HARMONIC MAPS FROM SURFACES TO COMPLEX PROJECTIVE SPACES AND CERTAIN LIE GROUPS

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Abstract

In this thesis we are concerned with harmonic maps from a Riemann surface to a complex projective space, the unitary group, the orthogonal group, or the symplectic group.

We describe and link two constructions of complex isotropic (equivalently, finite uniton number) harmonic maps from a Riemann surface to complex projective spaces; all harmonic maps from the 2-sphere are complex isotropic. We then specialise to harmonic maps from the 2-sphere to the complex projective plane and show that there is no restriction on the ramification behaviour in some situations and that the opposite is true in other situations.

We find the dimension of the spaces of holomorphic sections and holomorphic differentials of certain line bundles. We use those results to give improved lower bounds on the index of complex isotropic harmonic maps from the 2-sphere and torus to a complex projective space of arbitrary dimension and from higher genus surfaces in some cases.

We give, up to dimension 6, algebraic parametrizations of all S^1-invariant extended solutions of harmonic maps of finite uniton number from a Riemann surface to the symplectic group, giving the corresponding harmonic maps explicitly. For arbitrary dimension we give an algorithm which parametrizes all such S^1-invariant extended solutions of harmonic maps which are of *standard type*, i.e., of the maximum possible uniton number.

Contents

Chapter 1

Introduction

A map between (compact) Riemannian manifolds is called *harmonic* if it is a critical point of the *energy functional*, a natural extension of Dirichlet's energy to Riemannian manifolds. The general theory of harmonic maps took off in 1958 when J. Eells investigated infinite-dimensional spaces of maps, looking at points on these spaces that are critical points of the energy functional [22]. Little was known about their existence until Eells and J. Sampson [26] famously proved that one can continuously deform any given map between Riemannian manifolds into a harmonic map in its homotopy class, provided the target manifold is non-positively curved. The same cannot be said for positively curved target manifolds, for example Eells and J. C. Wood [27] proved that there does not exist a harmonic map from a 2-torus to the 2-sphere of degree ± 1. Although there is no general theorem on the existence of harmonic maps for positively curved manifolds, many examples of harmonic maps and existence results have been found. Examples of harmonic maps include but are not limited to: constant mappings between Riemannian manifolds; harmonic functions; geodesics and holomorphic maps between Kähler manifolds (see [35, §2.2], [51, Chapter 4] and [25, §3] for a comprehensive list). We direct the reader to the articles and book [24, 25, 51] for descriptions of existence and classification for particular domain and target manifolds.

A key tool in the study of harmonic maps is the *first variation of the energy*. The first variation of the energy gives a formula involving the *tension field* which is the Euler-Lagrange operator associated to the energy. The tension field also determines the direction in which the energy decreases most rapidly. If the tension field is identically zero then the map is a *critical point of the energy functional* and therefore a harmonic map [23, 51]. In [23] harmonic maps are linked to the physical action of rubber being stretched over marble, which can be realised as a mapping of a plane domain to a 2-sphere. The tension field at a point on the rubber represents the tension in the rubber: therefore if the tension field is zero, the the rubber is in elastic equilibrium.

The *second variation of the energy* [40, 47] is useful for determining the behaviour of the energy near a harmonic map: it gives a way of assessing the *stability* of a harmonic map by calculating its *index*. The index is the maximal dimension of the subspace on which the Hessian of the energy is negative definite. A harmonic map is called *(weakly) stable* if the $\mathrm{index} = 0$: a harmonic map that is a local minimum of the energy has $\mathrm{index} = 0$ and is therefore stable [25, §6]. Generally the index of a harmonic map is hard to calculate, for example, the index of the identity map on a (compact) Einstein manifold (the Ricci curvature is a constant multiple of the metric) is zero if and only if the first eigenvalue of the Laplacian is greater than twice the conformal factor [47]. Even for a constant mapping, which has $\mathrm{index} = 0$, the second variation is non-trivial [51, Chapter 5 §1.3]. Another notable result is due to A. Lichnerowicz [39]: holomorphic maps between compact Kähler manifolds are harmonic and have $\mathrm{index} = 0$. These maps are therefore stable, in fact they are global minima of the energy functional. In many cases only estimates have been found for the index of a harmonic map, for example, any harmonic map from a compact Riemann surface to a complex projective space which is neither holomorphic nor antiholomorphic is unstable [5, 6].

Harmonic maps into a complex projective space have been researched extensively [2, 3, 10, 12, 18, 19, 28, 37, 38, 53]. In [27] Eells and Wood showed that a harmonic map from a Riemann surface of genus g to the the 2-sphere $S^2 = \mathbb{CP}^1$ is holomorphic if the degree of the map is greater than the genus g, all such maps are therefore stable by [39]. For $g = 0$, in [19, 20], A.M. Din and W.J. Zakrzewski first gave the explicit solutions of the harmonic map equation. These were given by successively differentiating a holomorphic map then constructing what would later be known as *osculating spaces* and *associated curves* in [28].

The work of [27] and [19] was expanded upon by Eells and Wood in [28] where they constructed harmonic maps with no restriction on the genus g. This led to a classification theorem relating *complex isotropic harmonic maps* $\phi : M \to \mathbb{CP}^n$ to pairs (f, ρ) where $f : M \to \mathbb{CP}^n$ is a full holomorphic map and ρ an integer, $0 \leq \rho \leq n$ [28]. Further, for $0 < \rho < n$, these maps are unstable. In fact Eells and Wood in [28, §9] gave a lower bound for the index of these harmonic maps ϕ by noting that, given a holomorphic vector field along ϕ, there is a smooth variation of ϕ that contributes to the index of ϕ. In Chapter 4 of this thesis we improve the estimates of Eells and Wood [28, §9] for all harmonic maps from the 2-sphere to a complex projective space and complex isotropic harmonic maps from the torus to a complex projective space. We also give new bounds on the index of complex isotropic harmonic maps from higher genus surfaces to a complex projective space which improve those in [28] in some cases. Also in [28], Eells and Wood proved that all harmonic maps had been obtained for $g = 0$, and for $g = 1$ with non-zero degree [28, Proposition 7.6]. For $g = 0$, D. Burns added to the discussion by providing physical motivations to the work of Din and Zakrzewski [19] and Eells and Wood [28]. Also for $g = 0$, S.S. Chern and J. Wolfson [14, 52] interpret the work of Din and Zakrzewski through a moving frames approach providing a classification theorem for minimal 2-spheres in $\mathbb{CP}^n$.

As described in [28] harmonic maps $M \to \mathbb{CP}^n$, where M is a compact Riemann surface, are constructed from a full holomorphic map $f : M \to \mathbb{CP}^n$ as follows: first they construct a family of holomorphic maps from a Riemann surface into a complex Grassmannian which are indexed by the integer ρ, $0 \leq \rho \leq n$: these are called the associated curves of f. For each ρ, by pairing the ρth associated curve with the $(\rho - 1)$st associated curve, one gets a map into a flag manifold $\mathcal{F}_{\rho-1,\rho}$ which is horizontal with respect to a Riemann submersion $\pi : \mathcal{F}_{\rho-1,\rho} \to \mathbb{CP}^n$. Composing this map with the Riemann submersion gives a harmonic map into a complex projective space now known as the *ρth Gauss transform of f*.

After the work of Eells and Wood, many authors turned their attention to classifying harmonic maps from a compact Riemann surface to a complex Grassmannian $G(k, n)$. In [44] J. Ramanathan determined all harmonic maps $S^2 \to G(2, 4)$. In [16] (the results of which were announced in [15]) Chern and Wolfson constructed all harmonic maps $S^2 \to G(2, n)$ for n arbitrary. Later F.E. Burstall and Wood gave an interpretation of the construction of [28] in [12] by considering maps from M to a Grassmannian as subbundles of the trivial bundle $M \times \mathbb{C}^{n+1}$, and developing a technique of analysing harmonic maps from a Riemann surface into a complex Grassmannian using "diagrams". In Chapter 2 we recall the construction of complex isotropic harmonic maps $\phi : M \to \mathbb{CP}^n$ given by Eells and Wood in [28] and by Burstall and Wood in [12] and link the two approaches.

In the constructions of complex isotropic maps given in [28], to ensure the associated curves of the full holomorphic map f are well defined and holomorphic, one needs to pay particular attention to the *ramification points* of f. Analogously in the construction of [12] to ensure the subbundles of $M \times \mathbb{C}^{n+1}$ are holomorphic one needs to "fill out the zeros" of the subbundle. This again amounts to paying attention to the ramification points of another related bundle map. In §2.2.1 we prove these two definitions of ramification point are the same.

In [18] T.A. Crawford showed that subspaces consisting of maps of a fixed degree and energy of the space of harmonic maps $S^2 \to \mathbb{CP}^2$ are path connected and that they can be given the structure of a complex manifold. This is done by proving that the Gauss transform as seen as a mapping from the complex manifold of full holomorphic maps of fixed degree and *total ramification index* to the space of harmonic maps of fixed degree and energy is a homeomorphism. Therefore the complex structure of the domain manifold can be transported to the target. L. Lemaire and Wood carry further the work of Crawford to prove in [37] that the Gauss transform is, in fact, a diffeomorphism making the space of harmonic maps with fixed degree and energy a smooth closed submanifold of the space of all maps $S^2 \to \mathbb{CP}^2$. In this work, the total ramification index, i.e. the sum of the points of ramification of the full holomorphic map counted according to multiplicity, must be fixed. Our work sheds some light on the phenomenon of *ramification coalescence*, where the points of ramification can come together in the deformation of a map; at such a point, the space of harmonic maps remains smooth [37].

At a similar time to the extensive work dedicated to harmonic maps from surfaces to a complex projective space, K. Uhlenbeck developed the theory of harmonic maps into Lie groups. In [50], Uhlenbeck introduced *polynomial extended solutions* of a harmonic map, that is, maps from a Riemann surface M into the loop group of the unitary group $\Omega U(n)$ that are polynomial in a "spectral" parameter λ. Uhlenbeck showed that such a polynomial extended solution can be factorized with respect to certain subbundles of $\underline{\mathbb{C}}^n := M \times \mathbb{C}^n$ called "unitons". In [46], G. Segal introduced the *Grassmannian model* of an extended solution which represents an extended solution by a subbundle W of the trivial bundle $M \times \mathcal{H}$ where $\mathcal{H}$ is a Hilbert space.

In [9] Burstall and M.A. Guest used *canonical elements* and certain maps into a loop group to classify all polynomial extended solutions for harmonic maps into the unitary group. Chapter 5 concerns canonical elements, giving justification to two theorems presented in [7] and [8] and giving concrete descriptions of canonical elements for $SU(n)$, $O(n)$ and $Sp(n)$. The extended solutions classified in [9] were given by integration, with equations which are easy to solve for $U(n)$, especially for low dimensions. By viewing $O(n)$ as a subgroup of $U(n)$, M.J. Ferreira, B.A. Simões and Wood [29] applied the method of Burstall and Guest [9] to give a classification of extended solutions for harmonic maps into the orthogonal group. This classification was given according to their canonical elements. Further, they gave a parametrization (at least locally) of these extended solutions in terms of free holomorphic data by replacing every instance of integration with differentiation and algebraic operations. Chapter 6 recalls the work of Uhlenbeck, Segal, Burstall, Guest, Ferreira, Simões and Wood. In Chapter 7, we study the problem of constructing harmonic maps into the symplectic group, which is considerably harder than for the orthogonal group, as there are additional equations to solve to ensure the map is into the symplectic group.

For completeness we give an introduction to harmonic maps and present two important pieces of harmonic map theory, namely the first and second variation mentioned above.

1.1 Harmonic Maps

The first variation gives us the Euler-Lagrange equation for the energy functional. The equation for this can be used directly to see, for example, that all geodesics $\phi : S^1 \to N$ are harmonic maps. The second variation formula is useful for assessing if a harmonic map is stable or not. Full details for the first and second variation can be found in [51], this is also the main reference for the first chapter; for more information we direct the reader to the survey articles [23, 25, 24] and [56].

1.1.1 Energy Density Function and The Energy Integral

Consider two compact Riemannian manifolds (M, g) and (N, h) with dimension m and n, respectively.

Definition 1.1.1. *The* ***energy density function*** *of a C^∞-mapping $\phi : (M, g) \to (N, h)$ is* *the function* $e(\phi) \in C^\infty(M)$ *given by*

$$e(\phi) = \frac{1}{2}\sum_{i=1}^{m}(\phi^* h)(u_i, u_i) = \frac{1}{2}\sum_{i=1}^{m} h(d\phi(u_i), d\phi(u_i))\,.$$

where $C^\infty(M,N)$ denotes the space of C^∞ maps from M to N and $C^\infty(M) = C^{infty}(M,\mathbb{C})$. In the definition above, $\phi^* h$ is the pull back of h by ϕ and we denote by $d\phi : T_xM \to T_{\phi(x)}N$ the differential of ϕ at a point $x \in M$. Also $\{u_i\}_{i=1}^m$ is an orthonormal basis for the tangent space T_xM at x.

To use this definition when we have local coordinates, we may apply the Gram-Schmidt process. To do this we need to take local coordinates $(x^1, x^2, ..., x^m)$ on U around x. At each point $x \in U$ we apply the Gram–Schmidt process to $\{\partial/\partial x^i\}_{i=1}^m$ to get a locally defined orthonormal frame $\{u_i\}_{i=1}^m$. Noting that for each point $x \in U$ we have that $\{u_i(x)\}_{i=1}^m$ is an orthonormal basis for T_zM, then the energy density function defined on a neighbourhood U around x is given by,

$$e(\phi) = \frac{1}{2}\sum_{i=1}^{m}(\phi^* h)(u_i, u_i)$$

where $e(\phi) \in C^\infty(U)$ if $\phi \in C^\infty(U, N)\,.$

In [51, §1.1] it is shown that if we take local coordinates $(x^1, x^2, ..., x^m)$ on a neighbourhood around $x \in M$, local coordinates $(y^1, y^2, ..., y^n)$ on a neighbourhood around $\phi(x) \in N$ and define $\phi^\alpha := y^\alpha \circ \phi$, for $\alpha = 1, ..., n$, then we have

$$e(\phi) = \frac{1}{2}\sum_{i,j,\alpha,\beta} g^{ij} h_{\alpha\beta} \frac{\partial \phi^\alpha}{\partial x^i}\frac{\partial \phi^\beta}{\partial x^j}\,.$$

Here g^{ij} is the inverse of $g_{ij} = g(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})$, and $h_{\alpha\beta} = h(\frac{\partial}{\partial y_\alpha}, \frac{\partial}{\partial y_\beta})$.

Definition 1.1.2. *Let $\phi \in C^\infty(M,N)$, then the* ***energy*** *of ϕ is defined as*

$$E(\phi) = \int_M e(\phi) v_g = \frac{1}{2} \int_M |d\phi|^2 v_g$$

where v_g is the volume measure given in local coordinates by $v_g = \sqrt{\det g_{ij}} dx^1 \cdots dx^n$.

Definition 1.1.3. *A map $\phi \in C^\infty(M,N)$ is called* ***harmonic*** *if ϕ is a critical point of the energy E. Specifically ϕ is a harmonic map if, for any smooth variation ϕ_t of ϕ where $-\epsilon < t < \epsilon$, we have*

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = 0\,.$$

Here a **smooth variation** ϕ_t is defined as a smooth map $F : (-\epsilon, \epsilon) \times M \to N$, given by $F(t, x) := \phi_t(x)$, for $-\epsilon < t < \epsilon,\ x \in M$, where $F(0, x) = \phi(x)$ for all $x \in M$.

1.1.2 The First Variation Formula

We consider the main steps in constructing the first variation formula for harmonic maps. The first variation formula is a useful tool for finding harmonic maps, as it describes the criterion for a map to be harmonic in terms of the Euler-Lagrange equation.

Definition 1.1.4. *For any smooth variation ϕ_t of ϕ where $-\epsilon < t < \epsilon$, we define the* ***variation vector field*** *along ϕ to be*

$$V(x) := \left.\frac{d}{dt}\right|_{t=0} \phi_t(x) \in T_{\phi(x)}N \quad (x \in M)\,.$$

Now consider $\phi^{-1}TN$, the **pullback bundle** of TN by ϕ. That is, for the bundle projection $\pi : TN \to N$,

$$\phi^{-1}TN = \{(x, u) \in M \times TN \mid \pi(u) = \phi(x)\}\,.$$

We can now view the variation vector fields V as smooth sections of the pullback bundle $\phi^{-1}TN$. Let ∇ and ${}^N\nabla$ be the Levi-Civita connections on (M,g) and (N,h), respectively, then we can define the **induced connection** $\widetilde{\nabla}$ on $\phi^{-1}TN$ as follows: it is the unique connection such that, for each $x \in M$ with $\phi(x) \in N$, $X \in T_xM$ and Z a smooth section of TN, we have

$$\widetilde{\nabla}_X(\phi^* Z) = \phi^*({}^N\nabla_{d\phi(X)} Z)$$

where $d\phi : T_xM \to T_{\phi(x)}N$ is the differential of ϕ. Here $\phi^*Y = Y \circ \phi$ for any section Y of TN and is a section of $\phi^{-1}TN$ called the **pullback of** Y **by** ϕ. For more details on this definition see [24].

Theorem 1.1.5. *Let (M,g) and (N,h) be compact Riemannian manifolds and $\phi \in C^\infty(M,N)$. For any smooth variation ϕ_t, $-\epsilon < t < \epsilon$, of ϕ, let $V(x) := d/dt|_{t=0}\phi_t(x)$ for $x \in M$ and let $\{e_i\}_{i=1}^m$ be any orthonormal frame field. Then*

$$\left.\frac{d}{dt}\right|_{t=0} E(\phi_t) = -\int_M h(V, \tau(\phi)) v_g,$$

where

$$\tau(\phi) = \sum_{i=1}^{m} (\widetilde{\nabla}_{e_i}(d\phi(e_i)) - d\phi(\nabla_{e_i} e_i)).$$

Here, $d\phi$ is considered a bundle mapping $TM \to \phi^{-1}TN$ and so $d\phi(e_i)$ is a local section of $\phi^{-1}TN$. Thus, $\phi \in C^\infty(M,N)$ is a harmonic map if and only if

$$\tau(\phi) = 0$$

everywhere on M.

Proof. See [26, 51]. $\square$

Here $\tau(\phi)$ is known as the **tension field** of ϕ and the above theorem shows that $\tau(\phi) = 0$ is the **Euler-Lagrange equation** of the energy functional.

1.1.3 The Second Variation Formula

The second variation formula is used to assess the stability of harmonic maps and is useful when studying the structure of the space of harmonic maps.

Let $\phi : M \to N$ be a harmonic map. Then in a similar way to the above let us take a smooth variation $\phi_{s,t} : M \to N$ of ϕ, with two parameters s and t. More concretely, we have a smooth map

$$F : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \times M \to N,$$

$$F : (s, t, x) \mapsto \phi_{s,t}(x),$$

where $F(0, 0, x) = \phi(x)$ for $x \in M$. This gives two variation vector fields, i.e. sections of $\phi^{-1}TN$, one for each parameter s and t,

$$V(x) := \left.\frac{d}{ds}\right|_{s,t=0} \phi_{s,t}(x), \qquad W(x) := \left.\frac{d}{dt}\right|_{s,t=0} \phi_{s,t}(x) \,.$$

Definition 1.1.6. *Let $\phi : (M, g) \to (N, h)$ be a harmonic map. Then the **Hessian** of the energy E at ϕ is defined by*

$$H(E)_\phi(V, W) = \left.\frac{\partial^2}{\partial s \partial t}\right|_{(s,t)=(0,0)} E(\phi_{s,t}),$$

for V and W variation vector fields.

Definition 1.1.7. *The **Riemann curvature tensor** on N is defined by*

$${}^N R(U, V)W = {}^N\nabla_U {}^N\nabla_V W - {}^N\nabla_V {}^N\nabla_U W - {}^N\nabla_{[U,V]} W,$$

for U, V, W, vector fields on N.

Theorem 1.1.8. *Let $\phi : (M, g) \to (N, h)$ be a harmonic map. Then the Hessian of the* *energy E at ϕ is given by*

$$H(E)_\phi(V,W) = \frac{\partial^2}{\partial s \partial t}\bigg|_{(s,t)=(0,0)} E(\phi_{s,t}) = \int_M h\Big(W, J_\phi(V)\Big) v_g,$$

for V and W variation vector fields. Here J_ϕ is a second order self-adjoint elliptic differential operator acting on the space of variation vector fields along ϕ given by:

$$J_\phi(V) := -\sum_{i=1}^{m} (\widetilde{\nabla}_{e_i}\widetilde{\nabla}_{e_i} - \widetilde{\nabla}_{\nabla_{e_i} e_i})V - \sum_{i=1}^{m} {}^N R(V, d\phi(e_i))d\phi(e_i)\,.$$

Proof. See [26, 51] . □

Using the second variation formula above we can now define the stability of harmonic maps.

Definition 1.1.9. *The **index** of a harmonic map $\phi : M \to N$ is defined as*

$$\text{index}(\phi) = \sup\{\dim(F) \mid F \textit{ is a vector subspace of } \Gamma(\phi^{-1}TN)$$
$$\textit{with } H(E)_\phi \textit{ negative definite on } F\}\,.$$

*Note here that $\Gamma(\cdot)$ denotes the space of smooth sections, and the supremum is finite by standard elliptic operator theory [51, Chapter 5 §1.2]. A harmonic map $\phi : M \to N$ is said to be **(weakly) stable** if*

$$\text{index}(\phi) = 0\,.$$

An immediate consequence of this definition is that $\text{index}(\phi) = 0$ if and only if $H(E)_\phi(V,V) \geq 0$ for all $V \in \Gamma(\phi^{-1}TN)$.

1.2 Thesis Overview

Besides the introductory chapter this thesis is split into two main parts which cover the topic of harmonic maps from a compact Riemann surface to a complex projective space or certain Lie groups. All chapters concern harmonic maps from a Riemann surface, Chapters 2–4 concern harmonic maps with target space a complex projective space and Chapters 5–7 harmonic maps with target space the unitary group $U(n)$, the orthogonal group $O(n)$, or the symplectic group $Sp(n)$.

In Chapter 2 we describe and link two constructions of complex isotropic harmonic maps from M to complex projective spaces $\mathbb{CP}^n$ given in [28, 12].

In Chapter 3 we specialise to $S^2 \to \mathbb{CP}^2$ and show that there is no restriction on the ramification behaviour of full holomorphic maps of degree k with total ramification index $0 \leq r_0 \leq k-2$ and we give a counter-example that shows the opposite for $k-2 < r_0 \leq (3/2)k-3$. Through an application of the Gauss transform these holomorphic maps give harmonic maps which form a finite-dimensional manifold [37].

In Chapter 4 we find the dimension of the spaces of holomorphic sections and holomorphic differentials of certain line bundles to give improved lower bounds on the index of complex isotropic harmonic maps $M_g \to \mathbb{CP}^n$ for $g = 0$ or 1; here M_g denotes a compact Riemann surface of genus g. We also give new bounds on the index of complex isotropic harmonic maps from higher genus surfaces which improve those in [28] in some cases.

In Chapter 5 we give explicit descriptions of canonical elements for the Lie groups $SU(n)$, $O(n)$ and $Sp(n)$, which are used in the construction and classification of harmonic maps into $U(n)$ and $O(n)$ given in [9] and [29], respectively.

In Chapter 6 we recall the underlying theory of harmonic maps into Lie groups due to K. Uhlenbeck [50] and go on to recall the work of G. Segal [46], F.E. Burstall and M.A. Guest [9] for harmonic maps into the unitary group $U(n)$. Finally we recall the work of

M.J. Ferreira, B.A. Simões and J.C. Wood [29] for harmonic maps into the orthogonal group $O(n)$.

In Chapter 7 we give a new method of using canonical elements of $Sp(n)$ to give algebraic parametrizations of S^1-invariant extended solutions of harmonic maps of finite uniton number from a surface to the symplectic group $Sp(n)$ up to complex dimension 6. This method was inspired by [29], but is harder as there is an extra equation to solve which was not present in the $O(n)$ case. For arbitrary dimension we give an algorithm which parametrizes all such S^1-invariant extended solutions of harmonic maps which are of **standard type**.

Chapter 2

Harmonic Maps from Surfaces to Complex Projective Spaces

We recall the construction of complex isotropic harmonic maps given in [28, 12] and link the two approaches (Proposition 2.2.9); for additional reading related to these constructions see [2, 18, 37, 38, 53] and for a moving frames approach see [14, 52].

2.1 Subbundles of $M \times \mathbb{C}^{n+1}$

Let M be a compact Riemann surface. We give $\mathbb{CP}^n$ its standard structure as a Kähler manifold of constant holomorphic sectional curvature $c > 0$ [51, p. 147]. Let us identify $\mathbb{CP}^n$ with the set of complex lines (i.e. one-dimensional complex subspaces in $\mathbb{C}^{n+1}$) in the usual way, so that each point $V \in \mathbb{CP}^n$ is identified with a complex line in $\mathbb{C}^{n+1}$. This leads to the definition of the following canonical bundle, which will be of vital use in our work.

Definition 2.1.1. *The **tautological bundle** T over $\mathbb{CP}^n$ is the subbundle of the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1} \to \mathbb{CP}^n$ whose fibre at $V \in \mathbb{CP}^n$ is the complex line V in $\mathbb{C}^{n+1}$.*

By decomposing the complexified tangent bundle $T^{\mathbb{C}}\mathbb{CP}^n$ using the complex structure in the usual way we have

$$T^{\mathbb{C}}\mathbb{CP}^n = T^{(1,0)}\mathbb{CP}^n \oplus T^{(0,1)}\mathbb{CP}^n.$$

There is a well-known connection-preserving isomorphism $h : T^{(1,0)}\mathbb{CP}^n \to L(T, T^{\perp})$ where $L(T, T^{\perp})$ is the bundle of linear bundle maps from the tautological bundle to its orthogonal complement in the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. This is given by

$$h(Z)\sigma = \pi_{T^{\perp}} Z(\sigma), \tag{2.1.1}$$

where σ is a local section of T, $Z \in T^{(1,0)}\mathbb{CP}^n$, $\pi_{T^{\perp}}$ denotes the orthogonal projection onto $T^{\perp}$ and $Z(\cdot)$ denotes differentiation with respect to Z. For information on this isomorphism see [2, 12, 28].

Consider a smooth map $\phi : M \to \mathbb{CP}^n$. We may decompose the $\mathbb{C}$-linear extension of its differential $d\phi$ into components:

$$\partial\phi : T^{(1,0)}M \to T^{(1,0)}\mathbb{CP}^n, \qquad \overline{\partial}\phi : T^{(0,1)}M \to T^{(1,0)}\mathbb{CP}^n. \tag{2.1.2}$$

Let $G_k(\mathbb{C}^{n+1})$ be the Grassmannian of k-dimensional subspaces of $\mathbb{C}^{n+1}$. To each map $\phi : M \to G_k(\mathbb{C}^{n+1})$, we may associate the pullback of the tautological bundle $\underline{\phi} := \phi^{-1}T$; this is the rank k subbundle of the trivial bundle $M \times \mathbb{C}^{n+1}$ over M whose fibre at z is the k-dimensional subspace $\phi(z)$. Conversely, any rank k subbundle $\underline{\phi}$ of $M \times \mathbb{C}^{n+1}$ corresponds to a map $\phi : M \to G_k(\mathbb{C}^{n+1})$ where $\phi(z)$ is the k-dimensional subspace given by the fibre $\underline{\phi}_z$ for $z \in M$. We shall call $\underline{\phi}$ the **associated subbundle** of ϕ. The orthogonal projection π_{ϕ} onto $\underline{\phi}$ applied to the standard derivation on the trivial bundle $M \times \mathbb{C}^{n+1}$ induces a connection ${}^{\phi}\nabla$ on $\underline{\phi}$; on a (local complex) chart (U, z) of M this is given by

$$^{\phi}\nabla_{\partial/\partial z} v = \pi_{\phi} \frac{\partial}{\partial z} v, \qquad ^{\phi}\nabla_{\partial/\partial \bar{z}} v = \pi_{\phi} \frac{\partial}{\partial \bar{z}} v, \tag{2.1.3}$$

for $v \in \Gamma(\underline{\phi}) = \Gamma(\phi^{-1}T)$.

We may regard $\mathbb{CP}^n$ as the complex Grassmannian $G_1(\mathbb{C}^{n+1})$ of complex 1-planes in Euclidean $(n+1)$-space $\mathbb{C}^{n+1}$. Then to each map $\phi : M \to \mathbb{CP}^n$, we may associate the pullback of the tautological bundle $\underline{\phi} := \phi^{-1}T$; this is the complex line subbundle of the trivial bundle $M \times \mathbb{C}^{n+1}$ over M whose fibre at z is the line $\phi(z)$. Conversely, any complex line subbundle $\underline{\phi}$ of $M \times \mathbb{C}^{n+1}$ corresponds to a map $\phi : M \to \mathbb{CP}^n$ where $\phi(z)$ is the line given by the fibre $\underline{\phi}_z$ for $z \in M$.

As in [12], given mutually orthogonal subbundles $\underline{\phi}$ and $\underline{\psi}$ on a coordinate chart (U, z) we define the (linear) bundle maps $A'_{\phi,\psi} : \underline{\phi} \to \underline{\psi}$ and $A''_{\phi,\psi} : \underline{\phi} \to \underline{\psi}$ by

$$A'_{\phi,\psi}(v) = \pi_{\psi} \frac{\partial}{\partial z} v \quad \text{and} \quad A''_{\phi,\psi}(v) = \pi_{\psi} \frac{\partial}{\partial \bar{z}} v,$$

where π_{ψ} is the orthogonal projection onto $\underline{\psi}$ (some authors e.g. [11] interchange ϕ with ψ in this notation). These two maps are "adjoint up to sign" [11], more concretely, with $\langle \, , \, \rangle_{\phi}$ the Hermitian metric on $\underline{\phi}$ induced from the flat hermitian metric $\langle \, , \, \rangle$ on the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ then

$$-\langle A'_{\phi,\psi} v, w \rangle_{\phi} = \langle v, A''_{\psi,\phi} w \rangle_{\phi} \quad \text{for} \quad v \in \underline{\phi}, \;\; w \in \underline{\psi}.$$

A very useful special case of the above is the following: we set

$$A'_{\phi} = A'_{\phi,\phi^{\perp}} : \underline{\phi} \to \underline{\phi}^{\perp} \quad \text{and} \quad A''_{\phi} = A''_{\phi,\phi^{\perp}} : \underline{\phi} \to \underline{\phi}^{\perp}.$$

Then, using the pullback of (2.1.1), we have the following isomorphism of bundles over M:

$$\phi^{-1} T^{(1,0)} \mathbb{CP}^n \cong L(\underline{\phi}, \underline{\phi}^{\perp}). \tag{2.1.4}$$

Since this is a pullback of a connection-preserving isomorphism it is again a connection-preserving isomorphism, which can be used to identify $\partial\phi(\partial/\partial z)$ and $\overline{\partial}\phi(\partial/\partial\overline{z})$ with the bundle maps A'_ϕ and A''_ϕ, respectively. We give all bundles their Koszul-Malgrange structure [36], i.e. that with $\overline{\partial}$-operator given by the $(0,1)$-part of the connection on the respective bundle. It follows that (2.1.4) is an isomorphism of **holomorphic** bundles. Then we have the following for a smooth map $\phi : M \to \mathbb{CP}^n$:

Lemma 2.1.2. *[12]*

(i) *The map ϕ is holomorphic (respectively, antiholomorphic) if and only if $A''_\phi = 0$ (respectively $A'_\phi = 0$).*

(ii) *The map ϕ is harmonic if and only if $A'_\phi : \underline{\phi} \to \underline{\phi}^\perp$ is holomorphic, i.e.,*

$$A'_\phi \circ {}^\phi\nabla_{\partial/\partial\overline{z}} = {}^{\phi^\perp}\nabla_{\partial/\partial\overline{z}} \circ A'_\phi,$$

or equivalently, $A''_\phi : \underline{\phi} \to \underline{\phi}^\perp$ is antiholomorphic, i.e.,

$$A''_\phi \circ {}^\phi\nabla_{\partial/\partial z} = {}^{\phi^\perp}\nabla_{\partial/\partial z} \circ A''_\phi,$$

Let $\phi : M \to \mathbb{CP}^n$ be a non-antiholomorphic harmonic map. After a process of filling out the zeros of A'_ϕ detailed in [12, p. 266], according to Lemma 2.1.2 the image of A'_ϕ becomes a holomorphic subbundle of $\underline{\phi}^\perp$, i.e. closed under ${}^{\phi^\perp}\nabla_{\partial/\partial\overline{z}}$, which we denote by $\underline{\mathrm{Im}}A'_\phi$. We call this holomorphic subbundle the **∂'-Gauss bundle** and denote it by $\underline{G}'(\phi)$. As $\underline{G}'(\phi)$ is a complex line subbundle of the trivial bundle $M \times \mathbb{C}^{n+1}$ over M it corresponds to a map $G'(\phi) : M \to \mathbb{CP}^n$ such that $\underline{G}'(\phi) =: G'(\phi)^{-1}T$. Explicitly $G'(\phi)(z)$ is the fibre at $z \in M$ of $\underline{G}'(\phi)$. Similarly let $\phi : M \to \mathbb{CP}^n$ be a non-holomorphic harmonic map, then the image of A''_ϕ is an antiholomorphic subbundle of $\underline{\phi}^\perp$, i.e. closed under ${}^{\phi^\perp}\nabla_{\partial/\partial z}$, denoted $\underline{G}''(\phi)$ and called the **∂''-Gauss bundle**. As before $\underline{G}''(\phi)$ is a complex line subbundle of the trivial bundle $M \times \mathbb{C}^{n+1}$ over M and so induces a map $G''(\phi) : M \to \mathbb{CP}^n$ characterised by $\underline{G}''(\phi) =: G''(\phi)^{-1}T$.

Lemma 2.1.3. *[12, Proposition 2.3 and Remark] Let $\phi : M \to \mathbb{CP}^n$ be a harmonic map. If ϕ is not antiholomorphic then $G'(\phi)$ is harmonic and $G''(G'(\phi)) = \phi$. If ϕ is not holomorphic then $G''(\phi)$ is harmonic and $G'(G''(\phi)) = \phi$.*

Definition 2.1.4. *A map $f : M \to \mathbb{CP}^n$ is said to be **full** if its image does not lie in a proper projective subspace of $\mathbb{CP}^n$.*

We shall now construct a "harmonic sequence" from a full holomorphic map using the above. Let $f_0 : M \to \mathbb{CP}^n$ be a full holomorphic map, then as above $\underline{G}'(f_0) := \underline{\mathrm{Im}}A'_{f_0} \subset \underline{f}_0^{\perp}$ (most authors omit the underlining in the bundle $\underline{G}'(\cdot)$; we add it for clarity); here $G'(f_0) : M \to \mathbb{CP}^n$ is a harmonic map by Lemma 2.1.3. Applying the procedure again to $G'(f_0) : M \to \mathbb{CP}^n$ we have $\underline{G}'(G'(f_0)) := \underline{\mathrm{Im}}A'_{G'(f_0)} \subset \underline{G}'(f_0)^{\perp}$, where $\underline{G}'(G'(f_0)) = G'(G'(f_0))^{-1}T$ for some smooth map $G'(G'(f_0)) : M \to \mathbb{CP}^n$ which is again a harmonic map by Lemma 2.1.3.

For $j = 0, 1, \ldots, n$, write $f_j := (G')^j(f_0) := G'(G'(\ldots G'(G'(f_0))\ldots))$ where G' is applied j times to f_0, and $\underline{f}_j := \underline{(G')^j}(f_0) := \underline{G}'(G'(\ldots G'(G'(f_0))\ldots))$, so $f_j : M \to \mathbb{CP}^n$ is a harmonic map and $\underline{f}_j := f_j^{-1}T$ its associated subbundle given by the pullback of the tautological bundle. By fullness, none of the subbundles $\underline{f}_j$ is zero for $j < n$, since otherwise f would lie in the constant proper subspace of $\mathbb{C}^{n+1}$ spanned by the $\underline{f}_j$. Note that $\underline{f}_j := \underline{\mathrm{Im}}A'_{f_{j-1}} \subset \underline{f}_{j-1}^{\perp}$.

Remark 2.1.5. *Similarly, given a full antiholomorphic map g_0 and by replacing A' and G' with A'' and G'', respectively, we obtain harmonic maps $g_k := (G'')^k(g_0) := G''(G''(\ldots G''(G''(g_0))\ldots))$ where G'' is applied k times to g_0.*

It was shown in [12], and through a different interpretation in [28], that the nth iteration of the procedure above gives $\underline{f}_n := \underline{\mathrm{Im}}A'_{f_{n-1}} \subset \underline{f}_{n-1}^{\perp}$ where $f_n : M \to \mathbb{CP}^n$ is a full antiholomorphic map. Using Lemma 2.1.2 we see that $A''_{f_0} = A'_{f_n} = 0$ since f_0 and f_n are holomorphic and antiholomorphic, respectively, therefore $\underline{G}''(f_0) = \underline{G}'(f_n) = \underline{0}$ and so do not define maps into $\mathbb{CP}^n$. Therefore we have the following sequence of associated subbundles of harmonic maps and bundle maps between them, called the **harmonic sequence (of** f_0**)** [2, 53]:

$$\underline{f}_0 \underset{A''_{f_1}}{\overset{A'_{f_0}}{\rightleftarrows}} \underline{f}_1 \underset{A''_{f_2}}{\overset{A'_{f_1}}{\rightleftarrows}} \cdots \underset{A''_{f_{\rho-1}}}{\overset{A'_{f_{\rho-2}}}{\rightleftarrows}} \underline{f}_{\rho-1} \underset{A''_{f_\rho}}{\overset{A'_{f_{\rho-1}}}{\rightleftarrows}} \underline{f}_\rho \underset{A''_{f_{\rho+1}}}{\overset{A'_{f_\rho}}{\rightleftarrows}} \underline{f}_{\rho+1} \underset{A''_{f_{\rho+2}}}{\overset{A'_{f_{\rho+1}}}{\rightleftarrows}} \cdots \underset{A''_{f_n}}{\overset{A'_{f_{n-1}}}{\rightleftarrows}} \underline{f}_n; \tag{2.1.5}$$

where $f_0 : M \to \mathbb{CP}^n$ is a full holomorphic map with associated bundle $\underline{f}_0 := {f_0}^{-1}T$, $f_i : M \to \mathbb{CP}^n$ is a full harmonic map with associated bundle $\underline{f}_i := {f_i}^{-1}T$ for each $i \in \{1, 2, \ldots, n-1\}$ and $f_n : M \to \mathbb{CP}^n$ is a full antiholomorphic map with associated bundle $\underline{f}_n := {f_n}^{-1}T$.

Extending the notation above to a harmonic map $\phi : M \to \mathbb{CP}^n$ we write the j-fold iterate of G' on ϕ as $(G')^j(\phi) := G'(G'(\ldots G'(G'(\phi))\ldots))$ where G' is applied j times to ϕ and $\underline{(G')}^j(\phi) := \underline{G}'(G'(\ldots G'(G'(\phi))\ldots))$. Similarly we write the j-fold iterate of G'' on ϕ as $(G'')^j(\phi) := G''(G''(\ldots G''(G''(\phi))\ldots))$ where G'' is applied j times to ϕ and $\underline{(G'')}^j(\phi) := \underline{G}''(G''(\ldots G''(G''(\phi))\ldots))$. Note that if $(G')^j(\phi)$ is holomorphic so $\underline{(G')}^{j+1}(\phi) = \underline{0}$ and therefore does not define a map into $\mathbb{CP}^n$. Similarly if $(G'')^j(\phi)$ is antiholomorphic so $\underline{(G'')}^{j+1}(\phi) = \underline{0}$ and again does not define a map into $\mathbb{CP}^n$.

Definition 2.1.6. *[12, §3] A harmonic map $\phi : M \to \mathbb{CP}^n$ is called **complex isotropic** if its associated subbundle $\underline{\phi}$ is orthogonal to $\underline{(G')}^j(\phi)$ for each $j \geq 1$.*

Lemma 2.1.7. *[12, Lemma 3.1] If $\phi : M \to \mathbb{CP}^n$ is complex isotropic then*

$$\underline{(G')}^j(\phi) \perp \underline{(G')}^k(\phi)$$

for all $j, k \in \{0, 1, \ldots\}$, $j \neq k$.

Lemma 2.1.8. *The harmonic maps f_j from (2.1.5), that is, the harmonic maps constructed from a full holomorphic map as iterated ∂'-Gauss bundles, $\underline{(G')}^j(f_0)$, are complex isotropic.*

Proof. By Lemma 2.1.7 we need only prove that for a full holomorphic map $f_0 : M \to \mathbb{CP}^n$ that $\underline{f}_0$ is orthogonal to $\underline{f}_j$ for all $j \geq 0$ i.e. that f_0 is complex isotropic. Let F_0 be a local holomorphic lift of the full holomorphic map $f_0 : M \to \mathbb{CP}^n$, then F_0 can be seen as a local nowhere zero section of $\underline{f}_0$. Define $A'_{0,j} = A'_{f_{j-1}} \circ A'_{f_{j-2}} \circ \cdots \circ A'_{f_0}$ so $A'_{0,j} : \underline{f}_0 \to \underline{f}_j$ and $A'_{0,1} = A'_{f_0}$. We will use induction to prove our claim that $\underline{f}_0 \perp \underline{f}_j$ for all $j = 1, 2, \ldots, n$. We have $\underline{f}_0 \perp \underline{f}_1$ as by definition $\underline{f}_1 := \underline{G}'(f_0) := \underline{\mathrm{Im}} A'_{G'(f_0)} \subset \underline{f}_0^{\perp}$. We use this as a base of induction on $j \in \{1, \ldots, n\}$: for an induction hypothesis let us assume that $\underline{f}_0 \perp \underline{f}_k$ for all $k \leq j$ and we will see that this implies $\underline{f}_0 \perp \underline{f}_{j+1}$.

Let $\langle\ ,\ \rangle$ denote the flat Hermitian metric on the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$ and $F_0 \in \Gamma(\underline{f}_0)$ be the local holomorphic lift of $f_0 : M \to \mathbb{CP}^n$. We have,

$$\begin{aligned}
\langle F_0, A'_{0,j+1}(F_0)\rangle &= \langle F_0, \pi^{\perp}_{f_j} \frac{\partial}{\partial z} A'_{0,j}(F_0)\rangle \\
&= \langle F_0, \frac{\partial}{\partial z} A'_{0,j}(F_0) - \pi_{f_j} \frac{\partial}{\partial z} A'_{0,j}(F_0)\rangle \\
&= \langle F_0, \frac{\partial}{\partial z} A'_{0,j}(F_0)\rangle - \langle F_0, \pi_{f_j} \frac{\partial}{\partial z} A'_{0,j}(F_0)\rangle.
\end{aligned}$$

By the induction hypothesis $\underline{f}_0 \perp \underline{f}_j$ so $\langle F_0, \pi_{f_j} \frac{\partial}{\partial z} A'_{0,j}(F_0)\rangle = 0$. Therefore,

$$\begin{aligned}
\langle F_0, A'_{0,j+1}(F_0)\rangle &= \langle F_0, \frac{\partial}{\partial z} A'_{0,j}(F_0)\rangle \\
&= \frac{\partial}{\partial z}\langle F_0, A'_{0,j}(F_0)\rangle - \langle \frac{\partial}{\partial \bar{z}} F_0, A'_{0,j}(F_0)\rangle = 0,
\end{aligned}$$

as F_0 is holomorphic, $A'_{0,j}(F_0) \in \Gamma(\underline{f}_j)$ and $\underline{f}_0 \perp \underline{f}_j$ by the induction hypothesis. Therefore $\underline{f}_0 \perp \underline{f}_{j+1}$ and the induction step is complete. □

Remark 2.1.9. (i) *All harmonic maps $S^2 \to \mathbb{CP}^n$ are given as above; for higher genera the construction gives all harmonic maps which are complex isotropic (see also [28, §5] and [12, §3 ff.]), or equivalently of finite uniton number cf. [1, §4.3]. The terms* of infinite isotropy order, strongly isotropic *and* pseudoholomorphic *[3] are also used.*

(ii) *It follows from Lemma 2.1.8 and Lemma 2.1.7 that all* $\underline{f}_i$ *in (2.1.5) are mutually orthogonal subbundles of the trivial bundle* $M \times \mathbb{C}^{n+1}$.

Definition 2.1.10. *Let* (U, z) *be a chart of* M *and let* $z_0 \in U$ *be a zero of* $A'_{f_{\rho-1}}$ *where* $\rho \in \{1, \ldots, n\}$*; then we can write*

$$A'_{f_{\rho-1}}(z) = (z - z_0)^k \lambda(z),$$

where λ *is a smooth section of* $L(\underline{f}_{\rho-1}, \underline{f}_{\rho-1}^{\perp})$*, non-zero at* z_0 *and* $k \in \mathbb{N}$ *(where* $\mathbb{N} = \{1, 2, \ldots\}$*). Then we say that* f_0 *is* ***ρth(-order) ramified*** *at the point* z_0 *with* ***ramification index*** k*. We call the sum of all ramification indices of the points of* ρ*th ramification the* ***ρth total ramification index*** *and denote it* $r_{\rho-1}$.

2.1.1 Degree of a Smooth Map

Let M be a compact Riemann surface. For use in later sections we present some results concerning the degree of a smooth map $\phi : M \to \mathbb{CP}^n$.

Definition 2.1.11. *Let* $\phi : M \to \mathbb{CP}^n$ *be a smooth map. The* ***degree*** *of* ϕ*, denoted* $\deg(\phi)$ *is the degree of the induced map* $\phi^* : H^2(\mathbb{CP}^n, \mathbb{Z}) \to H^2(M, \mathbb{Z})$ *on second cohomology.*

More explicitly, the deRham cohomology class $[\omega^N]$ of the Kähler form ω^N of $N = \mathbb{CP}^n$ gives a generator of $H^2(\mathbb{CP}^n, \mathbb{Z}) \cong \mathbb{Z}$ and the deRham cohomology class $[\omega^M]$ of the volume form ω^M gives a generator of $H^2(M, \mathbb{Z}) \cong \mathbb{Z}$; then $[\phi^*\omega^N] = \deg(\phi)[\omega^M]$.

Lemma 2.1.12. *[12, Lemma 5.1] Let* $\phi : M \to \mathbb{CP}^n$ *be a smooth map and* $\underline{\phi}$ *its associated subbundle of the trivial bundle* $M \times \mathbb{C}^{n+1}$*. Then* $\deg(\phi) = -c_1(\underline{\phi})$ *where* $c_1(\underline{\phi})$ *is the first Chern class of* $\underline{\phi}$*.*

Recall for a smooth map $\phi : M \to \mathbb{CP}^n$ we may decompose the $\mathbb{C}$-linear extension of its differential $d\phi$ (2.1.2). This gives a decomposition of the energy density $e(\phi)$ of ϕ given in Definition 1.1.1 in the following way:

$$e(\phi) = e^{(1,0)}(\phi) + e^{(0,1)}(\phi),$$

where $e^{(1,0)}(\phi) = \frac{1}{2}|\partial\phi|^2$ and $e^{(0,1)}(\phi) = \frac{1}{2}|\overline{\partial}\phi|^2$. This, in turn, gives a decomposition of the energy $E(\phi)$ of ϕ from Definition 1.1.2 given by

$$E(\phi) = E^{(1,0)}(\phi) + E^{(0,1)}(\phi),$$

where $E^{(1,0)}(\phi) = \frac{1}{2}\int_M |\partial\phi|^2 v_g$ and $E^{(0,1)}(\phi) = \frac{1}{2}\int_M |\overline{\partial}\phi|^2 v_g$.

Lemma 2.1.13. *Let $\phi : M \to \mathbb{CP}^n$ be a smooth map. Then*

$$E^{(1,0)}(\phi) - E^{(0,1)}(\phi) = \frac{4\pi}{c}\deg(\phi),$$

where $c > 0$ is the value of the constant holomorphic sectional curvature of $\mathbb{CP}^n$.

Proof. See [28, p. 247] and [55, p. 141]. □

This lemma shows that a holomorphic or antiholomorphic map gives a harmonic map of minimum energy in its homotopy class, see [39].

2.2 Associated Curves and the Gauss Transforms

The construction of harmonic maps from surfaces to complex projective spaces was first given in a twistorial way in [28] using **associated curves**: it was later interpreted using the Gauss transform in [12]. We give a description of this construction with an aim to link the two approaches from [12] and [28].

Let $F : U \to \mathbb{C}^{n+1}\backslash\{0\}$ be a local holomorphic lift of a full holomorphic map $f : M \to \mathbb{CP}^n$ on a complex chart (U, z) of M. That is, $f|_U = \pi \circ F$ for $\pi : \mathbb{C}^{n+1}\backslash\{0\} \to$

$\mathbb{CP}^n$ defined by $\pi(w_0, w_1, \ldots, w_n) = [w_0, w_1, \ldots, w_n]$ where [ ] denote homogeneous coordinates. We define the derivatives of F by $F^{(\rho)} = d^\rho F/dz^\rho$.

Definition 2.2.1. *We define the* ρ***th osculating space of*** f *at* $z \in M$ *to be the space*

$$\theta_\rho = \theta_\rho(z) = \mathrm{Span}\{F^{(\alpha)}(z) \mid 0 \leq \alpha \leq \rho\}.$$

Note that, by the chain rule, θ_ρ is well-defined under change of local coordinates (from z to w) up to a non-zero factor (given by a power of dw/dz).

The ρth osculating space of f may vary in dimension with $z \in M$, however [28, Lemma 3.1] if $f : M \to \mathbb{CP}^n$ is full then there exists a point $z \in M$ such that the nth osculating space of f at z has dimension $n+1$ and so is the whole of $\mathbb{C}^{n+1}$.

Consider the wedge product $F \wedge F' \wedge F'' \wedge \cdots \wedge F^{(\rho)} : U \to \wedge^{\rho+1}\mathbb{C}^{n+1}$, where $F' = dF/dz$ and $0 \leq \rho \leq n$. Again by the chain rule, this wedge product is well defined up to scalar multiples under change of local coordinate. If $\rho = n$ then the zeros of the wedge product are exactly the points $z \in M$ such that $\dim \theta_n(z) < n+1$, we denote this collection of discrete points by B, similarly to [28, (3.1)] we have

$$B = \{z \in M \mid \dim \theta_n(z) < n+1\}. \tag{2.2.1}$$

These zeros have a special significance that we discuss later. If the wedge product $F \wedge F' \wedge F'' \wedge \cdots \wedge F^{(\rho)}$ is nowhere zero then it defines a $(\rho+1)$-dimensional subspace in $\mathbb{C}^{n+1}$ for each $z \in M$. If $z_0 \in B$ is a zero of $F \wedge F'$ of order $\kappa_0 \in \mathbb{N}$, then it is also a zero of $F \wedge F' \wedge F'' \wedge \cdots \wedge F^{(\rho)}$ of order $\kappa_\rho \in \mathbb{N}$ for each ρ, $1 \leq \rho \leq n$ where $\kappa_i \leq \kappa_j$ for $1 \leq i < j \leq n$. Let U be an open neighbourhood of z_0 then we write

$$F \wedge F' \wedge F'' \wedge \cdots \wedge F^{(\rho)}(z) = (z - z_0)^{\kappa_\rho}\gamma(z), \tag{2.2.2}$$

for all $z \in U$ and $\gamma(z) \in \wedge^{\rho+1}\mathbb{C}^{n+1}$ is non-zero. As $\gamma(z)$ is decomposable for all $z \neq z_0$ it is decomposable for $z = z_0$ and therefore defines a $(\rho+1)$-dimensional subspace in $\mathbb{C}^{n+1}$ for each $z \in U$. For convenience, we will define $\gamma(z)$ to be $F \wedge F' \wedge F'' \wedge \cdots \wedge F^{(\rho)}(z)$ at points that aren't zeros. As the wedge product $F \wedge F' \wedge F'' \wedge \cdots \wedge F^{(\rho)}$ is independent upto scalar multiples of choice of chart U and local holomorphic lift F we have the following:

Definition 2.2.2. *The* ***ρth associated curve*** *is the holomorphic map $f_{(\rho)} : M \to G_{\rho+1}(\mathbb{C}^{n+1})$ where $f_{(\rho)}(z)$ is the $(\rho+1)$-dimensional subspace defined by γ.*

See Remark 2.2.7 for the relationship with osculating space. Here $G_{\rho+1}(\mathbb{C}^{n+1})$ denotes the Grassmannian of $(\rho+1)$-dimensional subspaces of $\mathbb{C}^{n+1}$. Note that $f_{(\rho)}$ is not the same as f_ρ in the previous section and by [28] the ρth associated curve is independent of lift F and local coordinate z, so $f_{(\rho)}$ is well-defined, and is clearly smooth.

Remark 2.2.3. *Associated curves $h_{(\rho)} : M \to G_{\rho+1}(\mathbb{C}^{n+1})$ for a full antiholomorphic map $h : M \to \mathbb{CP}^n$ can be defined similarly by replacing $F^\rho = d^\rho F/dz^\rho$ by $H^\rho = d^\rho F/d\overline{z}^\rho$ where $H : U \to \mathbb{C}^{n+1}\backslash\{0\}$ is some local antiholomorphic lift over some chart U of M.*

Definition 2.2.4. *Let $f : M \to \mathbb{CP}^n$ be a full holomorphic (resp. antiholomorphic) map. The* ***polar of f*** *is defined by $g = f^\perp_{(n-1)} : M \to \mathbb{CP}^n$ (where $f^\perp_{(n-1)}$ denotes the line orthogonal to the hyperplane $f_{(n-1)}$) and is a full antiholomorphic (resp. holomorphic) map as shown in [28, §3B ff.].*

In §2.1 we defined the ∂'-Gauss bundle; we now define a related notion.

Definition 2.2.5. *The* ***first ∂'-Gauss transform*** *$\phi = G^{(1)}(f) : M \to \mathbb{CP}^n$ of a full holomorphic map is defined by*

$$\phi(z) = f(z)^\perp \cap f_{(1)}(z).$$

The ***ρth ∂'-Gauss transform*** *of f is defined by*

$$G^{(\rho)}(f)(z) = f_{(\rho-1)}(z)^\perp \cap f_{(\rho)}(z).$$

The ***first*** ∂''***-Gauss transform*** $\phi = G^{(-1)}(g) : M \to \mathbb{CP}^n$ *of a full antiholomorphic map is defined by*

$$\phi(z) = g(z)^{\perp} \cap g_{(1)}(z).$$

The ρ***th*** ∂''***-Gauss transform*** *of* g *is defined by*

$$G^{(-\rho)}(g)(z) = g_{(\rho-1)}(z)^{\perp} \cap g_{(\rho)}(z).$$

It was shown in [28] that for any $\rho = 1, \ldots, n$ the ρth ∂'-Gauss transform (resp. ρth ∂''-Gauss transform) of a full holomorphic (resp. full antiholomorphic) map defines a smooth and full harmonic map. Further, $G^{(n)}(f)$ is antiholomorphic and is the polar of f and similarly for $g : M \to \mathbb{CP}^n$ a full antiholomorphic map, $G^{(-n)}(g)$ is holomorphic and is the polar of g. Let $f : M \to \mathbb{CP}^n$ be a full holomorphic map.

Definition 2.2.6. *We define the* ρ***th osculating subbundle of*** f *to be the rank* $\rho + 1$ *subbundle of* $M \times \mathbb{C}^{n+1}$ *over* M *defined by* $\underline{\theta}_\rho = f_{(\rho)}^{-1}T$. *Here* T *is the tautological bundle over the complex Grassmannian* $G_{\rho+1}(\mathbb{C}^{n+1})$, *i.e. the subbundle of the trivial bundle* $G_{\rho+1}(\mathbb{C}^{n+1}) \times \mathbb{C}^{n+1} \to G_{\rho+1}(\mathbb{C}^{n+1})$ *whose fibre at* $V \in G_{\rho+1}(\mathbb{C}^{n+1})$ *is the* $(\rho+1)$*-dimensional subspace* V *in* $\mathbb{C}^{n+1}$.

Remark 2.2.7. *The* ρ*th osculating subbundle of* f *is the subbundle resulting from filling out the zeros [12, p.266] of the* ρ*th osculating space of* f. *In fact the process of filling out the zeros is exactly the same process as was done here; by defining the* ρ*th associated curve of* f *using the* ρ*th osculating space of* f *then defining the* ρ*th osculating subbundle of* f *to be the subbundle associated to the* ρ*th associated curve of* f.

Definition 2.2.8. *The* ρ***th*** ∂'***-Gauss bundle of*** f *is defined to be the line subbundle of* $M \times \mathbb{C}^{n+1}$ *defined by*

$$\underline{G}^{(\rho)}(f) = \pi^{\perp}_{\underline{\theta}_{\rho-1}} \underline{\theta}_\rho.$$

We define $G^{(\rho)}(f)$ *to be the (unique) map such that* $\underline{G}^{(\rho)}(f) := G^{(\rho)}(f)^{-1}T$, *this is the* ρ*th*

∂'-Gauss transform of f defined in Definition 2.2.5.

We now see that this definition coincides with the definition of ∂'-Gauss bundle given in §2.1.

Proposition 2.2.9. *Let $f_0 : M \to \mathbb{CP}^n$ be a full holomorphic map and let $\underline{f}_\rho$ for $0 \leq \rho \leq n$ be the subbundles of $M \times \mathbb{C}^{n+1}$ from (2.1.5) then*

$$\underline{G}^{(\rho)}(f_0) = \underline{\mathrm{Im}} A'_{f_{\rho-1}}.$$

Proof. Let F_0 be a local holomorphic lift of the full holomorphic map $f_0 : M \to \mathbb{CP}^n$, then F_0 can be seen as a local section of $\underline{f}_0$. Consider

$$A'_{f_0}(F_0) = \pi^{\perp}_{f_0} F'_0 = F'_0 - \pi_{f_0} F'_0. \tag{2.2.3}$$

Recall for a local holomorphic lift of f_0 that $\theta_0 = \mathrm{Span}\{F_0\}$ and $\theta_1 = \mathrm{Span}\{F_0, F'_0\}$ and so $A'_{f_0}(F_0) \in \theta_1$ and $A'_{f_0}(F_0)$ is orthogonal to θ_0 by (2.2.3). So we have $A'_{f_0}(F_0) \in \theta_0^{\perp} \cap \theta_1$ and so $\mathrm{Im}\, A'_{f_0} \subset \theta_0^{\perp} \cap \theta_1$. After filling out the zeros, both sides are one-dimensional subbundles so we have $\underline{\mathrm{Im}} A'_{f_0} = \pi^{\perp}_{\underline{\theta}_0} \underline{\theta}_1$. We use this as a base for an induction on ρ; as an induction hypothesis, assume that $\underline{f}_{\rho-1} := \underline{\mathrm{Im}} A'_{f_{\rho-2}} = \pi^{\perp}_{\underline{\theta}_{\rho-2}} \underline{\theta}_{\rho-1}$ for all $\rho - 1 \in \{1, 2, \dots, n-1\}$ and consider $A'_{f_{\rho-1}}(F_{\rho-1})$ for $F_{\rho-1}$ a local nowhere zero section of $\underline{f}_{\rho-1}$. By the induction hypothesis $F_{\rho-1}$ is orthogonal to $\theta_{\rho-2}$ and $F_{\rho-1} \in \theta_{\rho-1}$, so $\theta_{\rho-1} = \mathrm{Span}\{F_0^{(j)}, F_{\rho-1} \mid 0 \leq j \leq \rho-2\}$ and $\theta_\rho = \mathrm{Span}\{F_0^{(j)}, F_{\rho-1}, F'_{\rho-1} \mid 0 \leq j \leq \rho-2\}$. We have

$$A'_{f_{\rho-1}}(F_{\rho-1}) = \pi^{\perp}_{f_{\rho-1}} F'_{\rho-1} = F'_{\rho-1} - \pi_{f_{\rho-1}} F'_{\rho-1} \in \theta_\rho, \tag{2.2.4}$$

in particular $A'_{f_{\rho-1}}(F_{\rho-1})$ is orthogonal to $F_{\rho-1}$. We claim $\langle F'_{\rho-1}, F_0^{(j)} \rangle = 0$ for all $j = 1, 2, \dots, \rho-2$ where $\langle\, ,\, \rangle$ denotes the flat metric on the trivial bundle $\mathbb{CP}^n \times \mathbb{C}^{n+1}$. Indeed

$$\langle F'_{\rho-1}, F_0^{(j)} \rangle = \frac{\partial}{\partial z} \langle F_{\rho-1}, F_0^{(j)} \rangle - \langle F_{\rho-1}, \frac{\partial}{\partial \overline{z}} F_0^{(j)} \rangle = 0,$$

as $F_{\rho-1}$ is orthogonal to $\theta_{\rho-2}$ and $F_0^{(j)}$ is holomorphic. If follows from (2.2.4) that $A'_{f_{\rho-1}}(F_{\rho-1})$ is orthogonal to $F_0^{(j)}$ for all $j = 1, 2, \ldots, \rho - 2$ and so orthogonal to $\theta_{\rho-2}$. We conclude that $A'_{f_{\rho-1}}(F_{\rho-1})$ is orthogonal to $\theta_{\rho-1}$, and so $A'_{f_{\rho-1}}(F_{\rho-1}) \in \theta_{\rho-1}^{\perp} \cap \theta_\rho$, giving $\operatorname{Im} A'_{f_{\rho-1}} \subset \theta_{\rho-1}^{\perp} \cap \theta_\rho$. After filling out the zeros, both $\operatorname{Im} A'_{f_{\rho-1}}$ and $\theta_{\rho-1}^{\perp} \cap \theta_\rho$ are one-dimensional subbundles so we have $\underline{\operatorname{Im}} A'_{f_{\rho-1}} = \pi^{\perp}_{\underline{\theta}_{\rho-1}} \underline{\theta}_\rho$, which completes the induction step. □

Corollary 2.2.10. *Let $f_0 : M \to \mathbb{CP}^n$ be a full holomorphic map with local holomorphic lift $F_0 : M \to \mathbb{C}^{n+1} \backslash \{0\}$ and let $\underline{f}_\rho := f_\rho^{-1} T$ for $0 \le \rho \le n$ be the subbundles of $M \times \mathbb{C}^{n+1}$ from (2.1.5) where, as before, $f_\rho = (G')^\rho(f_0)$ where G' is applied ρ times to f_0. Define $A'_{0,j} = A'_{f_{j-1}} \circ A'_{f_{j-2}} \circ \cdots \circ A'_{f_0}$ so $A'_{0,j} : \underline{f}_0 \to \underline{f}_j$ and $A'_{0,1} = A'_{f_0}$, then*

(i) $G^{(\rho)}(f_0) = (G')^\rho(f_0)$*;*

(ii) *the wedge product satisfies*

$$F_0 \wedge F_0' \wedge \cdots \wedge F_0^{(\rho)} = F_0 \wedge A'_{0,1}(F_0) \wedge A'_{0,2}(F_0) \wedge \cdots \wedge A'_{0,\rho}(F_0).$$

Proof. (i) This follows from Proposition 2.2.9 and the definitions of the associated bundles: $\underline{G}^{(\rho)}(f) := G^{(\rho)}(f)^{-1} T$ and $\underline{f}_\rho = f_\rho^{-1} T$.

