CLASSIFICATION OF REAL AND COMPLEX ANALYTIC MAP-GERMS ON THE GENERALIZED CROSS CAP

by

Mohammed Salim Jbara Al-Bahadeli

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ABSTRACT

This thesis consists of four parts. In the first part, we prove the following conjecture [HL09].

Conjecture: Let $\varphi_d : (\mathbb{C}^{2d-2}, 0) \to (\mathbb{C}^{2d-1}, 0)$ be the minimal cross cap of multiplicity $d \geq 2$ and V be its image. Let $\Theta = \left\{\xi_j^1, \xi_j^2, \xi_j^3, \xi_e\right\}_{j=1}^{d-1}$ be the module of vector fields liftable over φ_d. Then the vector fields ξ_j^1, ξ_j^2, ξ_j^3 for $1 \leq j \leq d-1$ generate $\mathrm{Der}_0(-\log(h))$.

In the second part, we develop computational method suitable for performing the classification theory. A computer package called `CAST` is developed. This is written in the `Singular` program and performs calculations such as complete transversals, finite determinacy and triviality. We discuss the package in detail and give examples of calculations performed in this thesis.

In the third part, we classify map-germs $(\mathbb{C}^p, 0) \to (\mathbb{C}^q, 0)$ under ${}_V\mathcal{K}$-equivalence: the restriction of $\mathcal{K}$-equivalences to those preserving a particular subset of the singularity's domain. We consider the case where V is the image of the minimal crosscap of multiplicity $d \geq 2$.

In the final part, we give an application to classification problems. We classify corank 1 $\mathcal{A}_e$-codimension 2 map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$.

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Chapter 1

Introduction

One of the fundamental ideas of local singularity theory is the classification of map-germs under various types of equivalence. In the early 1950's, Whitney started the classification of stable map-germs (small perturbations of the map do not change the differential geometric properties of the singularities of the map) $(\mathbb{R}^2, 0) \to (\mathbb{R}^2, 0)$ and $(\mathbb{R}^n, 0) \to (\mathbb{R}^{2n+1}, 0)$, see [Whi44] and [Whi55]. The foundational work of Mather in the end of the 1960's defined the new standard equivalence relations (namely $\mathcal{R}$, $\mathcal{L}$, $\mathcal{A}$, $\mathcal{C}$ and $\mathcal{K}$) central to local singularity theory (see [Mat68] and [Mat69]). At around the same time, Thom classified the stable map-germs in low codimensions through the development of his catastrophe theory. In the 1970's, Arnol'd gave his famous list of function-germs under $\mathcal{R}$-equivalence; a general reference is [AGV85].

Since the 1980's, Bruce, Gaffney, Mond, Rieger, Ruas, Ratcliffe, Cooper, Houston, Kirk and many others have made significant progress in the classification of map-germs, especially in the classification of map-germs under $\mathcal{A}$-equivalence. (See [BG82], [Mon85], [Mon87], [Rie87], [RR91], [Rat90], [Rat95], [Coo93] and [HK99]).

In general, it is difficult to classify map-germs under $\mathcal{A}$-equivalence. However, Houston and Wik Atique are shown in [HW] that the $\mathcal{A}$-classification of map-germs is intimately related to the ${}_V\mathcal{K}$-classification of some related map-germs. The latter equivalence relation introduced by Damon in [Dam83] and it arises from the restriction of $\mathcal{K}$-equivalences to those preserving a particular subset of the domain of the singularity.

In order to apply ${}_V\mathcal{K}$-equivalence, in practice one needs a set of vector fields that, when integrated, preserve this subset. In [HL09] the vector fields liftable over a normal form for corank 1 map-germs from $(\mathbb{C}^n, 0)$ to $(\mathbb{C}^{n+1}, 0)$ are described. These liftables integrate to diffeomorphisms preserving the image of the normal form. By classifying map-germs on the image up to ${}_V\mathcal{K}$-equivalence we get an $\mathcal{A}$-classification.

In this thesis, we classify map-germs with ${}_V\mathcal{K}_e$-codimension at most 2, where V is the image of the minimal cross cap of multiplicity $d \geq 2$ and from these normal forms we get the classification of a corank 1 $\mathcal{A}_e$-codimension 2 map-germ $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. (We mention that we get some partial results which will be useful in classifying $\mathcal{A}_e$-codimension ≤ 4 map-germs).

This thesis is structured as follows.

In Chapter 2, we review the definitions and results of singularity theory which are used in the thesis. These include $\mathcal{A}$-equivalence, $\mathcal{K}$-equivalence, ${}_V\mathcal{R}$-equivalence and ${}_V\mathcal{K}$-equivalence, the module of vector fields tangent to a subset, the minimal cross cap mapping of multiplicity d, the sharp pullback and simple map-germs.

In Chapter 3, we review the module of vector fields liftable over the minimal cross cap mapping of multiplicity $d \geq 2$. We find the defining equation for the image of the minimal cross cap of multiplicity $d \geq 2$ by using Mond-Pellikaan algorithm and prove that the cross cap liftable vector fields without the Euler vector field annihilate this defining equation. we then show that these vector fields generate $\mathrm{Der}_0(-\log(h))$, where h is the defining equation for the image of the minimal cross cap of multiplicity $d \geq 2$.

In Chapter 4, we give the techniques of a classification method which we use to find the classification of map-germ such as complete transversals, finite determinacy and triviality. The classification method can be described as the following:

i) From a k-jet produce a list of possible $(k+1)$-jets.

ii) Reduce the list by removing redundancies and by scaling.

iii) For each possible $(k+1)$-jet, check determinacy. If not $(k+1)$-determined, then repeat the method for each $(k+1)$-jet by finding the possible $(k+2)$-jets.

In Chapter 5, we describe the `CAST` package. It is written in the `Singular` program and consists of a number of procedures (Appendix A). The liftable vector fields over the minimal cross cap mapping of multiplicity $d \geq 2$ are programmed in this package. A number of examples of calculations performed in this chapter are given.

In Chapter 6, we consider the classification of map-germs under ${}_{\Theta}\mathcal{K}$-equivalence, where Θ is the module of liftable vector fields over the minimal cross cap of multiplicity $d \geq 2$. We classify map-germs with ${}_{\Theta}\mathcal{K}_e$-codimension at most 2 and ${}_{V}\mathcal{K}_e$-codimension at most 2, where V is the image of the minimal cross cap of multiplicity $d \geq 2$.

In Chapter 7, we classify corank 1 $\mathcal{A}_e$-codimension 2 map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$.

In Chapter 8, we give some ideas for future work such as the classification of corank 1 $\mathcal{A}_e$-codimension ≤ 4 map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$, the geometry of the map-germs on the generalized cross cap and the relationship between $\mathcal{A}$ and ${}_{V}\mathcal{K}$ determinacy.

Chapter 2

Notation and Preliminary Material

In this chapter we introduce some basic notation and preliminary results which will be used throughout the whole thesis. As standard references we cite the survey articles of Wall, ([Wal81], [Wal95] and [Wal09]). In addition we refer to [BW98], [Dam91], [Dam06], [Gib79] and [Mar82]. Our notation will be based on these references.

2.1 Notation

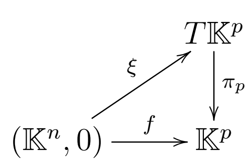
Throughout this thesis, $\mathbb{K}$ usually refers to the real number field $\mathbb{R}$ or the complex number field $\mathbb{C}$. In the latter case the assumption in statements that a map is smooth of course means that we have a complex analytic map.

Two subsets A and B of $\mathbb{K}^n$ are called **equivalent** at $x \in \mathbb{K}^n$, if there is an open neighbourhood $U \subset \mathbb{K}^n$ of x, such that $A \cap U = B \cap U$. It is easy to check that this is indeed an equivalence relation. The class of all sets equivalent to $A \subset \mathbb{K}^n$ at x will be denoted by (A, x) and is called the **germ** of A at x. A is called a representative of the germ. If $A \subset \mathbb{K}^n$ we sometimes say that (A, x) is a subgerm of $(\mathbb{K}^n, x)$, which we denote by $(A, x) \subset (\mathbb{K}^n, x)$.

Let S be a finite set in $\mathbb{K}^n$. A multi-germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, f(S))$ will mean an equivalence class of pairs (U, f) where U is a neighbourhood of S in $\mathbb{K}^n$ and $f : \mathbb{K}^n \to \mathbb{K}^p$ is a smooth map. Two such pairs (U_1, f_1) and (U_2, f_2) are equivalent if f_1 and f_2 agree on some neighbourhood of S contained in $U_1 \cap U_2$. In the special case $S = \{x\} \subset \mathbb{K}^n$ and $f(S) = \{y\} \subset \mathbb{K}^p$ we call such germs mono-germs. If $S = \{x_1, \ldots, x_r\}$, with $x_i \neq x_j$ for $i \neq j$ and $f(S) = \{0\}$, then a multi-germ $f : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ is called a r-multigerm and will be denoted $(f_1; \ldots; f_r)$, where $f_i : (\mathbb{K}^n, x_i) \to (\mathbb{K}^p, 0)$ is a smooth function-germ. The integer r is called the multiplicity of the multi-germ, and the f_i its branches.

We let $\mathcal{E}_n$ be the set of all smooth function-germs $(\mathbb{K}^n, 0) \to \mathbb{K}$. Clearly $\mathcal{E}_n$ is a ring under the obvious operations of addition and multiplication. This ring contains a unique maximal ideal, consisting of functions vanishing at the origin, denoted by $\mathbf{m}_n$. The set of all smooth map-germs $f : (\mathbb{K}^n, 0) \to \mathbb{K}^p$ is an $\mathcal{E}_n$-module and will be denoted $\mathcal{E}(n, p)$. (In case of $\mathbb{K} = \mathbb{C}$ this is often denoted $\mathcal{O}(n, p)$). We put $\mathcal{E}(n, 1) = \mathcal{E}_n$. The corresponding module of map-germs $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is denoted $\mathbf{m}_n \mathcal{E}(n, p)$.

Let $(T\mathbb{K}^p, 0)$ be the tangent bundle of $\mathbb{K}^p$ and $\pi_p : T\mathbb{K}^p \to \mathbb{K}^p$ be the natural projection. A vector field along f is a map-germ $\xi : (\mathbb{K}^n, 0) \to T\mathbb{K}^p$ such that the following diagram commutes:



The set of all vector fields along f is written $\theta(f)$; it is a free $\mathcal{E}_n$-module of rank p, i.e. $\theta(f) \cong \mathcal{E}(n, p)$.

We define $\theta_n = \theta(\mathrm{Id}_n)$ and $\theta_p = \theta(\mathrm{Id}_p)$ where Id_n and Id_p denote the germs at 0 of the identity maps on $\mathbb{K}^n$ and $\mathbb{K}^p$, respectively. Associated with these modules are certain important homomorphisms: the $\mathcal{E}_n$-homomorphism $tf : \theta_n \to \theta(f)$ defined by

$tf(\xi) = df \circ \xi$ and the $\mathcal{E}_p$-homomorphism(via $f^* : \theta_p \to \theta_n$, $\alpha \longmapsto \alpha \circ f$ for $\alpha \in \theta_p$), $\omega f : \theta_p \to \theta(f)$ defined by $\omega f(\eta) = \eta \circ f$. We define the local algebra of f to be

$$Q(f) := \frac{\mathcal{E}_n}{f^*(\mathbf{m}_p)} = \frac{\mathcal{E}_n}{\langle f_1, \dots, f_p \rangle}.$$

If we truncate the power series expansion of $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ at the origin by ignoring terms of degree greater than k, we obtain the k-jet of f, denoted by $j^k f$. The set of all k-jets forms a vector space $J^k(n,p)$. For each k there is a canonical projection $\pi_k : \mathcal{E}(n,p) \to J^k(n,p)$ which assigns to each map-germ in $\mathcal{E}(n,p)$ its k-jet at 0.

2.2 Unfoldings and Discriminants

Definition 2.2.1. *An* r-**parameter unfolding** *of a map-germ* $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ *is a map-germ* $F : (\mathbb{K}^n \times \mathbb{K}^r, 0) \to (\mathbb{K}^p \times \mathbb{K}^r, 0)$ *of the form* $F(x,u) = (\widetilde{F}(x,u), u)$ *such that* $\widetilde{F}(x, 0) = f(x)$*. Here* x,u *denotes local coordinates for* $(\mathbb{K}^n, 0)$ *and* $(\mathbb{K}^r, 0)$ *respectively.*

The notation $f_u(x) = \widetilde{F}(x,u)$ is often employed; f_u can be thought of as a deformation of f, parametrized smoothly by $u \in \mathbb{K}^r$.

Definition 2.2.2. *A* **trivial unfolding** *of a map-germ* $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ *is a map-germ* $F : (\mathbb{K}^n \times \mathbb{K}^r, 0) \to (\mathbb{K}^p \times \mathbb{K}^r, 0)$ *given by* $F(x,u) = (f(x), u)$*.*

Definition 2.2.3. *Let* $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ *be a smooth map-germ. The* **critical set** *of* f*, denote* Σf*, is the set of points* x *in* $\mathbb{K}^n$ *such that the rank of the Jacobian matrix of* f *at* x *is less than* p*.*

The **discriminant** *of* f*, denote* $D(f)$*, is defined to be the image of the critical set,* $f(\Sigma f)$*. When* $n < p$*, then* $D(f)$ *is the image of* f*.*

2.3 Equivalence relations of map-germs

We shall now look at the equivalence relations of interest. In fact, in singularity theory there are a number of standard equivalence relations. For example, $\mathcal{A}$-equivalence and $\mathcal{K}$-equivalence. These equivalence relations were defined by Mather (see [Mat68]).

Definition 2.3.1. *Let $f, g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be two smooth map-germs. We say that f and g are* $\mathcal{A}$**-equivalent**, *denoted by $f \sim_{\mathcal{A}} g$, if there exist diffeomorphism germs ψ and φ for which the following diagram commutes*

$$\begin{array}{ccc} (\mathbb{K}^n, 0) & \stackrel{f}{\longrightarrow} & (\mathbb{K}^p, 0) \\ \varphi \downarrow & & \psi \downarrow \\ (\mathbb{K}^n, 0) & \stackrel{g}{\longrightarrow} & (\mathbb{K}^p, 0) \end{array}$$

i.e. $\psi \circ f = g \circ \varphi$.

This is also known as Right-Left-equivalence. If the diffeomorphism germ in the target is the identity in the definition above, then we say that f and g are $\mathcal{R}$-equivalent.

Definition 2.3.2. *We say that a smooth map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is k-$\mathcal{A}$-determined if f is $\mathcal{A}$-equivalent to any other smooth map-germ $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ such that $j^k f = j^k g$. If f is k-$\mathcal{A}$-determined for some k, then f is said to be $\mathcal{A}$ finitely determined.*

Definition 2.3.3. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be smooth map-germ.*

i) The **extended $\mathcal{A}$-tangent space** *of f is defined by*

$$T\mathcal{A}_e(f) = tf(\theta_n) + \omega f(\theta_p).$$

ii) The **$\mathcal{A}_e$-codimension** *of f is defined by*

$$\mathcal{A}_e\text{−cod}(f) = \dim_{\mathbb{K}} \frac{\theta(f)}{T\mathcal{A}_e(f)}.$$

Definition 2.3.4. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a smooth map-germ. We say that f is* **stable** *if $\theta(f) = T\mathcal{A}_e(f)$.*

In fact, Mather in [Mat69], showed that a map-germ is stable in the above sense if and only if there exists a neighbourhood in the space of smooth maps (with the Whitney topology) such that all map-germs in the neighbourhood are $\mathcal{A}$-equivalent.

Example 2.3.5. *Let $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ be given by $f(x, y) = (x, y^2, xy)$. This is known as the cross cap or the Whitney umbrella. This map is stable.*

We shall use the coordinates (x, y) on the source and (X, Y, Z) on the target. Let $\eta \in\theta(f)$. We can write η as follows:

$$\eta = \eta_1 \, \mathbf{e}_1 + \eta_2 \, \mathbf{e}_2 + \eta_3 \, \mathbf{e}_3,$$

where for all $1 \leq i \leq 3$, $\eta_i \in \mathcal{E}_2$ and $\mathbf{e}_i$ is the standard basis vector in $\mathbb{K}^3$, i.e., the column vector with a 1 the i-th row and a 0 in all other rows.

Suppose that $\alpha = \begin{pmatrix} 1 \\ 0 \\ y \end{pmatrix}$ *and* $\beta = \begin{pmatrix} 0 \\ 2y \\ x \end{pmatrix}$.

Then we have

$$i) \;\; x^a y^b \, \mathbf{e}_1 = \begin{cases} (X^a Y^k \, \mathbf{e}_1) \circ f, & \text{if } b = 2k, \\ x^a y^{2k+1} \alpha - (X^a Y^{k+1} \, \mathbf{e}_3) \circ f, & \text{if } b = 2k+1; \end{cases}$$

$$ii) \;\; x^a y^b \, \mathbf{e}_2 = \begin{cases} (X^a Y^k \, \mathbf{e}_2) \circ f, & \text{if } b = 2k, \\ \frac{1}{2} x^a y^{2k} \beta - \frac{1}{2} (X^a Y^k \, \mathbf{e}_3) \circ f, & \text{if } b = 2k+1; \end{cases}$$

$$iii) \;\; x^a y^b \, \mathbf{e}_3 = \begin{cases} (X^a Y^b \, \mathbf{e}_3) \circ f, & \text{if } b = 2k, \\ x^a y^{2k} \alpha - (X^a Y^k \, \mathbf{e}_1) \circ f, & \text{if } b = 2k+1. \end{cases}$$

It follows $\theta(f) = T\mathcal{A}_e(f)$. Hence f is stable.

The minimal cross cap mapping

We will now give a generalization of the cross cap. The resulting maps will be used in this thesis. In fact, the main subject of this thesis is to find the classification of map-germs on the image of these maps.

Definition 2.3.6. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a smooth map-germ. We say a map-germ f is* **corank** 1 *if the rank of the Jacobian matrix at* 0 *is equal to* $\min(n, p) - 1$.

Example 2.3.7. *The following map-germs are corank* 1.

1. *The cross cap,*
2. *The cusp, i.e., $f : (\mathbb{K}^2, 0) \to (\mathbb{K}^2, 0)$ such that $f(x, y) = (x, xy + y^3)$.*

Definition 2.3.8. *For $d \geq 2$ the* **minimal cross cap mapping of multiplicity** d *is the map $\varphi_d : (\mathbb{K}^{2d-2}, 0) \to (\mathbb{K}^{2d-1}, 0)$ given by*

$$\varphi_d(u_1, \ldots, u_{d-2}, v_1, \ldots, v_{d-1}, y) = \left(u_1, \ldots, u_{d-2}, v_1, \ldots, v_{d-1}, y^d + \sum_{i=1}^{d-2} u_i y^i, \sum_{i=1}^{d-1} v_i y^i \right).$$

We shall label the coordinates of the target $u_1, \ldots, u_{d-2}, v_1, \ldots, v_{d-1}, w_1$ and w_2, respectively. The sets of coordinates will be abbreviated to $\underline{u}$, $\underline{v}$ and $\underline{w}$ respectively.

For $d = 2$ this is just the Whitney umbrella as in example 2.3.5. The minimal cross cap mapping of multiplicity d is stable and corank 1 for all $d \geq 2$.

Theorem 2.3.9 ([Mor65]). *A map-germ $F : (\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ is a stable corank* 1 *germ if and only if there exists a d such that F is $\mathcal{A}$-equivalent to the trivial unfolding of the minimal cross cap mapping of multiplicity d.*

$\mathcal{A}$-equivalence seems to be the most natural equivalence relation because this say that f coincides with g under suitable coordinate transformations of the source and the target spaces. However, it is difficult to classify map-germs under this relation. Mather introduced $\mathcal{K}$-equivalence (or contact equivalence) as a technical tool to aid with the classification of map-germs up to $\mathcal{A}$-equivalence.

Definition 2.3.10. *Suppose that $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ and $\widetilde{h} : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ are smooth map-germs. We say that h and $\widetilde{h}$ are $\mathcal{K}$-equivalent if there is a diffeomorphism germ $\varphi : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ and a germ of an invertible matrix $M : (\mathbb{K}^p, 0) \to GL(\mathbb{K}^q)$ such that $\widetilde{h}(x) = M(x)h(\varphi(x))$ for each $x \in (\mathbb{K}^p, 0)$.*

There is another definition of $\mathcal{K}$-equivalence, but is in fact equivalent to definition 2.3.10. We will give an alternative definition in the following theorem (see [Gib79], chapter V).

Theorem 2.3.11. *Let $h, \widetilde{h} : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be two smooth map-germs. Then h and $\widetilde{h}$ are $\mathcal{K}$-equivalent if there are a diffeomorphism germs Ψ of $(\mathbb{K}^p \times \mathbb{K}^q, 0)$ and ψ of $(\mathbb{K}^p, 0)$ such that*

$$\Psi(x, h(x)) = \Psi(\psi(x), \widetilde{h}(\psi(x))) \quad \text{for each } x \in (\mathbb{K}^p, 0).$$

Up to now we have only considered the standard equivalence relations $\mathcal{R}$, $\mathcal{A}$ and $\mathcal{K}$. We now turn our attention to other equivalence relations of great interest in singularity theory, namely ${}_V\mathcal{R}$- and ${}_V\mathcal{K}$-equivalence. These were introduced by Damon in [Dam83]. For more details see [Wal09].

Definition 2.3.12. *Let $(V, 0)$ be a subgerm of $(\mathbb{K}^p, 0)$. A diffeomorphism germ $\varphi : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ is said to* **preserve** *V if $\varphi(V) \subseteq V$.*

Definition 2.3.13. *Suppose that $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ and $\widetilde{h} : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ are smooth map-germs. Let $(V, 0)$ be a subgerm of $(\mathbb{K}^p, 0)$. We say that h and $\widetilde{h}$ are ${}_V\mathcal{R}$-equivalent if there is a diffeomorphism germ $\varphi : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ such that* *i)* $\widetilde{h} = h \circ \varphi \left(i.e., \widetilde{h} \sim_{\mathcal{R}} h\right)$,

ii) φ *preserves* V.

In [BR88] and [BW98] the notation for ${}_V\mathcal{R}$ is $\mathcal{R}(\mathrm{X})$ (where our V is their X).

Like ${}_V\mathcal{R}$-equivalence a similar definition can be made for $\mathcal{K}$-equivalences that preserve some subset V.

Definition 2.3.14. *Suppose that* $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ *and* $\widetilde{h} : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ *are smooth map-germs. Let* $(V, 0)$ *be a subgerm of* $(\mathbb{K}^p, 0)$*. We say that* h *and* $\widetilde{h}$ *are* ${}_V\mathcal{K}$*-equivalent if*

i) h *and* $\widetilde{h}$ *are* $\mathcal{K}$*-equivalent,*

ii) the resulting diffeomorphism of the source preserves V.

Example 2.3.15. *Let* V *be the complex Whitney umbrella, i.e., the image of* $\varphi_2 : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ *given by* $\varphi_2(v_1, y) = (v_1, y^2, v_1 y)$*. In fact,* $V = f^{-1}(0)$*, where* $f(v_1, w_1, w_2) = w_2^2 - v_1^2 w_1$.

Let $h, \widetilde{h} : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ *be smooth map-germs defined by*

$$h(v_1, w_1, w_2) = (v_1 - w_2, w_1) \quad \text{and}$$
$$\widetilde{h}(v_1, w_1, w_2) = (v_1, w_1).$$

We want to show that h *and* $\widetilde{h}$ *are* ${}_V\mathcal{K}$*-equivalent. We take a diffeomorphism germ*

$$\varphi(v_1, w_1, w_2) = (v_1 + w_2, w_1, w_2 + v_1 w_1)$$

and the matrix

$$M = \begin{pmatrix} 1 & v_1 \\ 0 & 1 \end{pmatrix}.$$

First we need to show that φ preserves V. Let $(v_1, w_1, w_2) \in V$, then we have

$$\begin{aligned} f \circ \varphi(v_1, w_1, w_2) &= f(v_1 + w_2, w_1, w_2 + v_1 w_1) \\ &= (w_2 + v_1 w_1)^2 - (v_1 + w_2)^2 w_1 \\ &= w_2^2 + 2v_1 w_1 w_2 + v_1^2 w_1^2 - v_1^2 w_1 - 2v_1 w_1 w_2 - w_1 w_2^2 \\ &= v_1^2 w_1^2 - w_1 w_2^2 \quad \textit{since } w_2^2 - v_1^2 w_1 = 0 \\ &= -w_1(w_2^2 - v_1^2 w_1) \\ &= 0 \quad \textit{since } w_2^2 - v_1^2 w_1 = 0. \end{aligned}$$

It follows $\varphi(V) \subseteq V$.

Now, for any point $(v_1, w_1, w_2) \in (\mathbb{C}^3, 0)$ we have

$$\begin{aligned} M(v_1, w_1, w_2) h(\varphi(v_1, w_1, w_2)) &= \begin{pmatrix} 1 & v_1 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} v_1 - v_1 w_1 \\ w_1 \end{pmatrix} \\ &= \begin{pmatrix} v_1 \\ w_1 \end{pmatrix} \\ &= \widetilde{h}. \end{aligned}$$

Remark 2.3.16. *Damon introduced other types of equivalence relation, namely ${}_V\mathcal{R}$-equivalence, ${}_V\mathcal{A}$-equivalence, $\mathcal{A}_V$-equivalence and $\mathcal{K}_V$-equivalence. For more details see [Dam83], [Dam87], [Dam91] and [Dam06]. Since our results in this thesis can be applied to $\mathcal{K}_V$-equivalence, we will give the definition of $\mathcal{K}_V$-equivalence only. We will use the definition of $\mathcal{K}$-equivalence as in Theorem 2.3.11.*

Definition 2.3.17. *Suppose that $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ and $\widetilde{h} : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ are smooth map-germs. Let $(V, 0)$ be a subgerm of $(\mathbb{K}^q, 0)$. We say that h and $\widetilde{h}$ are $\mathcal{K}_V$-equivalence if*

(i) h and $\widetilde{h}$ are $\mathcal{K}$-equivalent, i.e., there are a diffeomorphism germs Ψ of $(\mathbb{K}^p \times \mathbb{K}^q, 0)$

and ψ of $(\mathbb{K}^p, 0)$ such that

$$\Psi(x, h(x)) = \Psi(\psi(x), \widetilde{h}(\psi(x))) \quad \text{for each } x \in (\mathbb{K}^p, 0).$$

(ii) $\Psi(\mathbb{K}^p \times V) \subseteq \mathbb{K}^p \times V$.

Remark 2.3.18. *Suppose that $\mathcal{G}$ is an equivalence relation on the set of multi-germs $(\mathbb{K}^n, S)$ to $(\mathbb{K}^p, 0)$, for example ${}_V\mathcal{R}$ or ${}_V\mathcal{K}$. To each of these we can associate an equivalence using jet spaces.*

Let $J^s\mathcal{G}$ be the s-jet space of $\mathcal{G}$. The precise definition of this will depend on $\mathcal{G}$. For example, ${}_V\mathcal{R}$ is generated by the diffeomorphisms on source that preserve the subset V, and so in this case we take the s-jets of this diffeomorphism as $J^s\mathcal{G}$.

2.4 Vector fields on Discriminants

As usual in singularity theory, one integrates vector fields to produce diffeomorphisms that preserve a subset. In fact, there are very important types of vector fields which we can integrate to produce diffeomorphisms that preserve a subset.

Definition 2.4.1. *Suppose that V is a $\mathbb{K}$-analytic variety of $(\mathbb{K}^p, 0)$. We denote by $I(V)$ the ideal of germs vanishing on V. A vector field $\xi \in \theta_p$ is said to be* **tangent** *to V if*

$$\xi\left(I\left(V\right)\right) \subseteq I\left(V\right).$$

The module of such vector fields is denoted $\mathrm{Der}(-\mathrm{logV})$.

