{1}{r}\frac{dh}{dr} - 2\left(\frac{e^h - 1}{e^h + 1} + \tau\right) = 0. \qquad (3.148)$$

For small r, h has the asymptotic behaviour $h = \pm\ln(r^2)$ and for big r, it approaches $\log\left(\frac{1-\tau}{1+\tau}\right)$. Linearizing about the limit at infinity, we have the equation,

$$\frac{d^2\hat{h}}{dr^2} + \frac{1}{r}\frac{d\hat{h}}{dr} - (1-\tau^2)\hat{h} = 0, \qquad \lim_{r\to\infty}\hat{h} = 0. \qquad (3.149)$$

If we make the change of variables $r' = (1-\tau^2)^{1/2}\, r$, then the function $\hat{h}(r')$ is a solution to the modified Bessel equation,

$$\frac{d^2\hat{h}}{dr'^2} + \frac{1}{r'}\frac{d\hat{h}}{dr'} - \hat{h} = 0, \qquad \lim_{r'\to\infty}\hat{h} = 0. \qquad (3.150)$$

whose general solution is a linear combination of modified Bessel's function of first and second kind, J_0 and K_0. Since J_0 diverges at infinity, we deduce the approximation,

$$h(r) = \log\left(\frac{1-\tau}{1+\tau}\right) + qK_0\left((1-\tau^2)^{1/2}r\right). \qquad (3.151)$$

The constant q has to be determined numerically, as in the approximation done for Ginzburg-Landau vortices [52]. We found this constant for several values of τ by solving the Bogomolny equations as explained above, with this data, we computed the pairs $\left(K_0((1-\tau^2)^{1/2}\, r), h(r)\right)$ and fitted a least squares line as a model, whose slope was q. We tested visually and by means of the coefficient of determination R^2 the goodness of fit of the model to the data, finding on average $R^2 = 0.9985$, meaning the linear model explained 99.85% of the data, hence the fit was good. The dependence of the constant q on τ can be seen in

τ	-0.909	-0.682	-0.454	-0.227	0	0.227	0.454	0.682	0.909
q	-1.2457	-1.5414	-1.7921	-2.0321	-2.271	-2.5134	-2.7568	-2.9784	-3.2504

Table 3.1: Constant q for different values of τ for a vortex at origin in Euclidean space. For an antivortex, q has positive sign.

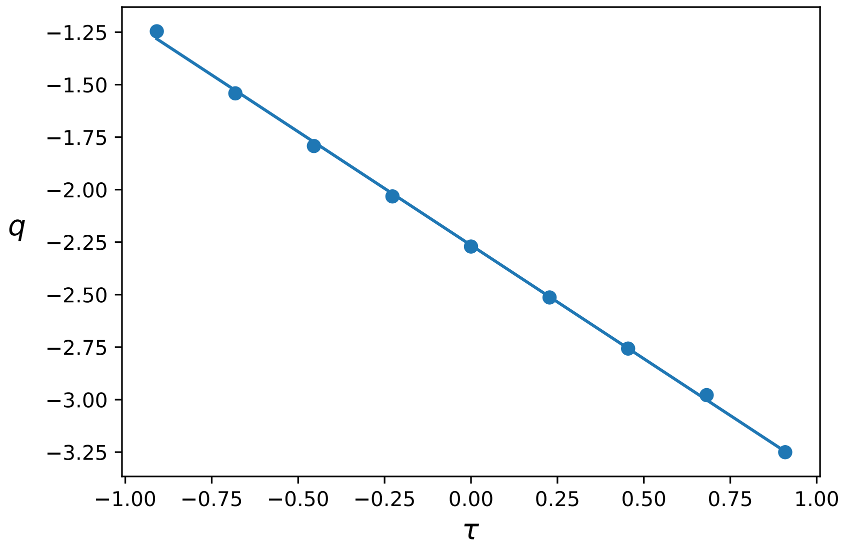


Figure 3.2: Dependency of the parameter q on the asymmetry τ of the vortex. For an antivortex q is positive, the pattern is reversed and q increases with τ.

figure 3.2. It is interesting to note that the graph suggests q depends linearly with τ, this is unexpected since q is not well understood even for the Ginzbug-Landau functional, where there is an argument by David Tong [57] proposing an explanation for the value of q based on string theory, but otherwise, the value of the constant is only known numerically and it is not clear whether such argument can be extended to the $O(3)$ Sigma model. The computed values of q are also displayed in table 3.1. For $\tau = 0$, the value of πq was computed by Romão-Speight [45, p. 23] as -7.1388, as can be seen in table 3.1, we found a value of $\pi q = -7.1346$, in agreement with the known data.

Let us consider an antivortex at position -2ϵ for big ϵ. The antivortex perturbs h in a neighbourhood of the origin, since the separation is large, we can assume that this is a small perturbation of the Hedgehog solution. Let h_0 be the single vortex solution at the origin. If h_1 is a small perturbation of h_0 caused by the antivortex in a neighbourhood of the origin, h_1 is a solution to the linearization of the Taubes equation,

$$-\Delta h_1 = \frac{4e^{h_0}\, h_1}{(1+e^{h_0})^2}. \tag{3.152}$$

The singularity at origin is carried by h_0 and since the operator in equation (3.152) if free of singularities, h_1 extends smoothly to the origin. Expanding in Fourier series h_1, we find,

$$h = h_0 + \frac{1}{2}f_0(r) + \sum_{n=1}^{\infty}\left(f_n(r)\cos(n\theta) + g_n(r)\sin(n\theta)\right). \tag{3.153}$$

The functions $f_n(r)$ and $g_n(r)$ are solutions to the equation

$$f_n'' + \frac{1}{r}f_n' - \left(\frac{4e^{h_0}}{(1+e^{h_0})^2} + \frac{n^2}{r^2}\right) f_n = 0, \tag{3.154}$$

and since h_1 is well defined at $r = 0$, to lowest order we have $f_n(r) = \alpha_n\, r^n$, $g_n = \beta_n\, r^n$.

To compute the coefficient b_1, we note that $h_0 = \log r^2 + \tilde{h}_0(r)$, where the regular part $\tilde{h}_0$ is a smooth function. Since $\log r^2$ is the fundamental solution of Laplace's equation on the plane, by (3.148), $\tilde{h}_0$ is a solution to the equation

$$\frac{d^2\tilde{h}_0}{dr^2} + \frac{1}{r}\frac{d\tilde{h}_0}{dr} - 2\left(\frac{e^{h_0}-1}{e^{h_0}+1} + \tau\right) = 0, \tag{3.155}$$

hence,

$$\begin{aligned}\overline{\partial}_x\tilde{h}_0(r) &= \tilde{h}_0'(r)\,\overline{\partial}_x\, r\\ &= \frac{1}{2}\tilde{h}_0'(r)\, e^{i\theta}\\ &= r\left(2\left(\frac{e^{h_0}-1}{e^{h_0}+1} + \tau\right) - \tilde{h}_0''\right).\end{aligned} \tag{3.156}$$

The function e^{h_0} has no singularity at the origin, moreover it is smooth, hence,

$$\overline{\partial}_x\tilde{h}_0(0) = \lim_{r\to\infty}\overline{\partial}_x\tilde{h}_0(r) = 0. \tag{3.157}$$

h is symmetric with respect to the line joining the two cores. These are located on the real axis, hence $h(x) = h(\overline{x})$ which translates into

$$2\,\partial_x|_{x=0}\left(h - \log r^2\right) = \partial_1|_{x=0}\left(h - \log r^2\right) = \alpha_1. \tag{3.158}$$

We conclude that $b_1 = \alpha_1$. To compute the nontrivial coefficient in the metric of the moduli space, we note that for large r, f_1 is a solution to the modified Bessel equation,

$$f_1'' + \frac{1}{r}f_1' - \left(1 - \tau^2 + \frac{1}{r^2}\right) f_1 = 0, \tag{3.159}$$

from here we can follow the computation done in [39] for Ginzburg-Landau vortices, the analysis is the same in the coordinate system x' and the conclusion is that the coefficient b_1 for a pair of distant vortices is,

$$b_1(\epsilon) = \frac{1}{2}q_1 q_2 \left(1-\tau^2\right)^{1/2} K_1\left(2\left(1-\tau^2\right)^{1/2}\epsilon\right). \tag{3.160}$$

By translation invariance, $b(\epsilon) = b_1(\epsilon)$, for b the nontrivial term in the conformal factor of the metric in the reduced moduli space. Using the properties of Bessel's functions given in equation (3.268), we find that at large separation the conformal factor can be approximated as,

$$\Omega(\epsilon) = 2\pi(1-\tau^2)\left(2 - q_1 q_2\, K_0\left(2\left(1-\tau^2\right)^{1/2}\epsilon\right)\right). \tag{3.161}$$

From this formula we observe the conformal factor vanishes at $\tau = \pm 1$, this can be understood because the effective mass of a vortex or antivortex is $2\pi(1 \mp \tau)$, hence as $\tau \to \pm 1$, most of the kinetic energy of a vortex-antivortex pair is concentrated at one of the cores which in the limit coincides with the centre of mass. Hence, by the decomposition of the L^2 metric in the centre of mass frame, proposition 3.4, one would expect this vanishing of the conformal factor.

3.2.1 The point-source formalism

Consider a single vortex or anti-vortex at origin, labelled 1, up to a local trivialization, the Higgs field is a map $\phi : U \subset \mathbb{R} \times \mathbb{R}^2 \to \mathbb{S}^2$ with coordinates $\phi(x_0, x_1, x_2) = (X_1, X_2, X_3)$. In the south pole projection, this field is equivalent to $\psi(x_0, x_1, x_2) = (X_1/(1+X_3), X_2/(1+X_3))$. We can choose a local gauge, the real gauge, in which ψ is real, or going back to the sphere, ϕ is constrained to the intersection circle of $\mathbb{S}^2$ with the plane $X_2 = 0$. Since the field has nontrivial winding, this gauge choice can be made only with exception of the core positions [39]. We aim to calculate a linear approximation to the field and vector potential far from the core, in which case we can make this assumption. It will be convenient to work in spherical coordinates, such that the Higgs field is parameterised as $\phi = (\sin(\varphi), 0, \cos(\varphi))$, with φ the azimuthal angle. In this gauge, the spherical covariant derivatives are

$$D_\mu \phi = \partial_\mu \phi - A_\mu \sin(\varphi)\, e_2. \tag{3.162}$$

In this section, we aim to show that if we have a collection of cores, vortices and antivortices well separated among each other, we can approximate the dynamics of the system as if at each core position there were a scalar monopole point-source and a magnetic dipole. For large r, the field approaches the vacuum manifold, perturbatively we can approximate ϕ as $(\sin(\varphi+\varphi_\infty), 0, \cos(\varphi+\varphi_\infty))$, where $\varphi_\infty = \cos^{-1}(\tau)$ and φ is small. Keeping linear terms in φ, we can make the approximation,

$$D_\mu \phi = (\cos(\varphi_\infty)\, \partial_\mu \varphi, -A_\mu(\sin(\varphi_\infty) + \cos(\varphi_\infty)\varphi), -\sin(\varphi_\infty)\, \partial_\mu \varphi)\,. \tag{3.163}$$

Retaining terms up to quadratic order, far from the vortex position, the Lagrangian density is approximately linear, corresponding to a non interacting field,

$$\mathcal{L}_{free} = \frac{1}{2}\partial_\mu \varphi\, \partial^\mu \varphi - \frac{1}{2}\sin^2(\varphi_\infty)\, \varphi^2 - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}\sin^2(\varphi_\infty)\, A_\mu A^\mu. \tag{3.164}$$

This is the Lagrangian density of two independent fields, whose extremals (φ, A) satisfy the real Klein-Gordon and Proca equations,

$$(\Box + \sin^2(\varphi_\infty))\, \varphi = 0, \tag{3.165}$$

$$(\Box + \sin^2(\varphi_\infty))\, A_\mu = \partial_\mu \partial^\nu A_\nu, \tag{3.166}$$

where $\Box = \partial_t^2 + \Delta$ is the D'Alambertian operator. We add a source term,

$$\mathcal{L}_{source} = \rho\,\varphi - j_\mu A^\mu, \tag{3.167}$$

to the free Lagrangian density, in order to match the expected behaviour at infinity of the fields, as in [52]. Therefore, the perturbed field equations are,

$$\left(\Box + \sin^2(\varphi_\infty)\right)\varphi = \rho, \tag{3.168}$$
$$\left(\Box + \sin^2(\varphi_\infty)\right)A_\mu = j_\mu + \partial_\mu\partial^\nu A_\nu. \tag{3.169}$$

Taking the divergence of the second equation, we find that,

$$\left(\Box + \sin^2(\varphi_\infty)\right)\partial^\mu A_\mu = \partial^\mu j_\mu + \Box\partial^\nu A_\nu, \tag{3.170}$$

hence, $\sin^2(\varphi_\infty)\,\partial^\mu A_\mu = \partial^\mu j_\mu$ and we infer,

$$\left(\Box + \sin^2(\varphi_\infty)\right)A_\mu = j_\mu + \frac{1}{\sin^2(\varphi_\infty)}\partial_\mu\partial^\nu j_\nu. \tag{3.171}$$

The sourced field equations of φ and A represent two massive fields of mass $\sin(\varphi_\infty) = \sqrt{1-\tau^2}$. In the real gauge, south pole stereographic projection of ϕ is $\psi = \phi_1/(1+\phi_3)$, hence, since φ is small,

$$\psi = \frac{\sin(\varphi_\infty) + \cos(\varphi_\infty)\,\varphi}{1 + \cos(\varphi_\infty) - \sin(\varphi_\infty)\,\varphi}, \tag{3.172}$$

moreover, to first order we have,

$$\psi = \frac{\sin(\varphi_\infty)}{1+\cos(\varphi_\infty)} + \frac{1}{1+\cos(\varphi_\infty)}\,\varphi. \tag{3.173}$$

On the other hand, if we fix one core and consider the field at a large distance from it but larger to the other cores, we have the approximation $\psi = \exp\left(\frac{1}{2}h_0\right)$, where h_0 is the solution to the radial Taubes equation, given by equation (3.151). To first order we have,

$$\begin{aligned}\psi &= \left(\frac{1-\tau}{1+\tau}\right)^{1/2}\left(1 + \frac{q_1}{2}K_0\left((1-\tau^2)^{1/2}\,r\right)\right)\\ &= \frac{\sin(\varphi_\infty)}{1+\cos(\varphi_\infty)}\left(1 + \frac{1}{2}q_1K_0\left(\sin(\varphi_\infty)\,r\right)\right).\end{aligned} \tag{3.174}$$

Hence, the asymptotic expansion of φ is,

$$\varphi = \frac{q_1}{2}\sin(\varphi_\infty)\, K_0\left(\sin(\varphi_\infty)\, r\right). \tag{3.175}$$

We are interested in static fields, in this case, the field equations reduce to the static Klein-Gordon equation with a mass term,

$$(\Delta + \sin^2(\varphi_\infty))\,\varphi = \rho, \quad (\Delta + \sin^2(\varphi_\infty))\, A_\mu = j_\mu + \frac{1}{\sin^2(\varphi_\infty)}\partial_\mu \partial^\nu j_\nu. \tag{3.176}$$

We have,

$$\mathcal{D}_k\phi = \partial_k\phi - A_k\, n\times\phi = \partial_k\varphi\cdot(\cos(\varphi)e_1 - \sin(\varphi)e_3) - A_k\sin(\varphi)\, e_2, \tag{3.177}$$

and

$$\phi\times(\partial_2\phi - A_2\, n\times\phi) = (\sin(\varphi)e_1 + \cos(\varphi)e_3) \tag{3.178}$$

$$\times\,(\partial_2\varphi\cdot\cos(\varphi)e_1 - A_2\sin(\varphi)\, e_2 - \partial_2\varphi\cdot\sin(\varphi)e_3) \tag{3.179}$$

$$= A_2\sin(\varphi)\cos(\varphi)e_1 + \partial_2\varphi\cdot e_2 - A_2\sin^2(\varphi)\, e_3. \tag{3.180}$$

In the gauge $A_0 = 0$ the first Bogomolny equation is,

$$(\partial_1\varphi + A_2\sin(\varphi))\,(\cos(\varphi)e_1 - \sin(\varphi)e_3) + (\partial_2\varphi - A_1\sin(\varphi))\, e_2 = 0, \tag{3.181}$$

which is equivalent to,

$$\partial_1\varphi + A_2\sin(\varphi) = 0, \qquad \partial_2\varphi - A_1\sin(\varphi) = 0. \tag{3.182}$$

In a region far from the core position, these equations can be linearized as

$$\partial_1\varphi + \sin(\varphi_\infty)A_2 = 0, \qquad \partial_2\varphi - \sin(\varphi_\infty)A_1 = 0. \tag{3.183}$$

In the gauge $A_0 = 0$ if the fields are static we have,

$$j_0 + \frac{1}{\sin^2(\varphi_\infty)}\partial_0\partial^\nu j_\nu = 0. \tag{3.184}$$

For the spatial components, note that

$$(A_1, A_2) = \frac{1}{\sin(\varphi_\infty)}\,(\partial_2\,\varphi, -\partial_1\,\varphi). \tag{3.185}$$

Introducing a fictitious unit vector $\mathbf{k}$ perpendicular to the plane in the positive orientation of $\mathbb{R}^3$ and defining $\mathbf{A} = (A_1, A_2)$, the spatial part of the linearized potential can be related to the Higgs field with the vector equation,

$$\mathbf{A} = -\frac{1}{\sin(\varphi_\infty)}\mathbf{k} \times \nabla\varphi. \tag{3.186}$$

To make our deduction of the point-source approximation, we will work in space-time coordinates; to this end, in this section we denote space-time coordinates as x and space coordinates as $\mathbf{x}$.

The static field equation of φ is,

$$(\Delta + \sin^2\varphi_\infty)\,\varphi = \rho. \tag{3.187}$$

Green's function for the static Klein-Gordon equation is $K_0(|\mathbf{x}|)$,

$$(\Delta + 1)\ K_0(|\mathbf{x}|) = 2\pi\delta(\mathbf{x}). \tag{3.188}$$

Substituting the asymptotic approximation to φ we found,

$$\begin{aligned}\left(\Delta + \sin^2\varphi_\infty\right)\varphi &= \frac{q_1}{2}\sin(\varphi_\infty)\ \left(\Delta + \sin^2\varphi_\infty\right) K_0(\sin(\varphi_\infty)\,r)\\ &= q_1\pi\ \sin^3(\varphi_\infty)\,\delta(\sin(\varphi_\infty)\,\mathbf{x})\\ &= q_1\pi\sin(\varphi_\infty)\,\delta(\mathbf{x}),\end{aligned} \tag{3.189}$$

where in the last inequality we have used that for any constant c, $\delta(c\mathbf{x}) = c^{-2}\delta(\mathbf{x})$. This suggests that the physics of a static vortex, seen far from the core is equivalent to a particle with charge $q_1\pi\sin(\varphi_\infty)$, therefore we define the one vortex source term,

$$\rho = q_1\pi\sin(\varphi_\infty)\,\delta(\mathbf{x}). \tag{3.190}$$

Applying the operator $(\Delta + \sin^2\varphi_\infty)$ to $\mathbf{A}$, we find,

$$\begin{aligned}\left(\Delta + \sin^2\varphi_\infty\right)\mathbf{A} &= -\frac{1}{\sin(\varphi_\infty)}\mathbf{k} \times \nabla\left(\Delta + \sin^2\varphi_\infty\right)\varphi\\ &= -q_1\pi\,\mathbf{k} \times \nabla\delta(\mathbf{x}).\end{aligned} \tag{3.191}$$

On the other hand, let us assume that the current is static, in the sense that $j_0 = 0$. From (3.176), we have that $\mathbf{A}$ satisfies the equation,

$$(\Delta + \sin^2\varphi_\infty)\mathbf{A} = \mathbf{j} - \frac{1}{\sin^2(\varphi_\infty)}\nabla\left(\nabla\cdot\mathbf{j}\right). \tag{3.192}$$

Thence,

$$\sin^2(\varphi_\infty)\,\mathbf{j} - \nabla\left(\nabla\cdot\mathbf{j}\right) = -q_1\pi\sin^2(\varphi_\infty)\,\mathbf{k}\times\nabla\delta(\mathbf{x}). \tag{3.193}$$

Taking the divergence of this equation we find that $\nabla\cdot\mathbf{j}$ is a solution in the sense of distributions, to the equation,

$$(\Delta + \sin^2\varphi_\infty)\nabla\cdot\mathbf{j} = 0. \tag{3.194}$$

We know that $\nabla\cdot\mathbf{j}$ is also a strong solution in $\mathbb{R}^2\setminus\{0\}$. It is sensible to assume that $\nabla\cdot\mathbf{j}$ is an L^2 solution to this equation. Under this assumption, by elliptic regularity $\nabla\cdot\mathbf{j}$ is smooth in the plane and since $\sin^2\varphi_\infty$ is in the resolvent set of geometers' Laplacian, $\nabla\cdot\mathbf{j} = 0$. Therefore, the current is conserved and we have that the core behaves as a magnetic dipole generated by a point current,

$$\mathbf{j} = -q_1\pi\,\mathbf{k}\times\nabla\delta(\mathbf{x}). \tag{3.195}$$

We will need later space-time coordinates, we define,

$$j_{static} = \left(0,\mathbf{j}\right), \tag{3.196}$$

as the space-time point current in the lab frame.

Having calculated expressions for the charge and current of the point particle approximation, we can calculate the interaction potential of a pair of vortices. For this, it is necessary to calculate the interaction Lagrangian, which is obtained as

$$\mathrm{L}_{int} = \int \mathcal{L}_{cross}\,dx, \tag{3.197}$$

where $\mathcal{L}_{cross}$ are the cross terms of $\mathcal{L}_{free} + \mathcal{L}_{source}$ in a superposition of two pairs of fields $(\varphi_k, \mathbf{A}_k)$, with sources $(\rho_k, \mathbf{j}_k)$. For a pair of cores, the interaction Lagrangian reduces to [52]

$$\mathrm{L}_{int} = \int \rho_1\varphi_2 - j^{(1)}_\mu A^\mu_{(2)}\,dx. \tag{3.198}$$

We aim to calculate the interaction Lagrangian for any number of separated moving cores whose separations are large. Let us consider a core moving slowly in the laboratory frame and let ξ be the coordinates on space-time with respect to this frame, which has coordinates x. If the vortex is moving at constant speed u in the direction of x_1 with respect to the lab frame, the coordinate change on tangent space at x is [4],

$$\left.\begin{aligned}
\xi_0 &= \gamma(u)\,(x_0 - u\,x_1),\\
\xi_1 &= \gamma(u)\,(-u\,x_0 + x_1),\\
\xi_2 &= x_2,
\end{aligned}\right\} \tag{3.199}$$

where $\gamma(u) = (1-u^2)^{-1/2}$ is the Lorentz contraction factor and the speed is relative to the speed of light, $|u| < 1$. Our aim is to write the charge and magnetic dipole of the moving core as seen in the laboratory frame. If the velocity with respect to the lab frame is not along the x_1 axes, we can always rotate the coordinates before and then boost in the x_1 direction. In the rest frame, the core is static, and therefore the charge density at large separation from their neighbours is $\rho(\boldsymbol{\xi}) = q\,\pi\sin(\varphi_\infty)\,\delta(\boldsymbol{\xi})$. Since we are interested in the infinitesimal behaviour of the charge, we can take $x_0 = 0$ in the Lorentz transformations relating rest and laboratory frames,

$$\begin{aligned}
\rho(\boldsymbol{\xi}) &= q\,\pi\sin(\varphi_\infty)\,\delta(\gamma\,x_1\,e_1 + x_2\,e_2)\\
&= \frac{1}{\gamma}\,q\,\pi\,\sin(\varphi_\infty)\,\delta(\mathbf{x}).
\end{aligned} \tag{3.200}$$

If the speed is much slower than the speed of light, γ^{-1} can be approximated as

$$\gamma(u)^{-1} = 1 - \frac{1}{2}\,u^2 + \mathcal{O}(u^4). \tag{3.201}$$

Discarding higher order terms in u, the instantaneous charge density of a slowly moving vortex is

$$\rho(x) = q\pi\sin(\varphi_\infty)\left(1 - \frac{u^2}{2}\right)\,\delta(\mathbf{x}). \tag{3.202}$$

If the core is at an arbitrary position $y(t)$ and $u = \dot{y}(t)$ is the speed of the moving core, we conclude the charge density as seen in the laboratory frame is,

$$\rho = q\pi\,\sin(\varphi_\infty)\left(1 - \frac{\dot{y}^2}{2}\right)\,\delta(\mathbf{x} - \mathbf{y}). \tag{3.203}$$

For an observer in an inertial frame, a slowly moving core $y(t)$ has the four-current,

$$\begin{aligned} j_0 &= q\pi\, \mathbf{k} \times \dot{\mathbf{y}} \cdot \nabla\delta(\mathbf{x} - \mathbf{y}), \\ \mathbf{j} &= q\pi\, \left(-\mathbf{k} \times \nabla + (\mathbf{k} \times \dot{\mathbf{y}})\, \dot{\mathbf{y}} \cdot \nabla + \mathbf{k} \times \ddot{\mathbf{y}}\right) \delta(\mathbf{x} - \mathbf{y}). \end{aligned} \tag{3.204}$$

(3.204) was computed by Speight for Ginzburg-Landau vortices, details of the computation can be found in [39, eqs. (3.20) (3.21)], for the O(3) Sigma model, the calculation is the same, except for the factor of π coming from our conventions on the constant q.

Since current is conserved, the components A_μ of the gauge potential are solutions to the equation,

$$\left(\Box + \sin^2(\varphi_\infty)\right) A_\mu = j_\mu. \tag{3.205}$$

If we define the primed coordinate system,

$$x' = \sin(\varphi_\infty)\, x, \tag{3.206}$$

and fields,

$$\varphi'(x') = \varphi(\sin(\varphi_\infty)^{-1}\, x'), \qquad A'_\mu(x') = A_\mu(\sin(\varphi_\infty)^{-1}\, x'), \tag{3.207}$$

with sources,

$$\begin{aligned} \rho'(x') &= \sin(\varphi_\infty)^{-2}\, \rho(\sin(\varphi_\infty)^{-1}\, x'), \\ j'_\mu(x') &= \sin(\varphi_\infty)^{-2}\, j_\mu(\sin(\varphi_\infty)^{-1}\, x'), \end{aligned} \tag{3.208}$$

then φ', A'_μ are solutions to the equations,

$$\begin{aligned} (\Box' + 1)\, \varphi' &= \rho', \\ (\Box' + 1)\, A'_\mu &= j'_\mu. \end{aligned} \tag{3.209}$$

Since $d\mathbf{y}/dt = d\mathbf{y}'/dt'$, defining $q' = q\pi \sin(\varphi_\infty)$, by (3.203),

$$\rho' = q' \left(1 - \frac{\dot{y}'^2}{2}\right) \delta(\mathbf{x}' - \mathbf{y}'), \tag{3.210}$$

whereas by (3.204),

$$\begin{aligned} j'_0 &= q'\, \mathbf{k} \times \dot{\mathbf{y}}' \cdot \nabla'\delta(\mathbf{x}' - \mathbf{y}'), \\ \mathbf{j}' &= q'\, \left(-\mathbf{k} \times \nabla' + (\mathbf{k} \times \dot{\mathbf{y}}')\, \dot{\mathbf{y}}' \cdot \nabla' + \mathbf{k} \times \ddot{\mathbf{y}}'\right) \delta(\mathbf{x}' - \mathbf{y}'). \end{aligned} \tag{3.211}$$

In the primed coordinate system, equations (3.209)-(3.211) are the same as those found in the asymptotic approximation of Ginzburg-Landau vortices by Speight and Manton, with the only exception that vortices and antivortices carry constants q of different values. Hence, by [52, Eq. (3.46)], for a pair of cores at positions labelled $\mathbf{x}_1$, $\mathbf{x}_2$,

$$\begin{aligned} \mathrm{L}_{int} &= -\frac{q_1' q_2'}{4\pi} \, |\dot{\mathbf{x}}_2' - \dot{\mathbf{x}}_1'|^2 \, K_0(|\mathbf{x}_2' - \mathbf{x}_1'|) \\ &= -\frac{q_1 q_2}{4} \, \pi \, \sin^2(\varphi_\infty) \, |\dot{\mathbf{x}}_2 - \dot{\mathbf{x}}_1|^2 \, K_0(\sin(\varphi_\infty) \, |\mathbf{x}_2 - \mathbf{x}_1|). \end{aligned} \tag{3.212}$$

Recall $m_r = 2\pi(1 + s_r\tau)$ is the effective mass of a core at position $\mathbf{x}_r$, where $s_r = \pm 1$ is the sign of the core, we conclude that if the cores are at large separation and moving slowly, their dynamics can be approximated by the Lagrangian,

$$\mathrm{L} = \sum_r \frac{m_r}{2} |\dot{\mathbf{x}}_r|^2 - \sum_{r \neq s} \frac{q_r q_s}{4} \, \pi \, \sin^2(\varphi_\infty) |\dot{\mathbf{x}}_r - \dot{\mathbf{x}}_s|^2 K_0 \left(\sin(\varphi_\infty) |\mathbf{x}_r - \mathbf{x}_s| \right). \tag{3.213}$$

For a vortex-antivortex pair at large separation, if $M = m_1 + m_2$, $\mathbf{X} = \frac{m_1}{M} \, \mathbf{x}_1 + \frac{m_2}{M} \, \mathbf{x}_2$ is the centre of mass of the pair and $\mathbf{x}_1 - \mathbf{x}_2 = 2\, \epsilon e^{i\theta}$, are coordinates relative to the centre of mass, the Lagrangian becomes,

$$\begin{aligned} \mathrm{L} &= \frac{M}{2} \, |\dot{\mathbf{X}}|^2 + \left(\frac{2 m_1 m_2}{M} - q_1 \, q_2 \, \pi \, \sin^2(\varphi_\infty) \, K_0(2 \sin(\varphi_\infty) \, \epsilon) \right) (\dot{\epsilon}^2 + \epsilon^2 \dot{\theta}^2) \\ &= \frac{M}{2} \, |\dot{\mathbf{X}}|^2 + (1 - \tau^2) \, \pi \, \left(2 - q_1 \, q_2 \, K_0(2(1 - \tau^2)^{1/2} \, \epsilon) \right) (\dot{\epsilon}^2 + \epsilon^2 \dot{\theta}^2). \end{aligned} \tag{3.214}$$

If we get rid of the centre of mass term, we find that the conformal factor in the reduced moduli space is again as in equation (3.161).

3.2.2 Approximating the conformal factor in a neighbourhood of the singularity

In this section we aim to derive an asymptotic approximation to the conformal factor for small ϵ, we do so finding the limit of the regular part of h_ϵ/ϵ as $\epsilon \to 0$, where h_ϵ is the solution to the Taubes equation with vortex at ϵ and antivortex at $-\epsilon$ and then we prove the convergence is uniform in disks centred at the origin.

Let us consider (ϵ, θ) coordinates, we know h_ϵ depends smoothly on ϵ and the function $\partial_\epsilon h$ is a solution of the equation,

$$-(\Delta + V(h_\epsilon))\,\partial_\epsilon h_\epsilon = 4\pi\partial_1\delta_\epsilon + 4\pi\partial_1\delta_{-\epsilon}. \tag{3.215}$$

If $\mu = \log((1-\tau)(1+\tau)^{-1})$ is the limit value of h_ϵ as $|z| \to \infty$, we know that as $\epsilon \to 0$, the potential function $V(h_\epsilon)$ converges pointwise to $V(\mu) = 1 - \tau^2 \in (0, 1]$ and uniformly outside of any neighbourhood of the origin. We also know each $\partial_\epsilon h_\epsilon$ decays exponentially fast as $|z| \to \infty$. Without loss of generality we assume $\tau = 0$ from now onwards. As the fundamental solution of the screened Poisson equation

$$-(\Delta + 1)G = \delta_0, \tag{3.216}$$

with convergence $G \to 0$ as $|z| \to \infty$, is $(2\pi)^{-1} K_0(|z|)$, if we denote by $\exp(i\theta_{\pm\epsilon})$ the argument of $z \mp \epsilon$, the function,

$$\begin{aligned} H_\epsilon &= 2\left(\partial_1 K_0(|z-\epsilon|) + \partial_1 K_0(|z+\epsilon|)\right), \\ &= -2\left(\cos(\theta_\epsilon)\, K_1(|z-\epsilon|) + \cos(\theta_{-\epsilon})\, K_1(|z+\epsilon|)\right), \end{aligned} \tag{3.217}$$

is the fundamental solution of the equation,

$$-(\Delta + 1)\, H_\epsilon = 4\pi\partial_1\delta_\epsilon + 4\pi\partial_1\delta_{-\epsilon}. \tag{3.218}$$

By (3.215) and (3.218),

$$(\Delta + V(h_\epsilon))\,(\partial_\epsilon h_\epsilon - H_\epsilon) = (1 - V(h_\epsilon))\, H_\epsilon. \tag{3.219}$$

Denoting by $f * g$ convolution on the plane,

$$\partial_\epsilon h_\epsilon - H_\epsilon = -\left((1 - V(h_\epsilon))\, H_\epsilon\right) * G_\epsilon, \tag{3.220}$$

where G_ϵ is Green's function of the operator $-(\Delta + V(h_\epsilon))$. We aim to prove $|\partial_\epsilon h_\epsilon - H_\epsilon| \to 0$ uniformly on the plane. To do this, we will use the concept of a doubling measure and prove a few common properties for the family of potentials $V(h_\epsilon)$.

A measure ν is called doubling if there exists a constant $C > 0$, such that for any $z \in \mathbb{C}$ and $r > 0$,

$$\nu(\mathbb{D}_{2R}(z)) \leq C\,\nu(\mathbb{D}_R(z)). \tag{3.221}$$

Suppose $D \subset \mathbb{R}^2$ is a measurable set with respect to the euclidean metric, we define,

$$\nu_\epsilon(D) = \int_D V(h_\epsilon)\mathrm{Vol}. \tag{3.222}$$

Given $\epsilon_0 > 0$, we will prove the existence of a uniform constant C_d such that (3.221) holds for any $\epsilon \in (0, \epsilon_0)$ and a uniform constant $\delta > 0$, such that,

$$\nu_\epsilon(\mathbb{D}_1(z)) > \delta \tag{3.223}$$

for any $\epsilon \in (0, \epsilon_0)$, then, by a result of Christ [9, Thm. 1.13] there are a function $\varrho : \mathbb{R}^2 \to \mathbb{R}^+$, a distance function $\rho : \mathbb{R}^2 \times \mathbb{R}^2 \to \mathbb{R}$ induced by a Riemannian metric $d\rho^2$, and constants C, γ all of them depending only on C_d, such that,

$$|G_\epsilon(z_1, z_2)| \leq C \begin{cases} \log(2\varrho(z_1)/|z_1 - z_2|), & |z_1 - z_2| \leq \varrho(z), \\ \exp(-\gamma\rho(z_1, z_2)), & |z_1 - z_2| \geq \varrho(z), \end{cases} \tag{3.224}$$

and $\rho(z_1, z_2) \geq c\,|z_1 - z_2|$ for some constant c depending on δ but not on C_d.

If $\tilde{h}_\epsilon = h_\epsilon + \log|z - \epsilon| - \log|z + \epsilon|$, we know that for any $\epsilon_0 > 0$, there are constants C_1, C_2, such that for any $z \in \mathbb{C}$ and $\epsilon \in (0, \epsilon_0)$,

$$C_1 \leq e^{\tilde{h}_\epsilon} \leq C_2, \tag{3.225}$$

hence,

$$\frac{4C_1|z-\epsilon|^2|z+\epsilon|^2}{(C_1|z-\epsilon|^2 + |z+\epsilon|^2)} \leq V(h_\epsilon) \leq \frac{4C_2|z-\epsilon|^2|z+\epsilon|^2}{(C_2|z-\epsilon|^2 + |z+\epsilon|^2)^2}. \tag{3.226}$$

Hence, there is a constant $C > 0$ independent of ϵ such that,

$$\frac{1}{C}\,\frac{4|z-\epsilon|^2|z+\epsilon|^2}{(|z-\epsilon|^2 + |z+\epsilon|^2)^2} \leq V(h_\epsilon) \leq C\,\frac{4|z-\epsilon|^2|z+\epsilon|^2}{(|z-\epsilon|^2 + |z+\epsilon|^2)^2}, \tag{3.227}$$

this implies the potential $V(h_\epsilon)$ induces a doubling measure if and only if

$$\tilde{V}_\epsilon = \frac{4|z-\epsilon|^2|z+\epsilon|^2}{(|z-\epsilon|^2 + |z+\epsilon|^2)^2}, \tag{3.228}$$

does.

Lemma 3.15. *Let $M \in (0,1)$, then $\tilde{V}_\epsilon^{-1}([0,M])$ consists of two connected components, whose boundaries are the circles centred at $\pm(1-M)^{-1/2}\epsilon$ of radii $M^{1/2}(1-M)^{-1/2}\epsilon$.*

Proof. Let $w = (z-\epsilon)(z+\epsilon)^{-1}$, if $\tilde{V}_\epsilon(z) = M$, then,

$$\frac{4|w|^2}{(|w|^2+1)^2} = M, \tag{3.229}$$

this equality implies,

$$|w|^2 - \frac{2}{M^{1/2}}|w| + 1 = 0. \tag{3.230}$$

The roots of this equation are,

$$r_\pm = \frac{1}{M^{1/2}}(1 \pm (1-M)^{1/2}). \tag{3.231}$$

If $z = x + y\,i$, for each root, the equation

$$\left|\frac{z-\epsilon}{z+\epsilon}\right| = r_\pm, \tag{3.232}$$

determines the circles

$$|z|^2 - 2\epsilon\left(\frac{1+r_\pm^2}{1-r_\pm^2}\right)x + \epsilon^2 = 0, \tag{3.233}$$

of centres

$$c_\pm = \epsilon\left(\frac{1+r_\pm^2}{1-r_\pm^2}\right) = \frac{\mp\epsilon}{(1-M)^{1/2}} \tag{3.234}$$

and squared radii

$$\begin{aligned} R_\pm^2 &= |c_\pm|^2 - \epsilon^2 \\ &= \frac{\epsilon^2}{1-M} - \epsilon^2 \\ &= \frac{M\epsilon^2}{1-M}. \end{aligned} \tag{3.235}$$

Mobius transformations map circles onto circles and $z = \epsilon$ is mapped to $w = 0$, while $z = -\epsilon$ is mapped to $w = \infty$, where for both points $4|w|^2(|w|^2+1)^{-2} = 0$, hence the disks $|z - c_\pm| \le r_\pm$ are the connected components of $\tilde{V}_\epsilon([0,M])$. □

Lemma 3.16. $\tilde{V}_1$ *defines a doubling measure.*

Proof. Assume otherwise towards a contradiction, then there exists a sequence $\{(z_n, r_n)\}$ such that,

$$\frac{\int_{\mathbb{D}_{2r_n}(z_n)} \tilde{V}_1 \, |dz|^2}{\int_{\mathbb{D}_{r_n}(z_n)} \tilde{V}_1 \, |dz|^2} \to \infty \tag{3.236}$$

After passing to a subsequence if necessary, we can assume $(z_n, r_n) \to (z_*, r_*) \in \overline{\mathbb{C}} \times [0, \infty]$, where z_* could be the point at infinity, meaning $|z_n| \to \infty$. Through the proof we will consider a fixed but arbitrary constant $M \in (0,1)$. If $z_* \in \mathbb{C}$ we consider four cases:

Case I. If $0 < r_* < \infty$, by the dominated convergence theorem,

$$\frac{\int_{\mathbb{D}_{2r_n}(z_n)} \tilde{V}_1 \, |dz|^2}{\int_{\mathbb{D}_{r_n}(z_n)} \tilde{V}_1 \, |dz|^2} \to \frac{\int_{\mathbb{D}_{2r_*}(z_*)} \tilde{V}_1 \, |dz|^2}{\int_{\mathbb{D}_{r_*}(z_*)} \tilde{V}_1 \, |dz|^2}, \tag{3.237}$$

hence (3.236) is not possible.

Case II. If $r_* = \infty$, by lemma 3.15, there is an $R > 0$ such that $\tilde{V}_1(z) \geq M$ for $|z| \geq R$. Let $\Omega_n = \mathbb{D}_{r_n}(z_n) \setminus \mathbb{D}_R(0)$. For n sufficiently large $\Omega_n \neq \emptyset$, moreover,

$$\frac{\int_{\mathbb{D}_{2r_n}(z_n)} \tilde{V}_1 \, |dz|^2}{\int_{\mathbb{D}_{r_n}(z_n)} \tilde{V}_1 \, |dz|^2} \leq \frac{4\pi r_n^2}{M|\Omega_n|} = \frac{4\pi r_n^2}{M(\pi r_n^2 - |\mathbb{D}_{r_n}(z_n) \cap \mathbb{D}_R(0)|)} \to \frac{4}{M}. \tag{3.238}$$

Case III. If $r_* = 0$ and $z_* \neq \pm 1$, by the mean value theorem for integrals,

$$\frac{\int_{\mathbb{D}_{2r_n}(z_n)} \tilde{V}_1 \, |dz|^2}{\int_{\mathbb{D}_{r_n}(z_n)} \tilde{V}_1 \, |dz|^2} = 4 \, \frac{\frac{1}{4\pi r_n^2} \int_{\mathbb{D}_{2r_n}(z_n)} \tilde{V}_1 \, |dz|^2}{\frac{1}{\pi r_n^2} \int_{\mathbb{D}_{r_n}(z_n)} \tilde{V}_1 \, |dz|^2} \to 4, \tag{3.239}$$

since each averaged integral converges to $\tilde{V}_1(z_*) \neq 0$.

Case IV. If $z_* = \pm 1$, assume without loss of generality $z_* = 1$, let $R \in (0, 1/2)$ be any constant, for n large enough, the disk $\mathbb{D}_{2r_n}(z_n)$ is contained in $\mathbb{D}_R(1)$, then there is a constant $C(R)$, such that for any $z \in \mathbb{D}_R(1)$,

$$\frac{1}{C} |z - 1|^2 \leq \tilde{V}_1(z) \leq C \, |z - 1|^2, \tag{3.240}$$

the function $|z - 1|^2$ defines a doubling measure because it is a non negative polynomial [62], implying for large n the quotient

$$\frac{\int_{\mathbb{D}_{2r_n}(z_n)} \tilde{V}_1 \, |dz|^2}{\int_{\mathbb{D}_{r_n}(z_n)} \tilde{V}_1 \, |dz|^2} \tag{3.241}$$

is bounded.

Therefore, if (3.236) holds, $|z_n| \to \infty$. If $r_n \to r_0$ for $r_0 \in [0, \infty)$, for n large the disk $\mathbb{D}_{2r_n}(z_n)$ is in the exterior of the disk $\mathbb{D}_R(0)$, hence $\tilde{V}_1 \in [M, 1]$, and

$$\frac{\int_{\mathbb{D}_{2r_n}(z_n)} \tilde{V}_1 \, |dz|^2}{\int_{\mathbb{D}_{r_n}(z_n)} \tilde{V}_1 \, |dz|^2} \leq \frac{4}{M}. \tag{3.242}$$

Finally, if $r_n \to \infty$, we can apply the same argument as in Case II to deduce that (3.236) is not possible. This concludes all the possibilities for the sequence and proves the lemma. □

If we define the change of variable $z = \epsilon w$, by lemma 3.16 we have,

$$\frac{\int_{\mathbb{D}_{2r}(z)} \tilde{V}_\epsilon \, |dz|^2}{\int_{\mathbb{D}_r(z)} \tilde{V}_\epsilon \, |dz|^2} = \frac{\int_{\mathbb{D}_{2r/\epsilon}(z/\epsilon)} \tilde{V}_1 \, |dw|^2}{\int_{\mathbb{D}_{r/\epsilon}(z/\epsilon)} \tilde{V}_1 \, |dw|^2} < C, \tag{3.243}$$

where C is independent of ϵ, proving the following corollary.

Corollary 3.17. *For any $\epsilon_0 > 0$, there is a constant C_d such that (3.221) holds for any $\epsilon \in (0, \epsilon_0)$.*

Lemma 3.18. *For any $\epsilon_0 > 0$, there is a constant $\delta > 0$ such that*

$$\int_{\mathbb{D}_1(z)} \tilde{V}_\epsilon > \delta, \tag{3.244}$$

for all $z \in \mathbb{C}$, $\epsilon \in (0, \epsilon_0)$.

