Geometry of Skeletal Structures and Symmetry Sets

Azeb Zain Jafar Alghanemi

Submitted in accordance with the requirements for the degree of Doctor of Philosophy

The University of Leeds
Department of Pure Mathematics

June 2012

Abstract

In this thesis we study the geometry of symmetry sets and skeletal structures. The relationship between a symmetry point (skeletal point) and the associated midlocus point is studied and the impact of the singularity of the radius function on this relationship is investigated. Moreover, the concept of the centroid set associated to a smooth submanifold of $\mathbb{R}^{n+1}$ is introduced and studied. Also, the relationship between the shape operator of a skeletal structure at a smooth point and the shape operator of its boundary at the associated point is studied.

Introduction

The idea of describing objects using the concept of medial axis was suggested by Harry Blum. Significant developments in describing many biological and physical objects using medial axis and symmetry set have been seen in the last century. In fact, the medial axis is a subset of a large set called the symmetry set. The concept of symmetry set and medial axis has been studied and developed by Peter Giblin, Bruce and others and a considerable mathematical investigation can be found in [4, 6, 11, 13]. In 2003, James Damon invented and developed the concept of skeletal structures of an object in an attempt to create its smooth boundary. A significant part of Damon's contribution is the radial shape operator which plays a central role in determining the differential geometry of the boundary of a skeletal structure [7, 8, 9].

In the real life, the symmetry set and the medial axis play a central role in many applications such as object recognition, object reconstruction and medical imaging and some of these applications can be found in [25]. In this thesis we focus only on the mathematical aspects of symmetry sets, medial axis, skeletal structures and their boundaries. The impact of the singularities of the radius function on the relationship between symmetry sets, medial axis, skeletal structures and their boundaries will be studied in this thesis. This thesis consists of six chapters and before describing those chapters we give the following definition.

Definition: *A map $f : N^n \longrightarrow M^m$ is singular at $x_0 \in N^n$ if the rank of the Jacobian matrix of f at x_0 is less than min (n, m).*

Now we give a brief description of each chapter of this thesis. In chapter one we give some basic definitions and theorems in the field of skeletal structures which will be used in subsequent chapters. Chapter two deals with the symmetry set of a smooth hypersurface in $\mathbb{R}^{n+1}$. It consists of three parts. The first part deals with the creating of the symmetry set from its boundary. The second main part of chapter two deals with the reconstruction of the boundary using the information given by the symmetry set and the associated radius function. In this part we study the impact of the singularity of the radius function on the relationship between the symmetry point and its associated midlocus point. In fact, this study is a generalization of what Peter Giblin pointed out in the relationship between the normals of a plane curve at tangency points associated to a smooth point of its symmetry set [16]. The third main part of this chapter deals with creating the symmetry set from the associated midlocus and radius function and in this part we generalize what Peter Giblin and Paul Warder did in [16, 32]. Before giving the main result of this chapter we give the following definition.

Definition B: *Let x_0 be a non-singular point of the symmetry set of a region Ω in $\mathbb{R}^{n+1}$, with smooth boundary X. Let x_1 and x_2 be the tangency points of the boundary associated to x_0. Then the midlocus point is given by $x_m = \frac{1}{2}(x_1 + x_2)$.*

The main result of chapter two is the following theorem.

Theorem A: *Let S be the symmetry set of a region Ω in $\mathbb{R}^{n+1}$, with smooth boundary X. Let x_0 be a non-singular point of S. Then x_0 and the associated midlocus point x_m coincide if and only if the radius function has a singularity at x_0.*

In chapter three we introduce the concept of the centroid set associated to a smooth submanifold M of $\mathbb{R}^{n+1}$ which is more general than the midlocus and depends on a multivalued radial vector field defined on M such that each value of the multivalued radial vector field forms a smooth radial vector field on M and each smooth radial vector field has a smooth radius function. This chapter consists of four main parts. In the first part the centroid set associated to a smooth submanifold of $\mathbb{R}^{n+1}$ is defined and the impact of the singularities of the associated radius function is studied. The second part of this chapter deals with the impact of the singularities of the radius function on the relationship between a smooth skeletal point and its associated midlocus point. In the third part of chapter three we define the pre-medial axis in $\mathbb{R}^{n+1}$ and study the relationship between the parameters of the boundary of a medial axis at the tangency points associated to a smooth point of the medial axis. The fourth part of this chapter deals with the classification of the singularity of the midlocus of a skeletal structures in $\mathbb{R}^3$. The main result of chapter three is the following.

Theorem B: *Let M be a smooth stratum of a skeletal structure $(\mathbb{S}, U)$ in $\mathbb{R}^3$ containing a smooth point x_0 and r be the radius function with a singularity at x_0 and λ_1 and λ_2 be the eigenvalues of the Hessian of r, and w_1 and w_2 are the associated eigenvectors such that $\lambda_1 \neq \lambda_2$, and $r(x_0) = \frac{1}{\lambda_1}$, $\lambda_1 \neq 0$. Then the midlocus at x_m associated to x_0 is $\mathcal{A}$-equivalent to the crosscap if and only if*

$$\lambda_1 k_{x_0}(w_1)\nabla^2_{w_1}\nabla_{w_2} r \neq 2\lambda_2 \tau_g \nabla^3_{w_1} r,$$

where $k_{x_0}(w_1)$ is the normal curvature of M in the direction w_1, τ_g is the geodesic torsion of M in the direction w_1, and $\nabla_{w_i} r$ is the directional derivative of the radius function in the direction w_i, $i = 1, 2$.

The fourth chapter of this thesis deals with the relationship between the radial shape operator of a skeletal structure and the differential geometric shape operator of the associated boundary. In [8] James Damon expressed the matrix representing the differential geometric shape operator in terms of the matrix representing the radial shape operator. In this chapter we express the matrix representing the radial shape operator in terms of the matrix representing the differential geometric shape operator of the boundary. Also, the relationship between the principal radial curvatures, Gaussian radial curvature, mean radial curvature of a skeletal structure and the associated principal curvatures, Gaussian curvature, mean curvature of the boundary is pointed out through this chapter. The main result of this chapter is the following.

Theorem C: *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^3$ such that for a choice of smooth value of U the associated compatibility 1-form η_U vanishes identically on a neighbourhood of a smooth point x_0 of $\mathbb{S}$, and $\frac{1}{r}$ is not an eigenvalue of the radial shape operator at x_0. Let $x_0^{'} = \Psi_1(x_0)$ and $V^{'}$ be the image of V under $d\Psi_1$ for a basis $\{v_1, v_2\}$, then*

$$S_{XV'} = \frac{1}{r^2K_r - 2rH_r + 1}(S_V - rK_rI)$$

or equivalently

$$S_V = \frac{1}{r^2K + 2rH + 1}(S_{XV'} + rKI).$$

Here S_V is the matrix representing the radial shape operator, K_r (resp. H_r) is the Gaussian (resp. mean) radial curvature of the skeletal structure, $S_{XV'}$ is the matrix representing the differential geometric shape operator of the boundary and K (resp. H) is the Gaussian (resp. mean) curvature of the boundary.

The fifth chapter of this thesis is devoted to study the relationship between the shape operator of skeletal structures and the associated shape operator of the boundary. In the first main part of this chapter we study the relationship between the curvature of a skeletal structure and the curvature of its associated boundary in the plane. Also, the relationship between the curvatures of the boundary at the tangency points associated to a smooth point of medial axis in the plane has been studied. In second main part of chapter five we express the matrix representing the shape operator of the medial axis in $\mathbb{R}^{n+1}$ at a smooth point in terms of the matrices representing the shape operators of the boundary at the associated tangency points. The main result of chapter five is given in the following.

Theorem D: *Let* $(\mathbb{S}, U)$ *be a skeletal structure such that for a choice of smooth value of* U *the associated compatibility 1-form* η_U *vanishes identically on a neighbourhood of a smooth point* x_0 *of* $\mathbb{S}$*, and* $\frac{1}{r}$ *is not an eigenvalue of the radial shape operator at* x_0*. Let* $x_{0'} = \Psi_1(x_0)$*, and* V' *be the image of* V *for a basis* $\{v_1, v_2, ..., v_n\}$*. Then the matrix* $S_{XV'}$ *representing the differential geometric shape operator of the boundary is given by*

$$S^T_{XV'} = \frac{1}{r}\{[I - r(\mathcal{H}^T_r + \rho S^T_m - \frac{1}{\rho} d\rho dr^T I^{-1}_m + \frac{1}{\rho} S^T_m dr dr^T I^{-1}_m)]^{-1} - I\},$$

where S_m *is the matrix representing the differential geometric shape operator of* $\mathbb{S}$ *at* x_0*,* I_m *is the first fundamental form of* $\mathbb{S}$ *at* x_0 *and* $\mathcal{H}_r$ *is the matrix representing the Hessian radial operator at* x_0*.*

The last chapter of this thesis deals with the focal point of the boundary associated to a skeletal structure. In this chapter we define the radial focal point of a skeletal structure and we show that this point coincides with the focal point of the boundary. Moreover, the location of the focal point of the boundary associated to a Blum medial axis in $\mathbb{R}^{n+1}$ is investigated through this chapter.

This work is dedicated with great respect and deep affection

to my parents, wife and lovely children.

Acknowledgements

No scientific progress is possible without standing on the shoulders of giants, and it was my good fortune to have Dr. Kevin Houston as a supervisor. I am greatly indebted to him for his inspiring, enthusiastic supervision, willingness to help and devoting so much of his valuable time during the time of preparing this thesis. Also, it is my pleasure to thank Dr. Martin Speight for being a co-supervisor. Special thanks go to professor Peter Giblin for giving me some of his papers in symmetry sets and for invaluable mathematical discussion during singularity days and a conference in Bedlewo.

I am also greatly indebted to my beloved parents and wife for their encouragement and support.

Last but not the least, I gratefully acknowledge financial support from King Abdulaziz University.

Contents

List of figures

Chapter 1

Background

Skeletal structures of a smooth boundary has been studied by James Damon in several papers [7, 8, 9] In this chapter we give some basic definitions and theorems in the field of Whitney stratifications and skeletal structures which will be used in the subsequent chapters . Also, the radial and edge shape operators will be reviewed in this chapter.

1.1 Whitney Stratification

Definition 1.1.1 [18] *Let $\mathbb{S}$ be a closed subset of a smooth manifold M and let $\mathbb{S}$ be decomposed into disjoint smooth submanifolds (possibly with boundary) S_i called strata. Then the decomposition is called a Whitney Stratification if the following conditions are met*

1. *$S_i \bigcap \overline{S_j} \neq \phi$ if and only if $S_i \subseteq \overline{S_j}$ for strata S_i, S_j with $i \neq j$, this is called the frontier condition.*

2. *Whitney condition (a): if x_i is a sequence of points in S_a converging to $y \in S_b$ and $T_{x_i}(S_a)$ converges to a plane τ (all this considered in the appropriate Grassmannian), then $T_y(S_b) \subseteq \tau$.*

3. *Whitney condition (b): if x_i and y_i are two sequences in S_a and S_b respectively converge to $y \in S_b$, l_i denotes the secant line between x_i and y_i and l_i converges to l then $l \subseteq \tau$.*

Remark 1.1.2 *Condition b implies condition a. Any Whitney stratified set can be triangulated (see [7]).*

Definition 1.1.3 *For a Whitney stratified set $\mathbb{S}$ we let*

1. *$\mathbb{S}_{reg}$ denote the points in the top-dimensional strata and these points are the smooth points of $\mathbb{S}$.*
2. *$\mathbb{S}_{sing}$ denote the remaining strata.*
3. *$\partial\mathbb{S}$ denote the subset of $\mathbb{S}_{sing}$ consisting of points of $\mathbb{S}$ at which $\mathbb{S}$ is locally an n-manifold with boundary. We refer these points as edge points in order to distinguish between $\partial\mathbb{S}$ and the boundary of the region of the skeletal structure.*
4. *$\overline{\partial\mathbb{S}}$ denote the closure of $\partial\mathbb{S}$.*

Definition 1.1.4 *let $\mathbb{S}$ be a Whitney stratified set and let $x_0 \in \mathbb{S}_{sing}$ then we define the following*

1. *The complementary local components for x_0 are the connected components of $\overline{B_\varepsilon}(x_0) \setminus \mathbb{S}$.*
2. *The neighbouring local components of x_0 are the connected components of $\overline{B_\varepsilon}(x_0) \bigcap \mathbb{S}_{reg}$,*

where $\overline{B_\varepsilon}(x_0)$ is a closed ball of radius ε about x_0 for sufficiently small $\varepsilon > 0$.

1.2 Skeletal Set and Skeletal Structure

Definition 1.2.1 [7] *An n-dimensional Whitney stratified set $\mathbb{S} \subseteq \mathbb{R}^{n+1}$ is a skeletal set if*

1. *For each local neighbouring component $\mathbb{S}_\alpha$ of $x_0 \in \mathbb{S}_\beta$ there is a unique limiting tangent space $T_{x_0}\mathbb{S}_\beta$ from sequence of points in $\mathbb{S}_\alpha$ (by properties of Whitney stratified set $T_{x_0}\mathbb{S}_\beta \subset T_{x_0}\mathbb{S}_\alpha$).*
2. *Locally in a neighbourhood of a singular point x_0, $\mathbb{S}$ may be expressed as a union of (smooth) n-manifolds with boundaries and corners $\mathbb{S}_j$, where two such intersect on boundary facets.*
3. *If $x_0 \in \overline{\partial\mathbb{S}}$ then those $\mathbb{S}_j$ in 2 meeting $\partial\mathbb{S}$ meet it in an $(n-1)$-dimensional facet.*

Facets means edges or faces in the triangulation of remark 1.1.2.

Definition 1.2.2 [7] *An edge coordinate parametrization at an edge point $x_0 \in \partial\mathbb{S}$ consists of an open neighbourhood W of x_0 in $\mathbb{S}$, an open neighbourhood $\widetilde{W}$ of 0 in $\mathbb{R}^n_+ = \{(x_1, x_2, ..., x_n) \in \mathbb{R}^n : x_n \geq 0\}$ and a differentiable homeomorphism $\Phi : \widetilde{W} \to W$ such that: both $\Phi|\{(x_1, x_2, ..., x_n) \in \widetilde{W} : x_n > 0\}$ and $\Phi|(\widetilde{W} \bigcap \mathbb{R}^{n-1})$ are diffeomorphisms on to their images.*

Definition 1.2.3 *Given an n-dimensional set $\mathbb{S} \subset \mathbb{R}^{n+1}$, a radial vector field U on $\mathbb{S}$ is nowhere zero multivalued vector satisfying the following conditions.*

1. *(Behaviour at smooth points) For each $x_0 \in \mathbb{S}_{reg}$, there are two values of U which are on opposite sides of $T_{x_0}\mathbb{S}$ i.e., their dot products with a normal vector are non zero with opposite signs. Moreover, on a neighbourhood of a point of $\mathbb{S}_{reg}$, the values of U corresponding to one side form a smooth vector field.*

2. *(Behaviour at a non-edge singular point) Let x_0 be a non-edge singular point with $\mathbb{S}_\alpha$ a local component of x_0. Then both smooth values of U on $\mathbb{S}_\alpha$ extend smoothly to values $U(x_0)$ on the stratum of x_0. If $\mathbb{S}_\alpha$ does not intersect $\partial\mathbb{S}$ in a neighbourhood of x_0, then $U(x_0)$ does not belong to $T_{x_0}\mathbb{S}$. Conversely to each value of U at a point $x_0 \in \mathbb{S}_{sing}$, there corresponds a local complementary component C_i of $\mathbb{S}$ of $\mathbb{S}$ at x_0 such that the value $U(x_0)$ locally points into C_i in the following sense. The value $U(x_0)$ extends smoothly to values $U(x)$ on the local complementary components of $\mathbb{S}$ for x_0 in ∂C_i. For a neighbourhood W of x_0 and an $\varepsilon > 0$, $x + tU(x) \in C_i$ for $0 < t < \varepsilon$ and $x \in (W \bigcap \mathbb{S})$.*

3. *(Tangency behaviour at edge points) At edge points $x_0 \in \partial\mathbb{S}$ there is a unique value for U tangent to the stratum of $\mathbb{S}_{reg}$ containing x_0 in the closure which points away from $\mathbb{S}$.*

Definition 1.2.4 *Given a skeletal set $\mathbb{S}$ and a smooth multivalued radial vector field U, the radial flow is defined by*

$$\Psi_t(x) = x + tU(x),$$

where $x \in \mathbb{S}$ and $t \in [0, 1]$.

Definition 1.2.5 *A radial vector field U on a skeletal set $\mathbb{S}$ satisfies the local initial conditions if it satisfies the following.*

1. *(Local separation property) For a local complementary component C_i of a non-edge point $x_0 \notin \partial\mathbb{S}$, let $\partial C_i = \bigcup \mathbb{S}_i$ denoting the local decomposition of ∂C_i into closed (in W) n-manifolds with boundaries and corners. Then the set $X = \{x + tU(x) : x \in \bigcup_i \partial\mathbb{S}_i, 0 \leq t \leq \varepsilon\}$ is an embedded Whitney stratified set such that distinct $int(\mathbb{S}_i)$ and $int(\mathbb{S}_j)$ lie in separate connected components of the complement of $\overline{C_i} \setminus X$.*

2. *(Local edge property) For each edge closure point* $x_0 \in \overline{\partial\mathbb{S}}$ *there is a neighbourhood* w *of* x_0 *in* $\mathbb{S}$ *and* $\varepsilon > 0$ *so that for each smooth value of* U*, the radial flow* $\Psi(x,t) = x + tU$ *is one-one on* $w \times [0, \varepsilon]$.

Definition 1.2.6 *For a radial vector field* U*, we put* $U = rU_1$*, for a positive multivalued function* r*, and a multivalued unit vector field* U_1 *on* $\mathbb{S}$*. We will call* r *the radius function.*

Now suppose C_i is a local complementary component of a singular point x_0. The local boundary of C_i in a small open neighbourhood can be expressed as a union of n-manifolds with boundary and corner $\{\mathbb{S}_i,\ i = 1, 2, ..., k\}$. The *abstract boundary* of C_i consists of a copy of $\mathbb{S}_i$ for each smooth value of U on $\mathbb{S}_i$ pointing into C_i [7].

Definition 1.2.7 *A skeletal structure* $(\mathbb{S}, U)$ *in* $\mathbb{R}^{n+1}$ *consists of an* n*-dimensional skeletal set and radial vector field* U *on* $\mathbb{S}$ *satisfying the local initial conditions, such that all abstract boundaries of local complementary components are homeomorphic to* n*-disks.*

Definition 1.2.8 *Given a skeletal structure* $(\mathbb{S}, U)$*, the associated boundary is defined by* $X = \{x + U(x) : x \in \mathbb{S}\}$*, where the definition includes all values of* $U(x)$ *for a given* x.

1.3 Radial Shape Operator

Definition 1.3.1 *Given a skeletal structure* $(\mathbb{S}, U)$ *in* $\mathbb{R}^{n+1}$ *we define for a regular point* $x_0 \in \mathbb{S}$ *and each smooth value of* U *defined in a neighbourhood of* x_0 *with associated unit vector field* U_1*, a radial shape operator*

$$S_{rad}(v) = -\text{proj}_U(\frac{\partial U_1}{\partial v}), \quad for \quad v \in T_{x_0}\mathbb{S}, \tag{1.1}$$

where $\dfrac{\partial U_1}{\partial v}$ *means* $\nabla_v U_1$ *and* proj_U *denotes projection onto* $T_{x_0}\mathbb{S}$ *along* U*. Also, if* $\{v_1, v_2, ..., v_n\}$ *is a basis for* $T_{x_0}\mathbb{S}$ *then,*

$$\frac{\partial U_1}{\partial v_i} = a_i U_1 - \sum_{j=1}^{n} s_{ji} v_j \tag{1.2}$$

which can be written in the vector form by

$$\frac{\partial U_1}{\partial V} = A_V.U_1 - S_V^T V. \tag{1.3}$$

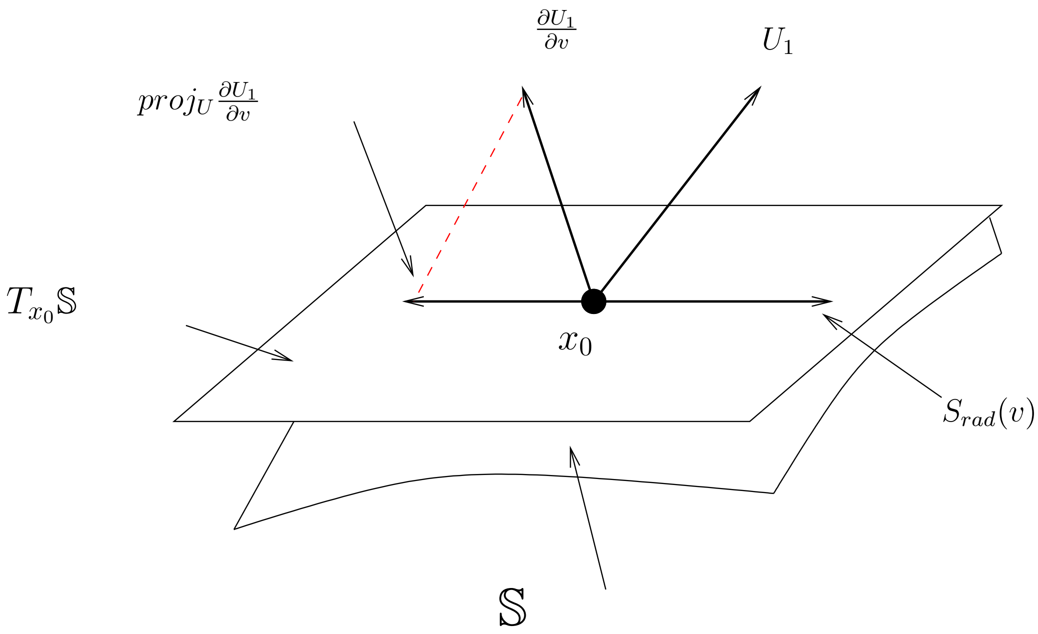

Figure 1.1: The radial shape operator in 3D. The dashed line denotes projection onto $T_{x_0}\mathbb{S}$ along U

Definition 1.3.2 *For $x_0 \in \mathbb{S}_{reg}$ and a given smooth value of U, we call the eigenvalues of the associated radial shape operator the principal radial curvatures at x_0 and denote them by κ_{ri}.*

Example 1.3.3 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^3$ and let $s_1(x, y) = (x, y, 1) \subset \mathbb{S}_{reg}$ such that $\{(x, y) \in \mathbb{R}^2 | x^2 + y^2 < \frac{1}{4}\}$. Now define the radial vector field on the image of s_1 by*

$$U = (x^2 + y^2 + 1)\left(-2x, -2y, \sqrt{1 - 4(x^2 + y^2)}\right) = rU_1,$$

where $r(x,y) = x^2 + y^2 + 1$ *is the radius function and*

$$U_1 = \left(-2x, -2y, \sqrt{1-4(x^2+y^2)}\right).$$

Let $v_1 = \frac{\partial s_1}{\partial x} = (1,0,0)$ *and* $v_2 = \frac{\partial s_1}{\partial y} = (0,1,0)$*, then the unit normal of* s_1 *is* $N = (0,0,1)$*. It is clear that* U_1 *is a smooth unit vector field on the image of* s_1 *and*

$$U_1 = -2xv_1 - 2yv_2 + \sqrt{1-4(x^2+y^2)}N. \tag{1.4}$$

Now we have

$$\frac{\partial U_1}{\partial x} = (-2, 0, \frac{-4x}{\sqrt{1-4(x^2+y^2)}}) = -2v_1 - \frac{-4x}{\sqrt{1-4(x^2+y^2)}}N.$$

But from equation 1.4 we

$$N = \frac{1}{\sqrt{1-4(x^2+y^2)}}U_1 + \frac{2x}{\sqrt{1-4(x^2+y^2)}}v_1 + \frac{2y}{\sqrt{1-4(x^2+y^2)}}v_2.$$

Therefore,

$$\frac{\partial U_1}{\partial x} = \frac{-4x}{1-4(x^2+y^2)}U_1 - (\frac{2-8y^2}{1-4(x^2+y^2)})v_1 - \frac{8xy}{1-4(x^2+y^2)}v_2.$$

Similarly

$$\frac{\partial U_1}{\partial y} = \frac{-4y}{1-4(x^2+y^2)}U_1 - \frac{8xy}{1-4(x^2+y^2)}v_1 - \frac{2-8x^2}{1-4(x^2+y^2)}v_2$$

Now we can apply definition 1.3.1 to evaluate the radial shape operator, thus the matrix representing the radial shape operator is given by

$$S_V = \begin{pmatrix} \frac{2-8y^2}{1-4(x^2+y^2)} & \frac{8xy}{1-4(x^2+y^2)} \\ \frac{8xy}{1-4(x^2+y^2)} & \frac{2-8x^2}{1-4(x^2+y^2)} \end{pmatrix},$$

and

$$A_V = \begin{pmatrix} \frac{-4x}{1-4(x^2+y^2)} \\ \frac{-4y}{1-4(x^2+y^2)} \end{pmatrix}.$$

Now from definition 1.3.2 we have

$$\kappa_{ri} = \frac{1}{2}\{tr(S_V) \pm \sqrt{tr^2(S_V) - 4det(S_V)}\}.$$

After some calculations we get $\kappa_{r1} = 2$ *and* $\kappa_{r2} = \frac{2}{1-4(x^2+y^2)}$.

1.4 Edge Radial Shape Operator

Definition 1.4.1 *Let $(\mathbb{S}, U)$ be a skeletal structure and let $x_0 \in \partial\mathbb{S}$ and let N be the unit normal vector field to $\mathbb{S}$ in a neighbourhood of x_0. Then, the edge shape operator is defined by*

$$S_E(v) = -\text{proj}_U(\frac{\partial U_1}{\partial v}), \tag{1.5}$$

for $v \in T_{x_0}\mathbb{S}$ and proj$_U$ *denotes projection onto $T_{x_0}\partial\mathbb{S} \bigoplus \langle N \rangle$.*

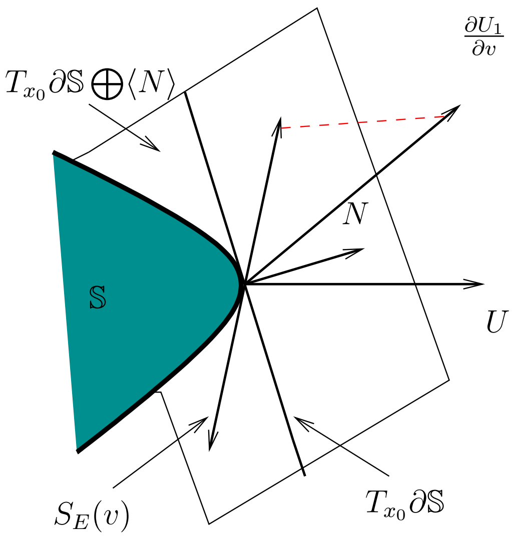

Figure 1.2: The edge shape operator in 3D. The dashed line denotes projection onto $T_{x_0}\partial\mathbb{S} \bigoplus \langle N \rangle$ along U .

Now given a basis $\{v_1, v_2, ..., v_{n-1}\}$ of $T_{x_0}\partial\mathbb{S}$ we choose a vector v_n in the edge coordinate system at x_0 so that $\{v_1, v_2, ..., v_n\}$ is a basis of $T_{x_0}\mathbb{S}$ in the edge coordinate system and so that v_n maps under the edge parametrization map to $cU_1(x_0)$ where $c \geq 0$. Then we can compute a matrix representation for the edge shape operator. Let N be a unit normal vector field to $\mathbb{S}$ on a neighbourhood w of x_0 then we have

$$\frac{\partial U_1}{\partial v_i} = a_i \cdot U_1 - c_i \cdot N - \sum_{j=1}^{n-1} b_{ji} v_j. \tag{1.6}$$

This equation can be written in vector form by

$$\frac{\partial U_1}{\partial V} = A_U \cdot U_1 - C_U \cdot N - B_{UV} \cdot \widetilde{V} \tag{1.7}$$

or

$$\frac{\partial U_1}{\partial V} = A_U \cdot U_1 - \begin{pmatrix} B_{UV} & C_U \end{pmatrix} \begin{pmatrix} \widetilde{V} \\ N \end{pmatrix} \tag{1.8}$$

or

$$\frac{\partial U_1}{\partial V} = A_U \cdot U_1 - S_{EV}^T \begin{pmatrix} \widetilde{V} \\ N \end{pmatrix}. \tag{1.9}$$

Therefore, S_{EV} is a matrix representation of the edge shape operator. Here, A_U and C_U are n-dimensional column vectors, B_{UV} is an $n \times (n-1)$-matrix, and $\widetilde{V}$ is the $(n-1)$-dimensional vector with entries $v_1, v_2, ..., v_{n-1}$.

Remark 1.4.2 *The basis $\{v_1, v_2, ..., v_n\}$ of $T_{x_0}\mathbb{S}$ in the edge coordinate system is called a special basis of $T_{x_0}\mathbb{S}$.*

Definition 1.4.3 *The principal edge curvatures are the generalized eigenvalues of the pair $(S_{EV}, I_{n-1,1})$ where, $I_{n-1,1}$ denotes the $(n \times n)$-diagonal matrix with $1's$ in the first $n-1$ diagonal positions and 0 otherwise.*

1.5 Compatibility 1-Form and Compatibility Condition

Definition 1.5.1 *Given a skeletal structure $(\mathbb{S}, U)$ the compatibility 1-form η_U is defined by*

$$\eta_U(v) = v \cdot U_1 + dr(v), \tag{1.10}$$

v is a tangent vector to $\mathbb{S}$.

$\mathbb{S}$ satisfies the compatibility condition at $x_0 \in \mathbb{S}$ with smooth value U if $\eta_U \equiv 0$ at x_0. The compatibility condition plays a central role in the investigation of the differential geometry of the boundary. A radial vector field plays the role of a normal vector of the boundary when $\mathbb{S}$ satisfies the compatibility condition.

Lemma 1.5.2 [7] *Let $(\mathbb{S}, U)$ be a skeletal structure. Suppose that $\mathbb{S}_\alpha$ is a local manifold component of x_0 on which is defined a smooth value of the radial vector field U. Suppose that either $\frac{1}{r}$ is an eigenvalue of the radial shape operator if $\mathbb{S}_\alpha$ is a non-edge component or $\frac{1}{r}$ is not a generalized eigenvalue of the pair $(S_{EV}, I_{n-1,1})$ if $\mathbb{S}_\alpha$ is an edge component. If the associated compatibility 1-form η_U vanishes at x_0 then $U(x_0)$ is orthogonal to the portion of the boundary X (given by $\Psi_1(\mathbb{S}_\alpha)$) at $\Psi_1(x_0)$.*

1.6 The Radial Map

Definition 1.6.1 *Given a skeletal structure $(\mathbb{S}, U)$ with boundary X then the radial map is given by:*

$$\Psi_1(x) = x + r(x)U_1(x), \; x \in \mathbb{S} \tag{1.11}$$

Example 1.6.2 *With the skeletal structure $(\mathbb{S}, U)$ as in example 1.3.3, we will calculate the boundary of this skeletal structure using the radial map Ψ_1; in fact, we see that for any point $x_0 \in s_1$ the associated boundary point x_1 is given by*

$$x_1 = x_0 + rU_1 = (x, y, 1) + (x^2 + y^2 + 1)\left(-2x, -2y, \sqrt{1 - 4(x^2 + y^2)}\right),$$

and after some calculations we obtain

$$x_1 = \left(-2x^3 - 2xy^2 - x, -2y^3 - 2x^2y - y, 1 + (x^2 + y^2 + 1)\sqrt{1 - 4(x^2 + y^2)}\right).$$

Now we will check the compatibility condition and to do so we have to check the dot product $\frac{\partial x_1}{\partial x} \cdot U_1 = \frac{\partial x_1}{\partial y} \cdot U_1 = 0$. Now

$$\frac{\partial x_1}{\partial x} = \left(-6x^2 - 2y^2 - 1, -4xy, \frac{-12x^3 - 12xy^2 - 2x}{\sqrt{1 - 4(x^2 + y^2)}}\right),$$

and

$$\frac{\partial x_1}{\partial y} = \left(-4xy, -6y^2 - 2x^2 - 1, \frac{-12y^3 - 12x^2y - 2y}{\sqrt{1 - 4(x^2 + y^2)}} \right).$$

Thus, this boundary is smooth and it is clear that

$$\frac{\partial x_1}{\partial x} \cdot U_1 = \frac{\partial x_1}{\partial y} \cdot U_1 = 0.$$

Therefore, using lemma 1.5.2 the compatibility 1-form vanishes identically in the given domain.

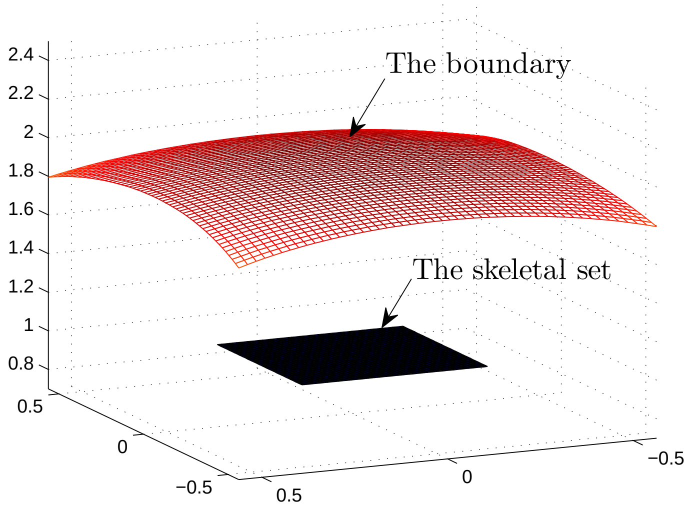

Figure 1.3: Skeletal set and associated boundary in example 1.5.2.

1.7 The Sufficient Conditions for Smooth Boundary

James Damon Discussed in [7, 8, 9] the sufficient conditions for the skeletal structure $(\mathbb{S}, U)$ to have a smooth boundary. These conditions are

1. (*Radial Curvature Condition*) For all points of $\mathbb{S}$ off $\partial\mathbb{S}$
   $r < min\{\frac{1}{\kappa_{ri}}\}$ for all positive principal radial curvature κ_{ri}.

2. (*Edge Condition*) For all points of of $\overline{\partial \mathbb{S}}$ (closure of $\partial \mathbb{S}$)
   $r < min\{\frac{1}{\kappa_{Ei}}\}$ for all positive principal edge curvature κ_{Ei}.

3. (*Compatibility Condition*) For all singular points of $\mathbb{S}$ (which includes edge points)
   $\eta_U = 0$.

Theorem 1.7.1 [7] *Let $(\mathbb{S}, U)$ be a skeletal structure which satisfies the above three conditions. Then*

1. *The associated boundary X is an immersed topological manifold which is smooth at all points except those point corresponding to points of $\mathbb{S}_{sing}$.*

2. *At points corresponding to points of $\mathbb{S}_{sing}$ it is weakly C^1 (this implies that it is C^1 on the points which are in the images of strata of codimension 1).*

3. *At smooth points, the projection along lines of U will locally map X diffeomorphically onto the smooth part of $\mathbb{S}$.*

4. *Also, if there is no nonlocal intersection, X will be an embedded manifold.*

1.8 Blum Medial Axis

Definition 1.8.1 *Given a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X, then the Blum medial axis of Ω is the locus of centers of hyperspheres tangent to the boundary X at least two points or having a single degenerate tangency such that these hyperspheres are contained in Ω.*

Definition 1.8.2 [7] *The pair $(\mathbb{S}, U)$ consisting of the Blum medial axis and associated multivalued radial vector field is a special case of a skeletal structure which satisfies the following*

At each smooth point x_0, the two values $U^{(1)}$ and $U^{(2)}$ must satisfy $\|U^{(1)}\| = \|U^{(2)}\|$ and $U^{(1)} - U^{(2)}$ is orthogonal to $T_{x_0}\mathbb{S}$.

Proposition 1.8.3 [7] *If $(\mathbb{S}, U)$ is a medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^{n+1}$ with generic smooth boundary X, then $(\mathbb{S}, U)$ satisfies both the radial curvature and edge conditions.*

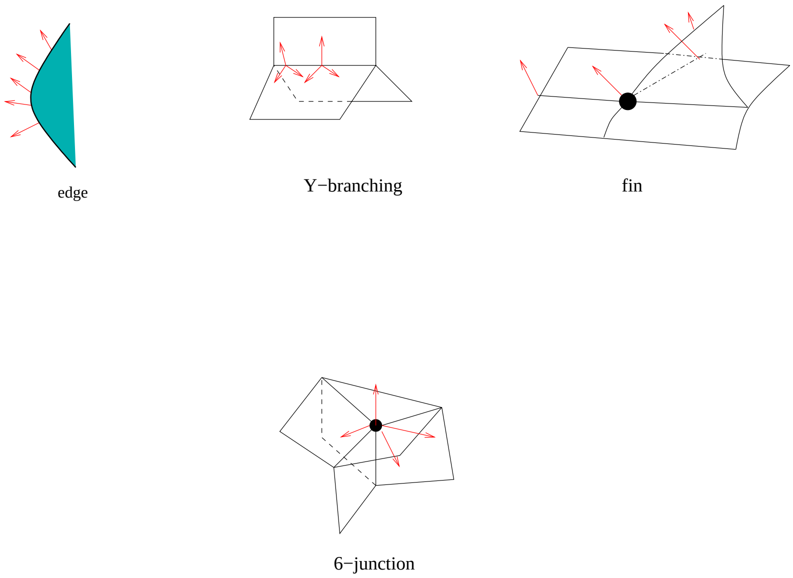

Figure 1.4: Local generic structure for Blum medial axis in $\mathbb{R}^3$ and the associated radial vector fields.

Chapter 2

Symmetry Set in $\mathbb{R}^{n+1}$

2.1 Introduction

This chapter is focused on the symmetry set of a smooth hypersurface in $\mathbb{R}^{n+1}$. It is divided into three main parts. The first part deals with the creating of the symmetry set using the boundary. In this part, we define the symmetry set in $\mathbb{R}^{n+1}$ and we study the smoothness of the symmetry set using the information provided by the principal curvatures of the boundary (**Theorem 2.2.3**). Also, the necessary and sufficient condition for two points on the boundary to form a symmetry point is discussed (**Theorem 2.2.4**). In the second part we consider the inverse procedure to that in the first part. In fact, this part deals with the reconstruction of the boundary using the information given by the symmetry set and the radius function (**Theorem 2.3.1**). Furthermore, the impact of the singularity of the radius function on the relationship between the symmetry set and the associated midlocus is investigated (**Theorem 2.3.4**). The last part of this chapter is focused on the creating of the symmetry set using the information provided by the midlocus and the radius function.

2.2 Creating Symmetry Set from the Boundary

The symmetry set and its smooth boundary in $\mathbb{R}^2$ and $\mathbb{R}^3$ have been studied intensively by Giblin and others in several papers such as [4, 6, 11, 13, 14, 15]. In this section we define the symmetry set of a smooth boundary in $\mathbb{R}^{n+1}$ in the same way as Giblin and then we generalize some results to the higher dimensions.

Definition 2.2.1 *Given a smooth hypersurface X in $\mathbb{R}^{n+1}$ the symmetry set S is the locus of centres of hyperspheres, bitangent to X. I.e., if $x_1 = X(s)$ and $x_2 = X(t)$ are two points of the tangency with a hypersphere then the corresponding point of the symmetry set S is given by $x_0 = x_1 + rN_1 = x_2 + rN_2$, where r is the radius function, $N_i, i = 1, 2$ are the unit normals of X at x_i, $s = (s_1, s_2, ..., s_n)$ and $t = (t_1, t_2, ..., t_n)$ pointing towards the centre of the hypersphere.*

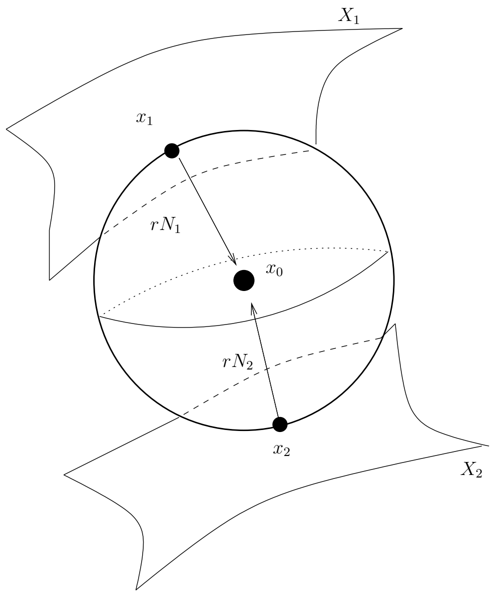

Figure 2.1: The symmetry point.

