Delay Systems with Variable Delay

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This work is dedicated

to my husband, my father, my mother and to

my mother in law

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Abstract

In this thesis, we investigate questions about the ways to analyse the stability of input-output systems with delays that may be variable.

After detailing the necessary background, the focus switches to analysing the H^∞-stability of the transfer function. This stability depends on whether the transfer function is bounded in the right half plane. Moreover, a generalisation of the Walton-Marshall method [39] is given for matrices and some operator cases.

Then, the focus becomes on different kinds of stability, BIBO stability and H^∞-stability, of variable delay systems. Via a convenient extension of results on such stability notions due to Bonnet and Partington [5], it becomes possible to consider a more general version of stability, which is L^p -stability for all $1 \leq p \leq \infty$.

Next, by changing the variables, the variable delay system can often be transformed into an ordinary system with weights. The transformed equation can sometimes be solved and we discuss the instability that makes the output of the system not lie in L^2 whereas its input is in L^2.

After that, the main focus is on the stability, which is BIBO stability and L^2-stability, of autonomous and non-autonomous systems without delay but with weights, g_1 and g_2, under suitable conditions on g_1 and it is illustrated by special examples. Furthermore, we give some results of types of stability that link the output of the ordinary delay system with the output of the variable delay system. Last, we get results involving stability using weighted L^2 spaces which correspond by the Laplace transform to Zen spaces on the half plane. We extend the theory of Zen spaces

(weighted Hardy/Bergman spaces on the right-hand half-plane) to the Hilbert-space valued case, and describe the multipliers on them; it is shown that the methods of H^∞ control can therefore be extended to certain weighted L^2 input and output spaces.

Finally, we give some suggestions for further research.

Contents

Chapter 1

Background

1.1 Introduction

In this thesis, we investigate questions about the ways to analyse the stability of input-output systems with delays that may be variable. This investigation divides the thesis into six chapters of which four present ways to analyse the stability and they are linked to each other directly or indirectly.

After detailing the necessary background in Chapter 1, the focus switches in Chapter 2 to analysing the H^∞-stability of the retarded ordinary delay system with constant delays and operator-valued transfer function. This is achieved by developing an extension of the Walton-Marshall technique [39, 47], originally presented in the purely scalar case, to matrices and some operator cases. Therefore, we start with the main result of the second chapter which is concerned with the transfer function in operator-valued H^∞. Then we adapt their methods to study a system with bounded operators, which requires us to consider the spectrum of the operators. From this we have a complex version of the Walton-Marshall formula. Additionally, we have a simple result about subnormal operators.

From analysing the H^∞-stability of ordinary delay systems, we move in the third chapter to looking at different kinds of stability of variable delay systems. This can be approached by considering the stability of an ordinary delay system that is very close to the variable delay one in order to ensure similar properties under specific conditions. Because of this we looked at a particular paper of Bonnet and Partington [5] in order to make extensions to the BIBO stability and H^∞-stability results covered in their paper and to consider a more general version of stability, namely L^p stability for $1 \leq p \leq \infty$. Additionally, we analyse the three main versions of stability for normal or subnormal operators.

Next, by changing the variables, the variable delay system mentioned in the previous chapter can often be transformed into an ordinary system with weights and constant delays, which may enable us to analyse the stability of the varying delay system. The transformed equation can sometimes be solved and we discuss the nature of instabilities that make the output of the system fail to lie in L^2 (or L^∞) whereas its input is in L^2 (or L^∞).

The main result of the fifth chapter is a theorem that combines two results. The first one is due to Jacob, Partington and Pott [29], which will be seen as a scalar version of our results. It involves stability using weighted L^2 spaces which correspond by the Laplace transform to Zen spaces on the half plane. However, we extend the theory of Zen spaces (weighted Hardy/Bergman spaces on the right-hand half-plane) to the Hilbert-space valued case. The second result that we are generalizing is Plancherel's Theorem [2, Thm. 1.8.2] which is for Hilbert-space valued functions, but with different spaces. Then we describe the multipliers on weighted Hardy/Bergman spaces on the right-hand half-plane. It is shown that the methods of H^∞ control can therefore be extended to certain weighted L^2 input and output spaces. Additionally, we focus on the BIBO stability and L^2-stability of autonomous and non-autonomous systems without delay but with weights, g_1 and g_2, under suitable conditions on g_1 and this is illustrated by special examples. Furthermore, we give some results on types of stability that link the output of the ordinary delay system with the output of the variable delay system.

Finally, we give some suggestions for further research.

Now we start by mentioning some essential background. The thesis combines ideas from operator theory, complex analysis, and linear systems theory, and so this chapter gives the basic concepts and results used later. These include Hardy spaces, Laplace transforms, semigroups, normal operators, subnormal operators, stability of basic linear systems and the theory of delay systems. There will be no new results and therefore we state most of the theorems without proof.

1.2 Hardy Spaces and Laplace Transforms

Some definitions are needed, which can be found in [2, 16, 27, 34, 37, 39, 41, 42, 48].

Definition 1.2.1. *If $1 \leq p < \infty$ and if f is a complex measurable function on X, where X is an arbitrary measure space with a positive measure μ, define*

$$\|f\|_p = \left\{ \int_X |f|^p \quad d\mu \right\}^{\frac{1}{p}},$$

and let $L^p(\mu)$ consist of all f for which

$$\|f\|_p < \infty,$$

where we identify two functions if they are equal almost everywhere. We call $\|f\|_p$ the L^p-norm of f.

Theorem 1.2.2. *(Hölder's Inequality) If p and q are conjugate exponents, i.e. $\dfrac{1}{p} + \dfrac{1}{q} = 1$, $1 < p < \infty$, and if $f \in L^p(\mu)$ and $g \in L^q(\mu)$, then $fg \in L^1(\mu)$, and*

$$\|fg\|_1 \leq \|f\|_p \|g\|_q.$$

Definition 1.2.3. *Let $\mathcal{X}$ be a Banach space; for $1 \leq p < \infty$ the Hardy space $H^p(\mathbb{C}_+; \mathcal{X})$ of the right half plane $\mathbb{C}_+$ may be defined as the set of all analytic functions $f : \mathbb{C}_+ \to \mathcal{X}$ such that*

$$\|f\|_p = \left(\sup_{x>0} \int_{-\infty}^{\infty} \|f(x+iy)\|^p \ dy \right)^{1/p} < \infty.$$

Likewise, the space $H^\infty(\mathbb{C}_+;\mathcal{X})$ consists of all analytic and bounded functions in $\mathbb{C}_+$, and the norm is given by

$$\|f\|_\infty = \sup_{z\in\mathbb{C}_+} \|f(z)\|.$$

Definition 1.2.4. *For suitable functions z defined on $(0,\infty)$ the Laplace transform $\hat{z} = \mathcal{L}z$, is given by*

$$\hat{z}(s) = \int_0^\infty z(t)\mathrm{e}^{-st}\, dt \quad (s \in \mathbb{C}_+).$$

Theorem 1.2.5. *(Paley-Wiener) Let H be a Hilbert space then*

$$\|\hat{z}\|_{H^2(\mathbb{C}_+;H)} = \sqrt{2\pi}\|z\|_{L^2(0,\infty;H)}.$$

Theorem 1.2.6. *Let $\mathcal{X}$ be a Banach space then*

$$\|\hat{z}\|_{H^\infty(\mathbb{C}_+;\mathcal{X})} \leq \|z\|_{L^1(0,\infty;\mathcal{X})}.$$

Proof. Suppose $z \in L^1(0,\infty;\mathcal{X})$ and $\hat{z}(s) = \int_0^\infty z(t)\mathrm{e}^{-st}\, dt$. Then for all $s \in \mathbb{C}_+$,

$$\begin{aligned}\|\hat{z}(s)\| &= \left\|\int_0^\infty z(t)\mathrm{e}^{-st}\, dt\right\| \\ &\leq \max_{t\geq 0}\left|\mathrm{e}^{-st}\right| \int_0^\infty \|z(t)\|\, dt \\ &\leq \int_0^\infty \|z(t)\|\, dt = \|z\|_{L^1(0,\infty;\mathcal{X})}.\end{aligned}$$

Since this holds for all $s \in \mathbb{C}_+$, we have the result. □

1.3 Semigroups

From [39, p. 22], we introduce:

Definition 1.3.1. *Let $\mathcal{X}$ be a Banach space. Then a strongly continuous semigroup (or C_0-semigroup) $(\mathrm{T}(t))$ is a collection of bounded operators $\{\mathrm{T}(t) : t \in \mathbb{R},\ t \geq 0\}$, satisfying the following conditions:*

1. $\mathrm{T}(0) = 1$, *the identity operator on* $\mathcal{X}$.

2. $\mathrm{T}(s)\mathrm{T}(t) = \mathrm{T}(s+t)$ *for every* $s,\ t \geq 0$.

3. *The mapping from* $\mathbb{R}_+$ *into* $\mathcal{X}$ *defined by* $t \to \mathrm{T}(t)x$ *is continuous for every* $x \in \mathcal{X}$.

One important and easy example is obtained by defining $\mathrm{T}(t) = \mathrm{e}^{At}$, *where* A *is a fixed bounded operator on* $\mathcal{X}$.

Definition 1.3.2. *Let* $\mathrm{T}(t)$ *be a* C_0 *semigroup on a Banach space* $\mathcal{X}$. *Then its infinitesimal generator is the linear operator* $A : \mathcal{D}(A) \to \mathcal{X}$ *defined by*

$$Ax = \lim_{h \to 0^+} \frac{\mathrm{T}(h)x - x}{h},$$

with domain $\mathcal{D}(A) \subseteq \mathcal{X}$ *given by* $\mathcal{D}(A) = \left\{ x \in \mathcal{X} : \lim_{h \to 0^+} \frac{\mathrm{T}(h)x - x}{h} \text{ exists} \right\}$.

Some facts in [17, p.16, 36, 57, 112, 302] are needed.

Proposition 1.3.3. *If* $A \in \mathrm{L}(\mathcal{X})$ *where* $\mathcal{X}$ *is a complex Banach space with norm* $\|\cdot\|$ *and* $\mathrm{L}(\mathcal{X})$ *is the Banach algebra of all bounded linear operators on* $\mathcal{X}$ *endowed with the operator norm, which again is denoted by* $\|\cdot\|$, *then* $(\mathrm{e}^{tA})_{t\geq 0}$ *is a semigroup on* $\mathcal{X}$ *such that*

$$\mathbb{R}_+ \ni t \mapsto T(t) := \mathrm{e}^{tA} \in (\mathrm{L}(\mathcal{X}), \|\cdot\|),$$

is continuous.

Definition 1.3.4. *A one-parameter semigroup* $(T(t))_{t\geq 0}$ *on a Banach space* $\mathcal{X}$ *is called uniformly continuous (or norm continuous) if*

$$\mathbb{R}_+ \ni t \mapsto T(t) \in \mathrm{L}(\mathcal{X}),$$

is continuous with respect to the uniform operator topology on $\mathrm{L}(\mathcal{X})$.

Definition 1.3.5. *A strongly continuous semigroup* $(T(t))_{t\geq 0}$ *is called eventually norm continuous if there exists* $t_0 \geq 0$ *such that the function* $t \mapsto T(t)$ *is norm continuous from* (t_0, ∞) *into* $\mathrm{L}(\mathcal{X})$*. The semigroup is called immediately norm continuous if* t_0 *can be chosen to be* $t_0 = 0$*.*

With this terminology, we can restate Proposition (1.3.3) by saying that $(\mathrm{e}^{tA})_{t\geq 0}$ is a uniformly continuous semigroup for any $A \in \mathrm{L}(\mathcal{X})$. The converse is also true.

1.4 Normal Operators

Definition 1.4.1. *To any linear operator* A *we associate its spectral bound defined by*

$$s(A) := \sup\{Re\lambda : \lambda \in \sigma(A)\},$$

where the spectrum of A *is*

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \quad \textit{not invertible}\}, \tag{1.1}$$

and if A *is a finite matrix, which is a bounded operator, then* $\sigma(A)$ *will be the set of eigenvalues.*

The following standard facts can be found in [33, Chapter 1].

Definition 1.4.2. *If* A *and* B *are two linear operators on the vector space* X *to the vector space* Y*, their linear combination* $\alpha B + \beta A$ *is defined by*

$$(\alpha B + \beta A)u = \alpha(Bu) + \beta(Au),$$

for all $u \in X$*, and is again a linear operator on* X *to* Y*. Let us denote by* $\mathrm{L}(X, Y)$ *the set of all operators on* X *to* Y*;* $\mathrm{L}(X, Y)$ *is a vector space with the bounded linear operations defined as above.*

Definition 1.4.3. *$A \in \mathrm{L}(H)$ is said to be normal if A and A^* commute :*

$$A^*A = AA^*.$$

This is equivalent to

$$\|A^*u\| = \|Au\| \quad \text{for all } u \in H.$$

An important property of a normal operator A is that

$$\|A^n\| = \|A\|^n, n = 1, 2, \ldots.$$

This implies in particular that (r denotes the spectral radius) $r(A) = \|A\|$ where

$$r(A) = \max\{|\lambda| : \lambda \in \sigma(A)\}.$$

We have also the following properties:

- If A is normal, $P(A)$ is normal for any polynomial P.
- A^{-1} is normal if A is normal and nonsingular.
- If A is a bounded normal operator, then its semigroup (e^{At}) consists of normal operators. Additionally, by the spectral mapping theorem in [14, Chapter VII]

$$\sigma(\mathrm{e}^{At}) = \{\mathrm{e}^{\lambda t} : \lambda \in \sigma(A)\}.$$

From the spectral theorem we have [15, p.911]:

Corollary 1.4.4. *Let A be a normal operator in the Hilbert space H. Then there exists a regular positive measure space (S, Σ, μ), a bounded μ-measurable scalar function f on S , and U unitary mapping H onto $L^2(S, \Sigma, \mu)$ which preserves the inner product and is such that*

$$A = U^* M_f U,$$

where $M_f : L^2(S) \to L^2(S)$ *is the multiplication operator such that* $M_f u(x) = f(x)u(x),\ x \in S$.

Remark 1.4.5. *For* A *a normal operator we can prove* $\|e^{At}\| = e^{at}$ *where* $s(A) = a$. *That requires the existence of a multiplication operator* M_f *such that* $A = U^* M_f U$, *so* A *is unitarily equivalent to* M_f, *on* X, *where* f *is a bounded function on* X*;*

$$\sigma(A) = \sigma(M_f) = \overline{\{f(x) : x \in X)\}} \subset \{\lambda \in \mathbb{C}_- : \ Re\lambda \leq s(A) = a\}.$$

Therefore, $\|e^{At}\| = \|M_{e^{f(t)}}\|$ *and so*

$$\begin{aligned} \|e^{At}\|_{t>0} =& \sup\{|e^{f(x)t}| : x \in X\}, \\ =& e^{t\sup(Ref)}, \\ =& e^{at}. \end{aligned}$$

Remark 1.4.6. *For the finite-dimensional case the matrix* A *is diagonalizable with respect to an orthonormal basis if and only if it is normal. An infinite diagonal matrix gives a normal operator.*

1.5 Subnormal Operators

The following standard facts can be found in [6] and [9].

Definition 1.5.1. *An operator* A *acting on a Hilbert space* K *is said to be subnormal if, on some space* H *containing* K, *there exists a normal operator* B *such that* $Bf = Af$ *for every f in* K*; then* B *is called a normal extension of* A. *Equivalently,* A *is a subnormal on* K, *a subspace of* H, *if the normal operator* B, *acting on* H, *leaves* K *invariant, and* A *is the restriction of* B *to* H.

Next, some relationships exist between various concepts associated with a subnormal operator and the corresponding constructs of its normal extension. The normal extension B acting on H, is a minimal normal extension of A if it has the property that no proper subspace L with $K \subset L \subset H$ satisfies that $B|L$ is a normal extension.

Proposition 1.5.2. *1. $\sigma(B) \subset \sigma(A)$ where $\sigma(T)$ denotes the spectrum of the operator T.*

2. $\sigma(A) \subset \sigma(B) \cup H(B)$ where $H(B) = \bigcup_{n=1}^{\infty} U_n$; where each U_n is a bounded component of $\mathbb{C} \setminus \sigma(B)$ (holes in $\sigma(B)$).

3. $\|f(A)\| = r(f(A)) = r(f(B)) = \|f(B)\|$ where f is any function in one variable that is analytic on a neighbourhood of $\sigma(A)$ where $r(T)$ denotes the spectral radius of the operator T.

1.6 Stability

As mentioned in [38], we are concerned here with linear operators (system operators) R defined on $L^p(0,\infty)$ for some $1 \leq p \leq \infty$. Conventionally, we shall write $y = Ru$, where $u, y \in L^p$, and u is called the input and y the output of the system.

A time-invariant convolution operator in continuous time on L^p can be defined by

$$y(t) = (R_g u)(t) = (g * u)(t) = \int_0^t g(t-\tau)u(\tau)\, d\tau,$$

where g is called the impulse response. Additionally, we associate $u(t)$ with its Laplace transform $\hat{u} = \mathcal{L}u$, where

$$\hat{u}(s) = \int_0^\infty \mathrm{e}^{-st}u(t)\, dt,$$

and similarly let $G = \hat{g} = \mathcal{L}g$ and $\hat{y} = \mathcal{L}y$. Then $\hat{y}(s) = G(s)\hat{u}(s)$ where $G(s)$ is called the transfer function. These definitions apply to both time-varying and time-invariant systems.

Definition 1.6.1. *BIBO Stability means if $u \in L^\infty(0,\infty;H)$ then $y \in L^\infty(0,\infty;H)$. It implies that $R_g : L^\infty \to L^\infty$ is bounded using the closed graph theorem.*

Definition 1.6.2. *L^1 Stability means if $u \in L^1(0,\infty;H)$ then $y \in L^1(0,\infty;H)$. It implies that $R_g : L^1 \to L^1$ is bounded.*

Definition 1.6.3. *L^2 Stability means if $u \in L^2(0,\infty;H)$ then $y \in L^2(0,\infty;H)$. It implies that $R_g : L^2 \to L^2$ is bounded.*

The next theorem says that for time invariant systems BIBO stability is the same as L^1 stability.

Theorem 1.6.4. *For $p = 1$ and ∞, the (continuous-time) convolution operator*

$$R_g : L^p(0,\infty;H) \to L^p(0,\infty;H),$$

is bounded if and only if $g \in L^1(0,\infty;\mathrm{L}(H))$: if so, then $\|R_g\| = \|g\|_1$. For $p = 2$ the operator R_g is bounded if and only if $G(s) \in H^\infty(\mathbb{C}_+,\mathrm{L}(H))$: if so, then $\|R_g\| = \|G\|_\infty$.

1.6.1 The Stability of Semigroups

There are different kinds of stability [17, p. 18, 296, 298].

Definition 1.6.5. *(uniform exponential stability)*
A semigroup $(T(t))_{t\geq 0}$ on a Banach space $\mathcal{X}$ is called uniformly exponentially stable if there exist constants $\epsilon > 0$, $M \geq 1$ such that

$$\|T(t)\| \leq M\mathrm{e}^{-\epsilon t},$$

for all $t \geq 0$.

Definition 1.6.6. *(exponential stability)*
A strongly continuous semigroup $(T(t))_{t\geq 0}$ with the generator $(A, D(A))$ is called exponentially stable if there exists $\epsilon > 0$ such that

$$\lim_{t\to\infty} \mathrm{e}^{\epsilon t}\|T(t)x\| = 0,$$

for all $x \in D(A)$.

Definition 1.6.7. *A strongly continuous semigroup $(T(t))_{t\geq 0}$ is called uniformly stable if*

$$\lim_{t\to\infty} \|T(t)\| = 0.$$

Remark 1.6.8. *From the previous definitions, the following can be deduced:*

(1) $(T(t))_{t\geq 0}$ is uniformly exponentially stable if and only if it is uniformly stable.

(2) If $(T(t))_{t\geq 0}$ is uniformly exponentially stable or uniformly stable, then it is exponentially stable.

Theorem 1.6.9. *An eventually norm-continuous semigroup $(T(t))_{t\geq 0}$ is uniformly exponentially stable if and only if the spectral bound $s(A)$ of its generator A satisfies*

$$s(A) < 0.$$

The following facts are given in [39, chapter 2].

Theorem 1.6.10. *Let A be the infinitesimal generator of a C_0 semigroup $(T(t))$ defined on $\mathcal{X}$. Then A is a closed operator, and $\mathcal{D}(A)$ is dense in $\mathcal{X}$.*

Definition 1.6.11. *(The Mild Solution)*
If A is the infinitesimal generator of a C_0 semigroup $(T(t))$ on $\mathcal{X}$, then the function $t \to z(t)$ is said to be a mild solution to the Cauchy problem

$$\frac{dz(t)}{dt} = Az(t) \qquad t > 0, \quad z(0) = z_0, \tag{1.2}$$

if $z_0 \in \mathcal{X}$ and $z(t) = T(t)z_0$.

Example 1.6.12. *[10, chapter 2](Here, A is an unbounded operator generating a semigroup in the Cauchy Problem, the Heat Equation).*

Equation (1.2) comes out of the heat equation

$$\begin{aligned}&\frac{\partial z}{\partial t}(y,t) = \frac{\partial^2 z}{\partial y^2}(y,t),\ z(y,0) = z_0(y),\\&\frac{\partial z}{\partial y}(0,t) = 0 = \frac{\partial z}{\partial y}(1,t),\end{aligned} \tag{1.3}$$

where $z(y,t)$ *represents the temperature at position* y *and time* t*, and* $z(.,t) \in L^2(0,1)$ *for each* $t \geq 0$*. Here* $z_0(y)$ *is the initial temperature, and we take the operator* A *on* $L^2(0,1)$ *to be*

$$\begin{aligned}Aw &= \frac{d^2w}{dy^2},\ \text{with}\\ D(A) &= \{w \in L^2(0,1)|w, \frac{dw}{dy}\ \text{are absolutely continous,}\\ &\quad \frac{d^2w}{dy^2} \in L^2(0,1)\ \text{and}\ \frac{dw}{dy}(0) = 0 = \frac{dw}{dy}(1)\}.\end{aligned}$$

A *has the eigenvalues* $\lambda_n = -n^2\pi^2,\ n \geq 0$ *and so* A *is given by an infinite diagonal matrix as follows:*

$$A = \begin{bmatrix} 0 & 0 & 0 & \dots & 0 & \dots \\ 0 & -\pi^2 & 0 & \dots & 0 & \dots \\ 0 & 0 & -4\pi^2 & 0 & \dots & \\ 0 & \dots & 0 & \ddots & 0 & \dots \\ & & \dots\dots\dots\dots & & & \\ 0 & 0 & 0 & 0 & \dots & \dots \end{bmatrix},$$

The set of eigenvalues is unbounded, which makes A *an unbounded operator. However, they lead to a uniformly bounded even compact semigroup*

$$T(t) = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 & \dots \\ 0 & e^{-\pi^2 t} & 0 & \dots & 0 & \dots \\ 0 & 0 & e^{-4\pi^2 t} & 0 & \dots & \\ 0 & \dots & 0 & \ddots & 0 & \dots \\ & & \dots\dots\dots\dots & & & \\ 0 & 0 & 0 & 0 & \dots & \dots \end{bmatrix},$$

where $\|T(t)\| \leq 1$. *Because of that, the solution of (1.2),*

$$z(t) = \mathrm{e}^{At} z_0, \quad z_0 \in \mathcal{D}(A),$$

is bounded. Similarly the equation

$$\frac{\partial z}{\partial t}(y,t) = \frac{\partial^2 z}{\partial y^2}(y,t) + u(y,t),$$

can be written as

$$\frac{dz(t)}{dt} = Az(t) + Bu(t),$$

where $B = I$.