When $I(V) = \langle h_1, \ldots, h_q \rangle$, we write

$$\mathrm{Der}(-\mathrm{logV}) = \Big\{\xi \in \theta_p : \exists g_{ij} \in \mathcal{E}_p \text{ such that } \xi\left(h_j\right) = \sum_{i=1}^{q} g_{ij} h_i, \quad j = 1, \ldots, q\Big\}.$$

Let $h : (\mathbb{K}^p, 0) \to (\mathbb{K}, 0)$ be any defining equation for V. Then we define

$$\mathrm{Der}_0(-\log(h)) = \left\{\xi \in \theta_p : \xi\left(h\right) = 0\right\}.$$

Example 2.4.2. *Let V be the discriminant of the cusp, i.e., the image of the critical set of the map-germ in example 2.3.7(2). We use coordinates (u_1, u_2) on the target, it can be easily checked that $V = h^{-1}(0)$, where $h(u_1, u_2) = 4u_1^3 + 27u_2^2$. Consider the vector fields*

$$\eta_1 = 9u_2\frac{\partial}{\partial u_1} - 2u_1^2\frac{\partial}{\partial u_2},$$
$$\eta_2 = 2u_1\frac{\partial}{\partial u_1} + 3u_2\frac{\partial}{\partial u_2}.$$

We check that

$$\begin{aligned} \eta_1(h) &= 9u_2\frac{\partial h}{\partial u_1} - 2u_1^2\frac{\partial h}{\partial u_2} \\ &= 9u_2(12u_1^2) - 2u_1^2(54u_2) \\ &= 0. \end{aligned}$$

Similarly,

$$\begin{aligned} \eta_2(h) &= 2u_1\frac{\partial h}{\partial u_1} + 3u_2\frac{\partial h}{\partial u_2} \\ &= 2u_1(12u_1^2) + 3u_2(54u_2) \\ &= 6h. \end{aligned}$$

Hence, $\eta_1, \eta_2 \in \mathrm{Der}(-\mathrm{logV})$ whilst only $\eta_1 \in \mathrm{Der}_0(-\log(h))$. In fact, $\mathrm{Der}(-\mathrm{logV}) = \langle \eta_1, \eta_2 \rangle$.

Example 2.4.3. *Suppose that V is the complex Whitney umbrella as in Example 2.3.15. From [Dam91] and [HL09] we have*

$$\mathrm{Der}(-\mathrm{logV}) = \left\langle \begin{pmatrix} w_2 \\ 0 \\ v_1 w_1 \end{pmatrix}, \begin{pmatrix} -v_1 \\ 2w_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 2w_2 \\ v_1^2 \end{pmatrix}, \begin{pmatrix} v_1 \\ 2w_1 \\ 2w_2 \end{pmatrix} \right\rangle.$$

These vector fields labelled ξ_1^1, ξ_1^2, ξ_1^3 and ξ_e respectively. We will discuss more details in chapter 3.

We will show that ξ_1^1, ξ_1^2 and ξ_1^3 are members of $\mathrm{Der}_0(-\log(h))$ whereas ξ_e is not. A defining equation for V is $f(v_1, w_1, w_2) = w_2^2 - v_1^2 w_1$.

Then,

$$\begin{aligned}
\xi_1^1(f) &= w_2 \frac{\partial f}{\partial v_1} + (0) \frac{\partial f}{\partial w_1} + v_1 w_1 \frac{\partial f}{\partial w_2} \\
&= w_2(-2v_1 w_1) + v_1 w_1 (2w_2) \\
&= 0.
\end{aligned}$$

It can be show in a similar way that $\xi_1^2(f) = 0$ and $\xi_1^3(f) = 0$. For ξ_e we have

$$\begin{aligned}
\xi_e(f) &= v_1 \frac{\partial f}{\partial v_1} + 2w_1 \frac{\partial f}{\partial w_1} + 2w_2 \frac{\partial f}{\partial w_2} \\
&= v_1(-2v_1 w_1) + 2w_1(-v_1^2) + 2w_2(2w_2) \\
&= -2v_1^2 w_1 - 2v_1^2 w_1 + 4w_2^2 \\
&= 4(w_2^2 - v_1^2 w_1) \\
&= 4f.
\end{aligned}$$

Definition 2.4.4. *A map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is said to be* **quasihomogeneous** *or* **weighted homogeneous** *of type $(a_1, \ldots, a_n; d_1, \ldots, d_p)$, with a_i, $d_j \in \mathbb{N}$ if the relation*

$$f_j(t^{a_1} x_1, \ldots, t^{a_n} x_n) = t^{d_j} f_j(x_1, , x_n)$$

holds for each coordinate function f_j of f for all $t \in (\mathbb{K}, 0)$. The number a_i is called the weight of the variable x_i and the number d_j is the degree of the function f_j.

Let $X_1, \ldots, X_p$ denote the standard coordinates on $\mathbb{K}^p$. Then the **Euler vector**

field *denoted by* ξ_e *is given by*

$$\xi_e = \begin{pmatrix} d_1X_1 \\ \vdots \\ d_pX_p \end{pmatrix}.$$

Remark 2.4.5. *Let* $V = h^{-1}(0)$, *where* $h : (\mathbb{C}^p, 0) \to (\mathbb{C}, 0)$ *is weighted homogeneous. It was shown by Damon and Mond in lemma 3.3 of [DM91] that*

$$\mathrm{Der}(-\mathrm{logV}) = \mathrm{Der}_0(-\log(h)) \oplus \langle \xi_e \rangle .$$

That is, we can conclude that one vector field is Euler and the other annihilate the defining equation.

Remark 2.4.6. *Suppose that* V *is a* $\mathbb{K}$*-analytic variety of* $(\mathbb{K}^p, 0)$. *In [Dam87], Damon shows that*

i) $\mathrm{Der}(-\mathrm{logV})$ *is a finitely generated* $\mathcal{E}_p$*-module.*

ii) If $\xi \in \mathrm{Der}(-\mathrm{logV})$ *and* φ_t *denotes the flow generated by* ξ, *then* φ_t *preserves* V.

iii) If $\xi \in \theta_p$ *with local flow* φ_t *and* $\varphi_t(V) \subset V$, *then* $\xi \in \mathrm{Der}(-\mathrm{logV})$.

These results above do not necessarily hold for the smooth case. Damon added the coherent condition on V to get the same results (see [Dam87], p.698).

Remark 2.4.7. *In ([BR88], Section 1), Bruce and Roberts show that in the complex case if* $\xi \in \theta_p$ *and vanishes at* 0 *then the flow* φ_t *generated by* ξ *preserves* V.

We are interested in the vector fields that can be integrated to give diffeomorphisms preserving a subset. Therefore, we make the following definition.

Definition 2.4.8. *Suppose that* $(V, 0)$ *is a subgerm of* $(\mathbb{K}^p, 0)$. *We say that a* **smooth vector field on** $(\mathbb{K}^p, 0)$ **preserves** V *if it can be integrated to give a diffeomorphism that preserves* V.

Example 2.4.9. *Suppose that V is the complex Whitney umbrella as in Example 2.4.3. We consider the vector field ξ_1^1. This vector field vanishes at 0 and it can be integrated to get a diffeomorphism*

$$\varphi(v_1, w_1, w_2) = (v_1 + w_2, w_1, w_2 + v_1 w_1).$$

We can see from Example 2.3.15 that this diffeomorphism preserves V.

Now, we need the following definition in our classification in Chapter 6.

Definition 2.4.10. *Suppose that $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ and $\widetilde{h} : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ are smooth map-germs. Let Θ be a module of smooth vector fields on $(\mathbb{K}^p, 0)$, i.e., a module over $\mathcal{E}_p$ of germs at 0 of smooth vector fields on $\mathbb{K}^p$. We say that h and $\widetilde{h}$ are ${}_\Theta\mathcal{K}$-equivalent if there is a vector field $\xi \in \Theta$, that can be integrated to give a diffeomorphism Φ so that h and $\widetilde{h}$ are $\mathcal{K}$-equivalent by Φ and a germ of an invertible matrix $M : (\mathbb{K}^p, 0) \to GL(\mathbb{K}^q)$.*

2.5 Tangent Spaces and Codimensions

2.5.1 Tangent Spaces

For $\mathcal{K}$-equivalence, the tangent space is an $\mathcal{E}_n$-module, but this is not the case for $\mathcal{A}$-equivalence and it is this that leads to many problems since we attempt to classify using algebraic methods.

We shall now describe our tangent spaces.

Definition 2.5.1. *Let $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be a smooth map-germ and let Θ be a module of smooth vector fields on the domain, i.e., $\Theta \subset \theta_p$.*

i) The **extended $\mathcal{R}$-tangent space with respect to Θ**, *denoted $T_\Theta\mathcal{R}_e(h)$, is* *the submodule of $\theta(h)$ given by*

$$T_\Theta\mathcal{R}_e(h) = \langle \xi(h) \,|\, \xi \in \Theta \rangle.$$

We also call this the **Jacobian of h with respect to Θ**, *denoted by $J_\Theta(h)$.*

ii) *The* **$\mathcal{R}$-tangent space with respect to Θ**, *denoted $T_\Theta\mathcal{R}(h)$, is the submodule of $\theta(h)$ given by*

$$T_\Theta\mathcal{R}(h) = \langle \xi(h) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta(h) \rangle$$

where $\mathbf{m}_p$ is the maximal ideal in $\mathcal{E}_p$.

iii) *The* **extended $\mathcal{K}$-tangent space with respect to Θ**, *denoted $T_\Theta\mathcal{K}_e(h)$, is the submodule of $\theta(h)$ given by*

$$T_\Theta\mathcal{K}_e(h) = T_\Theta\mathcal{R}_e(h) + h^*(\mathbf{m}_q)\theta(h).$$

iv) *The* **$\mathcal{K}$-tangent space with respect to Θ**, *denoted $T_\Theta\mathcal{K}(h)$, is the submodule of $\theta(h)$ given by*

$$T_\Theta\mathcal{K}(h) = T_\Theta\mathcal{R}(h) + h^*(\mathbf{m}_q)\theta(h).$$

Example 2.5.2. *Consider Example 2.4.3. Let Θ be the set of vector fields tangent to V and let $h(v_1, w_1, w_2) = v_1 + w_1^{k+1}$ with $k \geq 1$. We have*

$$\begin{aligned} T_\Theta\mathcal{K}_e(h) &= J_\Theta(h) + \langle h \rangle \\ &= \langle \xi_1^1(h), \xi_1^2(h), \xi_1^3(h), \xi_e(h) \rangle + \langle h \rangle \\ &= \langle w_2, -v_1 + 2(k+1)w_1^{k+1}, 2(k+1)w_1^k w_2, v_1 + 2(k+1)w_1^{k+1} \rangle \\ &\quad\quad + \langle v_1 + w_1^{k+1} \rangle \\ &= \langle v_1, w_1^{k+1}, w_2 \rangle. \end{aligned}$$

Remark 2.5.3. *If all the elements of Θ vanish at the origin, the $T_\Theta\mathcal{R}(h) = T_\Theta\mathcal{R}_e(h)$ and $T_\Theta\mathcal{K}(h) = T_\Theta\mathcal{K}_e(h)$. The first equality follows from the definitions and the second follows from the first.*

Remark 2.5.4. *Suppose that Θ is the set of all vector fields on $(\mathbb{K}^p, 0)$. Then ${}_V\mathcal{K}$-equivalence and ${}_V\mathcal{R}$-equivalence are just the standard $\mathcal{K}$- and $\mathcal{R}$-equivalences respectively.*

Example 2.5.5. *Consider the D_4 singularity $h(x,y,z) = x^2 + y^2z + z^3$ with respect to the vector field module $\Theta = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z\rangle$. We have*

$$\begin{aligned} T_\Theta\mathcal{K}_e(h) &= \langle \frac{\partial h}{\partial x}, \frac{\partial h}{\partial y}, \frac{\partial h}{\partial z}\rangle + \langle h\rangle \\ &= \langle 2x, 2yz, y^2 + 3z^2\rangle + \langle x^2 + y^2z + z^3\rangle \\ &= \langle x, y^2, yz, z^3\rangle. \end{aligned}$$

Similarly we have

$$\begin{aligned} T_\Theta\mathcal{K}(h) &= \langle x\frac{\partial h}{\partial x}, x\frac{\partial h}{\partial y}, x\frac{\partial h}{\partial z}, y\frac{\partial h}{\partial x}, y\frac{\partial h}{\partial y}, y\frac{\partial h}{\partial z}, z\frac{\partial h}{\partial x}, z\frac{\partial h}{\partial y}, z\frac{\partial h}{\partial z}\rangle + \langle h\rangle \\ &= \langle 2x^2, 2xyz, xy^2 + 3z^2, 2xy, 2y^2z, y^3 + 3yz^2, 2xz, 2yz^2, y^2z + 3z^3\rangle \\ &\quad + \langle x^2 + y^2z + z^3\rangle \\ &= \langle x^2, xy, xz, y^3, y^2z, yz^2, z^3\rangle. \end{aligned}$$

Remark 2.5.6. *Let $(V, 0)$ be a subgerm of $(\mathbb{K}^p, 0)$. If Θ is the module of vector fields tangent to V, then $T_V\mathcal{R}(h) = T_\Theta\mathcal{R}(h)$ where $T_V\mathcal{R}(h)$ is the standard Singularity Theory tangent space for the equivalence ${}_V\mathcal{R}$. Similar definitions can be made for ${}_V\mathcal{K}$-equivalence and the extended versions of the tangent spaces. See for example [Dam06] and [Wal09].*

2.5.2 Codimensions

We can, in the standard way, define the codimension and extended codimension for the equivalences by taking the dimension of the $\mathbb{K}$-vector space given by the quotient of $\theta(h)$ by the relevant tangent space as follows.

Definition 2.5.7. *Let $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be a smooth map-germ and let Θ be a module of smooth vector fields on $(\mathbb{K}^p, 0)$. Suppose $\mathcal{G} = \mathcal{K}$ or $\mathcal{R}$.*

*i) The ${}_\Theta\mathcal{G}(f)$-**codimension** of h is*

$$ {}_\Theta\mathcal{G}\text{−cod}(h) = \dim_{\mathbb{K}} \frac{\theta(h)}{T_\Theta\mathcal{G}(h)}. $$

*ii) The **extended** ${}_\Theta\mathcal{G}(h)$-**codimension** of f is*

$$ {}_\Theta\mathcal{G}_e\text{−cod}(h) = \dim_{\mathbb{K}} \frac{\theta(h)}{T_\Theta\mathcal{G}_e(h)}. $$

Example 2.5.8. *Suppose that V is the complex Whitney umbrella as in Example 2.5.2, then we have*

$$ T_\Theta\mathcal{K}_e(h) = \langle v_1, w_1^{k+1}, w_2 \rangle. $$

Then,

$$ \begin{aligned} {}_\Theta\mathcal{K}_e\text{−cod}(h) &= \dim_{\mathbb{K}} \frac{\mathcal{E}_3}{T_\Theta\mathcal{K}_e(h)} \\ &= \dim_{\mathbb{K}} \frac{\mathcal{E}_3}{\langle v_1, w_1^{k+1}, w_2 \rangle} \\ &= \dim_{\mathbb{K}} \langle 1, w_1, w_1^2, \ldots, w_1^k \rangle \\ &= k + 1. \end{aligned} $$

Remark 2.5.9. *We can make similar definitions as in Definition 2.5.7 for ${}_V\mathcal{G}$-codimension and ${}_V\mathcal{G}_e$-codimension.*

2.6 $\mathcal{A}$-equivalence vs ${}_V\mathcal{K}$-equivalence

If one can describe the diffeomorphisms that preserve V one can work with ${}_V\mathcal{R}$- and ${}_V\mathcal{K}$-equivalence. One can easily see that vector fields tangent to V can be integrated to give diffeomorphisms preserving V.

Now, as stated earlier, $\mathcal{A}$-equivalence classifications are hard to do but, armed with the liftable vector fields, classifications of maps on discriminants under ${}_V\mathcal{K}$-equivalence are much easier as they are similar to $\mathcal{K}$-classifications. More importantly, $\mathcal{A}$-classifications and ${}_V\mathcal{K}$-classifications are intimately related. First we need a definition.

Definition 2.6.1 ([HW])**.** *Let S be a finite set. Suppose that $F : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$ and $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ are smooth map-germs. We define the* **sharp pullback***, denoted $h^\sharp(F)$, to be the multi-germ given by $F|(h \circ F)^{-1}(0), S) \to (h^{-1}(0), 0)$.*

Example 2.6.2. *Consider the minimal cross cap mapping of multiplicity $d = 3$, i.e.,*

$$\varphi_3(u_1, v_1, v_2, y) = \left(u_1, v_1, v_2, y^3 + u_1 y, v_1 y + v_2 y^2\right).$$

Suppose that $h(u_1, v_1, v_2, w_1, w_2) = (v_2 - w_1, u_1)$. Thus $h \circ \varphi_3(u_1, v_1, v_2, y) = 0$ gives $u_1 = 0$ and $v_2 = y^3$.

Using coordinates $X = v_1$ and $Y = y$ on $(h \circ \varphi_3)^{-1}(0)$, we see that the map $\varphi_3|(h \circ \varphi_3)^{-1}(0), 0) \to (h^{-1}(0), 0)$ becomes

$$(X, Y) \mapsto (X, Y^3, XY + Y^5).$$

This map-germ is the H_2 singularity of Mond (see [Mon85], theorem 1.1). We will give more details in Chapter 7.

The connection between $\mathcal{A}$-equivalence and ${}_V\mathcal{K}$-equivalence is the following.

Theorem 2.6.3 ([HW])**.** *Suppose that $F : (\mathbb{K}^n, S) \to (\mathbb{K}^p, 0)$, $n < p$, is a stable map with discriminant V. Let $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ and $\widetilde{h} : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be submersions with $h^{-1}(0)$ and $\widetilde{h}^{-1}(0)$ transverse to F. (These conditions ensure that $h^\sharp(F)$ and $\widetilde{h}^\sharp(F)$ are maps between manifolds). Let $h^\sharp(F)$ and $\widetilde{h}^\sharp(F)$ be finitely $\mathcal{A}$-determined. Then*

$$h^\sharp(F) \sim_{\mathcal{A}} \widetilde{h}^\sharp(F) \iff h \sim_{{}_V\mathcal{K}} \widetilde{h}.$$

One advantage of Theorem 2.6.3 is that one can classify map-germs under ${}_V\mathcal{K}$-equivalence and then do a sharp pullback to get an $\mathcal{A}$-equivalence classification.

2.7 Simple Map-Germs and Moduli

In any classification of map-germs the simple singularities are extremely important. This notion was introduced by Arnol'd for right equivalence in a series of papers. More details on the modality under various circumstances can be found in [AGV85].

Definition 2.7.1 ([AGV85])**.** *Let X be a manifold and G a Lie group acting on X. The modality of a point $x \in X$ under the action of G on X is the least number m such that a sufficiently small neighbourhood of x may be covered by a finite number of m-parameter families of orbits . The point x is said to be simple if its modality is 0, that is, a sufficiently small neighbourhood intersects only a finite number of orbits.*

Chapter 3

A basis for Der_0(-log) of the minimal cross cap

We are interested in the vector fields that can be integrated to give diffeomorphisms preserving a subset. In this chapter we discuss the module of vector fields tangent to V, where V is the image of the image of the minimal cross cap of multiplicity $d \geq 2$. In fact, we will give a basis for $\mathrm{Der}_0(-\log(h))$.

3.1 Vector fields liftable over corank 1 stable maps

In this section we will give the explicit description of vector fields liftable over the minimal cross cap of multiplicity $d \geq 2$ [HL09].

Definition 3.1.1. *Let $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a smooth map-germ. A vector field ξ on $(\mathbb{K}^p, 0)$ is* **liftable** *over f if there is a vector field η on $(\mathbb{K}^n, 0)$ such that $df \circ \eta = \xi \circ f$.*

That is, the following diagram commutes

$$\begin{array}{ccc} T(\mathbb{K}^n,0) & \xrightarrow{df} & T(\mathbb{K}^p,0) \\ \eta\uparrow & & \uparrow\xi \\ (\mathbb{K}^n,0) & \xrightarrow{f} & (\mathbb{K}^p,0) \end{array}$$

In these circumstances η is called **lowerable**. *The set of vector field germs liftable over f is denoted* $\mathrm{Lift}(f)$ *and is an $\mathcal{E}_p$-module.*

The notion of liftable and tangent vector fields on the discriminant are equivalent for stable map-germs with $\mathbb{K} = \mathbb{C}$ ([Dam91], Lemma 2.2). In [Arn76], Arnol'd shows that there exist liftable vector fields that are not tangent when $\mathbb{K} = \mathbb{R}$.

In [HL09], Houston and Littlestone give three families of vector fields with the Euler vector field that are liftable over the minimal cross cap mapping of multiplicity $d \geq 2$. In fact, they proved that these vector fields generate the module of liftable vector fields over φ_d in the case of $\mathbb{K} = \mathbb{C}$. In the case of $\mathbb{K} = \mathbb{R}$, they show that the module of polynomial vector fields liftable over φ_d is generated by these vector fields. We shall now describe these families with the Euler vector field as in [HL09].

For $1 \leq f \leq 3$ and $1 \leq j \leq d-1$, we denote the members of the families by

$$\xi_j^f = \sum_{i=1}^{d-2} A_{i,j}^f \frac{\partial}{\partial u_i} + \sum_{i=1}^{d-1} B_{i,j}^f \frac{\partial}{\partial v_i} + \sum_{i=1}^{2} C_{i,j}^f \frac{\partial}{\partial w_i}.$$

We can consider the members as

$$\xi_j^f = \begin{pmatrix} A_{1,j}^f \\ \vdots \\ A_{d-2,j}^f \\ B_{1,j}^f \\ \vdots \\ B_{d-1,j}^f \\ C_{1,j}^f \\ C_{2,j}^f \end{pmatrix}$$

We have $u_{d-1} = v_d = 0$, $u_d = 1$ and $u_r = v_r = 0$ for $r \leq 0$ and for $r > d$.

i) **First family:** The components of each vector field in this family are given by

$$\begin{aligned}
A^1_{i,j} &= (d-i)(d-j)u_i u_j, \qquad 1 \leq i \leq d-2,\\
B^1_{i,j} &= d\sum_{r=1}^{i-1} u_{i+j-r}v_r - d\sum_{r=1}^{i} u_r v_{i+j-r} - (i-1)(d-j)u_j v_i\\
&\quad + dv_{i+j}w_1 - du_{i+j}w_2, \qquad 1 \leq i \leq d-1,\\
C^1_{1,j} &= d(d-j)u_j w_1,\\
C^1_{2,j} &= -dv_j w_1 + (d-j)u_j w_2.
\end{aligned}$$

ii) **Second family:** The components of each vector field in this family are given by

$$\begin{aligned}
A^2_{i,j} &= -d(d+i-j+1)u_{d+i-j+1}w_1 + d\sum_{r=1}^{i}(d+i-j-2r+1)u_r u_{d+i-j-r+1}\\
&\quad -j(i+1)u_{i+1}u_{d-j}, \qquad 1 \leq i \leq d-2,\\
B^2_{i,j} &= -d(k+i-j+1)v_{d+i-j+1}w_1 + d\sum_{r=1}^{i}(d+i-j-r+1)u_r v_{d+i-j-r+1}\\
&\quad -d\sum_{r=1}^{i} r u_{d+i-j-r+1}v_r - j(i+1)u_{d-j}v_{i+1}, \qquad 1 \leq i \leq d-1,\\
C^2_{1,j} &= d(d-j+1)u_{d-j+1}w_1 + ju_1 u_{d-j},\\
C^2_{2,j} &= d(d-j+1)v_{d-j+1}w_1 + jv_1 u_{d-j}.
\end{aligned}$$

iii) **Third family:** The components of each vector field in this family are given by

$$\begin{aligned}
A^3_{i,j} &= -d(d+i-j+1)u_{d+i-j+1}w_2 + d\sum_{r=1}^{i}(d+i-j-r+1)u_{d+i-j-r+1}v_r \\
&\quad -d\sum_{r=1}^{i} r u_r v_{d+i-j-r+1} - d(i+1)u_{i+1}v_{d-j}, \qquad 1 \leq i \leq d-2, \\
B^3_{i,j} &= -d(d+i-j+1)v_{d+i-j+1}w_2 + d\sum_{r=1}^{i}(d+i-j-2r+1)v_r v_{d+i-j-r+1} \\
&\quad -d(i+1)v_{i+1}v_{d-j}, \qquad 1 \leq i \leq d-1, \\
C^3_{1,j} &= d(d-j+1)u_{d-j+1}w_2 + du_1v_{d-j} \\
C^3_{2,j} &= d(d-j+1)v_{d-j+1}w_2 + dv_1v_{d-j}.
\end{aligned}$$

iv) **The Euler vector field for the map φ_d:**

$$\xi_e = \sum_{i=1}^{d-2}(d-i)u_i\frac{\partial}{\partial u_i} + \sum_{i=1}^{d-1}(d-i)v_i\frac{\partial}{\partial v_i} + d\sum_{i=1}^{2} w_i\frac{\partial}{\partial w_i}.$$

Theorem 3.1.2 ([HL09]). *Let $\varphi_d : (\mathbb{C}^{2d-2}, 0) \to (\mathbb{C}^{2d-1}, 0)$ be given by the normal form for a corank 1 minimal stable map of multiplicity $d \geq 2$ and V be its image. Then,*

$$\mathrm{Der}(-\mathrm{logV}) = \left\langle \xi^1_j, \xi^2_j, \xi^3_j, \xi_e \right\rangle_{j=1}^{d-1}.$$

This result above does not necessarily hold for the real analytic or smooth vector fields in the real case. However, Houston and Littlestone give the following theorem for polynomial vector fields.

Theorem 3.1.3 ([HL09]). *Let $\varphi_d : (\mathbb{R}^{2d-2}, 0) \to (\mathbb{R}^{2d-1}, 0)$ be given by the normal form for a corank 1 minimal stable map of multiplicity $d \geq 2$ and V be its image.*

Then, the module of polynomial vector fields liftable over φ_d is generated by the vector fields ξ^1_j, ξ^2_j, ξ^3_j for $1 \leq j \leq d-1$, together with the Euler vector field ξ_e.

In [HL09], K. Houston and D. Littlestone made the following conjecture.

Conjecture 3.1.4. *Let $\varphi_d : (\mathbb{C}^{2d-2}, 0) \to (\mathbb{C}^{2d-1}, 0)$ be the minimal cross cap of multiplicity $d \geq 2$ and V be its image. Then the vector fields ξ_j^1, ξ_j^2, ξ_j^3 for $1 \leq j \leq d-1$ generate* $\mathrm{Der}_0(-\log(h))$.

From Proposition 2.5 in [BW98], the first statement is true for $d = 2$. In this chapter, we shall give an answer to the conjecture above. The essential idea of our proof is to find the defining equation for the image of the minimal cross cap of multiplicity $d \geq 2$ and prove that the cross cap liftable vector fields annihilate this defining equation.

3.2 A defining equation for the image of the minimal cross cap

In this section we shall compute the defining equation for the image of the minimal cross cap of multiplicity $d \geq 2$.

Definition 3.2.1. *A map-germ $F : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is* **finite** *if it is continuous, closed and the fiber $F^{-1}(y)$ is finite for all $y \in (\mathbb{C}^p, 0)$.*

Let X be a Cohen-Macaulay space of dimension n and $F : (X, x) \to (\mathbb{C}^{n+1}, 0)$ be a finite map-germ. We can use the algorithm of Mond and Pellikaan to determine the corresponding defining equation for the image (see [MP89], section 2). An algorithm consists basically of the following steps:

1. Choose a projection $\pi : (\mathbb{C}^{n+1}, 0) \to (\mathbb{C}^n, 0)$ such that $\widetilde{F} = \pi \circ F$ is finite.

2. After a coordinate change we may suppose that $F(x) = (\widetilde{F}(x), F_{n+1}(x))$. Let X_{n+1} denote the last component of the coordinate system on $\mathbb{C}^{n+1}$ so that $F_{n+1} = X_{n+1} \circ F$.

3. Let $1, g_1, g_2, \ldots, g_k$ be generators of $Q(\widetilde{F})$, where $Q(\widetilde{F})$ is the local algebra of $\widetilde{F}$. Put $g_0 = 1$ and find elements $\alpha_{i,j} \in \mathcal{O}_n$, $0 \leq i, j \leq k$, such that

$$g_j F_{n+1} = \sum_{i=0}^{k} \left(\alpha_{i,j} \circ \widetilde{F}\right) g_i.$$

4. Define a matrix $\lambda = (\lambda_{i,j})$ by letting

   - $\lambda_{i,j} = \alpha_{i,j} \circ \pi$ for $i \neq j$,
   - $\lambda_{i,i} = \alpha_{i,i} \circ \pi - X_{n+1}$.