Proof. Pick $M \in (0, 1)$ such that $r_0 = M^{1/2}(1 - M)^{-1/2}\epsilon_0$ satisfies $2r_0^2 < 1$. By lemma 3.15 there are two disks D_1, D_2 of radius $r < r_0$ such that in the exterior of the disks $\tilde{V}_\epsilon \geq M$. The complement $\Omega = \mathbb{D}_1(z) \setminus (D_1 \cup D_2)$ is non empty for any $z \in \mathbb{C}$ and it has bounded area,

$$|\Omega| \geq \pi - 2\,|D_1| \geq \pi\,(1 - 2r_0^2), \tag{3.245}$$

hence,

$$\begin{aligned}\int_{\mathbb{D}_1(z)} \tilde{V}_\epsilon \, |dz|^2 &\geq \int_\Omega \tilde{V}_\epsilon \, |dz|^2 \\ &\geq M\,\pi\,(1 - 2r_0^2). \end{aligned} \tag{3.246}$$

Selecting any $\delta < M\,\pi\,(1 - 2r_0^2)$ proves the lemma. □

Therefore, for any $p \geq 2$, there is a constant $C > 0$ such that,

$$||G_\epsilon||_{\mathrm{L}^p} < C, \qquad \forall \epsilon \in (0, \epsilon_0), \tag{3.247}$$

let us choose any $p > 2$ and let $p_* = p\,(p-1)^{-1}$ be Hölder's conjugate of p, by (3.220) and Hölder's inequality,

$$\begin{aligned} |\partial_\epsilon h_\epsilon - H_\epsilon| &\leq ||(1 - V(h_\epsilon))H_\epsilon||_{\mathrm{L}^{p_*}}\, ||G_\epsilon||_{\mathrm{L}^p} \\ &\leq C\,(||(1 - V(h_\epsilon))K_1(|z-\epsilon|)||_{\mathrm{L}^{p_*}} \\ &\quad + ||(1 - V(h_\epsilon))K_1(|z+\epsilon|)||_{\mathrm{L}^{p_*}}). \end{aligned} \tag{3.248}$$

Lemma 3.19.

$$\lim_{\epsilon \to 0} ||(1 - V(h_\epsilon))\, K_1(|z-\epsilon|)||_{\mathrm{L}_{p_*}} = 0, \tag{3.249}$$

and a similar statement holds for $K_1(|z + \epsilon|)$.

Proof. Let $w = z - \epsilon$, then,

$$||(1 - V(h_\epsilon))\, K_1(|z-\epsilon|)||^{p_*}_{\mathrm{L}_{p_*}} = \int_{\mathbb{R}^2} (1 - V(h_\epsilon(w+\epsilon)))^{p_*}\, K_1(|w|)^{p_*}\, |dw|^2, \tag{3.250}$$

the function $K_1(|w|)$ is in L^{p_*} for any $p_* < 2$, and $(1 - V(h_\epsilon(w+\epsilon)))$ is a bounded function converging pointwise to 0, by the dominated convergence theorem,

$$\int_{\mathbb{R}^2} (1 - V(h_\epsilon(w+\epsilon)))^{p_*}\, K_1(|w|)^{p_*}\, |dw|^2 \to 0, \tag{3.251}$$

this proves the lemma for $K_1(|z - \epsilon|)$, for $K_1(|z + \epsilon|)$ the proof is analogous. □

By (3.248), $|\partial_\epsilon h_\epsilon - H_\epsilon| \to 0$ uniformly on the plane as $\epsilon \to 0$. Note that the function

$$H^\tau_\epsilon = (1 - \tau^2)^{1/2}\, H_{(1-\tau^2)^{1/2}\epsilon}((1-\tau^2)^{1/2}\,|z|), \tag{3.252}$$

is the fundamental solution to

$$-(\Delta + (1 - \tau^2))\, H^\tau_\epsilon = 4\pi\, \partial_1 \delta_\epsilon + 4\pi\, \partial_1 \delta_{-\epsilon}. \tag{3.253}$$

All the previous lemmas extend straightforwardly to conclude for any $\tau \in (-1,1)$, the convergence $|h_\epsilon - H_\epsilon^\tau| \to 0$ uniform on the plane. Let us define the function

$$f_\epsilon = \frac{1}{\epsilon}\left(h - \log|z-\epsilon|^2 + \log|z+\epsilon|^2 - \log\frac{1-\tau}{1+\tau}\right), \tag{3.254}$$

For this function at $\tau = 0$, Romão and Speight conjectured in [45], the uniform limit $f_\epsilon \to f_*$ where for general τ, f_* is a solution to the problem,

$$-(\Delta + 1 - \tau^2)\, f_* = -4(1-\tau^2)\,\frac{z_1}{|z|^2}, \tag{3.255}$$

$$\lim_{|z|\to\infty} f_* = 0, \tag{3.256}$$

$$\lim_{|z|\to 0} f_* = 0. \tag{3.257}$$

The equation for f_* can be solved exactly, for $\tau = 0$, they found,

$$\begin{aligned} f_* &= 4\,\frac{z_1}{|z|^2}\,(1 - |z|K_1(|z|)) \\ &= 4\,\cos(\theta)\left(\frac{1}{|z|} - K_1(|z|)\right). \end{aligned} \tag{3.258}$$

If we define,

$$f_*^\tau(z) = (1-\tau^2)^{1/2}\, f_*((1-\tau^2)^{1/2}\, z), \tag{3.259}$$

then f_*^τ is the conjectured limit for general τ.

Numerical evidence suggests C^1 uniform convergence as can be seen in Figure 3.3. In the next proposition, we prove that in fact, the convergence is uniform at least in $C^0(\mathbb{D}_R(0))$ for any disk centred at the origin.

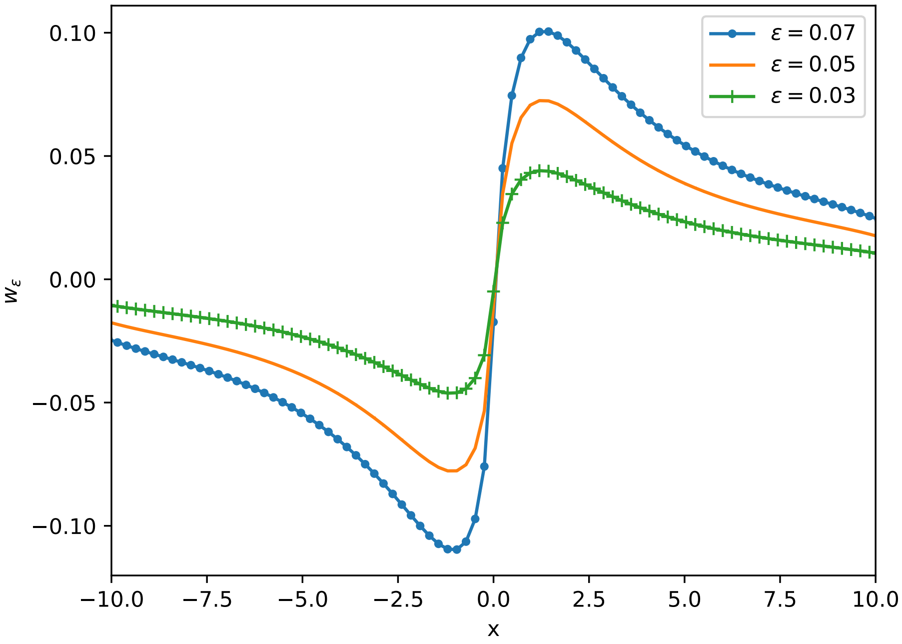
Proposition 3.20. *For any $R > 0$, $f_\epsilon \to f_*^\tau$ uniformly on $\overline{\mathbb{D}_R(0)}$.*

Proof. Let

$$\hat{h}_\epsilon = h - \log|z-\epsilon|^2 + \log|z+\epsilon|^2 - \log\frac{1-\tau}{1+\tau}, \tag{3.260}$$

we know $\hat{h}$ is smooth with respect to both z and ϵ and $\hat{h}_\epsilon \to 0$ in $C^1(\overline{\mathbb{D}_R(0)})$ as $\epsilon \to 0$. Let $\epsilon > 0$ be a given positive number, by the mean value theorem, for any z there is another $\epsilon' \in (0,\epsilon)$ that may depend on z, such that,

$$f_\epsilon(z) = \partial_\epsilon \hat{h}_\epsilon|_{\epsilon'}(z), \tag{3.261}$$



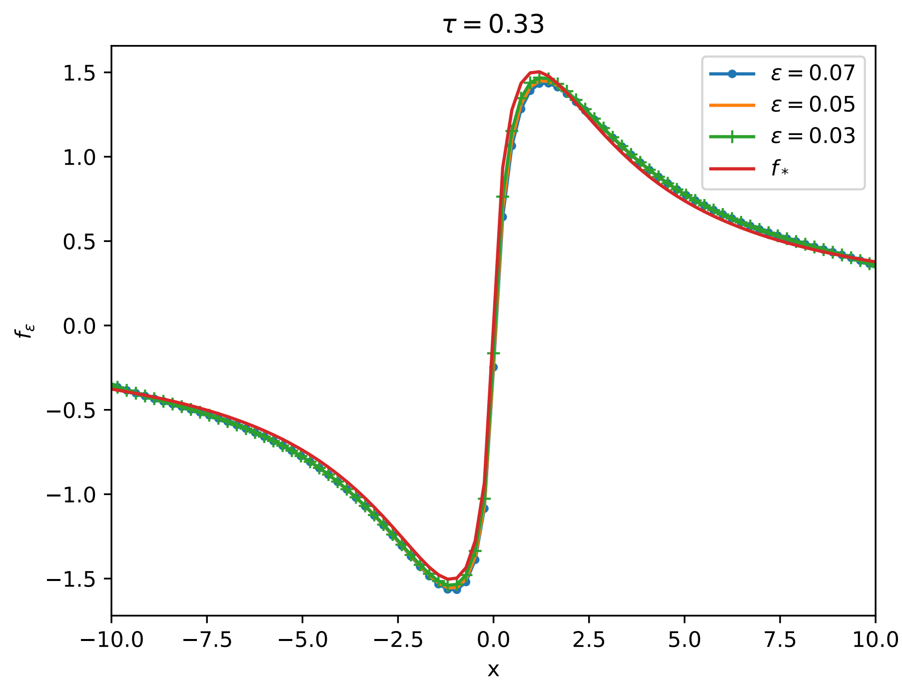


Figure 3.3: **Top.** Real profile of the functions $\tilde{h}_\epsilon - \log(1-\tau)(1+\tau)^{-1}$ converging uniformly to 0 on the real axis. **Bottom.** Real profile of the functions f_ϵ on the real axis and the conjectured asymptotic limit. In both cases, $\tau = 0.33$

hence, to prove the statement it is sufficient to show that $\partial_\epsilon \hat{h}_\epsilon \to f_*^\tau$ uniformly on $\overline{\mathbb{D}_R(0)}$. Since,

$$\partial_\epsilon \hat{h}_\epsilon = \partial_\epsilon h_\epsilon + \frac{2\cos(\theta_\epsilon)}{|z-\epsilon|} + \frac{2\cos(\theta_{-\epsilon})}{|z+\epsilon|}, \tag{3.262}$$

the convergence,

$$\left| \partial_\epsilon \hat{h}_\epsilon - H_\epsilon^\tau - \frac{2\cos(\theta_\epsilon)}{|z-\epsilon|} - \frac{2\cos(\theta_{-\epsilon})}{|z+\epsilon|} \right| \to 0 \tag{3.263}$$

is uniform in $\overline{\mathbb{D}_R(0)}$. Let,

$$\tilde{f}^\tau(z) = 2\cos(\theta)\left(\frac{1}{|z|} - (1-\tau^2)^{1/2} K_1((1-\tau^2)^{1/2}|z|)\right), \tag{3.264}$$

$\tilde{f}^\tau$ is a continuous function defined on the compact set $\overline{\mathbb{D}_R(0)}$, hence, it is equicontinuous, moreover, note that,

$$H_\epsilon^\tau + 2\left(\frac{\cos(\theta_\epsilon)}{|z-\epsilon|} + \frac{\cos(\theta_{-\epsilon})}{|z+\epsilon|}\right) = \tilde{f}^\tau(z-\epsilon) + \tilde{f}^\tau(z+\epsilon), \tag{3.265}$$

since $\tilde{f}^\tau$ is equicontinuous, on $\overline{\mathbb{D}_R(0)}$ we have the uniform convergence,

$$\lim_{\epsilon\to 0}(\tilde{f}^\tau(z-\epsilon) + \tilde{f}^\tau(z+\epsilon)) = 2\,\tilde{f}^\tau(z) = f_*^\tau(z). \tag{3.266}$$

By (3.263) and (3.266) $\partial \hat{h}_\epsilon \to f_*^\tau$ uniformly on $\overline{\mathbb{D}_R(0)}$ as claimed. This concludes the proof of the proposition. □

Numerics together with proposition 3.20 suggest we can extend our claim about uniform convergence to higher order derivatives. In the following, we assume the asymptotic expansion,

$$h_\epsilon = \epsilon f_*^\tau + \log|z-\epsilon|^2 - \log|z+\epsilon|^2 + \log\frac{1-\tau}{1+\tau}, \tag{3.267}$$

is also valid for derivatives of h_ϵ. With this expression, it is possible to derive an asymptotic approximation to the conformal factor for small ϵ as well.

Using the identities

$$K_0' = -K_1, \qquad\qquad K_1' = -K_0 - \frac{1}{x_1}K_1, \tag{3.268}$$

and defining $\nu = (1-\tau^2)^{1/2}$ to shorten the notation, we obtain the approximation

$$\Omega(\epsilon) = 4\pi\nu^2\left(1 + 2\left((2-\nu)K_0(\nu\epsilon) - \nu\epsilon K_1(\nu\epsilon)\right)\right), \tag{3.269}$$

valid for small ϵ.

3.3 Numerical approximation to the metric

To approximate the conformal factor numerically, we define $\tilde{h} = h - \log|x - \epsilon|^2 + \log|x + \epsilon|^2$. $\tilde{h}$ is the solution of the regularised equation,

$$-\Delta\tilde{h} = 2\left(\frac{|x-\epsilon|^2 e^{\tilde{h}} - |x+\epsilon|^2}{|x+\epsilon|^2 e^{\tilde{h}} + |x+\epsilon|^2} + \tau\right), \qquad \lim_{|x|\to\infty} \tilde{h} \; = \log\frac{1-\tau}{1+\tau}. \tag{3.270}$$

Since $\tilde{h}$ is symmetric with respect to the x_1 axis, the regularised Taubes equation was solved with an over-relaxation method on the domain $-10 \leq x_1 \leq 10$, $0 \leq x_2 \leq 10$. The domain was discretized with a square grid of size 0.1 as in [47]. The initial condition was taken as a superposition of an approximated vortex and an antivortex as,

$$\tilde{h}_0 = \log(\rho^2(R_+)) - \log(R_+^2) - \log(\rho^2(R_-)) + \log(R_+^2) - \log(\mu), \tag{3.271}$$

where $\rho = \tanh(0.6r)$, $R_\pm$ is the distance of a point in the grid to $\pm\epsilon$ and $\mu = (1-\tau)(1+\tau)^{-1}$.

The non-trivial term in the metric was computed as,

$$\frac{d}{d\epsilon}\left(\epsilon\, b(\epsilon)\right) = \frac{d}{d\epsilon}\left(\epsilon\partial_1\tilde{h}(\epsilon)\right). \tag{3.272}$$

Figure 3.4 shows the conformal factor for various values of τ. Motivated by the asymptotic approximations, conformal factor data was interpolated by a curve

$$\hat{\Omega} = A + B\, K_0(2\epsilon). \tag{3.273}$$

The interpolation showed to explain $99\% - 96\%$ of the data, depending on the value of τ. As can be seen in the figure, the metric flattens as $\tau \to 1$, preserving the singularity at the origin.

Figure 3.5 shows the short and long range approximations to the conformal factor for the symmetric case and for $\tau = 0.909$. As can be seen in the figure, the approximations are consistent with the data, with the long range approximation slightly better in the range of ϵ that the Taubes equation was solved.

Figure 3.6 shows the Gaussian curvature computed from the conformal factor,

$$K = -\frac{1}{2\epsilon\,\Omega}\frac{d}{d\epsilon}\left(\epsilon\frac{d}{d\epsilon}\log\Omega\right). \tag{3.274}$$

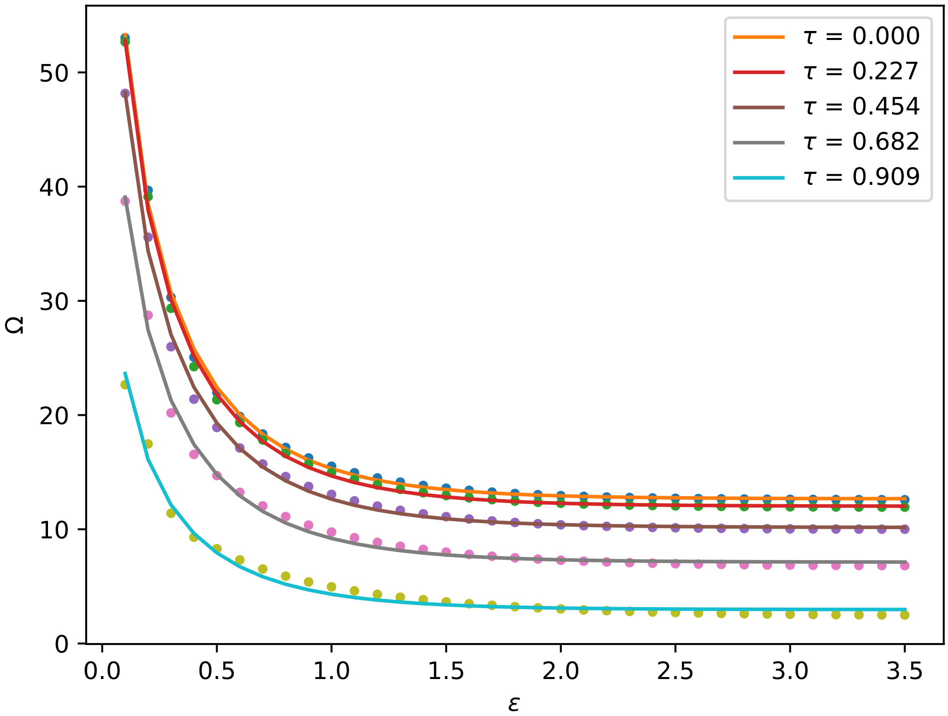


Figure 3.4: Conformal factor of the metric for some values of τ. The graph shows that as τ increases from 0, the metric flattens, maintaining its singularity at the origin.

As can be seen in figure 3.6, the curvature diverges to ∞ as $\epsilon \to 0$, while on the other hand, for large ϵ, it is negative and decays exponentially fast to 0 as $\epsilon \to \infty$. The moduli space can be realised as an embedded surface in $\mathbb{R}^3$, we used proposition 2.3 of [21] to compute the embedding shown in figure 3.7.

Assuming the asymptotic approximations for large and small ϵ, total Gaussian curvature can be shown to be zero, since total curvature is,

$$2\pi \int_0^\infty K(\epsilon)\,\epsilon\,\Omega(\epsilon)\,d\epsilon = -\pi \left(\epsilon\,\frac{\Omega'(\epsilon)}{\Omega(\epsilon)}\right)\Bigg|_0^\infty. \tag{3.275}$$

By (3.161), $\lim_{\epsilon\to\infty}\Omega(\epsilon) = 4\pi(1-\tau^2)$, while

$$\Omega' = 4\pi(1-\tau^2)^{3/2} q_1 q_2 K_1(2(1-\tau^2)^{1/2}\epsilon), \tag{3.276}$$

hence $\lim_{\epsilon\to\infty}\epsilon\,\Omega'\,\Omega^{-1} = 0$ since K_1 decays exponentially. For small ϵ, we know Ω diverges as $|\log\epsilon|$ according to (3.269) while $\epsilon\,\Omega'$ remains bounded, since Ω' diverges as ϵ^{-1}, then $\lim_{\epsilon\to 0}\epsilon\,\Omega'\,\Omega^{-1} = 0$. By (3.275) the total Gaussian curvature in $\mathcal{M}_0^{1,1}$ is 0.

3.3.1 Scattering

In this section we study the scattering of vortex-antivortex pairs in the centre of mass frame. In the centre of mass frame, total momentum is zero and the system preserves energy and angular momentum. For a trajectory on (ϵ, θ) coordinates,

$$\mathrm{E} = \frac{1}{2}\Omega(\epsilon)\,(\dot\epsilon^2 + \epsilon^2\dot\theta^2), \qquad\qquad \ell = \Omega(\epsilon)\,\epsilon^2\,\dot\theta. \tag{3.277}$$

Hence, $\epsilon(t)$ is a solution to the autonomous system,

$$\dot\epsilon = \left(\frac{2\mathrm{E}}{\Omega} - \frac{\ell^2}{\Omega^2\epsilon^2}\right)^{1/2}. \tag{3.278}$$

Equation (3.278) yields a necessary condition for the existence of closed geodesics, if ϵ_0 is the radial position of a closed geodesic,

$$2\mathrm{E}\,\Omega(\epsilon_0)\,\epsilon_0^2 = \ell^2, \tag{3.279}$$

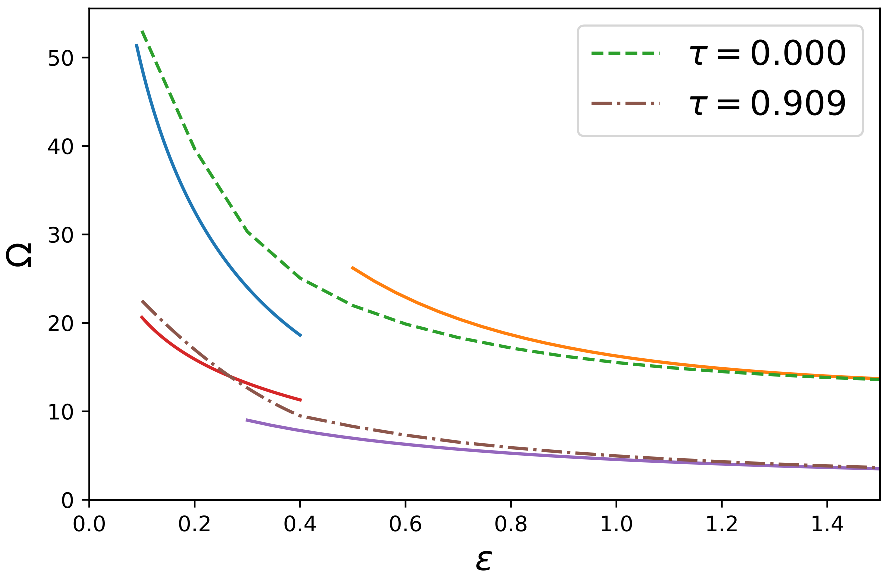


Figure 3.5: Short and long range approximation to the conformal factor for the symmetric case and a highly asymmetric configuration. The graph shows how the approximations fit the numerical data in these cases.

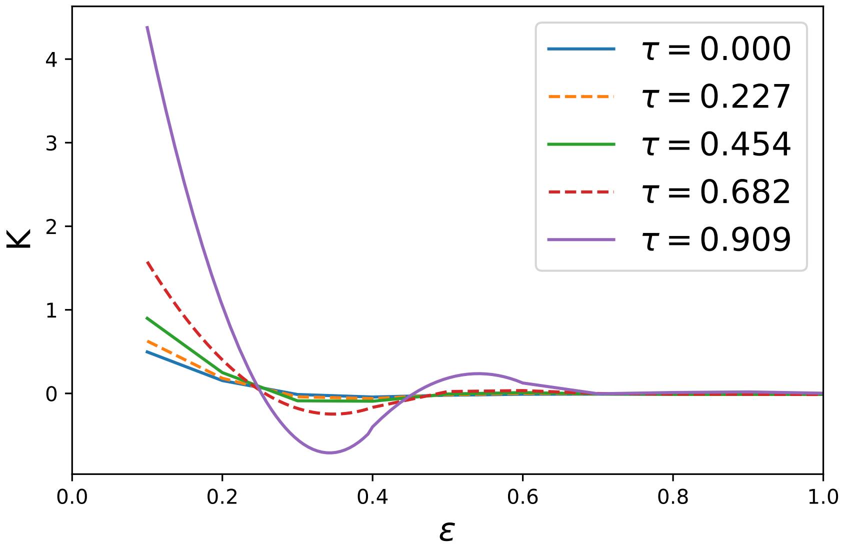


Figure 3.6: Curvature of the conformal factors

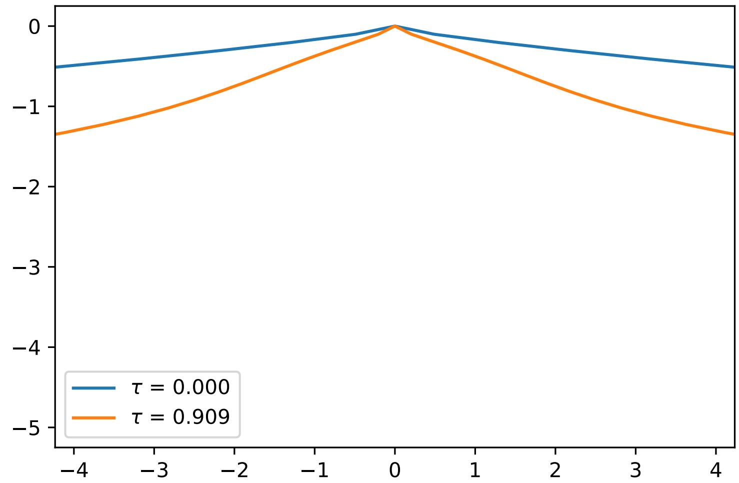


Figure 3.7: The image shows the profile of the moduli space as an embedded revolution surface in $\mathbb{R}^3$. The data shows that the moduli spaces embed as flat disks at infinity, with infinite gaussian curvature at the origin.

however, the right hand side of equation (3.278) is not differentiable at ϵ_0 and therefore the fundamental theorem of existence and uniqueness of solutions of ordinary differential equations is not applicable and (3.279) is not sufficient.

Based on our calculations, we assume for large separations 2ϵ, the conformal factor is approximately constant,

$$\Omega_\infty = 4\pi(1-\tau^2), \tag{3.280}$$

Suppose on the centre of mass frame a vortex moves from very far on the left with initial speed v parallel to the x-axis towards an antivortex. Hence the antivortex seems to move from far on the right towards the vortex with initial speed $v' = (1-\tau)(1+\tau)^{-1}v$. We define our impact parameter a as the distance of the instantaneous initial trajectory of the vortex to the x-axis as shown in the following diagram.

The total energy and angular momentum of the system are,

$$\mathrm{E} = \frac{1}{2}\,\Omega_\infty\, v^2, \qquad\qquad \ell = \Omega_\infty\, a\, v. \tag{3.281}$$

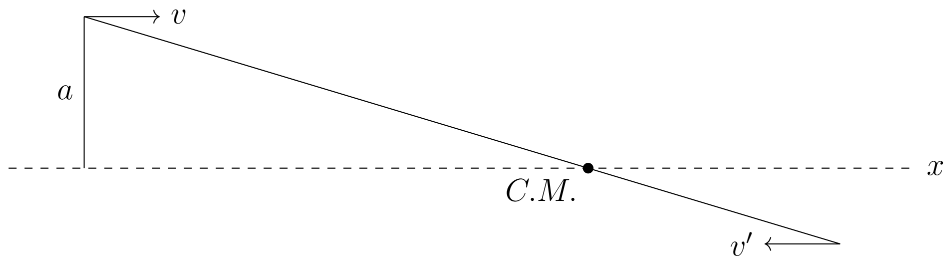


Figure 3.8: Scattering geometry with respect to the centre of mass.

Energy and angular momentum relate as,

$$\mathrm{E} = \frac{1}{2}\frac{\ell^2}{\Omega_\infty a^2}, \tag{3.282}$$

and if we assume θ depends implicitly on time as a function $\theta(\epsilon)$, we have,

$$\frac{d\theta}{d\epsilon} = \frac{\dot\theta}{\dot\epsilon} = \frac{\ell}{\epsilon\,(2\mathrm{E}\,\Omega\,\epsilon^2 - \ell^2)^{1/2}}. \tag{3.283}$$

The total deviation angle of the trajectory, $\Delta\theta$, from the initial time to $\epsilon_{\min}$, at the moment of minimum approach of the pair, therefore is,

$$\Delta\theta = -\int_{\epsilon_{\min}}^{\infty} \frac{d\epsilon}{\epsilon\left(\frac{\Omega\,\epsilon^2}{\Omega_\infty a^2} - 1\right)^{1/2}}. \tag{3.284}$$

As for a classical mechanical system, we define the deflection angle as [16],

$$\Theta = \pi + 2\,\Delta\theta. \tag{3.285}$$

To compute $\epsilon_{\min}$ we used a secant method to solve the equation

$$\Omega(\epsilon)\,\epsilon^2 - \Omega_\infty\,a^2 = 0. \tag{3.286}$$

Then we used the numerical library `scipy` to compute the integral based on the approximation $\hat\Omega$. In practice, we chose a small $\delta\epsilon$ and a value $\epsilon_{\max}$ for which our data showed the conformal factor was almost constant. Then we computed the integral,

$$\Delta\theta_1 = -\int_{\epsilon_{\min}+\delta\epsilon}^{\epsilon_{\max}} \frac{d\epsilon}{\epsilon\left(\frac{\Omega\,\epsilon^2}{\Omega_\infty a^2} - 1\right)^{1/2}}, \tag{3.287}$$

and added the result to

$$\Delta\theta_2 = -\int_{\epsilon_{\max}}^{\infty} \frac{a\,d\epsilon}{\epsilon\,\left(\epsilon^2 - a^2\right)^{1/2}} = -\frac{\pi}{2} + \arctan\left(\frac{(\epsilon_{\max}^2 - a^2)^{1/2}}{a}\right). \tag{3.288}$$

The result of our computations can be seen on figure 3.10. The deflection angle is negative, hence a vortex-antivortex pair behaves as a pair of attractive point particles, however, we would not expect bound orbits because as the impact parameter decreased, the angle also decreased until reaching a minimum, then is started growing again. The behaviour of the scattering angle can be explained based on the approximation (3.273). We assume Ω is a monotonous, decreasing function, such that,

$$\begin{aligned} &\Omega(\epsilon) \geq \Omega_\infty, \\ &\Omega(\epsilon) \approx -C\log\epsilon, \qquad \epsilon << 1. \end{aligned} \tag{3.289}$$

where $C > 0$ is some constant, and such that there are positive constants C_1, C_2 such that,

$$-C_1 \leq \Omega'(\epsilon)\,\epsilon < 0, \qquad\qquad 0 < \Omega''(\epsilon)\,\epsilon^2 \leq C_2. \tag{3.290}$$

Note that the approximation $\hat{\Omega}$ and the asymptotic approximations for small and large ϵ are consistent with these assumptions. Since for small ϵ,

$$\frac{d}{d\epsilon}\left(\Omega(\epsilon)\,\epsilon^2\right) = \left(\Omega'(\epsilon)\,\epsilon + 2\,\Omega(\epsilon)\right)\epsilon > 0, \tag{3.291}$$

with these assumptions, there is a continuous bijection between small impact parameters a and solutions $\epsilon_{\min}$ to the equation,

$$\Omega(\epsilon)\,\epsilon^2 = \Omega_\infty\,a^2. \tag{3.292}$$

If we use the approximation $\hat{\Omega}$ instead of Ω, this is actually a global bijection valid for any $a > 0$. We aim to show that,

$$\lim_{a\to 0} \Delta\theta = -\frac{\pi}{2}, \tag{3.293}$$

where $\Delta\theta$ is the integral (3.284). From now onwards we denote $\epsilon_{\min}$ as m to shorten the following computations. With the change of variables $u = \epsilon/m$, the integral transforms into,

$$\Delta\theta = -\int_1^\infty \frac{du}{u\left(\frac{\Omega(m\cdot u)\, m^2\, u^2}{\Omega_\infty\, a^2} - 1\right)^{1/2}}$$
$$= -\int_1^\infty \frac{du}{u\left(\frac{\Omega(m\cdot u)}{\Omega(m)}\, u^2 - 1\right)^{1/2}}, \tag{3.294}$$

where in the last step we used (3.284). By (3.289), for any $u \geq 1$, we have pointwise convergence,

$$\lim_{m\to 0} \frac{\Omega(m\cdot u)}{\Omega(m)} = 1. \tag{3.295}$$

To compute the integral by the dominated convergence theorem, we need to exhibit a function integrable in $[1,\infty)$ and bigger than each of the functions in the integrand of (3.294). To this end, let us define the function

$$f(u) = \frac{\Omega(m\cdot u)}{\Omega(m)}\, u^2 - 1, \tag{3.296}$$

as a short cumputation shows,

$$f'(1) = 2 + \frac{m\,\Omega'(m)}{\Omega(m)} \tag{3.297}$$
$$f''(u) = \frac{1}{\Omega(m)}\left(2\,\Omega(m\cdot u) + 4m\,\Omega'(m\cdot u)\,u + m^2\,\Omega''(m\cdot u)\,u^2\right). \tag{3.298}$$

Assume $f''(u) \geq 0$ for any $u \geq 1$. By Taylor's theorem, for any $u > 1$, there is some $\xi \in (1,u)$, such that

$$f(u) = f'(1)\,(u-1) + \frac{1}{2} f''(\xi)(u-1)^2 > f'(1)(u-1). \tag{3.299}$$

Since $f'(1) > 2$ for any $m > 0$, we deduce,

$$\int_1^\infty \frac{du}{u\left(\frac{\Omega(m\cdot u)}{\Omega(m)}\, u^2 - 1\right)^{1/2}} = \int_1^\infty \frac{du}{u\, f(u)^{1/2}}$$
$$< \frac{1}{\sqrt{2}} \int_1^\infty \frac{du}{u\,(u-1)^{1/2}}$$
$$= \frac{\pi}{\sqrt{2}}. \tag{3.300}$$

Hence, by the dominated convergence theorem,

$$\begin{aligned}\lim_{a\to 0}\Delta\theta &= -\lim_{m\to 0}\int_1^\infty \frac{du}{u f(u)^{1/2}}\\ &= -\int_1^\infty \frac{du}{u\,(u^2-1)^{1/2}}\\ &= -\frac{\pi}{2},\end{aligned} \tag{3.301}$$

provided $f''(u)$ is non-negative, or equivalently by (3.298), if

$$2\Omega(x) + 4x\Omega'(x) + x^2\Omega''(x) \geq 0, \tag{3.302}$$

for all $x > 0$. By the asymptotic properties of Ω, we know this is the case for small and large x, which shows it is sensible to assume this is the case, at least for not very large τ, as figure 3.9 shows.

Therefore, the total deflection satisfies,

$$\lim_{a\to 0}\Theta = \pi + 2\lim_{a\to 0}\Delta\theta = 0, \tag{3.303}$$

as shown in figure 3.10. Finally, equation (3.300) shows $\Delta\theta > -\pi/\sqrt{2}$ at least up to some τ, hence, the data suggests the lower bound,

$$\Theta > -(1-\sqrt{2})\pi \approx -74.5^\circ, \tag{3.304}$$

as can be seen in the figure.

Scattering at large separation

We also approximated the scattering angle of a vortex-antivortex pair at large separation with the method Manton and Speight [39]. Suppose $x(s)$ is a geodesic in Cartesian coordinates, with initial position $x(0)$, such that $a = x_2(0)$ is very big, $x_1(0) \ll 0$ and the initial velocity is $\dot{x}(0) = v\,\partial_1$. The geodesic equation for x_2 is

$$\ddot{x}_2 + \frac{\Omega'}{\Omega}\left(\dot{\epsilon}\dot{x}_2 - \frac{x_2}{2\epsilon}(\dot{x}_1^2 + \dot{x}_2^2)\right) = 0. \tag{3.305}$$

Since a is big, the metric is almost flat across the trajectory of the geodesic, the small deflection in the x_2 axis is caused by the small correction on $\dot{x}_2$ due to

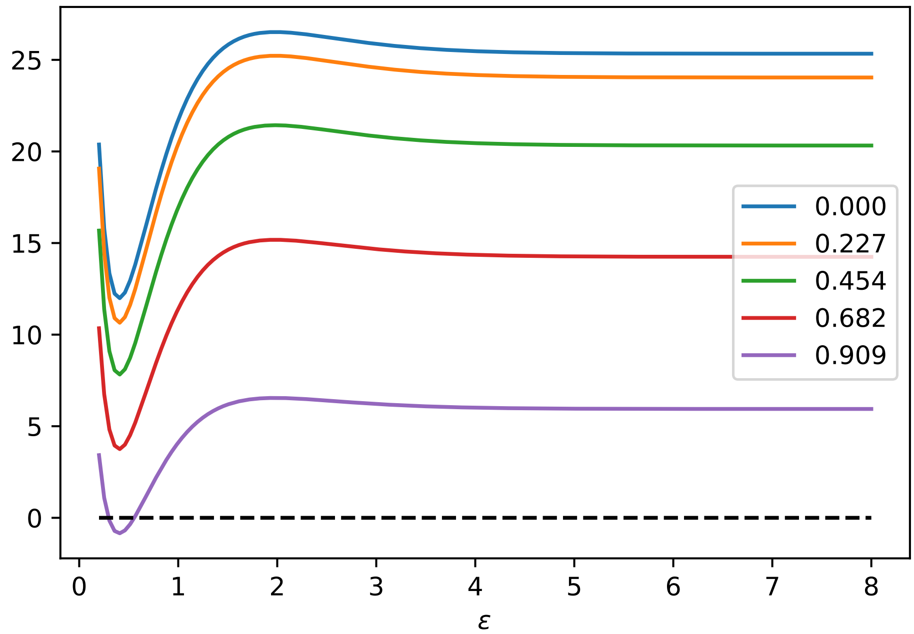


Figure 3.9: The graph shows the function $2\hat{\Omega}(x) + 4x\hat{\Omega}'(x) + x^2\hat{\Omega}''(x)$ for various values of τ, where $\hat{\Omega}(x) = A\,K_0(2x) + B$, and the coefficients A, B are chosen such that $\hat{\Omega}$ interpolates the values of Ω computed solving Taube's equation. The data shows equation (3.302) is expected to hold for τ up to some value $\tau_{\max}$, implying the deflection angle converges to 0 as the impact parameter decreases.

the conformal factor derivative. To leading order, Ω is constant but we take Ω' varying as in the long range approximation.

$$\frac{\Omega'}{2\Omega} = \frac{1}{2}(1-\tau^2)^{1/2} q_1 q_2 K_1(2(1-\tau^2)^{1/2}\epsilon). \tag{3.306}$$

We approximate x_2 and $\dot{x}_1$ as constants, $\dot{x}_2$ as a small number, such that the leading order term for $\ddot{x}_2$ is,

$$\ddot{x}_2 = \frac{\Omega'}{2\Omega}\frac{av^2}{\epsilon}. \tag{3.307}$$

For big a the deflection is small, the deviation angle can be approximated as

$$\Theta = \frac{\Delta \dot{x}_2}{v}. \tag{3.308}$$

The difference in $\dot{x}_2$ is,

$$\Delta \dot{x}_2 = \int_{-\infty}^{\infty} \frac{\Omega'}{2\Omega}\frac{av^2}{\epsilon} ds = av \int_{-\infty}^{\infty} \frac{\Omega'}{2\Omega\,\epsilon} dx_1. \tag{3.309}$$

Hence,

$$\Theta = \frac{a}{2}(1-\tau^2)^{1/2} q_1 q_2 \int_{-\infty}^{\infty} \frac{K_1(2(1-\tau^2)^{1/2}\,\epsilon)}{\epsilon} dx_1. \tag{3.310}$$

Recall $\epsilon = (a^2 + x_1^2)^{1/2}$ and let us make the change of variables

$$u = (1-\tau^2)^{1/2} x_1, \qquad a_\tau = (1-\tau^2)^{1/2} a. \tag{3.311}$$

The deflection angle is

$$\Theta = \frac{a_\tau}{2} q_1 q_2 \int_{-\infty}^{\infty} \frac{K_1(2\,(a_\tau^2 + u^2)^{1/2})}{(a_\tau^2+u^2)^{1/2}} du \tag{3.312}$$

$$= -\frac{q_1 q_2}{4}\frac{d}{da_\tau}\int_{-\infty}^{\infty} K_0(2\,(a_\tau^2+u^2)^{1/2}) du \tag{3.313}$$

The last integral was calculated in [39], using their result, the deflection angle is,

$$\Theta = -\frac{q_1 q_2}{4}\frac{d}{da_\tau}\left(\frac{\pi}{2}\exp(-2a_\tau)\right) = q_1 q_2\, \frac{\pi}{4}\, \exp(-2a_\tau). \tag{3.314}$$

The constant $q_1 q_2$ is negative, hence, the geodesic are slightly deflected towards the origin, which indicates a vortex-antivortex pair behaves as a pair of attractive particles in the long distance approximation. On figure 3.10 we can see the large distance approximation fits the scattering data for the symmetric case. Since $\Theta \to 0$ as $a \to 0$, the fact that for large a, Θ is negative explains the existence of a minimum negative deflection as seen in figure 3.10.

3.4 Ricci magnetic geodesic motion

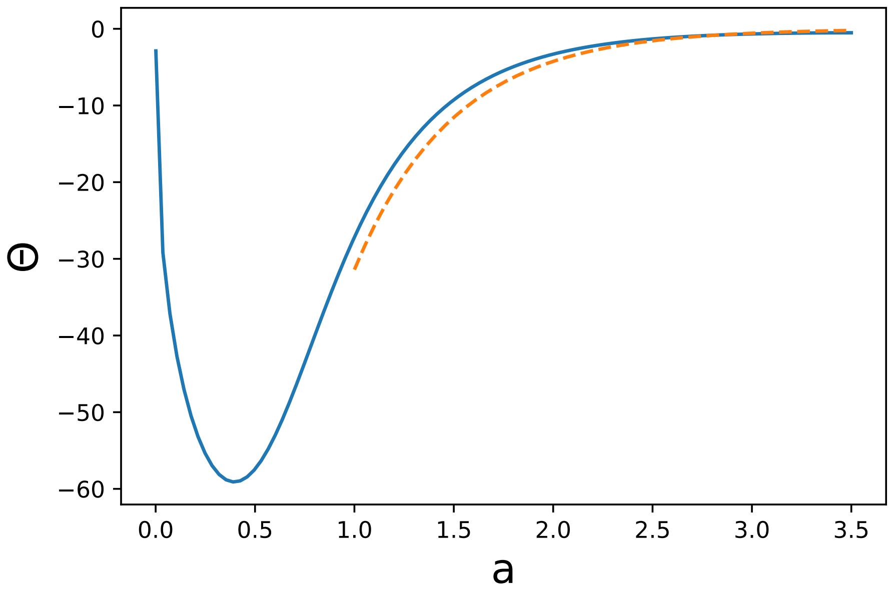
The metric on $\mathcal{M}^{1,1}(\mathbb{R}^2)$ can be split isometrically in a product with one flat term isometric to $\mathbb{R}^2$, the centre of mass coordinate. Since this term is flat, in the reduced moduli space we have that the global Ricci tensor coincides with the Ricci tensor as a Riemann surface. Therefore, the Ricci form in $\mathcal{M}_0^{1,1}(\mathbb{R}^2)$ is the restriction of the global Ricci form to the centre of mass frame,

$$\rho = K\, \epsilon\, d\epsilon \wedge d\theta, \tag{3.315}$$

where K is the Gauss curvature of the reduced moduli space. Interaction of vortices with a magnetic field can be modelled by means of Ricci magnetic geodesics, abbreviated RMGs. RMGs on the moduli space were introduced for the Ginzburg-Landau model with a Chern-Simons term by Collie and Tong [10], who proposed that the Ricci form was the magnetic form of the Chern-Simons term. Later, mathematical properties of RMGs were investigated by Krusch-Speight on hyperbolic space [28]. Although in our case RMG dynamics is not physically motivated, these curves are of mathematical interest: Krusch-Speight conjectured that geodesic completeness and RMG completeness were equivalent until Alqahtani-Speight found examples of incomplete surfaces which are RMG complete [1]. A curve γ is a Ricci magnetic geodesic if there is a constant scalar λ such that,

$$\nabla_{\dot\gamma}\dot\gamma = \lambda\, (\iota_{\dot\gamma}\rho)^\sharp, \tag{3.316}$$

where $\iota_{\dot\gamma}\rho = \rho(\dot\gamma, \cdot)$ is the interior product. Unlike geodesic flow, RMG trajectories are speed dependent, with changes in initial speed being reflected in the constant



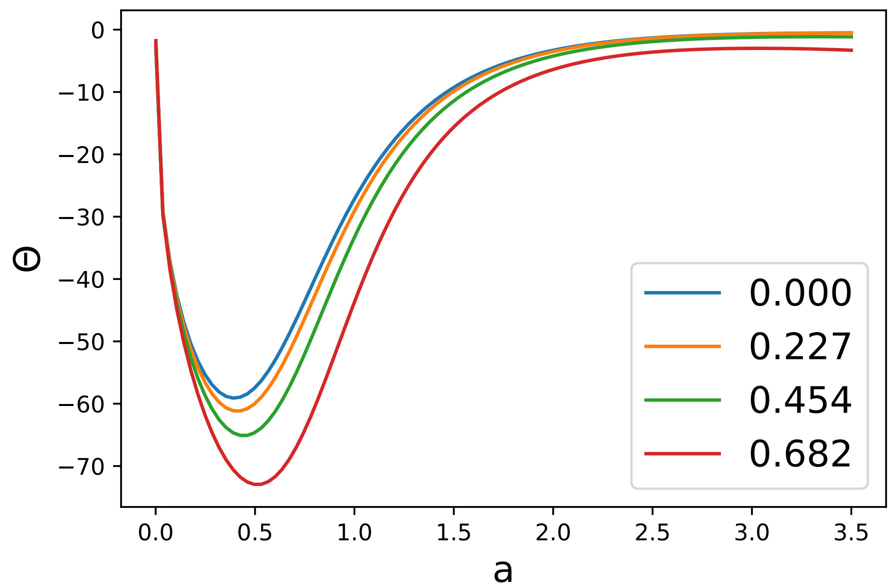


Figure 3.10: Above. Deflection angle at $\tau = 0$ and asymptotic approximation. Below. Comparison of the deflection angle for different values of τ.