(ii) We have that $A'_{0,\rho-1}(F_0)$ is a section of $\underline{f}_\rho$ for all $\rho \in \{1, \ldots, n-1\}$ and so $\theta_\rho = \operatorname{Span}\{F_0^{(j)} \mid 0 \le j \le \rho\} = \operatorname{Span}\{F_0, A'_{0,j}(F_0) \mid 1 \le j \le \rho\}$.

□

2.2.1 Ramification

Recall the space B from (2.2.1). Using Corollary 2.2.10, we show that B is the space of all zeros of A'_{f_j} for $j = 0, 1, \ldots, n$ as defined in Definition 2.1.10.

Definition 2.2.11. *Let $f : M \to \mathbb{CP}^n$ be a full holomorphic map. We say that f is **first-ramified** at a point $z \in M$ if $df(z) = 0$. The **(first) ramification index** of f at z is the order of the zero of $df(z)$ at z. The **first total ramification index,** r_0 of f is the sum of all the first ramification indices (cf. [31, p. 264]). Here by the **order** of the zero of $df(z_0)$ at z_0 we mean the natural number k such that in local coordinates*

$$\frac{dw}{dz} = (z - z_0)^k \tilde{f}(z),$$

where $\tilde{f}$ is smooth and non-zero at z_0 and z , $w = (w_1, \ldots, w_n)$, are local complex coordinates of M and $\mathbb{CP}^n$, respectively.

Recall from §2.2 and (2.2.1) the collection B of all points such that the wedge product $F \wedge F' \wedge F'' \wedge \cdots \wedge F^{(n)}$ is zero. We will describe the ρth ramification points from Definition 2.1.10 in terms of the wedge product $F_0 \wedge F_0' \wedge \cdots \wedge F^{(\rho)}$ to show the significance of the space B.

Let $F_0 : U \to \mathbb{C}^{n+1} \backslash \{0\}$ be a (local) nowhere zero holomorphic lift of a full holomorphic map $f_0 : M \to \mathbb{CP}^n$ on an open set U of M; we often view F_0 as a (local) section of $\underline{f}_0$. Let $z_0 \in U$ be a first ramification point of f_0 with ramification index k_1 then

$$F_0 \wedge F_0'(z) = (z - z_0)^{k_1} \, \gamma_1(z),$$

where γ_1 is non-zero at z_0. Definition 2.2.11 and Definition 2.1.10 coincide as Corollary 2.2.10 (ii) gives $F_0 \wedge F_0'(z) = F_0 \wedge A'_{f_0}(F_0)(z)$ and so the zeros of $F_0 \wedge F_0'$ are equal to the zeros of $A'_{f_0}(F_0)$ and have the same order.

Recall from Definition 2.1.10 that if f_0 is also ρth ramified at the point z_0 with ramification index k_ρ; then

$$A'_{f_{\rho-1}}(F_{\rho-1})(z) = (z - z_0)^{k_\rho} F_\rho(z),$$

where $F_{\rho-1}$ is a section of $\underline{f}_{\rho-1}$ non-zero at z_0 and F_ρ a section of $\underline{f}_\rho$ non-zero at z_0.

More generally, let $z_0 \in U$ and let $1 \leq \rho \leq n$. For each j, $1 \leq j \leq \rho$, let k_j be the jth ramification index of $z_0 \in M$, possibly zero. Then using the notation from Corollary 2.2.10 we have

$$A'_{0,\rho}(F_0)(z) = (z - z_j)^{\sum_{j=1}^{\rho} k_j} G_\rho(z),$$

where G_ρ a non-zero section of $\underline{f}_\rho$. Note that f_0 is not jth ramified at z_0 if $k_j = 0$ for some j, $1 \leq j \leq \rho$. Therefore, by Corollary 2.2.10 we have that:

Proposition 2.2.12. *Let $f_0 : M \to \mathbb{CP}^n$ be a full holomorphic map, with holomorphic lift F_0, let $z_0 \in M$ and let $0 \leq \rho \leq n$. For each j with $1 \leq j \leq \rho$, let k_j be the jth ramification index, possibly zero. Then, locally,*

$$\begin{aligned} F_0(z) \wedge F_0'(z) \wedge \cdots \wedge F_0^{(\rho)}(z) &= F_0(z) \wedge A'_{0,1}(F_0)(z) \wedge A'_{0,2}(F_0)(z) \wedge \cdots \wedge A'_{0,\rho}(F_0)(z) \\ &= (z - z_0)^{s_\rho} F_0 \wedge G_1(z) \wedge G_2(z) \wedge \cdots \wedge G_\rho(z), \end{aligned}$$

for $s_\rho = \sum_{j=1}^{\rho} (\rho - j + 1) k_j$ and $F_0 \wedge G_1(z) \wedge G_2(z) \wedge \cdots \wedge G_\rho(z)$ non-zero at z_0 with F_0, G_l, $l \in \{1, \ldots, \rho\}$ defined above.

Remark 2.2.13. *We note that by Proposition 2.2.12, the space B defined by (2.2.1) is the space of all points where A'_{f_j} is zero for some $j \in \{0, 1, \ldots, n\}$. Also from (2.2.2) we have $\kappa_\rho = s_\rho$ and $F_0 \wedge G_1(z) \wedge G_2(z) \wedge \cdots \wedge G_\rho$ is the decomposition of γ.*

Chapter 3

Harmonic 2-Spheres in the Complex Projective Plane and Ramification Points

In this chapter we will focus on harmonic maps $\phi : S^2 \to \mathbb{CP}^2$, in particular on their first-ramification points and how these can be used to describe the space of full harmonic maps. We give some results found in [18, 37] and look at natural questions that arise from [37], concerning the "coalescing of ramification points". We see an ansatz and a counterexample that provide answers to these natural questions. We call a map that is holomorphic or antiholomorphic **$\pm$-holomorphic**. Note that all harmonic maps $S^2 \to \mathbb{CP}^1$ are $\pm$-holomorphic, see [35, 54].

3.1 Harmonic 2-Spheres in the Complex Projective Plane

Let $f : S^2 \to \mathbb{CP}^2$ be a holomorphic map. It is well known that such a map can be represented by a triple of polynomials by first identifying S^2 with $\mathbb{C} \cup \{\infty\}$ via stereographic projection, and defining a map $p : \mathbb{C} \to \mathbb{C}^3 \backslash \{0\}$ by $p(z) = (p_0(z), p_1(z), p_2(z))$, where $(p_0,\ p_1,\ p_2)$ is a triple of coprime polynomials with $\max(\deg(p_0), \deg(p_1), \deg(p_2)) = \deg(f)$. We write $f = [\ p_0, p_1, p_2]$ where the square brackets represent homogeneous coordinates.

Let $f : S^2 \to \mathbb{CP}^2$ be full, by Definition 2.1.4 all harmonic maps that are not full lie in a $\mathbb{CP}^1$ and are holomorphic or antiholomorphic as above, see [35, 54] for details. So all harmonic maps $f : S^2 \to \mathbb{CP}^2$ that are not holomorphic or antiholomorphic are full.

Definition 3.1.1. *We denote by* $\mathrm{Hol}_k^*(\mathbb{CP}^2)$ *the space of all full holomorphic maps from* S^2 *to* $\mathbb{CP}^2$ *of degree* k*.*

Lemma 3.1.2. *The space* $\mathrm{Hol}_k^*(\mathbb{CP}^2)$ *is a complex manifold of dimension* $3k + 2$*.*

Proof. Let $f \in \mathrm{Hol}_k^*(\mathbb{CP}^2)$ and U a neighbourhood of f, then each $g \in U$ can be represented as a triple of coprime polynomials as above. Using the equivalence relation defining the homogeneous coordinates we have that for $g \in U$ where $g = [\ q_0, q_1, q_2]$ and a any non zero coefficient of q_0, q_1 or q_2 then $[\ q_0, q_1, q_2] \sim [\ q_0/a, q_1/a, q_2/a]$. The coefficients of the polynomials $q_0/a, q_1/a, q_2/a$ are complex numbers and so, after disregarding the unit coefficient after division by a, give us a mapping into $\mathbb{C}^{3k+2}$. □

By a first-ramification (resp. second-ramification) point of a holomorphic map f from S^2 to $\mathbb{CP}^2$ we mean a point where the holomorphic map f is first-ramified (resp. second-ramified) as defined in 2.2.11.

Definition 3.1.3. *Let* $f : S^2 \to \mathbb{CP}^2$ *be a full holomorphic map with first-ramification points* $\{z_1, z_2, \ldots, z_l\}$ *with ramification indices* $\{k_1, k_2, \ldots, k_l\}$ *for some* $l \in \mathbb{N}$ *and* $k_j \in \mathbb{N}$ *for all* $j \in \{1, 2, \ldots, l\}$ *then the* ***(first-)ramification divisor*** $R(f)$ *is the monic polynomial*

$$R(f) = \prod_{j=1}^{l} (z - z_j)^{k_j}.$$

Note that, by stereographic projection, we are regarding S^2 as the extended complex plane $\mathbb{C}_\infty$, i.e., $\{z \in \mathbb{C}\}$ with a point at infinity ∞. Also, note that the degree of the ramification divisor is equal to the first total ramification index if f is not ramified at the point at infinity, otherwise it has lower degree.

Consider f_t, a smooth variation in $\mathrm{Hol}^*_k(\mathbb{CP}^2)$ of the full holomorphic map $f : S^2 \to \mathbb{CP}^2$ by a parameter t. Then the first total ramification index or equivalently the degree of the ramification divisor could change for different values of t. This means that finding the first associated curve of the family f_t requires division by the ramification divisor $R(f)$ where the degree of $R(f)$ could change with t which may be a discontinuous process, see [37] and below for examples of when it is discontinuous. Recall the degree of a smooth map defined in Definition 2.1.11.

Definition 3.1.4. *Let* $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ *be the submanifold of* $\mathrm{Hol}^*_k(\mathbb{CP}^2)$ *of full holomorphic maps of degree* k *and first total ramification index* r_0*; we also define* $\mathrm{Harm}_{d,E}(\mathbb{CP}^2)$ *the space of all harmonic maps of degree* d *and energy* $4\pi E$*.*

Theorem 3.1.5. *[37] The map*

$$G'_{k,r_0} : \mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2) \to C^j(S^2, \mathbb{CP}^2)$$

is a smooth embedding onto $\mathrm{Harm}_{d,E}(\mathbb{CP}^2)$ *for any* $j \geq 2$ *where* $d = k - r_0 - 2$ *and* $E = 3k - r_0 - 2$*. Each component* $\mathrm{Harm}_{d,E}(\mathbb{CP}^2)$ *of* $\mathrm{Harm}(\mathbb{CP}^2)$ *is a closed smooth submanifold of* $C^j(S^2, \mathbb{CP}^2)$ *of dimension* $6E + 4$ *if* $E = |d|$ *and of dimension* $2E + 8$ *otherwise.*

Remark 3.1.6. *Here* G'_{k,r_0} *is the restriction to* $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ *of the first* ∂'*-Gauss transform* $G^{(1)}$ *defined in Definition 2.2.5 and shown to be equivalent to* G' *by Corollary 2.2.10.*

Our aim is to better understand the space $\mathrm{Harm}_{d,E}(\mathbb{CP}^2)$. From [18] we know that the first ∂'-Gauss transform maps $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ to $\mathrm{Harm}_{k-r_0-2,3k-r_0-2}(\mathbb{CP}^2)$ homeomorphically and by Theorem 3.1.5 this is a diffeomorphism. Therefore having a better understanding of the space $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ automatically gives us more information about the space $\mathrm{Harm}_{d,E}(\mathbb{CP}^2)$ via the ∂'-Gauss transform. One direction that could be pursued for a better understanding of $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ is to investigate the **coalescing of ramification points**.

3.2 Coalescing of Ramification Points

Definition 3.2.1. *Let f_t be a family of maps in $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ depending smoothly on $-\epsilon < t - t_0 < \epsilon$ where $t_0 \in \mathbb{R}$ and let $2 \le \rho \le r_0$ be an integer. Denote the (first-)ramification points of f_t by $z_i = z_i(t)$, $i = 1, \dots, \rho$. We say f_t has **ramification coalescence at** t_0, or the ramification points **coalesce** as $t \to t_0$, if for a ramification point z_i of f_{t_0} there exists $i_1, \dots, i_\delta$, $2 \le \delta \le \rho$, such that each z_{i_j} $(j \in \{1, \dots, \delta\})$ is a ramification point of f_t and $z_{i_j} \to z$ for all $j \in \{1, \dots, \delta\}$. In this case we say that the z_{i_j} coalesce to z. We also say the set $\{z_1(t), \dots, z_\rho(t)\}$ of ramification points (of f_t) coalesce to $\{z_1(t_0), \dots, z_\rho(t_0)\}$ (note that some elements of the second set may be identical).*

Remark 3.2.2. *Without loss of generality, by choosing a new coordinate z on S^2 by stereographic projection we can suppose that all our ramification points are in one chart and that the point at infinity given by this chart, f_{t_0} has no ramification. In this case the set $\{z_1(t), \dots, z_\rho(t)\}$ of ramification points coalesce to $\{z_1(t_0), \dots, z_{\widetilde{\rho}}(t_0)\}$ if and only if the roots of the (first-)ramification divisor $R(f_t)$ tend to the roots of $R(f_{t_0})$.*

To demonstrate this definition of ramification coalescence it is beneficial to consider an example; we present one which is different from that in [37].

Example 3.2.3. *Identifying S^2 with $\mathbb{C} \cup \{\infty\}$ by stereographic projection and let $f_t : S^2 \to \mathbb{CP}^2$ be the smooth map defined by $f_t(z) = [F_t(z)]$ (as described above), where*

$$F_t(z) = (z^4 - 2t^2z^2 - 1, z^3 - 3t^2z, z^4 - 2t^2z^2),$$

$(z \in \mathbb{C}, t \in \mathbb{R})$. *We have that* $f_t(\infty) = [1, 0, 1]$. *Using the local coordinate* $\hat{z} = \frac{1}{z}$ *centred on the point at infinity* $z = \infty$ *shows this is, in fact, smooth.*

Identifying $\wedge^2\mathbb{C}^3$ *with* $\mathbb{C}^3$, *after a short calculation, we have*

$$(F_t \wedge F_t')(z) = (z^2 - t^2)\psi(z),$$

where the ramification divisor is $R(f_t)(z) = (z^2 - t^2)$ *and*

$$\psi(z) = (z^4 - 6t^2z^2, 4z, 6t^2z^2 - z^4 - 3).$$

Therefore, if $t \neq 0$, *then* f_t *has two ramification points* $z = \pm t$, *both of index 1, but if* $t = 0$ *then these ramification points coalesce into one ramification point* $z = 0$ *of ramification index 2. In this example, the degree of the ramification divisor does not change with* t *and so by [37], the associated curve* $f_{t(1)}$ *varies smoothly with* t. *Note that* $f_t \in \mathrm{Hol}^*_{4,2}(\mathbb{CP}^2)$, *for all* t.

Two natural questions arise from [37]. The first question is, given two finite sets $\{k_1, \ldots, k_\rho\}$ and $\{\tilde{k}_1, \ldots, \tilde{k}_{\tilde{\rho}}\}$ of positive integers such that $r_0 = \sum_{i=1}^{\rho} k_i = \sum_{j=1}^{\tilde{\rho}} \tilde{k}_j$, does there exist a family of maps $f_t \in \mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ that have the points of ramification $\{z_1(t), \ldots, z_\rho(t)\}$ with $z_i(t)$ of ramification index k_i $(i = 1, \ldots, \rho)$, such that, when $t \to t_0$ then $\{z_1(t), \ldots, z_\rho(t)\}$ coalesce to $\{z_1(t_0), \ldots, z_{\tilde{\rho}}(t_0)\}$ with $z_i(t_0)$ of ramification index $\tilde{k}_i$ $(i = 1, \ldots, \tilde{\rho})$? The second question is, can we also specify the points $\{z_1(t), \ldots, z_\rho(t)\}$?

Proposition 3.2.4. *[37] The space* $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ *is non-empty precisely for the range* $k \geq 2, 0 \leq r_0 \leq \frac{3}{2}k - 3$.

Definition 3.2.5. *Let* $f : S^2 \to \mathbb{CP}^2$ *be a full holomorphic map, then the* ***conjugate polar*** h *of* f *is the complex conjugate* $\overline{g}$ *of the polar* g *of* f *defined in Definition 2.2.4 i.e.* $h(z) = \overline{f_{(1)}(z)^{\perp}}$ *where* $z \in S^2$.

Lemma 3.2.6. *[37, Proposition 2.6] For each pair of integers $k \geq 2$, $0 \leq r_0 \leq \frac{3}{2}k - 3$, the map $f \mapsto$ conjugate polar of f restricts to a bijection*

$$\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2) \to \mathrm{Hol}^*_{k',r_1}(\mathbb{CP}^2),$$

where $k' = 2k - r_0 - 2$, $r_1 = 3k - 2r_0 - 6$.

From Lemma 3.2.6 detailed in [37], maps in $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ for $k - 2 \leq r_0 \leq \frac{3}{2}k - 3$ are precisely the conjugate polars of the maps in $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ for $0 \leq r_0 \leq k-2$. We answer the natural questions above by considering ramification coalescence in $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ for these two ranges.

3.2.1 The Lower Range

Let us consider $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ for $0 \leq r_0 \leq k - 2$. We will construct a family of examples in this range that are ramified at any chosen points. Just as above, let $f : S^2 \to \mathbb{CP}^2$ be defined by $f(z) = [F(z)]$, with

$$F(z) = \left(\int R(f)(z)p_0(z)dz, \int R(f)(z)p_1(z)dz, \int R(f)(z)p_2(z)dz \right), \qquad (3.2.1)$$

where $R(f)$ is the desired ramification divisor of f constructed from the given points of ramification, (p_0, p_1, p_2) is a triple of coprime polynomials with at least one of degree $k - r_0 - 1$, the other two of possibly different degrees $\leq k - r_0 - 1$ and $z \in \mathbb{C}$. The maximum degree of the three integrands of (3.2.1) is $r_0 + k - r_0 - 1 = k - 1$, so the degree of f is the maximum degree of the three integrals which is k. Note the constants of integration of each component of (3.2.1) can be chosen so that f is full and the components of f are coprime.

As before identifying $\wedge^2\mathbb{C}^3$ with $\mathbb{C}^3$, we have

$$(F \wedge F')(z) = R(f)(z)\psi(z)$$

and

$$\begin{aligned}\psi(z) = \Big(p_2 \int R(f)(z)p_1(z)dz - p_1 \int R(f)(z)p_2(z)dz,\\ p_0 \int R(f)(z)p_2(z)dz - p_2 \int R(f)(z)p_0(z)dz,\\ p_1 \int R(f)(z)p_0(z)dz - p_0 \int R(f)(z)p_1(z)dz\Big).\end{aligned}$$

Also to ensure that the map f has no ramification at infinity, then the top coefficient of one of the components of $\psi(z)$ needs to be non-zero. Equivalently, one of the components of $\psi(z)$ must be of degree $2k - r_0 - 2$. Let

$$p_0(z) = \sum_{i=0}^{k-r_0-1} a_i z^i, \qquad p_1(z) = \sum_{i=0}^{k-r_0-1} b_i z^i, \qquad p_2(z) = \sum_{i=0}^{k-r_0-1} c_i z^i,$$

then to ensure that f is not ramified at infinity one of the following must be true:

$$\begin{aligned} b_{k-r_0-1}c_{k-r_0-2} - b_{k-r_0-2}c_{k-r_0-1} \neq 0,\\ c_{k-r_0-1}a_{k-r_0-2} - c_{k-r_0-2}a_{k-r_0-1} \neq 0,\\ a_{k-r_0-1}b_{k-r_0-2} - a_{k-r_0-2}b_{k-r_0-1} \neq 0,\end{aligned} \tag{3.2.2}$$

which our polynomials p_0, p_1, p_2 can be chosen to satisfy. Note also that the polynomials and constants of integration of the components of (3.2.1) can be chosen to ensure components of $\psi(z)$ are coprime, so that $R(f)$ really is the ramification divisor.

Lemma 3.2.7. *The ansatz (3.2.1) gives maps* $f = [F] \in \mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ *precisely for the range* $0 \leq r_0 \leq k - 2$.

Proof. Consider (3.2.1) for $R(f)(z)$ a monic polynomial of degree $r_0 = k - 1$. For $f = [F]$ to be of degree k there are only two possibilities:

(i) All p_0, p_1, p_2 have degree 0.

(ii) At least one of p_0, p_1, p_2 has degree 1 with others of possibly smaller degree with the constants of integration all zero (indeed, f is of degree k in this instance as each component of F is divisible by z).

For (i), regardless of the choice of constants of integration the components of F are linearly dependent. More concretely, let $\hat{R}$ be the anti-derivative of $R(f)$, then with arbitrary constants of integration $c_0, c_1, c_2 \in \mathbb{C}$ we have

$$F(z) = \left(\hat{R}(z) + c_0, \hat{R}(z) + c_1, \hat{R}(z) + c_2\right)$$

so that each component of F can be written as a linear combination of the other two. For (ii), if $\deg(p_0), \deg(p_1), \deg(p_2) \leq 1$ with equality holding for at least one of the p_0, p_1, p_2 then the components of F are again linearly dependent. To see this, write

$$p_0(z) = a_1 z + a_0, \qquad p_1(z) = b_1 z + b_0, \qquad p_2(z) = c_1 z + c_0,$$

for $a_0, a_1, b_0, b_1, c_0, c_1 \in \mathbb{C}$. Again, let $\hat{R}$ be the anti-derivative of $R(f)$ and using integration by parts and rearranging we have

$$
\begin{aligned}
F(z) &= \Big(\int R(f)(z)p_0(z)dz, \int R(f)(z)p_1(z)dz, \int R(f)(z)p_2(z)dz \Big) \\
&= \Big(\int R(f)(z)(a_1 z + a_0)dz, \int R(f)(z)(b_1 z + b_0)dz, \int R(f)(z)(c_1 z + c_0)dz \Big) \\
&= \Big(a_1 \Big(z\hat{R}(z) - \int \hat{R}(z)dz \Big) + a_0 \hat{R}(z) \ , \ b_1 \Big(z\hat{R}(z) - \int \hat{R}(z)dz \Big) + b_0 \hat{R}(z), \\
&\qquad c_1 \Big(z\hat{R}(z) - \int \hat{R}(z)dz \Big) + c_0 \hat{R}(z) \Big) \\
&= \begin{pmatrix} a_1 & a_0 \\ b_1 & b_0 \\ c_1 & c_0 \end{pmatrix} \begin{pmatrix} z\hat{R}(z) - \int \hat{R}(z)dz \\ \hat{R}(z) \end{pmatrix},
\end{aligned}
$$

which shows that the components of F are clearly linearly dependent. Also, for $R(f)$ of degree greater than $k-1$, then the degree of f will be greater than k. Therefore the ansatz (3.2.1) cannot be extended to allow $r_0 > k-2$. The examples

$$
F(z) = \Big(\int R(f)(z) z^{k-r_0-1} dz, \int R(f)(z)(z+1)^{k-r_0-2} dz, 1 \Big), \tag{3.2.3}
$$

where $R(f)(z)$ is some monic polynomial of degree r_0, so $p_0 = z^{k-r_0-1}$, $p_1 = (z+1)^{k-r_0-2}$ and $p_2 = 0$ in ansatz (3.2.1) provide maps $f = [F] \in \mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ for $0 \leq r_0 \leq k-2$. This can be seen as by identifying $\wedge^2 \mathbb{C}^3$ with $\mathbb{C}^3$, we have

$$
(F \wedge F')(z) = R(f)(z)\psi(z)
$$

with

$$
\begin{aligned}
\psi(z) = \Big(&-(z+1)^{k-r_0-2}, z^{k-r_0-1} - (z+1)^{k-r_0-2} \int R(f)(z) z^{k-r_0-1} dz, \\
&(z+1)^{k-r_0-2} \int R(f)(z) z^{k-r_0-1} dz - z^{k-r_0-1} \int R(f)(z)(z+1)^{k-r_0-2} dz \Big).
\end{aligned}
$$

Note that this map is not ramified at infinity as the third equation in (3.2.2) is satisfied. □

Remark 3.2.8. *For $r_0 = k-2$ in (3.2.3) above, we have the polynomials $p_0 = z, p_1 = 1$ and $p_2 = 0$ so*

$$F(z) = \Big(\int zR(f)(z)dz, \int R(f)(z)dz, 1\Big).$$

The components of F are clearly coprime. For any constants of integration the first component of F is of degree k, the second component of degree $k-1$ and the third component of degree 0. Let $s_0, s_1, s_2 \in \mathbb{C}$, by equating coefficients we see that

$$s_0 \int zR(f)(z)dz + s_1 \int R(f)(z)dz + s_2 = 0,$$

if and only if $s_0 = s_1 = s_2 = 0$ therefore the components of F are linearly independent. We have that the corresponding $f = [F]$ has degree k and is a full holomorphic map into $\mathbb{CP}^2$ so $f \in \mathrm{Hol}^*_k(\mathbb{CP}^2)$.

We differentiate $F(z)$ to get

$$F'(z) = (zR(f)(z), R(f)(z), 0),$$

and by identifying $\wedge^2\mathbb{C}^3$ with $\mathbb{C}^3$, we have

$$\begin{aligned}(F\wedge F')(z) &= (-R(f)(z), zR(f)(z), R(f)(z)\int zR(f)(z)dz - zR(f)(z)\int R(f)(z)dz)\\ &= R(f)(z)(-1, z, \int zR(f)(z)dz - z\int R(f)(z)dz)\\ &= R(f)(z)\psi(z)\end{aligned}$$

where $\psi(z) = (-1, z, \int zR(f)(z)dz - z\int R(f)(z)dz)$. As the third component of ψ is of degree $2k - r_0 - 2 = 2k - (k-2) - 2 = k$, or equivalently as the third equation of (3.2.2) is satisfied, then $f = [F]$ has no ramification at infinity. Therefore $f = [F]$ is, in fact, a map in $\mathrm{Hol}^*_{k,k-2}(\mathbb{CP}^2)$ and so in both the lower range, $0 \le r_0 \le k-2$, and the *upper range,* $k - 2 \leq r_0 \leq \frac{3}{2}k - 3$.

We see that by using the ansatz (3.2.1) we have

Proposition 3.2.9. *For any number of given points and ramification indices with sum* r_0 *where* $0 \leq r_0 \leq k-2$ *we can construct, using (3.2.1), a map* $f \in \mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$*, that is ramified at those points with the chosen ramification indices.*

Example 3.2.10. *Let* 0 *and* $-i$ *be the points in which we wish our degree* 6 *map* f *is to be ramified with ramification index* 2 *and* 1*, respectively. Then using the method described above the ramification divisor is* $R(f)(z) = z^2(z+i)$ *and using one of the simplest choices of polynomials fulfilling the criteria above:* $p_0 = z^2, p_1 = z, p_2 = 1$ *we can construct, using the ansatz (3.2.1), a map* $f \in \mathrm{Hol}^*_{6,3}(\mathbb{CP}^2)$*, ramified at the points given above, given by* $f = [F]$ *where*

$$F(z) = \left(\frac{1}{6}z^6 + \frac{i}{5}z^5,\ \frac{1}{5}z^5 + \frac{i}{4}z^4 + 1,\ \frac{1}{4}z^4 + \frac{i}{3}z^3\right).$$

Note that the constants of integration we have chosen are $c_0 = 0, c_1 = 1$ *and* $c_2 = 0$ *where* c_j *is the constant of integration of* $\int R(f)(z)p_j(z)dz$*. Also note that* f *is not ramified at infinity as the polynomials* p_0, p_1, p_2 *satisfy at least one of (3.2.2) above. The exterior product is given by*

$$(F \wedge F')(z) = R(f)(z)\psi(z),$$

where

$$\psi(z) = \left(-\frac{1}{20}z^5 - \frac{i}{12}z^4 + 1,\ \frac{1}{12}z^6 + \frac{2i}{15}z^5,\ -\frac{1}{30}z^7 - \frac{i}{20}z^6 - z^2\right).$$

Therefore $f(z) = [F(z)]$ *is a map that has the points of ramification chosen above and the components of* $\psi(z)$ *are coprime, e.g., as is easily checked by substituting the zeros of the second component into the others, meaning that there are no further points of ramification.*

Further to Proposition 3.2.9 we can construct a family of maps in $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ for $0 \leq r_0 \leq k-2$ that has ramification coalescence at any point. This can be done by constructing a smooth family of ramification divisors dependent on $t \in [0,1]$ that acts as a curve connecting two ramification divisors. For example let our ramification divisors be R_0 and R_1 both of the same degree, then $R_t = tR_1 + (1-t)R_0$ is a curve connecting the two. Following the above we can construct a smooth family of maps f_t such that, for $t = 0$, f_0 has ramification divisor R_0 and for $t = 1$, f_1 has ramification divisor R_1. Therefore for any "configuration" of ramification points that coalesce we can construct a smooth family of maps that has those points of ramification coalescing at given values of t.

Remark 3.2.11. *If R_0 and R_1 are not of the same degree then the associated curve $f_{t(1)}$ and first Gauss transform $G^{(1)}(f_t)$ are not continuous in t, if R_0 and R_1 are of the same degree then $f_{t(1)}$ and $G^{(1)}(f_t)$ are continuous in t [37].*

Example 3.2.12. *Let $2, 3$ be the points in which our degree 4 map f_1 is to be ramified and let us ask that these points coalesce into a ramification point at 0 of ramification index 2 for f_0. We have our smooth family of ramification divisors $R(f_t)(z) = t(z-2)(z-3) + (1-t)z^2$ and after one of the simplest choices of polynomials fulfilling the criteria above: $p_0 = z, p_1 = z, p_2 = 1$ we can construct, using the ansatz (3.2.1), a smooth family of maps $f_t = [F_t] \in \mathrm{Hol}^*_{4,2}(\mathbb{CP}^2)$ with prescribed ramification coalescence:*

$$F_t(z) = \left(\frac{1}{4}z^4 - \frac{5t}{3}z^3 + 3tz^2 + 1,\ \frac{1}{4}z^4 - \frac{5t}{3}z^3 + 3tz^2,\ \frac{1}{3}z^3 - \frac{5t}{2}z^2 + 6tz\right).$$

The exterior product is given by

$$(F_t \wedge F_t')(z) = R(f_t)(z)\psi_t(z)$$

for

$$\psi_t(z) = \left(-\frac{1}{12}z^4 + \frac{5t}{6}z^3 - 3tz^2,\ \frac{1}{12}z^4 - \frac{5t}{6}z^3 + 3tz^2 - 1,\ z\right).$$

Therefore $f_t(z) = [F_t(z)]$ is a smooth family of maps that have the ramification coales-

cence chosen above and the components of $\psi_t(z)$ are coprime which means there are no further points of ramification.

Proposition 3.2.13. *Given $r_0 \in \mathbb{N}$ with $0 \leq r_0 \leq k-2$ and a finite set of pairs $\{(z_i, k_i)\}$ where $z_i \in S^2$ and k_i is a positive integer with $\sum_{i=1}^{\rho} k_i = r_0$, there exists a holomorphic map $f : S^2 \to \mathbb{CP}^2$ that is ramified at $z_1, \ldots, z_\rho$, with ramification indices $k_1, \ldots, k_\rho$, respectively. Also, for any given points $\{z_1(t_0), \ldots, z_\rho(t_0)\}$ there exists a smooth 1-parameter family of holomorphic maps $f_t : S^2 \to \mathbb{CP}^2$ with ramification coalescence at those points (see Definition 3.2.1).*

3.2.2 The Upper Range

To answer fully the natural questions from [37] posed above we also need to consider $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ for $k-2 \leq r_0 \leq \frac{3}{2}k-3$. We will find two counterexamples for this upper range of ramification index that show that not all ramification is possible in the upper range.

Example 3.2.14. *We prove that there does not exist a map $\hat{f} \in \mathrm{Hol}^*_{4,3}(\mathbb{CP}^2)$ that is ramified of order 3 at zero.*

By Lemma 3.2.6 the conjugate polar map of Definition 3.2.5 restricts to a bijection $\mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2) \to \mathrm{Hol}^*_{k',r_1}(\mathbb{CP}^2)$ for $k \geq 2, 0 \leq r_0 \leq \frac{3}{2}k-3$ where $k' = 2k - r_0 - 2, r_1 = 3k - 2r_0 - 6$ (which is in fact an application of (4.1.7) below). Let $f \in \mathrm{Hol}^*_{3,0}(\mathbb{CP}^2)$ be the conjugate polar of $\hat{f} \in \mathrm{Hol}^*_{4,3}(\mathbb{CP}^2)$ and write $f = [F] = [p_0, p_1, p_2]$. Due to f not having any ramification points, $F \wedge F'$ defines the map $\hat{f}$ and so to find the associated curve $\hat{f}_{(1)}$ of $\hat{f}$ we use the wedge product $(F \wedge F') \wedge (F \wedge F')' = (F \wedge F') \wedge (F' \wedge F' + F \wedge F'') = F \wedge F' \wedge F''$ (so $\hat{f}_{(1)}$ is the second associated curve of f) and by (3.1.3) the greatest *common divisor of this is the first-ramification divisor of $\hat{f}$. Write*

$$p_0(z) = \sum_{i=0}^{3} a_i z^i, \qquad p_1(z) = \sum_{i=0}^{3} b_i z^i, \qquad p_2(z) = \sum_{i=0}^{3} c_i z^i,$$

then

$$F \wedge F' \wedge F'' = \det \begin{pmatrix} p_0 & p_1 & p_2 \\ p_0' & p_1' & p_2' \\ p_0'' & p_1'' & p_2'' \end{pmatrix} = 2\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} z^3 + 6\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} z^2$$
$$+ 6\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{pmatrix})z + 2\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix}.$$

We see that the coefficients of $F \wedge F' \wedge F''$ are linear combinations of the 3×3 matrix minors of the matrix

$$\begin{pmatrix} a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix}. \tag{3.2.4}$$

Now let us assume that $\hat{f}$ is ramified at $z = 0$ of order 3 (so f is second-ramified at $z = 0$ of order 3), then $z^3 \big| F \wedge F' \wedge F''$ and therefore we have that the coefficients of the terms of degree 0 to 2 of $F \wedge F' \wedge F''$ must be zero. Therefore all but one matrix minor must be zero. For ease let us write

$$\ell_{ijk} = \det \begin{pmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{pmatrix} \qquad 0 \le i, j, k \le 3.$$

Then $z^3 \big| F \wedge F' \wedge F''$ if and only if $\ell_{012} = \ell_{013} = \ell_{023} = 0$. Without loss of generality, *using that* $\ell_{023} = 0$*, let*

$$\sigma_3 = \alpha_0\sigma_0 + \alpha_1\sigma_2 \qquad \textit{for} \qquad \alpha_0, \alpha_1 \in \mathbb{C} \qquad \textit{and} \qquad \sigma_i = (a_i, b_i, c_i). \tag{3.2.5}$$

Now as $\ell_{012} = 0$ *we have one of the following equations:*

$$\begin{aligned} \sigma_0 &= \alpha_2\sigma_1 + \alpha_3\sigma_2, \\ \sigma_1 &= \alpha_4\sigma_0 + \alpha_5\sigma_2, \\ \sigma_2 &= \alpha_6\sigma_0 + \alpha_7\sigma_1, \end{aligned} \tag{3.2.6}$$

where $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \in \mathbb{C}$*. Each equation (3.2.6) together with (3.2.5) implies* $\ell_{123} = 0$*. Therefore all* 3×3 *matrix minors of (3.2.4) are zero, which implies that* f *is not a full map, a contradiction.*

We present another counterexample using similar arguments to the counterexample above.

Example 3.2.15. *We prove that there does not exist a map* $\hat{f} \in \mathrm{Hol}^*_{6,6}(\mathbb{CP}^2)$ *that is ramified of order* 5 *at zero.*

Let f *be the conjugate polar of* $\hat{f} \in \mathrm{Hol}^*_{6,6}(\mathbb{CP}^2)$ *so* $f \in \mathrm{Hol}^*_{4,0}(\mathbb{CP}^2)$*, and we write* $f = [F] = [p_0, p_1, p_2]$*. Due to* f *not having any ramification points,* $F \wedge F'$ *defines the map* $\hat{f}$ *and so to find the associated curve* $\hat{f}_{(1)}$ *of* $\hat{f}$ *we again use the wedge product* $F \wedge F' \wedge F''$ *(so* $\hat{f}_{(1)}$ *is the second associated curve of* f*) and again by (3.1.3) the greatest common divisor of this is the ramification divisor of* $\hat{f}$*. Write*

$$p_0(z) = \sum_{i=0}^{4} a_i z^i, \qquad p_1(z) = \sum_{i=0}^{4} b_i z^i, \qquad p_2(z) = \sum_{i=0}^{4} c_i z^i,$$

then

$$
\begin{aligned}
F \wedge F' \wedge F'' = \det \begin{pmatrix} p_0 & p_1 & p_2 \\ p'_0 & p'_1 & p'_2 \\ p''_0 & p''_1 & p''_2 \end{pmatrix} &= 2\det \begin{pmatrix} a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix} z^6 + 6\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix} z^5 \\
&+ \left(12\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_3 & b_3 & c_3 \\ a_4 & b_4 & c_4 \end{pmatrix} + 6\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{pmatrix} \right) z^4 \\
&+ \left(16\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 \\ a_4 & b_4 & c_4 \end{pmatrix} + 2\det \begin{pmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \right) z^3 \\
&+ \left(12\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_4 & b_4 & c_4 \end{pmatrix} + 6\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{pmatrix} \right) z^2 \\
&+ 6\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_3 & b_3 & c_3 \end{pmatrix} z + 2\det \begin{pmatrix} a_0 & b_0 & c_0 \\ a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \end{pmatrix} .
\end{aligned}
$$

We see that the coefficients of $F \wedge F' \wedge F''$ *are linear combinations of the* 3×3 *matrix minors of the matrix*

$$
\begin{pmatrix} a_4 & b_4 & c_4 \\ a_3 & b_3 & c_3 \\ a_2 & b_2 & c_2 \\ a_1 & b_1 & c_1 \\ a_0 & b_0 & c_0 \end{pmatrix}, \tag{3.2.7}
$$

and so are subject to some Plücker relations [30]. Similarly to before let us write

$$\ell_{ijk} = \det \begin{pmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{pmatrix} \qquad 0 \leq i, j, k \leq 4.$$

Then the Plücker relations for this situation are

$$\ell_{124}\ell_{034} - \ell_{024}\ell_{134} + \ell_{014}\ell_{234} = 0, \tag{3.2.8}$$

$$\ell_{123}\ell_{034} - \ell_{023}\ell_{134} + \ell_{013}\ell_{234} = 0,$$

$$\ell_{123}\ell_{024} - \ell_{023}\ell_{124} + \ell_{012}\ell_{234} = 0, \tag{3.2.9}$$

$$\ell_{123}\ell_{014} - \ell_{013}\ell_{124} + \ell_{012}\ell_{134} = 0,$$

$$\ell_{023}\ell_{014} - \ell_{013}\ell_{024} + \ell_{012}\ell_{034} = 0. \tag{3.2.10}$$

Now let us assume that $\hat{f}$ is ramified at $z = 0$ of order 5 (so f is second-ramified at $z = 0$ of order 5), then $z^5 \big| F \wedge F' \wedge F''$ and therefore we have that the coefficients of the terms of degree 0 to 4 of $F \wedge F' \wedge F''$ must be zero. This amounts to another set of equations:

$$6\ell_{012} = 0, \tag{3.2.11}$$

$$6\ell_{013} = 0, \tag{3.2.12}$$

$$12\ell_{014} + 6\ell_{023} = 0, \tag{3.2.13}$$

$$16\ell_{024} + 2\ell_{123} = 0, \tag{3.2.14}$$

$$12\ell_{034} + 6\ell_{124} = 0. \tag{3.2.15}$$

We have from (3.2.11) and (3.2.12) that $\ell_{012} = \ell_{013} = 0$, substituting into (3.2.10) and together with (3.2.13) we have $\ell_{023} = \ell_{014} = 0$, now substituting into (3.2.9) and together with (3.2.14) we have $\ell_{123} = \ell_{024} = 0$, now finally substituting into (3.2.8) and together *with (3.2.15) we have* $\ell_{124} = \ell_{034} = 0$*. Without loss of generality, using that* $\ell_{034} = 0$*, let*

$$\sigma_4 = \alpha_0\sigma_0 + \alpha_1\sigma_3 \quad \text{for} \quad \alpha_0, \alpha_1 \in \mathbb{C} \quad \text{and} \quad \sigma_i = (a_i, b_i, c_i). \tag{3.2.16}$$

Now as $\ell_{013} = 0$ *we have one of the following equations:*

$$\begin{aligned} \sigma_0 &= \alpha_2\sigma_1 + \alpha_3\sigma_3, \\ \sigma_1 &= \alpha_4\sigma_0 + \alpha_5\sigma_3, \\ \sigma_3 &= \alpha_6\sigma_0 + \alpha_7\sigma_1, \end{aligned} \tag{3.2.17}$$

where $\alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6, \alpha_7 \in \mathbb{C}$*. Each equation (3.2.17) together with (3.2.16) implies* $\ell_{134} = 0$*. Also, as* $\ell_{023} = 0$ *we have one of the following equations:*

$$\begin{aligned} \sigma_0 &= \beta_0\sigma_2 + \beta_1\sigma_3, \\ \sigma_2 &= \beta_2\sigma_0 + \beta_3\sigma_3, \\ \sigma_3 &= \beta_4\sigma_0 + \beta_5\sigma_2, \end{aligned} \tag{3.2.18}$$

where $\beta_0, \beta_1, \beta_2, \beta_3, \beta_4, \beta_5 \in \mathbb{C}$*. Each equation (3.2.18) together with (3.2.16) implies* $\ell_{234} = 0$*. Therefore all* 3×3 *matrix minors of (3.2.7) are zero, which implies that* f *is not a full map, a contradiction.*

Hence, we have proved:

Proposition 3.2.16. *There does not exist a holomorphic map in* $\mathrm{Hol}^*_{4,3}(\mathbb{CP}^2)$ *that is ramified of order* 3 *at* 0*. Nor does there exist a holomorphic map in* $\mathrm{Hol}^*_{6,6}(\mathbb{CP}^2)$ *that is ramified of order* 5 *at* 0*.*

Remark 3.2.17. *It does not seem easy to generalise Example 3.2.14 and 3.2.15 as the procedure relies on starting with a map* $f \in \mathrm{Hol}^*_{k,r_0}(\mathbb{CP}^2)$ *which has an unramified conjugate polar. Maps satisfying this have even degree* k *and total ramification index* $r_0 = \frac{3}{2}k - 3$*.*

The next map satisfying this criterion is $f \in \mathrm{Hol}^*_{8,9}(\mathbb{CP}^2)$ *which has* 20 *matrix minors and* 35 *Plücker relations.*

Chapter 4

On the Index of Harmonic Maps from Surfaces into a Complex Projective Space

In [28] an estimate was given for the index of non-$\pm$-holomorphic harmonic maps $\phi : M_g \to \mathbb{CP}^n$ where M_g is a closed Riemann surface of genus g. As before, we call a map that is holomorphic or antiholomorphic $\pm$-holomorphic.

Proposition 4.0.1. *[28] Let $\phi : M_g \to \mathbb{CP}^n$ be a non-$\pm$-holomorphic harmonic map. Then*

$$\mathrm{index}(\phi) \geq \deg(\phi)(n+1) + n(1-g).$$

Here $\deg(\phi)$ denotes the degree of ϕ on second cohomology as defined in Definition 2.1.11.

In this chapter we shall give improvements to this estimate for genus 0, for complex isotropic (see Definition 2.1.6) harmonic maps for genus 1, and in some cases for complex isotropic harmonic maps for higher genus. Elements of this chapter were written up in the paper [41].

4.1 The Space of Holomorphic Sections and Holomorphic Differentials

We familiarise the reader with definitions that we use to improve Proposition 4.0.1.

Definition 4.1.1. *[31, §5] Let X be a smooth manifold. A (smooth)* ***complex vector bundle*** *(of rank k) on X consists of a family of k-dimensional complex vector spaces $\{E_x\}_{x\in X}$ parametrized by X, together with a smooth manifold structure on $E = \cup_{x\in X} E_x$, such that:*

(i) *The projection map $\pi : E \to X$, $\pi(E_x) = x$, is smooth, and*

(ii) *any point $x_0 \in X$ has an open neighbourhood U, such that there exists a diffeomorphism*

$$\phi_U : \pi^{-1}(U) \to U \times \mathbb{C}^k,$$

taking the vector space E_x isomorphically onto $\{x\} \times \mathbb{C}^k$, called a ***local trivialisation***

Definition 4.1.2. *Let M be a Riemann surface with open covering $(U_i)_{i\in I}$. Let $\pi : E \to M$ be a complex vector bundle of rank k on M and*

$$\mathfrak{A} = \{\phi_i : \pi^{-1}(U_i) \to U_i \times \mathbb{C}^n \mid i \in I\}$$

be an atlas for E i.e. a collection of local trivialisations. Let

$$\varphi_{ij} := \phi_i \circ \phi_j^{-1} : (U_i \cap U_j) \times \mathbb{C}^n \to (U_i \cap U_j) \times \mathbb{C}^n.$$

The atlas $\mathfrak{A}$ is ***holomorphic*** *if the associated transition functions*

$$g_{ij} : U_i \cap U_j \to GL(n, \mathbb{C})$$

defined by

$$\varphi_{ij}(x,t) = (x, g_{ij}(x)t) \qquad \textit{for} \qquad (x,t) \in (U_i \cap U_j) \times \mathbb{C}^n$$

*are holomorphic. Two atlases $\mathfrak{A}$ and $\mathfrak{A}'$ are **holomorphically compatible** if $\mathfrak{A} \cup \mathfrak{A}'$ is a holomorphic atlas. The equivalence class of all holomorphically compatible atlases is called a **holomorphic structure**.*

Remark 4.1.3. *For a complex vector bundle, the transition functions are just smooth.*

Definition 4.1.4. *A **holomorphic vector bundle** on a Riemann surface M is a complex vector bundle together with a holomorphic structure.*

On a holomorphic vector bundle, we have a $\overline{\partial}$-operator [31, §5]. By [36] any complex vector bundle $E \to M$ over a Riemann surface equipped with a linear connection ∇ can be given a unique holomorphic structure (called the Koszul-Malgrange holomorphic structure) with $\overline{\partial}$-operator equal to the $(0,1)$-part of ∇.

The estimate in Proposition 4.0.1 was constructed by noting that given a holomorphic vector field along ϕ we have a smooth variation of ϕ that contributes to the index of ϕ:

Lemma 4.1.5. *[28, p. 258] Let $\phi : M \to \mathbb{CP}^n$ be a non-$\pm$-holomorphic harmonic map then*

$$\mathrm{index}(\phi) \geq \dim H^0(M, \phi^{-1}T^{(1,0)}\mathbb{CP}^n)$$

where $H^0(M, \phi^{-1}T^{(1,0)}\mathbb{CP}^n)$ is the space of holomorphic sections of $\phi^{-1}T^{(1,0)}\mathbb{CP}^n$ defined on the whole of M.

Here $\dim$ denotes complex dimension. Let M_g be a closed Riemann surface of genus g.

Theorem 4.1.6 (Riemann–Roch [33])**.** *Let $W \to M_g$ be a holomorphic vector bundle of rank n over a Riemann surface M_g of genus g then*

$$\dim H^0(M_g, W) - \dim H^1(M_g, W) = c_1(\wedge^n W) + n(1-g),$$

where c_1 is the first Chern class (evaluated on the canonical generator of $H_2(M_g, \mathbb{Z})$) and $H^1(M_g, W)$ is (by Serre duality) the space of holomorphic (1,0)-forms of M_g with values in the dual, W^*, of W [33, Theorem 9].

Corollary 4.1.7. *Let $W \to M_g$ be a complex vector bundle over a Riemann surface M_g which can be given more than one distinct holomorphic structure, then* $\dim H^0(M, W) - \dim H^1(M, W)$ *is independent of the choice of holomorphic structure.*

Proof. By Theorem 4.1.6 we have $\dim H^0(M_g, W) - \dim H^1(M_g, W) = c_1(\wedge^n W) + n(1 - g)$, where the right-hand side depends only on the complex structure. □

Let $\phi^{-1}T^{(1,0)}\mathbb{CP}^n \to M_g$ be the pullback of the complex vector bundle $T^{(1,0)}\mathbb{CP}^n \to \mathbb{CP}^n$ of rank n. Give the former the Koszul-Malgrange holomorphic structure from the $(0, 1)$-part of the pullback of the Levi-Civita connection defined on $\mathbb{CP}^n$. Let $\phi : M_g \to \mathbb{CP}^n$ be a non-$\pm$-holomorphic harmonic map. Then using Riemann-Roch for the holomorphic vector bundle $\phi^{-1}T^{(1,0)}\mathbb{CP}^n \to M_g$ of rank n we get

$$\dim H^0(M_g, \phi^{-1}T^{(1,0)}\mathbb{CP}^n) - \dim H^1(M_g, \phi^{-1}T^{(1,0)}\mathbb{CP}^n) = \deg(\phi)(n + 1) + n(1 - g)$$

and Proposition 4.0.1 follows directly by disregarding the non-negative number $\dim H^1(M_g, \phi^{-1}T^{(1,0)}\mathbb{CP}^n)$. We improve the estimate in Proposition 4.0.1 by looking at $\dim H^0(M_g, \phi^{-1}T^{(1,0)}\mathbb{CP}^n)$ more closely and finding an improved estimate for its dimension by using the connection-preserving isomorphism (2.1.4).

Considering the harmonic sequence (2.1.5) above:

Definition 4.1.8. *[12, 28] We say a full harmonic map $\phi : M_g \to \mathbb{CP}^n$ has **directrix** (f, ρ) if $\phi = G^{(\rho)}(f)$ for $\rho \in \{0, 1, \ldots, n\}$ and $f : M_g \to \mathbb{CP}^n$ a full holomorphic map.*

This is possible if and only if ϕ is complex isotropic (see Lemma 2.1.8). Given a harmonic map $\phi : M_g \to \mathbb{CP}^n$ with directrix (f, ρ) then by (2.1.4) we have the following decomposition into complex vector bundles:

$$\begin{aligned}\phi^{-1}T^{(1,0)}\mathbb{CP}^n &\cong L(\underline{f}_\rho, \underline{f}_\rho^\perp)\\ &= L(\underline{f}_\rho, \underline{f}_0 \oplus \underline{f}_1 \oplus \cdots \oplus \underline{f}_{\rho-1} \oplus \underline{f}_{\rho+1} \oplus \cdots \oplus \underline{f}_n)\\ &\cong L(\underline{f}_\rho, \underline{f}_0) \oplus \cdots \oplus L(\underline{f}_\rho, \underline{f}_{\rho-1}) \oplus L(\underline{f}_\rho, \underline{f}_{\rho+1}) \oplus \cdots \oplus L(\underline{f}_\rho, \underline{f}_n)\\ &= A_- \oplus A_+,\end{aligned}$$

where

$$A_- = \sum_{j=0}^{\rho-1} L(\underline{f}_\rho, \underline{f}_j), \qquad A_+ = \sum_{j=\rho+1}^{n} L(\underline{f}_\rho, \underline{f}_j).$$

To respect the Leibniz rule, we give $L(\underline{f}_\rho, \underline{f}_\rho^\perp)$ the connection ${}^L\nabla$ induced from $\underline{f}_\rho$ and $\underline{f}_\rho^\perp$ defined by

$$({}^L\nabla_{\frac{\partial}{\partial \bar{z}}} u)(s) = {}^{f_\rho^\perp}\nabla_{\frac{\partial}{\partial \bar{z}}} u(s) - u({}^{f_\rho}\nabla_{\frac{\partial}{\partial \bar{z}}} s), \tag{4.1.1}$$

where ${}^{f_\rho^\perp}\nabla$ and ${}^{f_\rho}\nabla$ are the connections defined in (2.1.3) on $\underline{f}_\rho^\perp$ and $\underline{f}_\rho$, respectively, $u \in \Gamma(L(\underline{f}_\rho, \underline{f}_\rho^\perp))$ and $s \in \Gamma(\underline{f}_\rho)$. We then give $L(\underline{f}_\rho, \underline{f}_\rho^\perp)$ the Koszul-Malgrange holomorphic structure from the $(0,1)$-part of that connection ${}^L\nabla$. As (2.1.1) is connection-preserving then $\phi^{-1}T^{(1,0)}\mathbb{CP}^n \cong L(\underline{f}_\rho, \underline{f}_\rho^\perp)$ is an isomorphism of holomorphic vector bundles where they are both given the Koszul-Malgrange holomorphic structures defined from their respective connections.

Lemma 4.1.9. *Let $L(\underline{f}_\rho, \underline{f}_\rho^\perp)$ be the holomorphic vector bundle over a Riemann surface M_g defined above, then A_+ and A_- are both holomorphic subbundles of $L(\underline{f}_\rho, \underline{f}_\rho^\perp)$.*

Proof. It suffices to show that both $\Gamma(A_+)$ and $\Gamma(A_-)$ are closed under ${}^L\nabla_{\frac{\partial}{\partial \bar{z}}}$. To show this we let $u \in \Gamma(A_+)$ and $s \in \Gamma(\underline{f}_\rho)$ then by (4.1.1) ,

$$({}^L\nabla_{\frac{\partial}{\partial \bar{z}}} u)(s) = {}^{f_\rho^\perp}\nabla_{\frac{\partial}{\partial \bar{z}}} u(s) - u({}^{f_\rho}\nabla_{\frac{\partial}{\partial \bar{z}}} s).$$

As $u(s) \in \Gamma(\sum_{j=\rho+1}^n \underline{f}_j)$, according to [2, p. 603] for suitable complex-valued functions

a and b,

$$\frac{\partial}{\partial \bar{z}} u(s) = a u(s) + b\nu,$$

where $\nu \in \Gamma(\sum_{j=\rho}^{n} \underline{f}_j)$. Therefore, by the definition of ${}^{f_\rho^\perp}\nabla$,

$${}^{f_\rho^\perp}\nabla_{\frac{\partial}{\partial \bar{z}}} u(s) = \pi_{f_\rho}^{\perp} \frac{\partial}{\partial \bar{z}} u(s) \in \Gamma(\sum_{j=\rho+1}^{n} \underline{f}_j).$$

Also, as ${}^{f_\rho}\nabla_{\frac{\partial}{\partial \bar{z}}} s \in \Gamma(\underline{f}_\rho)$ and since $u \in \Gamma(A_+) = \Gamma(\sum_{j=\rho+1}^{n} L(\underline{f}_\rho, \underline{f}_j))$ then $u({}^{f_\rho}\nabla_{\frac{\partial}{\partial \bar{z}}} s) \in \Gamma(\sum_{j=\rho+1}^{n} \underline{f}_j)$ also. Hence ${}^{L}\nabla_{\frac{\partial}{\partial \bar{z}}} u \in \Gamma(A_+)$. A similar argument can be made for A_-. □

For each $j \in \{\rho+1, \dots, n\}$, $L(\underline{f}_\rho, \underline{f}_j)$ is a complex subbundle of $L(\underline{f}_\rho, \underline{f}_\rho^\perp)$ and so can be given an induced (subbundle) holomorphic structure, i.e. that with $\overline{\partial}$-operator given by $\pi_{f_j}{}^{L}\nabla_{\frac{\partial}{\partial \bar{z}}}$. Using this we give A_+ a second 'direct sum' holomorphic structure $\overline{\partial}_{\text{sum}}$ defined by

$$\overline{\partial}_{\text{sum}}(\sigma) = \sum_{j=\rho+1}^{n} \pi_{f_j}{}^{L}\nabla_{\frac{\partial}{\partial \bar{z}}}(\sigma_j) \tag{4.1.2}$$

for $\sigma = \sigma_{\rho+1} + \sigma_{\rho+2} + \cdots + \sigma_n \in \Gamma(A_+)$ and $\sigma_j \in \Gamma(L(\underline{f}_\rho, \underline{f}_j))$ for all $j \in \{\rho+1, \dots, n\}$.

Lemma 4.1.10. *Let A_+ be the holomorphic bundle over Riemann surface M defined above equipped with the holomorphic structure $\overline{\partial}_{sum}$ and let each complex subbundle $L(\underline{f}_\rho, \underline{f}_j)$ of $L(\underline{f}_\rho, \underline{f}_\rho^\perp)$ be equipped with the induced (subbundle) holomorphic structure as above. Then*

(i) $H^0(M, A_+) = \sum_{j=\rho+1}^{n} H^0(M, L(\underline{f}_\rho, \underline{f}_j))$,

(ii) $H^1(M, A_+) = \sum_{j=\rho+1}^{n} H^1(M, L(\underline{f}_\rho, \underline{f}_j))$.

Proof. (i) Let $\sigma \in \Gamma(A_+)$ then σ may be decomposed uniquely as $\sigma = \sigma_{\rho+1} + \sigma_{\rho+2} + \cdots + \sigma_n$ where $\sigma_j \in \Gamma(L(\underline{f}_\rho, \underline{f}_j))$ for all $j \in \{\rho+1, \dots, n\}$. As $\overline{\partial}_{\text{sum}}|_{L(\underline{f}_\rho, \underline{f}_j)} =$

$\pi_{f_j}{}^L\nabla_{\frac{\partial}{\partial \bar{z}}}$ then $L(\underline{f}_\rho, \underline{f}_j)$ are holomorphic subbundles of $(A_+, \overline{\partial}_{\text{sum}})$, so

$$\begin{aligned}\overline{\partial}_{\text{sum}}(\sigma) &= \overline{\partial}_{\text{sum}}(\sigma_{\rho+1}) + \overline{\partial}_{\text{sum}}(\sigma_{\rho+2}) + \cdots + \overline{\partial}_{\text{sum}}(\sigma_n) \\ &= \pi_{f_{\rho+1}}{}^L\nabla_{\frac{\partial}{\partial \bar{z}}}(\sigma_{\rho+1}) + \pi_{f_{\rho+2}}{}^L\nabla_{\frac{\partial}{\partial \bar{z}}}(\sigma_{\rho+2}) + \cdots + \pi_{f_n}{}^L\nabla_{\frac{\partial}{\partial \bar{z}}}(\sigma_n).\end{aligned}$$

Therefore $\sigma \in H^0(M, A_+)$ if and only if $\sigma_j \in H^0(M, L(\underline{f}_\rho, \underline{f}_j))$ for each $j \in \{\rho+1, \ldots, n\}$.

(ii) Using Serre duality [33, Theorem 9] we have $H^1(M, A_+) \cong H^0(M, A_+^* \otimes T^*M_g)$, then (ii) follows from (i).