5. A defining equation for the image of F is given by the determinant of the matrix λ.

Example 3.2.2. *Consider the cross cap mapping, i.e., $\varphi_2(v_1, y) = (v_1, y^2, v_1 y)$. We choose a projection $\pi : (\mathbb{C}^3, 0) \to (\mathbb{C}^2, 0)$ such that $\pi(v_1, w_1, w_2) = (v_1, w_1)$. Then we have*

$$\begin{aligned} \widetilde{\varphi}_2(v_1, y) &= \pi \circ \varphi_2(v_1, y) \\ &= (v_1, y^2). \end{aligned}$$

We find that $Q(\widetilde{\varphi}_2)$ is generated by 1 and y. By solving the following equations

$$\begin{aligned} v_1 y &= \alpha_{0,0}(v_1, y^2) + \alpha_{1,0}(v_1, y^2) y \quad \text{and} \\ v_1 y^2 &= \alpha_{0,1}(v_1, y^2) + \alpha_{1,1}(v_1, y^2) y. \end{aligned}$$

We find $\alpha_{0,0}(v_1, y^2) = 0$, $\alpha_{1,0}(v_1, y^2) = v_1$, $\alpha_{0,1}(v_1, y^2) = v_1 y^2$ and $\alpha_{1,1}(v_1, y^2) = 0$. Now,

$$\begin{aligned} \lambda_{0,0} &= \alpha_{0,0} \circ \pi(v_1, w_1) - w_2 \\ &= 0 - w_2 \\ &= -w_2, \end{aligned}$$

$$\begin{aligned}\lambda_{1,1} &= \alpha_{1,1}\circ\pi(v_1,w_1) - w_2\\ &= 0 - w_2\\ &= -w_2,\end{aligned}$$

$$\begin{aligned}\lambda_{1,0} &= \alpha_{1,0}\circ\pi(v_1,w_1)\\ &= v_1,\end{aligned}$$

$$\begin{aligned}\lambda_{0,1} &= \alpha_{0,1}\circ\pi(v_1,w_1)\\ &= v_1w_1.\end{aligned}$$

We obtain the matrix

$$\lambda = \begin{pmatrix} -w_2 & v_1 \\ v_1w_1 & -w_2 \end{pmatrix}.$$

A defining equation for the image of φ_2 is given by the determinant of the matrix λ, i.e.,

$$\begin{aligned}H &= \det(\lambda)\\ &= w_2^2 - v_1^2w_1.\end{aligned}$$

In general, we shall use the algorithm above to find a defining equation for the image of the minimal cross cap of multiplicity $d \geq 2$.

Theorem 3.2.3. *Let $\varphi_d : (\mathbb{C}^{2d-2}, 0) \to (\mathbb{C}^{2d-1}, 0)$ be the minimal cross cap of multiplicity $d \geq 2$ and V be its image. Then, a defining equation for V is given by the determinant of the matrix $\lambda = M - w_2\mathrm{I_d}$ where $\mathrm{I_d}$ is the identity matrix and $M = [m_{i,j}]_{d\times d}$ is such that*

1.

$$m_{l,d} = v_{d-l} - \sum_{k=1}^{l-2} m_{k,d}u_{d-l+k} \quad \textit{for } 1 \leq l \leq d,$$

2.

$$m_{i,j} = v_{j-i} + m_{i-j,d}w_1 - \sum_{l=1}^{i-1} m_{l,d}u_{j-i+l} \quad \textit{for } 1 \leq i, j \leq d,$$

with $m_{r,d} = 0$ *for all* $r \leq 0$.

Proof. We shall use the algorithm of Mond and Pellikaan. Since $u_{d-1} = 0$ and $u_d = 1$, then we have

$$w_1 = \sum_{s=1}^{d} u_s y^s, \quad w_2 = \sum_{t=1}^{d-1} v_t y^t.$$

We choose a projection $\pi : (\mathbb{C}^{2d-1}, 0) \to (\mathbb{C}^{2d-2}, 0)$ such that

$$\pi(\underline{u}, \underline{v}, w_1, w_2) = (\underline{u}, \underline{v}, w_1).$$

Then we have

$$\begin{aligned} \widetilde{\varphi}_d(\underline{u}, \underline{v}, y) &= \pi \circ \varphi_d(\underline{u}, \underline{v}, y) \\ &= (\underline{u}, \underline{v}, w_1) \\ &= \left(\underline{u}, \underline{v}, \sum_{s=1}^{d} u_s y^s\right). \end{aligned}$$

We find that $Q(\widetilde{\varphi}_d)$ is generated by $1, y, \ldots, y^{d-1}$.

We can rewrite

$$m_{l,d} = v_{d-l} - \sum_{k=1}^{l-2} m_{k,d}u_{d-l+k} \quad \text{for } 1 \leq l \leq d,$$

As follows

$$\sum_{k=1}^{l} m_{k,d}u_{d-l+k} = v_{d-l} \quad \text{for } 1 \leq l \leq d.$$

Now we need to show that

$$\sum_{j=1}^{d} m_{i,j}y^{j-1} = y^{i-1}w_2 \qquad \text{for all } 1 \leq i \leq d.$$

We have

$$\sum_{j=1}^{d} m_{i,j}y^{j-1} = \sum_{j=1}^{d}\left(v_{j-i} + m_{i-j,d}w_1 - \sum_{l=1}^{i-1} m_{l,d}u_{j-i+l}\right)y^{j-1}$$

$$= \sum_{j=1}^{d} v_{j-i}y^{j-1} + \sum_{j=1}^{d} m_{i-j,d}w_1 y^{j-1} - \sum_{j=1}^{d}\left(\sum_{l=1}^{i-1} m_{l,d}u_{j-i+l}\right)y^{j-1}.$$

By substituting $w_1 = \sum_{s=1}^{d} u_s y^s$ in the second term on the RHS, we get

$$\begin{aligned}
\sum_{j=1}^{d} m_{i,j}y^{j-1} &= \sum_{j=1}^{d} v_{j-i}y^{j-1} + \sum_{j=1}^{d} m_{i-j,d}\left(\sum_{s=1}^{d} u_s y^s\right)y^{j-1} - \sum_{j=1}^{d}\left(\sum_{l=1}^{i-1} m_{l,d}u_{j-i+l}\right)y^{j-1} \\
&= \sum_{j=1}^{d} v_{j-i}y^{j-1} + \sum_{j=1}^{d} m_{i-j,d}\left(\sum_{s=1}^{d} u_s y^{s+j-1}\right) - \sum_{j=1}^{d}\left(\sum_{l=1}^{i-1} m_{l,d}u_{j-i+l}\right)y^{j-1} \\
&= \sum_{j=1}^{d} v_{j-i}y^{j-1} + \sum_{l=1}^{i-1} m_{l,d}\left(\sum_{s=1}^{d} u_s y^{s+i-l+1}\right) - \sum_{l=1}^{i-1} m_{l,d}\left(\sum_{j=1}^{d} u_{j-i+l}y^{j-1}\right) \\
&= \sum_{j=1}^{d} v_{j-i}y^{j-1} + \sum_{l=1}^{i-1} m_{l,d}\left(\sum_{s=1}^{d} u_s y^{s+i-l+1}\right) - \sum_{l=1}^{i-1} m_{l,d}\left(\sum_{s=1}^{d-i+l} u_s y^{s+i-l-1}\right) \\
&= \sum_{j=1}^{d} v_{j-i}y^{j-1} + \sum_{l=1}^{i-1} m_{l,d}\left(\sum_{s=1}^{d-i+l} u_s y^{s+i-l+1}\right) \\
&\quad + \sum_{l=1}^{i-1} m_{l,d}\left(\sum_{s=d-i+l+1}^{d} u_s y^{s+i-l+1}\right) - \sum_{l=1}^{i-1} m_{l,d}\left(\sum_{s=1}^{d-i+l} u_s y^{s+i-l-1}\right) \\
&= \sum_{j=1}^{d} v_{j-i}y^{j-1} + \sum_{l=1}^{i-1} m_{l,d}\left(\sum_{s=d-i+l+1}^{d} u_s y^{s+i-l+1}\right) \\
&= \sum_{j=1}^{d} v_{j-i}y^{j-1} + \sum_{l=1}^{i-1} m_{l,d}\left(\sum_{t=1}^{i-l} u_{d-i+l+t}y^{d+t-1}\right)
\end{aligned}$$

We rewrite the second term on the RHS above as follows:

$$\begin{aligned}
\sum_{l=1}^{i-1} m_{l,d}\left(\sum_{t=1}^{i-l} u_{d-i+l+t}y^{d+t-1}\right) &= m_{1,d}\left(\sum_{t=1}^{i-1} u_{d-i+t+1}y^{d+t-1}\right) + m_{2,d}\left(\sum_{t=1}^{i-2} u_{d-i+t+2}y^{d+t-1}\right) \\
&\quad + \cdots + m_{i-1,d}\left(\sum_{t=1}^{1} u_{d+t-1}y^{d+t-1}\right) \\
&= \sum_{t=1}^{i-1}\Big(m_{1,d}u_{d-i+t+1} + m_{2,d}u_{d-i+t+2} + \cdots + m_{i-2,d}u_{d+t-2} \\
&\quad + m_{i-1,d}u_{d+t-1}\Big)y^{t+i-1}
\end{aligned}$$

$$= \sum_{t=1}^{i-1}\left(\sum_{k=1}^{i-t} m_{k,d}u_{d-i+t+k}\right)y^{t+i-1}.$$

Therefore, we have

$$\begin{aligned}\sum_{j=1}^{d} m_{i,j}y^{j-1} &= \sum_{j=1}^{d} v_{j-i}y^{j-1} + \sum_{t=1}^{i-1}\left(\sum_{k=1}^{i-t} m_{k,d}u_{d-i+t+k}\right)y^{d+t-1} \\ &= \sum_{j=1}^{d} v_{j-i}y^{j-1} + \sum_{t=1}^{i-1} v_{d-i+t}y^{d+t-1} \\ &= \sum_{t=1}^{d-i} v_t y^{s+i-1} + \sum_{t=1}^{i-1} v_{d-i+t}y^{d+t-1} \\ &= \sum_{t=1}^{d-1} v_t y^{s+i-1} \\ &= y^{i-1}w_2.\end{aligned}$$

□

Now, from Theorem 3.2.3, a defining equation for the image of the minimal cross cap of multiplicity $d \geq 2$ is given by

$$H_d = w_2^d - T(M)w_2^{d-1} + G,$$

where T is the trace of M and G is a polynomial in $u_1, \ldots, u_{d-2}, v_1, \ldots, v_{d-1}, w_1$ and w_2 whose degree in w_2 is $\leq d-2$.

Corollary 3.2.4. *We have*

i.

$$T = \sum_{l=1}^{d-2}(d-l)m_{l,d}u_l,$$

ii. *T is not dependent on v_1 and w_1.*

Proof.

i. By definition,

$$\begin{aligned}
T &= \sum_{i=1}^{d} m_{i,i} \\
&= \sum_{i=1}^{d}\Big(-\sum_{l=1}^{i-1} m_{l,d}u_l\Big), \quad \text{by Theorem 3.2.3(2)}, \\
&= -\Big(\sum_{l=1}^{0} m_{l,d}u_l + \sum_{l=1}^{1} m_{l,d}u_l + \cdots + \sum_{l=1}^{d-1} m_{l,d}u_l\Big) \\
&= -m_{1,d}u_1 - (m_{1,d}u_1 + m_{2,d}u_2) - (m_{1,d}u_1 + m_{2,d}u_2 + m_{3,d}u_3) \\
&\qquad - \cdots - (m_{1,d}u_1 + m_{2,d}u_2 + \cdots + m_{d-1,d}u_{d-1}) \\
&= -(d-1)m_{1,d}u_1 - (d-2)m_{2,d}u_2 - \cdots - 2m_{d-2,d}u_{d-2} \\
&= -\sum_{l=1}^{d-2}(d-l)m_{l,d}u_l \\
&= \sum_{l=1}^{d-2}(l-d)m_{l,d}u_l.
\end{aligned}$$

ii. For $d = 2$ we have $T = 0$. For $d \geq 3$, we will use induction. From Theorem 3.2.3(1) we see that $m_{1,d} = v_{d-1}$, $m_{2,d} = v_{d-2}$ and $m_{3,d} = v_{d-3} - v_{d-1}u_{d-2}$ are not dependent on v_1 and w_1. Then, we can suppose that $m_{n,d}$ is not dependent on v_1 and w_1.

We have
$$m_{n+1,d} = v_{d-n-1} - \sum_{k=1}^{n-1} m_{k,d}u_{d-n+k-1}$$
with $1 \leq n < n+1 \leq d-2$. We know that $m_{1,d}$, $m_{2,d}$, $\ldots$, $m_{n-1,d}$ are not dependent on v_1 and w_1. If $d-n-1 = 1$, then $n = d-2$ and this is a contradiction. Therefore, $m_{n+1,d}$ is not dependent on v_1 and w_1. Hence T is not dependent on v_1 and w_1.

□

In the following example, we shall obtain a defining equation of the minimal cross cap φ_3 by using Theorem 3.2.3.

Example 3.2.5. *Consider the minimal cross cap $\varphi_3 : (\mathbb{C}^4, 0) \to (\mathbb{C}^5, 0)$. Then we have*

$$
\begin{aligned}
m_{1,3} &= v_{3-1} - \sum_{k=1}^{1-2} m_{k,3} u_{3-1+k} \\
&= v_2, \\
m_{2,3} &= v_{3-2} - \sum_{k=1}^{2-2} m_{k,3} u_{3-2+k} \\
&= v_1, \\
m_{3,3} &= v_{3-3} - \sum_{k=1}^{3-2} m_{k,3} u_{3-3+k} \\
&= v_0 - m_{1,3} u_1 \\
&= -v_2 u_1,
\end{aligned}
$$

$$
\begin{aligned}
m_{1,1} &= v_{1-1} + m_{1-1,3} w_1 - \sum_{l=1}^{1-1} m_{l,3} u_{1-1+l} \\
&= v_0 + m_{0,3} w_1 - \sum_{l=1}^{0} m_{l,3} u_l \\
&= 0, \\
m_{1,2} &= v_{2-1} + m_{1-2,3} w_1 - \sum_{l=1}^{1-1} m_{l,3} u_{2-1+l} \\
&= v_1 + m_{-1,3} w_1 - \sum_{l=1}^{0} m_{l,3} u_2 \\
&= v_1, \\
m_{2,1} &= v_{1-2} + m_{2-1,3} w_1 - \sum_{l=1}^{2-1} m_{l,3} u_{1-2+l} \\
&= v_{-1} + m_{1,3} w_1 - \sum_{l=1}^{1} m_{l,3} u_{l-1} \\
&= 0 + v_2 w_1 - v_2 u_0 \\
&= v_2 w_1.
\end{aligned}
$$

In the same way we find $m_{2,2} = -v_2u_1$, $m_{3,1} = v_1w_1$ and $m_{3,2} = v_2w_1 - v_1u_1$.

It follows

$$\lambda = \begin{pmatrix} -w_2 & v_1 & v_2 \\ v_2w_1 & -w_2 - u_1v_2 & v_1 \\ v_1w_1 & v_2w_1 - u_1v_1 & -w_2 - u_1v_2 \end{pmatrix}.$$

A defining equation for the image of φ_3 is given by the determinant of the matrix λ, i.e.,

$$\begin{aligned} H &= \det(\lambda) \\ &= -w_2^3 + v_1^3w_1 - u_1v_1^2w_2 + 3v_1v_2w_1w_2 - 2u_1v_2w_2^2 + u_1v_1v_2^2w_1 + v_2^3w_1^2 - u_1^2v_2^2w_2 \\ &= -w_2^3 - 2u_1v_2w_2^2 + (3v_1v_2w_1 - u_1v_1^2 - u_1^2v_2^2)w_2 + (v_1^3w_1 + u_1v_1v_2^2w_1 + v_2^3w_1^2). \end{aligned}$$

This equation agrees with the calculation in Example 5.2.18 of Chapter 5 by using the `CAST` package.

3.3 Der$_0$(-log) of the minimal cross cap

In this section we shall show Conjecture 3.1.4 is true.

Proposition 3.3.1. *For $1 \leq i \leq d-1$ we have*

1.

$$\sum_{i=1}^{d-1} u_{i+j}\frac{\partial T}{\partial v_i} + (d-j)u_j = 0.$$

2.

$$\sum_{i=1}^{d-2}(d+i-j+1)u_{d+i-j+1}\frac{\partial T}{\partial u_i} + \sum_{i=1}^{d-1}(d+i-j+1)v_{d+i-j+1}\frac{\partial T}{\partial v_i} + d(d-j+1)v_{d-j+1} = 0.$$

Proof. For $1 \leq j \leq d-1$ all vector fields in the second family are tangent to V, i.e., $\xi_j^2(H_d) = g_j^2(\underline{u}, \underline{v}, \underline{w})H_d$ for some polynomials g_j^2.

We can see that none of the coefficients of the vector fields in this family contain w_2. It follows we have that

$$\begin{aligned}\xi_j^2(H_d) &= \sum_1^{d-2} A_{i,j}^2 \frac{\partial H_d}{\partial u_i} + \sum_1^{d-1} B_{i,j}^2 \frac{\partial H_d}{\partial v_i} + C_{1,j}^2 \frac{\partial H_d}{\partial w_1} + C_{2,j}^2 \frac{\partial H_d}{\partial w_2} \\ &= \sum_1^{d-2} A_{i,j}^2 \left(-\frac{\partial T}{\partial u_i} w_2^{d-1} + \frac{\partial G}{\partial u_i}\right) + \sum_1^{d-1} B_{i,j}^2 \left(-\frac{\partial T}{\partial v_i} w_2^{d-1} + \frac{\partial G}{\partial v_i}\right) \\ &\quad + C_{1,j}^2 \left(\frac{\partial G}{\partial w_1}\right) + C_{2,j}^2 \left(d w_2^{d-1} - (d-1) T w_2^{d-2} + \frac{\partial G}{\partial w_2}\right)\end{aligned}$$

From Corollary 3.2.4, we can see that T does not depend on w_2. Therefore, we get

$$\xi_j^2(H_d) = \left(-\sum_1^{d-2} A_{i,j}^2 \frac{\partial T}{\partial u_i} - \sum_1^{d-1} B_{i,j}^2 \frac{\partial T}{\partial v_i} + d C_{2,j}^2\right) w_2^{d-1} + G^*,$$

where G^* is a polynomial in several variables whose degree in w_2 is $\leq d-2$.

It follows that we have $\xi_j^2(H_d)$ is a polynomial in $u_1, \dots, u_{d-2}, v_1, \dots, v_{d-1}, w_1$ and w_2 whose degree in w_2 is $\leq d-1$. However, $\gamma_j^2(\underline{u}, \underline{v}, \underline{w}) H_d$ is a polynomial in several variables whose degree in w_2 is $\geq d$. Therefore, we have that $g_j^2 = 0$. Hence $\xi_j^2(H_d) = 0$ for all $1 \leq j \leq d-1$.

It follows that

$$-\sum_1^{d-2} A_{i,j}^2 \frac{\partial T}{\partial u_i} - \sum_1^{d-1} B_{i,j}^2 \frac{\partial T}{\partial v_i} + d C_{2,j}^2 = 0.$$

Then we have

$$\begin{aligned}&-\overbrace{\left(\sum_{i=1}^{d-2}(d+i-j+1)u_{d+i-j+1}\frac{\partial T}{\partial u_i} + \sum_{i=1}^{d-1}(d+i-j+1)v_{d+i-j+1}\frac{\partial T}{\partial v_i} + d(d-j+1)v_{d-j+1}\right)}^{\Omega_1 :=} w_1 \\ &+d\overbrace{\left(\sum_{i=1}^{d-1} u_{d+i-j}\frac{\partial T}{\partial v_i} + j u_{d-j}\right)}^{\Omega_2 :=} v_1 - \overbrace{\left(\sum_{i=1}^{d-2}\left(d\sum_{r=1}^{i}(d+i-j-2r+1)u_r u_{d+i-j-r+1} + j(i+1)u_{i+1}u_{d-j}\right)\right)\frac{\partial T}{\partial u_i}}^{\Omega_3 :=} \\ &-\overbrace{\left(\sum_{i=1}^{d-1}\left(d\sum_{r=1}^{i}(d+i-j-r+1)u_r v_{d+i-j-r+1} + d\sum_{r=2}^{i} r u_{d+i-j-r+1}v_r + j(i+1)u_{d-j}v_{i+1}\right)\right)\frac{\partial T}{\partial v_i}}^{\Omega_4 :=} \quad = \quad 0.\end{aligned}$$

From Corollary 3.2.4, T does not contain v_1 and w_1. Hence $\frac{\partial T}{\partial u_i}$ and $\frac{\partial T}{\partial v_i}$ do not contain v_1 and w_1. Therefore Ω_2, Ω_3, Ω_4 do not contain w_1. Also we can see that Ω_1, Ω_3, Ω_4 do not contain v_1 because $3 \leq d+i-j+1$ and if $d+i-j-r+1 = 1$, then $r = d+i-j$. Now, if $1 \leq r \leq i$, then $1 \leq d+i-j \leq i$ and this means $1-i \leq d-j \leq 0$ and this is a contradiction.

Therefore, we have $\Omega_1 = 0$ and this proves part 2. Also $\Omega_2 = 0$, i.e.,

$$\sum_{i=1}^{d-1} u_{d+i-j}\frac{\partial T}{\partial v_i} + ju_{d-j} = 0.$$

Let $l = d-j$, then $1 \leq l \leq d-1$ and we get

$$\sum_{i=1}^{d-1} u_{i+})\frac{\partial T}{\partial v_i} + (d-l)u_l = 0.$$

This proves part 1. □

In the following theorem we shall give the answer of the first part of the Conjecture 3.1.4.

Theorem 3.3.2. *Let $\varphi_d : (\mathbb{C}^{2d-2}, 0) \to (\mathbb{C}^{2d-1}, 0)$ be given by the normal form for a corank 1 minimal stable map of multiplicity $d \geq 2$ and V be its image. Then,*

$$\mathrm{Der}_0(-\log(h)) = \left\langle \xi_j^1, \xi_j^2, \xi_j^3 \right\rangle_{j=1}^{d-1}.$$

Proof. From the proof of Proposition 3.3.1 we have $\xi_j^2(H_d) = 0$ for all $1 \leq j \leq d-1$. We want to show that for all $1 \leq j \leq d-1$, $\xi_j^1(H_d) = 0$ and $\xi_j^3(H_d) = 0$.

We consider the liftable vector fields ξ_j^1 and ξ_j^3. Then they are tangent vector fields, that is for $i = 1$ or $i = 3$ we have

$$\xi_j^i(H_d) = g_j^i(\underline{u}, \underline{v}, \underline{w})H_d \quad \text{for some polynomials } g_j^i.$$

Now,

$$\begin{aligned}
\xi_j^1(H_d) &= \sum_1^{d-2} A_{i,j}^1 \frac{\partial H_d}{\partial u_i} + \sum_1^{d-1} B_{i,j}^1 \frac{\partial H_d}{\partial v_i} + C_{1,j}^1 \frac{\partial H_d}{\partial w_1} + C_{2,j}^1 \frac{\partial H_d}{\partial w_2} \\
&= d^2\left(\sum_{i=1}^{d-1} u_{i+j}\frac{\partial T}{\partial v_i} + (d-j)u_j\right)w_2^d + G^{**},
\end{aligned}$$

where G^{**} is a polynomial in $u_1, \ldots, u_{d-2}, v_1, \ldots, v_{d-1}, w_1$ and w_2 whose degree in w_2 is $\leq d-1$.

From Proposition 3.3.1(1) we have

$$\xi_j^1(H_d) = d^2(0)w_2^d + G^{**}.$$

This means $\xi_j^1(H_d)$ is a polynomial in $u_1, \ldots, u_{d-2}, v_1, \ldots, v_{d-1}, w_1$ and w_2 whose degree in w_2 is $\leq d-1$. However, $g_j^1(\underline{u}, \underline{v}, \underline{w})H_d$ is a polynomial in several variables whose degree in w_2 is $\geq d$.

It follows $g_j^1 = 0$. Hence $\xi_j^1(H_d) = 0$.

Similarly, we have

$$\begin{aligned}\xi_j^3(H_d) &= \sum_1^{d-2} A_{i,j}^3 \frac{\partial H_d}{\partial u_i} + \sum_1^{d-1} B_{i,j}^3 \frac{\partial H_d}{\partial v_i} + C_{1,j}^3 \frac{\partial H_d}{\partial w_1} + C_{2,j}^3 \frac{\partial H_d}{\partial w_2} \\ &= d\Bigg(\sum_{i=1}^{d-2}(d+i-j+1)u_{d+i-j+1}\frac{\partial T}{\partial u_i} + \sum_{i=1}^{d-1}(d+i-j+1)v_{d+i-j+1}\frac{\partial T}{\partial v_i} \\ &\qquad +d(d-j+1)v_{d-j+1}\Bigg)w_2^d + G^{***},\end{aligned}$$

where G^{***} is a polynomial in several variables whose degree in w_2 is $\leq d-1$.

From Proposition 3.3.1(2) we have

$$\xi_j^3(H_d) = d(0)w_2^d + G^{***}.$$

It follows $g_j^3 = 0$. Therefore, we have $\xi_j^3(H_d) = 0$.

Now, let $\eta \in \mathrm{Der}_0(-\log(\mathrm{h})) \subset \mathrm{Der}(-\log \mathrm{V})$. Then from Theorem 3.1.2 we have

$$\eta = g_e\xi_e + \sum_{i=1}^{3}\sum_{j=1}^{d-1} g_{i,j}\xi_j^i \quad \text{for some polynomials } g_e,\ g_{i,j}.$$

It follows that

$$\eta(H_d) = g_e\xi_e(H_d) + \sum_{i=1}^{3}\sum_{j=1}^{d-1} g_{i,j}\xi_j^i(H_d).$$

Thus,

$$0 = d^2 g_e H_d + 0.$$

Therefore $g_e = 0$ and hence

$$\eta = \sum_{i=1}^{3} \sum_{j=1}^{d-1} g_{i,j} \xi_j^i.$$

□

Chapter 4

Determinacy and The Complete Transversal Method

In this chapter we will give the techniques of a classification method which we will use in this thesis to find the classification of map-germ under ${}_{\Theta}\mathcal{K}$-equivalence, when Θ is the module of liftable vector fields over the minimal cross cap of multiplicity $d \geq 2$. This method depend on finite determinacy and the complete transversal method.

Finite determinacy is a very powerful and practical idea. It allows us to study a smooth map-germ by replacing it with a polynomial which is $\mathcal{G}$-equivalent to it (where $\mathcal{G}$ is an equivalence relation.)

The complete transversal method is a very systematic and an efficient method of classification. The method is due to Bruce, Kirk and du Plessis in [BKdP97] and is a direct generalisation of the work of Dimca and Gibson for $\mathcal{K}$-equivalence, [DG83]. Independently, a similar method has been developed by D. Ratcliffe using 'triviality theorems' (Thom-Levine) as the main technical tool (see [Rat90] and [Rat95]). For more details see ([Bru01], [BG92] , [Wal95] and [BW98]).

The general method of classification can be described as the following:

i) From a k-jet produce a list of possible $(k+1)$-jets.

ii) Reduce the list by removing redundancies and by scaling.

iii) For each possible $(k+1)$-jet, check determinacy. If not $(k+1)$-determined, then repeat the method for each $(k+1)$-jet by finding the possible $(k+2)$-jets.

The theorems developed in this chapter have been used in the classification of map-germs $(\mathbb{K}^{2d-1}, 0) \to (\mathbb{K}^q, 0)$ under ${}_\Theta\mathcal{K}$-equivalence (see Definition 2.4.10), where Θ is the module of the liftable vector fields over the minimal cross cap mapping of multiplicity $d \geq 2$.

4.1 Finite Determinacy of Map-Germs

The aim of this section is to find algebraic criteria for a map-germ to be determined by its Taylor series expansion up to a sufficiently high order.

Definition 4.1.1. *Let $\mathcal{G}$ be an equivalence relation. We say that a smooth map-germ $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is k-$\mathcal{G}$-determined if f is $\mathcal{G}$-equivalent to any other smooth map-germ $g : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ such that $j^k f = j^k g$. If f is k-$\mathcal{G}$-determined for some k, then f is said to be $\mathcal{G}$ finitely determined.*

Once we know a map-germ is k-$\mathcal{G}$-determined for some k, it is sufficient to work in the k-jet-space to classify $\mathcal{G}$-classes.

The following lemma is very important tool in singularity theory.

Lemma 4.1.2 (Nakayama's lemma)**.** *Let R be a commutative ring, M an ideal such that for $x \in M$, $1 + x$ is a unit. Let C be an R-module, A and B R-submodules of C with A finitely generated. If $A \subset B + M.A$ then $A \subset B$.*

Proof. See [Wal81], Lemma 1.4 or [Wal95], page 929. □

4.1.1 Trivial unfoldings

Let ξ be a vector field on $(\mathbb{K}^p, 0)$. Then we can consider ξ as a vector field on $(\mathbb{K}^p \times \mathbb{K}, 0 \times 0)$ just by trivial extension. Similarly the maximal ideal $\mathbf{m}_p$ can be considered as the ideal which it generates in $\mathbf{m}_{p+1}$.