λ. On a surface of revolution, RMG equations are determined by the Lagrangian,

$$\mathrm{L} = \frac{1}{2}\Omega(\dot{\epsilon}^2 + \epsilon^2\dot{\theta}^2) + \frac{\lambda}{2}\left(\frac{\epsilon\,\Omega'}{\Omega}\right)\dot{\theta}. \tag{3.317}$$

This is a conservative Lagrangian symmetric with respect to translations in time and rotations of space, therefore, RMG trajectories on the reduced moduli space preserve energy and angular momentum,

$$\mathrm{E} = \frac{1}{2}\Omega\,(\dot{\epsilon}^2 + \epsilon^2\dot{\theta}^2), \qquad \ell = \Omega\,\epsilon^2\dot{\theta} + \frac{\lambda}{2}\frac{\epsilon\,\Omega'}{\Omega}. \tag{3.318}$$

Eliminating $\dot{\theta}$ from these equations, a RMG is a solution to the first order equation,

$$\mathrm{E} = \frac{1}{2}\,\Omega\dot{\epsilon}^2 + \mathrm{V}_{eff}, \tag{3.319}$$

where the effective potential is defined as,

$$\mathrm{V}_{eff} = \frac{1}{2\epsilon^2\,\Omega}\left(\ell - \lambda\,\frac{\epsilon\,\Omega'}{2\,\Omega}\right)^2. \tag{3.320}$$

Figure 3.11 shows V_{eff} for several values of $\tau = 0$. Data confirms $\mathrm{V}_{eff} \to \infty$ as $\epsilon \to 0$, consistently with the asymptotic approximation to the conformal factor, likewise, for $\epsilon \to \infty$, $\mathrm{V}_{eff} \to 0$ since $\Omega \to \Omega_\infty$ and $\Omega' \to 0$. The effective potential can be seen in figure 3.11, the shape depends on the relative value of ℓ/λ. A large computation reveals

$$\mathrm{V}'_{eff} = \frac{-1}{2\epsilon^3\Omega}\left(\ell - \lambda\frac{\epsilon\,\Omega'}{2\Omega}\right)\left(\frac{\lambda}{2}\left(\frac{\epsilon^2\Omega''}{\Omega} - 3\frac{\epsilon^2\,\Omega'^2}{\Omega^2}\right) + 2\ell\left(1 + \frac{\epsilon\,\Omega'}{2\Omega}\right)\right), \tag{3.321}$$

by virtue of the asymptotic approximations, both $\epsilon\Omega'$ and $\epsilon^2\Omega''$ are bounded functions, while Ω is positive and bounded below, hence for given λ if $|\ell|$ is large, V_{eff} is a positive decreasing function.

In this case RMGs are all unbounded curves. If ℓ is not very large, V_{eff} has relative extrema, giving rise to both unbounded and bounded trajectories orbiting around the singularity at $\epsilon = 0$. By equations (3.318) and (3.320), trajectories for which $\mathrm{E} = \mathrm{V}_{eff}$ at constant ϵ_0 are circular if $\mathrm{V}_{eff}(\epsilon_0) \neq 0$ or constant if $\mathrm{V}_{eff}(\epsilon_0) = 0$. If the perturbation is around a zero of V_{eff}, the angular velocity

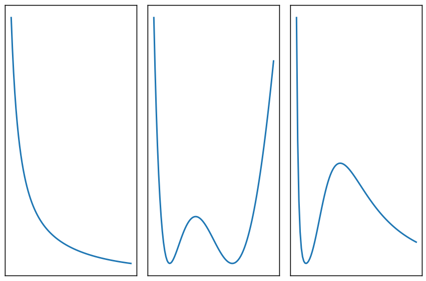


Figure 3.11: Typical types of effective potentials for $\tau = 0$ (description in text). In the three cases, $\lambda = 1$, in the second case, $V_{eff} \to 0$ as $\epsilon \to \infty$ although is not apparent in the figure because of the scale.

alternates sign, the pattern is as seen on the bounded curves on the first row of figure 3.12. If the perturbation is around a local minimum of V_{eff} which is not a zero, the angular velocity keeps the same sign and gives rise to the patterns seen on the second row of the figure.

As numerics show, the moduli space is RMG complete, even though it is geodesically incomplete, because the divergence of V_{eff} at the origin prevents RMGs of hitting the singularity.

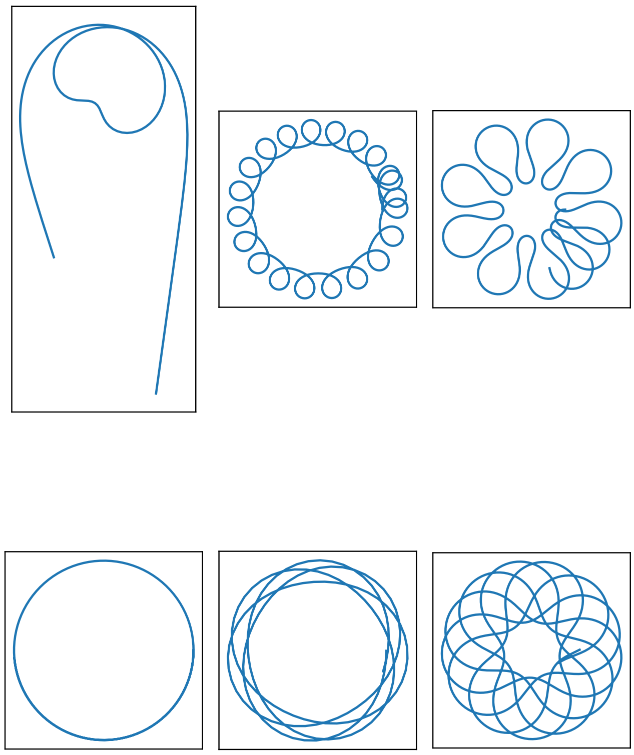


Figure 3.12: RMGs for $\tau = 0$ (Description in text).

Chapter 4

Asymmetric vortex-antivortex pairs on a compact surface

In this chapter we study vortex-antivortex systems on a compact surface. We aim to prove that the moduli space is incomplete and to compute the volume of the moduli space for the round sphere and flat tori. On a general compact domain, the problem of the statistical mechanics of Ginzburg-Landau vortices was addressed by Manton [37] and by Manton-Nasir [38]. As shown in [37], it can be described if we know the volume of the moduli space. For the abelian $O(3)$ Sigma model however, the problem of the volume of the moduli space is constrained by the fact that vortices and antivortices cannot coalesce, however, computing the volume is necessary for the partition function of a gas of BPS vortices [37, 38, 45]. There is a conjectured formula for the volume by Speight and Romão that depends on topological data, the volume of the domain, τ and the size of the sets P, Q of core positions [45]. The content of the chapter is as follows.

In section 4.1, we prove that the Taubes equation has exactly one solution for any $\tau \in (-1, 1)$.

The main result of section 4.2 is theorem 4.15 which asserts that the moduli space of vortex-antivortex pairs is incomplete. We prove the theorem after proving several lemmas necessary to bound the derivatives of solutions to the Taubes equation.

In section 4.3 we compute the volume of the moduli space of vortex-antivortex pairs for the round sphere and flat tori and compare our results with the conjecture.

4.1 Existence of vortices

In this section we will prove the existence of solutions to the Taubes equation on a compact surface. In [51] Sibner-Signer-Yang proved existence and uniqueness of solutions of the gauged $O(3)$ Sigma model on a compact manifold for $\tau = 0$. We prove the following generalisation of their results.

Theorem 4.1. *On any compact Riemann surface there exists exactly one solution u to the Taubes equation* (2.36)*, provided the condition*

$$-\frac{1+\tau}{2\pi}|\Sigma| < k_+ - k_- < \frac{1-\tau}{2\pi}|\Sigma| \tag{4.1}$$

holds. Moreover, u is of class C^2 except for the core positions.

We prove the theorem at the end of the section. The inequality (4.1) is a Bradlow type restriction [3], constraining the relative number of vortices and antivortices on a compact surface. It arises naturally from the second Bogomolny equation (2.21), since the total magnetic flux is,

$$\begin{aligned} 2\pi(k_+ - k_-) &= \int_\Sigma B \\ &= \int_\Sigma \langle N, \phi\rangle \,\mathrm{Vol} - \tau\,|\Sigma|, \end{aligned} \tag{4.2}$$

where N is the north pole section on the target sphere and hence $\langle N, \phi\rangle \in [-1, 1]$, it follows that (4.1) is a necessary condition for a pair (ϕ, A) of a field and a connection to be a solution to the Bogomolny equations.

We will define the function $F : \mathbb{R} \to \mathbb{R}$,

$$F(t) = 2\left(\frac{e^t - 1}{e^t + 1} + \tau\right), \tag{4.3}$$

and the constant,

$$F^{\pm\infty} = 2(\pm 1 + \tau), \tag{4.4}$$

in order to simplify notation in the proof of theorem 4.1. Let us define $F_0 : \mathbb{R} \to \mathbb{R}$ as the function,

$$\begin{aligned} F_0(t) &= 2\left(\frac{e^t-1}{e^t+1}+\tau\right)+\frac{4\pi(k_+-k_-)}{|\Sigma|} \\ &= \frac{4e^t}{e^t+1}-C_0, \end{aligned} \tag{4.5}$$

where the constant C_0 is,

$$C_0 = 2(1-\tau) - \frac{4\pi}{|\Sigma|}(k_+ - k_-). \tag{4.6}$$

For a given configuration of non-coalescent vortices, recall the function $v : \Sigma \to \mathbb{R} \cup \{\pm\infty\}$, defined on equation (2.165), if u is the solution of the Taubes equation, and we define $\tilde{h} = u - v$, then the regularized Taubes equation on a compact surface, equation (2.167), is equivalent to,

$$-\Delta \tilde{h} = F_0(v + \tilde{h}). \tag{4.7}$$

Equation (4.7) shows why Bradlow's bound is necessary: If a smooth solution exists, by the divergence theorem a necessary condition for C_0 is,

$$C_0 = \frac{1}{|\Sigma|}\int_\Sigma \frac{4\,e^{v+\tilde{h}}}{e^{v+\tilde{h}}+1}\,\mathrm{Vol} \in [0,4], \tag{4.8}$$

Bradlow's bound is equivalent to (4.8). Let

$$\mathcal{X} = \left\{ u \in \mathrm{H}^1(\Sigma)\ :\ \int_\Sigma u\,\mathrm{Vol} = 0 \right\} \tag{4.9}$$

be the subspace of Sobolev's space $\mathrm{H}^1(\Sigma)$ of functions of zero average. Since Σ is compact, $\mathrm{H}^1(\Sigma)$ can be decomposed as

$$\mathrm{H}^1(\Sigma) = \mathcal{X} \oplus \mathbb{R}. \tag{4.10}$$

Any $h \in \mathrm{H}^1(\Sigma)$ can be decomposed as a pair $(u, \tilde{c}) \in \mathcal{X} \times \mathbb{R}$, such that $h = u + \tilde{c}$. Hence, u is a solution to the equation,

$$-\Delta u = F_0(v + u + \tilde{c}). \tag{4.11}$$

We will use Leray-Schauder theory to prove existence of solutions to the Taubes equation as in the proof of Sibner et al. [51] for $\tau = 0$. Given $\tilde{h} \in \mathcal{X}$, the function

$$c \mapsto \int_\Sigma F_0(v + \tilde{h} + c)\,\mathrm{Vol}, \tag{4.12}$$

is a well defined, monotonous, continuous function. By Bradlow's bound, there exists a unique number $\tilde{c}$ such that

$$\int_\Sigma F_0(v + \tilde{h} + \tilde{c})\,\mathrm{Vol} = 0. \tag{4.13}$$

Lemma 4.2. *The function $\mathcal{C} : \mathcal{X} \to \mathbb{R}$, $\mathcal{C}(\tilde{h}) = \tilde{c}$ is weakly sequentially continuous in $\mathcal{X}$.*

Proof. We will highlight the steps different from [51] in the general case. If $\tilde{h}_n \rightharpoonup \tilde{h}_0$ in $\mathcal{X}$, then $\tilde{h}_n$ is a bounded sequence in $\mathcal{X}$, and by the Rellich lemma, after passing to a sub-sequence if necessary, we can assume $\tilde{h}_n \to \tilde{h}_0$ in L^p for $p \geq 1$. Let $\tilde{c}_n = \tilde{c}(\tilde{h}_n)$, $\tilde{c}_0 = \tilde{c}(\tilde{h}_0)$ and assume towards a contradiction that $\tilde{c}_n$ does not converge to $\tilde{c}_0$. In this case we can assume the existence of a constant ϵ_0 such that,

$$|\tilde{c}_n - \tilde{c}_0| \geq \epsilon_0, \tag{4.14}$$

for all n. We claim the sequence $\{\tilde{c}_n\}$ is bounded. Assume the contrary, after passing to a sub-sequence if necessary, we can assume the limit $\tilde{c}_n \to \infty$. Let K be any bound for F_0. By Egorov's theorem [30] and the strong convergence in L^p, there exists a measurable set Σ_ϵ and a constant K_ϵ, such that $|\Sigma_\epsilon| < \epsilon K^{-1}$, the sequence $\tilde{h}_n$ converges uniformly to $\tilde{h}_0$ in $\Sigma \setminus \Sigma_\epsilon$ and $|\tilde{h}_n| \leq K_\epsilon$ in $\Sigma \setminus \Sigma_\epsilon$.

On the one hand, the equality

$$\int_{\Sigma\setminus\Sigma_\epsilon} F_0(v + \tilde{c}_n + \tilde{h}_n)\,\mathrm{Vol} = -\int_{\Sigma_\epsilon} F_0(v + \tilde{c}_n + \tilde{h}_n)\,\mathrm{Vol}, \tag{4.15}$$

implies,

$$\left|\int_{\Sigma\setminus\Sigma_\epsilon} F_0(v + \tilde{c}_n + \tilde{h}_n)\,\mathrm{Vol}\right| \leq \epsilon, \tag{4.16}$$

and on the other hand, by monotony of F_0,

$$\int_{\Sigma\setminus\Sigma_\epsilon} F_0(v + \tilde{c}_n - K_\epsilon)\,\mathrm{Vol} \leq \int_{\Sigma\setminus\Sigma_\epsilon} F_0(v + \tilde{c}_n + \tilde{h}_n)\,\mathrm{Vol}. \tag{4.17}$$

Taking the limit as $n \to \infty$, from these two equations we have,

$$(F^{\infty} - C_0)(|\Sigma| - |\Sigma_{\epsilon}|) \leq \epsilon. \tag{4.18}$$

Hence,

$$(F^{\infty} - C_0)|\Sigma| \leq \epsilon + K|\Sigma_{\epsilon}| < 2\epsilon, \tag{4.19}$$

a contradiction since ϵ is arbitrary. A similar argument shows $\tilde{c}_n$ is bounded below. Therefore, $\tilde{c}_n$ is a bounded sequence of real numbers. By the Bolzano-Weierstrass theorem, we can assume towards a contradiction $\tilde{c}_n \to \tilde{c}$, but $\tilde{c} \neq \tilde{c}_0$ by (4.14). Let

$$\alpha = \left| \int_{\Sigma} F_0(v + \tilde{h}_0 + \tilde{c}) \,\mathrm{Vol} \right| > 0, \tag{4.20}$$

bearing in mind the definition of $\tilde{h}_n$,

$$\begin{aligned} \alpha &= \left| \int_{\Sigma} F_0(v + \tilde{h}_0 + \tilde{c}) - F_0(v + \tilde{h}_n + \tilde{c}_n) \,\mathrm{Vol} \right| \\ &\leq \sup_{t \in \mathbb{R}} \{F'(t)\} \cdot \left(|\tilde{c} - \tilde{c}_n| \cdot |\Sigma| + C \,||\tilde{h}_0 - \tilde{h}_n||_0 \cdot |\Sigma|^{1/2} \right) \to 0. \end{aligned} \tag{4.21}$$

Hence $\alpha = 0$, a contradiction. Therefore (4.14) is false and $\tilde{c}_n \to \tilde{c}_0$. This proves the lemma. □

Let us consider the operator $T : \mathfrak{X} \to \mathfrak{X}$, mapping each $\tilde{h} \in \mathfrak{X}$ to the weak solution $H \in \mathfrak{X}$ of the equation

$$-\Delta H = F_0(v + \tilde{c} + \tilde{h}). \tag{4.22}$$

Given that $\int_{\Sigma} F_0(v + \tilde{c} + \tilde{h}) \,\mathrm{Vol} = 0$, existence of a weak H^1 solution to (4.22) is a well established analysis fact [2, Thm. 4.7], moreover, any two weak solutions to the equation differ by a constant, by taking $H \in \mathfrak{X}$ we guarantee it is unique.

Recall a compact operator is an operator that maps bounded sequences to sequences with convergent subsequences. We aim to use Schäfer's alternative, theorem 2.17, to prove T has a fixed point.

Lemma 4.3. *The operator $T : \mathfrak{X} \to \mathfrak{X}$ is compact in the strong topology of $\mathfrak{X}$ as a subspace of $\mathrm{H}^1(\Sigma)$.*

Proof. Let $\{\tilde{h}_n\} \subset \mathcal{X}$ be a bounded sequence, after passing to a subsequence if necessary, we can assume $\tilde{h}_n \rightharpoonup \tilde{h}_0$ in $\mathcal{X}$ and strongly in L^2. Let $H_n = T\tilde{h}_n$, $n \geq 0$, by lemma 4.2 $\tilde{c}_n \to \tilde{c}_0$. Moreover,

$$\begin{aligned} ||\nabla H_n - \nabla H_0||^2_{\mathrm{L}^2} &= \int_\Sigma (H_n - H_0)\, \Delta(H_n - H_0) \,\mathrm{Vol} \\ &= \int_\Sigma (H_n - H_0) \left(F(v + \tilde{c}_n + \tilde{h}_n) - F(v + \tilde{c}_0 + \tilde{h}_0) \right) \mathrm{Vol} \\ &\leq \sup_{t\in\mathbb{R}} \{F'(t)\} \int_\Sigma \left(|\tilde{c}_n - \tilde{c}_0| + |\tilde{h}_n - \tilde{h}_0| \right) |H_n - H_0| \,\mathrm{Vol} \\ &\leq \sup_{t\in\mathbb{R}} \{F'(t)\} \left(|\tilde{c}_n - \tilde{c}_0| \cdot |\Sigma|^{1/2} + ||\tilde{h}_n - \tilde{h}_0||_{\mathrm{L}^2} \right) ||H_n - H_0||_{\mathrm{L}^2}. \end{aligned} \tag{4.23}$$

The last inequality is a consequence of the Cauchy-Schwarz inequality. By the Poincaré inequality, there are constants C_1, C_2 such that

$$||H_n - H_0||_{\mathrm{H}^1} \leq C_1 |\tilde{c}_n - \tilde{c}_0| + C_2 ||\tilde{h}_n - \tilde{h}_0||_{\mathrm{L}^2} \to 0. \tag{4.24}$$

This proves compactness of T. □

Let us consider the set

$$S = \left\{ \tilde{h} \in \mathcal{X} \,:\, \exists\, t \in [0,1] \; s.t. \; \tilde{h} = t \cdot T\,\tilde{h} \right\}. \tag{4.25}$$

If $\tilde{h} \in S$, then it is a solution of the equation,

$$\Delta \tilde{h} = tF_0(v + \tilde{c} + \tilde{h}), \tag{4.26}$$

where $\tilde{c} = \mathcal{C}(\tilde{h})$ was defined on lemma 4.2.

By the Cauchy-Schwarz inequality,

$$||\nabla \tilde{h}_t||^2_{\mathrm{L}^2} = \langle \tilde{h}_t, \Delta \tilde{h}_t \rangle \leq C \int_\Sigma |\tilde{h}_t| \,\mathrm{Vol} \leq C\, |\Sigma|^{1/2}\, ||\tilde{h}_t||_{\mathrm{L}^2}. \tag{4.27}$$

By the Poincaré inequality we conclude the existence of a constant C such that

$$||\tilde{h}_t||_{\mathrm{H}^1} \leq C. \tag{4.28}$$

Proof of Theorem 4.1. Since S is bounded, by Schäfer's alternative there is a fixed point $\tilde{h}$ of T. Let $h = \tilde{h} + \tilde{c}$, where $\tilde{c} = \mathcal{C}(\tilde{h})$, then h is a weak solution to the regularised Taubes equation. By the elliptic estimates h is also a strong solution in H^2. We follow a bootstrap argument to prove $h \in C^2$: By Sobolev's embedding we know h is continuous, hence $h \in \mathrm{L}^p$ for any $p \geq 1$. By (4.7) and the elliptic estimates $h \in \mathrm{W}^{2,p}$ for some $p > 2$, once more by Sobolev's embedding $h \in C^1$. Let $u = h + v$, the derivative $dh \in \Gamma(T^*\Sigma)$ is a weak solution of the linearized equation,

$$-\Delta\, dh = \frac{4\, e^u}{(e^u + 1)^2}\, dh + \frac{4\, e^u}{(e^u + 1)^2}\, dv. \tag{4.29}$$

The potential function $e^u(e^u + 1)^{-2}$ is continuous and with zeros of the same order than the singularities of dv at the cores, hence $\Delta(dh) \in \mathrm{L}^p$, $p > 2$. Since dh is continuous, it is also an L^p form. By the elliptic estimates and Sobolev's embedding we conclude $h \in C^2$. Since F is monotonous, h is unique by the strong maximum principle. Finally, u is the necessarily unique solution to the Taubes equation. □

4.2 Incompleteness of the moduli space

In [45] Romão and Speight prove that the moduli space of symmetric vortex-antivortex pairs on the sphere is incomplete. In this section we extend their result to general τ on a compact manifold. In order to prove this, we find bounds for the derivatives $\partial_{z_j} \nabla h_\epsilon$ on a holomorphic chart, where the cores are at positions z_1, z_2. Let $\mu = \log(1-\tau) - \log(1+\tau)$, first we prove a pair of technical lemmas.

Lemma 4.4. *Let Δ be the diagonal set of $\Sigma \times \Sigma$ and let $\{\mathbf{x}_n\} \subset \mathcal{M}^{1,1}(\Sigma)$ be a sequence such that $\mathbf{x}_n \to \mathbf{x} \in \Delta$ in the product metric. Let $\tilde{h}_n$ be the solution of the regular Taubes equation corresponding to each $\mathbf{x}_n$, then $\tilde{h}_n \rightharpoonup \mu$ in H^1 and $\tilde{h}_n \to \mu$ strongly in L^2.*

Proof. Let $v_n = v_{\mathbf{x}_n}$ for each point $\mathbf{x}_n$ in the given sequence. Let us decompose each solution to the regular Taubes equation as $\tilde{h}_n = u_n + \tilde{c}_n \in \mathcal{X} \oplus \mathbb{R}$. We claim the sequence $\{\tilde{c}_n\}$ is bounded. Assume towards a contradiction $\tilde{c}_n \to \infty$. Notice that in the vortex-antivortex case the functions F and F_0 coincide. We know that,

$$-\Delta u_n = F(u_n + \tilde{c}_n + v_n). \tag{4.30}$$

By the standard elliptic estimates, there is a constant C such that

$$||u_n||_{\mathrm{H}^2} \leq C||\Delta u_n||_{\mathrm{L}^2}. \tag{4.31}$$

Since F is a bounded function, $\{u_n\}$ is bounded in H^2 and by Sobolev's embedding also in C^0.

Assume $\mathbf{x} = (x_*, x_*)$ and notice that,

$$|v_n(x)| = 4\pi|G(x, x_{1n}) - G(x, x_{2n})|, \tag{4.32}$$

where $\mathbf{x}_n = (x_{1n}, x_{2n})$, since $G(x, y)$ is continuous away of the diagonal set, $v_n(x) \to 0$ for $x \neq x_*$, whence, we also have the convergence,

$$F(u_n + \tilde{c}_n + v_n) \to 2(1 + \tau), \tag{4.33}$$

pointwise almost everywhere. Applying the dominated convergence theorem and equation (4.30),

$$\int_\Sigma F(u_n + \tilde{c}_n + v_n)\,\mathrm{Vol} = 0 \to 2(1 + \tau)|\Sigma|, \tag{4.34}$$

a contradiction. If $\tilde{c}_n \to -\infty$ a similar argument holds. Therefore the sequence of averages $\tilde{c}_n$ is bounded, implying $\{\tilde{h}_n\}$ is bounded in C^0. Hence, the sequence is also bounded in L^p for any positive p. By the elliptic estimate

$$||\tilde{h}_n||_{\mathrm{H}^2} \leq C\left(||\Delta\tilde{h}_n||_{\mathrm{L}^2} + ||\tilde{h}_n||_{\mathrm{L}^2}\right), \tag{4.35}$$

$\{\tilde{h}_n\}$ is also bounded in H^1. By the Alaoglu and Rellich theorems, after passing to a subsequence if necessary, we can assume $\tilde{h}_n \rightharpoonup h_* \in \mathrm{H}^1$ and strongly in L^2. We claim that h_* is the constant function μ. To see this, let $\varphi \in \mathrm{H}^1$. From the regularized Taubes equation we have,

$$\begin{aligned} \langle\tilde{h}_n, \varphi\rangle_{\mathrm{H}^1} &= \langle\tilde{h}_n, \varphi\rangle_{\mathrm{L}^2} + \langle\nabla\tilde{h}_n, \nabla\varphi\rangle_{\mathrm{L}^2}, \\ &= \langle\tilde{h}_n, \varphi\rangle_{\mathrm{L}^2} + \langle\Delta\tilde{h}_n, \varphi\rangle_{\mathrm{L}^2} \\ &= \langle\tilde{h}_n, \varphi\rangle_{\mathrm{L}^2} - \langle F(\tilde{h}_n + v_n), \varphi\rangle_{\mathrm{L}^2}. \end{aligned} \tag{4.36}$$

Since $\tilde{h}_n \to h_*$ strongly in L^2, after passing to a subsequence if necessary, we can assume $\tilde{h}_n \to h_*$ pointwise almost everywhere. By the weak convergence of $\tilde{h}_n$ in H^1, together with the strong convergence in L^2 and the dominated convergence theorem,

$$\begin{aligned}\langle h_*, \varphi \rangle_{H^1} &= \lim \langle \tilde{h}_n, \varphi \rangle_{H^1} \\ &= \lim \langle \tilde{h}_n, \varphi \rangle_{L^2} - \lim \langle F(\tilde{h}_n + v_n), \varphi \rangle_{L^2} \\ &= \langle h_*, \varphi \rangle_{L^2} - \langle F(h_*), \varphi \rangle_{L^2}. \end{aligned} \tag{4.37}$$

From this equation, we infer

$$\langle \nabla h_*, \nabla \varphi \rangle_{L^2} = -\langle F(h_*), \varphi \rangle_{L^2}. \tag{4.38}$$

Therefore, h_* is a weak solution to the equation

$$-\Delta h_* = F(h_*). \tag{4.39}$$

By elliptic regularity, h_* is also a strong solution, and by the maximum principle, h_* is constant since F is an increasing function. Since the only zero of F is at $t = \mu$, we conclude $h_* = \mu$. If $\tilde{h}_{n_k}$ is any subsequence of $\tilde{h}_n$, this argument shows it has a subsequence weakly converging to μ in H^1 and strongly in L^2, the claim of the lemma follows. □

Lemma 4.5. *$\tilde{h}_n \to \mu$ strongly in $W^{2,p}$ for any positive p.*

Proof. We will prove that any subsequence of $\tilde{h}_n$ has another subsequence converging to μ in $W^{2,p}$, implying the lemma. To simplify notation, we denote subsequences of $\tilde{h}_n$ by the same symbol. From the previous lemma, $\tilde{h}_n \to \mu$ strongly in L^2. After passing to a subsequence if necessary, we can assume that $\tilde{h}_n \to \mu$ pointwise almost everywhere. We apply the dominated convergence theorem to deduce the limit,

$$||\Delta \tilde{h}_n||_{L^p} = ||F(\tilde{h}_n + v_n)||_{L^p} \to ||F(\mu)||_{L^p} = 0. \tag{4.40}$$

If $p = 2$, by the standard elliptic estimates, there is a constant C, such that,

$$||\tilde{h}_n - \mu||_{H^2} \leq C \left(||\Delta \tilde{h}_n||_{L^2} + ||\tilde{h}_n - \mu||_{L^2} \right) \to 0. \tag{4.41}$$

By Sobolev's embedding, $\tilde{h}_n \to \mu$ uniformly in C^0, hence also in L^p for any positive p. We apply one more time the elliptic estimate,

$$||\tilde{h}_n - \mu||_{\mathrm{W}^{2,p}} \leq C \left(||\Delta \tilde{h}_n||_{\mathrm{L}^p} + ||\tilde{h}_n - \mu||_{\mathrm{L}^p} \right) \to 0. \tag{4.42}$$

□

As a consequence of this lemma and Sobolev's embedding, we have the convergence,

$$||\tilde{h}_n - \mu||_{C^1} \to 0, \tag{4.43}$$

for any arbitrary sequence $\{\mathbf{x}_n\} \subset \mathcal{M}^{1,1}(\Sigma)$, such that $\mathbf{x}_n \to \mathbf{x} \in \Delta$. This proves the following corollary,

Corollary 4.6. *The limit,*

$$\lim_{d(x_1,x_2)\to 0} \left|\left| \tilde{h}(x; x_1, x_2) - \mu \right|\right|_{C^1(\Sigma)} = 0, \tag{4.44}$$

holds, where $d(x_1, x_2)$ is the Riemannian distance in Σ.

Let $\Sigma^2_\Delta = (\Sigma \times \Sigma) \setminus \Delta$ endowed with the product metric. As differentiable manifolds, $\mathcal{M}^{1,1}$ and Σ^2_Δ are equivalent. In what follows, we will consider $\tilde{h}$ as a function $\Sigma \times \Sigma^2_\Delta \to \mathbb{R}$. Let $U \subset \Sigma$ be an open and dense subset and let $\varphi : U \to V \subset \mathbb{C}$ be a holomorphic chart. In what follows we denote points on the surface as x and points on $\mathbb{C}$ as z, so $z = \varphi(x)$ for $x \in U$. We also assume vortices and antivortices are both located in U, such that up to a holomorphic chart, $\tilde{h} : \Sigma \times V^2_\Delta \to \mathbb{R}$, where $V^2_\Delta = V^2 \setminus \Delta_V$ and $\Delta_V \subset \mathbb{C}^2$ is the diagonal set. On this chart partial derivatives $\partial_{z_j} \tilde{h}$ are well defined functions

$$\partial_{z_j} \tilde{h} : \Sigma \times V^2_\Delta \to \mathbb{C}. \tag{4.45}$$

We denote the covariant derivative and Laplacian with respect to the first variable by ∇ and Δ and emphasize that the metric on V^2_Δ is the push forward of the metric induced by the surface. Our aim is to estimate the rate at which the second derivatives $\nabla \partial_{z_j} \tilde{h}$ grow as a sequence $\mathbf{z}_n \in V^2_\Delta$ diverges to the diagonal set. This will allow us to prove that the moduli space is incomplete. Since, Δ and ∂_{z_j} commute, $\partial_{z_j}\tilde{h}$ is the solution to the elliptic problem,

$$-\Delta\partial_{z_j}\tilde{h} = V(h)\partial_{z_j}\tilde{h} + s_j\, V(h)\, \partial_{z_j} v_j, \tag{4.46}$$

where $v_j(x) = 4\pi\, G(x, \varphi^{-1}(z_j))$. Let $d_j(x) = d(x, x_j)$, $x_j = \varphi^{-1}(z_j)$, we know there is a uniform constant C, such that the derivative of Green's function is bounded [2],

$$|\nabla G(x, x_j)| < \frac{C}{d_j}, \qquad\qquad |\nabla_2 G(x, x_j)| < \frac{C}{d_j}, \tag{4.47}$$

where $\nabla_2 G$ is the covariant derivative with respect to the second variable. Recall in holomorphic coordinates the metric is $e^{\Lambda(z)}|dz|^2$, hence, if z_j is restricted to a bounded domain,

$$|\partial_{z_j} v_j| \leq 4\pi e^{-\Lambda(z_j)}\, |\nabla_2 G(x, \varphi^{-1}(z_j))| < \frac{C}{d_j}. \tag{4.48}$$

Lemma 4.7. *For any positive constant C_1, there is another constant C, such that, for all $x, x_1, x_2 \in U$,*

$$\frac{d_{12}^2}{C_1 d_1^2 + d_2^2} \leq C, \tag{4.49}$$

$$\frac{d_j d_k^2}{(C_1 d_1^2 + d_2^2)^2} \leq \frac{C}{d_{12}}, \tag{4.50}$$

where $\{d_j, d_k\} = \{d_1, d_2\}$ and $d_{12} = d(x_1, x_2)$.

Proof. By the triangle inequality and Cauchy-Schwarz,

$$d_{12} \leq d_1 + d_2 \leq C\,(d_1^2 + d_2^2)^{1/2}, \tag{4.51}$$

on the other hand, any two norms in a finite dimensional vector space are equivalents, hence, there is another constant such that,

$$(d_1^2 + d_2^2)^{1/2} \leq C\,(C_1 d_1^2 + d_2^2)^{1/2}, \tag{4.52}$$

from these two inequalities we obtain the first claim of the lemma. For the second claim, it is enough to prove that the inequality

$$\frac{d_1 d_2^2}{(C_1 d_1^2 + d_2^2)^2} \leq \frac{C}{d_{12}}, \tag{4.53}$$

holds, the remaining case being equivalent to this one after relabelling d_1 and d_2. Let us note that since,

$$d_1 d_2 \leq \frac{1}{2}(d_1^2 + d_2^2) \leq C\,(C_1 d_1^2 + d_2^2), \tag{4.54}$$

is sufficient to prove that,

$$\frac{d_2}{C_1 d_1^2 + d_2^2} \leq \frac{C}{d_{12}}. \tag{4.55}$$

If $d_2 \leq d_1$, by the triangle inequality we have,

$$\begin{aligned} d_2 d_{12} &\leq d_1 d_2 + d_2^2 \\ &\leq d_1^2 + d_2^2 \\ &\leq C\,(C_1 d_1^2 + d_2^2), \end{aligned} \tag{4.56}$$

hence (4.55). On the other hand, if $d_1 \leq d_2$, repeating the previous step, we find that

$$d_1 d_{12} \leq C\,(C_1 d_1^2 + d_2^2), \tag{4.57}$$

this inequality, together with (4.49) and the triangle inequality, implies,

$$\frac{d_2}{C_1 d_1^2 + d_2^2} \leq \frac{d_1}{C_1 d_1^2 + d_2^2} + \frac{d_{12}}{C_1 d_1^2 + d_2^2} \leq \frac{C}{d_{12}}. \tag{4.58}$$

In any case, we conclude that equation (4.50) holds. □

Lemma 4.8. *There is a constant C such that for any pair of distinct points $x_1, x_2 \in \Sigma$,*

$$\left| G(x_1, x_2) - \frac{1}{2\pi} \log\, d(x_1, x_2) \right| \leq C. \tag{4.59}$$

Proof. We cover Σ with a finite cover of metric disks $\mathbb{D}_{R_j/2}(p_j)$ such that $R_j < \delta$, where δ is the injectivity radius of the metric and for each disk there is a holomorphic chart $\varphi_j : U_j \to \mathbb{C}$, $\mathbb{D}_{R_j}(p_j) \subset U_j$. Let $R = \min\{R_j\}$, for any pair of distinct points $x_1, x_2 \in \Sigma$, such that $d(x_1, x_2) < R/2$, there is a disk such that $x_1, x_2 \in \mathbb{D}_{R_j}(p_j)$. For any disk in the cover, let R'_j be a positive radius, such that,

$$|\varphi_j(x) - \varphi_j(p_j)| < R'_j, \qquad \forall x \in \mathbb{D}_{R_j}(p_j). \tag{4.60}$$

Let $z_j = \varphi_j(p_j)$ and let us denote by $D_{R'_j}(z_j) \subset \mathbb{C}$ the holomorphic disk of radius R'_j centred at z_j. For any small $\epsilon > 0$ there are continuous functions $\tilde{G}_j : D_{R'_j+\epsilon}(z_j) \times D_{R'_j+\epsilon}(z_j) \to \mathbb{R}$ such that if $x_1, x_2 \in \mathbb{D}_{R_j}(p_j)$,

$$G(x_1, x_2) = \frac{1}{2\pi} \log |\varphi_j(x_1) - \varphi_j(x_2)| + \tilde{G}_j(\varphi_j(x_1), \varphi_j(x_2)). \tag{4.61}$$

If $\exp \Lambda_j(z)$ is the conformal factor of the metric in the chart φ_j, let

$$\begin{aligned} M_j &= \max \left\{ e^{\Lambda_j(z)/2} \,:\, z \in \overline{D_{R'_j}(z_j)} \right\}, \\ m_j &= \min \left\{ e^{\Lambda_j(z)/2} \,:\, z \in \overline{D_{R'_j}(z_j)} \right\}, \end{aligned} \tag{4.62}$$

and $M = \max_j \{M_j\}$, $m = \min_j \{m_j\}$. Since each $\mathbb{D}_{R_j}(p_j)$ is geodesically convex, for any $x_1, x_2 \in \mathbb{D}_{R_j}(p_j)$,

$$m \, |\varphi_j(x_1) - \varphi_j(x_2)| \leq d(x_1, x_2) \leq M \, |\varphi_j(x_1) - \varphi_j(x_2)|. \tag{4.63}$$

Taking the log of this inequality we find a positive constant such that,

$$|\, d(x_1, x_2) - \log |\varphi_j(x_1) - \varphi_j(x_2)| \,| \leq C, \tag{4.64}$$

whenever $x_1, x_2 \in \mathbb{D}_{R_j}(p_j)$. Since each function $\tilde{G}_j$ is continuous in the compact set $\overline{D_{R'_j}(z_j)}$, we find another constant such that,

$$\begin{aligned} &\left| G(x_1, x_2) - \frac{1}{2\pi} \log \, d(x_1, x_2) \right| = \\ &\quad \left| \frac{1}{2\pi} \left(\log |\varphi_j(x_1) - \varphi_j(x_2)| - \log \, d(x_1, x_2) \right) + \tilde{G}_j(\varphi_j(x_1), \varphi_j(x_2)) \right| \leq C. \end{aligned} \tag{4.65}$$

This proves the inequality whenever $d(x_1, x_2) < R/2$. Since G and the distance function are continuous on the compact set,

$$\left\{ (x_1, x_2) \in \Sigma \times \Sigma \,:\, d(x_1, x_2) \geq \frac{R}{2} \right\}, \tag{4.66}$$

we can find a second constant satisfying the inequality whenever $d(x_1, x_2) \geq R/2$. Taking the maximum of both constants concludes the lemma. □

Lemma 4.9. *Let D be any bounded domain on $\mathbb{C}$. For any $p > 0$, there is a constant C, independent of $z_1, z_2 \in D$, $z_1 \neq z_2$ such that, if $x_j = \varphi^{-1}(z_j)$,*

$$||V(h)\partial_{z_j} v_j||_{\mathrm{L}^p} \leq \frac{C}{d(x_1, x_2)}. \tag{4.67}$$

Proof. By lemma 4.8, there is a constant, such that for all $x, y \in \Sigma$, $x \neq y$,

$$\left|G(x,y) - \frac{1}{2\pi}\log d(x,y)\right| \leq C. \tag{4.68}$$

Hence,

$$|V(h)\partial_{z_j} v_j| = \left|\frac{4e^{v_1}e^{v_2}e^{\tilde{h}}}{(e^{v_1}e^{\tilde{h}} + e^{v_2})^2}\partial_{z_j} v_j\right| \leq C\left|\frac{4d_1^2 d_2^2\, e^{\tilde{h}}}{(d_1^2 e^{\tilde{h}} + d_2^2)^2}\frac{1}{d_j}\right|, \tag{4.69}$$

where the constant depends on D. Since $\tilde{h}$ is uniformly bounded on Σ, there are constants C, C_1, such that by lemma 4.7.

$$|V(h)\partial_{z_j} v_j| \leq C\left|\frac{d_1^2 d_2^2}{(d_1^2 C_1 + d_2^2)^2}\frac{1}{d_j}\right| \leq \frac{C}{d(x_1,x_2)}, \tag{4.70}$$

this inequality implies the claim. □

The proof of the lemma depends only on properties of Green's function, we could repeat the proof of lemma 4.9 using ∇v_j instead of $\partial_{z_j} v_j$ to prove for any given domain $D \subset \mathbb{C}$ the existence of a constant, independent of $z_1, z_2 \in D$, such that,

$$||V(h)\nabla v_j||_{\mathrm{L}^p} \leq \frac{C}{d(x_1,x_2)}. \tag{4.71}$$

In the next lemmas we prove that the bilinear form,

$$\mathrm{B} : \mathrm{H}^1 \times \mathrm{H}^1 \to \mathbb{R}, \qquad \mathrm{B}(\phi,\psi) = \langle \nabla\phi, \nabla\psi\rangle_{\mathrm{L}^2}^2 + \langle V(h)\phi, \psi\rangle_{\mathrm{L}^2}, \tag{4.72}$$

is coercive with a uniform coercivity constant.