□

Proposition 4.1.11. *Let M be a Riemann surface, and $\phi : M \to \mathbb{CP}^n$ a harmonic map with directrix (f, ρ); let $A_+ = \sum_{j=\rho+1}^n L(\underline{f}_\rho, \underline{f}_j)$ be equipped with the holomorphic structure $\overline{\partial}_{sum}$ and let $L(\underline{f}_\rho, \underline{f}_j)$ be holomorphic subbundles of $L(\underline{f}_\rho, \underline{f}_\rho^\perp)$ equipped with the induced (subbundle) holomorphic structures. Then*

$$\dim H^0(M, \phi^{-1}T^{(1,0)}\mathbb{CP}^n) \geq \sum_{j=\rho+1}^{n} \dim H^0(M, L(\underline{f}_\rho, \underline{f}_j)) - \dim H^1(M, L(\underline{f}_\rho, \underline{f}_j)).$$

Proof. Recall that $\phi^{-1}T^{(1,0)}\mathbb{CP}^n \cong L(\underline{f}_\rho, \underline{f}_\rho^\perp)$ is an isomorphism of holomorphic vector bundles where $\phi^{-1}T^{(1,0)}\mathbb{CP}^n$ and $L(\underline{f}_\rho, \underline{f}_\rho^\perp)$ have Koszul-Malgrange holomorphic structures defined from their respective connections. By Lemma 4.1.9, A_+ and A_- are holomorphic subbundles of $L(\underline{f}_\rho, \underline{f}_\rho^\perp)$, so we have

$$\begin{aligned}\dim H^0(M, \phi^{-1}T^{(1,0)}\mathbb{CP}^n) &= \dim H^0(M, A_+) + \dim H^0(M, A_-) \\ &\geq \dim H^0(M, A_+) \\ &\geq \dim H^0(M, A_+) - \dim H^1(M, A_+), \qquad (4.1.3)\end{aligned}$$

as $\dim H^0(M, A_-) \geq 0$ and $\dim H^1(M, A_+) \geq 0$. By Corollary 4.1.7 the right-hand side of (4.1.3) is independent of choice of the holomorphic structure on A_+. Using this, we replace the holomorphic structure of A_+ induced by the Koszul-Malgrange holomorphic structure of $L(\underline{f}_\rho, \underline{f}_\rho^\perp)$ with the holomorphic structure defined by (4.1.2). Applying Lemma 4.1.10 to $\dim H^0(M, A_+) - \dim H^1(M, A_+)$ with A_+ equipped with the holomorphic structure $\overline{\partial}_{\text{sum}}$, we have

$$\begin{aligned}\dim H^0(M, \phi^{-1}T^{(1,0)}\mathbb{CP}^n) &\geq \dim H^0(M, A_+) - \dim H^1(M, A_+) \\ &= \sum_{j=\rho+1}^{n} \{\dim H^0(M, L(\underline{f}_\rho, \underline{f}_j)) - \dim H^1(M, L(\underline{f}_\rho, \underline{f}_j))\}.\end{aligned}$$

□

Recall from Lemma 2.1.12 that the degree of ϕ is minus the first Chern class c_1 of the bundle $\underline{\phi}$, we have from [28, p. 246], given a harmonic map $\phi : M_g \to \mathbb{CP}^n$ with directrix (f, ρ), where M_g is a Riemann surface of genus g, then

$$-\deg(\phi) = c_1(\underline{\phi}) = \sum_{\alpha=0}^{\rho-1} r_\alpha - \deg(f) + \rho(2 - 2g). \tag{4.1.4}$$

We deduce the following.

Theorem 4.1.12. *Let M_g be a Riemann surface of genus g, $\phi : M_g \to \mathbb{CP}^n$ a full non-$\pm$-holomorphic complex isotropic harmonic map with directrix (f, ρ), and r_α the $(\alpha + 1)$st total ramification index of f (see §2.2.1). Then*

$$\operatorname{index}(\phi) \geq (n+1)\deg(f) - \sum_{\alpha=0}^{\rho-1}(n-\alpha)r_\alpha + (2n\rho - \rho^2 + 2\rho - n)(g-1). \tag{4.1.5}$$

Proof. For each $j \in \{\rho+1, \dots, n\}$, we have using (4.1.4) that

$$\begin{aligned} c_1(L(\underline{f}_\rho, \underline{f}_j)) &= c_1(\underline{f}_\rho^* \otimes \underline{f}_j) \\ &= c_1(\underline{f}_\rho^*) + c_1(\underline{f}_j) \\ &= -c_1(\underline{f}_\rho) + c_1(\underline{f}_j) \\ &= -(j-\rho)(2g-2) + \sum_{\alpha=\rho}^{j-1} r_\alpha. \end{aligned}$$

Therefore for each $j \in \{\rho+1, \dots, n\}$ Theorem 4.1.6 (Riemann-Roch) gives

$$\begin{aligned} \dim H^0(M_g, L(\underline{f}_\rho, \underline{f}_\alpha)) - \dim H^1(M_g, L(\underline{f}_\rho, \underline{f}_\alpha)) &= c_1(L(\underline{f}_\rho, \underline{f}_j)) + 1 - g \\ &= -(2j-2\rho+1)(g-1) + \sum_{\alpha=\rho}^{j-1} r_\alpha. \end{aligned}$$

Using this together with Proposition 4.1.11 we have

$$\begin{aligned} \dim H^0(M_g, \phi^{-1}T^{(1,0)}\mathbb{CP}^n) &\geq \sum_{j=\rho+1}^{n} \{\dim H^0(M_g, L(\underline{f}_\rho, \underline{f}_j)) - \dim H^1(M_g, L(\underline{f}_\rho, \underline{f}_j))\} \\ &= \sum_{j=\rho+1}^{n} \{c_1(L(\underline{f}_\rho, \underline{f}_j)) + 1 - g\} \\ &= \sum_{j=\rho+1}^{n} \{-(2j-2\rho+1)(g-1) + \sum_{\alpha=\rho}^{j-1} r_\alpha\} \\ &= \Big((n-\rho)(2\rho-1) - n(n+1) + \rho(\rho+1)\Big)(g-1) + \sum_{j=\rho+1}^{n} \sum_{\alpha=\rho}^{j-1} r_\alpha \\ &= \Big((n-\rho)(2\rho-1) - n(n+1) + \rho(\rho+1)\Big)(g-1) + \sum_{\alpha=\rho}^{n-1} (n-\alpha) r_\alpha. \end{aligned} \tag{4.1.6}$$

From [31, p. 271] we have a useful relation involving the total ramification indices r_α:

$$\sum_{\alpha=0}^{n-1}(n-\alpha)r_\alpha = (n+1)\deg(f) + n(n+1)(g-1). \tag{4.1.7}$$

We split this sum and rearrange to give

$$\sum_{\alpha=\rho}^{n-1}(n-\alpha)r_\alpha = (n+1)\deg(f) + n(n+1)(g-1) - \sum_{\alpha=0}^{\rho-1}(n-\alpha)r_\alpha. \tag{4.1.8}$$

By substituting (4.1.8) into (4.1.6) we have the following:

$$\begin{aligned}\dim H^0(M_g,\phi^{-1}T^{(1,0)}\mathbb{CP}^n) \geq (n+1)\deg(f) - \sum_{\alpha=0}^{\rho-1}(n-\alpha)r_\alpha + \\ \Big((n-\rho)(2\rho-1) - n(n+1) + \rho(\rho+1) + n(n+1)\Big)(g-1) \\ = (n+1)\deg(f) - \sum_{\alpha=0}^{\rho-1}(n-\alpha)r_\alpha + (2n\rho - \rho^2 + 2\rho - n)(g-1).\end{aligned}$$

By Lemma 4.1.5 the theorem is proven. □

Corollary 4.1.13. *Let M_g be a Riemann surface of genus g, $\phi : M_g \to \mathbb{CP}^n$ a full non-$\pm$-holomorphic complex isotropic harmonic map with directrix (f, ρ), and r_α the $(\alpha+1)$st total ramification index of f (see §2.2.1). Then*

$$\operatorname{index}(\phi) \geq (n+1)\deg(\phi) + \sum_{\alpha=0}^{\rho-1}(\alpha+1)r_\alpha + (n+\rho^2)(1-g).$$

Proof. By (4.1.4) we have

$$\deg(f) = \deg(\phi) + \sum_{\alpha=0}^{\rho-1} r_\alpha - \rho(2g-2). \tag{4.1.9}$$

Substituting (4.1.9) into (4.1.5) and rearranging we have

$$\begin{aligned}
\operatorname{index}(\phi) &\geq (n+1)\Big(\deg(\phi) + \sum_{\alpha=0}^{\rho-1} r_\alpha - \rho(2g-2)\Big) - \sum_{\alpha=0}^{\rho-1}(n-\alpha)r_\alpha \\
&\quad + (2n\rho - \rho^2 + 2\rho - n)(g-1) \\
&= (n+1)\deg\phi + (n+1)\sum_{\alpha=0}^{\rho-1} r_\alpha - \sum_{\alpha=0}^{\rho-1}(n-\alpha)r_\alpha \\
&\quad - 2\rho(n+1)(g-1) + (2n\rho - \rho^2 + 2\rho - n)(g-1) \\
&= (n+1)\deg\phi + (n+1)\sum_{\alpha=0}^{\rho-1} r_\alpha - \sum_{\alpha=0}^{\rho-1}(n-\alpha)r_\alpha + (n+\rho^2)(1-g) \\
&= (n+1)\deg\phi + \sum_{\alpha=0}^{\rho-1}(\alpha+1)r_\alpha + (n+\rho^2)(1-g).
\end{aligned}$$

□

Remark 4.1.14. *Theorem 4.1.12 is an improvement on Proposition 4.0.1 [28] if and only if*

$$\sum_{\alpha=0}^{\rho-1}(\alpha+1)r_\alpha > \rho^2(g-1),$$

which clearly holds for $g = 0$*.*

Corollary 4.1.15. *Let* $\phi : S^2 \to \mathbb{CP}^n$ *be a full non-±-holomorphic harmonic map with directrix* (f, ρ) *and* r_α *the* $(\alpha+1)$*st total ramification index of* f*. Then*

$$\begin{aligned}
\operatorname{index}(\phi) &\geq (n+1)\deg(f) - \sum_{\alpha=0}^{\rho-1}(n-\alpha)r_\alpha - 2n\rho + \rho^2 - 2\rho + n \\
&= (n+1)\deg(\phi) + \sum_{\alpha=0}^{\rho-1}(\alpha+1)r_\alpha + n + \rho^2.
\end{aligned}$$

Proof. This follows immediately from Theorem 4.1.12 and Corollary 4.1.13 by putting $g = 0$ □

Corollary 4.1.16. *Let* $\phi : S^2 \to \mathbb{CP}^2$ *be a full non-±-holomorphic harmonic map with* *directrix* $(f, 1)$ *and let* r_0 *be the first total ramification index of* f. *Then*

$$\mathrm{index}(\phi) \geq 3\deg(f) - 2r_0 - 3 = 3\deg(\phi) + r_0 + 3.$$

Proof. Let $\phi : S^2 \to \mathbb{CP}^2$ be a non-$\pm$-holomorphic harmonic map, then we have $\rho = 1$ and $n = 2$; by substituting these values into Corollary 4.1.15 we get the desired result. □

Remark 4.1.17. *For a harmonic map* $\phi : S^2 \to \mathbb{CP}^n$ *with directrix* (f, ρ)*, Corollary 4.1.15 is an improvement on known estimates (Proposition 4.0.1) by the amount* $\rho^2 + \sum_{\alpha=0}^{\rho-1}(\alpha + 1)r_\alpha$. *In particular, Corollary 4.1.16 is an improvement for a harmonic map* $\phi : S^2 \to \mathbb{CP}^2$ *with directrix* $(f, 1)$ *by the amount* $1 + r_0$.

Corollary 4.1.18. *Let* $\phi : M_1 \to \mathbb{CP}^n$ *be a full non-*$\pm$*-holomorphic complex isotropic harmonic map with directrix* (f, ρ) *and* r_α *the* $(\alpha + 1)$*st total ramification index of* f. *Then*

$$\begin{aligned}\mathrm{index}(\phi) &\geq (n + 1)\deg(f) - \sum_{\alpha=0}^{\rho-1}(n - \alpha)r_\alpha \\ &= (n + 1)\deg(\phi) + \sum_{\alpha=0}^{\rho-1}(\alpha + 1)r_\alpha.\end{aligned}$$

Proof. This follows immediately from Theorem 4.1.12 and Corollary 4.1.13 by putting $g = 1$. □

Corollary 4.1.19. *Let* $\phi : M_1 \to \mathbb{CP}^2$ *be a full non-*$\pm$*-holomorphic complex isotropic harmonic map with directrix* $(f, 1)$ *and let* r_0 *be the first total ramification index of* f. *Then*

$$\mathrm{index}(\phi) \geq 3\deg(\phi) + r_0 = 3\deg(f) - 2r_0.$$

Proof. Let $\phi : M_1 \to \mathbb{CP}^2$ be a non-$\pm$-holomorphic harmonic map, then we have $\rho = 1$ and $n = 2$. By substituting these values into Corollary 4.1.18 we get the desired result. □

Remark 4.1.20. *For a harmonic map $\phi : M_1 \to \mathbb{CP}^n$ with directrix (f, ρ), Corollary 4.1.18 is an improvement on known estimates (Proposition 4.0.1) by the amount $\sum_{\alpha=0}^{\rho-1}(\alpha+1)r_\alpha$. In particular, Corollary 4.1.19 is an improvement for a harmonic map $\phi : M_1 \to \mathbb{CP}^2$ with directrix $(f, 1)$ by the amount r_0.*

4.2 Examples

We present examples of harmonic maps for genus $0, 1$, and higher genera for which the known estimates on the index are improved by Corollary 4.1.16, Corollary 4.1.19 and Theorem 4.1.12, respectively.

4.2.1 Genus 0

We define the family of maps $\eta_k : S^2 \to S^2$ given by $\eta_k(z) = z^k$ for $k \in \mathbb{Z}$: all η_k are holomorphic.

Example 4.2.1. *cf. [28, Example 8.1] Let $F : S^2 \to \mathbb{C}^3 \setminus \{0\}$, where $F(z) = (1, z, z^2)$. Then let $f = [F] : S^2 \to \mathbb{CP}^2$, so $f(z) = [1, z, z^2]$ and is a full holomorphic map with $\deg(f) = 2$ and $r_0 = 0$. Following §2.1 we have that $G'(f) : S^2 \to \mathbb{CP}^2$ is a full non-$\pm$-holomorphic (complex isotropic) harmonic map of degree 0 and directrix $(f, 1)$. For each $k \in \mathbb{N}$ the composition $f \circ \eta_k : S^2 \to \mathbb{CP}^2$ gives a full holomorphic map with $\deg(f \circ \eta_k) = 2k$ and $r_0 = 2(k-1)$. For each $k \in \mathbb{N}$, $G'(f \circ \eta_k) : S^2 \to \mathbb{CP}^2$ gives a full non-$\pm$-holomorphic harmonic map of degree 0 and directrix $(f \circ \eta_k, 1)$. By Corollary 4.1.16, $\operatorname{index}(G'(f \circ \eta_k)) \geq 3\deg(f \circ \eta_k) - 2r_0 - 3 = 6k - 4(k-1) - 3 = 2k+1$. By Remark 4.1.17, for each $k \in \mathbb{N}$, Corollary 4.1.16 improves the estimate in [28] (see Proposition 4.0.1 above) by $2k-1$.*

Example 4.2.2. *cf. [28, Example 8.2] Let $F : S^2 \to \mathbb{C}^3 \setminus \{0\}$, where $F(z) = (1, z + z^3, z^2)$. Then let $f = [F] : S^2 \to \mathbb{CP}^2$, so $f(z) = [1, z + z^3, z^2]$ and is a full holomorphic* *map with* $\deg(f) = 3$ *and* $r_0 = 0$. *Again following* §2.1 *we have that* $G'(f) : S^2 \to \mathbb{CP}^2$ *is a full non-*$\pm$*-holomorphic (complex isotropic) harmonic map of degree* 1 *and directrix* $(f, 1)$. *For each* $k \in \mathbb{N}$ *the composition* $f \circ \eta_k : S^2 \to \mathbb{CP}^2$ *gives a full holomorphic map with* $\deg(f \circ \eta_k) = 3k$ *and* $r_0 = 2(k-1)$. *For each* $k \in \mathbb{N}$, $G'(f \circ \eta_k) : S^2 \to \mathbb{CP}^2$ *gives a full non-*$\pm$*-holomorphic harmonic map of degree* k *and directrix* $(f \circ \eta_k, 1)$. *By Corollary 4.1.16,* $\text{index}(G'(f \circ \eta_k)) \geq 3\deg(f \circ \eta_k) - 2r_0 - 3 = 9k - 4(k-1) - 3 = 5k + 1$. *By Remark 4.1.17, for each* $k \in \mathbb{N}$*, Corollary 4.1.16 improves the estimate in [28] (see Proposition 4.0.1 above) by* $2k - 1$.

4.2.2 Genus 1

Let M_1, M_1' be tori, i.e. compact Riemann surfaces of genus 1 and $\psi : M_1 \to M_1'$ a holomorphic covering map of degree k.

Example 4.2.3. *Let* $f : M_1' \to \mathbb{CP}^2$ *be the degree 5 full holomorphic map with first total ramification index* 4 *constructed in [28, Lemma 8.7]. The composition* $f \circ \psi$ *is a full holomorphic map with* $\deg(f \circ \psi) = 5k$ *and* $r_0 = 4k$. *As an application of Corollary 4.1.19 let* $G'(f \circ \psi) : M_1 \to \mathbb{CP}^2$ *be the degree* k *harmonic non-*$\pm$*-holomorphic map with directrix* $(f \circ \psi, 1)$ *then* $\text{index}(G'(f \circ \psi)) \geq 3\deg(f \circ \psi) - 2r_0 = 15k - 8k = 7k$. *By Remark 4.1.20, for each* $k \in \mathbb{N}$*, Corollary 4.1.19 improves the estimate in [28] (see Proposition 4.0.1 above) by* $4k$.

4.2.3 Higher Genera

Let M_g be a compact Riemann surface of genus $g > 1$.

Example 4.2.4. *By [28, Theorem 8.10] there exist full non-*$\pm$*-holomorphic complex isotropic harmonic maps* $\phi : M_g \to \mathbb{CP}^2$ *of degree* $k > g$. *Indeed there exist holomorphic maps* $h : M_g \to \mathbb{CP}^1$ *of all degrees* $k > g$. *Composing such a map with the full harmonic* *map of degree* 1 *with directrix* $(f, 1)$ *from Example 4.2.2 gives a full non-*$\pm$*-holomorphic complex isotropic harmonic map of degree* $k > g$*. This is the Gauss transform of the full holomorphic map* $f \circ h$ *which has degree* $3k$*. From (4.1.9)* $r_0 = 2k + 2g - 2 > g - 1$*. Therefore by Remark 4.1.14, Theorem 4.1.12 improves the estimate in [28] (see Proposition 4.0.1 above) for all these maps, giving examples in all degrees* $> g$*.*

Chapter 5

Canonical Elements

In this chapter we discuss certain elements of a semi-simple Lie algebra called “canonical elements”. As described in [10] these elements give rise to natural fibrations of a class of homogeneous spaces over a Riemann surface. These homogeneous spaces are of the form G/H for G a semi-simple compact Lie group and H the centraliser of a torus of G: the homogeneous spaces G/H are called flag manifolds. The natural fibration of the flag manifold described in [10] is a “twistor fibration” and has a rich theory for describing harmonic maps. For more general information on twistor fibrations we direct the reader to the survey [25]. Before our discussion of canonical elements we will describe certain bases of $\mathbb{C}^n$ called “null bases” that will aid in the discussion of canonical elements and aid in the calculations of later sections, particularly the sections pertaining to the orthogonal group and symplectic group.

5.1 Null Bases

Let $\{e_i\}$ be the standard basis for $\mathbb{C}^n$ where $e_1 = (1, 0, \ldots, 0)^T$, $e_2 = (0, 1, 0, \ldots, 0)^T$, $\ldots$, $e_n = (0, \ldots, 0, 1)^T$ and let $(x, y) = x^T I y = \sum_{i=1}^n x_i y_i$ be the standard symmetric bilinear form on $\mathbb{C}^n$ for $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$.

Definition 5.1.1. *A basis* $\{\tilde{e}_i\}$ *for* $\mathbb{C}^n$ *which satisfies* $(\tilde{e}_i, \tilde{e}_j) = \delta_{i\bar{j}}$ *for any* $i, j \in \{1, 2, \ldots, n\}$ *and where* $\bar{j} = n + 1 - j$ *is called a* ***null basis****.*

5.1.1 The Orthogonal Group

Consider the orthogonal group

$$\begin{aligned} O(n) &= \{A \in GL(n, \mathbb{R}) \mid (Ax, Ay) = (x, y) \ \forall \ x, y \in \mathbb{R}^n\} \\ &= \{A \in GL(n, \mathbb{R}) \mid A^T A = I\}. \end{aligned} \tag{5.1.1}$$

The complexification of $O(n)$ denoted $O(n, \mathbb{C}) = O(n)^{\mathbb{C}}$ is given by

$$\begin{aligned} O(n) &= \{A \in GL(n, \mathbb{C}) \mid (Ax, Ay) = (x, y) \ \forall \ x, y \in \mathbb{C}^n\} \\ &= \{A \in GL(n, \mathbb{C}) \mid A^T A = I\}, \end{aligned}$$

for A^T the linear map characterised by $(Ax, y) = (x, A^T y)$ for the standard symmetric bilinear form on $\mathbb{C}^n$ and $x, y \in \mathbb{C}$. For more information see [34].

Definition 5.1.2. *For any subspace* $V \in \mathbb{C}^n$ *we say that* V *is* ***isotropic*** *if* $(x, y) = 0$ *for all* $x, y \in V$ *or equivalently if* $V^\perp \subset \overline{V}$ *where* $\overline{} : \mathbb{C}^n \to \mathbb{C}^n$ *denotes complex conjugation given by* $\overline{v} = \sum_{i=1}^n \overline{v_i} e_i$ *for* $v = \sum_{i=1}^n v_i e_i$*. We say that* V *is* ***maximally isotropic*** *if* $V^\perp = \overline{V}$*.*

We consider the orthogonal group for the particular null basis $\{\tilde{e}_i\}$ for $\mathbb{C}^n$ given by

$$\tilde{e}_j = \frac{1}{\sqrt{2}}(e_{2j} - ie_{2j-1}), \qquad \tilde{e}_{\bar{j}} = \frac{1}{\sqrt{2}}(e_{2j} + ie_{2j-1}), \tag{5.1.2}$$

for $j \leq \frac{n}{2}$ and with $\tilde{e}_{(n+1)/2} = e_n$ when n is odd. Let P be the matrix with columns given by the null basis (5.1.2),

$$P = \begin{pmatrix} | & | & & | \\ \tilde{e}_1 & \tilde{e}_2 & \dots & \tilde{e}_n \\ | & | & & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & & 0 & 0 & 0 & 0 & & i \\ 1 & & \vdots & \vdots & \vdots & \vdots & & 1 \\ 0 & & 0 & 0 & 0 & 0 & & 0 \\ 0 & \cdots & -i & 0 & 0 & i & \cdots & 0 \\ 0 & & 1 & 0 & 0 & 1 & & 0 \\ \vdots & & 0 & -i & i & 0 & & \vdots \\ 0 & & 0 & 1 & 1 & 0 & & 0 \end{pmatrix},$$

for n even; for n odd, P has a column $(0, 0, \dots, 0, 1)^T$ in the middle. The standard symmetric bilinear form on $\mathbb{C}^n$ in this null basis is given by

$$(x, y) = x^T P^T I P y = x^T Q y = \sum_{j=1}^{n} x_j y_{\bar{j}} \tag{5.1.3}$$

for $x = \sum_{j=1}^{n} x_j \tilde{e}_j$, $y = \sum_{j=1}^{n} y_j \tilde{e}_j$ and $Q = Q_n = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ is of size $n \times n$.

Definition 5.1.3. *Let $A = (a_{ij})$ be an $m \times n$ matrix. The* ***second transpose****, $A^{\mathfrak{T}}$ of A is the $n \times m$ matrix defined by $A^{\mathfrak{T}} = (a_{\bar{j}\bar{i}})$.*

As $Q^{-1} = Q$ we have from this definition that for an $n \times n$ matrix $A^T = (a_{ji})$,

$$A^{\mathfrak{T}} = Q A^T Q \quad \text{and} \quad A^T = Q A^{\mathfrak{T}} Q. \tag{5.1.4}$$

Note that $A^{\mathfrak{T}}$ is obtained from A by reflection in the **second diagonal** (sometimes called the antidiagonal), that is, all elements of $A = (a_{ij})$ such that $i + j = n + 1$. Also from

Definition 5.1.3, we have for $x = (x_1, x_2, \dots, x_n)^T$ that

$$x^{\mathfrak{T}} = x^T Q \quad \text{and} \quad x^T = x^{\mathfrak{T}} Q. \tag{5.1.5}$$

In this null basis the orthogonal group is given by $O(n, \mathbb{C}) = \{A \in GL(n, \mathbb{C}) \mid A^{\mathfrak{T}} A = I\}$. A lies in $O(n, \mathbb{C})$ if and only if it preserves the inner product (5.1.3), i.e. $A^T Q A = Q$; since $Q^{-1} = Q$ this is equivalent to $A^{\mathfrak{T}} A = I$.

Lemma 5.1.4. *[29] Let A be an $n \times n$ matrix in the null basis (5.1.2) and let c_j the jth column of A. Then $A \in O(n, \mathbb{C})$ if and only if*

$$(c_i, c_j) = \delta_{i\bar{j}}.$$

Proof. This follows immediately from $A^{\mathfrak{T}} A = I$, where

$$A = \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \dots & c_n \\ | & | & & | \end{pmatrix} \quad \text{and} \quad A^{\mathfrak{T}} = \begin{pmatrix} \text{---} & c_n^{\mathfrak{T}} & \text{---} \\ & \vdots & \\ \text{---} & c_2^{\mathfrak{T}} & \text{---} \\ \text{---} & c_1^{\mathfrak{T}} & \text{---} \end{pmatrix}.$$

□

5.1.2 The Symplectic Group

Define a skew-symmetric bilinear form on $\mathbb{C}^{2m}$ by

$$\omega(x, y) = \sum_{j=1}^{m} x_j y_{m+j} - x_{m+j} y_j = (x, \Omega y) = x^T \Omega y, \tag{5.1.6}$$

for $x = (x_1, \ldots, x_n)^T$, $y = (y_1, \ldots, y_n)^T$ and Ω the $2m \times 2m$ matrix

$$\Omega = \begin{pmatrix} 0 & I_m \\ -I_m & 0 \end{pmatrix}.$$

Here I_m denotes the $m \times m$ identity matrix. Let $\langle x, y\rangle = x^T I \overline{y} = \sum_{i=1}^n x_i \overline{y}_i$ be the standard Hermitian form on $\mathbb{C}^n$ for $x = (x_1, \ldots, x_n)^T$ and $y = (y_1, \ldots, y_n)^T$. Note that $\langle x, y\rangle = (x, \overline{y})$. Let $J : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ be the conjugate-linear map characterised by

$$\omega(x, y) = \langle x, J(y)\rangle \qquad (x, y \in \mathbb{C}^{2m}). \tag{5.1.7}$$

From (5.1.6) we have that the conjugate-linear map J can be represented by a matrix in the following way

$$(x, \Omega y) = \omega(x, y) = \langle x, J(y)\rangle = (x, \overline{J(y)}),$$

so $J(y) = \overline{\Omega y}$. For the standard basis $\{e_i\}$ we have

$$J(e_j) = \Omega e_j = -e_{m+j}, \qquad J(e_{m+j}) = \Omega e_{m+j} = e_j \qquad (j \in \{1, \ldots, m\}). \tag{5.1.8}$$

The symplectic group over the field of complex numbers is a non-compact Lie group defined by

$$\begin{aligned} Sp(2m, \mathbb{C}) &= \{A \in GL(n, \mathbb{C}) \mid \omega(Ax, Ay) = \omega(x, y) \ \forall\ x, y \in \mathbb{C}^n\} \\ &= \{A \in GL(n, \mathbb{C}) \mid A^T \Omega A = \Omega\}. \end{aligned}$$

The (compact) symplectic group is defined to be the intersection $Sp(m) := Sp(2m, \mathbb{C}) \cap U(2m)$.

Lemma 5.1.5. *Let A be an $2m \times 2m$ matrix and let c_j be the jth column of A. Then*

$A \in Sp(2m, \mathbb{C})$ *if and only if*

$$\omega(c_j, c_k) = \begin{cases} \delta_{j,k-m}, & \text{if } j \leq k, \\ -\delta_{j-m,k}, & \text{if } j > k. \end{cases}$$

Proof. Consider $A^T \cdot \Omega \cdot A$ for $A = \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \dots & c_{2m} \\ | & | & & | \end{pmatrix}$,

$$\begin{pmatrix} \text{---} & c_1^T & \text{---} \\ & \vdots & \\ \text{---} & c_{2m-1}^T & \text{---} \\ \text{---} & c_{2m}^T & \text{---} \end{pmatrix} \begin{pmatrix} & I_m \\ -I_m & \end{pmatrix} \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \dots & c_{2m} \\ | & | & & | \end{pmatrix} = (b_{jk}),$$

where (b_{jk}) is the matrix with entries given by

$$b_{jk} = c_j^T \cdot \Omega \cdot c_k = \omega(c_j, c_k).$$

Therefore, we have $A^T \cdot \Omega \cdot A = (b_{jk}) = \Omega$, if and only if $b_{jk} = \begin{cases} \delta_{j,k-m}, & \text{if } j \leq k, \\ -\delta_{j-m,k}, & \text{if } j > k. \end{cases}$ ☐

For more information of the symplectic group see [34, 45].

Definition 5.1.6. *For any subspace $V \in \mathbb{C}^{2n}$ we say that V is **J-isotropic** if $\omega(x, y) = 0$ for all $x, y \in V$ or equivalently if $V^\perp \subset JV$. We say that V is **maximally J-isotropic** if $V^\perp = JV$.*

Consider the null basis $\{\hat{e}_i\}$ for $\mathbb{C}^{2m}$ given by

$$\hat{e}_j = \frac{1}{\sqrt{2}}(e_{m+j} - ie_j), \qquad \hat{e}_{\bar{j}} = \frac{1}{\sqrt{2}}(e_{m+j} + ie_j) \tag{5.1.9}$$

for $j \leq m$. Let $\hat{P}$ be the matrix with columns given by the components of the null basis (5.1.9) with respect to the standard basis,

$$\hat{P} = \begin{pmatrix} | & | & & | \\ \hat{e}_1 & \hat{e}_2 & \dots & \hat{e}_n \\ | & | & & | \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} -iI_m & iQ_m \\ I_m & Q_m \end{pmatrix}, \tag{5.1.10}$$

where $Q_m = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}$ is of size $m \times m$. Note that $Q_m = Q_m^T = Q_m^{\mathfrak{T}} = Q_m^{-1}$. In the null basis (5.1.9) the skew-symmetric bilinear form (5.1.6) above is given by

$$\begin{aligned} \omega(x, y) &= x^T \hat{P}^T \Omega \hat{P} y = x^T \hat{\Omega} y = x^{\mathfrak{T}} Q_{2m} \hat{\Omega} y = x^{\mathfrak{T}} \Omega_{\text{null}} y \\ &= i \sum_{j=1}^{m} x_{\bar{j}} y_j - x_j y_{\bar{j}}, \end{aligned} \tag{5.1.11}$$

where $x = \sum_{j=1}^{2m} x_j \hat{e}_j$, $y = \sum_{j=1}^{2m} y_j \hat{e}_j$, $x^T = x^{\mathfrak{T}} Q_{2m}$ from (5.1.5),

$$\hat{\Omega} := \hat{P}^T \Omega \hat{P} = \begin{pmatrix} & -iQ_m \\ iQ_m & \end{pmatrix} \quad \text{and}$$

$$\Omega_{\text{null}} := Q_{2m} \hat{\Omega} = \begin{pmatrix} iI_m & \\ & -iI_m \end{pmatrix}. \tag{5.1.12}$$

Similarly to before we may represent the conjugate-linear map $J : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ by a matrix in this null basis. Let $\{\hat{e}_i\}$ be the null basis from (5.1.11) then using (5.1.8) we have

$$
\begin{aligned}
J(\hat{e}_j) &= \frac{1}{\sqrt{2}} J(e_{m+j} - ie_j) = \frac{1}{\sqrt{2}}\Big(J(e_{m+j}) - J(ie_j)\Big) \\
&= \frac{1}{\sqrt{2}}\Big(J(e_{m+j}) + iJ(e_j)\Big) = \frac{1}{\sqrt{2}}(e_j - ie_{m+j}) \\
&= i\frac{1}{\sqrt{2}}(e_{m+j} + ie_j) = -i\hat{e}_{\bar{j}} = i\overline{\hat{e}_{\bar{j}}}
\end{aligned}
$$

and

$$
\begin{aligned}
J(\hat{e}_{\bar{j}}) &= \frac{1}{\sqrt{2}} J(e_{m+j} + ie_j) = \frac{1}{\sqrt{2}}\Big(J(e_{m+j}) + J(ie_j)\Big) \\
&= \frac{1}{\sqrt{2}}\Big(J(e_{m+j}) - iJ(e_j)\Big) = \frac{1}{\sqrt{2}}(e_j + ie_{m+j}) \\
&= i\frac{1}{\sqrt{2}}(e_{m+j} - ie_j) = i\hat{e}_j = -\overline{i\hat{e}_{\bar{j}}}.
\end{aligned}
$$

Let $v = \sum_{j=1}^{2m} v_j \hat{e}_j \in \mathbb{C}^{2m}$ for $\{\hat{e}_j\}$ the null basis (5.1.9). We split the sum using the notation introduced in Definition 5.1.1 so $v = \sum_{k=1}^{m} v_k \hat{e}_k + \sum_{l=1}^{m} v_{\bar{l}} \hat{e}_{\bar{l}}$. Upon applying the conjugate-linear map $J : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ we have,

$$
\begin{aligned}
J(v) &= J(\sum_{k=1}^{m} v_k \hat{e}_k + \sum_{l=1}^{m} v_{\bar{l}} \hat{e}_{\bar{l}}) = \sum_{k=1}^{m} \overline{v_k} J(\hat{e}_k) + \sum_{l=1}^{m} \overline{v_{\bar{l}}} J(\hat{e}_{\bar{l}}) \\
&= \sum_{k=1}^{m} \overline{v_k}(\overline{i\hat{e}_k}) + \sum_{l=1}^{m} \overline{v_{\bar{l}}}(-\overline{i\hat{e}_{\bar{l}}}) = \sum_{k=1}^{m} \overline{iv_k \hat{e}_k} - \sum_{l=1}^{m} \overline{iv_{\bar{l}} \hat{e}_{\bar{l}}} \qquad (5.1.13) \\
&= \overline{\sum_{k=1}^{m} iv_k \hat{e}_k - \sum_{l=1}^{m} iv_{\bar{l}} \hat{e}_{\bar{l}}} = \overline{\Omega_{\text{null}} v}.
\end{aligned}
$$

Also note that for $v = \sum_{j=1}^{2m} v_j \hat{e}_j \in \mathbb{C}^{2m}$ then

$$
\overline{v} = \sum_{j=1}^{2m} \overline{v_j}\,\overline{\hat{e}_j} = \sum_{j=1}^{2m} \overline{v_j}\hat{e}_{\bar{j}} = \sum_{j=1}^{2m} \overline{v_{\bar{j}}}\hat{e}_j,
$$

where we rename the indices in the last equality. A lies in $Sp(2m, \mathbb{C})$ if and only if it preserves the inner product (5.1.11); as $\omega(x,y) = x^T \hat{\Omega} y$ then $A \in Sp(2m,\mathbb{C})$ if and only if $A^T \hat{\Omega} A = \hat{\Omega}$. Using (5.1.4) this is equivalent to $A^{\mathfrak{T}} \Omega_{\text{null}} A = \Omega_{\text{null}}$ and so we write the symplectic group in the null basis (5.1.9) as

$$Sp(2m,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) \mid A^{\mathfrak{T}} \Omega_{\text{null}} A = \Omega_{\text{null}}\}. \tag{5.1.14}$$

Lemma 5.1.7. *Let A be an $n \times n$ matrix in the null basis (5.1.9) given above and let c_j be the jth column of A. Then $A \in Sp(n,\mathbb{C})$ if and only if*

$$\omega(c_j, c_k) = \begin{cases} i\delta_{\bar{j}k}, & \text{if } j \geq k, \\ -i\delta_{\bar{j}k}, & \text{if } j < k. \end{cases}$$

Proof. Similarly to Lemma 5.1.5 we consider $A^{\mathfrak{T}} \cdot \Omega_{\text{null}} \cdot A$ for $A = \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \dots & c_n \\ | & | & & | \end{pmatrix}$,

$$\begin{pmatrix} \text{---} & c_n^{\mathfrak{T}} & \text{---} \\ & \vdots & \\ \text{---} & c_2^{\mathfrak{T}} & \text{---} \\ \text{---} & c_1^{\mathfrak{T}} & \text{---} \end{pmatrix} \begin{pmatrix} iI_m & \\ & -iI_m \end{pmatrix} \begin{pmatrix} | & | & & | \\ c_1 & c_2 & \dots & c_n \\ | & | & & | \end{pmatrix} = (b_{jk}),$$

where (b_{jk}) is the matrix with entries given by

$$b_{jk} = c_{\bar{j}}^{\mathfrak{T}} \cdot \Omega_{\text{null}} \cdot c_k = \omega(c_{\bar{j}}, c_k).$$

Therefore, we have $A^{\mathfrak{T}} \cdot \Omega_{\text{null}} \cdot A = (b_{jk}) = \Omega_{\text{null}}$, if and only if $b_{jk} = \begin{cases} i\delta_{jk}, & \text{if } \bar{j} \geq k, \\ -i\delta_{jk}, & \text{if } \bar{j} < k. \end{cases}$ $\square$

5.2 Canonical Elements

Let G be a compact (real) Lie group with semi-simple Lie algebra $\mathfrak{g}$. We give descriptions and examples of particular elements of the Lie algebra $\mathfrak{g}$ called **canonical elements**. We state and give justification to two results, Theorem 5.2.14 and Theorem 5.2.16 which describe canonical elements for $U(n)$ and $O(n)$, respectively. We go on to give concrete descriptions of canonical elements for $SU(n)$ and $O(n)$ following the theory of [10], summarised in Proposition 5.2.17 for the orthogonal group. Finally, we give a description of canonical elements for $Sp(m)$ summarised in Proposition 5.2.20. For more information on canonical elements see [7, 10, 17] and for the general theory below see [34, §6] and [45, §3].

Definition 5.2.1. *Let $\mathfrak{g}$ be a semi-simple Lie algebra then a* ***maximal commutative subalgebra*** *of $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ satisfying the following:*

(i) *for all $H_1, H_2 \in \mathfrak{h}$, $[H_1, H_2] = 0$ (commutative);*

(ii) *for all $X \in \mathfrak{g}$, if $[H, X] = 0$ for all $H \in \mathfrak{h}$, then $X \in \mathfrak{h}$ (maximal).*

Definition 5.2.2. *Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semi-simple Lie algebra then a* ***Cartan subalgebra*** *$\mathfrak{a}$ of $\mathfrak{g}^{\mathbb{C}}$ is a maximal commutative subalgebra of $\mathfrak{g}^{\mathbb{C}}$ such that $ad(\xi)$ is diagonalizable for each $\xi \in \mathfrak{a}$.*

Here diagonalizable means that there is a basis of $\mathfrak{g}^{\mathbb{C}}$ such that $ad(\xi)$ is represented by a diagonal matrix.

Proposition 5.2.3. *[34, Proposition 6.12] Let $\mathfrak{g}$ be a real semi-simple Lie algebra and $\mathfrak{t}$ some maximally commutative subalgebra. Let $\mathfrak{t}^{\mathbb{C}} = \mathfrak{t} \otimes \mathbb{C}$ and $\mathfrak{g}^{\mathbb{C}} = \mathfrak{g} \otimes \mathbb{C}$ be the complexicifations of $\mathfrak{t}$ and $\mathfrak{g}$, respectively. Then $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra for $\mathfrak{g}^{\mathbb{C}}$.*

Remark 5.2.4. *We will adopt the convention of many authors by calling a maximal commutative subalgebra $\mathfrak{t}$ a* ***maximal toral subalgebra****. This name is due to the following: let*

*T be a connected Lie group whose Lie algebra is a maximal commutative subalgebra $\mathfrak{t}$ of $\mathfrak{g}$, then T can be shown to be isomorphic to $S^1 \times S^1 \times \cdots \times S^1$, which is a **maximal torus** of G.*

Definition 5.2.5. *Let $\mathfrak{g}^{\mathbb{C}}$ be a complex semi-simple Lie algebra and $\mathfrak{a}$ be a cartan subalgebra for $\mathfrak{g}^{\mathbb{C}}$. We call the non-zero linear map $\lambda \in \mathfrak{a}^*$ where there exists non-zero $v \in \mathfrak{g}^{\mathbb{C}}$ such that $ad(\xi)v = \lambda(\xi)v$ for all $\xi \in \mathfrak{a}$ a **root** of $\mathfrak{g}^{\mathbb{C}}$ relative to $\mathfrak{a}$. We call the corresponding $v \in \mathfrak{g}^{\mathbb{C}}$ a **root vector** of λ. The collection of all root vectors $v \in \mathfrak{g}^{\mathbb{C}}$ together with 0 is called the **root space** of λ. The set of all roots $\lambda \in \mathfrak{a}^*$ is called the **root system** of $\mathfrak{g}$ relative to $\mathfrak{a}$ and is denoted Δ.*

Remark 5.2.6. *If $\lambda \in \Delta$ then $-\lambda \in \Delta$, [10, p. 26].*

Definition 5.2.7. *A **positive root system** Δ^+ is a subset of Δ such that*

(i) *For $\lambda, -\lambda \in \Delta$ then Δ^+ contains either λ or $-\lambda$;*

(ii) *Δ^+ is closed i.e., For $\lambda_1, \lambda_2 \in \Delta^+$ such that $\lambda_1 + \lambda_2 \in \Delta$ then $\lambda_1 + \lambda_2 \in \Delta^+$.*

*We call elements of Δ^+ **positive roots**.*

Definition 5.2.8. *[10] Given a positive root system Δ^+, a positive root is called **simple** if it cannot be expressed as a sum of two other positive roots.*

Remark 5.2.9. *There is a choice for the set of simple roots for each choice of positive root system [34, §6.8] and so when thinking of simple roots one must think of a choice of simple roots relative to a choice of positive root system (which is generally not unique).*

Definition 5.2.10. *Let $\lambda_1, \ldots, \lambda_l$ be simple roots relative to some positive root system Δ^+. Then the **dual vectors** $A_1, \ldots, A_l \in \mathfrak{a}$ to the simple roots $\lambda_1, \ldots, \lambda_l$ are characterised by*

$$\lambda_j(A_k) = i\delta_{jk}.$$

Definition 5.2.11. *[10] Let $\mathfrak{a}$ be a Cartan subalgebra for $\mathfrak{g}^{\mathbb{C}}$ and Δ^+ a positive root system with simple roots $\lambda_1, \ldots, \lambda_l$ and corresponding dual vectors $A_1, \ldots, A_l \in \mathfrak{a}$. Then a* ***canonical element*** *ξ of $\mathfrak{g}$ is the element of $\mathfrak{g}$ given by*

$$\xi = \sum_{i \in I} A_i,$$

where I is a subset of $\{1, 2, \ldots, l\}$.

Let G be a Lie group with Lie algebra $\mathfrak{g}$. We will often call a canonical element $\xi \in \mathfrak{g}$ a **canonical element for the Lie group** G.

For use in later chapters we also give the following definition.

Definition 5.2.12. *Fix a canonical element $\xi \in \mathfrak{g}$, then the homomorphism $\gamma_\xi : S^1 \to G$ defined by $\gamma_\xi(e^{it}) = \exp(t\xi)$ is called a* ***canonical geodesic****.*

This homomorphism is well defined see [9, p. 549ff.].

Remark 5.2.13. *In keeping with the notation of [9, 29] we often write $\lambda = e^{it} \in S^1$.*

5.2.1 Canonical Elements for $SU(n)$

For convenience, the definition of canonical elements (Definition 5.2.11) for $\mathfrak{g} = \mathfrak{su}(n)$ can be extended to allow the dual vectors A_i to be in $\mathfrak{u}(n) = \mathfrak{su}(n) \oplus \mathbb{R}$. They are then determined up to addition of an element $i\delta_0 \cdot I$, for I the $n \times n$ identity matrix and $\delta_0 \in \mathbb{R}$. We have:

Theorem 5.2.14. *[8, Proposition A.1] An element $\xi \in \mathfrak{u}(n)$ is canonical if and only if*

$$\xi = i(\delta_0 \cdot I + \sum_{j=1}^{m} j \cdot P_{E_j})$$

for some orthogonal decomposition $\mathbb{C}^n = \sum_{j=1}^{m} E_j$ *into subspaces, with orthogonal projection matrix* P_{E_j} *and any* $\delta_0 \in \mathbb{R}$.

Note also that $\dim E_j = \mathrm{rank}(P_{E_j})$.

Corollary 5.2.15. *An element* $\xi \in \mathfrak{su}(n)$ *is canonical if and only if*

$$\xi = i(\frac{-1}{n}\sum_{j=1}^{m} j\,\mathrm{rank}(P_{E_j}) \cdot I + \sum_{j=1}^{m} j \cdot P_{E_j})$$

for some orthogonal decomposition $\mathbb{C}^n = \sum_{j=1}^{m} E_j$, *with orthogonal projection matrix* P_{E_j}.

Proof. Consider $\mathfrak{su}(n) = \{A \in \mathfrak{u}(n) \mid \mathrm{trace}(A) = 0\}$. As $\mathfrak{su}(n) \subset \mathfrak{u}(n)$ then a canonical element $\xi = i(\delta_0 \cdot I + \sum_{j=1}^{m} j \cdot P_{E_j}) \in \mathfrak{u}(n)$ is a canonical element in $\mathfrak{su}(n)$ if and only if $\xi \in \mathfrak{su}(n)$. So,

$$\xi \in \mathfrak{su}(n) \iff \mathrm{trace}(\xi) = n\delta_0 + \sum_{j=1}^{m} j\,\mathrm{rank}(P_{E_j}) = 0.$$

□

We will give some justification for the corollary above by finding canonical elements with respect to some positive root system of a Cartan subalgebra and show that these are equal to the canonical elements in Corollary 5.2.15 for some orthogonal decomposition $\mathbb{C}^n = \sum_{j=1}^{m} E_j$.

Let $\mathfrak{t}$ be a maximal toral subalgebra (maximally commutative subalgebra) for $\mathfrak{su}(n)$ given by

$$\mathfrak{t} = \{i\,\mathrm{diag}(a_1, \ldots, a_n) \mid a_j \in \mathbb{R}\ \forall j, \sum_{j=1}^{n} a_j = 0\},$$

so a Cartan subalgebra for $\mathfrak{su}(n)^{\mathbb{C}}$ is given by

$$\mathfrak{t}^{\mathbb{C}} = \{\mathrm{diag}(\eta_1, \dots, \eta_n) \mid \eta_j \in \mathbb{C}\ \forall j,\ \sum_{j=1}^{n} \eta_j = 0\}.$$

The roots of $\mathfrak{su}(n)^{\mathbb{C}}$ relative to $\mathfrak{t}^{\mathbb{C}}$ are linear maps $\lambda_{jk} : \mathfrak{t}^{\mathbb{C}} \to \mathbb{C}$ defined by $\lambda_{jk}(\eta) = \eta_j - \eta_k$ where $\eta \in \mathfrak{t}^{\mathbb{C}}$ and $j < k$. The corresponding root spaces to the roots are spanned by $E_{jk} \in \mathfrak{su}(n)$ where E_{jk} has zeros in every entry except for the (j,k) entry which has a 1. We choose the simple roots of the positive root system Δ^+ to be $\lambda_{j,j+1}$ with associated root spaces $E_{j,j+1}$, $j = 1, 2, \dots, n-1$. Dual vectors to these simple roots are $A_1, \dots, A_{n-1} \in \mathfrak{t}$ such that

$$\lambda_{j,j+1}(A_k) = i\delta_{jk},$$

for all $k = 1, \dots, n-1$. These are of the form

$$A_k = i(\frac{-k}{n} \cdot I + D_k),$$

where $a_k \in \mathbb{R}$ and D_k is a 2×2 block matrix of the form

$$D_k = \begin{pmatrix} I_k & 0_{k,n-k} \\ 0_{n-k,k} & 0_{n-k,n-k} \end{pmatrix},$$

where I_k denotes a $k \times k$ identity matrix and $0_{j,k}$ denotes a $j \times k$ matrix with all zero entries. According to Definition 5.2.11 the canonical elements of $\mathfrak{su}(n)$ are of the form

$$\xi = i\big(\sum_{j\in J} \frac{-j}{n} \cdot I + D_j\big) = i\big(\sum_{j\in J} \frac{-j}{n} \cdot I + \sum_{j\in J} D_j\big), \tag{5.2.1}$$

where $J \subseteq \{1, 2, \dots, n-1\}$. We give an ordering to the elements of J, so $j_1 < j_2 < \cdots < j_l$ for $j_k \in J$, $k = 1, 2, \dots, n-1$ and $l \leq n-1$. Considering the second sum of

(5.2.1) we have,

$$\sum_{j\in J} D_j = \sum_{k=1}^{l} D_{j_k} = \sum_{k=1}^{l}(l+1-k)\cdot B_k,$$

for

$$B_k = \operatorname{diag}\Big(\overbrace{0,\dots,0}^{j_{k-1}},\overbrace{1,\dots,1}^{j_k-j_{k-1}},\overbrace{0,\dots,0}^{n-j_k}\Big),$$

where we set $j_0 = 0$. Now by adding the identity matrix I to both sides and rearranging we have,

$$I + \sum_{j\in J} D_j = I + \sum_{k=1}^{l}(l+1-k)\cdot B_k = \sum_{k=1}^{l+1} k\cdot B_{l+2-k}, \tag{5.2.2}$$

where

$$B_{l+1} = \operatorname{diag}\Big(\overbrace{0,\dots,0}^{j_l},\overbrace{1,\dots,1}^{n-j_l}\Big).$$

Let us consider the first sum of (5.2.1) minus the identity matrix,

$$-I + \sum_{j\in J}\frac{-j}{n}\cdot I = (-1 + \sum_{j\in J}\frac{-j}{n})\cdot I = \frac{-1}{n}(n + \sum_{j\in J} j)\cdot I = \frac{-1}{n}(n + \sum_{k=1}^{l} j_k)\cdot I.$$

By noting that $\operatorname{trace}(\sum_{k=1}^{l+1} B_k) = \operatorname{rank}(\sum_{k=1}^{l+1} B_k) = n$ and $j_k = \sum_{h=1}^{k}\operatorname{rank}(B_h) = \sum_{h=1}^{k}\operatorname{trace} B_h$, then

$$\begin{aligned}
\frac{-1}{n}(n + \sum_{k=1}^{l} j_k) &= \frac{-1}{n}(\operatorname{rank}(\sum_{k=1}^{l+1} B_k) + \sum_{k=1}^{l}\sum_{h=1}^{k}\operatorname{rank}(B_h))\\
&= \frac{-1}{n}(\operatorname{rank}(\sum_{k=1}^{l+1} B_k) + \sum_{k=1}^{l} k\operatorname{rank}(B_{l+1-k}))\\
&= \frac{-1}{n}(\operatorname{rank}(B_{l+1}) + \sum_{k=1}^{l}(k+1)\operatorname{rank}(B_{l+1-k}))\\
&= \frac{-1}{n}(\sum_{k=1}^{l+1} k\operatorname{rank}(B_{l+2-k})).
\end{aligned}$$

So,

$$-I+\sum_{j\in J}\frac{-j}{n}\cdot I=\frac{-1}{n}(\sum_{k=1}^{l+1}k\,\mathrm{rank}(B_{l+2-k}))\cdot I. \tag{5.2.3}$$

Using (5.2.1),(5.2.2) and (5.2.3) we have

$$\xi=i\Big(\frac{-1}{n}\sum_{k=1}^{l+1}k\,\mathrm{rank}(B_{l+2-k})\cdot I+\sum_{k=1}^{l+1}k\cdot B_{l+2-k}\Big),$$

where the $B_1, B_2, \ldots, B_{l+1}$ are orthogonal projection matrices giving a decomposition of $\mathbb{C}^n$, exactly as in Corollary 5.2.15.

5.2.2 Canonical Elements for $O(n)$

Theorem 5.2.16. *[7, Proposition 4.1] ξ is a canonical element for $\mathfrak{so}(n)$ if and only if either*

(i) *For some $k\in\mathbb{N}$ with $2k+1\le n$, ξ has eigenvalues $\pm il$, $0\le l\le k$, or*

(ii) *For some $k\in\mathbb{N}$ with $2k+2\le n$, ξ has eigenvalues $\pm i(l+\frac{1}{2})$, $0\le l\le k$, and the eigenvalues $\pm\frac{i}{2}$ have multiplicity at least 2.*

Note that this theorem classifies canonical elements up to conjugacy and is independent of the choice of simple roots of a positive root system with respect to some Cartan subalgebra. We give some justification for this theorem by describing the canonical elements with respect to some Cartan subalgebra as in [10]. First we deal with $O(n)$ for $n=2m$ and $m\in\mathbb{N}$.

Consider a maximal torus for $O(2m)$ (or $SO(2m)$)

$$T_1=\left\{\mathrm{diag}(R_1,R_2,\ldots,R_m)\;\middle|\;R_j=\begin{pmatrix}\cos(a_j) & \sin(a_j)\\ -\sin(a_j) & \cos(a_j)\end{pmatrix},\ a_j\in\mathbb{R}\right\},$$

which has Lie algebra

$$\mathfrak{t}_1 = \left\{ \left(\begin{array}{ccccc} 0 & a_1 & & & \\ -a_1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & a_m \\ & & & -a_m & 0 \end{array} \right) \middle| \, a_j \in \mathbb{R} \right\},$$

and is a maximal toral subalgebra of $\mathfrak{so}(n)$. We will regard T_1 as a real subgroup of $O(2m, \mathbb{C})$ and $\mathfrak{t}_1$ as a real subalgebra of $\mathfrak{so}(2m, \mathbb{C})$ in the obvious way and write them with respect to the null basis (5.1.2) given above. So we have in the null basis (5.1.2) that

$$T_1 = \left\{ \mathrm{diag}(e^{ia_1}, \ldots, e^{ia_m}, e^{-ia_m}, \ldots, e^{-ia_1}) \,\middle|\, a_j \in \mathbb{R} \right\}$$

and

$$\mathfrak{t}_1 = \{ \mathrm{diag}(ia_1, \ldots, ia_m, -ia_m, \ldots, -ia_1) \,|\, a_j \in \mathbb{R} \} .$$

Therefore a Cartan subalgebra for $\mathfrak{so}(2m, \mathbb{C})$ is

$$\mathfrak{t}_1^{\mathbb{C}} = \{ \mathrm{diag}(\eta_1, \ldots, \eta_m, -\eta_m, \ldots, -\eta_1) \,|\, \eta_j \in \mathbb{C} \} .$$

According to Definition 5.2.5 to find the roots of $\mathfrak{so}(n, \mathbb{C})$ with respect to $\mathfrak{t}_1^{\mathbb{C}}$ we need to consider $ad(\eta)v$ for $\eta \in \mathfrak{t}_1^{\mathbb{C}}$ and $v \in \mathfrak{so}(n, \mathbb{C}) = \{A \in \mathfrak{gl}(n, \mathbb{C}) \mid A^{\mathfrak{T}} = -A\}$. Let

$\eta = \operatorname{diag}(\eta_1, \ldots, \eta_m, -\eta_m, \ldots, -\eta_1) \in \mathfrak{t}_1^{\mathbb{C}}$ and, always using the null basis 5.1.2,

$$v = \begin{pmatrix} v_{1,1} & v_{1,2} & \cdots & \cdots & v_{1,2m-1} & 0 \\ v_{2,1} & v_{2,2} & & & 0 & -v_{1,2m-1} \\ \vdots & & & \iddots & & \vdots \\ \vdots & & \iddots & & & \vdots \\ v_{2m-1,1} & 0 & & & -v_{2,2} & -v_{1,2} \\ 0 & -v_{2m-1,1} & \cdots & \cdots & -v_{2,1} & -v_{1,1} \end{pmatrix} \in \mathfrak{so}(2m, \mathbb{C}).$$

Then considering $ad(\eta)v = \lambda(\eta)v$ we see that the roots λ are

$$i(\eta_j + \eta_k), \qquad -i(\eta_j + \eta_k), \qquad i(\eta_j - \eta_k), \qquad -i(\eta_j - \eta_k),$$

where $\eta_j, \eta_k \in \mathbb{C}$ and $j, k \in \{1, 2, \ldots, m\}$, $j < k$. The corresponding root spaces to these roots are spanned by

$$\begin{pmatrix} 0 & E_{j\bar{k}} - E_{k\bar{j}} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ E_{\bar{k}j} - E_{\bar{j}k} & 0 \end{pmatrix}, \quad \begin{pmatrix} E_{jk} & 0 \\ 0 & -E_{\bar{k}\bar{j}} \end{pmatrix}, \quad \begin{pmatrix} E_{kj} & 0 \\ 0 & -E_{\bar{j}\bar{k}} \end{pmatrix},$$

where E_{jk} is a $m \times m$ matrix of zeros in all positions other than the (j, k) position, which has a 1. We choose the positive root system Δ^+ to be the set containing the roots $i(\eta_j - \eta_k)$ and $i(\eta_j + \eta_k)$ for all $j, k \in \{1, 2, \ldots, m\}$, $j < k$, and we choose the simple roots of this positive root system to be $\lambda_j(\eta) = i(\eta_j - \eta_{j+1})$ for $j = 1, 2, \ldots, m-1$ together with $\lambda_m(\eta) = i(\eta_{m-1} + \eta_m)$. We see that any element of Δ^+ can be expressed as follows:

$$i(\eta_j - \eta_k) = \sum_{\alpha=j}^{k-1} \lambda_\alpha, \quad j, k \in \{1, 2, \ldots, m\},\ j < k,$$

$$i(\eta_j + \eta_k) = \sum_{\alpha=j}^{k-1} \lambda_\alpha + 2 \sum_{\beta=k}^{m-2} \lambda_\beta + \lambda_{m-1} + \lambda_m, \quad j, k \in \{1, 2, \ldots, m-1\},\ j < k,$$

$$i(\eta_j + \eta_m) = \sum_{\alpha=j}^{m-2} \lambda_\alpha + \lambda_m, \quad j \in \{1, 2, \dots, m-2\}.$$

We now wish to find the dual vectors to the simple roots, i.e. the unique $A_1, \dots, A_m \in \mathfrak{t}$ such that

$$\lambda_j(A_k) = i\delta_{jk}, \quad \forall k \in \{1, 2, \dots, m\}, \quad j \in \{1, 2, \dots, m\}.$$

By inspection these are the diagonal matrices of the form,

$$A_k = i \operatorname{diag}\Big(\overbrace{1, \dots, 1}^{k}, \overbrace{0, \dots, 0,}^{2(m-k)} \overbrace{-1, \dots, -1,}^{k}\Big), \tag{5.2.4}$$

$$A_{m-1} = \frac{i}{2} \operatorname{diag}\Big(\overbrace{1, \dots, 1}^{m-1}, -1, 1, \overbrace{-1, \dots, -1,}^{m-1}\Big), \qquad A_m = \frac{i}{2}\begin{pmatrix} I_m & \\ & -I_m \end{pmatrix},$$

for $k = 1, \dots, m-2$. According to Definition 5.2.11 canonical elements $\xi \in \mathfrak{so}(2m)$ are of the form

$$\xi = \sum_{i \in I} A_i,$$

for $I \subseteq \{1, 2, \dots, m\}$ and A_i above. It is easy to see that these satisfy Theorem 5.2.16.