Now let X_1 and X_2 be two pieces of smooth hypersurface X parametrized locally by $s = (s_1, s_2, ..., s_n)$ and $t = (t_1, t_2, ..., t_n)$ respectively. Define the function

$$f : \mathbb{R}^{2n+1} \longrightarrow \mathbb{R}^{n+1}$$

by

$$f(s, t, r) = (X_1(s) - X_2(t)) + r(N_1(s) - N_2(t)).$$

Then $f = 0$ when a hypersphere of radius r is tangent to X_1 and X_2. We expect $f^{-1}(0)$ to be a smooth manifold with dimension n. Define the centre map C by:

$$C : f^{-1}(0) \longrightarrow \mathbb{R}^{n+1}$$

$$C(s, t, r) = X_2(t) + rN_2(t).$$

Then clearly, $C(f^{-1}(0))$ is the symmetry set. Hence C projects $f^{-1}(0)$ to $\mathbb{R}^{n+1}$, therefore the condition for $f^{-1}(0)$ to project to a smooth hypersurface in $\mathbb{R}^{n+1}$ is C to be an immersion. Now since X_1 and X_2 are oriented, we can choose orthonormal bases for their tangent spaces using the principal directions.

Proposition 2.2.2 *Assume as above, then*

1. $f^{-1}(0)$ *is a smooth submanifold of* $\mathbb{R}^{2n+1}$ *parametrized by* $s = (s_1, s_2, ..., s_n)$ *provided* $\kappa_i \neq \frac{1}{r}$, $i = 1, 2, ..., n$.

2. $f^{-1}(0)$ *is a smooth submanifold of* $\mathbb{R}^{2n+1}$ *parametrized by* $t = (t_1, t_2, ..., t_n)$ *provided* $\lambda_i \neq \frac{1}{r}$, $i = 1, 2, ..., n$, *where* κ_i (*resp.* λ_i) *are the principal curvatures of* X_2 (*resp.* X_1).

Proof

Let p_0 and q_0 be tangency points ($p_0 \in X_1$ and $q_o \in X_2$) let $\{v_i\}, i = 1, 2, ..., n$ (resp. $\{u_i\}, i = 1, 2, ..., n$) be a basis for the tangent space of X_2 (resp. X_1) formed by the principal directions. Now using the fact that differentiating the normal in the principal direction produces the principal curvature times the principal direction. Therefore, the Jacobian matrix of f has the following column vectors

$$-(1-r\kappa_i)v_i, \quad (1-r\lambda_i)u_i, \; and \; (N_1-N_2).$$

$N_1 - N_2$ is parallel to $p_0 - q_0$, thus $N_1 - N_2 \neq 0$. Also, $N_1 - N_2, u_1, u_2, ..., u_n$ are linearly independent as well as $N_1 - N_2, v_1, v_2, ..., v_n$. Therefore, using the implicit function theorem the result holds. □

Now we will give the necessary and sufficient condition for the tangency points to give a smooth point on the symmetry set.

Theorem 2.2.3 *The symmetry set* $C(f^{-1}(0))$ *is a smooth hypersurface if* $\kappa_i \neq \frac{1}{r}$ *and* $\lambda_i \neq \frac{1}{r}$, $i = 1, 2, ..., n$.

Proof

$$C(s,t,r) = X_2 + rN_2.$$

The condition for $f^{-1}(0)$ to project to a smooth hypersurface in $\mathbb{R}^{n+1}$ is that C is an immersion, or equivalently the kernel of Df intersects the kernel of DC in zero. Now let

$$C(s,t,r) = X_2 + rN_2.$$

Then the Jacobian matrix of C has the column vectors

$$(1-r\kappa_i)v_i, \quad 0 \; and \; N_2,$$

and the Jacobian matrix of f has the column vectors

$$-(1-r\kappa_i)v_i, \quad (1-r\lambda_i)u_i \quad \text{and} \quad N_1 - N_2.$$

Now consider the following

$$\begin{pmatrix} (1-r\kappa_i)v_i & (1-r\lambda_i)u_i & N_1 - N_2 \\ (1-r\kappa_i)v_i & 0 & N_2 \end{pmatrix} \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_n \\ \vdots \\ \xi_{2n+1} \end{pmatrix} = 0$$

Therefore, the condition for C to be an immersion implies that $\xi_j = 0$, $j = 1, 2, ..., n, n+1, ..., 2n, 2n+1$. Since $N_1 \neq N_2$, we have $\xi_{2n+1} = 0$. From proposition 2.2.2, we have $f^{-1}(0)$ is smooth provided $\lambda_i \neq \frac{1}{r}$ or $\kappa_i \neq \frac{1}{r}$. Now assume that $\kappa_i \neq \frac{1}{r}$, then

$$\sum_{i=1}^{n} \xi_i (1 - r\kappa_i) v_i = 0.$$

Now since v_i are linearly independent then $\xi_i = 0$, ($i = 1, 2, ..., n$).
Hence, if $\lambda_i \neq \frac{1}{r}$ then, $\xi_j = 0$ where $j = n+1, n+2, ..., 2n$, and by this the proof is completed. □

Now the natural question is: what is the necessary and sufficient condition for two points x_1 and x_2 on the boundary with normals N_1 and N_2 respectively to form a symmetry point? In [14] this matter has been investigated in the case of the plane curve and to generalize that result we will take the case when the normals of the boundary (which are oriented inside the object figure 2.1) intersect each other and this case is a generic. The answer of this question is given in the following theorem.

Theorem 2.2.4 *Let x_1 and x_2 be two points on the boundary with unit normals N_1 and N_2 respectively. Suppose that $N_1 \neq -N_2$ and the normal lines intersect. Then a necessary* *and sufficient condition for* x_1 *and* x_2 *to form a symmetry point is that*

$$(x_1 - x_2) \cdot (N_1 + N_2) = 0. \tag{2.1}$$

Proof

First, assume that x_1 and x_2 form a symmetry point A , then

$$A = x_1 + rN_1 = x_2 + rN_2.$$

Therefore,

$$x_1 - x_2 = -r(N_1 - N_2).$$

So,

$$(x_1 - x_2) \cdot (N_1 + N_2) = -r(N_1 - N_2) \cdot (N_1 + N_2) = 0.$$

Second, assume that equation (2.1) holds since $N_1 \neq \pm N_2$ and the lines of normals intersect, then there exist ($a, b \in \mathbb{R}$) such that

$$x_1 + aN_1 = x_2 + bN_2.$$

Therefore

$$x_1 - x_2 + aN_1 - bN_2 = 0.$$

Hence,

$$(x_1 - x_2).(N_1 + N_2) + (aN_1 - bN_2).(N_1 + N_2) = 0.$$

Now using equation (2.1), we have

$$(aN_1 - bN_2).(N_1 + N_2) = 0$$

So,

$$a - b + aN_1 \cdot N_2 - bN_1 \cdot N_2 = 0.$$

Hence

$$(a - b)(1 + N_1 \cdot N_2) = 0.$$

Since $N_1 \neq -N_2$, we have $1 + N_1 \cdot N_2 \neq 0$ therefore, $a - b = 0 \Rightarrow a = b$ hence x_1 and x_2 form a symmetry point and $r = |a|$. $\square$

2.3 Creating the Boundary from the Symmetry Set

In this section we turn to the reconstructing the boundary from its symmetry set, and we will investigate the relationship between the symmetry set and the associated midlocus. Furthermore, we will study the radius function and its singularity. Also, the relationship between the singularity of the radius function and that of the midlocus in the plane will be investigated. In the rest of this chapter S_{reg} refers to the set of all smooth points of the the symmetry set that means the set of all points of type A_1^2 (*an* A_1^2 *point of the symmetry set is the centre of a bitangent hypersphere which has ordinary contacts with the boundary*).

Theorem 2.3.1 *Let S be the symmetry set of a region $\Omega \in \mathbb{R}^{n+1}$, with smooth boundary X and let $S_{reg} \subseteq S$. Then for any point $x_0 \in S_{reg}$, the associated tangency points on the boundary X are given by*

$$X_j = x_0 - r\nabla r \pm r\sqrt{1 - \|\nabla r\|^2}N, \qquad j = 1, 2 \tag{2.2}$$

such that $(1 - \|\nabla r\|^2 \geq 0)$, where N is the unit normal of S at x_0 and ∇r is the Riemannian gradient of r.

Proof

Let S_1 be a smooth patch of the symmetry set containing x_0, and consider the function

$$F = \|X - S_1\|^2 - r^2. \tag{2.3}$$

Now let $\{v_1, v_2, ..., v_n\}$ be a basis of $T_{x_0}S_1$, then

$$\frac{\partial F}{\partial v_j} = -2v_j \cdot (X - S_1) - 2rdr(v_j).$$

Therefore, the envelope of hyperspheres centred on S_1 is given by

$$\{X \in \mathbb{R}^{n+1} : F = \frac{\partial F}{\partial v_j} = 0, \quad j = 1, 2, ..., n\}.$$

Therefore,

$$-v_j \cdot (X - S_1) = rdr(v_j). \tag{2.4}$$

Now since $\{v_1, v_2, ..., v_n, N\}$, where N is the unit normal of S_1 at x_0, is a basis for $\mathbb{R}^{n+1}$, every point $Z \in \mathbb{R}^{n+1}$ can be written as

$$Z = \sum_{i=1}^{n} \lambda_i v_i + \lambda_{n+1} N.$$

Therefore,

$$X - x_0 = \sum_{i=1}^{n} \lambda_i v_i + \lambda_{n+1} N,$$

and from $F = 0$ we have $\lambda_{n+1} = \pm\sqrt{r^2 - (\sum_{i=1}^{n} \lambda_i v_i)^2} N$, thus

$$X = x_0 + \sum_{i=1}^{n} \lambda_i v_i \pm \sqrt{r^2 - (\sum_{i=1}^{n} \lambda_i v_i)^2} N. \tag{2.5}$$

But from equation (2.4), we have $v_j \cdot (X - S_1) = -rdr(v_j)$, where $dr(v_j)$ is the directional derivative of r in the direction v_j therefore $dr(v_j)$ can be written as

$$dr(v_j) = v_j \cdot \left(\frac{-1}{r} \sum_{i=1}^{n} \lambda_i v_i \right).$$

Since S_1 is smooth, then as S_1 is a submanifold of $\mathbb{R}^{n+1}$ so it has a Riemannian structure. Therefore, dr can be written again as:

$$dr(v_j) = \langle v_j, \frac{-1}{r} \sum_{i=1}^{n} \lambda_i v_i \rangle.$$

Therefore, $\frac{-1}{r} \sum_{i=1}^{n} \lambda_i v_i$ is the Riemannian gradient of the radius function r. Let ∇r denotes the Riemannian gradient of r, then $\frac{-1}{r} \sum_{i=1}^{n} \lambda_i v_i = \nabla r$. Therefore,

$$\sum_{i=1}^{n} \lambda_i v_i = -r \nabla r.$$

Hence by substitution in equation (2.5) the proof is completed. $\square$

Definition 2.3.2 *Given a smooth hypersurface X in $\mathbb{R}^{n+1}$ as in definition (2.2.1) the midlocus $\mathbb{M}$ of X is the closure of the set of midpoints of chord joining contact points of all hyperspheres bitangent to X. Thus if x_1 and x_2 are two points of tangency then the corresponding point of the midlocus is $x_m = \frac{1}{2}(x_1 + x_2)$.*

Now from theorem 2.3.1 we have the following.

Corollary 2.3.3 *Assume as in theorem 2.3.1 then, the midlocus point is given by*

$$x_m = x_0 - r\nabla r,$$

where x_0 is a smooth point of the symmetry set and ∇r is the Riemannian gradient of the radius function at x_0.

Proof

The proof comes directly from theorem 2.3.1. □

From corollary 2.3.3 we find that if S is a smooth part of the symmetry set, then the associated midlocus $\mathbb{M}$ is given by

$$\mathbb{M} = S - r\nabla r.$$

Peter Giblin pointed out that in the case of 3D if the tangent planes of the boundary at the tangency points are parallel then the radius function has a singularity [11]. In fact, the radius function plays a central role in the relationship between the boundary, the symmetry and the midlocus. Also there is a very complicated relationship between the differential geometry of the boundary and that of the symmetry set involving the radius function and its derivatives. The following theorem gives the answer to the question: At what condition does the symmetry point coincide with the associated midlocus point?

Theorem 2.3.4 *Let S be the symmetry set of a region Ω in $\mathbb{R}^{n+1}$, with smooth boundary X. Let $x_0 \in S_{reg}$, then x_0 and the associated midlocus point x_m coincide if and only if the radius function has a singularity at x_0.*

Proof

Let $x_m = x_0$, then $\nabla r = 0$ and since ∇r is the Riemannian gradient, it can be written as

$$\nabla r = g^{ij} dr(v_j) v_i,$$

where g^{ij} is the inverse of the matrix representing the Riemannian metric, $dr(v_j)$ is the partial derivative of the radius function r and v_i is the basis of the tangent space of the symmetry set S at x_0. Therefore, $\nabla r = 0 \Rightarrow g^{ij} dr(v_j) v_i = 0$, hence $dr(v_j) = 0$. Conversely assume that the radius function r has a singularity, then $\nabla r = 0$, therefore, $x_0 = x_m$ which completes the proof. □

The above theorem tells us the impact of the singularity of the radius function on the relationship between a smooth symmetry point and its associated midlocus point. But what about the relation between a smooth symmetry point and its associated tangency points on the boundary. Does the singularity of the radius function affect it? The answer to this question is given in the following proposition.

Proposition 2.3.5 *Let S be the symmetry set of a region Ω in $\mathbb{R}^{n+1}$ with smooth boundary X. Let $x_0 \in S_{reg}$, then the tangency points associated to x_0 are given by:*

$$x_j = x_0 \pm rN, \quad j = 1, 2$$

if and only if the radius function r has a singularity at x_0, where N is the unit normal of S at x_0.

Proof

From theorem 2.3.1 the tangency points associated to x_0 are given by

$$x_j = x_0 - r\nabla r \pm r\sqrt{1 - \|\nabla r\|^2} N, \qquad j = 1, 2.$$

Now assume that r has a singularity, then $\nabla r = 0$. Therefore, we get

$$x_j = x_0 \pm rN, \quad j = 1, 2.$$

Conversely, assume that $x_j = x_0 \pm rN,\ j = 1, 2$. Then, $-r\nabla r = 0$ and $(1 - \|\nabla r\|^2) = 1$. Therefore, $\nabla r = 0$, hence the radius function r has a singularity, and by this the proof is completed. □

Lemma 2.3.6 *Let S be the symmetry set of a region Ω in $\mathbb{R}^{n+1}$ with smooth boundary X. Let x_0 be a smooth point in S. Then the Riemannian gradient of the radius function at x_0 is given by:*

$$\nabla r = \frac{1}{2}(N_1 + N_2),$$

where N_1 and N_2 are the unit normals of the boundary at the tangency points.

Proof

From definition 2.2.1, we have

$$x_0 = x_1 + rN_1 = x_2 + rN_2$$

and from theorem 2.3.1 we have

$$x_1 = x_0 - r\nabla r + r\sqrt{1 - \|\nabla r\|^2}N$$

and,

$$x_2 = x_0 - r\nabla r - r\sqrt{1 - \|\nabla r\|^2}N.$$

Therefore,

$$N_1 = \nabla r - \sqrt{1 - \|\nabla r\|^2}N \qquad and \qquad N_2 = \nabla r + \sqrt{1 - \|\nabla r\|^2}N.$$

Hence, $\nabla r = \frac{1}{2}(N_1 + N_2)$. □

Proposition 2.3.7 *Assume as in lemma 2.3.6. Then r has a singularity if and only if $N_1 = -N_2$. where N_1 and N_2 are the unit normals of the boundary at the tangency points.*

Proof

From lemma 2.3.6 we have,

$$\nabla r = \frac{1}{2}(N_1 + N_2).$$

If the radius function has a singularity, then $\nabla r = 0$, and so $N_1 = -N_2$. Now if $N_1 = -N_2$ then $\nabla r = 0$ which implies that the radius function has a singularity. □

Example 2.3.8 *Let $S(s,t)$ be the symmetry set of a smooth surface X in $\mathbb{R}^3$, and $r(s,t)$ be the radius function. If $x_0 = S(s_0,t_0)$ be a smooth point, then the associated tangency points on the boundary X are given by*

$$\begin{aligned} x_j =& x_0 - \frac{r}{\|\epsilon_1 \times \epsilon_2\|}\{(r_s\|\epsilon_2\|^2 - r_t\epsilon_1.\epsilon_2)\epsilon_1 + (r_t\|\epsilon_1\|^2 - r_s\epsilon_1.\epsilon_2)\epsilon_2\} \\ &\pm r\sqrt{1 - \frac{1}{\|\epsilon_1 \times \epsilon_2\|^2}\{(r_s\|\epsilon_2\|^2 - r_t\epsilon_1.\epsilon_2)\epsilon_1 + (r_t\|\epsilon_1\|^2 - r_s\epsilon_1.\epsilon_2)\epsilon_2\}^2}N, \quad j = 1,2, \end{aligned}$$

where $\epsilon_1 = \frac{\partial S}{\partial s}|_{(s_0,t_0)}$ and, $\epsilon_2 = \frac{\partial S}{\partial t}|_{(s_0,t_0)}$ and $r_s = \frac{\partial r}{\partial s}$, $r_t = \frac{\partial r}{\partial t}$.
The midlocus point is given by:

$$x_m = x_0 - \frac{r}{\|\epsilon_1 \times \epsilon_2\|}\{(r_s\|\epsilon_2\|^2 - r_t\epsilon_1.\epsilon_2)\epsilon_1 + (r_t\|\epsilon_1\|^2 - r_s\epsilon_1.\epsilon_2)\epsilon_2\}.$$

In the above example we give a general method to calculate the tangency points and the midlocus point associated to a smooth point on the symmetry set of a smooth surface $X \subset \mathbb{R}^3$. In the following we will present specific examples to illustrate the ideas of the theorems mentioned in this section.

Example 2.3.9 *Let* $S(s,t) = (s, t, st)$ *and* $r(s,t) = s^2 + t^2 + 1$*, where* $\frac{-1}{4} \leq s \leq \frac{1}{4}$ *and* $\frac{-1}{4} \leq t \leq \frac{1}{4}$*. Then the midlocus is given by*

$$\mathbb{M}(s,t) = -\left(s + 2s^3 - 2st^2, t + 2t^3 - 2s^2t, 3st\right).$$

It is clear that $r_s(0,0) = r_t(0,0) = 0$*, which means that the radius function has a singularity at* $(0,0)$*. Also, it is obvious that* $S(0,0) = \mathbb{M}(0,0) = (0,0,0)$*.*

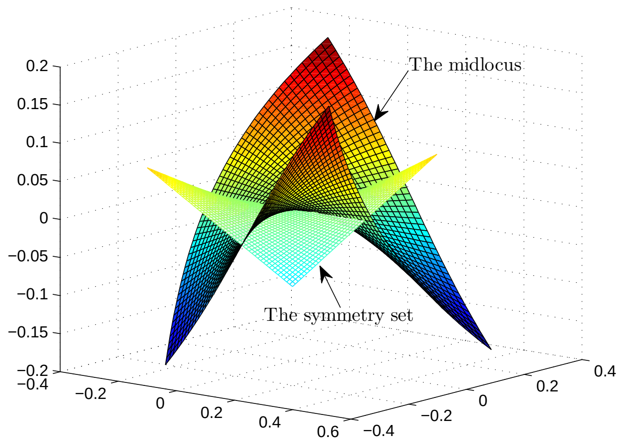

Figure 2.2: Symmetry set and associated midlocus in example 2.3.9.

Example 2.3.10 *Let* $S(s,t) = (s, t, s^2 + t^2)$ *and* $r(s,t) = s^2 + t^2 + 1$*. Then the midlocus is given by*

$$\mathbb{M}(s,t) = \left(\frac{2s^3 + 2st^2 - s}{4(s^2+t^2)+1}, \frac{2t^3 + 2s^2t - t}{4(s^2+t^2)+1}, \frac{-3(s^2+t^2)}{4(s^2+t^2)+1}\right).$$

It is clear that, r *has a singularity at* $(0,0)$ *and* $S(0,0) = \mathbb{M}(0,0) = (0,0,0)$*.*

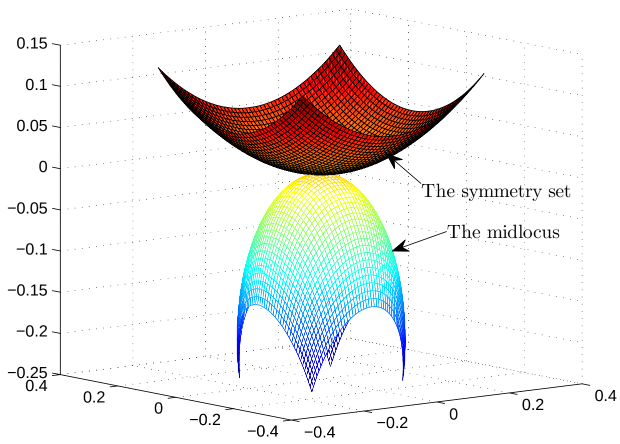

Figure 2.3: Symmetry set and associated midlocus in example 2.3.10.

Example 2.3.11 *Let S be the symmetry set of a plane curve and let S be smooth at x_0. Now parameterize S by the arc-length then the Riemannian gradient of the radius function is given by*

$$\nabla r = r'T = \frac{1}{2}(N_1 + N_2).$$

Therefore,

$$r'^2 = \frac{1}{4}(N_1 + N_2)^2.$$

Also,

$$r' = \frac{1}{2}(T.N_1 + T.N_2).$$

Thus

$$r' = \cos\theta,$$

where θ is the angle between N_i and T, $i = 1, 2$. If $N_1 \perp N_2$ then $r'^2 = \frac{1}{2}$.

Let γ be a smooth plane curve and suppose that x_0 is the centre of a bitangent circle to γ at x_1 and x_2. Let γ_1 and γ_2 be small pieces of γ close to x_1 and x_2 respectively as shown in the figure 2.4. The relation between the arc-lengths of the boundary was studied by Giblin. Our target is to study the relationship between the arc-lengths of the symmetry set and that of the boundary.

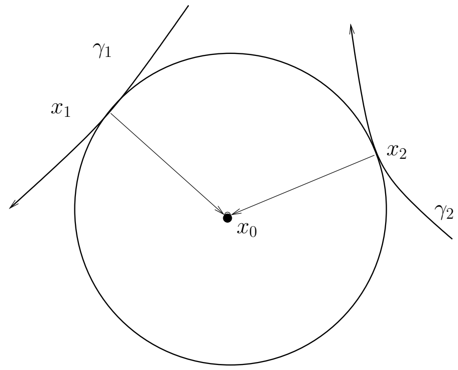

Figure 2.4: The symmetry point and the associated tangency points in the case of curve.

Lemma 2.3.12 *Let S be the symmetry set of a plane curve γ and $x_0 \in S_{reg}$ and let $x_1 \in \gamma_1$ and $x_2 \in \gamma_2$ be the associated tangency points on the boundary. Let s_1 and s_2 be the arc-lengths on γ_1 and γ_2 respectively. Then we have*

$$(1 - r\kappa_1)\frac{ds_1}{ds} = -(1 - r\kappa_2)\frac{ds_2}{ds}$$

where s is the arc-length of a smooth part of S close to x_0.

Proof

From definition of symmetry set the part S_1 of S associated to γ_1 and γ_2 is given by

$$\begin{aligned} S_1(s) &= \gamma_1(s_1) + r(s_1)N_1(s_1) \\ &= \gamma_2(s_2) + r(s_2)N_2(s_2). \end{aligned}$$

Therefore we have

$$T = T_1\frac{ds_1}{ds} + r'N_1 - r\kappa_1 T_1\frac{ds_1}{ds}. \tag{2.6}$$

Also, we have

$$T = T_2\frac{ds_2}{ds} + r'N_2 - r\kappa_2 T_2\frac{ds_2}{ds} \tag{2.7}$$

where T, T_1 and T_2 are the unit tangents of the symmetry set and the boundary respectively. Now (2.6)-(2.7) gives the following equation

$$(1 - r\kappa_1)\frac{ds_1}{ds}T_1 - (1 - r\kappa_2)\frac{ds_2}{ds}T_2 + r'(N_1 - N_2) = 0. \tag{2.8}$$

Now the inner product on both sides of equation 2.8 with $T_1 - T_2$ gives the following

$$(1 - r\kappa_1)(1 - T_1 \cdot T_2)\frac{ds_1}{ds} + (1 - r\kappa_2)(1 - T_1 \cdot T_2)\frac{ds_2}{ds} = 0.$$

Since $T_1 \neq T_2$ because of the orientation, then $1 - T_1 \cdot T_2 \neq 0$.
Therefore, $(1 - r\kappa_1)\frac{ds_1}{ds} + (1 - r\kappa_2)\frac{ds_2}{ds} = 0$. Hence $(1 - r\kappa_1)\frac{ds_1}{ds} = -(1 - r\kappa_2)\frac{ds_2}{ds}$, which completes the proof. □

Now we have the following corollary [13, 32].

Corollary 2.3.13 *Assume as in lemma 2.3.12. Then*

$$\frac{ds_2}{ds_1} = -\left(\frac{1 - r\kappa_1}{1 - r\kappa_2}\right).$$

Proof

The proof of this corollary comes directly from the above lemma. □

In the next proposition we will give the relationship between the arc-length of the symmetry set and those on the boundary. First of all we need this lemma.

Lemma 2.3.14 [14] *Let S be the symmetry set of a smooth plane curve γ. The tangents T_1 and T_2 and normals N_1 and N_2 of γ at the tangency points associated to a smooth point $x_0 \in \gamma$ are given by*

$$T_1 = -\sqrt{1 - r'^2}T - r'N,$$

$$T_2 = \sqrt{1 - {r'}^2}T - r'N,$$

$$N_1 = r'T - \sqrt{1 - {r'}^2}N,$$

and

$$N_2 = r'T + \sqrt{1 - {r'}^2}N$$

where, T and N are the unit tangent and unit normal of the symmetry set at x_0.

Proposition 2.3.15 *Let S be the symmetry set of a smooth plane curve γ and s be the arc-length on S_{reg}. Then we have*

$$\frac{ds_1}{ds} = -\frac{\sqrt{1 - r'^2}}{1 - r\kappa_1} \qquad and \qquad \frac{ds_2}{ds} = \frac{\sqrt{1 - r'^2}}{1 - r\kappa_2}.$$

Proof

Let S_1 be the smooth part of S associated to γ_1 and γ_2 parametrized by the arc-length s, thus $S_1 = \gamma_1 + rN_1$ which gives that

$$T = (1 - r\kappa_1)\frac{ds_1}{ds}T_1 + r'N_1. \tag{2.9}$$

But from lemma 2.3.14 we have

$$T_1 = -\sqrt{1 - r'^2}T - r'N \qquad and \qquad N_1 = r'T - \sqrt{1 - r'^2}N.$$

Therefore, substituting in (2.9) we get the following equation

$$T = \left(-(1 - r\kappa_1)\sqrt{1 - r'^2}\frac{ds_1}{ds}) + r'^2\right)T - r'\left((1 - r\kappa_1)\frac{ds_1}{ds} + \sqrt{1 - r'^2}\right)N.$$

Now equating the tangential part we obtain

$$-(1 - r\kappa_1)\sqrt{1 - r'^2}\frac{ds_1}{ds} + r'^2 = 1.$$

Therefore this equation gives

$$\frac{ds_1}{ds} = -\frac{\sqrt{1 - r'^2}}{1 - r\kappa_1}. \tag{2.10}$$

Also, from corollary 2.3.12 we have

$$(1 - r\kappa_1)\frac{ds_1}{ds} = -(1 - r\kappa_2)\frac{ds_2}{ds}$$

hence from this equation and equation 2.10 we find that

$$\frac{ds_2}{ds} = \frac{\sqrt{1 - r'^2}}{1 - r\kappa_2},$$

and by this the proof is completed. □

In the following proposition we will turn to the relationship between the singularity of the radius function and that of the midlocus in the case of plane curve.

Lemma 2.3.16 $[13, 32]$ *Given a smooth curve* γ *with* x_1 *and* x_2 *being points of contact of bitangent circle, then the midlocus is smooth here provided* $T_1 \neq -T_2$ *or* $\kappa_1 + \kappa_2 \neq \frac{2}{r}$ *where,* T_i *and* κ_i*,* $i = 1, 2$ *are the tangents and curvatures of* γ *at* x_1 *and* x_2*.*

This lemma tells us conditions for the smoothness of the midlocus. At a smooth point of the symmetry set the condition $T_1 \neq -T_2$ means that the radius function has no singularity and this can be obtained from proposition 2.3.5. Thus we can determine the conditions that allow the midlocus to have a singularity in terms of the singularity of the radius function.

Proposition 2.3.17 *Let* S *be the symmetry set of a smooth plane curve* γ*. Let* $x_0 \in S_{reg}$*. Then the midlocus is singular at the point associated to* x_0 *if and only if the radius function has a singularity at* x_0 *and* $r''(x_0) = \frac{1}{r(x_0)}$*.*

Proof

Let S_1 be the smooth part of S close to x_0 parametrized by the arc-length s then the associated midlocus $\mathbb{M}$ is given by

$$\mathbb{M} = S_1 - rr'T.$$

Therefore

$$\mathbb{M}' = (1 - r'^2 - rr'')T - rr'\kappa N.$$

Now assume that $r' = 0$ and $r''(x_0) = \frac{1}{r(x_0)}$ then $\mathbb{M}' = 0$ which means that the midlocus is singular. Conversely assume that the midlocus is singular, then the radius function has a singularity, so from the above equation we have $1 - rr'' = 0$ which implies that $r'' = \frac{1}{r}$. □

The above proposition tells us the relationship between the singularity of the radius function and that of the midlocus. Also, it determines the type of the singularity of the radius function. Recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to have an A_1 singularity at t_0 if $f'(t_0) = 0$ and $f''(t_0) \neq 0$.

Corollary 2.3.18 *Let S be the symmetry set of a smooth plane curve γ. Let $x_0 \in S_{reg}$. If the midlocus is singular at the point associated to x_0, then the radius function has only an A_1 singularity at x_0.*

Proof

From proposition 2.3.17 we have if the midlocus is singular then $r'' = \frac{1}{r}$ and $r' = 0$ which means that the radius function has an A_1 singularity. □

Now we will end this section by calculating the area of the triangle formed by a smooth symmetry point and its associated tangency points on the boundary.

Let $x_0 \in S_{reg}$, then by theorem 2.3.1 the tangency points are given by

$$x_1 = x_0 - r\nabla r + r\sqrt{1 - \|\nabla r\|^2}N,$$

and,

$$x_2 = x_0 - r\nabla r - r\sqrt{1 - \|\nabla r\|^2}N.$$

Therefore,

$$x_1 - x_2 = 2r\sqrt{1 - \|\nabla r\|^2}N.$$

This implies

$$\|x_1 - x_2\| = 2r\sqrt{1 - \|\nabla r\|^2}.$$

Also, the height of the triangle in figure 3.2 is given by

$$h = \|S - \mathbb{M}\| = r\|\nabla r\|.$$

Hence the area of this triangle is given by

$$A = r^2\|\nabla r\|\sqrt{1 - \|\nabla r\|^2}.$$

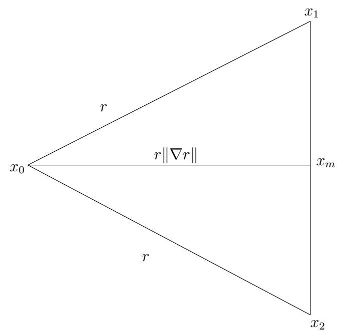

Figure 2.5: Triangle formed by symmetry point and its associated tangency points.

So, we can summarize this in the following proposition.

Proposition 2.3.19 *Let S be the symmetry set of a region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary X. Let x_0 be a smooth point of S, then the area of the triangle formed by x_0 and the associated tangency points is given by:*

$$A = r^2\|\nabla r\|\sqrt{1 - \|\nabla r\|^2}.$$

Corollary 2.3.20 *Assume as in proposition 2.3.19. If the radius function has a singularity, then $A = 0$.*

Proof

If the radius function has a singularity then we have $\nabla r = 0$. This implies that $A = 0$. □

2.4 Singularity of the Radius Function at a Singular Point of the Symmetry Set

It was pointed out by Giblin that in the case of the plane curve if the symmetry set has a cusp then the radius function has a singularity. In this section we discuss this phonemena in the case of higher dimensions and we study the relationship between the singularity of the radius function and the coincidence of the symmetry point and its associated midlocus point. Before studying this phonemena we recall that a point x_0 of a symmetry set is of type

- $A_1A_1 = A_1^2$ if the bitangent hypersphere has an ordinary contact with the boundary at the associated tangency points, i.e, the hypersphere is not the hypersphere of the curvature.
- $A_1A_{k\geq 2}$ if the bitangent hypersphere has an ordinary contact with the boundary at one point (A_1) and it is the hypersphere of the curvature at the other tangency point of the boundary ($A_{k\geq 2}$).
- A_3 if the hypersphere has a single contact with the boundary. This point is a limiting case of the two points in the A_1^2 case above.

Let S be the symmetry set of a smooth hypersurface X and x_1 and x_2 be the tangency points corresponding to $x_0 \in S$. Suppose that the contact at x_1 is of type A_1 and the contact at x_2 is of type $A_{k\geq 2}$ then, the symmetry set is singular at x_0. Now we will investigate the singularity of the radius function. Let $X_1 \subset X$ and $X_2 \subset X$ be two pieces of X around x_1 and x_2 respectively. Let $s = (s_1, s_2, ..., s_n)$ and $t = (t_1, t_2, ..., t_n)$ be the local parameters of X_1 and X_2 since the contact at x_1 is of type A_1 then using the implicit function theorem s is a smooth function of t. The part S_1 of the symmetry set associated to X_1 and X_2 is given by

$$S_1 = X_1 + rN_1 = X_2 + rN_2. \tag{2.11}$$

From equation 2.11 we have

$$S_1 = X_2 + rN_2$$

differentiate this equation with respect to t_i we get

$$\frac{\partial S_1}{\partial t_i} = v_i + r\frac{\partial N_2}{\partial t_i} + \frac{\partial r}{\partial t_i}N_2, \quad v_i = \frac{\partial X_2}{\partial t_i}.$$

This equation can be written in vector form

$$\frac{\partial S_1}{\partial t} = V - rS_{x_2}^T + \frac{\partial r}{\partial t}N_2 = (I - rS_{x_2}^T)V + \frac{\partial r}{\partial t}N_2, \tag{2.12}$$

where S_{x_2} is the matrix representation of the shape operator of the boundary at x_2,

$$V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad and \quad \frac{\partial r}{\partial t} = \begin{pmatrix} \frac{\partial r}{\partial t_1} \\ \frac{\partial r}{\partial t_2} \\ \vdots \\ \frac{\partial r}{\partial t_n} \end{pmatrix}.$$

Now applying dot product with N_2 to each entry in equation 2.12, we obtain

$$\frac{\partial S_1}{\partial t} \cdot N_2 = \frac{\partial r}{\partial t}. \tag{2.13}$$

Also, from equation 2.11 we have

$$S_1 = X_1 + rN_1. \tag{2.14}$$

Now differentiate X_1 with respect to t_i we get

$$\frac{\partial X_1}{\partial t_i} = \frac{\partial X_1}{\partial s_1} \cdot \frac{\partial s_1}{\partial t_i} + \frac{\partial X_1}{\partial s_2} \cdot \frac{\partial s_2}{\partial t_i} + ... + \frac{\partial X_1}{\partial s_n} \cdot \frac{\partial s_1}{\partial t_i} = \begin{pmatrix} \frac{\partial s_1}{\partial t_i} & \frac{\partial s_2}{\partial t_i} & \cdots & \frac{\partial s_n}{\partial t_i} \end{pmatrix} \begin{pmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{pmatrix},$$

where $v_i^* = \frac{\partial X_1}{\partial s_i}$. Therefore,

$$\frac{\partial X_1}{\partial t} = \begin{pmatrix} \frac{\partial s_1}{\partial t_1} & \frac{\partial s_2}{\partial t_1} & \cdots & \frac{\partial s_n}{\partial t_1} \\ \frac{\partial s_1}{\partial t_2} & \frac{\partial s_2}{\partial t_2} & \cdots & \frac{\partial s_n}{\partial t_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial s_1}{\partial t_n} & \frac{\partial s_2}{\partial t_n} & \cdots & \frac{\partial s_n}{\partial t_n} \end{pmatrix} \begin{pmatrix} v_1^* \\ v_2^* \\ \vdots \\ v_n^* \end{pmatrix} = \Gamma V^*.$$

Similarly,

$$\frac{\partial N_1}{\partial t} = -\Gamma S_{x_1}^T V^*,$$

where S_{x_1} is the matrix representation of the shape operator of the boundary at x_1. Now differentiating equation 2.14 with respect to t gives

$$\frac{\partial S_1}{\partial t} = \Gamma(I - rS_{x_1}^T)V^* + \frac{\partial r}{\partial t}N_1. \tag{2.15}$$

Now applying dot product with N_1 to each entry in equation 2.15 , we obtain

$$\frac{\partial S_1}{\partial t} \cdot N_1 = \frac{\partial r}{\partial t}. \tag{2.16}$$

Therefore, the radius function has a singularity if and only if N_1 and N_2 are perpendicular to the set of vectors $\Theta = \frac{\partial S_1}{\partial t_i}, i = 1, 2, ..., n$.

Therefore, we state the following.

Theorem 2.4.1 *Let S be the symmetry set of a smooth hypersurface $X \subseteq \mathbb{R}^{n+1}$. Let $x_0 \in S$ be a singular point of type $A_1A_{k\geq 2}$ of the symmetry set. Then the radius function has a singularity if and only if the unit normals of the tangency points corresponding to x_0 are perpendicular to the tangent space of the symmetry set at x_0.*

Example 2.4.2 *Let γ be a smooth closed curve and S be its symmetry set. Suppose that $x_0 \in S$ be an A_1A_2 point then the symmetry set is a cusp at x_0, therefore $\frac{dS}{dt}|_{x_0} = (0, 0)$ which is perpendicular to any vector hence the unit normals are perpendicular to this vector. Thus the radius function has a singularity.*

Remark 2.4.3 *The radius function has no singularity when the symmetry set has an edge point A_3 which is a limiting point of two points of type A_1. The normal of the boundary in this case is tangent the smooth stratum containing A_3.*

Now we will discuss what happens when the midlocus and the singular point of the symmetry set coincide.

Proposition 2.4.4 *Let S be the symmetry set of a smooth hypersurface $X \subseteq \mathbb{R}^{n+1}$. Let $x_0 \in S$ be a singular point of type $A_1A_{k\geq 2}$ of the symmetry set and x_m be the associated midlocus. If x_0 and x_m coincide then the radius function has a singularity.*

Proof

From the definition of the symmetry set we have

$$x_0 = x_1 + rN_1 = x_2 + rN_2.$$

Thus we get

$$2x_0 = x_1 + rN_1 + x_2 + rN_2 = 2x_m + r(N_1 + N_2).$$

Now if $x_0 = x_m$, then we have $N_1 + N_2 = 0$ and from equations 2.13 and 2.16 we have $2\frac{\partial r}{\partial t} = \frac{\partial S_1}{\partial t} \cdot (N_1 + N_2) = 0$. Thus $\frac{\partial r}{\partial t} = 0$. $\square$

2.5 Creating the Symmetry Set from the Midlocus

In this section we will discuss the possibility of creating the boundary using the information provided by the midlocus and the radius function. In fact, Peter Giblin and John Paul Warder [16, 32] created the symmetry set of a plane curve using the midlocus and radius function. Our task is to generalize this idea to the higher dimensions. Now if we are given the symmetry set S as a smooth hypersurface parametrized by $(x_1, x_2, ..., x_n)$

and let $\mathbb{M}$ be the associated smooth midlocus, then from corollary 2.3.3 for each $x \in S$, the associated midlocus point x_m is given by

$$x_m = x + \sum_{i=1}^{n} \lambda_i v_i, \tag{2.17}$$

where $v_i = \frac{\partial S}{\partial x_i}$ evaluated at x, also, we have

$$-v_j \cdot \left(\sum_{i=1}^{n} \lambda_i v_i \right) = r^j r, \quad r^j = \frac{\partial r}{\partial x_j}.$$

Now from equation 2.17 we have

$$x - x_m = -\sum_{i=1}^{n} \lambda_i v_i.$$

Therefore,

$$v_i \cdot (x - x_m) = -v_i \cdot \sum_{i=1}^{n} \lambda_i v_i = r^i r. \tag{2.18}$$

Now there are a lot of solutions of equation (2.18) i.e., there are many vectors $v_i \in \mathbb{R}^{n+1}$ such that the equation holds, but there is only one solution on the form $\alpha(x - x_m)$, where $\alpha \in \mathbb{R}$. This solution is of the form:

$$\frac{r^i r(x - x_m)}{\|x - x_m\|^2}$$

provided $x \neq x_m$. Therefore, we summarize this in the following theorem.

Theorem 2.5.1 *Given the midlocus and the radius function (smooth function) describing the radius of the hypersphere generating each point of the midlocus $\mathbb{M}$ of a smooth hypersurface X in $\mathbb{R}^{n+1}$. Then the symmetry set associated to this midlocus is a solution of the PDEs*

$$\frac{\partial S}{\partial x_i} = \frac{r^i r(S - \mathbb{M})}{\|S - \mathbb{M}\|^2}$$

provided $S \neq \mathbb{M}$, where $r^i = \frac{\partial r}{\partial x_i}$.