Lemma 4.10. *If $V_n : \Sigma \to \mathbb{R}$ is a sequence of continuous, uniformly bounded functions converging pointwise to the continuous function V_*, and $\phi_n \to \phi_*$ in L^2, then*

$$\langle V_n, \phi_n^2\rangle_{\mathrm{L}^2} \to \langle V_*, \phi_*^2\rangle_{\mathrm{L}^2}. \tag{4.73}$$

Proof. We have,

$$|\langle V_n, \phi_n^2\rangle_{\mathrm{L}^2} - \langle V_*, \phi_*^2\rangle_{\mathrm{L}^2}| \leq |\langle V_n, \phi_n^2 - \phi_*^2\rangle_{\mathrm{L}^2}| + |\langle V_n - V_*, \phi_*^2\rangle_{\mathrm{L}^2}|. \tag{4.74}$$

Since the functions V_n are uniformly bounded, there is a constant C such that,

$$\begin{aligned} |\langle V_n, \phi_n^2 - \phi_*^2 \rangle_{\mathrm{L}^2}| &\leq C \,\langle |\phi_n - \phi_*|, |\phi_n + \phi_*| \rangle_{\mathrm{L}^2} \\ &\leq C \,||\phi_n - \phi_*||_{\mathrm{L}^2} \,||\phi_n + \phi_*||_{\mathrm{L}^2}, \end{aligned} \tag{4.75}$$

by the convergence $\phi_n \to \phi_*$ in L^2, we obtain the limit

$$|\langle V_n, \phi_n^2 - \phi_*^2 \rangle_{\mathrm{L}^2}| \to 0. \tag{4.76}$$

Since there is a constant C such that the functions $(V_n - V_*)\phi_*^2$ are bounded by the measurable function $C\phi_*^2$ and $V_n - V_* \to 0$ pointwise, by the dominated convergence theorem,

$$|\langle V_n - V_*, \phi_*^2 \rangle_{\mathrm{L}^2}| \to 0. \tag{4.77}$$

Therefore,

$$|\langle V_n, \phi_n^2 \rangle_{\mathrm{L}^2} - \langle V_*, \phi_*^2 \rangle_{\mathrm{L}^2}| \to 0, \tag{4.78}$$

concluding the proof of the lemma. □

Lemma 4.11. *There is a positive constant C, independent of $(x_1, x_2) \in \Sigma^2_\Delta$, such that for any $\phi \in \mathrm{H}^1$,*

$$C||\phi||^2_{\mathrm{H}^1} \leq \mathrm{B}(\phi, \phi). \tag{4.79}$$

Proof. By the bilinearity of B, it is sufficient to prove the lemma assuming $||\phi||_{\mathrm{H}^1} = 1$. Let us assume towards a contradiction the statement is false, in this case there is a sequence $(\phi_n, \mathbf{x}_n) \subset \mathrm{H}^1 \times \Sigma^2_\Delta$, with $||\phi_n||_{\mathrm{H}^1} = 1$, such that,

$$\mathrm{B}(\phi_n, \phi_n) = ||\nabla \phi_n||^2_{\mathrm{L}^2} + \langle V_n, \phi_n^2 \rangle_{\mathrm{L}^2} \to 0, \tag{4.80}$$

where $V_n = V(h_n)$ is the potential function determined by h_n, the solution to the Taubes equation with data $\mathbf{x}_n$. Since the functions V_n are non-negative,

$$||\nabla \phi_n||_{\mathrm{L}^2} \to 0, \qquad\qquad \langle V_n, \phi_n^2 \rangle_{\mathrm{L}^2} \to 0. \tag{4.81}$$

Passing to a subsequence if necessary, we can assume $\phi_n \rightharpoonup \phi_*$ in H^1 and strongly in L^2 and $\mathbf{x}_n \to \mathbf{x}_*$ in $\Sigma \times \Sigma$. Since the functions

$$e^{v_j} : \Sigma \times \Sigma \to \mathbb{R} \tag{4.82}$$

are continuous and $\tilde{h}_n$ varies continuously with the initial data, if $\mathbf{x}_* \notin \Delta$, we have the uniform convergence $V_n \to V_* = V(h_*)$, where h_* is the solution to the Taubes equation determined by $\mathbf{x}_*$. On the other hand, if $\mathbf{x}_* \in \Delta$, we know that $\tilde{h}_n \to \mu$ in C^1, hence, we have pointwise convergence $V_n \to V_* \equiv 4\exp(\mu)(\exp(\mu)+1)^{-2}$. In any case, by our previous lemma,

$$\langle V_n, \phi_n^2 \rangle_{\mathrm{L}^2} \to \langle V_*, \phi_*^2 \rangle_{\mathrm{L}^2}, \tag{4.83}$$

but this limit is zero, hence $\phi_* = 0$ almost everywhere and $\phi_n \to 0$ in H^1 strongly, a contradiction. □

Proposition 4.12. *Let $D \subset V$ be any bounded domain. There is a positive constant $C(D)$, such that*

$$||\partial_{z_j}\tilde{h}||_{C^1} \leq \frac{C}{d_{12}}, \qquad \textit{and} \qquad ||\nabla\tilde{h}||_{C^1} \leq \frac{C}{d_{12}}, \tag{4.84}$$

for all $z_1, z_2 \in D$ with $z_1 \neq z_2$, where $\tilde{h}(x) = \tilde{h}(x; \varphi^{-1}(z_1), \varphi^{-1}(z_2))$ and $d_{12} = d(x_1, x_2)$.

Proof. $\partial_{z_j}\tilde{h}$ is a solution to the equation

$$-\Delta\partial_{z_j}\tilde{h} = V(h)\partial_{z_j}\tilde{h} + s_j V(h)\partial_{z_j} v_j. \tag{4.85}$$

By lemma 4.11, there is a positive constant C_1 independent of z_1, z_2, such that

$$C_1\, ||\phi||_{\mathrm{H}^1}^2 \leq ||\nabla\phi||_{\mathrm{L}^2}^2 + \langle V(h)\,\phi, \phi\rangle_{\mathrm{L}^2}, \tag{4.86}$$

for all $\phi \in \mathrm{H}^1$. As in the proof of lemma 4.9, a second uniform constant, dependent on D, can be found such that,

$$||V(h)\partial_{z_j} v_j||_{\mathrm{L}^2} \leq \frac{C_2}{d_{12}}. \tag{4.87}$$

By the Lax-Milgram theorem, we obtain the bound,

$$||\partial_{z_j}\tilde{h}||_{\mathrm{H}^1} \leq \frac{C}{d_{12}}, \tag{4.88}$$

where $C = C_2/C_1$. Now we follow a recursive argument: by Schauder's estimates, $||\partial_{z_j}\tilde{h}||_{\mathrm{H}^2}$ is also bounded by $C\, d_{12}^{-1}$ for some constant C. By Sobolev's embedding, there is another constant such that $||\partial_{z_j}\tilde{h}||_{C^0}$ is also bounded by $C\,d_{12}^{-1}$. Thence, for any given $p > 2$,

$$||\partial_{z_j}\tilde{h}||_{\mathrm{L}^p} \leq \frac{C}{d_{12}}. \tag{4.89}$$

By the elliptic estimates,

$$\begin{aligned} ||\partial_{z_j}\tilde{h}||_{\mathrm{W}^{2,p}} &\leq C\,(||\Delta\partial_{z_j}\tilde{h}||_{\mathrm{L}^p} + ||\tilde{h}||_{\mathrm{L}^p}) \\ &\leq C\,(||V(h)\partial_{z_j}\tilde{h}||_{\mathrm{L}^p} + ||V(h)\partial_{z_j}v_j||_{\mathrm{L}^p} + ||\partial_{z_j}\tilde{h}||_{\mathrm{L}^p}) \\ &\leq \frac{C}{d_{12}}, \end{aligned} \tag{4.90}$$

for the last inequality we have used that the function $V(t)$ is bounded. Sobolev's embedding implies the claimed bound,

$$||\partial_{z_j}\tilde{h}||_{C^1} \leq \frac{C}{d_{12}}. \tag{4.91}$$

This argument is also valid for $\nabla\tilde{h}$, because it is a solution to the elliptic problem,

$$-\Delta(\nabla\tilde{h}) = V(h)\nabla\tilde{h} + V(h)(\nabla v_1 - \nabla v_2), \tag{4.92}$$

and the upper bound

$$||V(h)\nabla v_j|| \leq \frac{C}{d_{12}} \tag{4.93}$$

also holds. □

For latter application, we need to translate this estimate to a holomorphic chart.

Lemma 4.13. *Let $\varphi : U \subset \Sigma \to V \subset \mathbb{C}$ be a holomorphic chart and let D be a geodesically convex neighbourhood such that $\overline{D} \subset U$, there is a positive constant C, such that for all $z_1, z_2 \in \varphi(D)$,*

$$C\,|z_1 - z_2| \leq d(\varphi^{-1}(z_1), \varphi^{-1}(z_2)). \tag{4.94}$$

Proof. The conformal factor is a continuous positive function on V and $\varphi(\overline{D})$ is compact, hence there is a constant $C > 0$, such that for all $z \in \varphi(D)$,

$$C^2 \leq e^{\Lambda(z)}. \tag{4.95}$$

Since D is geodesically convex, for any pair $z_1, z_2 \in \varphi(D)$, there is a curve $\gamma : [0,1] \to \varphi(D)$ joining z_1 to z_2 such that $\varphi^{-1} \circ \gamma$ is a minimizing geodesic joining $\varphi^{-1}(z_1)$ to $\varphi^{-1}(z_2)$, hence,

$$C \int_0^1 |\dot{\gamma}|\, ds \leq \int_0^1 e^{\Lambda/2} |\dot{\gamma}|\, ds = d(\varphi^{-1}(z_1), \varphi^{-1}(z_2)). \tag{4.96}$$

By the triangle inequality,

$$|z_1 - z_2| = \left| \int_0^1 \dot{\gamma} \right| \leq \int_0^1 |\dot{\gamma}|\, ds, \tag{4.97}$$

yielding the result. □

The advantage of the holomorphic chart is that it makes computations possible, on the other hand, the Riemannian distance is a geometric invariant defined globally on the surface and better suited to prove analytical properties of the solutions to the Taubes equation. For the next lemma, notice that if $\Sigma_1 \times \Sigma_2$ is a product of Riemmann surfaces, for any function $f : \Sigma_1 \times \Sigma_2 \to \mathbb{C}$ in local coordinates $\varphi_j : U_j \to \mathbb{C}$, $\varphi_j(x_j) = z_j$,

$$\partial_{x_1} \partial_{x_2} f = \partial_{z_1 z_2} f \, dz^1 \otimes dz^2 \in \Omega^{(2,0)}(\Sigma_1 \times \Sigma_2). \tag{4.98}$$

In the product metric, dz^1 and dz^2 are orthogonal, hence,

$$|\partial_{x_1} \partial_{x_2} f| = |\partial_{z_1, z_2} f| \, |dz^1| \, |dz^2|. \tag{4.99}$$

Lemma 4.14. *For any holomorphic chart $\varphi : U \subset \Sigma \to V \subset \mathbb{C}$ and any geodesically convex neighbourhood D such that $\overline{D} \subset U$, there is a constant $C > 0$ such that, for all $z_1, z_2 \in \varphi(D)$, $z_1 \neq z_2$,*

$$|\partial_{z_1} b_1(z_1, z_2)| \leq \frac{C}{|z_1 - z_2|}, \tag{4.100}$$

where the coefficient b_1 appearing in the metric of $\mathcal{M}^{1,1}(\Sigma)$ is defined as in (2.78).

Proof. If $z_1, z_2 \in \varphi(D)$, there is a smooth function $\tilde{v} : \varphi(\overline{D}) \times \varphi(\overline{D}) \times \varphi(\overline{D}) \to \mathbb{R}$, such that for all triples z, z_1, z_2 of points in the domain with $z_1 \neq z_2$,

$$v(\varphi^{-1}(z)) = \log |z - z_1|^2 - \log |z - z_2|^2 + \tilde{v}(z, z_1, z_2). \tag{4.101}$$

Hence,

$$\begin{aligned} b_1(z_1, z_2) &= 2\, \bar{\partial}\big|_{z=z_1} \left(h(\varphi^{-1}(z)) - \log |z - z_1|^2\right) \\ &= 2\, \bar{\partial}\big|_{z=z_1} \left(\tilde{h}(\varphi^{-1}(z)) - \log |z - z_2|^2 + \tilde{v}(z, z_1, z_2)\right) \\ &= 2\, \bar{\partial}_z \tilde{h}(\varphi^{-1}(z_1); \varphi^{-1}(z_1), \varphi^{-1}(z_2)) - \frac{2}{\bar{z}_1 - \bar{z}_2} + 2\, \bar{\partial}_z \tilde{v}(z_1, z_1, z_2), \end{aligned} \tag{4.102}$$

where $\bar{\partial}_z$ refers to complex derivatives with respect to the first entry. In the following calculation we denote $\tilde{h}(\varphi^{-1}(z_1); \varphi^{-1}(z_1), \varphi^{-1}(z_2))$ by $\tilde{h}$ and $\tilde{v}(z_1, z_1, z_2)$ by $\tilde{v}$, whence,

$$\begin{aligned} \partial_{z_1} b_1 &= 2\left(\partial_z \bar{\partial}_z \tilde{h} + \partial_{z_1} \bar{\partial}_z \tilde{h} + \partial_z \bar{\partial}_z \tilde{v} + \partial_{z_1} \bar{\partial}_z \tilde{v}\right) \\ &= 2\left(-\frac{e^{\Lambda(z_1)}}{2} \Delta_\Sigma \tilde{h} + \bar{\partial}_z \partial_{z_1} \tilde{h} + \partial_z \bar{\partial}_z \tilde{v} + \partial_{z_1} \bar{\partial}_z \tilde{v}\right) \\ &= 2\left(\frac{e^{\Lambda(z_1)}}{2} F(h) + \bar{\partial}_z \partial_{z_1} \tilde{h} + \partial_z \bar{\partial}_z \tilde{v} + \partial_{z_1} \bar{\partial}_z \tilde{v}\right). \end{aligned} \tag{4.103}$$

Since $\varphi(\overline{D})$ is compact, $\Lambda(z_1)$ and the last two terms are bounded functions on $\varphi(D)$ by continuity. Since function $F(t)$ is also bounded, we conclude the same statement for the first term. For the second term, if $x = \varphi^{-1}(z)$ and $x_j = \varphi^{-1}(z_j)$, we have by lemma 4.13 and proposition 4.12,

$$\begin{aligned} \left|\bar{\partial}_z \partial_{z_1} \tilde{h}\right| &= e^{\Lambda(z_1)/2} \left|\bar{\partial}_z \partial_{z_1} \tilde{h}\right| |dz| \\ &= |\bar{\partial}_x \partial_{z_1} \tilde{h}(x_1, \varphi^{-1}(z_1), \varphi^{-1}(z_2))| \\ &\leq \frac{C}{d(x_1, x_2)} \\ &\leq \frac{C}{|z_1 - z_2|}. \end{aligned} \tag{4.104}$$

Therefore the lemma is proved. □

Theorem 4.15. *The moduli space is incomplete. There is a Cauchy sequence* $\{\mathbf{x}_n\} \subset \mathcal{M}^{1,1}(\Sigma)$ *such that* $\mathbf{x}_n \to \mathbf{x} \in \Delta$ *as a sequence in* $\Sigma \times \Sigma$.

Proof. Let $\varphi : U \subset \Sigma \to \mathbb{C}$ be an holomorphic chart defined on an open and dense neighbourhood U. Let $z_1 \in \mathbb{C}$ be chosen such that $\varphi^{-1}(sz_1)$, $0 \leq s \leq 1$ is contained in a geodesically convex neighbourhood of $\varphi^{-1}(0)$. Let us define the curve,

$$\gamma : (0,1] \to \mathbb{C}^2_\Delta, \qquad\qquad \gamma(s) = (s\, z_1, 0). \tag{4.105}$$

Let $z(s) = sz_1$ and let $\varphi_*^{-1}\gamma(s) = (\varphi^{-1}(z(s)), \varphi^{-1}(0))$, be the push forward of the curve γ to the moduli space, hence,

$$|\varphi_*^{-1}\dot{\gamma}|^2_{\mathcal{M}} = (e^{\Lambda(z)}(1-\tau) + \partial_{z_1} b_1)\,|z_1|^2, \tag{4.106}$$

where we denote by $|\cdot|_{\mathcal{M}}$ the norm of vectors in $T_{\varphi_*^{-1}\gamma(s)}\mathcal{M}^{1,1}$.

By Lemma 4.14 there is a constant C, such that,

$$|\partial_{z_1} b_1| \leq \frac{C}{|z|} = \frac{C}{s\,|z_1|}. \tag{4.107}$$

Since the conformal factor is a continuous positive function defined on the whole plane, there is another constant, also denoted C, such that,

$$|\varphi_*^{-1}\dot{\gamma}|_{\mathcal{M}} \leq \frac{C}{s^{1/2}}. \tag{4.108}$$

Let $\ell[\gamma, a, b]$ be the arc-length of the segment $\gamma|_{[a,b]}$, $a, b \in (0,1)$, there is another constant, also denoted by C, such that,

$$\ell[\gamma, a, b] = \int_a^b |\varphi_*^{-1}\dot{\gamma}|_{\mathcal{M}}\, ds \leq C(b^{1/2} - a^{1/2}), \tag{4.109}$$

whence,

$$d(\varphi_*^{-1}\gamma(b), \varphi_*^{-1}\gamma(a)) \leq C\,(b^{1/2} - a^{1/2}). \tag{4.110}$$

This inequality shows if $\{s_n\} \subset (0,1]$ is any converging sequence $s_n \to 0$, the new sequence,

$$\mathbf{x}_n = \varphi_*^{-1}\gamma(s_n) \in \mathcal{M}^{1,1}(\Sigma), \tag{4.111}$$

is Cauchy, however γ is continuous which implies $\mathbf{x}_n \to (\varphi^{-1}(0), \varphi^{-1}(0)) \in \Delta_\Sigma$. Therefore, the moduli space is incomplete. □

4.3 The volume of the moduli space

We conclude this chapter computing the volume of the moduli space $\mathcal{M}^{1,1}(\Sigma)$ for the round sphere and flat tori. As it will turn out, the existence of a Lie group of isometries will play an important role in the calculations. Symmetries were studied for their relation to conservation laws in a Schrodinger-Chern-Simons model by Manton and Nasir in [36], for the Riemann sphere, symmetries of the coefficients of the L^2 metric for vortices of a non-relativistic Chern-Simons model were treated by Romão [46]. We follow similar ideas for asymmetric vortices of the $O(3)$ Sigma model. There is a general conjecture for the volume of the moduli space by Romão-Speight [45], which can be stated as follows,

Conjecture 4.16 (The volume conjecture). *Given a compact Riemann surface Σ of genus g and total area $|\Sigma|$, let,*

$$\begin{aligned} J_\pm &= 2\pi(1 \mp \tau)|\Sigma| - 4\pi^2(k_\pm - k_\mp), \\ K_\pm &= \mp 2\pi^2, \end{aligned}$$

then the total volume of the moduli space $\mathcal{M}^{k_+,k_-}(\Sigma)$ is,

$$\mathrm{Vol}(\mathcal{M}^{k_+,k_-}(\Sigma)) = \sum_{l=0}^{g} \frac{g!(g-l)!}{(-1)^l l!} \prod_{\sigma=\pm} \sum_{j_\sigma=l}^{g} \frac{(2\pi)^{2l} J_\sigma^{k_\sigma - j_\sigma} K_\sigma^{j_\sigma - l}}{(j_\sigma - l)!(g-j_\sigma)!(k_\sigma - j_\sigma)!}.$$

For $\Sigma = \mathbb{S}^2{}_{round}$, they corroborated it for a vortex-antivortex pair and $\tau = 0$. We aim to confirm the conjecture on the round sphere and flat tori for vortex-antivortex pairs and general τ.

4.3.1 The Riemann sphere

On the round sphere, the three dimensional Lie group of orthogonal transformations, $O(3)$, acts by isometries. The vortex equations are invariant under isometric actions on the domain, if $\mathcal{I} : \Sigma \to \Sigma$ is an isometry and u is the solution of the Taubes equation with vortex set P and antivortex set Q, then $u \circ \mathcal{I}$ is the solution with data $\mathcal{I}^{-1}(P)$, $\mathcal{I}^{-1}(Q)$. We will make use of this symmetry to obtain conservation laws for the non-trivial coefficients b_j and an explicit formula in the subspace of vortices and antivortices located at antipodal positions. This formula will lead us to the volume formula. We will prove the following theorem,

Recall the conformal factor of the sphere of radius R in a stereographic projection chart with coordinate z is,

$$\Omega = \frac{4R^2}{\left(1+|z|^2\right)^2}. \tag{4.112}$$

We can give an explicit description of the coefficients in the metric in the case of only k_+ coincident vortices or k_- coincident antivortices. By rotational symmetry, the function u depends only on the chordal distance to either the vortex or antivortex [34], the coefficients $b_\pm$ in this case are,

$$b_\pm = -\frac{2k_\pm z_\pm}{1+|z_\pm|^2}. \tag{4.113}$$

The proof relies on the rotational symmetry of the configuration and is analogous to the proof for n coincident Ginzburg-Landau vortices on the sphere that can be found in [37]. With this identity at hand, we prove the following theorem,

Theorem 4.17. *The volume of the moduli space* $\mathcal{M}^{k_+,0}\left(\mathbb{S}^2\right)$ *is,*

$$\mathrm{Vol}\left(\mathcal{M}^{k_+,0}(\mathbb{S}^2)\right) = \frac{\left(4\pi^2 R^2\left(2\left(1-\tau\right)-\frac{k_+}{R^2}\right)\right)^{k_+}}{k_+!}, \tag{4.114}$$

and the volume of $\mathcal{M}^{0,k_-}(\mathbb{S}^2)$ *can be obtained from equation* (4.114) *by changing* τ *into* $-\tau$*. For a vortex-antivortex pair, the volume of* $\mathcal{M}^{1,1}\left(\mathbb{S}^2\right)$ *is*

$$\mathrm{Vol}\left(\mathcal{M}^{1,1}(\mathbb{S}^2)\right) = \left(8\pi^2 R^2\right)^2\left(1-\tau^2\right). \tag{4.115}$$

For $k_+ = 0$ or $k_- = 0$ we follow ideas of Manton-Nasir [38], as their proof relies on the topology of the symmetric product $(\mathbb{S}^2)^N/S_N$, S_N being the N symmetric group, and can be adapted easily to vortices of the $O(3)$ Sigma model of the same type. For the case $k_+ = k_- = 1$, we extend the proof given by Romão-Speight [45, Thm. 5.2] for the symmetric case. For general τ we no longer have the symmetry $(z_1, z_2) \mapsto (z_2, z_1)$, instead, we complement the symmetries induced by $SO(3)$ in the moduli space with the symmetry $(z_1, z_2) \mapsto (\overline{z}_1, \overline{z}_2)$ to deduce a suitable formula for the volume of a general Kähler metric on ${\mathbb{S}^2}_\Delta$.

k_+ vortices of the same type

If there are k_+ vortices on $\mathbb{S}^2$ and no antivortices, the moduli space is isomorphic to $\mathbb{P}^n$, the complex projective space of dimension k_+ [34]. The subspace $\mathcal{M}_0^{k_+,0}(\mathbb{S}^2) \subset \mathcal{M}^{k_+,0}(\mathbb{S}^2)$ of k_+ coincident vortices on the other hand is isomorphic to $\mathbb{P}^1$, and can be parametrized with the coordinate z_+ of the coincident vortices. By equation (4.113) we know how to compute the coefficient b_+ in $\mathcal{M}_0^{k_+,0}(\mathbb{S}^2)$,

$$b_+ = -\frac{2k_+ z_+}{1+|z_+|^2}. \tag{4.116}$$

The metric in $\mathcal{M}_0^{k_+,0}(\mathbb{S}^2)$ therefore is,

$$\begin{aligned} ds^2 &= 2k_+\pi \left((1-\tau)\Omega + \frac{\partial b_+}{\partial z_+}\right) |dz_+|^2 \\ &= k_+\pi \left(2(1-\tau) - \frac{k_+}{R^2}\right) \Omega |dz_+|^2, \end{aligned} \tag{4.117}$$

as can be seen, the metric is a multiple of the round metric, hence, the volume of $\mathcal{M}_0^{k_+,0}(\mathbb{S}^2)$ is,

$$4\pi^2 R^2 k_+ \left(2(1-\tau) - \frac{k_+}{R^2}\right), \tag{4.118}$$

this volume is k_+ times the volume of the generating cycle in $\mathbb{P}^1$,

$$4\pi^2 R^2 \left(2(1-\tau) - \frac{k_+}{R^2}\right). \tag{4.119}$$

The total volume of the moduli space therefore is,

$$\mathrm{Vol}\left(\mathcal{M}^{k_+,0}(\mathbb{S}^2)\right) = \frac{\left(8\pi^2 R^2(1-\tau) - 4\pi^2 k_+\right)^{k_+}}{k_+!}, \tag{4.120}$$

the proof of the volume formula in $\mathcal{M}^{0,k_-}(\mathbb{S}^2)$ is analogous,

$$\mathrm{Vol}\left(\mathcal{M}^{0,k_-}(\mathbb{S}^2)\right) = \frac{\left(8\pi^2 R^2(1+\tau) - 4\pi^2 k_-\right)^{k_-}}{k_-!}. \tag{4.121}$$

The moduli space of vortex-antivortex pairs

In general, there is no explicit expression for the coefficients b_j of the metric if the cores are at general position, however, we can deduce from the invariance of the

Taubes equation under the action of $O(3)$ several constraints on the coefficients due to symmetry. Before doing so, we need a general lemma that will also be necessary for flat tori in the next section.

Lemma 4.18. *Let $\varphi : U \subset \Sigma \to V \subset \mathbb{C}$ be a holomorphic chart, containing the core set $\mathcal{Z}$ of a point in the moduli space $\mathcal{M}^{1,1}(\Sigma)$. For any bounded domain $D \subset V$, such that $\mathcal{Z} \subset \varphi^{-1}(D)$, there are continuous functions $\tilde{b}_j : D \times D \to \mathbb{C}$, $j = 1, 2$, such that:*

1. *If $\varphi(\mathcal{Z}) = \{z_1, z_2\}$, where z_1 (z_2) is the vortex (antivortex),*

$$b_j(z_1, z_2) = \frac{-2\, s_j}{\bar{z}_1 - \bar{z}_2} + \tilde{b}_j(z_1, z_2), \tag{4.122}$$

where b_j, $j = 1, 2$, are the non-trivial coefficients in the L^2 metric, defined in lemma 2.3.

2.

$$\lim_{|z_1 - z_2| \to 0} \tilde{b}_j(z_1, z_2) = 0. \tag{4.123}$$

Proof. On $\varphi^{-1}(D)$, Green's function can be written as

$$G(x_1, x_2) = \frac{1}{2\pi} \log |\varphi(x_1) - \varphi(x_2)| + \tilde{G}(x_1, x_2), \tag{4.124}$$

with a smooth regular part $\tilde{G} : \varphi^{-1}(D) \times \varphi^{-1}(D) \to \mathbb{R}$. Therefore, the solution h to the Taubes equation can be written as

$$h(x; x_1, x_2) = \tilde{h}(x; x_1, x_2) + \log |\varphi(x) - \varphi(x_1)|^2 - \log |\varphi(x) - \varphi(x_2)|^2 + \tilde{v}(x; x_1, x_2), \tag{4.125}$$

where

$$\tilde{v}(x; x_1, x_2) = 4\pi\, \tilde{G}(x, x_1) - 4\pi\, \tilde{G}(x, x_2), \tag{4.126}$$

and $\tilde{h}(x; x_1, x_2)$ can be extended in C^1 to the coincidence set $x_1 = x_2$ by corollary 4.6. Denoting $h(\varphi^{-1}(z); \varphi^{-1}(z_1), \varphi^{-1}(z_2))$ and $\tilde{h}(\varphi^{-1}(z); \varphi^{-1}(z_1), \varphi^{-1}(z_2))$ as

h, $\tilde{h}$, etcetera,

$$\begin{aligned} b_j(z_1,z_2) &= 2\,\overline{\partial}|_{z=z_j}\,(s_j\,h - \log|z-z_j|) \\ &= 2\,\overline{\partial}_{z=z_j}\left(s_j\tilde{h} - \log|z-z_k| + s_j\tilde{v}\right) \\ &= \frac{-2}{\overline{z}_j - \overline{z}_k} + 2\,s_j\overline{\partial}|_{z=z_j}(\tilde{h}+\tilde{v}) \\ &= \frac{-2\,s_j}{\overline{z}_1 - \overline{z}_2} + \tilde{b}_j, \end{aligned} \tag{4.127}$$

where the regular part $\tilde{b}_j$ is continuous in $D \times D$. This proves the first statement. The second statement is a consequence of corollary 4.6 and the fact that by (4.126),

$$\lim_{|z_1-z_2|\to 0} \overline{\partial}|_{z=z_j}\left(\tilde{v}(\varphi^{-1}(z);\varphi^{-1}(z_1),\varphi^{-1}(z_2))\right) = 0. \tag{4.128}$$

□

Suppose $\gamma : U_1 \subset \mathbb{C} \to U_2 \subset \mathbb{C}$ is a holomorphic change of coordinates in ambient space, such that $z_k \in U_1$ for all cores. There are pairs of corresponding coefficients $b_s(z_1,\dots,z_n)$, $b'_s(z'_1,\dots,z'_n)$ in each of the charts. Let $z' = \gamma(z)$, $z'_k = \gamma(z_k)$, as in [46], we have the transformation rule

$$b'_j = \frac{1}{\overline{\gamma'_j}} b_j - \frac{\overline{\gamma''_j}}{\left(\overline{\gamma'_j}\right)^2}. \tag{4.129}$$

Manton and Nasir noted in [35] that equation (4.129) is similar to the transformation rule for the Levi-Civita connection on $\mathbb{S}^2$ and resembles the topological nature of the coefficients b_j. In the sphere, the group of isometries is large, in the sense that it is a Lie group, and each of this isometries induces a holomorphic change of coordinates on the moduli space. We exploit this remark to prove the following lemmas.

Lemma 4.19. *In the projective chart, the coefficients b_j satisfy the identities,*

$$\sum_k (2\,\overline{z}_k + \overline{z}_k^2\, b_k + \overline{b}_k) = C, \tag{4.130}$$

$$\sum_k \overline{z}_k b_k \in \mathbb{R}, \tag{4.131}$$

for some constant C. For a vortex-antivortex pair, $C = 0$.

Romão deduced similar identities for vortices of a modified Chern-Simons model on the sphere in [46], employing the action of $SO(3)$ on the moduli space.

Proof. Let us consider a rotation $\gamma : \mathbb{S}^2 \to \mathbb{S}^2$. In a stereographic projection chart, γ can be represented as a Möbius transformation,

$$\gamma(z) = \frac{az + b}{-\overline{b}z + \overline{a}}, \tag{4.132}$$

for some coefficients $a, b \in \mathbb{C}$, such that $|a|^2 + |b|^2 = 1$. Since $\gamma : \mathbb{C} \setminus \left\{\overline{a}/\overline{b}\right\} \to \mathbb{C} \setminus \left\{-a/\overline{b}\right\}$ is a holomorphic change of coordinates, a rotation of the core positions in the sphere reads,

$$b_j' = (-b\overline{z}_j + a)^2 b_j - 2b(-b\overline{z}_j + a). \tag{4.133}$$

Invariance of the solutions to the Taubes equation under the group of isometries means that the vector fields generated by $SO(3)$ in the moduli space by diagonally acting on the cores' positions are Killing fields. These fields are generated by the 1-parameter families of matrices,

$$\begin{aligned} U_X(\alpha) &= \begin{pmatrix} \cos\left(\frac{\alpha}{2}\right) & -i\sin\left(\frac{\alpha}{2}\right) \\ -i\sin\left(\frac{\alpha}{2}\right) & \cos\left(\frac{\alpha}{2}\right) \end{pmatrix}, \quad U_Y(\beta) = \begin{pmatrix} \cos\left(\frac{\beta}{2}\right) & -\sin\left(\frac{\beta}{2}\right) \\ \sin\left(\frac{\beta}{2}\right) & \cos\left(\frac{\beta}{2}\right) \end{pmatrix}, \\ U_Z(\gamma) &= \begin{pmatrix} e^{-i\frac{\gamma}{2}} & 0 \\ 0 & e^{i\frac{\gamma}{2}} \end{pmatrix}, \end{aligned} \tag{4.134}$$

$\alpha, \beta, \gamma \in \mathbb{R}$. We can compute conservation equations corresponding to the generators of the Lie algebra $\mathfrak{su}(2)$. These equations correspond to conservation of angular momentum in the moduli space. The generating Killing fields in the moduli space are,

$$\begin{aligned} \xi_X &= \frac{i}{2}\sum_j (z_j^2 - 1)\partial_{z_j} - (\overline{z}_j^2 - 1)\partial_{\overline{z}_j}, \\ \xi_Y &= -\frac{1}{2}\sum_j (z_j^2 + 1)\partial_{z_j} + (\overline{z}_j^2 + 1)\partial_{\overline{z}_j}, \\ \xi_Z &= -i\sum_j z_j\partial_{z_j} - \overline{z}_j\partial_{\overline{z}_j}. \end{aligned} \tag{4.135}$$

By (4.133), the Lie derivatives of the coefficients are,

$$\begin{aligned}\mathcal{L}_{\xi_X} b_j &= i(\overline{z}_j b_j + 1),\\ \mathcal{L}_{\xi_Y} b_j &= \overline{z}_j b_j + 1,\\ \mathcal{L}_{\xi_Z} b_j &= -i b_j.\end{aligned} \tag{4.136}$$

Hence the coefficients b_j satisfy the identities,

$$\frac{1}{2}\sum_k (z_k^2 - 1)\partial_{z_k} b_j - (\overline{z}_k^2 - 1)\partial_{\overline{z}_k} b_j = \overline{z}_j b_j + 1 \tag{4.137}$$

$$-\frac{1}{2}\sum_k (z_k^2 + 1)\partial_{z_k} b_j + (\overline{z}_k^2 + 1)\partial_{\overline{z}_k} b_j = \overline{z}_j b_j + 1, \tag{4.138}$$

$$\sum_k z_k \partial_{z_k} b_j - \overline{z}_k \partial_{\overline{z}_k} b_j = b_j. \tag{4.139}$$

Recall the coefficients b_j have the symmetries,

$$\partial_{z_k} b_j = \partial_{\overline{z}_j} \overline{b}_k, \qquad \partial_{\overline{z}_k} b_j = \partial_{\overline{z}_j} b_k. \tag{4.140}$$

Hence,

$$\sum_k (z_k^2 - 1)\partial_{\overline{z}_j} \overline{b}_k - (\overline{z}_k^2 - 1)\partial_{\overline{z}_j} b_k = 2(\overline{z}_j b_j + 1), \tag{4.141}$$

$$\sum_k (z_k^2 + 1)\partial_{\overline{z}_j} \overline{b}_k + (\overline{z}_k^2 + 1)\partial_{\overline{z}_j} b_k = -2(\overline{z}_j b_j + 1), \tag{4.142}$$

$$\sum_k z_k \partial_{\overline{z}_j} \overline{b}_k - \overline{z}_k \partial_{\overline{z}_j} b_k = b_j. \tag{4.143}$$

Adding equations (4.141) and (4.142) and also subtracting and conjugating the same pair of equations,

$$\sum_k z_k^2 \partial_{\overline{z}_j} \overline{b}_k + \partial_{\overline{z}_j} b_k = 0, \tag{4.144}$$

$$\sum_k \partial_{z_j} b_k + z_k^2 \partial_{z_j} \overline{b}_k = -2(z_j \overline{b}_j + 1). \tag{4.145}$$

From these two equations, we deduce,

$$\partial_{z_j} \sum_k (2\overline{z}_k + \overline{z}_k^2 b_k + \overline{b}_k) = 0, \qquad \partial_{\overline{z}_j} \sum_k (2\overline{z}_k + \overline{z}_k^2 b_k + \overline{b}_k) = 0, \tag{4.146}$$

hence $\sum_k (2\bar{z}_k + \bar{z}_k^2 b_k + \bar{b}_k)$ is constant.

Equation (4.143) implies,

$$\partial_{z_j} \sum_k \left(z_k \bar{b}_k - \bar{z}_k b_k\right) = 0. \tag{4.147}$$

From this equation and its conjugate, $\sum_k \left(\bar{z}_k b_k - z_k \bar{b}_k\right)$ is constant, but this quantity must be zero when all the vortices and antivortices are located on the real line. Therefore,

$$\sum_k \bar{z}_k b_k \in \mathbb{R}. \tag{4.148}$$

Finally, for a vortex-antivortex pair at positions $z_\pm = \pm\epsilon$, we have $b_\pm(\epsilon, -\epsilon) \in \mathbb{R}$ and by (4.130),

$$b_+(\epsilon, -\epsilon) + b_-(\epsilon, -\epsilon) = \frac{C}{1+\epsilon^2}. \tag{4.149}$$

By lemma 4.18, there are continuous functions $\tilde{b}_\pm : \mathbb{R} \to \mathbb{R}$ such that,

$$b_\pm(\epsilon, -\epsilon) = \mp\frac{1}{\epsilon} + \tilde{b}_\pm(\epsilon), \tag{4.150}$$

and $\lim_{\epsilon\to 0} \tilde{b}_\pm(\epsilon) = 0$, hence,

$$\lim_{\epsilon\to 0}(b_+(\epsilon, -\epsilon) + b_-(\epsilon, -\epsilon)) = \lim_{\epsilon\to 0}(\tilde{b}_+(\epsilon) + \tilde{b}_-(\epsilon)) = 0. \tag{4.151}$$

Therefore, $C = 0$ for a vortex-antivortex pair.

□

Let $\mathbb{S}^2_\Delta$ be the diagonal in the product $\mathbb{S}^2 \times \mathbb{S}^2$. The orthogonal group acts diagonally on the moduli space $\mathcal{M}^{1,1}(\mathbb{S}^2) \cong (\mathbb{S}^2 \times \mathbb{S}^2) \setminus \mathbb{S}^2_\Delta$ by isometries. We can always assume there is a projective chart such that the pair is located with the vortex at $z_1 = \epsilon$ and the antivortex at $z_2 = -\epsilon$. From (4.130) and the fact that

$$b_j(\epsilon, -\epsilon) = \overline{b_j}(\epsilon, -\epsilon), \tag{4.152}$$

we conclude,

$$b_1(\epsilon, -\epsilon) + b_2(\epsilon, -\epsilon) = 0. \tag{4.153}$$

The L^2 metric in $\mathcal{M}^{1,1}(\mathbb{S}^2)$ is Kähler and invariant under the diagonal action of $O(3)$, given any pair $(z_1, z_2) \in \mathcal{M}^{1,1}(\mathbb{S}^2)$, we can always find a rotation of $\mathbb{S}^2$

such that in south pole stereographic projection, $z_1 = \epsilon$, $z_2 = -\epsilon$. In this way, we have a diffeomorphism,

$$(\mathbb{S}^2 \times \mathbb{S}^2) \setminus \mathbb{S}^2_\Delta \cong (0,1] \times SO(3), \tag{4.154}$$

hence, the moduli space can be parametrized as $(0,1] \times SO(3)$.

Lemma 4.20. *Let g be a Kähler metric in $\mathbb{S}^2 \times \mathbb{S}^2$ such that if $o \in O(3)$ and $(z_1, z_2) \in \mathbb{S}^2 \times \mathbb{S}^2$, then the action*

$$o * (z_1, z_2) = (o * z_1, o * z_2), \tag{4.155}$$

is by isometries. Let $E_0 = \partial_\epsilon$ and let $E_j \in \mathfrak{so}(3)$ be the left invariant vector field corresponding to rotations with respect to the j-th coordinate axis in $\mathbb{R}^3$. Then there exists a function

$$A : (0,1] \to \mathbb{R}, \tag{4.156}$$

and a real constant c such that in the parametrization (4.154),

$$g = A\left(\frac{1-\epsilon^2}{1+\epsilon^2}(\sigma^1)^2 + \frac{1+\epsilon^2}{1-\epsilon^2}(\sigma^2)^2\right) - \frac{1}{\epsilon}\frac{dA}{d\epsilon}\left((\sigma^0)^2 + \epsilon^2(\sigma^3)^2\right) + \frac{c}{1+\epsilon^2}\left(\sigma^0\sigma^2 + \frac{\epsilon(1-\epsilon^2)}{1+\epsilon^2}\sigma^1\sigma^3\right), \tag{4.157}$$

where $\sigma^k \in T^*((0,1] \times SO(3))$ is the co-vector dual to E_k, $k = 0, \ldots, 3$. For this metric, the volume is,

$$\mathrm{Vol}\left(\mathbb{S}^2 \times \mathbb{S}^2\right) = 4\pi^2 \lim_{\epsilon \to 0} A(\epsilon)^2 - c^2\pi^2. \tag{4.158}$$

Proof. This lemma is similar to [45, Prop. 5.1], but for $\tau \neq 0$, the swapping map $(z_1, z_2) \mapsto (z_2, z_1)$ is no longer a symmetry of the metric, instead, we consider the action of orientation reversing isometries of the sphere on the moduli space.

A general symmetric bilinear form in $T((0,1) \times SO(3))$ invariant under the diagonal $SO(3)$ action, will be a linear combination

$$A_{rs}\sigma^r\sigma^s, \tag{4.159}$$

with $A_{rs} = A_{sr}$. Let $q(\epsilon) = (\epsilon, -\epsilon)$, $\epsilon \in (0,1]$. Denoting by (X, Y, Z) coordinates in $\mathbb{R}^3$, the basis E_j can be represented in the canonical embedding of $\mathbb{S}^2$ as the unit sphere in $\mathbb{R}^3$ as

$$E_0 = \frac{2(1-\epsilon^2)}{(1+\epsilon^2)^2}\left(\frac{\partial}{\partial X_1} - \frac{\partial}{\partial X_2}\right) - 4\frac{\epsilon}{(1+\epsilon^2)^2}\left(-\frac{\partial}{\partial Z_1} - \frac{\partial}{\partial Z_2}\right), \tag{4.160}$$

$$E_1 = -\frac{1-\epsilon^2}{1+\epsilon^2}\left(\frac{\partial}{\partial Y_1} + \frac{\partial}{\partial Y_2}\right), \tag{4.161}$$

$$E_2 = -\frac{1-\epsilon^2}{1+\epsilon^2}\left(\frac{\partial}{\partial X_1} + \frac{\partial}{\partial X_2}\right) + \frac{2\epsilon}{1+\epsilon^2}\left(-\frac{\partial}{\partial Z_1} + \frac{\partial}{\partial Z_2}\right), \tag{4.162}$$

$$E_3 = \frac{2\epsilon}{1+\epsilon^2}\left(\frac{\partial}{\partial Y_1} - \frac{\partial}{\partial Y_2}\right). \tag{4.163}$$

A short calculation yields,

$$\mathrm{J}E_0 = \frac{1}{\epsilon}E_3, \qquad \mathrm{J}E_1 = \frac{1-\epsilon^2}{1+\epsilon^2}E_2, \tag{4.164}$$

where J is the pseudo-complex structure on $T((\mathbb{S}^2 \times \mathbb{S}^2) \setminus \mathbb{S}^2_\Delta)$. If the metric is Kähler, we deduce,

$$A_{03} = A_{12} = 0, \qquad A_{33} = \epsilon^2 A_{00}, \qquad A_{11} = \left(\frac{1-\epsilon^2}{1+\epsilon^2}\right)^2 A_{22}. \tag{4.165}$$

Let $C : \mathbb{S}^2 \to \mathbb{S}^2$ be the reflection map $Y \mapsto -Y$ on the XZ plane. C acts on σ^k as follows,

$$C^*\sigma^0 = \sigma^0, \qquad C^*\sigma^1 = -\sigma^1, \qquad C^*\sigma^2 = \sigma^2, \qquad C^*\sigma^3 = -\sigma^3. \tag{4.166}$$

From reflection invariance we further obtain,

$$A_{01} = A_{23} = 0. \tag{4.167}$$

Let $A = A_{00}$, $B = (1+\epsilon^2)^{-2}\,A_{22}$, then the metric is,

$$g = A\left((\sigma^0)^2 + \epsilon^2(\sigma^3)^2\right) + B\left((1-\epsilon^2)^2\,(\sigma^1)^2 + (1+\epsilon^2)^2\,(\sigma^2)^2\right) + A_{02}\,\sigma^0\sigma^2 + A_{13}\,\sigma^1\sigma^3. \tag{4.168}$$

If $\omega = g(\mathrm{J}\cdot,\cdot)$ is the Kähler form of the metric, then

$$\omega = \epsilon\,A\,\sigma^0\wedge\sigma^3 + (1-\epsilon^4)\,B\,\sigma^1\wedge\sigma^2 + \frac{1}{\epsilon}A_{13}\,\sigma^0\wedge\sigma^1 - \frac{1+\epsilon^2}{1-\epsilon^2}A_{13}\,\sigma^2\wedge\sigma^3, \tag{4.169}$$

provided $\epsilon A_{02} = (1+\epsilon^2)(1-\epsilon^2)^{-1}A_{13}$, to account for skew-symmetry of ω. The $SO(3)$ valued forms $\sigma^1, \sigma^2, \sigma^3$, are related by $d\sigma^1 = -\sigma^2 \wedge \sigma^3$ and cyclic permutations of this identity. Kähler forms are closed. For ω this is true provided the coefficients in (4.169) are solutions to the equations,

$$\epsilon A = -\frac{d}{d\epsilon}\left((1-\epsilon^4)\,B\right), \qquad \frac{1}{\epsilon}A_{13} = \frac{d}{d\epsilon}\left(\epsilon A_{02}\right). \tag{4.170}$$

Regularity of the metric as $\epsilon \to 1$ implies $\lim_{\epsilon\to 1}(1-\epsilon^4)B(\epsilon) = 0$. From the second equation in (4.170) and the algebraic relation of the coefficients A_{13}, A_{02}, we infer

$$A_{02} = \frac{c}{1+\epsilon^2}, \tag{4.171}$$

for some real constant c. Redefining the function $(1-\epsilon^4)B$ as $A(\epsilon)$, the metric has the form (4.157). Since $\int_{SO(3)} \sigma^1 \wedge \sigma^2 \wedge \sigma^3 = 8\pi^2$ [45] and

$$\int_0^1 \frac{\epsilon\,(1-\epsilon^2)}{(1+\epsilon^2)^3}d\epsilon = \frac{1}{8}, \tag{4.172}$$

for this metric, the volume form is

$$\mathrm{Vol} = -\left(A\,A' + \frac{c^2\epsilon(1-\epsilon^2)}{(1+\epsilon^2)^3}\right) d\epsilon \wedge \sigma^1 \wedge \sigma^2 \wedge \sigma^3. \tag{4.173}$$

After integration, the total volume of the metric is

$$\mathrm{Vol}\left(\mathbb{S}^2 \times \mathbb{S}^2\right) = 4\pi^2 \lim_{\epsilon\to 0} A(\epsilon)^2 - c^2\pi^2. \tag{4.174}$$

□

Applying lemma 4.20 to the L^2 metric, we obtain,

Lemma 4.21. *The L^2 metric on $\mathcal{M}^{1,1}(\mathbb{S}^2)$ has the structure provided by Lemma 4.20, with*

$$A = 2\pi\left(\frac{4R^2}{1+\epsilon^2} - \epsilon\, b_1 - 2R^2 - 1\right), \tag{4.175}$$

$$c = 8\pi R^2 \tau. \tag{4.176}$$

Proof. To compute the constant c, we calculate $g\left(E_0, E_2\right)$. Tangent vectors E_0, E_2 in projective coordinates $(z_1, z_2) \in \mathbb{S}^2 \times \mathbb{S}^2$ with respect to the south pole are,

$$E_0 = \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \qquad E_2 = \frac{1+\epsilon^2}{2}\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right). \tag{4.177}$$

where $z_k = x_k + iy_k$. Thence,

$$\begin{aligned} g\left(E_0, E_2\right) &= \frac{1+\epsilon^2}{2} g\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right) \\ &= \frac{1+\epsilon^2}{2} 2\pi\left(\Omega(1+\tau) - \Omega(1-\tau) + \frac{\partial b_1}{\partial z_1} + \frac{\partial b_2}{\partial z_1} - \frac{\partial b_1}{\partial z_2} - \frac{\partial b_2}{\partial z_2}\right) \end{aligned} \tag{4.178}$$

To simplify (4.178), we use the symmetries of the coefficients b_j, lemma 4.19,

$$\begin{aligned} \sum_j \left(\frac{\partial}{\partial z_1} b_j - \frac{\partial}{\partial z_2} b_j\right) &= \frac{1}{2}\sum_j \frac{d}{d\epsilon} b_j(\epsilon, -\epsilon) - \frac{i}{2}\sum_j\left(\frac{\partial}{\partial y_1} b_j - \frac{\partial}{\partial y_2} b_j\right) \\ &= \frac{1}{2}\frac{d}{d\epsilon}(b_1 + b_2) + \frac{1}{2\epsilon}(b_1 + b_2) \\ &= 0. \end{aligned} \tag{4.179}$$

Hence,

$$g(E_0, E_2) = \frac{8\pi R^2 \tau}{1+\epsilon^2} \tag{4.180}$$

and consequently $c = 8\pi R^2 \tau$. Let us compute $g(E_0, E_0)$,

$$\begin{aligned} g(E_0, E_0) &= g\left(\frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_1} - \frac{\partial}{\partial x_2}\right) \\ &= 2\pi\left(\Omega(1+\tau) + \Omega(1-\tau) + \frac{\partial}{\partial z_1} b_1 - \frac{\partial}{\partial z_1} b_2 - \frac{\partial}{\partial z_2} b_1 + \frac{\partial}{\partial z_2} b_2\right). \end{aligned} \tag{4.181}$$

Again by symmetry,

$$\frac{\partial}{\partial z_1} b_j - \frac{\partial}{\partial z_2} b_j = \frac{1}{2}\frac{db_j}{d\epsilon} + \frac{1}{2\epsilon} b_j. \tag{4.182}$$

Hence,

$$g(E_0, E_0) = 2\pi\left(\frac{8R^2}{(1+\epsilon^2)^2} + \frac{db_1}{d\epsilon} + \frac{1}{\epsilon} b_1\right). \tag{4.183}$$

Comparing (4.183) and (4.157),

$$-\frac{1}{\epsilon}\frac{dA}{d\epsilon} = 2\pi\left(\frac{8R^2}{(1+\epsilon^2)^2} + \frac{db_1}{d\epsilon} + \frac{1}{\epsilon}b_1\right), \tag{4.184}$$

Solving this equation, we find,

$$A = \frac{8\pi R^2}{1+\epsilon^2} - 2\pi\epsilon b_1 + \text{const.} \tag{4.185}$$

From the regularity condition $\lim_{\epsilon\to 1} A(\epsilon) = 0$ used to compute the formula for the volume of the moduli space and the explicit formula (4.113) for b_1 in the antipodal case, the constant is

$$\text{const.} = -2\pi(2R^2+1). \tag{4.186}$$

Therefore,

$$A = 2\pi\left(\frac{4R^2}{1+\epsilon^2} - \epsilon b_1 - 2R^2 - 1\right). \tag{4.187}$$

□

We claim that

$$\lim_{\epsilon\to 0}\epsilon b_1 = -1 \tag{4.188}$$

as can be seen numerically in figure 4.2 for the symmetric case in the unit sphere.