A similar argument can be made for odd dimensional orthogonal groups. Consider a maximal torus for $O(2m+1)$,

$$T_2 = \left\{ \operatorname{diag}(R_1, \dots, R_m, 1) \,\middle|\, R_j = \begin{pmatrix} \cos(a_j) & \sin(a_j) \\ -\sin(a_j) & \cos(a_j) \end{pmatrix}, a_j \in \mathbb{R} \right\}.$$

Again by seeing this real space as a subspace of $O(2m+1, \mathbb{C})$ we may write T_2 with respect to the null basis (5.1.2), we have

$$T_2 = \left\{ \operatorname{diag}(e^{ia_1}, \dots, e^{ia_m}, 1, e^{-ia_m}, \dots, e^{-ia_1}) \,\middle|\, a_j \in \mathbb{R} \right\},$$

with Lie algebra

$$\mathfrak{t}_2 = \left\{ \operatorname{diag}(ia_1, \dots, ia_m, 0, -ia_m, \dots, -ia_1) \,|\, a_j \in \mathbb{R} \right\}.$$

Therefore a Cartan subalgebra for $\mathfrak{so}(2m+1, \mathbb{C})$ is

$$\mathfrak{t}_2^{\mathbb{C}} = \left\{ \operatorname{diag}(\eta_1, \dots, \eta_m, 0, -\eta_m, \dots, -\eta_1) \,|\, \eta_j \in \mathbb{C} \right\}.$$

Again from Definition 5.2.5 we find the roots λ of $\mathfrak{so}(2m+1, \mathbb{C})$ with respect to $\mathfrak{t}_2^{\mathbb{C}}$ by considering $ad(\eta)v = \lambda(\eta)v$ for $\eta \in \mathfrak{t}_2^{\mathbb{C}}$ and $v \in \mathfrak{so}(2m+1, \mathbb{C}) = \{A \in \mathfrak{gl}(n, \mathbb{C}) \mid A^{\mathfrak{T}} = -A\}$. We see that the roots are

$$i(\eta_j + \eta_k), \qquad -i(\eta_j + \eta_k), \qquad i(\eta_j - \eta_k), \qquad -i(\eta_j - \eta_k), \qquad i\eta_k, \qquad -i\eta_k,$$

where $\eta_j, \eta_k \in \mathbb{C}$ and $j, k \in \{1, 2, \dots, m\}$, $j < k$. Simple roots to a positive root system can be chosen to be $\lambda_j(\eta) = i(\eta_j - \eta_{j+1})$, $j \in \{1, \dots, m-1\}$ together with $\lambda_m = i\eta_m$. As before we wish to find dual vectors to these positive roots i.e. $A_1, \dots, A_m \in \mathfrak{t}$ such that

$$\lambda_j(A_k) = i\delta_{jk}, \quad \forall k \in \{1, 2, \dots, m\}, \quad j \in \{1, 2, \dots, m\}.$$

By inspection these are diagonal matrices of the form,

$$A_k = i \operatorname{diag}\Big(\overbrace{1, \dots, 1}^{k}, \overbrace{0, \dots, 0}^{2(m-k)+1}, \overbrace{-1, \dots, -1}^{k}\Big), \tag{5.2.5}$$

for $k = 1, \dots, m$. According to Definition 5.2.11 canonical elements ξ of $\mathfrak{so}(2m+1)$ are of the form

$$\xi = \sum_{i \in I} A_i,$$

for $I \subseteq \{1, 2, \dots, m\}$ and A_i above. Note that these canonical elements have eigenvalues $\pm il$ for $1 \leq l \leq m$ and therefore satisfy Theorem 5.2.16. We therefore have the result

Proposition 5.2.17. *Let* T_1, T_2, $\mathfrak{t}_1$, $\mathfrak{t}_2$, *and* $\mathfrak{t}_1^{\mathbb{C}}$, $\mathfrak{t}_2^{\mathbb{C}}$ *be the maximal tori, maximal toral subalgebras and the Cartan subalgebras of* $O(2m, \mathbb{C})$ *and* $O(2m+1, \mathbb{C})$*, respectively, with the usual choices of simple roots of the positive root systems* Δ^+ *as above. Then*

(i) ξ *is a canonical element for* $\mathfrak{so}(2m+1)$ *if and only if*

$$\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_m, \xi_{m+1}, -\xi_m, -\xi_{m-1}, \ldots, -\xi_1),$$

where ξ_j *are positive integers such that* $\xi_j - \xi_{j+1} = 1$ *or* 0*, for all* $j = 1, 2, \ldots, m$*, and* $\xi_{m+1} = 0$.

(ii) ξ *is a canonical element for* $\mathfrak{so}(2m)$ *if and only if*

$$\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_m, -\xi_m, -\xi_{m-1}, \ldots, -\xi_1),$$

where ξ_j *are positive integers or half integers such that if* ξ_1 *is an integer then* $\xi_j - \xi_{j+1} = 1$ *or* 0*, for all* $j = 1, 2, \ldots, m-1$*,* $\xi_m = 0$*, and if* ξ_1 *is a half integer then* $\xi_j - \xi_{j+1} = 1$ *or* 0*, for all* $j = 1, 2, \ldots, m-2$*, and either* $\xi_{m-1} = \xi_m = 1/2$ *or* $\xi_{m-1} = 1/2, \xi_m = -1/2$.

Proof. For (i) we consider the A_j from (5.2.5) for $j \in I \subseteq \{1, 2, \ldots, m\}$. We order the elements of $I = \{j_1, j_2, \ldots, j_\alpha\}$ so $j_\beta < j_{\beta+1}$ for all $\beta = 1, 2, \ldots, \alpha - 1$, and so $|I| = \alpha \le m$. Then

$$\xi = \sum_{j \in I} A_j = i\,\mathrm{diag}\Big(\overbrace{\alpha, \ldots, \alpha}^{j_1}, \overbrace{\alpha - 1, \ldots, \alpha - 1}^{j_2 - j_1}, \ldots, \overbrace{1, \ldots, 1}^{j_\alpha - j_{\alpha-1}}, \overbrace{0, \ldots, 0}^{2(m-j_\alpha)+1},$$
$$\overbrace{-1, \ldots, -1}^{j_\alpha - j_{\alpha-1}}, \ldots, \overbrace{1 - \alpha, \ldots, 1 - \alpha}^{j_2 - j_1}, \overbrace{-\alpha, \ldots, -\alpha}^{j_1}\Big)$$

and by relabelling we have $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_m, \xi_{m+1}, -\xi_m, -\xi_{m-1}, \ldots, -\xi_1)$ where ξ_j are positive integers such that $\xi_l - \xi_{l+1} = 1$ or 0, for all $l = 1, 2, \ldots, m$ and $\xi_{m+1} = 0$.

For (ii) we consider the A_j from (5.2.4) for $j \in I \subseteq \{1, 2, \ldots, m\}$. Suppose that $m \notin I$ and $m - 1 \notin I$ and order the elements of $I = \{j_1, j_2, \ldots, j_\alpha\}$ so $j_\beta < j_{\beta+1}$ for all $\beta = 1, 2, \ldots, \alpha - 1$, $j_\alpha \leq m - 2$ and so $|I| = \alpha \leq m - 2$. Then

$$\xi = \sum_{j \in I} A_j = i \operatorname{diag}\Big(\overbrace{\alpha, \ldots, \alpha}^{j_1}, \overbrace{\alpha - 1, \ldots, \alpha - 1}^{j_2 - j_1}, \ldots, \overbrace{1, \ldots, 1}^{j_\alpha - j_{\alpha-1}}, \overbrace{0, \ldots, 0}^{2(m - j_\alpha)},$$
$$\overbrace{-1, \ldots, -1}^{j_\alpha - j_{\alpha-1}}, \ldots, \overbrace{1 - \alpha, \ldots, 1 - \alpha}^{j_2 - j_1}, \overbrace{-\alpha, \ldots, -\alpha}^{j_1}\Big)$$

and by relabelling we have $\xi = i \operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_m, -\xi_m, -\xi_{m-1}, \ldots, -\xi_1)$ where ξ_l are positive integers such that $\xi_l - \xi_{j+1} = 1$ or 0, for all $l = 1, 2, \ldots, m - 1$ and $\xi_m = 0$.

Now let $m, m - 1 \in I$ then we have $I = \{j_1, j_2, \ldots, j_\alpha, m - 1, m\}$, and so $2 \leq |I| = \alpha + 2 \leq m$. Note that

$$A_{m-1} + A_m = i \operatorname{diag}\Big(\overbrace{1, \ldots, 1}^{m-1}, 0, 0, \overbrace{1, \ldots, 1}^{m-1}\Big)$$

and so

$$\xi = \sum_{j \in I} A_j = i \operatorname{diag}\Big(\overbrace{\alpha + 1, \ldots, \alpha + 1}^{j_1}, \overbrace{\alpha, \ldots, \alpha}^{j_2 - j_1}, \ldots, \overbrace{1, \ldots, 1}^{m - 1 - j_\alpha}, 0, 0,$$
$$\overbrace{-1, \ldots, -1}^{m - 1 - j_\alpha}, \ldots, \overbrace{-\alpha, \ldots, -\alpha}^{j_2 - j_1}, \overbrace{-\alpha - 1, \ldots, -\alpha - 1}^{j_1}\Big).$$

By relabelling, similarly to above, we have
$\xi = i \operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_m, -\xi_m, -\xi_{m-1}, \ldots, -\xi_1)$ where ξ_l are positive integers such that $\xi_l - \xi_{j+1} = 1$ or 0, for all $l = 1, 2, \ldots, m - 1$ and $\xi_m = 0$.

For $m \notin I,\ m - 1 \in I$ we have that $I = \{j_1, j_2, \ldots, j_\alpha, m - 1\}$ and

$$\xi = \sum_{j \in I} A_j = i \operatorname{diag}\Big(\overbrace{\alpha + \tfrac{1}{2}, \ldots, \alpha + \tfrac{1}{2}}^{j_1}, \overbrace{\alpha - \tfrac{1}{2}, \ldots, \alpha - \tfrac{1}{2}}^{j_2 - j_1}, \ldots, \overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{m - 1 - j_\alpha}, -\tfrac{1}{2}, \tfrac{1}{2},$$

$$\overbrace{-\tfrac{1}{2},\ldots,-\tfrac{1}{2}}^{m-1-j_\alpha},\ldots,\overbrace{-\alpha,\ldots,-\alpha}^{j_2-j_1},\overbrace{-\alpha-1,\ldots,-\alpha-1}^{j_1}\Big).$$

Relabelling, we have $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_m, -\xi_m, -\xi_{m-1}, \ldots, -\xi_1)$ where ξ_l are positive half integers such that $\xi_l - \xi_{j+1} = 1$ or 0, for all $l = 1, 2, \ldots, m-2$ and $\xi_{m-1} = -\xi_m = \frac{1}{2}$.

Finally, let $m \in I$ and $m-1 \notin I$ then we have $I = \{j_1, j_2, \ldots, j_\alpha, m\}$ and

$$\xi = \sum_{j\in I} A_j = i\,\mathrm{diag}\Big(\overbrace{\alpha+\tfrac{1}{2},\ldots,\alpha+\tfrac{1}{2}}^{j_1},\overbrace{\alpha-\tfrac{1}{2},\ldots,\alpha-\tfrac{1}{2}}^{j_2-j_1},\ldots,\overbrace{\tfrac{1}{2},\ldots,\tfrac{1}{2}}^{m-j_\alpha}$$

$$\overbrace{-\tfrac{1}{2},\ldots,-\tfrac{1}{2}}^{m-j_\alpha},\ldots,\overbrace{-\alpha,\ldots,-\alpha}^{j_2-j_1},\overbrace{-\alpha-1,\ldots,-\alpha-1}^{j_1}\Big).$$

Relabelling we have $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_m, -\xi_m, -\xi_{m-1}, \ldots, -\xi_1)$ where ξ_l are positive half integers such that $\xi_l - \xi_{l+1} = 1$ or 0, for all $l = 1, 2, \ldots, m-2$ and $\xi_{m-1} = \xi_m = \frac{1}{2}$. □

Example 5.2.18. *For $\mathfrak{so}(6, \mathbb{C})$ the elements of Δ^+ are $\lambda_1(\eta) = i(\eta_1 - \eta_2)$, $\lambda_2(\eta) = i(\eta_2 - \eta_3)$, $\lambda_3(\eta) = i(\eta_2 + \eta_3)$. The dual vectors to these simple roots are*

$$A_1 = i\,\mathrm{diag}(1, 0, 0, 0, 0, -1), \qquad A_2 = \frac{i}{2}\,\mathrm{diag}(1, 1, -1, 1, -1, -1),$$

$$A_3 = \frac{i}{2}\,\mathrm{diag}(1, 1, 1, -1, -1, -1),$$

and so the canonical elements of $\mathfrak{so}(6)$ are

$$I = \{1\} : \xi = A_1 = i\,\mathrm{diag}(1, 0, 0, 0, 0, -1),$$

$$I = \{2\} : \xi = A_2 = \frac{i}{2}\,\mathrm{diag}(1, 1, -1, 1, -1, -1),$$

$$I = \{3\} : \xi = A_3 = \frac{i}{2}\,\mathrm{diag}(1, 1, 1, -1, -1, -1),$$

$$I = \{1+2\} : \xi = A_1 + A_2 = \frac{i}{2}\operatorname{diag}(3,1,-1,1,-1,-3),$$

$$I = \{1+3\} : \xi = A_1 + A_3 = \frac{i}{2}\operatorname{diag}(3,1,1,-1,-1,-3),$$

$$I = \{2+3\} : \xi = A_2 + A_3 = i\operatorname{diag}(1,1,0,0,-1,-1),$$

$$I = \{1+2+3\} : \xi = A_1 + A_2 + A_3 = i\operatorname{diag}(2,1,0,0,-1,-2).$$

Example 5.2.19. *For $\mathfrak{so}(5,\mathbb{C})$ the elements of Δ^+ are $\lambda_1(\eta) = i(\eta_1 - \eta_2)$ and $\lambda_2(\eta) = i(\eta_2 - \eta_3)$. The dual vectors to these simple roots are*

$$A_1 = i\operatorname{diag}(1,0,0,0,-1), \qquad A_2 = i\operatorname{diag}(1,1,0,-1,-1),$$

and so the canonical elements of $\mathfrak{so}(5)$ are

$$I = \{1\} : \xi = A_1 = i\operatorname{diag}(1,0,0,0,-1),$$

$$I = \{2\} : \xi = A_2 = i\operatorname{diag}(1,1,0,-1,-1),$$

$$I = \{1+2\} : \xi = A_1 + A_2 = i\operatorname{diag}(2,1,0,-1,-2).$$

5.2.3 Canonical Elements for $Sp(m)$

For use in later sections we will give a description of the canonical elements for the symplectic group, $Sp(m)$. Also, for the benefit of later sections we will give all matrices with respect to the null basis (5.1.9) given above. Recall in this basis that $Sp(2m,\mathbb{C}) = \{A \in GL(2m,\mathbb{C}) \mid A^{\mathfrak{T}}\Omega_{\text{null}}A = \Omega_{\text{null}}\}$, therefore its Lie algebra in this basis is given by $\mathfrak{sp}(2m,\mathbb{C}) = \{A \in \mathfrak{gl}(2m,\mathbb{C}) \mid A^{\mathfrak{T}}\Omega_{\text{null}} + \Omega_{\text{null}}A = 0\}$ and $Sp(m) = Sp(2m,\mathbb{C}) \cap U(2m)$. Consider the torus $T \subset Sp(m)$:

$$T = \left\{\operatorname{diag}(e^{ia_1},\ldots,e^{ia_m},e^{-ia_m},\ldots,e^{-ia_1}) \,\middle|\, a_j \in \mathbb{R}\right\}, \tag{5.2.6}$$

with Lie algebra

$$\mathfrak{t} = \{i\,\mathrm{diag}(a_1, \ldots, a_m, -a_m, \ldots, -a_1) \mid a_j \in \mathbb{R},\ \forall j\}. \tag{5.2.7}$$

Considering Definitions 5.2.1, 5.2.2 and Proposition 5.2.3 we see that this is a maximal toral subalgebra of $\mathfrak{sp}(m)$, T is a maximal torus of $Sp(m)$ and $\mathfrak{t}^{\mathbb{C}}$ is a Cartan subalgebra for $\mathfrak{sp}(m)^{\mathbb{C}} = \mathfrak{sp}(2m, \mathbb{C})$ where

$$\mathfrak{t}^{\mathbb{C}} = \{\mathrm{diag}(\eta_1, \ldots, \eta_m, -\eta_m, \ldots, -\eta_1) \mid \eta_j \in \mathbb{C}\ \forall i\}. \tag{5.2.8}$$

By Definition 5.2.5 the roots of $\mathfrak{sp}(2m, \mathbb{C})$ relative to $\mathfrak{t}^{\mathbb{C}}$ are linear maps $\lambda : \mathfrak{t}^{\mathbb{C}} \to \mathbb{C}$ satisfying $ad(\xi)v = \lambda(\xi)v$ for all $\xi \in \mathfrak{t}^{\mathbb{C}}$ and for some non-zero $v \in \mathfrak{sp}(2m, \mathbb{C})$. The elements of $\mathfrak{sp}(2m, \mathbb{C})$ represented in the null basis are precisely the $2m \times 2m$ matrices of the form

$$\begin{pmatrix} A & B \\ C & -A^{\mathfrak{T}} \end{pmatrix},$$

where A is an arbitrary $m \times m$ matrix with both B and C **second-symmetric** matrices i.e. matrices that are symmetric with respect to the second diagonal, so $B^{\mathfrak{T}} = B$ and $C^{\mathfrak{T}} = C$. For a description of $\mathfrak{sp}(2m, \mathbb{C})$ in the standard basis see [34, p.41].

The roots of $\mathfrak{sp}(n)^{\mathbb{C}}$ relative to $\mathfrak{t}^{\mathbb{C}}$ are linear maps $\mathfrak{t}^{\mathbb{C}} \to \mathbb{C}$ defined by

$$i(\eta_j + \eta_k), \qquad -i(\eta_j + \eta_k), \qquad i(\eta_j - \eta_k), \qquad 2i\eta_k, \qquad -2i\eta_k,$$

where $\eta_j, \eta_k \in \mathbb{C}$ and $j, k \in \{1, 2, \ldots, m\}$, $j \neq k$. The corresponding root spaces to these roots are spanned by

$$\begin{pmatrix} 0 & E_{jk} + E_{kj} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ E_{jk} + E_{kj} & 0 \end{pmatrix}, \quad \begin{pmatrix} E_{jk} & 0 \\ 0 & E_{kj} \end{pmatrix},$$

$$\begin{pmatrix} 0 & E_{j\bar{j}} \\ 0 & 0 \end{pmatrix}, \quad \begin{pmatrix} 0 & 0 \\ E_{\bar{j}j} & 0 \end{pmatrix},$$

where E_{jk} is an $m \times m$ matrix of zeros in all positions other than the (j, k) position, which has a 1. We choose the simple roots of the positive root system Δ^+ to be $\lambda_j(\eta) = i(\eta_j - \eta_{j+1})$ and $\lambda_m(\eta) = 2i\eta_m$ with associated root spaces v_j, v_m, respectively, where

$$v_j = \begin{pmatrix} E_{jj+1} & 0 \\ 0 & E_{j+1j} \end{pmatrix}, \qquad v_m = \begin{pmatrix} 0 & E_{m\bar{m}} \\ 0 & 0 \end{pmatrix}.$$

The dual vectors to the simple roots are $A_1, \ldots, A_m \in \mathfrak{t}$ such that

$$\lambda_j(A_k) = i\delta_{jk} \quad \text{and} \quad \lambda_m(A_k) = i\delta_{mk} \quad \forall k = 1, \ldots, m, \quad j = 1, \ldots, m-1.$$

These are diagonal matrices of the form

$$A_k = i \operatorname{diag}\Big(\overbrace{1, \ldots, 1}^{k}, \overbrace{0, \ldots, 0}^{2(m-k)}, \overbrace{-1, \ldots, -1}^{k}\Big), \quad A_m = \frac{i}{2}\begin{pmatrix} I_m & \\ & -I_m \end{pmatrix}, \tag{5.2.9}$$

for $k = 1, \ldots, m-1$. According to Definition 5.2.11 canonical elements $\xi \in \mathfrak{sp}(m)$ are of the form

$$\xi = \sum_{j \in I} A_j,$$

for $I \subseteq \{1, 2, \ldots, m\}$ and A_j above.

Proposition 5.2.20. *Let T be a maximal torus of $Sp(m)$ given by (5.2.6), $\mathfrak{t}$ the corresponding maximal toral subalgebra of $\mathfrak{sp}(2m)$ given by (5.2.7) and $\mathfrak{t}^{\mathbb{C}}$ a Cartan subalgebra of $\mathfrak{sp}(2m, \mathbb{C})$ given by (5.2.8), with the usual choice of simple roots of the positive root system Δ^+ as above. Then $\xi \in \mathfrak{t}$ is a canonical element for $\mathfrak{sp}(2m)$ if and only if $\xi = i \operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_m, -\xi_m, -\xi_{m-1}, \ldots, -\xi_1)$ where ξ_j are positive integers and half* *integers such that*

$$\xi_j - \xi_{j+1} = 1 \ \textit{or}\ 0, \quad \textit{for all}\ j = 1, 2, \ldots, m-1, \quad \xi_m = 0 \ \textit{or}\ \frac{1}{2}.$$

Proof. Considering the A_j from (5.2.9) for $j \in I \subseteq \{1, 2, \ldots, m\}$, let $m \notin I$ and order the elements of $I = \{j_1, j_2, \ldots, j_\alpha\}$ so $j_\beta < j_{\beta+1}$ for all $\beta = 1, 2, \ldots, \alpha - 1$ and $|I| = \alpha \leq m - 1$. Then

$$\xi = \sum_{j \in I} A_j = i\,\mathrm{diag}\Big(\overbrace{\alpha, \ldots, \alpha}^{j_1}, \overbrace{\alpha - 1, \ldots, \alpha - 1}^{j_2 - j_1}, \ldots, \overbrace{1, \ldots, 1}^{j_\alpha - j_{\alpha-1}}, \overbrace{0, \ldots, 0}^{2(m - j_\alpha)},$$
$$\overbrace{-1, \ldots, -1}^{j_\alpha - j_{\alpha-1}}, \ldots, \overbrace{1 - \alpha, \ldots, 1 - \alpha}^{j_2 - j_1}, \overbrace{-\alpha, \ldots, -\alpha}^{j_1}\Big)$$

and by relabelling we have $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_m, -\xi_m, -\xi_{m-1}, \ldots, -\xi_1)$ where ξ_j are positive integers such that $\xi_j - \xi_{j+1} = 1$ or 0, for all $j = 1, 2, \ldots, m-1$ and $\xi_m = 0$.

If $m \in I$ then we have $I = \{j_1, j_2, \ldots, j_\alpha, m\}$ and

$$\xi = \sum_{j \in I} A_j = i\,\mathrm{diag}\Big(\overbrace{\alpha + \tfrac{1}{2}, \ldots, \alpha + \tfrac{1}{2}}^{j_1}, \overbrace{\alpha - \tfrac{1}{2}, \ldots, \alpha - \tfrac{1}{2}}^{j_2 - j_1}, \ldots, \overbrace{\tfrac{3}{2}, \ldots, \tfrac{3}{2}}^{j_\alpha - j_{\alpha-1}}, \overbrace{\tfrac{1}{2}, \ldots, \tfrac{1}{2}}^{2(m - j_\alpha)},$$
$$\overbrace{-\tfrac{3}{2}, \ldots, -\tfrac{3}{2}}^{j_\alpha - j_{\alpha-1}}, \ldots, \overbrace{\tfrac{1}{2} - \alpha, \ldots, \tfrac{1}{2} - \alpha}^{j_2 - j_1}, \overbrace{-\alpha - \tfrac{1}{2}, \ldots, -\alpha - \tfrac{1}{2}}^{j_1}\Big).$$

By rebelling we have $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_m, -\xi_m, -\xi_{m-1}, \ldots, -\xi_1)$ where ξ_j are half integers such that $\xi_j - \xi_{j+1} = 1$ or 0, for all $j = 1, 2, \ldots, m-1$ and $\xi_m = \frac{1}{2}$. □

Example 5.2.21. *For $\mathfrak{sp}(6, \mathbb{C})$ the elements of Δ^+ are $\lambda_1(\eta) = i(\eta_1 - \eta_2)$, $\lambda_2(\eta) = i(\eta_2 - \eta_3)$, $\lambda_3(\eta) = 2i(\eta_3)$. The dual vectors to these simple roots are*

$$A_1 = i\,\mathrm{diag}(1, 0, 0, 0, 0, -1), \qquad A_2 = i\,\mathrm{diag}(1, 1, 0, 0, -1, -1),$$

$$A_3 = \frac{i}{2}\,\mathrm{diag}(1, 1, 1, -1, -1, -1).$$

and so the canonical elements of $\mathfrak{sp}(3)$ *are as follows:*

$$I = \{1\} : \xi = A_1 = i\,\mathrm{diag}(1, 0, 0, 0, 0, -1),$$

$$I = \{2\} : \xi = A_2 = i\,\mathrm{diag}(1, 1, 0, 0, -1, -1),$$

$$I = \{3\} : \xi = A_3 = \frac{i}{2}\,\mathrm{diag}(1, 1, 1, -1, -1, -1),$$

$$I = \{1 + 2\} : \xi = A_1 + A_2 = i\,\mathrm{diag}(2, 1, 0, 0, -1, -2),$$

$$I = \{1 + 3\} : \xi = A_1 + A_3 = \frac{i}{2}\,\mathrm{diag}(3, 1, 1, -1, -1, -3),$$

$$I = \{2 + 3\} : \xi = A_2 + A_3 = \frac{i}{2}\,\mathrm{diag}(3, 3, 1, -1, -3, -3),$$

$$I = \{1 + 2 + 3\} : \xi = A_1 + A_2 + A_3 = \frac{i}{2}\,\mathrm{diag}(5, 3, 1, -1, -3, -5).$$

Chapter 6

Harmonic Maps from Surfaces to Lie Groups

In [50], K. Uhlenbeck developed the theory of harmonic maps into Lie groups by introducing "polynomial extended solutions" of a harmonic map. Uhlenbeck showed that such a polynomial extended solution can be factorized with respect to certain subbundles of $\underline{\mathbb{C}}^n := M \times \mathbb{C}^n$ called "unitons". The Grassmannian model of an extended solution was introduced by G. Segal in [46]. The Grassmannian model represents an extended solution by a subbundle W of the trivial bundle $M \times \mathcal{H}$ for $\mathcal{H}$ a Hilbert space.

In [9], F.E. Burstall and M.A. Guest used canonical elements and certain maps into a loop group to classify all polynomial extended solutions for harmonic maps into the unitary group. These extended solutions were given by integration, with equations which are easy to solve for $U(n)$, especially for low dimensions. By viewing $O(n)$ as a subgroup of $U(n)$, M.J. Ferreira, B.A. Simões and J.C. Wood in [29] applied the work of F.E. Burstall and M.A. Guest to give a classification of extended solutions for harmonic maps into the orthogonal group according to their canonical elements. Further, they gave a parametrization (at least locally) of these extended solutions in terms of free holomorphic data by replacing every instance of integration with differentiation and algebraic operations.

In this chapter we give a description of the theory of harmonic maps into Lie groups, setting up the theory needed in Chapter 7.

6.1 Harmonic Maps into Lie Groups

Let $\varphi : M \to G$ be a smooth map from a Riemann surface to a Lie group. Define a 1-form, A^φ, with values in its Lie algebra $\mathfrak{g}$ by

$$A^\varphi = \frac{1}{2}\varphi^{-1}d\varphi;$$

decomposing into $(1,0)$- and $(0,1)$- parts we have

$$A^\varphi = A^\varphi_z dz + A^\varphi_{\overline{z}} d\overline{z}$$

where

$$A^\varphi_z = \frac{1}{2}\varphi^{-1}\varphi_z\,, \quad A^\varphi_{\overline{z}} = \frac{1}{2}\varphi^{-1}\varphi_{\overline{z}},$$

for z a local complex coordinate. Note that A^φ is half the pullback of the Maurer-Cartan form on $\mathfrak{g}$, and both A^φ_z and $A^\varphi_{\overline{z}}$ are local sections of the endomorphism bundle $\mathrm{End}(\underline{\mathbb{C}}^n)$ where $\underline{\mathbb{C}}^n := M \times \mathbb{C}^n$ is the complex trivial bundle over M. We define a unitary connection D^φ by

$$D^\varphi = d + A^\varphi$$

and, again decomposing into $(1,0)$- and $(0,1)$- parts, we have

$$D^\varphi_z = \partial_z + A^\varphi_z\,, \quad D^\varphi_{\overline{z}} = \partial_{\overline{z}} + A^\varphi_{\overline{z}}.$$

Recall from Chapter 4 that any complex vector bundle $E \to M$ over a Riemann surface equipped with a linear connection ∇ can be given a unique holomorphic structure (called the Koszul-Malgrange complex structure [36]) from the $(0,1)$-part of ∇ by realising it as a $\overline{\partial}$-operator. Therefore we may give $\underline{\mathbb{C}}^n$ a unique holomorphic structure from $D^{\varphi}_{\bar{z}}$, by which we mean that any local section σ of $\underline{\mathbb{C}}^n$ is holomorphic if and only if $D^{\varphi}_{\bar{z}}\sigma = 0$ for any complex coordinate z. We denote the holomorphic bundle by $(\underline{\mathbb{C}}^n, D^{\varphi}_{\bar{z}})$.

Theorem 6.1.1. *[50] Let G denote $U(n)$ or a compact Lie subgroup of $U(n)$. Then a smooth map $\varphi : M \to G$ is harmonic if and only if on each coordinate domain, A^{φ}_{z} is a holomorphic endomorphism of the holomorphic vector bundle $(\underline{\mathbb{C}}^n, D^{\varphi}_{\bar{z}})$ i,e,. $A^{\varphi}_{z} \circ D^{\varphi}_{\bar{z}} = D^{\varphi}_{\bar{z}} \circ A^{\varphi}_{z}$.*

6.1.1 Unitons

Definition 6.1.2. *Let $\varphi : M \to U(n)$ be harmonic and let α be a smooth subbundle of the trivial bundle $\underline{\mathbb{C}}^n$. We say that α is a **uniton for** φ if, for all $\sigma \in \Gamma(\alpha)$,*

(i) $D^{\varphi}_{\bar{z}}(\sigma) \in \Gamma(\alpha)$, *i.e. α is a holomorphic subbundle of $\underline{\mathbb{C}}^n$;*

(ii) $A^{\varphi}_{z}(\sigma) \in \Gamma(\alpha)$, *i.e. α is closed under A^{φ}_{z}.*

For any subspace $\alpha \in \mathbb{C}^n$, we denote by π_α and $\pi^{\perp}_{\alpha}$ the orthogonal projection onto α and its orthogonal compliment $\alpha^{\perp}$, respectively.

Theorem 6.1.3. *[50] Let $\varphi : M \to U(n)$ be a harmonic map and α a uniton for φ, then the map $\tilde{\varphi} : M \to U(n)$ given by $\tilde{\varphi} = \varphi(\pi_\alpha - \pi^{\perp}_{\alpha})$ is harmonic. This is known as **adding a uniton**.*

Uhlenbeck considered harmonic maps constructed in this way by starting from a constant map $\varphi_0 : M \to U(n)$ and adding a uniton α_1 for φ_0 to get a harmonic map

$$\varphi_1 = \varphi_0(\pi_{\alpha_1} - \pi^{\perp}_{\alpha_1}).$$

Continuing the process Uhlenbeck defined more complicated harmonic maps inductively as follows: for each $i = 1, 2, \ldots, r$ for some r let

$$\varphi_i = \varphi_{i-1}(\pi_{\alpha_i} - \pi_{\alpha_i}^{\perp}),$$

where α_i is a uniton for φ_{i-1}. Expanding, we have for each i, the **uniton factorization** of the harmonic map $\varphi = \varphi_i$:

$$\varphi_i = \varphi_0(\pi_{\alpha_1} - \pi_{\alpha_1}^{\perp})(\pi_{\alpha_2} - \pi_{\alpha_2}^{\perp}) \cdots (\pi_{\alpha_i} - \pi_{\alpha_i}^{\perp}) \tag{6.1.1}$$

and a sequence of unitons $\alpha_1, \alpha_2, \ldots, \alpha_i$. Harmonic maps φ_i of this form are said to be **of finite uniton number**. The minimal number of uniton factors required is called the **(minimal) uniton number** of φ.

Theorem 6.1.4. *[50] All harmonic maps $\varphi : S^2 \to U(n)$ are of finite uniton number.*

6.1.2 Extended Solutions

In Uhlenbeck's seminal work [50], smooth maps into loop groups of the Lie group were developed by introducing a parameter which she calls a **spectral parameter** $\lambda \in S^1$ in the following way.

Definition 6.1.5. *The* ***free*** *and* ***based loop groups*** *of any Lie group G are denoted ΛG and ΩG, respectively where*

$$\Lambda G = \{\gamma : S^1 \to G \mid \gamma \text{ is smooth } \}$$

and

$$\Omega G = \{\gamma \in \Lambda G \mid \gamma(1) = e\}$$

for e the identity element of G. The loop algebras $\Lambda\mathfrak{g}$ and $\Omega\mathfrak{g}$ are defined similarly.

Following [50] we restrict G to the Lie group $U(n)$ or a Lie subgroup of $U(n)$ with corresponding Lie algebra $\mathfrak{g}$.

Consider $A^\varphi = \frac{1}{2}\varphi^{-1}d\varphi$ for some smooth map $\varphi : M \to G$. Taking the exterior derivative of the 1-form A^φ, we have the pullback of the Maurer-Cartan equation: $dA^\varphi + [A^\varphi, A^\varphi] = 0$, where $[A^\varphi, A^\varphi]$ is a $\mathfrak{g}$-valued two form defined by $[A^\varphi, A^\varphi](X,Y) = [A^\varphi(X), A^\varphi(Y)]$.

Lemma 6.1.6. *Given a $\mathfrak{g}$-valued 1-form A, locally there exist smooth maps $\varphi : M \to G$ such that $A^\varphi \equiv \frac{1}{2}\varphi^{-1}d\varphi = A$ if and only if $dA + [A,A] = 0$.*

Proof. See [50]. □

We decompose our given $\mathfrak{g}$-valued 1-form A for a local complex coordinate z so $A = A_z dz + A_{\overline{z}} d\overline{z}$ and

$$dA + [A,A] = \partial_{\overline{z}} A_z - \partial_z A_{\overline{z}} + 2[A_{\overline{z}}, A_z]. \tag{6.1.2}$$

The vanishing of the tension field, is equivalent to $\partial_{\overline{z}} A_z + \partial_z A_{\overline{z}} = 0$ [50, (9)]. Adding this to (6.1.2) and using Lemma 6.1.6 gives

$$\partial_{\overline{z}} A_z + [A_{\overline{z}}, A_z] = 0, \tag{6.1.3}$$

which is an interpretation of the harmonic equation from Theorem 6.1.1. We introduce a "spectral" parameter, $\lambda \in S^1$, by setting

$$A_\lambda = \frac{1}{2}(1-\lambda^{-1})A_z dz + \frac{1}{2}(1-\lambda)A_{\overline{z}} d\overline{z}. \tag{6.1.4}$$

Theorem 6.1.7. *[50] Let M be a Riemann surface. Given a $\mathfrak{g}$-valued 1-form $A : M \to T^*M \otimes \mathfrak{g}$, then locally there exists an S^1-family of smooth maps $\Phi_\lambda : M \to G$, a harmonic map $\varphi : M \to G$ such that $\Phi_{-1} = c\varphi$, $c \in G$, and $A_\lambda = \frac{1}{2}\Phi_\lambda^{-1} d\Phi_\lambda$ if and only if*

(i) $\partial_{\overline{z}} A_z - \partial_z A_{\overline{z}} + 2[A_{\overline{z}}, A_z] = 0$;

(ii) $\partial_{\bar{z}} A_z + [A_{\bar{z}}, A_z] = 0.$

Proof (Sketch). There are locally defined smooth maps Φ such that $A_\lambda = \frac{1}{2}\Phi_\lambda^{-1} d\Phi_\lambda$ if and only if A_λ satisfies the integrability condition

$$dA_\lambda + [A_\lambda, A_\lambda] = 0.$$

With A_λ defined by (6.1.4), expanding this out in powers of λ, the constant term gives (i), the coefficient of λ^{-1} gives (ii) and the coefficient of λ gives the conjugate of (ii). □

Definition 6.1.8. *[50] Let $\varphi : M \to G$ be a harmonic map then we call a smooth map $\Phi = \Phi_\lambda : M \to \Omega G$ an **extended solution for** φ if $\Phi_1 = e$ and Φ_λ satisfies*

$$\frac{1}{2}\Phi_\lambda^{-1} d\Phi_\lambda = A_\lambda,$$

for A_λ the $\Omega\mathfrak{g}$-valued 1-form (6.1.4) above.

Remark 6.1.9. *Definition 6.1.8 implies that $\Phi_{-1} = c\varphi$ for some constant $c \in G$.*

Definition 6.1.10. *We call two harmonic maps φ and $\tilde{\varphi}$ **equivalent** if $\tilde{\varphi} = c\varphi$ for some $c \in G$. We also call two extended solutions Φ and $\tilde{\Phi}$ **equivalent** if $\tilde{\Phi}_{-1} = a\Phi_{-1}$ for some $a \in G$. Note that this implies that $\tilde{\Phi} = \eta\Phi$ for some $\eta \in \Omega G$.*

Proposition 6.1.11. *[50, Corollary 12.2] Let $\Phi : M \to \Omega G$ be an extended solution. A subbundle α of $\underline{\mathbb{C}}^n$ is a uniton for Φ if and only if $\tilde{\Phi} = \Phi(\pi_\alpha + \lambda\pi_\alpha^\perp)$ is an extended solution.*

Definition 6.1.12. *An extended solution $\Phi = \Phi_\lambda : M \to \Omega G$ is called S^1**-invariant** if $\Phi_\lambda\Phi_\mu = \Phi_{\lambda\mu}$, $\lambda, \mu \in S^1$.*

Recall the notion of uniton factorisation, (6.1.1).

Proposition 6.1.13. *[48, Proposition 2.10] An extended solution $\Phi : M \to \Omega G$ is S^1-invariant if and only if it has a uniton factorisation with nested unitons:*

$$0 = \alpha_0 \subset \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_r \subset \alpha_{r+1} = M \times \mathbb{C}^n,$$

for some $r \in \mathbb{N}$.

6.1.3 Grassmannian Model

In [46], G. Segal described the Grassmannian model of an extended solution; this is a sub-bundle W of the trivial bundle $M \times \mathcal{H}$ for $\mathcal{H}$ the Hilbert space $L^2(S^1, \mathbb{C}^n)$. By expanding into Fourier series, we have

$$\mathcal{H} = \overline{\text{Span}}\{\lambda^i e_j \mid i \in \mathbb{Z},\, j = 1, 2, \ldots, n\},$$

for $\{e_1, e_2, \ldots, e_n\}$ the standard basis for $\mathbb{C}^n$ and where $\overline{\text{Span}}$ denotes the closed linear span.

We specialise to $G = U(n)$ which has a natural action on $\mathbb{C}^n$. This action induces an action of $\Omega U(n)$ on $\mathcal{H}$ where $\gamma \in \Omega U(n)$ acts on $v \in \mathcal{H} = L^2(S^1, \mathbb{C}^n)$ by

$$(\gamma \cdot v)(\lambda) = \gamma(\lambda) v(\lambda), \tag{6.1.5}$$

where $\lambda \in S^1$. This group action is isometric with respect to the L^2 inner product defined by $\langle v, w \rangle_{L^2} = \sum_i \langle v_i, w_i \rangle$, where $v = \sum_i \lambda^i v_i \in \mathcal{H}$, $w = \sum_i \lambda^i w_i \in \mathcal{H}$ and $\langle \cdot, \cdot \rangle$ is the standard hermitian innner product on $\mathbb{C}^n$. Let $\mathcal{H}_+$ be a closed subspace of $\mathcal{H}$ defined by

$$\mathcal{H}_+ = \overline{\text{Span}}\{\lambda^i e_j \mid i \in \mathbb{N}_0,\, j = 1, 2, \ldots, n\}, \tag{6.1.6}$$

where $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. The action of $\Omega U(n)$ on $\mathcal{H}$ (6.1.5) induces an action on $\mathcal{H}_+$, the orbit of this is denoted Gr (see [43] for descriptions of the elements of Gr) and gives a bijection

$$\Omega U(n) \ni \Phi \mapsto W := \Phi\mathcal{H}_+ \in Gr. \tag{6.1.7}$$

We note here that, since $\lambda\mathcal{H}_+ \subset \mathcal{H}_+$, for any $W \in Gr$, $\lambda W \subset W$.

Definition 6.1.14. *$W \in Gr$ is called the **Grassmannian model** for the extended solution Φ if $W = \Phi\mathcal{H}_+$.*

We define a subgroup of the based loop group $\Omega U(n)$ (see Definition 6.1.5).

Definition 6.1.15. *The **algebraic loop group** of $U(n)$ is*

$$\Omega_{alg}U(n) = \{\gamma \in \Omega U(n) \mid \gamma = \sum_{i=s}^{t} \lambda^k S_k,\ S_k \in \mathfrak{gl}(n,\mathbb{C}),\ s,t \in \mathbb{Z},\ s \leq t\}.$$

In particular for $r \in \mathbb{N}_0$ we define $\Omega_r U(n) \subset \Omega_{\text{alg}}U(n)$ to be

$$\Omega_r U(n) = \{\gamma \in \Omega_{\text{alg}}U(n) \mid \gamma = \sum_{i=0}^{r} \lambda^k S_k,\ S_k \in \mathfrak{gl}(n,\mathbb{C})\ \}. \tag{6.1.8}$$

Remark 6.1.16. *For $\gamma = \sum_{i=s}^{t} \lambda^k S_k \in \Omega_{alg}U(n)$, we write $\overline{\gamma} = \sum_{i=s}^{t} \lambda^{-k}\overline{S_k}$ and $\gamma^T = \sum_{i=s}^{t} \lambda^k S_k^T$.*

Definition 6.1.17. *An extended solution $\Phi : M \to \Omega U(n)$ which takes values in $\Omega_r U(n)$ is called a **polynomial extended solution**. We say that the **degree of the polynomial extended solution** is at most r.*

Lemma 6.1.18. *[48, §2] Let Φ be a polynomial extended solution of degree at most r then*

$$\lambda^r\mathcal{H}_+ \subset \Phi\mathcal{H}_+ \subset \mathcal{H}_+.$$

Note that Lemma 6.1.18 shows that $\Phi\mathcal{H}_+ \subset \mathcal{H}_+/\lambda^r\mathcal{H}_+ = \mathbb{C}^n + \lambda\mathbb{C}^n + \cdots + \lambda^{r-1}\mathbb{C}^n$.

Definition 6.1.19. *Let $\Phi : M \to \Omega_r U(n)$ be a polynomial extended solution. A* ***uniton factorization*** *of Φ is*

$$\Phi = \Phi_r = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^\perp)\cdots(\pi_{\alpha_r} + \lambda\pi_{\alpha_r}^\perp),$$

where α_j is a uniton for $\Phi_{j-1} = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^\perp)\cdots(\pi_{\alpha_{j-1}} + \lambda\pi_{\alpha_{j-1}}^\perp)$, for $j = 1, \ldots, r$ and $\Phi_0 = I$.

Proposition 6.1.20. *Let $\Phi : M \to \Omega_r U(n)$ be an S^1-invariant polynomial extended solution of a harmonic map $\varphi : M \to U(n)$ with uniton factorisation*

$$\Phi = \Phi_r = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^\perp)\cdots(\pi_{\alpha_r} + \lambda\pi_{\alpha_r}^\perp), \qquad (6.1.9)$$

and $\psi_j = \alpha_j^\perp \cap \alpha_{j+1}$. Then the Grassmannian model of Φ is given by both

(i) $W = \Phi\mathcal{H}_+ = (\pi_{\psi_0} + \lambda\pi_{\psi_1} + \lambda^2\pi_{\psi_3} + \cdots + \lambda^r\pi_{\psi_r})\mathcal{H}_+$ *and*

(ii) $W = \Phi\mathcal{H}_+ = \alpha_1 + \lambda\alpha_2 + \lambda^2\alpha_3 + \cdots + \lambda^{r-1}\alpha_r + \lambda^r\mathcal{H}_+$.

Proof. By Proposition 6.1.13 the unitons of φ satisfy

$$0 = \alpha_0 \subset \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_r \subset \alpha_{r+1} = M \times \mathbb{C}^n. \qquad (6.1.10)$$

We expand out the brackets of (6.1.9). The term of degree zero in λ is $\pi_{\alpha_1}\circ\pi_{\alpha_2}\circ\cdots\circ\pi_{\alpha_r} = \pi_{\alpha_1}$ by (6.1.10). The term of degree 1 in λ is

$$\pi_{\alpha_1}^\perp \circ \pi_{\alpha_2} \circ \cdots \circ \pi_{\alpha_r} + \pi_{\alpha_1} \circ \pi_{\alpha_2}^\perp \circ \cdots$$
$$\cdots \circ \pi_{\alpha_r} + \cdots + \pi_{\alpha_1} \circ \pi_{\alpha_2} \circ \cdots \circ \pi_{\alpha_r}^\perp = \pi_{\alpha_1}^\perp \circ \pi_{\alpha_2} = \pi_{\alpha_1^\perp \cap \alpha_2} \qquad (6.1.11)$$

again by (6.1.10). For the term of degree 2 in λ we have

$$\sum_{1\le i<j\le r} \pi_{\alpha_1}\circ\pi_{\alpha_1}\circ\cdots\circ\pi_{\alpha_{i-1}}\circ\pi^{\perp}_{\alpha_i}\circ\pi_{\alpha_{i+1}}\circ\cdots\circ\pi_{\alpha_{j-1}}\circ\pi^{\perp}_{\alpha_j}\circ\pi_{\alpha_{j+1}}\circ\cdots\circ\pi_{\alpha_r}$$
$$= \pi^{\perp}_{\alpha_2}\circ\pi_{\alpha_3} = \pi_{\alpha_2^{\perp}\cap\alpha_3}$$

by (6.1.10), indeed all but the first term of the sum are zero. It is easy to see that the term of degree j in λ is of the form

$$\pi^{\perp}_{\alpha_1}\circ\pi^{\perp}_{\alpha_2}\circ\cdots\circ\pi^{\perp}_{\alpha_j}\circ\pi_{\alpha_{j+1}}\circ\cdots\circ\pi_{\alpha_r} = \pi^{\perp}_{\alpha_j}\circ\pi_{\alpha_{j+1}} = \pi_{\alpha_j^{\perp}\cap\alpha_{j+1}}.$$

By defining

$$\psi_j = \alpha_j^{\perp}\cap\alpha_{j+1}, \tag{6.1.12}$$

then the above calculations show that $\psi_0 = \alpha_1$ and $\Phi = \pi_{\psi_0} + \lambda\pi_{\psi_1} + \lambda^2\pi_{\psi_2} + \cdots + \lambda^r\pi_{\psi_r}$. Therefore the Grassmannian model of Φ is given by

$$W = \Phi\mathcal{H}_+ = (\pi_{\psi_0} + \lambda\pi_{\psi_1} + \lambda^2\pi_{\psi_3} + \cdots + \lambda^r\pi_{\psi_r})\mathcal{H}_+. \tag{6.1.13}$$

To interpret this, by Lemma 6.1.18 we have $\Phi\mathcal{H}_+ \subset \mathcal{H}_+/\lambda^r\mathcal{H}_+ = \mathbb{C}^n + \lambda\mathbb{C}^n + \cdots + \lambda^{r-1}\mathbb{C}^n$. Therefore to expand (6.1.13) we need only consider $\sum_{0\le j\le r-1}\Phi\lambda^j\mathbb{C}^n$. We have

$$\begin{aligned}
\Phi\mathbb{C}^n &= \psi_0 + \lambda\psi_1 + \cdots + \lambda^r\psi_r\\
\Phi\lambda\mathbb{C}^n &= \lambda\psi_0 + \lambda^2\psi_1 + \cdots + \lambda^r\psi_{r-1} + \lambda^r + 1\psi_r\\
\Phi\lambda^2\mathbb{C}^n &= \lambda^2\psi_0 + \lambda^3\psi_1 + \cdots + \lambda^r\psi_{r-2} + \lambda^{r+1}\psi_{r-1} + \lambda^{r+2}\psi_r\\
&\vdots \qquad\qquad\qquad\qquad \vdots\\
\Phi\lambda^{r-1}\mathbb{C}^n &= \lambda^{r-1}\psi_0 + \lambda^r\psi_1 + \cdots + \lambda^{2r-1}\psi_r.
\end{aligned}$$

Summing we have

$$\sum_{0\leq j\leq r-1} \Phi\lambda^j\mathbb{C}^n = \psi_0 + \lambda(\psi_0+\psi_1) + \lambda^2(\psi_0+\psi_1+\psi_2) + \cdots + \lambda^j(\psi_0+\cdots+\psi_j),$$

where $\psi_0+\psi_1+\cdots+\psi_j = \alpha_0^\perp\cap\alpha_1 + \alpha_1^\perp\cap\alpha_2 + \alpha_2^\perp\cap\alpha_3 + \cdots + \alpha_j^\perp\cap\alpha_{j+1} = \alpha_{j+1}$ by (6.1.10). So

$$W = \Phi\mathcal{H}_+ = \alpha_1 + \lambda\alpha_2 + \lambda^2\alpha_3 + \cdots + \lambda^{r-1}\alpha_r + \lambda^r\mathcal{H}_+.$$

□

Proposition 6.1.21. *[50] Let $\varphi : M \to U(n)$ be a harmonic map with a polynomial extended solution Φ then*

(i) *Φ has a uniton factorization;*

(ii) *φ is of finite uniton number.*

The bijection (6.1.7) restricts to a bijection from $\Omega_{\text{alg}}U(n)$ to λ-closed subspaces W of $\mathcal{H}_+$ which lie in $Gr_r^s = \{W \in Gr \mid \lambda^r\mathcal{H}_+ \subset W \subset \lambda^s\mathcal{H}_+,\ r \geq s\}$. This bijection further restricts to a bijection

$$\Omega_r U(n) \ni \Phi \mapsto W := \Phi\mathcal{H}_+ \in Gr_r,$$

where $Gr_r \subset Gr$ given by $Gr_r = \{W \in Gr \mid \lambda^r\mathcal{H}_+ \subset W \subset \mathcal{H}_+\}$.

Lemma 6.1.22. *[46] Let $\Phi : M \to \Omega U(n)$ be a smooth map and set $W = \Phi\mathcal{H}_+ : M \to Gr$. Then Φ is an extended solution if and only if W satisfies two conditions:*

1. *$\partial_{\bar z}(\Gamma(W)) \subset \Gamma(W)$, i.e. W is a holomorphic subbundle of $M \times \mathcal{H}$;*

2. *$\lambda\partial_z(\Gamma(W)) \subset \Gamma(W)$, i.e. W is closed under the operator $\lambda\partial_z$.*

6.1.4 Complex Extended Solutions

As in [9] we want to consider complex extended solutions, the idea of which comes from the infinite-dimensional complex structure of ΩG for G a Lie group [43]. This complex structure is described using the Iwasawa decomposition of the loop group $\Lambda G^{\mathbb{C}}$ in Proposition 6.1.23 below.

Let $\Lambda^+ G^{\mathbb{C}}$ (resp. $\Lambda^* G^{\mathbb{C}}$) be the subgroup of $\Lambda G^{\mathbb{C}}$ consisting of smooth maps $S^1 \to G^{\mathbb{C}}$ which extend holomorphically to $\{\lambda \in \mathbb{C} \mid |\lambda| < 1\}$ (resp. $\{\lambda \in \mathbb{C} \mid 0 < |\lambda| < 1\}$), we define $\Lambda^+ \mathfrak{g}^{\mathbb{C}}$ similarly.

Proposition 6.1.23. *[43, Theorem 8.1.1] The product map $\Omega G \times \Lambda^+ G^{\mathbb{C}} \to \Lambda G^{\mathbb{C}}$ is a diffeomorphism. Therefore any $\gamma \in \Lambda G^{\mathbb{C}}$ can be written uniquely in the form $\gamma = \gamma_u \cdot \gamma_+$, where $\gamma_u \in \Omega G$ and $\gamma_+ \in \Lambda^+ G^{\mathbb{C}}$. This is known as the* ***Iwasawa decomposition****.*

Both $\Lambda^+ G^{\mathbb{C}}$ and $\Lambda G^{\mathbb{C}}$ are complex Lie groups and so the homogeneous space $\Lambda G^{\mathbb{C}} / \Lambda^+ G^{\mathbb{C}}$ is a complex manifold. A consequence of Proposition 6.1.23 is the identification of ΩG with the complex homogeneous space $\Lambda G^{\mathbb{C}} / \Lambda^+ G^{\mathbb{C}}$ giving ΩG a complex structure, see [43, §8.1].

Definition 6.1.24. *[9, §1] A holomorphic map $\Psi : M \to \Lambda^* G^{\mathbb{C}}$ is called a* ***complex extended solution*** *if, on each coordinate domain (U, z),*

$$\operatorname{Im} \lambda \Psi^{-1} \Psi_z \subseteq \Lambda^+ \mathfrak{g}^{\mathbb{C}}.$$

Proposition 6.1.25. *[21, 9] Let $[\ \ ]$ denote the projection onto the first factor of the Iwasawa decomposition. If $\Psi : M \to \Lambda^* G^{\mathbb{C}}$ is a complex extended solution then its projection $\Phi = [\Psi]$ onto ΩG is an extended solution. Conversely if $\Phi : M \to \Omega G$ is an extended solution and z_0 is a point of M then there exists a neighbourhood U_0 of z_0 and a complex extended solution $\Psi : U_0 \to \Lambda^* G^{\mathbb{C}}$ such that $\Phi|_{U_0} = [\Psi]$.*

It follows from Proposition 6.1.23 that the Grassmannian model $W = \Phi\mathcal{H}_+$ is also given by $W = \Psi\mathcal{H}_+$ [29].

Definition 6.1.26. *[29, §2.3] A meromorphic map $\Psi : M \to \Lambda^* G^{\mathbb{C}}$ is called a **meromorphic extended solution** if it is an extended solution away from the poles.*

As in [29], we may extend $W = \Psi\mathcal{H}_+$ and $\Phi = [\Psi]$ smoothly over the poles and we will continue to write $\Phi = [\Psi]$ even when Ψ is meromorphic.

6.2 Harmonic Maps into the Unitary Group

In [9], a general theory was introduced to classify harmonic maps into a Lie group G by using the canonical elements from Definition 5.2.11 to give complex extended solutions. We will describe this theory for $G = U(n)$. Let $\tilde{\xi}$ be a canonical element of $\mathfrak{su}(n)$, then by recalling (5.2.1),

$$\tilde{\xi} = i\Big(\sum_{j\in J} \frac{-j}{n} \cdot I + \sum_{j\in J} D_j\Big) = i\,\mathrm{diag}\Big(\frac{1}{n}\delta + \xi_1, \frac{1}{n}\delta + \xi_2, \ldots, \frac{1}{n}\delta + \xi_n\Big),$$

where $\delta = -\sum_{j\in J} j$ and the ξ_j are non-negative integers satisfying

$$\xi_j - \xi_{j+1} = 0 \text{ or } 1, \quad \xi_n = 0.$$

By Definition 5.2.12 the canonical geodesic $\gamma_{\tilde{\xi}} : S^1 \to SU(n)$ is given by

$$\gamma_{\tilde{\xi}} = \exp(t\tilde{\xi}) = \mathrm{diag}(e^{(it/n)\delta}e^{it\xi_1}, e^{(it/n)\delta}e^{it\xi_2}, \ldots, e^{(it/n)\delta}e^{it\xi_n}). \tag{6.2.1}$$

By regarding $SU(n) \subset U(n)$ we can consider (6.2.1) as a map into $U(n)$, and as in [9, p. 562] (6.2.1) gives a representative of a geodesic in the projective unitary group $PU(n) = U(n)/Z(U(n))$. The centre of $U(n)$, $Z(U(n))$ consists of diagonal matrices of the form $c \cdot I$ for $c \in \mathbb{C}$, and $|c| = 1$, so $PU(n)$ is a group of equivalence classes of unitary matrices under multiplication by $c \cdot I$. By fixing representatives of each canonical geodesic in an equivalence class and recalling the notation from Remark 5.2.13 we may define a homomorphism $\gamma_\xi : S^1 \to U(n)$,

$$\gamma_\xi = e^{\frac{-it}{n}\delta}\gamma_{\tilde{\xi}} = \mathrm{diag}(e^{it\xi_1}, e^{it\xi_2}, \ldots, e^{it\xi_n}) = \mathrm{diag}(\lambda^{\xi_1}, \lambda^{\xi_2}, \ldots, \lambda^{\xi_n}),$$

where $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ and the ξ_j are non-negative integers satisfying $\xi_j - \xi_{j+1} = 0$ or 1, $\xi_n = 0$. Note here that $\gamma_\xi \in \Omega_r U(n)$ with $\xi_1 = r$.

Definition 6.2.1. *[29, §2.6] Let ξ be a diagonal matrix $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ where ξ_j are non-negative integers satisfying $\xi_j - \xi_{j+1} = 0$ or 1, $\xi_n = 0$ and $\xi_1 = r$. Then we call ξ* ***a canonical element of*** *$\Omega_r U(n)$ and $\gamma_\xi = \mathrm{diag}(\lambda^{\xi_1}, \lambda^{\xi_2}, \ldots, \lambda^{\xi_n}) \in \Omega_r U(n)$ the corresponding* ***canonical geodesic****.*

Definition 6.2.2. *Let $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ be a canonical element of $\Omega_r U(n)$. By* ***type*** *of the canonical element we mean the $(r+1)$-tuple $(t_0, t_1, \ldots, t_r)$ where $t_j := \#\{l \mid \xi_l = j\}$.*

Remark 6.2.3. *Note that $\sum_{j=0}^{r} t_j = n$.*

Definition 6.2.4. *We call an $n \times n$ matrix A a* ***block matrix*** *if it is of the form*

$$A = \begin{pmatrix} A_{1,1} & A_{1,2} & A_{1,3} & \ldots & A_{1,r} & A_{1,r+1} \\ A_{2,1} & A_{2,2} & A_{2,3} & \ldots & A_{2,r} & A_{2,r+1} \\ A_{3,1} & A_{3,2} & A_{3,3} & \ldots & A_{3,r} & A_{3,r+1} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ A_{r,1} & A_{r,2} & A_{r,3} & \ldots & A_{r,r} & A_{r,r+1} \\ A_{r+1,1} & A_{r+1,2} & A_{r+1,3} & \ldots & A_{r+1,r} & A_{r+1,r+1} \end{pmatrix},$$

where all entries $A_{j,l}$ $1 \le j, l \le r+1$ are matrices and $A_{l,l}$ $1 \le l \le r+1$ are square matrices of possibly different sizes. Note that matrices $A_{j,l}$ for $j \neq l$ are of possibly *different sizes and are not necessarily square. We call the entries of A of the form $A_{j,j+\kappa}$ $-r \leq \kappa \leq r$ the κ**th block superdiagonal** of A. We call A **upper block triangular** if $A_{j,l}$ has all zero entries for all $1 < l < j \leq r+1$ and we call an upper block triangular matrix A **block unitriangular** if the matrices $A_{l,l}$ $1 \leq l \leq r+1$ are identity matrices.*

Let $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ be a canonical element of $\Omega_r U(n)$ of type $(t_0, t_1, \ldots, t_r)$. Relabelling we have

$$\xi = i\,\mathrm{diag}\left(\overbrace{\tilde{\xi}_1, \ldots, \tilde{\xi}_1}^{t_r}, \overbrace{\tilde{\xi}_2, \ldots, \tilde{\xi}_2}^{t_r-1}, \ldots, \overbrace{\tilde{\xi}_{r+1}, \ldots, \tilde{\xi}_{r+1}}^{t_0}\right).$$

Therefore ξ has distinct eigenvalues $i\tilde{\xi}_1, i\tilde{\xi}_2, \ldots, i\tilde{\xi}_{r+1}$, where $i\tilde{\xi}_k$ has multiplicity t_{r+1-k} with associated eigenspaces of $\mathbb{C}^n$

$$E_1 = \{\left(\overbrace{*, \ldots, *}^{t_r}, \overbrace{0, \ldots, 0}^{n-t_r}\right)^T\},$$

$$E_2 = \{\left(\overbrace{0, \ldots, 0}^{t_r}, \overbrace{*, \ldots, *}^{t_{r-1}}, \overbrace{0, \ldots, 0}^{n-T_{r-1}}\right)^T\},$$

$$\vdots$$

$$E_j = \{\left(\overbrace{0, \ldots, 0}^{T_{r+2-j}}, \overbrace{*, \ldots, *}^{t_{r+1-j}}, \overbrace{0, \ldots, 0}^{n-T_{r+1-j}}\right)^T\},$$

$$\vdots$$

$$E_{r+1} = \{\left(\overbrace{0, \ldots, 0}^{n-t_1}, \overbrace{*, \ldots, *}^{t_1}\right)^T\},$$

where $T_k = \sum_{l=k}^{r} t_l$ and the *s denote arbitrary complex numbers.

Define $X_{jk} = E_j \otimes E_k$ where $\otimes$ denotes the Kronecker tensor product [4], so X_{jk} is an $n \times n$ matrix comprised of $(r+1) \times (r+1)$ blocks where the (j,k)th block is of size $t_{r+1-j} \times t_{r+1-k}$; the (j,k)th block of X_{jk} has arbitrary entries and all other blocks of X_{jk} have all entries 0. Let $e_{jk} \in X_{jk}$ then we have the matrix products $\xi e_{jk} = i\tilde{\xi}_j e_{jk}$,

$e_{jk}\xi = i\tilde{\xi}_k e_{jk}$. Therefore

$$\mathrm{ad}(\xi)e_{jk} = \xi e_{jk} - e_{jk}\xi = i\tilde{\xi}_j e_{jk} - i\tilde{\xi}_k e_{jk} = i(\tilde{\xi}_j - \tilde{\xi}_k)e_{jk},$$

and so the eigenvalues of $\mathrm{ad}(\xi)$ on $\mathfrak{g} = \mathfrak{u}(n)$ are $i(\tilde{\xi}_j - \tilde{\xi}_k)$ which are are of the form $i\kappa$ for $\kappa \in \mathbb{Z}$, $-r \le \kappa \le r$. In line with the notation of [29] we denote the eigenspaces of each eigenvalue $i\kappa$ by $\mathfrak{g}_\kappa^{\mathbb{C}} = \mathfrak{g}_\kappa^{\mathbb{C}}(\xi)$ (the superscript $\mathbb{C}$ here is to distinguish it from the $\mathfrak{o}(n)$ and $\mathfrak{sp}(2m)$ cases in §6.3 and §7.1, respectively), where

$$\mathfrak{g}_\kappa^{\mathbb{C}} = \mathrm{Span}\{X_{jk} \mid k - j = \kappa\}.$$

So $\mathfrak{g}_\kappa^{\mathbb{C}}$ consists of matrices with entries zero unless they are on the κth block superdiagonal and $\mathfrak{gl}(n,\mathbb{C}) = \mathfrak{u}(n)^{\mathbb{C}} = \sum_{\kappa=-r}^{r} \mathfrak{g}_\kappa^{\mathbb{C}}$.