Now we discuss this theorem in $\mathbb{R}^2$ and $\mathbb{R}^3$. Now let $\mathbb{M}(t) = (m_1(t), m_2(t))$ be the midlocus of a smooth curve in $\mathbb{R}^2$ and $r(t)$ be the smooth function describing the radius of each circle generating each point of $\mathbb{M}$. The above theorem indicates that

$$\begin{aligned}\frac{dS}{dt} &= \left(\frac{ds_1}{dt}, \frac{ds_2}{dt}\right) = \frac{rr'(S-\mathbb{M})}{\|S-\mathbb{M}\|^2} \\ &= \left(\frac{rr'(s_1-m_1)}{(s_1-m_1)^2+(s_2-m_2)^2}, \frac{rr'(s_2-m_2)}{(s_1-m_1)^2+(s_2-m_2)^2}\right).\end{aligned}$$

Now put $X = s_1 - m_1$, $Y = s_2 - m_2$ and $rr' = R$, then we have $X' = \dfrac{RX}{X^2+Y^2} - m_1'$ and $Y' = \dfrac{RY}{X^2+Y^2} - m_2'$ which are the same ordinary differential equations obtained by Giblin and Warder [16, 32] and an interesting example can be found in [16]. Now we discuss this theorem in $\mathbb{R}^3$ and we will have partial differential equations instead of ordinary differential equations. Let $\mathbb{M}(x, y) = (m_1, m_2, m_3)$ be the midlocus of a smooth surface in $\mathbb{R}^3$ and $r(x, y)$ be the radius function which is a smooth function describing the radius of each sphere generating each point of $\mathbb{M}$. Our target is to find the associated symmetry set $S(x, y) = (s_1, s_2, s_3)$, and from theorem 2.5.1 we can create the partial differential equations which hold for the symmetry set, thus we have

$$\frac{\partial S}{\partial x} = \frac{rr_x(S-\mathbb{M})}{\|S-\mathbb{M}\|^2} \quad and \quad \frac{\partial S}{\partial y} = \frac{rr_y(S-\mathbb{M})}{\|S-\mathbb{M}\|^2}.$$

Therefore,

$$\frac{\partial s_i}{\partial x} = \frac{rr_x(s_i-m_i)}{\sum\limits_{j=1}^{3}(s_j-m_j)^2} \quad and \quad \frac{\partial s_i}{\partial y} = \frac{rr_y(s_i-m_i)}{\sum\limits_{j=1}^{3}(s_j-m_j)^2}, \quad i = 1, 2, 3.$$

Now put $F_i = s_i - m_i$, $j = 1, 2, 3$, then we have

$$\frac{\partial F_i}{\partial x} = \frac{rr_x F_i}{\sum\limits_{j=1}^{3} F_j^2} - \frac{\partial m_i}{\partial x} \quad and \quad \frac{\partial F_i}{\partial y} = \frac{rr_y F_i}{\sum\limits_{j=1}^{3} F_j^2} - \frac{\partial m_i}{\partial y}.$$

After solving these PDEs the required symmetry set is given by

$$S = F + \mathbb{M} = (F_1 + m_1, F_2 + m_2, F_3 + m_3).$$

Chapter 3

Centroid Set, Skeletal Structure and the Singularity of the Radius Function

3.1 Introduction

The symmetry set of a hypersurface $X \subset \mathbb{R}^{n+1}$ and its associated midlocus were studied in chapter 2 as well as the impact of the singularity of the radius function on the relationship between them. In this chapter we will study a more general concept than the midlocus. Precisely this chapter consists of four main parts. In the first part the centroid set associated to a smooth submanifold M of $\mathbb{R}^{n+1}$ will be defined. The centroid set is more general than the midlocus and it depends on a multivalued radial vector field U defined on M such that each value of U forms a smooth radial vector field on M and has associated radius function. The impact of the singularity of the radius function on the relationship between M and the associated centroid set will be studied in this part (**Theorem 3.2.9** and **Theorem 3.2.12**). Moreover, the condition for the centroid set to have a singularity when the radius function has a singularity will be studied (**Proposition 3.2.10**). The second main part of this chapter deals with the skeletal structure and the singularity of the radius function. In this part the relationship between the singularity of the radius function and the orthogonality of the radial vector field on the tangent space of the skeletal structure will be studied (**Proposition 3.3.2**). Furthermore, the relationship between a smooth point of a skeletal structure and its associated midlocus point will be considered when the radius function has a singularity as well as the relationship between the tangent spaces of the boundary at the tangency points. The third part of this chapter deals with the pre-medial axis of a smooth hypersurface. In this part the pre-medial axis is defined and the relationship between the parameters of a skeletal structure in a neighbourhood of a smooth point and the parameters of the boundary in a neighbourhood of the associated point will be studied (**Lemma 3.4.2**). The main result of this part is (**Proposition 3.4.3**) which gives the relationship between the parameters of the boundary of a Blum medial axis at the tangency points associated to a smooth point. The fourth part of this chapter deals with the classification of the singularity of the midlocus associated to a skeletal structure in $\mathbb{R}^3$. The impact of the eigenvalues of the Hessian of the radius function on the corank of the singularity of the midlocus will be studied (**Theorem 3.5.6**). The main result of this part is (**Theorem 3.5.12**) which gives the necessary and sufficient condition for the midlocus to have a crosscap singularity.

3.2 Centroid Points

Let M be a smooth submanifold of $\mathbb{R}^{n+1}$ such that on this submanifold we pick a multivalued vector field $U = (u_1, u_2, ..., u_l)$ such that each U_i forms a smooth vector field on M. We put $u_i = r_i U_i$ where U_i is a smooth unit vector field on M and r_i is a smooth function on M i.e., $r_i : M \to \mathbb{R}$, and we assume that $r_i > 0$. Now let $T_{x_0}M$ be the tangent space of M at x_0 and $v \in T_{x_0}M$. For each smooth vector field U_i we equip M with the 1-form

$$\eta_i(v) = dr_i(v) + U_i \cdot v,$$

where $dr_i(v)$ is the directional derivative of r_i in the direction of v. Now since M is a smooth submanifold of $\mathbb{R}^{n+1}$ it has a Riemannian structure induced from $\mathbb{R}^{n+1}$ and the tangent space $T_{x_0}M$ of M at $x_0 \in M$ is considered to be embedded in the tangent space $T_{x_0}\mathbb{R}^{n+1}$ of $\mathbb{R}^{n+1}$ at x_0. Recall that the directional derivative of a smooth function on a Riemannian manifold in the direction of a tangent vector v_j is given by $dr_i(v_j) = \langle \nabla r_i, v_j \rangle$, where ∇r_i is the Riemannian gradient of r_i, and $\langle , \rangle$ is the Euclidean inner product. Therefore, η_i can be written as

$$\eta_i(v_j) = \langle \nabla r_i, v_j \rangle + \langle U_i, v_j \rangle = \langle \nabla r_i + U_i, v_j \rangle .$$

Definition 3.2.1 *Let M be a smooth k-dimensional submanifold of $\mathbb{R}^{n+1}$, then*

1. *The tangent space to M at $x_0 \in M$ is the vector subspace $T_{x_0}M \subset T_{x_0}\mathbb{R}^{n+1}$, which is defined by*

$$T_{x_0}M := df_p(\{p\} \times \mathbb{R}^k) = df_p(T_p\mathbb{R}^k)$$

for a parametrization $f : \mathcal{U} \longrightarrow M$ with $f(p) = x_0$, where $\mathcal{U} \subseteq \mathbb{R}^k$ is an open set and df is the differential of f. The vector space $T_{x_0}M$ does not depend on the choice of f.

2. *The normal space to M at $x_0 \in M$ is the vector subspace $N_{x_0}M \subset T_{x_0}\mathbb{R}^{n+1}$, which is the orthogonal complement of $T_{x_0}M$:*

$$T_{x_0}\mathbb{R}^{n+1} = T_{x_0}M \oplus N_{x_0}M.$$

Here $\oplus$ denotes the orthogonal direct sum with respect to the Euclidean inner product.

Lemma 3.2.2 *Let M be a smooth submanifold of $\mathbb{R}^{n+1}$ as above, then the 1-form η_i vanishes at $x_0 \in M$ if and only if $\nabla r_i + U_i \in N_{x_0}M$, where $N_{x_0}M$ is the normal space of M at x_0.*

Proof

Recall that a vector $z \in T_{x_0}\mathbb{R}^{n+1}$ is a normal vector of a submanifold M at x_0 if and only if $\langle z, v_j \rangle = 0$ for all $v_j \in T_{x_0}M$. Thus, $\eta_i(v_j) = \langle \nabla r_i + U_i, v_j \rangle = 0$ for all j if and only if $(\nabla r_i + U_i) \in N_{x_0}M$, where $N_{x_0}M$ is the normal space of M at x_0. □

Remark 3.2.3 *Now let $U_i = U_i^T + U_i^N$, where U_i^T is the tangential component of U_i and U_i^N is the normal component. Then, the 1-form $\eta_i = 0$ if and only if $U_i^T = -\nabla r_i$.*

Theorem 3.2.4 *Let (M, U) be a smooth submanifold of $\mathbb{R}^{n+1}$ and multivalued vector field as above such that $\eta_i = 0$ at $x_0 \in M$. Then*

1. *r_i has a singularity at x_0 if and only if $U_i(x_0) \in N_{x_0}M$.*
2. *If r_i has a singularity at x_0 for all i, then $\sum_{i=1}^{l} U_i(x_0) \in N_{x_0}M$.*

Proof

1. Since $\eta_i = 0$, then $\nabla r_i + U_i \in NM$, by lemma 3.2.2 r_i has a singularity if and only if $\nabla r_i = 0$ if and only if $U_i \in NM$.

2. Follows trivially from 1. □

Corollary 3.2.5 *Let (M, U) be a smooth submanifold of $\mathbb{R}^{n+1}$ and multivalued vector field as above such that $\eta_i = 0$ and $r_i = r$ for all i, then the following are equivalent*

1. *r has a singularity at x_0.*
2. *$\sum_{i=1}^{l} U_i(x_0) \in N_{x_0}M$.*

Proof

$(1 \Leftrightarrow 2)$ Since $\eta_i = 0$, then $(\nabla r + u_i) \in NM$. Thus $(l\nabla r + \sum_{i=1}^{l} U_i) \in NM$, hence r has a singularity if and only if $\nabla r = 0$ if and only if $\sum_{i=1}^{l} U_i \in NM$. $\square$

Definition 3.2.6 *Let M be a smooth submanifold of $\mathbb{R}^{n+1}$ such that for each $x_0 \in M$ there exist a multivalued vector field $U = (u_1, u_2, ..., u_l)$ such that each $u_i = r_i U_i$ forms a smooth vector field on M, where U_i is a smooth unit vector field on M and r_i is a smooth real valued function on M. We define the **centroid point** associated to x_0 by*

$$x_c = x_0 + \frac{1}{l}\sum_{i=1}^{l} r_i(x_0)U_i(x_0).$$

*The **centroid set** of (M, U) is given by*

$$C(M,U) = \{y \in \mathbb{R}^{n+1} | y = x + \frac{1}{l}\sum_{i=1}^{l} r_i(x)U_i(x),\ for\ some\ x \in M\}.$$

Example 3.2.7 *Let S be the smooth part of the symmetry set of a smooth boundary X, and r be the radius function, then for each $x_0 \in S$, we can define the multivalued vector field $U = (rU_1, rU_2)$ to be $U_1 = -r\nabla r + r\sqrt{1 - \|\nabla r\|^2}N$ on one side of S and $U_2 = -r\nabla r - r\sqrt{1 - \|\nabla r\|^2}N$ on the other side as shown in the figure 3.1, where N is the unit normal of S at x_0 and ∇r is the Riemannian gradient of the radius function r. It is obvious to observe that in this case $l = 2$, $r_1 = r_2$ and the centroid point is nothing but the midlocus point. An example of $l = 3$ is the Y-junction in a skeletal structure.*

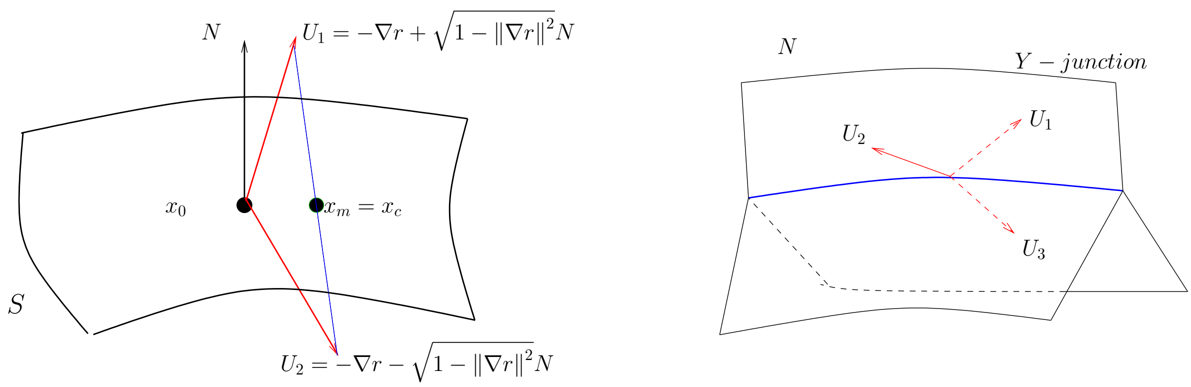

Figure 3.1: Figure of example 3.2.7.

Now we will give the definition of the centroid set associated to a skeletal set.

Definition 3.2.8 *Let* $(\mathbb{S}, U)$ *be a skeletal set and multivalued radial vector field with stratification of the form* $\mathbb{S} = \{M_\lambda\}_{\lambda \in \Lambda}$ *for some set* Λ*, and* $U = \{U_\lambda\}_{\lambda \in \Lambda}$*, then the centroid set associated to* $\mathbb{S}$ *is defined by*

$$C(\mathbb{S}, U) = \bigcup_{\lambda \in \Lambda} C(M_\lambda, U_\lambda).$$

Example 3.2.9 *Let* $\mathbb{S} = \mathbb{R} \subset \mathbb{R}^2$ *such that* $\mathbb{S} = \{(x, 0) \mid x \in \mathbb{R}\}$ *with stratification* $\mathbb{S} = \{\{0\}, \mathbb{R} \setminus \{0\}\}$ *and* $u_{\mathbb{R}\setminus\{0\}} = \pm(0, 1)$ *and* $u_{\{0\}} = \{(0, 1), (0, -2)\}$*. Then* $C(\{0\}, u_{\{0\}}) = \{(0, -1)\}$ *and* $C(\mathbb{R} \setminus \{0\}, u_{\mathbb{R}\setminus\{0\}}) = \mathbb{R} \setminus \{0\}$*. Thus* $C(\mathbb{S}, U) = \mathbb{R} \setminus \{0\} \cup \{(0, -1)\}$*. Observe that* $C(\{0\}, u_{\{0\}}) \nsubseteq \overline{C}(\mathbb{R} \setminus \{0\}, u_{\mathbb{R}\setminus\{0\}})$*, where* $\overline{C}$ *denotes the closure of* C*.*

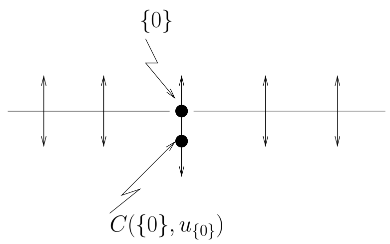

Figure 3.2: A schematic diagram of example 3.2.9.

From example 3.2.9 we can see that if the stratum X is in the closure of Y, then $C(X, u_X)$ is not necessarily in the closure of $C(Y, u_Y)$.

Now we will study the impact of the singularity of the radius function on the relationship between a point $x_0 \in M$ and its associated centroid point. In fact we will assume as in corollary 3.2.5, i.e., we will have the same radius function and in this case we have 1-form η on M.

Theorem 3.2.10 *Let (M, U) be a smooth submanifold of $\mathbb{R}^{n+1}$ and multivalued vector field as in corollary 3.2.5. Let $x_0 \in M$ and x_c be its associated centroid point, then*

1. *if $x_0 = x_c$, then the radius function has a singularity at x_0.*

2. *$x_c - x_0 \in N_{x_0}M$ if and only if the radius function has a singularity at x_0.*

Proof

1. Assume that $x_0 = x_c$, then from the definition of the centroid set we have $\frac{1}{l}\sum_{i=1}^{l} r_i U_i = 0$, but by our assumption we have $r_i = r$ and the 1-form η vanishes in a neighbourhood of x_0. Now for any $v_j \in T_{x_0}M$, then $\frac{r}{l}\sum_{i=1}^{l} U_i \cdot v_j = 0$. Thus $rdr(v_j) = 0$ and hence the radius function has a singularity at x_0.

2. Since we have the same radius function, then the centroid point x_c associated to x_0 is given by

$$x_c = x_0 - r\nabla r + \frac{r}{l}\sum_{i=1}^{l} U_i^N.$$

Thus $x_c - x_0 \in N_{x_0}M$ if and only if $\nabla r = 0$ if and only if r has a singularity at x_0. $\square$

The above theorem is a generalization of proposition 2.4.4. Now let M^k (k indicates the dimension of M) be a smooth submanifold of $\mathbb{R}^{n+1}$. For any point $x_0 \in M^k$ we put $\{v_1, v_2, ..., v_k\}$ as a basis for the tangent space of M^k at x_0 and $\{w_1^N, w_2^N, ..., w_m^N\}$ is a basis for the normal space of M^k. In the following proposition the radius function is the same for each u_i of multivalued radial vector field $U = (u_1, u_2, ..., u_l)$ and the sum of normal parts of u_i is zero, i.e., the centroid point associated to x_0 is given by

$$x_c = x_0 - r\nabla r. \tag{3.1}$$

Moreover, V is the matrix with i-th row entry v_i, N is the matrix with i-th row w_i^N, $\mathcal{H}_r$ is the Hessian matrix of r, β is the matrix of the normal coefficients of $\dfrac{\partial \nabla r}{\partial V}$, $dr(V)$ is a column matrix with i-th entry $\dfrac{\partial r}{\partial v_i}$, and V_c is the Jacobian matrix of the map $x \longmapsto x - r(x)\nabla r(x)$, $x \in M^k$ and all those terms are evaluated at x_0.

Proposition 3.2.11 *Let (M^k, U) be a smooth submanifold of $\mathbb{R}^{n+1}$ and multivalued vector field such that the radius function is the same for each u_i and $\eta = 0$, and the sum of the normal parts of $U = (u_1, u_2, ..., u_l)$ is zero. Then the centroid is singular at x_c associated to a point $x_0 \in M^k$ if and only if the rank of the matrix*

$$(V - dr(V)\nabla r - r\mathcal{H}_r^T V - r\beta N)$$

is less than k.

Proof

In this case the centroid point associated to a given point $x_0 \in M^k$ is given by

$$x_c = x_0 - r\nabla r.$$

Thus if $\{v_1, v_2, ..., v_k\}$ is a basis for the tangent space of M^k at x_0, then

$$v_{cj} = v_j - \frac{\partial r}{\partial v_j}\nabla r - r\frac{\partial \nabla r}{\partial v_j}, \quad j = i, 2, ..., k.$$

Here v_{cj} is the directional derivative of the map $x \longmapsto x - r(x)\nabla r(x)$, $x \in M^k$ in the direction of v_j. This equation can be written in vector form as the following

$$V_c = V - dr(V)\nabla r - r\mathcal{H}_r^T V - r\beta N. \tag{3.2}$$

Thus the centroid set is singular if and only if the rank of the matrix $V - dr(V)\nabla r - r\mathcal{H}_r^T V - r\beta N$ is less than k. $\square$

Corollary 3.2.12 *Let (M^k, U) be a smooth submanifold of $\mathbb{R}^{n+1}$ as in proposition 3.2.11. If the radius function has a singularity at x_0, then the centroid set is singular at x_c if and only if $\frac{1}{r}$ is an eigenvalue of $\mathcal{H}_r$.*

Proof

If the radius function r has a singularity, then $\dfrac{\partial}{\partial v_j}(\nabla r) \in T_{x_0}M^k$, thus $\beta = 0$. Therefore,

$$V_c = (I - r\mathcal{H}_r^T)V.$$

Now since V is a $(k \times (n+1))$ matrix with rank k and $(I - r\mathcal{H}_r^T)$ is a $k \times k$ matrix, then the rank of V_c is equal to the rank of $(I - r\mathcal{H}_r^T)$. Thus the centroid is singular if and only if the rank of $(I - r\mathcal{H}_r^T)$ is less than k if and only if $\frac{1}{r}$ is an eigenvalue of $\mathcal{H}_r$. $\square$

Now using the new set-up of the centroid point the natural question is what is the impact of the singularity of the radius function on the relationship between a given point $x_0 \in M^k$, and its associated centroid point? The answer of this question is given in the following theorem which is a generalization of theorem 2.3.4.

Theorem 3.2.13 *Let (M^k, U) be a smooth submanifold of $\mathbb{R}^{n+1}$ and multivalued vector field as in proposition 3.2.11. Let $x_0 \in M^k$ be any point and x_c its associated centroid point, then the radius function r has a singularity at x_0 if and only if x_0, and x_c coincide.*

Proof

The centroid point x_c associated to a point $x_0 \in M^k$ is given by equation 3.1, thus using theorem 2.3.4 the result holds. $\square$

3.3 The Skeletal Structure and the Singularity of the Radius Function

In the previous sections of this chapter the concept of the centroid set has been introduced and its singularity has been discussed in a general case. Also, the impact of the singularity of the radius function on the singularity of the centroid has been investigated and we found that the singularity of the radius function occurs when the radial vectors all lie in the normal space. In this section such a study will be carried out for the case of the skeletal structure. It is important to note that the centroid in the previous sections does not have a boundary but in the case of the midlocus associated to a skeletal structure with a smooth boundary it is subjected to the condition that allows the radial map to be a diffeomorphism.

Lemma 3.3.1 *Let $(\mathbb{S}, U)$ be a skeletal structure of a region Ω in $\mathbb{R}^{n+1}$ with smooth boundary X and let $x_0 \in \mathbb{S}$ be a non-edge point. Let U be a smooth value (on a non-edge local manifold component $\mathbb{S}_\alpha$), for which $\frac{1}{r}$ is not an eigenvalue of S_V at x_0. Then the radius function r has a singularity at x_0 if and only if*

$$\frac{\partial \Psi_1}{\partial v_i} \cdot U_1 = v_i \cdot U_1, \quad i = 1, 2, ..., n.$$

Proof

James Damon pointed out in [7] that if $\frac{1}{r}$ is not an eigenvalue of S_V at x_0, then Ψ_1 is a local diffeomorphism. Therefore, we choose a neighbourhood W of the local manifold component $\mathbb{S}_\alpha$ so that Ψ_1 is a diffeomorphism on W. Therefore, for $v \in T_{x_0}\mathbb{S}_\alpha$ we have

$$\frac{\partial \Psi_1}{\partial v} = v + dr(v)U_1 + r\frac{\partial U_1}{\partial v}.$$

Therefore,

$$dr(v) = \frac{\partial \Psi_1}{\partial v} \cdot U_1 - v \cdot U_1.$$

Hence $dr(v) = 0$ if and only if $\frac{\partial \Psi_1}{\partial v} \cdot U_1 = v \cdot U_1$ which completes the proof. □

In the following proposition we will study the relationship between the singularity of the radius function and the orthogonality of the radial vector field on the skeletal structure. Also, the relationship between the tangent space of the skeletal structure and its associated tangent space of the boundary will be studied in the case when the radius function has a singularity.

Proposition 3.3.2 *Suppose $(\mathbb{S}, U)$ is a skeletal structure and let x_0 be a non-edge point. Let U be a smooth value for which $\frac{1}{r}$ is not an eigenvalue of the radial shape operator and the compatibility 1-form η_U vanishes at x_0. Then the following are equivalent.*

1. *The radius function has a singularity at x_0.*
2. *The radial vector field U is orthogonal to the tangent space $T_{x_0}\mathbb{S}$ of $\mathbb{S}$ at x_0.*
3. *The space $T_{x_0}\mathbb{S}$ is parallel to the associated tangent space of the boundary $T_{x'}X$.*

Proof

(1⇔ 2) Can be proved directly from theorem 3.2.4.

(1⇔ 3) Assume that $\frac{1}{r}$ is not an eigenvalue of S_V at x_0, then Ψ_1 is a local diffeomorphism. Therefore, we choose a neighbourhood W of the local manifold component $\mathbb{S}_\alpha$ so that Ψ_1 is a diffeomorphism on W. Let $B = \{v_1, v_2, ..., v_n\}$ be a basis for the tangent space $T_{x_0}\mathbb{S}_\alpha$. Then, $\{v_1^{'}, v_2^{'}, ..., v_n^{'}\}$ such that

$$v_i^{'} = \frac{\partial \Psi_1}{\partial v_i} = v_i + dr(v_i)U_1 + r\frac{\partial U_1}{\partial v_i} \tag{3.3}$$

is a basis for the tangent space of the boundary. Now the dot product with U_1 for both sides of equation 3.3, gives

$$v_i^{'} \cdot U_1 = \frac{\partial \Psi_1}{\partial v_i} \cdot U_1 = v_i \cdot U_1 + dr(v_i) = \eta_U(v_i).$$

From the definition, the compatibility 1-form vanishes if and only if

$$\eta_U(v) = v \cdot U + dr(v) = 0,$$

which means that the radial vector field is perpendicular to the tangent space of the boundary. Now assume that the radius function has a singularity at x_0, then $v_i \cdot U_1 = 0$ for $i = 1, 2, ..., n$, thus the tangent spaces $T_{x'}X$ and $T_{x_0}\mathbb{S}$ are parallel. Conversely assume $T_{x'}X$ and $T_{x_0}\mathbb{S}$ are parallel, then U_1 is perpendicular to $T_{x_0}\mathbb{S}$, and from the compatibility condition the radius function has a singularity. □

Corollary 3.3.3 *Suppose $\Omega \subseteq \mathbb{R}^{n+1}$ is a region with smooth boundary X and Blum medial axis and radial vector field $(\mathbb{S}, U)$. Let $x_1 \in X$ be a point for which the projection on the medial axis along the normal to X is a local diffeomorphism (with x_1 mapping to x_0 in $\mathbb{S}$). Then the radius function r has a singularity if and only if $U \perp T_{x_0}\mathbb{S}$.*

Proof

By the Blum condition we have $U \perp X$ and since the projection along normal is a local diffeomorphism, then its inverse, which is in this case Ψ_1, is a local diffeomorphism. Also, $\frac{1}{r}$ is not an eigenvalue of S_V. Hence the result comes directly from proposition. □

Now we will generalize what Peter Giblin pointed out when the radius function has a singularity in the case of symmetry sets in $\mathbb{R}^3$ [11] to skeletal structures in $\mathbb{R}^{n+1}$. In general the radius functions need not to be same at the both sides of a skeletal structure on a neighbourhood of a smooth point. But if we have the same radius function on both sides of the skeletal structure, does the singularity of the radius function affect the relationship between the skeletal point and its associated midlocus? Also, what is the relationship between the tangent spaces of the boundary at the tangency points in the case when the radius function has a singularity? The answer of these questions is given in the following.

Proposition 3.3.4 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$ such $\frac{1}{r}$ is not an eigenvalue of the radial shape operators and the compatibility 1-form vanishes identically on a neighbourhood of a smooth point $x_0 \in \mathbb{S}$ and suppose that the radius function is the same for the two sides of $\mathbb{S}$ on a neighbourhood of x_0. Then the following are equivalent*

1. *The radius function has a singularity at x_0.*
2. *x_0 and the associated midlocus point x_m coincide.*
3. *The tangent spaces of the boundary at the tangency points are parallel.*

Proof

(1⇔2) can be proved directly from theorem 3.2.13.

(1⇔3) Let Ψ_1 be the radial map on one side of $\mathbb{S}$ and Ψ_2 be the radial map on the other side of $\mathbb{S}$. Since $\frac{1}{r}$ is not an eigenvalue of the radial shape operator on both sides then Ψ_1 and Ψ_2 are local diffeomorphisms at x_0. Therefore, we can choose a neighbourhood W of x_0 so that Ψ_1 and Ψ_2 are diffeomorphisms on W. Let $B = \{v_1, v_2, ..., v_n\}$ be a basis for $T_{x_0}\mathbb{S}$ then, $B_1 = \{v_1^{'}, v_2^{'}, ..., v_n^{'}\}$ and $B_2 = \{v_1^{''}, v_2^{''}, ..., v_n^{''}\}$ are bases for the tangent spaces of the boundary at the tangency points such that

$$v_i^{'} = \frac{\partial \Psi_1}{\partial v_i} = v_i + dr(v_i)U_1 + r\frac{\partial U_1}{\partial v_i},$$

and

$$v_i^{''} = \frac{\partial \Psi_2}{\partial v_i} = v_i + dr(v)U_2 + r\frac{\partial U_2}{\partial v_i},$$

where U_1 is the smooth value of the unit radial vector field on one side of $\mathbb{S}$ and U_2 is the smooth value of the radial vector field on the other side. Now from proposition 3.3.2 we have that $T_{x_0}\mathbb{S}$ is parallel to the tangent spaces of the boundary at the tangency points, thus the tangent spaces of the boundary at the tangency points are parallel. □

From this proposition we can see the impact of the singularity of the radius function on the relationship between the radial vector field and the normal of the skeletal structure at a smooth point (see figure 3.3).

Corollary 3.3.5 *Suppose* $\Omega \subseteq \mathbb{R}^{n+1}$ *is a region with smooth boundary* X *and Blum medial axis and radial vector field* $(\mathbb{S}, U)$*. Let* x_1 *and* x_2 *be two points on* X *for which the projections onto the medial axis along normals are local diffeomorphisms (with* x_1 *and* x_2 *mapping to* $x_0 \in \mathbb{S}$*). Then the following are equivalent*

1. *The radius function has a singularity.*

2. x_0 *and the associated midlocus* x_m *coincide.*

3. *The tangent spaces of the boundary at* x_1 *and* x_2 *are parallel.*

Proof

By Blum condition we have $U \perp X$ and since the projections along the normals are local diffeomorphisms. Then, their inverses which are in this case Ψ_1 and Ψ_2 are local diffeomorphisms. Thus, $\frac{1}{r}$ is neither an eigenvalue of S_{V_1} nor S_{V_2}, where S_{V_1} (resp. S_{V_2}) is the matrix representing the radial shape operator on one side (resp. the matrix representing the radial shape operator on the other side). Therefore, we can apply proposition 3.3.4. □

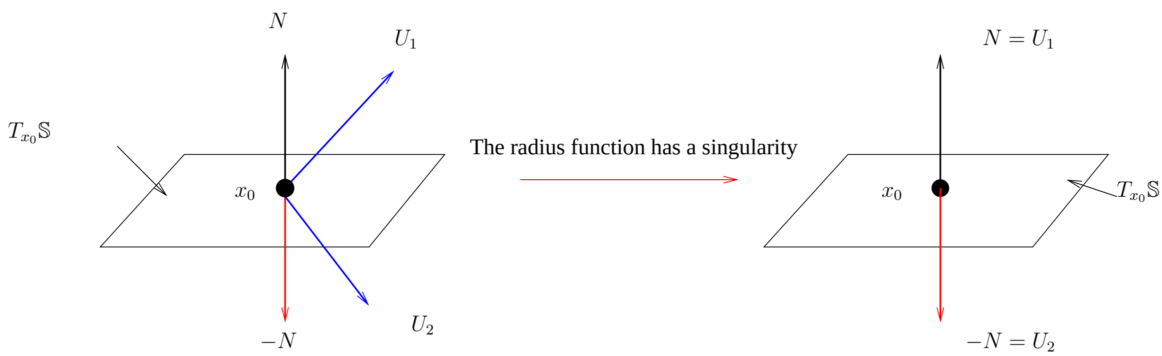

Figure 3.3: The case when the radius function has a singularity.

In proposition 3.3.4 we discussed the effect of the singularity of the radius function on the relationship between a smooth skeletal point and the associated midlocus. But, we assume that the radius functions are same for the two sides of the skeletal set in a neighbourhood of a smooth point. The logical question is : given a smooth skeletal point, when does this point and its associated midlocus coincide? The answer of this question is given in the following proposition.

Proposition 3.3.6 *Let* $(\mathbb{S}, U)$ *be a skeletal structure and* $x_0 \in \mathbb{S}$ *be a smooth point. Then,* x_0 *and the associated midlocus coincide if and only if* $r_1(x_0) = r_2(x_0)$ *and* $U_1(x_0) = -U_2(x_0)$.

Proof

In this case the midlocus is nothing but the centroid point, thus

$$x_m = x_0 + \frac{1}{2}(r_1(x_0)U_1(x_0) + r_2(x_0)U_2(x_0)).$$

Now assume that x_0 and the associated midlocus coincide, then we have

$$r_1(x_0)U_1(x_0) + r_2(x_0)U_2(x_0) = 0.$$

Therefore, $r_1U_1 = -r_2U_2$ which implies that

$$\mid r_1U_1 \mid = \mid -r_2U_2 \mid . \qquad (3.4)$$

Now since U_1 and U_2 are unit vectors and r_1 and r_2 are positive. Then equation 3.4 holds when $r_1 = r_2$ and $U_1 = -U_2$. The converse is obvious. □

Lemma 3.3.7 *Let* $(\mathbb{S}, U)$ *be a skeletal structure such that the compatibility condition holds on a neighbourhood of a smooth point* $x_0 \in \mathbb{S}$. *If the radius function has a singularity, then* $A_V = 0$.

Proof

Let $\{v_1, v_2, ..., v_n\}$ be a basis for the tangent space of the skeletal set at x_0. James Damon shows in [7] that $A_V = S_V^T V \cdot U_1$ and since the compatibility condition holds, then $V \cdot U = -dr(V)$, thus $A_V = -S^T dr$. If the radius function has a singularity we have $A_V = 0$. □

Now we will define a function that plays a central role in the relationship between the matrices representing the radial shape operator and the geometric shape operator of the skeletal structure. Now let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$. Define the function

$$\rho : \mathbb{S} \longrightarrow \mathbb{R}$$

by

$$\rho = U_1 \cdot N, \tag{3.5}$$

where N is the unit normal of $\mathbb{S}$ at x_0 (x_0 is a non-edge point). Originally this function was introduced by James Damon [8] and he called it the normal component function for U_1. Let $\omega = \{v_1, v_2, ..., v_n\}$ be a basis for the tangent space of $\mathbb{S}$ at x_0 (for a non-edge singular point ω is a basis for the limiting tangent space). Differentiate equation 3.5 with respect to v_i we obtain

$$\frac{\partial \rho}{\partial v_i} = \frac{\partial U_1}{\partial v_i} \cdot N + \frac{\partial N}{\partial v_i} \cdot U_1, \qquad i = 1, 2, ..., n.$$

This equation can be written in vector form by

$$\begin{aligned}
\frac{\partial \rho}{\partial V} &= \frac{\partial U_1}{\partial V} \cdot N + \frac{\partial N}{\partial V} \cdot U_1 \\
&= (A_V U_1 - S_V^T V) \cdot N - S_m^T V \cdot U_1 \\
&= A_V U_1 \cdot N - S_m^T V \cdot U_1 \\
&= \rho A_V - S_m^T V \cdot U_1
\end{aligned}$$

$$= (\rho S_V^T - S_m^T)V \cdot U_1,$$

where S_V (resp. S_m) is the matrix representing the radial shape operator (resp. the differential geometric shape operator) on $\mathbb{S}$ and

$$V = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix} \quad and \quad d\rho(V) = \begin{pmatrix} d\rho(v_1) \\ d\rho(v_2) \\ \vdots \\ d\rho(v_n) \end{pmatrix}.$$

Therefore, we can summarize this in the following proposition.

Proposition 3.3.8 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$. Let $x_0 \in \mathbb{S}$ (x_0 be a non-edge point) and define $\rho = U_1 \cdot N$ where N is the unit normal of $\mathbb{S}$ at x_0. Then*

$$\frac{\partial \rho}{\partial V} = (\rho S_V^T - S_m^T)V \cdot U_1,$$

where S_V and S_m are the matrices representing the radial shape operator and the differential geometric shape operator of $\mathbb{S}$ at x_0 respectively.

Corollary 3.3.9 *Let $(\mathbb{S}, U)$ be a skeletal structure such that for a choice of smooth value of the radial vector field U the compatibility 1-form η_U vanishes in a neighbourhood of a non-edge point x_0. Define the function ρ as in proposition 3.3.8, then*

$$\frac{\partial \rho}{\partial V} = d\rho(V) = -(\rho S_V^T - S_m^T)dr(V).$$

Proof

Since the compatibility condition holds, then $\eta_U = 0$. Therefore,

$$0 = dr(v_i) + v_i \cdot U_1, \qquad i = 1, 2, ..., n.$$

Thus

$$dr(V) = -V \cdot U_1.$$

Hence

$$d\rho(V) = -(\rho S_V^T - S_m^T)dr(V).$$

Therefore, the proof is completed. □

Corollary 3.3.10 *Let* $(\mathbb{S}, U)$ *be a skeletal structure as in corollary 3.3.9. If the radius function has a singularity then* ρ *has a singularity but the converse is not true.*

Proof

If the radius function has a singularity, then it is obvious that ρ has a singularity. The converse is not true (see example 3.3.12). □

Corollary 3.3.11 *Assume as in corollary 3.3.9. If the radius function* r *has no singularity and* ρ *has a singularity at* x_0*, then* $\rho(x_0)$ *is a generalized eigenvalue of the pair* (S_m, S_V)*.*

Proof

Recall that $a \neq 0$ is a generalized eigenvalue of the pair (A, B) if $det(A - aB) = 0$. Now assume that the radius function has no singularity at x_0 and the function ρ has a singularity, then we have

$$0 = -(\rho S_V^T - S_m^T)dr(V)$$

or

$$0 = (\rho S_V^T - S_m^T)dr(V)$$

or

$$0 = (S_m^T - \rho S_V^T)dr(V)$$

and since $dr(V) \neq 0$, then the matrix $(S_m - \rho S_V)^T$ is not invertible.
i.e., $det(S_m - \rho S_V) = 0$. Thus ρ is a generalized eigenvalue of the pair (S_m, S_V). □

Example 3.3.12 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^3$ and*

$$s_1(x,y) = (x, y, \frac{1}{2}\kappa_1 x^2 + \frac{1}{2}\kappa_2 y^2 + h.o.t.) \subset \mathbb{S}_{reg}$$

and let $r(x,y) = r_0 + ax + \frac{1}{2}by^2$, $(a, b \in \mathbb{R}$, s.t $a^2 < 1)$ be the radius function we define the unit radial vector field by

$$U_1 = -\nabla r + \sqrt{1 - \|\nabla r\|^2} N,$$

where ∇r is the Riemannian gradient of r and N is the unit normal of s_1. In this case the compatibility condition holds. Now at the origin, direct calculations show that

$$\rho = \sqrt{1-a^2}, \quad dr = \begin{pmatrix} a \\ 0 \end{pmatrix}, \quad d\rho = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \quad S_m^T = \begin{pmatrix} \kappa_1 & 0 \\ 0 & \kappa_2 \end{pmatrix} \quad \textit{and}$$

$$S_V^T = \begin{pmatrix} \dfrac{\kappa_1}{\sqrt{1-a^2}} & 0 \\ 0 & b + \kappa_2\sqrt{1-a^2} \end{pmatrix}.$$

Now

$$-(\rho S_V^T - S_m^T)dr = -\begin{pmatrix} 0 & 0 \\ 0 & -\kappa_2 a^2 + b\sqrt{1-a^2} \end{pmatrix} \begin{pmatrix} a \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} = d\rho.$$

It is clear that if $a \neq 0$, then the radius function has no singularity, but as shown from the calculations ρ has a singularity at the origin which means that the singularity of ρ does not imply the singularity of the radius function and this supports our result in corollary 3.3.10. Moreover, at the origin ρ is a generalized eigenvalue of the pair (S_m, S_V).

Corollary 3.3.13 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary X. Let $x_0 \in \mathbb{S}$ be a non-edge point then*

$$d\rho(V) = -(\rho S_V^T - S_m^T)dr(V).$$

Proof

Since $\mathbb{S}$ is a Blum medial axis, then by Blum condition the radial vector field is perpendicular to the boundary. Therefore, $\eta_U = 0$. Thus, $dr(V) = -V \cdot U_1$ which implies that

$$d\rho(V) = -(\rho S_V^T - S_m^T) dr(V).$$

Thus the proof is completed. □

3.4 Pre-medial Axis

The pre-symmetry sets of 2D and 3D shapes had been studied by Giblin and Diatta [10]. In this section we study the relationship between the parameters of the skeletal structure in a neighbourhood of a smooth point and the parameters of the boundary in a neighbourhood of the associated point. By this way we are able to transfer to the relationship between the parameters of the boundary.

Definition 3.4.1 *Given a smooth hypersurface $X \subset \mathbb{R}^{n+1}$, the pre-symmetry set is the closure of the set of pairs of distinct points $(p, q) \in X \times X$ for which there exists a hypersphere tangent to X at p and at q.*

Lemma 3.4.2 *Let $(\mathbb{S}, U)$ be a skeletal structure of a region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary X such that for a choice of smooth value of U the compatibility condition holds and $\frac{1}{r}$ is not an eigenvalue of the radial shape operator S_{rad}. Let x_0 be a smooth point of $\mathbb{S}$ and $\varepsilon_0(x_0) \subset \mathbb{S}$ be a neighbourhood of x_0. Also, let $\varepsilon_1(x_1) \subset X$ be a neighbourhood of $x_1 = x_0 + rU_1$. If $\varepsilon_0(x_0)$ parametrized by $(s_1, s_2, ..., s_n)$ and $\varepsilon_1(x_1)$ parametrized by $(t_1, t_2, ..., t_n)$, then the map:*

$$\varphi : (s_1, s_2, ..., s_n) \longmapsto (t_1, t_2, ..., t_n)$$

is a local diffeomorphism.