For a vortex-antivortex pair,

$$b_1(\epsilon,-\epsilon) = 2\,\frac{\partial}{\partial x}\bigg|_{z=\epsilon} h_\epsilon - \frac{1}{\epsilon}. \tag{4.189}$$

Since $h_\epsilon \to \mu$ in C^1 as $\epsilon \to 0$,

$$\lim_{\epsilon\to 0}\epsilon\, b_1(\epsilon,-\epsilon) = -1. \tag{4.190}$$

Applying lemmas 4.20 and 4.21, the volume of the moduli space is

$$\mathrm{Vol}\left(\mathcal{M}^{1,1}(\mathbb{S}^2)\right) = \left(8\pi^2R^2\right)^2(1-\tau^2). \tag{4.191}$$

Notice that another way to express the volume is as $4\pi^2(1-\tau^2)\mathrm{Vol}(\mathbb{S}^2)$, which corresponds to the volume of a product of spheres, each factor weighted by $2\pi(1\pm\tau)$, the effective mass of a core, hence, it is expected that as $\tau\to\pm1$, the volume vanishes, because of the negligible weight of one of the factors.

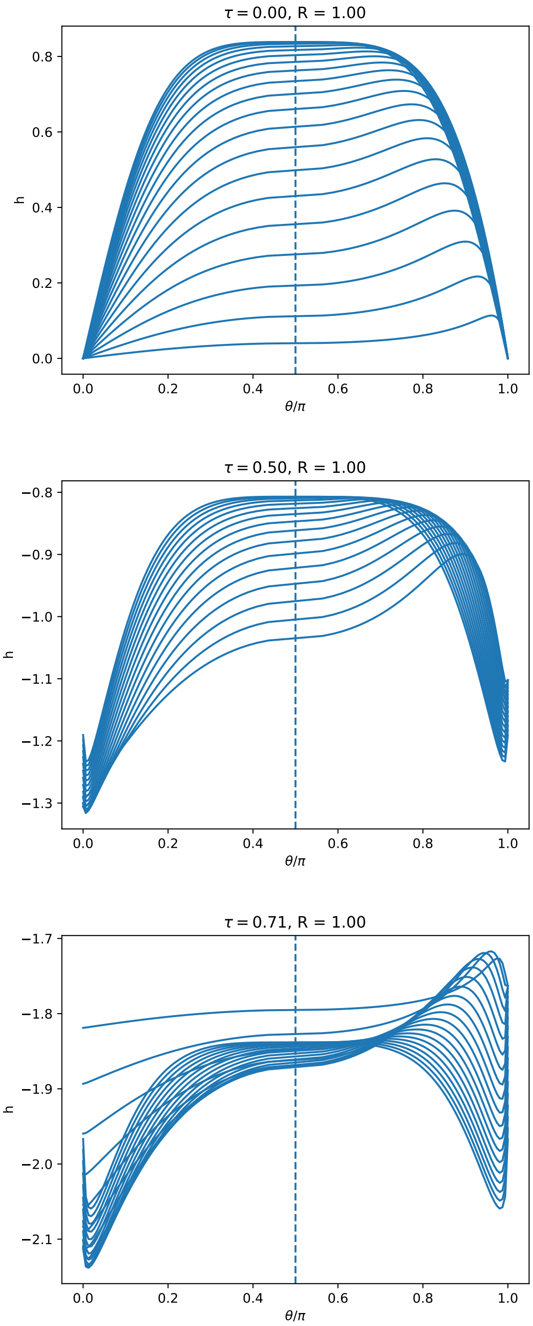


Figure 4.1: Three views of the declination data of $\tilde{h}_\epsilon$, the regular part of the solution to the Taubes equation, for three different values of the asymmetry parameter τ on the unit sphere. **Top.** Vortex and antivortex are symmetric, with the same effective mass. **Middle and bottom**. The antivortex becomes more massive. We solved from $\epsilon = 1$ down to 0.05 in steps of 0.05, except that for $\tau = 0.5$, the computation stopped at $\epsilon = .20$ due to algorithm divergence. As $\epsilon \to 0$ the data shows how $\tilde{h}_\epsilon$ flattens as expected.

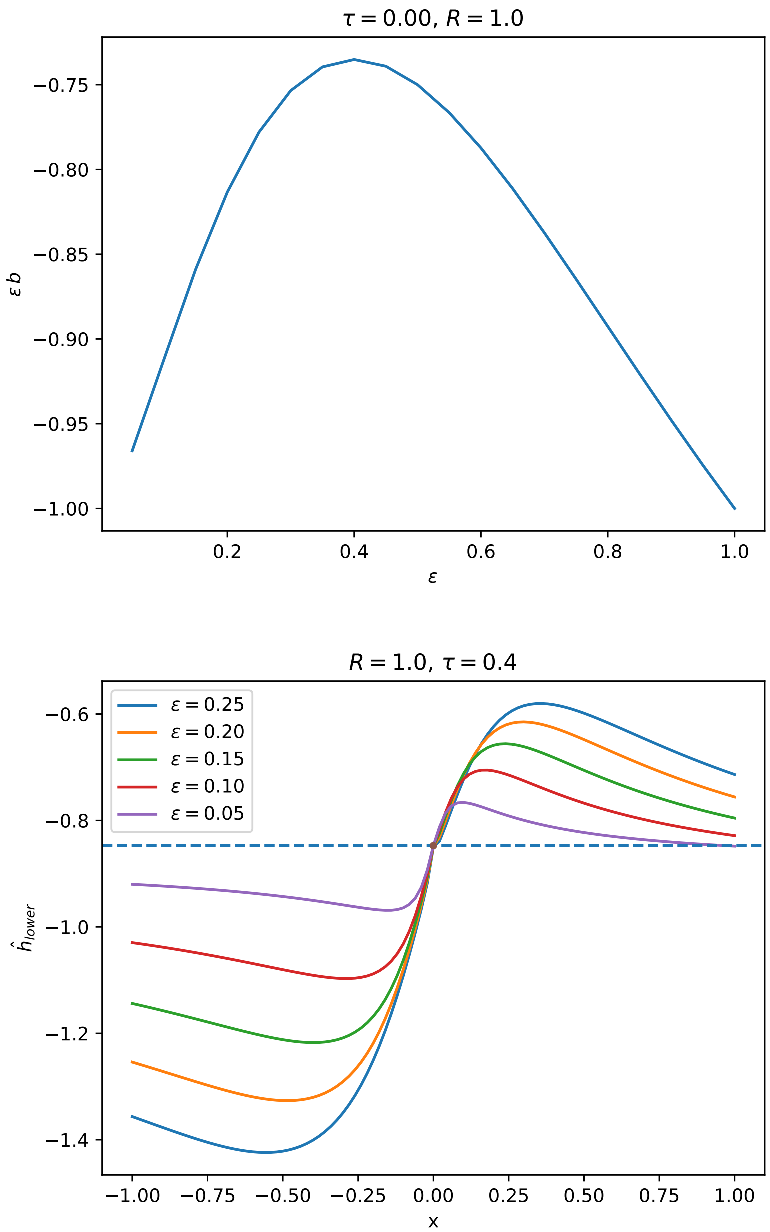


Figure 4.2: **Top**. Real profile of ϵb in the symmetric case. The limit $\lim_{\epsilon\to 0} \epsilon b = -1$ is apparent in the numerical data. **Bottom**. Real profile of a vortex-antivortex pair located at $\pm\epsilon$ on the real axis of the extended complex plane for several values of ϵ. In both cases, the domain is the unit sphere, the bottom plot shows the behaviour of the real profile of $\tilde{h}$ as $\epsilon \to 0$ in the south pole of the domain. The dashed horizontal line is $\log\left((1-\tau)(1+\tau)^{-1}\right)$. The data shows how the regular part of the solution to the Taubes equation converges to this constant value as the pair collides at the north pole.

4.3.2 Flat tori

In this section we compute the volume of the moduli space for a flat tori, to this end, we extend the coefficients b_q in the L^2 metric to a global object and relate it to the volume of $\mathcal{M}^{1,1}(\mathbb{T}^2)$ in lemma 4.22. Consider a holomorphic chart $\varphi : U \subset \mathbb{T}^2 \to \mathbb{C}$ on an open and dense set U, with coordinates $z = \varphi(x)$, $x \in U$. Let us define,

$$b_U = b_j \, d\bar{z}^j \in \Omega^{(0,1)}((U \times U) \setminus \Delta_U). \tag{4.192}$$

In general b_U is only well defined on a chart, however, flat tori admit atlases such that the holomorphic changes of coordinates are translations. Since translations have trivial second derivatives, by (4.129) b_U extends to a global form $b \in \Omega^{(0,1)}(\mathcal{M}^{1,1}(\mathbb{T}^2))$. By the symmetries of the coefficients b_j, this form is holomorphic, as the following short calculation shows in coordinates:

$$\begin{aligned} \bar{\partial} b_U &= \sum_{i,j} \bar{\partial}_{z_i} b_j \, d\bar{z}^i \wedge d\bar{z}^j \\ &= -\sum_{i,j} \bar{\partial}_{z_j} b_i \, d\bar{z}^j \wedge d\bar{z}^i \\ &= -\bar{\partial} b_U, \end{aligned} \tag{4.193}$$

hence, $\bar{\partial} b_U = 0$.

To compute the volume of flat tori, we will use the $(1,1)$-form ∂b to define another form in the moduli space which is more convenient for calculations. Let $\Pi_j : \mathbb{T}^2 \times \mathbb{T}^2 \to \mathbb{T}^2$ be the canonical projection map onto the j-th factor of the product. Let us define the form

$$\omega_0 = 2\pi \,(1-\tau)\, \Pi_1^* \, \omega_{\mathbb{T}^2} + 2\pi \,(1+\tau)\, \Pi_2^* \, \omega_{\mathbb{T}^2}. \tag{4.194}$$

The Kähler form on the moduli space can be written as,

$$\omega = \omega_0 + \pi i \, \partial b \in \Lambda^{1,1}(\mathcal{M}^{1,1}(\mathbb{T}^2)). \tag{4.195}$$

Notice that,

$$\begin{aligned} \mathrm{Vol} &= \frac{1}{2} \, \omega \wedge \omega \\ &= \mathrm{Vol}_0 + \pi i \, \omega_0 \wedge \partial b - \frac{\pi^2}{2} \partial b \wedge \partial b, \end{aligned} \tag{4.196}$$

where $\mathrm{Vol}_0 = \frac{1}{2}\omega_0 \wedge \omega_0$ is the restriction of the volume form in the product $\mathbb{T}^2 \times \mathbb{T}^2$ to the moduli space.

Lemma 4.22. *Let Δ_ϵ be the ϵ-tubular neighbourhood of the diagonal set of $\mathbb{T}^2 \times \mathbb{T}^2$ for small ϵ. The volume of the moduli space can be computed as,*

$$\begin{aligned}\mathrm{Vol}(\mathcal{M}^{1,1}(\mathbb{T}^2)) = 4\pi^2(1-\tau^2)\,\mathrm{Vol}(\mathbb{T}^2)^2 \\ + \lim_{\epsilon\to 0}\int_{\mathbb{T}^2\times\mathbb{T}^2\setminus\Delta_\epsilon}\left(\pi i\,\omega_0 \wedge \partial b - \frac{\pi^2}{2}\partial b \wedge \partial b\right).\end{aligned} \tag{4.197}$$

Proof.

$$\begin{aligned}\mathrm{Vol}(\mathcal{M}^{1,1}(\mathbb{T}^2)) &= \lim_{\epsilon\to 0}\int_{\mathbb{T}^2\times\mathbb{T}^2\setminus\Delta_\epsilon}\mathrm{Vol} \\ &= \int_{\mathbb{T}^2\times\mathbb{T}^2}\mathrm{Vol}_0 + \lim_{\epsilon\to 0}\int_{\mathbb{T}^2\times\mathbb{T}^2\setminus\Delta_\epsilon}\left(\pi i\,\omega_0\wedge\partial b - \frac{\pi^2}{2}\partial b\wedge\partial b\right).\end{aligned} \tag{4.198}$$

On the other hand,

$$\mathrm{Vol}_0 = 4\pi^2(1-\tau^2)\,\Pi_1^*\,\omega_{\mathbb{T}^2} \wedge \Pi_2^*\,\omega_{\mathbb{T}^2}. \tag{4.199}$$

Applying Fubini and the change of variables theorems,

$$\int_{\mathbb{T}^2\times\mathbb{T}^2}\mathrm{Vol}_0 = 4\pi^2(1-\tau^2)\left(\int_{\mathbb{T}^2}\omega_{\mathbb{T}^2}\right)^2 = 4\pi^2(1-\tau^2)\mathrm{Vol}(\mathbb{T}^2)^2. \tag{4.200}$$

This concludes the proof of the lemma. □

According to lemma 4.22, to compute the volume of $\mathcal{M}^{1,1}(\mathbb{T})$, we must compute the two non-trivial terms in (4.197).

Lemma 4.23. *Let $\pi : \mathbb{C} \to \mathbb{T}^2$ be the canonical covering map and let $R \subset \mathbb{C}$ be an open parallelogram such that $\pi|_R : R \to \mathbb{T}^2$ is a bi-holomorphism onto its image and $U = \pi|_R(R)$ is open and dense. On the local coordinates $\pi|_R^{-1} : U \to R$, there is a constant $c \in \mathbb{C}$ such that for any pair of different points $z_1, z_2 \in R$,*

$$b_1(z_1, z_2) + b_2(z_1, z_2) = c, \tag{4.201}$$

Proof. If $\mathcal{I} : \mathbb{T}^2 \to \mathbb{T}^2$ is an isometry, the Taubes equation is invariant under $\mathcal{I}$,

$$h(\mathcal{I}(x); \mathcal{I}(x_1), \mathcal{I}(x_2)) = h(x; x_1, x_2), \tag{4.202}$$

$x, x_1, x_2 \in \mathbb{T}^2$, $x_1 \neq x_2$. By construction, there is a $v \in \mathbb{C}$ such that $\mathcal{I}_\varphi = \varphi \circ \mathcal{I} \circ \varphi^{-1}(z) = z + v$ for $z \in \varphi(\mathcal{I}^{-1}(U) \cap U)$. For small v, the translation $\mathcal{I}_\varphi$ maps a neighbourhood, not necessarily connected, $N \subset R$ of x_1 and x_2 into R. This implies b_j has the symmetries,

$$b_j(z_1 + v, z_2 + v) = b_j(z_1, z_2), \tag{4.203}$$

v small. Hence,

$$\partial_{z_1} b_j + \partial_{z_2} b_j = \bar{\partial}_{z_1} b_j + \bar{\partial}_{z_2} b_j = 0. \tag{4.204}$$

Applying the symmetries of the coefficients b_j,

$$\partial_{z_j}(b_1 + b_2) = \bar{\partial}_{z_1} \bar{b}_j + \bar{\partial}_{z_2} \bar{b}_j = 0. \tag{4.205}$$

Similarly,

$$\bar{\partial}_{z_j}(b_1 + b_2) = 0. \tag{4.206}$$

Hence $b_1 + b_2$ is constant on the connected neighbourhood R. □

Proposition 4.24. *In a flat torus $\mathbb{T}^2$, for the $(1,1)$ form ∂b we have,*

$$\partial b \wedge \partial b = 0. \tag{4.207}$$

Proof. We apply the previous lemma to prove the proposition. By lemma 4.23, there is an open and dense set $U \subset \mathbb{T}^2$ and a chart $\varphi : U \to R \subset \mathbb{C}$, R an open parallelogram, such that in this local coordinates $b_1 + b_2$ is a constant. Denoting points in R as z_j, a direct calculation shows,

$$\begin{aligned} b_U \wedge \partial b_U &= (b_2\, \partial_{z_1} b_1 - b_1\, \partial_{z_1} b_2)\, dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 \\ &\quad + (-b_2\, \partial_{z_2} b_1 + b_1\, \partial_{z_2} b_2)\, d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 \\ &= -c\, \partial_{z_1} b_2\, dz_1 \wedge d\bar{z}_1 \wedge d\bar{z}_2 - c\, \partial_{z_2} b_1\, d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2. \end{aligned} \tag{4.208}$$

Since b_1 and b_2 add to a constant,

$$\partial b_U \wedge \partial b_U = -c\, (\partial_{z_2} \partial_{z_1} b_2 + \partial_{z_1} \partial_{z_2} b_1)\, dz_1 \wedge d\bar{z}_1 \wedge dz_2 \wedge d\bar{z}_2 = 0. \tag{4.209}$$

Since U is dense, we conclude $\partial b \wedge \partial b \equiv 0$. □

By this proposition and lemma 4.22, to compute the volume of the moduli space, we have to integrate $\omega_0 \wedge \partial b$.

Theorem 4.25. *For a flat torus* $\mathbb{T}^2$*, the volume of the moduli space is,*

$$\mathrm{Vol}(\mathcal{M}^{1,1}(\mathbb{T}^2)) = 4\pi^2(1-\tau^2)\,\mathrm{Vol}(\mathbb{T}^2)^2 + 16\pi^3\,\mathrm{Vol}(\mathbb{T}^2). \tag{4.210}$$

Notice that the first term of the formula is similar to the case of the sphere (4.191), however, the second term is new, bearing in mind the volume conjecture, 4.16, one can argue the extra term is related to the genus of the base surface, however, it is not clear how to relate our computation to this fact and the relation is open to future work.

Proof. Let,

$$\mathbb{T}^2(\epsilon) = (\mathbb{T}^2 \times \mathbb{T}^2) \setminus \Delta_\epsilon, \tag{4.211}$$

$$\omega_j = \Pi_j^* \omega_{\mathbb{T}^2}, \qquad j = 1, 2, \tag{4.212}$$

and let k be the complementary index of j, such that $\{j,k\} = \{1,2\}$. By Fubini's theorem,

$$\int_{\mathbb{T}^2(\epsilon)} \omega_0 \wedge \partial b = 2\pi \sum_j (1 - s_j\tau) \int_{\mathbb{T}^2} \left(\int_{\mathbb{T}^2 \setminus \mathbb{D}_\epsilon(x_j)} \iota_k^* \partial b \right) \omega_{\mathbb{T}^2}, \tag{4.213}$$

where for any given $x_j \in \mathbb{T}^2$, $\iota_k : \mathbb{T}^2 \hookrightarrow \mathbb{T}^2 \times \mathbb{T}^2$ is the inclusion of the torus as the k-th factor of the product anchored at x_j. Since b is well defined globally,

$$\int_{\mathbb{T}^2 \setminus \mathbb{D}_\epsilon(x_j)} \iota_k^* \partial b = \int_{\partial \mathbb{T}^2 \setminus \mathbb{D}_\epsilon(x_j)} \iota_k^* b = -\int_{\partial \mathbb{D}_\epsilon(x_j)} \iota_k^* b, \tag{4.214}$$

where we always orient a submanifold by the outward pointing normal. Let $\varphi : U \to \mathbb{C}$ be a holomorphic chart defined on an open and dense set U. If $x_j \in U$, for small ϵ, $\mathbb{D}_\epsilon(x_j) \subset U$. Assume $j = 1$, $k = 2$, in the chart,

$$(\varphi^{-1})^* \iota_2^* b = b_2 d\bar{z}. \tag{4.215}$$

If $z_1 = \varphi(x_1)$ and $D(z_1) \subset \mathbb{C}$ is a bounded domain and neighbourhood of z_1, by lemma 4.18,

$$b_2 = \frac{2}{\bar{z}_1 - \bar{z}_2} + \tilde{b}_2(z_1, z_2), \qquad z_2 \in D(z_1). \tag{4.216}$$

If $D_\epsilon(z_j) = \varphi(\mathbb{D}_\epsilon(x_j))$, by Cauchy's residue theorem,

$$\int_{\partial\mathbb{D}_\epsilon(x_1)} \iota_2^* b = -2\int_{\partial D_\epsilon(z_1)} \frac{d\overline{z}}{\overline{z}-\overline{z}_1} + \int_{\partial D_\epsilon(z_1)} \tilde{b}_2(z_1,z)d\overline{z} = 4\pi i + \int_{\partial D_\epsilon(z_1)} \tilde{b}_2(z_1,z)d\overline{z}. \tag{4.217}$$

If $j=2$, $k=1$, we find a similar result,

$$\int_{\partial\mathbb{D}_\epsilon(x_2)} \iota_1^* b = 4\pi i + \int_{\partial D_\epsilon(z_2)} \tilde{b}_1(z,z_2)d\overline{z}. \tag{4.218}$$

Since $\tilde{b}_k$ is a continuous function in a neighbourhood of each $z_j \in \mathbb{C}$,

$$\lim_{\epsilon\to 0} \int_{\partial D_\epsilon(z_j)} \tilde{b}_k \, d\overline{z} = 0. \tag{4.219}$$

Hence, since U is dense in $\mathbb{T}^2$,

$$\begin{aligned} lim_{\epsilon\to 0} \int_{\mathbb{T}^2} \left(\int_{\mathbb{T}^2\setminus\mathbb{D}_\epsilon(x_j)} \iota_k^* \partial b \right) \omega_{\mathbb{T}^2} &= -4\pi i \mathrm{Vol}(\mathbb{T}^2) \\ &\quad - \frac{i}{2}\int_{\mathbb{C}} lim_{\epsilon\to 0} \left(\int_{\partial D_\epsilon(z_j)} \tilde{b}_k \right) e^{\Lambda(z_j)} dz_j \wedge d\overline{z}_j \\ &= -4\pi i \mathrm{Vol}(\mathbb{T}^2). \end{aligned} \tag{4.220}$$

Finally,

$$\begin{aligned} \int_{\mathcal{M}^{1,1}(\mathbb{T}^2)} \omega_0 \wedge \partial b &= 2\pi \sum_j (1-s_j\tau) lim_{\epsilon\to 0} \int_{\mathbb{T}^2} \left(\int_{\mathbb{T}^2\setminus\mathbb{D}_\epsilon(x_j)} \iota_k^* \partial b \right) \omega_{\mathbb{T}^2}. \\ &= 2\pi \sum_j (1-s_j\tau)\left(-4\pi i \mathrm{Vol}(\mathbb{T}^2)\right) \\ &= -16\pi^2\, i \,\mathrm{Vol}(\mathbb{T}^2). \end{aligned} \tag{4.221}$$

By lemma 4.22, we conclude the volume formula. □

Chapter 5

Chern-Simons deformations of vortices

In this chapter we consider Chern-Simons deformations of vortices of the $O(3)$ Sigma model and of the Abelian Higgs model. We will consider deformations relying on a deformation constant κ. There are several results in the literature about existence of solutions to the field equations, for both types of models.

In section 5.1 we address existence and uniqueness of solutions to the field equations for deformations of the $O(3)$ Sigma model. On the plane, Han and Nam prove in [19] that the field equations admit a solution up to some upper and lower bound for κ. If there are only vortices or antivortices, Han and Song prove in [20] existence of solutions for any κ. On a flat torus, Chae-Nam [5] and Chiacchio-Ricciardi prove [7] the existence of a bound on the constant for the existence of solutions as well as the existence of multiple solutions if the number of vortices and antivortices on the surface is different. We extend the technique used by Flood and Speight in [14] for Chern-Simons deformations of the Abelian Higgs model to show the existence of a minimal deformation constant, independent of the position of the vortices, if the surface is compact. We know from chapter 4 that for $\kappa = 0$, the moduli space is incomplete, imposing some technical difficulties in the techniques used for deformations of the Abelian Higgs model. In subsection 5.1.1, we show that on a compact surface, small deformations of the solution to the Taubes equation vary smoothly with κ. In subsection 5.1.2, we show the existence of a positive lower bound for $|\kappa|$, independent of the position of the cores. In subsection 5.1.3 we focus on the unbalanced case where the number of vortices and antivortices differ and show that the possible constants κ for which the field equations admit a solution are bounded. By means of several bounds in the norm of solutions to the governing elliptic problem, we show the existence of multiple solutions in subsection 5.1.4. We finalize in subsection 5.1.5 with numerical evidence on the sphere supporting a conjecture about the existence of solutions to the field equations for any κ if the number of vortices and antivortices coincide.

In section 5.2 we study low energy dynamics of both the Abelian Higgs and the $O(3)$ Sigma model vortices with a Chern-Simons deformation. The results discussed in section 5.1 guarantee this is a well posed problem for small κ. Previous work in this direction includes the models by Kim-Lee [25] and by Collie-Tong [10] for deformations of Abelian vortices. From the work of Alqahtani-Speight [1] we know the model of Kim-Lee cannot extend to the coincidence set. We show our formula can be extended and compare it with the model of Collie-Tong for deformations of Abelian vortices, showing that our computation leads to different dynamics for pairs of vortices on the plane.

In subsection 5.2.1 we introduce the Maxwell-Higgs-Chern-Simons model, as we will see, the introduction of a Chern-Simons term in the field equations induces a connection term affecting the dynamics in moduli space. In subsections 5.2.2 and 5.2.3 we find the localization formula (5.215) for this term and compare it with the Collie-Tong connection. In subsection 5.2.4 we show how to extend our arguments to include the $O(3)$ Sigma model. Finally, in subsection 5.2.5, we show that the connection term can be extended to coalescence points, in the case of the $O(3)$ Sigma model, provided the cores are of the same type.

5.1 Chern-Simons deformations of the O(3) Sigma model

Recall the construction of the $O(3)$ Lagrangian (2.15) in section 2.1. To the pair (ϕ, A) of a field and a connection, we add an additional neutral field $N \in C^\infty(\Sigma)$

and to avoid a name collision, in this chapter we denote the north pole section as n, so that we modify the $O(3)$ Lagrangian as,

$$L_{O(3),CS} = \frac{1}{2}\Big(||D_t\phi||^2 + ||e||^2 + ||\dot{N}||^2 - \big(||D\phi||^2 + ||B||^2 + ||dN||^2 \\ + ||\kappa N + \tau - \phi_3||^2 + ||NX_\phi||^2\big)\Big), \quad (5.1)$$

where ϕ_3 is the gauge invariant product $\langle n, \phi\rangle$ and X_ϕ is defined in a local trivialization $\phi_\alpha : U_\alpha \to \mathbb{S}^2$ as the section such that,

$$X_{\phi_\alpha} = \mathbf{e}_3 \times \phi_\alpha. \quad (5.2)$$

We add a Chern-Simons term to the Lagrangian,

$$L_{CS} = \frac{1}{2}\left(\langle a, *e\rangle + \langle a_0, *B\rangle\right). \quad (5.3)$$

This Chern-Simons term is not gauge invariant, however, any two gauge related terms differ by a divergence. The product is the L^2 product induced in the exterior algebra by the metric in Σ. With this notation, the Kim-Lee-Lee Lagrangian [27] is,

$$L = L_{O(3),CS} + \kappa L_{CS}. \quad (5.4)$$

For the Abelian Higgs model, there are two ways to introduce a Chern-Simons term in the theory, one is due to Jackiew-Lee-Weinberg [22] and the other to Lee-Lee-Min [29]. In the first case, the connection term is replaced by a Chern-Simons term and the potential term is replaced by a sextic potential that admits a set of Bogomolny equations. It is known that several difficulties arise to study solutions to this model [14]. For the second model, the extension to Chern-Simons deformations of the $O(3)$ Sigma model given in (5.4) is well established, yet the model has only being explored for compact tori due to the difficulties that non-compactness of the moduli space impose. We address those difficulties in this section, let us start considering variations with respect to a_0 yielding the Gauss law,

$$d^*e = -\langle D_t\phi, X_\phi\rangle + \kappa * B, \quad (5.5)$$

where $d^* = - * d *$ is the codifferential. Instead of computing the field equations by the variational method, we note that these equations admit the use of the

Bogomolny trick. Assume Gauss's law holds, the total energy of a tuple (ϕ, A, N) is,

$$\mathrm{E} = \frac{1}{2}\Big(||D_t\phi||^2 + ||e||^2 + ||\dot{N}||^2 + ||D\phi||^2 + ||B||^2 + ||dN||^2 + ||\kappa N + \tau - \phi_3||^2 + ||NX_\phi||^2\Big)\,. \tag{5.6}$$

Let us define the quadratic form,

$$\mathrm{Q} = \frac{1}{2}\Big(||D_t\phi - N\,X_\phi||^2 + ||e - dN||^2 + ||\dot{N}||^2 + ||{*B} + \kappa N + \tau - \phi_3||^2\Big) + ||\bar{\partial}_A\phi||^2, \tag{5.7}$$

where $\bar{\partial}_A\phi$ is the $(0,1)$ component of $D\phi$ with respect to the almost complex structures of Σ and the target $\mathbb{S}^2$. We can simplify Q as follows,

$$\begin{aligned}\mathrm{Q} &= \mathrm{E} - \langle D_t\phi, NX_\phi\rangle - \langle e, dN\rangle + \langle {*B}, \kappa N + \tau - \phi_3\rangle + ||\bar{\partial}_A\phi||^2 - \frac{1}{2}||D\phi||^2 \\ &= \mathrm{E} - \langle\, \langle D_t\phi, X_\phi\rangle + d^*e - \kappa * B, N\rangle + \langle {*B}, \tau - \phi_3\rangle + ||\bar{\partial}_A\phi||^2 - \frac{1}{2}||D\phi||^2 \\ &= \mathrm{E} + \langle {*B}, \tau - \phi_3\rangle + ||\bar{\partial}_A\phi||^2 - \frac{1}{2}||D\phi||^2,\end{aligned} \tag{5.8}$$

where we have used Gauss's law in the second equation.

Since,

$$D\phi = \partial_A\phi + \bar{\partial}_A\phi, \tag{5.9}$$

we deduce,

$$\mathrm{Q} = \mathrm{E} + \langle {*B}, \tau - \phi_3\rangle + \frac{1}{2}||\bar{\partial}_A\phi||^2 - \frac{1}{2}||\partial_A\phi||^2. \tag{5.10}$$

Consider a trivialization $\psi : \pi^{-1}(U) \to U \times \mathbb{S}^2$, such that $\psi \circ \phi(x) = (x, \tilde{\phi}(x))$, where $\tilde{\phi} : U \to \mathbb{S}^2$ and U is an open, simply connected, dense set such that $Z = \phi_3^{-1}(\pm 1) \subset U$. Assume in this chart the connection is represented by a form $a \in \Omega^1(U)$, let $\theta : \mathbb{S}^2 \setminus \{\pm n\} \to \mathbb{R}$ be the azimuthal angle on the sphere, Romão and Speight show in [45] that there is a well defined, gauge invariant form $\xi \in \Omega^1(\Sigma \setminus Z)$, such that on U,

$$\xi = \phi_3 \cdot (a - \tilde{\phi}^* d\theta). \tag{5.11}$$

A short computation in local coordinates shows

$$\frac{1}{2} * (|\partial_A \phi|^2 - |\bar{\partial}_A \phi|^2) = d\xi - \tau\, B + (\tau - \phi_3)\, B, \tag{5.12}$$

hence,

$$\begin{aligned} \mathrm{Q} &= \mathrm{E} - \int_\Sigma (d\xi - \tau B) \\ &= \mathrm{E} - (2\pi\,(1-\tau)k_+ - 2\pi\,(1+\tau)k_-), \end{aligned} \tag{5.13}$$

where the last integral was also computed in [45]. Therefore, as for the $O(3)$ Sigma model without deformation,

$$\mathrm{E} \geq 2\pi\,(1-\tau)k_+ - 2\pi\,(1+\tau)k_-, \tag{5.14}$$

with equality if and only the following Bogomolny equations are satisfied,

$$\dot{N} = 0, \tag{5.15}$$

$$e = dN, \tag{5.16}$$

$$\mathcal{D}_t \phi = N X_\phi, \tag{5.17}$$

$$\overline{\partial}_A \phi = 0, \tag{5.18}$$

$$*B = -(\kappa N + \tau - \langle n, \phi \rangle). \tag{5.19}$$

Since equation (5.18) holds, by the result [51, p. 8] of Sibner et al., Z is finite, moreover, if we consider a holomorphic chart $\varphi : U \subset \Sigma \to V \subset \mathbb{C}$ about $c \in Z$, and a trivialisation $\psi : \pi^{-1}(U) \to U \times \mathbb{S}^2$, such that $\psi \circ \phi|_U = (\mathrm{id}, \tilde{\phi})$, then the degree of the map $\tilde{\phi} \circ \varphi^{-1} : V \to \widehat{\mathbb{C}}$ at $\varphi(c)$ is independent of the holomorphic coordinates chosen, as for the $O(3)$ Sigma model. We call this the degree of the section ϕ at c. As in the $O(3)$ Sigma model, we define the sets $P = \phi_3^{-1}(1)$, $Q = \phi_3^{-1}(-1)$ of vortices and antivortices and denote by $k_+ = |P|$, $k_- = |Q|$ the size of the core sets, counted with multiplicity, where a core c is repeated as many times as the degree of $\phi(c)$. If we choose the gauge $a_0 = -N$, the fields become stationary: $\dot{\phi} = \dot{a} = 0$. Defining the function $u : \Sigma \setminus P \cup Q \to \mathbb{R}$,

$$u = \log\left(\frac{1-\phi_3}{1+\phi_3}\right), \tag{5.20}$$

from the Bogomolny equations and the Gauss's law, we find that (u, N) is a solution to the elliptic problem,

$$\begin{aligned} -\Delta u &= 2\left(\kappa N + \tau + \frac{e^u - 1}{e^u + 1}\right) + 4\pi \sum_{p \in P} \delta_p - 4\pi \sum_{q \in Q} \delta_q, \\ -\Delta N &= \kappa\left(\kappa N + \tau + \frac{e^u - 1}{e^u + 1}\right) + \frac{4e^u}{(e^u + 1)^2} N, \end{aligned} \tag{5.21}$$

As a consequence of the second equation, if $\kappa = 0$, $N \equiv 0$ and the equation for u reduces to the elliptic problem of the abelian $O(3)$ Sigma model, in which case we know that there exists exactly one solution, provided Bradlow's bound holds. Given a set of disjoint divisors $D = (P, Q)$, we define

$$\kappa_*(D) = \sup \{\kappa > 0 \mid \text{there exists a solution of the field equations}\}. \tag{5.22}$$

This is a non-negative number we aim to prove that satisfies the inequality,

$$\inf \left\{\kappa_*(D) \mid D \in \mathcal{M}^{k_+, k_-}\right\} > 0. \tag{5.23}$$

Moreover, if $|k_+ - k_-| > 0$ and Bradlow's bound is fulfilled, it will turn out that the supremum

$$\sup \left\{\kappa_*(D) \mid D \in \mathcal{M}^{k_+, k_-}\right\}, \tag{5.24}$$

is bounded and for small but positive κ, there are two solutions to the field equations, one close to the solution $(u_0, 0)$ of BPS solitons and another with arbitrarily large norm, in a sense to be defined on the following sections. Similar statements hold for negative κ.

5.1.1 Small deformations of κ

In this section we prove that small deformations of the solution h_0 to the Taubes equation vary smoothly with κ. In order to do this, we will define a suitable operator and use it together with the implicit function theorem. Recall for any holomorphic chart $\varphi : U \subset \Sigma \to \mathbb{C}$ and bounded domain $D \subset \mathbb{C}$, there is a smooth function $\tilde{G} : \varphi^{-1}(D) \to \mathbb{R}$ such that if $x, y \in \varphi^{-1}(D)$,

$$G(x, y) = \frac{1}{2\pi} \log |\varphi(x) - \varphi(y)| + \tilde{G}(x, y), \tag{5.25}$$

Hence, for the functions,

$$v_{\mathbf{c}} = 4\pi \sum_i G(x, c_i), \qquad \mathbf{c} = (c_1, \ldots, c_{k_\pm}), \tag{5.26}$$

$e^{v_{\mathbf{c}}}$ varies smoothly with $\mathbf{c}$. We denote by Δ_{k_+,k_-} the (k_+, k_-) diagonal of $\Sigma^{k_+} \times \Sigma^{k_-}$. The space of solutions of the elliptic problem at $\kappa = 0$ is the moduli space of asymmetric vortices and antivortices described on Chapter 2. We define the function

$$v : \mathcal{M}^{k_+,k_-} \to C^\infty(\Sigma, \overline{\mathbb{R}}), \qquad\qquad v(\mathbf{p}, \mathbf{q}) = v_{\mathbf{p}} - v_{\mathbf{q}}, \tag{5.27}$$

where $C^\infty(\Sigma, \overline{\mathbb{R}})$ means the set of smooth functions, except at a finite set of points at which we have divergences to $\pm\infty$. If $F : \mathbb{R} \to \mathbb{R}$ is the function

$$F(t) = \frac{e^t - 1}{e^t + 1} + \tau, \tag{5.28}$$

then solving equation (5.21) is equivalent to finding a pair of functions (h, N) such that,

$$\Delta h + 2(\kappa N + F(v + h)) + \frac{4\pi}{|\Sigma|}(k_+ - k_-) = 0, \tag{5.29}$$

$$\Delta N + \kappa\,(\kappa N + F(v + h)) + 2F'(v + h)\,N = 0. \tag{5.30}$$

We introduce the potential function $V(t) = 2F'(t)$ such that, if $(\mathbf{p}, \mathbf{q}) \in \mathcal{M}^{k_+,k_-}$, $h \in \mathrm{H}^r$ and $v = v(\mathbf{p}, \mathbf{q})$, then,

$$V(v + h) = \frac{4e^{v_{\mathbf{p}} + v_{\mathbf{q}} + h}}{(e^{v_{\mathbf{p}} + h} + e^{v_{\mathbf{q}}})^2}. \tag{5.31}$$

As shown in the proof of Theorem 2.12, by equation (2.173) the functions $e^{v_{\mathbf{p}}}$, $e^{v_{\mathbf{q}}}$ are smooth and vary smoothly with $(\mathbf{p}, \mathbf{q})$. We observe that the spaces H^r, $r \geq 2$, are algebras, a prove can be found in [14] where Flood and Speight use this result to prove that e^h is a smooth function $\mathrm{H}^r \to \mathrm{H}^r$, the claim follows because e^h is the limit of the absolutely converging power series $\sum_{n=0}^\infty h^n/n!$. As a consequence, $V(v + h) \in \mathrm{H}^r$. Likewise, $F(v + h) \in \mathrm{H}^r$ if $h \in \mathrm{H}^r$. For any given pair of disjoint sets P, Q, let us define the operator,

$$\mathrm{T} : \mathbb{R} \times \mathrm{H}^r \times \mathrm{H}^r \to \mathrm{H}^{r-2} \times \mathrm{H}^{r-2}, \tag{5.32}$$

$$T(\kappa, h, N) = \Big(\Delta h + 2\left(\kappa N + F(v+h)\right) + \frac{4\pi}{|\Sigma|}(k_+ - k_-), \tag{5.33}$$
$$\Delta N + \kappa\left(\kappa N + F(v+h)\right) + V(v+h)N \Big) \tag{5.34}$$

T is a smooth mapping between Hilbert spaces. For any given $h \in \mathrm{H}^r$, $r \geq 2$, we define the operator

$$\mathrm{L} : \mathrm{H}^r \to \mathrm{H}^{r-2}, \qquad \mathrm{L} = \Delta + V(v+h). \tag{5.35}$$

The derivative of the restriction

$$\mathrm{T}| : \mathrm{H}^r \times \mathrm{H}^r \to \mathrm{H}^{r-2} \times \mathrm{H}^{r-2}, \qquad \mathrm{T}|(h, N) = \mathrm{T}(0, h, N), \tag{5.36}$$

at a point $(h, 0)$ is $d\mathrm{T}|_{(h,0)} = \mathrm{L} \oplus \mathrm{L}$.

Lemma 5.1. *For any set of core points* $(\mathbf{p}, \mathbf{q})$ *in the moduli space, the operator* L *is a Hilbert space isomorphism* $\mathrm{H}^r \to \mathrm{H}^{r-2}$.