Example 6.2.5. *Let* $\xi = i\,\mathrm{diag}(2,1,1,0)$ *be a canonical element of* $\Omega_2 U(4)$ *which is of type* $(1,2,1)$ *and so has eigenvalues* $\tilde{\xi}_1 = 2i$, $\tilde{\xi}_2 = i$, *and* $\tilde{\xi}_3 = 0$ *of multiplicities* $1, 2$, *and* 1, *respectively. These have corresponding eigenspaces of the form,*

$$E_1 = \{(*,0,0,0)^T\}, \quad E_2 = \{(0,*,*,0)^T\}, \quad E_3 = \{(0,0,0,*)^T\}.$$

We display the X_{jk}

$$X_{11} = \left(\begin{array}{c|cc|c} * & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad X_{22} = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & * & * & 0 \\ 0 & * & * & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad X_{33} = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & * \end{array}\right),$$

$$X_{12} = \left(\begin{array}{c|cc|c} 0 & * & * & 0 \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad X_{23} = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & * \\ 0 & 0 & 0 & * \\ \hline 0 & 0 & 0 & 0 \end{array}\right), \quad X_{13} = \left(\begin{array}{c|cc|c} 0 & 0 & 0 & * \\ \hline 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 \end{array}\right),$$

and $X_{21} = X_{12}^T$, $X_{31} = X_{31}^T$, $X_{32} = X_{23}^T$. *So the eigenspaces for each of the eigenvalues are*

$$\mathfrak{g}_0^{\mathbb{C}} = \mathrm{Span}\{X_{11}, X_{22}, X_{33}\}, \quad \mathfrak{g}_1^{\mathbb{C}} = \mathrm{Span}\{X_{12}, X_{23}\},$$

$$\mathfrak{g}_2^{\mathbb{C}} = \mathrm{Span}\{X_{13}\}, \quad \mathfrak{g}_{-1}^{\mathbb{C}} = \mathrm{Span}\{X_{21}, X_{32}\}, \quad \mathfrak{g}_{-2}^{\mathbb{C}} = \mathrm{Span}\{X_{31}\}.$$

Note that $g_1^{\mathbb{C}}$ *and* $g_2^{\mathbb{C}}$ *are the* 1*st and* 2*nd block superdiagonal, respectively and* $\mathfrak{gl}(4, \mathbb{C}) = \mathfrak{u}(4)^{\mathbb{C}} = \sum_{\kappa=-2}^{2} \mathfrak{g}_\kappa^{\mathbb{C}}$.

Let $\Lambda_{\mathrm{alg}}^+ U(n)^{\mathbb{C}} = \Lambda_{\mathrm{alg}}^+ GL(n, \mathbb{C}) := \Lambda^+ U(n)^{\mathbb{C}} \cap \Lambda_{\mathrm{alg}} U(n)^{\mathbb{C}}$, and similarly $\Lambda_{\mathrm{alg}}^+ \mathfrak{u}(n)^{\mathbb{C}} = \Lambda_{\mathrm{alg}}^+ \mathfrak{gl}(n, \mathbb{C}) := \Lambda^+ \mathfrak{u}(n)^{\mathbb{C}} \cap \Lambda_{\mathrm{alg}} \mathfrak{u}(n)^{\mathbb{C}}$.

Definition 6.2.6. *[29] Let* ξ *be a canonical element of* $\Omega_r U(n)$. *We define a finite-dimensional Lie subgroup* $\mathfrak{A}_\xi^{\mathbb{C}}$ *of* $\Lambda_{\mathrm{alg}}^+ GL(n, \mathbb{C})$ *by*

$$\mathfrak{A}_\xi^{\mathbb{C}} = \{A = (a_{jk}) \in \Lambda_{\mathrm{alg}}^+ GL(n, \mathbb{C}) \mid a_{jk} = \delta_{jk} \textit{ if } \xi_j \le \xi_k, \\ \textit{otherwise } a_{jk} \textit{ is a polynomial in } \lambda \in S^1 \textit{ of degree at most } \xi_j - \xi_k - 1\}.$$

We also define the Lie subgroup $(\mathfrak{A}_\xi^{\mathbb{C}})_0$ *of* $\mathfrak{A}_\xi^{\mathbb{C}}$ *by* $(\mathfrak{A}_\xi^{\mathbb{C}})_0 = \mathfrak{A}_\xi^{\mathbb{C}} \cap U(n)$.

The elements of $\mathfrak{A}_\xi^{\mathbb{C}}$ are block unitriangular. Also the κth block superdiagonal has entries polynomial in λ of degree at most $\kappa - 1$ for $\kappa > 0$.

Example 6.2.7. (i) *Let* $\xi = i\,\mathrm{diag}(2, 1, 1, 0)$, *a canonical element of* $\Omega_2 U(4)$, *then*

$A \in \mathfrak{A}_\xi^{\mathbb{C}}$ if and only if

$$A = \left(\begin{array}{c|cc|c} 1 & a_{12} & a_{13} & \alpha_1\lambda + \alpha_0 \\ \hline 0 & 1 & 0 & a_{24} \\ 0 & 0 & 1 & a_{34} \\ \hline 0 & 0 & 0 & 1 \end{array}\right),$$

for some $a_{12}, a_{13}, a_{24}, a_{34}, \alpha_1, \alpha_0 \in \mathbb{C}$, and $\lambda \in S^1$.

(ii) *Let $\xi = i\,\mathrm{diag}(3,2,1,0)$, a canonical element of $\Omega_3 U(4)$, then $A \in \mathfrak{A}_\xi^{\mathbb{C}}$ if and only if*

$$A = \left(\begin{array}{c|c|c|c} 1 & a_{12} & \alpha_1\lambda + \alpha_0 & \delta_2\lambda^2 + \delta_1\lambda + \delta_0 \\ \hline 0 & 1 & a_{23} & \beta_1\lambda + \beta_0 \\ \hline 0 & 0 & 1 & a_{34} \\ \hline 0 & 0 & 0 & 1 \end{array}\right),$$

for some $a_{12}, a_{23}, a_{34}, \alpha_1, \alpha_0, \beta_1, \beta_0, \delta_2, \delta_1, \delta_0 \in \mathbb{C}$ and $\lambda \in S^1$.

Recall the definition of canonical geodesic from Definition 5.2.12, the projection onto $\Omega U(n)$ given by the Iwasawa decomposition from Proposition 6.1.23 and the definition of equivalent extended solution from Definition 6.1.10.

Proposition 6.2.8. *[29, Proposition 2.2] Let $\tilde{\Phi} : M \to \Omega_{\tilde{r}} U(n)$ be a polynomial extended solution for some $\tilde{r} \in \mathbb{N}_0$. Then there is an equivalent extended solution $\Phi : M \to \Omega_r U(n)$ with $0 \le r \le \tilde{r}$, a canonical element $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ of $\Omega_r U(n)$ and a meromorphic map $A : M \to \mathfrak{A}_\xi^{\mathbb{C}}$ such that $\Phi = [A\gamma_\xi]$. A and ξ are uniquely determined by Φ, and all harmonic maps $\varphi : M \to U(n)$ of finite uniton number have such an extended solution.*

Following [29], let c_j for $j = 1, \ldots, n$ denote the columns of $A \in \mathfrak{A}_\xi^{\mathbb{C}}$, so $c_k = (a_{1k}, a_{2k}, \ldots, a_{nk})^T$, where a_{jk} are polynomials in λ of maximum degree determined by

Definition 6.2.6. Let $\sum_{j:P(j)}$ denote the sum over all j satisfying the condition $P(j)$, and let x' denote the derivative of x with respect to any local complex coordinate on M. We state the converse of Proposition 6.2.8 for which we recall the definition of complex extended solution in Definition 6.1.24.

Proposition 6.2.9. *[29, Proposition 2.4] Let ξ be a canonical element of $\Omega_r U(n)$, $A : M \to \mathfrak{A}_\xi^{\mathbb{C}}$ be a holomorphic map and $\Psi = A\gamma_\xi$. Then Ψ is a complex extended solution if and only if the columns c_j for $j = 1, \ldots, n$ of A satisfy*

$$c'_k = \sum_{j:\xi_j > \xi_k} \lambda^{\xi_j - \xi_k - 1} \rho'_{jk} c_j, \qquad r > \xi_k \geq 0, \tag{6.2.2}$$

where ρ_{jk} is the coefficient of the term of degree $\xi_j - \xi_k - 1$ in a_{jk}. This is equivalent to

$$a'_{ik} = \sum_{j:\xi_i \geq \xi_j > \xi_k} \lambda^{\xi_j - \xi_k - 1} \rho'_{jk} a_{ij}, \qquad r \geq \xi_i > \xi_k \geq 0, \tag{6.2.3}$$

and holds if and only if it holds $\mod \lambda^{\xi_i - \xi_k - 1}$, so is equivalent to

$$a'_{ik} = \sum_{j:\xi_i > \xi_j > \xi_k} \lambda^{\xi_j - \xi_k - 1} \rho'_{jk} a_{ij} \mod \lambda^{\xi_i - \xi_k - 1}, \qquad r \geq \xi_i > \xi_k + 1 \geq 1. \tag{6.2.4}$$

Any of the above (6.2.2), (6.2.3) and (6.2.4) are known as the ***extended solution equations*** *for A.*

Definition 6.2.10. *We denote the space of meromorphic maps $A : M \to \mathfrak{A}_\xi^{\mathbb{C}}$ satisfying (6.2.2) by $\mathrm{Sol}_\xi^{\mathbb{C}}$ and the subspace of $\mathrm{Sol}_\xi^{\mathbb{C}}$ of meromorphic maps $A : M \to (\mathfrak{A}_\xi^{\mathbb{C}})_0$ by $(\mathrm{Sol}_\xi^{\mathbb{C}})_0$.*

Now we have a corollary to both Propositions 6.2.8 and 6.2.9.

Corollary 6.2.11. *[29] Let ξ be a canonical element of $\Omega_r U(n)$. The assignment $A \mapsto \Phi = [A\gamma_\xi]$ defines a one-to-one correspondence between $\mathrm{Sol}_\xi^{\mathbb{C}}$ and the space of extended* *solutions* $\Phi = [A\gamma_\xi] : M \to \Omega_r U(n)$ *and restricts to a one-to-one correspondence between* $(\mathrm{Sol}_\xi^{\mathbb{C}})_0$ *and the set of* S^1*-invariant extended solutions* $\Phi = [A\gamma_\xi] : M \to \Omega_r U(n)$.

We describe how S^1-invariant polynomial extended solutions $\Phi : M \to \Omega_r U(n)$ give harmonic maps $\varphi : M \to U(n)$.

Proposition 6.2.12. *[23, §4] Let* $\phi : M \to N$ *be a harmonic map between Riemannian manifolds and* $\psi : N \to P$ *be a totally geodesic map i.e. a map with vanishing second fundamental form. Then the composition* $\psi \circ \phi : M \to P$ *is harmonic.*

Proposition 6.2.13. *[13, p. 66] Let* $G_k(\mathbb{C}^n)$ *be the Grassmannian of* k*-dimensional subspaces of* $\mathbb{C}^n$, $G_*(\mathbb{C}^n) = \cup_{k=0,1,\dots,n} G_k(\mathbb{C}^n)$ *and* V *a subspace of* $\mathbb{C}^n$. *The map* $\iota : G_*(\mathbb{C}^n) \to U(n)$ *defined by* $\iota(V) = \pi_V - \pi_V^\perp$ *is a totally geodesic embedding that is isometric up to a constant factor. The embedding* ι *is known as the* ***Cartan embedding***.

Example 6.2.14. *Given an* S^1*-invariant polynomial extended solution* $\Phi : M \to \Omega_r U(n)$, *recall from Proposition 6.1.20 that we can write this as*

$$\Phi = \pi_{\psi_0} + \lambda \pi_{\psi_1} + \lambda^2 \pi_{\psi_2} + \cdots + \lambda^r \pi_{\psi_r},$$

for some $r \in \{0, 1, \dots\}$ *where* $\psi_j = \alpha_j^\perp \cap \alpha_{j+1}$*; by Proposition 6.1.13 the unitons* α_j *are nested. Also from Proposition 6.1.20 the Grassmannian model is given by*

$$W = \Phi \mathcal{H}_+ = \alpha_1 + \lambda \alpha_2 + \lambda^2 \alpha_3 + \cdots + \lambda^{r-1} \alpha_r + \lambda^r \mathcal{H}_+.$$

On putting $\lambda = -1$ *we see that the corresponding harmonic map* $\varphi : M \to U(n)$ *is given by*

$$\varphi = \sum_{j \ \text{even}} \pi_{\psi_j} - \sum_{j \ \text{odd}} \pi_{\psi_j}.$$

This can be rewritten as

$$\varphi = \pi_\phi - \pi_\phi^\perp, \tag{6.2.5}$$

where

$$\pi_\phi = \sum_{j \;\; even} \pi_{\psi_j} \quad \textit{and} \quad \pi_\phi^\perp = \sum_{j \;\; odd} \pi_{\psi_j}.$$

Recall from §2.1 that to each map $\phi : M \to G_*(\mathbb{C}^{n+1})$, we may associate the pullback of the tautological bundle $\underline{\phi} := \phi^{-1}T$. Conversely, any subbundle $\underline{\phi}$ of $M \times \mathbb{C}^{n+1}$ corresponds to a map $\phi : M \to G_*(\mathbb{C}^{n+1})$. By Propositions 6.2.12 and 6.2.13 we see that (6.2.5) is given by

$$\varphi = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$$

where $\phi : M \to G_*(\mathbb{C}^n)$ is given by

$$\underline{\phi} = \sum_{j \;\; even} \psi_j. \tag{6.2.6}$$

In [9, §4] and [32, Ch. 22] the equations in Proposition 6.2.9 were easily solved giving low-dimensional examples. In [29], the above was adapted for $O(n)$ and (6.2.2) was solved giving a classification and parametrization of harmonic maps into $O(n)$ in terms of the canonical elements from Definition 5.2.16, we turn our attention to this now.

6.3 Harmonic Maps into the Orthogonal Group

Recall the orthogonal group $O(n)$ (5.1.1). We consider $O(n)$ as a subgroup of $U(n)$ given by $O(n) = \{A \in U(n) \mid \overline{A} = A\} = \{A \in U(n) \mid A^T A = I\}$, where for $A = (a_{jk})$, $\overline{A} = (\overline{a}_{jk})$. We may also regard the loop group $\Omega O(n) = \Omega SO(n)$ as a subgroup of $\Omega U(n)$ given by $\Omega O(n) = \{\Phi \in \Omega U(n) \mid \overline{\Phi} = \Phi\} = \{\Phi \in \Omega U(n) \mid \Phi^T \Phi = I\}$.

Lemma 6.3.1. *[43, Proposition 8.5.1] Let $\Phi \in \Omega U(n)$ and, recalling the bijection (6.1.7), set $W = \Phi \mathcal{H}_+$. Then $\Phi \in \Omega O(n)$ if and only if $\overline{W}^\perp = \lambda W$.*

Recall the subset $\Omega_r U(n)$ of the algebraic loop group $\Omega_{\mathrm{alg}} U(n)$ (6.1.8). We define a subset

$\Omega_r U(n)^{\mathbb{R}} \subset \Omega_r U(n)$ by

$$\Omega_r U(n)^{\mathbb{R}} = \left\{\Phi \in \Omega_r U(n) \,\middle|\, \overline{\Phi} = \lambda^{-r}\Phi\right\} = \left\{\Phi \in \Omega_r U(n) \,\middle|\, \Phi^T\Phi = \lambda^r I\right\}. \tag{6.3.1}$$

Lemma 6.3.2. *[48, §6] Let $W = \Phi\mathcal{H}_+$ for $\Phi \in \Omega_r U(n)$, then $\Phi \in \Omega_r U(n)^{\mathbb{R}}$ if and only if $\overline{W}^{\perp} = \lambda^{1-r}W$. We call an element $\Phi \in \Omega_r U(n)^{\mathbb{R}}$ (and the corresponding W)* ***real of degree*** *r.*

Let $\Phi : M \to \Omega_r U(n)^{\mathbb{R}}$ be an S^1-invariant polynomial extended solution. It is easy to see from (6.3.1) that if r is even then $\overline{\Phi} = \Phi$ so $\phi = \pm\Phi_{-1}$ are harmonic maps into a real Grassmannian. Then by Propositions 6.2.12 and 6.2.13 we have the harmonic map $\varphi = \iota \circ \phi : M \to O(n)$. For r odd, by [48, Theorem 6.8] n must be even, so $\overline{\Phi} = -\Phi$ and following [48, §6.3] $\phi = \pm i\Phi_{-1}$ are harmonic maps into the symmetric space $O(2m)/U(m)$ for $n = 2m$ which can be identified with the space of maximally isotropic subspaces of $\mathbb{C}^{2m}$. Again by Propositions 6.2.12 and 6.2.13 we have the harmonic map $\varphi = \iota \circ \phi : M \to O(2m)$.

Similarly to [29] we shall write all matrices and vectors with respect to a null basis (5.1.2) and we note that with respect to this null basis we have

$$\Omega_r U(n)^{\mathbb{R}} = \left\{\Phi \in \Omega_r U(n) \,\middle|\, \Phi^{\mathfrak{T}}\Phi = \lambda^r I\right\}. \tag{6.3.2}$$

Recall the canonical elements ξ of $\mathfrak{so}(2m)$ and $\mathfrak{so}(2m+1)$ given in Proposition 5.2.17. Let $\tilde{\xi}$ be a canonical element of $\mathfrak{so}(2m)$ so we have that
$\tilde{\xi} = i\,\mathrm{diag}(\tilde{\xi}_1, \ldots, \tilde{\xi}_m, -\tilde{\xi}_m, \ldots, -\tilde{\xi}_1)$. By Definition 5.2.12 we may define a canonical geodesic $\gamma_{\tilde{\xi}} : S^1 \to O(2m)$ by

$$\begin{aligned}\gamma_{\tilde{\xi}} = \exp(t\tilde{\xi}) &= \mathrm{diag}(e^{it\tilde{\xi}_1}, e^{it\tilde{\xi}_2}, \ldots, e^{it\tilde{\xi}_m}, e^{-it\tilde{\xi}_m}, \ldots, e^{-it\tilde{\xi}_1}) \\ &= \mathrm{diag}(\lambda^{\tilde{\xi}_1}, \lambda^{\tilde{\xi}_2}, \ldots, \lambda^{\tilde{\xi}_m}, \lambda^{-\tilde{\xi}_m}, \ldots, \lambda^{-\tilde{\xi}_1}).\end{aligned} \tag{6.3.3}$$

We regard $O(2m) \subset U(2m)$ and so can view the canonical geodesic $\gamma_{\tilde{\xi}}$ as a map into $U(2m)$. Again by following [9, p. 562] (6.3.3) gives a representative of a geodesic in the projective unitary group $PU(2m) = U(2m)/Z(U(2m))$. Recall that $PU(2m)$ is a group of equivalence classes of unitary matrices under multiplication by $c \cdot I$, for $c \in \mathbb{C}$. Fixing representatives of each canonical geodesic in an equivalence class we define the homomorphism $\gamma_\xi : S^1 \to U(2m)$,

$$\begin{aligned}\gamma_\xi = e^{it\tilde{\xi}_1}\gamma_{\tilde{\xi}} &= \operatorname{diag}(e^{2it\tilde{\xi}_1}, e^{it(\tilde{\xi}_2+\tilde{\xi}_1)}, \dots, e^{it(\tilde{\xi}_m+\tilde{\xi}_1)}, e^{it(\tilde{\xi}_1-\tilde{\xi}_m)}, \dots, e^{it(\tilde{\xi}_1-\tilde{\xi}_2)}, e^0)\\ &= \operatorname{diag}(\lambda^{2\tilde{\xi}_1}, \lambda^{(\tilde{\xi}_2+\tilde{\xi}_1)}, \dots, \lambda^{(\tilde{\xi}_m+\tilde{\xi}_1)}, \lambda^{(\tilde{\xi}_1-\tilde{\xi}_m)}, \dots, \lambda^{(\tilde{\xi}_1-\tilde{\xi}_2)}, \lambda^0)\\ &= \operatorname{diag}(\lambda^{\xi_1}, \lambda^{\xi_2}, \dots, \lambda^{\xi_m}, \lambda^{\xi_{m+1}}, \dots, \lambda^{\xi_{2m-1}}, \lambda^{\xi_{2m}})\end{aligned}$$

where $\xi_j = \begin{cases} \tilde{\xi}_j + \tilde{\xi}_1, & \text{if } 1 \leq j \leq m, \\ \tilde{\xi}_1 - \tilde{\xi}_{2m+1-j}, & \text{if } m+1 \leq j \leq 2m, \end{cases}$ and $\xi = i\operatorname{diag}(\xi_1, \xi_2, \dots, \xi_{2m})$. Note that, for ξ_1 even, then the ξ_j are non-negative integers such that $\xi_j - \xi_{j+1} = 0$ or 1, $\xi_{2m} = 0$, $\xi_1 = 2\tilde{\xi}_1$ and $\xi_{\bar{j}} = \xi_1 - \xi_j$. For ξ_1 odd then either $\xi_j - \xi_{j+1} = 0$ or 1, and $\xi_{m-1} = \xi_m$ or $\xi_j - \xi_{j+1} = 0$, or 1, for $j = 1, 2, \dots, m-2, m+2, \dots, 2m$ and $\xi_{m-1} - \xi_m = 1$, $\xi_m - \xi_{m+1} = -1$, $\xi_{m+1} - \xi_{m+2} = 1$. Also note that

$$\begin{aligned}\gamma_\xi^{\mathfrak{T}}\gamma_\xi &= \Big(\operatorname{diag}(\lambda^{\xi_{2m}}, \lambda^{\xi_{2m-1}}, \dots, \lambda^{\xi_1})\Big)\Big(\operatorname{diag}(\lambda^{\xi_1}, \dots, \lambda^{\xi_{2m}})\Big)\\ &= \operatorname{diag}(\lambda^{\xi_1+\xi_{2m}}, \lambda^{\xi_2+\xi_{2m-1}}, \dots, \lambda^{\xi_{2m}+\xi_1})\\ &= \lambda^{\xi_1} I\end{aligned}$$

as $\xi_{\bar{j}} = \xi_1 - \xi_j$. Therefore by (6.3.2), γ_ξ takes values in $\Omega_r U(n)^{\mathbb{R}}$. A similar argument can be applied to odd dimension: let $\tilde{\xi}$ be a canonical element of $\mathfrak{so}(2m+1)$ so we have that $\tilde{\xi} = i\operatorname{diag}(\tilde{\xi}_1, \dots, \tilde{\xi}_m, 0, -\tilde{\xi}_m, \dots, -\tilde{\xi}_1)$, with canonical geodesic $\gamma_{\tilde{\xi}} = \operatorname{diag}(\lambda^{\tilde{\xi}_1}, \dots, \lambda^{\tilde{\xi}_m}, 1, \lambda^{-\tilde{\xi}_{m+2}}, \dots, \lambda^{-\tilde{\xi}_1})$. Again fixing representatives of each canonical geodesic in an equivalence class in $PU(2m+1)$ we define the homomorphism $\gamma_\xi : S^1 \to$

$U(2m+1)$ by

$$\gamma_\xi = \operatorname{diag}(\lambda^{\xi_1}, \lambda^{\xi_2}, \ldots, \lambda^{\xi_m}, \lambda^{\xi_{m+1}}, \lambda^{\xi_{m+2}} \ldots, \lambda^{\xi_{2m}}, \lambda^{\xi_{2m+1}}),$$

where $\xi_j = \begin{cases} \tilde{\xi}_j + \tilde{\xi}_1, & \text{if } 1 \leq j \leq m, \\ \tilde{\xi}_1 & \text{if } j = m+1, \\ \tilde{\xi}_1 - \tilde{\xi}_{2m+2-j}, & \text{if } m+2 \leq j \leq 2m+1, \end{cases}$

and $\xi = i\operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_{2m+1})$. Note the ξ_j are non-negative integers such that $\xi_j - \xi_{j+1} = 0$ or 1, $\xi_{2m+1} = 0$, $\xi_1 = 2\tilde{\xi}_1$ and $\xi_{\bar{j}} = \xi_1 - \xi_j$. We arrive at the definition:

Definition 6.3.3. *[29, Definition 3.1] Let ξ be a diagonal matrix $\xi = i\operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ where ξ_j are non-negative integers satisfying $\xi_j - \xi_{j+1} = 0$ or $1, \xi_n = 0$, $\xi_{\bar{j}} = r - \xi_j$, $\xi_1 = r$ and when r is odd $\xi_{n/2-1} = \xi_{n/2}$. Then we call ξ* ***a canonical element of*** *$\Omega_r U(n)^{\mathbb{R}}$ and $\gamma_\xi = \operatorname{diag}(\lambda^{\xi_1}, \lambda^{\xi_2}, \ldots, \lambda^{\xi_n}) \in \Omega_r U(n)^{\mathbb{R}}$ the corresponding* ***canonical geodesic****.*

Remark 6.3.4. (i) *It was shown in [48, Theorem 6.8] that if n is odd then r is even. This can be seen from Proposition 5.2.17 as our representatives of the canonical geodesics in $PU(n)$ are chosen by adding ξ_1 to ξ, a canonical element of $\mathfrak{so}(n)$. Therefore if n is odd then ξ_1 is always an integer and we have $r = 2\xi_1$.*

(ii) *The definition from [29] above seems to miss the case when $n = 2m$, r is odd and $\xi_{m-1} - \xi_m = 1$, $\xi_m - \xi_{m+1} = -1$, $\xi_{m+1} - \xi_{m+2} = 1$. This is not the case as performing the change of basis that swaps the entries ξ_m and ξ_{m+1}, gives canonical elements such that $\xi_{m-1} = \xi_m$, $\xi_m - \xi_{m+1} = 1$, $\xi_{m+1} = \xi_{m+2}$ which is covered in the definition above. (As in Proposition 6.2.8, we are only interested in finding extended solutions and harmonic maps up to equivalence (Definition 6.1.10).)*

(iii) *Recalling Definition 6.2.2 where the 'type' of a canonical element was defined, this definition extends to the canonical elements of Definition 6.3.3. The possible types of these canonical elements are $(t_0, t_1, \ldots, t_r)$ where the t_j are positive integers with $t_j = t_{r-j}$, and when r is odd, $t_{(r-1)/2} = t_{(r+1)/2} \geq 2$.*

Similarly to §6.2, the corresponding eigenspaces of $\mathrm{ad}(\xi)$ for ξ a canonical element of $\Omega_r U(n)^{\mathbb{R}}$ are denoted $\mathfrak{g}^{\mathbb{R}}_\kappa = \mathfrak{g}^{\mathbb{R}}_\kappa(\xi)$ and consist of matrices with entries zero unless on the κth block superdiagonal. If ξ is a canonical element of $\Omega_r U(n)^{\mathbb{R}}$ then it is a canonical element of $\Omega_r U(n)$ and we have $\mathfrak{g}^{\mathbb{R}}_\kappa(\xi) = \mathfrak{g}^{\mathbb{C}}_\kappa(\xi) \cap \mathfrak{o}(n, \mathbb{C})$. Recall the space $\mathfrak{A}^{\mathbb{C}}_\xi$ from Definition 6.2.6.

Definition 6.3.5. (i) *Let ξ be a canonical element of $\Omega_r U(n)^{\mathbb{R}}$. We define the finite-dimensional Lie subgroup $\mathfrak{A}^{\mathbb{R}}_\xi$ of $\Lambda^+_{\mathrm{alg}} GL(n, \mathbb{C})$ to be the intersection $\mathfrak{A}^{\mathbb{R}}_\xi = \mathfrak{A}^{\mathbb{C}}_\xi \cap \Omega O(n, \mathbb{C})$. We also define the Lie subgroup $(\mathfrak{A}^{\mathbb{R}}_\xi)_0$ of $\mathfrak{A}^{\mathbb{R}}_\xi$ by $(\mathfrak{A}^{\mathbb{R}}_\xi)_0 = \mathfrak{A}^{\mathbb{R}}_\xi \cap O(n, \mathbb{C})$.*

(ii) *We denote the space of meromorphic maps $A : M \to \mathfrak{A}^{\mathbb{R}}_\xi$ satisfying (6.2.2) by $\mathrm{Sol}^{\mathbb{R}}_\xi$ and the subspace of $\mathrm{Sol}^{\mathbb{R}}_\xi$ of meromorphic maps $A : M \to (\mathfrak{A}^{\mathbb{R}}_\xi)_0$ by $(\mathrm{Sol}^{\mathbb{R}}_\xi)_0$.*

Proposition 6.3.6. *[29, Proposition 3.4] Let $\tilde{\Phi} : M \to \Omega_r U(n)^{\mathbb{R}}$ be a polynomial extended solution for some $\tilde{r} \in \mathbb{N}_0$. Then there is an equivalent extended solution $\Phi : M \to \Omega_r U(n)^{\mathbb{R}}$ with $0 \leq r \leq \tilde{r}$, a canonical element $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ of $\Omega_r U(n)^{\mathbb{R}}$ and a meromorphic map $A : M \to \mathfrak{A}^{\mathbb{R}}_\xi$ such that $\Phi = [A\gamma_\xi]$. A and ξ are uniquely determined by Φ, and all harmonic maps $\varphi : M \to O(n)$ of finite uniton number have such an extended solution.*

Example 6.3.7. *Let $\Phi \in \Omega_r U(n)^{\mathbb{R}}$ be an S^1-invariant polynomial extended solution. Recall from Proposition 6.1.20 that*

$$\Phi = \pi_{\psi_0} + \lambda \pi_{\psi_1} + \lambda^2 \pi_{\psi_2} + \cdots + \lambda^r \pi_{\psi_r}, \tag{6.3.4}$$

where $\psi_j = \alpha_j^\perp \cap \alpha_{j+1}$, and by Proposition 6.1.13 the unitons are nested, i.e.

$$0 = \alpha_0 \subset \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_r \subset \alpha_{r+1} = M \times \mathbb{C}^n.$$

Also from Proposition 6.1.20 the Grassmannian model is given by

$$W = \Phi \mathcal{H}_+ = \alpha_1 + \lambda \alpha_2 + \lambda^2 \alpha_3 + \cdots + \lambda^{r-1} \alpha_r + \lambda^r \mathcal{H}_+. \tag{6.3.5}$$

As shown in Example 6.2.14, on putting $\lambda = -1$ into (6.3.4) we see that the corresponding harmonic map $\varphi = M \to U(n)$ is given by

$$\varphi = \Phi_{-1} = \sum_{j \text{ even}} \pi_{\psi_j} - \sum_{j \text{ odd}} \pi_{\psi_j}.$$

This can be rewritten as

$$\varphi = \pi_\phi - \pi_\phi^\perp \tag{6.3.6}$$

where

$$\pi_\phi = \sum_{j \text{ even}} \pi_{\psi_j} \quad \textit{and} \quad \pi_\phi^\perp = \sum_{j \text{ odd}} \pi_{\psi_j}.$$

By Propositions 6.2.12 and 6.2.13, we see that (6.3.6) is given by

$$\varphi = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$$

where $\phi : M \to G_*(\mathbb{C}^n)$ and is given by the subbundle

$$\underline{\phi} = \sum_{j \text{ even}} \psi_j. \tag{6.3.7}$$

Recall from Definition 6.3.2 that $\overline{W}^\perp = \lambda^{1-r}W$ or equivalently from (6.3.1) the extended solution satisfies $\overline{\Phi} = \lambda^{-r}\Phi$. Using (6.3.4) with $\overline{W}^\perp = \lambda^{1-r}W$ or equivalently (6.3.5) with $\overline{\Phi} = \lambda^{-r}\Phi$ we see that $\overline{\alpha}_j^\perp = \alpha_{r+1-j}$ and $\psi_j = \overline{\psi}_{r-j}$ for all j.

For r even, as $\psi_j = \overline{\psi}_{r-j}$ for all j, we see that $\overline{\underline{\phi}} = \underline{\phi}$ and so the corresponding map ϕ is, in fact, a map into a real Grassmannian $\phi : M \to G_*(\mathbb{R}^n)$. Note that

$$G_k(\mathbb{R}^n) = \frac{O(n)}{O(k) \times O(n-k)},$$

so we will often write ϕ from (6.3.7) as a map into $O(n)/(O(k) \times O(n-k))$ for an appropriate k. The Cartan embedding from Proposition 6.2.13 restricts to the totally *geodesic embedding* $\iota : G_*(\mathbb{R}^n) \to O(n)$*, therefore* $\varphi = \Phi_{-1} = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp : M \to O(n)$*.*

For r *odd, then as earlier by [48, Theorem 6.8] or Remark 6.3.4,* n *is even, say* $n = 2m$*. As* r *is odd we have from (6.3.7) that*

$$\underline{\phi} = \sum_{j \ \ even} \psi_j = \psi_0 + \psi_2 + \psi_4 + \cdots + \psi_{r-1}.$$

We see that

$$\underline{\phi}^\perp = \psi_1 + \psi_3 + \cdots + \psi_r = \overline{\psi}_{r-1} + \overline{\psi}_{r-3} + \cdots + \overline{\psi}_0 = \overline{\underline{\phi}}$$

so $\overline{\underline{\phi}}^\perp = \underline{\phi}$ *and by Definition 5.1.2 is maximally isotropic. We identify the space of maximally isotropic subspaces of* $\mathbb{C}^{2m}$ *with the space* $O(2m)/U(m)$ *[48, §6.3]. Therefore for the corresponding map* $\phi : M \to G_*(\mathbb{C}^{2m})$ *and* $x \in M$*,* $\phi(x)$ *is a maximally isotropic subspace of* $\mathbb{C}^{2m}$ *and so* ϕ *is a harmonic map into* $O(2m)/U(m)$*. There is a totally geodesic embedding of* $O(2m)/U(m)$ *into* $G_m(\mathbb{C}^{2m})$ *which upon composition with the Cartan embedding from Proposition 6.2.13 has image in* $\{g \in U(2m) \mid \overline{g} = -g\}$ *[48, §6.3]. The minus sign in* $\overline{g} = -g$ *reflects the fact that* $\Phi_{-1} = \pi_\phi - \pi_\phi^\perp = \pi_\phi - \pi_{\overline{\phi}}$ *so it is more natural to consider the associated harmonic map* $\varphi = i\Phi_{-1} = i(\pi_\phi - \pi_{\overline{\phi}})$ *which is real, i.e., has values in* $O(2m) = \{g \in U(2m) : \overline{g} = +g\}$*.*

6.3.1 Adding a Border

We will now describe a procedure introduced in [29] which is used to give a parametrization of harmonic maps $\varphi : M \to O(n)$.

Let $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_n)$ be a canonical element of $\Omega_r U(n)^{\mathbb{R}}$ of type $(t_0, t_1, \ldots, t_r)$ for some $r \in \mathbb{N}_0$, $m \geq 2$. We define $\tilde{\xi} = i\,\mathrm{diag}(\xi_2, \ldots, \xi_{2m-1})$ which, if ξ is not of type

$(2,2)$, is a canonical element of $\Omega_{\tilde{r}}U(n-2)^{\mathbb{R}}$ of type

$$(\tilde{t}_0, \tilde{t}_1, \dots, \tilde{t}_{\tilde{r}}) = \begin{cases} (t_0 - 1, t_1, \dots, t_{r-1}, t_r - 1), \text{ with } \tilde{r} = r & \text{if } t_0 \geq 2, \\ (t_1, \dots, t_{r-1}), \text{ with } \tilde{r} = r - 2 & \text{otherwise.} \end{cases} \tag{6.3.8}$$

If ξ is of type $(2,2)$ then, $\tilde{\xi}$ is of type $(1,1)$, $\tilde{r} = r = 1$ and is therefore not a canonical element, so we must treat this case separately. Let $A : M \to \mathfrak{A}_{\xi}^{\mathbb{R}}$ be given by $A = (a_{jk})_{j,k=1,2,\dots,n}$ then we define a matrix $\tilde{A} = (a_{jk})_{j,k=2,\dots,n-1}$, this is called **removing the border**.

Now let $\tilde{A} = (a_{jk})_{j,k=2,\dots,n-1} : M \to \mathfrak{A}_{\tilde{\xi}}^{\mathbb{R}}$ be given, we add a new first column $(a_{11}, a_{21}, \dots, a_{n1})^T$ where $a_{j1} = \delta_{j1}$, a new bottom row $(a_{n2}, a_{n3}, \dots, a_{nn})$ where $a_{nk} = \delta_{nk}$. We also add a new top row $(a_{12}, a_{13}, \dots, a_{1,n-1})$, new last column $(a_{2n}, a_{3n}, \dots, a_{n-1,n})^T$ and new top-right element a_{1n} so we have

$$A = \left(\begin{array}{c|cccc|c} 1 & a_{12} & a_{13} & \dots & a_{1,n-1} & a_{1n} \\ \hline 0 & & & & & a_{2n} \\ 0 & & & \tilde{A} & & a_{3n} \\ 0 & & & & & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

This process is called **adding a border** and the resulting A defines a map $A : M \to \mathfrak{A}_{\xi}^{\mathbb{C}}$. To ensure that A takes values in $\mathfrak{A}_{\xi}^{\mathbb{R}}$ we use Lemma 5.1.4 on the columns c_j of A, in fact, given the new top row (resp. new last column) we can use Lemma 5.1.4 to find expressions for the elements of the new last column (new top row) in terms of those in the new top row (resp. new last column), we may also find the new top-right element using $(c_n, c_n) = 0$. Applying Lemma 5.1.4 in this way to complete the border so that A takes values in $\mathfrak{A}_{\xi}^{\mathbb{R}}$ is called (completing the border) **by algebra**.

Lemma 6.3.8. *[29, Lemma 3.6] Let ξ be a canonical element of $\Omega_r U(n)^{\mathbb{R}}$ not of type*

$(2,2)$, *and let* $\tilde{\xi}$ *be the canonical element of* $\Omega_{\tilde{r}}U(n-2)^{\mathbb{R}}$ *defined above. Let* $A = (a_{jk})_{j,k=1,2,\dots,n} : M \to \mathfrak{A}_{\xi}^{\mathbb{R}}$ *be a holomorphic map, and let* $\tilde{A} : (a_{jk})_{j,k=2,\dots,n-1} : M \to \mathfrak{A}_{\tilde{\xi}}^{\mathbb{R}}$ *be the holomorphic map defined above. Suppose* A *satisfies (6.2.2), then so does* $\tilde{A}$*. Conversely, suppose* $\tilde{A}$ *satisfies (6.2.2) then* A *satisfies (6.2.2) if and only if the new top row satisfies (6.2.3), and this holds if and only if the new last column satisfies (6.2.3).*

Therefore given an extended solution $\tilde{\Phi} = [\tilde{A}\gamma_{\tilde{\xi}}] : M \to \Omega_{\tilde{r}}U(n-2)^{\mathbb{R}}$ we may find another (non-equivalent) extended solution $\Phi = [A\gamma_{\xi}] : M \to \Omega_r U(n)^{\mathbb{R}}$ by adding a border to $\tilde{A}$, then solving (6.2.3) for the elements of the new top row and finding the new last column and new top-right element by algebra.

6.3.2 Solving the Extended Solution Equation

To solve the extended solution equations (6.2.3) for the new top row and therefore parametrize the extended solutions for $O(n)$, Ferreira, Simões and Wood in [29, §3.4] introduced new parameters which change the problem of integrating (6.2.3) to differentiating the new parameters and doing some algebraic operations.

Definition 6.3.9. *Let* ν *and* β_j *for* $j = 1, 2, \dots, l,\ l \in \mathbb{N}$, *be meromorphic functions on the Riemann surface* M*. We define the* ***generalised derivative of*** ν ***with respect to*** β_1 *to be the quotient* $\nu^{(1)} = \nu'/\beta_1'$ *where* $'$ *denotes the derivative with respect to some local complex coordinate* z *on* M*. Higher generalised derivatives are defined inductively by* $\nu^{(k)} = (\nu^{(k-1)})'/\beta_k'$, *for* $0 \le k \le l$, *and we set* $\nu^{(0)} = \nu$*. Note that these are well-defined under change of complex coordinates.*

Let $\mathcal{M}(M)$ denote the space of meromorphic functions on the Riemann surface M. Let $p_1 = p_1(\xi) = \dim \mathfrak{g}_1^{\mathbb{R}}(\xi)$ and $p = p(\xi) = \sum_{j=1}^{r} \dim \mathfrak{g}_j^{\mathbb{R}}(\xi)$, for ξ a canonical element of $\Omega_r U(n)^{\mathbb{R}}$ for some $r \in \mathbb{N}_0$. Using the process of adding a border and using Definition 6.3.9 with Lemma 6.3.8, Ferreira, Simões and Wood gave an algorithm which defines a map $h = h_\xi : \mathcal{M}(M)^p \to \mathrm{Sol}^{\mathbb{R}}_\xi$. The map h restricts to a map $h_0 = (h_0)_\xi : \mathcal{M}(M)^{p_1} \to (\mathrm{Sol}^{\mathbb{R}}_\xi)_0$. We will now see an example of this algorithm in practice and we direct the reader to [29, Proposition 3.7] for full details of the map $h : \mathcal{M}(M)^p \to \mathrm{Sol}^{\mathbb{R}}_\xi$, and for more interpretation of the following example.

Example 6.3.10. *Consider the 1×1 matrix $\tilde{\xi} = (0)$ which is a canonical element of $\Omega_0 U(1)^{\mathbb{R}}$ of type (1) and let us define a 1×1 matrix $\tilde{A} = (1) : M \to \mathfrak{A}^{\mathbb{R}}_{\tilde{\xi}}$. Let $\xi = i\,\mathrm{diag}(2,1,0)$ be the canonical element of $\Omega_2 U(3)^{\mathbb{R}}$ of type $(1,1,1)$ defined from $\tilde{\xi}$ as in (6.3.8). We will add a border to $\tilde{A}$ to get a 3×3 matrix A as above so we have*

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} \\ 0 & \boxed{1} & a_{23} \\ 0 & 0 & 1 \end{pmatrix}.$$

The extended solution equations (6.2.3) for A are

$$a'_{12} = \sum_{j:2 \geq \xi_j > 0} \lambda^{\xi_j - 2} \rho'_{j2} a_{1j} = \lambda^0 a'_{12} a_{11} = a'_{12},$$

and therefore the extended solution equations are automatically satisfied. To ensure that $A : M \to \mathfrak{A}^{\mathbb{R}}_{\tilde{\xi}}$ we use Lemma 5.1.4 which says that $A \in O(3, \mathbb{C})$ if and only if $(c_j, c_k) = \delta_{j\bar{k}}$ where c_j are the columns of A. The only non-trivial equations are

$$(c_2, c_3) = 0 \implies a_{12} = -a_{23},$$

$$(c_3, c_3) = 0 \implies a_{13} = \frac{1}{2} a_{23}^2.$$

Therefore we have

$$A = \begin{pmatrix} 1 & a_{12} & -\frac{1}{2} a_{12}^2 \\ 0 & \boxed{1} & -a_{12} \\ 0 & 0 & 1 \end{pmatrix}.$$

By Proposition 6.3.6, $\Phi = [A\gamma_\xi] : M \to \Omega_2 U(3)^{\mathbb{R}}$ *is an extended solution for the harmonic map* $\phi = \Phi_{-1} : M \to G_2(\mathbb{R}^3)$. *The unitons of* ϕ, α_j, $j = 1, 2$ *are given by* $\alpha_1 = \mathrm{Span}\{c_3\}$ *and* $\alpha_2 = \mathrm{Span}\{c_2, c_3\}$ *where* c_2 *and* c_3 *denote the second and third columns of* A*, respectively. We see that* $\alpha_0 \subset \alpha_1 \subset \alpha_2 \subset \alpha_3$ *and so by Proposition 6.1.13 the extended solution is* S^1*-invariant and by Proposition 6.1.20 its Grassmannian model is given by* $W = \Phi\mathcal{H}_+ = \alpha_1 + \lambda\alpha_2 + \lambda^2\mathcal{H}_+$. *As* r *is even, by Example 6.3.7, the corresponding harmonic map takes values in a real Grassmannian* $\phi : M \to G_2(\mathbb{R}^3)$. *By Propositions 6.2.12 and 6.2.13, we have the harmonic map* $\varphi = \iota \circ \phi : M \to O(3)$ *given by* $\varphi = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$.

Chapter 7

Harmonic Maps from Surfaces into the Symplectic Group

In this chapter we adapt the work of [9, 29] to the symplectic group. We do this by modifying the process of adding a border in §6.3.1 and solving the extended solution equations by introducing generalised derivatives. This process is more difficult compared to the $O(n)$ case as, when solving the extended solution equations we must replace parameters *inside* the border, as opposed to *on* the border as in [29], this is because the algebra is based on an antisymmetric form ω which does not determine as much as the symmetric form used in the $O(n)$ case.

We give a parametrization of all S^1-invariant extended solutions up to dimension 6, and give a theorem parametrizing S^1-invariant extended solutions of canonical type for all dimensions.

7.1 Harmonic Maps into The Symplectic Group

Throughout this chapter we will use the null basis (5.1.9) from §5.1.2 and from now on we will write all vectors and matrices with respect to this null basis. Recall the symplectic group §5.1.2, that is $Sp(m) := Sp(2m,\mathbb{C}) \cap U(2m)$, where on using a null basis,

$$Sp(2m,\mathbb{C}) = \{A \in GL(n,\mathbb{C}) \mid A^{\mathfrak{T}}\Omega_{\text{null}}A = \Omega_{\text{null}}\},$$

for

$$\Omega_{\text{null}} = \begin{pmatrix} iI_m & \\ & -iI_m \end{pmatrix}.$$

We will regard $Sp(m)$ as a subgroup of $U(2m)$ in the obvious way:

$$Sp(m) = \{A \in U(2m) \mid A^{\mathfrak{T}}\Omega_{\text{null}}A = \Omega_{\text{null}}\}.$$

Similarly,

$$\Omega Sp(m) = \{\Phi \in \Omega U(2m) \mid \Phi^{\mathfrak{T}}\Omega_{\text{null}}\Phi = \Omega_{\text{null}}\},$$

where $\Phi^{\mathfrak{T}}$ is defined by Remark 6.1.16, replacing ^{T} by $^{\mathfrak{T}}$. Recall the bijection (6.1.7), and the conjugate-linear map $J : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ defined by (5.1.13).

Lemma 7.1.1. *[43, Proposition 8.5.4] Given $\Phi \in \Omega U(2m)$, set $W = \Phi\mathcal{H}_+ \in Gr$, then $\Phi \in \Omega Sp(m)$ if and only if $JW^\perp = \lambda W$.*

Lemma 7.1.2. *In the null basis (5.1.9), given $\Phi \in \Omega U(2m)$, set $W = \Phi\mathcal{H}_+ \in Gr$, then $\Phi \in \Omega Sp(m)$ if and only if $\Omega_{\text{null}}\overline{W}^\perp = \lambda W$.*

Proof. From Lemma 7.1.1 it suffices to show that $JW^\perp = \Omega_{\text{null}}\overline{W}^\perp$. Recall from (5.1.13) that $J(v) = \overline{\Omega_{\text{null}}v}$ for $v \in \mathbb{C}^{2m}$. Therefore we have

$$JW^\perp = \overline{\Omega_{\text{null}}W^\perp} = \overline{\Omega}_{\text{null}}\overline{W}^\perp = \Omega_{\text{null}}\overline{W}^\perp.$$

□

Note that the formulation above highlights the link between the Grassmannian model W and its **polar** $\overline{W}^{\perp}$, not to be confused with the polar of f from Definition 2.2.4. Recall the skew-symmetric bilinear form $\omega(\cdot,\cdot)$ given by (5.1.11). In keeping with notation of [48, 49] we define a subset of $\Omega_r U(n)$ similar to that of (6.3.1).

Definition 7.1.3. *Let $\Omega_r U(2m)^J$ be the subset of $\Omega_r U(2m)$ defined by*

$$\Omega_r U(2m)^J = \{\Phi \in \Omega_r U(2m) \mid \omega(\Phi x, \Phi y) = \lambda^r \omega(x,y)\}.$$

where $x, y \in \mathbb{C}^{2m}$.

We will show that this definition is equivalent to the definition given in [48, §6.8][49, §7.1] as well as give another equivalent definition.

Lemma 7.1.4. *The subset $\Omega_r U(2m)^J$ of $\Omega_r U(2m)$ can be equivalently defined by*

(i) $\Omega_r U(2m)^J = \{\Phi \in \Omega_r U(2m) \mid J\Phi J^{-1} = \lambda^{-r}\Phi\}$ *in the standard basis and*

(ii) $\Omega_r U(2m)^J = \{\Phi \in \Omega_r U(2m) \mid \Phi^{\mathfrak{T}}\Omega_{\text{null}}\Phi = \lambda^r \Omega_{\text{null}}\}$ *in the null basis (5.1.9).*

Proof. Recall the standard Hermitian form $\langle\cdot,\cdot\rangle$ on $\mathbb{C}^{2m}$ given in §5.1.2. In the null basis (5.1.9) this is given by

$$\langle x, y\rangle = x^T \hat{P}^T I \overline{\hat{P}} \overline{y} = x^T I \overline{y}, \tag{7.1.1}$$

where $\hat{P}$ is the unitary matrix given by (5.1.10). Note the similarity between the expressions of $\langle x, y\rangle$ given in both the standard basis and the null basis (5.1.9), indeed both bases are Hermitian. Also note that in the null basis (5.1.9) (as well as the standard basis) we have

$$\langle \Phi x, y\rangle = \langle x, \overline{\Phi}^T y\rangle \tag{7.1.2}$$

where Φ an $2m \times 2m$ matrix. From (7.1.1) we have that, in the null basis (5.1.9), $\Phi \in U(2m)$ if and only if $\overline{\Phi}^T \Phi = I$.

For (i), as $\Phi \in \Omega_r U(2m) \subset \Omega U(2m)$, then $\overline{\Phi}^T \Phi = I$, together with (5.1.7) we have

$$\omega(\Phi x, \Phi y) = \langle \Phi x, J\Phi y \rangle = \langle x, \overline{\Phi}^T J\Phi y \rangle = \langle x, \Phi^{-1} J\Phi y \rangle$$

and

$$\lambda^r \omega(x, y) = \lambda^r \langle x, Jy \rangle = \langle x, \lambda^{-r} Jy \rangle,$$

as $\overline{\lambda^j} = \lambda^{-j}$ for all j. Therefore $\omega(\Phi x, \Phi y) = \lambda^r \omega(x, y)$ if and only if $\Phi^{-1} J\Phi = \lambda^{-r} J$ if and only if $J\Phi J^{-1} = \lambda^{-r}\Phi$. For (ii) we have from (5.1.11) that

$$\omega(\Phi x, \Phi y) = x^T \Phi^T \hat{\Omega} \Phi y.$$

Therefore $\omega(\Phi x, \Phi y) = \lambda^r \omega(x, y)$ if and only if $\Phi^T \hat{\Omega} \Phi = \lambda^r \hat{\Omega}$. Now recall from (5.1.4) that $\Phi^{\mathfrak{T}} = Q\Phi^T Q$ for $Q = Q^{-1} = \begin{pmatrix} & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & \end{pmatrix}$ and from (5.1.12) that $\Omega_{\text{null}} := Q\hat{\Omega}$. Due to $QQ = I$, the following are equivalent

$$\begin{aligned} \Phi^T \hat{\Omega} \Phi &= \lambda^r \hat{\Omega} \\ \Phi^T QQ\hat{\Omega} \Phi &= \lambda^r QQ\hat{\Omega} \\ Q\Phi^T QQ\hat{\Omega} \Phi &= \lambda^r Q\hat{\Omega} \\ \Phi^{\mathfrak{T}} \Omega_{\text{null}} \Phi &= \lambda^r \Omega_{\text{null}}. \end{aligned}$$

Therefore $\omega(\Phi x, \Phi y) = \lambda^r \omega(x, y)$ if and only if and only if $\Phi^{\mathfrak{T}} \Omega_{\text{null}} \Phi = \lambda^r \Omega_{\text{null}}$. $\square$

Lemma 7.1.5. *Given $\Phi \in \Omega_r U(2m)$, set $W = \Phi \mathcal{H}_+ \in Gr_r$, then $\Phi \in \Omega_r U(2m)^J$ if and only if $JW^\perp = \lambda^{1-r} W$.*

Proof. It suffices to prove that $JW^\perp = \lambda^{1-r}W$ if and only if $J\Phi J^{-1} = \lambda^{-r}\Phi$. As $W = \Phi\mathcal{H}_+$, by multiplying by λ^{1-r} we have

$$\lambda^{1-r}W = \lambda^{1-r}\Phi\mathcal{H}_+. \tag{7.1.3}$$

Recall from §6.1.3 that the natural action of $\Omega U(2m)$ is isometric with respect to the L^2 inner product defined by $\langle v, w\rangle_{L^2} = \sum_i \langle v_i, w_i\rangle$, where $v = \sum_i \lambda^i v_i \in \mathcal{H}$, $w = \sum_i \lambda^i w_i \in \mathcal{H}$ and $\langle\cdot,\cdot\rangle$ is the standard hermitian innner product on $\mathbb{C}^n$. Therefore, as $\Phi \in \Omega U(2m)$ then Φ preserves the inner product and we deduce $W^\perp = \Phi\mathcal{H}_+^\perp$. Upon applying the conjugate-linear map $J : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ we have

$$JW^\perp = J\Phi\mathcal{H}_+^\perp = J\Phi J^{-1}J\mathcal{H}_+^\perp = \lambda J\Phi J^{-1}\mathcal{H}_+, \tag{7.1.4}$$

as $J\mathcal{H}_+^\perp = \lambda\mathcal{H}_+$. Comparing (7.1.3) and (7.1.4) we have $JW^\perp = \lambda^{1-r}W$ if and only if $\lambda^{1-r}\Phi\mathcal{H}_+ = \lambda J\Phi J^{-1}\mathcal{H}_+$ if and only if $J\Phi J^{-1} = \lambda^{-r}\Phi$ as required. □

We give a version of Lemma 7.1.5 for the null basis where J is calculated using Ω_{null}.

Lemma 7.1.6. *In the null basis (5.1.9), given $\Phi \in \Omega_r U(2m)$, set $W = \Phi\mathcal{H}_+ \in Gr_r$, then $\Phi \in \Omega_r U(2m)^J$ if and only if $\Omega_{\text{null}}\overline{W}^\perp = \lambda^{1-r}W$.*

Proof. Similarly to Lemma 7.1.2, by Lemma 7.1.1, it suffices to show that $JW^\perp = \Omega_{\text{null}}\overline{W}^\perp$. Recall from (5.1.13) that $J(v) = \overline{\Omega_{\text{null}}v}$ for $v \in \mathbb{C}^{2m}$. As in the proof of Lemma 7.1.2,

$$JW^\perp = \overline{\Omega_{\text{null}}W^\perp} = \overline{\Omega}_{\text{null}}\overline{W}^\perp = \Omega_{\text{null}}\overline{W}^\perp.$$

Then Lemma 7.1.6 follows from Lemma 7.1.5. □

We now define canonical elements of $\Omega_r U(2m)^J$ from those of $\mathfrak{sp}(m)$ similarly to what we did for $\mathfrak{o}(m)$ in §6.3. Let $\tilde{\xi}$ be a canonical element of $\mathfrak{sp}(m)$, so by Proposition 5.2.20,

$$\tilde{\xi} = i\,\mathrm{diag}(\tilde{\xi}_1, \dots, \tilde{\xi}_m, -\tilde{\xi}_m, \dots, \tilde{\xi}_1),$$

where $\tilde{\xi}_j - \tilde{\xi}_{j+1} = 1$ or 0, for all $j = 1, 2, \dots, m-1$, and $\tilde{\xi}_m = 0$ or 1. By Definition 5.2.12 the canonical geodesics $\gamma_{\tilde{\xi}} : S^1 \to Sp(m)$ are of the form

$$\gamma_{\tilde{\xi}} = \exp(t\tilde{\xi}) = \mathrm{diag}(e^{it\tilde{\xi}_1}, e^{it\tilde{\xi}_2}, \dots, e^{it\tilde{\xi}_m}, e^{-it\tilde{\xi}_m}, \dots, e^{-it\tilde{\xi}_1}). \tag{7.1.5}$$

As above we regard $Sp(m) \subset U(2m)$ and so can view the canonical geodesic $\gamma_{\tilde{\xi}}$ as a map into $U(2m)$. Again by following [9, p.562] we will fix representatives of the geodesics (7.1.5) in the projective unitary group $PU(n) = U(n)/Z(U(n))$. Recall that $PU(n)$ is a group of equivalence classes of unitary matrices under multiplication by $c \cdot I$, for $c \in \mathbb{C}$. By choosing suitable representatives of each canonical geodesic in an equivalence class and recalling the notation from Remark 5.2.13 we define the homomorphism $\gamma_\xi : S^1 \to Sp(m)$,

$$\begin{aligned}\gamma_\xi = e^{it\tilde{\xi}_1}\gamma_{\tilde{\xi}} &= \mathrm{diag}(e^{2it\tilde{\xi}_1}, e^{it(\tilde{\xi}_2+\tilde{\xi}_1)}, \dots, e^{it(\tilde{\xi}_m+\tilde{\xi}_1)}, e^{it(\tilde{\xi}_1-\tilde{\xi}_m)}, \dots, e^{it(\tilde{\xi}_1-\tilde{\xi}_2)}, e^0)\\ &= \mathrm{diag}(\lambda^{2\tilde{\xi}_1}, \lambda^{(\tilde{\xi}_2+\tilde{\xi}_1)}, \dots, \lambda^{(\tilde{\xi}_m+\tilde{\xi}_1)}, \lambda^{(\tilde{\xi}_1-\tilde{\xi}_m)}, \dots, \lambda^{(\tilde{\xi}_1-\tilde{\xi}_2)}, \lambda^0)\\ &= \mathrm{diag}(\lambda^{\xi_1}, \lambda^{\xi_2}, \dots, \lambda^{\xi_m}, \lambda^{\xi_{m+1}}, \dots, \lambda^{\xi_{2m-1}}, \lambda^{\xi_{2m}})\end{aligned}$$

where $\xi_j = \begin{cases} \tilde{\xi}_j + \tilde{\xi}_1, & \text{if } 1 \le j \le m, \\ \tilde{\xi}_1 - \tilde{\xi}_{2m+1-j}, & \text{if } m+1 \le j \le 2m, \end{cases}$ and $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \dots, \xi_{2m})$. Note that ξ_j are non-negative integers satisfying $\xi_j - \xi_{j+1} = 0$ or 1, $\xi_{2m} = 0$, $\xi_1 = 2\tilde{\xi}_1$ and

$\xi_{\bar{j}} = \xi_1 - \xi_j$. We also have

$$\begin{aligned}\gamma_\xi^{\mathcal{I}}\Omega_{\text{null}}\gamma_\xi &= \operatorname{diag}(\lambda^{\xi_{2m}}, \ldots, \lambda^{\xi_{m+1}}, -\lambda^{\xi_m}, \ldots, -\lambda^{\xi_1}) \operatorname{diag}(\lambda^{\xi_1}, \ldots, \lambda^{\xi_m}, \lambda^{\xi_{m+1}}, \ldots, \lambda^{\xi_{2m}}) \\ &= \operatorname{diag}(\lambda^{\xi_{2m}+\xi_1}, \ldots, \lambda^{\xi_{m+1}+\xi_m}, -\lambda^{\xi_m+\xi_{m+1}}, \ldots, -\lambda^{\xi_1+\xi_{2m}}) \\ &= \lambda^{\xi_1}\Omega_{\text{null}},\end{aligned}$$

as $\xi_{\bar{j}} = \xi_1 - \xi_j$, which shows that γ_ξ takes values in $\Omega_r U(2m)^J$ where $r = \xi_1$. We arrive at a definition similar to Definition 6.2.1:

Definition 7.1.7. *Let ξ be a diagonal matrix $\xi = i\operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_{2m})$ where ξ_j are non-negative integers satisfying $\xi_j - \xi_{j+1} = 0$ or $1, \xi_{2m} = 0,$ $\xi_{\bar{j}} = r - \xi_j$ and $\xi_1 = r$. Then we call ξ* ***a canonical element of*** *$\Omega_r U(2m)^J$ and $\gamma_\xi = \operatorname{diag}(\lambda^{\xi_1}, \lambda^{\xi_2}, \ldots, \lambda^{\xi_{2m}}) \in \Omega_r U(2m)^J$ the corresponding* ***canonical geodesic****.*

Remark 7.1.8. (i) *Recall Definition 6.3.3 and Remark 6.3.4.*

Let $\xi = i\operatorname{diag}(\xi_1, \xi_2, \ldots, \xi_{2m})$ be a canonical element of $\Omega_r U(2m)^J$. If r is odd, then ξ is a canonical element of $\Omega_r U(2m)^{\mathbb{R}}$ if and only if $\xi_{m-1} = \xi_m$.