Proof

From the boundary point definition we have $x_1 = x_0 + rU_1$ and since $\frac{1}{r}$ is not an eigenvalue of the radial shape operator S_{rad}, then the radial map is a diffeomorphism. As a notation let $\frac{\partial x_1}{\partial s_i} = \frac{\partial \varepsilon_1}{\partial s_1}|_{x_1}$. Therefore, we have

$$v_1^{'} = \frac{\partial x_1}{\partial s_1} = \frac{\partial x_1}{\partial t_1}\frac{\partial t_1}{\partial s_1} + \frac{\partial x_1}{\partial t_2}\frac{\partial t_2}{\partial s_1} + ... + \frac{\partial x_1}{\partial t_n}\frac{\partial t_n}{\partial s_1} = v_1 + \frac{\partial r}{\partial s_1}U_1 + r\frac{\partial U_1}{\partial s_1}$$

or

$$\begin{pmatrix} \frac{\partial t_1}{\partial s_1} & \frac{\partial t_2}{\partial s_1} & . & . & . & \frac{\partial t_n}{\partial s_1} \end{pmatrix} \begin{pmatrix} v_1^{'} \\ v_2^{'} \\ . \\ . \\ . \\ v_n^{'} \end{pmatrix} = v_1 + \frac{\partial r}{\partial s_1}U_1 + r\frac{\partial U_1}{\partial s_1}.$$

Therefore,

$$\begin{pmatrix} \frac{\partial t_1}{\partial s_1} & \frac{\partial t_2}{\partial s_1} & \cdots & \frac{\partial t_n}{\partial s_1} \\ \frac{\partial t_1}{\partial s_2} & \frac{\partial t_2}{\partial s_2} & \cdots & \frac{\partial t_n}{\partial s_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial t_1}{\partial s_n} & \frac{\partial t_2}{\partial s_n} & \cdots & \frac{\partial t_n}{\partial s_n} \end{pmatrix} \begin{pmatrix} v_1^{'} \\ v_2^{'} \\ \vdots \\ v_n^{'} \end{pmatrix} = V + dr(V)U_1 + r\frac{\partial U_1}{\partial V}.$$

Now since the radial map is a local diffeomorphism then the matrix

$$A = \Gamma V^{'} = \begin{pmatrix} \frac{\partial t_1}{\partial s_1} & \frac{\partial t_2}{\partial s_1} & \cdots & \frac{\partial t_n}{\partial s_1} \\ \frac{\partial t_1}{\partial s_2} & \frac{\partial t_2}{\partial s_2} & \cdots & \frac{\partial t_n}{\partial s_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial t_1}{\partial s_n} & \frac{\partial t_2}{\partial s_n} & \cdots & \frac{\partial t_n}{\partial s_n} \end{pmatrix} \begin{pmatrix} v_1^{'} \\ v_2^{'} \\ \vdots \\ v_n^{'} \end{pmatrix}$$

has a maximal rank, i.e., $rank(A) = n$. Also, since the boundary X is smooth then the matrix V' has rank n. Therefore, $rank(A) = rank(\Gamma) = n$. Hence the map

$$\varphi : (s_1, s_2, ..., s_n) \longmapsto (t_1, t_2, ..., t_n)$$

is a local diffeomorphism. $\square$

Proposition 3.4.3 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary X. Let x_0 be a smooth point of $\mathbb{S}$ and $\varepsilon_0(x_0) \subset \mathbb{S}$ be a neighbourhood of x_0. Also, let $\varepsilon_1(x_1) \subset X$ be a neighbourhood of $x_1 = x_0 + rU_1$. If $\varepsilon_0(x_0)$ parametrized by $(s_1, s_2, ..., s_n)$ and $\varepsilon_1(x_1)$ parametrized by $(t_1, t_2, ..., t_n)$, then the map:*

$$\varphi : (s_1, s_2, ..., s_n) \longmapsto (t_1, t_2, ..., t_n)$$

is a local diffeomorphism.

Proof

Since we are in the Blum case the compatibility condition holds and $\frac{1}{r}$ is not an eigenvalue of the radial shape operator. Therefore, we can apply lemma 3.4.2. $\square$

Lemma 3.4.2 and proposition 3.4.3 give us enough tools to study the relationship between the boundary parameters at the tangency points associated to a smooth point on the medial axis.

Proposition 3.4.4 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary X. Let x_1 and x_2 be the tangency points associated to a smooth point in $x_0 \in \mathbb{S}$ with neighbourhoods $\varepsilon_1(x_1)$ and $\varepsilon_2(x_2)$ respectively. If $\varepsilon_1(x_1)$ parametrized by $(s_1, s_2, ..., s_n)$ and $\varepsilon_2(x_2)$ parametrized by $(t_1, t_2, ..., t_n)$, then the map:*

$$\varphi : (s_1, s_2, ..., s_n) \longmapsto (t_1, t_2, ..., t_n)$$

is a local diffeomorphism.

Proof

Let $\varepsilon_0(x_0)$ be a neighbourhood of $x_0 \in \mathbb{S}$ parametrized by $(z_1, z_2, ..., z_n)$. Then by proposition 3.4.3

$$\varphi_1 : (z_1, z_2, ..., z_n) \longmapsto (s_1, s_2, ..., s_n)$$

and

$$\varphi_2 : (z_1, z_2, ..., z_n) \longmapsto (t_1, t_2, ..., t_n)$$

are local diffeomorphism. But

$$\varphi = \varphi_2 \circ \varphi_1^{-1}.$$

Therefore, the map

$$\varphi : (s_1, s_2, ..., s_n) \longmapsto (t_1, t_2, ..., t_n)$$

is a local diffeomorphism. □

3.5 Singularity of the Midlocus in the Case of Skeletal Structure in $\mathbb{R}^3$

In the rest of this chapter we focus on the singularity of the midlocus in $\mathbb{R}^3$. Precisely we take a smooth point $x_0 \in \mathbb{S}_{reg}$ and we take $M(x, y)$ as a local parametrization of the smooth stratum containing x_0 around x_0. In fact, we will study corank one singularities and to do so we need a general form of the midlocus to deal with and lemma 3.5.1 fulfils this need.

Note: In this section we are dealing with the case when the radius function is the same for both sides of M and the compatibility condition holds and in this case the midlocus point associated to x_0 is given by $x_m = x_0 - r(x_0)\nabla r(x_0)$.

Lemma 3.5.1 *Let $M(x,y) = (x, y, f(x,y))$ be a local parametrization of a smooth stratum of skeletal set around a smooth point $x_0 \in \mathbb{S}_{reg}$ and $r(x,y)$ be the radius function, then the midlocus is given by $\mathbb{M}(x,y) = (g, h, l)$, where*

$$g = \frac{x + xf_x^2 + xf_y^2 - rr_x - rr_xf_y^2 + rr_yf_xf_y}{1 + f_x^2 + f_y^2},$$

$$h = \frac{y + yf_x^2 + yf_y^2 - rr_y - rr_yf_x^2 + rr_xf_xf_y}{1 + f_x^2 + f_y^2},$$

and

$$l = \frac{f + ff_x^2 + ff_y^2 - rr_xf_x - rr_yf_y}{1 + f_x^2 + f_y^2}.$$

Proof

Let $M(x,y) = (x, y, f(x,y))$ and r be the radius function, then the midlocus point x_m associated to $x_0 \in M$ is given by $x_m = x_0 - r\nabla r$, $\nabla r = dr^T I_m^{-1} V$, where

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & f_x \\ 0 & 1 & f_y \end{pmatrix},$$

and

$$I_m = \begin{pmatrix} 1 + f_x^2 & f_xf_y \\ f_xf_y & 1 + f_y^2 \end{pmatrix}.$$

Thus

$$I_m^{-1} = \begin{pmatrix} \frac{1 + f_y^2}{1 + f_x^2 + f_y^2} & \frac{-f_xf_y}{1 + f_x^2 + f_y^2} \\ \frac{-f_xf_y}{1 + f_x^2 + f_y^2} & \frac{1 + f_x^2}{1 + f_x^2 + f_y^2} \end{pmatrix}.$$

Therefore,

$$\nabla r = \left(\frac{r_x + r_xf_y^2 - r_yf_xf_y}{1 + f_x^2 + f_y^2}, \frac{r_y + r_yf_x^2 - r_xf_xf_y}{1 + f_x^2 + f_y^2}, \frac{r_xf_x + r_yf_y}{1 + f_x^2 + f_y^2} \right).$$

Thus, after some calculations the result follows. □

In the forthcoming results and examples we need some concepts from the theory of surface in $\mathbb{R}^3$, these concepts are given in the following definition.

Definition 3.5.2 *Let $\mathcal{S}$ be a regular surface parametrized by $X(x, y)$, then*

1. *The shape operator (or Weingarten map) at each point $p \in \mathcal{S}$ is defined by $S_p : T_p\mathcal{S} \to T_p\mathcal{S}$, $u \mapsto -\nabla_u n$, where n is the unit normal of the $\mathcal{S}$.*

2. *The first and the second fundamental forms of $\mathcal{S}$ are the quadratic forms on the tangent plane defined by $I(u, v) = u \cdot v$ and $II(u, v) = u \cdot S_p(v)$ respectively, they are represented by the matrices $I = \begin{pmatrix} X_x \cdot X_x & X_x \cdot X_y \\ X_x \cdot X_y & X_y \cdot X_y \end{pmatrix} = \begin{pmatrix} E & F \\ F & G \end{pmatrix}$ and $II = \begin{pmatrix} X_{xx} \cdot n & X_{xy} \cdot n \\ X_{yx} \cdot n & X_{yy} \cdot n \end{pmatrix} = \begin{pmatrix} L & M \\ M & N \end{pmatrix}$.*

3. *The normal curvature k_p in the tangent direction $w = aX_x + bX_y$ is defined by*

$$k_p(w) = \frac{II(w)}{I(w)} = \frac{a^2L + 2abM + b^2N}{a^2E + 2abF + b^2G}.$$

4. *The geodesic torsion in the direction of a unit vector w is defined by*

$$\tau_g(w) = II(w, w^{\perp}),$$

   where $w^{\perp}$ is the unit vector perpendicular to w.

5. *If $(\cos\theta, \sin\theta)$ is a direction on $\mathcal{S}$ with respect to a principal coordinate system (A principal coordinate system is one where the x-axis and y-axis are always the principal directions) then*

$$k_p = \kappa_1 \cos^2\theta + \kappa_2 \sin^2\theta \text{ and } \tau_g = (\kappa_2 - \kappa_1) \sin\theta \cos\theta.$$

Definition 3.5.3 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map, then the k-jet $j^k f$ at a point p is the Taylor expansion about p truncated at degree k.*

Definition 3.5.4 *Let $f : \mathbb{R}^n \to \mathbb{R}^m$ be a smooth map, then the corank δ of f is defined by $\delta = min(n, m) - rank(df)$, where df is the differential of f.*

Example 3.5.5 *Consider the $f : \mathbb{R}^2 \to \mathbb{R}^3$ such that $(x, y) \mapsto (x, xy, y^2)$, then the differential of f is given by* $df = \begin{pmatrix} 1 & 0 \\ y & x \\ 0 & 2y \end{pmatrix}$.

Theorem 3.5.6 *Let M be a smooth stratum of skeletal structure $(\mathbb{S}, U)$ in $\mathbb{R}^3$ containing a smooth point $x_0 \in \mathbb{S}_{reg}$ and r be a radius function with singularity at x_0. Let λ_1 and λ_2 be the eigenvalues of the Hessian of r at x_0 with $r(x_0) = \dfrac{1}{\lambda_1}$, $\lambda_1 \neq 0$ and let x_m be the associated midlocus point to x_0, then*

1. *The midlocus is parametrized by a corank two singularity at x_m if and only if $\lambda_1 = \lambda_2$.*

2. *The midlocus is parametrized by a corank one singularity at x_m if and only if $\lambda_1 \neq \lambda_2$.*

Proof

Let M be a smooth stratum of skeletal structure $(\mathbb{S}, U)$ in $\mathbb{R}^3$ containing a smooth point x_0 and r be the radius function with singularity at x_0 and $r(x_0) = \dfrac{1}{\lambda_1}$, $\lambda_1 \neq 0$ where λ_1 is an eigenvalue of the Hessian of r at x_0. Now we parameterize M locally at x_0 such that $(0, 0) \mapsto x_0 = (0, 0, 0)$ and M is in Monge form i.e., $M(\widetilde{x}, \widetilde{y}) = (\widetilde{x}, \widetilde{y}, \frac{1}{2}\kappa_1\widetilde{x}^2 + \frac{1}{2}\kappa_2\widetilde{y}^2 + h.o.t)$ and the radius function is given by

$$r(\widetilde{x}, \widetilde{y}) = b_{00} + \frac{1}{2}\widetilde{b_{20}}\widetilde{x}^2 + \frac{1}{2}\widetilde{b_{02}}\widetilde{y}^2 + \widetilde{b_{11}}\widetilde{x}\widetilde{y} + \frac{1}{2}\widetilde{b_{12}}\widetilde{x}\widetilde{y}^2 + \frac{1}{2}\widetilde{b_{21}}\widetilde{x}^2\widetilde{y} + h.o.t.$$

Now we rotate the new coordinates in the source by

$$\begin{bmatrix} \widetilde{x} \\ \widetilde{y} \end{bmatrix} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

such that the radius function transforms to $r(x, y) = b_{00} + \frac{1}{2}b_{20}x^2 + \frac{1}{2}b_{02}y^2 + h.o.t.$ This rotation transforms M to

$$M(x, y) = (x \cos t + y \sin t, -x \sin t + y \cos t, \frac{1}{2}a_{20}x^2 + \frac{1}{2}a_{02}y^2 + a_{11}xy + h.o.t),$$

where $a_{20} = \kappa_1 \cos^2 t+\kappa_2 \sin^2 t$, $a_{0,2} = \kappa_1 \sin^2 t+\kappa_2 \cos^2 t$, and $a_{11} = (\kappa_1-\kappa_2) \sin t \cos t$. Now we rotate the coordinates in the target around z- axis by

$$\begin{bmatrix} \overline{X} \\ \overline{Y} \\ \overline{Z} \end{bmatrix} = \begin{bmatrix} \cos(-t) & \sin(-t) & 0 \\ -\sin(-t) & \cos(-t) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} X \\ Y \\ Z \end{bmatrix},$$

where X, Y, Z are the old coordinates and $\overline{X}$, $\overline{Y}$, $\overline{Z}$ are new coordinates. Thus, this transforms M to $M(x, y) = (x, y, \frac{1}{2}a_{20}x^2 + \frac{1}{2}a_{02}y^2 + a_{11}xy + h.o.t)$. Now we use lemma 3.5.1 to find the form of the midlocus, and by using Maple (*see the linear parts of equations A.8, A.9 and A.10 in the appendix*), we get

$$j^1\mathbb{M} = ((1 - b_{00}b_{20})x, (1 - b_{00}b_{02})y, 0)\,.$$

Observe that, corollary 3.2.12 tells us the midlocus is singular when $\frac{1}{r}$ is an eigenvalue of $\mathcal{H}_r$. This description of $j^1\mathbb{M}$ allows a verification of this result in $\mathbb{R}^3$. Now the Hessian matrix of the radius function is $\mathcal{H}_r = \begin{pmatrix} b_{20} & 0 \\ 0 & b_{02} \end{pmatrix}$, without loss of generality we put $\lambda_1 = b_{02}$, and $\lambda_2 = b_{20}$ and since $b_{00} = r(x_0)$ the 1-jet of the midlocus is now $(\dfrac{\lambda_1 - \lambda_2}{\lambda_1}x, 0, 0)$. Therefore, the Jacobian matrix of the midlocus is $dx_m = \begin{pmatrix} \dfrac{\lambda_1 - \lambda_2}{\lambda_1} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$. Thus, dx_m has rank zero if and only if $\lambda_1 = \lambda_2$, and has rank one if and only if $\lambda_1 \neq \lambda_2$, hence the results have been proved. □

Now in the rest of this section we will study the singularity of the midlocus in $\mathbb{R}^3$ in the case when it has corank one — this means that the eigenvalues of the Hessian of the radius function are distinct. We will use the finite determinacy to study this singularity. First of all we state some needed definitions.

Definition 3.5.7 *Two map-germs* $f_i : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ $(i = 1, 2)$ *are* $\mathcal{A}$*-equivalent if there exist germs of* C^∞*-diffeomorphisms* ϑ *and* φ *such that* $\varphi \circ f_1 = f_2 \circ \vartheta$ *holds, where* $\vartheta : (\mathbb{R}^n, 0) \to (\mathbb{R}^n, 0)$ *and* $\varphi : (\mathbb{R}^p, 0) \to (\mathbb{R}^p, 0)$.

Definition 3.5.8 *A map-germ* $f : (\mathbb{R}^n, 0) \to (\mathbb{R}^p, 0)$ *is* k-determined *if whenever* $j^k g(0) = j^k f(0)$*, then* g *is* $\mathcal{A}$*-equivalent to* f.

Definition 3.5.9 *The crosscap or Whitney umbrella is a map-germ* $\mathcal{A}$*-equivalent to* $(x, y) \mapsto (x, xy, y^2)$ *at the origin.*

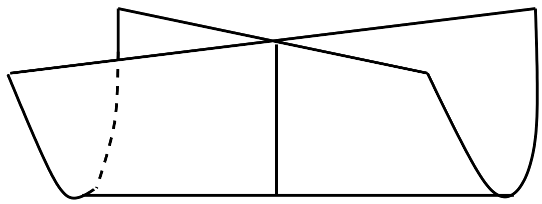

Figure 3.4: Crosscap or Whitney umbrella.

Since we are dealing with finite determinacy we focus on the 2-jet and 3-jet of the midlocus. First of all, we classify the second jet and for this we need the following theorem which was proved by Mond in [22].

Theorem 3.5.10 *The map-germ* $(x, y) \mapsto (x, xy, y^2)$ *is stable and 2-determined.*

Lemma 3.5.11 *If the map-germ* $f : (\mathbb{R}^2, 0) \longrightarrow (\mathbb{R}^3, 0)$ *has a corank one singularity with* $j^2 f = (x, axy + by^2, cxy + dy^2)$*, then* f *is* $\mathcal{A}$*-equivalent to the crosscap if and only if* $ad - cb \neq 0$.

Proof

Assume that $ad - cb \neq 0$, then from the Mond classification in [22] (proposition 4.2) $j^2 f$ transforms by an appropriate coordinate change to (x, xy, y^2) and since the crosscap is 2-determined thus f is $\mathcal{A}$-equivalent to the crosscap. Conversely, assume that $ad - cb = 0$ and f is $\mathcal{A}$-equivalent to the crosscap. Therefore, $j^2 f$ is $\mathcal{A}$-equivalent to (x, xy, y^2). But since $ad - cb = 0$, then from Mond classification $j^2 f$ transforms to one element of the set $\{(x, 0, 0), (x, xy, 0), (x, y^2, 0)\}$ and no one of these elements is $\mathcal{A}$-equivalent to (x, xy, y^2) which is a contradiction. Thus, f is $\mathcal{A}$-equivalent to the crosscap if and only if $ad - cb \neq 0$. $\square$

Now we state a theorem which gives a necessary and sufficient conditions for a midlocus to be $\mathcal{A}$-equivalent to the crosscap.

Theorem 3.5.12 *Let* M *be a smooth stratum of a skeletal structure* $(\mathbb{S}, U)$ *in* $\mathbb{R}^3$ *containing a smooth point* x_0 *and* r *be the radius function with a singularity at* x_0 *and* λ_1 *and* λ_2 *be the eigenvalues of the Hessian of* r *and* w_1 *and* w_2 *are the associated eigenvectors such that* $\lambda_1 \neq \lambda_2$*, and* $r(x_0) = \frac{1}{\lambda_1}$*,* $\lambda_1 \neq 0$*, then the midlocus at* x_m *associated to* x_0 *is* $\mathcal{A}$*-equivalent to the crosscap if and only if*

$$\lambda_1 k_{x_0}(w_1) \nabla^2_{w_1} \nabla_{w_2} r \neq 2\lambda_2 \tau_g \nabla^3_{w_1} r,$$

where $k_{x_0}(w_1)$ *is the normal curvature of* M *in the direction* w_1*,* τ_g *is the geodesic torsion of* M *in the direction* w_1*, and* $\nabla_{w_i} r$ *is the directional derivative of the radius function in the direction* w_i*,* $i = 1, 2$.

Proof

We repeat the same procedure of the proof of theorem 3.5.6 and thus we have

$$r = b_{00} + +\frac{1}{2}b_{20}x^2 + \frac{1}{2}b_{02}y^2 + \frac{1}{3}b_{30}x^3 + \frac{1}{2}b_{21}x^2y + \frac{1}{2}b_{12}xy^2 + \frac{1}{3}b_{03}y^3 + b_{40}x^4 + b_{31}x^3y$$
$$+ b_{22}x^2y^2 + b_{13}xy^3 + b_{04}y^4 + h.o.t$$

and $M(x,y) = (x, y, \frac{1}{2}a_{20}x^2 + a_{11}xy + \frac{1}{2}a_{02}y^2 + h.o.t)$. As in the proof of theorem 3.5.6 without loss of generality, we put $\lambda_1 = b_{02}$ and $\lambda_2 = b_{20}$ and by using lemma 3.5.1 we find the form of the centroid set and using Maple in calculations (*see equations A.8, A.9 and A.10 in the appendix*) we get $j^2\mathbb{M}_c = (p, q, s)$, where

$$p = (1 - b_{00}b_{20})x - b_{00}b_{30}x^2 - b_{00}b_{21}xy - \frac{1}{2}b_{00}b_{12}y^2,$$
$$q = -\frac{1}{2}b_{00}b_{21}x^2 - b_{00}b_{12}xy - b_{00}b_{03}y^2 \; and$$
$$s = \left(\frac{1}{2}a_{20} - b_{00}b_{20}a_{20}\right)x^2 - b_{00}b_{20}a_{11}xy - \frac{1}{2}a_{02}y^2.$$

Now consider the parameter change in the source

$$x = u + \frac{b_{12}}{b_{02-b_{20}}}uy + \frac{b_{30}}{b_{02} - b_{20}}u^2 + \frac{b_{12}}{2(b_{02} - b_{20})}y^2.$$

This parameter change transforms $j^2\mathbb{M}$ (*see equations A.11, A.12 and A.13 in the appendix*) to $(\overline{p}, \overline{q}, \overline{s})$, where

$$\overline{p} = \frac{b_{02} - b_{20}}{b_{02}}u, \quad \overline{q} = -\frac{b_{21}}{2b_{02}}u^2 - \frac{b_{12}}{b_{02}}uy - \frac{b_{03}}{b_{02}}y^2,$$

and

$$\overline{s} = \frac{a_{20}(b_{02} - b_{20})}{2b_{02}}u^2 - \frac{a_{11}b_{20}}{b_{02}}uy - \frac{1}{2}a_{02}y^2.$$

Now consider the coordinate change in the target

$$\overline{X} = \frac{b_{02}}{b_{02} - b_{20}}X, \quad \overline{Y} = Y + \frac{b_{21}}{2b_{02}}\left(\frac{b_{02}}{b_{02} - b_{20}}\right)^2 X^2, \quad \overline{Z} = Z - \frac{a_{20}b_{02}}{2(b_{02} - b_{20})}X^2,$$

where X, Y, Z are the old coordinates and $\overline{X}, \overline{Y}, \overline{Z}$ are new coordinates. This coordinate change transforms $j^2\mathbb{M}$ into

$$j^2\mathbb{M} = \left(u, -\frac{b_{12}}{b_{02}}uy - \frac{b_{03}}{b_{02}}y^2, -\frac{a_{11}b_{20}}{b_{02}}uy - \frac{1}{2}a_{02}y^2\right).$$

Now using lemma 3.5.11 the midlocus is $\mathcal{A}$-equivalent to the crosscap if and only if

$$\begin{vmatrix} -\dfrac{b_{12}}{b_{02}} & -\dfrac{b_{03}}{b_{02}} \\ -\dfrac{a_{11}b_{20}}{b_{02}} & -\dfrac{1}{2}a_{02} \end{vmatrix} \neq 0$$

if and only if $b_{02}a_{02}b_{12} - 2b_{20}a_{11}b_{03} \neq 0$, if and only if $b_{02}a_{02}b_{12} \neq 2b_{20}a_{11}b_{03}$ but $b_{02} = \lambda_1$, $b_{20} = \lambda_2$, $a_{02} = \kappa_2 \cos^2 t + \kappa_1 \sin^2 t = k_{x_0}(w_1)$, $a_{11} = (\kappa_1 - \kappa_2) \sin t \cos t = \tau_g$, $b_{12} = r_{yyx}(0,0) = \nabla^2_{w_1} \nabla_{w_2} r$, and $b_{03} = r_{yyy}(0,0) = \nabla^3_{w_1} r$, thus the result has been proved. □

Remark 3.5.13 *It is obvious from the above theorem that the conditions are generic and the geometry of the surface M plays a central role for the centroid to have a crosscap singularity.*

Corollary 3.5.14 *Assume as in theorem 3.5.12. If x_0 is a planar point, i.e., $\kappa_1 = \kappa_2 = 0$, then the midlocus is not $\mathcal{A}$-equivalent to the crosscap.*

Proof

If $\kappa_1 = \kappa_2 = 0$, then $k_{x_0}(w_1) = \tau_g = 0$, thus the midlocus is not $\mathcal{A}$-equivalent to the crosscap. □

Example 3.5.15 *Let $M(x,y) = (x, y, y^2)$ and $r(x,y) = 1 + x^3 + xy^2 + \frac{1}{2}y^2$, then the midlocus is singular at the origin and direct calculation gives*
$\mathbb{M}(x,y) = (p, q, s)$, *where*

$$p = x - (xy^4 + \frac{1}{2}y^2 + 4x^3y^2 + \frac{3}{2}x^2y^2 + 3x^2 + 3x^3),$$

$$q = \frac{y(7y^2 - 4x - 4x^4 - 2x^3 - 4x^2y^2 - 4xy^2)}{2 + 8y^2}$$

and

$$s = \frac{y^2(3y^2 - 4x - 4x^4 - 2x^3 - 4x^2y^2 - 4xy^2 - 1)}{2 + 8y^2}.$$

Now we calculate the values of the terms are in the non-inequality of theorem 3.5.12. It is clear that the Hessian matrix at the origin is given by $\mathcal{H}_r = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$ *and its eigenvalues are* $\lambda_1 = 1$ *and* $\lambda_2 = 0$ *with associated eigenvectors* $w_1 = t_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ *and* $w_2 = t_2 \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ *respectively. Now we do not need to calculate* $\nabla^3_{w1} r$ *and* τ_g *since* $\lambda_2 = 0$*. The tangent vector in the direction of* w_1 *is given by* $0.M_x + M_y = M_y$*, thus the normal curvature in the direction of* w_1 *is given by*

$$k_0(w_1) = \frac{II(w_1)}{I(w_1)} = 2.$$

Now we calculate $\nabla^2_{w1} \nabla_{w2} r$*,*

$$\nabla_{w_2} r = (r_x, r_y) \cdot (1, 0) = r_x = 3x^2 + y^2,$$

$$\nabla_{w_1} \nabla_{w_2} r = (6x, 2y) \cdot (0, 1) = 2y \quad and \quad \nabla^2_{w_1} \nabla_{w_2} r = 2.$$

Therefore, $\lambda_1 k_{x_0}(w_1) \nabla^2_{w_1} \nabla_{w_2} r \neq 2\lambda_2 \tau_g \nabla^3_{w_1} r$*. That is, the singularity of the midlocus is a crosscap.*

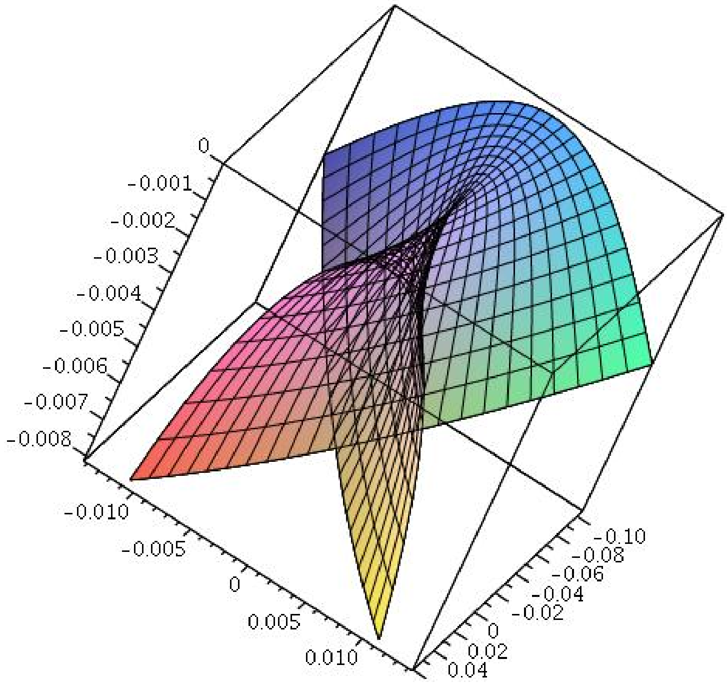

Figure 3.5: Figure of example 3.5.15.

Example 3.5.16 *Let $M(x,y) = (x,y,y^2)$ and consider the family of radius functions $r(x,y) = \frac{2}{5} + y^2 + \mu y^3 + xy + \frac{1}{4}x^2$. Now we discuss the conditions in theorem 3.5.12 in this example and from the first look at this example someone could ask, " Do we need y^3 in the radius function to allow the midlocus to have a crosscap singularity?" The answer to this question will be given through the following geometric discussion. The Hessian matrix of the radius function at the origin is given by $\mathcal{H}_r = \begin{pmatrix} \frac{1}{2} & 1 \\ 1 & 2 \end{pmatrix}$, and the eigenvalues are $\lambda_1 = \frac{5}{2}$ and $\lambda_2 = 0$ and the associated eigenvectors are $w_1 = t_1 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$ and $w_2 = t_2 \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ respectively. It is clear that $r(0,0) = \frac{1}{\lambda_1}$ and the tangent vector of M in the direction of w_1 is given by $w = M_x + 2M_y$, hence $k_0(w_1) = \dfrac{II(w_1)}{I(w_1)} = \dfrac{8}{5}$. Since $\lambda_2 = 0$, we do not need to calculate τ_g and $\nabla^3_{w_1} r$, so in this case the midlocus* *is a crosscap if and only if* $\nabla^2_{w_1}\nabla_{w_2}r \neq 0$*. Now direct calculations show that* $\nabla_{w_2}r = (r_x, r_y)\cdot(-2,1) = -2y-x+x+2y+3\mu y^2 = 3\mu y^2$, $\nabla_{w_1}\nabla_{w_2}r = (0, 6\mu y)\cdot(1,2) = 12\mu y$*, and finally* $\nabla^2_{w_1}\nabla_{w_2}r = (0, 12\mu)\cdot(1,2) = 24\mu \neq 0$ *if and only if* $\mu \neq 0$*. Thus, it is vitally important that the radius function should have a non zero coefficient for the* y^3 *term. Now take* $\mu = 1$*, then the midlocus is* $\mathcal{A}-$*equivalent to the crosscap at the origin and the direct calculation gives* $\mathbb{M}(x,y) = (g,h,l)$*, where*

$$g = \frac{4}{5}x - \frac{1}{8}x^3 - \frac{3}{4}x^2y - \frac{2}{5}y - \frac{3}{2}xy^2 - y^3 - y^4 - \frac{1}{2}xy^3,$$

$$h = \frac{(4y + 40y^3 - 24y^2 - 8x - 100y^4 - 60xy^2 - 60y^5 - 80y^3x - 30x^2y - 15x^2y^2 - 5x^3)}{20(1+4y^2)},$$

and

$$l = \frac{-y(6y + 24y^2 + 8x + 100y^4 + 60xy^2 + 60y^5 + 80xy^3 + 30x^2y + 15x^2y^2 + 5x^3)}{10(1+4y^2)}.$$

Remark 3.5.17 *Example 3.5.16 shows that if* r *is* $\mathcal{R}$*-equivalent to* $\widetilde{r}$*, then we do not necessarily have the midlocus* $\widetilde{\mathbb{M}}$ *associated to* $\widetilde{r}$ *is* $\mathcal{A}$*-equivalent to* $\mathbb{M}$*.*

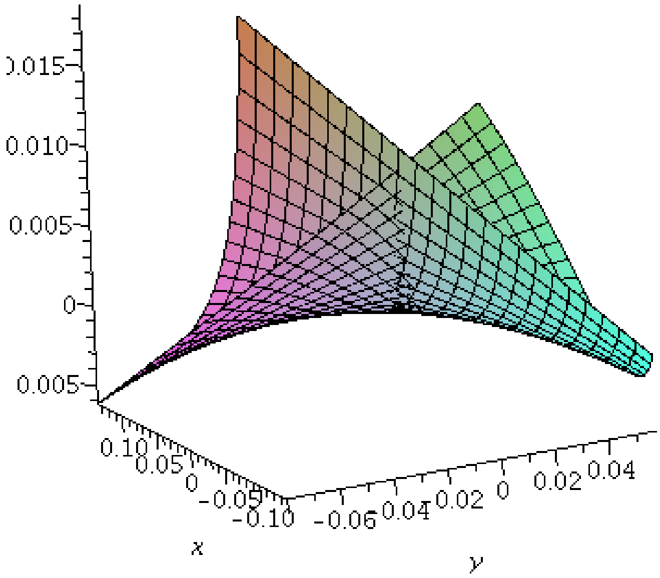

Figure 3.6: Figure of example 3.5.16.

Now we will study the singularity of the midlocus when it fails to have a crosscap. First of all, we state the following theorem.

Theorem 3.5.18 [22] *A map germ* $(\mathbb{R}^2, 0) \to (\mathbb{R}^3, 0)$ *whose 2-jet is equivalent to* $(x, y^2, 0)$ *is equivalent to a germ of the form*

$$(x, y) \mapsto (x, y^2, yP(x, y^2))$$

for smooth P.

Now we state the following theorem which indicates the form of the midlocus when it is not a crosscap.

Theorem 3.5.19 *Assume as in theorem 3.5.12 and the midlocus is not $\mathcal{A}$-equivalent to the crosscap. If $k_{x_0}(w_1) \neq 0$ or $\nabla^3_{w_1} r \neq 0$, then the midlocus parametrized locally by $(u, v) \mapsto (u, v^2, vP(u, v^2))$ for smooth P.*

Proof

From the proof of theorem 3.5.12 we have

$$j^2\mathbb{M} = \left(u, -\frac{b_{12}}{b_{02}}uy - \frac{b_{03}}{b_{02}}y^2, -\frac{a_{11}b_{20}}{b_{02}}uy - \frac{1}{2}a_{02}y^2 \right).$$

Since the midlocus is not $\mathcal{A}$-equivalent to the crosscap, then

$$\begin{vmatrix} -\dfrac{b_{12}}{b_{02}} & -\dfrac{b_{03}}{b_{02}} \\ -\dfrac{a_{11}b_{20}}{b_{02}} & -\dfrac{1}{2}a_{02} \end{vmatrix} = 0.$$

Now if $k_{x_0}(w_1) = a_{02} \neq 0$, then we can complete the square in the third component, and the corresponding change in the v variable then transforms $j^2\mathbb{M}$ into $(u, 0, v^2)$ and by the coordinate change $\overline{X} = X$, $\overline{Y} = Z$, $\overline{Z} = Y$, we have $j^2\mathbb{M} = (u, v^2, 0)$. Now if $\nabla^3_{w_1} r = b_{0,3} \neq 0$, then we can complete the square in the second component and the corresponding change in the v variable then transforms $j^2\mathbb{M}$ into $(u, v^2, 0)$. Thus in both case we apply theorem 3.5.18 and the result holds. □

Example 3.5.20 *Let $M(x, y) = (x, y, x^2)$ and $r(x, y) = 1 + \frac{1}{2}x^2 + y^3$, then the midlocus is singular at the origin and the direct calculation gives $\mathbb{M}(x, y) = (g, h, l)$, where*

$$g = \frac{x(7x^2 - 2y^3)}{2(1 + 4x^2)}, \quad h = \frac{-1}{2}y(3yx^2 + 6y^4 + 6y - 3) \text{ and } l = \frac{x^2(3x^2 - 2y^3 - 1)}{1 + 4x^2}.$$

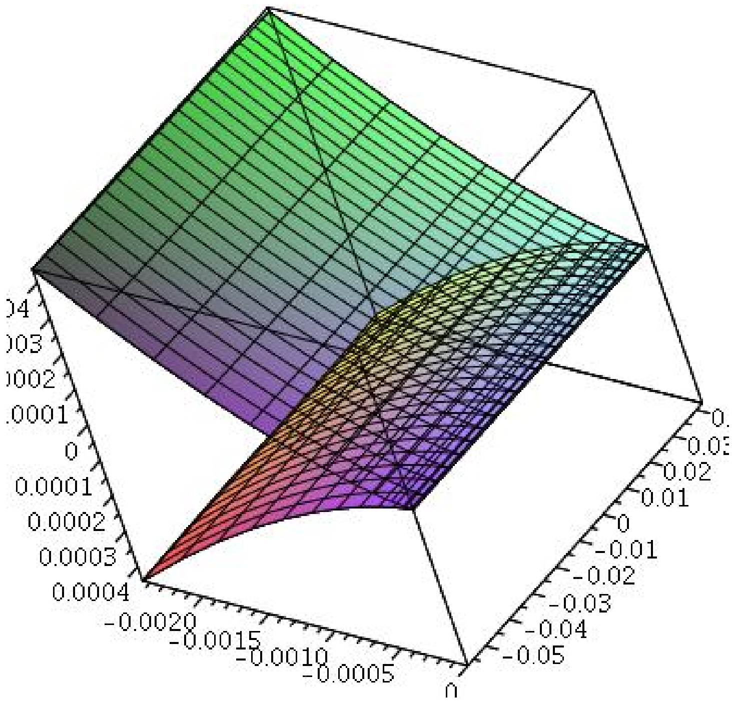

Figure 3.7: Figure of example 3.5.20.

Theorem 3.5.19 gives the general form of the midlocus under the mentioned conditions. Now we will study the 3-jet of the midlocus when it fails to have a crosscap singularity and $k_{x_0}(w_1) \neq 0$. Particularly, we will discuss the conditions for the midlocus to have $S_1^{\pm}$ singularity, which is a map-germ $\mathcal{A}$-equivalent to

$$S_1^{\pm} : (x, y) \mapsto (x, y^2, x^2y \pm y^3)$$

at the origin. Also, the conditions for the 3-jet of the midlocus to be $\mathcal{A}$-equivalent to the cuspidal edge will be investigated. A cuspidal edge is a map-germ $\mathcal{A}$-equivalent to

$$CE : (x, y) \mapsto (x, y^2, y^3)$$

at the origin.

Lemma 3.5.21 *Let M be a smooth stratum of skeletal structure $(\mathbb{S}, U)$ in $\mathbb{R}^3$ containing a smooth point x_0 and r be the radius function with a singularity at x_0. Let λ_1 and λ_2 $(\lambda_1 \neq \lambda_2)$ be the eigenvalues of the Hessian of r and w_1 and w_2 be the associated eigenvectors. Assume that $r(x_0) = \frac{1}{\lambda_1}$, $\lambda_1 \neq 0$. If the midlocus fails to have a crosscap singularity and $k_{x_0}(w_1) \neq 0$, then the 3-jet of the midlocus is given by*

$$j^3\mathbb{M} = (u, v^2, w_{1,2}u^2v + w_{3,0}v^3),$$

where $w_{1,2}$ and $w_{3,0}$ are equations $B.15$ and $B.14$ respectively in appendix.