Proof. By Sobolev's embedding, $h \in C^0(\Sigma)$, hence $V \geq 0$ is a continuous function which is only zero at the finite set $P \cup Q$. By Lemma 2.10, for any $\psi \in \mathrm{H}^{r-2}$ there is exactly one $\varphi \in \mathrm{H}^2$ such that,

$$\mathrm{L}\varphi = \psi, \tag{5.37}$$

but by Schauder's estimates,

$$||\varphi||_{\mathrm{H}^r} \leq C\left(||\Delta\varphi||_{\mathrm{H}^{r-2}} + ||\varphi||_{\mathrm{L}^2}\right), \tag{5.38}$$

for some constant C, hence $\varphi \in \mathrm{H}^r$ and L is a bijective bounded operator. By the open mapping theorem, L^{-1} is also continuous, hence bounded and the claim follows. ☐

Proposition 5.2. *Assume Bradlow's bound holds, then there is a positive constant* $\kappa_0(P, Q)$ *such that if* $|\kappa| < \kappa_0$*, the elliptic problem* (5.21) *has a solution. Moreover, for any open neighbourhood* U *of* $P \cup Q$*, the restriction of the solutions* (h, N) *to* $\Sigma \setminus \overline{U}$ *varies smoothly with* κ.

Proof. Let $h_0 \in \mathrm{H}^r$ be the solution of equation (5.29) with $\kappa = N = 0$, i.e. h_0 is the solution to the regularised Taubes equation of the abelian Sigma model, since $\mathrm{T}(0, h_0, 0) = (0, 0)$, by the implicit function theorem, there is an interval $(-\kappa_0, \kappa_0)$ such that the map,

$$(-\kappa_0, \kappa_0) \ni \kappa \mapsto (\kappa, h_\kappa, N_\kappa) \in (-\kappa_0, \kappa_0) \times \mathrm{H}^r \times \mathrm{H}^r, \tag{5.39}$$

is smooth and $\mathrm{T}(\kappa, h_\kappa, N_\kappa) = (0, 0)$. Therefore, each pair (h_κ, N_κ) is a solution to the regular elliptic problem (5.29), (5.30) in $\mathrm{H}^r \times \mathrm{H}^r$. By a bootstrap argument, each (h_κ, N_κ) is in $\mathrm{H}^k \times \mathrm{H}^k$ for any $k \geq r$. Hence, by Sobolev's embedding, (h_κ, N_κ) is smooth, moreover, the function $u_\kappa = v + h_\kappa$ varies smoothly as a function of κ and $(\mathbf{p}, \mathbf{q})$ if $p_j, q_k \in U$ are such that $p_j \neq q_k$ for each vortex and antivortex position. □

Thus if κ is small, there is a family of solutions to the field equations close to the BPS soliton determined by h_0, in the sense that $(h_\kappa - h_0, N_\kappa)$ is small in the $\mathrm{H}^r \times \mathrm{H}^r$ norm for any $r > 0$.

5.1.2 A positive gap for $\kappa_*(D)$

By proposition 5.2, $\kappa_*(D) > 0$ for any distribution of the divisors. On this section we will prove the existence of a positive lower bound for κ_*, independent of the core positions. Thus, localization of vortex-antivortex systems makes sense globally for small deformations κ of the BPS model as is for the case of Ginzburg-Landau vortices [14], even though in this case the moduli space should be incomplete as is the case for the BPS model at $\kappa = 0$. We prove several technical lemmas first, in order to find bounds for the norm of T', the derivative of the operator defined in the previous section.

Lemma 5.3. *The solutions h of equation (5.29) with $\kappa = 0$ are uniformly bounded on H^2.*

Proof. Let,

$$c(h) = \frac{1}{|\Sigma|} \int_\Sigma h \,\mathrm{Vol}, \tag{5.40}$$

be the average of h on Σ. $h - c$ is orthogonal to the kernel of Δ, Schauder's estimates in this case give,

$$||h - c||_{\mathrm{H}^2} \leq C||\Delta h||_{\mathrm{L}^2}, \tag{5.41}$$

where we denote by C a positive constant, independent of the function h. The function $F(t)$ is bounded, hence

$$\Delta h = -2F(v + h) - \frac{4\pi}{|\Sigma|}(k_+ - k_-) \tag{5.42}$$

is uniformly bounded in L^2. Therefore, the set of functions $\{h - c\}$ is bounded in H^2 and by Sobolev's embedding also in $C^0(\Sigma)$. We claim that the averages are also bounded. Assume otherwise towards a contradiction. Let $\tilde{h} = h - c$, then there are sequences v_n, $\tilde{h}_n$, c_n such that $|c_n| \to \infty$. Suppose $c_n \to \infty$, and let $(\mathbf{p}_n, \mathbf{q}_n) \in \mathcal{M}^{k_+,k_-}$ be the points defining v_n. Since Σ is compact, we can assume the convergence $(\mathbf{p}_n, \mathbf{q}_n) \to (\mathbf{p}_*, \mathbf{q}_*) \in \Sigma^{k_+} \times \Sigma^{k_-}$. We have pointwise convergence $v_n \to v_* = v_{\mathbf{p}_*} - v_{\mathbf{q}_*}$, except possibly at points on the surface which are in $\mathbf{p}_*$ and $\mathbf{q}_*$ if there is any. Since the functions $\tilde{h}_n$ are uniformly bounded, we also have,

$$2F(v_n + \tilde{h}_n + c_n) \to 2(1 + \tau), \qquad \text{pointwise a.e.} \tag{5.43}$$

By the dominated convergence theorem,

$$\int_\Sigma 2F(v_n + \tilde{h}_n + c_n)\mathrm{Vol} \to 2(1 + \tau)|\Sigma|, \tag{5.44}$$

but by the divergence theorem,

$$\int_\Sigma 2F(v_n + \tilde{h}_n + c_n)\mathrm{Vol} = -\int_\Sigma \left(\Delta h + \frac{4\pi}{|\Sigma|}(k_+ - k_-)\right)\mathrm{Vol} = -4\pi(k_+ - k_-), \tag{5.45}$$

and this contradicts Bradlow's bound. If $c_n \to -\infty$ the same argument gives another contradiction. Therefore the set of averages $\{c(h)\}$ is bounded. The lemma follows. □

Lemma 5.4. *For any $\epsilon > 0$ there is a positive constant $C(\epsilon)$, such that for any set of divisors and any $h \in \mathrm{H}^2$ with $||h||_{\mathrm{H}^2} < \epsilon$,*

$$\langle V(v + h), 1\rangle_{\mathrm{L}^2} \geq C. \tag{5.46}$$

Proof. We will omit the subindex in the product $\langle V, 1\rangle$ since it is clear that we refer to $\mathrm{L}^2(\Sigma)$. The potential is a non negative function, hence $\langle V(v+h), 1\rangle \geq 0$. Assume towards a contradiction the existence of sequences $\{v_n\}$, $\{h_n\}$, where $||h_n||_{\mathrm{H}^2} < \epsilon$ and with vortices and antivortices at positions $\mathbf{p}_n$, $\mathbf{q}_n$, such that for the sequences of potentials,

$$V_n = V(v_n + h_n), \tag{5.47}$$

we have $\langle V_n, 1\rangle \to 0$. As in the previous lemma, we can assume $\mathbf{p}_n \to \mathbf{p}_* \in \Sigma^{k_+}$ and $\mathbf{q}_n \to \mathbf{q}_* \in \Sigma^{k_-}$ together with pointwise convergence $v_n \to v_*$, except possibly at points $x \in \Sigma$ belonging to the set of coordinates of $\mathbf{p}_*$ or $\mathbf{q}_*$. Let $C_0 > 0$ be Sobolev's constant, such that,

$$||h||_{C^0(\Sigma)} \leq C_0\, ||h||_{\mathrm{H}^2}. \tag{5.48}$$

Hence,

$$0 \leq \langle V(|v_{\mathbf{p}_n}| + |v_{\mathbf{q}_n}| + C_0\epsilon), 1\rangle \leq \langle V_n, 1\rangle \to 0. \tag{5.49}$$

On the other hand, $V(|v_{\mathbf{p}_n}| + |v_{\mathbf{q}_n}| + C_0\epsilon)$ is a sequence of bounded functions converging pointwise to the continuous function $V(|v_{\mathbf{p}_*}| + |v_{\mathbf{q}_*}| + C_0\epsilon)$. By the dominated convergence theorem,

$$\langle V(|v_{\mathbf{p}_*}| + |v_{\mathbf{q}_*}| + C_0\,\epsilon), 1\rangle = 0, \tag{5.50}$$

a contradiction. □

Given any pair $(D, h) \in \mathcal{M}^{k_+,k_-} \times \mathrm{H}^r$, $r \geq 2$, the potential $V(v+h)$ is a non negative continuous function such that $0 < \langle V(v+h), 1\rangle$. If $\psi \in \mathrm{L}^2$, by Lemma 2.10, there is exactly one $\varphi \in \mathrm{H}^2$ such that,

$$(\Delta + V(v+h))\,\varphi = \psi, \tag{5.51}$$

and a positive constant C', independent of $V(v+h)$, φ and ψ, such that,

$$||\varphi||_{\mathrm{H}^1} \leq \frac{C'}{\langle V(v+h), 1\rangle}\left(\frac{||V(v+h)||^2_{\mathrm{L}^2}}{\langle V(v+h), 1\rangle} + ||V(v+h)||_{\mathrm{L}^2} + 1\right)||\psi||_{\mathrm{L}^2}. \tag{5.52}$$

We let the point in the moduli space vary in order to define the operator,

$$\mathcal{L} : \mathcal{M}^{k_+,k_-} \times \mathrm{H}^2 \to B(\mathrm{H}^2, \mathrm{L}^2), \qquad \mathcal{L}(\mathbf{p}, \mathbf{q}, h) = L, \tag{5.53}$$

where $L = \Delta + V(v+h)$ was defined previously. $\mathcal{L}$ is a continuous map such that each $\mathcal{L}(\mathbf{p}, \mathbf{q}, h)$ is invertible. Since inversion of bounded invertible operators is continuous, the map

$$\mathcal{L}' : \mathcal{M}^{k_+,k_-} \times \mathrm{H}^2 \to B(\mathrm{L}^2, \mathrm{H}^2), \qquad \mathcal{L}'(\mathbf{p}, \mathbf{q}, h) = \mathrm{L}^{-1}, \tag{5.54}$$

is also continuous.

Lemma 5.5. *Given $\epsilon > 0$, let $\Omega = \mathcal{M}^{k_+,k_-} \times B_\epsilon(0)$ and let,*

$$C_*(\epsilon) = \sup_\Omega ||\mathcal{L}'||, \tag{5.55}$$

then C_* is finite.

Proof. By lemma 5.4, there is a constant $C(\epsilon)$ such that $\langle V(v+h), 1\rangle \geq C$ for any $(D, h) \in \Omega$. If $\varphi = \mathcal{L}'\psi$, with $||\psi||_{\mathrm{L}^2} = 1$ by (5.52) we have the bound

$$||\varphi||_{\mathrm{H}^1} \leq \frac{C'}{C}\left(\frac{|\Sigma|}{C} + |\Sigma|^{1/2} + 1\right). \tag{5.56}$$

By Schauder's estimates,

$$\begin{aligned} ||\varphi||_{\mathrm{H}^2} &\leq C\left(||\Delta\varphi||_{\mathrm{L}^2} + ||\varphi||_{\mathrm{L}^2}\right) \\ &= C\left(||-V(v+h)\varphi + \psi||_{\mathrm{L}^2} + ||\varphi||_{\mathrm{L}^2}\right) \\ &\leq C, \end{aligned} \tag{5.57}$$

where the last constant is not necessarily equal to the first one. Therefore, $||\mathcal{L}'|_\Omega|| \leq C$, hence $C_* \leq C$. □

For given κ, let us define $\mathrm{T}_\kappa| : \mathrm{H}^2 \times \mathrm{H}^2 \to \mathrm{L}^2 \times \mathrm{L}^2$ as the restriction $\mathrm{T}_\kappa|(h, N) = \mathrm{T}(\kappa, h, N)$, then we have,

$$d\mathrm{T}_\kappa|_{(h,N)} = \mathrm{L} \oplus \mathrm{L} + \mathrm{T}', \tag{5.58}$$

where,

$$\mathrm{T}'(h', N') = \left(2\kappa N', \kappa^2 N' + \frac{\kappa}{2} V(v+h)h' + V'(v+h)Nh'\right). \tag{5.59}$$

Since $V'(t)$ and $V(t)$ have range $[0, 1]$,

$$||\mathrm{T}'(h', N')||_{\mathrm{L}^2\times\mathrm{L}^2} \leq (2|\kappa| + \kappa^2)||N'||_{\mathrm{L}^2} + \left(\frac{|\kappa|}{2} + ||N||_{\mathrm{L}^2}\right)||h'||_{\mathrm{L}^2}. \tag{5.60}$$

By Cauchy-Schwarz,

$$||T'|| \leq \left(\kappa^2(2+|\kappa|)^2 + (|\kappa|/2 + ||N||_{\mathrm{L}^2})^2\right)^{1/2}. \tag{5.61}$$

We use lemma 5.3 and choose ϵ_0 such that $||h_0||_{\mathrm{H}^2} < \epsilon_0$ for any solution $(h_0, 0)$ of equation (5.29) with $\kappa = 0$. We also take the constant $C_*(\epsilon_0)$ of lemma 5.5 and define

$$\epsilon = \min\left\{\epsilon_0, \frac{1}{2\,C_* \cdot (\frac{7}{2} + \epsilon_0)}\right\}. \tag{5.62}$$

Proposition 5.6. *For any set of divisors $D \in \mathcal{M}^{k_+,k_-}$,*

$$\kappa_*(D) \geq \epsilon \min\left\{1, \frac{1}{2\,C_*(2\epsilon(1+\epsilon) + \max\{1 \pm \tau\}\,|\Sigma|^{1/2})}\right\}. \tag{5.63}$$

Proof. For $|\kappa| < \epsilon$, $||h||_{\mathrm{L}^2} < \epsilon$ and $||N||_{\mathrm{L}^2} < \epsilon$, we have,

$$||\mathrm{T}'|| < \frac{1}{2C_*} \leq \frac{1}{2||(\mathrm{L} \oplus \mathrm{L})^{-1}||}, \tag{5.64}$$

hence, as in the proof of Lemma 5 in [14], we can conclude that $d\mathrm{T}_\kappa|$ is invertible, independently of the point in the moduli space and

$$||(d\mathrm{T}_\kappa|)^{-1}|| \leq 2C_*. \tag{5.65}$$

If $\chi(\kappa) = (h_\kappa, N_\kappa)$ is the curve of solutions to equations (5.29),(5.30), guaranteed to exist by proposition 5.2, then by the implicit function theorem, this curve can be extended whenever $d\mathrm{T}_\kappa|$ is invertible at $\chi(\kappa)$. This is the case if $|\kappa| < \epsilon$ and $||h||, ||N|| < \epsilon$. So, for any $D \in \mathcal{M}^{k_+,k_-}$, let $\kappa_0 > 0$ be the right end of the maximal interval $[0, \kappa_0)$ on which this curve can be extended. Either $\kappa_0 \geq \epsilon$, or there exists a κ_1 with $|\kappa_1| < \kappa_* < \epsilon$ such that $||\dot{\chi}(\kappa)||_{\mathrm{H}^2\times\mathrm{H}^2} \geq \epsilon\kappa_*^{-1}$. In the later case,

$$\begin{aligned} ||\dot{\chi}_{\kappa_1}||_{\mathrm{H}^2\times\mathrm{H}^2} &= ||(d\mathrm{T}_\kappa|_{\chi(\kappa)})^{-1}\partial_\kappa \mathrm{T}|_{(\kappa_1,\chi(\kappa_1))}||_{\mathrm{H}^2\times\mathrm{H}^2} \\ &\leq 2C_*||\partial_\kappa \mathrm{T}|_{(\kappa_1,\chi(\kappa_1))}||_{\mathrm{L}^2\times\mathrm{L}^2}. \end{aligned} \tag{5.66}$$

But,

$$\partial_\kappa \mathrm{T} = (2N, 2\kappa N + F(v+h)), \tag{5.67}$$

hence,

$$||\partial_\kappa \mathrm{T}||_{\mathrm{L}^2\times\mathrm{L}^2} \leq 2||N||_{\mathrm{L}^2} + 2|\kappa|||N||_{\mathrm{L}^2} + \sup\{|F|\}\,|\Sigma|^{1/2}$$
$$\leq 2\epsilon(1+\epsilon) + \max\{1\pm\tau\}\,|\Sigma|^{1/2}. \tag{5.68}$$

Therefore, either $\kappa_0 \geq \epsilon$ or

$$\kappa_0 \geq \frac{\epsilon}{2C_*(2\epsilon(1+\epsilon) + \max\{1\pm\tau\}\,|\Sigma|^{1/2})}. \tag{5.69}$$

Since $\kappa_*(D) \geq \kappa_0$ we conclude the claimed lower bound. □

5.1.3 The unbalanced case

In this section we assume $k_+ \neq k_-$. In this case the family of deformation constants is bounded, contrasting with the euclidean case, where there are examples for which the elliptic problem can be solved for any κ [6]. We will prove the existence of multiple solutions of the field equations, the first step will be to describe the possible limit points of sequences $(h_{\kappa_n}, \kappa_n N_{\kappa_n})$ as $\kappa_n \to 0$.

It will be convenient to redefine the neutral field as follows. Let $N' = \kappa N$, equation (5.30) can be rewritten as,

$$\Delta N' + (\kappa^2 + 2F'(v+h))\,N' + \kappa^2 F(v+h) = 0. \tag{5.70}$$

Proposition 5.7. *If $|k_+ - k_-| > 0$, $\kappa_*(D)$ is uniformly bounded,*

$$\kappa_*(D) \leq \left(\frac{\max\{1\mp\tau\}\,|\Sigma|}{2\pi\,|k_+ - k_-|}\right)^{1/2}. \tag{5.71}$$

Proof. Let $\underline{x}$ be such that $N'(\underline{x}) = \min_\Sigma N'$ and likewise, let $\overline{x}$ be such that $N'(\overline{x}) = \max_\Sigma N'$.

Let us denote $F(v(x)+h(x))$ as $F(x)$. Likewise, we denote $F'(v(x)+h(x))$ as $F'(x)$.

By the maximum principle,

$$-\frac{\kappa^2 F(\underline{x})}{\kappa^2 + 2F'(\underline{x})} \leq N'(x) \leq -\frac{\kappa^2 F(\overline{x})}{\kappa^2 + 2F'(\overline{x})}. \tag{5.72}$$

Since $F'(x) \geq 0$, we conclude the uniform bounds,

$$-(1+\tau) \leq N' \leq (1-\tau). \tag{5.73}$$

From equations (5.29) and (5.70) we obtain,

$$\Delta N' - \frac{\kappa^2}{2}\left(\Delta h + \frac{4\pi}{|\Sigma|}(k_+ - k_-)\right) + 2F'(v+h)N' = 0. \tag{5.74}$$

Integrating this equation,

$$-2\pi\kappa^2(k_+ - k_-) + 2\langle F'(v+h), N'\rangle = 0. \tag{5.75}$$

Using the fact that F' is a positive function bounded by $1/2$ and the uniform bound for N',

$$\kappa^2 = \frac{\langle F'(v+h), N'\rangle}{\pi\,(k_+ - k_-)} \leq \frac{\max\{1 \mp \tau\}\,|\Sigma|}{2\pi\,|k_+ - k_-|}. \tag{5.76}$$

□

As a consequence of this proposition, we have the following lemma,

Lemma 5.8. *If $|k_+ - k_-| > 0$, and (h_κ, N'_κ) denotes a solution to the pair of equations (5.29), (5.70) with deformation parameter κ, for any $p \geq 2$ there is a uniform constant $C(p)$, such that,*

$$||h_\kappa - c(h_\kappa)||_{\mathrm{W}^{2,p}} + ||N'_\kappa||_{\mathrm{W}^{2,p}} \leq C, \tag{5.77}$$

where $c(h_\kappa)$ is the average of h_κ on the surface.

Proof. We will assume the constant C can change from one line to the next. Since κ_* is bounded, from (5.70) we have a uniform bound for the norm of N' in Sobolev's space,

$$||\Delta N'||_{\mathrm{L}^p} \leq C. \tag{5.78}$$

By Calderon-Sygmund theory, this implies,

$$||N'||_{\mathrm{W}^{2,p}} \leq C\,(||\Delta N'||_{\mathrm{L}^p} + ||N'||_{\mathrm{L}^p}) \leq C. \tag{5.79}$$

Similarly, from (5.29) we deduce the existence of an upper bound for the set $\{h_\kappa - c(h_\kappa)\}$ in $\mathrm{W}^{2,p}$. □

If we fix any $p > 2$, Sobolev's theory says that the embedding $\mathrm{W}^{2,p} \to C^1(\Sigma)$ is continuous, thence there exists a constant independent of κ, such that,

$$||h_\kappa - c(h_\kappa)||_{C^1} + ||N'||_{C^1} \leq C. \tag{5.80}$$

Let (h_n, N'_n) denote a sequence of solutions to the elliptic equations (5.29)-(5.70) with a corresponding sequence of parameters $\{\kappa_n\}$. We are interested in describing the behaviour of these solutions as $\kappa_n \to 0$. Although the sequence $(h - c(h), N')$ is uniformly bounded in C^1 we cannot rule out the possibilities $c(h_n) \to \pm\infty$. In the following lemmas we deal with the three cases arising on this analysis.

Lemma 5.9. *If (h_n, N'_n) is a sequence of solutions to the elliptic equations (5.29)-(5.70) with parameters $\kappa_n \to 0$ such that the sequence is bounded in $\mathrm{H}^1 \times \mathrm{H}^1$, then for any $p \geq 2$ the sequence converges to $(h_0, 0)$ strongly in $\mathrm{W}^{2,p} \times \mathrm{W}^{2,p}$, where h_0 is the solution to the regularized Taubes equation.*

In particular, this means the convergence is uniform in $C^1 \times C^1$.

Proof. By the Banach-Alaoglu theorem, for any subsequence (h_{n_k}, N'_{n_k}), after passing to another subsequence if necessary, we can assume $(h_{n_k}, N'_{n_k}) \to (h_*, N'_*)$ weakly in $\mathrm{H}^1 \times \mathrm{H}^1$ and by the Rellich-Kondrachov lemma, strongly in $\mathrm{L}^2 \times \mathrm{L}^2$. Let $(u, w) \in \mathrm{H}^1 \times \mathrm{H}^1$, equations (5.29) - (5.70) can be expressed in weak form as,

$$\begin{aligned} \langle \nabla u, \nabla h_{n_k} \rangle + \left\langle u, 2\,(N'_{n_k} + F(v + h_{n_k})) + \frac{4\pi}{|\Sigma|}\,(k_+ - k_-) \right\rangle = 0, \\ \langle \nabla w, \nabla N'_{n_k} \rangle + \langle w, (\kappa_{n_k}^2 + 2\,F'(v + h_{n_k}))\,N'_{n_k} + \kappa_{n_k}^2\,F(v + h_{n_k}) \rangle = 0. \end{aligned} \tag{5.81}$$

Weak convergence in W^1 plus strong convergence in L^2 imply

$$\langle \nabla u, \nabla h_{n_k} \rangle \to \langle \nabla u, \nabla h_* \rangle, \qquad \langle \nabla w, \nabla N'_{n_k} \rangle \to \langle \nabla w, \nabla N'_* \rangle. \tag{5.82}$$

After passing to another subsequence if necessary, we can assume $h_{n_k} \to h_*$ pointwise almost everywhere. By the dominated convergence theorem,

$$\langle u, F(v + h_{n_k}) \rangle = \langle u, F(v + h_*) \rangle, \tag{5.83}$$

and simmilarly for w. Therefore, (h_*, N'_*) is a weak solution to the equations,

$$\Delta h_* + 2\,(N'_* + F(v + h_*)) + \frac{4\pi}{|\Sigma|}\,(k_+ - k_-) = 0, \tag{5.84}$$

$$\Delta N'_* + 2\,F'(v + h_*)\,N'_* = 0. \tag{5.85}$$

Ellipticity guarantees the solution is in fact strong, hence, by the usual elliptic estimates, $(h_*, N'_*) \in \mathrm{H}^2 \times \mathrm{H}^2$ and the solution is continuous. This together with (5.85) implies $N'_* \equiv 0$. Therefore (5.84) is the regularised Taubes equation, whose unique solution is $h_* \equiv h_0$.

Since any subsequence of (h_n, N'_n) can be refined to a convergent subsequence to $(h_0, 0)$ in $\mathrm{L}^2 \times \mathrm{L}^2$, we obtain the limit,

$$||h_n - h_0||_{\mathrm{L}^2} + ||N'_n||_{\mathrm{L}^2} \to 0. \tag{5.86}$$

In particular, $N'_n \to 0$ in L^2, this limit and the boundedness of the functions $F(t)$ and $F'(t)$ imply by means of equation (5.70) the limit,

$$||\Delta N'_n||_{\mathrm{L}^2} \to 0. \tag{5.87}$$

Hence, by the usual combination of Schauder's estimates and Sobolev's embedding, we find two constants such that,

$$||N'_n||_{C^0} \leq C_1\, ||N'_n||_{\mathrm{H}^2} \leq C_2\, (||\Delta N'_n||_{\mathrm{L}^2} + ||N'_n||_{\mathrm{L}^2}) \to 0. \tag{5.88}$$

Whereas by equation (5.29),

$$-\Delta(h_n - h_0) = 2\, N'_n + 2\, (F(v + h_n) - F(v + h_0)). \tag{5.89}$$

Note that by the mean value theorem, for any $x \notin Z$,

$$F(v(x) + h_n(x)) - F(v(x) + h_0(x)) = F'(\xi)\, (h_n(x) - h_0(x)), \tag{5.90}$$

for some ξ between $h_n(x)$ and $h_0(x)$, whereas for $x \in Z$,

$$F(v(x) + h_n(x)) - F(v(x) + h_0(x)) = 0, \tag{5.91}$$

since $F(v(x) + h_n(x)) = F(v(x) + h_0(x)) = \pm 1 + \tau$ in this case, hence, there is a constant $C > 0$, such that,

$$|F(v + h_n) - F(v + h_0)| \leq C\, |h_n - h_0|, \tag{5.92}$$

by (5.89) and (5.92),

$$\begin{aligned} ||\Delta(h_n - h_0)||_{\mathrm{L}^2} &\leq 2\, ||N'_n||_{\mathrm{L}^2} + 2\, ||F(v + h_n) - F(v + h_0)||_{\mathrm{L}^2} \\ &\leq C\, (||N'_n||_{\mathrm{L}^2} + ||h_n - h_0||_{\mathrm{L}^2}). \end{aligned} \tag{5.93}$$

We deduce $||\Delta(h_n - h_0)||_{\mathrm{L}^2} \to 0$. Repeating the elliptic estimate argument we find,

$$||h_n - h_0||_{\mathrm{H}^2} \to 0, \tag{5.94}$$

and consequently also $h_n \to h_0$ in $C^0(\Sigma)$. Finally, we follow a bootstrap argument. Knowing the convergence $(h_n, N_n') \to (h_0, 0)$ is uniform in C^0, we can repeat the previous computations for the norm of the Laplacian, this time in the L^p norm and deduce the claimed limit. □

In case (h_n, N_n') is not bounded, necessarily $\{c(h_n)\}$ has a subsequence diverging to $\pm\infty$. We consider each possibility in the following lemma.

Lemma 5.10. *If $\{(h_n, N_n')\}$ is a sequence of solutions to equations (5.29) (5.70) such that $\kappa_n \to 0$ and $c(h_n) \to \infty$, and if $p \geq 2$ is a fixed but otherwise arbitrary constant, the following limit holds,*

$$||h_n - c(h_n)||_{\mathrm{W}^{2,p}} + ||N_n' - \alpha_+||_{\mathrm{W}^{2,p}} \to 0, \tag{5.95}$$

where

$$\alpha_+ = -1 - \tau - \frac{2\pi\,(k_+ - k_-)}{|\Sigma|}, \tag{5.96}$$

and the condition $k_+ - k_- < 0$ is necessary. Similar statements hold if $c(h_n) \to -\infty$, where the constant α in this case is,

$$\alpha_- = 1 - \tau - \frac{2\pi\,(k_+ - k_-)}{|\Sigma|}, \tag{5.97}$$

and the condition $k_+ - k_- > 0$ is necessary.

Proof. We proceed as in lemma 5.9. Let $\tilde{h}_n = h_n - c(h_n)$, we consider a subsequence $(\tilde{h}_{n_k}, N_{n_k}')$. By lemma 5.8, after passing to another subsequence if necessary, we can assume $(\tilde{h}_{n_k}, N_{n_k}') \to (\tilde{h}_*, N_*')$ weakly in $\mathrm{H}^1 \times \mathrm{H}^1$ and strongly in $\mathrm{L}^2 \times \mathrm{L}^2$. Repeating the argument of lemma 5.9, this time using the fact that $c(h_{n_k}) \to \infty$, we find $(\tilde{h}_*, N_*')$ is a strong solution to the equations,

$$\Delta\tilde{h}_* + 2\,(N_*' + 1 + \tau) + \frac{4\pi}{|\Sigma|}\,(k_+ - k_-) = 0, \tag{5.98}$$
$$\Delta N_*' = 0.$$

Since N'_* is in the kernel of the Laplacian, it has to be a constant function α. By a bootstrap argument, $\tilde{h}_*$ is smooth. To determine the constant, we integrate equation (5.98) by means of the divergence theorem and find,

$$\alpha = -1 - \tau - \frac{2\pi}{|\Sigma|}(k_+ - k_-), \tag{5.99}$$

and,

$$\Delta \tilde{h}_* = 0. \tag{5.100}$$

Since each $\tilde{h}_n$ has zero average the same holds for $\tilde{h}_*$, therefore $\tilde{h}_* \equiv 0$.

We follow once more the pattern of lemma 5.9 to conclude strong convergence $(\tilde{h}_{n_k}, N'_{n_k}) \to (0, \alpha)$ in $\mathrm{L}^2 \times \mathrm{L}^2$ and deduce that for any $p \geq 2$,

$$||\tilde{h}_n||_{\mathrm{W}^{2,p}} + ||N'_n - \alpha||_{\mathrm{W}^{2,p}} \to 0. \tag{5.101}$$

Finnally, by (5.73),

$$-(1+\tau) \leq \alpha \leq 1 - \tau, \tag{5.102}$$

hence,

$$-(1+\tau) \leq -(1+\tau) - \frac{2\pi}{|\Sigma|}(k_+ - k_-), \tag{5.103}$$

implying $k_+ < k_-$. It is clear these arguments can be rearranged for the case $c(h_n) \to -\infty$. □

Lemmas 5.9, 5.10 thus prove the following proposition.

Proposition 5.11. *If $\{(h_n, N'_n)\}$ is a sequence of solutions to equations (5.29) (5.70) such that $\kappa_n \to 0$, the only possible limit points are $(h_0, 0)$ and $(0, \alpha_\pm)$, for $\alpha_\pm$ defined on lemma 5.10. If $\{(h_n, N'_n)\}$ has a bounded (unbounded) subsequence, then $(h_0, 0)$ ($(0, \alpha_\pm)$) is a limit point.*

5.1.4 Existence of multiple solutions

In this section we prove the existence of multiple solutions to the field equations using theorem 2.21. In order to do this, we define an operator Φ satisfying the hypothesis of the theorem trough a series of technical lemmas.

Lemma 5.12. *For any* $0 < \kappa_0 < \kappa_*(D)$ *and* $p \geq 2$*, the set* $\{(h_\kappa, N'_\kappa) \mid \kappa > \kappa_0\}$ *is bounded in* $\mathrm{W}^{2,p} \times \mathrm{W}^{2,p}$.

Proof. Assume otherwise towards a contradiction. If (h_n, N'_n) is an unbounded sequence, by lemma 5.8 we can suppose $c(h_n) \to \pm\infty$ depending on the sign of $k_+ - k_-$. Without loss of generality, let us assume $k_+ - k_- > 0$. In this case $c(h_n) \to -\infty$ by lemma 5.10. Since $\{\kappa_n\}$ is bounded, we can assume $\kappa_n \to k_* \neq 0$. Let $\tilde{h}_n = h_n - c(h_n)$, going through the steps of the proof of lemma 5.9, we deduce the existence of a strong limit $(\tilde{h}_n, N'_n) \to (\tilde{h}_*, N'_*)$ in $\mathrm{W}^{2,p} \times \mathrm{W}^{2,p}$, such that $(\tilde{h}_*, N'_*)$ is a solution to the problem,

$$(\Delta + \kappa_*^2)\, N'_* + \kappa_*^2\,(-1 + \tau) = 0, \tag{5.104}$$

$$\Delta\tilde{h}_* + 2N'_* + 2\left(-1 + \tau + \frac{2\pi}{|\Sigma|}(k_+ - k_-)\right) = 0. \tag{5.105}$$

By elliptic regularity the pair $(\tilde{h}_*, N'_*)$ is smooth. Since $\kappa_*^2 > 0$, integrating the first equation, we obtain,

$$\langle N'_*, 1\rangle + (-1 + \tau)|\Sigma| = 0. \tag{5.106}$$

Integrating the second equation, we have,

$$\langle N'_*, 1\rangle + (-1 + \tau)\,|\Sigma| + 2\pi(k_+ - k_-) = 0. \tag{5.107}$$

Hence $2\pi\,(k_+ - k_-) = 0$, a contradiction. $\square$

We will prove the existence of multiple solutions to the field equations in the unbalanced case, which can be seen in figure 5.2 on the right column, adapting the argument from [18] which relies on Leray-Schauder's degree.

We define the operators,

$$\mathrm{L} = (-\Delta - \lambda, -\Delta - \lambda), \qquad \Phi_\kappa(h, N) = (f_\kappa(h, N), g_\kappa(h, N)), \tag{5.108}$$

where

$$f_\kappa(h, N) = 2(N + F(v + h)) + \frac{4\pi}{|\Sigma|}(k_+ - k_-) - \lambda h, \tag{5.109}$$

$$g_\kappa(h, N) = (\kappa^2 + 2F'(v + h))N + \kappa^2 F(v + h) - \lambda N, \tag{5.110}$$

and λ is a positive constant. Recall a continuous non-linear map $T : X \to Y$ of Banach spaces is said to be compact if it maps any bounded subset $A \subset X$ to a precompact set $TA \subset Y$,

Lemma 5.13. *The operator* $\Phi : [0, \kappa_*] \times \mathrm{H}^2 \times \mathrm{H}^2 \to \mathrm{L}^2 \times \mathrm{L}^2$ *such that* $\Phi(\kappa, \cdot, \cdot) = \Phi_\kappa$ *is continuous.*

Proof. We show the component functions f_κ and g_κ are continuous as follows. Notice,

$$\begin{aligned}||f_{\kappa_2}(h_2, N_2) - f_{\kappa_1}(h_1, N_1)||_{\mathrm{L}^2} &\leq 2\,||N_2 - N_1||_{\mathrm{L}^2} + 2\,||F'(\xi)\,(h_2 - h_1)||_{\mathrm{L}^2} + \lambda\,||h_2 - h_1||_{\mathrm{L}^2} \\ &\leq C\,(||N_2 - N_1||_{\mathrm{L}^2} + ||h_2 - h_1||_{\mathrm{L}^2}), \qquad (5.111)\end{aligned}$$

where ξ is well defined almost everywhere. For g_κ we have,

$$\begin{aligned}||g_{\kappa_2}(h_2, N_2) - g_{\kappa_1}(h_1, N_1)||_{\mathrm{L}^2} &\leq ||\kappa_2^2\,N_2 - \kappa_1^2\,N_1||_{\mathrm{L}^2} \\ &\quad + ||\kappa_2^2\,F(v + h_2) - \kappa_1^2\,F(v + h_1)||_{\mathrm{L}^2} \\ &\quad + 2\,||N_2\,F'(v + h_2) - N_1\,F'(v + h_1)||_{\mathrm{L}^2} \\ &\quad + \lambda\,||N_2 - N_1||_{\mathrm{L}^2} \\ &\leq |\kappa_2^2 - \kappa_1^2|\,||N_2||_{\mathrm{L}^2} + \kappa_1^2\,||N_2 - N_1||_{\mathrm{L}^2} \\ &\quad |\kappa_2^2 - \kappa_1^2|\,||F(v + h_2)||_{\mathrm{L}^2} + \kappa_1^2\,||F(v + h_2) - F(v + h_1)||_{\mathrm{L}^2} \\ &\quad 2\,||(N_2 - N_1)\,F'(v + h_2)||_{\mathrm{L}^2} \\ &\quad + 2\,||N_1\,(F'(v + h_2) - F'(v + h_1))||_{\mathrm{L}^2} \\ &\quad + \lambda\,||N_2 - N_1||_{\mathrm{L}^2}. \qquad (5.112)\end{aligned}$$

$F(v + h_2)$ and $F'(v + h_2)$ are uniformly bounded by a constant independent of h_2. Also, there exist functions ξ, η well defined except at core positions, such that,

$$|F(v + h_2) - F(v + h_1)| = |F'(\xi)\,(h_2 - h_1)| \leq C\,|h_2 - h_1|, \qquad (5.113)$$

$$|F'(v + h_2) - F'(v + h_1)| = |F''(\eta)\,(h_2 - h_1)| \leq C\,|h_2 - h_1|. \qquad (5.114)$$

By Sobolev's embedding, N_1 is continuous and the norm $||N_1||_{C^0}$ is controlled by $||N_1||_{\mathrm{H}^2}$, hence,

$$\begin{aligned}||N_1\,(F'(v + h_2) - F'(v + h_1))||_{\mathrm{L}^2} &\leq ||N_1||_{C^0}\,||F'(v + h_2) - F'(v + h_1)||_{\mathrm{L}^2} \\ &\leq C\,||N_1||_{\mathrm{H}^2}\,||h_2 - h_1||_{\mathrm{L}^2}. \qquad (5.115)\end{aligned}$$

Hence, there is a constant C, independent of (κ_j, h_j, N_j), such that,

$$\begin{aligned}||g_{\kappa_2}(h_2, N_2) - g_{\kappa_1}(h_1, N_1)||_{\mathrm{L}^2} &\leq C\,(|\kappa_2^2 - \kappa_1^2|\,(1 + ||N_2||_{\mathrm{L}^2}) \\ &\quad + ||N_2 - N_1||_{\mathrm{L}^2}\,(1 + \kappa_1^2) + ||h_2 - h_1||_{\mathrm{L}^2}\,(\kappa_1^2 + ||N_1||_2)). \qquad (5.116)\end{aligned}$$

(5.111) and (5.116) prove the component functions are continuous. □

Proposition 5.14. *The operator* $T = \mathrm{L}^{-1} \circ \Phi : [0, \kappa_*] \times \mathrm{H}^2 \times \mathrm{H}^2 \to \mathrm{H}^2 \times \mathrm{H}^2$ *is compact.*

Proof. By the Cauchy-Schwarz inequality and the standard elliptic estimates, if $u, f \in \mathrm{L}^2$, and

$$(\Delta + \lambda)\, u = f \tag{5.117}$$

in the weak sense, then $u \in \mathrm{H}^2$ and there is a constant C, independent of (u, f) such that,

$$||u||_{\mathrm{H}^2} \leq C\, ||f||_{\mathrm{L}^2}. \tag{5.118}$$

This shows $(\Delta + \lambda)^{-1} : \mathrm{L}^2 \to \mathrm{H}^2$ is continuous, therefore L is also continuous and by lemma 5.13 also T. If $T(\kappa, h, N) = (u, w)$, then we have,

$$(\Delta + \lambda)\, u = -f_\kappa, \qquad\qquad (\Delta + \lambda)\, w = -g_\kappa. \tag{5.119}$$

Let $A \subset [0, \kappa_*] \times \mathrm{H}^2 \times \mathrm{H}^2$ be bounded and closed and let $R > 0$ be sufficiently large such that if $(\kappa, h, N) \in A$, then $||h||_{\mathrm{H}^2} + ||N||_{\mathrm{H}^2} \leq R$. If $\{(u_n, w_n)\}$ is a sequence in $T(A)$, such that $(u_n, w_n) = T(\kappa_n, h_n, N_n)$ for $(\kappa_n, h_n, N_n) \in A$, we can find a subsequence (u_{n_k}, w_{n_k}) such that $\kappa_{n_k} \to \kappa' \in [0, \kappa_*]$ and $(h_{n_k}, N_{n_k}) \to (h_*, N_*)$ weakly in $\mathrm{H}^1 \times \mathrm{H}^1$ and strongly in $\mathrm{L}^2 \times \mathrm{L}^2$. By equations (5.111) and (5.116) and the fact that $\{N_n\}$ is bounded in H^2, the sequence $\{\Phi(\kappa_{n_k}, h_{n_k}, N_{n_k})\}$ is Cauchy in $\mathrm{L}^2 \times \mathrm{L}^2$. By (5.119), $\{(u_{n_k}, w_{n_k})\}$ is Cauchy in $\mathrm{H}^2 \times \mathrm{H}^2$, therefore convergent in $\overline{T(A)}$. □

By this proposition, for any bounded open set $\Omega \subset \mathrm{H}^2 \times \mathrm{H}^2$ such that $(I - \mathrm{L}^{-1} \circ \Phi_\kappa)^{-1}(0) \notin \partial\Omega$, the degree

$$\deg(I - \mathrm{L}^{-1} \circ \Phi_\kappa, \Omega, 0), \tag{5.120}$$

is well defined and a homotopical invariant for κ restricted to any subinterval $[a, b] \subset [0, \kappa_*]$ such that,

$$(I - \mathrm{L}^{-1} \circ \Phi_\kappa)^{-1}(0) \notin \partial\Omega \qquad \forall \kappa \in [a, b]. \tag{5.121}$$

Notice $(h, N) \in (I - \mathrm{L}^{-1} \circ \Phi_\kappa)^{-1}(0)$ if and only if it is a solution to the governing elliptic problem, (5.29), (5.70).

Lemma 5.15. *For any ball $B \subset \mathrm{H}^2 \times \mathrm{H}^2$ and for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $(h, N) \in B$, $|\kappa_2 - \kappa_1| < \delta$ implies $|T(\kappa_2, h, N) - T(\kappa_1, h, N)| < \epsilon$.*

It is said that the operator T is continuous in κ uniformly with respect to (h, N) in balls in $\mathrm{H}^2 \times \mathrm{H}^2$

Proof. In order to prove the lemma, it is sufficient to consider balls centred at the origin. Let $R > 0$, and $(h, N) \in B_R(0)$. If $(u, w) = \mathrm{L}^{-1} \circ \Phi(\kappa, h, N)$, then u and w are solutions to (5.119). Let $(u_j, w_j) = \mathrm{L}^{-1} \circ \Phi(\kappa_j, h, N)$, $j = 1, 2$, then $u_1 = u_2$ because $f_\kappa(h, N)$ is independent of κ. For w_j, by (5.116) there is a constant $C = C(R)$ independent of (h, N), such that,

$$||g_{\kappa_2}(h, N) - g_{\kappa_1}(h, N)||_{\mathrm{L}^2} \leq C\,|\kappa_2 - \kappa_1|^2. \tag{5.122}$$

By (5.119),

$$\begin{aligned} ||\nabla(w_2 - w_1)||^2_{\mathrm{L}^2} + \lambda\,||w_2 - w_1||^2_{\mathrm{L}^2} &= -\langle (w_2 - w_1), g_{\kappa_2}(h, N) - g_{\kappa_1}(h, N)\rangle \\ &\leq ||w_2 - w_1||_{\mathrm{L}^2} \cdot ||g_{\kappa_2}(h, N) - g_{\kappa_1}(h, N)||_{\mathrm{L}^2} \end{aligned} \tag{5.123}$$

Whence, there exists another constant, independent of (h, N), such that,

$$||w_2 - w_1||_{\mathrm{L}^2} \leq C\,||g_{\kappa_2}(h, N) - g_{\kappa_1}(h, N)||_{\mathrm{L}^2}. \tag{5.124}$$

By Schauder's estimates,

$$\begin{aligned} ||w_2 - w_1||_{\mathrm{H}^2} &\leq C\,\left(||\Delta(w_2 - w_1)||_{\mathrm{L}^2} + ||w_2 - w_1||_{\mathrm{L}^2}\right) \\ &\leq C\,(||g_{\kappa_2}(h, N) - g_{\kappa_1}(h, N)||_{\mathrm{L}^2} + (\lambda + 1)\,||w_2 - w_1||_{\mathrm{L}^2}) \\ &\leq C\,||g_{\kappa_2}(h, N) - g_{\kappa_1}(h, N)||_{\mathrm{L}^2} \\ &\leq C|\kappa_2^2 - \kappa_1^2|. \end{aligned} \tag{5.125}$$

Therefore, $||(u_2, w_2) - (u_1, w_1)||_{\mathrm{H}^2 \times \mathrm{H}^2} \to 0$ uniformly as $\kappa_2 \to \kappa_1$. □

If $0 < \kappa_0 < \kappa_*(D)$, we know by lemma 5.12 that there exists an $R > 0$ such that for any $\kappa \in [\kappa_0, \kappa_*(D)]$ the solution to equations (5.29), (5.70) is in the interior of the disk $\mathbb{D}(0, R) \subset \mathrm{H}^2 \times \mathrm{H}^2$. Since for any $\epsilon > 0$ there is no solution to the equations for $\kappa_*(D)+\epsilon$, by the homotopy invariance of the degree, we conclude,

$$\deg_{\rm LS}(\mathrm{I}-\mathrm{L}^{-1}\circ\Phi_\kappa,\mathbb{D}(0,R),0)=0,\qquad \kappa\in[\kappa_0,\kappa_*(D)]. \tag{5.126}$$

By proposition 5.2 we know there is a neighbourhood U of $(h_0,0)$ such that for κ small enough, there is exactly one solution (h_κ,N'_κ) to equations (5.29) and (5.70) in U and this solution varies smoothly in $\mathrm{H}^2\times\mathrm{H}^2$ with κ.