(ii) *We may define the "type" (Definition 6.2.2) of a canonical element ξ of $\Omega_r U(2m)^J$ in a similar to that of canonical elements of $U(n)$ and $O(n)$. We say that a canonical element ξ of $\Omega_r U(2m)^J$ of type $(1, 1, \ldots, 1)$ is of* ***standard type****.*

Similarly to §6.2 and §6.3 we denote the corresponding eigenspaces of $\operatorname{ad}(\xi)$ for ξ a canonical element of $\Omega_r U(n)^J$ by $\mathfrak{g}_\kappa^J = \mathfrak{g}_\kappa^J(\xi)$; this consists of matrices with entries zero unless on the κth block superdiagonal. If ξ is a canonical element of $\Omega_r U(n)^J$ then, similarly to §6.3 it is a canonical element of $\Omega_r U(n)$ so the eigenspaces satisfy $\mathfrak{g}_\kappa^J(\xi) = \mathfrak{g}_\kappa^{\mathbb{C}}(\xi) \cap \mathfrak{sp}(n, \mathbb{C})$. Recall the space $\mathfrak{A}_\xi^{\mathbb{C}}$ from Definition 6.2.6.

Definition 7.1.9. (i) *Let ξ be a canonical element of $\Omega_r U(2m)^J$. We define the finite-dimensional Lie subgroup $\mathfrak{A}_\xi^J$ of $\Lambda_{alg}^+ GL(n, \mathbb{C})$ to be the intersection $\mathfrak{A}_\xi^J = \mathfrak{A}_\xi^{\mathbb{C}} \cap$*

$\Omega Sp(2m, \mathbb{C})$. *We denote its Lie algebra by* $\mathfrak{a}^J_\xi$. *We also define the Lie subgroup* $(\mathfrak{A}^J_\xi)_0$ *of* $\mathfrak{A}^J_\xi$ *to be* $(\mathfrak{A}^J_\xi)_0 = \mathfrak{A}^J_\xi \cap Sp(2m, \mathbb{C})$.

(ii) *We denote the space of meromorphic maps* $A : M \to \mathfrak{A}^J_\xi$ *satisfying (6.2.2) by* Sol^J_ξ *and the subspace of* Sol^J_ξ *consisting of meromorphic maps* $A : M \to (\mathfrak{A}^J_\xi)_0$ *by* $(\mathrm{Sol}^J_\xi)_0$.

We give a proposition similar to Proposition 6.2.8 and Proposition 6.3.6. We also direct the reader to [42] where a related proposition in a different context was given.

Proposition 7.1.10. *Let* $\tilde{\Phi} : M \to \Omega_{\tilde{r}} U(2m)^J$ *be a polynomial extended solution for some* $\tilde{r} \in \mathbb{N}$. *Then there is an equivalent extended solution* $\Phi : M \to \Omega_r U(2m)^J$ *with* $0 \le r \le \tilde{r}$, *a canonical element* $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_{2m})$ *of* $\Omega_r U(2m)^J$ *and a meromorphic map* $A : M \to \mathfrak{A}^J_\xi$ *such that* $\Phi = [A\gamma_\xi]$. A *and* ξ *are uniquely determined by* Φ, *and all harmonic maps* $\varphi : M \to Sp(m)$ *of finite uniton number have such an extended solution.*

Proof. Let $\tilde{\Phi} : M \to \Omega_{\tilde{r}} U(n)^J$ be a polynomial extended solution for some $\tilde{r} \in \mathbb{N}$, then according to [9, §4], in the fashion of [29], we may write $\tilde{\Phi} = [B\gamma_\tau]$ for $B : M \to \Lambda^+_{\mathrm{alg}} GL(2m, \mathbb{C})$ meromorphic, $\tau = i\,\mathrm{diag}(\tau_1, \tau_2, \ldots, \tau_{2m})$ for τ_j non-negative integers such that $\tau_j - \tau_{j+1} = \eta_j \in \mathbb{N}$, $\tau_{2m} = 0$, $\tau_{\bar{j}} = \tau_1 - \tau_j$ and $\gamma_\tau(t) = \exp(t\tau)$, note that τ is not necessarily a canonical element. (Recall from Proposition 6.1.23 and Proposition 6.1.25 that $[\quad]$ is the projection onto the first factor of the Iwasawa decomposition.)

We relabel the τ_j, for $j = 1, 2, \ldots, 2m$ so we have

$$\tau = i\,\mathrm{diag}(\tilde{\tau}_1, \tilde{\tau}_1, \ldots, \tilde{\tau}_1, \tilde{\tau}_2, \ldots, \tilde{\tau}_2, \ldots, \tilde{\tau}_\delta, \ldots, \tilde{\tau}_\delta),$$

where $1 \le \delta \le \tilde{\tau}_1$, $\tilde{\tau}_j - \tilde{\tau}_{j+1} = \tilde{\eta}_j \in \mathbb{N}$, $\tilde{\tau}_\delta = 0$, $\tilde{\tau}_{\bar{j}} = \tilde{\tau}_1 - \tilde{\tau}_j$. Note here that by relabelling we collect together all τ_j such that $\eta_j = 0$ so $\tilde{\eta}_j \ge 1$ for all $j \in \{1, \ldots, \delta\}$.

Define $W = B\gamma_\tau \mathcal{H}_+ \in Gr_{\tilde{r}}$ and consider the filtration of W given by

$$W = W \cap \lambda^0 \mathcal{H}_+ \supseteq W \cap \lambda^1 \mathcal{H}_+ \supseteq W \cap \lambda^2 \mathcal{H}_+ \supseteq \ldots \supseteq W \cap \lambda^{\tilde{\tau}_1} \mathcal{H}_+ \supseteq W \cap \lambda^{\tilde{\tau}_1+1} \mathcal{H}_+ = \lambda^{\tilde{\tau}_1+1} \mathcal{H}_+.$$

Let $P_j : \mathcal{H} \to \mathbb{C}^n$ be the natural projection defined by $P_j(v) = v_j$ for $v = \sum_j \lambda^j v_j \in \mathcal{H}$. Following [42, §3.3] and [48, §3.4] define

$$A_j = \frac{W \cap \lambda^j \mathcal{H}_+}{(\lambda W \cap \lambda^j \mathcal{H}_+) + (W \cap \lambda^{j+1} \mathcal{H}_+)} \cong \frac{P_j(W \cap \lambda^j \mathcal{H}_+)}{P_{j-1}(W \cap \lambda^{j-1} \mathcal{H}_+)}. \tag{7.1.6}$$

According to [42, Theorem 2] if $A_j = 0$ for some $j \in \{0, 1, \ldots, \tilde{\tau}_1\}$ (and so by the symmetry of τ we have $A_{\tilde{\tau}_1 - j} = 0$) then there exists a constant loop $\mu \in \Omega_2 U(2m)^J$ such that $\mu B\gamma_\tau \mathcal{H}_+ \in Gr_{\tilde{r}-2}$ and further, $[\mu B\gamma_\tau] = \mu[B\gamma_\tau]$ is an equivalent extended solution to $[B\gamma_\tau]$, see Definition 6.1.10. We see from (7.1.6) that $A_j = 0$ if and only if $P_j(W \cap \lambda^j \mathcal{H}_+) = P_{j-1}(W \cap \lambda^{j-1} \mathcal{H}_+)$, also if we let $\tilde{\tau}_k < j \leq \tilde{\tau}_{k+1}$ for some $k \in \{1, \ldots, \delta\}$. Then $P_j(W \cap \lambda^j \mathcal{H}_+) \neq P_{j-1}(W \cap \lambda^{j-1} \mathcal{H}_+)$ if any only if $j = \tilde{\tau}_k + 1$. We conclude that $A_j \neq 0$ for all j such that $\tilde{\tau}_k < j \leq \tilde{\tau}_{k+1}$ for some $k \in \{1, \ldots, \delta\}$ if and only if $\tilde{\tau}_k - \tilde{\tau}_{k+1} = 1$ and so τ is a canonical element of $\Omega_{\tilde{r}} U(2m)^J$. If, on the other hand, $A_j = 0$ for each j in some subset $D \subseteq \{0, 1, \ldots, \tilde{\tau}_1\}$ we iterate the procedure of [42, §3.3] for each $j \in D$ by multiplying by some constant loop $\mu_j \in \Omega_2 U(2m)^J$ to get

$$\left(\prod_{j \in D} \mu_j\right) B\gamma_\tau = C\gamma_\xi$$

for some $C : M \to \Lambda^+_{\text{alg}} GL(2m, \mathbb{C})$ and ξ a canonical element of $\Omega_r U(2m)^J$ for some $r \leq \tilde{r}$. We have that $[C\gamma_\xi]$ is an equivalent extended solution to $[B\gamma_\tau]$.

Following [9, p.560 ff.] we need only consider $C : M \to \mathfrak{A}^J_\xi \subset \Lambda^+_{\text{alg}} GL(2m, \mathbb{C})$ as $[C\gamma_\xi] : M \to \Omega_r U(2m)^J$ is holomorphic and $\{[\exp(\eta)\gamma_\xi] \mid \eta \in \mathfrak{a}^J_\xi\}$ is a proper algebraic subvariety of $\{[A\gamma_\xi] \mid A \in \Lambda^+_{\text{alg}} GL(2m, \mathbb{C})\}$. □

We now give an interpretation of S^1-invariant polynomial extended solutions. Recall the Grassmannian model of an S^1-invariant extended solution from Proposition 6.1.20.

Lemma 7.1.11. *Let* $\Phi = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^{\perp})(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^{\perp})\cdots(\pi_{\alpha_r} + \lambda\pi_{\alpha_r}^{\perp}) \in \Omega_r U(2m)^J$ *be an* S^1*-invariant polynomial extended solution,* $W = \Phi\mathcal{H}_+$ *be its corresponding Grassmannian model and* $J : \mathbb{C}^{2m} \to \mathbb{C}^{2m}$ *be the conjugate-linear map defined by (5.1.13). Then the following are equivalent:*

(i) $JW^{\perp} = \lambda^{1-r}W$,

(ii) $J\alpha_j^{\perp} = \alpha_{r+1-j}$ *for all* $1 \leq j \leq r$ *and*

(iii) $J\psi_j = \psi_{r-j}$ *where* $\psi_j = \alpha_j^{\perp} \cap \alpha_{j+1}$ *for all* $1 \leq j \leq r$.

Proof. Recall $\mathcal{H}_+ \subset \mathcal{H} = L^2(S^1, \mathbb{C}^{2m})$ from §6.1.3. Then $W = \Phi\mathcal{H}_+ \subset \mathcal{H}$ and by Proposition 6.1.20, $W = \Phi\mathcal{H}_+ = \alpha_1 + \lambda\alpha_2 + \lambda^2\alpha_3 + \cdots + \lambda^{r-1}\alpha_r + \lambda^r\mathcal{H}_+$. Let $v = \sum_j \lambda_j v_j \in \mathcal{H}$, then $v \in W^{\perp}$ if and only if $v_j \perp \alpha_j$ for all $0 < j \leq r-1$ and $v_j = 0$ for all $j \geq r$. Hence

$$W^{\perp} = \alpha_1^{\perp} + \lambda\alpha_2^{\perp} + \lambda^2\alpha_3^{\perp} + \cdots + \lambda^{r-1}\alpha_r^{\perp} + \mathcal{H}_+^{\perp}.$$

As J is conjugate-linear we have

$$JW^{\perp} = J\alpha_1^{\perp} + \lambda^{-1}J\alpha_2^{\perp} + \lambda^{-2}J\alpha_3^{\perp} + \cdots + \lambda^{1-r}J\alpha_r^{\perp} + \lambda\mathcal{H}_+, \qquad (7.1.7)$$

where $\lambda\mathcal{H}_+ = JH_+^{\perp}$ for $H_+^{\perp} = \overline{\text{Span}}\{\lambda^{-i}e_j \mid i \in \mathbb{N},\, j = 1, 2, \ldots, 2m\}$ the orthogonal complement of $\mathcal{H}_+$ from (6.1.6). On the other hand,

$$\lambda^{1-r}W = \alpha_r + \lambda^{-1}\alpha_{r-1} + \lambda^{-2}\alpha_{r-2} + \cdots + \lambda^{1-r}\alpha_1 + \lambda\mathcal{H}_+. \qquad (7.1.8)$$

By comparing (7.1.7) and (7.1.8) we see that $JW^{\perp} = \lambda^{1-r}W$ if and only if $J\alpha_j^{\perp} = \alpha_{r+1-j}$ for all $1 \leq j \leq r$.

Now suppose $J\alpha_j^\perp = \alpha_{r+1-j}$ for all $1 \le j \le r$ and let $\psi_j = \alpha_j^\perp \cap \alpha_{j+1}$. Then for each j, $1 \le j \le r,\ J\psi_j = J\alpha_j^\perp \cap J\alpha_{j+1} = \alpha_{r+1-j} \cap \alpha_{r-j}^\perp = \psi_{r-j}$.

Conversely, suppose $J\psi_j = \psi_{r-j}$ for all $1 \le j \le r$. By Proposition 6.1.13 as Φ is S^1-invariant we have

$$0 = \alpha_0 \subset \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_r \subset \alpha_{r+1} = M \times \mathbb{C}^n,$$

giving $\alpha_j = \sum_{k=0}^{j-1} \alpha_k^\perp \cap \alpha_{k+1} = \sum_{k=0}^{j-1} \psi_k$ and $\alpha_j^\perp = \sum_{k=j}^{r} \psi_k$. Upon applying J we have

$$J\alpha_j^\perp = \sum_{k=j}^{r} J\psi_k = \sum_{k=j}^{r} \psi_{r-k} = \sum_{l=0}^{r-j} \psi_l = \alpha_{r+1-j},$$

and we conclude $J\alpha_j^\perp = \alpha_{r+1-j}$ if and only if $J\psi_j = \psi_{r-j}$ for all $j,\ 1 \le j \le r$. □

Remark 7.1.12. *Recall from (5.1.13) that $Jv = \overline{\Omega_{\text{null}} v}$ for $v \in \mathbb{C}^n$. By Lemma 7.1.11 we also have $\Omega_{\text{null}}\overline{W}^\perp = \lambda^{1-r} W$ if and only if $\Omega_{\text{null}}\overline{\alpha}_j^\perp = \alpha_{r+1-j}$ if and only if $\Omega_{\text{null}}\overline{\psi}_j = \psi_{r-j}$ in the null basis (5.1.9).*

Example 7.1.13. *Let $n = 2m$ and $\Phi \in \Omega_r U(n)^J$ be an S^1-invariant extended polynomial extended solution. Recall from Proposition 6.1.20 that*

$$\Phi = \pi_{\psi_0} + \lambda\pi_{\psi_1} + \lambda^2\pi_{\psi_2} + \cdots + \lambda^r\pi_{\psi_r}, \tag{7.1.9}$$

where $\psi_j = \alpha_j^\perp \cap \alpha_{j+1}$, and by Proposition 6.1.13 the unitons are nested, i.e.

$$0 = \alpha_0 \subset \alpha_1 \subset \alpha_2 \subset \cdots \subset \alpha_r \subset \alpha_{r+1} = M \times \mathbb{C}^n.$$

Also from Proposition 6.1.20 the Grassmannian model is given by

$$W = \Phi\mathcal{H}_+ = \alpha_1 + \lambda\alpha_2 + \lambda^2\alpha_3 + \cdots + \lambda^{r-1}\alpha_r + \lambda^r\mathcal{H}_+. \tag{7.1.10}$$

As shown in Example 6.2.14, on putting $\lambda = -1$ into (7.1.9) we see that the corresponding harmonic map $\varphi = M \to U(n)$ is given by

$$\varphi = \Phi_{-1} = \sum_{j \quad even} \pi_{\psi_j} - \sum_{j \quad odd} \pi_{\psi_j}.$$

This can be rewritten as

$$\varphi = \pi_\phi - \pi_\phi^\perp \tag{7.1.11}$$

where

$$\pi_\phi = \sum_{j \quad even} \pi_{\psi_j} \quad \text{and} \quad \pi_\phi^\perp = \sum_{j \quad odd} \pi_{\psi_j}.$$

By Propositions 6.2.12 and 6.2.13, we see that (7.1.11) is given by

$$\varphi = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$$

where $\phi : M \to G_*(\mathbb{C}^n)$ and is given by the subbundle

$$\underline{\phi} = \sum_{j \quad even} \psi_j. \tag{7.1.12}$$

As $\Phi \in \Omega_r U(n)^J$ we have by Lemma 7.1.5 and Lemma 7.1.11 that $JW^\perp = \lambda^{1-r}W$, $J\alpha_j^\perp = \alpha_{r+1-j}$ and $J\psi_j = \psi_{r-j}$ for all $1 \leq j \leq r$.

For r even, as $J\psi_j = \psi_{r-j}$ for all j, we see that (7.1.12) is the sum of J-closed subspaces of $\mathbb{C}^n$, $\psi_j + \psi_{r-j}$. Therefore the corresponding map ϕ is, in fact, a map into a quaternionic Grassmannian $\phi : M \to G_*(\mathbb{H}^m)$. Note that

$$G_k(\mathbb{H}^m) = \frac{Sp(m)}{Sp(m-k) \times Sp(k)},$$

so we will often write ϕ from (7.1.12) as a map into $Sp(m)/(Sp(m-k) \times Sp(k))$ for an appropriate k. The Cartan embedding from Proposition 6.2.13 restricts to the totally *geodesic embedding $\iota : G_*(\mathbb{H}^m) \to Sp(m)$, giving a harmonic map $\varphi = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp : M \to Sp(m)$.*

For r odd, considering (7.1.12) we have

$$\underline{\phi} = \sum_{j \ \ even} \psi_j = \psi_0 + \psi_2 + \psi_4 + \cdots + \psi_{r-1}. \tag{7.1.13}$$

We see that

$$J\underline{\phi} = J\psi_0 + J\psi_2 + \cdots + J\psi_{r-1} = \psi_r + \psi_{r-2} + \cdots + \psi_1 = \underline{\phi}^\perp$$

so $J\underline{\phi} = \underline{\phi}^\perp$ and by Definition 5.1.6 $\underline{\phi}$ is maximally J-isotropic. We identify the space of maximally J-isotropic subspaces of $\mathbb{C}^{2m}$ with the space $Sp(m)/U(m)$ [48, §6.8] (cf. [42, §4]). Therefore for the corresponding map $\phi : M \to G_*(\mathbb{H}^m)$ and $x \in M$, $\phi(x)$ is a maximally J-isotropic subspace of $\mathbb{C}^{2m} \cong \mathbb{H}^m$ and so ϕ is a harmonic map into $Sp(m)/U(m)$. There is a totally geodesic embedding of $Sp(m)/U(m)$ into $G_m(\mathbb{C}^{2m})$ which upon composition with the Cartan embedding from Proposition 6.2.13 has image in $\{g \in U(2m) \mid JgJ^{-1} = -g\}$. As in the $O(2m)$ case (Example 6.3.7) to obtain the image in $Sp(m) = \{g \in U(2m) \mid JgJ^{-1} = +g\}$, we consider the associated harmonic map $\varphi = i\Phi_{-1} = i(\pi_\phi - \pi_\phi^\perp) = i(\pi_\phi - \pi_{J\phi})$ which gives a harmonic map map into $Sp(m)$.

7.1.1 Adding a Border in $Sp(m)$

Similarly to §6.3.1 we will discuss a method of adding a border to $\tilde{A} : M \to \mathfrak{A}^J_{\tilde{\xi}}$ for $\tilde{\xi}$ a canonical element of $\Omega_r U(2m)^J$. This will give us a method of parametrizing complex extended solutions of a harmonic map $\varphi : M \to Sp(m)$ of finite uniton number. We remind the reader that throughout this chapter we use the null basis (5.1.9) from §5.1.2 and write all vectors and matrices with respect to this null basis.

Let $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_{2m})$ be a canonical element of $\Omega_r U(2m)^J$ of type $(t_0, t_1, \ldots, t_r)$ for some $r \in \mathbb{N}$, $m \geq 1$. Similarly to §6.3.1 let us define $\tilde{\xi} = i\,\mathrm{diag}(\xi_2, \ldots, \xi_{2m-1})$, which is a canonical element of $\Omega_{\tilde{r}} U(2m-2)^J$ of type

$$(\tilde{t}_0, \tilde{t}_1, \ldots, \tilde{t}_{\tilde{r}}) = \begin{cases} (t_0 - 1, t_1, \ldots, t_r - 1), \text{ with } \tilde{r} = r & \text{if } t_0 \geq 2, \\ (t_1, \ldots, t_{r-1}), \text{ with } \tilde{r} = r - 2 & \text{otherwise.} \end{cases}$$

Now let $\tilde{A} = (a_{jk})_{j,k=2,\ldots,n-1} : M \to \mathfrak{A}^J_{\tilde{\xi}}$ be given then we add a border to $\tilde{A}$ as in §6.3.1 to get

$$A = \left(\begin{array}{c|cccc|c} 1 & a_{12} & a_{13} & \cdots & a_{1,2m-1} & a_{1,2m} \\ \hline 0 & & & & & a_{2,2m} \\ 0 & & & \tilde{A} & & a_{3,2m} \\ 0 & & & & & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right).$$

The resulting A defines a map $A : M \to \mathfrak{A}^{\mathbb{C}}_{\xi}$. To ensure that A takes values in $\mathfrak{A}^J_{\xi}$ we need

Lemma 7.1.14. *$A \in Sp(2m, \mathbb{C})$ if and only if $A^{\mathfrak{T}} \in Sp(2m, \mathbb{C})$.*

Proof. Recall (5.1.14) which says that $A \in Sp(2m, \mathbb{C})$ if and only if $A^{\mathfrak{T}} \Omega_{\text{null}} A = \Omega_{\text{null}}$. My multiplying both sides by Ω_{null}, A, A^{-1} and $\Omega^{-1}_{\text{null}}$ in the following way

$$A\Omega_{\text{null}}(A^{\mathfrak{T}}\Omega_{\text{null}}A)A^{-1}\Omega^{-1}_{\text{null}} = A\Omega_{\text{null}}(\Omega_{\text{null}})A^{-1}\Omega^{-1}_{\text{null}}$$

by noting that $-\Omega_{\text{null}} = \Omega^{-1}_{\text{null}} = \Omega^{\mathfrak{T}}_{\text{null}}$ we get that $A^{\mathfrak{T}}\Omega_{\text{null}}A = \Omega_{\text{null}}$ if and only if $A\Omega_{\text{null}}A^{\mathfrak{T}} = \Omega_{\text{null}}$. □

We give a version of Lemma 6.3.8 for $Sp(2m, \mathbb{C})$; the proof is a modified version of the proof of [29, Lemma 3.6].

Proposition 7.1.15. *Let ξ be a canonical element of $\Omega_r U(2m)^J$, and let $\tilde{\xi}$ be the canonical element of $\Omega_{\tilde{r}} U(2m-2)^J$ defined above. Let $A = (a_{jk})_{j,k=1,2,\ldots,2m} : M \to \mathfrak{A}^J_\xi$ be a holomorphic map, and let $\tilde{A} : (a_{jk})_{j,k=2,\ldots,2m-2} : M \to \mathfrak{A}^J_{\tilde{\xi}}$ be the holomorphic map defined above. Suppose A satisfies (6.2.2), then so does $\tilde{A}$. Conversely, suppose $\tilde{A}$ satisfies (6.2.2) then the following are equivalent:*

(a) A satisfies (6.2.2);

(b) the new top row satisfies (6.2.3) i.e.,

$$a'_{1k} = \sum_{l\,:\,\xi_l > \xi_k} \lambda^{\xi_l - \xi_k - 1} \rho'_{lk} a_{1l} \mod \lambda^{r-\xi_k-1} \quad k = t_r, t_r+1, \ldots, 2m; \tag{7.1.14}$$

(c) the new last column satisfies (6.2.3) i.e.,

$$a'_{j,2m} = \sum_{l\,:\,\xi_l > 0} \lambda^{\xi_l - 1} \rho'_{l,2m} a_{jl} \mod \lambda^{\xi_j - 1} \quad j = 1, 2, \ldots, \sum_{k=1}^{r} t_k. \tag{7.1.15}$$

Proof. First, let us consider the top-right element $a_{1,2m}$ of A, then (7.1.14) and (7.1.15) both read

$$a'_{1,2m} = \sum_{l\,:\,\xi_l > 0} \lambda^{\xi_l - 1} \rho'_{l,2m} a_{1l} \mod \lambda^{r-1}$$

as $\xi_1 = r$, $\xi_{2m} = 0$ and $\xi_1 \geq \xi_l$ for all $l = 1, \ldots, 2m$. Therefore we need only consider (7.1.14) for $k = t_r, t_r+1, \ldots, 2m-1$, and (7.1.15) for $j = 2, 3, \ldots, \sum_{k=1}^r t_k$.

Now assume that (7.1.15) holds, then by Proposition 6.2.9, (7.1.15) is equivalent to

$$\tilde{c}'_{2m} = \sum_{l\,:\,\xi_l > 0} \lambda^{\xi_l - 1} \rho'_{l,2m} \tilde{c}_l, \tag{7.1.16}$$

where $\tilde{c}_l$ are the columns c_l, $l = 1, 2, \ldots, 2m$, of A with the top and bottom elements omitted, and so $\tilde{c}_l$ for $l = 2, \ldots, 2m-1$ are the columns of $\tilde{A}$. Recall (5.1.11) and Lemma 5.1.7, then as $A \in Sp(2m, \mathbb{C})$ we have $\omega(c_k, c_{2m}) = 0$ for all $k \geq 2$. Therefore by expanding $\omega(c_k, c_{2m})$ we have

$$0 = \omega(c_k, c_{2m}) = -ia_{1k} + \omega(\tilde{c}_k, \tilde{c}_{2m}). \tag{7.1.17}$$

Differentiating gives

$$a'_{1k} = -i\Big(\omega(\tilde{c}'_k, \tilde{c}_{2m}) + \omega(\tilde{c}_k, \tilde{c}'_{2m})\Big). \tag{7.1.18}$$

Using (7.1.16) and by Lemma 5.1.7 we have

$$\begin{aligned}\omega(\tilde{c}_k, \tilde{c}'_{2m}) = \sum_{l\,:\,\xi_l>0} \lambda^{\xi_l-1}\rho'_{l,2m}\omega(\tilde{c}_k, \tilde{c}_l) &= \begin{cases} i\lambda^{\xi_{\overline{k}}-1}\rho'_{\overline{k},2m}, & \text{if } k>l, \\ -i\lambda^{\xi_{\overline{k}}-1}\rho'_{\overline{k},2m}, & \text{if } k>l, \end{cases} \\ &= 0 \mod \lambda^{r-\xi_k-1},\end{aligned}$$

as $\xi_{\overline{k}} = r - \xi_k$. Now consider $\omega(\tilde{c}'_k, \tilde{c}_{2m})$, as $\tilde{A}$ satisfies (6.2.2) and using (7.1.17) we have that

$$\omega(\tilde{c}'_k, \tilde{c}_{2m}) = \sum_{l\geq 2\,:\,\xi_l>\xi_k} \lambda^{\xi_l-\xi_k-1}\rho'_{lk}\omega(\tilde{c}_l, \tilde{c}_{2m}) = i\sum_{l\geq 2\,:\,\xi_l>\xi_k} \lambda^{\xi_l-\xi_k-1}\rho'_{lk}a_{1k}.$$

Substituting these into (7.1.18) we have

$$\begin{aligned} a'_{1k} &= -i(i\sum_{l\geq 2\,:\,\xi_l>\xi_k} \lambda^{\xi_l-\xi_k-1}\rho'_{lk}a_{1k}) \mod \lambda^{r-\xi_k-1} \\ &= \sum_{l\geq 2\,:\,\xi_l>\xi_k} \lambda^{\xi_l-\xi_k-1}\rho'_{lk}a_{1k} \mod \lambda^{r-\xi_k-1} \\ &= \sum_{l\,:\,\xi_1\geq\xi_l>\xi_k} \lambda^{\xi_l-\xi_k-1}\rho'_{lk}a_{1k}. \end{aligned}$$

Therefore (7.1.15) implies (7.1.14), moreover (7.1.15) implies that all columns of A satisfy (6.2.2) and therefore implies (a).

Now assume that (7.1.14) holds and recall that we need only consider (7.1.14) for $k =$

$t_r, t_r+1, \ldots, 2m-1$. We will prove that (7.1.15) holds by downwards induction on $j \in \{2, 3, \ldots, \sum_{k=1}^{r} t_k\}$. Let us consider the base case $\sum_{k=2}^{r} t_k < j \leq \sum_{k=1}^{r} t_k$ then for $\xi = i\,\mathrm{diag}(\xi_1, \xi_2, \ldots, \xi_{2m})$ of type $(t_0, t_1, \ldots, t_r)$ we have $\xi_j = 1$ for all such j and so (7.1.15) reads $a'_{j,2m} = a'_{j,2m}$ as $\rho'_{j,2m} = a'_{j,2m}$ and $a_{jj} = 1$, and therefore (7.1.15) holds. Let us introduce the notation

$$\hat{c}_k^j = \begin{pmatrix} a_{1k} \\ a_{2k} \\ \vdots \\ a_{j-1,k} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \qquad \check{c}_k^{\bar{j}} = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_{\bar{j}+1,k} \\ a_{\bar{j}+2,k} \\ \vdots \\ a_{2m,k} \end{pmatrix},$$

where $\hat{c}_k^j$ and $\check{c}_k^{\bar{j}}$ are of length $2m$ for all $j, k \in \{1, 2, \ldots, 2m\}$. For an induction hypothesis suppose that (7.1.15) holds for all $j > I$ for some $I \in \{2, 3, \ldots, \sum_{k=2}^{r} t_k\}$, therefore

$$(\check{c}_{2m}^I)' = \sum_{l\,:\,\xi_l > 0} \lambda^{\xi_l - 1} \rho'_{l,2m} \check{c}_l^I. \tag{7.1.19}$$

We show that (7.1.15) holds for $j = I$. Define $\varepsilon_{\bar{I}} = \begin{cases} -1, & \text{if } \bar{I} > m, \\ 1, & \text{if } \bar{I} \leq m, \end{cases}$ then we can expand $\omega(c_{\bar{I}}, c_{2m})$ as follows:

$$\omega(c_{\bar{I}}, c_{2m}) = i\varepsilon_{\bar{I}} a_{I,2m} + \omega(\hat{c}_{\bar{I}}^{\bar{I}}, \check{c}_{2m}^I).$$

By Lemma 5.1.7 we have, $i\varepsilon_{\bar{I}} a_{I,2m} = -\omega(\hat{c}_{\bar{I}}^{\bar{I}}, \check{c}_{2m}^I)$ and differentiating we get

$$i\varepsilon_{\bar{I}} a'_{I,2m} = -\omega((\hat{c}_{\bar{I}}^{\bar{I}})', \check{c}_{2m}^I) - \omega(\hat{c}_{\bar{I}}^{\bar{I}}, (\check{c}_{2m}^I)'). \tag{7.1.20}$$

As $\tilde{A}$ satisfies (6.2.2) and $I \geq 2$ we have

$$\omega((\hat{c}_{\overline{I}}^{\overline{I}})', \check{c}_{2m}^{I}) = \sum_{j\,:\,\xi_j > \xi_{\overline{I}}} \lambda^{\xi_i - \xi_{\overline{I}} - 1} \rho'_{j,\overline{I}} \omega(\hat{c}_j^{\overline{I}}, \check{c}_{2m}^{I}),$$

where, as $j < \overline{I}$ in the sum above, then

$$\hat{c}_j^{\overline{I}} = (a_{1j}, a_{2j}, \dots, a_{j-1,j}, 1, 0, \dots, 0)^T,$$

$$\check{c}_{2m}^{I} = (0, \dots, 0, a_{I+1,2m}, a_{I+2,2m}, \dots, a_{2m-1,2m}, 1)^T.$$

For each j in the sum above we have

$$\omega(\hat{c}_j^{\overline{I}}, \check{c}_{2m}^{I}) = \omega(c_j, c_{2m}) = i\delta_{\overline{j},2m},$$

as $j < 2m$ for all j. We see that $\omega(\hat{c}_j^{\overline{I}}, \check{c}_{2m}^{I}) \neq 0$ if and only if $j = \overline{2m} = 1$, and so

$$\omega((\hat{c}_{\overline{I}}^{\overline{I}})', \check{c}_{2m}^{I}) = i\lambda^{\xi_1 - \xi_{\overline{I}} - 1} \rho'_{1,\overline{I}} = i\lambda^{\xi_I - 1} \rho'_{1,\overline{I}} = 0 \mod \lambda^{\xi_I - 1},$$

as $\xi_1 = r$, and $\xi_{\overline{I}} = r - \xi_I$. Now (7.1.20) reads

$$i\varepsilon_{\overline{I}} a'_{I,2m} = -\omega(\hat{c}_{\overline{I}}^{\overline{I}}, (\check{c}_{2m}^{I})') \mod \lambda^{\xi_I} - 1, \tag{7.1.21}$$

and by the induction hypothesis (7.1.19) we have

$$\omega(\hat{c}_{\overline{I}}^{\overline{I}}, (\check{c}_{2m}^{I})') = \sum_{l\,:\,\xi_l > 0} \lambda^{\xi_l - 1} \rho'_{l,2m} \omega(\hat{c}_{\overline{I}}^{\overline{I}}, \check{c}_l^{I}). \tag{7.1.22}$$

We can expand $\omega(c_{\overline{I}}, c_l)$ similarly to earlier to give

$$\omega(c_{\overline{I}}, c_l) = i\varepsilon_{\overline{I}} a_{Il} + \omega(\hat{c}_{\overline{I}}^{\overline{I}}, \check{c}_l^{I}),$$

again using that $A \in Sp(2m, \mathbb{C})$ and Lemma 5.1.7 we have $\omega(c_{\overline{I}}, c_l) = i\varepsilon_{\overline{I}}\delta_{\overline{I}l}$ and so if $I \neq l$ then

$$-i\varepsilon_{\overline{I}}a_{Il} = \omega(\hat{c}_{\overline{I}}^{\overline{I}}, \check{c}_l^I),$$

and if $I = l$ then $\check{c}_{\overline{I}}^{\overline{I}}$ is a column of zeros and so $\omega(\hat{c}_{\overline{I}}^{\overline{I}}, \check{c}_I^I) = 0$. Putting these together with (7.1.21) and (7.1.22) we have

$$\begin{aligned} i\varepsilon_{\overline{I}}a'_{I,2m} &= -\omega(\hat{c}_{\overline{I}}^{\overline{I}}, (\check{c}_{2m}^I)') \mod \lambda^{\xi_I} - 1 \\ &= i\varepsilon_{\overline{I}} \sum_{l\,:\,\xi_l>0} \lambda^{\xi_l-1}\rho'_{l,2m}a_{Il} \mod \lambda^{\xi_I-1}. \end{aligned}$$

Therefore $a'_{I,2m} = \sum_{l\,:\,\xi_l>0} \lambda^{\xi_l-1}\rho'_{l,2m}a_{Il} \mod \lambda^{\xi_I-1}$ which completes the induction step and therefore the lemma is proven. □

By Lemma 7.1.14, given the new top row (resp. new last column) we solve the equations from Lemma 5.1.7 to find expressions for the elements of the new last column (resp. new top row). Unlike the $O(n)$ case detailed in §6.3.1 where elements of the new last column and new top-right element were found by algebra, we cannot find the new top-right element $a_{1,2m}$ in the $Sp(m)$ case by solving $\omega(c_{2m}, c_{2m}) = 0$ from Lemma 5.1.7, this is because $\omega(c_{2m}, c_{2m}) = 0$ is automatically satisfied. So we must find the new top-right element by integrating (7.1.14). We use '**by algebra**' to describe the method of solving the equations of Lemma 5.1.7 to find expressions for the new last column in the $Sp(m)$ case as well as for the $O(n)$ case in §6.3.1.

Completing the border by algebra is where using the null basis (5.1.9) is particularly useful. If were to find expressions for the new last column from the new top row in the standard basis we would use Lemma 5.1.5 as opposed to Lemma 5.1.7 which, for a unitriangular matrix $A = (a_{jk})_{j,k=1,2,\ldots 2m}$ (see Definition 6.2.4), does not give us an expression for $a_{m,2m}$; this does not fit well with our key construction of adding a border. Also in the null basis (5.1.9) we replace the non-diagonal matrix Ω in (5.1.6) with the diagonal matrix Ω_{null} in (5.1.11).

7.2 Classification up to Dimension 6

Recall from Definition 6.1.12 that, for $\Phi = \Phi_\lambda : M \to \Omega_r U(n)^J$ a polynomial extended solution, then Φ is called S^1-invariant if $\Phi_\lambda \Phi_\mu = \Phi_{\lambda\mu}$ for all $\lambda, \mu \in S^1$. If $\Phi = [A\gamma_\xi]$ for ξ a canonical element of $\Omega_r U(2m)^J$ and $A \in \text{Sol}_\xi^J$ then Φ is S^1-invariant if and only if $A \in (\text{Sol}_\xi^J)_0$, that is, A is independent of $\lambda \in S^1$. By using Proposition 7.1.10 we will find a classification of all S^1-invariant extended solutions with respect to the canonical elements of $\Omega_r U(2m)^J$, up to $m = 3$. We will find these extended solutions by successively adding borders, solving the extended solution equations (6.2.3) by introducing generalised derivatives from Definition 6.3.9 and solving the equations arising from Lemma 5.1.7. We will conclude

Theorem 7.2.1. *Let M be a Riemann surface, let $\xi = i\,\text{diag}(\xi_1, \xi_1, \ldots, \xi_n)$ for $n = 2m \leq 6$ be a canonical element of $\Omega_r U(n)^J$ for some $r \in \mathbb{N}_0$ and let $p_1 = p_1(\xi) = \dim \mathfrak{g}_1^J(\xi)$. There exists a bijective map $h_0 : \mathcal{M}(M)^{p_1} \to (Sol_\xi^J)_0$ where, for $(\nu_1, \nu_2, \ldots, \nu_p) \in \mathcal{M}(M)^{p_1}$, each entry of $h_0(\nu_1, \nu_2, \ldots, \nu_p)$ is a rational function of ν_j and their derivatives.*

7.2.1 $m = 1$

For $m = 1$, note that as $0 \leq r \leq 2m$ then $r = 0$, or 1. We wish to find parametrizations for extended solutions $\Phi : M \to \Omega_r U(2)^J$ where ξ a canonical element of $\Omega_r U(2)^J$ and $A \in (\text{Sol}_\xi^J)_0$. By Definition 7.1.7 the canonical element of $\Omega_0 U(2)^J$ is $\xi = i\,\text{diag}(0, 0)$ and the canonical element of $\Omega_1 U(2)^J$ is $\xi = i\,\text{diag}(1, 0)$; these are of type (2) and $(1, 1)$, respectively. By Proposition 7.1.10, and Proposition 6.2.9 every extended solution $\tilde{\Phi} : M \to \Omega_r U(2)^J$, $r = 0$ or 1 is equivalent to one of the two extended solutions given below.

Type (2)

We will first consider the canonical element $\xi = i\,\mathrm{diag}(0,0)$ of $\Omega_0 U(2)^J$ so that $\gamma_\xi = I$. By Definition 7.1.9, $A : M \to (\mathfrak{A}_\xi^J)_0$ is the identity matrix which automatically satisfies the symplecticity condition in Lemma 5.1.7 and the extended solution equations (6.2.3). This leads to the extended solution $\Phi = [A\gamma_\xi] = I$; we call the extended solutions in $\Omega_0 U(2m)^J$ with $A = I$ for any m **trivial solutions**.

Type $(1,1)$

We consider the canonical element $\xi = i\,\mathrm{diag}(1,0)$ of $\Omega_1 U(2)^J$. By Definition 7.1.9 $A : M \to (\mathfrak{A}_\xi^J)_0$ is of the form

$$A = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}, \tag{7.2.1}$$

for $f : M \to \mathbb{C}$ meromorphic. It is easy to see by Lemma 5.1.7 that $A \in Sp(2,\mathbb{C})$; further A automatically satisfies the extended solution equations (6.2.3) as these read $a'_{12} = \lambda^0 \rho'_{12} a_{11} = a'_{12}$, and therefore $A \in (\mathrm{Sol}_\xi^J)_0$. So $\Phi = [A\gamma_\xi] = \pi_{\alpha_1} + \lambda \pi_{\alpha_1}^\perp$ is an extended solution, where $\alpha_1 = \mathrm{Span}\{c_2\}$ and c_2 is the second column of A. By Example 7.1.13 the corresponding Grassmannian Model is given by $W = \Phi\mathcal{H}_+ = \alpha_1 + \lambda\mathcal{H}_+$, with $\Phi = \pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp$. As r is odd, by (7.1.13), this corresponds to the harmonic map $\phi : M \to Sp(1)/U(1)$ associated to the subbundle $\underline{\phi} = \psi_0 = \alpha_1$ of $M \times \mathbb{C}^2$. By Example 7.1.13 we have the harmonic map $\varphi = i\Phi_{-1} = i(\iota \circ \phi) : M \to Sp(1)$ where $\Phi_{-1} = \iota \circ \phi = \pi_{\alpha_1} - \pi_{\alpha_1}^\perp$.

7.2.2 $m = 2$

The possible canonical elements of $\Omega_r U(4)^J$ from Definition 7.1.7
are $\xi = i\,\mathrm{diag}(3,2,1,0)$ for $\Omega_3 U(4)^J$, $\xi = i\,\mathrm{diag}(2,1,1,0)$ for $\Omega_2 U(4)^J$, $\xi =$

$i\,\mathrm{diag}(1,1,0,0)$ for $\Omega_1 U(4)^J$ and $\xi = i\,\mathrm{diag}(0,0,0,0)$ for $\Omega_0 U(4)^J$. These are of type $(1,1,1,1)$, $(1,2,1)$, $(2,2)$ and (4), respectively, and the canonical element of type (4) gives a trivial solution: $\Phi = [A\gamma_\xi] = I : M \to \Omega_0 U(4)^J$.

Type $(1,1,1,1)$

Consider the canonical element $\xi = i\,\mathrm{diag}(3,2,1,0)$ of $\Omega_3 U(4)^J$. This gives us the canonical geodesic $\gamma_\xi = \mathrm{diag}(\lambda^3, \lambda^2, \lambda^1, \lambda^0) \in \Omega_3 U(4)^J$. Let $\tilde{A} = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix}$ be the A from the type $(1,1)$ example above. We wish to add a border to $\tilde{A}$ as in §7.1.1 to define $A : M \to (\mathfrak{A}^{\mathbb{C}}_\xi)_0$. We have

$$A = \left(\begin{array}{c|cc|c} 1 & a_{12} & a_{13} & a_{14} \\ \hline 0 & \multicolumn{2}{c|}{\multirow{2}{*}{$\tilde{A}$}} & a_{24} \\ 0 & & & a_{34} \\ \hline 0 & 0 & 0 & 1 \end{array}\right) = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & f & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{7.2.2}$$

Now to ensure that $A : M \to (\mathfrak{A}^{\mathbb{C}}_\xi)_0$ takes values in $(\mathfrak{A}^J_\xi)_0$ and further to ensure $A \in (\mathrm{Sol}^J_\xi)_0$, we use Proposition 7.1.15, which states that we need only solve the extended solution equations (6.2.3) for the new top row and new top-right element then we complete the border by algebra. The extended solution equations for the new top row and new top-right element are

$$a'_{12} = a'_{12}, \qquad a'_{13} = a_{12}a'_{23} = a_{12}f', \qquad a'_{14} = a_{13}a'_{34}.$$

The first equation is automatically satisfied so let us relabel $a_{12} = g$ and turn our attention to the second equation $a'_{13} = gf'$, using integration by parts we get

$$a_{13} = \int a'_{13} = \int gf' = gf - \int g'f. \tag{7.2.3}$$

We introduce a new parameter τ with $\tau' = g'f$, (note that τ is defined up to an arbitrary constant) then we have $f = \tau^{(1)} := \tau'/g'$ so f is now equal to the generalised derivative of τ with respect to g. The integral now reads

$$a_{13} = g\tau^{(1)} - \tau.$$

To ensure $A : M \to (\mathfrak{A}_\xi^J)_0 \subset (\mathfrak{A}_\xi^{\mathbb{C}})_0$ we solve the equations in Lemma 5.1.7;

$$a_{34} = -a_{12} \qquad\qquad a_{24} = a_{13} + a_{23}a_{34}. \tag{7.2.4}$$

Solving these gives

$$A = \begin{pmatrix} 1 & g & g\tau^{(1)} - \tau & a_{14} \\ 0 & 1 & \tau^{(1)} & -\tau \\ 0 & 0 & 1 & -g \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Finally we find a_{14} by integrating $a'_{14} = a_{13}a'_{34} = (g\tau^{(1)} - \tau)(-g)'$. We do this by using integration by parts to reduce the order of the (generalised) derivative of τ,

$$\begin{aligned} a_{14} &= \int a'_{14} = \int (g\tau^{(1)} - \tau)(-g)' = \int -g'\left(\frac{\tau'}{g'}\right)g + g'\tau \\ &= \int -\tau' g + g'\tau = -g\tau + 2\int g'\tau. \end{aligned} \tag{7.2.5}$$

Similarly as before we introduce a new parameter ν such that $\nu' = g'\tau$ so $\tau = \nu^{(1)} := \nu'/g'$ and τ is equal to the generalised derivative or ν with respect to g. Note also that $\tau^{(1)} = \nu^{(2)} := (\nu^{(1)})'/g'$. The integral now reads

$$a_{14} = -g\nu^{(1)} + 2\nu,$$

and so

$$A = \begin{pmatrix} 1 & g & g\nu^{(2)} - \nu^{(1)} & 2\nu - g\nu^{(1)} \\ 0 & 1 & \nu^{(2)} & -\nu^{(1)} \\ 0 & 0 & 1 & -g \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.2.6}$$

for meromorphic functions g and ν, we call such functions **data** or **parameters**. The reader can easily check that this satisfies the extended solution equations of Proposition 6.2.9, which for type $(1, 1, \ldots, 1)$ say each column differentiates into a multiple of the one before. Note here that this gives rise to a bijective map $h_\xi : \mathcal{M}(M)^2 \to (\mathrm{Sol}_\xi^J)_0$ given by $(g, \nu) \mapsto A$. Conversely given an $A = (a_{jk})_{j,k=1,2,3,4} \in (\mathrm{Sol}_\xi^J)_0$, then we can retrieve the data g and ν by setting $g = a_{12}$ and $\nu = (1/2)(a_{14} - a_{12}a_{24})$. Set

$$\Phi = [A\gamma_\xi] = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^\perp)(\pi_{\alpha_3} + \lambda\pi_{\alpha_3}^\perp),$$

for $\alpha_1 = \mathrm{Span}\{c_4\}$, $\alpha_2 = \mathrm{Span}\{c_4, c_3\}$, $\alpha_3 = \mathrm{Span}\{c_4, c_3, c_2\}$ for c_2, c_3, c_4 the second, third and fourth columns of A, respectively. Then Φ is an extended solution with Grassmannian model $W = \Phi\mathcal{H}_+ = \alpha_1 + \lambda\alpha_2 + \lambda^2\alpha_3 + \lambda^3\mathcal{H}_+$. As r is odd, by (7.1.13), this corresponds to the harmonic map $\phi : M \to Sp(2)/U(2)$ associated to the subbundle $\underline{\phi} = \psi_0 + \psi_2 = \alpha_1 + \alpha_2^\perp \cap \alpha_3$ of $M \times \mathbb{C}^4$. By Example 7.1.13 we have the harmonic map $\varphi = i\Phi_{-1} = i(\iota \circ \phi) : M \to Sp(2)$ where $\Phi_{-1} = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$ which is of uniton number at most 3.

Type $(1, 2, 1)$

Consider the canonical element $\xi = i\,\mathrm{diag}(2, 1, 1, 0)$ of $\Omega_2 U(4)^J$, with the corresponding canonical geodesic $\gamma_\xi = \mathrm{diag}(\lambda^2, \lambda^1, \lambda^1, \lambda^0) \in \Omega_2 U(4)^J$. We find $A \in (\mathrm{Sol}_\xi^J)_0$ from the general solution of type (2) which is the 2×2 identity matrix. By adding a border we have

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & 0 & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Similarly to the above, we need only solve the extended solution equations (6.2.3) for the new top row and new top-right element and then complete the border by algebra. The extended solution equations in this case are

$$a_{12}' = a_{12}', \quad a_{13}' = a_{13}', \quad a_{14}' = a_{24}' a_{12} + a_{34}' a_{13}.$$

For first two equations are trivial and we therefore relabel our data thus; $a_{12} = g$, $a_{13} = h$. Before we integrate the third equation to parametrize a_{14}, we need to find a_{24} and a_{34} by algebra: the equations derived from Lemma 5.1.7 give $a_{24} = a_{13} = h$ and $a_{34} = -a_{12} = -g$. To find the new top-right element we integrate

$$a_{14} = \int a_{14}' = \int h'g + g'h = hg - 2\int g'h. \tag{7.2.7}$$

By introducing a parameter ν such that $\nu' = g'h$ we have $h = \nu^{(1)} = \nu'/g'$. By substituting these into (7.2.7) we have that $A \in (\mathrm{Sol}_\xi^J)_0$ can be written

$$A = \begin{pmatrix} 1 & g & \nu^{(1)} & g\nu^{(1)} - 2\nu \\ 0 & 1 & 0 & \nu^{(1)} \\ 0 & 0 & 1 & -g \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.2.8}$$

where g and ν are meromorphic functions. The reader can easily check that this satisfies the extended solution equations of Proposition 6.2.9, which for type $(1, 2, 1)$ say that column c_4 differentiates into a linear combination of columns c_3 and c_2 and columns c_3

and c_2 differentiate into a multiple of column c_1. This gives the extended solution

$$\Phi = [A\gamma_\xi] = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^\perp),$$

where $\alpha_1 = \operatorname{Span}\{c_4\}$, $\alpha_2 = \operatorname{Span}\{c_4, c_3, c_2\}$, for c_2, c_3, c_4 the second, third and fourth columns of A, respectively. Following Example 7.1.13, the corresponding Grassmannian model and harmonic map are $W = \Phi\mathcal{H}_+ = \alpha_1 + \lambda\alpha_2 + \lambda^2\mathcal{H}_+$. As r is even, by (7.1.12), this corresponds to the harmonic map into the quaternionic Grassmannian $\phi : M \to G_1(\mathbb{H}^2) = \mathbb{HP}^1$ associated to the subbundle $\underline{\phi} = \psi_0 + \psi_2 = \psi_0 + J\psi_0$ of $M \times \mathbb{C}^4$. So $\phi : M \to Sp(2)/(Sp(1) \times Sp(1))$. By Example 7.1.13 we have the harmonic map $\varphi = \Phi_{-1} = \iota \circ \phi : M \to Sp(2)$ defined by $\varphi = \Phi_{-1} = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$. The data, g and ν, can also be retrieved from $A = (a_{jk})_{j,k=1,2,3,4}$ by setting $g = a_{12}$ and $\nu = -(1/2)(a_{14} - a_{12}a_{24})$, therefore the process above gives rise to a bijection $h_\xi : \mathcal{M}(M)^2 \to (\mathrm{Sol}_\xi^J)_0$.

Type $(2,2)$

The solution comes from the general solution (7.2.1) of type $(1,1)$ by adding a border. Recall the canonical element of type $(2,2)$ of the form $\xi = i\,\mathrm{diag}(1,1,0,0)$ so $r = 1$. Adding a border we have

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} \\ 0 & 1 & f & a_{24} \\ 0 & 0 & 1 & a_{34} \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

From the block structure of elements in $(\mathfrak{A}_\xi^{\mathbb{C}})_0$, then $a_{12} = a_{34} = 0$. The extended solution equations (6.2.3) give $a_{13}' = a_{13}'$ and $a_{14}' = a_{14}'$ and so all we need is to solve the equations arising from Lemma 5.1.7 to ensure that $A : M \to (\mathfrak{A}_\xi^J)_0$. We get

$$A = \begin{pmatrix} 1 & 0 & g & h \\ 0 & 1 & f & g \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.2.9}$$

for f, g and h meromorphic. The map $h_\xi : \mathcal{M}(M)^3 \to (\mathrm{Sol}_\xi^J)_0$ defined by $(f, g, h) \mapsto A$ is obviously bijective, and gives the extended solution $\Phi = [A\gamma_\xi] = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)$, for $\alpha_1 = \mathrm{Span}\{c_4, c_3\}$. As r is odd, by (7.1.13), this corresponds to the harmonic map $\phi : M \to Sp(2)/U(2)$ associated to the subbundle $\underline{\phi} = \psi_0 = \alpha_1$ of $M \times \mathbb{C}^4$. By Example 7.1.13 we have the harmonic map $\varphi = i\Phi_{-1} = i(\iota \circ \phi) : M \to Sp(2)$ defined by $\Phi_{-1} = \iota \circ \phi = \pi_{\alpha_1} - \pi_{\alpha_1}^\perp$.

7.2.3 $m = 3$

All solutions for the different types of canonical elements of $\Omega_r U(6)^J$ for $r = 1, \ldots, 6$ are found by adding a border to the solutions in §7.2.2, i.e. type $(1, 1, 1, 1)$, $(1, 2, 1)$, $(2, 2)$ and (4). There are seven non-trivial classes of solutions indexed by the type of the canonical elements of $\Omega_r U(6)^J$, these are type $(1, 1, 1, 1, 1, 1)$, $(1, 2, 2, 1)$, $(1, 4, 1)$ $(1, 1, 2, 1, 1)$, $(2, 2, 2)$ $(3, 3)$ and $(2, 1, 1, 2)$, with the trivial solution $\Phi = [A\gamma_\xi] = I : M \to \Omega_0 U(6)^J$ arising from the canonical element $\xi = i\,\mathrm{diag}(0, 0, 0, 0, 0, 0)$ of $\Omega_0 U(6)^J$ which has type (6).

Type $(1, 1, 1, 1, 1, 1)$

The canonical element of type $(1, 1, 1, 1, 1, 1)$ is the canonical element
$\xi = i\,\mathrm{diag}(5, 4, 3, 2, 1, 0)$ of $\Omega_5 U(6)^J$. To find $A \in (\mathrm{Sol}_\xi^J)_0$ we add a border to the general

solution (7.2.6) of type $(1,1,1,1)$ above so we have

$$A=\begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & 1 & g & g\nu^{(2)}-\nu^{(1)} & 2\nu-g\nu^{(1)} & a_{26} \\ 0 & 0 & 1 & \nu^{(2)} & -\nu^{(1)} & a_{36} \\ 0 & 0 & 0 & 1 & -g & a_{46} \\ 0 & 0 & 0 & 0 & 1 & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{7.2.10}$$

where g and ν are meromorphic functions and $\nu^{(1)} = \nu'/g'$, $\nu^{(2)} = \nu^{(1)}/g'$. Similarly to the above we find a_{12}, a_{13}, a_{14}, a_{15} and a_{16} by integrating the extended solution equations and introducing new parameters to give generalised derivatives. We find a_{26}, a_{36}, a_{46} and a_{56}, by algebra to ensure A takes values in $(\mathfrak{A}^J_\xi)_0$. The extended solution equations (6.2.3) are

$$a'_{12}=a'_{12},\quad a'_{13}=a_{12}a'_{23},\quad a'_{14}=a_{13}a'_{34},\quad a'_{15}=a_{14}a'_{45},\quad a'_{16}=a_{15}a'_{56}.$$

Note that the first equation is automatically satisfied and the last equation must be solved after we have completed the last column by algebra. Set $a_{12} = h$ then we find a_{13} by integration:

$$a_{13}=\int a'_{13}=\int hg'=hg-\int h'g. \tag{7.2.11}$$

We introduce a new parameter α with $\alpha' = h'g$ so $g = \alpha^{(1)} := \alpha'/h'$. The integral then reads $a_{13} = h\alpha^{(1)} - \alpha$. Note that the generalised derivatives $\nu^{(1)}$ and $\nu^{(2)}$ are now with respect to $\alpha^{(1)}$. We now find a_{14} by integration:

$$a_{14}=\int a'_{14}=\int a_{13}a'_{34}=\int (h\alpha^{(1)}-\alpha)(\nu^{(2)})'. \tag{7.2.12}$$

We will use integration by parts to lower the order of the (generalised) derivative of ν to put the right-hand side of the above in a form where we may introduce a new parameter.