Proof

We repeat the same procedure of theorems 3.5.6 and 3.5.12 and after using Maple in calculations we get $j^3\mathbb{M} = (p, q, s)$ and we define a parameter change $x = x(u, y)$ such that this parameter change transforms p into $p = \frac{\lambda_1 - \lambda_2}{\lambda_1}u$ and again we define a parameter change $y = y(u, v)$ such that this parameter change transforms s into

$$s = k_{0,2}u^2 + k_{0,3}u^3 - \frac{1}{2}a_{02}v^2$$

and q into

$$q = w_{0,2}u^2 + w_{2,0}v^2 + w_{1,1}uv + w_{2,1}v^2u + w_{1,2}u^2v + w_{3,0}v^3 + w_{0,3}u^3.$$

Now we put $j^3\mathbb{M} = (p, s, q)$, and we define the coordinate change in the target

$$\overline{X} = \frac{\lambda_1}{\lambda_1 - \lambda_2}X, \quad \overline{Y} = Y - \frac{\lambda_1^2 k_{0,2}}{(\lambda_1 - \lambda_2)^2}X^2 - \frac{\lambda_1^3 k_{0,3}}{(\lambda_1 - \lambda_2)^3}X^3,$$

$$\overline{Z} = Z - \frac{\lambda_1^2 w_{0,2}}{(\lambda_1 - \lambda_2)^2}X^2 - \frac{\lambda_1^3 w_{0,3}}{(\lambda_1 - \lambda_2)^3}X^3.$$

This coordinate change transforms $j^3\mathbb{M}$ into

$$j^3\mathbb{M} = \left(u, -\frac{1}{2}a_{02}v^2, w_{1,1}uv + w_{2,1}uv^2 + w_{1,2}u^2v + w_{3,0}v^3\right)$$

since the midlocus is not $\mathcal{A}$-equivalent to the crosscap, then $w_{1,1} = 0$. Now consider the coordinate change in the target

$$\widetilde{X} = X, \quad \widetilde{Y} = -\frac{2}{a_{0,2}}Y, \quad \widetilde{Z} = Z + \frac{2w_{2,1}}{a_{02}}XY,$$

where X, Y, Z are the old coordinates and $\widetilde{X}, \widetilde{Y}, \widetilde{Z}$ are new coordinates. This coordinate change transforms $j^3\mathbb{M}$ into

$$j^3\mathbb{M} = (u, v^2, w_{1,2}u^2v + w_{3,0}v^3)$$

and thus the proof is completed. □

Proposition 3.5.22 *Assume as in lemma 3.5.21, then*

1. *If $w_{1,2} \neq 0$ and $w_{3,0} \neq 0$, then the 3-jet of the midlocus is equivalent to $(u, v^2, u^2v \pm v^3)$ and consequently the midlocus is $\mathcal{A}$-equivalent to an $S_1^{\pm}$ singularity.*
2. *If $w_{1,2} = 0$ and $w_{3,0} \neq 0$, then $j^3\mathbb{M} = (u, v^2, v^3)$.*

Proof

1. Assume that $w_{1,2} \neq 0$ and $w_{3,0} \neq 0$, then using the parameter change $u = \sqrt{|\frac{w_{3,0}}{w_{1,2}}|}\widetilde{u}$, thus this transforms the 3-jet into $j^3\mathbb{M} = \left(\sqrt{|\frac{w_{3,0}}{w_{1,2}}|}\widetilde{u}, v^2, |w_{3,0}|(\widetilde{u}^2v \pm v^3)\right)$. Now consider the coordinate change in the target

$$\overline{X} = \sqrt{|\frac{w_{1,2}}{w_{3,0}}|}X, \quad \overline{Y} = Y, \quad \overline{Z} = \frac{1}{|w_{3,0}|}Z.$$

Thus the 3-jet transformed by this to $(\widetilde{u}, v^2, \widetilde{u}^2v \pm v^3)$, and since $S_1^{\pm}$ is 3-determined [17] the result has been proved. The second part is obvious. □

Now we will give an example when the midlocus is $\mathcal{A}$-equivalent to $S_1^{\pm}$.

Example 3.5.23 *Let $M(x, y) = (x, y, \pm y^2)$ and $r(x, y) = \frac{1}{2} + y^2 \pm x^2y^2$ then the centroid set is $\mathcal{A}$-equivalent to $S_1^{\pm}$ singularity at the origin and the direct calculation gives $\mathbb{M}(x, y) = (g^{\pm}, h^{\pm}, l^{\pm})$, where*

$$g^{\pm} = -x(\pm 2y^4 \pm y^2 + 2x^2y^4 - 1), \quad h^{\pm} = \frac{-y(-2y^2 \pm x^2 \pm 4x^2y^2 + 2x^4y^2)}{1 + 4y^2},$$

and

$$l^{\pm} = \frac{-y^2(2x^2 + 8x^2y^2 \pm 4x^4y^2 \pm 1)}{1 + 4y^2}.$$

In the case of $+$ *the midlocus is equivalent to* S_1^-*, and in the case of* $-$ *the midlocus is equivalent to* S_1^+*. Also, the direct calculations give that* $w_{3,0}^{\pm} = 2$ *and* $w_{1,2}^{\pm} = \mp 1$*.*

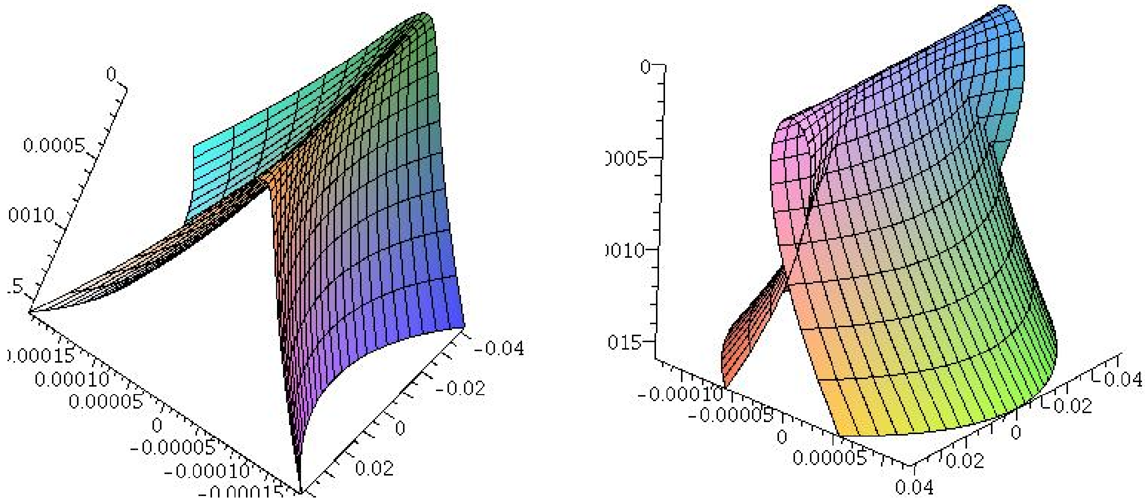

Figure 3.8: Figure of example 3.5.23 in the case $-$, $+$.

Chapter 4

Relation Between Radial Geometry of Skeletal Structure and Differential Geometry of its Boundary

4.1 Introduction

This chapter deals with the relationship between the radial geometry of the skeletal structure and the differential geometry of the associated boundary. In fact, James Damon studied this phenomenon in [8, 9] and he obtained a relationship between S_V and $S_{XV'}$. Moreover, he expressed $S_{XV'}$ in terms of S_V and found out the link between the principal radial curvatures of the skeletal structure and the associated principal curvature of the boundary. In this chapter we express S_V in terms of $S_{XV'}$ (**Proposition 4.2.4**). Also, some algebraic properties between S_V and $S_{XV'}$ are investigated through out this chapter (**Proposition 4.2.6**). Moreover, the relationship between the Gauss radial curvature K_r of a skeletal structure and its associated Gauss curvature K on the boundary has been studied as well as the relationship between the mean radial curvature H_r and its associated mean curvature on the boundary (**Proposition 4.2.10**). The final part of this chapter deals with the situation when K_r and K coincide. We study the relationship between the radial skew curvature of a skeletal structure and the skew curvature of the boundary in situation when $K_r = K \neq 0$ in the case of skeletal structure in $\mathbb{R}^3$.

4.2 Skeletal Structures in $\mathbb{R}^{n+1}$

Our aim in this section is to express the matrix of the radial shape operator in terms of the matrix of the differential geometric shape operator of the boundary. First of all, we give the following theorem which was proved by James Damon.

Theorem 4.2.1 ([8], **Theorem 3.2**) *Let* $(\mathbb{S}, U)$ *be a skeletal structure such that for a choice of smooth value of* U *the associated compatibility 1-form* η_U *vanishes identically on a neighbourhood of a non-edge point* x_0 *of* $\mathbb{S}$*, and* $\frac{1}{r}$ *is not an eigenvalue of the radial shape operator at* x_0*. Let* $x_{0'} = \Psi_1(x_0)$*, and* V' *be the image of* V *for a basis* $\{v_1, v_2, ..., v_n\}$*, then*

1. *The differential geometric shape operator* $S_{XV'}$ *of the boundary* X *at* x_0' *has a matrix representation with respect to* V' *given by*
$$S_{XV'} = (I - rS_V)^{-1} S_V. \tag{4.1}$$
2. *There is a bijection between the principal curvatures* κ_i *of* X *at* x_0' *and the principal radial curvatures* κ_{ri} *of* $\mathbb{S}$ *at* x_0 *(counted with multiplicities) given by*
$$\kappa_i = \frac{\kappa_{ri}}{1 - r\kappa_{ri}} \quad or \quad equivalently \quad \kappa_{ri} = \frac{\kappa_i}{1 + r\kappa_i}. \tag{4.2}$$
3. *The principal radial directions corresponding to* κ_{ri} *are mapped by* $d\Psi_1$ *to the principal directions corresponding to* κ_i*.*

One of the aims of this chapter is to express S_V in terms of $S_{XV'}$ and to do so we need the following lemmas.

Lemma 4.2.2 *Let A be an $(n \times n)$ matrix and suppose that $\frac{1}{\alpha}, (\alpha \neq 0)$ is not eigenvalue of A. If $B = (I - \alpha A)^{-1}A$, then $B = \frac{1}{\alpha}[(I - \alpha A)^{-1} - I]$, where I is the identity matrix.*

Proof

To prove this lemma it is enough to show that

$$(I - \alpha A)^{-1}A - \frac{1}{\alpha}[(I - \alpha A)^{-1} - I] = 0.$$

Now

$$\begin{aligned}
(I - \alpha A)^{-1}A - \frac{1}{\alpha}(I - \alpha A)^{-1} + \frac{1}{\alpha}I &= (I - \alpha A)^{-1}(A - \frac{1}{\alpha}I) + \frac{1}{\alpha}I \\
&= -\frac{1}{\alpha}(I - \alpha A)^{-1}(I - \alpha A) + \frac{1}{\alpha}I \\
&= -\frac{1}{\alpha}I + \frac{1}{\alpha}I \\
&= 0.
\end{aligned}$$

Therefore, the proof is completed. □

Lemma 4.2.3 *Let A be an $(n \times n)$ matrix and suppose that $\frac{1}{\alpha}, (\alpha \neq 0)$ is not eigenvalue of A. If $B = (I - \alpha A)^{-1}A$, then $A = (I + \alpha B)^{-1}B$.*

Proof

From lemma 4.2.2 we have

$$B = \frac{1}{\alpha}[(I - \alpha A)^{-1} - I].$$

Therefore,

$$A = \frac{1}{\alpha}[I - (I + \alpha B)^{-1}]. \tag{4.3}$$

Now our task is to show that

$$(I + \alpha B)^{-1}B - A = 0. \tag{4.4}$$

Now

$$\begin{aligned}(I+\alpha B)^{-1}B - A &= (I+\alpha B)^{-1}B - \frac{1}{\alpha}[I-(I+\alpha B)^{-1}] \\ &= \frac{1}{\alpha}(I+\alpha B)^{-1}(I+\alpha B) - \frac{1}{\alpha}I \\ &= \frac{1}{\alpha}I - \frac{1}{\alpha}I \\ &= 0.\end{aligned}$$

Hence equation 4.4 is satisfied. □

Now we are in the position to do express S_V in terms of $S_{XV'}$. Using the above lemmas and theorem 4.2.1 we have the following:

Proposition 4.2.4 *Let $(\mathbb{S}, U)$ be a skeletal structure as in theorem 4.2.1, then the matrix S_V representing the radial shape operator and the matrix $S_{XV'}$ representing the differential geometric shape operator have the following relation*

$$S_V = (I + rS_{XV'})^{-1} S_{XV'}$$

or equivalently

$$S_V = \frac{1}{r}[I - (I + rS_{XV'})^{-1}].$$

Proof

The proof of this theorem comes directly from theorem 4.2.1 and the above lemmas. □

This proposition leads to the following corollary.

Corollary 4.2.5 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary X. Let $x_0^{'} = \Psi_1(x_0)$ be the associated boundary point to a non-edge point $x_0 \in \mathbb{S}$, and $V^{'}$ be the image of V under $d\Psi_1$ for a basis $\{v_1, v_2, ..., v_n\}$. Then the matrix S_V representing the radial shape operator at x_0 and the* *matrix $S_{XV'}$ representing the differential geometric shape operator of the boundary at $x_0^{'}$ have the following relation*

$$S_V = (I + rS_{XV'})^{-1} S_{XV'}.$$

Proof

Since a Blum medial axis is a special case of the skeletal structure for which the compatibility 1-form vanishes and $\frac{1}{r}$ is not an eigenvalue of the radial shape operator then, we can apply proposition 4.2.4 to get the result. □

The following proposition gives us two relations between S_V and $S_{VX'}$ under the same conditions of theorem 4.2.1.

Proposition 4.2.6 *Let $(\mathbb{S}, U)$ be a skeletal structure as in theorem 4.2.1, then*

1. $S_{XV'} - S_V = rS_{XV'}S_V$.
2. $S_{XV'}S_V = S_V S_{XV'}$, *i.e., the operators commute.*

Proof

1-From theorem 4.2.1 we have

$$\begin{aligned} S_{XV'} - S_V &= (I - rS_V)^{-1} S_V - S_V \\ &= ((I - rS_V)^{-1} - I) S_V \\ &= rS_{XV'} S_V. \end{aligned}$$

2- From (1) we have

$$\begin{aligned} rS_{XV'}S_V &= S_{XV'} - S_V \\ &= S_{XV'} - (I + rS_{XV'})^{-1} S_{XV'} \ (by\ proposition\ 4.2.5) \\ &= (I - (I + rS_{XV'})^{-1}) S_{XV'} \\ &= rS_V S_{XV'} \ (by\ proposition\ 4.2.5). \end{aligned}$$

Hence from above we have $S_V S_{XV'} = S_{XV'} S_V$. $\square$

Remark 4.2.7 *If the S_V is invertible then $det(S_{XV'}) = det((1 - rS_V)^{-1}S_V) \neq 0$ thus $S_{XV'}$ is invertible and vice versa. Thus if S_V is invertible then, $S_V^{-1}S_{XV'} = S_{XV'}S_V^{-1}$ and $S_{XV'}^{-1}S_V = S_V S_{XV'}^{-1}$.*

Now we will give an example to illustrate the results in proposition 4.2.6.

Example 4.2.8 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^3$ and let $S_1(x, y) = (x, y, \frac{1}{2}\kappa_{m1}x^2 + \frac{1}{2}\kappa_{m2}y^2 + h.o.t) \subset \mathbb{S}_{reg.}$ and $r = r_0 + \frac{1}{2}ax^2 + \frac{1}{2}by^2$ be the radius function on S_1 such that $\frac{1}{r_0} \notin \{\kappa_{m1} + a, \kappa_{m2} + b\}$. Now we define the unit radial vector field by*

$$U_1 = -\nabla r + \sqrt{1 - \|\nabla r\|^2}N,$$

where ∇r is the Riemannian gradient of the radius function and N is the unit normal of S_1. Direct calculation shows that at the origin we have

$$S_V = \begin{pmatrix} \kappa_{m1} + a & 0 \\ 0 & \kappa_{m2} + b \end{pmatrix}, \quad and \quad S_{XV'} = \begin{pmatrix} \dfrac{\kappa_{m1} + a}{1 - r_0(\kappa_{m1} + a)} & 0 \\ 0 & \dfrac{\kappa_{m2} + b}{1 - r_0(\kappa_{m2} + b)} \end{pmatrix}.$$

It is clear that

$$S_V S_{XV'} = S_{XV'} S_V = \begin{pmatrix} \dfrac{(\kappa_{m1} + a)^2}{1 - r_0(\kappa_{m1} + a)} & 0 \\ 0 & \dfrac{(\kappa_{m2} + b)^2}{1 - r_0(\kappa_{m2} + b)} \end{pmatrix}.$$

Now

$$S_{XV'} - S_V = \begin{pmatrix} \dfrac{r_0(\kappa_{m1} + a)^2}{1 - r_0(\kappa_{m1} + a)} & 0 \\ 0 & \dfrac{r_0(\kappa_{m2} + b)^2}{1 - r_0(\kappa_{m2} + b)} \end{pmatrix} = r_0 S_V S_{XV'}.$$

Definition 4.2.9 *Let S_V be the matrix representation of the radial shape operator of a skeletal structure and $S_{XV'}$ be the matrix representation of the differential geometric shape operator of the associated boundary, then the Gaussian radial curvature K_r and the mean radial curvature H_r are given by*

$$K_r = det(S_V) \quad and \quad H_r = \frac{1}{n} tr(S_V),$$

and the Gaussian curvature K and the mean curvature of the boundary are given by

$$K = det(S_{XV'}) \quad and \quad H = \frac{1}{n} tr(S_{XV'}).$$

The $i-$th mean radial curvature K_{ri} is defined by

$$K_{ri} = \binom{n}{i}^{-1} \sum_{j_1<j_2<\ldots<j_i} \kappa_{rj_1} \ldots . \kappa_{rj_i}$$

and the associated $i-$th mean curvature of the boundary is defined by

$$K_i = \binom{n}{i}^{-1} \sum_{j_1<j_2<\ldots<j_i} \kappa_{j_1} \ldots . \kappa_{j_i}.$$

Now we will turn to the relation between the Gaussian radial curvature K_r of the skeletal structure at a non-edge point and the Gaussian curvature K of the boundary at the associated point. Also, the relation between the mean radial curvature H_r of the skeletal structure and the associated mean curvature H of the boundary will be investigated. In fact Anthony Pollitt studied in [24] the relationship between the principal radial curvatures and the associated principal curvatures on the boundary in the case of medial axis in $\mathbb{R}^3$. In the following proposition we generalize the result obtained by Pollitt to the higher dimensions in the case of skeletal structure which is more general than medial axis and we give other results as well.

Proposition 4.2.10 *Let $(\mathbb{S}, U)$ be a skeletal structure as in theorem 4.2.1, then*

1. $K = \dfrac{K_r}{1 - rnH_r + \sum_{i=2}^{n-1} (-1)^i r^i \binom{n}{i} K_{ri} + (-1)^n r^n K_r}.$

2. $H = \dfrac{H_r - rnH_r^2 + \frac{r}{n}\sum\limits_{i=1}^{n}\kappa_{ri}^2 + \sum\limits_{j=3}^{n}(-1)^{j-1}r^{j-1}\binom{n}{j}K_{ri}}{1 - rnH_r + \sum\limits_{i=2}^{n-1}(-1)^i r^i\binom{n}{i}K_{ri} + (-1)^n r^n K_r}.$

3. $K_r = \dfrac{K}{1 + rnH + \sum\limits_{i=2}^{n-1} r^i\binom{n}{i}K_i + r^nK}.$

4. $H_r = \dfrac{H + rnH - \frac{r}{n}\sum\limits_{i=1}^{n}\kappa_i^2 + \sum\limits_{j=3}^{n} r^{j-1}\binom{n}{j}K_j}{1 + rnH + \sum\limits_{i=2}^{n-1} r^i\binom{n}{i}K_i + r^nK}.$

5. $\kappa_{rl}(1 + rnH + \sum\limits_{i=2}^{n-1} r^i\binom{n}{i}K_i + r^nK) = \kappa_l + rn\kappa_l H - r\kappa_l^2 + \sum\limits_{j=3}^{n}\binom{n}{j}r^{j-1}K_j.$

6. $\kappa_l(1 - rnH_r + \sum\limits_{i=2}^{n-1}(-1)^i r^i\binom{n}{i}K_{ri} + (-1)^n r^n K_r) = \kappa_{rl} - rn\kappa_{rl}H_r + r\kappa_{rl}^2 + \sum\limits_{j=3}^{n}(-1)^{j-1}r^{j-1}K_{rj}.$

7. *If* $K_r \neq 0$, *then*

$\dfrac{H_r}{K_r} - \dfrac{H}{K} = \dfrac{rnH_r^2 - \dfrac{r}{n}\sum\limits_{i=1}^{n}\kappa_{ri}^2 - \sum\limits_{j=3}^{n}(-1)^{j-1}r^{j-1}\binom{n}{j}K_{rj}}{K_r}.$

Proof

1. From theorem 4.2.1 we have $\kappa_i = \dfrac{\kappa_{ri}}{1 - r\kappa_{ri}}$. Therefore,

$$K = \prod_{i=1}^{n}\frac{\kappa_{ri}}{1 - r\kappa_{ri}} = \frac{K_r}{\prod\limits_{i=1}^{n}(1 - r\kappa_{ri})}. \tag{4.5}$$

Now from the theory of symmetric polynomials we can expand the denominator of the above equation to get $\prod\limits_{i=1}^{n}(1 - r\kappa_{ri}) = 1 - rH_r + \sum\limits_{i=2}^{n-1}(-1)^i r^i\binom{n}{i}K_{ri} + (-1)^n r^n K_r$. Thus by substituting in equation 4.5 the result is proved.

2. We have

$$nH = \sum_{i=1}^{n}\kappa_i = \sum_{i=1}^{n}\frac{\kappa_{ri}}{1 - r\kappa_{ri}}.$$

Thus

$$nH = \frac{\kappa_{r1}\prod_{i=2}^{n}(1-r\kappa_{ri}) + \kappa_{r2}(1-r\kappa_{r1})\prod_{i=3}^{n}(1-r\kappa_{ri}) + ... + \kappa_{rn}\prod_{i=1}^{n-1}(1-r\kappa_{ri})}{1 - rH_r + \sum_{i=2}^{n-1}(-1)^i r^i \binom{n}{i} K_{ri} + (-1)^n r^n K_r}.$$

Now we will simplify the numerator of this equation using the concept of symmetric polynomial. First of all we have

$$\begin{aligned}\kappa_{r1}\prod_{i=2}^{n}(1-r\kappa_{ri}) &= \kappa_{r1}(1-r\kappa_{r2})(1-r\kappa_{r3})...(1-r\kappa_{rn}) \\ &= \kappa_{r1} - rn\kappa_{r1}H_r + r\kappa_{r1}^2 + \sum_{j=3}^{n}(-1)^{j-1}r^{j-1}\binom{n}{j}K_{rj}.\end{aligned}$$

Similarly, we have

$$\begin{aligned}\kappa_{rl}(1-r\kappa_{r1})...(1-r\kappa_{rl-1})(1-r\kappa_{rl+1})...(1-r\kappa_{rn}) &= \kappa_{rl} - rn\kappa_{rl}H_r + r\kappa_{rl}^2 \\ &\quad + \sum_{j=3}^{n}(-1)^{j-1}r^{j-1}\binom{n}{j}K_{rj}.\end{aligned}$$

Therefore, after simplification the numerator becomes

$$n\left(H_r - rnH_r^2 + \frac{r}{n}\sum_{i=1}^{n}\kappa_{ri}^2 + \sum_{j=3}^{n}(-1)^{j-1}r^{j-1}\binom{n}{j}K_{rj}\right).$$

Thus

$$H = \frac{H_r - rnH_r^2 + \frac{r}{n}\sum_{i=1}^{n}\kappa_{ri}^2 + \sum_{j=3}^{n}(-1)^{j-1}r^{j-1}\binom{n}{j}K_{ri}}{1 - rnH_r + \sum_{i=2}^{n-1}(-1)^i r^i \binom{n}{i} K_{ri} + (-1)^n r^n K_r}.$$

Similarly we can prove 3 and 4.

5- We have

$$\begin{aligned}\kappa_{rl}(1+rnH+\sum_{i=2}^{n-1} r^i\binom{n}{i}K_i+r^nK) &= \kappa_{rl}\prod_{i=1}^{n}(1+r\kappa_i)\\ &= \frac{\kappa_l}{1+r\kappa_l}\prod_{i=1}^{n}(1+r\kappa_i)\\ &= \kappa_l(1+r\kappa_1)...(1+r\kappa_{l-1})(1+r\kappa_{l+1})...(1+r\kappa_n)\\ &= \kappa_l+rn\kappa_lH-r\kappa_l^2+\sum_{j=3}^{n}\binom{n}{j}r^{j-1}K_j.\end{aligned}$$

Similarly we can prove 6.

7. If $K_r \neq 0$ then $K \neq 0$ and we have

$$\begin{aligned}\frac{H_r}{K_r}-\frac{H}{K} &= \frac{H_r}{K_r}-\frac{H_r-rnH_r^2+\frac{r}{n}\sum_{i=1}^{n}\kappa_{ri}^2+\sum_{j=3}^{n}(-1)^{j-1}r^{j-1}\binom{n}{j}K_{ri}}{K_r}\\ &= \frac{rnH_r^2-\frac{r}{n}\sum_{i=1}^{n}\kappa_{ri}^2-\sum_{j=3}^{n}(-1)^{j-1}r^{j-1}\binom{n}{j}K_{rj}}{K_r}.\end{aligned}$$

Thus the proof is completed. □

If the radius function is a constant on a smooth stratum S_1 of the skeletal structure containing a smooth point x_0, then this stratum and its associated boundary are parallel and the radial vector field becomes the normal of that stratum. Thus if we replace r in proposition 4.2.10 by a constant, then K_r and H_r is the Gaussian curvature and mean curvature of S_1 at x_0 respectively. Thus proposition 4.2.10 indicates that for each smooth point x_0 of the skeletal structure the smooth hypersurface containing x_0 and parallel to the boundary has Gaussian curvature K_r and mean curvature H_r.

James Damon gave a relationship between the matrix representing the radial shape operator of the skeletal structure and the matrix representing the differential geometric shape operator of the skeletal structure (theorem 4.2.1). Our aim in this thesis is to find the relationship between the radial shape operator of the skeletal structure and the differential geometric shape operator of the associated boundary and we will look at this relation when the radius function has a singularity. Now let S_{rad} denotes the radial shape operator and S_{Bond} is the associated shape operator of the boundary, if the radius function has a singularity then this has strong consequence for the relationship between S_{rad} and S_{Bond}. In particular we have the following.

Proposition 4.2.11 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$ as in theorem 4.2.1. If the radius function r has a singularity at x_0 then*

1. $S^T_{XV'}V' = S^T_V V$,
2. $S_{Bond}(v^{'}) = S_{rad}(v)$,
3. $V = (I + rS_{XV'})^T V'$.

Proof

1. From equation 1.3 we have

$$\frac{\partial U_1}{\partial V} = A_V U_1 - S^T_V V. \tag{4.6}$$

Also, the Jacobian matrix of the radial map is given by

$$V^{'} = \frac{\partial \Psi_1}{\partial V} = (dr(V) + rA_V)U_1 + (1 - rS_V)^T V. \tag{4.7}$$

Now since the radius function r has a singularity then using lemma 3.3.7 we have $A_V = 0$. Thus, equation 4.7 becomes

$$V^{'} = (I - rS_V)^T V.$$

Also, since $\frac{1}{r}$ is not an eigenvalue of the radial shape operator we can solve for V to get

$$V = (I - rS^T_V)^{-1} V^{'}. \tag{4.8}$$

From the proof of the proposition 2.1 in [8] we have

$$\frac{\partial U_1}{\partial V} = \frac{\partial U_1}{\partial V'}.$$

Now substitute in equation (4.6) we have

$$\frac{\partial U_1}{\partial V'} = -S_V^T (I - rS_V^T)^{-1} V' = -S_V^T V.$$

Now we will show that

$$S_V^T (I - rS_V^T)^{-1} = (I - rS_V^T)^{-1} S_V^T.$$

To do so it is enough to show that

$$S_V (I - rS_V)^{-1} - \frac{1}{r}[(I - rS_V)^{-1} - I] = 0.$$

Now

$$\begin{aligned}
[S_V (I - rS_V)^{-1} - \frac{1}{r}[(I - rS_V)^{-1} - I] &= S_V (I - rS_V)^{-1} - \frac{1}{r}(I - rS_V)^{-1} + \frac{1}{r} I \\
&= -\frac{1}{r} I + \frac{1}{r} I \\
&= 0.
\end{aligned}$$

Therefore,

$$\frac{\partial U_1}{\partial V'} = -(I - rS_V^T)^{-1} S_V^T V'.$$

Hence $S_{XV'}^T V' = S_V^T V$.

2. The unit normal of the boundary is U_1 and the shape operator S_{Bond} of the boundary is given by $-\text{proj}_{U_1}\left(\frac{\partial U_1}{\partial v'}\right)$, where proj_{U_1} is the projection along U_1 to the tangent space of the boundary. Also, the radial shape operator S_{rad} is given by $-\text{proj}_{U_1}\left(\frac{\partial U_1}{\partial v}\right)$ where proj_{U_1} is the projection along U_1 to the tangent space of the skeletal structure. From 1 we have $-\text{proj}_{U_1}\left(\frac{\partial U_1}{\partial V'}\right) = S_{XV'}^T V' = S_V^T V = -\text{proj}_{U_1}\left(\frac{\partial U_1}{\partial V}\right)$. Thus, $S_{Bond}(v') = S_{rad}(v)$.

3. From equation (4.7) $V = (I - rS_V^T)^{-1} V'$ and from proposition 4.2.4 it is easy to obtain that $(I - rS_V^T)^{-1} = (I + rS_{XV'})^T$. Thus the proof is completed. □

Example 4.2.12 *Assume as in example 4.2.8, then at the origin the radius function has a singularity and*

$$V = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \; and \, V^{'} = \begin{pmatrix} v_1^{'} \\ v_2^{'} \end{pmatrix} = \begin{pmatrix} 1 - r_0(\kappa_{m1} + a) & 0 & 0 \\ 0 & 1 - r_0(\kappa_{m2} + b) & 0 \end{pmatrix}.$$

Thus

$$S_V^T V = \begin{pmatrix} \kappa_{m1} + a & 0 \\ 0 & \kappa_{m2} + b \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} = \begin{pmatrix} \kappa_{m1} + a & 0 & 0 \\ 0 & \kappa_{m2} + b & 0 \end{pmatrix}.$$

Also,

$$\begin{aligned} S_{XV'}^T V^{'} &= \begin{pmatrix} \dfrac{\kappa_{m1} + a}{1 - r_0(\kappa_{m1} + a)} & 0 \\ 0 & \dfrac{\kappa_{m2} + b}{1 - r_0(\kappa_{m2} + b)} \end{pmatrix} \begin{pmatrix} 1 - r_0(\kappa_{m1} + a) & 0 & 0 \\ 0 & 1 - r_0(\kappa_{m2} + b) & 0 \end{pmatrix} \\ &= \begin{pmatrix} \kappa_{m1} + a & 0 & 0 \\ 0 & \kappa_{m2} + b & 0 \end{pmatrix}. \end{aligned}$$

Therefore,

$$S_V^T V = S_{XV'}^T V^{'},$$

$$S_{rad}(v_1) = (\kappa_{m1} + a, 0, 0) = S_{Bond}(v_1^{'})$$

and

$$S_{rad}(v_2) = (0, \kappa_{m2} + b, 0) = S_{Bond}(v_2^{'}).$$

Moreover,

$$\begin{aligned} (I + r_0 S_{XV'}^T) V^{'} &= \begin{pmatrix} \frac{1}{1 - r_0(\kappa_{m1} + a)} & 0 \\ 0 & \frac{1}{1 - r_0(\kappa_{m2} + b)} \end{pmatrix} \begin{pmatrix} 1 - r_0(\kappa_{m1} + a) & 0 & 0 \\ 0 & 1 - r_0(\kappa_{m2} + b) & 0 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \\ &= V. \end{aligned}$$

Now we turn to the relationship between the differential geometric shape operator of the skeletal structure at a smooth point and the differential geometric shape operator of the boundary at the associated point. First we have the following definition.

Definition 4.2.13 *Let $(\mathbb{S}, U)$ be a skeletal structure and radial vector field such that the compatibility holds in a neighbourhood of a smooth x_0. We define the radial Hessian operator by*

$$\mathbb{H}_r : T_{x_0}\mathbb{S} \to T_{x_0}\mathbb{S}$$

such that $\mathbb{H}_r(v) = -\text{proj}_N\left(\frac{\partial U_{1tan}}{\partial v}\right)$, where proj_N denotes orthogonal projection onto $T_{x_0}\mathbb{S}$ and U_{1tan} is the tangential component of the unit radial vector field U_1.

Proposition 4.2.14 ([8], **proposition 4.1**) *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$ which satisfies the compatibility condition on an open set $W \subset \mathbb{S}_{reg}$ Let U be a smooth value on W. Then, on W there is the following relation*

$$S_{rad} = \rho S_{med} + \mathbb{H}_r + Z, \tag{4.9}$$

where $Z(v) = \rho^{-1}(\frac{\partial U_1}{\partial v} \cdot N)U_{1tan}$, N is the unit normal of the skeletal set and U_{1tan} is the tangential parts of the unit radial vector field U_1 and S_{med} is the differential geometric shape operator of the skeletal structure.

Damon discusses in [8] that the operator Z is difficult to work with and interpret. But again when the radius function has a singularity we get a situation with strong consequences for the various operators.

Proposition 4.2.15 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$ as in proposition 4.2.14. If the radius function r has a singularity at x_0 then the differential geometric shape operator S_{med} of $\mathbb{S}$ at x_0 and the differential geometric shape operator S_{Bond} of the boundary at $x_0^{'} = \Psi_1(x_0)$ are related by*

$$S_{Bond} = S_{med} + \mathbb{H}_r.$$

Proof

If the radius function r has a singularity then the unit radial vector field is normal to the tangent space of the skeletal set. Thus $Z = 0$ and $\rho = N \cdot U_1 = 1$. From proposition 4.2.11 we have $S_{Bond} = S_{rad}$. Therefore, equation (4.9) becomes

$$S_{Bond} = S_{med} + \mathbb{H}_r,$$

which completes the proof. □

This proposition gives the relationship between the geometric shape operator of the boundary and that of the skeletal structure and it is obvious from this proposition to obtain that the tangent space of the skeletal structure at x_0 is parallel to the tangent space of the boundary at the associated point. This means that the skeletal structure and the hypersurface containing x_0 and parallel to the boundary have the same tangent space at x_0.

Example 4.2.16 *Assume as in example 4.2.8. Then the radius function has a singularity at the origin and $S_{med}(v_i) = \kappa_i v_i$, $i = 1, 2$. Also, $\mathbb{H}_r(v_1) = av_1$ and $\mathbb{H}_r(v_2) = bv_2$ and from example 4.2.12 we have*

$$S_{Bond}(v_1^{'}) = S_{rad}(v_1) = (\kappa_{m1} + a, 0, 0) = \kappa_{m1} v_1 + av_1 = S_{med}(v_1) + \mathbb{H}_r(v_1),$$

and

$$S_{Bond}(v_2^{'}) = S_{rad}(v_2) = (0, \kappa_{m2} + b, 0) = \kappa_2 v_2 + bv_2 = S_{med}(v_2) + \mathbb{H}_r(v_2).$$

4.3 Skeletal Structures in $\mathbb{R}^3$

In this section we will give a special form of the shape operator of the boundary in terms of the radial shape operator of the skeletal structure in $\mathbb{R}^3$ and vice versa.

Theorem 4.3.1 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^3$ such that for a choice of smooth value of U the associated compatibility 1-form η_U vanishes identically on a neighbourhood of a smooth point x_0 of $\mathbb{S}$, and $\frac{1}{r}$ is not an eigenvalue of the radial shape operator at x_0. Let $x_0^{'} = \Psi_1(x_0)$ and $V^{'}$ be the image of V under $d\Psi_1$ for a basis $\{v_1, v_2\}$ then*

$$S_{XV'} = \frac{1}{r^2 K_r - 2rH_r + 1}(S_V - rK_r I) \tag{4.10}$$

or equivalently

$$S_V = \frac{1}{r^2 K + 2rH + 1}(S_{XV'} + rKI). \tag{4.11}$$

Proof

From lemma 4.2.2 we have

$$S_{XV'} = \frac{1}{r}[(I - rS_V)^{-1} - I]. \tag{4.12}$$

Now let

$$S_V = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Therefore,

$$(I - rS_V) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} - \begin{pmatrix} ra & rb \\ rc & rd \end{pmatrix} = \begin{pmatrix} 1 - ra & -rb \\ -rc & 1 - rd \end{pmatrix}.$$

Therefore,

$$\begin{aligned} det(I - rS_V) &= (1 - ra)(1 - rd) - r^2 cb \\ &= 1 - r(a + b) + r^2(ad - cb) \\ &= 1 - 2rH_r + r^2 K_r. \end{aligned}$$

Now

$$(I - rS_V)^{-1} = \frac{1}{1 - 2rH_r + r^2 K_r} \begin{pmatrix} 1 - rd & rb \\ rc & 1 - ra \end{pmatrix}.$$

Thus

$$(I - rS_V)^{-1} - I = \frac{1}{1 - 2rH_r + r^2K_r} \begin{pmatrix} 1 - rd & rb \\ rc & 1 - ra \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix},$$

hence this matrix becomes

$$\frac{1}{1 - 2rH_r + r^2K_r} \begin{pmatrix} 1 - rd - 1 + r(a+d) - r^2K_r & rb \\ rc & 1 - ra - 1 + r(a+d) - r^2K_r \end{pmatrix},$$

which gives the following

$$(I - rS_V)^{-1} - I = \frac{r}{1 - 2rH_r + r^2K_r} \begin{pmatrix} a - rK_r & b \\ c & d - rK_r \end{pmatrix}$$

$$= \frac{r}{1 - 2rH_r + r^2K_r}(S_V - rK_rI).$$

Now by substituting in equation 4.12 we get

$$S_{XV'} = \frac{1}{1 - 2rH_r + r^2K_r}(S_V - rK_rI).$$

Similarly, we can prove equation 4.11. □

Now we have the following corollary from proposition 4.2.10.

Corollary 4.3.2 *Let* $(\mathbb{S}, U)$ *be a skeletal structure in* $\mathbb{R}^3$ *as in theorem 4.3.1. Then, the radial geometric factors (the principal radial curvatures* κ_{ri}*, the Gaussian radial curvature* K_r *and mean radial curvature* H_r *) of* $\mathbb{S}$ *at* x_0 *and the differential geometric factors (the principal curvatures* κ_i*, the Gaussian curvature* K *and the mean curvature* H*) of the boundary at* x_0' *satisfy the following*

1. $K = \dfrac{K_r}{1 - 2rH_r + r^2K_r}$.
2. $H = \dfrac{H_r - rK_r}{1 - 2rH_r + r^2K_r}$.

3. $K_r = \dfrac{K}{1 + 2rH + r^2K}$.

4. $H_r = \dfrac{H + rK}{1 + 2rH + r^2K}$.

5. $\kappa_{rl}(1 + 2rH + r^2K) = \kappa_l + rK$, $l = 1, 2$.

6. $\kappa_l(1 - 2rH_r + r^2K_r) = \kappa_{rl} - rK_r$, $l = 1, 2$.

7. *If* $K_r \neq 0$ *then,*
$\dfrac{H_r}{K_r} - \dfrac{H}{K} = r$.

Proof

The proof of this corollary comes directly from proposition 4.2.10 just by taking $n = 2$. □

Example 4.3.3 *Assume as in example 4.2.8, then at the origin the principal radial curvatures, Gaussian radial curvature and mean radial curvature are*

$$\kappa_{r1} = \kappa_{m1} + a,\ \kappa_{r2} = \kappa_{m2} + b,\ K_r = (\kappa_{m1} + a)(\kappa_{m2} + b) \text{ and } H_r = \frac{1}{2}(\kappa_{m1} + \kappa_{m2} + a + b).$$

Also the associated the principal curvatures, Gaussian curvature and mean curvature of the boundary are

$$\kappa_1 = \frac{\kappa_{m1} + a}{1 - r_0(\kappa_{m1} + a)},\ \kappa_2 = \frac{\kappa_{m2} + b}{1 - r_0(\kappa_{m2} + b)},$$

$$K = \frac{(\kappa_{m1} + a)(\kappa_{m2} + b)}{1 - r_0(\kappa_{m1} + \kappa_{m2} + a + b) + r_0^2(\kappa_{m1} + a)(\kappa_{m2} + b)}$$

and

$$H = \frac{\frac{1}{2}(\kappa_{m1} + \kappa_{m2} + a + b) - r_0(\kappa_{m1} + a)(\kappa_{m2} + b)}{1 - r_0(\kappa_{m1} + a + \kappa_{m2} + b) + r_0^2(\kappa_{m1} + a)(\kappa_{m2} + b)}.$$

Thus it is easy to check that this example satisfies relations 1, 2, 3, 4, 5 and 6 in corollary 4.3.2. Now assume that $K_r \neq 0$, then we have

$$\begin{aligned}\frac{H_r}{K_r} - \frac{H}{K} &= \frac{\kappa_{m1} + \kappa_{m2} + a + b}{2(\kappa_{m1} + a)(\kappa_{m2} + b)} - \frac{\kappa_{m1} + \kappa_{m2} + a + b - 2r_0(\kappa_{m1} + a)(\kappa_{m2} + b)}{2(\kappa_{m1} + a)(\kappa_{m2} + b)} \\ &= \frac{2r_0(\kappa_{m1} + a)(\kappa_{m2} + b)}{2(\kappa_{m1} + a)(\kappa_{m2} + b)} \\ &= r_0.\end{aligned}$$

Proposition 4.3.4 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^3$ as in corollary 4.3.2. If $K_r \neq 0$ or equivalently $K \neq 0$, then*

$$\frac{1}{K_r} S_V - \frac{1}{K} S_{XV'} = rI.$$

Proof

From equation 4.10 we have

$$S_{XV'} = \frac{1}{r^2 K_r - 2rH_r + 1}(S_V - rK_r I).$$

Therefore,

$$\frac{1}{K_r} S_V - \frac{1}{K} S_{XV'} = \frac{1}{K_r} S_V - \frac{1}{K}\left(\frac{1}{r^2 K_r - 2rH_r + 1}(S_V - rK_r I)\right).$$

But from corollary 4.3.2 we have

$$K = \frac{K_r}{r^2 K - 2rH_r + 1}.$$

Hence

$$\frac{1}{K_r} S_V - \frac{1}{K} S_{XV'} = \frac{1}{K_r} S_V - \frac{r^2 K_r - 2rH_r + 1}{K_r}\left(\frac{1}{r^2 K_r - 2rH_r + 1} S_V - \frac{rK_r}{r^2 K_r - 2rH_r + 1} I\right).$$

Therefore, we have the following

$$\frac{1}{K_r} S_V - \frac{1}{K} S_{XV'} = \frac{1}{K_r} S_V - \frac{1}{K_r} S_V + rI = rI.$$

Therefore, the proof is completed. □

Now we will study the answer of the question: what happens if $K_r = K$?