Lemma 5.16. $|\mathrm{ind}(I-T(0,\cdot,\cdot),(h_0,0),0)|=1$.

Proof. At $\kappa=0$, the derivative of Φ_0 at $(h_0,0)$ has components,

$$f'_0(h_0,0)\cdot(\delta h,\delta N)=2\,(\delta N+F'(v+h_0)\,\delta h)-\lambda\,\delta h, \tag{5.127}$$
$$g'_0(h_0,0)\cdot(\delta h,\delta N)=2\,F'(v+h_0)\,\delta N-\lambda\,\delta N. \tag{5.128}$$

The operator L^{-1} is linear, hence, the derivative of $T_0=T(0,\cdot,\cdot)$ at $(h_0,0)$ is,

$$T'_0(h_0,0)=\mathrm{L}^{-1}\circ\,\Phi'_0(h_0,0). \tag{5.129}$$

If $(\delta h,\delta N)\in\mathrm{Ker}(I-T'_0(h_0,0))$, then $(\delta h,\delta N)$ is the solution to the elliptic problem,

$$-\Delta\,\delta h=2\,\delta N+2\,F'(v+h_0)\,\delta h, \tag{5.130}$$
$$-\Delta\,\delta N=2\,F'(v+h_0)\,\delta N. \tag{5.131}$$

By lemma 2.10, the operator $\Delta+2\,F'(v+h_0):\mathrm{H}^2\to\mathrm{L}^2$ is an isomorphism. Therefore, $\delta h=\delta N=0$. By theorem 2.19,

$$\mathrm{ind}(I-T(0,\cdot,\cdot),(h_0,0),0)=\pm1, \tag{5.132}$$

where the sign depends on the multiplicities of the eigenvalues $\lambda>1$ of $I-T'_0(h_0,0)$. ∎

Proposition 5.17. *There is a $\kappa_0>0$ such that, if $0<\kappa<\kappa_0$, equations (5.29),(5.70) have exactly two continuous families of solutions. As $\kappa\to0$ one of the families is convergent to $(h_0,0)$, the solution to of the regularised Taube's equation and the second family is such that $(h_\kappa-c(h_\kappa),N'_\kappa)\to(0,\alpha_\pm)$ and $c(h_\kappa)\to\mp\infty$, where $\alpha_\mp$ and the sign of the divergence depend on the sign of k_+-k_- as in lemma 5.10. Moreover, for any $R>0$, there is a $\kappa'>0$ such that if $0<\kappa<\kappa'$, then $||(h_\kappa,N_\kappa)||_{\mathrm{H}^2\times\mathrm{H}^2}>R$ for at least one pair of solutions to the equations.*

Proof. Proposition 5.14, and Lemma 5.15 show T satisfies the hypotesis of theorem 2.21. By proposition 5.2, there exists $\kappa_0 \in (0, \kappa_*]$ and an open bounded set $U \subset \mathrm{H}^2 \times \mathrm{H}^2$ of $(h_0, 0)$ such that the restriction of (h_κ, N_κ) to U varies smoothly with $\kappa \in [0, \kappa_0)$. By lemma 5.16, if $\mathrm{diam}(U)$ is small, $\deg(I - T(0, \cdot, \cdot), U, 0) \neq 0$ and there is no other solution for $\kappa = 0$ in $\overline{U}$. By theorem 2.21, there is a connected closed set $\mathcal{C} \subset [0, \kappa_*] \times \mathrm{H}^2 \times \mathrm{H}^2$, such that $(0, h_0, 0) \in \mathcal{C}$ and either $\mathcal{C}$ is unbounded or $\mathcal{C} \cap (\{0\} \times (\mathrm{H}^2 \times \mathrm{H}^2 \setminus \overline{U})) \neq \emptyset$. Since for $\kappa = 0$ there is only one solution to equations (5.29) (5.70), we rule out the second possibility. As $\kappa_* < \infty$, by lemma 5.12 there is a second family (h_n, N_n) of solutions to the equations, such that $\kappa_n \to 0$ and $||(h_\kappa, N_\kappa)||_{\mathrm{H}^2 \times \mathrm{H}^2} \to \infty$. By proposition 5.11, $(0, \alpha_\pm)$ is a limit point. In order to prove the last claim, assume towards a contradiction, the existence of $R > 0$ and a sequence $\kappa_n \to 0$ such that $||(h_{\kappa_n} - h_0, N_{\kappa_n})||_{\mathrm{H}^2 \times \mathrm{H}^2} \leq R$ for all solutions with parameter κ_n. By lemma 5.9, the set of solutions $\{(h_\kappa, N_\kappa) \mid ||(h_\kappa - h_0, N_\kappa)|| = R\}$ can not accumulate at $\kappa = 0$. Let $\kappa_R > 0$ be such that if $||(h_\kappa - h_0, N_\kappa)||_{\mathrm{H}^2 \times \mathrm{H}^2} = R$, then $\kappa > \kappa_R$ and let us choose n such that $\kappa_n < \kappa_R$. Consider the relatively open set,

$$V = \left\{(\kappa, h, N) \in [0, \kappa_*] \times \mathrm{H}^2 \times \mathrm{H}^2 \mid ||(h - h_0, N)||_{\mathrm{H}^2 \times \mathrm{H}^2} > R,\ 0 < \kappa < \kappa_n\right\} \cap \mathcal{C}. \tag{5.133}$$

V is not empty because there is a divergent sequence in $\mathcal{C}$ with deformation parameter converging to 0, we claim V is also closed, because if $\{(\mu_n, h_n, N_n)\} \subset V$ has an accumulation point (μ_*, h_*, N_*), then $(\mu_*, h_*, N_*) \in \mathcal{C}$ because this set is closed. At the same time, $||(h_* - h_0, N_*)||_{\mathrm{H}^2 \times \mathrm{H}^2} \geq R$ and $0 \leq \mu_* \leq \kappa_n$. Since $||(h_* - h_0, N_*)||_{\mathrm{H}^2 \times \mathrm{H}^2} = R$ implies $\mu_* > \kappa_R$, we can discard this case. If $\mu_* = 0$ then $(h_*, N_*) = (h_0, 0)$ which is impossible. If $\mu_* = \kappa_n$, then we also have $||(h_* - h_0, N_*)||_{\mathrm{H}^2 \times \mathrm{H}^2} \leq R$, which is absurd. Therefore $(\mu_*, h_*, N_*) \in V$ and this set is open and closed. Since $\mathcal{C}$ is connected, $V = \mathcal{C}$. A contradiction. □

In view of this proposition, we can define $\kappa_{\mathcal{C}}(D)$ as the supremum,

$$\kappa_{\mathcal{C}}(D) = \sup\left\{\kappa > 0 \mid (\kappa, h_\kappa, N_\kappa) \in \mathcal{C}\right\}. \tag{5.134}$$

Proposition 5.17 shows $\kappa_{\mathcal{C}}$ is actually a maximum, moreover, since the set of solutions for which $(h_\kappa, N_\kappa) \in U$ is contained in $\mathcal{C}$, the same lower bound for κ_* is also valid for κ_C. We summarise the results of this section in the following theorem.

Theorem 5.18. *Let $(\kappa, \phi_\kappa, A_\kappa, N_\kappa)$ be a solution of the Bogomolny equations of the BPS soliton equations with Chern-Simons deformation constant κ. Assume $k_+ - k_- \neq 0$ and Bradlow's bound is satisfied, then the following properties hold,*

1. *κ is bounded by a constant independent of the position of the divisors as given in proposition 5.7,*
2. *If κ is small, for each divisor there are at least two gauge inequivalent families of solutions to the Bogomolny equations.*
3. *There are an $\epsilon > 0$ and $\kappa_0 > 0$ such that, if $|\kappa| < \kappa_0$ there is exactly one gauge equivalence class $(\kappa, \phi_\kappa, A_\kappa, N_\kappa)$ of solutions to the Bogomolny equations, such that*

$$||h_\kappa - h_0||_{C^1} + ||N_\kappa||_{C^1} < \epsilon. \tag{5.135}$$

   Outside any closed neighbourhood $\overline{U}$ of the core set $P \cup Q$, this family of solutions varies smoothly with κ.

4. *For any neighbourhood U of the core set, we have the following property: For any $\epsilon > 0$, there is a $\kappa' > 0$, such that if $|\kappa| < \kappa'$, there is a solution $(\kappa, \phi_\kappa, A_\kappa, N_\kappa)$ to the Bogomolny equations, such that*

$$||\phi_3^\kappa \mp 1||_{C^1(\Sigma \setminus U)} + ||N_\kappa - \alpha_\pm||_{C^1} < \epsilon, \tag{5.136}$$

   where the signs chosen and the constant $\alpha_\pm$ depend on the sign of the difference $k_+ - k_-$ as defined on lemma 5.10.

5.1.5 Symmetric deformations on the sphere

In this section we study the deformation constant in the sphere. It is known that in the Euclidean plane, there is a solution to the elliptic problem for any $\kappa \in \mathbb{R}$. Hence, the existence of an upper bound for κ_* in a compact surface is a nontrivial task. We will suppose all the vortices are located at the north pole of the domain sphere, and all the antivortices at the south pole. We choose trivialisations $\phi_\pm : U_\pm \to \mathbb{S}^2$ at $U_\pm = \mathbb{S}^2 \setminus \{(0, 0, \mp 1)\}$, which stereographically project from the south or north pole respectively, as $\varphi_\pm : U_\pm \to \mathbb{C}$. This projections are related by a gauge transformation, which by spherical symmetry is,

$$\varphi_+ = \frac{e^{in\theta}}{\varphi_-}, \qquad x \in U_+ \cap U_-. \tag{5.137}$$

Whereas for the connection, if it is represented locally by $a_\pm \in \Omega^1(U_\pm)$,

$$a_+ = a_- + nd\theta, \qquad x \in U_+ \cap U_-. \tag{5.138}$$

Stereographic coordinates in the domain sphere will be denoted accordingly $x_\pm = r_\pm e^{i\theta_\pm}$. Hence, $x_+x_- = 1$ in $U_+ \cap U_-$ and $\theta_+ = -\theta_-$. We choose the ansatz,

$$\varphi_\pm = f_\pm(r_\pm)e^{ik_\pm\theta_\pm} \qquad\qquad a_\pm = a_\pm(r_\pm)\, d\theta_\pm, \tag{5.139}$$

which is justified by the equivariant rotational symmetry of the problem. Compatibility of the fields then requires $n = k_+ - k_-$. The Bogomolny equations reduce to a system of ODES which we aim to integrate,

$$f_\pm' = \frac{1}{r}(k_\pm \mp a_\pm)\, f_\pm, \tag{5.140}$$

$$a_\pm' = r\Omega(r)B_\pm, \tag{5.141}$$

$$N_\pm'' = -\Omega(r)\left(\kappa B_\pm - \frac{4f_\pm^2 N_\pm}{(1+f_\pm^2)^2}\right) - \frac{1}{r}N_\pm', \tag{5.142}$$

where,

$$\Omega(r) = \frac{4R^2}{(1+r^2)^2}, \tag{5.143}$$

$$B_\pm = -\left(\kappa N_\pm + \tau \pm 1 \mp \frac{2}{1+f_\pm^2}\right). \tag{5.144}$$

We solved the Bogomolny equations in the punctured disk $\mathbb{D}_1(0) \setminus \{0\}$ adding the compatibility conditions,

$$\begin{aligned} f_+(1)f_-(1) &= 1, & a_+(1) + a_-(1) &= n, \\ N_+(1) &= N_-(1), & N_+'(1) &= -N_-'(1), \end{aligned} \tag{5.145}$$

together with the lowest order approximation to the fields at $r = 0$,

$$f_\pm = q_\pm r^{k_\pm} + \mathcal{O}\left(r^{k_\pm+1}\right), \tag{5.146}$$

$$B_\pm = \begin{cases} -\left(\kappa p_\pm + \tau \pm 1 \mp \frac{2}{1+q_\pm^2}\right) + \mathcal{O}(r), & k_\pm = 0, \\ -(\kappa p_\pm + \tau \pm 1 \mp 2) + \mathcal{O}(r), & k_\pm \neq 0, \end{cases} \tag{5.147}$$

$$a_\pm = 2B_\pm R^2 r^2 + \mathcal{O}(r^3), \tag{5.148}$$

$$N_\pm = \begin{cases} p_\pm + \left(-\kappa B_\pm + \frac{4q_\pm^2 p_\pm}{(1+q_\pm^2)^2}\right) R^2 r^2 + \mathcal{O}(r^3), & k_\pm = 0, \\ p_\pm - \kappa B_\pm R^2 r^2 + \mathcal{O}(r^3), & k_\pm \neq 0. \end{cases} \tag{5.149}$$

To find the initial stable solution, we used the shooting method in the interval $[\delta, 1]$ for a small value $\delta > 0$. Given initial conditions $Z = (q_+, q_-, p_+, p_-)$ for the parameters, we solved the Bogomolny equations and defined a map $M : \mathbb{R}^4 \to \mathbb{R}^4$,

$$Z \mapsto (f_+(1)f_-(1) - 1, a_+(1) + a_-(1) - n, N_+(1) - N_-(1), N_+'(1) + N_-'(1)), \tag{5.150}$$

whose zero determines suitable initial conditions for a solution to the Bogomolny equations compatible at the boundary of the disk. Next, we applied the pseudo-arclength continuation method, as described in [14]. Given initial data $(\kappa_0, Z_0) \in \mathbb{R}^5$, we sought a nearby point (κ, Z) such that,

$$\dot{Z}_0 \cdot (Z - Z_0) + \dot{\kappa}_0 (\kappa - \kappa_0) = \delta s, \tag{5.151}$$

for a small positive constant δs. We restricted ourselves to positive κ and solved the Bogomolny equation in the vortex-antivortex case and the case $k_+ = 2$, $k_- = 0$. We solved both cases on a sphere of radius 2. The results can be seen on Figure 5.1. We found that for the vortex-antivortex case, the data suggests κ is unbounded. This would be the case if for all the solutions, the function $h = \log f^2$ have bounded average. If rotationally symmetric solutions are unique, invariance of Taube's equation under isometries of the sphere implies the average is actually zero. Therefore, we conjecture κ unbounded for this configuration of cores' on the sphere. On the $(2, 0)$ case, arclength continuation started growing fairly quickly until it reached a maximum value and started decreasing towards zero as expected. In Figure 5.2 on the right column can be seen the two limiting solutions of theorem 5.18. At $\kappa = 0$ we obtained the solution to the Taubes

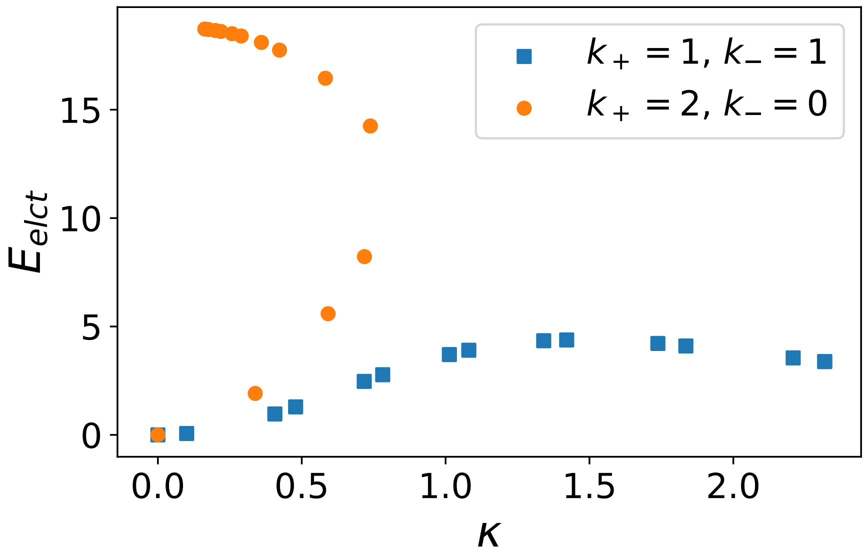


Figure 5.1: Comparison of the electrostatic energy in the balanced and unbalanced cases. The existence of two types of solutions if $k_+ \neq k_-$ is evident from the graph, while the energy also suggests uniqueness of the solution (h_κ, N_κ) for each κ in the balanced case.

equation as expected and a limiting solution, as the averages of h diverged towards infinity, the gauge invariant component ϕ_3 of the Higgs field ϕ started converging to constant 1, in other words, ϕ converged to the north pole section, while κN converged to the expected limit

$$\alpha_- = \frac{3}{4}. \tag{5.152}$$

5.2 Dynamics of the moduli space of Ginzburg-Landau vortices with a Chern-Simons term

In section 5.1 we proved the existence of a minimum constant κ_{min} such that regardless of core positions on the moduli space $\mathcal{M}^{k_+,k_-}(\Sigma)$ of vortices and antivortices of the $U(1)$-gauged $O(3)$ Sigma model with a Chern-Simons deformation, there exists a solution to the Bogomolny equations close to the solution at

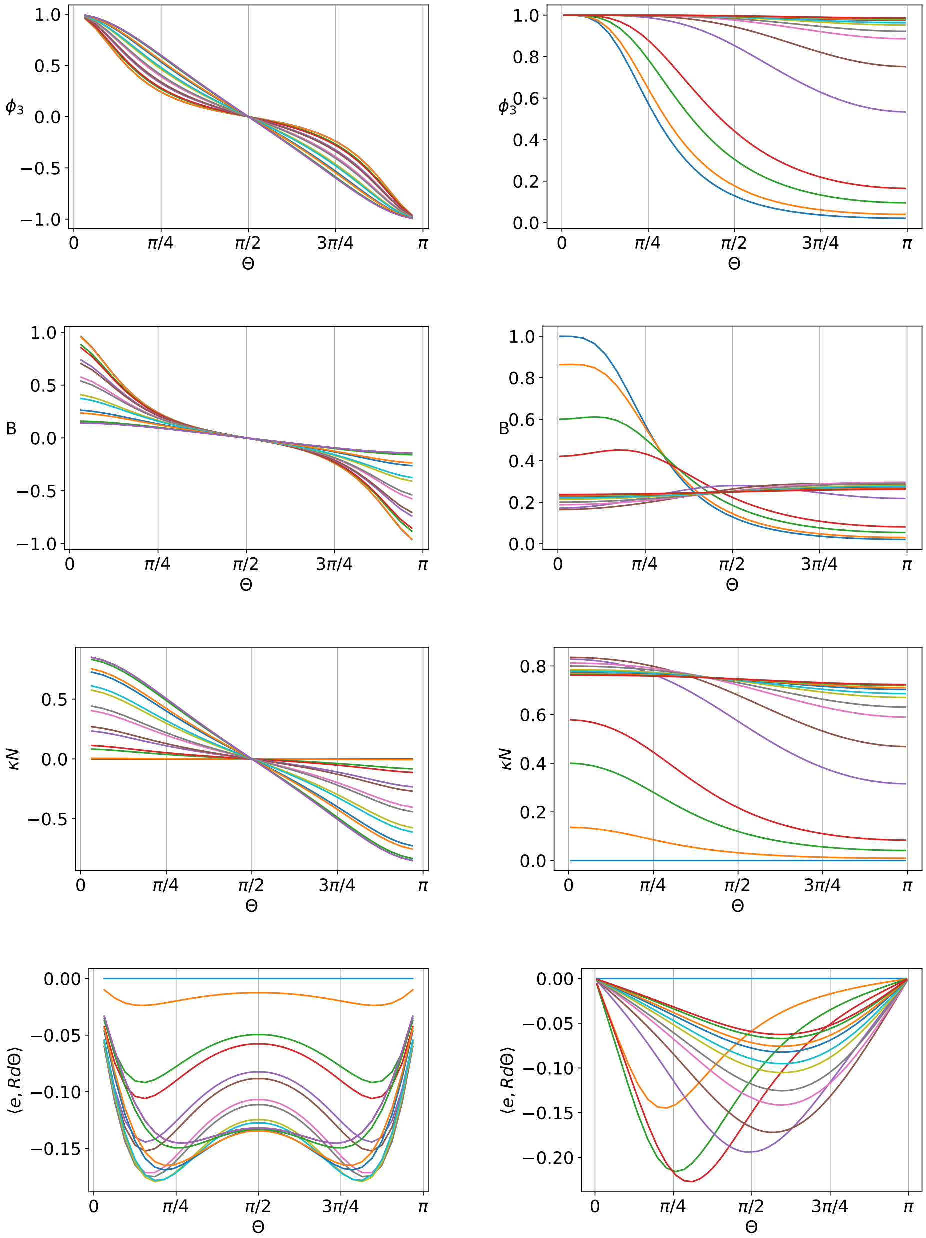


Figure 5.2: Snapshots of solutions to the Bogomolny equations on the sphere along the declination angle Θ. The radius was set to $R = 2$. **Left.** Vortex-antivortex case. **Right.** Two vortices at north pole and no antivortices. The asymmetry parameter was set to $\tau = 0$.

$\kappa = 0$. This result extends similar claims for the Ginzburg-Landau model obtained in [14] and justifies the possible existence of a localization formula similar to the one obtained in section 2.2.1 for BPS solitons of the gauged $O(3)$ Sigma model, with the addition of a term dependent on the Chern-Simons constant κ. In this section we apply the general framework of section 2.2 to compute the extra κ term in the localization formula. As the calculations are similar for both models, we compute a localization formula for each one and finalise discussing about the extension of our formula to the coincidence set.

Previous work on the subject for the Ginzburg-Landau functional includes models of Kim-Min and Kim-Lee [24, 25], where the authors considered a related model with a different type of the Chern-Simons interaction, [26], where Kim and Lee analysed the dynamics of the Ginzburg-Landau model with a neutral field on the plane and [10] where Collie and Tong addressed motion on the moduli space of abelian vortices in the presence of a magnetic field and concluded that the extra Chern-Simons term in the localization formula is the Ricci form of the metric on the reduced moduli space. Later, in [1] Alqahtani and Speight showed that the deformation term of Kim-Lee cannot extend to the coincidence set of modulli space, whereas the term from Collie-Tong can, and thus Kim-Lee and Collie-Tong deformations of the Abelian Higgs model are different.

5.2.1 The Maxwell Higgs Chern Simons model

We work on a Riemann surface Σ that can be either compact or the Euclidean plane. The setup is as in section 2.2. We assume the existence of a principal bundle,

$$U(1) \to P \to \mathbb{R} \times \Sigma \tag{5.153}$$

and denote by ρ the representation of $U(1)$ as isometries of the complex plane,

$$\rho : U(1) \to \mathrm{Aut}(\mathbb{C}). \tag{5.154}$$

Let $F = (\mathbb{R} \times \Sigma) \times_\rho \mathbb{C}$ be the hermitian bundle associated to ρ. We fix the metric in $\mathbb{R} \times \Sigma$ as the product

$$dt^2 - g, \tag{5.155}$$

where g denotes the Riemannian metric in Σ. Let $\tilde{\mathcal{D}}$ be the connection induced by ρ,

$$\tilde{\mathcal{D}} : \Gamma F \to \Gamma\left((\mathbb{R} \oplus T^*\Sigma) \otimes TF\right). \tag{5.156}$$

Given $\phi \in \Gamma F$, in a local trivialisation this is the map

$$\tilde{\mathcal{D}}\phi = d\phi - i\tilde{A} \otimes \phi. \tag{5.157}$$

As in section 5.1, we add a neutral scalar field,

$$N \in C^\infty(\mathbb{R} \times \Sigma). \tag{5.158}$$

It will be convenient to make the division of space and time explicit, we denote by $\mathcal{D}_t\phi \in C^\infty(\mathbb{R} \times \Sigma)$ the time component of $\tilde{\mathcal{D}}\phi$ and from now onwards $d\phi : \mathbb{R} \to \Gamma(T^*\Sigma \otimes TF)$ will be the spatial component of $d\phi$ as a function of time. The spatial component of $\tilde{\mathcal{D}}\phi$, denoted by $\mathcal{D}\phi$, is

$$\mathcal{D}\phi : \mathbb{R} \to \Gamma\left(T^*\Sigma \otimes TF\right), \qquad \mathcal{D}\phi = d\phi - iA \otimes \phi, \tag{5.159}$$

The Maxwell-Higgs Lagrangian is,

$$L_{MH} = \frac{1}{2}\left(||\mathcal{D}_t\phi||^2 + ||e||^2 + ||\dot{N}||^2 - ||\mathcal{D}\phi||^2 - ||B||^2 - ||dN||^2\right) - \langle 1, U\rangle,$$

the norms and the product in the Lagrangian are in the L^2 sense. The potential function U is given by,

$$U = \frac{1}{8}\left(-2\kappa N + 1 - |\phi|^2\right)^2 + \frac{1}{2}|N\phi|^2. \tag{5.160}$$

We add a Chern-Simons term to the Lagrangian,

$$L_{CS} = \frac{1}{2}\left(\langle A, *e\rangle + \langle \tilde{A}_0, *B\rangle\right), \tag{5.161}$$

the Maxwell-Higgs-Chern-Simons Lagrangian is,

$$L = L_{MH} + \kappa L_{CS}. \tag{5.162}$$

As for the $O(3)$ Sigma model, the Chern-Simons term is not gauge invariant, however if two gauge potentials differ by a gauge transformation, the corresponding Lagrangians will differ by a total divergence, hence they will yield the same field equations. If Σ is compact, A and $\tilde{A}_0$ are only defined locally, however, we can always choose an open and dense subset of Σ, diffeomorphic to the unit disk by the Riemann mapping theorem, in which the connection is trivializable. In any case, the Lagrangian is well defined up to gauge equivalence. Variating L with respect to $\tilde{A}_0$, Gauss's law is,

$$d^*e = -\langle \mathcal{D}_t\phi, i\phi\rangle + \kappa * B. \tag{5.163}$$

In terms of the gauge potential this is the same as,

$$-\left(\Delta + |\phi|^2\right)\tilde{A}_0 = \frac{1}{2i}\left(\phi\dot{\phi}^\dagger - \phi^\dagger\dot{\phi}\right) - d^*\dot{A} + \kappa * B. \tag{5.164}$$

Let T and V be the kinetic and potential energy of the fields,

$$\begin{aligned} T &= \frac{1}{2}(||\mathcal{D}_t\phi||^2 + ||e||^2 + ||\dot{N}||^2), \\ V &= \frac{1}{2}(||\mathcal{D}\phi||^2 + ||B||^2 + ||dN||^2) + \langle 1, U\rangle, \end{aligned} \tag{5.165}$$

the total conserved energy of the fields is

$$E = T + V, \tag{5.166}$$

If Σ is the Euclidean plane, we assume the following convergence at infinity,

$$\dot{A}, A \in H^1\left(\Omega^1(\mathbb{R}^2)\right), \qquad \tilde{A}_0, N \in H^1\left(\mathbb{R}^2\right), \tag{5.167}$$

$$\dot{N}, \dot{\phi} \in \mathrm{L}^2\left(\mathbb{R}^2\right), \qquad \lim_{|x|\to\infty} |\phi(x)|^2 = 1. \tag{5.168}$$

Let $\partial_A\phi$ be the projection of $\mathcal{D}\phi$ in the sub-space $(1,0)$ of the complexification of the bundle $T^*\Sigma \otimes TF$, in local coordinates,

$$\overline{\partial}_A\phi = \frac{1}{2}\left(\mathcal{D}_1\phi + i\,\mathcal{D}_2\phi\right). \tag{5.169}$$

We apply the Bogomolny trick to obtain a set of equations obeyed by the fields,

$$\begin{aligned}
0 \leq \frac{1}{2}\Big(&||\mathcal{D}_t\phi - iN\phi||^2 + ||\dot{N}||^2 + ||-dN + e||^2 \\
&+ \left|\left|*B - \frac{1}{2}\left(-2\kappa N + 1 - |\phi|^2\right)\right|\right|^2\Big) + ||\overline{\partial}_A\phi||^2 \\
= {}& E - \langle \mathcal{D}_t\phi, iN\phi\rangle - \langle dN, e\rangle - \frac{1}{2}\langle *B, -2\kappa N + 1 - |\phi|^2\rangle \\
&+ ||\overline{\partial}_A\phi||^2 - \frac{1}{2}||\mathcal{D}\phi||^2 \\
= {}& E - \langle N, - * d * e + \langle \mathcal{D}_t\phi, i\phi\rangle - \kappa * B\rangle - \frac{1}{2}\langle *B, 1\rangle \\
&+ \frac{1}{2}\langle *B, |\phi|^2\rangle + ||\overline{\partial}_A\phi||^2 - \frac{1}{2}||\mathcal{D}\phi||^2 \\
= {}& E - n\pi.
\end{aligned} \tag{5.170}$$

To obtain the last equation, we discarded several divergences and used Gauss's law and the identity,

$$*B\,|\phi|^2 = |\partial_A\phi|^2 - |\overline{\partial}_A\phi|^2. \tag{5.171}$$

We also used that,

$$\int_\Sigma B = 2n\pi, \qquad n \in \mathbb{Z}. \tag{5.172}$$

If Σ is compact, this is due to the fact that B is the curvature of the line bundle F, in the case that Σ is the Euclidean plane, this comes from the assumptions of decaying of the gauge potential at infinity and the nontrivial winding of A at the circle at infinity. Hence, a set of fields $\Phi = (\phi, N, A) \in \mathcal{A}$, is a minimal with energy

$$E = n\pi, \tag{5.173}$$

provided it satisfies the Bogomolny equations,

$$\dot{N} = 0, \tag{5.174}$$

$$e = dN, \tag{5.175}$$

$$\mathcal{D}_t\phi = iN\phi, \tag{5.176}$$

$$\overline{\partial}_A\phi = 0, \tag{5.177}$$

$$*B = \frac{1}{2}\left(-2\kappa N + 1 - |\phi|^2\right). \tag{5.178}$$

From equations (5.175), (5.176) and Gauss's law, N is a solution to the elliptic problem,

$$\left(\Delta + |\phi|^2\right) N = \kappa * B. \tag{5.179}$$

If Φ is a solution to the Bogomolny equations and Gauss's law, by conservation of Energy and the Bogomolny equations,

$$L = 2T + \kappa L_{CS} - E = \frac{\kappa}{2}\left(\tilde{A}_0 + N, *B\right) - n\pi. \tag{5.180}$$

If we take the radiation gauge, $\tilde{A}_0 = -N$, Φ is an extremal of the Lagrangian. The solution is stationary by the Bogomolny equations. Hence Φ is a solution to the field equations.

5.2.2 Low energy dynamics with a Chern-Simons term

To apply the low energy approximation, we will work in the space $\mathcal{A}'$ of fields $\Phi = (N, \phi, \tilde{A})$ which are solutions to the Bogomolny equations. We have two models of the moduli space, the $O(3)$ Sigma model, for which localization was discussed on section 2.2 and the Ginzburg-Landau model, studied by Samols on the plane [47]. The Bogomolny equations for both soliton types have a similar structure, allowing to compute the contribution to the L^2 metric by the Chern-Simons term in both models. In this section, we do so and compute the energy contribution for the space of solutions to the Bogomolny equations $\mathcal{A}'$, our computation will be valid in both cases. In the next section we specialise into the MHCS moduli space and later extend our result to the $O(3)$ Sigma model. Given $\Phi \in \mathcal{A}'$, the formal tangent space $T_\Phi\mathcal{A}'$ is the space of solutions to the linearization of the Bogomolny equations at Φ. We introduce the L^2 metric on $T_\Phi\mathcal{A}'$ induced by the metrics in Σ and target space. We have the inclusion

$$\mathscr{G} \hookrightarrow T_\Phi\mathcal{A}', \qquad \alpha \in \mathscr{G} \mapsto (N, e^{i\alpha}\phi, A + d\alpha) \in T_\Phi\mathcal{A}', \tag{5.181}$$

defining the vertical bundle $\mathscr{G} \to \mathcal{V} \to \mathcal{A}'$. Suppose $\Phi_s : \mathbb{R} \to \mathcal{A}'$ is a differentiable curve in $\mathcal{A}'$, meaning that as a function in an open dense set $U \subset \Sigma$, $\mathbb{R} \times U \to \mathbb{R} \times F \times \Omega^1(U)$ is differentiable. Let us define $\beta = \tilde{A}_0 + N$, by Gauss's law and (5.179), β is a solution to

$$\left(\Delta + |\phi|^2\right)\beta = -\frac{1}{2i}\left(\phi\dot{\phi}^\dagger - \phi^\dagger\dot{\phi}\right) + d^*\dot{A}. \tag{5.182}$$

Equation (5.182) means β is the orthogonal projection of $\dot{\Phi}$ onto $\mathcal{V}$. Recall in $\mathcal{A}'$ the energy is conserved. By the Bogomolny equations energy is given by the expression,

$$E = \frac{1}{2}\left(||N\phi||^2 + ||dN||^2\right) + V = n\pi. \tag{5.183}$$

As in section 2.2, we assume variations of fields in $\mathcal{A}'$ are good approximations to slowly moving vortices. We work perturbatively in the deformation parameter. Assume κ is small, by equation (5.179),

$$N = \kappa\, N_\kappa + \mathcal{O}(\kappa^2). \tag{5.184}$$

Discarding terms of order κ^2 and a divergence, the kinetic energy of a field $\Phi \in \mathcal{A}'$ is,

$$\begin{aligned} T &= \frac{1}{2}\left(||\dot{\phi}^\perp - iN\phi||^2 + ||\dot{A}^\perp - dN||^2\right) \\ &= \frac{1}{2}\left(||\dot{\phi}^\perp||^2 + ||\dot{A}^\perp||^2 + ||N\phi||^2 + ||dN||^2\right) \end{aligned} \tag{5.185}$$

For the Chern-Simons term we have,

$$\begin{aligned} L_{CS} &= \frac{1}{2}\left(\langle A, *e\rangle + \langle \tilde{A}_0, *B\rangle\right) \\ &= \frac{1}{2}\left(\langle A, *\dot{A} - *d\tilde{A}_0\rangle + \langle \tilde{A}_0, *B\rangle\right) \\ &= \frac{1}{2}\langle A, *\dot{A}\rangle + \langle \tilde{A}_0, *B\rangle \end{aligned} \tag{5.186}$$

To first order in κ, the Lagrangian can be approximated as,

$$\begin{aligned} L' &= T - V + \kappa L_{CS} \\ &= \frac{1}{2}\left(||\dot{\phi}^\perp||^2 + ||\dot{A}^\perp||^2\right) + ||N\phi||^2 + ||dN||^2 - n\pi + \frac{\kappa}{2}\langle A, *\dot{A}\rangle + \kappa\,\langle \tilde{A}_0, *B\rangle \\ &= \frac{1}{2}(||\dot{\phi}^\perp||^2 + ||\dot{A}^\perp||^2) - n\pi + \kappa\,\langle \beta, *B\rangle + \frac{\kappa}{2}\langle A, *\dot{A}\rangle, \end{aligned} \tag{5.187}$$

where we discarded another divergence and used (5.179). Let us introduce the kinetic and connection terms, $K, \Omega_{\mathcal{M}} : T\mathcal{A}' \to \mathbb{R}$, defined as,

$$K = \frac{1}{2}(||\dot{\phi}^\perp||^2 + ||\dot{A}^\perp||^2), \qquad \Omega_{\mathcal{M}} = \kappa\left(\langle \beta, *B\rangle + \frac{1}{2}\langle A, *\dot{A}\rangle\right), \tag{5.188}$$

in geometric terms, K plays the role of a metric on tangent space, on the other hand, $\Omega_{\mathcal{M}}$ is a connection, deviating the motion of the fields from geodesic motion, as will be evident in the next subsection when we obtain a formula for this term. Therefore, the effective Lagrangian at low energy is,

$$L_{eff} = K + \Omega_{\mathcal{M}}. \tag{5.189}$$

K is gauge invariant and if $\kappa = 0$, it is the kinetic energy term in the Samols approximation to Ginzburg-Landau theory or the kinetic energy term computed in section 2.2. However if $\kappa \neq 0$, this term does not render the same energy as the extra κ term in the Bogomolny equations deforms the fields. Although $\Omega_{\mathcal{M}}$ is gauge dependent because of the β factor, L_{eff} determines the dynamics in a gauge invariant way, because any gauge transformation contributes a total divergence.

5.2.3 A formula for the connection term

In this section we focus on the Maxwell-Higgs-Chern-Simons model. Let $P \subset \Sigma$ be the set of zeros of ϕ. We assume the zeros are simple. If the energy of a solution to the Bogomolny equations is $n\pi$, there are n vortices on Σ. We work in a chart $\varphi : U \to \mathbb{C}$ defined on an open and dense subset $U \subset \Sigma$ and assume that $P \subset U$. We denote by $z = \varphi(x)$ points on $\mathbb{C}$ and assume the metric takes the form

$$g = e^{\Lambda(z)} \left(dz_1^2 + dz_2^2\right), \qquad z \in \mathbb{C}. \tag{5.190}$$

Since U is contractible, the restriction $F|_U$ is trivial. Let $U' = U \setminus P$, we define the fields $h, \chi \in C^\infty(U')$, $\eta \in C^\infty(U', \mathbb{C})$, such that,

$$\phi = e^{\frac{h}{2} + i\chi}, \qquad\qquad \eta = \frac{\dot{h}}{2} + i\dot{\chi}. \tag{5.191}$$

As for the $O(3)$ Sigma model χ is only well defined modulo 2π, however, h, η and $d\chi$ are well defined functions on U'. Since the zeros of ϕ are simple, for any $p \in P$ there is a coordinate neighbourhood U_p and a smooth function $\tilde{\phi}_p \in C^\infty(U_p, \mathbb{C} \setminus \{0\})$, such that,

$$\phi(x) = \left(\varphi(x) - \varphi(p)\right) \tilde{\phi}(x), \qquad x \in U_p. \tag{5.192}$$

Let $r_p = \log|\varphi(x) - \varphi(p)|$, $\theta_p = \mathrm{Arg}(\varphi(x) - \varphi(p))$, $x \in U'_p = U_p \setminus \{p\}$, we have local expansions,

$$h(x) = \log r_p^2 + \tilde{h}_p(x), \qquad \chi = \theta_p(x) + \tilde{\chi}_p(x), \tag{5.193}$$

where the regular parts are functions $\tilde{h}_p, \tilde{\chi}_p \in C^\infty(U_p)$. Locally, by (5.177) the gauge potential can be expressed in terms of $d\chi$ and dh,

$$A = d\chi - \frac{1}{2} * dh. \tag{5.194}$$

Hence in U', h and χ satisfy the equations,

$$\Delta h = 2 * B, \qquad \Delta\chi = d^* A. \tag{5.195}$$

By Gauss's law, on U' we have the following relation between β and $\dot{\chi}$,

$$\left(\Delta + |\phi|^2\right)\dot{\chi} = \left(\Delta + |\phi|^2\right)\beta. \tag{5.196}$$

Note that β is a smooth function defined on U whereas $\dot{\chi}$ has divergences at vortex positions. Let $\mathbb{D}_\epsilon$ denote a collection of small ϵ geodesic disks, each one centred at one vortex position. The orientation in each geodesic disk given by the outward unit normal. Let $U_\epsilon = U \setminus \mathbb{D}_\epsilon$ be the surface with the holes left by removing the disks. The orientation of ∂U_ϵ is given by the outward unit normal and if Σ is the euclidean plane, we assume the fields β and N converge fast enough at infinity. Using Green's second identity and discarding divergences in the following integral, we find,

$$\begin{aligned}
\kappa\int_\Sigma \beta B &= \int_\Sigma \beta\,(\Delta + |\phi|^2)N \\
&= \int_\Sigma (\Delta + |\phi|^2)\beta \cdot N \\
&= \lim_{\epsilon\to 0}\int_{U_\epsilon} (\Delta + |\phi|^2)\,\dot{\chi}\cdot N \\
&= \lim_{\epsilon\to 0}\kappa\int_{U_\epsilon}\dot{\chi}B + \lim_{\epsilon\to 0}\int_{\partial\mathbb{D}_\epsilon}(-\dot{\chi} * dN + N * d\dot{\chi}).
\end{aligned} \tag{5.197}$$

On the other hand,

$$
\begin{aligned}
A \wedge \dot{A} &= A \wedge d\dot{\chi} - \frac{1}{2} A \wedge * d\dot{h} \\
&= A \wedge d\dot{\chi} + \frac{1}{2} \dot{A} \wedge * dh - \frac{1}{2}\frac{d}{dt}\left(A \wedge * dh\right) \\
&= A \wedge d\dot{\chi} + \frac{1}{2}\left(d\dot{\chi} \wedge * dh - \frac{1}{2} * d\dot{h} \wedge * dh\right) - \frac{1}{2}\frac{d}{dt}\left(A \wedge * dh\right) \\
&= 2A \wedge d\dot{\chi} - d\chi \wedge d\dot{\chi} + \frac{1}{4} dh \wedge d\dot{h} - \frac{1}{2}\frac{d}{dt}\left(A \wedge * dh\right) \\
&= 2A \wedge d\dot{\chi} + d\left(\dot{\chi} d\chi\right) - \frac{1}{4} d\left(\dot{h} dh\right) - \frac{1}{2}\frac{d}{dt}\left(A \wedge * dh\right) \\
&= 2\dot{\chi} B - 2d\left(\dot{\chi} A\right) + d\left(\dot{\chi} d\chi\right) - \frac{1}{4} d\left(\dot{h} dh\right) - \frac{1}{2}\frac{d}{dt}\left(A \wedge * dh\right). \qquad (5.198)
\end{aligned}
$$

Discarding the time derivative, we find,

$$
\int_\Sigma A \wedge \dot{A} = 2 \lim_{\epsilon \to 0} \int_{U_\epsilon} \dot{\chi} B + \lim_{\epsilon \to 0} \int_{\partial \mathbb{D}_\epsilon} \left(2\dot{\chi} A - \dot{\chi} d\chi + \frac{1}{4} \dot{h} dh\right). \qquad (5.199)
$$

Thence,

$$
\begin{aligned}
\Omega_{\mathcal{M}} &= \kappa \int_\Sigma \beta B - \frac{\kappa}{2} \int_U A \wedge \dot{A} \\
&= \lim_{\epsilon \to 0} \int_{\partial \mathbb{D}_\epsilon} \left(-\dot{\chi} * dN + N * d\dot{\chi} - \kappa \dot{\chi} A + \frac{\kappa}{2} \dot{\chi} d\chi - \frac{\kappa}{8} \dot{h} dh\right). \qquad (5.200)
\end{aligned}
$$

If $\alpha \in \Omega^1(U_p)$, in local, polar coordinates at U_p,

$$
\alpha = \alpha_r dr_p + \alpha_\theta r_p d\theta_p, \qquad \dot{\theta}_p = -\frac{-\sin\theta_p\, \dot{p}_1 + \cos\theta_p\, \dot{p}_2}{r_p}, \qquad (5.201)
$$

where for a time varying point, $p(s)$, in local coordinates $\varphi_* \dot{p} = \dot{p}_1\, \partial_1 + \dot{p}_2\, \partial_2$. We deduce,

$$
\lim_{\epsilon \to 0} \int_{\partial \mathbb{D}_\epsilon(p)} \dot{\theta}_p\, \alpha = -\pi \left(\alpha_1(p)\, \dot{p}_1 + \alpha_2(p)\, \dot{p}_2\right). \qquad (5.202)
$$

Let $\mathcal{M}' \subset \mathcal{M}$ be the open subset of non coalescent vortices, which we can identify with the set,

$$
\{(p_1, \ldots, p_n) \in \Sigma^n \mid p_j \neq p_k \; if \; j \neq k\}. \qquad (5.203)
$$

We define the projector,

$$\Pi_j : \mathcal{M}' \to \mathbb{C}, \qquad \Pi_j(\mathbf{p}) = p_j. \tag{5.204}$$

Thence,

$$\lim_{\epsilon\to 0} \sum_j \int_{\partial\mathbb{D}_\epsilon(p_j)} \dot{\theta}_{p_j}\, \alpha = -\pi \sum_j \left\langle \Pi_j^* \alpha, \dot{\mathbf{p}} \right\rangle, \tag{5.205}$$

where $\langle\cdot,\cdot\rangle$ is the pairing of the pullback of α with the tangent vector $\dot{\mathbf{p}} \in T_{\mathbf{p}}\mathcal{M}'$.