Definition 4.1.3. *Let Θ be a finitely generated $\mathcal{E}_p$-module of vector fields on $(\mathbb{K}^p, 0)$ and $H : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^q, 0)$ be a one-parameter family of smooth map-germs with $H(0,t) = 0$ for small t.*

1) *We say that H is ${}_{\Theta}\mathcal{R}$-**trivial** if there exists vector field $\xi \in \Theta$ that can be integrated to give a one-parameter family of diffeomorphisms $\Phi : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^p, 0)$ with $\Phi(x, 0) = x$ (for all x), $\Phi(0,t) = 0$ (for small t) and $H(\Phi(x,t), t) = H(x, 0)$.*

2) *We say that H is ${}_{\Theta}\mathcal{K}$-**trivial** if there exists vector field $\xi \in \Theta$ that can be integrated to give a one-parameter family of diffeomorphisms $\Phi : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^p, 0)$ with $\Phi(x, 0) = x$ (for all x), $\Phi(0,t) = 0$ (for small t) and a germ of invertible matrix $M : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to GL(\mathbb{K}^q)$ such that $M(x,t)H(\Phi(x,t), t) = H(x, 0)$.*

We can now state a condition which ensures the triviality of a family. The statements and proofs are very similar to standard results in singularity theory, but see in particular Proposition 3.9 of [BW98] where a very similar ${}_V\mathcal{R}$-trivial result is stated and proved.

Theorem 4.1.4. *Let Θ be a finitely generated $\mathcal{E}_p$-module of vector fields on $(\mathbb{K}^p, 0)$ such that every vector field in Θ can be integrated to give a one-parameter family of diffeomorphisms. Let $H : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^q, 0)$ be a smooth map-germ with $H(0,t) = 0$ for small t.*

1) The family H is ${}_\Theta\mathcal{R}$-trivial if

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta_p \rangle.$$

2) The family H is ${}_\Theta\mathcal{K}$-trivial if

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta_p \rangle + H^*(\mathbf{m}_q)\theta(H).$$

Proof. We will give the proof of ${}_\Theta\mathcal{K}$-triviality as it does not appear to be in the literature in the form we describe. There is a version from a different perspective in Proposition 2 of [GH85]. For the ${}_V\mathcal{R}$-triviality proof see Proposition 3.9(i) in [BW98].

We suppose that

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta_p \rangle + H^*(\mathbf{m}_q)\theta(H).$$

In other words

$$\frac{\partial H}{\partial t} = \sum_{i=1}^{r} \alpha_i \xi_i(H) + \sum_{i=1}^{q}\sum_{j=1}^{q} \beta_{ij} H_i \mathbf{e}_j.$$

where $\xi_1, \xi_2, \ldots, \xi_r$ are vector fields in $\Theta \cap \mathbf{m}_p\theta_p$ and for $1 \leq i, j \leq q$ we have $\beta_{ij} \in \mathcal{E}_{p+1}$ and $\mathbf{e}_j = (0, 0, \ldots, 1, 0, \ldots, 0)^T \in \mathbb{K}^q$ which has zeroes except at position j, where it has a 1.

By the fundamental theorem on the existence of solutions to ordinary differential equations (see [Hur64]) and integration of vector fields we have the following.

i) If

$$\eta = \sum_{i=1}^{r} \alpha_i \xi_i = \sum_{i=1}^{p} \eta_i \frac{\partial}{\partial x_i},$$

then the differential equation

$$\frac{\partial \Phi}{\partial t}(x, t) = \eta(\Phi(x, t), t), \qquad \Phi(x, 0) = x,$$

has a unique solution Φ as a family of diffeomorphisms of $\mathbb{K}^p$. That is, we can find $\Phi_t : (\mathbb{K}^p, 0) \to (\mathbb{K}^p, 0)$ for each small t. Note that

$$\frac{\partial \Phi}{\partial t}(0,t) = \eta(\Phi(0,t),t)$$

has the unique solution $\Phi(0,t) \equiv 0$, since the vector fields vanish at the origin.

ii) From $\sum_{i=1}^{q}\sum_{j=1}^{q} \beta_{ij} H_i e_j$, we define a $q \times q$ matrix $A = [\beta_{ij}]$.

Then the differential equation

$$\frac{\partial M}{\partial t}(x,t) = MA(\Phi(x,t),t), \qquad M(x,0) = Id_q$$

has a unique solution $M : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to GL(\mathbb{K}^q)$. (Here Id_q is the $q \times q$ identity matrix.)

Now we define a new family $G : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^q, 0)$ by

$$G(x,t) = M(x,t)H(\Phi(x,t),t).$$

Differentiating with respect to t we obtain

$$\begin{aligned}
\frac{\partial G}{\partial t}(x,t) &= \frac{\partial M}{\partial t}(x,t)H(\Phi(x,t)) \\
&\quad + M(x,t)\left(\sum_{i=1}^{p} \frac{\partial \Phi_i}{\partial t}(x,t)\frac{\partial H}{\partial x_i}(\Phi(x,t),t) + \frac{\partial H}{\partial t}(\Phi(x,t),t)\right) \\
&= (M(x,t)A(\Phi(x,t),t))H(\Phi(x,t)) \\
&\quad + M\left(\sum_{i=1}^{p} \eta_i(\Phi(x,t),t)\frac{\partial H}{\partial x_i}(\Phi(x,t),t) + \frac{\partial H}{\partial t}(\Phi(x,t),t)\right) \\
&= M(x,t)\left(A(x,t)H(x,t) + \sum_{i=1}^{p} \eta_i \frac{\partial H}{\partial x_i} + \frac{\partial H}{\partial t}\right)(\Phi(x,t),t) \equiv 0.
\end{aligned}$$

Fixing x we see that $G(x,t)$ is constant, i.e., $G(x,t) = G(x,0)$ for all x and t. In other words $M(x,t)H(\Phi(x,t),t) = H(x,0)$. Hence H is ${}_{\Theta}\mathcal{K}$-trivial. □

Example 4.1.5. *Let V be the image of φ_3 and H_λ is the 1-parameter family given by $H_\lambda(u_1, v_1, v_2, w_1, w_2) = v_2 + w_1 + \lambda u_1^2$.*

Then we have

$$\begin{aligned} T_\Theta\mathcal{K}(H_\lambda) &= \langle \xi_j^1(H_\lambda), \xi_j^2(H_\lambda), \xi_j^3(H_\lambda), \xi_e(H_\lambda)\rangle_{j=1}^2 + \langle H_\lambda \rangle \\ &= \langle 8\lambda u_1^3 - 3w_2 - 5u_1v_2 + 6u_1w_1, 3v_1, 12\lambda u_1^2 - 6v_2 + 9w_1, \\ &\quad -18\lambda u_1 w_1 - 3v_1 + 2u_1^2 + 18\lambda u_1 v_1 + 9w_2 + 3u_1v_2, -18\lambda u_1 w_2 \\ &\quad -6\lambda u_1^2 v_2 + 3u_1v_1, 4\lambda u_1^2 + v_2 + 3w_1\rangle + \langle v_2 + w_1 + \lambda u_1^2 \rangle \\ &= \langle v_1, v_2, w_1, w_2, u_1^2 \rangle. \end{aligned}$$

Thus,

$$\frac{\partial H_\lambda}{\partial \lambda} \in T_\Theta\mathcal{K}(H_\lambda).$$

and H_λ is ${}_\Theta\mathcal{K}$-trivial.

The converse of theorem 4.1.4 is not true in general. However, if Θ is the module of all smooth vector fields on $(\mathbb{K}^p, 0)$ preserving a subset V, then the converse is true. In fact, there is a version from a different perspective in ([BG92], Proposition 11.11) and ([GH85], Proposition 2(a)).

Theorem 4.1.6. *Let $(V, 0) \subseteq (\mathbb{K}^p, 0)$ and Θ be the module of all smooth vector fields on $(\mathbb{K}^p, 0)$ preserving V. Let $H : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^q, 0)$ be a smooth map-germ with $H(0, t) = 0$ for small t.*

1. *If the family H is ${}_\Theta\mathcal{R}$-trivial, then*

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \rangle.$$

2. *If the family H is ${}_\Theta\mathcal{K}$-trivial, then*

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \rangle + H^*(\mathbf{m}_q)\theta(H).$$

Proof. We will give the proof of ${}_\Theta\mathcal{K}$-triviality. For the proof of ${}_\Theta\mathcal{R}$-triviality we just delete the term $H^*(\mathbf{m}_q)\theta(H)$.

We suppose that H is ${}_\Theta\mathcal{K}$-trivial family, so that there is a one-parameter family of diffeomorphisms $\Phi : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^p, 0)$ with $\Phi(x, 0) = x$, $\Phi(0, t) = 0$ (for small t) and a germ of invertible matrix $M : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to GL(\mathbb{K}^q)$ such that

$$M(x,t)H(\Phi(x,t),t) = H(x,0).$$

Differentiating with respect to t we obtain

$$\frac{\partial M}{\partial t}(x,t)H(\Phi(x,t)) + M(x,t)\left(\sum_{i=1}^{p}\frac{\partial \Phi_i}{\partial t}(x,t)\frac{\partial H}{\partial x_i}(\Phi(x,t),t) + \frac{\partial H}{\partial t}(\Phi(x,t),t)\right) = 0.$$

Multiplication on the left by M^{-1} and composition on the right with Φ_t^{-1} where $\Phi_t(x) = \Phi(x,t)$. Then we get

$$M^{-1}\frac{\partial M}{\partial t}(x,t)H(x,t) + \left(\sum_{i=1}^{p}\frac{\partial \Phi_i}{\partial t}(\Phi_t^{-1}(x),t)\frac{\partial H}{\partial x_i}(x,t) + \frac{\partial H}{\partial t}(x,t)\right) = 0.$$

We may take $\xi_i(x,t) = \dfrac{\partial \Phi_i}{\partial t}(\Phi_t^{-1}(x),t)$ for $1 \le i \le p$.

Note that

$$\begin{aligned}
\xi_i(0,t) &= \frac{\partial \Phi_i}{\partial t}(\Phi_t^{-1}(0),t) \\
&= \frac{\partial \Phi_i}{\partial t}(0,t) \quad \text{since } \Phi_t^{-1}(0) = 0 \\
&= 0.
\end{aligned}$$

In other words, there is a vector field $\xi : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^p, 0)$ defined by $\xi = \sum_{i=1}^{p} \xi_i \dfrac{\partial}{\partial x_i}$ with $\xi(0) = 0$ such that

$$M^{-1}\frac{\partial M}{\partial t}(x,t)H(x,t) + \left(\sum_{i=1}^{p}\xi_i(x,t)\frac{\partial H}{\partial x_i}(x,t) + \frac{\partial H}{\partial t}(x,t)\right) = 0.$$

By integrating ξ we obtain a diffeomorphism which preserves V, i.e., $\xi \in \Theta$.

It follows

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \rangle + H^*(\mathbf{m}_q)\theta(H).$$

□

Example 4.1.7. *Let V be the image of the cross cap. Suppose that $H(v_1, w_1, w_2, \nu) = w_2 + \nu w_1^2$.*

Then we have

$$\begin{aligned} T_{\Theta}\mathcal{K}(H) &= \langle \xi_1^1(H), \xi_1^2(H), \xi_1^3(H), \xi_e(H) \rangle + \langle h \rangle \\ &= \langle v_1 w_1, 4\nu w_1^2, 4\nu w_1 w_2 + v_1^2, 4\nu w_1^2 + 2w_2 \rangle + \langle w_2 + \nu w_1^2 \rangle \\ &= \langle v_1 w_1, \nu w_1^2, 4\nu w_1 w_2 + v_1^2, w_2 \rangle. \\ &= \langle v_1 w_1, \nu w_1^2, v_1^2, w_2 \rangle. \end{aligned}$$

Thus,

$$\frac{\partial H}{\partial \nu} \notin T_{\Theta}\mathcal{K}(H).$$

Hence H is non ${}_{\Theta}\mathcal{K}$-trivial along ν.

Remark 4.1.8. *Suppose that $\mathcal{G} = \mathcal{K}$ or $\mathcal{R}$. Let $(V, 0)$ be a subgerm of $(\mathbb{K}^p, 0)$ and $H : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^q, 0)$ be a smooth map-germ with $H(0, t) = 0$ for small t. Let Θ be a finitely generated $\mathcal{E}_p$-module of tangent vector fields on $(\mathbb{K}^p, 0)$ to V. If all members of Θ vanish at the origin, then from theorem 4.1.4 and theorem 4.1.6 we can see that H is ${}_{\Theta}\mathcal{G}$-trivial if and only if $\frac{\partial H}{\partial t} \in T_{\Theta}\mathcal{G}(H)$. In fact, this is very important result in our classification because when we get non ${}_{\Theta}\mathcal{K}$-triviality along a parameter (or family of parameters) and we can not scale this parameter by any diffeomorphism which preserves V, then this parameter is a modulus and in this case H is not a simple map-germ.*

Now as a corollary of Theorem 4.1.4, we obtain the following theorem.

Theorem 4.1.9 (Finite determinacy theorem). *Let $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be smooth map-germ. Suppose $\mathcal{G} = \mathcal{K}$ or $\mathcal{R}$. If Θ is a finitely generated module of vector fields on $(\mathbb{K}^p, 0)$ such that every vector field in Θ can be integrated to give a one-parameter family of diffeomorphisms and*

$$\mathbf{m}_p^k \theta(h) \subseteq T_\Theta \mathcal{G}(h),$$

then h is k-${}_\Theta\mathcal{G}$-determined.

Proof. We will give the proof for ${}_\Theta\mathcal{K}$-equivalence. The idea of the proof is the same as Gibson's proof ([Gib79], page 117), based on the method of homotopy.

Let $g : (\mathbb{K}^p, S) \to (\mathbb{K}^q, 0)$ be a smooth map-germ with $j^k(g) = j^k(h)$. We set $\Omega(x) = g(x) - h(x)$ and

$$H(x, t) = H_t(x) = h(x) + t\Omega(x).$$

Note that $H_0 = h$ and $H_1 = g$. Obviously, it suffices to show that for every $t \in [0, 1]$ the family H_t is ${}_\Theta\mathcal{K}$-trivial. Thus, we have to show that

$$\frac{\partial H_t}{\partial t} = g - h \in T_\Theta \mathcal{G}(H_t).$$

We have $\Omega \in \mathbf{m}_p^{k+1}\theta(h)$, then

$$\begin{aligned} \mathbf{m}_p T_\Theta \mathcal{G}(\Omega) &= \mathbf{m}_p(\langle \xi(\Omega) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta(\Omega)\rangle + \Omega^*(\mathbf{m}_q)\theta(\Omega)) \\ &\subseteq \mathbf{m}_p^{k+2}\theta(H_t). \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{m}_p T_\Theta \mathcal{G}(H_t) + \mathbf{m}_p^{k+2}\theta(H_t) &= \mathbf{m}_p T_\Theta \mathcal{G}(h + t\Omega) + \mathbf{m}_p^{k+2}\theta(H_t) \\ &= \mathbf{m}_p T_\Theta \mathcal{G}(h) + \mathbf{m}_p^{k+2}\theta(H_t). \end{aligned}$$

The latter module contains $\mathbf{m}_p^{k+1}\theta(H_t)$, then

$$\mathbf{m}_p^{k+1}\theta(H_t) \subseteq \mathbf{m}_p T_\Theta \mathcal{G}(H_t) + \mathbf{m}_p^{k+2}\theta(H_t).$$

By using Nakayama's lemma, we get

$$\mathbf{m}_p^{k+1}\theta(H_t) \subseteq \mathbf{m}_p T_\Theta \mathcal{G}(H_t)$$

Since

$$\mathbf{m}_p T_\Theta \mathcal{G}(H_t) \subset T_\Theta \mathcal{G}(H_t)$$

Then, we have

$$\mathbf{m}_p^{k+1}\theta(H_t) \subset T_\Theta \mathcal{G}(H_t)$$

But $\mathbf{m}_p^{k+1}\theta(H_t)$ contains $g-h$. Therefore we have H_t is a ${}_\Theta\mathcal{G}$-trivial for every $t \in [0,1]$. Since $[0,1]$ is compact, then we can choose a finite cover $U_1,\ldots,U_\tau$ for $[0,1]$. Also $[0,1]$ is connected, then we can find a finite sequence $\{t_0 = 0, t_1, \ldots, t_m = 1\} \subset [0,1]$ such that for each $i = 0,\ldots,m-1$ the intervals $\{t_i, t_{i+1}\}$ is contained in one of U_i and then

$$H_0 \sim_{{}_\Theta\mathcal{K}} H_{t_1} \sim_{{}_\Theta\mathcal{K}} \ldots \sim_{{}_\Theta\mathcal{K}} H_1 \Rightarrow g \sim_{{}_\Theta\mathcal{K}} h.$$

For the proof of ${}_\Theta\mathcal{R}$-equivalence we just delete the terms $h^*(\mathbf{m}_q)\theta(h)$, $H^*(\mathbf{m}_q)\theta(H)$ and $\Omega^*(\mathbf{m}_q)\theta(\Omega)$. □

Example 4.1.10. *Let V be the image of cross cap and Θ be the set of vector fields tangent to V. Let $h(v_1,w_1,w_2) = w_2 + w_1^{k+1}$ with $k \geq 1$. we have*

$$\begin{aligned} T_\Theta\mathcal{K}(h) &= \langle v_1w_1, 2(k+1)w_1^{k+1}, 2(k+1)w_2w_1^k + v_1^2, 2(k+1)w_1^{k+1} + 2w_2\rangle \\ &\quad +\langle w_2 + w_1^{k+1}\rangle \\ &= \langle v_1w_1, w_1^{k+1}, 2(k+1)w_2w_1^k + v_1^2, w_2\rangle \\ &= \langle v_1w_1, w_1^{k+1}, v_1^2, w_2\rangle. \end{aligned}$$

Hence, for $k \geq 1$, $\mathbf{m}_3^{k+1} \subseteq T_\Theta\mathcal{K}(h)$. That is $w_2 + w_1^{k+1}$ is $(k+1)$-${}_\Theta\mathcal{K}$-determined.

Example 4.1.11. *Let V be the image of φ_3 and $h : (\mathbb{C}^5, 0) \to (\mathbb{C}, 0)$ is given by $h(u_1, v_1, v_2, w_1, w_2) = v_2 + w_1$.*

Then we have

$$\begin{aligned} T_{\Theta}\mathcal{K}(h) &= \langle \xi_j^1(h), \xi_j^2(h), \xi_j^3(h), \xi_e(h)\rangle_{j=1}^2 + \langle h\rangle \\ &= \langle -3w_2 - 5u_1v_2 + 6u_1w_1, 3v_1, -6v_2 + 9w_1, -3v_1 + 2u_1^2 \\ &\qquad 9w_2 + 3u_1v_2, 3u_1v_1, v_2 + 3w_1\rangle + \langle v_2 + w_1\rangle \\ &= \langle v_1, v_2, w_1, w_2, u_1^2\rangle. \end{aligned}$$

Obviously, $\mathbf{m}_5^2 \subseteq \langle v_1, v_2, w_1, w_2, u_1^2\rangle$*. Hence* h *is* 2-${}_{\Theta}\mathcal{K}$*-determined.*

We can do the preceding at the jet level.

Definition 4.1.12. *Let* Θ *be a finitely generated* $\mathcal{E}_p$*-module of vector fields on* $(\mathbb{K}^p, 0)$ *and* $H : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^q, 0)$ *be a smooth map-germ with* $H(0,t) = 0$ *for small* t*. Let* $k \geq 1$ *be an integer.*

1) *We say that* H *is* k-${}_{\Theta}\mathcal{R}$-**trivial** *if there exists vector field* $\xi \in \Theta$ *that can be integrated to give a one-parameter family of diffeomorphisms* $\Phi : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^p, 0)$ *with* $\Phi(x, 0) = x$ *(for all* x*),* $\Phi(0,t) = 0$ *(for small* t*) and* $H(\Phi(x,t), t) = H(x, 0) + \psi(x,t)$ *for some* $\psi \in \mathbf{m}_p^{k+1}\theta(H)$*.*

2) *We say that* H *is* k-${}_{\Theta}\mathcal{K}$-**trivial** *if there exists vector field* $\xi \in \Theta$ *that can be integrated to give a one-parameter family of diffeomorphisms* $\Phi : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^p, 0)$ *with* $\Phi(x, 0) = x$ *(for all* x*),* $\Phi(0,t) = 0$ *(for small* t*) and a germ of invertible matrix* $M : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to GL(\mathbb{K}^q)$ *such that* $M(x,t)H(\Phi(x,t), t) = H(x, 0) + \psi(x,t)$ *for some* $\psi \in \mathbf{m}_p^{k+1}\theta(H)$*.*

Obviously an ${}_{\Theta}\mathcal{R}$-trivial (resp. ${}_{\Theta}\mathcal{K}$-trivial) family is k-${}_{\Theta}\mathcal{R}$-trivial (resp. k-${}_{\Theta}\mathcal{K}$-trivial) for any k.

Theorem 4.1.13. *Let* Θ *be a finitely generated* $\mathcal{E}_p$*-module of vector fields on* $(\mathbb{K}^p, 0)$ *such that every vector field in* Θ *can be integrated to give a one-parameter family of* *diffeomorphisms. Let* $H : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^q, 0)$ *be a smooth map-germ with* $H(0,t) = 0$ *for small* t.

i) The family H *is* k-${}_\Theta\mathcal{R}$*-trivial if*

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta(H) \rangle + \mathbf{m}_p^{k+1}\theta(H).$$

ii) The family H *is* k-${}_\Theta\mathcal{K}$*-trivial if*

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta(H) \rangle + H^*(\mathbf{m}_q)\theta(H) + \mathbf{m}_p^{k+1}\theta(H).$$

Proof. For the proof of k-${}_\Theta\mathcal{R}$-triviality see Proposition 3.9(ii) in [BW98]. For k-${}_\Theta\mathcal{K}$-triviality is very similar.

Suppose that

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta(H) \rangle + H^*(\mathbf{m}_q)\theta(H) + \mathbf{m}_p^{k+1}\theta(H).$$

By using the same argument as that the proof of theorem 4.1.4, there is a one-parameter family of diffeomorphisms $\Phi : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^p, 0)$ with $\Phi(x,0) = x$, $\Phi(0,t) = 0$ (for small t) and a germ of invertible matrix $M : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to GL(\mathbb{K}^q)$.

Let $G(x,t) = M(x,t)H(\Phi(x,t),t)$. Differentiating with respect to t we obtain

$$\begin{aligned}
\frac{\partial G}{\partial t}(x,t) &= \frac{\partial M}{\partial t}(x,t)H(\Phi(x,t),t) \\
&\quad + M(x,t)\Big(\sum_{i=1}^{p} \frac{\partial \Phi_i}{\partial t}(x,t)\frac{\partial H}{\partial x_i}(\Phi(x,t),t) + \frac{\partial H}{\partial t}(\Phi(x,t),t)\Big) \\
&= \Big(\frac{\partial M}{\partial t}(x,t)H(x,t) \\
&\quad + M(x,t)\Big(\sum_{i=1}^{p} \frac{\partial \Phi_i}{\partial t}(x,t)\frac{\partial H}{\partial x_i}(x,t) + \frac{\partial H}{\partial t}(x,t)\Big)\Big)(\Phi(x,t),t)
\end{aligned}$$

The term

$$\frac{\partial M}{\partial t}(x,t)H(x,t) + M(x,t)\Big(\sum_{i=1}^{p} \frac{\partial \Phi_i}{\partial t}(x,t)\frac{\partial H}{\partial x_i}(x,t) + \frac{\partial H}{\partial t}(x,t)\Big)$$

lies in $\mathbf{m}_p^{k+1}\theta(H)$ and hence so does $\frac{\partial G}{\partial t}(x,t)$. In particular we can write $\frac{\partial G}{\partial t}(x,t)$ as a sum $\sum G_I(x,t)x^I$, where I is a multi-index with $|I| = k+1$. So

$$G(x,t) - G(x,0) = \int_0^t \frac{\partial G}{\partial u}(x,u)du = \sum \left(\int_0^t \frac{\partial G_I}{\partial u}(x,u)du \right) x^I \in \mathbf{m}_p^{k+1}\theta(H)$$

Since $G(x,0) = H(x,0)$, note that $M(x,t).H(\Phi(x,t),t) = H(x,0) + \psi(x,t)$ for some $\psi \in \mathbf{m}_p^{k+1}\theta(H)$. □

Example 4.1.14. *Let V be the image of the cross cap $\varphi_2 : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ and Θ be the set of vector fields tangent to V. Let $H(v_1, w_1, w_2, \lambda, \mu) = w_2 + \lambda v_1^2 + \mu v_1 w_1 + \nu w_1^2$. Then we have*

$$\xi_1^3(H) = v_1^2 + \phi \text{ where } \phi \in \mathbf{m}^3.$$

Thus,

$$\frac{\partial H}{\partial \lambda} = v_1^2 \in T_\Theta\mathcal{K}(H) + \mathbf{m}^3$$

and H is 2-${}_\Theta\mathcal{K}$-trivial along λ. A similar calculation gives that H is 2-${}_\Theta\mathcal{K}$-trivial along μ.

Now, we consider H as a 1-parameter family of function-germs with μ as the parameter. Then from Example 4.1.7, H is non ${}_\Theta\mathcal{K}$-trivial along ν and we can fix $\nu = 1$.

In general the converse of Theorem 4.1.13 is not true. However, if Θ is module of all vector fields on $(\mathbb{K}^p, 0)$ preserving a subset V, then the converse is true. In fact, there is a version from a different perspective in ([BG92], Proposition 11.28).

Theorem 4.1.15. *Let $(V,0) \subseteq (\mathbb{K}^p, 0)$ and Θ be the module of all smooth vector fields on $(\mathbb{K}^p, 0)$ preserving V. Let $H : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^q, 0)$ be a smooth map-germ with $H(0,t) = 0$ for small t.*

i) If the family H is k-${}_\Theta\mathcal{R}$-trivial, then

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta(H) \rangle + \mathbf{m}_p^{k+1}\theta(H)$$

ii) If the family H is k-${}_\Theta\mathcal{K}$-trivial, then

$$\frac{\partial H}{\partial t} \in \langle \xi(H) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta(H)\rangle + H^*(\mathbf{m}_q)\theta(H) + \mathbf{m}_p^{k+1}\theta(H)$$

Proof. We will give the proof for ${}_\Theta\mathcal{K}$-trivial. For the proof of ${}_\Theta\mathcal{R}$-trivial we just delete the term $H^*(\mathbf{m}_q)\theta(H)$.

We suppose that H is ${}_\Theta\mathcal{K}$-trivial family, so that there is a one-parameter family of diffeomorphisms $\Phi : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to (\mathbb{K}^p, 0)$ preserving V with $\Phi(x, 0) = x$, $\Phi(0, t) = 0$ (for small t) and a germ of invertible matrix $M : (\mathbb{K}^p \times \mathbb{K}, 0 \times 0) \to GL(\mathbb{K}^q)$ such that

$$M(x,t)H(\Phi(x,t),t) = H(x,0) + \psi(x,t) \text{ for some } \psi \in \mathbf{m}_p^{k+1}\theta(H).$$

Differentiating with respect to t we obtain

$$\frac{\partial M}{\partial t}(x,t)H(\Phi(x,t)) + M(x,t)\left(\sum_{i=1}^{p}\frac{\partial \Phi_i}{\partial t}(x,t)\frac{\partial H}{\partial x_i}(\Phi(x,t),t) + \frac{\partial H}{\partial t}(\Phi(x,t),t)\right) = \frac{\partial \psi}{\partial t}(x,t).$$

Since $\psi \in \mathbf{m}_p^{k+1}\theta(H)$, then we have $\dfrac{\partial \psi}{\partial t} \in \mathbf{m}_p^{k+1}\theta(H)$. Hence, we have

$$M^{-1}\frac{\partial \psi}{\partial t}(\Phi^{-1}(x,t),t) \in \mathbf{m}_p^{k+1}\theta(H).$$

By using the same argument as in the proof of Theorem 4.1.6 we can get the result. □

4.2 Complete Transversal Method

In this section we will discuss the use of the complete transversal method. This method has been used in several classifications in the past, see for example, [DG83], [DG85], [Rat90], [BKdP97], [BW98], [HK99], [Kir00].

A **complete transversal** is a list of homogeneous maps (that satisfy a certain condition). This list provides a complete list of the $(k+1)$-jets (associated to a k-jet) that we need to investigate when looking for distinct singularities.