This gives

$$\begin{aligned}\int (h\alpha^{(1)} - \alpha)(\nu^{(2)})' &= (h\alpha^{(1)} - \alpha)\nu^{(2)} - \int (h\alpha^{(1)} - \alpha)' \frac{(\nu^{(1)})'}{(\alpha^{(1)})'} \\ &= (h\alpha^{(1)} - \alpha)\nu^{(2)} - \int \frac{\alpha' + h(\alpha^{(1)})' - \alpha'}{(\alpha^{(1)})'}(\nu^{(1)})' \qquad (7.2.13) \\ &= (h\alpha^{(1)} - \alpha)\nu^{(2)} - h\nu^{(1)} + \int h' \frac{\nu'}{(\alpha^{(1)})'} \\ &= (h\alpha^{(1)} - \alpha)\nu^{(2)} - h\nu^{(1)} + \frac{h'}{(\alpha^{(1)})'}\nu - \int \left(\frac{h'}{(\alpha^{(1)})'}\right)' \nu.\end{aligned}$$

For ease of notation let us write $h^{(1)} := h'/(\alpha^{(1)})'$. We introduce a new parameter τ such that $\tau' = (h^{(1)})'\nu$ so $\nu = \tau^{(1)} := \tau'/(h^{(1)})'$, then

$$a_{14} = (h\alpha^{(1)} - \alpha)\tau^{(3)} - h\tau^{(2)} + h^{(1)}\tau^{(1)} - \tau,$$

where

$$\tau^{(1)} := \frac{\tau'}{(h^{(1)})'}, \quad \tau^{(2)} := \frac{(\tau^{(1)})'}{(\alpha^{(1)})'}, \quad \tau^{(3)} := \frac{(\tau^{(2)})'}{(\alpha^{(1)})'}, \quad h^{(1)} := \frac{h'}{(\alpha^{(1)})'}, \quad \alpha^{(1)} := \frac{\alpha'}{h'}. \qquad (7.2.14)$$

Next we integrate $a'_{15} = a_{14}a'_{45}$ to find a_{15}:

$$a_{15} = \int a'_{15} = \int a_{14}a'_{45} = \int (h\alpha^{(1)} - \alpha)\tau^{(3)} - h\tau^{(2)} + h^{(1)}\tau^{(1)} - \tau)(-\alpha^{(1)})'. \quad (7.2.15)$$

Again we will use integration by parts to lower the order of the (generalised) derivatives of τ to put the right-hand side of the above in a form where we may introduce a new parameter:

$$a_{15} = (\alpha - h\alpha^{(1)})\tau^{(3)} + h\tau^{(2)} - h^{(1)}\tau^{(1)} + \tau)\alpha^{(1)}$$

$$
\begin{aligned}
&-\int(\alpha-h\alpha^{(1)})\tau^{(3)}+h\tau^{(2)}-h^{(1)}\tau^{(1)}+\tau)'\alpha^{(1)} \qquad (7.2.16)\\
&=(\alpha-h\alpha^{(1)})\tau^{(3)}+h\tau^{(2)}-h^{(1)}\tau^{(1)}+\tau)\alpha^{(1)}-\int\Big\{(\alpha'-h'\alpha^{(1)}-h(\alpha^{(1)})')\tau^{(3)}\\
&\quad+(\alpha-h\alpha^{(1)})(\tau^{(3)})'+h'\tau^{(2)}+h(\tau^{(2)})'-(h^{(1)})'\tau^{(1)}-h^{(1)}(\tau^{(1)})'+\tau'\Big\}\alpha^{(1)}.
\end{aligned}
$$

Using (7.2.14) we see $(\alpha'-h'\alpha^{(1)}-h(\alpha^{(1)})')\tau^{(3)}+h'\tau^{(2)}+h(\tau^{(2)})'-(h^{(1)})'\tau^{(1)}-h^{(1)}(\tau^{(1)})'+\tau'=0$. So we have

$$
\begin{aligned}
a_{15}&=\Big((\alpha-h\alpha^{(1)})\tau^{(3)}+h\tau^{(2)}-h^{(1)}\tau^{(1)}+\tau\Big)\alpha^{(1)}-\int(\alpha-h\alpha^{(1)})\alpha^{(1)}(\tau^{(3)})'\\
&=\Big((\alpha-h\alpha^{(1)})\tau^{(3)}+h\tau^{(2)}-h^{(1)}\tau^{(1)}+\tau\Big)\alpha^{(1)}-(\alpha-h\alpha^{(1)})\alpha^{(1)}\tau^{(3)}\\
&\quad+\int((\alpha-h\alpha^{(1)})\alpha^{(1)})'\tau^{(3)}\\
&=h\alpha^{(1)}\tau^{(2)}-h^{(1)}\alpha^{(1)}\tau^{(1)}+\alpha^{(1)}\tau+\int((\alpha-h\alpha^{(1)})\alpha^{(1)})'\tau^{(3)}\\
&=h\alpha^{(1)}\tau^{(2)}-h^{(1)}\alpha^{(1)}\tau^{(1)}+\alpha^{(1)}\tau+\int(\alpha-2h\alpha^{(1)})(\alpha^{(1)})'\frac{(\tau^{(2)})'}{(\alpha^{(1)})'}\\
&=(\alpha-h\alpha^{(1)})\tau^{(2)}-h^{(1)}\alpha^{(1)}\tau^{(1)}+\alpha^{(1)}\tau-\int(\alpha-2h\alpha^{(1)})'\tau^{(2)}\\
&=(\alpha-h\alpha^{(1)})\tau^{(2)}-h^{(1)}\alpha^{(1)}\tau^{(1)}+\alpha^{(1)}\tau-\int\Big(-\frac{\alpha'}{(\alpha^{(1)})'}-2h\Big)(\tau^{(1)})' \qquad (7.2.17)\\
&=(\alpha-h\alpha^{(1)})\tau^{(2)}+\Big(\frac{\alpha'}{(\alpha^{(1)})'}+2h-h^{(1)}\alpha^{(1)}\Big)\tau^{(1)}+\alpha^{(1)}\tau\\
&\quad+\int\Big(-\frac{\alpha'}{(\alpha^{(1)})'}-2h\Big)'\frac{\tau'}{(h^{(1)})'}
\end{aligned}
$$

$$= (\alpha - h\alpha^{(1)})\tau^{(2)} + 2h\tau^{(1)} + \alpha^{(1)}\tau + \int \left(\frac{-(h^{(1)}\alpha^{(1)})' - 2h'}{(h^{(1)})'} \right) \tau'$$

$$= (\alpha - h\alpha^{(1)})\tau^{(2)} + 2h\tau^{(1)} + \frac{-h^{(1)}(\alpha^{(1)})' - 2h'}{(h^{(1)})'}\tau$$

$$- \int \left(-\alpha^{(1)} + \frac{-h^{(1)}(\alpha^{(1)})' - 2h'}{(h^{(1)})'} \right)' \tau$$

$$= (\alpha - h\alpha^{(1)})\tau^{(2)} + 2h\tau^{(1)} - \frac{3h'}{(h^{(1)})'}\tau + \int \left(\alpha^{(1)} + \frac{3h'}{(h^{(1)})'} \right)' \tau.$$

We introduce a new parameter β such that

$$\beta' = \left(\alpha^{(1)} + \frac{3h'}{(h^{(1)})'} \right)' \tau \quad \text{so} \quad \tau = \beta^{(1)} := \frac{\beta'}{\left(\alpha^{(1)} + \frac{3h'}{(h^{(1)})'} \right)'},$$

and for ease of notation let us set

$$a = \alpha^{(1)} + \frac{3h'}{(h^{(1)})'},$$

so $\beta^{(1)} := \beta'/a'$. Therefore we my write

$$a_{15} = (\alpha - h\alpha^{(1)})\beta^{(3)} + 2h\beta^{(2)} + (\alpha^{(1)} - a)\beta^{(1)} + \beta,$$

where

$$\beta^{(2)} := \frac{\beta^{(1)}}{(h^{(1)})'}, \quad \beta^{(3)} := \frac{(\beta^{(2)})'}{(\alpha^{(1)})'}, \quad \beta^{(4)} := \frac{(\beta^{(3)})'}{(\alpha^{(1)})'}, \quad h^{(1)} := \frac{h'}{(\alpha^{(1)})'}, \quad \alpha^{(1)} := \frac{\alpha'}{h'}.$$

Now all other entries of A can be rewritten in terms of h, α and β, in fact so far we have

$$A = \begin{pmatrix} 1 & h & h\alpha^{(1)} - \alpha & \begin{matrix}(h\alpha^{(1)} - \alpha)\beta^{(4)} - h\beta^{(3)} \\ +h^{(1)}\beta^{(2)} - \beta^{(1)}\end{matrix} & \begin{matrix}(\alpha - h\alpha^{(1)})\beta^{(3)} + 2h\beta^{(2)} \\ +(\alpha^{(1)} - a)\beta^{(1)} + \beta\end{matrix} & a_{16} \\ 0 & 1 & \alpha^{(1)} & \alpha^{(1)}\beta^{(4)} - \beta^{(3)} & 2\beta^{(2)} - \alpha^{(1)}\beta^{(3)} & a_{26} \\ 0 & 0 & 1 & \beta^{(4)} & -\beta^{(3)} & a_{36} \\ 0 & 0 & 0 & 1 & -\alpha^{(1)} & a_{46} \\ 0 & 0 & 0 & 0 & 1 & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We now turn our attention to the elements in the new last column, to ensure $A \in (\mathfrak{A}^J_\xi)_0$ we solve the equations arising from Lemma 5.1.7, these equations are

$$a_{56} = -a_{12}, \quad a_{46} = -a_{13} - a_{23}a_{56}, \quad a_{36} = a_{14} + a_{24}a_{56} + a_{34}a_{46},$$

$$a_{26} = a_{15} + a_{25}a_{56} - a_{45}a_{36} + a_{35}a_{46}.$$

Solving these give

$$a_{56} = -h, \quad a_{46} = \alpha, \quad a_{36} = h^{(1)}\beta^{(2)} - \beta^{(1)}, \tag{7.2.18}$$

$$a_{26} = h^{(1)}\alpha^{(1)}\beta^{(2)} - a\beta^{(1)} + \beta.$$

Now we need only find the new top-right element a_{16} which we do by integrating $a'_{16} = a_{15}a'_{56}$, introducing a new parameter and a new generalised derivative similar to the calculation for a_{15} above.

$$a_{16} = \int a'_{16} = \int a_{15}a'_{56} = \int \Big((\alpha - h\alpha^{(1)})\beta^{(3)} + 2h\beta^{(2)} + (\alpha^{(1)} - a)\beta^{(1)} + \beta\Big)(-h')$$

$$= \frac{(h\alpha^{(1)} - \alpha)h'}{(\alpha^{(1)})'}(\beta^{(2)})' - \frac{2hh'}{(h^{(1)})'}(\beta^{(1)})' - \frac{(\alpha^{(1)} - a)h'}{a'}\beta' + h'\beta$$

$$= (h\alpha^{(1)} - \alpha)h^{(1)}\beta^{(2)} - \frac{2hh'}{(h^{(1)})'}\beta^{(1)} - \frac{(\alpha^{(1)} - a)h'}{a'}\beta$$

$$- \int (h\alpha^{(1)} - \alpha)h^{(1)})'\beta^{(2)} - \left(\frac{2hh'}{(h^{(1)})'}\right)'\beta^{(1)} - \left\{\left(\frac{(\alpha^{(1)} - a)h'}{a'}\right)' + h'\right\}\beta$$

$$= (h\alpha^{(1)} - \alpha)h^{(1)}\beta^{(2)} - \frac{2hh'}{(h^{(1)})'}\beta^{(1)} - \frac{(\alpha^{(1)} - a)h'}{a'}\beta$$

$$- \int \frac{(h\alpha^{(1)} - \alpha)h^{(1)})'}{(h^{(1)})'}(\beta^{(1)})' - \frac{\left(\frac{2hh'}{(h^{(1)})'}\right)'}{a'}\beta' - \left\{\left(\frac{(\alpha^{(1)} - a)h'}{a'}\right)' + h'\right\}\beta$$

$$= (h\alpha^{(1)} - \alpha)h^{(1)}\beta^{(2)} - \frac{2hh'}{(h^{(1)})'}\beta^{(1)} - \frac{(\alpha^{(1)} - a)h'}{a'}\beta$$

$$- \int \frac{(h\alpha^{(1)} - \alpha)h^{(1)})'}{(h^{(1)})'}(\beta^{(1)})' - \frac{\left(\frac{2hh'}{(h^{(1)})'}\right)'}{a'}\beta' - \left\{\left(\frac{(\alpha^{(1)} - a)h'}{a'}\right)' + h'\right\}\beta \tag{7.2.19}$$

$$= (h\alpha^{(1)} - \alpha)h^{(1)}\beta^{(2)} - \left\{\frac{(h\alpha^{(1)} - \alpha)h^{(1)})'}{(h^{(1)})'} + \frac{2hh'}{(h^{(1)})'}\right\}\beta^{(1)}$$

$$+ \left\{\frac{\left(\frac{2hh'}{(h^{(1)})'}\right)'}{a'} - \frac{(\alpha^{(1)} - a)h'}{a'}\right\}\beta$$

$$+ \int \left(\frac{(h\alpha^{(1)} - \alpha)h^{(1)})'}{(h^{(1)})'}\right)'\beta^{(1)} + \left\{\left(\frac{-\left(\frac{2hh'}{(h^{(1)})'}\right)'}{a'}\right)' + \left(\frac{(\alpha^{(1)} - a)h'}{a'}\right)' + h'\right\}\beta$$

$$= (h\alpha^{(1)} - \alpha)h^{(1)}\beta^{(2)} - \left\{\frac{(h\alpha^{(1)} - \alpha)h^{(1)})'}{(h^{(1)})'} + \frac{2hh'}{(h^{(1)})'}\right\}\beta^{(1)}$$

$$+\left\{\frac{\left(\frac{2hh'}{(h^{(1)})'}\right)'}{a'}-\frac{(\alpha^{(1)}-a)h'}{a'}\right\}\beta$$

$$+\int\frac{\left(\frac{(h\alpha^{(1)}-\alpha)h^{(1)})'}{(h^{(1)})'}\right)'}{a'}\beta'+\left\{\left(\frac{-\left(\frac{2hh'}{(h^{(1)})'}\right)'}{a'}\right)'+\left(\frac{(\alpha^{(1)}-a)h'}{a'}\right)'+h'\right\}\beta$$

$$=(h\alpha^{(1)}-\alpha)h^{(1)}\beta^{(2)}-\left\{\frac{(h\alpha^{(1)}-\alpha)h^{(1)})'}{(h^{(1)})'}+\frac{2hh'}{(h^{(1)})'}\right\}\beta^{(1)}$$

$$+\left\{\frac{\left(\frac{(h\alpha^{(1)}-\alpha)h^{(1)})'}{(h^{(1)})'}\right)'}{a'}+\frac{\left(\frac{2hh'}{(h^{(1)})'}\right)'}{a'}-\frac{(\alpha^{(1)}-a)h'}{a'}\right\}\beta$$

$$+\int\left\{\left(\frac{\left(\frac{(h\alpha^{(1)}-\alpha)h^{(1)})'}{(h^{(1)})'}\right)'}{a'}\right)'+\left(\frac{-\left(\frac{2hh'}{(h^{(1)})'}\right)'}{a'}\right)'+\left(\frac{(\alpha^{(1)}-a)h'}{a'}\right)'+h'\right\}\beta.$$

If we set

$$b=(h\alpha^{(1)}-\alpha)h^{(1)},\quad c=\frac{2hh'+b'}{(h^{(1)})'},\quad d=\frac{-(\alpha^{(1)}-a)h'+c'}{a'},$$

and introduce a new parameter γ with

$$\gamma'=(d+h)'\beta\quad\text{so}\quad\beta=\gamma^{(1)}:=\frac{\gamma'}{(d+h)'}.$$

We have

$$a_{16}=b\gamma^{(3)}-c\gamma^{(2)}+d\gamma^{(1)}+\gamma,$$

where

$$\gamma^{(2)}:=\frac{\gamma^{(1)}}{a'},\qquad\gamma^{(3)}:=\frac{\gamma^{(2)}}{(h^{(1)})'},\qquad\gamma^{(4)}:=\frac{(\gamma^{(3)})'}{(\alpha^{(1)})'},$$
$$\gamma^{(5)}:=\frac{(\gamma^{(4)})'}{(\alpha^{(1)})'},\qquad h^{(1)}:=\frac{h'}{(\alpha^{(1)})'},\qquad\alpha^{(1)}:=\frac{\alpha'}{h'}.$$

Therefore rewriting all elements of A we have

$$A = \begin{pmatrix} 1 & h & h\alpha^{(1)} - \alpha & \begin{matrix}(h\alpha^{(1)} - \alpha)\gamma^{(5)} - h\gamma^{(4)} \\ +h^{(1)}\gamma^{(3)} - \gamma^{(2)}\end{matrix} & \begin{matrix}(\alpha - h\alpha^{(1)})\gamma^{(4)} + 2h\gamma^{(3)} \\ +(\alpha^{(1)} - a)\gamma^{(2)} + \gamma^{(1)}\end{matrix} & a_{16} \\ 0 & 1 & \alpha^{(1)} & \alpha^{(1)}\gamma^{(5)} - \gamma^{(4)} & 2\gamma^{(3)} - \alpha^{(1)}\gamma^{(4)} & a_{26} \\ 0 & 0 & 1 & \gamma^{(5)} & -\gamma^{(4)} & a_{36} \\ 0 & 0 & 0 & 1 & -\alpha^{(1)} & \alpha \\ 0 & 0 & 0 & 0 & 1 & -h \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

for a_{16} as given above, $a_{26} = h^{(1)}\alpha^{(1)}\gamma^{(3)} - \alpha\gamma^{(2)} + \gamma^{(1)}$ and $a_{36} = h^{(1)}\gamma^{(3)} - \gamma^{(2)}$ with h, α and γ meromorphic functions. This gives the extended solution

$$\Phi = [A\gamma_\xi] = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^\perp)(\pi_{\alpha_3} + \lambda\pi_{\alpha_3}^\perp)(\pi_{\alpha_4} + \lambda\pi_{\alpha_4}^\perp)(\pi_{\alpha_5} + \lambda\pi_{\alpha_5}^\perp),$$

where $\alpha_1 = \mathrm{Span}\{c_6\}$, $\alpha_2 = \mathrm{Span}\{c_6, c_5\}$, $\alpha_3 = \mathrm{Span}\{c_6, c_5, c_4\}$,
$\alpha_4 = \mathrm{Span}\{c_6, c_5, c_4, c_3\}$, $\alpha_5 = \mathrm{Span}\{c_6, c_5, c_4, c_3, c_2\}$, where $c_1, c_2, c_3, c_4, c_5, c_6$ are the columns of A. As r is odd, by (7.1.13), this corresponds to the harmonic map $\phi : M \to Sp(3)/U(3)$ associated to the subbundle $\underline{\phi} = \psi_0 + \psi_2 + \psi_4 = \alpha_1 + \alpha_2^\perp \cap \alpha_3 + \alpha_4^\perp \cap \alpha_5$ of $M \times \mathbb{C}^6$. By Example 7.1.13 we have the harmonic map $\varphi = i\Phi_{-1} = i(\iota \circ \phi) : M \to Sp(3)$ defined by $\Phi_{-1} = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$ which is of uniton number at most 5. The map $h_\xi : \mathcal{M}(M)^3 \to (\mathrm{Sol}_\xi^J)_0$ defined by $(h, \alpha, \gamma) \mapsto A$ described above is bijective as given an $A = (a_{jk})_{j,k=1,\ldots,6} \in (\mathrm{Sol}_\xi^J)_0$ we can recover the data by setting $h = a_{12}$, $\alpha = a_{46}$ and $\gamma = a_{16} - b\gamma^{(3)} + c\gamma^{(2)} - d\gamma^{(1)}$ for

$$\gamma^{(3)} = (1/2)(a_{25} - a_{23}a_{35}), \qquad \gamma^{(2)} = -a_{35} + (a_{12}'/a_{23}')\gamma^{(3)},$$

$$\gamma^{(1)} = a_{26} - (a'_{46}/a'_{23})\gamma^{(3)} + (a'_{46}/a'_{12} + 3a'_{12}/(a'_{12}/a'_{23}))\gamma_{(2)},$$

$$b = (a_{13}a_{23} - a_{46})\frac{a'_{12}}{a'_{23}}, \qquad c = \frac{2a_{12}a'_{12} + b'}{\left(\frac{a'_{12}}{a'_{23}}\right)'}, \qquad d = \frac{-\frac{3(a'_{12})^2}{\left(\frac{a'_{12}}{a'_{23}}\right)'} + c'}{\left(\frac{a'_{46}}{a'_{12}} + \frac{3a'_{12}}{\left(\frac{a'_{12}}{a'_{23}}\right)'}\right)'}.$$

Type $(1,2,2,1)$

The canonical element of type $(1,2,2,1)$ is $\xi = i\,\mathrm{diag}(3,2,2,1,1,0) \in \Omega_3 U(6)^J$. To find $A \in (\mathrm{Sol}^J_\xi)_0$ we add a border to the general solution (7.2.9) of type $(2,2)$ above so we have

$$A = \begin{pmatrix} 1 & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & 1 & 0 & g & h & a_{26} \\ 0 & 0 & 1 & f & g & a_{36} \\ 0 & 0 & 0 & 1 & 0 & a_{46} \\ 0 & 0 & 0 & 0 & 1 & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where g, h and f are meromorphic functions. First we find a_{12}, a_{13}, a_{14}, a_{15} and a_{16} by integrating the extended solution equations and introducing new parameters to give generalised derivatives. The extended solution equations (6.2.3) are

$$a'_{12} = a'_{12}, \quad a'_{13} = a'_{13}, \quad a'_{14} = a_{12}a'_{24} + a_{13}a'_{34},$$

$$a'_{15} = a_{12}a'_{25} + a_{13}a'_{35}, \quad a'_{16} = a_{14}a'_{46} + a_{15}a'_{56}.$$

Note that the first two equations are automatically satisfied so we set $a_{12} = \alpha$ and $a_{13} = \beta$ then we find a_{14} by integration:

$$a_{14} = \int a'_{14} = \int a_{12}a'_{24} + a_{13}a'_{34} = \int \alpha g' + \beta f' = \alpha g + \beta f - \int \alpha' g + \beta' f.$$

We introduce two new parameters ν_1 and ν_2 such that $\nu_1' = g\alpha'$, and $\nu_2' = f\beta'$ so $g = \nu_1^{(1)} := \nu_1'/\alpha'$ and $f = \nu_2^{(1)} := \nu_2'/\beta'$ and so $a_{14} = \alpha\nu_1^{(1)} + \beta\nu_2^{(1)} - \nu_1 - \nu_2$. We notice at this point that the map we are constructing is not injective, as we may replace ν_1 and ν_2 by $\nu_1 + c$ and $\nu_2 - c$. To ensure this map is bijective we will replace ν_1 and ν_2 by $\tilde{\nu}_1$ and $\tilde{\nu}_2$ with $\tilde{\nu}_1 = \nu_1^{(1)}$ and $\tilde{\nu}_2 = \nu_1 + \nu_2$ then we have

$$A = \begin{pmatrix} 1 & \alpha & \beta & -\tilde{\nu}_2 + \alpha\tilde{\nu}_1 + \frac{\beta(\tilde{\nu}_2' - \alpha'\tilde{\nu}_1)}{\beta'} & a_{15} & a_{16} \\ 0 & 1 & 0 & \tilde{\nu}_1 & h & a_{26} \\ 0 & 0 & 1 & \frac{\tilde{\nu}_2' - \alpha'\tilde{\nu}_1}{\beta'} & \tilde{\nu}_1 & a_{36} \\ 0 & 0 & 0 & 1 & 0 & a_{46} \\ 0 & 0 & 0 & 0 & 1 & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

Now we find a_{15} by integration:

$$a_{15} = \int a_{15}' = \int a_{12}a_{25}' + a_{13}a_{35}' = \int \alpha h' + \beta\tilde{\nu}_1' = \alpha h + \beta\tilde{\nu}_1 - \int \alpha' h + \beta'\tilde{\nu}_1.$$

Again introducing new parameters ν_3 and ν_4 with $\nu_3' = h\alpha'$ and $\nu_4' = \tilde{\nu}_1\beta'$ so we have new generalised derivatives $h = \nu_3^{(1)} := \nu_3'/\alpha'$ and $\tilde{\nu}_1 = \nu_4^{(1)} := \nu_4'/\beta'$, and so $a_{15} = \alpha\nu_3^{(1)} + \beta\nu_4^{(1)} - \nu_3 - \nu_4$. Similarly to earlier when finding a_{14}, to ensure our parametrization is bijective, we replace the parameters ν_3 and ν_4 with $\tilde{\nu}_3$ and $\tilde{\nu}_3$ by setting $\tilde{\nu}_3 = \nu_3^{(1)}$ and $\tilde{\nu}_4 = \nu_3 + \nu_4$ then

$$A = \begin{pmatrix} 1 & \alpha & \beta & a_{14} & a_{15} & a_{16} \\ 0 & 1 & 0 & \frac{\tilde{\nu}_4' - \alpha'\tilde{\nu}_3}{\beta'} & \tilde{\nu}_3 & a_{26} \\ 0 & 0 & 1 & \frac{\tilde{\nu}_2' - \frac{\alpha'(\tilde{\nu}_4' - \alpha'\tilde{\nu}_3)}{\beta'}}{\beta'} & \frac{\tilde{\nu}_4 - \alpha'\tilde{\nu}_3}{\beta'} & a_{36} \\ 0 & 0 & 0 & 1 & 0 & a_{46} \\ 0 & 0 & 0 & 0 & 1 & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$a_{14} = -\tilde{\nu}_2 + \frac{\alpha(\tilde{\nu}_4' - \alpha\tilde{\nu}_3)}{\beta'} + \frac{\beta\Big(\tilde{\nu}_2' - \frac{\alpha'(\tilde{\nu}_4' - \alpha'\tilde{\nu}_3)}{\beta'}\Big)}{\beta'},$$

$$a_{15} = -\tilde{\nu}_4 + \alpha\tilde{\nu}_3 + \frac{\beta(\tilde{\nu}_4' - \alpha'\tilde{\nu}_3)}{\beta'}.$$

Finding the new last column by algebra means solving the equations from Lemma 5.1.7 which give

$$a_{56} = -\alpha, \quad a_{46} = -\beta, \quad a_{36} = -\tilde{\nu}_2, \quad a_{26} = -\tilde{\nu}_4.$$

We turn our attention to finding the new top-right element a_{16}:

$$\begin{aligned} a_{16} &= \int a_{16}' = \int a_{14}a_{46}' + a_{15}a_{56}' \\ &= \int \Bigg(\tilde{\nu}_2 + \frac{\alpha(\tilde{\nu}_4' - \alpha\tilde{\nu}_3)}{\beta'} + \frac{\beta\Big(\tilde{\nu}_2' - \frac{\alpha'(\tilde{\nu}_4' - \alpha'\tilde{\nu}_3)}{\beta'}\Big)}{\beta'}\Bigg)(-\beta') \\ &\quad + \Big(-\tilde{\nu}_4 + \alpha\tilde{\nu}_3 + \frac{\beta(\tilde{\nu}_4' - \alpha'\tilde{\nu}_3)}{\beta'}\Big)(-\alpha') \\ &= \int \beta'\tilde{\nu}_2 - \beta\tilde{\nu}_2' + \alpha'\tilde{\nu}_4 - \alpha\tilde{\nu}_4' = -\beta\tilde{\nu}_2 - \alpha\tilde{\nu}_4 + \int 2\beta'\tilde{\nu}_2 + 2\alpha'\tilde{\nu}_4. \end{aligned}$$

We introduce new parameters ν_5 and ν_6 such that $\nu_5' = \beta'\tilde{\nu}_2$ and $\nu_6' = \alpha'\tilde{\nu}_4$ so we have

$\tilde{\nu}_2 = \nu_5^{(1)} := \nu_5'/\beta'$ and $\tilde{\nu}_4 = \nu_6^{(1)} := \nu_6'/\alpha'$ which gives $a_{16} = -\beta\nu_5^{(1)} - \alpha\nu_6^{(1)} + 2\nu_5 + 2\nu_6$. Renaming parameters once again to ensure that the parametrization is bijective we set $\tilde{\nu}_6 = \nu_6^{(1)}$ and $\tilde{\nu}_5 = \nu_5 + \nu_6$ which gives

$$A = \begin{pmatrix} 1 & \alpha & \beta & a_{14} & a_{15} & a_{16} \\ 0 & 1 & 0 & \frac{\tilde{\nu}_6' - \alpha'\tilde{\nu}_3}{\beta'} & \tilde{\nu}_3 & -\tilde{\nu}_6 \\ 0 & 0 & 1 & a_{34} & \frac{\tilde{\nu}_6' - \alpha'\tilde{\nu}_3}{\beta'} & \frac{\alpha'\tilde{\nu}_6 - \tilde{\nu}_5'}{\beta'} \\ 0 & 0 & 0 & 1 & 0 & -\beta \\ 0 & 0 & 0 & 0 & 1 & \alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$a_{14} = \frac{\alpha'\tilde{\nu}_6 - \tilde{\nu}_5'}{\beta'} + \frac{\alpha(\tilde{\nu}_6' - \alpha'\tilde{\nu}_3)}{\beta'} + \frac{\beta\left(\left(\frac{\tilde{\nu}_5' - \alpha'\tilde{\nu}_6}{\beta'}\right)' - \frac{\alpha'(\nu_6' - \alpha'\tilde{\nu}_3)}{\beta'}\right)}{\beta'},$$

$$a_{34} = \frac{\left(\frac{\tilde{\nu}_5' - \alpha'\tilde{\nu}_6}{\beta'}\right)' - \frac{\alpha'(\nu_6' - \alpha'\tilde{\nu}_3)}{\beta'}}{\beta'}, \quad a_{15} = -\tilde{\nu}_6 + \alpha\tilde{\nu}_3 + \frac{\beta(\tilde{\nu}_6' - \alpha'\tilde{\nu}_3)}{\beta'},$$

$$a_{16} = 2\tilde{\nu}_5 - \alpha\tilde{\nu}_6 - \frac{\beta(\tilde{\nu}_5' - \alpha'\tilde{\nu}_6)}{\beta'},$$

The parameters α, β, $\tilde{\nu}_3$, $\tilde{\nu}_5$ and $\tilde{\nu}_6$ are meromorphic functions which can be recovered for a given $A = (a_{jk})_{j,k=1,\dots,6} \in (\mathrm{Sol}_\xi^J)_0$ by setting $\alpha = a_{12}$, $\beta = a_{13}$, $\tilde{\nu}_3 = a_{15}$, $\tilde{\nu}_5 = (1/2)(a_{16} - a_{12}a_{26} - a_{13}a_{36})$ and $\tilde{\nu}_6 = -a_{26}$.

We have the extended solution

$$\Phi = [A\gamma_\xi] = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^\perp)(\pi_{\alpha_3} + \lambda\pi_{\alpha_3}^\perp),$$

where $\alpha_1 = \mathrm{Span}\{c_6\}$, $\alpha_2 = \mathrm{Span}\{c_6, c_5, c_4\}$, $\alpha_3 = \mathrm{Span}\{c_6, c_5, c_4, c_3, c_2\}$, and c_2, c_3, c_4, c_5, c_6 denote the columns of A. As r is odd, by (7.1.13), this corresponds to the harmonic map $\phi : M \to Sp(3)/U(3)$ associated to the subbundle $\underline{\phi} = \psi_0 + \psi_2 = \alpha_1 + \alpha_2^\perp \cap \alpha_3$ of $M \times \mathbb{C}^6$. By Example 7.1.13 we have the harmonic map $\varphi = i\Phi_{-1} =$

$i(\iota \circ \phi) : M \to Sp(3)$ where $\Phi_{-1} = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$ which is of uniton number at most 3.

Type $(1,4,1)$

The canonical element of $\Omega_2 U(6)^J$ of type $(1,4,1)$ is of the form
$\xi = i\,\mathrm{diag}(2,1,1,1,1,0)$ and to describe $A : M \to (\mathfrak{A}_\xi^J)_0$ we add a border to that of the general solution of type (4) i.e. the identity matrix. Then as before to ensure $A \in (\mathrm{Sol}_\xi^J)_0$ we solve the extended solution equations (6.2.3) and ensure that A satisfies Lemma 5.1.7. The extended solution equations are

$$a'_{12} = a'_{12}, \quad a'_{13} = a'_{13}, \quad a'_{14} = a'_{14}, \quad a'_{15} = a'_{15},$$

$$a'_{16} = a'_{26}a_{12} + a'_{36}a_{13} + a'_{46}a_{14} + a'_{56}a_{15}.$$

Let $a_{12} = \alpha$, $a_{13} = \beta$, $a_{14} = \gamma$, and $a_{15} = \delta$ then as these automatically satisfy the extended solution equations, we need only solve the extended solution equation for a_{16}. Before we find this new top-right element we will use Lemma 5.1.7 to find the new last column of A which gives

$$a_{26} = \delta, \quad a_{36} = \gamma, \quad a_{46} = -\beta, \quad a_{56} = -\alpha.$$

We now use similar methods to the above to solve the following integration:

$$\begin{aligned} a_{16} &= \int a'_{16} = \int a'_{26}a_{12} + a'_{36}a_{13} + a'_{46}a_{14} + a'_{56}a_{15} \\ &= \int \delta'\alpha + \gamma'\beta - \beta'\gamma - \alpha'\delta = \delta\alpha + \gamma\beta - 2\int \gamma\beta' + \delta\alpha'. \end{aligned}$$

We introduce new parameters ν_1 and ν_2 such that $\nu_1' = \gamma\beta'$ and $\nu_2' = \delta\alpha'$ with generalised derivatives $\gamma = \nu_1^{(1)} := \nu_1'/\beta'$ and $\delta = \nu_2^{(1)} := \nu_2'/\alpha'$ so the integral gives $a_{16} = \beta\nu_1^{(1)} +$

$\alpha\nu_2^{(1)} - 2(\nu_1+\nu_2)$. Therefore we have $A \in (\mathrm{Sol}_\xi^J)_0$ for

$$A = \begin{pmatrix} 1 & \alpha & \beta & \nu_1^{(1)} & \nu_2^{(1)} & \beta\nu_1^{(1)} + \alpha\nu_2^{(1)} - 2(\nu_1+\nu_2) \\ 0 & 1 & 0 & 0 & 0 & \nu_2^{(1)} \\ 0 & 0 & 1 & 0 & 0 & \nu_1^{(1)} \\ 0 & 0 & 0 & 1 & 0 & -\beta \\ 0 & 0 & 0 & 0 & 1 & -\alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

with meromorphic functions α, β, ν_1 and ν_2. As before we wish for our parametrization to be given by a bijective map, it is clear that given an $A = (a_{jk})_{j,k=1,\dots,6} \in (\mathrm{Sol}_\xi^J)_0$ one would not be able to retrieve the data ν_1 and ν_2 without integration therefore we follow the procedure detailed in the type $(1,2,2,1)$ example above. We set $\nu_1^{(1)} = \tilde{\nu}_1$, and $\nu_1+\nu_2 = \tilde{\nu}_2$ so our $A \in (\mathrm{Sol}_\xi^J)_0$ is now given by

$$A = \begin{pmatrix} 1 & \alpha & \beta & \tilde{\nu}_1 & \frac{\tilde{\nu}_2'-\beta'\tilde{\nu}_1}{\alpha'} & \beta\tilde{\nu}^{(1)} + \frac{\alpha(\tilde{\nu}_2'-\beta'\tilde{\nu}_1)}{\alpha'} - 2\tilde{\nu}_2 \\ 0 & 1 & 0 & 0 & 0 & \frac{\tilde{\nu}_2'-\beta'\tilde{\nu}_1}{\alpha'} \\ 0 & 0 & 1 & 0 & 0 & \tilde{\nu}_1 \\ 0 & 0 & 0 & 1 & 0 & -\beta \\ 0 & 0 & 0 & 0 & 1 & -\alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where α, β, $\tilde{\nu}_1$ and $\tilde{\nu}_2$ are meromorphic functions which can be retrieved by setting $\alpha = a_{12},\ \beta = a_{13},\ \tilde{\nu}_1 = a_{14}$ and $\tilde{\nu}_2 = (1/2)(a_{16} - a_{12}a_{26} - a_{13}a_{36})$. The resulting extended solution is given by

$$\Phi_\lambda = \Phi = [A\gamma_\xi] = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^\perp).$$

The unitons above are given by $\alpha_1 = \mathrm{Span}\{c_6\}$ and $\alpha_2 = \mathrm{Span}\{c_6, c_5, c_4, c_3\}$, for c_2, c_3, c_4, c_5, c_6 the columns of A. As r is even, by (7.1.12), this corresponds to the harmonic map into the quaternionic Grassmannian $\phi : M \to G_1(\mathbb{H}^3) = \mathbb{HP}^2$ associated to the subbundle $\underline{\phi} = \psi_0 + \psi_2 = \psi_0 + J\psi_0$ of $M \times \mathbb{C}^6$. So $\phi : M \to Sp(3)/(Sp(2) \times Sp(1))$. By Example 7.1.13 we have the harmonic map $\varphi = \Phi_{-1} = \iota \circ \phi : M \to Sp(3)$ defined by $\varphi = \Phi_{-1} = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$.

Type $(1,1,2,1,1)$

We obtain this from the general solution (7.2.8) of type $(1,2,1)$ by adding a border, solving the extended solution equations and completing the border by algebra described in §7.1.1. The canonical element of type $(1,1,2,1,1)$ is of the form $\xi = i\,\mathrm{diag}(4,3,2,2,1,0) \in \Omega_4 U(6)^J$, and the extended solution equations are

$$a'_{12} = a'_{12}, \quad a'_{13} = a_{12}a'_{23}, \quad a'_{14} = a_{12}a'_{24}, \quad a'_{15} = a_{13}a'_{35} + a_{14}a'_{45}, \quad a_{16} = a_{15}a'_{56}.$$

We set $a_{12} = \alpha$, so we have

$$A = \begin{pmatrix} 1 & \alpha & a_{13} & a_{14} & a_{15} & a_{16} \\ 0 & 1 & g & \nu^{(1)} & g\nu^{(1)} - 2\nu & a_{26} \\ 0 & 0 & 1 & 0 & \nu^{(1)} & a_{36} \\ 0 & 0 & 0 & 1 & -g & a_{46} \\ 0 & 0 & 0 & 0 & 1 & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where α, g and ν are meromorphic functions, with generalised derivative $\nu^{(1)} = \nu'/g'$. To find a_{13} we integrate $a_{12}a'_{23}$ as follows:

$$a_{13} = \int a'_{13} = \int a_{12}a'_{23} = \int \alpha g' = \alpha g - \int \alpha' g.$$

Choose h with $h' = \alpha' g$ so that h is a new parameter that replaces g, we also have the generalised derivative $g = h^{(1)} := h'/\alpha'$, so $a_{13} = \alpha h^{(1)} - h$ and the generalised derivative of ν with respect to g is now with respect to $h^{(1)}$. We will use integration by parts on $a_{12}a'_{24}$ to lower the order of the (generalised) derivative of ν to find a_{14}. We have

$$\begin{aligned} a_{14} = \int a'_{14} = \int a_{12}a'_{24} = \int \alpha(\nu^{(1)})' = \alpha\nu^{(1)} - \int \alpha' \frac{\nu'}{(h^{(1)})'} \\ = \alpha\nu^{(1)} - \nu\frac{\alpha'}{(h^{(1)})'} + \int \Big(\frac{\alpha'}{(h^{(1)})'}\Big)'\nu, \end{aligned}$$

for ease of notation let us set $a := \alpha'/(h^{(1)})'$ and we introduce a new parameter τ where $\tau' = a'\nu$, so we have the generalised derivative $\nu = \tau^{(1)} := \tau'/a'$, and $a_{14} = \alpha\tau^{(2)} - a\tau^{(1)} + \tau$ as $\nu^{(1)} = \tau^{(2)} := (\tau^{(1)})'/(h^{(1)})'$. Now A becomes

$$A = \begin{pmatrix} 1 & \alpha & \alpha h^{(1)} - h & \alpha\tau^{(2)} - a\tau^{(1)} + \tau & a_{15} & a_{16} \\ 0 & 1 & h^{(1)} & \tau^{(2)} & h^{(1)}\tau^{(2)} - 2\tau^{(1)} & a_{26} \\ 0 & 0 & 1 & 0 & \tau^{(2)} & a_{36} \\ 0 & 0 & 0 & 1 & -h^{(1)} & a_{46} \\ 0 & 0 & 0 & 0 & 1 & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $h^{(1)} := h'/\alpha'$, $\tau^{(2)} := \tau^{(1)}/(h^{(1)})'$, $\tau^{(1)} := \tau'/a'$. We now find a_{15} by integration by parts which lowers the order of (generalised) derivative of τ,

$$\begin{aligned} a_{15} &= \int a'_{15} = \int a_{13}a'_{35} + a_{14}a'_{45} \\ &= \int (\alpha h^{(1)} - h)(\tau^{(2)})' + (\alpha\tau^{(2)} - a\tau^{(1)} + \tau)(-h^{(1)})' \\ &= (\alpha h^{(1)} - h)\tau^{(2)} + \int -2\alpha(h^{(1)})'\tau^{(2)} + \alpha'\tau^{(1)} - (h^{(1)})'\tau \end{aligned}$$

$$= (\alpha h^{(1)} - h)\tau^{(2)} + \int \frac{-2\alpha(h^{(1)})'}{(h^{(1)})'}(\tau^{(1)})' + \frac{\alpha'}{a'}\tau' - (h^{(1)})'\tau$$

$$= (\alpha h^{(1)} - h)\tau^{(2)} - 2\alpha\tau^{(1)} + \frac{\alpha'}{a'}\tau + \int 2\alpha'\tau^{(1)} - \Big\{\Big(\frac{\alpha'}{a'}\Big)' + (h^{(1)})'\Big\}\tau$$

$$= (\alpha h^{(1)} - h)\tau^{(2)} - 2\alpha\tau^{(1)} + \frac{\alpha'}{a'}\tau + \int \frac{2\alpha'}{a'}\tau' - \Big\{\Big(\frac{\alpha'}{a'}\Big)' + (h^{(1)})'\Big\}\tau$$

$$= (\alpha h^{(1)} - h)\tau^{(2)} - 2\alpha\tau^{(1)} + \frac{3\alpha'}{a'}\tau - \int \Big\{\Big(\frac{3\alpha'}{a'}\Big)' + (h^{(1)})'\Big\}\tau.$$

We introduce a new parameter β with

$$\beta' = \Big(\frac{3\alpha'}{a'} + h^{(1)}\Big)'\tau, \quad \text{so} \quad \tau = \beta^{(1)} := \frac{\beta'}{(\frac{3\alpha'}{a'} + h^{(1)})'},$$

and so we have

$$a_{15} = (\alpha h^{(1)} - h)\beta^{(3)} - 2\alpha\beta^{(2)} + \frac{3\alpha'}{a'}\beta^{(1)} - \beta.$$

Therefore substituting the above formulae into A gives

$$A = \begin{pmatrix} 1 & \alpha & \alpha h^{(1)} - h & \alpha\beta^{(3)} - a\beta^{(2)} + \beta^{(1)} & \begin{matrix}(\alpha h^{(1)} - h)\beta^{(3)} - 2\alpha\beta^{(2)} \\ + \frac{3\alpha'}{a'}\beta^{(1)} - \beta\end{matrix} & a_{16} \\ 0 & 1 & h^{(1)} & \beta^{(3)} & h^{(1)}\beta^{(3)} - 2\beta^{(2)} & a_{26} \\ 0 & 0 & 1 & 0 & \beta^{(3)} & a_{36} \\ 0 & 0 & 0 & 1 & -h^{(1)} & a_{46} \\ 0 & 0 & 0 & 0 & 1 & a_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\beta^{(2)} := (\beta^{(1)})'/a'$, $\beta^{(3)} := (\beta^{(2)})'/(h^{(1)})'$, $h^{(1)} := h'/\alpha'$.

We now find the new last column by algebra to get,

$$a_{56} = -a_{12} = -\alpha, \quad a_{46} = -a_{13} - a_{23}a_{56} = h, \quad a_{36} = a_{14} + a_{24}a_{56} = \beta^{(1)} - a\beta^{(2)},$$

$$a_{26} = a_{15} + a_{25}a_{56} - a_{45}a_{36} + a_{35}a_{46} = -ah^{(1)}\beta^{(2)} + \Big(h^{(1)} + \frac{3\alpha'}{a'}\Big)\beta^{(1)} - \beta,$$

and so A now takes values in $(\mathfrak{A}^J_\xi)_0$. Finally we complete the border by integrating $a'_{16} = a_{15}a'_{56}$ to find a'_{16}, we do this by using integration by parts to lower the order of the (generalised) derivatives of β and then simplifying the expressions. We have

$$a_{16} = \int a'_{16} = \int a_{15}a'_{56} = \int \Big((\alpha h^{(1)} - h)\beta^{(3)} - 2\alpha\beta^{(2)} + \frac{3\alpha'}{a'}\beta^{(1)} - \beta\Big)(-\alpha')$$

$$= \int -\alpha'(\alpha h^{(1)} - h)\beta^{(3)} + 2\alpha\alpha'\beta^{(2)} - \frac{3\alpha'\alpha'}{a'}\beta^{(1)} + \alpha'\beta$$

$$= \int -a(\alpha h^{(1)} - h)(\beta^{(2)})' + \frac{2\alpha\alpha'}{a'}(\beta^{(1)})' - \frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'} + h^{(1)})'}\beta' + \alpha'\beta$$

$$= -a(\alpha h^{(1)} - h)\beta^{(2)} + \frac{2\alpha\alpha'}{a'}\beta^{(1)} - \frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'} + h^{(1)})'}\beta$$

$$+ \int (a(\alpha h^{(1)} - h))'\beta^{(2)} - \Big(\frac{2\alpha\alpha'}{a'}\Big)'\beta^{(1)} + \left\{\left(\frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'} + h^{(1)})'}\right)' + \alpha'\right\}\beta$$

$$= -a(\alpha h^{(1)} - h)\beta^{(2)} + \frac{2\alpha\alpha'}{a'}\beta^{(1)} - \frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'} + h^{(1)})'}\beta$$

$$+ \int \frac{(a(\alpha h^{(1)} - h))'}{a'}(\beta^{(1)})' - \frac{(\frac{2\alpha\alpha'}{a'})'}{(\frac{3\alpha'}{a'} + h^{(1)})'}\beta' + \left\{\left(\frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'} + h^{(1)})'}\right)' + \alpha'\right\}\beta$$

$$= -a(\alpha h^{(1)} - h)\beta^{(2)} + \Big\{\frac{2\alpha\alpha'}{a'} + \frac{(a(\alpha h^{(1)} - h))'}{a'}\Big\}\beta^{(1)}$$

$$- \Big\{\frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'} + h^{(1)})'} + \frac{(\frac{2\alpha\alpha'}{a'})'}{(\frac{3\alpha'}{a'} + h^{(1)})'}\Big\}\beta$$

$$
\begin{aligned}
&+\int -\Big(\frac{(a(\alpha h^{(1)}-h))'}{a'}\Big)'\beta^{(1)}+\left\{\left(\frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'}+h^{(1)})'}\right)'+\left(\frac{(\frac{2\alpha\alpha'}{a'})'}{(\frac{3\alpha'}{a'}+h^{(1)})'}\right)'+\alpha'\right\}\beta\\
&=-a(\alpha h^{(1)}-h)\beta^{(2)}+\Big\{\frac{2\alpha\alpha'}{a'}+\frac{(a(\alpha h^{(1)}-h))'}{a'}\Big\}\beta^{(1)}\\
&-\Big\{\frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'}+h^{(1)})'}+\frac{(\frac{2\alpha\alpha'}{a'})'}{(\frac{3\alpha'}{a'}+h^{(1)})'}\Big\}\beta\\
&+\int\frac{-\Big(\frac{(a(\alpha h^{(1)}-h))'}{a'}\Big)'}{(\frac{3\alpha'}{a'}+h^{(1)})'}\beta'+\left\{\left(\frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'}+h^{(1)})'}\right)'+\left(\frac{(\frac{2\alpha\alpha'}{a'})'}{(\frac{3\alpha'}{a'}+h^{(1)})'}\right)'+\alpha'\right\}\beta\\
&=-a(\alpha h^{(1)}-h)\beta^{(2)}+\Big\{\frac{2\alpha\alpha'}{a'}+\frac{(a(\alpha h^{(1)}-h))'}{a'}\Big\}\beta^{(1)}\\
&-\left\{\frac{\Big(\frac{(a(\alpha h^{(1)}-h))'}{a'}\Big)'}{(\frac{3\alpha'}{a'}+h^{(1)})'}+\frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'}+h^{(1)})'}+\frac{(\frac{2\alpha\alpha'}{a'})'}{(\frac{3\alpha'}{a'}+h^{(1)})'}\right\}\beta\\
&+\int\left\{\left(\frac{\Big(\frac{(a(\alpha h^{(1)}-h))'}{a'}\Big)'}{(\frac{3\alpha'}{a'}+h^{(1)})'}\right)'+\left(\frac{\frac{3\alpha'\alpha'}{a'}}{(\frac{3\alpha'}{a'}+h^{(1)})'}\right)'+\left(\frac{(\frac{2\alpha\alpha'}{a'})'}{(\frac{3\alpha'}{a'}+h^{(1)})'}\right)'+\alpha'\right\}\beta.
\end{aligned}
$$

If we set

$$
b=a(\alpha h^{(1)}-h),\quad c=\frac{2\alpha\alpha'+b'}{a'},\quad d=\frac{\frac{3\alpha'\alpha'}{a'}+c'}{(\frac{3\alpha'}{a'}+h^{(1)})'},
$$

and introduce a new parameter γ with

$$
\gamma'=(d+\alpha)'\beta,\quad \text{so}\quad \beta=\gamma^{(1)}:=\frac{\gamma'}{(d+\alpha)'},
$$

and we have

$$
a_{16}=-b\gamma^{(3)}+c\gamma^{(2)}-d\gamma^{(1)}+\gamma.
$$

Therefore rewriting all elements of A we have

$$A = \begin{pmatrix} 1 & \alpha & \alpha h^{(1)} - h & \begin{matrix}\alpha\gamma^{(4)} - a\gamma^{(3)} \\ +\gamma^{(2)}\end{matrix} & \begin{matrix}(\alpha h^{(1)} - h)\gamma^{(4)} - 2\alpha\gamma^{(3)} \\ +\frac{3\alpha'}{a'}\gamma^{(2)} - \gamma^{(1)}\end{matrix} & \begin{matrix}-b\gamma^{(3)} + c\gamma^{(2)} \\ -d\gamma^{(1)} + \gamma\end{matrix} \\ 0 & 1 & h^{(1)} & \gamma^{(4)} & h^{(1)}\gamma^{(4)} - 2\gamma^{(3)} & \begin{matrix} ah^{(1)}\gamma^{(3)} - \gamma^{(1)} \\ +\left(h^{(1)} + \frac{3\alpha'}{a'}\right)\gamma^{(2)}\end{matrix} \\ 0 & 0 & 1 & 0 & \gamma^{(4)} & \gamma^{(2)} - a\gamma^{(3)} \\ 0 & 0 & 0 & 1 & -h^{(1)} & h \\ 0 & 0 & 0 & 0 & 1 & -\alpha \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\gamma^{(2)} := \frac{\gamma^{(1)}}{(\frac{3\alpha'}{a'} + h^{(1)})'}, \quad \gamma^{(3)} := \frac{\gamma^{(2)}}{a'}, \quad \gamma^{(4)} := \frac{(\gamma^{(3)})'}{(h^{(1)})'}, \quad h^{(1)} := \frac{h'}{\alpha'},$$

for h, α and γ meromorphic functions. This gives the extended solution

$$\Phi = [A\gamma_\xi] = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^\perp)(\pi_{\alpha_3} + \lambda\pi_{\alpha_3}^\perp)(\pi_{\alpha_4} + \lambda\pi_{\alpha_4}^\perp),$$

where $\alpha_1 = \text{Span}\{c_6\}$, $\alpha_2 = \text{Span}\{c_6, c_5\}$, $\alpha_3 = \text{Span}\{c_6, c_5, c_4, c_3\}$, $\alpha_4 = \text{Span}\{c_6, c_5, c_4, c_3, c_2\}$, for $c_1, c_2, c_3, c_4, c_5, c_6$ the columns of A. As r is even, by (7.1.12), this corresponds to the harmonic map into the quaternionic Grassmannian $\phi : M \to G_2(\mathbb{H}^3)$ associated to the subbundle $\underline{\phi} = \psi_0 + \psi_2 + \psi_4 = \psi_0 + J\psi_2 + J\psi_0$ of $M \times \mathbb{C}^6$. So $\phi : M \to Sp(3)/(Sp(1) \times Sp(2))$. By Example 7.1.13 we have the harmonic map $\varphi = \Phi_{-1} = \iota \circ \phi : M \to Sp(3)$ defined by $\varphi = \Phi_{-1} = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$. Also, $\underline{\phi}^\perp = \psi_1 + \psi_3 = \psi_1 + J\psi_1$, and so is a harmonic map into the quaternionic Grassmannian $\phi^\perp : M \to G_1(\mathbb{H}^3) = \mathbb{HP}^2$. By Example 7.1.13 we have the harmonic map

$\iota \circ \phi^{\perp} : M \to Sp(3)$ where $\iota \circ \phi^{\perp} = -\iota \circ \phi = -\Phi_{-1} = \pi_{\phi}^{\perp} - \pi_{\phi}$.

Unlike type $(1,2,2,1)$ and type $(1,4,1)$ the parameters do not need to be modified to ensure the parametrization $(\alpha, h, \gamma) \mapsto A$ is bijective as, similarly to type $(1,1,1,1,1,1)$, given an $A = (a_{jk})_{j,k=1,\dots,6} \in (\mathrm{Sol}_{\xi}^{J})_0$ we can recover the data by setting $\alpha = a_{12}$, $h = a_{46}$, and $\gamma = a_{16} + b\gamma^{(3)} - c\gamma^{(2)} + d\gamma^{(1)}$, where

$$\gamma^{(3)} = \frac{-1}{2}(a_{25} - a_{23}a_{24}), \quad \gamma^{(2)} = \frac{-a_{12}'}{2a_{23}'}(a_{25} - a_{23}a_{14}),$$

$$\gamma^{(1)} = -\Big(a_{26} - \frac{a_{12}'a_{23}}{a_{23}'}\gamma^{(3)} - \Big(a_{23} + \frac{3a_{12}'}{(\frac{a_{12}'}{a_{23}})'}\Big)\gamma^{(2)}\Big),$$

$$a = \frac{a_{12}'}{a_{23}'}, \quad b = aa_{13}, \quad c = \frac{2a_{12}a_{12}' + b'}{a'}, \quad d = \frac{3a_{12}'a_{12}' + c'}{(\frac{3a_{12}'}{a'} + a_{23})'}.$$

Type $(2,2,2)$

This has $r = 2$ with canonical element $\xi \in \Omega_2 U(6)^J$ of the form
$\xi = i\,\mathrm{diag}(2,2,1,1,0,0)$. We obtain $A : M \to (\mathfrak{A}_{\xi}^{J})_0$ from the general solution (7.2.8) of type $(1,2,1)$ by adding a border, ensuring $A \in (\mathrm{Sol}_{\xi}^{J})_0$ by solving the extended solution equations and completing the border by algebra. The extended solution equations for type $(2,2,2)$ are

$$a_{13}' = a_{13}', \quad a_{14}' = a_{14}', \quad a_{15}' = a_{13}a_{35}' + a_{14}a_{45}', \quad a_{16}' = a_{13}a_{36}' + a_{14}a_{46}'.$$

We see that a_{13} and a_{14} automatically satisfies these and so by setting $a_{13} = \alpha$, and $a_{14} = \beta$, and recalling the form of $(\mathfrak{A}_\xi^J)_0$ from Definition 7.1.9 we have

$$A = \begin{pmatrix} 1 & 0 & \alpha & \beta & a_{15} & a_{16} \\ 0 & 1 & g & \nu^{(1)} & g\nu^{(1)} - 2\nu & a_{26} \\ 0 & 0 & 1 & 0 & \nu^{(1)} & a_{36} \\ 0 & 0 & 0 & 1 & -g & a_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We integrate to find a_{15}:

$$\begin{aligned} a_{15} &= \int a'_{15} = \int a_{13}a'_{35} + a_{14}a'_{45} = \int \alpha(\nu^{(1)})' - \beta g' \\ &= \alpha\nu^{(1)} - \beta g + \int -\frac{\alpha'}{g'}\nu' + \beta' g \\ &= \alpha\nu^{(1)} - \frac{\alpha'}{g'}\nu' - \beta g + \int +\Big(\frac{\alpha'}{g'}\Big)'\nu + \beta' g. \end{aligned}$$

We introduce the parameters h and τ such that $h' = \beta' g$ so $g = h^{(1)} := h'/\beta'$ and $\tau = (\alpha'/(h^{(1)})')'\nu$, so $\nu = \tau^{(1)} := \tau'/((\alpha'/(h^{(1)})')'$, for ease of notation let us set $\alpha^{(1)} := \alpha'/(h^{(1)})'$, so $\tau^{(1)} := \tau'/(\alpha^{(1)})'$ and $a_{15} = \alpha\tau^{(2)} - \alpha^{(1)} - \beta h^{(1)} + h + \tau$. Similarly to type $(1,2,2,1)$ and type $(1,4,1)$ we will modify the parameters to ensure our algorithm is bijective and so we have an algorithm that gives a bijective map between the space of meromorphic functions on M of dimension 4 and $(\mathrm{Sol}_\xi^J)_0$. Let $\tilde{h} = h^{(1)}$ and $\tilde{\tau} = \tau + h$, then

$$a_{15} = \tilde{\tau} - \beta\tilde{h} - \frac{\alpha^{(1)}(\tilde{\tau}' - \beta'\tilde{h})}{(\alpha^{(1)})'} + \alpha\frac{\left(\frac{\tilde{\tau}' - \beta'\tilde{h}}{(\alpha^{(1)})'}\right)'}{\tilde{h}'}$$

and

$$A = \begin{pmatrix} 1 & 0 & \alpha & \beta & a_{15} & a_{16} \\ 0 & 1 & \tilde{h} & \frac{\left(\frac{\tilde{\tau}'-\beta'\tilde{h}}{(\alpha^{(1)})'}\right)'}{\tilde{h}'} & \frac{\tilde{h}\left(\frac{\tilde{\tau}'-\beta'\tilde{h}}{(\alpha^{(1)})'}\right)'}{\tilde{h}'} - \frac{2(\tilde{\tau}'-\beta'\tilde{h})}{(\alpha^{(1)})'} & a_{26} \\ 0 & 0 & 1 & 0 & \frac{\left(\frac{\tilde{\tau}'-\beta'\tilde{h}}{(\alpha^{(1)})'}\right)'}{\tilde{h}'} & a_{36} \\ 0 & 0 & 0 & 1 & -\tilde{h} & a_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}.$$

We now find the new last column by algebra, we get $a_{46} = -\alpha$, $a_{36} = \beta$ and $a_{26} = \tilde{\tau} - \alpha^{(1)}(\tilde{\tau}' - \beta'\tilde{h})/(\alpha^{(1)})'$. Finally we find the new top-right element a_{16} by integration as before

$$\begin{aligned} a_{16} &= \int a_{16}' = \int a_{13}a_{36}' + a_{14}a_{46}' = \int \beta'\alpha - \alpha'\beta \\ &= \alpha\beta - 2\int \alpha'\beta. \end{aligned}$$

We introduce a new parameter γ such that $\gamma' = \alpha'\beta$ which gives the generalised derivative $\beta = \gamma^{(1)} := \gamma'/\alpha'$ and so $a_{16} = \alpha\gamma^{(1)} - 2\gamma$. Replacing the parameters we obtain

$$A = \begin{pmatrix} 1 & 0 & \alpha & \gamma^{(1)} & \begin{matrix}\tilde{\tau} - \gamma^{(1)}\tilde{h} \\ -\frac{\alpha^{(1)}(\tilde{\tau}' - (\gamma^{(1)})'\tilde{h})}{(\alpha^{(1)})'} + \alpha\frac{\left(\frac{\tilde{\tau}' - (\gamma^{(1)})'\tilde{h}}{(\alpha^{(1)})'}\right)'}{\tilde{h}'}\end{matrix} & \alpha\gamma^{(1)} - 2\gamma \\ 0 & 1 & \tilde{h} & \frac{\left(\frac{\tilde{\tau}' - (\gamma^{(1)})'\tilde{h}}{(\alpha^{(1)})'}\right)'}{\tilde{h}'} & \frac{\tilde{h}\left(\frac{\tilde{\tau}' - (\gamma^{(1)})'\tilde{h}}{(\alpha^{(1)})'}\right)'}{\tilde{h}'} - \frac{2(\tilde{\tau}' - (\gamma^{(1)})'\tilde{h})}{(\alpha^{(1)})'} & \tilde{\tau} - \frac{\alpha^{(1)}(\tilde{\tau}' - (\gamma^{(1)})'\tilde{h})}{(\alpha^{(1)})'} \\ 0 & 0 & 1 & 0 & \frac{\left(\frac{\tilde{\tau}' - (\gamma^{(1)})'\tilde{h}}{(\alpha^{(1)})'}\right)'}{\tilde{h}'} & \gamma^{(1)} \\ 0 & 0 & 0 & 1 & -\tilde{h} & -\alpha \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where α, γ, $\tilde{h}$, and $\tilde{\tau}$ are meromorphic functions with $\alpha^{(1)} = \alpha'/\tilde{h}'$. Given $A = (a_{jk})_{j,k=1,\ldots,6} \in (\mathrm{Sol}^J_\xi)_0$ we may retrieve the meromorphic functions by setting $\alpha = a_{13}$, $\gamma = (-1/2)(a_{16} - a_{13}a_{14})$, $\tilde{h} = a_{23}$ and $\tilde{\tau} = a_{26} - (a'_{13}/2a'_{23})(a_{25} - a_{23}a_{24})$.

We have the extended solution

$$\Phi = [A\gamma_\xi] = (\pi_{\alpha_1} + \lambda\pi^\perp_{\alpha_1})(\pi_{\alpha_2} + \lambda\pi^\perp_{\alpha_2}),$$

where $\alpha_1 = \mathrm{Span}\{c_6, c_5\}$, $\alpha_2 = \mathrm{Span}\{c_6, c_5, c_4, c_3\}$, for c_3, c_4, c_5, c_6 the columns of A. As r is even, by (7.1.12), this corresponds to the harmonic map into the quaternionic Grassmannian $\phi : M \to G_2(\mathbb{H}^3) = \mathbb{HP}^2$ associated to the subbundle $\underline{\phi} = \psi_0 + \psi_2 = \psi_0 + J\psi_0$ of $M \times \mathbb{C}^6$. So $\phi : M \to Sp(3)/(Sp(2) \times Sp(1))$. By Example 7.1.13 we have the harmonic map $\varphi = \Phi_{-1} = \iota \circ \phi : M \to Sp(3)$ defined by $\varphi = \Phi_{-1} =$

$\iota \circ \phi = \pi_\phi - \pi_\phi^\perp$. Also, $\underline{\phi}^\perp = \psi_1 = J\psi_1$, and so is a harmonic map into the quaternionic Grassmannian $\phi^\perp : M \to G_1(\mathbb{H}^3) = \mathbb{HP}^2$. By Example 7.1.13 we have the harmonic map $\iota \circ \phi^\perp : M \to Sp(3)$ where $\iota \circ \phi^\perp = -\iota \circ \phi = -\Phi_{-1} = \pi_\phi^\perp - \pi_\phi$.

Type $(3, 3)$

We add a border to the general solution (7.2.9) of type $(2, 2)$ to find $A : M \to (\mathfrak{A}_\xi^J)_0$ with canonical element $\xi = i\,\mathrm{diag}(1, 1, 1, 0, 0, 0)$ of $\Omega_1 U(6)^J$. All extended solution equations for type $(3, 3)$ are automatically satisfied and so we let $a_{14} = \alpha$, $a_{15} = \beta$, $a_{16} = \gamma$, and due to the block structure of $(\mathfrak{A}_\xi^J)_0$ we have $a_{12} = a_{13} = a_{46} = a_{56} = 0$. To ensure $A \in (\mathfrak{A}_\xi^J)_0$ we use Lemma 5.1.7 to complete the border to get

$$A = \begin{pmatrix} 1 & 0 & 0 & \alpha & \beta & \gamma \\ 0 & 1 & 0 & g & h & \beta \\ 0 & 0 & 1 & f & g & \alpha \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

for α, β, γ, g, f, h, meromorphic functions. We have the extended solution

$$\Phi = [A\gamma_\xi] = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp),$$

where $\alpha_1 = \mathrm{Span}\{c_6, c_5, c_4\}$ for c_6, c_5, c_4, c_3 the columns of A. As r is odd, by (7.1.13), this corresponds to the harmonic map $\phi : M \to Sp(3)/U(3)$ associated to the subbundle $\underline{\phi} = \psi_0 = \alpha_1$ of $M \times \mathbb{C}^6$. By Example 7.1.13 we have the harmonic map $\varphi = i\Phi_{-1} = i(\iota \circ \phi) : M \to Sp(3)$ where $\Phi_{-1} = \iota \circ \phi = \pi_{\alpha_1} - \pi_{\alpha_1}^\perp$ which is of uniton number at most 1. The map is clearly bijective, and so gives a bijection between the space of 6-tuples of meromorphic functions on M of $(\mathrm{Sol}_\xi^J)_0$.