Proposition 4.3.5 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^3$ as in corollary 4.3.2. Then we have*

1. *If $K_r = K = 0$, then*
$$H = \frac{H_r}{1 - 2rH_r},$$
or equivalently
$$H_r = \frac{H}{1 + 2rH}.$$
2. *If $K_r = K \neq 0$, then $H = -H_r = \dfrac{-r}{2}K \neq 0$.*

Proof

1-From corollary 4.3.2 we have

$$H = \frac{H_r - rK_r}{r^2K_r - 2rH_r + 1} \text{ and } H_r = \frac{H + rK}{r^2K + 2rH + 1}.$$

Therefore, if $K_r = K = 0$ we get the result.

2-Assume that $K_r = K \neq 0$, then we have $1 + 2rH + r^2K = 1 - 2rH_r + r^2K_r$ which gives that $r^2K + 2rH = 0$ and $r^2K_r - 2rH_r = 0$ thus the result holds. □

The second part of proposition 4.3.5 indicates that if $K_r = K \neq 0$, then the boundary has non zero mean curvature.

Example 4.3.6 *Assume as in example 4.2.8 such that $r_0 = 1$, $\kappa_{m1} = 2$, $\kappa_{m2} = \frac{1}{2}$ and $a = b = 1$, then at the origin we have*

$$S_V = \begin{pmatrix} 3 & 0 \\ 0 & \frac{3}{2} \end{pmatrix} \text{ and } S_{XV'} = \begin{pmatrix} \frac{-3}{2} & 0 \\ 0 & -3 \end{pmatrix}.$$

Thus $K_r = K = \frac{9}{2}$, $H_r = \frac{9}{4}$ *and* $H = -\frac{9}{4}$.

In the next theorem we will discuss at what conditions does $K_r = K \neq 0$ in the case of the skeletal structure in $\mathbb{R}^3$?

Theorem 4.3.7 *Let* $(\mathbb{S}, U)$ *be a skeletal structure in* $\mathbb{R}^3$ *such that for a choice of smooth value of* U *the associated compatibility 1-form* η_U *vanishes identically on a neighbourhood of a non-edge point* x_0 *of* $\mathbb{S}$*, and* $\frac{1}{r}$ *is not an eigenvalue of the radial shape operator at* x_0*. Let* $u_r = \frac{1}{2}(\kappa_{r2} - \kappa_{r1}) = \sqrt{H_r^2 - K_r}$ *is the radial skew curvature. If* x_0 *is not a radial umbilic point* (*i.e.,* $\kappa_{r1} \neq \kappa_{r2}$)*, then* $K_r = K$ *if and only if* $u = u_r$ *where* $u = \frac{1}{2}(\kappa_2 - \kappa_1) = \sqrt{H^2 - K}$.

Proof

Let $K_r = K \neq 0$ then from proposition 4.3.5 we have $H = -H_r$. Therefore,

$$u_r = \sqrt{H_r^2 - K_r} = \sqrt{(-H)^2 - K} = u.$$

Conversely, assume that $u = u_r$ then

$$\kappa_2 - \kappa_1 = \kappa_{r2} - \kappa_{r1} = \frac{\kappa_2}{1 + r\kappa_2} - \frac{\kappa_1}{1 + r\kappa_1}.$$

Therefore,

$$\kappa_2 - \kappa_1 = \frac{\kappa_2 - \kappa_1}{r^2 K + 2rH + 1}.$$

Since x_0 is not a radial umbilic point, then $x_0' = \Psi_1(x_0)$ is not an umbilic point on the boundary (i.e., $\kappa_1 \neq \kappa_2$). Therefore, $r^2 K + 2rH + 1 = 1$ which gives that $K_r = K$. □

Example 4.3.8 *Assume as in example 1.3.3, then the matrix representing the differential geometric shape operator is given by*

$$S_{XV'} = \frac{1}{r^2 K_r - 2rH_r + 1}(S_V - rK_r I),$$

and after some calculations and simplifications we get

$$S_{XV'} = \frac{-2}{4(x^2+y^2)[3(x^2+y^2)+2]+1} \begin{pmatrix} 1+2x^2+6y^2 & -4xy \\ -4xy & 1+6x^2+2y^2 \end{pmatrix}.$$

Now the principal curvatures of the boundary are given by

$$\kappa_1 = \frac{-2}{1+2x^2+2y^2} \quad and \quad \kappa_2 = \frac{-2}{1+6x^2+6y^2}.$$

Thus the Gaussian and mean curvatures of the boundary are given by

$$K = \frac{4}{(1+6x^2+6y^2)(1+2x^2+2y^2)} \quad and \quad H = \frac{-(2+8x^2+8y^2)}{(1+6x^2+6y^2)(1+2x^2+2y^2)}.$$

From direct calculation we have

$$\frac{H_r}{K_r} - \frac{H}{K} = x^2+y^2+1 = r.$$

From this example we can see that the radial curvature condition is not necessary for the smoothness of the boundary. For instance when the radius function has a singularity we have $\kappa_{r1} = \kappa_{r2} = 2$ but the radius function $r = 1$.

Chapter 5

The Relationship Between the Differential Geometry of the Skeletal Structure and that of the Boundary

5.1 Introduction

In chapter 4 we studied the relationship between the radial shape operator of a skeletal structure and the differential geometric shape operator of its associated boundary. This chapter focuses on the relationship between the differential geometric shape operator of a skeletal structure and the differential geometric shape operator of its boundary. To find out this relationship we first study the relationship between the differential geometric shape operator of a skeletal structure and its radial shape operator. First we study this relationship in the case of a skeletal structure in $\mathbb{R}^2$ (**Theorem 5.2.4**). After this we study the relationship between the curvatures of the boundary at the tangency points associated to a smooth point of a Blum medial axis in $\mathbb{R}^2$ (**Theorem 5.2.13**). Second we study the relationship between the radial shape operator of a skeletal structure in $\mathbb{R}^{n+1}$ and its differential geometric shape operator (**Theorem 5.3.17**). This gives us enough tools to study the required relationship between the differential geometric shape operator of a skeletal structure and the differential geometric shape operator of its boundary which is given in theorem 5.3.23.

5.2 The Differential Geometry of the Skeletal Structure and its Boundary in the Plane

In this section we will study the relationship between the curvature of the skeletal set and the curvature of its boundary. Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^2$ and let $x_0 \in \mathbb{S}$ be a smooth point such that the compatibility 1-form vanishes identically on a neighbourhood of x_0. Now let γ be the smooth stratum containing x_0 parametrized by the arc-length s. Define the following functions

$$\rho_1 = U_1 \cdot N \qquad and \qquad \rho_2 = U_1 \cdot T,$$

where N and T are the unit normal and the unit tangent such that U_1 and N oriented in the same direction. The smooth choice of the radial vector field is: $U_1 = -r^{'}T + \sqrt{1 - r^{'2}}N$ and in this case $\rho_1 = \sqrt{1 - r^{'2}}$ and $\rho_2 = -r^{'}$. Recall that at a smooth point x_0 $U_1 \neq T$ and if the radius function has no singularity at x_0, then the possible positions of U_1 are illustrated in figure 5.1. If the radius has a singularity at x_0, then $U_1 = N$.

Remark 5.2.1 *Let ρ_1 and ρ_2 defined as above, then*

1. $\rho_1^2 + \rho_2^2 = 1$;
2. *if the radius function has no singularity at x_0 then ρ_1 has a singularity at x_0 if and only if ρ_2 has a singularity at x_0.*

Figure 5.1: The possible positions of the radial vector field when the radius function has no singularity.

Lemma 5.2.2 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^2$ such that for a choice of smooth value of U, the associated compatibility 1-form vanishes identically on a neighbourhood of a smooth point $x_0 \in \mathbb{S}$. If the function ρ_1 has a singularity at x_0 and the radius function has no singularity at x_0, then*

$$\kappa_r = \frac{1}{\rho_1}\kappa_m,$$

where κ_r (resp. κ_m) is the radial curvature of $\mathbb{S}$ at x_0 (resp. curvature of $\mathbb{S}$ at x_0).

Proof

Let $\gamma(s)$ be the smooth stratum containing x_0 parametrized by the arc-length s, then

$$\frac{\partial U_1}{\partial s} = aU_1 - \kappa_r T.$$

Therefore,

$$\frac{\partial U_1}{\partial s} \cdot U_1 = 0 = a - \kappa_r \rho_2 \Rightarrow a = \kappa_r \rho_2.$$

Now the derivative of the function $\rho_1 = U_1 \cdot N$ with respect to s is given by

$$\frac{\partial \rho_1}{\partial s} = \frac{\partial U_1}{\partial s} \cdot N + \frac{\partial N}{\partial s} \cdot U_1$$

$$= a\rho_1 - \kappa_m \rho_2.$$

Now since $\frac{\partial \rho_1}{\partial s} = 0$ this implies $a\rho_1 = \kappa_m \rho_2$ or $a = \frac{\rho_2}{\rho_1}\kappa_m$. But $a = \rho_2 \kappa_r$ therefore, $\rho_2 \kappa_r = \frac{\rho_2}{\rho_1}\kappa_m$ which gives $\kappa_r = \frac{1}{\rho_1}\kappa_m$. □

The above lemma gives us a good tool to study the relationship between the curvature of the skeletal structure and the curvature of its boundary. This relation is given in the following proposition.

Proposition 5.2.3 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^2$ such that for a choice of smooth value of U, the associated compatibility 1-form vanishes identically on a neighbourhood of a smooth point $x_0 \in \mathbb{S}$ and $\kappa_r \neq \frac{1}{r}$. If the function ρ_1 has a singularity at x_0 and the radius function has no singularity, then*

$$\kappa = \frac{\kappa_m}{\rho_1 - r\kappa_m},$$

where κ_m (resp. κ) is the curvature of $\mathbb{S}$ at x_0 (resp. curvature of the associated boundary at $x_0^{'} = \Psi_1(x_0)$), where Ψ_1 is the radial map.

Proof

From lemma 5.2.2 we have $\kappa_r = \frac{1}{\rho_1}\kappa_m$ also from theorem 4.2.1 we have

$$\kappa = \frac{\kappa_r}{1 - r\kappa_r}.$$

Therefore, replacing κ_r by $\frac{1}{\rho_1}\kappa_m$ gives the result. □

The previous proposition tells us the relationship between κ_m and κ_r under specific conditions depend on the singularity of ρ_1 when the radius function has no singularity. The next result gives this relation in general without controlling it by any conditions regarding to the singularity of ρ_1.

Theorem 5.2.4 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^2$ such that for a choice of smooth value of U, the associated compatibility 1-form vanishes identically on a neighbourhood of a smooth point. Then*

$$\kappa_r = \frac{\rho_1\kappa_m - d\rho_2}{\rho_1^2}$$

or equivalently

$$\kappa_r = \frac{\sqrt{1-r^{'2}}\kappa_m + r^{''}}{1-r^{'2}},$$

where κ_m *(resp.* κ_r *) is the curvature of* $\mathbb{S}$ *at* x_0 *(resp. radial curvature of* $\mathbb{S}$ *at* x_0 *).*

Proof

Let $\gamma(s)$ be the smooth stratum containing x_0 parametrized by the arc-length, then we have

$$\rho_2 = T \cdot U_1. \tag{5.1}$$

Now differentiate equation (5.1) with respect to the arc-length we obtain

$$d\rho_2 = \rho_1\kappa_m + a\rho_2 - \kappa_r.$$

But from the proof of lemma 5.2.2 we have $a = \rho_2\kappa_r$, thus

$$d\rho_2 = \rho_1\kappa_m + a\rho_2 - \kappa_r = \rho_1\kappa_m + \rho_2^2\kappa_r - \kappa_r.$$

Therefore,

$$\kappa_r = \frac{d\rho_2 - \rho_1\kappa_m}{\rho_2^2 - 1} = \frac{\rho_1\kappa_m - d\rho_2}{\rho_1^2}$$

or equivalently

$$\kappa_r = \frac{\sqrt{1-r^{'2}}\kappa_m + r^{''}}{1-r^{'2}}.$$

Therefore the proof is completed. □

Corollary 5.2.5 *Assume as in theorem 5.2.4. If the radius function has a singularity, then*

$$\kappa_r = \kappa_m - d\rho_2 = \kappa_m + r^{''}.$$

Proof

From theorem 5.2.4 we have

$$\kappa_r = \frac{\rho_1\kappa_m - d\rho_2}{\rho_1^2}$$

and since the radius function has a singularity then $\rho_1 = 1$ therefore, the above equation becomes

$$\kappa_r = \kappa_m - d\rho_2 = \kappa_m + r''$$

which completes the proof. □

Corollary 5.2.6 *Assume as in theorem 5.2.4. If the radius function has a singularity, then ρ_2 has a singularity if and only if $\kappa_r = \kappa_m$.*

Proof

The proof comes directly from corollary 5.2.5. □

Now we will turn to the relationship between the curvature of the skeletal structure and the curvature of its boundary. Theorem 5.2.4 and theorem 4.2.1 give enough information to discuss the requested relation in the following theorem.

Theorem 5.2.7 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^2$ such that for a choice of smooth value of U, the associated compatibility 1-form vanishes identically on a neighbourhood of a smooth point $x_0 \in \mathbb{S}$ and $\kappa_r \neq \frac{1}{r}$. Then the curvature κ of the boundary at $x_0' = \Psi_1(x_0)$ is given by*

$$\kappa = \frac{\rho_1 \kappa_m - d\rho_2}{\rho_1^2 + r d\rho_2 - r\rho_1 \kappa_m} \tag{5.2}$$

or equivalently

$$\kappa = \frac{\sqrt{1 - r'^2}\kappa_m + r''}{1 - r'^2 - rr'' - r\sqrt{1 - r'^2}\kappa_m}. \tag{5.3}$$

Proof

From theorem 4.2.1 we have

$$\kappa = \frac{\kappa_r}{1 - r\kappa_r} \tag{5.4}$$

and from theorem 5.2.4 we have

$$\kappa_r = \frac{\rho_1 \kappa_m - d\rho_2}{\rho_1^2}$$

or equivalently

$$\kappa_r = \frac{\sqrt{1 - {r'}^2}\kappa_m + r''}{1 - {r'}^2}.$$

Now substituting in equation 5.4 the result holds. □

It can be seen from equation 5.3 that the type of the singularity of the radius function plays a central role in the relation between κ and κ_m.
Recall that a function $f : \mathbb{R} \longrightarrow \mathbb{R}$ is said to have an A_k singularity at t_0 if

$$f'(t_0) = f''(t_0) = ... = f^{(k)}(t_0) = 0, \; f^{(k+1)}(t_0) \neq 0.$$

Corollary 5.2.8 *Assume as in theorem 5.2.7.*

1. *If the radius function has an A_1 singularity, then*

$$\kappa = \frac{\kappa_m + r''}{1 - rr'' - r\kappa_m}. \tag{5.5}$$

2. *If the radius function has an A_2 singularity, then*

$$\kappa = \frac{\kappa_m}{1 - r\kappa_m}. \tag{5.6}$$

Proof

The proof of this corollary comes directly from equation 5.3. □

In the rest of this section we will discuss the relationship between the curvature of the Blum medial axis and the curvatures κ_1 and κ_2 of the boundary at tangency points, on another words if we know the curvatures of the boundary, could we find the curvature of the Blum medial axis? Also, we will investigate the relationship between κ_1 and κ_2.

Proposition 5.2.9 *Let* $(\mathbb{S}, U)$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^2$ *with smooth boundary* X. *Let* $x_0 \in \mathbb{S}$ *be a smooth point.*

1. *The radial curvatures* κ_{r1} *and* κ_{r2} *are given by*

$$\kappa_{r1} = \frac{\sqrt{1-r'^2}\kappa_m + r''}{1-r'^2}, \tag{5.7}$$

$$\kappa_{r2} = \frac{-\sqrt{1-r'^2}\kappa_m + r''}{1-r'^2}. \tag{5.8}$$

2. *The curvatures* κ_1 *and* κ_2 *of the boundary at the tangency points are given by*

$$\kappa_1 = \frac{\sqrt{1-r'^2}\kappa_m + r''}{1-r'^2 - rr'' - r\sqrt{1-r'^2}\kappa_m}, \tag{5.9}$$

$$\kappa_2 = \frac{-\sqrt{1-r'^2}\kappa_m + r''}{1-r'^2 - rr'' + r\sqrt{1-r'^2}\kappa_m}. \tag{5.10}$$

Proof

The proof of this proposition comes directly from theorems 5.2.4 and 5.2.7. □

Corollary 5.2.10 *Let* $(\mathbb{S}, U)$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^2$ *with smooth boundary* X. *Let* $x_0 \in \mathbb{S}$ *be a smooth point, then*

$$\frac{1}{2}(\kappa_{r1} + \kappa_{r2}) = \frac{r''}{1-r'^2}. \tag{5.11}$$

Proof

This result comes by adding equations 5.7 and 5.8. □

Proposition 5.2.11 *Let* $(\mathbb{S}, U)$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^2$ *with smooth boundary* X. *Let* $x_0 \in \mathbb{S}$ *be a smooth point, then the curvature of* $\mathbb{S}$ *at* x_0 *is given by*

$$\kappa_m = \frac{1}{2}\left(\frac{\kappa_1}{1+r\kappa_1} - \frac{\kappa_2}{1+r\kappa_2}\right)\sqrt{1-r'^2} \tag{5.12}$$

where κ_1 *and* κ_2 *are the curvatures of the boundary at tangency points associated to* x_0.

Proof

From equations 5.7 and 5.8 we have

$$\kappa_{r1} = \frac{\sqrt{1-{r'}^2}\kappa_m + r''}{1-{r'}^2}$$

and

$$\kappa_{r2} = \frac{-\sqrt{1-{r'}^2}\kappa_m + r''}{1-{r'}^2}.$$

Therefore,

$$\kappa_{r1} - \kappa_{r2} = 2\frac{\sqrt{1-{r'}^2}\kappa_m}{1-{r'}^2}.$$

Thus

$$\kappa_m = \frac{1}{2}(\kappa_{r1} - \kappa_{r2})\sqrt{1-{r'}^2}.$$

But we have

$$\kappa_{ri} = \frac{\kappa_i}{1+r\kappa_i}, i = 1, 2$$

and by substituting in the above equation the proof is completed. □

Corollary 5.2.12 *Assume as in proposition 5.2.11. If the radius function has a singularity, then*

$$\kappa_m = \frac{1}{2}\left(\frac{\kappa_1}{1+r\kappa_1} - \frac{\kappa_2}{1+r\kappa_2}\right).$$

Proof

The proof is obvious. □

Now we are in the position to study the relationship between the curvatures of the boundary. This relation is given in the following theorem.

Theorem 5.2.13 *Let* $(\mathbb{S}, U)$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^2$ *with smooth boundary* X*. Let* $x_0 \in \mathbb{S}$ *be a smooth point, then the curvatures of* *the boundary at the tangency points associated to x_0 are related by the following equation*

$$\kappa_1 = \frac{2r'' - \kappa_2(1 - r'^2 - 2rr'')}{(1 - r'^2 - 2rr'') + 2r\kappa_2(1 - r'^2 - rr'')}. \tag{5.13}$$

Proof

From corollary 5.2.10 we have

$$\kappa_{r1} + \kappa_{r2} = \frac{2r''}{1 - r'^2}$$

or

$$\frac{\kappa_1}{1 + r\kappa_2} + \frac{\kappa_2}{1 + r\kappa_2} = \frac{2r''}{1 - r'^2}.$$

Thus

$$\frac{\kappa_1 + 2r\kappa_1\kappa_2 + \kappa_2}{1 + r\kappa_1 + r\kappa_2 + r^2\kappa_1\kappa_2} = \frac{2r''}{1 - r'^2}.$$

Therefore,

$$\kappa_1(1 + 2r\kappa_2)(1 - r'^2) + \kappa_2(1 - r'^2) = 2r'' + 2rr''\kappa_1(1 + r\kappa_2) + 2rr''\kappa_2.$$

Thus

$$\kappa_1(1 + 2r\kappa_2 - r'^2 - 2rr'^2\kappa_2 - 2rr'' - 2r^2r''\kappa_2) = 2r'' + \kappa_2(2rr'' - 1 + r'^2).$$

Hence

$$\kappa_1[(1 - r'^2 - 2rr'') + 2r\kappa_2(1 - r'^2 - rr'')] = 2r'' - \kappa_2(1 - r'^2 - 2rr'').$$

Therefore,

$$\kappa_1 = \frac{2r'' - \kappa_2(1 - r'^2 - 2rr'')}{(1 - r'^2 - 2rr'') + 2r\kappa_2(1 - r'^2 - rr'')}$$

which completes the proof. □

Corollary 5.2.14 *Assume as in theorem 5.2.13. If $r'' = 0$, then*

$$\kappa_1 = \frac{-\kappa_2}{1 + r\kappa_2}.$$

Proof

The proof comes directly from equation 5.13. □

5.3 Shape Operator of the Blum Medial Axis and those of its Boundary at Tangency Points

In this section we will turn to higher dimensions; in particular we will investigate the Hessian operator in terms of the radial shape operators of the Blum medial axis and then we are able to find the the expression of the Hessian operator in terms of the differential geometric shape operators of the boundary at the tangency points. Moreover, we are going to find out the relationship between the shape operator of the Blum medial axis and the the differential geometric shape operators of the boundary at the tangency points. Recall that for each smooth point $x_0 \in \mathbb{S}$ of skeletal structure we have two values of the radial vector field U which are on opposite sides of $T_{x_0}\mathbb{S}$. The values of U corresponding to one side form a smooth vector field. Also, for each side we have a radial shape operator.

Theorem 5.3.1 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X. Let $x_0 \in \mathbb{S}$ be a smooth point and $\{v_1, v_2, ..., v_n\}$ be a basis for the tangent space of $\mathbb{S}$ at x_0, then the radial Hessian operator is given by*

$$\mathbb{H}_r(V) = \frac{1}{2}(S_{V_1} + S_{V_2})^T (I - dr dr^T I_m^{-1})V, \tag{5.14}$$

where I_m is the matrix representing the first fundamental form of $\mathbb{S}$ at x_0 and S_{V_i}, $i = 1, 2$ are the matrices representing the radial shape operators, V is the matrix with i-th row entry v_i and dr is a column matrix with i-row entry $dr(v_i)$, where $dr(v_i)$ is the directional derivative of the radius function in the direction of v_i.

Proof

The radial Hessian operator is a map

$$\mathbb{H}_r : T_{x_0}\mathbb{S} \to T_{x_0}\mathbb{S}$$

such that

$$\mathbb{H}_r(v_i) = -\nabla_{v_i}(U_{1tan}) = -\text{proj}_N\left(\frac{\partial U_{1tan}}{\partial v_i}\right),$$

where ∇_{v_i} is the covariant derivative with respect to the basis of the tangent space of the Blum medial axis at a smooth point. But we have $U_{1tan} = -\nabla r = \frac{1}{2}(U_1 + U_2)$ and ∇r is the Riemannian gradient of the radius function. Therefore,

$$\mathbb{H}_r(v_i) = -\nabla_{v_i}(-\nabla r) = \nabla_{v_i}(\nabla r) = \text{proj}_N\left(\frac{-1}{2}\frac{\partial}{\partial v_i}(U_1 + U_2)\right).$$

Thus

$$\mathbb{H}_r(v_i) = \frac{-1}{2}\text{proj}_N\left(\frac{\partial U_1}{\partial v_i} + \frac{\partial U_2}{\partial v_i}\right).$$

Now using equation (1.2) we have

$$\left(\frac{\partial U_1}{\partial v_i} + \frac{\partial U_2}{\partial v_i}\right) = a1_i U_1 - \sum_{j=1}^{n} s1_{ji} v_j + a2_i U_2 - \sum_{j=1}^{n} s2_{ji} v_j.$$

Now we write this equation in vector notation to get

$$\frac{\partial U_1}{\partial V} + \frac{\partial U_2}{\partial V} = A_1 U_1 - S_{V1}^T V + A_2 U_2 - S_{V2}^T V,$$

but from the proof of lemma 3.3.7 we have $A_i = -S_{Vi}^T dr$, $i = 1, 2$. Therefore,

$$\frac{\partial U_1}{\partial V} + \frac{\partial U_2}{\partial V} = -(S_{V1} + S_{V2})^T V - S_{V1}^T dr U_1 - S_{V2}^T dr U_2.$$

But the possible choices for the radial vector fields are

$$U_1 = -\nabla r + \sqrt{1 - \|\nabla r\|^2} N \quad and \quad U_2 = -\nabla r - \sqrt{1 - \|\nabla r\|^2} N.$$

Thus

$$\frac{\partial U_1}{\partial V} + \frac{\partial U_2}{\partial V} = -(S_{V1} + S_{V2})^T V + (S_{V1} + S_{V2})^T dr \nabla r - \sqrt{1 - \|\nabla r\|^2}(S_{V1} - S_{V2})^T dr N.$$

Now the projection of this equation to the tangent space along the normal is given by

$$\text{proj}_N\left(\frac{\partial U_1}{\partial V}+\frac{\partial U_2}{\partial V}\right)=-(S_{V1}+S_{V2})^T(V-dr\nabla r), \tag{5.15}$$

but the Riemannian gradient of the radius function is given locally by $\nabla r = dr^T I_m^{-1} V$ where I_m is the first fundamental form of the Blum medial axis. Thus equation 5.15 becomes

$$\text{proj}_N\left(\frac{\partial U_1}{\partial V}+\frac{\partial U_2}{\partial V}\right)=-(S_{V1}+S_{V2})^T(I-drdr^T I_m^{-1})V.$$

Therefore,

$$\mathbb{H}_r(V)=\frac{1}{2}(S_{V_1}+S_{V_2})^T(I-drdr^T I_m^{-1})V.$$

Hence by this the proof is completed. □

Now let $\mathcal{H}_r$ be the matrix representing the radial Hessian operator $\mathbb{H}_r$. In the following corollary we express $\mathcal{H}_r$ in terms of S_{V_1} and S_{V_2}.

Corollary 5.3.2 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X. Let $x_0 \in \mathbb{S}$ be a smooth point, then the matrix $\mathcal{H}_r$ representing the radial Hessian operator is given by*

$$\mathcal{H}_r=\frac{1}{2}(S_{V_1}+S_{V_2})^T(I-drdr^T I_m^{-1}). \tag{5.16}$$

Proof

The proof of this corollary comes directly from equation 5.14. □

Proposition 5.3.3 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X. Let $x_0 \in \mathbb{S}$ be a smooth point. If the radius function has a singularity at x_0, then the matrix $\mathcal{H}_r$ representing the radial Hessian operator is given by*

$$\mathcal{H}_r=\frac{1}{2}(S_{V_1}+S_{V_2})^T. \tag{5.17}$$

Proof

Assume that the radius function has a singularity then equation 5.16 becomes

$$\mathcal{H}_r = \frac{1}{2}(S_{V_1} + S_{V_2})^T.$$

Hence the proof is completed. □

Now we define the *mean Hessian curvature* H^* by

$$H^* = \frac{1}{n} tr(\mathcal{H}_r).$$

Using this we have the following.

Corollary 5.3.4 *Assume as in proposition 5.3.3, then the mean Hessian curvature is given by*

$$H^* = \frac{1}{2}(H_{r1} + H_{r2}), \tag{5.18}$$

where H_{r1} and H_{r2} are the mean radial curvatures of the Blum medial axis.

Proof

If we take the trace for both sides of equation 5.17 we obtain

$$tr(\mathcal{H}_r) = \frac{1}{2}(tr(S_{V_1}) + tr(S_{V_2})).$$

Thus equation 5.18 is satisfied. □

Our task now is to find out the connection between the radial Hessian operator and the shape operators of the boundary at the tangency points.

Theorem 5.3.5 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X. Let $x_0 \in \mathbb{S}$ be a smooth point. Then the matrix $\mathcal{H}_r$ representing the radial Hessian operator is given by*

$$\mathcal{H}_r = \frac{1}{2}\{(I + rS_{XV_1'})^{-1}S_{XV_1'} + (I + rS_{XV_2''})^{-1}S_{XV_2''}\}^T (I - dr dr^T I_m^{-1}), \tag{5.19}$$

where $S_{XV_1'}$ and $S_{XV_2''}$ are the matrices representing the differential geometric shape operators of the boundary at the tangency points associated to x_0.

Proof

From corollary 5.3.2 we have

$$\mathcal{H}_r = \frac{1}{2}(S_{V_1} + S_{V_2})^T (I - drdr^T I_m^{-1})$$

and from proposition 4.2.4 we have

$$S_{V_1} = (I + rS_{XV_1'})^{-1} S_{XV_1'} \quad \textit{and} \quad S_{V_2} = (I + rS_{XV_2''})^{-1} S_{XV_2''}$$

and by substituting this in the above equation the proof is completed. □

Corollary 5.3.6 *Assume as in theorem 5.3.5. If the radius function has a singularity at x_0, then*

$$\mathcal{H}_r = \frac{1}{2}\{(I + rS_{XV_1'})^{-1} S_{XV_1'} + (I + rS_{XV_2''})^{-1} S_{XV_2''}\}^T.$$

Proof

The proof of this result comes directly from equation 5.19. □

Now we will give a special form for the matrix representing the radial Hessian operator in the case of a Blum medial axis in $\mathbb{R}^3$.

Theorem 5.3.7 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^3$ with smooth boundary X. Let $x_0 \in \mathbb{S}$ be a smooth point. Then the matrix $\mathcal{H}_r$ representing the radial Hessian operator is given by*

$$\begin{aligned}\mathcal{H}_r =& \frac{1}{2}\{\frac{1}{r^2K_1 + 2rH_1 + 1} S_{XV_1'} - \frac{1}{r^2K_2 + 2rH_2 + 1} S_{XV_2''} \\ &+ \left(\frac{rK_1}{r^2K_1 + 2rH_1 + 1} - \frac{rK_2}{r^2K_2 + 2rH_2 + 1}\right) I\}^T (I - drdr^T I_m^{-1}).\end{aligned}$$

Proof

From theorem 4.3.1 we have

$$S_{V1} = \frac{1}{r^2K_1 + 2rH_1 + 1} S_{XV_1'} - \frac{rK_1}{r^2K_1 + 2rH_1 + 1} I,$$

and

$$S_{V2} = \frac{1}{r^2K_2 + 2rH_2 + 1} S_{XV_2''} - \frac{rK_1}{r^2K_1 + 2rH_1 + 1} I.$$

Now substitute by these in equation 5.16 the result holds. □

Corollary 5.3.8 *Assume as in theorem 5.3.7. If the radius function has a singularity at x_0, then*

$$\begin{aligned}\mathcal{H}_r =& \frac{1}{2}\{\frac{1}{r^2K_1 + 2rH_1 + 1} S_{XV_1'} - \frac{1}{r^2K_2 + 2rH_2 + 1} S_{XV_2''} \\ &+ \left(\frac{rK_1}{r^2K_1 + 2rH_1 + 1} - \frac{rK_2}{r^2K_2 + 2rH_2 + 1}\right) I\}^T.\end{aligned}$$

Proof

The proof of this corollary comes directly from theorem 5.3.7. □

Example 5.3.9 *Let $(\mathbb{S}, U)$ be a Blum medial axis in $\mathbb{R}^3$ and let $S_1(x,y) = (x, y, \frac{1}{2}\kappa_{m1}x^2 + \frac{1}{2}\kappa_{m2}y^2 + h.o.t) \subset \mathbb{S}_{reg}$ and $r = r_0 + \frac{1}{2}ax^2 + \frac{1}{2}by^2$ be the radius function on S_1 such that $\frac{1}{r_0} \notin \{\kappa_{m1} + a, \kappa_{m2} + b\}$. Now we define the unit radial vector fields by*

$$U_1 = -\nabla r + \sqrt{1 - \|\nabla r\|^2} N \text{ and } U_2 = -\nabla r - \sqrt{1 - \|\nabla r\|^2} N,$$

where ∇r is the Riemannian gradient of the radius function and N is the unit normal of S_1. The radius function has a singularity at the origin and direct calculations show that at the origin we have

$$S_{V_1} = \begin{pmatrix} \kappa_{m1} + a & 0 \\ 0 & \kappa_{m2} + b \end{pmatrix}, \; S_{V_2} = \begin{pmatrix} a - \kappa_{m1} & 0 \\ 0 & b - \kappa_{m2} \end{pmatrix},$$

$$S_{XV_1'} = \begin{pmatrix} \dfrac{\kappa_{m1}+a}{1-r_0(\kappa_{m1}+a)} & 0 \\ 0 & \dfrac{\kappa_{m2}+b}{1-r_0(\kappa_{m2}+b)} \end{pmatrix},$$

$$S_{XV_2'} = \begin{pmatrix} \dfrac{a-\kappa_{m1}}{1-r_0(a-\kappa_{m1})} & 0 \\ 0 & \dfrac{b-\kappa_{m2}}{1-r_0(b-\kappa_{m2})} \end{pmatrix} \textit{ and } \mathcal{H}_r = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}.$$

Now it is easy to check that

$$\mathcal{H}_r = \frac{1}{2}(S_{V_1}+S_{V_2})^T = \frac{1}{2}\{(I+rS_{XV_1'})^{-1}S_{XV_1'} + (I+rS_{XV_2''})^{-1}S_{XV_2''}\}^T.$$

In the rest of this section we will focus on the relationship between the shape operator of the Blum medial axis and the shape operators of its boundary at tangency points corresponding to a smooth point on the medial axis. Now let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X. Assume that x_1 and x_2 are the tangency points associated to a smooth point $x_0 \in \mathbb{S}$ then we have only two choices for the smooth value of the radial vector field U these choices are $U_1 = -\nabla r + \sqrt{1-\|\nabla r\|^2}N$ and $U_2 = -\nabla r - \sqrt{1-\|\nabla r\|^2}N$ such that U_1 and N have the same direction and $x_1 = x_0 + rU_1$ and $x_2 = x_0 + rU_2$ and $\rho = \sqrt{1-\|\nabla r\|^2}$. Now it is clear that

$$U_1 = U_2 + 2\rho N. \tag{5.20}$$

Therefore, with this equation we have a good tool to investigate the relation mentioned above in particularly we have the following results.

Theorem 5.3.10 *Let* $(\ \mathbb{S}, U\ )$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^{n+1}$ *with smooth boundary* X*. Let* $x_0 \in \mathbb{S}$ *be a smooth point and* $\{v_1, v_2, ..., v_n\}$ *be a basis for the tangent space of* $\mathbb{S}$ *at* x_0*. Then the differential geometric shape operator* S_{med} *of* $\mathbb{S}$ *at* x_0 *is given by*

$$S_{med}(V) = \frac{1}{2\rho}(S_{V1} - S_{V2})^T (I - dr dr^T I_m^{-1})V. \tag{5.21}$$

Proof

From equation 5.20 we have

$$U_1 = U_2 + 2\rho N.$$

Let $\{v_1, v_2, ..., v_n\}$ be a basis for the tangent space of $\mathbb{S}$ at x_0, now differentiate both sides of the above equation with respect to v_i we get

$$\frac{\partial U_1}{\partial v_i} = \frac{\partial U_2}{\partial v_i} + 2\frac{\partial \rho}{\partial v_i} N + 2\rho \frac{\partial N}{\partial v_i}, \quad i = 1, 2, ..., n.$$

This equation can be written in vector forms as the following

$$\frac{\partial U_1}{\partial V} = \frac{\partial U_2}{\partial V} + 2d\rho(V)N + 2\rho \frac{\partial N}{\partial V}$$

or

$$A_1 U_1 - S_{V1}^T V = A_2 U_1 - S_{V2}^T V + 2d\rho N + 2\rho \frac{\partial N}{\partial V}$$

or

$$-S_{V1}^T (I - drdr^T I_m^{-1})V - \rho S_{V1}^T drN = -S_{V2}^T (I - drdr^T I_m^{-1})V + \rho S_{V2}^T drN + 2d\rho N + 2\rho \frac{\partial N}{\partial V}.$$

Now apply the projection to the tangent space along normal ($-\text{proj}_N$) we obtain the following

$$S_{med}(V) = \frac{1}{2\rho}(S_{V1} - S_{V2})^T (I - drdr^T I_m^{-1})V$$

which completes the proof. □

Corollary 5.3.11 *Let ($\mathbb{S}, U$) be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X. Let $x_0 \in \mathbb{S}$ be a smooth point. Then the matrix representing the shape operator of the Blum medial axis at x_0 is given by*

$$S_m^T = \frac{1}{2\rho}(S_{V1} - S_{V2})^T (I - drdr^T I_m^{-1}). \tag{5.22}$$

Proof

The proof of this corollary comes directly from equation 5.21. □

Corollary 5.3.12 *Assume as in corollary 5.3.11. If the radius function has a singularity at x_0, then*

$$S_m^T = \frac{1}{2}(S_{V1} - S_{V2})^T. \tag{5.23}$$

Proof

If the radius function has a singularity, then $\rho = 1$. Therefore, equation 5.22 becomes $S_m^T = \frac{1}{2}(S_{V1} - S_{V2})^T$ which completes the proof. □

Example 5.3.13 *Let $(\mathbb{S}, U)$ be a Blum medial axis in $\mathbb{R}^3$ and let $S_1(x, y) = (x, y, y^2 - x^2) \subset \mathbb{S}_{reg}$ and $r = 0.1 + xy + y^2$ be the radius function on S_1 such that $x^2 + 4xy + 5y^2 < 1$. At the origin the radius function has a singularity and we have*

$$S_m = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}, \; S_{V_1} = \begin{pmatrix} -2 & 1 \\ 1 & 4 \end{pmatrix} \text{ and } S_{V_2} = \begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}.$$

It is clear that $S_m^T = \frac{1}{2}(S_{V1} - S_{V2})^T$.

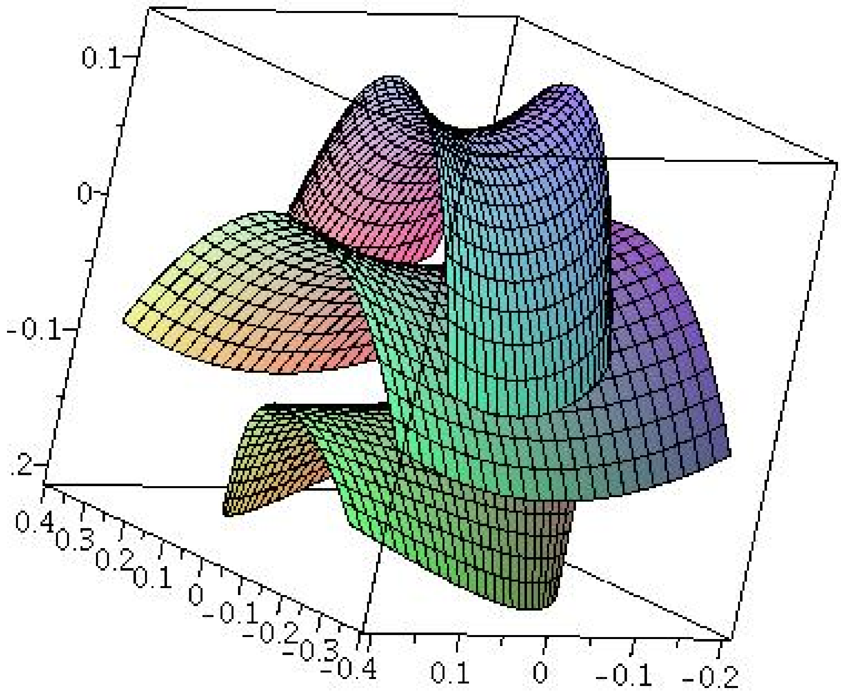

Figure 5.2: The Blum medial axis and associated boundary of example 5.3.13.

Now we will turn to one of the main aims of this chapter which is the relationship between the differential geometric shape operator of the Blum medial axis and the differential geometric shape operators of its boundary. In fact, corollary 5.3.11 gives us a good tool as well as proposition 4.2.4 to investigate this relationship which given in the following theorem.

Theorem 5.3.14 *Let* $(\mathbb{S}, U)$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^{n+1}$ *with smooth boundary* X*. Let* $x_0 \in \mathbb{S}$ *be a smooth point.*

1. *The matrix representation of the shape operator of the Blum medial axis at* x_0 *is*

given by

$$S_m^T = \frac{1}{2\rho}\{(I + rS_{XV_1'})^{-1}S_{XV_1'} - (I + rS_{XV_2''})^{-1}S_{XV_2''}\}^T(I - drdr^T I_m^{-1}). \quad (5.24)$$

2. *If the radius function has a singularity at x_0, then*

$$S_m^T = \frac{1}{2}\{(I + rS_{XV_1'})^{-1}S_{XV_1'} - (I + rS_{XV_2''})^{-1}S_{XV_2''}\}^T, \quad (5.25)$$

where $S_{XV_1'}$ and $S_{XV_2''}$ are the matrices representing the differential geometric shape operators of the boundary at the tangency points associated to x_0.