The engine that drives the complete transversal method in our situation is the following theorem

Theorem 4.2.1 (Complete transversal theorem)**.** *Suppose that* $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ *is a smooth map-germ with* $(V, 0) \subseteq (\mathbb{K}^p, 0)$*, that* $\mathcal{G}$ *is either* $\mathcal{K}$ *or* $\mathcal{R}$ *and* Θ *is a finitely generated* $\mathcal{E}_p$*-module of vector fields preserving* V*.*

If $G_1, \ldots, G_r$ *are homogeneous polynomial maps of degree* $k+1$ *such that*

$$\mathbf{m}_p^{k+1}\theta(h) \subseteq \mathbf{m}_p T_\Theta \mathcal{G}(h) + \mathrm{span}\{G_1, \ldots, G_r\} + \mathbf{m}_n^{k+2}\theta(h),$$

then every g *with* $j^k(h) = j^k(g)$ *is* ${}_\Theta\mathcal{G}$*-equivalent to*

$$j^k(h) + \sum_{i=1}^{r} \alpha_i G_i + \phi,$$

for some $\phi \in \mathbf{m}_p^{k+2}\theta(h)$ *and* $\alpha_i \in \mathbb{K}$*.*

Proof. We will use the method of homotopy, with an argument very similar to the proof of Theorem 4.1.9.

Given a map-germ $g : (\mathbb{K}^p, S) \to (\mathbb{K}^q, 0)$ with $j^k(g) = j^k(h)$. We set

$$\begin{aligned}
\Omega_1(x) &= \sum_{i=1}^{r} \alpha_i G_i(x), \\
\Omega_2(x) &= g(x) - h(x) - \sum_{i=1}^{r} \alpha_i G_i(x) \quad \text{and} \\
H_t(x) &= H(x,t) = h(x) + \Omega_1(x) + t\Omega_2(x).
\end{aligned}$$

Note that $H_0 = h + \sum_{i=1}^{r} \alpha_i G_i$ and $H_1 = g$. Our aim is to show that for every $t \in [0, 1]$ the family H_t is $(k+1)$-${}_\Theta\mathcal{K}$-trivial. Thus, we have to show that

$$\Omega_2 \in T_\Theta \mathcal{K}(H_t) + \mathbf{m}_p^{k+2}\theta(H_t).$$

We have $\Omega_1 \in \mathbf{m}_p^{k+1}\theta(h)$, then

$$\begin{aligned}
\mathbf{m}_p T_\Theta \mathcal{K}(\Omega_1) &= \mathbf{m}_p(\langle \xi(\Omega_1) \,|\, \xi \in \Theta \cap \mathbf{m}_p\theta(\Omega_1) \rangle + \Omega_1^*(\mathbf{m}_q)\theta(\Omega_1)) \\
&\subseteq \mathbf{m}_p^{k+2}\theta(H_t).
\end{aligned}$$

Similarly $\Omega_2 \in \mathbf{m}_p^{k+1}\theta(h)$, then we have $\mathbf{m}_p T_\Theta \mathcal{K}(\Omega_2) \subseteq \mathbf{m}_p^{k+2}\theta(H_t)$.

Hence,

$$\begin{aligned} \mathbf{m}_p T_\Theta \mathcal{K}(H_t) + \mathbf{m}_p^{k+2}\theta(H_t) &= \mathbf{m}_p T_\Theta \mathcal{K}(h + \Omega_1 + t\Omega_2) + \mathbf{m}_p^{k+2}\theta(H_t) \\ &= \mathbf{m}_p T_\Theta \mathcal{K}(h) + \mathbf{m}_p^{k+2}\theta(H_t). \end{aligned}$$

By assumption

$$\Omega_2 = g - h - \sum_{i=1}^{r} \alpha_i G_i \in \mathbf{m}_p T_\Theta \mathcal{K}(h) + \mathbf{m}_p^{k+2}\theta(h).$$

It follows that

$$\Omega_2 \in \mathbf{m}_p T_\Theta \mathcal{K}(H_t) + \mathbf{m}_p^{k+2}\theta(H_t) \subseteq T_\Theta \mathcal{K}(H_t) + \mathbf{m}_p^{k+2}\theta(H_t)$$

Therefore we have H_t is a $(k+1)$-${}_\Theta\mathcal{K}$-trivial for every $t \in [0,1]$. The result follows by the compactness and connectedness of $[0,1]$.

$$H_0 \sim_{\Theta\mathcal{K}} H_{t_1} \sim_{\Theta\mathcal{K}} \ldots \sim_{\Theta\mathcal{K}} H_1 \Rightarrow g \sim_{\Theta\mathcal{K}} h + \sum_{i=1}^{r} \alpha_i G_i.$$

For ${}_\Theta\mathcal{R}$-equivalence we just delete the terms $h^*(\mathbf{m}_q)\theta(h)$, $H^*(\mathbf{m}_q)\theta(H)$, $\Omega_1^*(\mathbf{m}_q)\theta(\Omega_1)$ and $\Omega_2^*(\mathbf{m}_q)\theta(\Omega_2)$. □

Remark 4.2.2. *The diffeomorphism generated in the equivalence has* 1*-jet equal to the identity.*

Definition 4.2.3. *The set* $\{G_1, G_2, \ldots, G_r\}$ *is called a* **complete transversal** *of degree* $k+1$*. Sometimes we call it a* $(k+1)$**-transversal** *or* $(k+1)$*-CT.*

Corollary 4.2.4. *If the* $(k+1)$*-transversal of* h *is empty, then* h *is* k*-*${}_\Theta\mathcal{G}$*-determined (for* $\mathcal{G} = \mathcal{K}$ *or* $\mathcal{R}$*).*

Proof. As the $(k+1)$-transversal is empty we have

$$\mathbf{m}^{k+1}\theta(h) \subseteq \mathbf{m}T_{\Theta}\mathcal{G}(h) + \mathbf{m}^{k+2}\theta(h).$$

By Nakayama's lemma and an obvious inclusion we have

$$\mathbf{m}^{k+1}\theta(h) \subseteq \mathbf{m}T_{\Theta}\mathcal{G}(h) \subset T_{\Theta}\mathcal{G}(h).$$

Hence h is $(k+1)$-determined.

If $j^k(g) = j^k(h)$, then by the complete transversal theorem g is ${}_{\Theta}\mathcal{G}$-equivalent to $j^k(h) + \phi$ where $\phi \in \mathbf{m}^{k+2}\theta(h)$. Hence,

$$j^{k+1}(g) = j^k(h) = j^{k+1}(h)$$

and so g and h are ${}_{\Theta}\mathcal{G}$-equivalent. □

In particular, this means that when we reach an empty complete transversal, we can terminate the classification at that branch.

Remark 4.2.5. *A similar statement is not true in the $\mathcal{A}$-equivalence case. In fact, if the $(k+1)$-transversal is empty, the $(k+2)$-transversal may be non empty. In the $\mathcal{R}$- or $\mathcal{K}$-equivalence cases this does not occur as we can apply Nakayama's Lemma.*

This explains why $\mathcal{A}$-classifications, such as in [HK99], are so complicated and explains why it is so important to change the $\mathcal{A}$-classification to an equivalent ${}_V\mathcal{K}$-classification.

Example 4.2.6. *Let V be the image of the cross cap $\varphi_2 : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ and Θ be the set of vector fields tangent to V. Suppose that $h : (\mathbb{C}^3, 0) \to (\mathbb{C}, 0)$ is given by $h(v_1, w_1, w_2) = v_1$.*

Then the $(k+1)$-transversal is calculated as follows: We have

$$\begin{aligned}
\mathbf{m}_3 T_{\Theta}\mathcal{K}(h) &= \mathbf{m}_3 J_{\Theta}(h) + \langle h \rangle \\
&= \mathbf{m}_3 \left(\langle w_2, -v_1, 0, v_1 \rangle + \langle v_1 \rangle \right) \\
&= \mathbf{m}_3^2 \backslash \langle w_1^2, w_1^3, w_1^4, \dots \rangle.
\end{aligned}$$

Thus $\{w_1^{k+1}\}$ *is a* $(k+1)$*-transversal.*

Hence any function g *with* k*-jet equal to* v_1 *is* ${}_{\Theta}\mathcal{K}$*-equivalent to some* F *with* $j^{k+1}F = v_1 + \lambda w_1^{k+1}$*, where* $\lambda \in \mathbb{C}$ *and the diffeomorphism giving this equivalence has 1-jet equal to the identity.*

Suppose that $\lambda \neq 0$ *and* $H(v_1, w_1, w_2, \lambda) = v_1 + \lambda w_1^{k+1}$*. Then we have*

$$\begin{aligned} T_{\Theta}\mathcal{K}(H) &= \langle \xi_1^1(h), \xi_1^2(h), \xi_1^3(h), \xi_e(h)\rangle + \langle h\rangle \\ &= \langle w_2, -v_1 + 2(k+1)\lambda w_1^{k+1}, 2(k+1)\lambda w_1^k w_2, v_1 + 2(k+1)\lambda w_1^{k+1}\rangle \\ &\qquad + \langle v_1 + w_1^{k+1}\rangle \\ &= \langle v_1, \lambda w_1^{k+1}, w_2\rangle. \end{aligned}$$

Thus,

$$\frac{\partial H}{\partial \lambda} = w_1^{k+1} \notin T_{\Theta}\mathcal{K}(H).$$

Hence H *is non* ${}_{\Theta}\mathcal{K}$*-trivial along* λ *with* $\lambda \neq 0$*. The vector field* $\frac{1}{2}(\xi_1^2 + \xi_e)$ *can be integrated to give the diffeomorphism*

$$(v_1, w_1, w_2) \mapsto (v_1, e^{2\alpha} w_1, e^{\alpha} w_2),$$

for some $\alpha \in \mathbb{C}$*. Thus* λ *can be scaled away and the map is* ${}_{\Theta}\mathcal{K}$*-equivalent to a germ with* $(k+1)$*-jet equal to* $v_1 + w_1^{k+1}$*.*

Now we assume that $h = v_1 + w_1^{k+1}$*. Then from Example 2.5.2 we have*

$$T_{\Theta}\mathcal{K}(h) = \langle v_1, w_1^{k+1}, w_2\rangle.$$

Obviously,

$$\mathbf{m}_3^{k+1} \subseteq T_{\Theta}\mathcal{K}(h).$$

Therefore, from Theorem 4.1.9 h *is* $(k+1)$*-*${}_{\Theta}\mathcal{K}$*-determined. Furthermore, from Example 2.5.8 we have* ${}_{\Theta}\mathcal{K}_e\mathrm{-cod}(h) = k+1$*.*

It follows that any finite codimension function-germ with k-jet equal to v_1 is ${}_{\Theta}\mathcal{K}$-equivalent to a function-germ of the form $v_1 + w_1^{k+1}$ and so is $(k+1)$-${}_{\Theta}\mathcal{K}$-determined and has ${}_{\Theta}\mathcal{K}_e$-codimension $k+1$.

Example 4.2.7. *Let V and Θ be as in Example 4.2.6 and let $h(v_1, w_1, w_2) = w_2$. Then we have*

$$\begin{aligned} T_{\Theta}\mathcal{K}(h) &= T_{\Theta}\mathcal{K}_e(h) \\ &= J_{\Theta}(h) + \langle h \rangle \\ &= \langle v_1w_1, 0, v_1^2, 2w_2 \rangle + \langle w_2 \rangle \\ &= \langle v_1w_1, v_1^2, w_2 \rangle. \end{aligned}$$

Hence,

$$\begin{aligned} \mathbf{m}_3 T_{\Theta}\mathcal{K}(h) &= \mathbf{m}_3 \langle v_1w_1, 0, v_1^2, 2w_2 \rangle \\ &= \langle v_1^2w_1, v_1^3, v_1w_2, v_1w_1^2, v_1^2w_1, w_1w_2, v_1w_1w_2, v_1^2w_2, w_2^2 \rangle. \end{aligned}$$

The 2-transversal in this case has a different format to the general $(k+1)$-transversal with $k > 1$. The 2-transversal is given by:

$$\begin{aligned} &\mathbf{m}_3 T_{\Theta}\mathcal{K}(h) + \mathbf{m}_3^3 \\ = \ &\langle v_1^2w_1, v_1^3, v_1w_2, v_1w_1^2, v_1^2w_1, w_1w_2, v_1w_1w_2, v_1^2w_2, w_2^2 \rangle + \mathbf{m}^3 \\ = \ &\langle v_1w_2, w_1w_2, w_2^2 \rangle + \mathbf{m}_3^3. \end{aligned}$$

Thus, $\{v_1^2, v_1w_1, w_1^2\}$ is a 2-transversal.

A similar calculation shows that for $k > 1$ the function h has $\{w_1^{k+1}\}$ as a $(k+1)$-transversal.

Let $H(v_1, w_1, w_2, \lambda, \mu) = w_2 + \lambda v_1^2 + \mu v_1w_1 + \nu w_1^2$. Then from Example 4.1.14, we have H is 2-${}_{\Theta}\mathcal{K}$-trivial along λ and μ and H non ${}_{\Theta}\mathcal{K}$-triviality along ν. The vector *field* ξ_1^2 *can be integrated to give the diffeomorphism*

$$(v_1, w_1, w_2) \mapsto (e^{-\alpha} v_1, e^{2\alpha} w_1, w_2),$$

for some $\alpha \in \mathbb{C}$. *Thus* ν *can be scaled away and the map is* ${}_\Theta\mathcal{K}$*-equivalent to a germ with 2-jet equal to* $w_2 + w_1^2$.

It can be show in a similar way that for all $k \geq 2$ *any map with k-jet equal to* w_2 *is* ${}_\Theta\mathcal{K}$*-equivalent to a germ with* $(k+1)$*-jet equal to* $w_2 + w_1^{k+1}$.

We suppose that $h = w_2 + w_1^{k+1}$. *Then, from Example 4.1.10 we have* h *is* $(k+1)$*-* ${}_\Theta\mathcal{K}$*-determined. Hence, we conclude that any finite codimension function-germ with k-jet equal to* w_2 *is* ${}_\Theta\mathcal{K}$*-equivalent to a function-germ of the form* $w_2 + w_1^{k+1}$.

Example 4.2.8. *Let* V *be the image of the minimal cross cap* $\varphi_3 : (\mathbb{C}^4, 0) \to (\mathbb{C}^5, 0)$ *and* Θ *be the set of vector fields tangent to* V. *Let* $h(u_1, v_1, v_2, w_1, w_2) = v_2 + w_1$.

Then we have

$$\begin{aligned} T_\Theta\mathcal{K}(h) &= \langle \xi_j^1(h), \xi_j^2(h), \xi_j^3(h), \xi_e(h)\rangle_{j=1}^2 + \langle h\rangle \\ &= \langle -3w_2 - 5u_1v_2 + 6u_1w_1, 3v_1, -6v_2 + 9w_1, 3u_1v_1, v_2 + 3w_1, \\ &\quad\; -3v_1 + 2u_1^2 + 9w_2 + 3u_1v_2\rangle + \langle v_2 + w_1\rangle \\ &= \langle v_1, v_2, w_1, w_2, u_1^2\rangle. \end{aligned}$$

Then we have

$$\mathbf{m}_5^2 \subseteq \mathbf{m}_5 T_\Theta\mathcal{K}(h) + \langle u_1^2\rangle.$$

And for $k \geq 2$,

$$\mathbf{m}_5^{k+1} \subseteq \mathbf{m}_5 T_\Theta\mathcal{K}(h).$$

This $\{u_1^2\}$ *is a 2-transversal and for all* $k \geq 2$, *the* $(k+1)$*-transversal is empty.*

Hence any function g *with 1-jet equal to* $v_2 + w_1$ *is* ${}_\Theta\mathcal{K}$*-equivalent to some* H *with* $j^2H = v_2 + w_1 + \lambda u_1^2$, *where* $\lambda \in \mathbb{C}$.

If we consider j^2H *as a* 1*-parameter family* H_λ*, then from Example 4.1.5 we get* H_λ *is a* ${}_\Theta\mathcal{K}$*-trivial.*

Therefore we deduce that H *is* ${}_\Theta\mathcal{K}$*-equivalent to* $h(u_1, v_1, v_2, w_1, w_2) = v_2 + w_1$*. From Example 4.1.11 we get* h *is* 2*-*${}_\Theta\mathcal{K}$*-determined. Also we have*

$$\begin{aligned}
{}_\Theta\mathcal{K}_e\text{−cod}(h) &= \dim_{\mathbb{K}} \frac{\mathcal{E}_5}{T_V\mathcal{K}_e(h)} \\
&= \dim_{\mathbb{K}} \frac{\mathcal{E}_5}{\langle v_1, v_2, w_1, w_2, u_1^2\rangle} \\
&= \dim_{\mathbb{K}} \langle 1, u_1\rangle \\
&= 2.
\end{aligned}$$

It follows that any finite codimension function-germ with 1*-jet equal to* $v_2 + w_1$ *is* ${}_\Theta\mathcal{K}$*-equivalent to* $v_2 + w_1$ *itself.*

Chapter 5

CAST: A `Singular` Package for Singularity Theory

In this chapter we shall describe our package (called `CAST`). It is written in the `Singular` Program (see [DGPS10]) and consists of a number of procedures (see Appendix A). Note that this package is much simpler than the considerable programming effort in Kirk's `Transversal` package, [Kir00]. `Transversal` was written in `Maple` in the 1990s and though readily available it cannot be run without modification due to the instability over time of `Maple` commands. Similar work by Ratcliffe dealt with use of computational methods in the $\mathcal{A}$-classification of map-germ $(\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$, significantly extending Mond's results, [Rat90]. The original program was written in Pascal and dealt with this particular classification. `Singular` has shown itself to be much more stable over time and is ideally suited to the calculations we require.

`CAST` has a number of commands which are given as procedures in `Singular`. Let Θ be a module of smooth vector fields and $\mathcal{G}$ either $\mathcal{R}$ or $\mathcal{K}$-equivalence.

i) `setphi` this sets the ring for use with the minimal cross cap mapping of multiplicity d;

ii) `phivfs` returns a module generated by liftable vector fields over the minimal cross cap mapping of multiplicity d;

iii) `phivfs0` returns a module generated by liftable vector fields over the minimal cross cap mapping of multiplicity d without the Euler vector field;

iv) `tthe` calculates $T_{\Theta}\mathcal{G}_e$, the extended ${}_{\Theta}\mathcal{G}$-tangent space of a map;

v) `tth` calculates $T_{\Theta}\mathcal{G}$, the ${}_{\Theta}\mathcal{G}$-tangent space of a map;

vi) `nthe` calculates $N_{\Theta}\mathcal{G}_e$, the extended ${}_{\Theta}\mathcal{G}$ normal space of a map;

vii) `codthe` calculates the extended ${}_{\Theta}\mathcal{G}_e$-codimension of a map;

viii) `ct` calculates a complete k-transversal of a module;

ix) `guessdet` gives an estimate for the determinacy of a map;

x) `trivunf` checks whether an unfolding is trivial or not;

xi) `def_eq` computes the defining function of the image of the minimal cross cap mapping of multiplicity d.

5.1 Getting Started

The `CAST` package is written in the `Singular` program. The code for `Singular` allows us to use the vector fields in a package designed for the investigation of singularities. Using the procedures is very simple. Put the file `CAST.lib` in any folder that can be accessed by `Singular`. Once `Singular` is running load the commands by executing `CAST` file. I.e., use

```
                     SINGULAR                             /
 A Computer Algebra System for Polynomial Computations   /  version 3-1-1
                                                       0<
     by: G.-M. Greuel, G. Pfister, H. Schoenemann        \  Feb 2010
FB Mathematik der Universitaet, D-67653 Kaiserslautern    \
> LIB"CAST.lib";
// ** loaded CAST.lib $Id$
// ** loaded ring.lib (1.31,2006/12/15)
// ** loaded primdec.lib (1.135,2007/04/20)
// ** loaded absfact.lib (1.6,2007/07/13)
// ** loaded triang.lib (1.11,2006/12/06)
// ** loaded matrix.lib (1.37,2007/04/20)
// ** loaded random.lib (1.17,2006/07/20)
// ** loaded poly.lib (1.46,2007/07/25)
// ** loaded elim.lib (1.21,2006/08/03)
// ** loaded general.lib (1.54,2007/01/08)
// ** loaded inout.lib (1.28,2006/07/20)
```

5.2 Description of Singular commands

We set the ring using the `setphi` command. The usage is `setphi(d)` where d is an integer. This sets the ring, called *phiring*, to have the variables $u_1, \ldots, u_{d-2}$, $v_1, \ldots, v_{d-1}$ and w_1, w_2, i.e., the variables used in the codomain of the minimal cross cap mapping of multiplicity $d \geq 2$, i.e., φ_d. We then use `phivfs(d)` to produce the module generated by the vector fields described in Chapter 3 for φ_d. The elements of the module are given in the order family 1, followed by family 2, family 3 and finally the Euler vector field.

Remark 5.2.1. *In the examples involving φ_d the user should first load the procedures in CAST.lib and set the ring and the vector fields.*

For example, for φ_3 enter the following commands:

```
> setphi(2);
> phivfs(2);
_[1]=w(2)*gen(1)+v(1)*w(1)*gen(3)
_[2]=-v(1)*gen(1)+2*w(1)*gen(2)
_[3]=2*w(2)*gen(2)+v(1)^2*gen(3)
_[4]=v(1)*gen(1)+2*w(1)*gen(2)+2*w(2)*gen(3)
> phivfs0(2);
_[1]=w(2)*gen(1)+v(1)*w(1)*gen(3)
_[2]=-v(1)*gen(1)+2*w(1)*gen(2)
_[3]=2*w(2)*gen(2)+v(1)^2*gen(3)
```

If we wish to use a different d, then we redefine the ring. The procedure `phivfs` returns a module and so we can define a variable to be module returned.

```
> setphi(3);
// ** redefining phiring
> module derlog=phivfs(3);
> derlog;
derlog[1]=-3*w(2)*gen(3)+4*u(1)^2*gen(1)-3*u(1)*v(1)*gen(2)
-5*u(1)*v(2)*gen(3)+6*u(1)*w(1)*gen(4)-3*v(1)*w(1)*gen(5)
+3*v(2)*w(1)*gen(2)+2*u(1)*w(2)*gen(5)
derlog[2]=3*v(1)*gen(3)-3*w(2)*gen(2)-3*u(1)*v(2)*gen(2)
```

```
-3*v(2)*w(1)*gen(5)
derlog[3]=6*u(1)*gen(1)-3*v(1)*gen(2)-6*v(2)*gen(3)+9*w(1)*gen(4)
derlog[4]=-3*v(1)*gen(3)-9*w(1)*gen(1)+2*u(1)^2*gen(4)
+2*u(1)*v(1)*gen(5)+2*u(1)*v(2)*gen(2)+6*v(2)*w(1)*gen(5)
derlog[5]=9*v(1)*gen(1)+9*w(2)*gen(4)+3*u(1)*v(2)*gen(4)
+3*v(1)*v(2)*gen(5)-6*v(2)^2*gen(2)
derlog[6]=-9*w(2)*gen(1)+3*u(1)*v(1)*gen(4)+3*v(1)^2*gen(5)
-3*u(1)*v(2)*gen(1)-3*v(1)*v(2)*gen(2)+6*v(2)*w(2)*gen(5)
derlog[7]=2*u(1)*gen(1)+2*v(1)*gen(2)+v(2)*gen(3)
+3*w(1)*gen(4)+3*w(2)*gen(5)
```

From Example 3.4 in [HL09] we can see that `derlog[1]` and `derlog[2]` are the two elements of the first family. Similarly `derlog[3]` and `derlog[4]` are the elements of the second family and `derlog[5]` and `derlog[6]` are the elements of the third family. The Euler vector field is `derlog[7]`.

5.2.1 The `tthe` command

The `tthe` command calculates the extended ${}_{\Theta}\mathcal{G}_e$ tangent space for the map h. The usage of `tthe` is `tthe`(*module*, *ideal*, *string*) where the module is Θ, the ideal is made of the components of h and the string G either R or K. It returns a module, $T_{\Theta}\mathcal{G}_e(h)$.

If $\mathcal{G} = \mathcal{R}$, then The `tthe` command calculates $J_{\Theta}(h)$, i.e., the Jacobian of f with respect to Θ.

```
proc tthe (module theta, ideal h, string G)
"
USAGE:  tthe( theta, h, string G); theta module, h ideal, G string
PURPOSE: Calculate the extended _Theta\GG-tangent space of h
```

```
with respect to a module of vector fields
RETURN:  Returns T_Theta\GG_{e}(h)
"
{
module dh = jacob (h);
module Ch = freemodule(ncols(h))*h;
module TVE;
def EQ=G[1];
if (EQ=="R")
{
 TVE = dh*theta;
}
if (EQ=="K")
{
 TVE = dh*theta+Ch;
}
return(TVE);
}
```

Example 5.2.2. *Let us take calculate the Jacobian of the D_5 singularity $h(x, y) = x^2y + y^4$ with respect to the vector field module $\Theta = \langle \partial/\partial x, \partial/\partial y\rangle$.*

```
> ring r = 0,(x,y),ds;
> module Theta = freemodule(2);
> ideal h = x^2*y+y^4;
> tthe(Theta,h,"R");
_[1]=2xy*gen(1)
_[2]=x2*gen(1)+4y3*gen(1)
```

Example 5.2.3. *Let us calculate the Jacobian of* $h(v_1, w_1, w_2) = w_2$ *and* $h(v_1, w_1, w_2) = (v_1, w_2)$ *with respect to the module of vector fields tangent to the cross cap (see Example 2.4.3).*

```
> setphi(2);
> module dv = phivfs(2);
> ideal h = w(2);
> module Jv = tthe(Theta,h,"R");
> Jv;
Jv[1]=v(1)*w(1)*gen(1)
Jv[2]=0
Jv[3]=v(1)^2*gen(1)
Jv[4]=2*w(2)*gen(1)
> ideal h = v(1),w(2);
// ** redefining h **
> Jv=tthe(Theta,h,"R");
> Jv;
Jv[1]=w(2)*gen(1)+v(1)*w(1)*gen(2)
Jv[2]=-v(1)*gen(1)
Jv[3]=v(1)^2*gen(2)
Jv[4]=v(1)*gen(1)+2*w(2)*gen(2)
> std(Jv);
_[1]=v(1)*gen(1)+2*w(2)*gen(1)
_[2]=w(2)*gen(1)
> kbase(std(Jv));
_[1]=0
> vdim(std(Jv));
```

```
-1
```

The last command calculates the codimension of $J_\Theta(h)$ and hence as the $\mathbb{C}$-dimension of $\mathcal{O}_3^2/J_\Theta(h)$ is infinite `Singular` returns -1. Note that the command previous to this produces a rather misleading answer – we might deduce (incorrectly) that $J_\Theta(h) = \mathcal{O}_3^2$.

Example 5.2.4. *We shall calculate $T_\Theta\mathcal{K}_e(h)$ where Θ is the module of vector fields liftable over φ_3 and the same module generated without the Euler vector field.*

```
> setphi(3);
> module derlog=phivfs(3);
> ideal h=w(2);
> tthe(derlog,h,"K");
_[1]=3*w(2)*gen(1)
_[2]=-3*v(1)*w(1)*gen(1)+2*u(1)*w(2)*gen(1)
_[3]=0
_[4]=3*v(1)*v(2)*gen(1)
_[5]=-3*v(2)*w(1)*gen(1)
_[6]=2*u(1)*v(1)*gen(1)+6*v(2)*w(1)*gen(1)
_[7]=3*v(1)^2*gen(1)+6*v(2)*w(2)*gen(1)
> module derlog0 = phivfs0(3);
> tthe(derlog0,h,"K");
_[1]=-3*v(1)*w(1)*gen(1)+2*u(1)*w(2)*gen(1)
_[2]=0
_[3]=3*v(1)*v(2)*gen(1)
_[4]=-3*v(2)*w(1)*gen(1)
_[5]=2*u(1)*v(1)*gen(1)+6*v(2)*w(1)*gen(1)
```

```
_[6]=3*v(1)^2*gen(1)+6*v(2)*w(2)*gen(1)
```

Example 5.2.5. *Let Θ be the module of vector fields liftable over φ_3 and*

$$h(u_1, v_1, v_2, w_1, w_2) = u_1 + v_2.$$

```
> setphi(3);
> module derlog=phivfs(3);
> ideal h=u(1)+v(2);
> module tv=tthe(derlog,h,"K");
> tv;
tv[1]=-3*w(2)*gen(1)+4*u(1)^2*gen(1)-5*u(1)*v(2)*gen(1)
tv[2]=3*v(1)*gen(1)
tv[3]=6*u(1)*gen(1)-6*v(2)*gen(1)
tv[4]=-3*v(1)*gen(1)-9*w(1)*gen(1)
tv[5]=-9*w(2)*gen(1)-3*u(1)*v(2)*gen(1)
tv[6]=2*u(1)*gen(1)+v(2)*gen(1)
tv[7]=u(1)*gen(1)+v(2)*gen(1)
> std(tv);
_[1]=u(1)*gen(1)
_[2]=v(1)*gen(1)
_[3]=v(2)*gen(1)
_[4]=w(1)*gen(1)
_[5]=3*w(2)*gen(1)
```

From the calculation of the standard basis we see that $T_\Theta\mathcal{K}_e(h) = \mathbf{m}_5$.