Type $(2,1,1,2)$

We have the canonical element $\xi = i\,\mathrm{diag}(3,3,2,1,0,0)$ of $\Omega_3 U(6)^J$ and we add a border to the general solution (7.2.6) of type $(1,1,1,1)$ taking into account the block structure of $(\mathfrak{A}^J_\xi)_0$. We have extended solution equations $a'_{13} = a'_{13}$, $a'_{14} = a'_{34}a_{13}$, $a'_{15} = a'_{45}a'_{14}$ and $a'_{16} = a'_{46}a_{14}$. As a_{13} automatically satisfies these we set $a_{13} = \alpha$ with α arbitrary so we have

$$A = \begin{pmatrix} 1 & 0 & \alpha & a_{14} & a_{15} & a_{16} \\ 0 & 1 & g & g\nu^{(2)} - \nu^{(1)} & 2\nu - g\nu^{(1)} & a_{26} \\ 0 & 0 & 1 & \nu^{(2)} & -\nu^{(1)} & a_{36} \\ 0 & 0 & 0 & 1 & -g & a_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where g and ν are meromorphic functions and $\nu^{(1)} = \nu'/g'$, $\nu^{(2)} = \nu^{(1)}/g'$. We integrate $a'_{14} = a'_{34}a_{13}$ using integration by parts to lower the order of the (generalised) derivatives of ν:

$$\begin{aligned} a_{14} &= \int a'_{14} = \int a'_{34}a_{13} = \int (\nu^{(2)})'\alpha = \nu^{(2)}\alpha - \int (\nu^{(1)})'\frac{\alpha'}{g'} \\ &= \alpha\nu^{(2)} - \frac{\alpha'}{g'}\nu^{(1)} + \int \frac{(\frac{\alpha'}{g'})'}{g'}\nu' \\ &= \alpha\nu^{(2)} - \frac{\alpha'}{g'}\nu^{(1)} + \frac{(\frac{\alpha'}{g'})'}{g'}\nu - \int \left(\frac{(\frac{\alpha'}{g'})'}{g'}\right)'\nu. \end{aligned}$$

For ease of notation let $\alpha^{(1)} := \alpha'/g'$ and $\alpha^{(2)} := (\alpha^{(1)})'/g'$ and let us introduce a new parameter τ such that $\tau' = (\alpha^{(2)})'\nu$ so we have the generalised derivative $\nu = \tau^{(1)} := \tau'/(\alpha^{(2)})'$. By replacing the new parameter we have $a_{14} = \alpha\tau^{(3)} - \alpha^{(1)}\tau^{(2)} + \alpha^{(2)}\tau^{(1)} - \tau$

and

$$A = \begin{pmatrix} 1 & 0 & \alpha & \alpha\tau^{(3)} - \alpha^{(1)}\tau^{(2)} + \alpha^{(2)}\tau^{(1)} - \tau & a_{15} & a_{16} \\ 0 & 1 & g & g\tau^{(3)} - \tau^{(2)} & 2\tau^{(1)} - g\tau^{(2)} & a_{26} \\ 0 & 0 & 1 & \tau^{(3)} & -\tau^{(2)} & a_{36} \\ 0 & 0 & 0 & 1 & -g & a_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\tau^{(2)} = (\tau^{(1)})'/g'$, $\tau^{(3)} = (\tau^{(2)})'/g'$. We follow the same procedure to find a_{15}:

$$\begin{aligned} a_{15} &= \int a_{15}' = \int a_{45}' a_{14}' = \int (-g')(\alpha\tau^{(3)} - \alpha^{(1)}\tau^{(2)} + \alpha^{(2)}\tau^{(1)} - \tau) \\ &= \int -\frac{g'\alpha}{g'}(\tau^{(2)})' + \frac{g'\alpha^{(1)}}{g'}(\tau^{(1)})' - \frac{(\alpha^{(1)})'}{(\alpha^{(2)})'}\tau' + g'\tau \\ &= -\alpha\tau^{(2)} + \alpha^{(1)}\tau^{(1)} - \frac{(\alpha^{(1)})'}{(\alpha^{(2)})'}\tau + \int \alpha^{(1)}(\tau^{(1)})' - \frac{(\alpha^{(1)})'}{(\alpha^{(2)})'}\tau' + \Big\{\Big(\frac{(\alpha^{(1)})'}{(\alpha^{(2)})'}\Big)' + g'\Big\}\tau \\ &= -\alpha\tau^{(2)} + 2\alpha^{(1)}\tau^{(1)} - \frac{2(\alpha^{(1)})'}{(\alpha^{(2)})'}\tau + \int -\frac{(\alpha^{(1)})'}{(\alpha^{(2)})'}\tau' + \Big\{\Big(\frac{2(\alpha^{(1)})'}{(\alpha^{(2)})'}\Big)' + g'\Big\}\tau \\ &= -\alpha\tau^{(2)} + 2\alpha^{(1)}\tau^{(1)} - \frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\tau + \int \Big\{\Big(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\Big)' + g'\Big\}\tau, \end{aligned}$$

and by introducing a new parameter γ such that

$$\gamma' = \bigg(\Big(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\Big)' + g'\bigg)\tau, \quad \text{and so} \quad \tau = \gamma^{(1)} := \frac{\gamma'}{\Big(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\Big)' + g'},$$

we have

$$a_{15} = -\alpha\gamma^{(3)} + 2\alpha^{(1)}\gamma^{(2)} - \frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\gamma^{(1)} + \gamma.$$

We replace all τ with $\gamma^{(1)}$ to get

$$A = \begin{pmatrix} 1 & 0 & \alpha & \begin{matrix} \alpha\gamma^{(4)} - \alpha^{(1)}\gamma^{(3)} \\ +\alpha^{(2)}\gamma^{(2)} - \gamma^{(1)} \end{matrix} & \begin{matrix} -\alpha\gamma^{(3)} + 2\alpha^{(1)}\gamma^{(2)} \\ -3\left(\frac{(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)\gamma^{(1)} + \gamma \end{matrix} & a_{16} \\ 0 & 1 & g & g\gamma^{(4)} - \gamma^{(3)} & 2\gamma^{(2)} - g\gamma^{(3)} & a_{26} \\ 0 & 0 & 1 & \gamma^{(4)} & -\gamma^{(3)} & a_{36} \\ 0 & 0 & 0 & 1 & -g & a_{46} \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\gamma^{(2)} := (\gamma^{(1)})'/(\alpha^{(2)})'$, $\gamma^{(3)} := (\gamma^{(2)})'/g'$, $\gamma^{(4)} := (\gamma^{(3)})'/g'$.

We now turn our attention to the new last column which we find, as usual, by algebra to get

$$a_{46} = -\alpha, \quad a_{36} = -\alpha^{(1)}\gamma^{(3)} + \alpha^{(2)}\gamma^{(2)} - \gamma^{(1)},$$

$$a_{26} = -g\alpha^{(1)}\gamma^{(3)} + (2\alpha^{(1)} + g\alpha^{(2)})\gamma^{(2)} - \Big(\frac{3(\alpha)^{(1)})'}{(\alpha^{(2)})'} + g\Big)\gamma^{(1)} + \gamma.$$

To complete the border we now need only find a_{16}, which we do by integrating the extended solution equation as before. We have

$$a_{16} = \int a'_{16} = \int a'_{46}a_{14} = \int -\alpha'(\alpha\gamma^{(4)} - \alpha^{(1)}\gamma^{(3)} + \alpha^{(2)}\gamma^{(2)} - \gamma^{(1)})$$

$$= \int -\alpha'\alpha\gamma^{(4)} + \alpha'\alpha^{(1)}\gamma^{(3)} - \alpha'\alpha^{(2)}\gamma^{(2)} + \alpha'\gamma^{(1)}$$

$$= \int -\alpha^{(1)}\alpha(\gamma^{(3)})' + \alpha^{(1)}\alpha^{(1)}(\gamma^{(2)})' - \frac{\alpha'\alpha^{(2)}}{(\alpha^{(2)})'}(\gamma^{(1)})' + \frac{\alpha'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma'$$

$$= -\alpha^{(1)}\alpha\gamma^{(3)} + \alpha^{(1)}\alpha^{(1)}\gamma^{(2)} - \frac{\alpha'\alpha^{(2)}}{(\alpha^{(2)})'}\gamma^{(1)} + \frac{\alpha'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma$$

$$+ \int (\alpha^{(1)}\alpha)'\gamma^{(3)} - (\alpha^{(1)}\alpha^{(1)})'\gamma^{(2)} + \Big(\frac{\alpha'\alpha^{(2)}}{(\alpha^{(2)})'}\Big)'\gamma^{(1)} - \left(\frac{\alpha'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\right)'\gamma$$

$$= -\alpha^{(1)}\alpha\gamma^{(3)} + \alpha^{(1)}\alpha^{(1)}\gamma^{(2)} - \frac{\alpha'\alpha^{(2)}}{(\alpha^{(2)})'}\gamma^{(1)} + \frac{\alpha'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma$$

$$+ \int (\alpha^{(2)}\alpha + \alpha^{(1)}\alpha^{(1)})(\gamma^{(2)})' - \frac{(\alpha^{(1)}\alpha^{(1)})'}{(\alpha^{(2)})'}(\gamma^{(1)})'$$

$$+ \frac{\left(\frac{\alpha'\alpha^{(2)}}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma' - \left(\frac{\alpha'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\right)'\gamma$$

$$= -\alpha^{(1)}\alpha\gamma^{(3)} + (\alpha^{(2)}\alpha + 2\alpha^{(1)}\alpha^{(1)})\gamma^{(2)} - \frac{3\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\gamma^{(1)} + \frac{\alpha' + \left(\frac{\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma$$

$$+ \int -(\alpha^{(2)}\alpha + \alpha^{(1)}\alpha^{(1)})'\gamma^{(2)} + \Big(\frac{2\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\Big)'\gamma^{(1)} - \left(\frac{\left(\alpha + \frac{\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\right)'\gamma$$

$$= -\alpha^{(1)}\alpha\gamma^{(3)} + (\alpha^{(2)}\alpha + 2\alpha^{(1)}\alpha^{(1)})\gamma^{(2)} - \frac{3\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\gamma^{(1)} + \frac{\alpha' + \left(\frac{\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma$$

$$+ \int -(\alpha + \frac{3\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}(\gamma^{(1)})' + \frac{\left(\frac{2\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma' - \left(\frac{\left(\alpha + \frac{\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\right)'\gamma$$

$$= -\alpha^{(1)}\alpha\gamma^{(3)} + (\alpha^{(2)}\alpha + 2\alpha^{(1)}\alpha^{(1)})\gamma^{(2)} - \Big(\alpha + \frac{6\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\Big)\gamma^{(1)}$$

$$+\frac{\alpha' + \left(3\frac{\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma + \int \left(\alpha + \frac{3\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'\gamma^{(1)} - \left(\frac{\left(\alpha + \frac{3\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\right)'\gamma$$

$$= -\alpha^{(1)}\alpha\gamma^{(3)} + (\alpha^{(2)}\alpha + 2\alpha^{(1)}\alpha^{(1)})\gamma^{(2)} - \left(\alpha + \frac{6\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)\gamma^{(1)}$$

$$+\frac{\alpha' + \left(3\frac{\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma + \int \frac{\left(\alpha + \frac{3\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma' - \left(\frac{\left(\alpha + \frac{3\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\right)'\gamma$$

$$= -\alpha^{(1)}\alpha\gamma^{(3)} + (\alpha^{(2)}\alpha + 2\alpha^{(1)}\alpha^{(1)})\gamma^{(2)} - \left(\alpha + \frac{6\alpha^{(1)}(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)\gamma^{(1)}$$

$$+\frac{2\left(\alpha + \frac{3\alpha^{(1)}\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\gamma - \int \left(\frac{2\left(\alpha + \frac{3\alpha^{(1)}\alpha^{(1)})'}{(\alpha^{(2)})'}\right)'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}\right)'\gamma.$$

Now let

$$a = \alpha^{(1)}\alpha, \quad b = \alpha^{(1)}\alpha^{(1)} + \frac{a'}{g'}, \quad c = \frac{3\alpha^{(1)}\alpha' + b'}{(\alpha^{(2)})'}, \quad d = \frac{\alpha + c'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'},$$

and introduce a new parameter β such that $\beta' = d'\gamma$ and so $\gamma = \beta^{(1)} := \beta'/d'$. Replacing all these we have $a_{16} = -a\beta^{(4)} + b\beta^{(3)} - c\beta^{(2)} + d\beta^{(1)} - \beta$ and

$$A = \begin{pmatrix} 1 & 0 & \alpha & \begin{array}{c}\alpha\beta^{(5)} - \alpha^{(1)}\beta^{(4)} \\ +\alpha^{(2)}\beta^{(3)} - \beta^{(2)}\end{array} & \begin{array}{c}-\alpha\beta^{(4)} + 2\alpha^{(1)}\beta^{(3)} \\ -3\left(\frac{(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)\beta^{(2)} + \beta^{(1)}\end{array} & \begin{array}{c}-a\beta^{(4)} + b\beta^{(3)} - c\beta^{(2)} \\ +d\beta^{(1)} - \beta\end{array} \\ 0 & 1 & g & g\beta^{(5)} - \beta^{(4)} & 2\beta^{(3)} - g\beta^{(4)} & \begin{array}{c}-g\alpha^{(1)}\beta^{(4)} + \beta^{(1)} \\ +(2\alpha^{(1)} + g\alpha^{(2)})\beta^{(3)} \\ -\left(\frac{3(\alpha)^{(1)})'}{(\alpha^{(2)})'} + g\right)\beta^{(2)}\end{array} \\ 0 & 0 & 1 & \beta^{(5)} & -\beta^{(4)} & -\alpha^{(1)}\beta^{(4)} + \alpha^{(2)}\beta^{(3)} - \beta^{(2)} \\ 0 & 0 & 0 & 1 & -g & -\alpha \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where

$$\beta^{(2)} := \frac{(\beta^{(1)})'}{\left(\frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\right)' + g'}, \quad \beta^{(3)} := \frac{(\beta^{(2)})'}{(\alpha^{(2)})'}, \quad \beta^{(4)} := \frac{(\beta^{(3)})'}{g'}, \beta^{(5)} := \frac{(\beta^{(4)})'}{g'},$$

for meromorphic functions g, α and β.

We have the extended solution

$$\Phi = [A\gamma_\xi] = (\pi_{\alpha_1} + \lambda\pi_{\alpha_1}^\perp)(\pi_{\alpha_2} + \lambda\pi_{\alpha_2}^\perp)(\pi_{\alpha_3} + \lambda\pi_{\alpha_3}^\perp),$$

where $\alpha_1 = \text{Span}\{c_6, c_5\}$, $\alpha_2 = \text{Span}\{c_6, c_5, c_4\}$, $\alpha_3 = \text{Span}\{c_6, c_5, c_4, c_3\}$, where c_3, c_4, c_5, c_6 denote the columns A. As r is odd, by (7.1.13), this corresponds to the harmonic map $\phi : M \to Sp(3)/U(3)$ associated to subbundle $\underline{\phi} = \psi_0 + \psi_2 = \alpha_1 + \alpha_2^\perp \cap \alpha_3$ of $M \times \mathbb{C}^6$. By Example 7.1.13 we have the harmonic map $\varphi = i\Phi_{-1} = i(\iota \circ \phi) : M \to$

$Sp(3)$ where $\Phi_{-1} = \iota \circ \phi = \pi_\phi - \pi_\phi^\perp$ which is of uniton number at most 3.

The process above gives rise to a bijection $(\alpha, g, \beta) \mapsto A$ between the space of meromorphic functions on M of dimension 3 to $(\mathrm{Sol}_\xi^J)_0$, as given some $A = (a_{jk})_{j,k=1,\dots,6} \in (\mathrm{Sol}_\xi^J)_0$ we retrieve the meromorphic functions by setting $g = a_{23}$, $\alpha = a_{13}$ and $\beta = -a_{16} - a\beta^{(4)} + b\beta^{(3)} - c\beta^{(2)} + d\beta^{(1)}$, where

$$\alpha^{(1)} = \frac{a'_{13}}{a'_{23}}, \quad \alpha^{(2)} = \frac{(\alpha^{(2)})'}{a'_{23}} \quad \beta^{(5)} = a_{34}, \quad \beta^{(4)} = -a_{35}, \quad \beta^{(3)} = \frac{1}{2}(a_{25} - g a_{35}),$$

$$\beta^{(2)} = -a_{36} - \frac{\alpha^{(1)}}{2}\beta^{(4)} + \alpha^{(2)}\beta^{(3)}, \quad \beta^{(1)} = a_{15} + g\beta^{(4)} - 2\alpha^{(1)}\beta(3) + \frac{3(\alpha^{(1)})'}{(\alpha^{(2)})'}\beta^{(2)},$$

and a, b, c, and d given above.

7.3 Standard Type Theorem

In the classification above a different algorithm was used for each type of canonical element to identify and isolate the parameters that are to be replaced; for canonical elements of higher dimension than those in §7.2 such an algorithm may not exist. For $O(n)$ an algorithm was given in [29] which classifies all extended solutions of finite uniton number. This algorithm, demonstrated in Example 6.3.10, adds a border to a lower dimensional solution and then parametrizes the new top row by solving the extended solution equations (6.2.2), the new last column and new top-right element found by algebra as detailed in §6.3.1. When solving the extended solution equations and finding the new top row, Ferreira, Simões and Wood introduced new parameters that replace the parameters in the new top row only, leaving all parameters inside the border unchanged, then went on to complete the border by finding the new last column and new top-right element by algebra. The resulting parameters were sometimes only local in character. For $Sp(m)$ finding such a general algorithm is more difficult as, unlike the $O(n)$ case, there is an additional problem that **we cannot find the new top-right element by algebra**. This is because of the equation $(c_n, c_n) = 0$ from Lemma 5.1.4, which determined the new top-right element in the $O(n)$ case, is now $\omega(c_n, c_n) = 0$ from Lemma 5.1.7, which is automatically satisfied: this forces us to introduce new parameters that replace certain parameters **inside** the border as well as parameters on the new top row, as opposed to only on the new top row as in the $O(n)$ case. In our work, the parameters introduced in this way are always globally defined. Further, if one does not carefully choose the parameters that are to be replaced when solving the extended solution equations, then problems may arise in finding the new top-right element of the border, for example consider the type $(1, 1, 1, 1)$ example from §7.2.2: If we introduce new parameters which replace the parameters in the new top row of the border only as in the $O(n)$ case, we get

$$A = \begin{pmatrix} 1 & \tau^{(1)} & \tau & a_{14} \\ 0 & 1 & f & -\tau \\ 0 & 0 & 1 & -\tau^{(1)} \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

where $\tau^{(1)} = \tau'/f'$. Then, when finding the new top-right element we get $a_{14} = \int -\tau^{(1)}\tau$ which cannot be solved by introducing a new parameter as before. This leads us to the method used in §7.2.2, and to a general result for standard type which we now discuss.

Recall the definition of "standard type" from Remark 7.1.8. We prove that, for a standard type canonical element ξ, when adding a border to an $A \in (\mathrm{Sol}^J_\xi)_0$ we may introduce new parameters which replace the parameters on the **superdiagonal** of A and then complete the border by algebra. More concretely, we give an algorithm that defines a parametrization $h_0 = (h_0)_\xi : \mathcal{M}(M)^m \to (\mathrm{Sol}^J_\xi)_0$ inductively for $m = 1, 2, \ldots$, where ξ is the canonical element of $\Omega_r U(2m)^J$ of standard type. To prove that the algorithm gives Theorem 7.3.1 below we use a series of nested inductions. Recall the definition of $(\mathfrak{A}^J_\xi)_0$ from Definition 7.1.9.

We prove:

Theorem 7.3.1. *Suppose M is a Riemann surface, and $m \in \mathbb{N}$. Let $\mathcal{M}(M)^m$ denote the space of m-tuples $(\nu_1, \ldots, \nu_m)$ of meromorphic functions on M, and for $r = 2m-1$ let ξ be the canonical element of $\Omega_r U(2m)^J$ of standard type, with corresponding canonical geodesic γ_ξ. Let $(Sol_\xi^J)_0$ be the space of meromorphic maps $A : M \to (\mathfrak{A}_\xi^J)_0$ which satisfy the extended solution equation (6.2.3) (away from the poles of A). Then there exists a bijection $h_0 : \mathcal{M}(M)^m \to (Sol_\xi^J)_0$ with the property that for every S^1-invariant extended solution $\Phi : M \to \Omega_r U(2m)^J$ of standard type there exist meromorphic parameters $(\nu_1, \ldots, \nu_m)$ such that $\Phi = [A\gamma_\xi]$ away from the poles of A, where $A = h_0(\nu_1, \ldots, \nu_m)$ and $[\quad]$ denotes the projection onto the first factor of the Iwasawa decomposition.*

Remark 7.3.2.
(i) *Our parameters are globally defined, i.e. defined on the whole of M.*

(ii) *Each entry of $h_0(\nu_1, \ldots, \nu_m)$ is a rational function of $\nu_1, \ldots, \nu_m$ and their (generalised) derivatives (Definition 6.3.9); recall that these are well-defined under change of complex coordinates.*

(iii) *The map from Theorem 7.3.1 is bijective as each parameter can be found from the elements of A in a way that generalises the classifications for type $(1,1)$, $(1,1,1,1)$ and $(1,1,1,1,1,1)$ in §7.2.1, §7.2.2 and §7.2.3, respectively.*

Proof. Our scheme of nested inductions starts with an 'overarching' induction on dimension (Induction Hypothesis 7.3.3). This proves that we may add a border to a given $\tilde{A} \in (\mathrm{Sol}_{\tilde{\xi}}^J)_0$ with prescribed superdiagonal to give a $A \in (\mathrm{Sol}_\xi^J)_0$ where $\tilde{\xi}$ and ξ are standard type canonical elements of $\Omega_{2m-2}U(2m-2)^J$ and $\Omega_{2m}U(2m)^J$, respectively. To achieve the induction step for the overarching induction hypothesis we use two separate inductions, Induction Hypothesis 7.3.6 and Induction Hypothesis 7.3.8. Induction Hypothesis 7.3.6 proves that we can solve the extended solution equations (7.3.5) below and therefore parametrize the first half of the new top row (i.e. the first m elements in the new top row) in terms of the elements of the superdiagonal of A and their derivatives. Similar to Induction Hypothesis 7.3.6, Induction Hypothesis 7.3.8 proves that we can also solve the extended solution equations and parametrize the last $m-1$ elements in the new top row in terms of the elements of the superdiagonal of A and their derivatives. The $(m+1)$st element of the new top row is treated separately. Finally to finish the induction step for the overarching induction hypothesis (Induction Hypothesis 7.3.3) we complete the border by algebra.

We use induction on m so the dimension $2m$ of $Sp(2m, \mathbb{C})$ is increased by 2 in the overarching induction step.

We start with $m = 1$ where we have $\xi = i\,\mathrm{diag}(1, 0)$. Let $f : M \to \mathbb{C}$ be a meromorphic function. Then elements of $(\mathrm{Sol}^J_\xi)_0$ are of the form

$$A = \begin{pmatrix} 1 & f \\ 0 & 1 \end{pmatrix},$$

so we have a bijective mapping $h_0 : \mathcal{M}(M)^1 \to (\mathrm{Sol}^J_\xi)_0$ given by $f \mapsto A$. We use this as a base for induction on the dimension m. Let $m \geq 1$ and let $\xi = i\,\mathrm{diag}(2m, 2m-1, \ldots, 1, 0)$ be a canonical element of $\Omega_{2m}U(2m)^J$. Note that ξ is of standard type and $\gamma_\xi \in \Omega_{2m}U(2m)^J$ (Definition 7.1.7). We define another canonical element of standard type $\tilde{\xi} = i\,\mathrm{diag}(2m-2, 2m-3, \ldots, 1, 0)$ of $\Omega_{2m-2}U(2m-2)^J$. To explain our algorithm, we need to introduce parameters α_{ij} depending on two indices.

Induction Hypothesis 7.3.3. *Let* $\sigma(m) = m(m-1)/2 - 1$ *and assume we have determined* $\tilde{h}_0 = \mathcal{M}(M)^{m-1} \to (\mathrm{Sol}^J_{\tilde{\xi}})_0$ *such that that for meromorphic data*

$$(\nu_1, \ldots, \nu_{m-1}) = (\alpha_{m-2,0},\, \alpha_{m-3,1},\, \alpha_{m-4,2}, \ldots,\, \alpha_{1,m-3},\, \alpha_{0,\sigma(m)}) \in \mathcal{M}(M)^{m-1} \quad (7.3.1)$$

the associated $\tilde{A} \in (\mathrm{Sol}_{\xi}^{J})_0$ *defined from* $\tilde{h}_0$ *has superdiagonal of the form*

$$(\alpha_{m-2,0},\, \alpha_{m-3,1}^{(1)},\, \alpha_{m-4,2}^{(2)},\, \ldots,\, \alpha_{1,m-3}^{(m-3)},\, \alpha_{0,\sigma(m)}^{(\sigma(m))},\\ -\alpha_{1,m-3}^{(m-3)},\, \ldots,\, -\alpha_{m-3,1}^{(1)},\, -\alpha_{m-2,0}). \quad (7.3.2)$$

Here,

all the generalised derivatives of each $\alpha_{i,j}$ *in the superdiagonal (7.3.2) are with respect to functions of 'previous' entries* $\alpha_{k,l}$ *in the superdiagonal, i.e. ones with* $m-2 \geq k > i$. (7.3.3)

We call this the ***overarching induction hypothesis****.*

Note that the last $m-2$ entries of (7.3.2) are minus the first $m-2$ entries and that the $m-1$st entry has a special formula.

Notation 7.3.4. *For* $\alpha_{a,b}$ *then "*a*" numbers the parameter and its position in the superdiagonal and "*b*" denotes the "generation" of the parameter. Each time during our inductions that we introduce a new parameter to replace an existing ('old') parameter such that the old parameter is a generalised derivative of the new one.*

Remark 7.3.5. (i) *For* $m = 2$*, (7.3.1) and (7.3.2) read* $\alpha_{0,0} \in \mathcal{M}(M)^1$ *and* $(\alpha_{0,0})$ *(as* $\alpha_{0,0}^{(0)} = \alpha_{0,0}$ *by Definition 6.3.9), respectively.*

(ii) *For* $m = 3$*, (7.3.1) and (7.3.2) read* $(\alpha_{1,0}, \alpha_{0,2}) \in \mathcal{M}(M)^2$ *and* $(\alpha_{1,0},\, \alpha_{0,2}^{(2)}, -\alpha_{1,0})$*, respectively.*

Now suppose Induction Hypothesis 7.3.3 holds for some $m-1$. We will show how to find $h_0 : \mathcal{M}(M)^m \to (\mathrm{Sol}_{\xi}^{J})_0$ from $\tilde{h}_0 = \mathcal{M}(M)^{m-1} \to (\mathrm{Sol}_{\xi}^{J})_0$ by adding a border to the $\tilde{A}$ associated to $\tilde{h}_0$, so we have

$$A = \left(\begin{array}{c|cccc|c} 1 & a_{12} & a_{13} & \dots & a_{1,2m-1} & a_{1,2m} \\ \hline 0 & & & & & a_{2,2m} \\ 0 & & & \tilde{A} & & a_{3,2m} \\ 0 & & & & & \vdots \\ \hline 0 & 0 & 0 & 0 & 0 & 1 \end{array}\right). \tag{7.3.4}$$

The form of (7.3.4) for types $(1,1,1,1)$ and $(1,1,1,1,1,1)$ are given by (7.2.2) and (7.2.10). We find a parametrization of the new top row $(a_{12}, a_{13}, \dots, a_{1,2m-1})$ and new top-right element $a_{1,2m}$ by solving the extended solution equations (6.2.3) which read in this case

$$a'_{1j} = a_{1,j-1} a'_{j-1,j}. \tag{7.3.5}$$

Here, as usual, $'$ denotes the derivative with respect to any local complex coordinate on M, but our equations are independent of choice of complex coordinate. **We find the first $m-1$ elements of the new top row**. The first element in the new top row a_{12} automatically satisfies (7.3.5) and so we parametrize this by $a_{12} = \alpha_{m-1,0}$ where $\alpha_{m-1,0}$ is meromorphic. For the next element a_{13} we have $a_{13} = \int a_{12} a'_{23}$. As a_{23} is the second element of the superdiagonal of A and therefore the first element in the superdiagonal of $\tilde{A}$ we have by (7.3.2) in our overarching induction hypothesis that $a_{23} = \alpha_{m-2,0}$. Using this and integration by parts we have

$$a_{13} = \int \alpha_{m-1,0}\, \alpha'_{m-2,0} = \alpha_{m-1,0}\, \alpha_{m-2,0} - \int \alpha'_{m-1,0}\, \alpha_{m-2,0}. \tag{7.3.6}$$

This calculation was presented for type $(1,1,1,1,1,1)$ earlier in (7.2.11). We introduce a new parameter $\alpha_{m-2,1}$ (which is a 'new generation' of $\alpha_{m-2,0}$ which replaces it) such that $\alpha'_{m-2,1} = \alpha_{m-2,0}\, \alpha'_{m-1,0}$ and so $\alpha_{m-2,0} = \alpha'_{m-2,1}/\alpha'_{m-1,0} =: \alpha^{(1)}_{m-2,1}$, substituting and integrating (7.3.6) we have

$$a_{13} = \alpha_{m-1,0}\, \alpha^{(1)}_{m-2,1} - \alpha_{m-2,1}.$$

We use this as a base for an induction on K, $3 \leq K < m$ inside our overarching induction hypothesis to prove that we can solve the extended solution equations (7.3.5) and parametrize the first m elements of the new top row in terms of certain elements of the superdiagonal of A:

Induction Hypothesis 7.3.6. *For some K with $3 \leq K < m$, assume there exists $\alpha_{m-K+1,K-2}$ such that*

$$a_{1K} = g_K(\alpha_{m-1,0}, \alpha_{m-2,1}, \dots, \alpha_{m-K+1,K-2}), \tag{7.3.7}$$

by which we mean that

> *a_{1K} is a rational function of $\alpha_{m-1,0}, \alpha_{m-2,1}, \dots, \alpha_{m-K+1,K-2}$ and their generalised derivatives. The generalised derivatives of each $\alpha_{i,j}$ are with respect to functions of 'previous' parameters $\alpha_{k,l}$, i.e. ones with $m-1 \geq k > i$.* (7.3.8)

By (7.3.3) in our overarching induction hypothesis (Induction Hypothesis 7.3.3) we know for $3 \leq K < m$ that

$$\alpha^{(j)}_{m-K+1,K-3} = \frac{(\alpha^{(j-1)}_{m-K+1,K-3})'}{f'_{m-K+1,j-1}}, \qquad 1 \leq j \leq K-2, \tag{7.3.9}$$

where $f_{m-K+1,j-1}$ are functions of $\alpha_{m-1,0}, \alpha_{m-2,1}, \dots, \alpha_{m-K+2,K-3}$ and their derivatives.

Remark 7.3.7. *Note that "$j-1$" in $f_{m-K+1,j-1}$ does not denote the generation of the parameter but indicates the order $j-1$ of the generalised derivative of $\alpha_{m-K+1,K-3}$. Also, by the overarching induction hypothesis (7.3.2), $a_{K,K+1} = \alpha^{(K-2)}_{m-K,K-2}$.*

Suppose Induction Hypothesis 7.3.6 holds for some K, $3 \leq K < m$. From the extended solution equation (7.3.5) for $j = K+1$ we have

$$a_{1,K+1} = \int a'_{1,K+1} = \int a_{1,K}\, a'_{K,K+1} = \int g_K (\alpha^{(K-2)}_{m-K,K-2})' \tag{7.3.10}$$

where g_k is given by (7.3.7). Integrating (7.3.10) by parts to lower the order of the (generalised) derivative of $\alpha^{(K-2)}_{m-K,K-2}$ and using (7.3.9) we have

$$a_{1,K+1} = g_K (\alpha^{(K-2)}_{m-K,K-2}) - \int \frac{g'_K}{f'_{m-K,K-3}} (\alpha^{(K-3)}_{m-K,K-2})'. \tag{7.3.11}$$

Using integration by parts again, we get

$$a_{1,K+1} = g_K (\alpha^{(K-2)}_{m-K,K-2}) - \frac{g'_K}{f'_{m-K,K-3}} (\alpha^{(K-3)}_{m-K,K-2}) + \int \frac{\left(\frac{g'_K}{f'_{m-K,K-3}}\right)}{f'_{m-K,K-4}} (\alpha^{(K-4)}_{m-K,K-2})'. \tag{7.3.12}$$

After repeating this use of integration by parts to lower the order of the (generalised) derivative and using (7.3.9) we obtain an expression of the form

$$a_{1,K+1} = \sum_{k=1}^{K-1} \gamma^{K+1}_k \alpha^{(k-1)}_{m-K,K-2} - \int (\gamma^{K+1}_1)' \alpha_{m-K,K-2}, \tag{7.3.13}$$

where $\gamma^{K+1}_{K-1} = g_K$, and inductively $\gamma^{K+1}_j = -(\gamma^{K+1}_{j+1})'/f'_{m-K,j-1}$, $j = 1, 2, \ldots, K-2$. Note here that the γ^{K+1}_j are all functions of $\alpha_{m-1,0}, \alpha_{m-2,1}, \ldots, \alpha_{m-K+1,K-2}$ and their derivatives as by (7.3.8) and (7.3.9) both g_K and $f_{m-K,j-1}$ are functions of these. Let $\alpha_{m-K,K-1}$ be a new generation of $\alpha_{m-K,K-2}$ satisfying $\alpha'_{m-K,K-1} = (\gamma^{K+1}_1)' \alpha_{m-K,K-2}$, and so $\alpha_{m-K,K-2} = (\alpha_{m-K,K-1})'/(\gamma^{K+1}_1)' =: \alpha^{(1)}_{m-K,K-1}$. Then by substituting this last formula into (7.3.13) and integrating we conclude

$$\begin{aligned} a_{1,K+1} &= \sum_{k=1}^{K-1} \gamma_k^{K+1}\, \alpha_{m-K,K-1}^{(k)} - \alpha_{m-K,K-1} \\ &=: g_{K+1}(\alpha_{m-1,0}, \alpha_{m-2,1}, \ldots, \alpha_{m-K,K-1}) \end{aligned}$$

for some g_{K+1}, which completes the induction step for Induction Hypothesis 7.3.6. Note that when we introduce a new generation $\alpha_{m-K,K-1}$ of $\alpha_{m-K,K-2}$ for some K we have the following relations of generalised derivatives:

$$\alpha_{m-K,K-1}^{(j)} = \alpha_{m-K,K-2}^{(j-1)}, \quad j = 2, 3, \ldots, K-1. \tag{7.3.14}$$

We have thus found a parametrization of the first m elements in the new top row.

Next, **we find the element $a_{1,m+1}$ of the new top row**. To do this we solve (7.3.5) in a similar way to before:

$$a_{1,m+1} = \int a_{1,m} a'_{m,m+1} = \int g_m (\alpha_{0,\sigma(m)}^{(\sigma(m))})',$$

where g_m is as in (7.3.7) and $a_{m,m+1} = \alpha_{0,\sigma(m)}^{(\sigma(m))}$ by (7.3.2) from the overarching induction hypothesis. Using integration by parts we have

$$a_{1,m+1} = g_m\, \alpha_{0,\sigma(m)}^{(\sigma(m))} - \int g'_m\, \alpha_{0,\sigma(m)}^{(\sigma(m))}. \tag{7.3.15}$$

From the overarching induction hypothesis (7.3.3) the generalised derivatives of $\alpha_{0,\sigma(m)}$ are of the form

$$\alpha_{0,\sigma(m)}^{(j)} = \frac{(\alpha_{0,\sigma(m)}^{(j-1)})'}{h'_{j-1}}, \qquad j = 1, 2, \ldots, \sigma(m), \tag{7.3.16}$$

where h_{j-1} are functions of $\alpha_{m-1,0}, \alpha_{m-2,1}, \ldots, \alpha_{1,m-2}$ and their derivatives. Substituting the expression (7.3.16) for $j = \sigma(m)$ into (7.3.15) we obtain

$$a_{1,m+1} = g_m\, \alpha_{0,\sigma(m)}^{(\sigma(m))} - \int \frac{g_m'}{(h_{\sigma(m)-1})'} (\alpha_{0,\sigma(m)}^{(\sigma(m)-1)})'. \tag{7.3.17}$$

Using integration by parts to evaluate the integral in (7.3.17) we have

$$a_{1,m+1} = g_m\, \alpha_{0,\sigma(m)}^{(\sigma(m))} - \frac{g_m'}{(h_{\sigma(m)-1})'} (\alpha_{0,\sigma(m)}^{(\sigma(m)-1)})$$
$$+ \int \frac{\left(\frac{g_m'}{(h_{\sigma(m)-1})'}\right)'}{(h_{\sigma(m)-2})'} (\alpha_{0,\sigma(m)}^{(\sigma(m)-2)})'.$$

With repeated use of integration by parts to lower the order of the (generalised) derivative of $\alpha_{0,\sigma(m)}$ and using (7.3.16) we end up with

$$a_{1,m+1} = \sum_{k=1}^{\sigma(m)+1} \gamma_k^{m+1}\, \alpha_{0,\sigma(m)}^{(k-1)} - \int (\gamma_1^{m+1})'\, \alpha_{0,\sigma(m)}, \tag{7.3.18}$$

where

$$\gamma_{\sigma(m)+1}^{m+1} = g_m, \qquad \gamma_j^{m+1} = -\frac{(\gamma_{j+1}^{m+1})'}{h_{j-1}'}, \qquad j = 1, 2, \ldots, \sigma(m). \tag{7.3.19}$$

Note here that the γ_j^{m+1} are all functions of $\alpha_{m-1,0}, \alpha_{m-2,1}, \ldots, \alpha_{1,m-2}$ and their derivatives as by (7.3.8) and (7.3.16) both g_m and h_j are functions of these. We introduce a new generation $\alpha_{0,\sigma(m)+1}$ of $\alpha_{0,\sigma(m)}$ in the following way: let $\alpha_{0,\sigma(m)+1} = (\gamma_1^{m+1})'\alpha_{0,\sigma(m)}$, which gives a new generalised derivative

$$\alpha_{0,\sigma(m)} = (\alpha_{0,\sigma(m)+1})/(\gamma_1^{m+1})' =: \alpha_{0,\sigma(m)+1}^{(1)}. \tag{7.3.20}$$

The relation between the generalised derivatives of the new generation and the old generation is

$$\alpha^{(j)}_{0,\sigma(m)+1} = \alpha^{(j-1)}_{0,\sigma(m)}, \qquad j = 1, 2, \ldots, \sigma(m) + 1. \tag{7.3.21}$$

Substituting (7.3.20) into (7.3.18) and evaluating the integral we get

$$a_{1,m+1} = \sum_{k=1}^{\sigma(m)+1} \gamma^{m+1}_k \, \alpha^{(k)}_{0,\sigma(m)+1} - \alpha_{0,\sigma(m)+1}, \tag{7.3.22}$$

where the γ^{m+1}_k are as in (7.3.19), and so we have a parametrization for the $(m+1)$st element of the new top row. The calculation above was presented in (7.2.3) for $m = 2$ and both (7.2.12) and (7.2.13) for $m = 3$.

We turn our attention to the element $a_{2m-1,2m}$ of the new last column. We find this element by algebra, that is we use Lemma 5.1.7 and solve

$$\omega(c_j, c_k) = \begin{cases} i\delta_{\bar{j}k}, & \text{if } j > k, \\ -i\delta_{\bar{j}k}, & \text{if } j < k, \end{cases}$$

where c_j is the jth column of A. Applying this to c_2, and c_{2m} we get

$$\omega(c_2, c_{2m}) = i(-a_{2m-1,2m} - \alpha_{m-1,0}) = 0$$

and so $a_{2m-1,2m} = -\alpha_{m-1,0}$.

We now find the last $m-2$ elements of the new top row and new top-right element. Firstly to find $a_{1,m+2}$ in the new top row we solve (7.3.5) by first using $a_{m+1,m+2} = -\alpha^{(m-2)}_{1,m-2}$ (from the overarching induction hypothesis (7.3.2)) and then (7.3.22). This gives

$$a_{1,m+2} = \int a'_{1,m+2} = \int a_{1,m+1} a'_{m+1,m+2} = \int a_{1,m+1} (\alpha_{1,m-2}^{(m-2)})' \qquad \text{by (7.3.5)}$$

$$= \int \Big(\sum_{k=1}^{\sigma(m)+1} \gamma_k^{m+1} \, \alpha_{0,\sigma(m)+1}^{(k)} - \alpha_{0,\sigma(m)+1} \Big) (-\alpha_{1,m-2}^{(m-2)})'. \qquad \text{by (7.3.22)}$$

Expanding the brackets and pulling out the first term of the sum, we see

$$a_{1,m+2} = \int \sum_{k=1}^{\sigma(m)+1} \gamma_k^{m+1} (-\alpha_{1,m-2}^{(m-2)})' \alpha_{0,\sigma(m)+1}^{(k)} - \int (-\alpha_{1,m-2}^{(m-2)})' \alpha_{0,\sigma(m)+1}$$

$$= \int \sum_{k=2}^{\sigma(m)+1} \gamma_k^{m+1} (-\alpha_{1,m-2}^{(m-2)})' \alpha_{0,\sigma(m)+1}^{(k)} + \int \gamma_1^{m+1} (-\alpha_{1,m-2}^{(m-2)})' \alpha_{0,\sigma(m)+1}^{(1)} \qquad (7.3.23)$$

$$- \int (-\alpha_{1,m-2}^{(m-2)})' \alpha_{0,\sigma(m)+1}.$$

Recall from (7.3.16) and (7.3.21) that

$$\alpha_{0,\sigma(m)+1}^{(1)} = \frac{(\alpha_{0,\sigma(m)+1})'}{(\gamma_1^{m+1})'}, \quad \alpha_{0,\sigma(m)+1}^{(j)} = \frac{(\alpha_{0,\sigma(m)+1}^{(j-1)})'}{h'_{j-2}}, \qquad (7.3.24)$$

for $j = 2, 3, \ldots, \sigma(m) + 1$. We substitute (7.3.24) for $j = k$ into the first integrand of (7.3.23) then use integration by parts and collect like terms as follows:

$$a_{1,m+2} = \int \sum_{k=2}^{\sigma(m)+1} \frac{\gamma_k^{m+1} (-\alpha_{1,m-2}^{(m-2)})'}{h'_{k-2}} (\alpha_{0,\sigma(m)+1}^{(k-1)})' + \int \frac{\gamma_1^{m+1} (-\alpha_{1,m-2}^{(m-2)})'}{(\gamma_1^{m+1})'} \alpha'_{0,\sigma(m)+1}$$

$$- \int (-\alpha_{1,m-2}^{(m-2)})' \alpha_{0,\sigma(m)+1}$$

$$= \frac{\gamma_1^{m+1} (-\alpha_{1,m-2}^{(m-2)})'}{(\gamma_1^{m+1})'} \alpha_{0,\sigma(m)+1} + \sum_{k=2}^{\sigma(m)+1} \frac{\gamma_k^{m+1} (-\alpha_{1,m-2}^{(m-2)})'}{h'_{k-2}} \alpha_{0,\sigma(m)+1}^{(k-1)}$$

$$-\int\sum_{k=2}^{\sigma(m)+1}\Big(\frac{\gamma_k^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_{k-2}}\Big)'\alpha_{0,\sigma(m)+1}^{(k-1)} \tag{7.3.25}$$

$$-\int\Big\{(-\alpha_{1,m-2}^{(m-2)})'+\Big(\frac{\gamma_1^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{(\gamma_1^{m+1})'}\Big)'\Big\}\alpha_{0,\sigma(m)+1}.$$

We now substitute (7.3.24) for $j=k-1$ into the first integrand of (7.3.25) and pull out the first term of the sum in the first integrand to get

$$a_{1,m+2}=\frac{\gamma_1^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{(\gamma_1^{m+1})'}\alpha_{0,\sigma(m)+1}+\sum_{k=2}^{\sigma(m)+1}\frac{\gamma_k^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_{k-2}}\alpha_{0,\sigma(m)+1}^{(k-1)}$$

$$-\int\sum_{k=3}^{\sigma(m)+1}\frac{\Big(\frac{\gamma_k^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_{k-2}}\Big)'}{h'_{k-3}}(\alpha_{0,\sigma(m)+1}^{(k-2)})' \tag{7.3.26}$$

$$-\int\frac{\Big(\frac{\gamma_k^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_0}\Big)'}{(\gamma_1^{m-1})'}\alpha'_{0,\sigma(m)+1}$$

$$-\int\Big\{(-\alpha_{1,m-2}^{(m-2)})'+\Big(\frac{\gamma_1^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{(\gamma_1^{m+1})'}\Big)'\Big\}\alpha_{0,\sigma(m)+1}.$$

Using integration by parts on the first two integrals into (7.3.26) and collecting the like terms, we see

$$a_{1,m+2}=\Big\{\frac{\gamma_1^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{(\gamma_1^{m+1})'}-\frac{\Big(\frac{\gamma_k^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_0}\Big)'}{(\gamma_1^{m-1})'}\Big\}\alpha_{0,\sigma(m)+1}$$

$$+\sum_{k=2}^{\sigma(m)}\Big\{\frac{\gamma_k^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_{k-2}}-\frac{\Big(\frac{\gamma_{k+1}^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_{k-1}}\Big)'}{h'_{k-2}}\Big\}\alpha_{0,\sigma(m)+1}^{(k-1)}$$

$$+\frac{\gamma_{\sigma(m)+1}^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_{\sigma(m)}}\alpha_{0,\sigma(m)+1}^{(\sigma(m))}$$

$$+\int \sum_{k=3}^{\sigma(m)+1} \left(\frac{\left(\frac{\gamma_k^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_{k-2}} \right)'}{h'_{k-3}} \right)' \alpha_{0,\sigma(m)+1}^{(k-2)}$$

$$-\int \left\{ (-\alpha_{1,m-2}^{(m-2)})' + \left(\frac{\gamma_1^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{(\gamma_1^{m+1})'} \right)' - \left(\frac{\left(\frac{\gamma_k^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_0} \right)'}{(\gamma_1^{m-1})'} \right)' \right\} \alpha_{0,\sigma(m)+1}.$$

If we continue to use (7.3.24), integrate by parts to lower the order of (generalised) derivative and then collect like terms we finally get

$$a_{1,m+2} = \sum_{k=1}^{\sigma(m)+1} \delta_k^{m+2}\, \alpha_{0,\sigma(m)+1}^{(k-1)} - \int (-\alpha_{1,m-2}^{(m-2)} + \delta_1^{m+2})' \alpha_{0,\sigma(m)+1}, \tag{7.3.27}$$

where the δ_k^{m+2} are inductively defined as follows:

$$\delta_{\sigma(m)+1}^{m+2} = \frac{\gamma_{\sigma(m)+1}^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_{\sigma(m)+1}}, \qquad \delta_j^{m+2} = \frac{\gamma_j^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{h'_{j-2}} - \frac{(\delta_{j+1}^{m+2})'}{h'_{j-2}},$$

$j = 2, 3, \ldots, \sigma(m)$ and

$$\delta_1^{m+2} = \frac{\gamma_1^{m+1}(-\alpha_{1,m-2}^{(m-2)})'}{(\gamma_1^{m+1})'} - \frac{(\delta_2^{m+2})'}{(\gamma_1^{m+1})'}.$$

As before we introduce a new generation $\alpha_{0,\sigma(m)+2}$, of $\alpha_{0,\sigma(m)+1}$ satisfying

$$\alpha'_{0,\sigma(m)+2} = (-\alpha_{1,m-2}^{(m-2)} + \delta_1^{m+2})'\, \alpha_{0,\sigma(m)+1},$$

so

$$\alpha_{0,\sigma(m)+1} = \frac{(\alpha_{0,\sigma(m)+2})'}{(-\alpha_{1,m-2}^{(m-2)} + \delta_1^{m+2})'} =: \alpha_{0,\sigma(m)+2}^{(1)}. \tag{7.3.28}$$

Substituting (7.3.28) into (7.3.27) and evaluating the integral we have

$$a_{1,m+2} = \sum_{k=1}^{\sigma(m)+1} \delta_k^{m+2}\, \alpha_{0,\sigma(m)+2}^{(k)} - \alpha_{0,\sigma(m)+2}.$$

Note here that all the δ_j^{m+2}, $j = 2, 3, \ldots, \sigma(m)$ are functions of $\alpha_{m-1,0}, \alpha_{m-2,1}, \ldots, \alpha_{1,m-2}$ and their derivatives as all γ_j^{m+1} and h_j are functions of them and their derivatives. This calculation was given in (7.2.5) for $m = 2$ and (7.2.15), (7.2.16) and (7.2.17) for $m = 3$.

We will use this as a basis for another induction, this time to find the elements a_{1I} with $m + 2 \leq I \leq 2m$.

Induction Hypothesis 7.3.8. *For each I, $m + 2 \leq I < 2m$, assume there exists $\alpha_{0,\sigma(m)+I-m}$ such that*

$$a_{1I} = \sum_{k=1}^{\sigma(m)+1} \delta_k^I \, \alpha_{0,\sigma(m)+I-m}^{(k)} - \alpha_{0,\sigma(m)+I-m}, \tag{7.3.29}$$

for

$$\delta_{\sigma(m)+1}^I = \frac{\delta_{\sigma(m)+1}^{I-1}(-\alpha_{I-m-1,2m-I}^{(2m-I)})'}{l'_{\sigma(m)}}, \qquad \delta_j^I = \frac{\delta_j^{I-1}(-\alpha_{I-m-1,2m-I}^{(2m-I)})'}{l'_{j-1}} - \frac{(\delta_{j+1}^I)'}{l'_{j-1}}, \tag{7.3.30}$$

$j = 1, 2, \ldots, \sigma(m)$,

where l_j are functions of $\alpha_{m-1,0}, \alpha_{m-2,1}, \ldots, \alpha_{1,m-2}$ and their derivatives with (7.3.31)

$$\alpha_{0,\sigma(m)+I-m}^{(j)} = \frac{\alpha_{0,\sigma(m)+I-m}^{(j-1)}}{l'_{j-1}}, \qquad j = 1, 2, \ldots, \sigma(m) + I - m. \tag{7.3.32}$$

We find a parametrization of $a_{1,I+1}$ by solving (7.3.5) for $j = I + 1$ which reads

$$a'_{1,I+1} = a_{1I} a'_{I,I+1}. \tag{7.3.33}$$

From (7.3.2) we have that $a_{I,I+1} = -\alpha^{(2m-I-1)}_{I-m,2m-I-1}$, substituting this into (7.3.33) we have

$$a'_{1,I+1} = a_{1I}(-\alpha^{(2m-I-1)}_{I-m,2m-I-1})'. \tag{7.3.34}$$

Suppose Induction Hypothesis 7.3.8 holds for some I, $m+2 \leq I < 2m$, then a_{1I} is given by (7.3.29). By substituting this into (7.3.34) we get

$$a'_{1,I+1} = \Big(\sum_{k=1}^{\sigma(m)+1} \delta^I_k\, \alpha^{(k)}_{0,\sigma(m)+I-m} - \alpha_{0,\sigma(m)+I-m}\Big)(-\alpha^{(2m-I-1)}_{I-m,2m-I-1})'. \tag{7.3.35}$$

We integrate both sides of (7.3.35) and expand the brackets to give

$$\begin{aligned} a_{1,I+1} &= \int \sum_{k=1}^{\sigma(m)+1} (\delta^I_k\, \alpha^{(k)}_{0,\sigma(m)+I-m} - \alpha_{0,\sigma(m)+I-m})(-\alpha^{(2m-I-1)}_{I-m,2m-I-1})' \\ &= \int \sum_{k=1}^{\sigma(m)+1} \delta^I_k(-\alpha^{(2m-I-1)}_{I-m,2m-I-1})'\alpha^{(k)}_{0,\sigma(m)+I-m} \\ &\quad - \int (-\alpha^{(2m-I-1)}_{I-m,2m-I-1})'\alpha_{0,\sigma(m)+I-m}. \end{aligned} \tag{7.3.36}$$

Substituting (7.3.32) from Induction Hypothesis 7.3.8 into (7.3.36) we get

$$\begin{aligned} a_{1,I+1} &= \int \sum_{k=1}^{\sigma(m)+1} \frac{\delta^I_k(-\alpha^{(2m-I-1)}_{I-m,2m-I-1})'}{l'_{k-1}}(\alpha^{(k-1)}_{0,\sigma(m)+I-m})' \\ &\quad - \int (-\alpha^{(2m-I-1)}_{I-m,2m-I-1})'\alpha_{0,\sigma(m)+I-m}. \end{aligned} \tag{7.3.37}$$

We use integration by parts on (7.3.37) then pull out the first term of the sum in the integrand to collect like terms as follows:

$$a_{1,I+1} = \sum_{k=1}^{\sigma(m)+1} \frac{\delta^I_k(-\alpha^{(2m-I-1)}_{I-m,2m-I-1})'}{l'_{k-1}}\alpha^{(k-1)}_{0,\sigma(m)+I-m}$$

$$- \int \sum_{k=2}^{\sigma(m)+1} \Big(\frac{\delta_k^I (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})'}{l'_{k-1}} \Big)' \alpha_{0,\sigma(m)+I-m}^{(k-1)} \tag{7.3.38}$$

$$- \int \Bigg\{ (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})' + \Big(\frac{\delta_1^I (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})'}{l'_0} \Big)' \Bigg\} \alpha_{0,\sigma(m)+I-m}.$$

We substitute (7.3.32) from Induction Hypothesis 7.3.8 into the sum in the integrand in (7.3.38) to get

$$a_{1,I+1} = \sum_{k=1}^{\sigma(m)+1} \frac{\delta_k^I (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})'}{l'_{k-1}} \alpha_{0,\sigma(m)+I-m}^{(k-1)}$$

$$- \int \sum_{k=2}^{\sigma(m)+1} \frac{\Big(\frac{\delta_k^I (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})'}{l'_{k-1}} \Big)'}{l'_{k-2}} (\alpha_{0,\sigma(m)+I-m}^{(k-2)})' \tag{7.3.39}$$

$$- \int \Bigg\{ (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})' + \Big(\frac{\delta_1^I (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})'}{l'_0} \Big)' \Bigg\} \alpha_{0,\sigma(m)+I-m}.$$

Similarly to before we use integration by parts on (7.3.39) then pull out the first term of the sum in the integrand to collect like terms as follows:

$$a_{1,I+1} = \sum_{k=1}^{\sigma(m)+1} \frac{\delta_k^I (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})'}{l'_{k-1}} \alpha_{0,\sigma(m)+I-m}^{(k-1)}$$

$$- \sum_{k=2}^{\sigma(m)+1} \frac{\Big(\frac{\delta_k^I (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})'}{l'_{k-1}} \Big)'}{l'_{k-2}} \alpha_{0,\sigma(m)+I-m}^{(k-2)}$$

$$- \int \sum_{k=3}^{\sigma(m)+1} \left(\frac{\Big(\frac{\delta_k^I (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})'}{l'_{k-1}} \Big)'}{l'_{k-2}} \right)' \alpha_{0,\sigma(m)+I-m}^{(k-2)}$$

$$- \int \Bigg\{ (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})' + \Big(\frac{\delta_1^I (-\alpha_{I-m,2m-I-1}^{(2m-I-1)})'}{l'_0} \Big)'$$

$$- \left(\frac{\left(\frac{\delta_2^I(-\alpha_{I-m,2m-I-1}^{(2m-I-1)})'}{l_1'} \right)'}{l_0'} \right)' \Bigg\} \alpha_{0,\sigma(m)+I-m}.$$

Continuing the programme of using (7.3.32) from Induction Hypothesis 7.3.8, integration by parts and pulling out the first term of the sum in the integrand to collect like terms we end up with

$$a_{1,I+1} = \sum_{k=1}^{\sigma(m)+1} \delta_k^{I+1} \, \alpha_{0,\sigma(m)+I-m}^{(k-1)} - \int (\alpha_{I-m,2m-I-1}^{(2m-I-1)})' \alpha_{0,\sigma(m)+I-m}, \tag{7.3.40}$$

for

$$\delta_{\sigma(m)+1}^{I+1} = \frac{\delta_{\sigma(m)+1}^I(-\alpha_{I-m,2m-I-1})'}{l_{\sigma(m)}'}, \qquad \delta_j^{I+1} = \frac{\delta_j^I(-\alpha_{I-m,2m-I-1})'}{l_{j-1}'} - \frac{\delta_{j+1}^{I+1}}{l_{j-1}'},$$

$j = 1, 2, \ldots, \sigma(m)$ defined inductively. We introduce a new generation $\alpha_{0,\sigma(m)+I-m+1}$ of $\alpha_{0,\sigma(m)+I-m}$ such that $\alpha_{0,\sigma(m)+I-m+1}' = (\alpha_{I-m,2m-I-1}^{(2m-I-1)})' \alpha_{0,\sigma(m)+I-m}$, therefore we have

$$\alpha_{0,\sigma(m)+I-m} = \frac{(\alpha_{0,\sigma(m)+I-m+1})'}{(\alpha_{I-m,2m-I-1}^{(2m-I-1)})'} =: \alpha_{0,\sigma(m)+I-m+1}^{(1)}. \tag{7.3.41}$$

Substituting (7.3.41) into (7.3.40) and evaluating the integral gives

$$a_{1,I+1} = \sum_{k=1}^{\sigma(m)+1} \delta_k^{I+1} \, \alpha_{0,\sigma(m)+I-m+1}^{(k)} - \alpha_{0,\sigma(m)+I-m+1}.$$

Note that the δ_k^{I+1} are all functions of $\alpha_{m-1,0}, \alpha_{m-2,1}, \ldots, \alpha_{1,m-2}$ and their derivatives as by (7.3.30) and (7.3.31) of Induction Hypothesis 7.3.8, δ_k^I and l_i are functions of these and their derivatives. For $I = 2m - 1 = 5$, so $m = 3$ and ξ is of type $(1, 1, 1, 1, 1, 1)$ this calculation was given in (7.2.19).

This completes the induction step for Induction Hypothesis 7.3.8 and so we have found a parametrization for the new top row, new top-right element and $a_{2m-1,2m}$ of the new last column.

We find the remaining elements $a_{j,2m}$, **for** $2 \leq j \leq 2m-2$ by algebra similar to how we parametrized $a_{2m-1,2m}$ earlier. More concretely we use (5.1.11) and Lemma 5.1.7 to get that the columns of A must satisfy

$$\omega(c_j, c_{2m}) = i \sum_{k=1}^{m} a_{\overline{k},j} a_{k,2m} - a_{K,j} a_{\overline{k},2m} = 0,$$

for $1 < j \leq 2m$ therefore we find $a_{j,2m}$ by evaluating $\omega(c_{2m+1-j}, c_{2m}) = 0$. For $m = 2$ and $m = 3$ these calculation are given in (7.2.4) and (7.2.18), respectively.

This completes the border on $\tilde{A}$ and the resulting A as in (7.3.4) is an element of $(\mathrm{Sol}_{\xi}^{J})_0$.

During this process the generation of each of the parameters $\alpha_{m-2,0}$, $\alpha_{m-3,1}$, $\alpha_{m-4,2}, \ldots, \alpha_{1,m-3}$ on the superdiagonal of $\tilde{A}$ increased by 1, as a new generalised derivative was introduced. We introduced a completely new parameter $\alpha_{m-1,0}$, and the generation of the parameter $\alpha_{0,\sigma(m)}$ was increased by m and so introducing m more generalised derivatives of $\alpha_{0,\sigma(m)}$. In fact, $\alpha_{m-2,0}$ is replaced by $\alpha^{(1)}_{m-2,1}$ and in general $\alpha^{(j)}_{m-2-j,j}$ is replaced by $\alpha^{(j+1)}_{m-2-j,j+1}$ for $j = 0, 1, \ldots, m-3$ with $\alpha^{(\sigma(m))}_{0,\sigma(m)}$ replaced by $\alpha^{(\sigma(m+1))}_{0,\sigma(m+1)}$. Therefore the superdiagonal of A has the form

$$\begin{aligned}(\alpha_{m-1,0}, \alpha^{(1)}_{m-2,1}, \alpha^{(2)}_{m-3,2}, \alpha^{(3)}_{m-4,3}, \ldots, \alpha^{(m-2)}_{1,m-2}, \alpha^{(\sigma(m+1))}_{0,\sigma(m+1)}, \\ -\alpha^{(m-2)}_{1,m-2}, \ldots, -\alpha^{(2)}_{m-3,2}, -\alpha^{(1)}_{m-2,1}, -\alpha_{m-1,0}).\end{aligned} \tag{7.3.42}$$

This completes the induction step for the overarching induction hypothesis (Induction Hypothesis 7.3.3). Given $A \in (\mathrm{Sol}_{\xi}^{J})_0$ then A has superdiagonal of the form (7.3.42). The meromorphic data

$$(\nu_1, \ldots, \nu_m) = (\alpha_{m-1,0},\ \alpha_{m-2,1},\ \alpha_{m-3,2}, \ldots,\ \alpha_{1,m-3},\ \alpha_{0,\sigma(m+1)}) \in \mathcal{M}(M)^m$$

can be found from the elements of A, see Remark 7.3.2. $\square$

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