Proof

1. From proposition 4.2.4 we have

$$S_{V_1} = (I + rS_{XV_1'})^{-1}S_{XV_1'} \quad and \quad S_{V_2} = (I + rS_{XV_2''})^{-1}S_{XV_2''}.$$

Now substitute by this in equation 5.22 the result holds immediately.

2. The proof is obvious. □

Example 5.3.15 *Let $(\mathbb{S}, U)$ be a Blum medial axis in $\mathbb{R}^3$ and let $S_1(x, y) = (x, y, x^3 - y^2) \subset \mathbb{S}_{reg}$ and $r = 0.1 + y^2$ be the radius function on S_1 such that $4y^2 < 1$. At the origin the radius function has a singularity and we have*

$$S_m = \begin{pmatrix} 0 & 0 \\ 0 & -2 \end{pmatrix}, \; S_{XV_1'} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \; and \; S_{V_2} = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix} = (I + r_0 S_{XV_2''})^{-1}S_{XV_2''}.$$

Thus it is clear that $S_m^T = \frac{1}{2}\{(I + rS_{XV_1'})^{-1}S_{XV_1'} - (I + rS_{XV_2''})^{-1}S_{XV_2''}\}^T$.

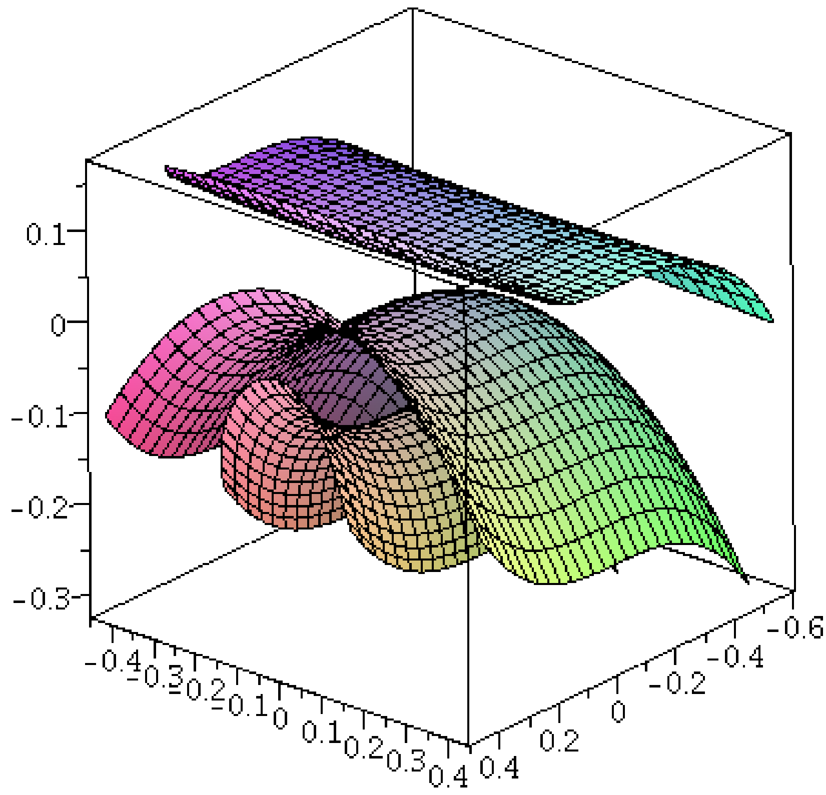

Figure 5.3: The Blum medial axis and associated boundary of example 5.3.15.

Now we will give a special form for the matrix representing the differential geometric shape operator in the case of a Blum medial axis in $\mathbb{R}^3$.

Theorem 5.3.16 *Let* $(\mathbb{S}, U)$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^3$ *with smooth boundary* X*. Let* $x_0 \in \mathbb{S}$ *be a smooth point. Then the matrix* S_m *representing the differential geometric shape operator is given*

$$\begin{aligned} S_m^T =& \frac{1}{2\rho}\{\frac{1}{r^2K_1 + 2rH_1 + 1} S_{XV_1'} - \frac{1}{r^2K_2 + 2rH_2 + 1} S_{XV_2''} \\ &+ \left(\frac{rK_1}{r^2K_1 + 2rH_1 + 1} - \frac{rK_2}{r^2K_2 + 2rH_2 + 1}\right) I\}^T (I - dr dr^T I_m^{-1}). \end{aligned}$$

Proof

The proof of this theorem comes directly by applying theorem 4.3.1 in corollary 5.3.11. □

In the following we will give the exact relationship between the matrix representing the radial shape operator of a skeletal structure and the matrix representing the differential geometric shape operator of the skeletal set and then we will give the exact relationship between the matrix representing the differential geometric shape operator of the skeletal set and the matrix representing the differential geometric shape operator of the boundary at a point associated to a smooth point of the skeletal set. Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$ such that the compatibility condition holds and let $x_0 \in \mathbb{S}_{reg}$ be a smooth point and $\{v_1, v_2, ..., v_n\}$ be a basis for the tangent space of $\mathbb{S}$ at x_0. Let r be the radius function, since x_0 is a smooth point and the compatibility condition holds then the unit radial vector field is given by [8]:

$$U_1 = -\nabla r + \sqrt{1 - \|\nabla r\|^2} N,$$

where ∇r is the Riemannian gradient of the radius function and N is the unit normal of the smooth stratum S containing x_0. We put $\rho = \sqrt{1 - \|\nabla r\|^2}$, so we have the following

$$U_1 = -\nabla r + \rho N. \tag{5.26}$$

From this equation we have the following equation

$$N = \frac{1}{\rho} U_1 + \frac{1}{\rho} \nabla r. \tag{5.27}$$

This equation is a useful tool to determine the coefficients of the unit radial vector field in $\frac{\partial U_1}{\partial v_i}$. From equation 5.26, we have

$$\frac{\partial U_1}{\partial v_i} = -\frac{\partial \nabla r}{\partial v_i} + \frac{\partial \rho}{\partial v_i} N + \rho \frac{\partial N}{\partial v_i}. \tag{5.28}$$

Now we put

$$\frac{\partial \nabla r}{\partial v_i} = \left(\frac{\partial \nabla r}{\partial v_i}\right)^T + \left(\frac{\partial \nabla r}{\partial v_i}\right)^N,$$

where $\left(\frac{\partial \nabla r}{\partial v_i}\right)^T$ (resp. $\left(\frac{\partial \nabla r}{\partial v_i}\right)^N$) is the tangential (resp. normal) part of $\frac{\partial \nabla r}{\partial v_i}$. Now since S is a hypersurface, then we put $\left(\frac{\partial \nabla r}{\partial v_i}\right)^N = \beta_i N$. Now using this together with equation 5.27, equation 5.28 becomes

$$\frac{\partial U_1}{\partial v_i} = -\left(\frac{\partial \nabla r}{\partial v_i}\right)^T + \frac{1}{\rho}\left(\frac{\partial \rho}{\partial v_i} - \beta_i\right) U_1 + \frac{1}{\rho}\left(\frac{\partial \rho}{\partial v_i} - \beta_i\right) \nabla r + \rho \frac{\partial N}{\partial v_i}. \tag{5.29}$$

Now writing this equation in vector form we get

$$\frac{\partial U_1}{\partial V} = \frac{1}{\rho}(d\rho - \beta)U_1 - (\mathcal{H}_r^T + \rho S_m^T - \frac{1}{\rho}(d\rho - \beta) dr^T I_m^{-1})V. \tag{5.30}$$

From equation 1.3 we have

$$\frac{\partial U_1}{\partial V} = A_V U_1 - S^T V. \tag{5.31}$$

Now since $\{U_1, v_1, v_2, ..., v_n\}$ is a basis for $\mathbb{R}^{n+1}$, then from equations 5.30 and 5.31 we obtain

$$A_V = \frac{1}{\rho}(d\rho - \beta) \quad and \quad S_V^T = \mathcal{H}_r^T + \rho S_m^T - \frac{1}{\rho}(d\rho - \beta) dr^T I_m^{-1}, \tag{5.32}$$

where $\mathcal{H}_r$ is the matrix representing the radial Hessian operator, S_m is the matrix representing the differential geometric shape operator of the skeletal set, $d\rho$ is a column matrix with i-th entry $\frac{\partial \rho}{\partial v_i}$, dr is a column matrix with i-th entry $\frac{\partial r}{\partial v_i}$, β is a column matrix with i-th entry β_i, I_m is the first fundamental form of the skeletal set and V is the matrix with i-th row v_i.

Our task now is to find the exact expression of the matrix β. In [7] James Damon pointed out that $A_V = S_V^T V \cdot U_1$ and since the compatibility condition holds, then $A_V = -S_V^T dr$. Thus from this and equation 5.32 we have $-S_V^T dr = \frac{1}{\rho}(d\rho - \beta)$. Thus

$$\beta = d\rho + \rho S_V^T dr. \tag{5.33}$$

Also, from corollary 3.3.9 we have $d\rho = -(\rho S_V^T - S_m^T) dr$. Hence we obtain that $\beta = S_m^T dr$. Therefore, equation 5.32 can be rewritten as the following

$$A_V = \frac{1}{\rho}(d\rho - S_m^T dr) \quad and \quad S_V^T = \mathcal{H}_r^T + \rho S_m^T - \frac{1}{\rho}(d\rho - S_m^T dr) dr^T I_m^{-1}. \tag{5.34}$$

Now we summarize the above discussion in the following theorem.

Theorem 5.3.17 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$ such the compatibility condition holds in a neighbourhood of smooth point $x_0 \in \mathbb{S}$, then*

$$A_V = \frac{1}{\rho}(d\rho - S_m^T dr) \tag{5.35}$$

and

$$S_V^T = \mathcal{H}_r^T + \rho S_m^T - \frac{1}{\rho} d\rho dr^T I_m^{-1} + \frac{1}{\rho} S_m^T dr dr^T I_m^{-1}. \tag{5.36}$$

Example 5.3.18 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$ suppose the image of $s_1(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n, 1) \subset \mathbb{S}_{reg}$ such that $\{(x_1, x_2, ..., x_n) \in \mathbb{R}^n | \sum_{i=1}^{n} x_i^2 < \frac{1}{4}\}$ and let $r(x_1, x_2, ..., x_n) = 1 + \sum_{i=1}^{n} x_i^2$ be the radius function. Now the radial vector field is given by $U_1 = -\nabla r + \rho N$, where ∇r is the Riemannian gradient, $\rho = \sqrt{1 - 4\sum_{i=1}^{n} x_i^2}$ and N is the unit normal of s_1. Now we will apply theorem 5.3.17 to calculate S_V and A_V. Since s_1 is a hyperplane, then $S_m = 0$. Also, $\nabla r = (2x_1, 2x_2, ..., 2x_n, 0)$ and $\mathcal{H}_r = 2I$, where I in the $(n \times n)-$identity matrix. From theorem 5.3.17 we have*

$$S_V^T = \mathcal{H}_r^T - \frac{1}{\rho} d\rho dr^T \text{ and } A_V = \frac{1}{\rho} d\rho.$$

For each $j \in \{1, 2, ..., n\}$ we have $\frac{\partial \rho}{\partial x_j} = \frac{-4x_j}{\sqrt{1 - 4\sum_{i=1}^{n} x_i^2}}$. Thus

$$\begin{pmatrix} \frac{\partial \rho}{\partial x_1} \\ \frac{\partial \rho}{\partial x_2} \\ \vdots \\ \frac{\partial \rho}{\partial x_n} \end{pmatrix} = \frac{-4}{\sqrt{1 - 4\sum_{i=1}^{n} x_i^2}} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Therefore,

$$A_V = \frac{-4}{1 - 4\sum_{i=1}^{n} x_i^2} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Also, after simplifying the matrix $\mathcal{H}_r^T - \frac{1}{\rho} d\rho dr^T$ *we obtain that*

$$S_V = \frac{2}{1-4\sum_{i=1}^{n} x_i^2} \begin{pmatrix} 1+4x_1^2-4\sum_{i=1}^{n} x_i^2 & 4x_1x_2 & 4x_1x_3 & \cdots & 4x_1x_n \\ 4x_1x_2 & 1+4x_2^2-4\sum_{i=1}^{n} x_i^2 & 4x_2x_3 & \cdots & 4x_2x_n \\ 4x_1x_3 & 4x_2x_3 & 1+4x_3^2-4\sum_{i=1}^{n} x_i^2 & \cdots & 4x_3x_n \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 4x_1x_{n-1} & 4x_2x_{n-1} & 4x_3x_{n-1} & \cdots & 4x_{n-1}x_n \\ 4x_1x_n & 4x_2x_n & 4x_3x_n & \cdots & 1+4x_n^2-4\sum_{i=1}^{n} x_i^2 \end{pmatrix}.$$

It is clear that at $(x_1, x_2, ..., x_n) = (0, 0, ..., 0)$ *we have* $S_V = 2I$.

In theorem 5.3.17 we expressed the matrix S_V representing the radial shape operator of a skeletal structure in terms of $\mathcal{H}_r$, S_m, $d\rho$ and dr. Our task now is to assume that if $x_0 \in \mathbb{S}$ is a smooth point then the smooth sheet of the skeletal set $\mathbb{S}$ say S_1 containing x_0 is in Monge form i.e., S_1 can be parametrized locally by $(x_1, x_2, ..., x_n)$ such that S_1 is given by the graph $S_1(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n, \frac{1}{2}\sum_{i=1}^{n} \kappa_{mi} x_i^2 + h.o.t)$. Now let r be the radius function on S_1 i.e., it is a smooth function of $(x_1, x_2, ..., x_n)$. In the following we will calculate S_V at the origin and to do so we just calculate $\mathcal{H}_r$, S_m, $d\rho$ at the origin and substitute in theorem 5.3.17. It is clear that at the origin we have $S_m = diag[\kappa_{mi}]$. To calculate $\mathcal{H}_r$ we will use the definition of Riemannian gradient which is given by $\nabla r = g^{ij}\partial_j r v_i$, where g^{ij} is the inverse of the matrix representing the Riemannian metric, ∂_j is the partial derivative of the radius function and v_i is a basis of the tangent space. Since S_1 is a smooth hypersurface in $\mathbb{R}^{n+1}$, we assume that the matrix I_m representing the first fundamental form is the matrix g_{ij} representing the Riemannian metric i.e., $I_m = g_{ij}$.

Now we have

$$g^{ij}g_{ij} = g^{ij}I_m \tag{5.37}$$

and

$$\frac{\partial \nabla r}{\partial x_l} = \frac{\partial g^{ij}}{\partial x_l}\partial_j r v_i + g^{ij}\frac{\partial}{\partial x_l}(\partial_j r)v_i + g^{ij}\partial_j r\frac{\partial v_i}{\partial x_l}. \tag{5.38}$$

Now we calculate this at the origin and first of all we calculate the first fundamental form I_m in general. It is clear that

$$v_1 = \frac{\partial S_1}{\partial x_1} = (1, 0, 0, ..., 0, \kappa_{m1}x_1 + h.o.t),$$

$$v_2 = \frac{\partial S_1}{\partial x_1} = (0, 1, 0, ..., 0, \kappa_{m2}x_2 + h.o.t),$$

$$\vdots$$

$$v_n = \frac{\partial S_1}{\partial x_1} = (0, 0, 0, ..., 1, \kappa_{mn}x_n + h.o.t).$$

Thus

$$g_{ij} = \begin{pmatrix} 1+\kappa_{m1}^2x_1^2 + h.o.t. & \kappa_{m1}\kappa_{m2}x_1x_2 + h.o.t. & \cdots & \kappa_{m1}\kappa_{mn}x_1x_n + h.o.t. \\ \kappa_{m1}\kappa_{m2}x_1x_2 + h.o.t. & 1+\kappa_{m2}^2x_2^2 + h.o.t. & \cdots & \kappa_{m2}\kappa_{mn}x_2x_n + h.o.t. \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{m1}\kappa_{mn}x_1x_n + h.o.t. & \kappa_{m2}\kappa_{mn}x_2x_n + h.o.t. & \cdots & 1+\kappa_{mn}^2x_n^2 + h.o.t. \end{pmatrix}.$$

At the origin $g_{ij} = I$ and hence $g^{ij} = I$. Also, $\partial g_{ij} = 0$ at the origin. In general

$$\partial(g^{ij}g_{ij}) = (\partial g^{ij})g_{ij} + g^{ij}(\partial g_{ij}) = \partial I = 0.$$

Now since $g_{ij} = I$ at the origin thus $\partial g^{ij} = 0$ at the origin. Therefore,

$$\frac{\partial \nabla r}{\partial x_l} = \frac{\partial}{\partial x_l}(\partial_i r)v_i + \partial_i r\frac{\partial v_i}{\partial x_l}. \tag{5.39}$$

At the origin is easy to check that

$$\frac{\partial v_i}{\partial x_l} = \begin{cases} \kappa_{mi}N & if \quad i = l \\ 0 & if \quad i \neq l, \end{cases}$$

where N is the unit normal of S_1. Thus

$$\mathcal{H}_r = \begin{pmatrix} r_{x_1x_1} & r_{x_1x_2} & \cdots & r_{x_1x_n} \\ r_{x_2x_1} & r_{x_2x_2} & \cdots & r_{x_2x_n} \\ \vdots & \vdots & \ddots & \vdots \\ r_{x_nx_1} & r_{x_nx_2} & \cdots & r_{x_nx_n} \end{pmatrix},$$

where $r_{x_ix_j} = \dfrac{\partial}{\partial x_i}\left(\dfrac{\partial r}{\partial x_j}\right)$. This matrix is given by $[a_{kl}] = [r_{x_kx_l}]$. Now we will calculate $d\rho$ and to do so we put

$$\rho^2 = 1 - \nabla r \cdot \nabla r.$$

Thus

$$\rho\frac{\partial \rho}{\partial x_l} = -\frac{\partial \nabla r}{\partial x_l} \cdot \nabla r.$$

At the origin $\dfrac{\partial v_i}{\partial x_l} \cdot \nabla r = 0$ for all $l = 1, 2, ..., n$. Therefore,

$$\rho\frac{\partial \rho}{\partial x_l} = -\frac{\partial}{\partial x_l}(\partial_i r)v_i \cdot \nabla r.$$

Thus

$$\rho d\rho = -\mathcal{H}_r \begin{pmatrix} v_1 \cdot \nabla r \\ v_2 \cdot \nabla r \\ \vdots \\ v_n \cdot \nabla r \end{pmatrix}.$$

Also, at the origin we have $\nabla r = (r_{x_1}, r_{x_2}, ..., r_{x_n}, 0)$, $v_i \cdot \nabla r = r_{x_i}$ and $\rho = \sqrt{1 - \sum\limits_{i=1}^{n} r_{x_i}^2}$. Hence $d\rho = -\frac{1}{\rho}\mathcal{H}_r dr$ and after simplification we get

$$d\rho = \frac{-1}{\sqrt{1 - \sum\limits_{i=1}^{n} r_{x_i}^2}} \begin{pmatrix} \sum\limits_{i=1}^{n} r_{x_i} r_{x_1x_i} \\ \sum\limits_{i=1}^{n} r_{x_i} r_{x_2x_i} \\ \vdots \\ \sum\limits_{i=1}^{n} r_{x_i} r_{x_nx_i} \end{pmatrix}.$$

Now we will calculate A_V which is given by $A_V = \frac{1}{\rho}(d\rho - S_m^T dr)$ after direct calculation and simplification we get

$$A_V = \frac{-1}{1-\sum\limits_{i=1}^{n} r_{x_i}^2}\begin{pmatrix} \sum\limits_{i=1}^{n} r_{x_i}r_{x_1x_i} + \kappa_{m1}r_{x_1}\sqrt{1-\sum\limits_{i=1}^{n} r_{x_i}^2} \\ \sum\limits_{i=1}^{n} r_{x_i}r_{x_2x_i} + \kappa_{m2}r_{x_2}\sqrt{1-\sum\limits_{i=1}^{n} r_{x_i}^2} \\ \vdots \\ \sum\limits_{i=1}^{n} r_{x_i}r_{x_nx_i} + \kappa_{mn}r_{x_n}\sqrt{1-\sum\limits_{i=1}^{n} r_{x_i}^2} \end{pmatrix}.$$

Thus A_V can be written as $[b_{k1}] = \frac{-1}{1-\sum\limits_{i=1}^{n} r_{x_i}^2}\left[\sum\limits_{i=1}^{n} r_{x_i}r_{x_kx_i} + \kappa_{mk}r_{x_k}\sqrt{1-\sum\limits_{i=1}^{n} r_{x_i}^2}\right]$. Now after some calculations we get

$$\frac{1}{\rho}d\rho dr^T = \frac{-1}{1-\sum\limits_{i=1}^{n} r_{x_i}^2}\begin{pmatrix} r_{x_1}\sum\limits_{i=1}^{n} r_{x_i}r_{x_1x_i} & r_{x_2}\sum\limits_{i=1}^{n} r_{x_i}r_{x_1x_i} & \cdots & r_{x_n}\sum\limits_{i=1}^{n} r_{x_i}r_{x_1x_i} \\ r_{x_1}\sum\limits_{i=1}^{n} r_{x_i}r_{x_2x_i} & r_{x_2}\sum\limits_{i=1}^{n} r_{x_i}r_{x_2x_i} & \cdots & r_{x_n}\sum\limits_{i=1}^{n} r_{x_i}r_{x_2x_i} \\ \vdots & \vdots & \ddots & \vdots \\ r_{x_1}\sum\limits_{i=1}^{n} r_{x_i}r_{x_nx_i} & r_{x_2}\sum\limits_{i=1}^{n} r_{x_i}r_{x_nx_i} & \cdots & r_{x_n}\sum\limits_{i=1}^{n} r_{x_i}r_{x_nx_i} \end{pmatrix}.$$

This matrix can be given by

$$[c_{kl}] = \left[\frac{-r_{x_l}\sum\limits_{i=1}^{n} r_{x_i}r_{x_kx_i}}{1-\sum\limits_{i=1}^{n} r_{x_i}^2}\right].$$

Also,

$$\frac{1}{\rho}S_m^T dr dr^T = \frac{1}{\sqrt{1-\sum\limits_{i=1}^{n} r_{x_i}^2}}\begin{pmatrix} \kappa_{m1}r_{x_1}^2 & \kappa_{m1}r_{x_1}r_{x_2} & \cdots & \kappa_{m1}r_{x_1}r_{x_n} \\ \kappa_{m2}r_{x_1}r_{x_2} & \kappa_{m2}r_{x_2}^2 & \cdots & \kappa_{m2}r_{x_2}r_{x_n} \\ \vdots & \vdots & \ddots & \vdots \\ \kappa_{mn}r_{x_1}r_{x_n} & \kappa_{mn}r_{x_2}r_{x_n} & \cdots & \kappa_{mn}r_{x_n}^2 \end{pmatrix}.$$

This matrix can be written as

$$[d_{kl}] = \left[\frac{\kappa_{mk}r_{x_k}r_{x_l}}{\sqrt{1-\sum\limits_{i=1}^{n} r_{x_i}^2}}\right].$$

Now let $A = \mathcal{H}_r - \frac{1}{\rho} d\rho dr^T + \frac{1}{\rho} S_m^T dr dr^T$. Thus A is given by

$$[\alpha_{kl}] = \left[\frac{r_{x_k x_l}(1 - \sum_{i=1}^{n} r_{x_i}^2) + r_{x_l} \sum_{i=1}^{n} r_{x_i} r_{x_k x_i} + \kappa_{mk} r_{x_k} r_{x_l} \sqrt{1 - \sum_{i=1}^{n} r_{x_i}^2}}{1 - \sum_{i=1}^{n} r_{x_i}^2} \right].$$

Now we summarize the above discussion in the following.

Proposition 5.3.19 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$ such that the compatibility condition holds in a neighbourhood of a smooth point $x_0 \in \mathbb{S}$. Assume that S_1 be the smooth sheet of $\mathbb{S}$ containing x_0 as the origin and S_1 is given in Monge form i.e., S_1 is given by $S_1(x_1, x_2, ..., x_n) = (x_1, x_2, ..., x_n, \frac{1}{2} \sum_{i=1}^{n} \kappa_{mi} x_i^2 + h.o.t)$, then at x_0 A_V is given by the matrix*

$$[b_{k1}] = \frac{-1}{1 - \sum_{i=1}^{n} r_{x_i}^2} \left[\sum_{i=1}^{n} r_{x_i} r_{x_k x_i} + \kappa_{mk} r_{x_k} \sqrt{1 - \sum_{i=1}^{n} r_{x_i}^2} \right]$$

and

$$S_V^T = A + S_m^*,$$

where A is given by

$$[\alpha_{kl}] = \left[\frac{r_{x_k x_l}(1 - \sum_{i=1}^{n} r_{x_i}^2) + r_{x_l} \sum_{i=1}^{n} r_{x_i} r_{x_k x_i} + \kappa_{mk} r_{x_k} r_{x_l} \sqrt{1 - \sum_{i=1}^{n} r_{x_i}^2}}{1 - \sum_{i=1}^{n} r_{x_i}^2} \right],$$

and $S_m^* = diag \left[\kappa_{mk} \sqrt{1 - \sum_{i=1}^{n} r_{x_i}^2} \right]$.

Corollary 5.3.20 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^3$ suppose the image of $S_1(x, y) = (x, y, \frac{1}{2}\kappa_{m1} x^2 + \frac{1}{2}\kappa_{m2} y^2 + h.o.t) \subset \mathbb{S}_{reg}$ and Let r be the radius function. Then*

1. *S_V and A_V are given by*

$$A_V = \frac{-1}{1 - r_x^2 - r_y^2} \begin{pmatrix} r_x r_{xx} + r_y r_{xy} + \kappa_{m1} r_x \sqrt{1 - r_x^2 - r_y^2} \\ r_x r_{xy} + r_y r_{yy} + \kappa_{m2} r_y \sqrt{1 - r_x^2 - r_y^2} \end{pmatrix}$$

and $S_V^T = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *where,*

$$a = \frac{r_{xx}(1-r_y^2) + \kappa_{m1}(1-r_x^2-r_y^2)^{\frac{3}{2}} + r_x r_y r_{xy} + \kappa_{m1} r_x^2\sqrt{1-r_x^2-r_y^2}}{1-r_x^2-r_y^2},$$

$$b = \frac{r_{xy}(1-r_x^2) + r_x r_y r_{xx} + \kappa_{m1} r_x r_y\sqrt{1-r_x^2-r_y^2}}{1-r_x^2-r_y^2},$$

$$c = \frac{r_{xy}(1-r_y^2) + r_x r_y r_{yy} + \kappa_{m2} r_x r_y\sqrt{1-r_x^2-r_y^2}}{1-r_x^2-r_y^2}$$

and

$$d = \frac{r_{yy}(1-r_x^2) + \kappa_{m2}(1-r_x^2-r_y^2)^{\frac{3}{2}} + r_x r_y r_{xy} + \kappa_{m2} r_y^2\sqrt{1-r_x^2-r_y^2}}{1-r_x^2-r_y^2}.$$

2. *If the radius function has a singularity at the origin then* $A_V = 0$ *and*

$$S_V^T = \begin{pmatrix} r_{xx} + \kappa_{m1} & r_{xy} \\ r_{xy} & r_{yy} + \kappa_{m2} \end{pmatrix}.$$

Proof

The proof of this corollary comes directly from proposition 5.3.19 just by putting $n = 2$.

□

Example 5.3.21 *Let* $(\mathbb{S}, U)$ *be a skeletal structure in* $\mathbb{R}^3$ *and let* $s_1(x,y) = (x, y, x^2) \subset \mathbb{S}_{reg}$ *such that* $\{(x,y) \in \mathbb{R}^2 | -0.45 < x < 0.45, -0.45 < y < 0.45\}$. *We define the positive function* r *on* s_1 *by* $r(x,y) = \frac{1}{2}x + 1$*, and we define the unit vector field* U_1 *on* s_1 *by:*

$$U_1 = -\nabla r + \sqrt{1 - \|\nabla r\|^2} N,$$

where ∇r *is the Riemannian gradient of* r *and* N *is the unit normal of* s_1*. Thus after some calculations we obtain that*

$$U_1 = \left(\frac{-1 - 2x\sqrt{3+16x^2}}{2(1+4x^2)}, 0, \frac{-2x + \sqrt{3+16x^2}}{2(1+4x^2)} \right).$$

It is clear that the compatibility condition holds and the associated boundary is given by

$$X(x,y) = \left(x - \frac{(x+2)(1+2x\sqrt{3+16x^2})}{4(1+4x^2)}, y, x^2 + \frac{(x+2)(-2x+\sqrt{3+16x^2})}{4(1+4x^2)} \right).$$

After some calculations we get

$$S_V = \begin{pmatrix} \dfrac{8x(1+4x^2) - 4(1+4x^2\sqrt{1+4x^2})\sqrt{3+16x^2}}{(1+4x^2)^2(3+16x^2)} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$A_V = \begin{pmatrix} \dfrac{-4x(1+4x^2) + 2(1+4x^2\sqrt{1+4x^2})\sqrt{3+16x^2}}{(1+4x^2)^2(3+16x^2)} \\ 0 \end{pmatrix}.$$

The radial principal curvatures are $\kappa_{r1} = \dfrac{8x(1+4x^2) - 4(1+4x^2\sqrt{1+4x^2})\sqrt{3+16x^2}}{(1+4x^2)^2(3+16x^2)}$ *and* $\kappa_{r2} = 0$*. Thus the Gaussian radial curvature* $K_r = 0$*.*

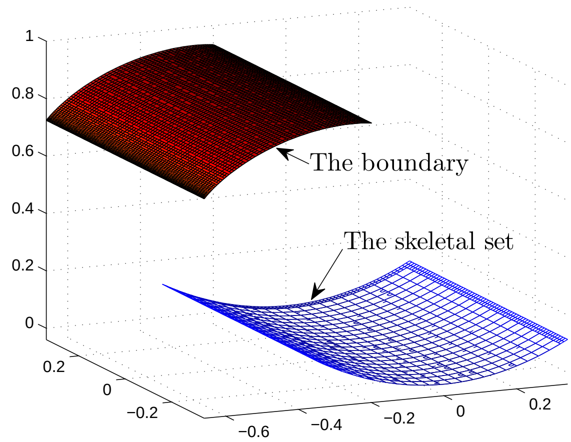

Figure 5.4: Skeletal set and associated boundary in example 5.3.21.

Example 5.3.22 *Let* $(\mathbb{S}, U)$ *be a skeletal structure in* $\mathbb{R}^3$ *and let* $s_1(x,y) = (x, y, y^3 - x^2) \subset \mathbb{S}_{reg}$ *and* $r = 0.1 + y^2$ *be the radius function such that* $4y^2 < 1$*. At the origin the* *radius function has a singularity and*

$$A_V = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \textit{ and } S_V = \begin{pmatrix} -2 & 0 \\ 0 & 2 \end{pmatrix}.$$

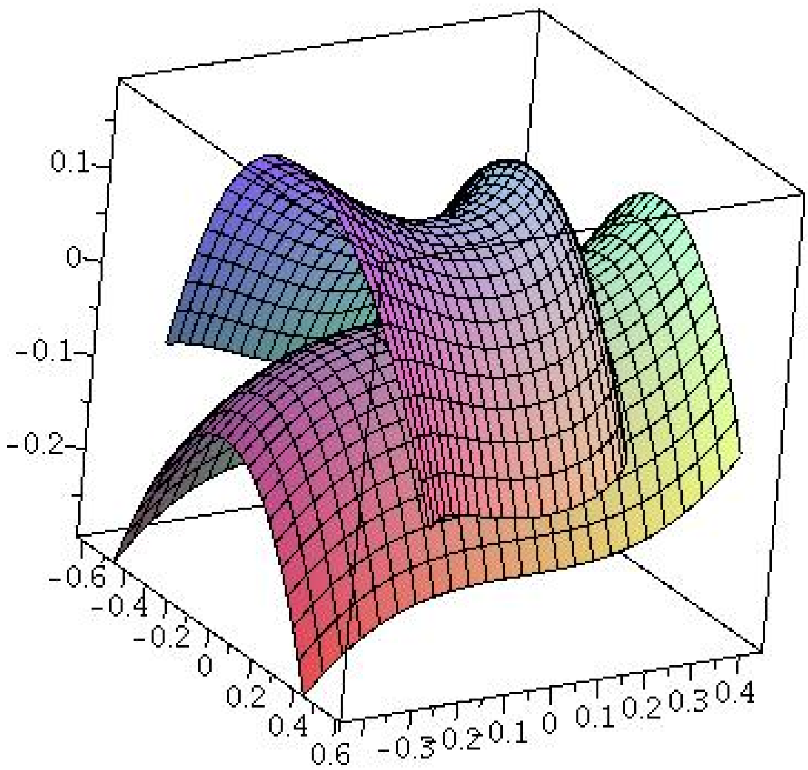

Figure 5.5: Skeletal set and associated boundary in example 5.3.22.

Theorem 5.3.17 gives the relationship between the matrix S_V representing the radial shape operator and the matrix S_m representing the differential geometric shape operator of the skeletal structure. In proposition 4.2.4 we express the the matrix S_V in terms of the matrix $S_{XV'}$ representing the differential geometric shape operator of the boundary. Thus now we are able to find the exact relationship between S_m and $S_{XV'}$, this relation is given in the following theorem.

Theorem 5.3.23 *Let $(\mathbb{S}, U)$ be a skeletal structure such that for a choice of smooth value of U the associated compatibility 1-form η_U vanishes identically on a neighbourhood of a smooth point x_0 of $\mathbb{S}$, and $\frac{1}{r}$ is not an eigenvalue of the radial shape operator at x_0. Let $x_{0'} = \Psi_1(x_0)$ and V' be the image of V for a basis $\{v_1, v_2, ..., v_n\}$. Then the matrix $S_{XV'}$ representing the differential geometric shape operator of the boundary is given by*

$$S_{XV'}^T = \frac{1}{r}\{[I - r(\mathcal{H}_r^T + \rho S_m^T - \frac{1}{\rho}d\rho dr^T I_m^{-1} + \frac{1}{\rho}S_m^T dr dr^T I_m^{-1})]^{-1} - I\}. \tag{5.40}$$

Proof

From proposition 4.2.4 we have

$$S_V^T = S_{XV'}^T (I + rS_{XV'}^T)^{-1}$$

and from lemma 4.2.3 we have

$$S_V^T = S_{XV'}^T (I + rS_{XV'}^T)^{-1} = \frac{1}{r}[I - (I + rS_{XV'}^T)^{-1}].$$

Thus equation 5.36 becomes

$$\frac{1}{r}[I - (I + rS_{XV'}^T)^{-1}] = \mathcal{H}_r^T + \rho S_m^T - \frac{1}{\rho}d\rho dr^T I_m^{-1} + \frac{1}{\rho}S_m^T dr dr^T I_m^{-1}.$$

After simplifying this equation we obtain

$$S_{XV'}^T = \frac{1}{r}\{[I - r(\mathcal{H}_r^T + \rho S_m^T - \frac{1}{\rho}d\rho dr^T I_m^{-1} + \frac{1}{\rho}S_m^T dr dr^T I_m^{-1})]^{-1} - I\}.$$

□

Corollary 5.3.24 *Assume as in theorem 5.3.23. If the radius function has a singularity, then*

$$S_{XV'}^T = \frac{1}{r}\{[I - r(\mathcal{H}_r^T + S_m^T)]^{-1} - I\}. \tag{5.41}$$

Proof

If the radius function has a singularity then, $\rho = 1$ and $d\rho = 0$. Thus equation 5.40 becomes

$$S^T_{XV'} = \frac{1}{r}\{[I - r(\mathcal{H}^T_r + S^T_m)]^{-1} - I\}.$$

□

5.4 Blum Medial Axis and the Singularity of the Associated Midlocus

In this section we will study the specific conditions for the midlocus to have a singularity at a point associated to a smooth point on the medial axis. Also, the impact of this singularity on the radial shape operators will be investigated. In lemma 2.3.16 Peter Giblin gave a condition for the midlocus to have a singularity at a point associated to a smooth point of the symmetry set. In the following theorem we give an equivalent condition depends on the radial curvatures.

Theorem 5.4.1 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^2$ with smooth boundary X. Let x_0 be a smooth point of $\mathbb{S}$ and let x_m be the associated midpoint then, the midlocus $\mathbb{M}$ is singular at x_m if and only if the radius function r has a singularity and the radial curvatures satisfy the equation*

$$\kappa_{r1} + \kappa_{r2} = \frac{2}{r}.$$

Proof

Let γ be the smooth stratum containing x_0 parametrized by the arc-length. The midlocus associated to γ is given by

$$\mathbb{M} = \gamma + \frac{r}{2}(U_1 + U_2). \tag{5.42}$$

Now the differentiating of the above equation with respect to the arc-length gives

$$\mathbb{M}^{'} = [1 - \frac{r}{2}(\kappa_{r1} + \kappa_{r2})]T + \frac{r^{'}}{2}(1 - r\kappa_{r1})U_1 + \frac{r^{'}}{2}(1 - r\kappa_{r2})U_2,$$

where T is the unit tangent of γ at x_0. Now assume that $r^{'} = 0$ and $\kappa_{r1} + \kappa_{r2} = \frac{2}{r}$, then the midlocus is singular. Conversely, assume that the midlocus is singular, then $\mathbb{M}^{'} = 0$ and hence $r^{'} = 0$. Therefore, $\kappa_{r1} + \kappa_{r2} = \frac{2}{r}$. □

Now we will study the impact of the singularity of the midlocus on the medial axis in particular the curvature of the medial axis.

Proposition 5.4.2 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^2$ with smooth boundary X. Let $x_0 \in \mathbb{S}$ and x_m be the associated midlocus point. If the midlocus is singular at x_m, then $\kappa_m \neq 0$, where κ_m is the curvature of $\mathbb{S}$ at x_0.*

Proof

Since the midlocus is singular, then by theorem 5.4.1 we have $\kappa_{r1} + \kappa_{r2} = \frac{2}{r}$ which gives $\kappa_{r2} = \frac{2}{r} - \kappa_{r1}$, and from proposition 5.2.11 we have

$$\kappa_m = \frac{1}{2}(\kappa_{r1} - \kappa_{r2})\sqrt{1 - r^{'2}}.$$

Also, since the midlocus is singular, then the radius function has a singularity i.e., $r^{'} = 0$ thus the above equation becomes

$\kappa_m = \frac{1}{2}(\kappa_{r1} - \kappa_{r2}) = \frac{1}{2}(\kappa_{r1} - \frac{2}{r} + \kappa_{r1}) = \kappa_{r1} - \frac{1}{r} \neq 0.$ □

In [13] Peter Giblin and Brassett pointed out that the singularity of the midlocus of a plane curve is generally a cusp. In the following proposition we give a sufficient condition for the midlocus to have a cusp singularity. Before stating the result recall that the criteria for a parametrized plane curve $\gamma : I \longrightarrow \mathbb{R}^2$ to have a cusp singularity at t_0 is that

- $\gamma^{'}(t_0) = (0, 0)$,
- $\gamma^{''}(t_0) \neq (0, 0)$, and
- $\gamma^{''}(t_0)$ and $\gamma^{'''}(t_0)$ are linearly independent.

Proposition 5.4.3 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^2$ with smooth boundary X. Let x_0 be a smooth point of $\mathbb{S}$. Assume that the midlocus is singular at x_m associated to x_0. If at x_0*

$$3r^4 (r^{'''})^2 \kappa_m + 2r^3 r^{'''} \kappa_m^{'} + 2r^2 \kappa_m^3 - 3\kappa_m - r^3 r^{(4)} \kappa_m \neq 0, \tag{5.43}$$

where κ_m is the curvature of $\mathbb{S}$ at x_0, then the singularity of the midlocus is a cusp.

Proof

Let γ be the smooth stratum containing x_0 and parametrized by the arc-length. The associated midlocus is given by

$$\mathbb{M} = \gamma - rr^{'}T,$$

where T is the unit tangent of γ. Since the midlocus is singular at x_0, then the radius function has a singularity at x_0 and $r^{''} = \frac{1}{r}$. Direct calculation shows that at x_0 we have

$$\mathbb{M}^{''}(x_0) = -r(x_0) r^{'''}(x_0) T(x_0) - \kappa_m(x_0) N(x_0),$$

where N is the unit normal of γ. Since the midlocus is singular at x_0, then from proposition 5.4.2 $\kappa_m(x_0) \neq 0$. Thus $\mathbb{M}^{''}(x_0) \neq 0$. Also, at x_0 we have

$$\mathbb{M}^{'''}(x_0) = \left(2\kappa_m^2(x_0) - \frac{3}{r^2(x_0)} - r(x_0) r^{(4)}(x_0)\right) T(x_0) - \left(3r(x_0) r^{'''}(x_0) \kappa_m(x_0) + 2\kappa_m^{'}(x_0)\right) N(x_0).$$

Now $\mathbb{M}^{''}(x_0)$ and $\mathbb{M}^{'''}(x_0)$ are linearly independent if and only if their vector product is non-zero vector if and only if

$$3r^4 (r^{'''})^2 \kappa_m + 2r^3 r^{'''} \kappa_m^{'} + 2r^2 \kappa_m^3 - 3\kappa_m - r^3 r^{(4)} \kappa_m \neq 0.$$

Thus if this condition holds then the midlocus is a cusp. □

In corollary 3.2.12 we gave the condition for the centroid to have a singularity when the radius function has a singularity. We found that condition depends on the Hessian operator. In the following theorem we will give an equivalent condition for the midlocus to have a singularity when the radius function has a singularity at a smooth point of the medial axis. This condition depends on the radial shape operators of the medial axis.