5.2.2 The tth command

The `tth` command calculates the ${}_\Theta\mathcal{G}$ tangent space for the map h. The usage of `tth` is `tth`(*module*, *ideal*, *string*) where the module is Θ, the ideal is made of the components of h and the string G either R or K. It returns a module, $T_\Theta\mathcal{G}(h)$.

```
proc tth (module theta, ideal h, string G)
"
USAGE:  tth( theta, h, string G); theta module, h ideal, G string
PURPOSE: Calculate the _Theta\GG-tangent space of h with
respect to a module of vector fields
RETURN:  Returns T_Theta\GG (h)
"
{
def EQ=G[1];
module theta1 = intersect(theta, maxideal(1)*freemodule(nrows(theta)));
module TV;
if (EQ=="R")
{
 TV = tthe(theta1,h, "R");
}
if (EQ=="K")
{
 TV= tthe(theta1,h, "K");
}
return(TV);
}
```

Example 5.2.6. *Consider the Morse singularity $h(x,y,z) = x^2 + y^2 + z^2$ with respect to the vector field module $\Theta = \langle \partial/\partial x, \partial/\partial y, \partial/\partial z \rangle$. Then, $T_\Theta\mathcal{K}(h) = T\mathcal{K}(h)$ and $T_\Theta\mathcal{K}(h) = T\mathcal{K}(h)$, i.e., the standard $\mathcal{R}$-tangent space and $\mathcal{K}$-tangent space.*

```
> ring r=0,(x,y,z),ds;
> ideal h=x^2+y^2+z^2;
> module Theta=freemodule(3);
> tth(Theta,h,"R");
_[1]=2xz*gen(1)
_[2]=2xy*gen(1)
_[3]=2x2*gen(1)
_[4]=2yz*gen(1)
_[5]=2y2*gen(1)
_[6]=2xy*gen(1)
_[7]=2z2*gen(1)
_[8]=2yz*gen(1)
_[9]=2xz*gen(1)
> tthe(Theta,h,"R");
_[1]=2x*gen(1)
_[2]=2y*gen(1)
_[3]=2z*gen(1)
> tth(Theta,h,"K");
_[1]=2xz*gen(1)
_[2]=2xy*gen(1)
_[3]=2x2*gen(1)
_[4]=2yz*gen(1)
_[5]=2y2*gen(1)
```

```
_[6]=2z2*gen(1)
_[7]=x2*gen(1)+y2*gen(1)+z2*gen(1)
> tthe(Theta,h,"K");
_[1]=2x*gen(1)
_[2]=2y*gen(1)
_[3]=2z*gen(1)
_[4]=x2*gen(1)+y2*gen(1)+z2*gen(1)
```

5.2.3 The nthe command

The `nthe` command is used in the same was as `tthe` and calculates the normal space and as such returns the $\mathbb{K}$-basis for the quotient $\theta(h)/T_{\Theta}\mathcal{G}_e(h)$ with $\mathcal{G}$ either ${}_{\Theta}\mathcal{R}$- or ${}_{\Theta}\mathcal{K}$-equivalence.

```
proc nthe (module theta, ideal h, string G)
"
USAGE:  nthe( theta, h, G); theta module, h ideal, G string
PURPOSE: Calculate the extended _Theta\GG_e-normal space
of h with respect to a module of vector fields
RETURN:  Returns N_Theta\GG_{e}(h)
"
{
def EQ=G[1];
module NTV;
if (EQ=="R")
{
 NTV =kbase(std(tthe(theta,h, "R")));
```

```
}
if (EQ=="K")
{
 NTV =kbase(std(tthe(theta,h, "K")));
}
return(NTV);
}
```

Example 5.2.7. *We calculate the* ${}_{\Theta}\mathcal{K}$*-normal space for* $v_1 + w_1^3$ *with respect to the vector fields tangent to the cross cap* φ_2*.*

```
> setphi(2);
> module dv=phivfs(2);
> ideal h=v(1)+w(1)^3;
> module NT=nthe(dv,h,"K");
> NT;
NT[1]=w(1)^2*gen(1)
NT[2]=w(1)*gen(1)
NT[3]=gen(1)
> std(NT);
_[1]=gen(1)
> kbase(std(NT));
_[1]=0
```

Note that just as in Example 5.2.3 `nthe` may produce a misleading answer. One should be aware that an answer of 0 may mean that the normal space is infinite as a $\mathbb{K}$-vector space. In this example however we can see that the normal space really is all of $\mathcal{O}_3$.

Example 5.2.8. *Consider $h(u_1, v_1, v_2, w_1, w_2) = u_1 + v_2$ where Θ is the module of vector fields liftable over φ_3.*

```
> setphi(3);
> module dv=phivfs(3);
> ideal h=u(1)+v(2);
> nthe(dv,h,"K");
_[1]=gen(1)
```

Thus h is a ${}_{\Theta}\mathcal{K}_e$-codimension 1 germ. Thus, $h^{\#}(\varphi_3)$ has $\mathcal{A}_e$-codimension 1 as shown in [HL09].

5.2.4 The `codthe` command

The `codthe` command is used in the same way as the two previous commands and returns an integer, the dimension of the $\mathbb{K}$-basis calculated by `nthe`.

```
proc codthe (module theta, ideal h, string G)
"
USAGE:  codthe( theta, h, G); theta module, h ideal, G string
PURPOSE: Calculate the extended _Theta\GG_e-codimension of h
with respect to a module of vector fields
RETURN:  Returns _Theta\GG_{e}\cod(h)
"
{
def EQ=G[1];
int COD;
if (EQ=="R")
{
```

```
COD = vdim(std(tthe(theta,h, "R")));
}
if (EQ=="K")
{
COD = vdim(std(tthe(theta,h, "K")));
}
return(COD);
}
```

Example 5.2.9. *Consider* $h(u_1, v_1, v_2, w_1, w_2) = u_1 + v_2^k$ *where* Θ *is the module of vector fields liftable over* φ_3*.*

```
> ideal h=u(1)+v(2);
> codthe(dv,h,"K");
1
> ideal h=u(1)+v(2)^2;
// ** redefining h **
> codthe(dv,h,"K");
2
> ideal h=u(1)+v(2)^3;
// ** redefining h **
> codthe(dv,h,"K");
3
> ideal h=u(1)+v(2)^4;
// ** redefining h **
> codthe(dv,h,"K");
4
```

Thus we would conjecture that the ${}_\Theta\mathcal{K}_e$-codimension of $u_1+v_2^k$ is k. An elementary (but tedious) calculation shows that this is the case.

5.2.5 The guessdet command

The guessdet command returns a possible value of the $k - {}_\Theta\mathcal{G}$-determinacy of a map with $\mathcal{G}$ either ${}_\Theta\mathcal{R}$- or ${}_\Theta\mathcal{K}$-equivalence. This may not be the best value (in theory a lower k could suffice) but in practice this has given very good results. Like the previous commands the guessdet command takes a module and a map (in the form of an ideal)

```
proc guessdet (module theta, ideal h, string G)
"
USAGE:  guessdet( theta, h, G); theta module, h ideal, G string
PURPOSE: Guess the k-_Theta\GG-determinacy of h with
respect to a module of vector fields
RETURN:  Returns k-_Theta\GG-determinacy
"
{
def EQ=G[1];
vector hc;
if (EQ=="R")
{
hc = highcorner(std(tth(theta,h, "R")));
}
if (EQ=="K")
{
```

```
hc = highcorner(std(tth(theta,h, "K")));
}
return(deg(hc)+1);
}
```

Example 5.2.10. *Let Θ be the module of vector fields liftable over the cross cap . We show that $h(v_1, w_1, w_2) = v_1 + w_1^2$ is 2-${}_\Theta\mathcal{K}$-determined.*

```
> setphi(2);
> module dv=phivfs(2);
> ideal h=v(1)+w(1)^2;
> guessdet(dv,h,"K");
2
```

Since v_1 is not 1-determined and h is of degree 2 we conclude that h is 2-determined.

5.2.6 The `ct` command

This command calculates a complete transversal of degree k. The usage is of the form `ct`(*module*, *integer*). The integer is k. The output is a `Singular` kbase made up of the terms of degree k. Note that the module is usually related to a tangent space, for example, $\mathbf{m}T_\Theta\mathcal{G}$ with $\mathcal{G}$ either ${}_\Theta\mathcal{R}$- or ${}_\Theta\mathcal{K}$-equivalence, but this is not a requirement.

```
proc ct (module tangent, int k)
"
USAGE:  ct( tangent, k); tangent module, k integer. The module is usually
 related to a tangent space module, eg, \M T_\Theta \GG. However,
 it can be any module, doesn't have to be a tangent space
```

```
PURPOSE: Compute a complete k-transversal
RETURN:  Returns a set of monomials of degree k which
         form the k-transversal
"
{
module Ch1 = freemodule(nrows(tangent))*maxideal(k+1);
module comp = std(tangent+Ch1);
return(kbase(comp,k));
}
```

Example 5.2.11. *Let Θ be the module of vector fields liftable over φ_3. Consider $h : (\mathbb{C}^5, 0) \to (\mathbb{C}^2, 0)$ given by $h(u_1, v_1, v_2, w_1, w_2) = (v_2, u_1)$.*

```
> setphi(3);
> module derlog=phivfs(3);
> ideal h=v(2),u(1);
> module TK=maxideal(1)*tth(derlog,h,"K");
> ct(TK,2);
_[1]=w(1)^2*gen(1)
> ct(TK,3);
_[1]=w(1)^3*gen(1)
> ct(TK,4);
_[1]=w(1)^4*gen(1)
> ct(TK,5);
_[1]=w(1)^5*gen(1)
```

Thus we would conjecture in general that a complete k-transversal for h is $\{(w_1^k, 0)\}$. In fact, we will show that in Chapter 5 this is exactly the k-transversal.

Example 5.2.12. *For the module of vector fields liftable over φ_5 we can see some interesting behavior in the structure of families of singularities.*

```
> setphi(5);
> module derlog=phivfs(5);
> ideal h=v(4)+u(2);
> module TK=maxideal(1)*tth(derlog,h,"K");
> ct(TK,2);
_[1]=u(3)^2*gen(1)
> ct(TK,3);
_[1]=u(3)^3*gen(1)
> ct(TK,4);
_[1]=0
> ct(TK,5);
_[1]=0
```

The first two transversals calculated might make one conjecture that there is a family $v_4 + u_2 + u_3^k$. The 4-transversal however is empty (and hence so is the 5-transversal as we have verified).

Example 5.2.13. *We can verify the calculations of the transversals in Example 4.2.7 for low values of k. That is, we use the vector fields liftable over φ_2 and the function $h(v_1, w_1, w_2) = w_2$.*

```
> setphi(2);
// ** redefining phiring
> module derlog=phivfs(2);
> ideal h=w(2);
> module TK=maxideal(1)*tth(derlog,h,"K");
```

```
> ct(TK,2);
_[1]=w(1)^2*gen(1)
_[2]=v(1)*w(1)*gen(1)
_[3]=v(1)^2*gen(1)
> ct(TK,3);
_[1]=w(1)^3*gen(1)
> ct(TK,4);
_[1]=w(1)^4*gen(1)
```

Example 5.2.14. *Let V as in Example 2.4.2.*

```
> ring r=0,(u(1..2)),ds;
> module derlog=[9*u(2),-2*u(1)^2],[2*u(1),3*u(2)];
> ideal h=u(1);
> module TR=maxideal(1)*tth(derlog,h,"R");
> ct(TR,2);
_[1]=0
> ct(TR,3);
_[1]=0
> ct(TR,4);
_[1]=0
> h=u(1)^2;
> module TR=maxideal(1)*tth(derlog,h,"R");
// ** redefining TR **
> ct(TR,3);
_[1]=u(2)^3*gen(1)
> ct(TR,4);
_[1]=u(2)^4*gen(1)
```

```
> ct(TR,5);
_[1]=u(2)^5*gen(1)
> h=u(1)*u(2);
> module TR=maxideal(1)*tth(derlog,h,"R");
// ** redefining TR **
> ct(TR,3);
_[1]=u(1)^3*gen(1)
> ct(TR,4);
_[1]=0
> ct(TR,5);
_[1]=0
```

Example 5.2.15. *Let V be the swallowtail discriminant, then the module of vector fields tangent to V is generated by*

$$\begin{aligned}\xi_1 &= (16u_3 - 4u_1^2)\frac{\partial}{\partial u_1} - 8u_1u_2\frac{\partial}{\partial u_2} - 3u_2^2\frac{\partial}{\partial u_3},\\ \xi_2 &= 6u_2\frac{\partial}{\partial u_1} + (8u_3 - 2u_1^2)\frac{\partial}{\partial u_2} - u_1u_2\frac{\partial}{\partial u_3},\\ \xi_3 &= 2u_1\frac{\partial}{\partial u_1} + 3u_2\frac{\partial}{\partial u_2} + 4u_3\frac{\partial}{\partial u_3}.\end{aligned}$$

```
> ring r=0,(u(1..3)),ds;
> module dv=[16*u(3)-4*u(1)^2,-8*u(1)*u(2),-3*u(2)^2],[6*u(2),
8*u(3)-2*u(1)^2,-u(1)*u(2)],[2*u(1),3*u(2),4*u(3)];
> ideal h=u(1);
> module TR=maxideal(1)*tth(dv,h,"R");
// ** redefining TR **
> ct(TR,2);
_[1]=0
> ct(TR,3);
```

```
_[1]=0
> ct(TR,4);
_[1]=0
> h=u(2);
> module TR=maxideal(1)*tth(dv,h,"R");
// ** redefining TR **
> ct(TR,2);
_[1]=u(1)^2*gen(1)
> ct(TR,3);
_[1]=u(1)^3*gen(1)
> ct(TR,4);
_[1]=u(1)^4*gen(1)
```

5.2.7 The trivunf command

This command uses Theorem 4.1.3(ii) to calculate when an unfolding is trivial. The usage is `trivunf`(*module ct*, *module tangent*) where the module ct is the complete transversal, the module tangent is the $_{\Theta}\mathcal{G}$ tangent space for the map h. The output is an element equal to the input element in ct if the unfolding is not trivial and is zero if the unfolding is trivial.

```
proc trivunf (module ct, module tangent)
"
USAGE:  trivunf(ct, tangent); ct module, tangent module
PURPOSE: when an unfolding is trivial
RETURN:  Returns an element equal to the input element in ct if
the unfolding is not trivial and is zero if the unfolding is trivial.
"
```

```
{
module NTV=reduce(ct,std(tangent));
return(NTV);
}
```

Example 5.2.16. *Consider again the situation in Example 5.2.13. We can apply the* `trivunf` *command to show that* v_1^2 *and* $v_1 w_1$ *are trivial unfoldings whereas* w_1^k *for* $k = 2, 3, 4$ *are not.*

```
> setphi(2);
> module dv=phivfs(2);
> ideal h=w(2);
> module tv=tth(dv,h,"K");
> module TK=maxideal(1)*tv;
> module t=ct(TK,2);
> t;
t[1]=w(1)^2*gen(1)
t[2]=v(1)*w(1)*gen(1)
t[3]=v(1)^2*gen(1)
> trivunf(t,std(tv));
_[1]=w(1)^2*gen(1)
_[2]=0
_[3]=0
> module t=ct(TK,3);
// ** redefining t **
> t;
t[1]=w(1)^3*gen(1)
> trivunf(t,std(tv));
```

```
_[1]=w(1)^3*gen(1)
> module t=ct(TK,4);
// ** redefining t **
> t;
t[1]=w(1)^4*gen(1)
> trivunf(t,std(tv));
_[1]=w(1)^4*gen(1)
```

Example 5.2.17. *Let us do an example for* φ_4.

```
> setphi(4);
> module derlog=phivfs(4);
> ideal h=u(1)+v(2);
> module tv=tth(derlog,h,"K");
> module TK=maxideal(1)*tv;
> module t=ct(TK,2);
> t;
t[1]=v(3)^2*gen(1)
t[2]=u(2)*v(3)*gen(1)
t[3]=u(2)^2*gen(1)
> trivunf(t,std(tv));
_[1]=v(3)^2*gen(1)
_[2]=3/2*v(3)^2*gen(1)
_[3]=u(2)^2*gen(1)
> std(trivunf(t,std(tv)));
_[1]=u(2)^2*gen(1)
_[2]=v(3)^2*gen(1)
```

This shows that although $u_1 + v_2 + \mu u_2 v_3$ is not a trivial unfolding of $u_1 + v_2$ it is a trivial unfolding of $u_1 + v_2 + \lambda v_3^2$. Hence, we need only one of the unfoldings.

This demonstrates that the output from this command can be extremely useful even when it is non-zero.

5.2.8 The `def_eq` command

To calculate a defining function of the image of the minimal cross cap mapping of multiplicity $d \geq 2$ we use procedure `def_eq`.

Example 5.2.18. *We shall apply all the vector fields in the cases of φ_2 and φ_3 to the relevant function defining the image.*

```
> setphi(2);
> poly H = def_eq(2);
> H;
w(2)^2-v(1)^2*w(1)
> module derlog = phivfs(2);
> derlog;
derlog[1]=w(2)*gen(1)+v(1)*w(1)*gen(3)
derlog[2]=-v(1)*gen(1)+2*w(1)*gen(2)
derlog[3]=2*w(2)*gen(2)+v(1)^2*gen(3)
derlog[4]=v(1)*gen(1)+2*w(1)*gen(2)+2*w(2)*gen(3)
> jacobth(derlog,H);
_[1]=0
_[2]=0
_[3]=0
_[4]=4*w(2)^2*gen(1)-4*v(1)^2*w(1)*gen(1)
```

```
> setphi(3);
// ** redefining phiring
> poly H = def_eq(3);
> H;
w(2)^3-v(1)^3*w(1)+u(1)*v(1)^2*w(2)-3*v(1)*v(2)*w(1)*w(2)
+2*u(1)*v(2)*w(2)^2-u(1)*v(1)*v(2)^2*w(1)-v(2)^3*w(1)^2
+u(1)^2*v(2)^2*w(2)
> module derlog = phivfs(3);
> jacobth(derlog,H);
_[1]=0
_[2]=0
_[3]=0
_[4]=0
_[5]=0
_[6]=0
_[7]=9*w(2)^3*gen(1)-9*v(1)^3*w(1)*gen(1)+9*u(1)*v(1)^2*w(2)*gen(1)
-27*v(1)*v(2)*w(1)*w(2)*gen(1)+18*u(1)*v(2)*w(2)^2*gen(1)
-9*u(1)*v(1)*v(2)^2*w(1)*gen(1)-9*v(2)^3*w(1)^2*gen(1)
+9*u(1)^2*v(2)^2*w(2)*gen(1)
```

Note that the final element in the answer is a constant times the defining function.

We can restrict ourselves to the three families by using `phivfs0`:

```
> setphi(5);
> poly H = def_eq(5);
> module derlog0 = phivfs0(5);
> jacobth(derlog0,H);
_[1]=0
```

```
_[2]=0
_[3]=0
_[4]=0
_[5]=0
_[6]=0
_[7]=0
_[8]=0
_[9]=0
_[10]=0
_[11]=0
_[12]=0
```

Thus when the vector fields in the three families are applied to a defining function they return 0.

In the following example we will show that the results in CAST package coincide with the results of Theorem 4.9 in [BKdP97].

Example 5.2.19. *Let V as in Example 2.4.2.*

```
> ring r=0,(u(1..2)),ds;
> module derlog=[9*u(2),-2*u(1)^2],[2*u(1),3*u(2)];
> ideal h=u(1);
> module tv=tth(derlog,h,"R");
> module TR=maxideal(1)*tv;
> ct(TR,2);
_[1]=0
> ct(TR,3);
_[1]=0
```

```
> codthe(derlog,h,"R");
1
> guessdet(derlog,h,"R");
1
> h=u(2);
> module tv=tth(derlog,h,"R");
// ** redefining tv **
> module TR=maxideal(1)*tv;
// ** redefining TR **
> module t=ct(TR,2);
> t;
t[1]=u(1)^2*gen(1)
> trivunf(t,std(tv));
_[1]=0
> module t=ct(TR,3);
// ** redefining t **
> t;
t[1]=0
> codthe(derlog,h,"R");
2
> guessdet(derlog,h,"R");
2
> h=u(1)*u(2);
> module tv=tth(derlog,h,"R");
// ** redefining tv **
> module TR=maxideal(1)*tv;
```

```
// ** redefining TR **
> module t=ct(TR,3);
// ** redefining t **
> t;
t[1]=u(1)^3*gen(1)
> trivunf(t,std(tv));
_[1]=u(1)^3*gen(1)
> module t=ct(TR,4);
// ** redefining t **
> t;
t[1]=0
> h=u(1)*u(2)+u(1)^3;
> codthe(derlog,h,"R");
5
> guessdet(derlog,h,"R");
4
> h=u(1)^2;
> module tv=tth(derlog,h,"R");
// ** redefining tv **
> module TR=maxideal(1)*tv;
// ** redefining TR **
> ct(TR,3);
_[1]=u(2)^3*gen(1)
> ct(TR,4);
_[1]=u(2)^4*gen(1)
```

Chapter 6

Classification of Map-Germs on the image of The Generalized Cross cap

The classification of map-germs on discriminant varieties has been discussed in a number of papers. Bruce, Kirk and du Plessis classified function-germs on the discriminants of the simple singularities: A_k, D_k and E_k in [BKdP97]. In [BW98], Bruce and J.M. West gave the classification of simple function-germs from 3-space to $\mathbb{R}$ up to change of coordinates in the source preserving the image of a cross cap, i.e. under ${}_V\mathcal{R}$-equivalence where V is the image of a cross cap.

In this chapter we present a list of all map-germs from $(\mathbb{K}^{2d-1}, 0)$ to $(\mathbb{K}^q, 0)$ up to codimension 2 under ${}_\Theta\mathcal{K}$-equivalence, where Θ is the module of liftable vector fields over the minimal cross cap of multiplicity $d \geq 2$. Also, we give the classification of map-germs $(\mathbb{K}^{2d-1}, 0)$ to $(\mathbb{K}^q, 0)$ under ${}_V\mathcal{K}$-equivalence, where V is the image of the minimal crosscap of multiplicity $d \geq 2$. In ${}_\Theta\mathcal{K}$-classification we use diffeomorphisms induced from integrating the liftable vector fields while in ${}_V\mathcal{K}$-classification we use diffeomorphisms induced from integrating the liftable vector fields and a diffeomorphism which preserves V and not necessarily induced from integrating the vector field.

The following theorem summaries our main results:

Theorem 6.0.20. *Let Θ be the module of liftable vector fields over the minimal cross cap of multiplicity $d \geq 2$ and $h : (\mathbb{K}^{2d-1}, 0) \to (\mathbb{K}^q, 0)$ be a submersion map-germ with ${}_\Theta\mathcal{K}_e$-codimension at most 2. Then $1 \leq q \leq 2$ and h is ${}_\Theta\mathcal{K}$-equivalent to one of the map-germs in the following:($\varepsilon_i = \pm 1$ when $\mathbb{K} = \mathbb{R}$ and $\varepsilon = 1$ when $\mathbb{K} = \mathbb{C}$.)*

Label	**Normal form**	${}_\Theta\mathcal{K}_e$**-codimension**	${}_\Theta\mathcal{K}$**-determinacy**	**Conditions**
$\mathrm{I}_{2,k+1}$	$v_1 + \varepsilon w_1^{k+1}$	$k+1$	$k+1$	$d = 2$ *and* $0 \leq k \leq 1$
$\mathrm{II}_{2,2}$	$w_1 + \varepsilon v_1^2$	2	2	$d = 2$
$\mathrm{III}_{d,k+1}$	$u_{d-2} + \varepsilon v_{d-1}^{k+1}$	$k+1$	$k+1$	$d \geq 3$ *and* $0 \leq k \leq 1$
$\mathrm{IV}_{d,2}$	$u_{d-3} + v_{d-1} + \varepsilon u_{d-2}^2$	2	2	$d \geq 4$
$\mathrm{V}_{3,2}$	$v_2 + \varepsilon w_1$	2	2	$d = 3$
$\mathrm{VI}_{2,1}$	(v_1, w_1)	2	1	$d = 2$
$\mathrm{VII}_{3,1}$	$(u_1, v_2 + \varepsilon w_1)$	2	1	$d = 3$
$\mathrm{VIII}_{4,1}$	$(u_2, u_1 + \varepsilon_1 v_3 + \varepsilon_2 w_1)$	2	1	$d = 4$

Each germ is labelled with a Roman numeral $X_{i,j}$ such that the i is the multiplicity d of the minimal cross cap and the j is equal to ${}_\Theta\mathcal{K}_e$-codimension. As a result of the theorem above, we find the classification of map-germs $(\mathbb{K}^{2d-1}, 0)$ to $(\mathbb{K}^q, 0)$ under ${}_V\mathcal{K}$-equivalence, where V is the image of the minimal crosscap of multiplicity $d \geq 2$. The only difference between ${}_\Theta\mathcal{K}$-equivalence and ${}_V\mathcal{K}$-equivalence is that the germs of our lists that differ by some sign in ${}_\Theta\mathcal{K}$-equivalence may form a single orbit in ${}_V\mathcal{K}$-equivalence.

Corollary 6.0.21. *Let V be the image of the minimal crosscap of multiplicity $d \geq 2$ and $h : (\mathbb{K}^{2d-1}, 0) \to (\mathbb{K}^q, 0)$ be a submersion map-germ with ${}_V\mathcal{K}_e$-codimension at most 2. Then $1 \leq q \leq 2$ and h is ${}_V\mathcal{K}$-equivalent to one of the map-germs in the following:*

Label	Normal form	${}_V\mathcal{K}_e$-codimension	${}_V\mathcal{K}$-determinacy	Conditions
$\mathrm{I}_{2,k+1}$	$v_1 + w_1^{k+1}$	$k+1$	$k+1$	$d = 2$ *and* $0 \leq k \leq 1$
$\mathrm{II}_{2,2}$	$w_1 + \varepsilon v_1^2$	2	2	$d = 2$
$\mathrm{III}_{d,1}$	$u_{d-2} + v_{d-1}$	1	1	$d \geq 3$
$\mathrm{III}_{d,2}$	$u_{d-2} + \varepsilon v_{d-1}^2$	2	2	$d \geq 3$
$\mathrm{IV}_{d,2}$	$u_{d-3} + v_{d-1} + \varepsilon u_{d-2}^2$	2	2	$d \geq 4$
$\mathrm{V}_{3,2}$	$v_2 + w_1$	2	2	$d = 3$
$\mathrm{VI}_{2,1}$	(v_1, w_1)	2	1	$d = 2$
$\mathrm{VII}_{3,1}$	$(u_1, v_2 + w_1)$	2	1	$d = 3$
$\mathrm{VIII}_{4,1}$	$(u_2, u_1 + v_3 + \varepsilon w_1)$	2	1	$d = 4$

The rest of the chapter is devoted to the proof of theorem 6.0.20. In fact, we apply the results of the previous chapters; that is the use of liftable vector fields, complete transversal method, ${}_\Theta\mathcal{K}$-determinacy and ${}_\Theta\mathcal{K}$-triviality. When $q = 2$ the majority of the calculations were done by the `CAST` package.

Before we start with the proof of theorem 6.0.20, we have to state and prove some technical results.

6.1 The 1-jets of Coordinate Changes

To perform the classification under ${}_\Theta\mathcal{K}$-equivalence, we need to determine some coordinate changes, i.e., we need diffeomorphisms induced from integrating the vector fields and matrices. As we know from chapter 2 that the module of the liftable vector fields over the minimal cross cap of multiplicity $d \geq 2$ is generated by ξ_j^1, ξ_j^2, ξ_j^3 and ξ_e for $1 \leq j \leq d-1$. All these vector fields vanish at the origin, and we can integrate these vector fields to get diffeomorphisms. In fact, these diffeomorphisms preserve V, where V is the image of the minimal cross cap of multiplicity $d \geq 2$.

We denote these diffeomorphisms by Φ_j^f, i.e., Φ_j^f means a diffeomorphism induced by integrating the vector field ξ_j^f. For ξ_e we denote this diffeomorphism by Φ_e. We use coordinates $(U_1, \ldots, U_{d-2}, V_1, \ldots, V_{d-1}, W_1, W_2)$ on the target of these diffeomorphisms. For ξ_e we denote this diffeomorphism by Φ_e.

In general, it not easy to find these diffeomorphisms. However, we can find the 1-jets of these diffeomorphisms by integrating the 1-jets of liftable vector fields (see [Mar82]).