1.7 Solutions of Basic Linear Systems

First, equations with known solutions are identified, primarily from [39].

1.7.1 Equation Without Delay

Finite-Dimensional Case

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \ t \in [0, \infty). \tag{1.4}$$

Here $x(t) \in \mathbb{C}^n$ is a vector, $u(t) \in \mathbb{C}^m$, A is an $n \times n$ matrix and B is an $n \times m$ matrix. The solution for equation (1.4) using a Laplace transform is

$$\begin{aligned} s\hat{x}(s) &= A\hat{x}(s) + B\hat{u}(s), \\ (sI - A)\hat{x}(s) &= B\hat{u}(s), \\ \hat{x}(s) &= (sI - A)^{-1} B\hat{u}(s), \end{aligned}$$

here $(sI - A)^{-1}B$ is the transfer function. $\hat{x}$ and $\hat{u}$ are the Laplace transforms of x and u respectively.

Additionally, equation (1.4) can be multiplied by the exponential e^{-At} and solved, where A is a matrix and the solution as follows:

$$\mathrm{e}^{-At}\left(\frac{dx}{dt} - Ax(t)\right) = \mathrm{e}^{-At}Bu(t),$$

$$\left(\mathrm{e}^{-At}x(t)\right)' = \mathrm{e}^{-At}Bu(t),$$

$$x(t) = \mathrm{e}^{At}\int_0^t \mathrm{e}^{-Ay}Bu(y) \quad dy,$$

$$x(t) = \int_0^t \mathrm{e}^{A(t-y)}Bu(y) \quad dy, \tag{1.5}$$

which is a convolution between the function $u(y)$ and the exponential.

Infinite-Dimensional Case

More generally we have equation (1.4) with $x(t) \in X, u(t) \in U$ (where X and U are Hilbert spaces), $A : X \to X$ and $B : U \to X$. If A is a bounded (continuous) operator, then $\mathrm{e}^{kA} = \sum_{n=0}^{\infty} \frac{k^n}{n!}A^n$ will be defined to allow A to be used to write e^{kA} where k is a scalar and (1.5) will be the solution of equation (1.4). However, later we want to do this when A is not bounded. That will be more difficult as we have to think what we mean by the exponential of an operator and use semigroups.

1.7.2 Equation with a Constant Delay

The theory of delay differential systems

$$\begin{aligned} &p_m x^{(m)}(t) + p_{m-1}x^{(m-1)}(t) + ... + p_1\dot{x}(t) + p_0 x(t) \\ &\qquad + q_n x^{(n)}(t-h) + q_{n-1}x^{(n-1)}(t-h) + ... + q_1\dot{x}(t-h) + q_0 x(t-h) = u(t), \end{aligned}$$

is analysed in [3].

The General Case

More generally we can consider the case

$$\dot{x}(t) = Ax(t-\tau) + Bu(t), \tag{1.6}$$

and so

$$(sI - Ae^{-s\tau})\hat{x}(s) = B\hat{u}(s),$$
$$\hat{x}(s) = (sI - Ae^{-s\tau})^{-1}B\hat{u}(s).$$

Here, $\hat{x}(s)$ is obtained when $\hat{u}(s)$ is known, but explicit calculation of $x(t)$ is more difficult. An example of such a case can be found in [39, p.123].

A Special Case

When $A = -1$, τ is constant, $B = 1$ and $u(t) = \mathrm{e}^{-t}$, equation (1.6) transforms into

$$\dot{x}(t) = -x(t-\tau) + \mathrm{e}^{-t}, \quad x(0) = 0.$$

This is a normal delay equation that can be solved as follows:

$$\begin{aligned}
s\hat{x}(s) &= -\mathrm{e}^{-s\tau}\hat{x}(s) + \frac{1}{1+s}, \\
\hat{x}(s) &= \frac{1}{(1+s)} \cdot \frac{1}{(\mathrm{e}^{-s\tau}+s)}, \\
x(t) &= \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \mathrm{e}^{st}\left(\frac{1}{(1+s)} \cdot \frac{1}{(\mathrm{e}^{-s\tau}+s)}\right)\, ds.
\end{aligned}$$

1.8 Stability of Basic Linear Systems

1.8.1 Undelayed Case

The stability of the equation

$$\dot{x}(t) = Ax(t) + Bu(t), \quad x(0) = 0, \tag{1.7}$$

was studied in [39, Chapter 6] and [38, Chapter 8] for different statuses of A and B. Here, we mention some of this study as follows:

- A, B Scalars

  Laplace transform of equation (1.7)

  $$\begin{aligned} s\hat{x}(s) &= A\hat{x}(s) + B\hat{u}(s), \\ \hat{x}(s) &= \frac{B\hat{u}(s)}{s - A}. \end{aligned}$$

  This shows that the property of stability can not hold if $A \geq 0$. Because of the singularity of A, $\hat{x}$ is not analytic in $\mathbb{C}_+$ which means $\hat{x} \notin H^2$ for some choices of $\hat{u}$. Thus $x \notin L^2(0, \infty)$.

- A, B Matrices

  The transfer function of equation (1.7)

  $$\hat{x}(s) = (sI - A)^{-1}B\hat{u}(s),$$

  shows that if A has eigenvalues in $\mathbb{C}_+$, $\hat{x}$ will not be stable.

- A, B Operators and A Generates a Semigroup $(T(t))_{t \geq o}$

The solution of equation (1.7) for operators will be

$$x(t) = \int_0^t T(t-y)Bu(y)\ dy,$$

where [39, p 22]

$$\|T(t)\| \le M\mathrm{e}^{\omega t},\ \ M > 0 \text{ constant},\ \ \omega \in \mathbb{R}.$$

If $\omega > 0$, then the stability can not be guaranteed. Whereas, if $\omega < 0$, then the stability can be obtained.

1.8.2 Delay System Case

- A Bounded Operator

The stability of delay systems was analysed in [36, p. 4], where the equation

$$\dot{x}(t) = Ax(t-\tau) + Bu(t), \tag{1.8}$$

was considered with A and B bounded operators. By taking Laplace transforms as in (1.6) we obtain

$$\hat{x}(s) = (sI - A\mathrm{e}^{-s\tau})^{-1}B\hat{u}(s).$$

So

$$\begin{aligned} \hat{x} \in H^2(\mathbb{C}_+;H) \text{ for all } \hat{u} \in H^2(\mathbb{C}_+;H) &\iff (sI - A\mathrm{e}^{-s\tau})^{-1}B \text{ is bounded in } \mathbb{C}_+, \\ &\iff (sI - A\mathrm{e}^{-s\tau})^{-1}B \in H^\infty(\mathbb{C}_+;H), \\ &\Rightarrow (sI - A\mathrm{e}^{-s\tau})^{-1}B \ \text{ exists } \ \forall s;\ \ s \in \mathbb{C}_+, \end{aligned}$$

if $B = I$

$$\begin{aligned} &\Rightarrow A\mathrm{e}^{-s\tau} - sI \quad \text{is always invertible for } s \in \mathbb{C}_+, \\ &\iff A - s\mathrm{e}^{s\tau} I \quad \text{is always invertible for } s \in \mathbb{C}_+, \\ &\iff s\mathrm{e}^{s\tau} \notin \sigma(A) \quad \text{for } s \in \mathbb{C}_+. \end{aligned}$$

- A Unbounded Operator

  We can get the stability of equation (1.8) for unbounded operators, by using the steps that have been followed when A is a bounded operator.

1.8.3 Classification of Simple Delay System

If $G(s) = 1/(P(s) + Q(s)\mathrm{e}^{-sh})$ we describe the system as follows (see [39]):

$$\begin{aligned} &\deg P < \deg Q \qquad \text{advanced type} \\ &\deg P = \deg Q \qquad \text{neutral type} \\ &\deg P > \deg Q \qquad \text{retarded type.} \end{aligned}$$

1.8.4 Some Relevant Literature

There is a large literature on the stability of delay systems, analysed by various techniques. We mention in particular the following:

- [7] is a self-contained monograph that provides a large collection of results on linear parameter-varying systems.

- In [12] stability conditions for systems with large time-varying delays are provided under the assumption of the closeness of the delays instead of the delays' smallness.

- In [19], the input–output approach is extended to consider the stability of neutral type systems with uncertain time-varying delays and norm-bounded uncertainties.

- In [20] a stability criterion is derived in the general multiple delay case without any constraints on the delay derivative.

- In [21] the main results state that if certain linear matrix inequalities hold, then the system is Lyapunov stable.

- [22] gives sufficient conditions involving linear matrix inequalities for the stability and state feedback H^∞ control of of neutral systems with time varying delays.

- In [23] stability is discussed for linear time-delay systems assuming that the time-varying delay consists of two parts, an ordinary constant-delay and a time-varying perturbation delay.

- [31, 32] the stability of systems is studied in the presence of bounded uncertain time-varying delays in the loop.

- In [44], the authors study delay systems of the form

$$x'(t) = Ax(t) + A_d \int_0^\infty K(\theta)x(t-\theta)\ d\theta,$$

where $x \in \mathbb{R}^n$, A and $A_d \in \mathbb{R}_{n\times n}$ and $K(t)$ is a scalar.

Chapter 2

Location of Poles

2.1 Introduction

In this chapter we consider the differential equation

$$\dot{x}(t) + Ax(t-h) = u(t), \quad \text{for } t \geq 0, \tag{2.1}$$

where $x(t) = 0$ for $t \leq 0$ and similar equations, where $x(t), u(t) \in \mathcal{X}$ (where $\mathcal{X}$ is a Banach space), $A : \mathcal{X} \to \mathcal{X}$ is a bounded operator and $h \geq 0$ is the delay, and we mainly focus on analysing the H^∞ stability of the delay system with more general transfer function $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)^{-1}$ where $P(s), Q(s)$ are complex polynomials (see Subsection 1.8.2). This will depends on whether it has poles in the right half plane or not. The Walton-Marshall method [39] did such analysis in the purely scalar case. The method is based on the observation that the zeros of $P(s) + Q(s)e^{-sh}$ vary continuously with h (by Rouché's theorem) and so that if they cross from the left half-plane to the right half-plane or vice-versa, then there will be values of h for which they cross the axis. Moreover, the crossing points do not depend on h. This method was clarified in a proposition which shows that if $(P(s) + Q(s)\mathrm{e}^{-sh})$ has a zero at point $s \in i\mathbb{R}$ and the real polynomials $P(s), Q(s)$ are not zero then such an

s satisfies the equation

$$P(s)P(-s) = Q(s)Q(-s). \tag{2.2}$$

Additionally, at such point s the direction in which the zeros cross the axis is identified by

$$sgnRe\frac{ds}{dh} = sgnRe\frac{1}{s}\left[\frac{Q'(s)}{Q(s)} - \frac{P'(s)}{P(s)}\right]. \tag{2.3}$$

However, we develop the Walton-Marshall method further to apply to matrices and some operator cases. Some of this work will be published in [1].

We basically start with the easier case where A is a finite square matrix to study the stability of the transfer function $(sI + A\mathrm{e}^{-sh})^{-1}$. Using a theorem of Schur [28, Theorem 2.3.1] [28] we manage to adopt the Walton-Marshall method. The theorem says that A can be transformed into a triangular matrix T whose eigenvalues $\lambda_k; 1 \leq k \leq n$ are its diagonal entries, although they might be complex numbers. Therefore, by knowing the zeros of the determinant of the transfer function as it is equal to $\prod_{k=1}^{n}(sI + \lambda_k \mathrm{e}^{-sh})$ we find that the zeros cross the axis at $s = \pm i|\lambda_k|$ and from this we deduce that the system is stable on $(0, h)$ where $h = min\left(\frac{1}{|\lambda_k|}arg\left(i\frac{|\lambda_k|}{\lambda_k}\right), \frac{1}{|\lambda_k|}arg\left(i\frac{\lambda_k}{|\lambda_k|}\right)\right)$.

Then, we generalize the Walton-Marshall method for operators in different systems of ordinary constant-delay to get the formula

$$P(s)P(-s) = |\lambda|^2 Q(s)Q(-s), \tag{2.4}$$

which applies to $P(s)+\lambda Q(s)\mathrm{e}^{-sh}$ and $\lambda \in \mathbb{C}$ where $P(s)$, $Q(s)$ are real polynomials. The formula (2.4) is the same of the Walton-Marshall formula (2.2), however, the right side of (2.4) is multiplied by $|\lambda|^2$. Moreover, the direction of the zeros crossing is deduced to be given by (2.3) as in the work of Walton and Marshall. Furthermore, the improved formula is obtained easily when A is a bounded normal operator to study the stability of the system with a constant delay. In [15, Corollary X.5.4], it is shown that A is unitarily equivalent to a multiplication operator that tells us the spectrum of A and then the location of the poles of the transfer function for the delay system.

At the last but not least we introduce the H^∞ stability theorem that analyses the bounded operator case to know where the transfer function $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)^{-1}$ is bounded in the right half plane where $P(s), Q(s)$ are complex polynomials. The theorem does not apply unless $\deg P > \deg Q$, the case of a retarded delay system. Bonnet and Partington [4] give an interesting example clarifying how the stability can not be determined from the location of the poles when $\deg P = \deg Q$, a neutral delay system.

Proposition 2.1.1. *(**Walton-Marshall Method**) [47], [39, p.132] Let $P(s)$ and $Q(s)$ be real polynomials. If*

$$R_h(s) = P(s) + Q(s)\mathrm{e}^{-sh}, \tag{2.5}$$

where h is the delay in the delay system, has a zero at a point $s \in i\mathbb{R}$, and $P(s)$ and $Q(s)$ are not zero there, then such an s satisfies the equation

$$P(s)P(-s) = Q(s)Q(-s). \tag{2.6}$$

Moreover, at such a point s we have

$$sgnRe\frac{ds}{dh} = sgnRe\frac{1}{s}\left[\frac{Q'(s)}{Q(s)} - \frac{P'(s)}{P(s)}\right].$$

Example 2.1.2. *The H^∞ stability of the scalar delay system*

$$\dot{x}(t) = Ax(t-h) + Bu(t),$$

where $A, B \in \mathbb{R}$ gives

$$\hat{x}(s) = \frac{B\hat{u}(s)}{s - A\mathrm{e}^{-sh}},$$

and depends on the transfer function $\dfrac{B}{s - A\mathrm{e}^{-sh}}$ being in $H^\infty(\mathbb{C}_+)$.

By using the previous proposition we can show specifically for $A = -\lambda \in \mathbb{R};\ \lambda > 0$

and $B = 1$ when exactly the transfer function $\dfrac{1}{s+\lambda e^{-sh}}$ is bounded in $\mathbb{C}_+$ as follows [39, p. 132]:

Consider the denominator of the transfer function to be $R_h(s) = s + \lambda e^{-sh}$, which for $h = 0$ has no right half-plane zeros. Equation (2.6) indicates that imaginary axis zeros can occur only if $-s^2 = \lambda^2$; that is, if $s = \pm\lambda i$. It is only necessary to consider one of the conjugate pair, say $s = \lambda i$ and solving for h we have

$$\lambda i + \lambda e^{-\lambda i h} = 0, \quad \text{that is,} \quad h = \frac{\pi}{2\lambda} + 2n\pi, \quad n \geq 0.$$

Further we have

$$sgnRe \frac{ds}{dh} = sgnRe\left(-\frac{1}{s^2}\right) > 0,$$

indicating that zeros cross from left to right. We may deduce that $R_h(s)$ is stable (has no right half-plane zeros) and so $\dfrac{1}{s+\lambda e^{-sh}} = \dfrac{1}{R_h(s)}$ is bounded if and only if $0 \leq h < \pi/2\lambda$.

2.2 H^∞ Stability

In this section, we consider the H^∞ stability of retarded delay system with transfer function of the form $G(s) = \left(P(s)I + Q(s)Ae^{-sh}\right)^{-1}$, where P and Q are complex polynomials and A a bounded operator on a Banach space $\mathcal{X}$. We shall see that, even in the operatorial case, for systems of retarded type invertibility of $P(s)I+Q(s)e^{-sh}A$ is equivalent to the inverse function being in H^∞: this is true for retarded systems, but not for systems of neutral type.

Theorem 2.2.1. *If A is a bounded operator in Banach space $\mathcal{X}$ such that $h \geq 0$ and $P(s), Q(s)$ are complex polynomials with $\deg P > \deg Q$ then the following three statements are equivalent:*

(i) $\left(P(s)I + Q(s)Ae^{-sh}\right)^{-1} \in H^\infty(\mathrm{L}(\mathcal{X}))$.

(ii) $\left(P(s)I + Q(s)Ae^{-sh}\right)$ *is invertible* $\forall s \in \overline{\mathbb{C}}_+$.

(iii) $P(s)I + Q(s)\lambda \mathrm{e}^{-sh} \neq 0 \quad \forall s \in \overline{\mathbb{C}}_+, \ \forall \lambda \in \sigma(A).$

Proof. $(i) \implies (ii)$: The condition $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)^{-1} \in H^\infty(\mathrm{L}(\mathcal{X}))$ implies that $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)$ is always invertible for s in the closed half plane.

$(ii) \implies (iii)$: This is straightforward. The operator $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)$ is invertible $\forall s \in \mathbb{C}_+$ if and only if $0 \notin \sigma\left[P(s)I + Q(s)A\mathrm{e}^{-sh}\right]$; but for fixed s, we get $\sigma\left[P(s)I + Q(s)A\mathrm{e}^{-sh}\right] = \left\{P(s) + Q(s)\lambda\mathrm{e}^{-sh};\ \lambda \in \sigma(A)\right\}$ which means $P(s) + Q(s)\lambda\mathrm{e}^{-sh} \neq 0 \quad \forall s \in \mathbb{C}_+, \ \forall \lambda \in \sigma(A)$.

$(iii) \implies (i)$: Suppose $P(s) + Q(s)\lambda\mathrm{e}^{-sh} \neq 0 \quad \forall s \in \overline{\mathbb{C}}_+, \ \forall \lambda \in \sigma(A)$ and so $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)$ is invertible $\forall s \in \overline{\mathbb{C}}_+$. We show that the inverse is bounded as a function of s;

First: there is an $R > 0$ such that for $s \in \overline{\mathbb{C}}_+$ with $|s| > R$, we have

$$|P(s)| > |Q(s)| \|A\| |\mathrm{e}^{-sh}| + 1,$$

and so for $x \in \mathcal{X}$ we get

$$\|P(s)x\| > \left(|Q(s)| \|A\| |\mathrm{e}^{-sh}|\right) \|x\| + \|x\|,$$

and so

$$\|P(s)Ix + Q(s)A\mathrm{e}^{-sh}x\| \geq \|P(s)x\| - \left(|Q(s)| \|A\| |\mathrm{e}^{-sh}|\right) \|x\| \geq \|x\|.$$

That means

$$\| \left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)^{-1} \| \leq 1,$$

and so $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)^{-1}$ is bounded for $|s| > R, \ s \in \overline{\mathbb{C}}_+$.

Second, to prove $\left(P(s)I + Q(s)A\mathrm{e}^{-sh}\right)^{-1}$ is uniformly bounded for $|s| \leq R, \ s \in \overline{\mathbb{C}}_+$, we suppose not, so $\exists (x_n) \subset \mathcal{X}, \ \|x_n\| = 1$ and a sequence $(s_n) \subset S$ where

$S = \{s \in \overline{\mathbb{C}}_+ : |s| \le R\}$ such that

$$\left(P(s_n)I + Q(s_n)A\mathrm{e}^{-s_n h}\right) x_n \to 0,$$

and because S is a compact set then there is a subsequence $(s_{n_k})_{k\ge 0}$ and $s_0 \in S$ such that $(s_{n_k}) \to s_0$. Now $\|P(s_n)I + Q(s_n)A\mathrm{e}^{-s_n h} - P(s_0)I - Q(s_0)A\mathrm{e}^{-s_0 h}\| \to 0$ and so $\left(P(s_0)I + Q(s_0)A\mathrm{e}^{-s_0 h}\right) x_n \to 0$ which means that

$$0 \in \sigma\left(P(s_0)I + Q(s_0)A\mathrm{e}^{-s_0 h}\right),$$

so there exists a $\lambda \in \sigma(A)$ such that $P(s_0)I + Q(s_0)\lambda\mathrm{e}^{-s_0 h} = 0$.

□

Remark 2.2.2. *1. The result does not hold in general if A is unbounded (and in this case the linear system may even be destabilised by an arbitrarily small delay). For example, if A is a diagonal operator on a Hilbert space with orthonormal eigenvectors and eigenvalues $\lambda_n = (ni + 1/n)(n \ge 1)$, then $s + \lambda_n$ has no zeros in $\mathbb{C}_+$ but $(sI + A)^{-1}$ is unbounded on $\mathbb{C}_+$.*

2. The location of the poles of a neutral delay system does not determine its H^∞ stability; as the following example indicates [4].

Consider $R(s) = \frac{1}{s+1+s\mathrm{e}^{-s}}$. If $Res > 0$ then we cannot have $\mathrm{e}^{-s} = -1 - \dfrac{1}{s}$, since the left-hand side has modulus < 1 and the right-hand side has modulus strictly > 1; thus this system has no poles in $\mathbb{C}_+$, nor indeed on $i\mathbb{R}$ (as is easily verified), although it does have a sequence of poles z_n with $Imz_n \approx (2n+1)\pi$ and $Rez_n \to 0$. The system is not stable, as an analysis of its values at $s = i\left[(2n+1)\pi + \dfrac{1}{(2n+1)\pi}\right], n \in \mathbb{Z}$, shows that it is not even in H^∞.