Theorem 5.4.4 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subset \mathbb{R}^{n+1}$ with smooth boundary X. Let x_0 be a smooth point of $\mathbb{S}$ and let x_m be the associated midpoint. If the radius function has a singularity at x_0 then the midlocus is singular at x_m if and only if $\frac{1}{r}$ is an eigenvalue of the matrix*

$$\frac{1}{2}(S_{V_1} + S_{V_2}).$$

Proof

In corollary 3.2.12 we have that the centroid set has a singularity when the radius has a singularity if and only if $\dfrac{1}{r}$ is an eigenvalue of the radial Hessian operator. Since the midlocus is a special case of the centroid set then we can apply this corollary. Also, from proposition 5.3.3 the matrix representing the radial Hessian operator is given by

$$\mathcal{H}_r = \frac{1}{2}(S_{V_1} + S_{V_2}),$$

where $S_{V_i},\ i = 1, 2$ are the matrices representing the radial shape operators. Thus if the radius function has a singularity at a smooth point of the Blum medial axis, then the associated midlocus is singular at the associated midlocus point if and only if $\dfrac{1}{r}$ is an eigenvalue of $\mathcal{H}_r$ if and only if $\frac{1}{r}$ is an eigenvalue of the matrix $\frac{1}{2}(S_{V_1} + S_{V_2})$. □

Chapter 6

Radial Focal Point of a Skeletal Set and Focal Point of the Boundary

6.1 Introduction

In this chapter we will study the focal point of the boundary and give the relation between the focal point and the radial one. First we will define the radial focal point of a skeletal structure and then study the relation between it and the associated focal point of the boundary.

6.2 Location of the Focal Point of the Boundary

Definition 6.2.1 *Let $\varphi : M \to \mathbb{R}^{n+1}$ be a parametrized n-surface, let $p \in M$, and let $\beta : \mathbb{R} \to \mathbb{R}^{n+1}$ be the normal line given by $\beta(s) = \varphi(p) + sN(p)$. Then the focal points of φ along β are the points $\beta(\frac{1}{\kappa_i(p)})$, where N is the unit normal and $\kappa_i(p)$ are the non-zero principal curvatures of φ at p.*

Now we will define the radial focal point of the skeletal structure using the same way as the above definition.

Definition 6.2.2 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$ such that for a choice of smooth value of the radial vector field U, at $x_0 \in \mathbb{S}$ the associated compatibility 1-form vanishes and $\frac{1}{r}$ is not an eigenvalue of the radial shape operator or the edge radial shape operator. The radial focal points of $\mathbb{S}$ at x_0 are defined by*

$$p_{ri} = x_0 + \frac{1}{\kappa_{ri}} U_1, \tag{6.1}$$

where κ_{ri} are the principal radial curvatures if x_0 is a non-edge point or edge principal radial curvatures if x_0 is an edge point.

Now we will give a precise relationship between radial focal point and its associated focal point of the boundary in the following proposition.

Proposition 6.2.3 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^{n+1}$ such that for a choice of smooth value of the radial vector field U the compatibility 1-form vanishes identically on a neighbourhood of a non-edge point $x_0 \in \mathbb{S}$ and $\frac{1}{r}$ is not an eigenvalue of the radial shape operator. Then, the radial focal points of $\mathbb{S}$ at x_0 and the associated focal points of the boundary at $x_0' = \Psi_1(x_0)$ coincide.*

Proof

From definition, the radial focal points of $\mathbb{S}$ at x_0 are given by

$$p_{ri} = x_0 + \frac{1}{\kappa_{ri}} U_1$$

and the associated focal points of the boundary at $x_0' = \Psi_1(x_0)$ are given by

$$p_{Xi} = x_0' + \frac{1}{\kappa_i} N_X,$$

where κ_i are the principal curvatures of the boundary at $x_0^{'}$ and N_X is the unit normal of the boundary at $x_0^{'}$. But $x_0^{'} = x_0 + rU_1$ and $N_X = U_1$ therefore,

$$\begin{aligned} p_{Xi} &= x_0^{'} + \frac{1}{\kappa_i} N_X \\ &= x_0 + \left(r + \frac{1}{\kappa_i} \right) U_1 \\ &= x_0 + \left(\frac{r\kappa_i + 1}{\kappa_i} \right) U_1 \\ &= x_0 + \frac{1}{\kappa_{ri}} U_1 \ (\textit{by equation } 4.2) \\ &= p_{ri} \end{aligned}$$

□

Corollary 6.2.4 *Let* $(\mathbb{S}, U)$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^{n+1}$ *with smooth boundary* X. *Let* $x_0 \in \mathbb{S}$ *be a non-edge point then, the radial focal points of* $\mathbb{S}$ *at* x_0 *and the associated focal points of the boundary at* $x_0^{'} = \Psi_1(x_0)$ *coincide.*

Proof

Since the Blum medial axis satisfies the radial condition and the compatibility condition, so we can apply proposition 6.2.3 which completes the proof. □

Our task now is to find the necessary and sufficient condition for the focal point of a smooth boundary to be in its interior. First of all, we discuss this phenomenon in the case when the boundary point associated to a non-edge point in the skeletal structure; after that we will discuss it at a boundary point associated to an edge point.

Lemma 6.2.5 *Let* $(\mathbb{S}, U)$ *be a skeletal structure in* $\mathbb{R}^{n+1}$ *such that for a choice of smooth value of the radial vector field* U *the compatibility 1-form vanishes identically on a* *neighbourhood of a non-edge point $x_0 \in \mathbb{S}$ and $\frac{1}{r}$ is not an eigenvalue of the radial shape operator. If there exists $\kappa_{r\alpha} < 0$ at x_0 for some index α such that $\frac{1}{|\kappa_{r\alpha}|} < r$, then the focal point of the boundary associated to $\kappa_{r\alpha}$ is closer to x_0 than $x_0^{'} = x_0 + rU_1$ along the radial line.*

Proof

The boundary point $x_0^{'}$ and the focal points lie on the radial line and the distance between the the boundary point $x_0^{'} = x_0 + rU_1$ and x_0 along the radial line is r. On the other hand, the distance between x_0 and the focal point $p_{r\alpha} = x_0 + \frac{1}{\kappa_{r\alpha}} U_1$ along the radial line is $\frac{1}{|\kappa_{r\alpha}|}$ and by our assumption we have $\frac{1}{|\kappa_{r\alpha}|} < r$. Thus the focal point of the boundary associated to $\kappa_{r\alpha}$ is closer to x_0 than $x_0^{'} = x_0 + rU_1$ along the radial line. □

This lemma gives us a good tool to examine the location of the focal point and leads us to the following theorem.

Theorem 6.2.6 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X. Let $x_0 \in \mathbb{S}$ be a smooth point such that there exists a principal radial curvature $\kappa_{r\alpha} < 0$ at x_0 with $\frac{1}{|\kappa_{r\alpha}|} < r$. Then the focal point of the boundary at $x_0^{'} = x_0 + rU_1$ associated to $\kappa_{r\alpha}$ is inside the boundary X.*

Proof

From lemma 6.2.5 the focal point is closer to x_0 than $x_0^{'}$ and since we are in the Blum medial case then the focal point lies on the diameter of the bitangent hypersphere also, from the definition of the Blum medial axis the bitangent hypersphere lies inside the boundary X. Hence the proof is completed. □

Lemma 6.2.7 [9] *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X. Let $x_0 \in \mathbb{S}$ be a smooth point with associated* *boundary point $x_1 \in X$. Then the principal radial curvatures at x_0 have the same sign as the corresponding principal curvatures of the boundary at x_1 and one is zero if and only if the other is.*

Proposition 6.2.8 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X. Let $x_1 \in X$ corresponding to a smooth point of $\mathbb{S}$. If there exists $\kappa_\alpha < 0$ such that $|\kappa_\alpha| > \dfrac{1}{2r}$, then the focal point associated to κ_α is inside the interior of X, where κ_α is a principal curvature of X at x_1.*

Proof

From theorem 6.2.6 if there exist a negative radial curvature satisfies the condition $\dfrac{1}{|\kappa_{r\alpha}|} < r$. Then the boundary has a focal point inside its interior, this focal point is that associated to κ_α which is the associated principal curvature of the boundary to $\kappa_{r\alpha}$. Also, from lemma 6.2.7 the principal radial curvatures and the associated principal curvatures of the boundary have the same sign. Therefore, $\dfrac{1 + r\kappa_\alpha}{|\kappa_\alpha|} < r$. Thus the result holds. □

Lemma 6.2.9 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth and convex boundary X. Let $x_1 \in X$ corresponding to a smooth point of $\mathbb{S}$. Then the corresponding focal points of the boundary associated to positive principal radial curvatures will be outside the interior of the boundary.*

Proof

From corollary 6.2.4 we have the radial focal point of the Blum medial axis at a smooth point is nothing but the focal point of the boundary at the associated tangency point. Also, the radial line points from the medial point to the boundary. Now since the Blum medial axis satisfies *The Radial Curvature Condition* (by proposition 1.8.3) and the boundary is convex, then the result holds. □

Now we will turn to the relationship between the focal point of the boundary at a point associated to an edge point in the Blum medial axis and the edge point. First of all, we prove the following lemma.

Lemma 6.2.10 *Let A be an $(n \times n)$ matrix. If $\alpha \neq 0$ is not a generalized eigenvalue of the pair $(A, I_{n-1,1})$, where $I_{n-1,1}$ is the $(n \times n)$-diagonal matrix with 1's in the first $(\ n-1\ )$ diagonal positions and 0 otherwise, then $\frac{-1}{\alpha}$ is an eigenvalue of the matrix*

$$B = (I_{n-1,1} - \alpha A)^{-1} A.$$

Proof

Since α is not a generalized eigenvalue of the pair $(A, I_{n-1,1})$, then the matrix $(I_{n-1,1} - \alpha A)$ is invertible. Now let

$$B^{'} = (I_{n-1,1} - \alpha A)^{-1} A + \frac{1}{\alpha} I.$$

Then,

$$(I_{n-1,1} - \alpha A) B^{'} = A + \frac{1}{\alpha}(I_{n-1,1} - \alpha A) I = \frac{1}{\alpha} I_{n-1,1}.$$

Therefore, the matrix $(I_{n-1,1} - \alpha A)B^{'}$ is not invertible and

$$det[(I_{n-1,1} - \alpha A)B^{'}] = det(I_{n-1,1} - \alpha A) det(B^{'}) = 0.$$

But $det(I_{n-1,1} - \alpha A) \neq 0$ hence $det(B^{'}) = 0$ which implies that $\frac{-1}{\alpha}$ is an eigenvalue of the matrix $B = (I_{n-1,1} - \alpha A)^{-1} A$. □

In the following lemma a crest point on the boundary is a point corresponds to an edge point of the medial axis.

Lemma 6.2.11 [8] *Suppose Ω is a region in $\mathbb{R}^{n+1}$ with smooth boundary X and Blum medial axis and radial vector field $(\mathbb{S}, U)$. Let x_1 be a crest point corresponding to an edge point $x_0 \in \partial\mathbb{S}$. We let V be a special basis for $T_{x_0}\mathbb{S}$ (as in section 1.4) with $V^{'}$*

the corresponding basis for $T_{x_1}X$. Then the differential geometric shape operator for the boundary X has a matrix representation with respect to V' given by

$$S_{XV'} = (I_{n-1,1} - rS_{EV})^{-1} S_{EV}. \tag{6.2}$$

The principal curvature κ_i and the principal directions of X at x_1 are the eigenvalues and eigenvectors of RHS of the above equation.

Theorem 6.2.12 *Let $(\mathbb{S}, U)$ be a Blum medial axis and radial vector field of a region $\Omega \subseteq \mathbb{R}^{n+1}$ with smooth boundary X. Then the edge point $x_0 \in \partial\mathbb{S}$ is a focal point of the boundary.*

Proof

From lemma 6.2.11 the matrix representation of the differential geometric shape operator is given by

$$S_{XV'} = (I_{n-1,1} - rS_{EV})^{-1} S_{EV}$$

and from lemma 6.2.10 we have $\frac{-1}{r}$ is an eigenvalue of the differential geometric shape operator of the boundary at $x_0' = x_0 + rU_1$. Since U is perpendicular to the boundary, then the focal point of the boundary corresponding to the principal curvature $\kappa = \frac{-1}{r}$ is given by

$$p = x_0' + \frac{1}{\kappa} N_X = x_0' + \frac{1}{\kappa} U_1 = x_0' - rU_1.$$

But $x_0' = x_0 + rU_1$. Therefore,

$$p = x_0 + rU_1 - rU_1 = x_0.$$

Thus the edge point is a focal point of the boundary. □

Corollary 6.2.13 *Let $(\mathbb{S}, U)$ as in theorem 6.2.12. Let $x_1 \in X$ be a crest point then there exists at least one focal point of X associated to x_1 inside its interior of X.*

Proof

From theorem 6.2.12 we proved that the edge point which is the point corresponding to the crest point is a focal point of the boundary. Therefore, there exists at least one focal point of the boundary inside its interior corresponding to the crest point. □

6.3 Creating the Focal Points of the Boundary from Skeletal Structures

In this section we will focus on the focal points of a smooth plane curve. In particular, we are going to create the focal point of a plane curve at a point associated to a smooth point of a skeletal set, Blum medial axis or symmetry set using only the information provided by the differential geometry (unit normal, unit tangent and curvature) and the radius function.

Theorem 6.3.1 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^2$ such that for a choice of smooth value of the radial vector field U, the associated compatibility 1-form vanishes identically on a neighbourhood of a smooth point x_0 of $\mathbb{S}$ and $\frac{1}{r} \neq \kappa_r$. Then the focal point of the boundary at a point $x_0^{'}$ associated to x_0 is given by*

$$p = x_0 + \frac{1 - r^{'2}}{r'' + \sqrt{1 - r^{'2}}\kappa_m}(-r^{'}T + \sqrt{1 - r^{'2}}N), \tag{6.3}$$

where T and N are the unit tangent and unit normal of $\mathbb{S}$ at x_0 respectively.

Proof

let $\gamma(s)$ be the smooth stratum containing x_0 parametrized by the arc-length s, and T and

N are the unit tangent and unit normal of $\gamma(s)$ at x_0 respectively. Then from proposition 6.2.3 we have

$$p = x_0 + \frac{1}{\kappa_r} U_1.$$

But

$$U_1 = -r^{'} T + \sqrt{1 - r^{'2}} N$$

and

$$\kappa_r = \frac{1 - r^{'2}}{r^{''} + \sqrt{1 - r^{'2}} \kappa_m}.$$

Thus the proof is completed. □

Theorem 6.3.2 *Let* $(\mathbb{S}, U)$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^2$ *with smooth boundary* X*. Let* $x_0 \in \mathbb{S}$ *be a smooth point. Then the focal points of the boundary at* $x_0^{'}$ *and* $x_0^{''}$ *associated to* x_0 *are given by*

$$p_1 = x_0 + \frac{1 - r^{'2}}{r^{''} + \sqrt{1 - r^{'2}} \kappa_m} (-r^{'} T + \sqrt{1 - r^{'2}} N), \tag{6.4}$$

and

$$p_2 = x_0 + \frac{1 - r^{'2}}{r^{''} - \sqrt{1 - r^{'2}} \kappa_m} (-r^{'} T - \sqrt{1 - r^{'2}} N). \tag{6.5}$$

Proof

let $\gamma(s)$ be the smooth stratum containing x_0 parametrized by the arc-length s, and T and N are the unit tangent and unit normal of $\gamma(s)$ at x_0 respectively. Now let $x_0^{'}$ and $x_0^{''}$ be the tangency points associated to x_0 and p_1 and p_2 be the focal points of the boundary associated to $x_0^{'}$ and $x_0^{''}$ respectively then from proposition 6.2.3 we have

$$p_1 = x_0 + \frac{1}{\kappa_{r1}} U_1 \qquad and \qquad p_2 = x_0 + \frac{1}{\kappa_{r2}} U_2 \tag{6.6}$$

and from proposition 5.2.9 we have

$$\kappa_{r1} = \frac{\sqrt{1 - r^{'2}} \kappa_m + r^{''}}{1 - r^{'2}} \qquad and \qquad \kappa_{r2} = \frac{-\sqrt{1 - r^{'2}} \kappa_m + r^{''}}{1 - r^{'2}}.$$

Also, we have

$$U_1 = -r^{'}T + \sqrt{1 - r^{'2}}N \qquad and \qquad U_2 = -r^{'}T - \sqrt{1 - r^{'2}}N.$$

Now substitute by these equations in equation 6.6 the results hold. □

Now we turn to the relationship between the focal point of a skeletal structure and that of the boundary. In particularly we will investigate the condition which makes the focal point of a skeletal structure and the associated focal point of its boundary coincide.

Theorem 6.3.3 *Let $(\mathbb{S}, U)$ be a skeletal structure in $\mathbb{R}^2$ such that for a choice of smooth value of the radial vector field U, the associated compatibility 1-form vanishes identically on a neighbourhood of a smooth point x_0 of $\mathbb{S}$ and $\frac{1}{r} \neq \kappa_r$. Then the focal point p_0 of the skeletal structure associated to x_0 and the focal point p of the boundary associated to $x_0^{'} = \Psi_1(x_0)$ coincide if and only if the radius function has an $A_{k\geq 2}$ singularity at x_0.*

Proof

let $\gamma(s)$ be the smooth stratum containing x_0 parametrized by the arc-length s, and T and N are the unit tangent and unit normal of $\gamma(s)$ at x_0 respectively. Now the focal point of $\gamma(s)$ at x_0 is given by

$$p_0 = x_0 + \frac{1}{\kappa_m}N. \tag{6.7}$$

Also, the focal point of the boundary is given by equation 6.3. Now assume that the focal point of the skeletal structure and the associated focal point of the boundary coincide then from equations 6.3 and 6.7 we have

$$r^{'} = 0 \qquad and \qquad r^{''} + \kappa_m = \kappa_m \Rightarrow r^{''} = 0,$$

which gives that the radius function has an $A_{k\geq 2}$ singularity at x_0. Conversely, assume that the radius function has an $A_{k\geq 2}$ singularity at x_0, then $p_1 = x_0 + \frac{1}{\kappa_m}N = p_0$ which completes the proof. □

Proposition 6.3.4 *Let* $(\mathbb{S}, U)$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^2$ *with smooth boundary* X*. Let* $x_0 \in \mathbb{S}$ *be a smooth point. If the radius function has an* $A_{k\geq 2}$ *singularity at* x_0*, then the focal points of the boundary at the tangency points associated to* x_0 *coincide.*

Proof

From theorem 6.3.2 the focal points of the boundary at the tangency points are given by

$$p_1 = x_0 + \frac{1 - r'^2}{r'' + \sqrt{1 - r'^2}\kappa_m}(-r'T + \sqrt{1 - r'^2}N),$$

and

$$p_2 = x_0 + \frac{1 - r'^2}{r'' - \sqrt{1 - r'^2}\kappa_m}(-r'T - \sqrt{1 - r'^2}N).$$

Therefore, if the radius function has an $A_{k\geq 2}$ singularity at x_0, then

$$p_1 = x_0 + \frac{1}{\kappa_m}N \quad and \quad p_2 = x_0 + \frac{1}{\kappa_m}N.$$

Thus $p_1 = p_2$. □

Proposition 6.3.5 *Let* $(\mathbb{S}, U)$ *be a Blum medial axis and radial vector field of a region* $\Omega \subseteq \mathbb{R}^2$ *with smooth and convex boundary* X*. Let* $x_0 \in \mathbb{S}$ *be a smooth point. If the radius function has an* $A_{k\geq 2}$ *at* x_0*, then the focal point of the boundary will be outside the interior of the boundary.*

Proof

From corollary 5.2.10, we have $\kappa_{r1} + \kappa_{r2} = \frac{2r''}{1 - r'^2}$. Therefore, if the radius has an $A_{k\geq 2}$ singularity at x_0, then $\kappa_{r1} = -\kappa_{r2}$. Thus $\kappa_1 > 0$ or $\kappa_2 > 0$, without loss of generality assume that $\kappa_1 > 0$, then from proposition 6.3.4 $p_1 = p_2$ and from lemma 6.2.9, the focal point will be outside the interior of the boundary. □

Appendices

A The Maple Code Used in Calculating the 2-jet of the Midlocus in Chapter 3

```
> f:=(1/2)*a[2,0]*x^2+(1/2)*a[0,2]*y^2+a[1,1]*x*y+a[3,0]*x^3
> +a[2,1]*x^2*y+a[1,2]*x*y^2+a[0,3]*y^3;
> r:=b[0,0]+(1/2)*b[2,0]*x^2+(1/2)*b[0,2]*y^2+(1/3)*b[3,0]*x^3
> +(1/2)*b[2,1]*x^2*y+(1/2)*b[1,2]*x*y^2
> +(1/3)*b[0,3]*y^3+b[4,0]*x^4+b[3,1]*x^3*y+b[1,3]*x*y^3
> +b[2,2]*x^2*y^2+b[0,4]*y^4;
> f1 := diff(f, x);
> f2 := diff(f, y);
> r1 := diff(r, x);
> r2 := diff(r, y);
> g:=(x+x*f1^2+x*f2^2-r*r1-r*r1*f2^2+r*r2*f1*f2)/(1+f1^2+f2^2);
> h:=(y+y*f1^2+y*f2^2-r*r2-r*r2*f1^2+r*r1*f1*f2)/(1+f1^2+f2^2);
> l:=(f+f*f1^2+f*f2^2-r*r1*f1-r*r2*f2)/(1+f1^2+f2^2);
```

```
> p := mtaylor(g, x, y, 3);
```

$$p := (1 - b_{0,0}b_{2,0})\,x - b_{0,0}b_{3,0}x^2 - b_{0,0}b_{2,1}yx - 1/2\,b_{0,0}b_{1,2}y^2 \tag{A.8}$$

```
> q := mtaylor(h, x, y, 3);
```

$$q := (1 - b_{0,0}b_{0,2})\,y - 1/2\,b_{0,0}b_{2,1}x^2 - b_{0,0}b_{1,2}yx - b_{0,0}b_{0,3}y^2 \tag{A.9}$$

```
> s := mtaylor(l, x, y, 3);
```

$$s := (1/2\,a_{2,0} - b_{0,0}b_{2,0}a_{2,0})\,x^2 + (-b_{0,0}b_{0,2}a_{1,1} + a_{1,1} - b_{0,0}b_{2,0}a_{1,1})\,yx + (1/2\,a_{0,2} - b_{0,0}b_{0,2}a_{0,2})\,y^2 \tag{A.10}$$

```
> b[0, 0] := 1/b[0, 2];
> x:=u+d[1,1]*u*y+d[2,0]*u^2+d[0,2]*y^2;
> p := mtaylor(p, u, y, 3);
> z1 := simplify(coeff(p, u^2));
> z2 := simplify(coeff(coeff(p,u),y));
> z3 := simplify(coeff(p, y^2));
> d[1,1]:=simplify(solve(z2=0,d[1,1]));
> d[2,0]:=simplify(solve(z1=0,d[2,0]));


> p := simplify(p);
```

$$p := \frac{(b_{0,2} - b_{2,0})\,u}{b_{0,2}} \tag{A.11}$$

```
> q := simplify(mtaylor(q, u, y,3));
```

$$q := -1/2\,\frac{b_{2,1}u^2 + 2\,b_{1,2}yu + 2\,b_{0,3}y^2}{b_{0,2}} \tag{A.12}$$

```
> s := simplify(mtaylor(s, u, y,3));
```

$$s := 1/2\,\frac{a_{2,0}u^2b_{0,2} - 2\,a_{2,0}u^2b_{2,0} - 2\,b_{2,0}a_{1,1}yu - a_{0,2}y^2b_{0,2}}{b_{0,2}} \tag{A.13}$$

B The Maple Code Used in Calculating the 3-jet of the Midlocus in Chapter 3

```
> f :=
> (1/2)*a[2,0]*x^2+(1/2)*a[0,2]*y^2+a[1,1]*x*y
> +a[3,0]*x^3+a[2,1]*x^2*y+a[1, 2]*x*y^2 +a[0, 3]*y^3;
> r :=
> b[0,0]+(1/2)*b[2,0]*x^2+(1/2)*b[0,2]*y^2+(1/3)*b[3,0]*x^3
> +(1/2)*b[2,1]*x^2*y+(1/2)*b[1,2]*x*y^2+(1/3)*b[0,3]*y^3
> +b[4,0]*x^4+b[3,1]*x^3*y+b[1,3]*x*y^3 +b[2,2]*x^2*y^2
> +b[0,4]*y^4;
> f1 := diff(f, x);
> f2 := diff(f, y);
> r1 := diff(r, x);
> r2 := diff(r, y);
> g :=
> (x+x*f1^2+x*f2^2-r*r1-r*r1*f2^2+r*r2*f1*f2)/(1+f1^2+f2^2);
> h :=
> (y+y*f1^2+y*f2^2-r*r2-r*r2*f1^2+r*r1*f1*f2)/(1+f1^2+f2^2);
> l :=
> (f+f*f1^2+f*f2^2-r*r1*f1-r*r2*f2)/(1+f1^2+f2^2);
> p := mtaylor(g, x, y, 4);
> q := mtaylor(h, x, y, 4);
> s := mtaylor(l, x, y, 4);
```

```
> b[0, 0] := 1/b[0, 2];
> x:=u+d[1,1]*u*y+d[2,0]*u^2+d[0,2]*y^2+d[2,1]*u^2*y
> +d[1,2]*u*y^2+d[3,0]*u^3+d[0,3]*y^3;
> p := mtaylor(p, u, y, 4);
> z1 := simplify(coeff(p, u^2));
> z2 := simplify(coeff(coeff(p, u), y));
> z3 := simplify(coeff(coeff(p, u^2), y));
> z4:=simplify(coeff(coeff(p,y^2),u));
> z5 := simplify(coeff(p, y^2));
> z6 := simplify(coeff(p, u^3));
> z7 := simplify(coeff(p, y^3));
> d[1,1]:=simplify(solve(z2=0,d[1,1]));
> d[2, 0] := simplify(solve(z1 = 0, d[2,0]));
> d[0, 2] := simplify(solve(z5 = 0, d[0,2]));
> d[2, 1] := simplify(solve(z3 = 0, d[2,1]));
> d[1, 2] := simplify(solve(z4 = 0, d[1,2]));
> d[3, 0] := simplify(solve(z6 = 0, d[3,0]));
> d[0, 3] := simplify(solve(z7 = 0, d[0,3]));
> p := simplify(p);
```

$$p := \frac{(b_{0,2} - b_{2,0})\, u}{b_{0,2}}$$

```
> q := simplify(mtaylor(q, u, y, 4));
> s := simplify(mtaylor(s, u, y, 4));
> y:=v+c[0,1]*u+c[1,1]*u*v+c[2,0]*v^2+c[0,2]*u^2;
> s := simplify(mtaylor(s, u, v, 4));
> e[1, 1]:=simplify(coeff(coeff(s,u),v));
```

```
> e[2, 1] := simplify(coeff(coeff(s,v^2),u));
```

```
> e[0, 2] := simplify(coeff(s, u^2));
> e[2, 0] := simplify(coeff(s, v^2));
> e[1,2]:=simplify(coeff(coeff(s,u^2),v));
> c[0, 1] :=simplify(solve(e[1,1]=0,c[0,1]));
```

```
> c[2, 0] :=simplify(solve(e[3,0]=0,c[2,0]));
> c[0, 2] :=simplify(solve(e[1,2]=0,c[0,2]));
> s := simplify(mtaylor(s, u, v, 4));
> k[2, 1] := simplify(coeff(coeff(s,v^2),u));
```

$$k_{2,1} := 0$$

```
> k[1, 2] := simplify(coeff(coeff(s,u^2),v));
```

$$k_{1,2} := 0$$

```
> k[2, 0] := simplify(coeff(s, v^2));
```

$$k_{2,0} := -1/2\, a_{0,2}$$

```
> k[3, 0] := simplify(coeff(s, v^3));
```

$$k_{3,0} := 0$$

```
> k[0, 3] := simplify(coeff(s, u^3));
> k[0, 2] := simplify(coeff(s, u^2));
```

$$k_{0,2} := 1/2\, \frac{a_{2,0} a_{0,2} {b_{0,2}}^2 - 2\, a_{2,0} a_{0,2} b_{0,2} b_{2,0} + {b_{2,0}}^2 {a_{1,1}}^2}{a_{0,2} {b_{0,2}}^2}$$

```
> q := simplify(mtaylor(q, u, v, 4));
> w[0, 3] := simplify(coeff(q, u^3));
> w[3,0] := simplify(coeff(q, v^3));
```

$$w_{3,0} := \frac{2{b_{0,2}}^3{a_{0,2}}^3 - 2{a_{0,2}}^3 b_{2,0}{b_{0,2}}^2 - a_{0,2}{b_{0,2}}^5 + a_{0,2}{b_{0,2}}^4 b_{2,0}}{2a_{0,2}{b_{0,2}}^2(b_{0,2} - b_{2,0})} + \frac{-8b_{0,4}{b_{0,2}}^2 a_{0,2} + 4a_{0,2}b_{0,2}{b_{0,3}}^2 + 8b_{0,4}b_{2,0}b_{0,2}a_{0,2}}{2a_{0,2}{b_{0,2}}^2(b_{0,2} - b_{2,0})} + \frac{-{b_{1,2}}^2 b_{0,2}a_{0,2} - 4a_{0,2}{b_{0,3}}^2 b_{2,0} + 8a_{0,3}{b_{0,2}}^2 b_{0,3}}{2a_{0,2}{b_{0,2}}^2(b_{0,2} - b_{2,0})} + \frac{-8a_{0,3}b_{0,2}b_{0,3}b_{2,0} + 2a_{1,1}b_{0,2}b_{0,3}b_{1,2}}{2a_{0,2}{b_{0,2}}^2(b_{0,2} - b_{2,0})} \tag{B.14}$$

```
>  w[2, 1] := simplify(coeff(coeff(q,v^2),u));

>  w[1,2] := simplify(coeff(coeff(q, u^2),v));
```

$$
\begin{aligned}
w_{1,2} :=& \frac{-2{b_{2,1}}^2{a_{0,2}}^3{b_{0,2}}^3+{b_{0,2}}^5{b_{2,0}}^2{a_{0,2}}^3}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}+\frac{-4b_{2,2}{b_{0,2}}^4{a_{0,2}}^3-{b_{0,2}}^6b_{2,0}{a_{0,2}}^3}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{2{a_{1,1}}^2{b_{0,2}}^5{a_{0,2}}^3+6b_{2,0}a_{1,1}b_{2,1}b_{1,2}{a_{0,2}}^2{b_{0,2}}^2}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}+\frac{-2{b_{2,0}}^2a_{2,0}{a_{0,2}}^4{b_{0,2}}^3}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-8{b_{0,2}}^4b_{2,0}{a_{1,1}}^2{a_{0,2}}^3-2{b_{2,0}}^3{a_{1,1}}^2{a_{0,2}}^3{b_{0,2}}^2}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}+\frac{8{b_{2,0}}^2{a_{1,1}}^2{a_{0,2}}^3{b_{0,2}}^3+4b_{2,2}b_{2,0}{a_{0,2}}^3{b_{0,2}}^3}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-2b_{3,0}b_{1,2}{a_{0,2}}^3{b_{0,2}}^3-3{b_{0,2}}^5{b_{2,0}}^2{a_{1,1}}^2a_{0,2}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}+\frac{3{b_{0,2}}^4{b_{2,0}}^3{a_{1,1}}^2a_{0,2}+2a_{2,0}{a_{0,2}}^4{b_{0,2}}^4b_{2,0}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-3{b_{2,0}}^2{a_{1,1}}^2{b_{1,2}}^2a_{0,2}b_{0,2}+24{b_{2,0}}^3{a_{1,1}}^2b_{0,4}a_{0,2}b_{0,2}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-24{b_{2,0}}^2{a_{1,1}}^2b_{0,4}a_{0,2}{b_{0,2}}^2+12b_{2,0}a_{1,1}b_{1,3}{a_{0,2}}^2{b_{0,2}}^3}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-12{b_{2,0}}^2a_{1,1}b_{1,3}{a_{0,2}}^2{b_{0,2}}^2+2{a_{0,2}}^3{b_{0,2}}^3b_{0,3}b_{2,1}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}+\frac{-2{a_{0,2}}^3{b_{0,2}}^2{b_{1,2}}^2b_{2,0}+2{a_{0,2}}^2a_{1,2}{b_{0,2}}^4b_{1,2}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-24{a_{0,2}}^2a_{1,1}{b_{0,2}}^2b_{0,3}b_{1,2}b_{2,0}+18{a_{0,2}}^2a_{1,1}b_{0,2}b_{0,3}b_{1,2}{b_{2,0}}^2}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{4{a_{0,2}}^2a_{2,0}{b_{0,2}}^2b_{0,3}b_{2,0}b_{2,1}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}+\frac{-12a_{0,2}{a_{1,1}}^2{b_{0,2}}^2b_{0,3}b_{2,0}b_{2,1}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}+\frac{-12a_{0,2}a_{1,1}a_{1,2}{b_{0,2}}^3b_{0,3}b_{2,0}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{12a_{0,2}a_{1,1}a_{1,2}b_{0,2}b_{0,3}{b_{2,0}}^3}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}+\frac{-12a_{0,2}a_{0,3}a_{1,1}{b_{0,2}}^3b_{1,2}b_{2,0}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{12a_{0,2}a_{0,3}a_{1,1}{b_{0,2}}^2b_{1,2}{b_{2,0}}^2}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}+\frac{48a_{0,3}{a_{1,1}}^2{b_{0,2}}^2b_{0,3}{b_{2,0}}^2}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-48a_{0,3}{a_{1,1}}^2b_{0,2}b_{0,3}{b_{2,0}}^3+12{a_{1,1}}^3b_{0,2}b_{0,3}b_{1,2}{b_{2,0}}^2}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{2{a_{0,2}}^3{b_{0,2}}^3{b_{1,2}}^2+6{a_{0,2}}^2a_{1,1}{b_{0,2}}^3b_{0,3}b_{1,2}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-2{a_{0,2}}^3{b_{0,2}}^2b_{0,3}b_{2,0}b_{2,1}+4{a_{0,2}}^2a_{1,1}{b_{0,2}}^3b_{0,3}b_{3,0}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{2{a_{0,2}}^2a_{1,1}{b_{0,2}}^3b_{1,2}b_{2,1}-2{a_{0,2}}^2a_{1,2}{b_{0,2}}^2b_{1,2}{b_{2,0}}^2}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{8{a_{0,2}}^2a_{2,1}{b_{0,2}}^3b_{0,3}b_{2,0}+{a_{0,2}}^2a_{2,0}{b_{0,2}}^2{b_{1,2}}^2b_{2,0}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-8{a_{0,2}}^2a_{2,1}{b_{0,2}}^2b_{0,3}{b_{2,0}}^2-12a_{0,2}{a_{1,1}}^2{b_{0,2}}^2{b_{0,3}}^2b_{2,0}}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-3a_{0,2}{a_{1,1}}^2{b_{0,2}}^2{b_{1,2}}^2b_{2,0}+36a_{0,2}{a_{1,1}}^2b_{0,2}{b_{0,3}}^2{b_{2,0}}^2}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}\\
&+\frac{-24a_{0,2}{a_{1,1}}^2{b_{0,3}}^2{b_{2,0}}^3-6a_{0,2}a_{1,1}a_{2,0}b_{0,2}b_{0,3}b_{1,2}{b_{2,0}}^2}{2{b_{0,2}}^4{a_{0,2}}^3\left(b_{0,2}-b_{2,0}\right)}
\end{aligned}
\tag{B.15}
$$

```
>  w[1, 1] := simplify(coeff(coeff(q, u), v));
```

$$w_{1,1} := -\frac{b_{1,2}a_{0,2}b_{0,2} - 2\,b_{0,3}a_{1,1}b_{2,0}}{a_{0,2}{b_{0,2}}^2}$$

```
>  w[0, 2] := simplify(coeff(q, u^2));

>  w[2, 0] := simplify(coeff(q, v^2));
```

Bibliography

[1] H. Blum and R. Negal, *Shape descriptions using weighted symmetric axis features*, Pattern Recognition **10** (1978), 167–180.

[2] J. W. Bruce, *On singularities, envelopes and elementary differential geometry*, Math. Proc. Cambridge Philo. Soc. **89** (1981), 43 – 48.

[3] J. W. Bruce and P. J. Giblin, *What is an envelope?*, Math Gazette **65** (1981), 186 – 192.

[4] J. W. Bruce and P. J. Giblin, *Growth, motion and one parameter families of symmetry sets*, Proceedings of the Royal Society of Edinburgh Sect. A **104** (1986), 179–204.

[5] J. W. Bruce and P. J. Giblin, *Curves and singularities*, Cambridge University Press, 1992.

[6] J. W. Bruce, P. J. Giblin, and C. G. Gibson, *Symmetry sets*, Proc.Roy. Soc. Edinburgh Sect. A **101** (1985), 163–186.

[7] J. Damon, *On the smoothness and geometry of boundaries associated to skeletal structures I: Sufficient conditions for smoothness*, Annales Inst. Fourier **53** (2003), 1941–1985.

[8] J. Damon, *On the smoothness and geometry of boundaries associated to skeletal structures II: Geometry in the blum case*, Compositio Mathematica **140** (2004), 1657–1674.

[9] J. Damon, *Determining the geometry of boundaries from medial data*, International Journal Of Computer Vision **63** (2005), 45–64.

[10] A. Diatta and P. J. Giblin, *Pre-symmetry sets of 3d shapes*, DSSCV workshop proceedings, Lectrue Notes In Computer Science **3753** (2005), 36–48.

[11] P. J. Giblin, *Symmetry sets and medial axis in two and three dimensions*, In R. Cipolla and R. Martin, editors. The Mathematics of Surfaces IX, Springer-Verlag (2000), 306– 321.

[12] P. J. Giblin, *Affinely invariant symmetry sets*, Geometry and Topology of Caustics - Caustics '06, Banach Centre Publications **82** (2008), 71–84.

[13] P. J. Giblin and S. A. Brassett, *Local symmetry of plane curves*, Amer. Math. Monthly **92** (1985), 689– 707.

[14] P. J. Giblin and B. B. Kimia, *On the intrinsic reconstructions of shape from its symmetries*, IEEE Transactions on Pattern, Analysis and Machine Intelligence **25** (2003), 895–911.

[15] P. J. Giblin and B. B. Kimia, *A formal classifications of 3d medial axis points and their local geometry*, IEEE Transactions on Pattern, Analysis and Machine Intelligence **26** (2004), 238–251.

[16] P. J. Giblin and J. P. Warder, *Reconstruction from medial representations*, Amer. Math. Monthly **118** (2011), 712– 725.

[17] C. A. Hobbs and N. P. Kirk, *On the classification and bifurcation of multigerms of maps from surfaces to 3-space*, Math. Scand **89** (2001), 57 – 96.

[18] K. Houston, *Topology of differentiable mappings*, Handbook of Global Analysis, Ed.D. Krupka and D. Saundes, Elsevier (2008), 493–532.

[19] Wolfgang Kühnel, *Differential geometry, curves - surfaces - manifolds*, American Mathematical Society, 2000.

[20] M. Van. Manen, *The geometry of confilict sets*, Ph.D. Thesis University of Utrecht, 2003.

[21] D. Mond, *On the tangent developable of a space curve*, Proc. Camb. Phil. Soc. **91** (1982), 351 – 355.

[22] D. Mond, *On the classification of germs of maps from* $\mathbb{R}^2$ *to* $\mathbb{R}^3$, Proc. London Math. Soc. **50** (1985), 333– 369.

[23] D. Mond, *Singularities of the tangent developable surface of a space curve*, Quart. J. Math. Oxford **40** (1989), 79 – 91.

[24] A. Pollitt, *Euclidean and affine symmetry sets and medial axes*, Ph.D. Thesis University of Liverpool, 2004.

[25] K. Siddiqi and M. Pizer, *Medial representations: Mathematics, algorithms and applications*, Springer ISBN 978-1-4020-8657-1, 2008.

[26] D. Siersma, *Properties of conflict sets in the plane*, Geometry and Topology Of Caustics 1998 Banach Center Publ. **50** (1999), 267–276.

[27] J. Sotomayor, D. Siersma, and R. Garcia, *Curvatures of conflict surfaces in Euclidean 3-space*, Geometry and Topology of Caustics-Caustics 1998, Banach Center Publ. **50** (1999), 277–285.

[28] M. Spivak, *A comperhensive introduction to differential geometry*, vol. I- V, Publish or Perish Inc., Berkeley, CA, 1975, 1979.

[29] F. Tari, *Some applications of singularity theory to the geometry of curves and surfaces*, Ph.D. Thesis University of Liverpool, 1990.

[30] J. A. Thrope, *Elementary topics in differential geometry*, Springer - Verlag New York Inc, 1979.

[31] C. T. C. Wall, *Finite determinacy of smooth map-germs*, Bull. London Math. Soc. **13** (1981), 481 – 539.

[32] J. P. Warder, *Symmetries of curves and surface*, Ph.D. Thesis University of Liverpool, 2009.