The 1-jets of the liftable vector fields in the first family are given in the following table

Linear part	ξ_1^1	ξ_2^1	ξ_3^1	ξ_4^1	$\ldots$	ξ_{d-3}^1	ξ_{d-2}^1	ξ_{d-1}^1
$\mathbf{A}_{i,j}^1$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\ldots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$
$\mathrm{B}_{1,j}^1$	0	0	0	0	$\ldots$	0	0	$-dw_2$
$\mathrm{B}_{2,j}^1$	0	0	0	0	$\ldots$	0	$-dw_2$	dv_1
$\mathrm{B}_{3,j}^1$	0	0	0	0	$\ldots$	$-dw_2$	dv_1	dv_2
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{B}_{d-3,j}^1$	0	0	$-dw_2$	dv_1	$\ldots$	dv_{d-6}	dv_{d-5}	dv_{d-4}
$\mathrm{B}_{d-2,j}^1$	0	$-dw_2$	dv_1	dv_2	$\ldots$	dv_{d-5}	dv_{d-4}	dv_{d-3}
$\mathrm{B}_{d-1,j}^1$	$-dw_2$	dv_1	dv_2	dv_3	$\ldots$	dv_{d-4}	dv_{d-3}	dv_{d-2}
$\mathbf{C}_{i,j}^1$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$	$\ldots$	$\mathbf{0}$	$\mathbf{0}$	$\mathbf{0}$

By integrating these 1-jets we get the 1-jets of the diffeomorphisms Φ_j^1 with coordinates

$$U_i = u_i \quad \text{for all } 1 \le i \le d-2,$$

$$V_i = v_i \quad \text{for all } 1 \le i \le d-j-1,$$

$$V_{d-j} = v_{d-j} - d\alpha w_2,$$

$$V_i = v_i + d\alpha v_{i-d+j} \quad \text{for all } d-j+1 \le i \le d-1 \text{ and}$$

$$W_i = w_i \quad \text{for all } 1 \le i \le 2,$$

where $\alpha \in \mathbb{K}$.

The 1-jets of the liftable vector fields in the second family are given in the following table

Linear part	ξ_1^2	ξ_2^2	ξ_3^2	ξ_4^2	$\dots$	ξ_{d-3}^2	ξ_{d-2}^2	ξ_{d-1}^2
$A_{1,j}^2$	$d(d-1)u_1$	$-d^2w_1$	0	0	$\dots$	0	0	0
$A_{2,j}^2$	$d(d-2)u_2$	$d(d-1)u_1$	$-d^2w_1$	0	$\dots$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$A_{d-3,j}^2$	$3du_{d-3}$	$4du_{d-4}$	$5du_{d-5}$	$6du_{d-6}$	$\dots$	$d(d-1)u_1$	$-d^2w_1$	0
$A_{d-2,j}^2$	$2du_{d-2}$	$3du_{d-3}$	$4du_{d-4}$	$5du_{d-5}$	$\dots$	$d(d-2)u_2$	$d(d-1)u_1$	$-d^2w_1$
$B_{1,j}^2$	$-dv_1$	0	0	0	$\dots$	0	0	0
$B_{2,j}^2$	$-2dv_2$	$-dv_1$	0	0	$\dots$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$B_{d-2,j}^2$	$-d(d-2)v_{d-2}$	$-d(d-3)v_{d-3}$	$-d(d-4)v_{d-4}$	$-d(d-5)v_{d-5}$	$\dots$	$-2dv_2$	$-dv_1$	0
$B_{d-1,j}^2$	$-d(d-1)v_{d-1}$	$-d(d-2)v_{d-2}$	$-d(d-3)v_{d-3}$	$-d(d-4)v_{d-4}$	$\dots$	$-3dv_3$	$-2dv_2$	$-dv_1$
$C_{1,j}^2$	d^2w_1	0	0	0	$\dots$	0	0	0
$C_{2,j}^2$	0	0	0	0	$\dots$	0	0	0

By integrating these 1-jets we get the 1-jet of Φ_1^2 with coordinates

$$U_i = e^{d(d-i)\beta} u_i \quad \text{for all } 1 \le i \le d-2,$$

$$V_i = e^{-id\beta} v_i \quad \text{for all } 1 \le i \le d-1,$$

$$W_1 = e^{d^2\beta} w_1 \quad \text{and}$$

$$W_2 = w_2.$$

The 1-jets of Φ^2_j for $2 \leq j \leq d-1$ are given by

$$\begin{aligned} U_i &= u_i \quad \text{for all } 1 \leq i \leq j-2, \\ U_{j-1} &= u_{j-1} - d^2\beta w_1, \\ U_i &= u_i + d(d-i+j-1)\beta u_{i-j+1} \quad \text{for all } j \leq i \leq d-2, \\ V_i &= v_i \quad \text{for all } 1 \leq i \leq j-1, \\ V_i &= v_i - d(i-j+1)\beta v_{i-j+1} \quad \text{for all } j \leq i \leq d-1 \text{ and} \\ W_i &= w_i \quad \text{for all } 1 \leq i \leq 2, \end{aligned}$$

where $\beta \in \mathbb{K}$.

The 1-jets of the liftable vector fields in the third family are given in the following table

Linear part	ξ^3_1	ξ^3_2	ξ^3_3	ξ^3_4	$\dots$	ξ^3_{d-3}	ξ^3_{d-2}	ξ^3_{d-1}
$\mathrm{A}^3_{1,j}$	d^2v_1	$-d^2w_2$	0	0	$\dots$	0	0	0
$\mathrm{A}^3_{2,j}$	d^2v_2	d^2v_1	$-d^2w_2$	0	$\dots$	0	0	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$\mathrm{A}^3_{d-3,j}$	d^2v_{d-3}	d^2v_{d-4}	d^2v_{d-5}	d^2v_{d-6}	$\dots$	d^2v_1	$-d^2w_2$	0
$\mathrm{A}^3_{d-2,j}$	d^2v_{d-2}	d^2v_{d-3}	d^2v_{d-4}	d^2v_{d-5}	$\dots$	d^2v_2	dv_1	$-d^2w_2$
$\mathbf{B}^3_{i,j}$	**0**	**0**	**0**	**0**	$\dots$	**0**	**0**	**0**
$\mathrm{C}^3_{1,j}$	d^2w_2	0	0	0	$\dots$	0	0	0
$\mathrm{C}^3_{2,j}$	0	0	0	0	$\dots$	0	0	0

By integrating these 1-jets we get the 1-jets of the diffeomorphisms Φ^3_j. The 1-jet of Φ^3_1 is given by

$$\begin{aligned} U_i &= u_i + d^2\gamma v_i \quad \text{for all } 1 \leq i \leq d-2, \\ V_i &= v_i \quad \text{for all } 1 \leq i \leq d-1, \\ W_1 &= w_1 + d^2\gamma w_2 \quad \text{and} \\ W_2 &= w_2. \end{aligned}$$

For $2 \leq j \leq d-1$, the 1-jets of Φ_j^3 are given by

$$
\begin{aligned}
U_i &= u_i \quad \text{for all } 1 \leq i \leq j-2, \\
U_{j-1} &= u_{j-1} - d^2\gamma w_2, \\
U_i &= u_i + d^2\gamma v_{i-j+1} \quad \text{for all } j \leq i \leq d-2, \\
V_i &= v_i \quad \text{for all } 1 \leq i \leq d-1 \text{ and} \\
W_i &= w_i \quad \text{for all } 1 \leq i \leq 2,
\end{aligned}
$$

where $\gamma \in \mathbb{K}$.

Finally, by integrating the Euler vector field we get the diffeomorphism Φ_e with coordinates

$$
\begin{aligned}
U_i &= e^{(d-i)\mu}u_i \quad \text{for all } 1 \leq i \leq d-2, \\
V_i &= e^{(d-i)\mu}v_i \quad \text{for all } 1 \leq i \leq d-1 \text{ and} \\
W_i &= e^{d\mu}w_i \quad \text{for all } 1 \leq i \leq 2,
\end{aligned}
$$

where $\mu \in \mathbb{K}$.

We can find other diffeomorphisms from these vector fields. In fact, we shall use these diffeomorphisms in our classification.

Example 6.1.1. *Let*

$$\xi = \xi_e - \frac{1}{d}\xi_1^2 = d\sum_{i=1}^{d-1} v_i \frac{\partial}{\partial v_i} + dw_2 \frac{\partial}{\partial w_2}.$$

Then by integrating this vector field we get the diffeomorphisms with coordinates

$$
\begin{aligned}
U_i &= u_i \quad \textit{for all } 1 \leq i \leq d-2, \\
V_i &= e^{d\alpha}v_i \quad \textit{for all } 1 \leq i \leq d-1 \textit{ and} \\
W_1 &= w_1 \quad \textit{and} \\
W_2 &= e^{d\alpha}w_2,
\end{aligned}
$$

where $\alpha \in \mathbb{K}$.

We shall use the diffeomorphism in the following proposition in the proof of Corollary 6.0.21.

Proposition 6.1.2. *Let* V *be the image of the minimal crosscap of multiplicity* $d \geq 2$. *The map-germ* $\Phi : (\mathbb{K}^{2d-1}, 0) \to (\mathbb{K}^{2d-1}, 0)$ *defined by*

$$\Phi(\underline{u}, v_1, \ldots, v_{d-1}, w_1, w_2) = (\underline{u}, -v_1, \ldots, -v_{d-1}, w_1, -w_2)$$

preserves V.

Proof. We will show that $\Phi(V) \subseteq V$. We have

$$\begin{aligned} \Phi \circ \varphi_d\left(\underline{u}, \underline{v}, y\right) &= \Phi\left(\underline{u}, \underline{v}, y^d + \sum_{i=1}^{d-2} u_i y^i, \sum_{i=1}^{d-1} v_i y^i\right) \\ &= \left(\underline{u}, -v_1, \ldots, -v_{d-1}, y^d + \sum_{i=1}^{d-2} u_i y^i, -\sum_{i=1}^{d-1} v_i y^i\right) \\ &= \varphi_d\left(\underline{u}, -v_1, \ldots, -v_{d-1}, y\right). \end{aligned}$$

In other words we have $\Phi(\operatorname{im}\varphi_d) \subseteq \operatorname{im}\varphi_d$. □

6.2 Classification Techniques

Theorem 6.2.1. *Let* Θ *be a module of smooth vector fields on* $(\mathbb{K}^p, 0)$ *such that all the vector fields vanish at* 0. *Let* $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ *be a map-germ. Suppose* $\mathcal{G} = \mathcal{K}$ *or* $\mathcal{R}$. *Then* ${}_\Theta\mathcal{G}_e\text{-}\mathrm{cod}(h) \geq q$.

Proof. Since every vector field in Θ vanishes at the origin, then $T_\Theta\mathcal{G}_e(h)$ can not contain $e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T \in \mathbb{K}^q$ which has zeros except at position j, where it has a 1 for all $1 \leq j \leq q$. Hence, ${}_\Theta\mathcal{G}_e\text{-}\mathrm{cod}(h) \geq q$. □

Definition 6.2.2 ([Wal09]). *Let Θ be a module of smooth vector fields on $(\mathbb{K}^p, 0)$ and $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be a smooth map-germ. We say that h is ${}_{\Theta}\mathcal{G}$-stable if $\theta(f) = T_{\Theta}\mathcal{G}_e(h)$.*

Corollary 6.2.3. *Let Θ be a module of smooth vector fields on $(\mathbb{K}^p, 0)$ such that all the vector fields in Θ vanish at 0. Let $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be a map-germ. Suppose that $\mathcal{G} = \mathcal{K}$ or $\mathcal{R}$. Then there are no ${}_{\Theta}\mathcal{G}$-stable map-germs.*

Proof. Since every vector field in Θ vanishes at the origin, then from Theorem 6.2.1 we have ${}_{\Theta}\mathcal{G}_e{-}\mathrm{cod}(h) > 0$. □

Corollary 6.2.4. *Let Θ be a module of smooth vector fields on $(\mathbb{K}^p, 0)$ generated by $\{\xi_1, \xi_2, \ldots, \xi_r\}$ such that all these vector fields vanish at 0. Let $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be a map-germ. Suppose that $\mathcal{G} = \mathcal{K}$ or $\mathcal{R}$. If $pq > r + q^2$, then ${}_{\Theta}\mathcal{G}_e{-}\mathrm{cod}(h) > q$.*

Proof. Since every vector field in Θ vanishes at the origin, then from Theorem 6.2.1, we have ${}_{\Theta}\mathcal{G}_e{-}\mathrm{cod}(h) \geq q$. Suppose that ${}_{\Theta}\mathcal{G}_e{-}\mathrm{cod}(h) = q$. Then, since $e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T \notin T_{\Theta}\mathcal{G}(h)$ for all $j = 1, 2, \ldots, q$. We have

$$T_{\Theta}\mathcal{G}(h) = \mathbf{m}_p \langle e_1, e_2, \ldots, e_q \rangle .$$

Then, from right hand side we have pq distinct generators and from left hand side we have at most $r + q^2$ generators. In other words, we have $pq \leq r + q^2$ and this is a contradiction. □

Corollary 6.2.5. *Let Θ be the module of liftable vector fields over the minimal cross cap of multiplicity d. If $d \geq 5$, then there is not a map-germ $h : (\mathbb{K}^{2d-1}, 0) \to (\mathbb{K}^2, 0)$ with ${}_{\Theta}\mathcal{K}_e{-}\mathrm{cod}(h) \leq 2$.*

Proof. The dimension of the target of the minimal cross cap of multiplicity d is $2d - 1$ and from Chapter 3 we know that Θ generated by $3d - 2$ vector fields.

Suppose that $d = 4 + t$ with $t \geq 1$, then we have

$$\begin{aligned}
(2d-1)\,q &= 4d-2 \\
&= 3d-2+d \\
&= 3d-2+4+t, \quad t \geq 1 \\
&= 3d-2+q^2+t, \quad t \geq 1 \\
&> 3d-2+q^2.
\end{aligned}$$

Therefore, from corollary 6.2.4, we have ${}_{\Theta}\mathcal{K}{-}\mathrm{cod}(h) > 2$. □

The following proposition is used as a technical tool in the classification of function-germs from $(\mathbb{K}^{2d-1}, 0)$ to $(\mathbb{K}, 0)$.

Proposition 6.2.6. *Let Θ be the module of liftable vector fields over the minimal cross cap of multiplicity $d \geq 2$. Let $h : (\mathbb{K}^{2d-1}, 0) \to (\mathbb{K}, 0)$ be a function-germ defined by*

$$h = \sum_{i=1}^{2} \alpha_{d-i-1} u_{d-i-1} + \sum_{i=1}^{2} \beta_{d-i} v_{d-i} + \gamma_1 w_1,$$

for some constants α_i, β_i and γ_1 in $\mathbb{K}$.

Then, for $1 \leq j \leq d-1$ we have

$$\begin{aligned}
\xi_j^1(h) = \sum_{i=1}^{2}(i+1)(d-j)\alpha_{d-i-1}u_{d-i}u_j + \sum_{i=1}^{2}\beta_{d-i}\Big(dv_{j-i} - du_{d+j-i}w_2 + dv_{d+j-i}w_1 \\
- \big((d-i)(d-j)-j\big)u_j v_{d-i} - du_{j-1}v_{d-i+1}\Big) + d(d-j)\gamma_1 u_j w_1.
\end{aligned}$$

$$\begin{aligned}
\xi_j^2(h) = \sum_{i=1}^{2}\alpha_{d-i-1}\Big(d(j+i)u_{d-j-i} - d(2d-j-i)u_{2d-j-i}w_1 - 2(d-j)u_{d-i}u_{d-j}\Big) \\
+ \sum_{i=1}^{2}\beta_{d-i}\Big(-d(d-j-i+1)v_{d-j-i+1} + (d-1)(d-j)u_{d-j}v_{d-i+1}\Big) \\
+ d(d-j+1)\gamma_1 u_{d-j+1}w_1 + j\gamma_1 u_1 u_{d-j}.
\end{aligned}$$

$$\xi_j^3(h) = \sum_{i=1}^{2} \alpha_{d-i-1}\Big(d^2 v_{d-j-i} - d(2d-j-i)u_{2d-j-i}w_2 - d(d-j)u_{d-j}v_{d-i}\Big)$$
$$+ \sum_{i=1}^{2} d(d-j)\beta_{d-i}v_{d-j}v_{d-i+1} + d(d-j+1)\gamma_1 u_{d-j+1}w_2 + d\gamma_1 u_1 v_{d-j}.$$

$$\xi_e(h) = \sum_{i=1}^{2}(i+1)\alpha_{d-i-1}u_{d-i-1} + \sum_{i=1}^{2} i\alpha_{d-i-1}v_{d-i} + d\gamma_1 w_1.$$

Proof. For all $1 \le j \le d-1$ and $1 \le m \le 3$ we have

$$\xi_j^m(h) = \sum_{i=1}^{2} \alpha_{d-i-1}A^m_{d-i-1,j} + \sum_{i=1}^{2} \beta_{d-i}B^m_{d-i,j} + \gamma_1 C^m_{1,j}$$

where $A^m_{d-i-1,j}$ are the entries of ξ_j^m that correspond to the coordinates u_{d-3}, u_{d-2}, $B^m_{d-i,j}$ are the entries of ξ_j^m that correspond to the coordinates v_{d-2}, v_{d-1} and $C^m_{1,j}$ are the the entries of ξ_j^m that correspond to the coordinate w_1.

From the first family of liftable vector fields we easily find

$$A^1_{d-2,j} = 2(d-j)u_{d-2}u_j,$$

$$A^1_{d-3,j} = 3(d-j)u_{d-3}u_j,$$

$$C^1_{1,j} = d(d-j)u_j w_1.$$

For $B^1_{d-1,j}$ and $B^1_{d-2,j}$ we have that

$$B^1_{d-1,j} = d\sum_{r=1}^{d-2} u_{d+j-r-1}v_r - d\sum_{r=1}^{d-1} u_r v_{d+j-r-1} - (d-2)(d-j)u_j v_{d-1}$$
$$+ dv_{d+j-1}w_1 - du_{d+j-1}w_2.$$

Now, $u_{d+j-r-1} = 0$ if $d+j-r-1 > d$, i.e., $r < j-1$. Similarly $v_{d+j-r-1} = 0$ if $r < j$ and also $v_{d+j-1} = 0$ for all $1 \le j \le d-1$.

Therefore we have

$$B^1_{d-1,j} = d\sum_{r=j-1}^{d-2} u_{d+j-r-1}v_r - d\sum_{r=j}^{d-1} u_r v_{d+j-r-1} - (d-2)(d-j)u_j v_{d-1} - du_{d+j-1}w_2.$$

We want to change the lower limit and upper limit in $\sum_{r=j-1}^{d-2} u_{d+j-r-1}v_r$. We write $s = d+j-r-1$. We find that

$$\sum_{r=j-1}^{d-2} u_{d+j-r-1}v_r = \sum_{s=d}^{j+1} u_s v_{d+j-s-1}$$

It follows that

$$B^1_{d-1,j} = d\sum_{s=d}^{j+1} u_s v_{d+j-s-1} - d\sum_{r=j}^{d-1} u_r v_{d+j-r+1} - (d-2)(d-j)u_j v_{d-1} - du_{d+j-1}w_2.$$

Since $u_d = 1$ and $u_{d-1} = 0$. We see that

$$\begin{aligned} B^1_{d-1,j} &= d\sum_{s=j+1}^{d-2} u_s v_{d+j-s-1} - d\sum_{r=j}^{d-2} u_r v_{d+j-r+1} - (d-2)(d-j)u_j v_{d-1} - du_{d+j-1}w_2. \\ &= dv_{j-1} + d\sum_{s=j+1}^{d-2} u_s v_{d+j-s-1} - du_j v_{d-1} - d\sum_{r=j}^{d-2} u_r v_{d+j-r+1} \\ &\qquad - (d-2)(d-j)u_j v_{d-1} - du_{d+j-1}w_2 \\ &= dv_{j-1} - \big(d + (d-2)(d-j)\big)u_j v_{d-1} - du_{d+j-1}w_2. \end{aligned}$$

It can be show in a similar way that

$$B^1_{d-2,j} = dv_{j-2} - \big(d + (d-3)(d-j)\big)u_j v_{d-2} - du_{d+j-2}w_2 - du_{j-1}v_{d-1} + dv_{d+j-2}w_1.$$

From the second family we have

$$\begin{aligned} A^2_{d-2,j} &= -d(d+d-2-j+1)u_{d+d-2-j+1}w_1 \\ &\quad +d\sum_{r=1}^{d-2}(d+d-2-j-2r+1)u_r u_{d+d-2-j-r+1} - j(d-2+1)u_{d-2+1}u_{d-j} \\ &= -d(2d-j-1)u_{2d-j-1}w_1 + d\sum_{r=1}^{d-2}(2d-j-2r-1)u_r u_{2d-j-r-1} \\ &\quad -j(d-1)u_{d-1}u_{d-j} \end{aligned}$$

Now, $u_{2d-j-r-1} = 0$ if $2d - j - r - 1 > d$, i.e., $r < d - j - 1$. Also, we have $u_d = 1$ and $u_{d-1} = 0$. It follows that

$$\begin{aligned}
A^2_{d-2,j} &= -d(2d-j-1)u_{2d-j-1}w_1 + d\sum_{r=d-j-1}^{d-2}(2d-j-2r-1)u_r u_{2d-j-r-1} \\
&= -d(2d-j-1)u_{2d-j-1}w_1 + d(2d-j-2d+2j+2-1)u_{d-j-1}u_{2d-j-d+j+1-1} \\
&\quad +d(2d-j-2d+2j-1)u_{d-j}u_{2d-j-d+j-1} \\
&\quad +d\sum_{r=d-j+1}^{d-2}(2d-j-2r-1)u_r u_{2d-j-r-1} \\
&= -d(2d-j-1)u_{2d-j-1}w_1 + d(j+1)u_{d-j-1}u_d + d(j-1)u_{d-j}u_{d-1} \\
&\quad +d\sum_{r=d-j+1}^{d-2}(2d-j-2r-1)u_r u_{2d-j-r-1} \\
&= -d(2d-j-1)u_{2d-j-1}w_1 + d(j+1)u_{d-j-1} \\
&\quad +d\sum_{r=d-j+1}^{d-2}(2d-j-2r-1)u_r u_{2d-j-r-1}.
\end{aligned}$$

We need to show that

$$\sum_{r=d-j+1}^{d-2}(2d-j-2r-1)u_r u_{2d-j-r-1} = 0.$$

Let $s = j - d + r$. Therefore we have

$$\sum_{r=d-j+1}^{d-2}(2d-j-2r-1)u_r u_{2d-j-r-1} = \sum_{s=1}^{j-2}(j-2s-1)u_{d+s-j}u_{d-s-1}.$$

a) If j is even, then we have

$$\begin{aligned}
\sum_{s=1}^{j-2}(j-2s-1)u_{d+s-j}u_{d-s-1} &= \sum_{s=1}^{\frac{j}{2}-1}(j-2s-1)u_{d+s-j}u_{d-s-1} \\
&\quad + \sum_{s=\frac{j}{2}}^{j-2}(j-2s-1)u_{d+s-j}u_{d-s-1}.
\end{aligned}$$

In the second summation, let $t = j - s - 1$. Therefore we have

$$\sum_{s=\frac{j}{2}} (j-2s-1)u_{d+s-j}u_{d-s-1} = \sum_{t=\frac{j}{2}-1}^{1} (j-2t-1)u_{d-t-j}u_{d+t-1}$$
$$= \sum_{t=1}^{\frac{j}{2}-1} (j-2t-1)u_{d+t-j}u_{d-t-1}.$$

It follows that

$$\sum_{s=1}^{j-2} (j-2s-1)u_{d+s-j}u_{d-s-1} = \sum_{s=1}^{\frac{j}{2}-1} (j-2s-1)u_{d+s-j}u_{d-s-1} - \sum_{t=1}^{\frac{j}{2}-1} (j-2t-1)u_{d+t-j}u_{d-t-1}$$
$$= 0.$$

b) If j is odd, then as $j-2s-1 = 0$ for $s = \frac{j-1}{2}$, we have

$$\sum_{s=1}^{j-2} (j-2s-1)u_{d+s-j}u_{d-s-1} = \sum_{s=1}^{\frac{j-1}{2}-1} (j-2s-1)u_{d+s-j}u_{d-s-1} + \sum_{s=\frac{j-1}{2}+1}^{j-2} (j-2s-1)u_{d+s-j}u_{d-s-1}.$$

Let $t = j - s - 1$ in the second summation, then we have

$$\sum_{s=\frac{j-1}{2}+1}^{j-2} (j-2s-1)u_{d+s-j}u_{d-s-1} = -\sum_{t=\frac{j-1}{2}-1}^{1} (j-2t-1)u_{d+t-j}u_{d-t-1}$$
$$= -\sum_{t=1}^{\frac{j-1}{2}-1} (j-2t-1)u_{d-t-j}u_{d+t-1}.$$

It follows that

$$\sum_{s=1}^{j-2} (j-2s-1)u_{d+s-j}u_{d-s-1} = \sum_{s=1}^{\frac{j-1}{2}-1} (j-2s-1)u_{d+s-j}u_{d-s-1} - \sum_{t=1}^{\frac{j-1}{2}-1} (j-2t-1)u_{d-t-j}u_{d+t-1}$$
$$= 0.$$

Therefore, we proved our claim and hence we have

$$A^2_{d-2,j} = -d(2d-j-1)u_{2d-j-1}w_1 + d(j+1)u_{d-j-1}.$$

It can be show in a similar way that

$$A^2_{d-3,j} = -d(2d-j-2)u_{2d-j-2}w_1 + d(j+2)u_{d-j-2} - 2(d-j)u_{d-2}u_{d-j}.$$

For $B^2_{d-1,j}$ and $B^2_{d-2,j}$ we have

$$\begin{aligned} B^2_{d-1,j} &= -d(d+d-1-j+1)v_{d+d-1-j+1}w_1 + d\sum_{r=1}^{d-1}(d+d-1-j-r+1)u_r v_{d+d-1-j-r+1} \\ &\quad -d\sum_{r=1}^{d-1} r u_{d+d-1-j-r+1}v_r - j(d-1+1)u_{d-j}v_{d-1+1} \\ &= -d(2d-j)v_{2d-j}w_1 + d\sum_{r=1}^{d-1}(2d-j-r)u_r v_{2d-j-r} \\ &\quad -d\sum_{r=1}^{d-1} r u_{2d-j-r}v_r - dju_{d-j}v_d. \\ &= -d(2d-j-1)u_{2d-j-1}w_1 + d(j+1)u_{d-j-1}. \end{aligned}$$

We can see that for all $1 \leq j \leq d-1$, $v_{2d-j} = v_d = 0$. Also, $v_{2d-j-r} = 0$ if $2d-j-r \geq d$, i.e., $r \leq d-j$ and similarly $u_{2d-j-r} = 0$ if $r \leq d-j$.

Therefore we have

$$\begin{aligned} B^2_{d-1,j} &= d\sum_{r=d-j+1}^{d-1}(2d-j-r)u_r v_{2d-j-r} - d\sum_{r=d-j}^{d-1} r u_{2d-j-r}v_r \\ &= d\sum_{r=d-j+1}^{d-1}(2d-j-r)u_r v_{2d-j-r} - d(d-j)u_d v_{d-j} \\ &\quad -d\sum_{r=d-j+1}^{d-1} r u_{2d-j-r}v_r \\ &= d\sum_{r=d-j+1}^{d-1}(2d-j-r)u_r v_{2d-j-r} - d(d-j)v_{d-j} \\ &\quad -d\sum_{r=d-j+1}^{d-1} r u_{2d-j-r}v_r. \end{aligned}$$

Let $s = 2d - j - r$ in the second summation, then we have

$$\sum_{r=d-j+1}^{d-1} r u_{2d-j-r} v_r = \sum_{s=d-j+1}^{d-1} (2d - j - s) u_s v_{2d-j-s}.$$

It follows that

$$\begin{aligned} B_{d-1,j}^2 &= d \sum_{r=d-j+1}^{d-1} (2d - j - r) u_r v_{2d-j-r} - d(d-j) v_{d-j} \\ &\quad - d \sum_{r=d-j+1}^{d-1} r u_{2d-j-r} v_r \\ &= -d(d-j) v_{d-j}. \end{aligned}$$

It can be show in a similar way that

$$B_{d-2,j}^2 = -d(d-j-1) v_{d-j-1} + (d-1)(d-j) u_{d-j} v_{d-1}.$$

Also, we have

$$C_{1,j}^2 = d(d-j+1) u_{d-j+1} w_1 + j u_1 u_{d-j}.$$

From the third family we have

$$\begin{aligned} A_{d-2,j}^3 &= -d(d+d-2-j+1) u_{d+d-2-j+1} w_2 \\ &\quad + d \sum_{r=1}^{d-2} (d+d-2-j-r+1) u_{d+d-2-j-r+1} v_r \\ &\quad - d \sum_{r=1}^{d-2} r u_r v_{d+d-2-j-r+1} - d(d-2+1) u_{d-2+1} v_{d-j} \\ &= -d(2d-j-1) u_{2d-j-1} w_2 + d \sum_{r=1}^{d-2} (2d-j-r-1) u_{2d-j-r-1} v_r \\ &\quad - d \sum_{r=1}^{d-2} r u_r v_{2d-j-r-1} - d(d-1) u_{d-1} v_{d-j}. \end{aligned}$$

Now, we can see that $u_{2d-j-r-1} = 0$ if $2d - j - r - 1 > d$, i.e., $r < d - j - 1$.