2.3 Generalization of the Walton-Marshall Method for Operators in Different Systems of Constant Delay

In this section, we use the Walton-Marshall method to study the stability of different retarded systems of ordinary constant-delay given by the equation

$$\begin{aligned} &p_m x^{(m)}(t) + p_{m-1}x^{(m-1)}(t) + \dots + p_1\dot{x}(t) + p_0 x(t) \\ &+ A[q_n x^{(n)}(t-h) + q_{n-1}x^{(n-1)}(t-h) + \dots + q_1\dot{x}(t-h) + q_0 x(t-h)] = u(t), \end{aligned} \tag{2.7}$$

where $u \in L^2(0,\infty;H)$, $p_0,\dots,p_m$, $q_0,\dots,q_n \in \mathbb{R}$ and $A : H \to H$ is a bounded operator. The Walton-Marshall method requires writing equation (2.7) in its Laplace transform

$$\begin{aligned} &(p_m s^m + p_{m-1}s^{m-1} + \dots + p_1 s + p_0)\hat{x}(s) \\ &\qquad\qquad + A(q_n s^n + q_{n-1}s^{n-1} + \dots + q_1 s + q_0)\hat{x}(s)\mathrm{e}^{-sh} = \hat{u}(s), \end{aligned}$$

by putting $P(s) = (p_m s^m + p_{m-1}s^{m-1} + \dots + p_1 s + p_0)$ and $Q(s) = (q_n s^n + q_{n-1}s^{n-1} + \dots + q_1 s + q_0)$ where $\deg P > \deg Q$, we get

$$[P(s)I + AQ(s)\mathrm{e}^{-sh}]\hat{x}(s) = \hat{u}(s),$$

and so

$$\hat{x}(s) = [P(s)I + AQ(s)\mathrm{e}^{-sh}]^{-1}\hat{u}(s).$$

Then, we need $\hat{x}(s) \in H^2(\mathbb{C}_+;H)$ when $\hat{u}(s) \in H^2(\mathbb{C}_+;H)$. That means that $[P(s)I + AQ(s)\mathrm{e}^{-sh}]^{-1} \in H^\infty(\mathbb{C}_+;L(H))$ where $[P(s)I + AQ(s)\mathrm{e}^{-sh}]^{-1}$ will be the operator such that $[P(s)I + AQ(s)\mathrm{e}^{-sh}]^{-1} : H^2(\mathbb{C}_+;H) \to H^2(\mathbb{C}_+;H)$. We can study the stability of system (2.7) by knowing for which $s \in i\mathbb{R}$

$$P(s) + \lambda Q(s)\mathrm{e}^{-sh} = 0 \ \text{ for some } \ \lambda \in \sigma(A). \tag{2.8}$$

From (2.8), we get $P(s) = -\lambda Q(s)\mathrm{e}^{-sh}$ that gives for $\bar{s} = -s;\ s \in i\mathbb{R}$ equation $P(-s) = -\bar{\lambda}Q(-s)\mathrm{e}^{sh}$ and so

$$P(s)P(-s) = |\lambda|^2 Q(s)Q(-s), \tag{2.9}$$

where $P(s)$, $Q(s)$ are real polynomials. Formula (2.9) is the same of the Walton-Marshall formula (2.6), however, the right side of formula (2.9) is multiplied by $|\lambda|^2$. Solving (2.9) gives the required values of s and then from (2.8) the delay h can be obtained and for each λ. Finally, we can deduce the suitable interval of h when system (2.7) is stable.

2.4 The Walton-Marshall Method for Bounded Normal Operators in Different Systems of Constant Delay

In this section, we use the Walton-Marshall method to study the stability of system (2.7) where $x, u \in H$; here H is Hilbert space and $A : H \to H$ is a bounded normal operator. We need $\hat{x}(s) \in H^2(\mathbb{C}_+; H)$ when $\hat{u}(s) \in H^2(\mathbb{C}_+; H)$. That means $[P(s)I + AQ(s)\mathrm{e}^{-sh}]^{-1} \in H^\infty(\mathbb{C}_+, \mathrm{L}(H))$. Having $[P(s)I + AQ(s)\mathrm{e}^{-sh}]$ invertible, requires knowing the spectrum of A, $\sigma(A)$, where

$$\sigma(A) = \{\lambda \in \mathbb{C} : \lambda I - A \text{ not invertible}\}.$$

Recall from Corollary 1.4.4 that a normal operator A is unitarily equivalent to a multiplication operator M_f on a space $L^2(\Omega)$ with $f \in L^\infty(\Omega)$, so

$$\begin{aligned}
& 0 \in \sigma(P(s)I + Q(s)A\mathrm{e}^{-sh}), \\
\iff\ & 0 \in \{(P(s) + Q(s)\lambda\mathrm{e}^{-sh});\ \lambda \in \sigma(A)\}, \\
\iff\ & 0 \in \{(P(s) + Q(s)\lambda\mathrm{e}^{-sh});\ \lambda \in \sigma(M_f)\}, \\
\iff\ & 0 \in \{(P(s) + Q(s)\lambda\mathrm{e}^{-sh});\ \lambda \in \overline{\{f(x) : x \in \Omega\}}\}.
\end{aligned}$$

Then formula (2.9) can be obtained. Solving (2.9) gives the required values of s and then from

$$P(s) + \lambda Q(s)\mathrm{e}^{-sh} = 0, \tag{2.10}$$

the delay h can be obtained for each λ. Additionally, to know the direction of the zeros crossing that can be got by deriving (2.10) with respect of h where s is a function of h as follows:

$$\begin{aligned} P'(s)\frac{ds}{dh} + \lambda Q'(s)\frac{ds}{dh}\mathrm{e}^{-sh} - s\lambda Q(s)\mathrm{e}^{-sh} - h\lambda Q(s)\mathrm{e}^{-sh}\frac{ds}{dh} &= 0, \\ \left[P'(s) + \lambda Q'(s)\mathrm{e}^{-sh} - h\lambda Q(s)\mathrm{e}^{-sh}\right]\frac{ds}{dh} - s\lambda Q(s)\mathrm{e}^{-sh} &= 0 \end{aligned}$$

with $\lambda \mathrm{e}^{-sh} = -\dfrac{P(s)}{Q(s)}$, gives:

$$\left[P'(s) - \frac{Q'(s)}{Q(s)}P(s) + hP(s)\right]\frac{ds}{dh} = -sP(s),$$

and so

$$\frac{1}{-s}\left[\frac{P'(s)}{P(s)} - \frac{Q'(s)}{Q(s)} + h\right]\frac{ds}{dh} = 1.$$

Then

$$\frac{ds}{dh} = -s\left[\frac{P'(s)}{P(s)} - \frac{Q'(s)}{Q(s)} + h\right]^{-1}.$$

Now $sgn\ Re\ u = sgn\ Re\ u^{-1}$ for any $u \in \mathbb{C}\backslash 0$ and $\frac{h}{s}$ purely imaginary so

$$sgn\ Re(\frac{ds}{dh}) = sgn\ Re\frac{1}{s}\left(\frac{Q'(s)}{Q(s)} - \frac{P'(s)}{P(s)}\right). \tag{2.11}$$

If (2.11) greater than zero then the zeros go to $\mathbb{C}_+$ as h increases, but if (2.11) less than zero then the zeros go to $\mathbb{C}_-$. Finally, we can deduce the suitable interval of h when system (2.7) is stable. To summarize since even real matrices may have complex spectrum, we require a complex version of Proposition 2.1.1, as follows:

Proposition 2.4.1. *Let $P(s)$ and $Q(s)$ be real polynomials. If*

$$P(s) + \lambda Q(s)\mathrm{e}^{-sh},$$

where h is the delay in the system, has a zero for some $h \in \mathbb{R}, \lambda \in \mathbb{C}$ and $s \in i\mathbb{R}$, *and $P(s), Q(s)$ are not zero there, then such an s satisfies the equation*

$$P(s)P(-s) = |\lambda|^2 Q(s)Q(-s). \tag{2.12}$$

Moreover, at such a point s we have

$$sgnRe\frac{ds}{dh} = sgnRe\frac{1}{s}\left[\frac{Q'(s)}{Q(s)} - \frac{P'(s)}{P(s)}\right].$$

Proof. The proposition has already been proven (see (2.9)). □

Remark 2.4.2. *If $P(s)$ and $Q(s)$ in Proposition 2.4.1 are real polynomials, the equation (2.12) becomes*

$$P(s)\overline{P(s)} = |\lambda|^2 Q(s)\overline{Q(s)}, \tag{2.13}$$

that is by putting $P(-s) = \overline{P(s)}$ and $Q(-s) = \overline{Q(s)}$ and so (2.13) is obtained.

Example 2.4.3. *(Continuous Spectrum) Suppose in (2.7), we have the normal multiplication operator $A = M_{\frac{2}{1+y^2}}$ on $L^2(\mathbb{R})$; $f(y) = \dfrac{2}{1+y^2}$; $\{f(y) : y \in \mathbb{R}\} = (0, 2]$. Let us take $P(s) = s + \frac{1}{2}, Q(s) = 1$ and $\sigma(A) = \sigma(M_{\frac{1}{1+y^2}}) = \overline{\{f(y) : y \in \mathbb{R}\}} = \{\lambda : \lambda \in [0, 2]\}$. Then we consider the equation*

$$s + \frac{1}{2} + \lambda e^{-sh} = 0, \quad \textit{for each } 0 \le \lambda \le 2. \tag{2.14}$$

First, we need to study the stability for $h = 0$ from (2.14) where we get

$$s = -(\frac{1}{2} + \lambda) \notin \overline{\mathbb{C}}_+, \quad 0 \le \lambda \le 2,$$

therefore the undelayed system

$$\dot{x}(t) + \frac{1}{2}x(t) + Ax(t) = u(t)$$

is stable. Then, we need to study the stability for $h > 0$ and $0 \le \lambda \le 2$ where in (2.9)

$$s^2 = \frac{1}{4} - \lambda^2$$

and so

$$s = \pm\sqrt{\frac{1}{4} - \lambda^2}, \tag{2.15}$$

as follows:

- *For* $0 \leq \lambda < \frac{1}{2}$ *from (2.15) we get* $s = \pm\sqrt{\frac{1}{4} - \lambda^2} \notin i\mathbb{R}$*, so the system*

$$\dot{x}(t) + \frac{1}{2}x(t) + \lambda x(t-h) = u(t), \tag{2.16}$$

  is stable.

- *For* $\lambda = \frac{1}{2}$ *from (2.15) we get* $s = 0$*, so from (2.14) we require* h *where* $\mathrm{e}^{-sh} = -1$; $s = 0$*: there is no solution which means the system (2.16) is stable.*

- *For* $\frac{1}{2} < \lambda \leq 2$ *from (2.15) we get* $s = \pm i\sqrt{\lambda^2 - \frac{1}{4}}$*, so from (2.14) we can find* h *where* $\mathrm{e}^{-sh} = \frac{-(\frac{1}{2}+i\sqrt{\lambda^2-\frac{1}{4}})}{\lambda}$; $s = i\sqrt{\lambda^2 - \frac{1}{4}}$ *that gives* $\cos\left(h\sqrt{\lambda^2 - \frac{1}{4}}\right) = \frac{-1}{2\lambda}$ *and* $\sin\left(h\sqrt{\lambda^2 - \frac{1}{4}}\right) = \frac{\sqrt{\lambda^2-\frac{1}{4}}}{\lambda}$ *such that* $\sin^2\left(h\sqrt{\lambda^2 - \frac{1}{4}}\right) + \cos^2\left(h\sqrt{\lambda^2 - \frac{1}{4}}\right) = 1$ *and so*

$$h = \frac{\cos^{-1}(\frac{-1}{2\lambda})}{\sqrt{\lambda^2 - \frac{1}{4}}} + 2n\pi,\ n \in \mathbb{Z}$$

  and so h_{min} *can be obtained when* $\lambda = 2$ *that gives*

$$h_{min} = \frac{2\cos^{-1}(\frac{-1}{4})}{\sqrt{15}} \approx 0.941.$$

  Further from (2.11) we have

$$sgn\ Re(\frac{ds}{dh}) = sgn\ Re\frac{-1}{s}\left(\frac{1}{(s+\frac{1}{2})}\right),$$
$$sgn\ Re(\frac{ds}{dh}) = sgn\ Re\left(\frac{-1}{(s^2+\frac{1}{2}s)}\right),$$
$$sgn\ Re(\frac{ds}{dh}) = sgn\ Re\left(\frac{-1}{s^2}\right) > 0,$$

indicating that zeros cross from left to right. Then we may deduce the system

$$\dot{x}(t) + \frac{1}{2}x(t) + Ax(t-h) = u(t),$$

is stable if and only if $0 \leq h < 0.941$*. For* $\lambda = 0$*, the only zero from (2.14) is at* $s = -\frac{1}{2}$ *and these calculations are not used.*

2.5 Subnormal Operators

Let A be a subnormal operator with minimal normal extension N, we have from Theorem 2.2.1 that $(P(s)I + Q(s)A\mathrm{e}^{-sh})^{-1} \in H^\infty(L(\mathcal{X}))$ if and only if $P(s) + Q(s)\lambda\mathrm{e}^{-sh} \neq 0$ for all $\lambda \in \sigma(A)$ and for all $s \in \bar{\mathbb{C}}_+$. We refer to Section 1.5 for the necessary notation.

Proposition 2.5.1. *Under the hypotheses of Theorem 2.2.1 if* $P(s)+Q(s)\lambda\mathrm{e}^{-sh} \neq 0$ *for all* $s \in \overline{\mathbb{C}}_+$ *and for all* $\lambda \in \sigma(N) \cup H(N)$ *then* $(P(s)+Q(s)A\mathrm{e}^{-sh})^{-1} \in H^\infty(L(\mathcal{X}))$*. Conversely, if* $(P(s) + Q(s)A\mathrm{e}^{-sh})^{-1} \in H^\infty(L(\mathcal{X}))$ *then* $P(s) + Q(s)\lambda\mathrm{e}^{-sh} \neq 0$ *for all* $s \in \overline{\mathbb{C}}_+$ *and for all* $\lambda \in \sigma(N)$ *so that it gives a way of testing whether the transfer function is in* H^∞*.*

Proof. This follows from Proposition 1.5.2 (Section 1.5) and Theorem 2.2.1. □

2.6 The Walton-Marshall Method for Finite Square Matrices

As we have mentioned in the previous two sections, the formula (2.6) of Walton-Marshall, which is

$$P(s)P(-s) = Q(s)Q(-s),$$

where $P(s)$ and $Q(s)$ are real polynomials, is applied on (2.5), which is

$$R_h(s) = P(s) + Q(s)\mathrm{e}^{-sh},$$

to identify which values of $s \in i\mathbb{R}$ that make the system unstable and then the delay h can be found.

In this section, we will again study the stability of system (2.1) in the special case that A is any finite square real matrix. Studying the stability as we have mentioned in the previous section will be via applying the formula (2.6) on the determinant of our transfer function $sI + A\mathrm{e}^{-sh}$ that equals zero but here it will be done by matrix methods.

2.6.1 Transformation of A into a Triangular Matrix

We recall that the eigenvalues of a triangular matrix are its diagonal entries. Furthermore, the next theorem [28, P.79], asserts that any finite matrix can be transformed into an upper triangular matrix by conjugation with a unitary matrix. Recall that a matrix U is unitary if $U^*U = I$, or equivalently its columns are orthonormal.

Theorem 2.6.1. *(Schur). Given $A \in M_n$ with eigenvalues $\lambda_1, ..., \lambda_n$ in any prescribed order, there is a unitary matrix $U \in M_n$ such that*

$$U^*AU = T = [t_{ij}],$$

is upper triangular, with diagonal entries $t_{ii} = \lambda_i,\ i = 1, ..., n$. That is, every square matrix A is unitarily equivalent to a triangular matrix whose diagonal entries are the eigenvalues of A in a prescribed order. Furthermore, if $A \in M_n(R)$ and if all the eigenvalues of A are real, then U may be chosen to be real and orthogonal.

Because of that and since the determinant of T is the product of its eigenvalues, we get

$$\det(sI + A\mathrm{e}^{-sh}) = \det(sI + T\mathrm{e}^{-sh}) = \prod_{k=1}^{n}(sI + \lambda_k \mathrm{e}^{-sh}), \tag{2.17}$$

where λ_k is a real or complex eigenvalue of T, and T the triangular matrix of A. That means factorizing the determinant of $(sI + A\mathrm{e}^{-sh})$ gives us for each eigenvalue λ the formula (2.5) for which we can use Walton-Marshall formula (2.6) for λ_k real number by putting $P(s) = sI,\ P(-s) = -sI,\ Q(s) = Q(-s) = \lambda_k$ and so $-s^2 = \lambda_k^2$, that gives

$$s = \pm i\lambda_k. \tag{2.18}$$

However, we need to extend the Walton-Marshall formula to complex eigenvalues, so we do the following

- We have $s + \lambda_k \mathrm{e}^{-sh} = 0$ and so

$$s = -\lambda_k \mathrm{e}^{-sh}. \tag{2.19}$$

- Because $s \in i\mathbb{R}$ satisfies (2.19) its complex conjugate $\bar{s} = -s$ satisfies

$$-s = -\bar{\lambda}_k \mathrm{e}^{sh}. \tag{2.20}$$

- From (2.19) and (2.20), we get a special case of (2.9), namely

$$\begin{aligned} (s)(-s) &= (-\lambda_k)(-\bar{\lambda}_k), \\ -s^2 &= |\lambda_k|^2, \end{aligned}$$

  and so

$$s = \pm i|\lambda_k|, \tag{2.21}$$

The formula (2.21) can be the general formula applied for each λ_k. Then we can find the smallest h that makes the determinant (2.17) vanish as follows:

By putting $s = i|\lambda_k|$ in (2.19), we get

$$e^{-i|\lambda_k|h} = -\frac{i|\lambda_k|}{\lambda_k},$$

and so

$$e^{i|\lambda_k|h} = -\frac{\lambda_k}{i|\lambda_k|} = i\frac{\lambda_k}{|\lambda_k|},$$

then we get

$$h = \frac{1}{|\lambda_k|} arg\left(i\frac{\lambda_k}{|\lambda_k|}\right), \tag{2.22}$$

whereas for $s = -i|\lambda_k|$ we get

$$h = \frac{1}{|\lambda_k|} arg\left(i\frac{|\lambda_k|}{\lambda_k}\right), \tag{2.23}$$

where $i\frac{\lambda_k}{|\lambda_k|}$ and $i\frac{|\lambda_k|}{\lambda_k}$ are complex numbers on the circle of modulus one and we look at the smallest positive angle that gives h. Note that, if λ_k is real, then (2.22) and (2.23) give the same answer.

Remark 2.6.2. *We assume the stability at $h = 0$ holds if and only if $s + \lambda_k$ has no roots in $\overline{\mathbb{C}}_+$, which means if and only if $Re\lambda_k > 0$.*

Example 2.6.3. *Suppose we have $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{bmatrix}$, which is not diagonalisable, but it is triangular. From (2.17) where $P(s) = s, Q(s) = \lambda_k$ and the eigenvalues $\lambda_k \in \{1, 2, 2\}$ we must check the zero sets of $s + e^{-sh}$ and $s + 2e^{-sh}$. The equation (2.18) giving the points where zeros cross the axis with increasing h are $s = \pm i$ and $s = \pm 2i$, respectively, and from (2.22) or (2.23) we arrive at stability ranges $[0, \pi/2)$ and $[0, \pi/4)$, respectively. Thus the system (2.1) is stable for $0 \leq h < \pi/4$.*

Example 2.6.4. *For the normal operator $A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ we have the transformation*

$$T = U^{-1} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} U = \begin{bmatrix} 1+i & 0 \\ 0 & 1-i \end{bmatrix},$$

where $U = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 & -i \\ -i & 1 \end{bmatrix}$ *which is unitary. From (2.17), we get*

$$(s + (1+i)\mathrm{e}^{-sh})(s + (1-i)\mathrm{e}^{-sh}) = 0. \tag{2.24}$$

To study the stability we need first to check it from (2.24) at $h = 0$*, which gives* $s = -(1 \pm i) \notin \mathbb{C}_+$ *and so the system (2.1) is stable at* $h = 0$*. Then, we apply (2.21), (2.22) and (2.23) for each factor as follows:*

- *For the factor* $(s + (1+i)\mathrm{e}^{-sh}) = 0$ *where* $\lambda = 1 + i$*, we get* $s = \pm i\sqrt{2}$ *and so*
  - *For* $s = i\sqrt{2}$, $h = \frac{1}{\sqrt{2}} \arg\left(i\frac{1+i}{\sqrt{2}}\right)$ *and so* $h = \frac{3\pi}{4\sqrt{2}}$.
  - *For* $s = -i\sqrt{2}$, $h = \frac{1}{\sqrt{2}} \arg\left(i\frac{\sqrt{2}}{1+i}\right)$ *and so* $h = \frac{\pi}{4\sqrt{2}}$ *which is rejected.*
- *For the factor* $(s + (1-i)\mathrm{e}^{-sh}) = 0$ *where* $\lambda = 1 - i$*, we get* $s = \pm i\sqrt{2}$ *and so*
  - *For* $s = i\sqrt{2}$, $h = \frac{1}{\sqrt{2}} \arg\left(i\frac{1-i}{\sqrt{2}}\right)$ *and so* $h = \frac{\pi}{4\sqrt{2}}$ *which is rejected.*
  - *For* $s = -i\sqrt{2}$, $h = \frac{1}{\sqrt{2}} \arg\left(i\frac{\sqrt{2}}{1-i}\right)$ *and so* $h = \frac{3\pi}{4\sqrt{2}}$.