In equation (5.200), all the regular parts of the forms will converge to zero as $\epsilon \to 0$, thus, the only terms to consider are the singular parts. We compute those singular parts in the following equations,

$$\begin{aligned}\lim_{\epsilon\to 0} \int_{\partial\mathbb{D}_\epsilon} \dot{\chi}(- * dN - \kappa A) &= \lim_{\epsilon\to 0} \sum_j \int_{\partial\mathbb{D}_\epsilon(p_j)} \dot{\theta}_{p_j}(- * dN - \kappa\, A) \\ &= -\pi \sum_j \langle \Pi_j^*(- * dN - \kappa A), \dot{\mathbf{p}}\rangle. \end{aligned} \tag{5.206}$$

We also have,

$$\begin{aligned}\lim_{\epsilon\to 0} \int_{\mathbb{D}_\epsilon} N * d\dot{\chi} &= \lim_{\epsilon\to 0} \sum_j \int_{\partial\mathbb{D}_\epsilon(p_{(j)})} N * d\dot{\theta}_{p_{(j)}} \\ &= \lim_{\epsilon\to 0} \sum_j \int_{\partial\mathbb{D}_\epsilon(p_{(j)})} N \left(\frac{-\sin\theta_{p_{(j)}}\, \dot{p}_{(j)1} + \cos\theta_{p_{(j)}}\, \dot{p}_{(j)2}}{\epsilon} \right) d\theta_{p_{(j)}}, \end{aligned} \tag{5.207}$$

where we denote p_j as $p_{(j)}$ to avoid confusion with the role of both subindexes. Taylor's expansion of $N(x)$ in a neighbourhood V of a point p is,

$$N(x) = N(p) + \partial_1(N\circ\varphi^{-1})(\varphi(p))\, r_p \cos\theta_p + \partial_2(N\circ\varphi^{-1})(\varphi(p))\, r_p \sin\theta_p + \mathcal{O}(r_p^2), \tag{5.208}$$

for $x \in V$. Let $N_\varphi = N\circ\varphi^{-1} : U \to \mathbb{R}$, thence,

$$\lim_{\epsilon\to 0} \int_{\partial\mathbb{D}_\epsilon(p)} N \left(\frac{-\sin\theta_p\, \dot{p}_1 + \cos\theta_p\, \dot{p}_2}{\epsilon} \right) d\theta_p = \pi \left(-\partial_2 N_\varphi(\varphi(p))\, \dot{p}_1 + \partial_1 N_\varphi(\varphi(p))\, \dot{p}_2 \right). \tag{5.209}$$

We conclude,

$$\lim_{\epsilon\to 0} \int_{\mathbb{D}_\epsilon} N * d\dot{\chi} = \pi \sum_j \left\langle \Pi_j^*(*dN), \dot{\mathbf{p}} \right\rangle. \tag{5.210}$$

Similarly, for the term $\dot{h}dh$ at $p \in P$ we have,

$$\dot{h} = -2\left(\frac{\cos(\theta_p)\dot{p}_1 + \sin(\theta_p)\dot{p}_2}{r_p}\right) + \partial_t \tilde{h}_p, \qquad dh = \frac{2dr_p}{r_p} + d\tilde{h}_p. \tag{5.211}$$

Discarding the integrals of regular parts,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\partial\mathbb{D}_\epsilon} \dot{h}dh &= -2\lim_{\epsilon\to 0}\sum_j \int_{\partial\mathbb{D}_\epsilon(p_{(j)})} \left(\frac{\cos(\theta_{p_{(j)}})\dot{p}_{(j)1} + \sin(\theta_{p_{(j)}})\dot{p}_{(j)2}}{\epsilon}\right) d\tilde{h}_p \\ &= -2\lim_{\epsilon\to 0}\sum_j \int_{\partial\mathbb{D}_\epsilon(p_{(j)})} \Big(-\partial_1(\tilde{h}_{p_{(j)}}\circ\varphi^{-1})(\varphi(p_{(j)}))\cdot \sin^2\theta_{p_{(j)}}\cdot \dot{p}_{(j)2} \\ &\qquad +\partial_2(\tilde{h}_{p_{(j)}}\circ\varphi^{-1})(\varphi(p_{(j)}))\cdot\cos^2\theta_{p_{(j)}}\cdot\dot{p}_{(j)1}\Big)\, d\theta_{p_{(j)}} \\ &= 2\pi\sum_j \left\langle \Pi_j^*(*d\tilde{h}_{p_{(j)}}), \dot{\mathbf{p}}\right\rangle .\end{aligned} \tag{5.212}$$

For the $\dot{\chi}\, d\chi$ term we have,

$$\dot{\chi} = \dot{\theta}_p + \partial_t\tilde{\chi}_p, \qquad d\chi = d\theta_p + d\tilde{\chi}_p. \tag{5.213}$$

We deduce the remaining integral is,

$$\begin{aligned}\lim_{\epsilon\to 0}\int_{\partial\mathbb{D}_\epsilon}\dot{\chi}d\chi &= \lim_{\epsilon\to 0}\sum_j\int_{\partial\mathbb{D}_\epsilon(p_j)}(\dot{\chi}d\theta_{p_j} + \dot{\chi}d\tilde{\chi}_{p_j}) \\ &= \lim_{\epsilon\to 0}\sum_j\int_{\partial\mathbb{D}_\epsilon(p_j)}\left(\partial_t\tilde{\chi}_{p_j}\cdot d\theta_{p_j} + \dot{\chi}d\tilde{\chi}_{p_j}\right) \\ &= \pi\sum_j\left(2\,\partial_t\tilde{\chi}_{p_j}(p_j) - \langle\Pi_j^*\left(d\tilde{\chi}_{p_j}\right), \dot{\mathbf{p}}\rangle\right).\end{aligned} \tag{5.214}$$

Collecting all the pieces, we find the following expression for the connection on the moduli space,

$$\Omega_{\mathcal{M}} = \pi\sum_j\left(\Pi_j^*\left(2 * dN + \kappa A - \frac{\kappa}{2}\, d\tilde{\chi}_{p_j} - \frac{\kappa}{4} * d\tilde{h}_{p_j}\right) + \kappa\partial_t\tilde{\chi}_{p_j}(p_j)\right). \tag{5.215}$$

Because of this term, motion of a set of vortices on the moduli space deviates from geodesic motion according to a force given by the two form,

$$
\begin{aligned}
d\Omega_{\mathcal{M}} &= \pi \sum_j \left(\Pi_j^* \left(2d * dN + \kappa B - \frac{\kappa}{4} d * d\tilde{h}_{p_j} \right) + \kappa \, d\partial_t \tilde{\chi}_{p_j}(p_j) \right) \\
&= \pi \sum_j \left(\Pi_j^* \left(2 \, |\phi|^2 * N - 2\kappa B + \kappa B - \frac{\kappa}{4}(-2B) \right) + \kappa \, d\partial_t \tilde{\chi}_{p_j}(p_j) \right) \\
&= \kappa\pi \sum_j \left(-\frac{1}{2} \Pi_j^* B + d\partial_t \tilde{\chi}_{p_j}(p_j) \right). \qquad (5.216)
\end{aligned}
$$

To simplify this equation, we used (5.179) and (5.194). if $z_j = \varphi(p_j)$, $\tilde{\chi}_{z_j} = \tilde{\chi}_{p_j} \circ \varphi^{-1}$ are local expression on the chart, discarding higher order terms in κ, from (5.178) we find,

$$
\kappa \, \Pi_j^* B = \Pi_j^* \left(\frac{\kappa}{2} (1 - |\phi|^2) \, e^{\Lambda(z)} dz^1 \wedge dz^2 \right) = \frac{\kappa}{2} e^{\Lambda(z_j)} \, dz_j^1 \wedge dz_j^2. \qquad (5.217)
$$

Hence,

$$
d\Omega_{\mathcal{M}} = \kappa\pi \sum_j \left(-\frac{1}{4} e^{\Lambda(z)} dz_j^1 \wedge dz_j^2 + d\partial_t \tilde{\chi}_{z_j}(z_j) \right). \qquad (5.218)
$$

To obtain an explicit formula for the remaining terms $d\partial_t \tilde{\chi}_{z_j}(z_j)$, we will work to lowest order in κ. From (5.178) and (5.195), h and N are solutions to the following system of equations in the sense of distribution,

$$
-\Delta h = e^h - 1 + 2\kappa N + 4\pi \sum_j \delta_{p_j}, \qquad (5.219)
$$

$$
-\Delta N = e^h N - \frac{\kappa}{2}(1 - e^h), \qquad (5.220)
$$

Let (h_0, N_0) be the solution to this system at $\kappa = 0$, we know $N_0 = 0$ by the Julia-Zee theorem [53] and h_0 is the solution to the Taubes equation for the Ginzburg-Landau functional [56],

$$
-\Delta h_0 = e^{h_0} - 1 + 4\pi \sum_j \delta_{p_j}, \qquad (5.221)
$$

if $(\partial_\kappa h, \partial_\kappa N)$ is the next order solution at $\kappa = 0$, then in U',

$$
-\Delta \partial_\kappa h = e^{h_0} \partial_\kappa h, \qquad (5.222)
$$

$$
-\Delta \partial_\kappa N = e^{h_0} \partial_\kappa N - \frac{1}{2} + \frac{\kappa}{2} e^{h_0} \partial_\kappa h. \qquad (5.223)
$$

We make the assumption $(\partial_\kappa h, \partial_\kappa N) \in \mathrm{L}^2 \times \mathrm{L}^2$, in this case elliptic regularity implies these are smooth functions, hence $\partial_\kappa h \equiv 0$ and to lowest order,

$$\left(\Delta + e^{h_0}\right) \partial_\kappa N = \frac{1}{2}. \tag{5.224}$$

Therefore $\partial_\kappa N \neq 0$, moreover,

$$h = h_0 + \mathcal{O}(\kappa^2), \qquad\qquad N = \kappa\, \partial_\kappa N + \mathcal{O}(\kappa^2). \tag{5.225}$$

For the remaining of the argument, we will assume h is the solution to the Taubes equation without further notice. We can get and explicit formula for the nontrivial term in (5.218) introducing complex coordinates on $\mathcal{M}'$. If we consider the singularity of h at p_j, we have the following equations in the sense of distributions,

$$-\left(\Delta + e^{h}\right) \partial_{z_j} h = 4\pi\, \partial \delta_{p_j}, \qquad -\left(\Delta + e^{h}\right) \overline{\partial_{z_j}} h = 4\pi\, \overline{\partial} \delta_{p_j}. \tag{5.226}$$

Let

$$\eta = \frac{1}{2}\dot{h} + i\dot{\chi}, \tag{5.227}$$

from (5.196) and (5.193), we deduce,

$$\eta = \sum_j \dot{p}_j\, \partial_{z_j} h + i\beta. \tag{5.228}$$

We can expand h in a neighbourhood of each $p_j \in P$ as in the $O(3)$ Sigma model,

$$h = \log r_j^2 + a_j + \frac{1}{2}\left(\overline{b}_j\,(z - z_j) + b_j(\overline{z} - \overline{z}_j)\right) + \mathcal{O}(r_j^2). \tag{5.229}$$

Hence,

$$\begin{aligned} \partial_t \tilde{\chi}_{p_i}(p_i) &= \Im\left(\dot{q} \cdot \partial_q \tilde{h}_{p_i}(p_i)\right) + \beta \\ &= \Im\left(\left(\sum_j \dot{q}_j \partial_{z_j} a_i\right) - \frac{1}{2}\,\overline{b_i}\,\dot{z}_i\right) + \beta. \end{aligned} \tag{5.230}$$

So far, we have not used our gauge freedom in $\mathcal{A}'$, we can do so now and discard the β term to ease the final expression of the computation. Let us define the complex form ω_c,

$$\omega_c = \sum_{i,j} \left(\partial_{z_j} a_i - \frac{1}{2}\overline{b}_i\, \delta_{ij}\right) dz^j = \partial a - \frac{1}{2}\overline{b}, \tag{5.231}$$

where,

$$\partial = \sum_i dz^i \otimes \partial_{z_i}, \qquad a = \sum_i a_i, \qquad \bar{b} = \sum_i \bar{b}_i \, dz^i, \tag{5.232}$$

The imaginary part of ω_c is the nontrivial term in $\Omega_{\mathcal{M}}$,

$$d\Im\left(\omega_c\right) = \Im\left(d\omega_c\right) = \Im\left(\bar{\partial}\partial a - \frac{1}{2}\left(\partial\bar{b} + \overline{\partial b}\right)\right). \tag{5.233}$$

The coefficients b_i have the symmetries,

$$\partial_{z_j}\bar{b}_i = \partial_{z_i}\bar{b}_j, \qquad \bar{\partial}_{z_j}\bar{b}_i = \partial_{z_i}b_j, \tag{5.234}$$

proved in [31] by Manton-Sutcliffe for the Euclidean plane. The proof can be adapted to compact manifolds and is essentially the same as the proof of lemma 2.3. Whence,

$$\partial\bar{b} = 0, \qquad \overline{\partial b} = -\partial b. \tag{5.235}$$

Hence $\partial b \in \Lambda^2(U, i\mathbb{R})$. Since a is real, it is also valid that $\bar{\partial}\partial a \in \Lambda^2(U, i\mathbb{R})$. Hence, the curvature induced by the Chern-Simons term is written locally as,

$$d\Omega_{\mathcal{M}} = -\kappa\pi i\left(\frac{1}{8}\sum_i e^{\Lambda(z_i)} dz^i \wedge d\bar{z}^i + \bar{\partial}\partial a + \frac{1}{2}\partial b.\right) \tag{5.236}$$

To lowest order, the metric is the L^2 metric,

$$ds^2 = \pi\sum_{i,j}\left(e^{\Lambda(z_i)}\delta_{ij} + 2\partial_{z_i}b_j\right) dz^i\, d\bar{z}^j. \tag{5.237}$$

whose symplectic form is [31, p. 212],

$$\omega_0 = \frac{i\pi}{2}\left(\sum_i e^{\Lambda(z_i)}\delta_{ij}\, dz^i \wedge d\bar{z}^j + 2\partial b\right). \tag{5.238}$$

Therefore, the Chern-Simons curvature in $\mathcal{M}'$ is related to the symplectic form of the L^2 metric by,

$$d\Omega_{\mathcal{M}} = -\frac{\kappa}{2}\omega_0 - \kappa\pi i\left(-\frac{1}{8}\sum_i e^{\Lambda(z_i)}\delta_{ij}\, dz^i \wedge d\bar{z}^j + \bar{\partial}\partial a\right). \tag{5.239}$$

Comparing with the Collie-Tong connection

With our choice of notation, Collie and Tong proposed for vortices of the Abelian Higgs model [10] that $d\Omega_{\mathcal{M}} = \kappa\rho$, where ρ is the Ricci form of the metric in the moduli space, then properties of the dynamics of vortices with a Chern-Simons interaction term were studied by Krusch-Speight [28] and Alqahtani-Speight [1] in all cases assuming the dynamics is modified by the Ricci form, however, little is known in the literature about how good the Ricci form approximation is. We do not compare the dynamics of the Collie-Tong proposal with the connection term found due to lack of time, the problem remains open for future work, instead, if we consider (5.239), we can see the connection term is not the Ricci form in the case of the moduli space $\mathcal{M}^2(\mathbb{R}^2)$ of the MHCS model. For a pair (z_1, z_2) of non-coalescent abelian vortices in $\mathbb{R}^2$, define centre of mass coordinates (Z, W), such that $z_1 = Z + W$, $z_2 = Z - W$, $W = \epsilon\, e^{i\theta}$, then the metric in the open and dense subset of non-coalescent vortices is,

$$g_{\mathrm{L}^2} = 2\pi\, dZ\, d\overline{Z} + f(\epsilon)\,(d\epsilon^2 + \epsilon^2\, d\theta^2), \tag{5.240}$$

where the conformal factor is,

$$f(\epsilon) = 2\pi\left(1 + \frac{1}{\epsilon}\frac{d}{d\epsilon}(\epsilon\tilde{b}(\epsilon))\right), \tag{5.241}$$

and the coefficient $\tilde{b}(\epsilon)$ is defined as $\tilde{b}(\epsilon) = b_1(\epsilon, -\epsilon)$. In centre of mass coordinates, the Ricci form is,

$$\begin{aligned} \rho &= i\,\partial\bar{\partial}\log\sqrt{|g_{\mathrm{L}^2}|} \\ &= i\,\partial\overline{\partial}\,\log f(\epsilon) \\ &= -K(\epsilon)\, f(\epsilon)\,\epsilon d\epsilon\wedge d\theta, \end{aligned} \tag{5.242}$$

where, $K(\epsilon)$ is the Gaussian curvature of the subspace of vortex pairs with $Z = 0$,

$$K = -\frac{1}{2\epsilon f(\epsilon)}\frac{d}{d\epsilon}\left(\epsilon\frac{d}{d\epsilon}\log f(\epsilon)\right). \tag{5.243}$$

On the other hand, by (5.239),

$$\begin{aligned} d\Omega_{\mathcal{M}} = -\frac{\kappa}{2}\, i\pi\, dZ\wedge d\overline{Z} - \frac{\kappa}{2} f(\epsilon)\,\epsilon d\epsilon\wedge d\theta \\ - \kappa\,\pi\, i\left(-\frac{1}{4}\, dZ\wedge d\overline{Z} + \frac{i}{2}\, r\, dr\wedge d\theta + \bar{\partial}\partial a\right). \end{aligned} \tag{5.244}$$

On the plane, (h, N), the solution to equations (5.219)-(5.220), is invariant under isometries. For small κ and small perturbations of $(h_0, 0)$, where h_0 is the solution to the Taubes equation for the Abelian Higgs model, the result of Flood-Speight [14] shows the existence of exactly one solution (h, N) to the field equations on a compact surface. It is sensible to assume the same statement holds on the plane, this implies a is invariant under isometries of $\mathbb{R}^2$, hence,

$$\partial_{z_1} a + \partial_{z_2} a = 0, \qquad \bar{\partial}_{z_1} a + \bar{\partial}_{z_2} a = 0. \tag{5.245}$$

From these equations, we deduce,

$$\partial a = \partial_{z_1} a \, dz^1 + \partial_{z_2} a \, dz^2 = -\partial_{z_1} a \, (dz^2 - dz^1). \tag{5.246}$$

Likewise,

$$\begin{aligned} \bar{\partial}\partial a &= -(\bar{\partial}_{z_1}\partial_{z_1} a \, d\bar{z}^1 + \bar{\partial}_{z_2}\partial_{z_1} a \, d\bar{z}^2) \wedge (dz^2 - dz^1) \\ &= \bar{\partial}_{z_1}\partial_{z_1} a \, (d\bar{z}^2 - d\bar{z}^1) \wedge (dz^2 - dz^1) \\ &= 2\, i\, \bar{\partial}_{z_1}\partial_{z_1} a \cdot \epsilon \, d\epsilon \wedge d\theta. \end{aligned} \tag{5.247}$$

Let $\tilde{a}(\epsilon) = a(\epsilon, -\epsilon)$, isometric invariance of a implies,

$$\bar{\partial}_{z_1}\partial_{z_1} a = \frac{1}{4\epsilon} \frac{d}{d\epsilon} \left(\epsilon \frac{d\tilde{a}}{d\epsilon} \right). \tag{5.248}$$

Going back to equation (5.244), we find,

$$\begin{aligned} d\Omega_{\mathcal{M}} = &-\frac{\kappa\, \pi\, i}{4} \, dZ \wedge d\bar{Z} \\ &- \frac{\kappa\pi}{2} \left(1 + \frac{2}{\epsilon} \frac{d}{d\epsilon}(\epsilon\, \tilde{b}(\epsilon)) - \frac{1}{\epsilon} \frac{d}{d\epsilon} \left(\epsilon \frac{d\tilde{a}}{d\epsilon} \right) \right) \epsilon \, d\epsilon \wedge d\theta. \end{aligned} \tag{5.249}$$

Equation (5.249) shows $d\Omega_{\mathcal{M}} \neq \kappa\rho$, since ρ has no $dZ \wedge d\bar{Z}$ component.

5.2.4 Chern-Simons localization on the O(3) Sigma model

We can adapt our previous arguments to the $O(3)$ Sigma model with some minor adjustments. Most of our previous deduction follows without change, since the Bogomolny equations have the same structure, except for the algebraic formula of $*B$. Gauss's law in this case has to be replaced as in section 5.1. In this case, the projection β of a solution $\Phi \in \mathcal{A}'$ to the static Bogomolny equations onto vertical space is a solution to equation (2.46), which for variations of the core positions is,

$$(\Delta + |X_\phi|^2)\,\beta = \langle \dot{\phi}, X_\phi \rangle + d^* \dot{A}, \tag{5.250}$$

In this case at each core $p_j \in P \cup Q$, we have to take into consideration the sign function s_j, otherwise our computation on sections 5.2.2, 5.2.3 follow the same pattern and we find that for non-coalescent vortices, to first order in κ,

$$\begin{aligned} d\Omega_{\mathcal{M}} &= \kappa\pi \sum_j s_j \left(-\frac{1}{2}\Pi_j^* B + d\partial_t \tilde{\chi}_{p_j}(p_j) \right) \\ &= \kappa\pi \sum_j s_j \left(\frac{1}{2}\Pi_j^* (*(\kappa N + \tau - \langle n, \phi \rangle)) + d\partial_t \tilde{\chi}_{p_j}(p_j) \right) \\ &= \kappa\pi \sum_j \left(-\frac{i}{4}(1 - s_j\tau)\, e^{\Lambda(z_j)}\, dz^j \wedge d\overline{z}^j + s_j\, d\partial_t \tilde{\chi}_{z_j}(z_j) \right). \end{aligned} \tag{5.251}$$

To deduce a formula for the second term in the sum, we know by section 5.1 that for $\kappa = 0$, the only solution to the governing elliptic problem is $(h_0, 0)$ where h_0 is the solution to the Taubes equation for the $O(3)$ Sigma model. As is shown in equation (2.77), η can be computed from the derivatives of h, in accordance to (5.228). If we recall equation (2.102), we find for any small holomorphic neighbourhood U_j of $p_j \in P \cup Q$, $z_j = \varphi(p_j)$,

$$\tilde{h}_{p_j}(\varphi(x)) = s_j\, a_j + \frac{1}{2}\, s_j\, \left(\overline{b}_j\,(z - z_j) + b_j\,(\overline{z} - \overline{z}_j) \right) + \mathcal{O}(r_j^2). \tag{5.252}$$

Comparing with (5.229), we deduce,

$$s_j \partial_t \tilde{\chi}_{p_j}(p_j) = \Im \left(\sum_i \dot{z}_i \partial_{z_i} a_j - \frac{1}{2}\, \overline{b}_j\, \dot{z}_j \right). \tag{5.253}$$

By lemma 2.3, b has the same symmetries than for Ginzburg-Landau solitons, therefore,

$$d\Omega_{\mathcal{M}} = -\kappa\pi\, i \sum_j \left(\frac{1}{4}(1 - s_j\tau)\, e^{\Lambda(z_j)}\, dz^j \wedge d\overline{z}^j + \overline{\partial}\partial\, a + \frac{1}{2}\, \partial b \right). \tag{5.254}$$

5.2.5 Extending to the coalescence points

In the previous sections we developed two formulae for the extra term in a localization formula of vortices in a model with a Chern-Simons term. We assumed κ small and found two related formulae on $\mathcal{M}'$, the open and dense subspace of the moduli space of non-coalescent vortices. To extend $d\Omega_{\mathcal{M}}$ to the coalescence points means to study the limit of the formula as any pair of vortices, (of the same type for deformations of the $O(3)$ Sigma model) coalesce, meaning as $d(p_j, p_k) \to 0$ for $p_j \neq p_k$ and p_j, p_k vortices of the same type. Let us consider first the $O(3)$ Sigma model. Let $\tilde{h} \in C^\infty(\Sigma)$ be the regular part of h, theorem 2.12 shows $\tilde{h}$ depends smoothly on vortex positions as long as vortices and antivortices do not coalesce. Recall in section 2.3 we defined smooth functions $v_j : \Sigma \setminus \{p_j\} \to \mathbb{R}$, such that,

$$h = \sum_j s_j \, v_j + \tilde{h}. \tag{5.255}$$

In the compact case, we assume $P \cup Q \subset U$, U an open and dense subset of the surface in which a holomorphic chart $\varphi : U \to \mathbb{C}$ is defined. Let $D \subset \mathbb{C}$ be a bounded domain containing $\varphi(P \cup Q)$. Since each v_j is a constant multiple of Green's function, there is a smooth function $\tilde{v} : D \times D \to \mathbb{R}$ such that for any $z \in D$,

$$v_j(\varphi^{-1}(z)) = \log |z - z_j|^2 + \tilde{v}(z, z_j). \tag{5.256}$$

Thus,

$$s_i a_i = \sum_{j \neq i} s_j \log |z_j - z_i|^2 + \sum_j s_j \tilde{v}(z_j, z_i) + \tilde{h}(\varphi(z_i)), \tag{5.257}$$

and,

$$\begin{aligned} s_i b_i &= 2\,\overline{\partial}_z \left(\sum_{j \neq i} s_j \log |z - z_j|^2 + \sum_j s_j \, \tilde{v}(z, z_j) + \tilde{h}(\varphi^{-1}(z)) \right)(z_i) \\ &= \sum_{j \neq i} \frac{2 s_j}{\overline{z}_i - \overline{z}_j} + 2 \sum_j s_j \, \overline{\partial}_z \tilde{v}(z_j, z_i) + 2\,\overline{\partial}_z \tilde{h}(z_i). \end{aligned} \tag{5.258}$$

Hence, for non-coalescent cores at D,

$$\begin{aligned}\overline{\partial}\partial\, a &= \sum_{i,j} s_i s_j\, \overline{\partial}\partial\, \tilde{v}(z_j, z_i) + \sum_i s_i\, \overline{\partial}\partial\, \tilde{h}(z_i),\\ \partial\, b &= 2\partial\left(\sum_{i,j}\left(s_i s_j \overline{\partial}_z \tilde{v}(z_j, z_i) + s_i \delta_{ij} \overline{\partial}_z \tilde{h}(z_i)\right) d\overline{z}^i\right).\end{aligned} \tag{5.259}$$

If $z \in D$ is such that for a fixed pair of indices j, k, the vortices at z_j, z_k are of the same type and both converge to z, equation (5.259) implies the limit $\lim_{|z_j - z_k| \to 0} d\Omega_{\mathcal{M}}$ exists and is unique, in fact, it corresponds to solving the regularised Taubes equation with configuration $\mathbf{p}$ such that $p_j = p_k = \varphi^{-1}(z)$.

For Ginzburg-Landau vortices the same argument is valid, it is simpler, since in this case all the vortices are of the same type and we can take $s_p = 1$ for all the cores. In this case we arrive to an algebraic expression similar to (5.259) and apply proposition 2.14 to conclude $d\Omega_{\mathcal{M}}$ can also be extended to the coalescence points.

Chapter 6

Conclusion

In this work we focused on geometric models of vortices and antivortices of the $O(3)$ Sigma model. We emphasised the geometric nature of the interaction of a vortex-antivortex pair on the moduli space.

We were able to prove that the L^2 metric in the moduli space is incomplete both on the euclidean plane and on a compact surface. We also analysed the dynamical properties of the interaction on the plane, focusing on scattering of vortex-antivortex pairs.

We also computed the volume of the moduli space on spheres and flat tori, corroborating the work of Speight and Rõmao who conjectured a formula for the volume of the moduli space for a general surface.

The fact that the moduli space is incomplete imposed some technical difficulties on the proofs, that we overcame by analysing the behaviour of solutions to the Taubes equations in the collision of a vortex and an antivortex.

Finally, we added a Chern-Simons interaction term to our model and applied the geodesic approximation ideas to determine the extra term in the metric of the moduli space for small perturbations due to the interaction. Our analysis indicates that the extra term can be extended to the coincidence set.

Some questions remain opened, representing an opportunity for future work. The short range approximation formula for the metric on the space of vortex-antivortex pairs of the euclidean plane relies on uniform convergence of the family $\tilde{h}_\epsilon/\epsilon$, as $\epsilon \to 0$, where $\tilde{h}_\epsilon$ is the regular part of the Taubes equation. Numerical evidence suggests this conjecture is true. Should it be the case, we would be able to prove formally that the Gaussian curvature of $\mathcal{M}_0^{1,1}(\mathbb{R}^2)$ diverges as $\epsilon \to 0$ as expected from the numerical evidence and we could also justify analytically the effective potential of Ricci magnetic geodesics. The equivalent conjecture for a compact surface would allow to compute the volume formula for a general surface, where we no longer have the extra symmetries that we used for the task.

In conclusion, geometric ideas to study field theory originated in the realm of superconductivity with the Ginzburg-Landau functional at critical coupling, but they have proved to be fruitful in a broader context. In particular, for asymmetric vortex-antivortex systems of the $O(3)$ Sigma model, where with these ideas one can understand dynamics from a geometric point of view.

References

[1] LS Alqahtani and JM Speight. Ricci magnetic geodesic motion of vortices and lumps. *Journal of Geometry and Physics*, 98:556–574, 2015. 102, 148, 177, 191

[2] Thierry Aubin. *Some nonlinear problems in Riemannian geometry*. Springer Science & Business Media, 2013. 34, 35, 111, 117

[3] Steven B Bradlow. Vortices in Holomorphic Line Bundles over Closed Kahler Manifolds. *Commun. Math. Phys*, 135(1):1–17, 1990. URL https://projecteuclid.org:443/euclid.cmp/1104201917. 108

[4] Sean Carroll. *Spacetime and geometry : an introduction to general relativity*. Addison Wesley, San Francisco, 2004. ISBN 0805387323. 79

[5] D Chae and Hee Seok Nam. On the condensate multivortex solutions of the self-dual maxwell-chern-simmons cp1 model. *Annales de l'Institut Henri Poincare*, (2):887–906, 2001. ISSN 03779017. doi: 10.1007/s11005-005-8443-0. 147

[6] Zhi-You Chen and Jann-Long Chern. The analysis of solutions for Maxwell–Chern–Simons O(3) sigma model. *Calculus of Variations and Partial Differential Equations*, 58(4):147, jul 2019. ISSN 1432-0835. doi: 10.1007/s00526-019-1590-4. URL https://doi.org/10.1007/s00526-019-1590-4. 160

[7] Francesco Chiacchio and Tonia Ricciardi. Multiple vortices for a self-dual CP(1) Maxwell-Chern-Simons model. *Nonlinear Differential Equations*

and Applications, 13(5-6):563–584, 2007. ISSN 10219722. doi: 10.1007/s00030-006-4024-x. 147

[8] Michel Chipot. *Handbook of differential equations: stationary partial differential equations*. Elsevier, 2011. 40, 41, 42

[9] Michael Christ. On the $\bar{\partial}$ equation in weighted L^2 norms in $\mathbb{C}^1$. *The Journal of Geometric Analysis*, 1(3):193–230, 1991. 83

[10] Benjamin Collie and David Tong. Dynamics of Chern-Simons vortices. *Physical Review D - Particles, Fields, Gravitation and Cosmology*, 2008. ISSN 15507998. doi: 10.1103/PhysRevD.78.065013. 3, 12, 102, 148, 177, 191

[11] Klaus Deimling. *Nonlinear functional analysis*. Courier Corporation, 2010. 40

[12] Sophia Demoulini and David Stuart. Adiabatic Limit and the Slow Motion of Vortices in a Chern-Simons-Schrödinger System. *Communications in Mathematical Physics*, 290(2):597–632, sep 2009. ISSN 0010-3616, 1432-0916. doi: 10.1007/s00220-009-0844-y. URL `https://0-link-springer-com.wam.leeds.ac.uk/article/10.1007/s00220-009-0844-y`. 12

[13] L.C. Evans. *Partial Differential Equations*. Graduate studies in mathematics. American Mathematical Society, 2010. ISBN 9780821849743. 27

[14] S. P. Flood and J. M. Speight. Chern-Simons deformation of vortices on compact domains. *Journal of Geometry and Physics*, 133:153–167, 2018. ISSN 03930440. doi: 10.1016/j.geomphys.2018.07.009. URL `https://doi.org/10.1016/j.geomphys.2018.07.009`. 147, 149, 153, 155, 159, 174, 177, 192

[15] David Gilbarg and Neil S Trudinger. *Elliptic partial differential equations of second order*. springer, 2015. 27, 28, 30

[16] Herbert Goldstein, Charles Poole, and John Safko. Classical mechanics, 2002. 96

[17] Jongmin Han. Existence of topological multivortex solutions in the self-dual gauge theories. *Proceedings of the Royal Society of Edinburgh Section A: Mathematics*, 130(6):1293–1309, 2000. 26

[18] Jongmin Han and Chang-Shou Lin. Multiplicity for Self-Dual Condensate Solutions in the Maxwell-Chern-Simons O(3) Sigma Model. *Communications in Partial Differential Equations*, 39(8):1424–1450, 2014. doi: 10.1080/03605302.2014.908909. URL https://doi.org/10.1080/03605302.2014.908909. 166

[19] Jongmin Han and Hee Seok Nam. On the topological multivortex solutions of the self-dual Maxwell-Chern-Simons gauged O(3) sigma model. *Letters in Mathematical Physics*, 73(1):17–31, 2005. ISSN 03779017. doi: 10.1007/s11005-005-8443-0. 147

[20] Jongmin Han and Kyungwoo Song. Existence and asymptotics of topological solutions in the self-dual Maxwell-Chern-Simons O(3) sigma model. *Journal of Differential Equations*, 250(1):204–222, 2011. ISSN 00220396. doi: 10.1016/j.jde.2010.08.003. URL http://dx.doi.org/10.1016/j.jde.2010.08.003. 147

[21] AD Hwang. A symplectic look at surfaces of revolution. *Enseignement Mathematique*, pages 1–17, 2003. doi: http://doi.org/10.5169/seals-66685. 93

[22] R Jackiw, Kimyeong Lee, and Erick J. Weinberg. Self-dual Chern-Simons solitons. 42(10):3488–3499, 1990. 149

[23] A Jaffe and CH Taubes. *Monopoles and Vortices*. Birkhäuser, Basel, 1980. 51

[24] Sung Ku Kim and Hyunsoo Min. Statistical interactions between chern-simons vortices. *Physics Letters B*, 281(1):81 – 85, 1992. ISSN 0370-2693. doi: https://doi.org/10.1016/0370-2693(92)90278-C. URL http://www.sciencedirect.com/science/article/pii/037026939290278C. 177

[25] Yoonbai Kim and Kimyeong Lee. Vortex dynamics in self-dual Chern-Simons-Higgs systems. *Physical Review D*, 49(4):2041–2054, feb 1994. doi: 10.1103/PhysRevD.49.2041. URL http://0.link.aps.org/doi/10.1103/PhysRevD.49.2041. 12, 148, 177

[26] Yoonbai Kim and Kimyeong Lee. First and second order vortex dynamics. *Physical Review D*, 66(4):045016, aug 2002. doi: 10.1103/PhysRevD.66.045016. URL http://0.link.aps.org/doi/10.1103/PhysRevD.66.045016. 2, 3, 177

[27] Kyoungtae Kimm, Kimyeong Lee, and Taejin Lee. Anyonic Bogomol'nyi solitons in a gauged O(3) σ model. *Phys. Rev. D*, 53(8):4436–4440, 1996. doi: 10.1103/PhysRevD.53.4436. URL https://link.aps.org/doi/10.1103/PhysRevD.53.4436. 149

[28] Steffen Krusch and J Martin Speight. Exact moduli space metrics for hyperbolic vortex polygons. *Journal of Mathematical Physics*, 51(2):022304, 2010. 102, 191

[29] Choonkyu Lee, Kimyeong Lee, and Hyunsoo Min. Self-dual Maxwell Chern-Simons solitons. *Physics Letters B*, 252(1):79–83, 1990. ISSN 0370-2693. doi: https://doi.org/10.1016/0370-2693(90)91084-O. URL http://www.sciencedirect.com/science/article/pii/037026939091084O. 149

[30] Elliott H. Lieb. *Analysis*. Graduate studies in mathematics, v. 14. American Mathematical Society, Providence, R.I, 1997. ISBN 0821806327. 110

[31] N. Manton and P. Sutcliffe. *Topological Solitons*. Cambridge Monographs on Mathematical Physics. Cambridge University Press, 2004. ISBN 978-1-139-45469-8. 15, 20, 22, 190

[32] N. S. Manton. A remark on the scattering of BPS monopoles. *Physics Letters B*, 110(1):54 – 56, 1982. ISSN 0370-2693. doi: https://doi.org/10.1016/0370-2693(82)90950-9. URL http://www.sciencedirect.com/science/article/pii/0370269382909509. 1, 12, 16

[33] N. S. Manton. First Order Vortex Dynamics. *Annals of Physics*, 256(1):114–131, may 1997. ISSN 0003-4916. doi: 10.1006/aphy.1997.5672. URL http://www.sciencedirect.com/science/article/pii/S0003491697956724. 12

[34] N S Manton and S M Nasir. Conservation laws in a first-order dynamical system of vortices. *Nonlinearity*, 12(4):851, 1999. URL http://stacks.iop.org/0951-7715/12/i=4/a=306. 128, 129

[35] N S Manton and S M Nasir. Volume of vortex moduli spaces. *Communications in mathematical physics*, 199(3):591–604, 1999. 131

[36] N S Manton and S M Nasir. Conservation laws in a first-order dynamical system of vortices. *Nonlinearity*, 12(4):851, 1999. URL http://stacks.iop.org/0951-7715/12/i=4/a=306. 127

[37] N.S. Manton. Statistical mechanics of vortices. *Nuclear Physics B*, 400(1):624 – 632, 1993. ISSN 0550-3213. doi: 10.1016/0550-3213(93)90418-O. 107, 128

[38] NS Manton and SM Nasir. Volume of vortex moduli spaces. *Communications in mathematical physics*, 199(3):591–604, 1999. 107, 128

[39] N.S. Manton and J.M. Speight. Asymptotic interactions of critically coupled vortices. *Communications in Mathematical Physics*, 236(3):535–555, Jun 2003. ISSN 1432-0916. doi: 10.1007/s00220-003-0842-4. URL https://doi.org/10.1007/s00220-003-0842-4. 67, 73, 74, 80, 99, 101

[40] Jean Mawhin. Leray-Schauder degree: a half century of extensions and applications. *Topological Methods in Nonlinear Analysis*, 14(2):195, 1999. ISSN 1230-3429. doi: 10.12775/tmna.1999.029. 42

[41] Marius Mitrea and Michael Taylor. Boundary layer methods for lipschitz domains in riemannian manifolds. *Journal of Functional Analysis*, 163(2):181–251, 1999. 32

[42] Marius Mitrea and Michael Taylor. Potential theory on lipschitz domains in riemannian manifolds: Holder continuous metric tensors. *Communications in Partial Differential Equations*, 25(7-8):1487–1536, 2000. 32

[43] Ákos Nagy. The Berry Connection of the Ginzburg-Landau Vortices. *Communications in Mathematical Physics*, 350(1):105–128, 2 2017. ISSN 0010-3616. doi: 10.1007/s00220-016-2701-0. URL `http://link.springer.com/10.1007/s00220-016-2701-0`. 13

[44] L. E. Payne and H. F. Weinberger. An optimal poincaré inequality for convex domains. *Archive for Rational Mechanics and Analysis*, 5(1):286–292, 1960. ISSN 1432-0673. doi: 10.1007/BF00252910. URL `https://doi.org/10.1007/BF00252910`. 60

[45] N. M. Romão and J. M. Speight. The Geometry of the Space of BPS Vortex–Antivortex Pairs. *Communications in Mathematical Physics*, 379(2): 723–772, 2020. ISSN 14320916. doi: 10.1007/s00220-020-03824-y. URL `https://doi.org/10.1007/s00220-020-03824-y`. 1, 2, 8, 22, 49, 65, 71, 88, 107, 113, 127, 128, 135, 137, 150, 151

[46] Nuno M Romao. Quantum chern–simons vortices on a sphere. *Journal of Mathematical Physics*, 42(8):3445–3469, 2001. 127, 131, 132

[47] T M Samols. Mathematical Physics Vortex Scattering. *Commun. Math. Phys*, 145(1):149–179, 1992. URL `https://projecteuclid.org:443/euclid.cmp/1104249538`. 12, 16, 46, 49, 65, 91, 181

[48] Helmut Schaefer. Über die Methode der a priori-Schranken. *Mathematische Annalen*, 129:415–416, 1955. 40

[49] B. J. Schroers. The spectrum of Bogomol'nyi solitons in gauged linear sigma models. *Nuclear Physics B*, 475(1):440 – 468, 1996. ISSN 0550-3213. doi: https://doi.org/10.1016/0550-3213(96)00348-3. URL `http://www.sciencedirect.com/science/article/pii/0550321396003483`. 8, 11, 12

[50] BJ Schroers. Bogomol'nyi solitons in a gauged O (3) sigma model. *Physics Letters B*, 356(2-3):291–296, 1995. 8, 68

[51] Lesley Sibner, Robert Sibner, and Yisong Yang. Abelian gauge theory on Riemann surfaces and new topological invariants. *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, 456 (1995):593–613, 2000. ISSN 1364-5021. doi: 10.1098/rspa.2000.0532. 8, 10, 26, 108, 110, 151

[52] J. M. Speight. Static intervortex forces. *Phys. Rev. D*, 55(6):3830–3835, mar 1997. doi: 10.1103/PhysRevD.55.3830. URL https://link.aps.org/doi/10.1103/PhysRevD.55.3830. 67, 68, 70, 75, 78, 81

[53] Joel Spruck and Yisong Yang. Proof of the Julia–Zee Theorem. *Communications in Mathematical Physics*, 291(2):347–356, oct 2009. ISSN 0010-3616, 1432-0916. doi: 10.1007/s00220-009-0791-7. URL https://0-link-springer-com.wam.leeds.ac.uk/article/10.1007/s00220-009-0791-7. 188

[54] I. A. B. Strachan. Low-velocity scattering of vortices in a modified Abelian Higgs model. *Journal of Mathematical Physics*, 33(1):102–110, 1992. doi: 10.1063/1.529949. URL https://doi.org/10.1063/1.529949. 12, 65

[55] D. Stuart. Dynamics of abelian Higgs vortices in the near Bogomolny regime. *Communications in Mathematical Physics*, 159(1):51–91, 1994. URL https://projecteuclid.org:443/euclid.cmp/1104254491. 12

[56] Clifford Henry Taubes. Arbitrary n-vortex solutions to the first order ginzburg-landau equations. *Comm. Math. Phys.*, 72(3):277–292, 1980. URL https://projecteuclid.org:443/euclid.cmp/1103907703. 12, 38, 50, 51, 52, 188

[57] David Tong. Ns5-branes, t-duality and worldsheet instantons. *Journal of High Energy Physics*, 2002(07):013, 2002. 71

[58] David Tong and Kenny Wong. Vortices and impurities. *Journal of High Energy Physics*, 2014(1):90, jan 2014. ISSN 1029-8479. doi: 10.1007/JHEP01(2014)090. URL http://link.springer.com/10.1007/JHEP01(2014)090. 12

[59] Yisong Yang. Strings of opposite magnetic charges in a gauge field theory. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 455, pages 601–629. The Royal Society, 1999. 51

[60] Yisong Yang. Strings of opposite magnetic charges in a gauge field theory. In *Proceedings of the Royal Society of London A: Mathematical, Physical and Engineering Sciences*, volume 455, pages 601–629. The Royal Society, 1999. 12, 50

[61] Yisong Yang. On a System of Nonlinear Elliptic Equations Arising in Theoretical Physics. *Journal of Functional Analysis*, 170(1):1 – 36, 2000. ISSN 0022-1236. doi: https://doi.org/10.1006/jfan.1999.3492. URL http://www.sciencedirect.com/science/article/pii/S0022123699934924. 38

[62] Shen Zhongwei. L^p estimates for Schrodinger operators with certain potentials. *Annales de l'Institut Henri Poincare*, 45(2):513–546, 1995. 85