Chapter 3

Delays with Small Variation

3.1 Introduction

In this chapter we look at the stability of the variable-delay input-output systems. There are other approaches in references [5, 19, 20, 22, 31, 32, 43, 49] that consider stability, but not from an input output point of view (eg. autonomous systems). Because of that, we are going to look at the particular paper of Bonnet and Partington [5] to make an extension to different cases of stability, BIBO stability and H^∞-stability, which are covered in their paper and to consider a more general version of stability, which is L^p-stability for all $1 \leq p < \infty$. Paper [5] showed that if the nominal system with constant delay and output $z(t) \in \mathbb{C}^n$ and input $w(t) \in \mathbb{C}^p$

$$\dot{v}(t) = Av(t)+\sum_{j=1}^{J} A_j v(t-h_j)+\int_0^D h(\theta)Iv(t-\theta)\, d\theta+Bu(t)+\sum_{k=1}^{K} B_k u(t-T_k) \quad (t > 0),$$

is stable where the delay θ and $h_1, ..., h_j$ are positive, then under certain conditions the time varying system

$$\dot{x}(t) = Ax(t)+\sum_{j=1}^{J} A_j x(t-\tau_j(t))+\int_0^{\delta(t)} h(\theta)Ix(t-\theta)\, d\theta+Bu(t)+\sum_{k=1}^{K} B_k u(t-\sigma_k(t)) \quad (t > 0),$$

is stable where $x(t) \in \mathbb{C}^n$ denotes the output and $u(t) \in \mathbb{C}^p$ the input, the matrices A, A_j, B, B_k and I (identity matrix) have the appropriate sizes, and the delays $\tau_j(t)$ and $\sigma_k(t)$ are positive. Bonnet and Partington give a theorem that gives us sufficient conditions for the variable-delay input-output systems to be BIBO and H^∞ stable. Their technique plays an essential role to get the BIBO stability, L^1-stability and even the L^p-stability; $1 < p < \infty$ for our system

$$\dot{x}(t) = Ax(t - \tau(t)) + Bu(t) \quad (t > 0),$$

where $x(t)$ is the output in the Hilbert space H, and $u(t)$ is the input in the Hilbert space K, the operators $A : H \to H$ and $B : K \to H$ are bounded, and the delay $\tau(t)$ is positive. Additionally, we introduce corollaries about the three versions of the stability which apply when A is a bounded normal operator. The corollaries are motivated by an example before being stated.

3.2 Stability of Time-Varying Delay Systems

In this section, we shall consider the variable-delay input-output system

$$\dot{x}(t) = Ax(t - \tau(t)) + Bu(t) \quad (t > 0), \tag{3.1}$$

where $x(t)$ is in the Hilbert space H denotes the state and $u(t)$ is the input in the Hilbert space K, both assumed zero for $t \leq 0$, the operators $A : H \to H$ and $B : K \to H$ are bounded, and the delay $\tau(t)$ is positive.

We have followed the approach in [5] to consider our system (3.1) as a small perturbation of the ordinary constant-delay system

$$\dot{v}(t) = Av(t - h) + Bu(t) \quad (t > 0), \tag{3.2}$$

where $h - \mu \leq \tau(t) \leq h + \mu$ and μ is a small real positive number. Then, we try to see how the solution of (3.1) changes and behaves when the delay is almost constant.

After that, we can go back and look at the other way around to know how to make (3.1) unstable by showing how big the delay we can put in (3.2). We also write $y = x - v$, which gives

$$\begin{aligned}
\dot{y} &= \dot{x} - \dot{v}, \\
&= Ax(t - \tau(t)) - Av(t - h), \\
&= A[x(t - \tau(t)) - x(t - h)] + A[x(t - h) - v(t - h)], \\
&= Ay(t - h) + A[x(t - \tau(t)) - x(t - h)].
\end{aligned}$$

We may write

$$\dot{y}(t) = Ay(t - h) + Af(t), \tag{3.3}$$

with $f(t) = x(t - \tau(t)) - x(t - h)$. We now need supplementary conditions to ensure that the function f lies in $L^\infty(0, \infty; H), L^1(0, \infty; H)$ or $L^p(0, \infty; H)$ for $1 < p < \infty$ (including $L^2(0, \infty; H)$). The calculations are slightly different in each case.

3.2.1 BIBO Stability

Proving BIBO stability of the system (3.1) requires us first introduce some constants which we call $M_\infty, M_\infty^d, M_\infty^{nom}$ and M_∞^{nomd}. The definitions of the constants are as follows:

- M_∞ is the maximum L^∞ gain from w to z for the BIBO stable system

$$\dot{z}(t) = Az(t - h) + w(t). \tag{3.4}$$

  Saying that the system (3.4) is BIBO stable means that the mapping $w \rightarrowtail z$ is bounded. Because of that, the norm of this mapping is the constant M_∞ and

$$\|z\|_\infty \leq M_\infty \|w\|_\infty. \tag{3.5}$$

- M_∞^d is the maximum L^∞ gain from w to $\dot{z}$. It can be obtained from (3.4) as follows:

$$\begin{aligned}\|\dot{z}\|_\infty &= \sup_{t\geq 0}\|Az(t-h)+w(t)\|,\\ &\leq \|A\|\|z\|_\infty + \|w\|_\infty.\end{aligned}$$

From (3.5), we get

$$\|\dot{z}\|_\infty \leq M_\infty\|A\|\|w\|_\infty + \|w\|_\infty,$$

and so

$$\frac{\|\dot{z}\|_\infty}{\|w\|_\infty} \leq M_\infty\|A\| + 1,$$

then

$$M_\infty^d = \sup_{w\neq 0}\frac{\|\dot{z}\|_\infty}{\|w\|_\infty} \leq M_\infty\|A\| + 1. \tag{3.6}$$

- M_∞^{nom} is the maximum L^∞ gain from u to v for the BIBO stable system (3.2) and so

$$\|v\|_\infty \leq M_\infty^{nom}\|u\|_\infty.$$

- M_∞^{nomd} is the maximum L^∞ gain from u to $\dot{v}$ for the BIBO stable system (3.2). It can be obtained as follows:

$$\begin{aligned}\|\dot{v}\|_\infty &= \sup_{t\geq 0}\|Av(t-h)+Bu(t)\|,\\ &\leq \|A\|\|v\|_\infty + \|B\|\|u\|_\infty,\\ &\leq M_\infty^{nom}\|A\|\|u\|_\infty + \|B\|\|u\|_\infty,\end{aligned}$$

and so

$$\frac{\|\dot{v}\|_\infty}{\|u\|_\infty} \leq M_\infty^{nom}\|A\| + \|B\|,$$

then

$$M_\infty^{nomd} = \sup_{u\neq 0}\frac{\|\dot{v}\|_\infty}{\|u\|_\infty} = M_\infty^{nom}\|A\| + \|B\|.$$

Remark 3.2.1. *It is easy to prove that if M_∞ for the system (3.4) is finite, then*

M_∞^{nom} for the system (3.2) will be finite. That is because by putting $z = v$ and $w = Bu$ in (3.5) we get

$$\begin{aligned}\|v\|_\infty &\le M_\infty \|Bu\|_\infty, \\ &\le (M_\infty \|B\|)\, \|u\|_\infty =: M_\infty^{nom} \|u\|_\infty.\end{aligned}$$

Theorem 3.2.2. *Suppose that the system (3.4) is BIBO stable. If $M_\infty^d \mu \|A\| < 1$ then the system (3.1) is BIBO stable. Also $M_\infty^d \le M_\infty \|A\| + 1$.*

Proof. We have $M_\infty^d \le M_\infty \|A\| + 1$ by (3.6). The basic calculation is as follows:

$$\begin{aligned}\|\dot{x}\|_\infty &\le \|\dot{v}\|_\infty + \|\dot{y}\|_\infty, \\ &\le \|\dot{v}\|_\infty + M_\infty^d \|A\| \|f\|_\infty.\end{aligned}$$

and then we shall bound $\|f\|_\infty$ in term of $\|\dot{x}\|_\infty$.

We have
$$\|f\|_\infty = \sup_t \left\| \int_{t-\tau(t)}^{t-h} \dot{x}(s) \;\; ds \right\| \le \mu \|\dot{x}\|_\infty.$$

So that, provided that $M = M_\infty^d \mu \|A\| < 1$, we have

$$\|\dot{x}\|_\infty \le \|\dot{v}\|_\infty + M_\infty^d \mu \|A\| \|\dot{x}\|_\infty, \tag{3.7}$$

$$\begin{aligned}&\le \|\dot{v}\|_\infty + M\|\dot{x}\|_\infty, \\ &\le M_\infty^{nomd} \|u\|_\infty + M\|\dot{x}\|_\infty, \\ &\le (1-M)^{-1} M_\infty^{nomd} \|u\|_\infty.\end{aligned}$$

and hence there is a bound on $\|f\|_\infty$. Now $x = v + y$ and so by (3.2) and (3.3) we have

$$
\begin{aligned}
\|x\|_\infty &\leq \|v\|_\infty + \|y\|_\infty, \\
&\leq M_\infty^{nom}\|u\|_\infty + M_\infty\|A\|\|f\|_\infty, \\
&\leq M_\infty^{nom}\|u\|_\infty + M_\infty\mu\|A\|\|\dot{x}\|_\infty, \\
&\leq M_\infty^{nom}\|u\|_\infty + M_\infty\mu\|A\|(1-M)^{-1}M_\infty^{nomd}\|u\|_\infty.
\end{aligned}
$$

which gives a finite L^∞ gain from u to x. □

3.2.2 L^1 Stability

Proving L^1 stability of the system (3.1) requires us first to introduce some constants which we call M_1, M_1^d, M_1^{nom} and M_1^{nomd}. The definitions of the constants are as follows:

- M_1 is the maximum L^1 gain from w to z for the L^1-stable system (3.4) and so

$$\|z\|_1 \leq M_1\|w\|_1. \tag{3.8}$$

- M_1^d is the maximum L^1 gain from w to $\dot{z}$. It can be obtained from (3.4) as follows:

$$\|\dot{z}\|_1 \leq \|A\|\|z\|_1 + \|w\|_1.$$

From (3.8), we get

$$\|\dot{z}\|_1 \leq M_1\|A\|\|w\|_1 + \|w\|_1$$

and so

$$\frac{\|\dot{z}\|_1}{\|w\|_1} \leq M_1\|A\| + 1,$$

then

$$M_1^d = \sup_{w\neq 0}\frac{\|\dot{z}\|_1}{\|w\|_1} \leq M_1\|A\| + 1. \tag{3.9}$$

- M_1^{nom} is the maximum L^1 gain from u to v for the L^1-stable system (3.2) and so
$$\|v\|_1 \leq M_1^{nom}\|u\|_1.$$

- M_1^{nomd} is the maximum L^1 gain from u to $\dot{v}$ for the L^1-stable system (3.2). It can be obtained as follows:
$$\|\dot{v}\| \leq M_1^{nom}\|A\|\|u\| + \|B\|\|u\|,$$
and so
$$\frac{\|\dot{v}\|_1}{\|u\|_1} \leq M_1^{nom}\|A\| + \|B\|,$$
then
$$M_1^{nomd} = \sup\frac{\|\dot{v}\|_1}{\|u\|_1} \leq M_1^{nom}\|A\| + \|B\|.$$

Remark 3.2.3. *It is easy to prove that if M_1 for the system (3.4) is finite, then M_1^{nom} for the system (3.2) will be finite. That is because by putting $z = v$ and $w = Bu$ in (3.8) we get*
$$\begin{aligned}\|v\|_1 &\leq M_1\|Bu\|_1,\\ &\leq (M_1\|B\|)\,\|u\|_1 =: M_1^{nom}\|u\|_1.\end{aligned}$$

Theorem 3.2.4. *Suppose that the system (3.4) is L^1-stable. If $M_1^d\mu\|A\| < 1$, then the system (3.1) is L^1-stable in the sense that there is a finite L^1 gain between u and x. Also $M_1^d \leq M_1\|A\| + 1$*

Proof. We have $M_1^d \leq M_1\|A\| + 1$ by (3.9).

First, we have, recalling that $x(t) = \dot{x}(t) = 0$ for $t \leq 0$,

$$
\begin{aligned}
\|f\|_1 = \int_{t=0}^{\infty} \left\| \int_{s=t-\tau(t)}^{t-h} \dot{x}(s) \ \ ds \right\| \ dt &\leq \int_{t=0}^{\infty} \int_{s=t-h-\mu}^{t-h} \|\dot{x}(s)\| \ ds \ dt, \\
&= \int_{s=0}^{\infty} \int_{t=s+h}^{s+h+\mu} \|\dot{x}(s)\| \ dt \ ds, \\
&= \mu \int_{s=0}^{\infty} \|\dot{x}(s)\| \ ds = \mu \|\dot{x}\|_1.
\end{aligned}
$$

That is from Fubini's theorem and the assumption that $\tau(t) \leq \mu + h$, which gives $t - \tau(t) \geq t - (\mu + h)$. In the case of L^1 stability, we start with the similar inequality to the proof of the previous theorem, which is (3.7). Hence again, provided that $M = M_1^d \mu \|A\| < 1$, we have

$$
\begin{aligned}
\|\dot{x}\|_1 &\leq \|\dot{v}\|_1 + M\|\dot{x}\|_1, \\
&\leq (1 - M)^{-1} M_1^{nomd} \|u\|_1,
\end{aligned}
$$

and hence there is a bound on $\|f\|_1$.

By (3.2) and (3.3) we have

$$
\begin{aligned}
\|x\|_1 &\leq \|v\|_1 + \|y\|_1, \\
&\leq M_1^{nom} \|u\|_1 + M_1 \mu \|A\| \|\dot{x}\|_1, \\
&\leq M_1^{nom} \|u\|_1 + M_1 \mu \|A\| (1 - M)^{-1} M_1^{nomd} \|u\|_1,
\end{aligned}
$$

which gives a finite L^1 gain from u to x. □

3.2.3 L^p Stability for $1 < p < \infty$

Proving L^p stability of the system (3.1) requires us first to introduce some constants which we call M_p, M_p^d, M_p^{nom} and M_p^{nomd}. The definitions of the constants are as follows:

- M_p is the maximum L^p gain from w to z for the L^p-stable system (3.4) and so

$$\|z\|_p \leq M_p\|w\|_p. \tag{3.10}$$

- M_p^d is the maximum L^p gain from w to $\dot{z}$. It can be obtained from (3.4) as follows:

$$\|\dot{z}\|_p \leq \|A\|\|z\|_p + \|w\|_p.$$

From (3.10), we get

$$\|\dot{z}\|_p \leq M_p\|A\|\|w\|_p + \|w\|_p$$

and so

$$\frac{\|\dot{z}\|_p}{\|w\|_p} \leq M_p\|A\| + 1,$$

then

$$M_p^d = \sup_{w\neq 0} \frac{\|\dot{z}\|_p}{\|w\|_p} \leq M_p\|A\| + 1. \tag{3.11}$$

- M_p^{nom} is the maximum L^p gain from u to v for the L^p-stable system (3.2) and so

$$\|v\|_p \leq M_p^{nom}\|u\|_p.$$

- M_p^{nomd} is the maximum L^p gain from u to $\dot{v}$ for the L^p-stable system (3.2). It can be obtained as follows:

$$\|\dot{v}\|_p \leq M_p^{nom}\|A\|\|u\|_p + \|B\|\|u\|_p,$$

and so

$$\frac{\|\dot{v}\|_p}{\|u\|_p} \leq M_p^{nom}\|A\| + \|B\|,$$

then

$$M_p^{nomd} = \sup \frac{\|\dot{v}\|_p}{\|u\|_p} \leq M_p^{nom}\|A\| + \|B\|.$$

Remark 3.2.5. *It is easy to prove that if M_p for the system (3.4) is finite, then M_p^{nom} for the system (3.2) will be finite. That is because by putting $z = v$ and*

$w = Bu$ in (3.5) we get

$$\begin{aligned}\|v\|_p &\leq M_p\|Bu\|_p,\\ &\leq (M_p\|B\|)\,\|u\|_p =: M_p^{nom}\|u\|_p.\end{aligned}$$

Theorem 3.2.6. *Suppose that the system (3.4) is L^p-stable, where $1 < p < \infty$. If $M_p^d\mu\|A\| < 1$, then the system (3.1) is L^p-stable in the sense that there is a finite L^p gain between u and x. Also $M_p^d \leq M_p\|A\| + 1$*

Proof. We have $M_p^d \leq M_p\|A\| + 1$ by (3.11).
First, we have, recalling that $x(t) = \dot{x}(t) = 0$ for $t \leq 0$ and letting $q = p/(p-1)$, it follows from the fact that $\tau(t) \leq \mu$ that

$$\begin{aligned}\|f\|_p^p &= \int_{t=0}^{\infty}\left(\left\|\int_{s=t-\tau(t)}^{t-h}\dot{x}(s)\ \ ds\right\|^p\right)\ dt,\\ &\leq \int_{t=0}^{\infty}\left(\left(\int_{s=t-h-\mu}^{t-h}1^q\ ds\right)^{\frac{p}{q}}\int_{s=t-h-\mu}^{t-h}\|\dot{x}(s)\|^p\ ds\right)\ dt,\\ &= \int_{t=0}^{\infty}\left(\mu^{\frac{p}{q}}\int_{s=t-h-\mu}^{t-h}\|\dot{x}(s)\|^p\ ds\right)\ dt,\\ &= \int_{s=0}^{\infty}\mu^{\frac{p}{q}}\int_{t=s+h}^{s+h+\mu}\|\dot{x}(s)\|^p\ dt\ ds,\\ &= \mu^{\frac{p}{q}+1}\int_0^{\infty}\|\dot{x}(s)\|^p\ ds = \mu^{\frac{p}{q}+1}\|\dot{x}\|_p^p = \mu^p\|\dot{x}\|_p^p,\end{aligned}$$

by Hölder's inequality. Therefore, $\|f\|_p \leq \mu\|\dot{x}\|_p$. In the case of L^p stability, we start with the similar inequality in the proof of the previous theorem, which is (3.7). Hence again, provided that $M = M_p^d\mu\|A\| < 1$, we have

$$\begin{aligned}\|\dot{x}\|_p &\leq \|\dot{v}\|_p + M\|\dot{x}\|_p,\\ &\leq (1-M)^{-1}M_p^{nomd}\|u\|_p,\end{aligned}$$

and hence there is a bound on $\|f\|_p$.

By (3.2) and (3.3) we have

$$\begin{aligned}\|x\|_p &\leq \|v\|_p + \|y\|_p, \\ &\leq M_p^{nom}\|u\|_p + M_p\mu\|A\|\|\dot{x}\|_p, \\ &\leq M_p^{nom}\|u\|_p + M_p\mu\|A\|(1-M)^{-1}M_p^{nomd}\|u\|_p,\end{aligned}$$

which gives a finite L^p gain from u to x. □

Example 3.2.7. *Putting* $h = 0, A = -\lambda;\ \lambda = a + bi \in \mathbb{C}_+$ *and* $\|A\| = |\lambda| = \sqrt{a^2 + b^2} = k$ *in (3.4), gives the system*

$$\dot{z}(t) = -\lambda z(t) + w(t). \tag{3.12}$$

If system (3.12) is BIBO stable and if $\mu < \frac{1}{M_\infty^d\|A\|}$, *(for example if* $\mu < \frac{1}{(M_\infty\|A\|+1)\|A\|}$*) then from Theorem 3.2.2 the system*

$$\dot{x}(t) = -\lambda x(t - \tau(t)) + u(t), \tag{3.13}$$

is BIBO stable for $0 \leq \tau(t) < \mu$. *So for* $A = -\lambda$, *we have*

$$\mu < \frac{1}{(M_\infty k + 1)k}. \tag{3.14}$$

M_∞ *can be obtained from the solution of system (3.12) as follows:*

$$z(t) = \int_0^t w(s)\ \mathrm{e}^{-\lambda(t-s)}\ ds, \tag{3.15}$$

and so

$$\begin{aligned}\|z\|_\infty &\leq \sup_{t>0}\|w\|_\infty \int_0^t \mathrm{e}^{-a(t-s)}\ ds, \\ &= \sup_{t>0}\|w\|_\infty\ \frac{1-\mathrm{e}^{-at}}{a}, \\ &\leq \frac{1}{a}\|w\|_\infty,\end{aligned}$$

and so $M_\infty \leq \frac{1}{a}$ *and then from (3.14), we get the condition*

$$\mu < \frac{1}{\frac{k^2}{a} + k}.$$

However, if system (3.12) is L^1*-stable and if* $\mu < \frac{1}{(M_1\|A\|+1)\|A\|}$ *and* $\mu < \frac{1}{M_1^d\|A\|}$ *, then from Theorem 3.2.4 the system (3.13) is* L^1*-stable for* $0 \leq \tau(t) < \mu$*. So we have*

$$\mu < \frac{1}{(M_1 k + 1)k}. \tag{3.16}$$

M_1 *can be obtained from (3.15) as follows:*

$$\begin{aligned}
\int_0^\infty |z(t)| \;\; dt &\leq \int_{t=0}^\infty \int_{s=0}^t |w(s)| \; |\mathrm{e}^{-\lambda(t-s)}| \; ds \; dt, \\
&= \int_{s=0}^\infty \int_{t=s}^\infty |w(s)| \; |\mathrm{e}^{-\lambda(t-s)}| \; dt \; ds, \\
&= \int_0^\infty |w(s)| \; \frac{1}{a} \; ds, \\
&= \frac{1}{a}\|w\|_1,
\end{aligned}$$

and so $M_1 \leq \frac{1}{a}$ *and then from (3.16), we get the condition*

$$\mu < \frac{1}{\frac{k^2}{a} + k}.$$

However, if system (3.12) is L^2*-stable and if* $\mu < \frac{1}{(M_2\|A\|+1)\|A\|}$ *and* $\mu < \frac{1}{M_2^d\|A\|}$ *, then from Theorem 3.2.6 the system (3.13) is* L^2*-stable for* $0 \leq \tau(t) < \mu$*. So we have*

$$\mu < \frac{1}{(M_2 k + 1)k}. \tag{3.17}$$

M_2 *can be obtained from the system (3.12) as follows:*

$$s\hat{z}(s) = -\lambda\hat{z}(s) + \hat{w}(s),$$

then

$$\hat{z}(s) = \frac{1}{s+\lambda}\hat{w}(s),$$

and so

$$\|\hat{z}\|_2 \leq \sup_{s\in\mathbb{C}_+} \|\hat{w}\|_2 \left|\frac{1}{s+\lambda}\right|,$$

so that

$$\|\hat{z}\|_2 \leq \frac{1}{a}\|\hat{w}\|_2,$$

and so $M_2 \leq \frac{1}{a}$ *and then from (3.17)*

$$\mu < \frac{1}{\frac{k^2}{a}+k}.$$

Example 3.2.8. *This example simulates the example mentioned in [5, p. 12].*

Consider the ordinary delay system

$$\dot{x}(t) + x(t-h) = u(t),$$

with transfer function $G_h(s) = 1/(s + \mathrm{e}^{-sh})$. *This is* L^2 *and BIBO stable provided that* $0 \leq h < \pi/2$ *(see, for example, [39, Chap.6]).*

Now we consider the perturbed system

$$\dot{x}(t) + x(t-\tau(t)) = u(t),$$

with $0 \leq \tau(t)$ *and* $|\tau(t) - h| < \mu$

By Theorem 3.2.6, we have L^2 *stability if* $\mu < (\|G_h\|_\infty + 1)^{-1}$ *or we can suppose* $\mu < (M_2^d)^{-1} = \mu_2$*, which is more precise, where* $M_2^d = \|s/(s+\mathrm{e}^{-sh})\|_\infty$.

For $h = 0, 0.5, 1$ *and* 1.5 *the values of* μ_2 *are* $1, 0.63, 0.32$ *and* 0.03 *respectively; naturally* $h + \mu_2 < \pi/2$ *in all cases.*

For BIBO stability a similar result holds, except that we require the BIBO norm of

G_h which is not easy to calculate as we do not have explicit form of impulse response.

One way round this is to use the Hardy-Littlewood inequality given in [24, p. 182] (see also [4]), namely that

$$\|G_h\|_{BIBO} \leq \frac{1}{2}\|G_h'\|_{L^1(i\mathbb{R})}.$$

The L^1 norm of G_h' can be found by the inequality

$$\|f\|_{L^1(i\mathbb{R})} \leq \|fg\|_\infty \|\frac{1}{g}\|_{L^1(i\mathbb{R})},$$

such that $f = G_h'$ and $g = (s+c)^2; c > 0$. Because of that to calculate the BIBO norm of G_h for each h, we pick an appropriate c to maximize the estimate stability margin. For $h = 0, 0.5, 1$ and 1.5 we take $c = 1, 2, 1.5$ and 1 respectively. Then the calculation steps will be as follows:

- $A = \left\|(iy+1.2)^2 \dfrac{1-h\mathrm{e}^{-hiy}}{(iy+\mathrm{e}^{-hiy})^2}\right\|_\infty = 1,\ 4.13,\ 27.21$ *and* 2369.57,
- $B = \|G_h'\|_{L^1(i\mathbb{R})} \leq A \left\|\dfrac{1}{(iy+1.2)^2}\right\|_{L^1(i\mathbb{R})} = A\left(\dfrac{\pi}{1.2}\right)$, *where* $A\left(\dfrac{\pi}{1.2}\right) = 3.14,\ 6.49,\ 56.99,\ 7444.22,$
- $C = \|G_h\|_{BIBO} = M_\infty \leq B\left(\dfrac{1}{2}\right) = 1.57,\ 3.24,\ 28.5,\ 3722.11.$

Then, BIBO stability can be obtained for $\mu < (M_\infty^d)^{-1} = \mu_\infty;\ \mu_\infty \geq \frac{1}{C+1}$. The values of μ_∞ are at most $0.39,\ 0.24,\ 0.034$ and 0.00027 respectively; naturally $h+\mu_\infty < \pi/2$ in all cases.

Example 3.2.9. *(An example with matrices) If $A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ is a diagonal matrix in*

$$\dot{x}(t) + Ax(t-\tau(t)) = u(t),$$

then $M_2^d = \|s/(s+A\mathrm{e}^{-sh})\|_\infty = max\left\{\|s/(s+\mathrm{e}^{-sh})\|_\infty, \|s/(s+2\mathrm{e}^{-sh})\|_\infty\right\}$ where $0 \leq h < \frac{\pi}{4}$.

Example 3.2.10. *From supposing that $h = 0$ in system (3.4) and taking A to be a bounded normal operator such that:*

- $\sigma(A) \subset \{\lambda \in \mathbb{C}_- : \; Re\lambda \leq -a\}$ *where* $s(A) = -a; \; a > 0$;
- $\|A\| = \max\{|z| : z \in \sigma(A)\} = q$;
- $T(t) = \mathrm{e}^{At}$ *is the semigroup of* A, *where*

$$\begin{aligned}\|T(t)\|_{t>0} = \|\mathrm{e}^{At}\|_{t>0} =& r(\mathrm{e}^{At}),\\ =& \sup\{|\mu| : \mu \in \sigma(\mathrm{e}^{At})\},\\ =& \sup\left\{|\mu| : \mu \in \{\mathrm{e}^{\lambda t} : \lambda \in \sigma(A)\}\right\},\\ =& \sup\{|\mathrm{e}^{\lambda t}| : \lambda \in \sigma(A)\},\\ =& \mathrm{e}^{t\sup(Re\lambda)},\\ =& \mathrm{e}^{-at}; \; \textit{with} \; \sup Re\; \sigma(A) = -a < 0,\end{aligned}$$

and

$$\|(sI - A)^{-1}\|_\infty \leq \max_{s\in\mathbb{C}_+} \max_{\lambda\in\sigma(A)} \left|\frac{1}{s-\lambda}\right| = \frac{1}{a},$$

we get the system

$$\dot{z}(t) = Az(t) + w(t). \tag{3.18}$$

If system (3.18) is BIBO stable and if $\mu < \frac{1}{(M_\infty\|A\|+1)\|A\|}$ *and* $\mu < \frac{1}{M_\infty^d\|A\|}$, *then from Theorem 3.2.2 the system*

$$\dot{x}(t) = Ax(t - \tau(t)) + u(t), \tag{3.19}$$

is BIBO stable for $0 \leq \tau(t) < \mu$. *So we have*

$$\mu < \frac{1}{(M_\infty q + 1)q}.$$

M_∞ can be obtained from the solution of system (3.18) as follows:

$$z(t) = \int_0^t T(t-s)w(s)\ ds; \tag{3.20}$$

by putting $t-s=y$, we get

$$\begin{aligned}
\|z\|_\infty &\leq \sup_{t>0} \|w\|_\infty \int_0^t \|T(y)\|\ dy, \\
&= \sup_{t>0} \|w\|_\infty \int_0^t \mathrm{e}^{-ay}\ dy, \\
&= \sup_{t>0} \|w\|_\infty \frac{1-\mathrm{e}^{-at}}{a}, \\
&\leq \frac{1}{a}\|w\|_\infty.
\end{aligned}$$

and so $M_\infty \leq \frac{1}{a}$ and then

$$\mu < \frac{1}{\frac{q^2}{a}+q}.$$

Now system (3.19) is BIBO stable for $0 \leq \tau(t) < \mu$.

However, if the system (3.18) is L^1-stable and if $\mu < \frac{1}{(M_1\|A\|+1)\|A\|}$ and $\mu < \frac{1}{M_1^d\|A\|}$, then from Theorem 3.2.4 the system (3.19) is L^1-stable for $0 \leq \tau(t) < \mu$. So we have

$$\mu < \frac{1}{(M_1 q+1)q}.$$

M_1 can be obtained from the system (3.20) as follows:

$$\begin{aligned}
\int_0^\infty \|z\|\ \ dt &\leq \int_{t=0}^\infty \int_{s=0}^t \|w(s)\|\ \|T(t-s)\|\ ds\ dt, \\
&= \int_{s=0}^\infty \int_{t=s}^\infty \|w(s)\|\ \|T(t-s)\|\ dt\ ds, \\
&= \int_{s=0}^\infty \int_{t=s}^\infty \|w(s)\|\ \mathrm{e}^{-a(t-s)}\ dt\ ds, \\
&= \int_0^\infty \|w(s)\|\ \frac{1}{a}\ ds, \\
&= \frac{1}{a}\|w\|_1,
\end{aligned}$$

and so $M_1 \leq \frac{1}{a}$ *and then*

$$\mu < \frac{1}{\frac{q^2}{a} + q}.$$

Now system (3.19) is L^1*-stable for* $0 \leq \tau(t) < \mu$.

However, if the system (3.18) is L^2*-stable and if* $\mu < \frac{1}{(M_2\|A\|+1)\|A\|}$ *and* $\mu < \frac{1}{M_2^d\|A\|}$, *then from Theorem 3.2.6 the system (3.19) is* L^2*-stable for* $0 \leq \tau(t) < \mu$. *So we have*

$$\mu < \frac{1}{(M_2 q + 1)q}.$$

M_2 *can be obtained from the system (3.18) as follows:*

$$s\hat{z}(s) = A\hat{z}(s) + \hat{w}(s),$$

then

$$\hat{z}(s) = (s - A)^{-1}\ \hat{w}(s),$$

and so

$$\|\hat{z}\|_2 \leq \sup_{s\in\mathbb{C}_+} \|\hat{w}\|_2\ \left\|(s - A)^{-1}\right\|,$$

so that

$$\|\hat{z}\|_2 \leq \frac{1}{a}\|\hat{w}\|_2,$$

and so $M_2 \leq \frac{1}{a}$ *and then*

$$\mu < \frac{1}{\frac{q^2}{a} + q}.$$

Now system (3.19) is L^2*-stable for* $0 \leq \tau(t) < \mu$.

Remark 3.2.11. *A bounded diagonal matrix is a special case of a bounded normal operator. If A in system (3.18) is a diagonal matrix with eigenvalues* (λ_n)*, then that requires to be*

$$Re\lambda_n \leq -a \quad \forall n \in \mathbb{N}; \ -a = \max Re\ \sigma(A) < 0,$$

and then we can get the same μ we got in the previous example.

The same result hold for subnormal A as if $\sup\{Re\lambda : \lambda \in \sigma(A)\} = -a$ then $\sigma(N) \subset \{\lambda \in \mathbb{C}_- : \ Re\lambda \leq -a\}$ if N is a minimal normal extension (see Section 1.5). Therefore

- If $\|e^{tN}\| = e^{-at}$, then $\|e^{tA}\| = e^{-at}$;
- If $\|(s-N)^{-1}\| \leq \frac{1}{a}$ Then $\|(s-A)^{-1}\| \leq \frac{1}{a}$; $\ Re\ s > 0$;

and so $M_\infty, M_1, M_2 \leq \frac{1}{a}$.

Using Example 3.2.10 we have the following corollaries:

Corollary 3.2.12. *Suppose that the system (3.4) with $h = 0$ is BIBO stable and A is a subnormal operator with $\sigma(A) \subset \{\lambda \in \mathbb{C}_- : \ Re\lambda \leq -a\}$ where $a > 0$. Then $M_\infty \leq \frac{1}{a}$ and $M_\infty^d \leq M_\infty\|A\|+1$ and if $M_\infty^d \mu\|A\| < 1$ then the system (3.1) is BIBO stable.*

Corollary 3.2.13. *Suppose that the system (3.4) with $h = 0$ is L^1-stable and A is a subnormal operator with $\sigma(A) \subset \{\lambda \in \mathbb{C}_- : \ Re\lambda \leq -a\}$ where $a > 0$. Then $M_1 \leq \frac{1}{a}$ and $M_1^d \leq M_1\|A\| + 1$ and if $M_1^d \mu\|A\| < 1$ then the system (3.1) is L^1-stable.*

Corollary 3.2.14. *Suppose that the system (3.4) with $h = 0$ is L^2-stable and A is a subnormal operator with $\sigma(A) \subset \{\lambda \in \mathbb{C}_- : \ Re\lambda \leq -a\}$ where $a > 0$. Then $M_2 \leq \frac{1}{a}$ and $M_2^d \leq M_2\|A\| + 1$ and if $M_2^d \mu\|A\| < 1$ then the system (3.1) is L^2-stable.*

Chapter 4

Changing Variables

4.1 Introduction

The variable delay system plays a significant role in representing many phenomena in physics. This chapter will focus on complicated delay equations, which have a multi-step solution. Thus, the following delay equation will first be considered

$$\dot{x}(t) = Ax(t - \tau(t)) + Bu(t), \; x(0) = 0, \; t \in [0, \infty) \;\text{ and }\; t \geq \tau(t), \tag{4.1}$$

where $x(t) = 0$ for $t < 0$; thus, $x(t - \tau(t)) = 0$ when $t - \tau(t) < 0$. Here we suppose that $x(t)$ is the output in the Hilbert space H, and $u(t)$ is the input in the Hilbert space K, the operators $A : H \to H$ and $B : K \to H$ are bounded, and the delay $\tau(t)$ is positive. By changing variables we obtain an alternative equation with constant delays, which may be easier to analyse. Additionally, at the end of the chapter we discuss examples of instability that make the output of the system not in L^∞ whereas its input is in L^∞.

4.2 An Example of the Equation in the Finite-Dimensional Case

In this section, equation (4.1) is examined such that $x : (0, \infty) \to \mathbb{C}^n, A : \mathbb{C}^n \to \mathbb{C}^n, B : \mathbb{C}^m \to \mathbb{C}^n$ and $u \in L^\infty(0, \infty; \mathbb{C}^m)$. A and B are bounded operators (given as matrices).

Here, we consider the case $t - \tau(t) = \lambda t, \quad 0 < \lambda < 1$ in equation (4.1) to get

$$\dot{x}(t) = Ax(\lambda t) + Bu(t), \quad 0 < \lambda < 1. \tag{4.2}$$

Then, the variable is changed to $t = \lambda^{-y}, \quad -\infty < y < \infty$, which gives:

1. $\lambda t = \lambda^{-(y-1)}$. Additionally, supposing $x(t) = x(\lambda^{-y}) = z(y)$ leads to be $x(\lambda t) = x(\lambda^{-(y-1)}) = z(y-1)$,
2. $\dfrac{dt}{dy} = \dfrac{d}{dy}\, \mathrm{e}^{\ln \lambda^{-y}} = -\ln \lambda \cdot \lambda^{-y}$.

We suppose also $u(t) = v(y)$. Since $\dfrac{dz}{dy} = \dfrac{dt}{dy}\dot{x}(t)$ which is $\dfrac{dz}{dy} = [-\ln \lambda \cdot \lambda^{-y}]\ \dot{x}(t)$, from (4.2) we get:

$$\frac{dz}{dy} = [-\ln \lambda \cdot \lambda^{-y}]\ \ [Az(y-1) + Bv(y)]. \tag{4.3}$$

This formula is a delay system with an extra function multiplied, whereas with a constant, the Laplace transform could be used. By taking

$$-\ln \lambda . \lambda^{-y} = -\ln \lambda e^{-y \ln \lambda} = ce^{cy},$$

where $c = -\ln \lambda > 0$ as $\lambda < 1$ and $\ln \lambda < 0$, equation (4.3) turns into

$$\frac{dz}{dy} = ce^{cy}\ \ [Az(y-1) + Bv(y)]. \tag{4.4}$$

Solving equation (4.4) may require looking at the operator $Gz(y) = ce^{cy}Az(y-1)$.

Some considerations for this case

- G is an unbounded operator on $L^\infty(0,\infty;\mathbb{C}^n)$ because of the exponential, which gives $\dfrac{dt}{dy} = ce^{cy}$;
- The assumption $t = \lambda^{-y}$ leads to $y = -\frac{\ln t}{\ln \lambda} = f(t)$;
- By changing the variable, $x(t-\tau(t)) = z(y-1)$, where

$$\begin{aligned} y - 1 &= f(t) - 1, \\ &= -\frac{\ln t}{\ln \lambda} - 1, \\ &= -\frac{\ln t + \ln \lambda}{\ln \lambda}, \\ &= -\frac{\ln(t-\tau(t))}{\ln \lambda}, \\ &= f(t-\tau(t)). \end{aligned}$$

4.3 Generalization of the Previous Case

In this section, equation (4.1) is studied in the case when we can use a change of variables as follows:

1. Choosing y to be the function $y = f(t)$, $\forall t \geqslant 0$, where f needs to be injective and C^1 (so y is increasing) with $f(t) \geqslant 0$ for $t \geqslant 0$;
2. $x(t) = z(y) = z(f(t))$ so that leads to:
   - $x(t-\tau(t)) = z(f(t-\tau(t)))$. Then, the following is needed:

   $$x(t-\tau(t)) = z(y-\alpha), \quad \alpha > 0,$$

   and $f(t_1) = \alpha$, where t_1 is the solution of the equation $t - \tau(t) = 0$

(i.e., $\tau(t_1) = t_1$), and so we must have $f(t - \tau(t)) = f(t) - \alpha, \quad \forall t \geqslant t_1$.

Thus, $t \geqslant t_1$ is necessary to give $t - \tau(t) \geqslant 0$;

- $\dfrac{dx}{dy} = \dfrac{dx}{dt}\dfrac{dt}{dy}$.

3. $u(t) = v(y)$.

Then the equation will be as follows:

- For $y \leq 0$ we have $z(y) = 0$;
- For $0 \leq y \leq \alpha$ we have $\dfrac{dz}{dy} = \dfrac{dt}{dy}Bv(y)$, giving the solution

$$z(y) = \int_0^y \frac{dt}{dy}Bv(y) \quad dy;$$

- For $y \geq \alpha$ we have:

$$\frac{dz}{dy} = \frac{dt}{dy} \quad [Az(y-\alpha) + Bv(y)],$$

$$\frac{dz}{dy} = \frac{dt}{dy}Az(y-\alpha) + \frac{dt}{dy}Bv(y). \tag{4.5}$$

It is clear that equation (4.1) turns from a variable-delay differential equation into equation (4.5) with constant delay, but time-varying. A suitable solution of equation (4.5) is based on the next theorem.

It is worth mentioning that there is possibility of writing equation (4.5) to be

$$\frac{dz}{dy} = Gz(y) + Hv(y), \tag{4.6}$$

such that:

- $Gz(y) = \dfrac{dt}{dy}Az(y-\alpha)$, where $z \longmapsto \dfrac{dt}{dy}Az(y-\alpha)$.
- $H = \dfrac{dt}{dy}B$.

We do not know how to solve equation (4.6) yet, as G is an operator taking functions to functions, while A is an operator taking X to X.

Theorem 4.3.1. *Given the delay equation (4.5) with A, B bounded such that the function $y = f(t)$ satisfies that $f(t) - \alpha = f(t - \tau(t))$, $\forall t \geqslant t_1$ and suppose the derivative $\dfrac{dt}{dy}$ is continuous and $v \in L^1_{loc}(0, \infty; U)$, then the equation is well-posed, and we can obtain the solution of (4.5) by iteratively solving the equation on intervals.*

Proof. The equation

$$\frac{dz}{dy} = g(y)Az(y-\alpha) + g(y)Bv(y), \qquad \frac{dt}{dy} = g(y),$$

with $z(y) = v(y) = 0$ for $y \leq 0$, can be solved iteratively on intervals. So

$$z(y) = \int_0^y g(p)Bv(p) \quad dp, \tag{4.7}$$

for $0 \leq y \leq \alpha$.

Then for $\alpha \leq y \leq 2\alpha$,

$$z(y) = z(\alpha) + \int_\alpha^y g(p)Az(p-\alpha) \quad dp + \int_\alpha^y g(p)Bv(p) \quad dp, \tag{4.8}$$

and we know $z(\alpha)$ and $z(p-\alpha)$ from (4.7).

Then for $2\alpha \leq y \leq 3\alpha$,

$$z(y) = z(2\alpha) + \int_{2\alpha}^y g(p)Az(p-\alpha) \quad dp + \int_{2\alpha}^y g(p)Bv(p) \quad dp,$$

and we know $z(2\alpha)$ and $z(p-\alpha)$ from (4.8). And so on.

□

Remark 4.3.2. *If $u \in L^2_{loc}(0, \infty; U)$ and $f' \in L^2_{loc}(0, \infty)$ then $v \in L^1_{loc}(0, \infty; U)$ by the Cauchy - Schwarz inequality. Note that $x(t) = z(y) = z(f(t))$, so we get a solution for equation (4.1). Also If $u \in L^\infty_{loc}(0, \infty; U)$ and $f' \in L^1_{loc}(0, \infty)$ then $v \in L^1_{loc}(0, \infty; U)$.*

Remark 4.3.3. *From the previous explanation, the equation can be rewritten as equation (4.6), where* $Gz(y) = \dfrac{dt}{dy} Az(y-\alpha)$ *and* $H = \dfrac{dt}{dy} B$*. It is clear that* G *is a bounded operator as* A *and* $\dfrac{dt}{dy}$ *are bounded. That means that*

$$\frac{\|Gz\|_{L^\infty}}{\|z\|_{L^\infty}} \leq \|A\| \left\| \frac{dt}{dy} \right\|_{L^\infty} \quad \text{and so} \quad \|G\| \leq \|A\| \left\| \frac{dt}{dy} \right\|_{L^\infty}.$$

Example 4.3.4. *Suppose we have the relation:*

$$y(t) = t^2.$$

We can see the following:

1. *The function* y *is increasing as* $\dfrac{dy}{dt} = 2t > 0, \quad \forall t \geqslant 0.$
2. $\dfrac{dt}{dy}$ *is bounded for* $t \geq 1$ *as* $t = \sqrt{y}$ *and* $\dfrac{dt}{dy} = \dfrac{1}{2\sqrt{y}} = \dfrac{1}{2t}$.
3. *By choosing* $t_1 = 1$ *we have* $f(t_1) = f(1) = \alpha = 1$.
4. *From* $f(t) - \alpha = f(t - \tau(t)), \quad \forall t \geq 1,$ *we can find that:*

$$\tau(t) = t - \sqrt{t^2 - \alpha} = \sqrt{y} - \sqrt{y - \alpha} \quad \text{and} \quad \tau(1) = 1.$$

We obtain

$$\frac{dz}{dy} = \frac{1}{2\sqrt{y}} Az(y-\alpha) + \frac{1}{2\sqrt{y}} Bv(y).$$

Theorem 4.3.1 can be applied here since

$$\begin{aligned} z(y) &= \int_0^y g(p) B v(p) \quad dp, \\ &= \int_0^{f(t)} Bu(s) \quad ds, \end{aligned}$$

which converges for $u \in L^1_{loc}(0, \infty; U)$.

Example 4.3.5. *Suppose we have the relation:*

$$y(t) = t - \frac{1}{2}\,\frac{t}{t+1}.$$

We can see the following:

1. *The function y is increasing as* $\dfrac{dy}{dt} = 1 - \dfrac{1}{2}\,\dfrac{1}{(t+1)^2} > 0, \quad \forall t \geqslant 0.$

2. $\dfrac{dy}{dt}$ *is bounded as* $\dfrac{1}{2} \leq \dfrac{dy}{dt} \leq 1$ *and so* $\dfrac{dt}{dy}$ *is bounded with* $1 \leq \dfrac{dt}{dy} \leq 2.$

3. *By choosing* $t_1 = 1$ *we have* $f(t_1) = f(1) = \alpha = \dfrac{3}{4}.$

4. *From (3) and* $f(t) - \alpha = f(t - \tau(t)), \quad \forall t \geq t_1,$ *we can find that:*

$$\tau(t) = \frac{\sqrt{16t^4 + 40t^3 + 17t^2 - 34t - 23} + 4t^2 + 11t + 5}{8t + 8}, \quad \forall t \geq 1$$

   and when $t = 1$, *it will be* $\tau(1) = 1$.

   We obtain

$$\frac{dz}{dy} = \frac{2(t+1)^2}{2(t+1)^2 - 1} Az(y - \alpha) + \frac{2(t+1)^2}{2(t+1)^2 - 1} Bv(y),$$

   which is

$$\frac{dz}{dy} = \frac{2y + \sqrt{4y(y+3)+1} + 3}{2\sqrt{4y(y+3)+1}} Az(y - \alpha) + \frac{2y + \sqrt{4y(y+3)+1} + 3}{2\sqrt{4y(y+3)+1}} Bv(y), \quad (4.9)$$

   where $g(y) = \frac{2y+\sqrt{4y(y+3)+1}+3}{2\sqrt{4y(y+3)+1}}$.

5. *If we put $\alpha = 0$, $A = -\lambda$ and $B = 1$ in equation (4.9), we get*

$$\frac{dz}{dy} = -\lambda\,\frac{2y + \sqrt{4y(y+3)+1} + 3}{2\sqrt{4y(y+3)+1}} z(y) + v(y). \quad (4.10)$$

Equation (4.10) can be solved by using the integrating factor

$$\exp\left(\lambda \int_1^y \frac{2y+\sqrt{4y(y+3)+1}+3}{2\sqrt{4y(y+3)+1}}\, dy\right),$$

to get the solution

$$z(y) = C\ \exp\left[\frac{-\lambda}{4}(\sqrt{4y(y+3)+1}+2y)\right]$$
$$+\exp\left[\frac{-\lambda}{4}(\sqrt{4y(y+3)+1}+2y)\right]\int_1^y \exp\left[\frac{\lambda}{4}(\sqrt{4\zeta(\zeta+3)+1}+2\zeta)\right] v(\zeta)\, d\zeta,$$

which is bounded when $v(y)$ is bounded.

4.4 Instability of Ordinary Delay Systems

In this section, we discuss instability of the ordinary-delay input-output system

$$z'(y) = Ag(y)z(y-\alpha) + Bv(y) \quad (y > 0), \tag{4.11}$$

where $z(y)$ is in the Hilbert space H and $v(y)$ is in the Hilbert space K, both assumed zero for $y \le 0$, the operators $A : H \to H, B : H \to K$ are bounded and we assume B is invertible, $g(y)$ is a continuous function and the delay α is positive. The instability is obtained by choosing a specific output $z(y) \notin L^2$ (resp. not in L^∞) that gives an input $v(y) \in L^2$ (resp. in L^∞). L^2 stability of the mapping from v to z is not always the same as L^2 stability of the mapping from u to x, because there is a change of variables. It corresponds to mappings between weighted L^2 spaces, which are discussed further in Chapter 5.

4.4.1 L^2-instability

Theorem 4.4.1. *Suppose z in equation (4.11) is a function on $(0,\infty)$ with $z(y) \notin L^2, z'(y) \in L^2$ and $g(y)z(y-\alpha) \in L^2$ then $v(y) \in L^2$ and the system is unstable.*

Proof. From the assumptions where the output $z(y) \notin L^2$ and as

$$v(y) = B^{-1}\left[z'(y) - Ag(y)z(y-\alpha)\right],$$

is the input where its terms in L^2 that gives $v(y) \in L^2$ which proves the instability. $\square$

Example 4.4.2. *Suppose that $0 < \alpha \leq 1$. Take $B = I = A, z(y) = (y+1)^\gamma$ and $g(y) = (y+1)^\beta$ where $-\frac{1}{2} \leq \gamma < \frac{1}{2}$, $\beta \geq -1$ and $\gamma + \beta < -\frac{1}{2}$ in (4.11). In this example we have $g(y) \notin L^1$, which we want since $g(y) = \frac{dt}{dy}$ and $t \to \infty$ as $y \to \infty$. Therefore*

$$v(y) = \gamma(y+1)^{\gamma-1} - (y+1)^\beta(y+1-\alpha)^\gamma,$$

which is in L^2. For such choices the L^2-instability is obtained.
In particular, $z(y) = g(y) = (y+1)^{-\frac{1}{2}}$ gives

$$t = 2(y+1)^{\frac{1}{2}} - 2, \tag{4.12}$$

and so

$$v(y) = -\frac{1}{2}(y+1)^{-\frac{3}{2}} - (y+1)^{-\frac{1}{2}}(y+1-\alpha)^{-\frac{1}{2}}.$$

For this particular example we can get $\tau(t)$ as follows:

First, from 4.12 where $y = f(t) = \frac{1}{4}(t+2)^2 - 1$, and from Section 4.3 where $f(t) - \alpha = f(t - \tau(t))$ that satisfies for every $t \geq t_1$, we get

$$\frac{1}{4}(t+2)^2 - 1 - \alpha = \frac{1}{4}(t - \tau(t) + 2)^2 - 1,$$

that gives

$$\tau^2(t) - 2(t+2)\tau(t) + 4\alpha = 0,$$

and so

$$\tau(t) = (t+2) - \sqrt{(t+2)^2 - 4\alpha} \quad \forall t \geq t_1.$$

Then, we can find t_1 *from Section 4.3 where*

$$t_1 = \tau(t_1),$$
$$t_1 = (t_1 + 2) - \sqrt{(t_1 + 2)^2 - 4\alpha},$$

and so

$$t_1 = \sqrt{4 + 4\alpha} - 2,$$

that becomes $t_1 = \sqrt{8} - 2$ *for* $\alpha = 1$*. Additionally, the* α *needs to satisfy* $f(t_1) = \alpha$ *and we can check that*

$$f(\sqrt{8} - 2) = 1.$$

Therefore from (4.1)

$$\dot{x}(t) = Ax\left(-2 + \sqrt{(t + 2)^2 - 4}\right) + Bu(t) \quad \forall t \geq t_1.$$

4.4.2 BIBO instability

Theorem 4.4.3. *Suppose* z *in equation (4.11) is a function on* $(0, \infty)$ *with* $z(y) \notin L^\infty, z'(y) \in L^\infty$ *and* $g(y)z(y - \alpha) \in L^\infty$ *then* $v(y) \in L^\infty$ *and the system is unstable.*

Proof. From the assumptions where the output $z(y) \notin L^\infty$ and as

$$v(y) = B^{-1}\left[z'(y) - Ag(y)z(y - \alpha)\right],$$

is the input where its terms in L^∞ that gives $v(y) \in L^\infty$ which proves the instability. □

Example 4.4.4. *Suppose that* $0 < \alpha \leq 1$*. Take* $B = I = A, z(y) = (y + 1)^\gamma$ *and* $g(y) = (y + 1)^\beta$ *where* $0 < \gamma \leq 1$*,* $\beta \geq -1$ *and* $\gamma + \beta < 0$ *in (4.11). Therefore*

$$v(y) = \gamma(y + 1)^{\gamma - 1} - (y + 1)^\beta (y + 1 - \alpha)^\gamma,$$

which is in L^∞*. For such choices the BIBO instability is obtained.*

In particular, $z(y) = (y+1)^{\frac{1}{2}}$ *and* $g(y) = (y+1)^{-\frac{1}{2}}$ *gives* $t = 2(y+1)^{\frac{1}{2}} - 2$, *and so*

$$v(y) = \frac{1}{2}(y+1)^{-\frac{1}{2}} - (y+1)^{\frac{1}{2}}(y+1-\alpha)^{-\frac{1}{2}}.$$

For this particular example $\tau(t)$ *with* $\alpha = 1$ *is the same as we obtained in Example 4.4.2.*

Chapter 5

Autonomous and Non-Autonomous Systems with Weight

5.1 Introduction

In this chapter we investigate the stability for the system

$$\frac{dz}{dy} = g_1(y)Az(y-\alpha) + g_2(y)Bv(y), \quad y \geq 0, \tag{5.1}$$

with $\alpha \geq 0$, $z(y) = f(y)$ in H a Hilbert space for $-\alpha \leq y \leq 0$, where g_1, g_2 are continuous functions on $[0, \infty)$ and A, B are matrices, bounded or unbounded operators. Stability means that if the input $v \in L^2(0, \infty; H)$ or $v \in Ł^\infty(0, \infty; H)$, the output z will be also in the same space.

We begin by giving conditions for stability of autonomous systems with and without delay, and then we consider non-autonomous systems. At the end of the chapter we discuss the significance of changing variables where we have to use weighted L^2 spaces. Additionally, we extend the recent theory of Zen spaces [25, 26, 39] to functions taking values in a Hilbert space H, where the Laplace transform $\mathcal{L}$ provides an isometric embedding from a weighted function space $L^2(0,\infty,w(t)dt;H)$ into a space $A^2_\nu(\mathbb{C}_+,H)$ of analytic operator-valued functions. From this we prove a result showing that the H^∞ norm can be used to measure the gain (operator-norm) in the context of a wide variety of weighted L^2 spaces, and thus show that various notions of stability are equivalent.

5.2 The Stability of Autonomous Systems Without Delay

5.2.1 Introduction

If we have an equation with non-zero initial conditions

$$\frac{dz}{dy} = g_1(y)Az(y) + g_2(y)Bv(y), \quad z(0) = z_0, \tag{5.2}$$

then if we can solve

$$\begin{aligned} \frac{dz_1}{dy} &= g_1(y)Az_1(y), \quad z_1(0) = z_0, \\ \frac{dz_2}{dy} &= g_1(y)Az_2(y) + g_2(y)Bv(y), \quad z_2(0) = 0, \end{aligned}$$

the solution to (5.2) is $z = z_1 + z_2$. So we start by looking at the stability of the autonomous equation

$$\frac{dz(y)}{dy} = g_1(y)Az(y), \quad z(0) = z_0 \neq 0. \tag{5.3}$$

The stability of this system means that the solution $z \in \mathbf{L}^2(0,\infty;H)$ or $z \in \mathbf{L}^\infty(0,\infty;H)$ for every $z_0 \in H$ where H is a Hilbert space. Because of that, a bounded solution of equation (5.3) makes the system BIBO stable. This will depend on A and its properties. A could be a scalar, a matrix or an operator.

5.2.2 A is an operator that generates a semigroup

For such A we need before discussing the stability to give the solution of equation (5.3).

Getting the solution

Let us take A in equation (5.3), which has an initial condition $z_0 \in H$, is an operator that generates a bounded semigroup $T(t)$ and $g_1(y) > 0$ for $y > 0$. From the solution of the scalar case, we can expect that the solution of this case should be

$$z(y) = T\left(\int_0^y g_1(t)\ dt\right) z_0. \tag{5.4}$$

Now we need to check that (5.4) is the right solution of equation (5.3) as follows:

First: Checking for $g_1(y) = 1$

For $g_1(y) = 1$, the mild solution is:

$$z(y) = T(y)z_0. \tag{5.5}$$

By differentiating (5.5) for $z_0 \in D(A)$, we get

$$\begin{aligned} z'(y) &= AT(y)z_0, \\ &= Az(y), \end{aligned}$$

that is exactly equation (5.3) with $g_1(y) = 1$. So (5.4) is the right solution.

Then: Checking for $g_1(y) \geq 0$

By differentiating (5.4), we get

$$\begin{aligned} z'(y) &= AT\left(\int_0^y g_1(t)\ dt\right) z_0 g_1(y), \\ &= Ag_1(y)z(y). \end{aligned}$$

We can conclude that if A in equation is an operator that generates a bounded semigroup, then (5.4) is the mild solution of equation (5.3) with $z_0 \in H$.

Theorem 5.2.1. *If $\delta > 0$ and $g_1(y) \geq \delta$ for all $y \geq 0$ in equation (5.3) and A an operator that generates a semigroup that is uniformly exponentially stable then the solution (5.4) of this equation will satisfy $z \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$.*

Proof. Suppose

$$\left\| T\left(\int_0^y g_1(t)\ dt\right)\right\| \leq M\mathrm{e}^{-\beta\int_0^y g_1(t)\ dt},$$

for all t and some $M \geq 1$, $\beta > 0$. Then

$$\begin{aligned} \|z(y)\| &= \left\| T\left(\int_0^y g_1(t)\ dt\right) z_0\right\|, \\ &\leq \|z_0\| M\mathrm{e}^{-\beta\int_0^y g_1(t)\ dt}, \\ &\leq \|z_0\| M\mathrm{e}^{-\beta\int_0^y \delta\ dt} = \|z_0\| M\mathrm{e}^{-\beta\delta y}, \end{aligned}$$

and so $z \in L^\infty(0,\infty;H) \cap L^2(0,\infty;H)$. □

5.2.3 Special Cases of A

- <u>**A is a scalar, a matrix or a bounded operator**</u>

  In this case we can take $z_0 \in H$ not just $D(A)$. Equation (5.3) can be solved directly using an integrating factor as follows:

Multiplying it by $F(y) = \mathrm{e}^{-\int_0^y Ag_1(t)\ dt}$ gives

$$
\begin{aligned}
F(y)\frac{dz(y)}{dy} - g_1(y)AF(y)z(y) = 0,\\
(F(y)z(y))' = 0.
\end{aligned}
$$

By the integration

$$
\begin{aligned}
F(y)z(y) &= constant,\\
z(y) &= \mathrm{e}^{(\int_0^y g_1(t)\ dt)A} z_0. \qquad (5.6)
\end{aligned}
$$

where $z_0 = z(0)$ is the starting point at $y = 0$. Here, the behaviour of $F(y)$ determines the stability of $z(y)$ in (5.6).

From the above, we can deduce that the absence of the input leads to the behaviour of z being controlled by F.

Example 5.2.2. *Suppose $H = \mathbb{C}$, $A = -\lambda$, where $\lambda \in \mathbb{C}_+$, $Re\ \lambda = m > 0$ and $\delta > 0$ and $g_1(y) \geq \delta$ for all $y \geq 0$. From (5.6), we get*

$$
\begin{aligned}
|z(y)| =& \ |\mathrm{e}^{-\lambda \int_0^y g_1(t)\ dt} z_0|,\\
\leq& \ |z_0|\mathrm{e}^{-m\delta y}. \qquad (5.7)
\end{aligned}
$$

The solution z is clearly bounded, and in fact in $L^2(0,\infty)$ as well, since (5.7) gives

$$
\begin{aligned}
\int_0^\infty |z(y)|^2\ dy &\leq |z_0|^2 \int_0^\infty \mathrm{e}^{-2m\delta y}\ dy,\\
&= \frac{|z_0|^2}{2m\delta}\\
&< \infty.
\end{aligned}
$$

Example 5.2.3. *Suppose*

$$
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},
$$

then it is clear that A does not have a basis of eigenvectors. Next, we will solve equation (5.3) depending on this A and with

$$z(y) = \begin{bmatrix} z_1(y) \\ z_2(y) \end{bmatrix} \in \mathbb{C}^2,$$

as follows:

We have the equation

$$\begin{bmatrix} \frac{dz_1(y)}{dy} \\ \frac{dz_2(y)}{dy} \end{bmatrix} = g_1(y) \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} z_1(y) \\ z_2(y) \end{bmatrix}, \quad z(0) = \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix},$$

where $g_1(y) \geq \delta$ for all y and $\delta > 0$. From the above, there are two differential equations

$$\frac{dz_1(y)}{dy} = g_1(y) z_2(y), \tag{5.8}$$

$$\frac{dz_2(y)}{dy} = 0. \tag{5.9}$$

The solution of equation (5.9) is $z_2(y) = z_2(0)$ and then the solution of equation (5.8) is $z_1(y) = z_2(0) \int_0^y g_1(y)\, dy + z_1(0)$. So it is possible here to solve equation (5.3), but it has a different solution as A does not have a basis of eigenvectors. We see this system is an unstable here.

<u>Another method</u>

From A we can get the semigroup

$$T(y) = \mathrm{e}^{Ay} = I + Ay + \frac{A^2 y^2}{2!} + ...,$$

but because $A^2 = 0$, then

$$\mathrm{e}^{Ay} = \begin{bmatrix} 1 & y \\ 0 & 1 \end{bmatrix},$$

is not bounded and then the solution given by (5.6) which is

$$\begin{bmatrix} z_1(y) \\ z_2(y) \end{bmatrix} = \begin{bmatrix} 1 & \int_0^y g_1(y)\ dy \\ 0 & 1 \end{bmatrix} \begin{bmatrix} z_1(0) \\ z_2(0) \end{bmatrix},$$

is not stable.

- **A is an unbounded operator generates a bounded semigroup**

We have to be careful when we solve equation (5.3) with an unbounded A that generates a semigroup $T(y)$. Here, A is not defined everywhere on the Banach space $\mathcal{X}$. However, the domain of A , $\mathcal{D}(A)$, is a dense subspace of $\mathcal{X}$. In addition, the mild solution of the equation is

$$z(y) = T\left(\int_0^y g_1(t)\ dt\right) z_0.$$

This is a classical solution if $z_0 \in D(A)$.

Example 5.2.4. *(Infinite Diagonal Matrix)*

Let us take H a Hilbert space, (x_n) an orthonormal basis in H and A an infinite diagonal matrix which means

$$Ax_n = \lambda_n x_n,$$

where λ_n is an eigenvalue for every n. Then, we can write

$$\mathrm{e}^{Ay} x_n = \mathrm{e}^{\lambda_n y} x_n, \quad y \geq 0. \tag{5.10}$$

To get the stability, we suppose that Re $\lambda_n \leq 0$ and so $|\mathrm{e}^{\lambda_n y}| \leq 1$.

To make what was written above clear, let take $z \in H$ and then it is possible to write z as infinite sum

$$z = \sum_{n=1}^{\infty} \langle z, x_n \rangle x_n,$$

where $\langle z, x_n\rangle$ *is the coefficient* z_n *and* z *depends on* y *and so* $\frac{d}{dy}z(y) = \sum_{n=1}^{\infty}\frac{d}{dy}\langle z(y), x_n\rangle x_n$. *From* $Az(y) = \sum_{n=1}^{\infty}\lambda_n\langle z(y), x_n\rangle x_n$, *and from equation (5.3), we can get*

$$\frac{d}{dy}z_n(y) = g_1(y)\lambda_n z_n(y). \tag{5.11}$$

Equation (5.11) is very easy to solve as it is just scalar equation and from (5.4) its solution is

$$z_n(y) = \mathrm{e}^{(\lambda_n \int_0^y g_1(t)\ dt)} z_n(0),$$

where $z_0 = \sum_0^{\infty} z_n(0)x_n = \sum_0^{\infty} c_n x_n$ *is the starting point. Then, the end is just*

$$\begin{aligned} z(y) &= \sum_0^{\infty} z_n(y)x_n, \\ &= \sum_0^{\infty} c_n \mathrm{e}^{(\lambda_n \int_0^y g_1(t)\ dt)} x_n. \end{aligned}$$

Because $|\mathrm{e}^{(\lambda_n \int_0^y g_1(t)\ dt)}| \leq 1$ *since Re* $\lambda_n \leq 0$ *and* $g_1(y) \geq 0$, *we get*

$$|c_n \mathrm{e}^{(\lambda_n \int_0^y g_1(t)\ dt)}| \leq |c_n|,$$

which ensures $z(y)$ *is a vector in* H *and then we get the solution of equation (5.3), which is BIBO stable since* $\|z(y)\| \leq \|z_0\|$. *If Re*$\lambda_n \leq -\epsilon$ *for all* n *and* $g_1(y) \geq \delta$ *for all* y *then* $\|z(y)\| \leq \mathrm{e}^{-\epsilon\delta y}\|z_0\|$ *so the solution is* L^2 *stable.*

Remark 5.2.5. *From (5.10)*

$$T(y)x = \mathrm{e}^{Ay}x = \sum_{n=1}^{\infty} \mathrm{e}^{\lambda_n y}\langle x, x_n\rangle x_n,$$

and so

$$\begin{aligned}\|T(y)x\|^2 &= \sum_{n=1}^{\infty} \quad |e^{\lambda_n y}|^2 \quad |\langle x, x_n\rangle x_n|^2,\\ &\leq \sum_{n=1}^{\infty} \quad |\langle x, x_n\rangle|^2,\\ &= \|x\|^2,\end{aligned}$$

and so

$$\begin{aligned}T(y) \quad \textit{(with fixed } y\textit{)} \quad \textit{is bounded} &\iff \sup_n |e^{\lambda_n y}| < \infty,\\ &\iff \sup Re\ \lambda_n < \infty.\end{aligned}$$

5.3 Autonomous Systems with delay

The autonomous version of equation (5.1), which is

$$\frac{dz}{dy} = g_1(y)Az(y-\alpha), \quad z(y) = f(y); \ -\alpha \leq y \leq 0, \tag{5.12}$$

where $f \in C[-\alpha, 0]$ can be solved in intervals as shown in the next theorem.

Theorem 5.3.1. *Given the delay equation (5.12) with A bounded such that $f \in C[-\alpha, 0]$ then we can obtain the solution of (5.12) by iteratively solving the equation on intervals. For $g_1 \in L^1_{loc}(0, \infty)$ the solution is locally bounded.*

Proof. The equation

$$\frac{dz}{dy} = g_1(y)Az(y-\alpha),$$

with $z(y) = 0$ for $y \leq -\alpha$, can be solved iteratively on intervals. So

$$z(y) = f(y), \tag{5.13}$$

for $-\alpha \leq y \leq 0$.

Then for $0 \leq y \leq \alpha$,

$$z(y) = z(0) + \int_0^y g_1(p) A z(p - \alpha) \quad dp, \tag{5.14}$$

and we know $z(\alpha)$ and $z(p-\alpha)$ from (5.13).

Then for $\alpha \leq y \leq 2\alpha$,

$$z(y) = z(\alpha) + \int_\alpha^y g_1(p) A z(p - \alpha) \quad dp,$$

and we know $z(\alpha)$ and $z(p-\alpha)$ from (5.14). And so on.

It is clear that z is bounded on each interval $[n\alpha, (n+1)\alpha]$, with $n \in \mathbb{N}$.

□

5.4 The Stability of Non-Autonomous Systems Without Delay

In this section we study BIBO stability, which is L^∞ stability, and L^2 stability of non-autonomous systems without delay

$$\frac{dz}{dy} = A g_1(y) z(y) + v(y), \quad z_0 = 0. \tag{5.15}$$

Because BIBO stability is the easiest case, we start with it.

5.4.1 BIBO Stability

Getting this stability requires starting with $v \in L^\infty(\mathbb{C}_+; H)$, we suppose that $\|v\|_\infty = \epsilon$, then to get $z \in L^\infty(\mathbb{C}_+; H)$ that depends on A and putting specific conditions on the continuous function $g_1(y)$.

- **A is a scalar or a diagonal matrix**

Putting $H = \mathbb{C}$, $A = -\lambda$, $\lambda \in \mathbb{C}_+$ and $Re\lambda = m > 0$ in equation (5.15) gives the equation

$$\frac{dz}{dy} = -\lambda g_1(y)z(y) + v(y), \quad z_0 = 0, \tag{5.16}$$

which can be solved using the integrating factor $F(y) = \mathrm{e}^{\lambda \int_0^y g_1(t)\ dt}$ as follows:

$$\begin{aligned} F(y)z'(y) + \lambda g_1(y)F(y)z(y) &= F(y)v(y), \\ (F(y)z(y))' &= F(y)v(y), \\ z(y) &= \frac{1}{F(y)} \int_0^y F(t)v(t)\ dt, \\ z(y) = \mathrm{e}^{-\lambda \int_0^y g_1(t)\ dt} &\int_0^y \mathrm{e}^{\lambda \int_0^t g_1(s)\ ds} v(t)\ dt. \end{aligned}$$

Then

$$|z(y)| \leq \epsilon \mathrm{e}^{-m \int_0^y g_1(t)\ dt} \int_0^y \mathrm{e}^{m \int_0^t g_1(s)\ ds}\ dt. \tag{5.17}$$

Getting $z(y) \in L^\infty$ requires

$$\mathrm{e}^{m \int_0^y g_1(t)\ dt} \geq \gamma \int_0^y \mathrm{e}^{m \int_0^t g_1(s)\ ds}\ dt, \quad \text{where } \gamma > 0, \tag{5.18}$$

which holds if

$$|F(y)| \geq \gamma \int_0^y |F(t)|\ dt,$$

and that can be obtained when $g_1(y)$ satisfies the conditions given in the following theorem, which is the main result of this section.

Theorem 5.4.1. *If $\delta > 0$ and $g_1(y) \geq \delta$ for all $y \geq 0$ in equation (5.16) and $v \in L^\infty(\mathbb{R}_+)$ then the solution of this equation will be bounded for $y \geq 0$, i.e., $z \in L^\infty(\mathbb{R}_+)$.*

Note Satisfying (5.18) is sufficient to prove BIBO stability of equation (5.16)

Proof. From the integrating factor of equation (5.16)

$$F(y) = \exp\left(\lambda \int_0^y g_1(t)\ dt\right),$$

we can take $t < y$ to get

$$\begin{aligned}
F(y) &= \exp\left(\lambda \int_0^y g_1(t)\ dt\right), \\
&= \exp\left(\lambda \int_0^t g_1(s)\ ds\ + \lambda \int_t^y g_1(s)\ ds\right), \\
&= \exp\left(\lambda \int_0^t g_1(s)\ ds\ \right) \exp\left(\lambda \int_t^y g_1(s)\ ds\right), \\
&= F(t) \exp\left(\lambda \int_t^y g_1(s)\ ds\right).
\end{aligned}$$

That means

$$\begin{aligned}
&|F(y)| = |F(t)| \exp\left(m \int_t^y g_1(s)\ ds\right), \\
&|F(y)| \geq |F(t)| \exp\left(m\delta(y-t)\right),
\end{aligned}$$

and so

$$|F(t)| \leq |F(y)| \exp\left(-m\delta(y-t)\right),$$

by integrating

$$\int_0^y |F(t)|\ dt \leq |F(y)| \int_0^y \exp\left(-m\delta(y-t)\right)\ dt,$$

putting $x = y - t$ gives

$$\begin{aligned}
\int_0^y |F(t)|\ dt &\leq |F(y)| \int_0^y \exp\left(-m\delta x\right)\ dx, \\
&\leq |F(y)| \int_0^\infty \exp\left(-m\delta x\right)\ dx, \\
&= \frac{|F(y)|}{m\delta}.
\end{aligned}$$

which means (5.18) is obtained by choosing $\gamma = m\delta$. Applying (5.18) in (5.17) gives

$$\begin{aligned} |z(y)| &\leq \epsilon \mathrm{e}^{-m\int_0^y g_1(t)\ dt} \frac{\mathrm{e}^{m\int_0^y g_1(t)\ dt}}{m\delta}, \\ &\to \frac{\epsilon}{m\delta} \quad \text{as} \quad y \to \infty, \end{aligned}$$

and so $z \in L^\infty$. □

Corollary 5.4.2. *Theorem 5.4.1 can be applied if A in equation (5.15) is a diagonal matrix with eigenvalues $-\lambda_{ii} \in \mathbb{C}_-$, $i = 1, 2, ...$ such that $\exists\ c > 0$ with $Re\ \lambda_{ii} \geq c$ for all i.*

Remark 5.4.3.

1. We know if $\delta > 0$ and $g_1(y) \geq \delta$ for all $y \geq 0$, equation (5.16) is stable. In general suppose $g_1(y) = G'(y)$, where $G(y)$ is an absolutely continuous and increasing function. Because of that, from the inverse function theorem, G is an invertible function and from Section 4.3 we have
$$g_1(y) = \frac{dt}{dy}; \quad y = f(t),$$
and to get more results we may suppose $g_1(y)$ not continuous but locally L^1, and so
$$t(y) = f^{-1}(y) = G(y). \tag{5.19}$$
Therefore
$$y(t) = G^{-1}(t) = f(t). \tag{5.20}$$

2. For the previous G, we can find $\tau(t)$ as in Section 4.3 as follows:
$$f(t) - \alpha = f(t - \tau(t)),$$
from (5.19) and (5.20), we get
$$\begin{aligned} G^{-1}(t) - \alpha &= G^{-1}(t - \tau(t)), \\ G(G^{-1}(t) - \alpha) &= t - \tau(t), \end{aligned}$$

and so

$$\tau(t) = t - G(G^{-1}(t) - \alpha). \tag{5.21}$$

Examples

Example 5.4.4. (The Heat Equation with a weight)
Let us take H a Hilbert space, (x_n) an orthonormal basis in H, A in equation (5.15) to be as we have mentioned in Example 1.6.12, where the eigenvalues of A are $-n^2\pi^2$ ($\lambda_n = n^2, n = 1, 2, ...$), $\delta > 0$ and $g_1(y) \geq \delta$ and the input $v \in L^\infty(0, \infty; H)$ to be

$$v(y) = \sum_{n=1}^{\infty} v_n(y)x_n,$$

such that

$$\|v\|^2 = \sum_{n=1}^{\infty} |v_n|^2.$$

Then, we get the output

$$z(y) = \sum_{n=1}^{\infty} z_n(y)x_n,$$

such that

$$\|z\|^2 = \sum_{n=1}^{\infty} |z_n|^2.$$

where z_n satisfies equation (5.15) with v_n and λ_n. From Theorem 5.4.1, we get

$$\sup_y \|z\| < \infty,$$

which means BIBO stability.

Example 5.4.5. For the equation

$$\frac{dz}{dy} = -\lambda\left(1 + \frac{1}{2\sqrt{y}}\right) z(y) + v(y),$$

it is obvious that $g_1(y) = 1 + \frac{1}{2\sqrt{y}} > 1,\ y > 0$. Then, for $\|v\|_\infty = \epsilon$ and

$Re\lambda = m > 0$, we have

$$
\begin{aligned}
|F(y)| &= \exp\left(m\int_0^y 1+\frac{1}{2\sqrt{t}}\ \ dt\right), \\
&= \exp\left(m\int_0^t 1+\frac{1}{2\sqrt{s}}\ \ ds\right)\ \exp\left(m\int_t^y 1+\frac{1}{2\sqrt{s}}\ \ ds\right), \\
&= \exp\left(m\int_0^t 1+\frac{1}{2\sqrt{s}}\ \ ds\right)\ \exp\left(m\int_t^y 1+\frac{1}{2\sqrt{s}}\ \ ds\right), \\
&\geq |F(t)|\ \exp(m(y-t)).
\end{aligned}
$$

And so

$$
\begin{aligned}
\int_0^y |F(t)|\ dt &\leq |F(y)|\ \int_0^y \exp(-m(y-t))\ dt, \\
&\leq |F(y)|\ \int_0^y \exp(-mx)\ dx, \\
&\leq \frac{|F(y)|}{m}.
\end{aligned}
\tag{5.22}
$$

Applying (5.22) in (5.17) gives the bounded solution of our equation as it satisfies

$$
|z(y)| \leq \frac{1}{m} \quad \to \frac{1}{m} \quad \text{as} \quad y \to \infty.
$$

For this example, we can obtain $y = f(t)$ as follows:

$$
g_1(y) = \frac{dt}{dy} = 1 + \frac{1}{2\sqrt{y}},
$$

and so

$$
t = y + \sqrt{y} = f^{-1}(y) = G(y), \tag{5.23}
$$

giving the quadratic equation

$$
y + \sqrt{y} - t = 0,
$$

where $a = 1, b = 1$ and $c = -t$. Therefore

$$
\sqrt{y} = \frac{-1 \pm \sqrt{1+4t}}{2},
$$

and for $y > 0$ we choose the positive sign of the square root to get

$$y = \left(\frac{-1+\sqrt{1+4t}}{2}\right)^2 = f(t) = G^{-1}(t). \tag{5.24}$$

In addition, we can find $\tau(t)$ by using (5.21) from (5.23) and (5.24) as follows:

$$\begin{aligned}\tau(t) &= t - G\left(\left(\frac{-1+\sqrt{1+4t}}{2}\right)^2 - \alpha\right),\\ &= t - \left(\left(\frac{-1+\sqrt{1+4t}}{2}\right)^2 - \alpha\right) - \sqrt{\left(\frac{-1+\sqrt{1+4t}}{2}\right)^2 - \alpha}.\end{aligned}$$

Example 5.4.6. For equation (5.16) we choose $g(y)$ to be

$$g_1(y) = \begin{cases} 2, & y \in (2n-2, 2n-1) \\ 1, & y \in (2n-1, 2n) \end{cases}, \quad n \in \mathbb{N}.$$

This is an example of a switching system (for other resources on switching systems see [8, 45, 35]) and we can obtain the solution on intervals, showing that the system is well-posed. Therefore we find the following functions:

1. Let I be the integrating factor, which is

$$I = \begin{cases} \mathrm{e}^{2\lambda(y-2(n-1))}, & y \in (2n-2, 2n-1) \\ \mathrm{e}^{\lambda(y-(2n-1))}, & y \in (2n-1, 2n) \end{cases}, n \in \mathbb{N}.$$

2. $z(y)$ the solution of equation (5.16) for
   - $y \in (2n-2, 2n-1)$ is given by

$$z(y) = z(2n-2)\mathrm{e}^{-2\lambda(y-2(n-1))} + \mathrm{e}^{-2\lambda(y-2(n-1))} \int_{2n-2}^{y} \mathrm{e}^{2\lambda(\zeta-2(n-1))} v(\zeta) d\zeta,$$

   - $y \in (2n-1, 2n)$ is given by

$$z(y) = z(2n-1)\mathrm{e}^{-\lambda(y-(2n-1))} + \mathrm{e}^{-\lambda(y-(2n-1))} \int_{2n-1}^{y} \mathrm{e}^{\lambda(\zeta-(2n-1))} v(\zeta)\, d\zeta,$$

where

$$\begin{cases} z(2n-1) = z(2n-2)\mathrm{e}^{-2\lambda} + \mathrm{e}^{-2\lambda}\displaystyle\int_{2n-2}^{2n-1} \mathrm{e}^{2\lambda(\zeta-(2n-2))} v(\zeta)\ d\zeta \\ \\ z(2n) = z(2n-1)\mathrm{e}^{-\lambda} + \mathrm{e}^{-\lambda}\displaystyle\int_{2n-1}^{2n} \mathrm{e}^{\lambda(\zeta-(2n-1))} v(\zeta)\ d\zeta \end{cases}.$$

By choosing $v(y) = 1$ in $z(y)$ and in the initial conditions, we get

$$z(y) = \begin{cases} z(2n-2)\mathrm{e}^{-2\lambda(y-2(n-1))} + \frac{1}{2\lambda}(1-\mathrm{e}^{-2\lambda(y-2(n-1))}), \\ \qquad\qquad\qquad\qquad y \in (2n-2, 2n-1) \\ \\ z(2n-1)\mathrm{e}^{-\lambda(y-(2n-1))} + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda(y-(2n-1))}), \\ \qquad\qquad\qquad\qquad y \in (2n-1, 2n) \end{cases},$$

where

$$\begin{cases} z(2n-1) = z(2n-2)\mathrm{e}^{-2\lambda} + \frac{1}{2\lambda}(1-\mathrm{e}^{-2\lambda}) \\ \\ z(2n) = z(2n-1)\mathrm{e}^{-\lambda} + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda}) \end{cases},$$

which means

$$z(y) = \begin{cases} \frac{1}{2\lambda}(1-\mathrm{e}^{-2\lambda y}), & y \in (0,1) \\ z(1)\mathrm{e}^{-\lambda(y-1)} + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda(y-1)}) & y \in (1,2) \\ z(2)\mathrm{e}^{-2\lambda(y-2)} + \frac{1}{2\lambda}(1-\mathrm{e}^{-2\lambda(y-2)}) & y \in (2,3) \\ z(3)\mathrm{e}^{-\lambda(y-3)} + \frac{1}{\lambda}(1-\mathrm{e}^{-\lambda(y-3)}) & y \in (3,4) \\ . \end{cases},$$

where

$$\begin{cases} z(0) = z_0 = 0 \\ z(1) = \frac{1}{2\lambda}(1 - \mathrm{e}^{-2\lambda}) \\ z(2) = z(1)\mathrm{e}^{-\lambda} + \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda}) \\ z(3) = z(2)\mathrm{e}^{-2\lambda} + \frac{1}{2\lambda}(1 - \mathrm{e}^{-2\lambda}) \\ . \end{cases} .$$

To prove that the solution $z(y)$ is bounded, we need to do the following steps:

(a) Prove that the sequence $(z(2n))$ is bounded where

$$\begin{aligned} z(2n) &= z(2n-1)\mathrm{e}^{-\lambda} + \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda}), \\ &= [z(2n-2)\mathrm{e}^{-2\lambda} + \frac{1}{\lambda}(1 - \mathrm{e}^{-2\lambda})]\mathrm{e}^{-\lambda} + \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda}), \end{aligned}$$

and so

$$z(2n) = z(2n-2)\mathrm{e}^{-3\lambda} + \frac{1}{\lambda}(1 - \mathrm{e}^{-2\lambda})\mathrm{e}^{-\lambda} + \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda}). \tag{5.25}$$

By putting $A = \mathrm{e}^{-3\lambda}$, $B = \frac{1}{\lambda}(1-\mathrm{e}^{-2\lambda})\mathrm{e}^{-\lambda}+\frac{1}{\lambda}(1-\mathrm{e}^{-\lambda})$ and $z(2n) = a_n$, we can get from (5.25)

– For $n = 1$

$$z(2) = a_1 = B.$$

– For $n = 2$

$$\begin{aligned} z(4) = a_2 &= a_1 A + B, \\ &= BA + B. \end{aligned}$$

– For $n = 3$

$$\begin{aligned} z(6) = a_3 &= a_2 A + B, \\ &= BA^2 + BA + B, \end{aligned}$$

and so on...

So the general formula of $z(2n)$ is

$$\begin{aligned} z(2n) = a_n &= BA^{n-1} + BA^{n-2} ... + BA + B, \\ &= B(A^{n-1} + A^{n-2} ... * + A + 1), \\ &= B \sum_{m=1}^{n} A^{m-1}, \end{aligned}$$

where $\sum_{m=1}^{n} A^{m-1}$ is geometric series with the sum $\frac{1-A^n}{1-A}$ and because

$$|A| = |\mathrm{e}^{-3\lambda}| = \mathrm{e}^{-3m} < 1, \quad \text{where} \quad m = Re\lambda > 0,$$

the sum converges as $n \to 0$ which proves the sequence $(z(2n))$ is bounded.

(b) Prove that the sequence $(z(2n-1))$ is bounded where

$$z(2n-1) = z(2n-2)\mathrm{e}^{-2\lambda} + \frac{1}{2\lambda}(1 - \mathrm{e}^{-2\lambda}). \tag{5.26}$$

By putting $C = \mathrm{e}^{-2\lambda},\ D = \frac{1}{2\lambda}(1 - \mathrm{e}^{-2\lambda})$ and $z(2n-1) = b_n$, we can get from (5.26)

- For $n = 1$

$$z(1) = b_1 = D.$$

- For $n = 2$

$$z(3) = b_2 = z(2)C + D.$$

- For $n = 3$

$$z(5) = b_3 = z(4)C + D,$$

and so on...

So the general formula of $z(2n-1)$ is

$$z(2n-1) = b_n = z(2n-2)C + D.$$

Therefore the sequence $(z(2n-1))$ is bounded because of the convergence of $(z(2n))$.

(c) Prove that the solution $z(y)$ is bounded between the intervals $(2n-1, 2n)$ where

$$z(y) = z(2n-1)\mathrm{e}^{-\lambda(y-(2n-1))} + \frac{1}{\lambda}(1 - \mathrm{e}^{-\lambda(y-(2n-1))}),$$

and we know that

$$|z(y)| \leq |z(2n-1)| + \frac{2}{|\lambda|}.$$

Therefore, $|z(y)|$ is bounded and the bound does not depend on y because $|z(2n-1)|$ is bounded and the bounds does not depend on n and so $|z(y)|$ is bounded in the interval $(2n-1, 2n)$. Similarly, $|z(y)|$ is bounded and the bound does not depend in n the interval $(2n-2, 2n-1)$. That proves $|z(y)|$ is bounded everywhere by a constant.

For this example, we can obtain $y = f(t)$ by finding $t(y)$ that equals to $G(y)$ as follows:

– On $y \in (2n-2, 2n-1)$, we get

$$\begin{aligned} t(y) &= 2[y - (2n-2)] + t(2n-2), \\ &= 2y - 4n + 4 + t(2n-2). \end{aligned} \tag{5.27}$$

– On $y \in (2n-1, 2n)$, we get

$$t(y) = y - (2n-1) + t(2n-1). \tag{5.28}$$

From (5.27) and (5.28) we can get $t(y)$ as follows:

$$t(y) = \begin{cases} 2y + t(0), & y \in (0,1) \quad \text{for } n = 1 \\ y - 1 + t(1), & y \in (1,2) \\ 2y - 4 + t(2), & y \in (2,3) \quad \text{for } n = 2 \\ y - 3 + t(3), & y \in (3,4) \\ . \end{cases},$$

and so

$$\begin{cases} t(1) = 2, \quad \text{where } t(0) = 0, \\ t(2) = 3, \\ t(3) = 5, \\ . \end{cases}.$$

Therefore, we can obtain that $t(2n-2) = 3n-3$ and $t(2n-1) = 3n-1$ which turns $t(y)$ in (5.27) and (5.28) into

$$t(y) = \begin{cases} 2y - (n-1), & y \in (2n-2, 2n-1) \\ y + n, & y \in (2n-1, 2n) \end{cases}, n \in \mathbb{N}.$$

Therefore $y(t)$ can be obtained from $t(y)$ as follows:

– As $t(y) = 2y - (n-1)$ for $y \in (2n-2, 2n-1)$, then

$$y(t) = \frac{1}{2}[t + (n-1)] \quad \text{for} \quad t \in (3n-3, 3n-1).$$

– As $t(y) = y + n$ for $y \in (2n-1, 2n)$, then

$$y(t) = t - n \quad \text{for} \quad t \in (3n-1, 3n).$$

and so

$$y(t) = f(t) = \begin{cases} \frac{1}{2}[t + (n-1)], & t \in (3n-3, 3n-1) \\ t - n, & t \in (3n-1, 3n) \end{cases}, n \in \mathbb{N}.$$

In addition, we can deduce $\tau(t)$ from $y = f(t)$ which satisfies

$$f(t) - \alpha = f(t - \tau(t)),$$

and such that for $t \in (3n-3, 3n-1)$ or $t \in (3n-1, 3n)$ and which $\alpha = \frac{1}{2}$ gives

$$\tau(t) = \begin{cases} 1, & t \in (3n-3, 3n-1) \\ \frac{1}{2}, & t \in (3n-1, 3n) \end{cases}, n \in \mathbb{N}.$$

Remark 5.4.7. *We can solve equation (5.16) with the constant delay $\alpha = \frac{1}{2}$, which is*

$$\frac{dz}{dy} = -\lambda g_1(y) z(y - \frac{1}{2}) + 1,$$

in each interval. However, the formula of the solution would be more complicated.

- **A is an operator that generates a semigroup**

For such A we need before discussing the stability to get first the solution of equation (5.15).

Getting the solution

Let us take H to be a Hilbert space, A in equation (5.15) is an operator that generates a bounded semigroup $T(\int_0^y g_1(t)dt)$ and $g_1(y) > 0$ where $y > 0$. From the solution of the scalar case, we can conjecture that the solution of this case should be

$$z(y) = \int_0^y T\left(\int_t^y g_1(s)ds\right) v(t)\ dt. \tag{5.29}$$

Now we need to check that (5.29) is the right solution of equation (5.15) as follows:

First: Checking for $g_1(y) = 1$

For $g_1(y) = 1$, the suggested solution is:

$$z(y) = \int_0^y T(y-t)v(t)\ dt. \tag{5.30}$$

By differentiating (5.30), we get

$$\begin{aligned} z'(y) &= \int_0^y AT(y-t)v(t)\ dt + v(y), \\ &= A\left(\int_0^y T(y-t)v(t)\ dt\right) + v(y), \\ &= Az(y) + v(y), \end{aligned}$$

which is exactly equation (5.15) with $g_1(y) = 1$. So (5.29) is the right solution.

Then: Checking for $g_1(y) \geq 0$

By differentiating (5.29), we get

$$\begin{aligned} z'(y) &= \int_0^y AT\left(\int_t^y g_1(s)\ ds\right) g_1(y)v(t)\ dt + v(y), \\ &= Ag_1(y)\int_0^y T\left(\int_t^y g_1(s)\ ds\right) v(t)\ dt + v(y), \\ &= Ag_1(y)z(y) + v(y). \end{aligned}$$

We can conclude that if A in equation (5.15) is an operator that generates a bounded semigroup, then (5.29) is the mild solution of the equation.

Proving the stability

For this A we can obtain uniform exponential stability as we will show in the next result, which is the first main result of this section.

Theorem 5.4.8. *If $\delta > 0$ and $g_1(y) \geq \delta$ in equation (5.15), $v \in L^\infty(0,\infty; H)$ and A is an operator that generates a semigroup that is uniformly exponentially stable then the solution of this equation will be bounded, i.e., $z \in L^\infty(0,\infty; H)$.*

Proof. Suppose

$$\left\| T\left(\int_x^y g_1(t)\ dt\right)\right\| \leq M\mathrm{e}^{-\beta \int_x^y g_1(t)\ dt}, \tag{5.31}$$

for all t and some $M \geq 1$, $\beta > 0$. Then

$$\begin{aligned}
\|z(y)\| &= \left\| \int_0^y T\left(\int_t^y g_1(s)ds\right) v(t)\ dt\right\|, \\
&\leq \int_0^y \left\| T\left(\int_t^y g_1(t)\ dt\right)\right\| \|v(t)\|\ dt, \\
&\leq \int_0^y M\mathrm{e}^{-\beta \int_t^y g_1(t)\ dt} \|v(t)\|\ dt, \qquad (5.32) \\
&\leq M\epsilon \int_{x=0}^y \mathrm{e}^{-\beta\delta x}\ dx, \text{ where } x = y - t \text{ and } \|v\|_\infty = \epsilon, \\
&\leq M\epsilon \int_{x=0}^\infty \mathrm{e}^{-\beta\delta x}\ dx, \\
&\leq \frac{M\epsilon}{\beta\delta}.
\end{aligned}$$

□

5.4.2 L^2 Stability

To obtain stability of (5.15) in this sense we suppose that $v \in L^2(0,\infty; H)$; then to get $z \in L^2(0,\infty; H)$ we use Theorem 1.6.4 and again the result depends on A and the specific conditions on $g_1(y)$.

Theorem 5.4.9. *If $\delta > 0$ and $g_1(y) \geq \delta$ in equation (5.15), $v \in L^2(0,\infty;H)$ and A is an operator that generates a semigroup that is uniformly exponentially stable then the solution of this equation satisfies $z \in L^2(0,\infty;H)$.*

Proof. By (5.32),

$$\|z(y)\| \leq \int_0^y M\mathrm{e}^{-\beta\delta(y-t)}\|v(t)\|\ dt.$$

Now let

$$I(y) = \int_0^y M\mathrm{e}^{-\beta\delta(y-t)}\|v(t)\|\ dt = \int_0^y Mg(y-t)\|v(t)\|\ dt;$$

where $g : x \to \mathrm{e}^{-\beta\delta x} \in L^1(0,\infty)$ and so from Theorem 1.2.6 $\quad G = \mathcal{L}g \in H^\infty(\mathbb{C}_+)$, and then $\hat{I} = MG\hat{w}$ where $w(t) = \|v(t)\|$. Then from Theorem 1.6.4

$$\begin{aligned} \|I\|_2 &\leq M\|G\|_\infty\|w\|_2 \\ &= \left\|\frac{M}{s+\beta\delta}\right\|_\infty \|v\|_2 \\ &= \frac{M}{\beta\delta}\|v\|_2. \end{aligned}$$

The proof is done. $\square$

5.5 Further Stability Results

5.5.1 Introduction

In the previous part, we have discussed the stability of the ordinary delay equation

$$\frac{dz}{dy} = g(y)Az(y-\alpha) + g(y)Bv(y), \quad z_0 = 0, \tag{5.33}$$

for $z(y) \in H$ and $v(y) \in K$ where H, U are Hilbert spaces; $A : H \to H$ or $A : D(A) \to H$ and $B : K \to H$ are operators where $D(A)$ is the domain of A and $g(y) = \frac{dt}{dy}$ is a continuous function such that $g(y) \in \mathbb{C}$ for each y. Recall that the stability of equation (5.33) means that if the input $v \in \mathbf{L}^2(0,\infty;H)$ or $v \in \mathbf{L}^\infty(0,\infty;H)$, the output z will be also in the same space. However, this section does not give results of the stability that links the output of equation (5.33) with its input. It gives results of the stability that links the output of equation (5.33) with the output of our original variable delay equation

$$\dot{x}(t) = Ax(t-\tau(t)) + Bu(t), \tag{5.34}$$

where $x(t) \in H$ and $u(t) \in K$ and $x(t) = 0$ for $t < 0$; thus, $x(t-\tau(t)) = 0$ when $t - \tau < 0$.

Equation (5.33) is obtained by changing the variables in equation (5.34) as we have done in Section 4.3 by putting $\dfrac{dt}{dy}$ to be $g(y)$ in equation (5.33), where $y = f(t)$ and $f : (0,\infty) \to (0,\infty)$ is a continuous bijection. .

Proposition 5.5.1. *If the solution of equation (5.34) is bounded then the solution of equation (5.33) will be so. That means,*

$$x \in L^\infty(0,\infty;H) \iff z \in L^\infty(0,\infty;H).$$

Proof. The formula relating x and z is $x(t) = z(y)$, where $y = f(t)$. That means x and z take the same values but we just change the time axis from t into y. Because of that,

$$\|x(t)\|_\infty = ess\sup|x(t)| = ess\sup|z(y)| = \|z(y)\|_\infty.$$

$\square$

We now discuss L^2 stability, which is a more complicated question because of the change of variables. The next results show why it is essential that $g, \dfrac{1}{g} \in L^\infty$ to get both of the solutions (outputs) of equation (5.33) and equation (5.34) in L^2 when one of them is in it.

Theorem 5.5.2. *If $g, \dfrac{1}{g} \in L^\infty$ then $f', \dfrac{1}{f'}$ are bounded and*

$$x \in L^2(0,\infty;H) \iff z \in L^2(0,\infty;H).$$

Proof. Because $f(t) = y$, we get $f'(t) = \dfrac{d}{dt}f(t) = \dfrac{dy}{dt} = 1/\left(\dfrac{dt}{dy}\right) = 1/g(y)$ and so f' is bounded. We can prove that $\dfrac{1}{f'}$ is bounded using the same way. Let us now suppose that $x \in L^2(0,\infty;H)$ and we will find

$$\begin{aligned}\|z\|_2^2 &= \int_0^\infty \|z(y)\|^2 dy,\\ &= \int_0^\infty \|x(t)\|^2 \frac{dy}{dt} dt,\end{aligned}$$

as $\dfrac{dy}{dt}$ is bounded this leads to $z \in L^2(0,\infty;H)$. Conversely, if $z \in L^2(0,\infty;H)$ then we will find

$$\begin{aligned}\|x\|_2^2 &= \int_0^\infty \|x(t)\|^2 dt,\\ &= \int_0^\infty \|z(y)\|^2 \frac{dt}{dy} dy,\end{aligned}$$

as $\dfrac{dt}{dy}$ bounded this leads to $x \in L^2(0,\infty;H)$. □

Let $w(t)$ be a positive measurable function. Then for a separable Hilbert space H we write $L^2(0,\infty,w(t)dt,H)$ for the space of measurable H-valued functions f such that the norm $\|f\|$, given by

$$\|f\|^2 = \int_0^\infty \|f\|^2 w(t) \ \ dt,$$

is finite. We write L^2_w for $L^2(0,\infty;w(t)\,dt)$.

Proposition 5.5.3. *If the solution of equation (5.33) is in L^2 then the solution of equation (5.34) will be in a weighted L^2 space and vice versa. That means*

$$x \in L^2(0,\infty;H) \iff z \in L^2(0,\infty;\frac{dt}{dy}dy;H).$$

Additionally,

$$z \in L^2(0,\infty;H) \iff x \in L^2(0,\infty;\frac{dy}{dt}dt;H).$$

Proof. As in the previous proof, it is enough to prove that

$$z \in L^2(0,\infty; \frac{dt}{dy}dy; H) \Rightarrow x \in L^2(0,\infty; H).$$

Suppose that $z \in L^2(0,\infty; \frac{dt}{dy}dy; H)$ and then we have

$$\begin{aligned} \|x\|_2^2 &= \int_0^\infty \|x(t)\|^2 dt, \\ &= \int_0^\infty \|z(y)\|^2 \frac{dt}{dy} dy, \end{aligned}$$

and so $x \in L^2(0,\infty; H)$, which means $\|x\|_{L^2(0,\infty;H)} = \|z\|_{L^2(0,\infty;\frac{dt}{dy}dy;H)}$. The second part is similar. $\square$

Remark 5.5.4. *Sometimes for the equation*

$$\frac{dz}{dy} = -\lambda g(y) z(y-\alpha) + v(y), \quad \lambda \in \mathbb{C}, \quad Re\lambda > 0,$$

we can find $\tau(t)$ when we know $f(t)$. The simple example for that is when $y = Ct$, where $C > 0$ is a constant and it is clear that $y = 0$ when $t = 0$. Then we can get

$$f'(t) = \frac{dy}{dt} = C \quad \textit{and so} \quad \frac{dt}{dy} = g(y) = \frac{1}{C}$$

where both of $\frac{dy}{dt}, \frac{dt}{dy}$ are bounded. And so from

$$\begin{aligned} f(t) - \alpha &= f(t - \tau(t)), \\ Ct - \alpha &= Ct - C\tau(t), \end{aligned}$$

we have $\tau(t) = \frac{\alpha}{C}$, which is a constant delay.

Example 5.5.5. *From Example 4.3.4, we suppose that*

$$y = f(t) = t^2.$$

And so

$$f' = 2t \quad \textit{and} \quad \frac{1}{f'} = \frac{1}{2t},$$

are not bounded for $t \geq 0$. So

$$x \in L^2(0,\infty;H) \iff z \in L^2(0,\infty;\frac{dy}{2\sqrt{y}};H).$$

Additionally,

$$z \in L^2(0,\infty;H) \iff x \in L^2(0,\infty;2t\ dt;H).$$

The next proposition from [29] shows an isometric map between L^2_w and A^2_ν, a weighted Bergman space.

Definition 5.5.6. A^2_ν

Let $\tilde{\nu}$ be a positive regular Borel measure on $[0,\infty)$ satisfying the following $\triangle_2$-condition:

$$\sup_{t>0} \frac{\tilde{\nu}[0,2t)}{\tilde{\nu}[0,t)} < \infty. \tag{5.35}$$

This is sometimes referred to as a doubling condition, and such measures have been studied in the theory of harmonic analysis and partial differential equations for many years. Let ν be the positive regular Borel measure on $\overline{\mathbb{C}}_+ = [0,\infty) \times \mathbb{R}$ given by $d\nu = d\tilde{\nu} \otimes d\lambda$, where λ denote Lebesgue measure. In this case, for $p = 2$, we call

$$A^2_\nu = \left\{ f : \mathbb{C}_+ \to \mathbb{C} \text{ analytic: } \sup_{\epsilon>0} \int_{\overline{\mathbb{C}_+}} |f(z+\epsilon)|^2\ d\nu(z) < \infty \right\},$$

a Zen space on $\mathbb{C}_+$. If $\tilde{\nu}\{0\} > 0$, then by standard Hardy space theory, f has a well-defined boundary function $\tilde{f} \in L^2(i\mathbb{R})$, and we can give meaning to the expression $\int_{\overline{\mathbb{C}_+}} |f(z)|^2\ d\nu(z)$. Therefore, we may write

$$\|f\|_{A^2_\nu} = \left(\int_{\overline{\mathbb{C}_+}} |f(z)|^2\ d\nu(z) \right)^{\frac{1}{2}}.$$

Proposition 5.5.7. *Let A^2_ν be a Zen space, and let $w : (0,\infty) \to \mathbb{R}_+$ be given by*

$$w(t) = 2\pi \int_0^\infty \mathrm{e}^{-2rt}\ d\check{\nu}(r) \qquad (t > 0).$$

Then the Laplace transform defines an isometric map $\mathcal{L} : L^2_w(0,\infty) \to A^2_\nu$.

Example 5.5.8. *There are isomorphisms between L^2_w and spaces of functions that are related to Hardy spaces such as*

- *The map between $L^2(\frac{dt}{t})$ and the Bergman space A^2 which is*

  $$z \in L^2(\frac{dt}{t}) \iff \hat{z} \in A^2,$$

  where

  $$A^2 = \left\{ f : \mathbb{C}_+ \to \mathbb{C} \quad \textit{analytic} \; ; \iint\limits_{\mathbb{C}_+} |f(x+iy)|^2 \;\; dxdy < \infty \right\}.$$

  See also [11].

- *The map between $L^2(t\ dt)$ and the Dirichlet space D^2 which is*

  $$x \in L^2(t\ dt) \iff \hat{x} \in D^2,$$

  where

  $$D^2 = \left\{ f : \mathbb{C}_+ \to \mathbb{C} \quad \textit{analytic} \; ; \iint\limits_{\mathbb{C}_+} |f'(x+iy)|^2 \;\; dxdy < \infty \right\}.$$

  However, this example is not a Zen space.

5.5.2 Stability on Weighted L^2 Spaces

There is an extensive literature on the use of the H^∞ norm of an analytic (operator-valued) function on the right-hand half-plane $\mathbb{C}_+$, which describes the gain of a linear time-invariant system from (vector-valued) $L^2(0,\infty)$ inputs to $L^2(0,\infty)$ outputs; we mention here some well-known books on the subject, namely, [13, 18, 46]. In this section we show that H^∞ methods can be applied to stability questions in a wide variety of weighted $L^2(0,\infty)$ spaces [1].

We start by showing that in many cases the Laplace transform induces an isometry between $L^2(0,\infty,w(t)dt,H)$ and a space of H-valued analytic functions on $\mathbb{C}_+$.
Let $\tilde{\nu}$ be a positive regular Borel measure on $[0,\infty)$ satisfying the doubling condition (5.35). Again let ν be the positive regular Borel measure on $\overline{\mathbb{C}}_+ = [0,\infty)\times\mathbb{R}$ given by $d\nu = d\tilde{\nu}\otimes d\lambda$, where λ denote Lebesgue measure. The Zen space $A^2_\nu(H)$ is defined to consist of all analytic H-valued functions F on $\mathbb{C}_+$ such that the norm, given by

$$\|F\|^2 = \sup_{\epsilon>0}\int_{\overline{\mathbb{C}_+}} \|F(s+\epsilon)\|^2 \;\; d\tilde{\nu}(x)\; dy$$

is finite where we write $s = x+iy$ for $x \geq 0$ and $y \in \mathbb{R}$.
The best-known examples here are:

1. For $\tilde{\nu} = \delta_0$, a Dirac mass at 0, we obtain the Hardy space $H^2(\mathbb{C}_+, H)$.
2. For $\tilde{\nu}$ equal to Lebesgue measure (dx), we obtain the Bergman space $A^2(\mathbb{C}_+, H)$.

Often we shall have $\tilde{\nu}\{0\} = 0$, in which case $\|F\|^2$ can be written simply as

$$\int_{\overline{\mathbb{C}_+}} \|F(s)\|^2 \;\; d\tilde{\nu}(x)\; dy.$$

Theorem 5.5.9. *Suppose that w is given as a weighted Laplace transform*

$$w(t) = 2\pi\int_0^\infty \mathrm{e}^{-2rt}\; d\tilde{\nu}(r), \qquad (t>0). \tag{5.36}$$

Then the Laplace transform provides an isometric map

$$\mathcal{L} : L^2(0,\infty,w(t)dt,H) \to A^2_\nu(H). \tag{5.37}$$

Proof. This result was given in the scalar case $H = \mathbb{C}$ in [29] (see also [30], where applications to admissibility and controllability were given, and [25, 26] for earlier related work) the general case follows using the standard method for proving the Hilbert space-valued case of Plancherel's theorem [2, Thm. 1.8.2]: $(\mathrm{e}_n)_{n=1}^\infty$ be an orthonormal basis for H, and write

$$f(t) = \sum_{n=1}^{\infty} f_n(t) \mathrm{e}_n,$$

where $f_n \in L^2(0, \infty, w(t)dt, \mathbb{C})$. Then $F := \mathcal{L}f = \sum_{n=1}^{\infty} F_n \mathrm{e}_n$, where $F_n = \mathcal{L}f_n \in A^2_\nu(\mathbb{C})$ and $\|f_n\| = \|F_n\|$ from [29, Prop. 2.3].

Now $\|f\|^2 = \sum_{n=1}^{\infty} \|f_n\|^2$ and $\|F\|^2 = \sum_{n=1}^{\infty} \|F_n\|^2$, so the result follows. $\square$

In this case that $\tilde{\nu} = \delta_0$, we have the vectorial version of the well-known Paley-Wiener result linking $L^2(0, \infty)$ and the Hardy space $H^2(\mathbb{C}_+)$; for $\tilde{\nu}$ equal to Lebesgue measure, we find that the weighted signal space $L^2(0, \infty, dt/t)$ is isometric (within a constant) to the Bergman space on $\mathbb{C}_+$.

We now have a result for input-output stability which generalizes the case $p = 2$ of Theorem 1.6.4.

Theorem 5.5.10. *Let $G \in H^\infty(\mathbb{C}_+, L(H))$. Then the multiplication operator M_G defined by*

$$(M_G F)(s) = G(s)F(s) \qquad (s \in \mathbb{C}_+, \quad F \in A^2_\nu(H))$$

is bounded on $A^2_\nu(H)$ with $\|M_G\| \leq \|G\|_\infty$. In the case when the Laplace transform (5.37) is surjective onto $A^2_\nu(H)$ we have equality.

Proof. It is clear that

$$\sup_{\epsilon>0} \int_{\overline{\mathbb{C}_+}} \|G(s+\epsilon)\|^2 \|F(s+\epsilon)\|^2 \ \ d\tilde{\nu}(x)dy \leq \|G\|^2_\infty \sup_{\epsilon>0} \int_{\overline{\mathbb{C}_+}} \|F(s+\epsilon)\|^2 \ \ d\tilde{\nu}(x)dy,$$

so that $\|M_G\| \leq \|G\|_\infty$.

For the converse inequality we begin by noting that by (5.36) we have the inequality $w(t) \geq 2\pi \mathrm{e}^{-2\epsilon t} \tilde{\nu}[0, \epsilon)$ for every $\epsilon > 0$. Hence, if $z = x + iy \in \mathbb{C}_+$, we have for $0 < \epsilon < x$ the inequality

$$\int_0^\infty |\mathrm{e}^{-\bar{z}t}/w(t)|^2 w(t) \ dt \leq \int_0^\infty \mathrm{e}^{-2xt} \frac{1}{2\pi\nu[0, \epsilon)} \mathrm{e}^{2\epsilon t} \ dt < \infty.$$

Thus the function $k_z : t \mapsto \mathrm{e}^{-\bar{z}t}/w(t)$ lies in $L^2(0,\infty,w(t)dt)$ for every $z \in \mathbb{C}_+$, and we have

$$\mathcal{L}f(z) = \langle f, k_z\rangle_{L^2(0,\infty,w(t)dt)}$$

for all $f \in L^2(0,\infty,w(t)dt)$. That is, $A^2_\nu = \mathcal{L}L^2(0,\infty,w(t)dt)$ is a reproducing kernel Hilbert space with kernel $K_z := \mathcal{L}k_z$ (see, for example, [40] for more on such spaces). For $x \in H$ we write $K_z \otimes x$ for the function $s \mapsto K_z(s)x$ in $A^2_\nu(H)$ and note that for a function $F \in A^2_\nu(H)$ we have $\langle F, K_z \otimes x\rangle_{A^2_\nu(H)} = \langle F(z), x\rangle_H$. Moreover $\|K_z \otimes x\|_{A^2_\nu(H)} = \|K_z\|_{A^2_\nu}\|x\|_H$.
Now for $F \in A^2_\nu(H)$ and $G \in H^\infty(L(H))$ we have, for every $x \in H$ and $z \in \mathbb{C}_+$, that

$$\begin{aligned}\langle F, M_G^*(K_z \otimes x)\rangle_{A^2_\nu(H)} &= \langle M_G F, K_z \otimes x\rangle_{A^2_\nu(H)} = \langle G(z)F(z), x\rangle_H \\ &= \langle F(z), G(z)^* x\rangle_H = \langle F, K_z \otimes G(z)^* x\rangle_{A^2_\nu(H)},\end{aligned}$$

and so $M_G^*(K_z \otimes x) = K_z \otimes G(z)^* x$, and $\|M_G\| = \|M_G^*\| \geq \|G^*\|_\infty = \|G\|_\infty$. $\square$

Summing up the ideas above, we see that getting L^2 stability in all $(0,\infty)$ might be inaccessible, but the stability of system (5.33) might be achieved if we work with weighted L^2 spaces.
Suppose $u \in L^2(0,\infty)$, then

$$\int_0^\infty \|u(t)\|^2 \, dt < \infty;$$

as $y = f(t)$, that leads to

$$\int_0^\infty \|v(y)\|^2 \, \frac{dt}{dy} dy < \infty,$$

which means $v \in L^2((0,\infty);\ |\frac{dt}{dy}|dy)$ and it might be $z \in L^2((0,\infty);\ \text{some weight})$. Then, by changing the variable again we would get $x \in L^2((0,\infty);\ \text{some weight})$.

Chapter 6

Possibilities for further research

In this section we identify some areas where questions remain open for investigation.

6.1 Chapter 2

We could study the stability of the delay systems

$$\frac{dz}{dy} = g(y)\lambda z(y-\alpha) + v(y), \tag{6.1}$$

where g is a continuous function on $[0,\infty)$ and $Re\ \lambda > 0$ using the Walton-Marshall method in Chapter 2. The Walton-Marshall method depends on having $\dfrac{\|\hat{z}\|_{H^2}}{\|\hat{v}\|_{H^2}} < \infty$ which means identifying the poles of $\dfrac{\hat{z}}{\hat{v}}$ or the zeros of $\dfrac{\hat{v}}{\hat{z}}$ on the imaginary axis $i\mathbb{R}$, to get when the system is unstable. Because of that, we would start by choosing $z(y)$ and $g(y)$ to be specific functions such as $z(y) = \mathrm{e}^{\sigma y}$ where $\sigma \in R$ and $g(y) = \frac{1}{\sqrt{y}}$. Then, by finding $\hat{z}$ and $\hat{v}$ from (6.1), we might apply the method. However, we have not yet concluded with a clear result on stability as $\dfrac{\hat{z}}{\hat{v}}$ includes different variables.

6.2 Chapter 4

We can study the stability of the equation

$$\frac{dz}{dy} = g_1(y)Az(y-\alpha) + g_2(y)Bv(y). \tag{6.2}$$

Additionally, we may do more research about the non-standard Laplace transform

$$Z(s) = \int_0^{\infty} e^{-sf(t)} z(t)\ dt;$$

to determine its inverse in order to obtain $x(t)$ explicitly. What is more, we could try to find the solution of equation (4.6), which we have tried to solve by using the generalized semigroup. In addition, we are still investigating how to express the solution of equation (4.6) using integral transforms, as we can when $g(y) \equiv 1$. Furthermore, we want to discuss stability properties of equation (4.1), e.g. if $u \in \mathbf{L}^2(0,\infty;U)$ does $x \in \mathbf{L}^2(0,\infty;X)$? and if we are given $z \notin L^2(0,\infty)$ and (6.2) is satisfied, then we can calculate v, and if $v \in L^2(0,\infty)$ the system is unstable. A suitable choice may be $z(y) = \mathrm{e}^{\beta y}$ with $\beta \in i\mathbb{R}$.

6.3 Chapter 5

We would like to investigate more switching systems with varying parameters and to study the stability of them with giving examples. Additionally, we still investigate the link between L^2 stability and weighted L^2 stability in the time-varying case.

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