Similarly $v_{2d-j-r-1} = 0$ if $r < d-j$. Therefore we have

$$\begin{aligned} A^3_{d-2,j} &= -d(2d-j-1)u_{2d-j-1}w_2 + d\sum_{r=d-j-1}^{d-2}(2d-j-r-1)u_{2d-j-r-1}v_r \\ &\qquad -d\sum_{r=d-j}^{d-2} ru_r v_{2d-j-r-1} - d(d-1)u_{d-1}v_{d-j} \\ &= -d(2d-j-1)u_{2d-j-1}w_2 + d^2 u_d v_{d-j-1} + d(d-1)u_{d-1}v_{d-j} \\ &\qquad +d\sum_{r=d-j+1}^{d-2}(2d-j-r-1)u_{2d-j-r-1}v_r \\ &\qquad\quad -d\sum_{r=d-j}^{d-2} ru_r v_{2d-j-r-1} - d(d-1)u_{d-1}v_{d-j} \\ &= -d(2d-j-1)u_{2d-j-1}w_2 + d^2 v_{d-j-1} \\ &\qquad +d\sum_{r=d-j+1}^{d-2}(2d-j-r-1)u_{2d-j-r-1}v_r - d\sum_{r=d-j}^{d-2} ru_r v_{2d-j-r-1}. \end{aligned}$$

Let $s = 2d-j-r-1$ in the second summation, then we have

$$\begin{aligned} A^3_{d-2,j} &= -d(2d-j-1)u_{2d-j-1}w_2 + d^2 v_{d-j-1} \\ &\quad + d\sum_{r=d-j+1}^{d-2}(2d-j-r-1)u_{2d-j-r-1}v_r \\ &\qquad - d\sum_{s=d-1}^{d-j+1}(2d-j-s-1)u_{2d-j-s-1}v_s \\ &= -d(2d-j-1)u_{2d-j-1}w_2 + d^2 v_{d-j-1} \\ &\quad + d\sum_{r=d-j+1}^{d-2}(2d-j-r-1)u_{2d-j-r-1}v_r \\ &\qquad - d\sum_{s=d-j+1}^{d-2}(2d-j-s-1)u_{2d-j-s-1}v_s - d(d-j)u_{d-j}v_{d-1} \\ &= -d(2d-j-1)u_{2d-j-1}w_2 + d^2 v_{d-j-1} - d(d-j)u_{d-j}v_{d-1}. \end{aligned}$$

Similarly, we have

$$A^3_{d-3,j} = -d(2d-j-2)u_{2d-j-2}w_2 + d^2 v_{d-j-2} - d(d-j)u_{d-j}v_{d-2}.$$

For $B^3_{d-1,j}$ and $B^3_{d-2,j}$ we have

$$\begin{aligned}
B^3_{d-1,j} &= -d(d+d-1-j+1)v_{d+d-1-j+1}w_2 \\
&\quad +d\sum_{r=1}^{d-1}(d+d-1-j-2r+1)v_r v_{d+d-1-j-r+1} - d(d-1+1)v_{d-1+1}v_{d-j} \\
&= -d(2d-j)v_{2d-j}w_2 + d\sum_{r=1}^{d-1}(2d-j-2r)v_r v_{2d-j-r} \\
&\quad -d^2 v_d v_{d-j}.
\end{aligned}$$

For all $1 \leq j \leq d-1$ we have $v_{2d-j} = 0$ and also $v_d = 0$. It follows

$$B^3_{d-1,j} = d\sum_{r=1}^{d-1}(2d-j-2r)v_r v_{2d-j-r}.$$

We will show that

$$\sum_{r=1}^{d-1}(2d-j-2r)v_r v_{2d-j-r} = 0.$$

Let $s = j-d+r$. Therefore we have

$$\sum_{r=1}^{d-1}(2d-j-2r)v_r v_{2d-j-r} = \sum_{s=1}^{j-1}(j-2s)v_{d-j+s}v_{d-s}.$$

a) If j is even, then we have

$$\begin{aligned}
\sum_{s=1}^{j-1}(j-2s)v_{d+s-j}v_{d-s} &= \sum_{s=1}^{\frac{j}{2}}(j-2s)v_{d-j+s}v_{d-s} \\
&\quad + \sum_{s=\frac{j}{2}+1}^{j-1}(j-2s)v_{d-j+s}v_{d-s} \\
&= \sum_{s=1}^{\frac{j}{2}-1}(j-2s)v_{d-j+s}v_{d-s} + (j-2(\frac{j}{2}))v_{d+\frac{j}{2}-j}v_{d-\frac{j}{2}} \\
&\quad + \sum_{s=\frac{j}{2}+1}^{j-1}(j-2s)v_{d-j+s}v_{d-s} \\
&= \sum_{s=1}^{\frac{j}{2}-1}(j-2s)v_{d+s-j}v_{d-s} \\
&\quad + \sum_{s=\frac{j}{2}+1}^{j-1}(j-2s)v_{d-j+s}v_{d-s}.
\end{aligned}$$

In the second summation, let $t = j - s$. Therefore we have

$$\begin{aligned}\sum_{s=1}^{j-1}(j-2s)v_{d-j+}v_{d-s} &= \sum_{s=1}^{\frac{j}{2}-1}(j-2s)v_{d-j+s}v_{d-s} \\ &\quad + \sum_{t=\frac{j}{2}-1}^{1}(j-2t)v_{d+t}v_{d-j+t} \\ &= 0.\end{aligned}$$

b) If j is odd, then we have

$$\begin{aligned}\sum_{s=1}^{j-1}(j-2s)v_{d-j+s}v_{d-s} &= \sum_{s=1}^{\frac{j-1}{2}}(j-2s)v_{d-j+s}v_{d-s} \\ &\quad + \sum_{s=\frac{j-1}{2}+1}^{j-1}(j-2s)v_{d-j+}v_{d-s}.\end{aligned}$$

Let $t = j - s$ in the second summation, then we have

$$\begin{aligned}\sum_{s=1}^{j-1}(j-2s)v_{d-j+s}v_{d-s} &= \sum_{s=1}^{\frac{j-1}{2}}(j-2s)v_{d+s-j}v_{d-s} \\ &\quad - \sum_{t=\frac{j-1}{2}}^{1}(j-2t)v_{d-j+t}v_{d-t} \\ &= 0.\end{aligned}$$

It follows that

$$\begin{aligned}\sum_{s=1}^{j-2}(j-2s-1)u_{d+s-j}u_{d-s-1} &= \sum_{s=1}^{\frac{j-1}{2}-1}(j-2s-1)u_{d+s-j}u_{d-s-1} \\ &\quad - \sum_{t=1}^{\frac{j-1}{2}-1}(j-2t-1)u_{d-t-j}u_{d+t-1} \\ &= 0.\end{aligned}$$

Therefore, we have proved that

$$\sum_{r=1}^{d-1}(2d-j-2r)v_r v_{2d-j-r} = 0.$$

Hence we have $B^3_{d-1,j} = 0$ for all $1 \le j \le d-1$. For $B^3_{d-2,j}$ we have

$$\begin{aligned} B^3_{d-2,j} &= -d(d+d-2-j+1)v_{d+d-2-j+1}w_2 \\ &\quad +d\sum_{r=1}^{d-2}(d+d-2-j-2r+1)v_r v_{d+d-2-j-r+1} - d(d-2+1)v_{d-2+1}v_{d-j} \\ &= -d(2d-j-1)v_{2d-j-1}w_2 + d\sum_{r=1}^{d-2}(2d-j-2r-1)v_r v_{2d-j-r-1} \\ &\quad -d(d-1)v_{d-1}v_{d-j}. \end{aligned}$$

We can see that $v_{2d-j-1} = 0$ for all $1 \le j \le d-1$. Also, $v_{2d-j-r-1} = 0$ if $2d-j-r-1 \le d$, i.e., $r \le d-j-1$. Therefore we have

$$\begin{aligned} B^3_{d-2,j} &= d\sum_{r=d-j}^{d-2}(2d-j-2r-1)v_r v_{2d-j-r-1} - d(d-1)v_{d-1}v_{d-j} \\ &= d(j-1)v_{d-j}v_{d-1} + d\sum_{r=d-j+1}^{d-2}(2d-j-2r-1)v_r v_{2d-j-r-1} \\ &\quad -d(d-1)v_{d-1}v_{d-j} \\ &= -d(d-j)v_{d-j}v_{d-1} + d\sum_{r=d-j+1}^{d-2}(2d-j-2r-1)v_r v_{2d-j-r-1}. \end{aligned}$$

In a similar way to the $B^3_{d-1,j}$ we can show that

$$\sum_{r=d-j+1}^{d-2}(2d-j-2r-1)v_r v_{2d-j-r-1} = 0.$$

Therefore, we have

$$B^3_{d-2,j} = -d(d-j)v_{d-j}v_{d-1}.$$

Also, from the third family we have

$$C^3_{1,j} = d(d-j+1)u_{d-j+1}w_2 + du_1 v_{d-j}.$$

Therefore, we deduce the stated form of $\xi^3_j(h)$. □

Proposition 6.2.7. *i) Any finite codimension $(k+1)$-jet with k-jet $u_{d-2}+\varepsilon v_{d-1}$ and $k \geq 1$ is ${}_{\Theta}\mathcal{K}$-equivalent to $u_{d-2}+\varepsilon v_{d-1}$. ($\varepsilon = \pm 1$ when $\mathbb{K}=\mathbb{R}$ and $\varepsilon = 1$ when $\mathbb{K}=\mathbb{C}$.)*

ii) The jet $u_{d-2}+\varepsilon v_{d-1}$ is 1-${}_{\Theta}\mathcal{K}$-determined and has ${}_{\Theta}\mathcal{K}_e$-codimension 1.

Proof.

i) Let h denote the k-jet $u_{d-2}+\varepsilon v_{d-1}$. Then from Proposition 6.2.6 we have for all $1 \leq j \leq d-1$:

1) $\xi^1_j(h) = 2(d-j)u_{d-2}u_j + \varepsilon d v_{j-1} - \varepsilon\Big(d+(d-2)(d-j)\Big)u_j v_{d-1} - \varepsilon d u_{d+j-1} w_2$,

2) $\xi^2_j(h) = d(j+1)u_{d-j-1} - d(2d-j-1)u_{2d-j-1}w_1 + \varepsilon d(d-j)v_{d-j}$,

3) $\xi^3_j(h) = d^2 v_{d-j-1} - d(2d-j-1)u_{2d-j-1}w_2 - d(d-j)u_{d-j}v_{d-1}$,

4) $\xi_e(h) = 2u_{d-2} + \varepsilon v_{d-1}$.

Since $h = u_{d-2}+\varepsilon v_{d-1}$ and $\xi_e(h) = 2u_{d-2}+\varepsilon v_{d-1}$, then u_{d-2} and v_{d-1} are in $T_{\Theta}\mathcal{K}(h)$. From $\xi^1_1(h)$ and the above we get $w_2 \in T_{\Theta}\mathcal{K}(h)$. From $\xi^3_j(h)$ we have $v_1, v_2, \ldots, v_{d-2}$ and w_2 are in $T_{\Theta}\mathcal{K}(h)$.

Now, from $\xi^1_j(h)$ we get $u_1, u_2, \ldots, u_{d-3}$ in $T_{\Theta}\mathcal{K}(h)$.

Therefore, we have

$$T_{\Theta}\mathcal{K}(h) = \mathbf{m}_{2d-1}.$$

We can see that the $(k+1)$-transversal is empty for all $k \geq 1$.

ii) Let h denote the $(k+1)$-jet $u_{d-2}+\varepsilon v_{d-1}$. Since $T_{\Theta}\mathcal{K}(h) = \mathbf{m}_{2d-1}$, then from Theorem 4.1.9 h is 1-${}_{\Theta}\mathcal{K}$-determined and we can see that ${}_{\Theta}\mathcal{K}_e\text{−cod}(h) = 1$.

□

From above if we have $h = u_{d-2} + \varepsilon v_{d-1}$, then h is a ${}_\Theta\mathcal{K}_e$-codimension 1 germ. Thus, $h^\sharp(\varphi_3)$ has $\mathcal{A}_e$-codimension 1 as shown in [HL09].

Proposition 6.2.8. *i) Any finite codimension $(k+1)$-jet with k-jet u_{d-2} and $k \geq 1$ is ${}_\Theta\mathcal{K}$-equivalent to $u_{d-2} + \varepsilon v_{d-1}^{k+1}$ or u_{d-2}. ($\varepsilon = \pm 1$ when $\mathbb{K} = \mathbb{R}$ and $\varepsilon = 1$ when $\mathbb{K} = \mathbb{C}$.)*

ii) The jet $u_{d-2} + \varepsilon v_{d-1}^{k+1}$ is $(k+1)$-${}_\Theta\mathcal{K}$-determined and has ${}_\Theta\mathcal{K}_e$-codimension $k+1$.

Proof.

i) Let h denote the k-jet u_{d-2}. Then from Proposition 6.2.6 we have for all $1 \leq j \leq d-1$:

1) $\xi_j^1(h) = 2(d-j)u_{d-2}u_j$,

2) $\xi_j^2(h) = d(j+1)u_{d-j-1} - d(2d-j-1)u_{2d-j-1}w_1$,

3) $\xi_j^3(h) = d^2 v_{d-j-1} - d(2d-j-1)u_{2d-j-1}w_2 - d(d-j)u_{d-j}v_{d-1}$,

4) $\xi_e(h) = 2u_{d-2}$.

Since $h = u_{d-2}$, then we have $u_{d-2} \in T_\Theta\mathcal{K}(h)$. From $\xi_j^2(h)$ we can see that $u_1, u_2, \ldots, u_{d-3}$ and w_1 are in $T_\Theta\mathcal{K}(h)$. Also, from $\xi_j^3(h)$ we have $v_1, v_2, \ldots, v_{d-2}$ and w_2 are in $T_\Theta\mathcal{K}(h)$.

It follows that

$$T_\Theta\mathcal{K}(h) = \langle u_1, u_2, \ldots, u_{d-2}, v_1, v_2, \ldots, v_{d-2}, w_1, w_2 \rangle.$$

Therefore, for all $k \geq 1$ we have

$$\mathbf{m}_{2d-1}^{k+1} \subseteq \mathbf{m}_{2d-1} T_\Theta\mathcal{K}(h) + \langle v_{d-1}^{k+1} \rangle.$$

Then from Theorem 4.2.1 we have a $(k+1)$-transversal that is spanned by $\left\{v_{d-1}^{k+1}\right\}$. Hence, any $(k+1)$-jet with k-jet h is ${}_\Theta\mathcal{K}$-equivalent to $u_{d-2} + \lambda v_{d-1}^{k+1}$

with $\lambda \in \mathbb{K}$. If $\lambda \neq 0$, then from integrating the vector field $\xi = \xi_e - \frac{1}{d}\xi_2^1$ we get a diffeomorphism Φ which fixes $\lambda = \pm 1$.

ii) Let h denote the $(k+1)$-jet $u_{d-2} + \varepsilon v_{d-1}^{k+1}$.

Then from Proposition 6.2.6 we can see that that

1) $\xi_j^1(h) = 2(d-j)u_{d-2}u_j + \varepsilon(k+1)\Big(dv_{j-1} - \big(d + (d-2)(d-j)\big)u_j v_{d-1} - du_{d+j-1}w_2\Big)v_{d-1}^k$,

2) $\xi_j^2(h) = d(j+1)u_{d-j-1} - d(2d-j-1)u_{2d-j-1}w_1 - \varepsilon d(k+1)(d-j)v_{d-j}v_{d-1}^k$,

3) $\xi_j^3(h) = d^2 v_{d-j-1} - d(2d-j-1)u_{2d-j-1}w_2 - d(d-j)u_{d-j}v_{d-1}$,

4) $\xi_e(h) = 2u_{d-2} + \varepsilon(k+1)v_{d-1}^k$.

It follows that

$$T_\Theta\mathcal{K}(h) = \langle u_1, u_2, \ldots, u_{d-2}, v_1, v_2, \ldots, v_{d-1}^{k+1}, w_1, w_2\rangle.$$

Thus

$$\mathbf{m}_{2d-1}^{k+1} \subseteq T_\Theta\mathcal{K}(h).$$

Therefore, from Theorem 4.1.9 h is $(k+1)$-${}_\Theta\mathcal{K}$-determined. Furthermore, from the description of $T_\Theta\mathcal{K}(h)$ we get ${}_\Theta\mathcal{K}_e{-}\mathrm{cod}(h) = k+1$.

□

Proposition 6.2.9. *i) Any $(k+1)$-jet of a finite codimension function-germ with k-jet $u_{d-3} + v_{d-1}$ $(d \geq 4)$ and $k \geq 1$ is ${}_\Theta\mathcal{K}$-equivalent to one of the following. ($\varepsilon = \pm 1$ when $\mathbb{K} = \mathbb{R}$ and $\varepsilon = 1$ when $\mathbb{K} = \mathbb{C}$.)*

a) $u_{d-3} + v_{d-1} + \varepsilon u_{d-2}^{k+1}$ with d is even and $1 \leq k \leq 3$ or $d = 5, 7, 9$ and $1 \leq k \leq \frac{d+1}{2}$.

b) $u_{d-3}+\varepsilon v_{d-1}$ *with* $d=5,7$ *and* $k\geq\frac{d+1}{2}-1$.

ii) *a)* *The jet* $u_{d-3}+v_{d-1}+\varepsilon u_{d-2}^{k+1}$ *is* $(k+1)$-${}_{\Theta}\mathcal{K}$*-determined and has* ${}_{\Theta}\mathcal{K}_e$*-codimension* $k+1$.

b) *The jet* $u_{d-3}+v_{d-1}$ *is* $(\frac{d+1}{2})$-${}_{\Theta}\mathcal{K}$*-determined and has* ${}_{\Theta}\mathcal{K}_e$*-codimension* $\frac{d+1}{2}$.

Proof.

i) Let h denote the k-jet $u_{d-3}+v_{d-1}$. Then from Proposition 6.2.6 we have for all $1\leq j\leq d-1$:

1) $\xi_j^1(h)=3(d-j)u_{d-3}u_j+dv_{j-1}-\big(d+(d-2)(d-j)\big)u_jv_{d-1} - du_{d+j-1}w_2$,

2) $\xi_j^2(h)=d(j+2)u_{d-j-2}-d(2d-j-2)u_{2d-j-2}w_1-2(d-j)u_{d-2}u_{d-j}-d(d-j)v_{d-j}$,

3) $\xi_j^3(h)=d^2v_{d-j-2}-d(2d-j-2)u_{2d-j-2}w_2-d(d-j)u_{d-j}v_{d-2}$,

4) $\xi_e(h)=3u_{d-3}+v_{d-1}$.

From h and $\xi_e(h)$ we can get u_{d-3} and v_{d-1} in $T_{\Theta}\mathcal{K}(h)$. From $\xi_j^1(h)$ we can get $v_1,v_2,\ldots,v_{d-2}$ and w_2 in $T_{\Theta}\mathcal{K}(h)$.

(a) If d is even, then from ξ_j^2 for j is odd we get $u_{d-3},u_{d-5},\ldots,u_1$ and for j is even we have

$$\langle u_{d-4}+\psi_{d-4},u_{d-6}+\psi_{d-6},\ldots,u_2+\psi_2,w_1+\psi_1\rangle\subseteq T_{\Theta}\mathcal{K}(h)+\mathbf{m}_{2d-1}^2$$

with $\psi_{d-4},\psi_{d-6},\ldots,\psi_2$ and ψ_1 in $\mathbf{m}_{2d-1}^2$.

Therefore, for all $k\geq 1$ we have

$$\mathbf{m}_{2d-1}^{k+1}\subseteq\mathbf{m}_{2d-1}T_{\Theta}\mathcal{K}(h)+\langle u_{d-2}^{k+1}\rangle+\mathbf{m}_{2d-1}^{k+2}.$$

Then from Theorem 4.2.1 we have a $(k+1)$-transversal that is spanned by $\{u_{d-2}^{k+1}\}$. Hence, any $(k+1)$-jet with k-jet h is ${}_\Theta\mathcal{K}$-equivalent to $u_{d-3}+v_{d-1}+\lambda u_{d-2}^{k+1}$ with $\lambda \in \mathbb{K}$. If $\lambda \neq 0$, then from integrating the vector field $\xi = (d-1)\xi_e + \frac{1}{d}\xi_2^1$ we get a diffeomorphism which fixes $\lambda = \pm 1$.

(b) If d is odd, then from ξ_j^2 for j is odd we get $u_{d-3}, u_{d-5}, \ldots, u_2, w_1$ and for j is even we have $u_{d-2}^{\frac{d+1}{2}} \in T_\Theta\mathcal{K}(h)$ and

$$\langle u_{d-4}+\psi_{d-4}, u_{d-6}+\psi_{d-6}, \ldots, u_3+\psi_3, u_1+\psi_1\rangle \subseteq T_\Theta\mathcal{K}(h) + \mathbf{m}_{2d-1}^2$$

with $\psi_{d-4}, \psi_{d-6}, \ldots, \psi_3$ and ψ_1 in $\mathbf{m}_{2d-1}^2$.

(b$_1$) For all $1 \leq k \leq \frac{d+1}{2} - 1$ we have a $(k+1)$-transversal that is spanned by $\{u_{d-2}^{k+1}\}$. In the same way in (a) we have any $(k+1)$-jet with k-jet h is ${}_\Theta\mathcal{K}$-equivalent to $u_{d-3}+v_{d-1} \pm u_{d-2}^{k+1}$ with $\lambda \in \mathbb{K}$.

(b$_2$) For all $k \geq \frac{d+1}{2}$ we can see that the $(k+1)$-transversal is empty. Therefore, any $(k+1)$-jet with k-jet h is ${}_\Theta\mathcal{K}$-equivalent to $u_{d-3}+v_{d-1}$.

ii) *a)* For d is even with $k \geq 1$ or d is odd with $1 \leq k \leq \frac{d+1}{2} - 1$. Let $h = u_{d-3}+v_{d-1} \pm u_{d-2}^{k+1}$, then we have

$$\mathbf{m}_{2d-1}^{k+1} \subseteq \mathbf{m}_{2d-1} T_\Theta\mathcal{K}(h) + \mathbf{m}_{2d-1}^{k+2}.$$

By Nakayama's lemma we have

$$\mathbf{m}_{2d-1}^{k+1} \subseteq T_\Theta\mathcal{K}(h).$$

Therefore, from Theorem 4.1.9 h is $(k+1)$-${}_\Theta\mathcal{K}$-determined. Furthermore, we have ${}_\Theta\mathcal{K}_e-\mathrm{cod}(h) = k+1$.

b) For d is odd with $k \geq \frac{d+1}{2}$. Let $h = u_{d-3}+v_{d-1}$, then we have

$$\mathbf{m}_{2d-1}^{\frac{d+1}{2}} \subseteq \mathbf{m}_{2d-1} T_\Theta\mathcal{K}(h).$$

Therefore, from Theorem 4.1.9 h is $(\frac{d+1}{2})$-${}_\Theta\mathcal{K}$-determined and we can see that ${}_\Theta\mathcal{K}_e{-}\mathrm{cod}(h) = \frac{d+1}{2}$. □

Proposition 6.2.10. *i) Let $d = 3$. Then any $(k+1)$-jet with k-jet $v_{d-1} + w_1$ and $k \geq 1$ is ${}_\Theta\mathcal{K}$-equivalent to $v_{d-1} + w_1$.*

ii) The jet $v_{d-1} + w_1$ is 2-${}_\Theta\mathcal{K}$-determined and has ${}_\Theta\mathcal{K}_e$-codimension 2.

Proof.

i) Let $h(u_1, v_1, v_2, w_1, w_2) = v_2 + w_1$. Then we have

$$\begin{aligned} T_\Theta\mathcal{K}(h) &= T_\Theta\mathcal{K}_e(h) \\ &= \langle \xi_j^1(h), \xi_j^2(h), \xi_j^3(h), \xi_e(h)\rangle_{j=1}^2 + \langle h\rangle \\ &= \langle -3w_2 - 5u_1v_2 + 6u_1w_1, 3v_1, -6v_2 + 9w_1, 3u_1v_1, v_2 + 3w_1, \\ &\quad -3v_1 + 2u_1^2 + 9w_2 + 3u_1v_2\rangle + \langle v_2 + w_1\rangle \\ &= \langle v_1, v_2, w_1, w_2, u_1^2\rangle. \end{aligned}$$

Then we have

$$\mathbf{m}_5^2 \subseteq \mathbf{m}_5 T_\Theta\mathcal{K}(h) + \langle u_1^2\rangle.$$

And for $k \geq 2$,

$$\mathbf{m}_5^{k+1} \subseteq \mathbf{m}_5 T_\Theta\mathcal{K}(h).$$

This $\{u_1^2\}$ is a 2-transversal and for all $k \geq 2$, the $(k+1)$-transversal is empty. Hence any function g with 1-jet equal to $v_2 + w_1$ is ${}_\Theta\mathcal{K}$-equivalent to some H with $j^2H = v_2 + w_1 + \lambda u_1^2$, where $\lambda \in \mathbb{K}$.

If we consider j^2H as a 1-parameter family H_λ, then we have

$$\begin{aligned} T_\Theta\mathcal{K}(H_\lambda) &= \langle \xi_j^1(H_\lambda), \xi_j^2(H_\lambda), \xi_j^3(H_\lambda), \xi_e(H_\lambda)\rangle_{j=1}^2 + \langle H_\lambda\rangle \\ &= \langle 8\lambda u_1^3 - 3w_2 - 5u_1v_2 + 6u_1w_1, 3v_1, 12\lambda u_1^2 - 6v_2 + 9w_1, \\ &\quad -18\lambda u_1w_1 - 3v_1 + 2u_1^2 + 18\lambda u_1v_1 + 9w_2 + 3u_1v_2, -18\lambda u_1w_2 \\ &\quad -6\lambda u_1^2v_2 + 3u_1v_1, 4\lambda u_1^2 + v_2 + 3w_1\rangle + \langle v_2 + w_1 + \lambda u_1^2\rangle \\ &= \langle v_1, v_2, w_1, w_2, u_1^2\rangle. \end{aligned}$$

Thus,

$$\frac{\partial H_\lambda}{\partial \lambda} \in T_\Theta\mathcal{K}(H_\lambda).$$

Hence, we get H_λ is a ${}_\Theta\mathcal{K}$-trivial.

ii) Obviously, $\mathbf{m}_5^2 \subseteq \mathbf{m}_5 T_\Theta\mathcal{K}(h)$. Hence h is 2-${}_V\mathcal{K}$-determined and from the description of $T_\Theta\mathcal{K}_e(h)$ we get ${}_\Theta\mathcal{K}_e$−cod$(h) = 2$.

□

Definition 6.2.11. *Let Θ be a module of smooth vector fields on $(\mathbb{K}^p, 0)$ such that all these vector fields vanish at 0 and $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be a map-germ. We define $T_{j^1\Theta}\mathcal{K}(j^1h)$ to be the vector space over $\mathbb{K}$ of the linear parts of the elements in $T_\Theta\mathcal{K}(h)$.*

We need the following proposition in the proof of theorem 6.0.20.

Proposition 6.2.12. *Let Θ be a module of smooth vector fields on $(\mathbb{K}^p, 0)$ such that all these vector fields vanish at 0 and $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be a smooth map-germ. Let g be a linear. If $g \notin T_{j^1\Theta}\mathcal{K}(j^1h)$, then $g \notin T_\Theta\mathcal{K}(h)$.*

Proof. Suppose that $g \in T_\Theta\mathcal{K}(h)$. We have

$$T_\Theta\mathcal{K}(h) \subseteq T_{j^1\Theta}\mathcal{K}(j^1h) + G \quad \text{with } G \subseteq \mathbf{m}_p^2\theta(h).$$

Therefore, we get that $g \in T_{j^1\Theta}\mathcal{K}(j^1 h) + G$ with $G \subseteq \mathbf{m}_p^2\theta(h)$.

It follows $g \in T_{j^1\Theta}\mathcal{K}(j^1 h)$. □

6.3 The proof of theorem 6.0.20

In this section we give the proof of Theorem 6.0.20. Let $h : (\mathbb{K}^{2d-1}, 0) \to (\mathbb{K}^q, 0)$ be a submersion map-germ with ${}_\Theta\mathcal{K}_e$-codimension at most 2. We divide the proof into the following steps:

First Step: From Chapter 3 we know that Θ is generated by the liftable vector fields ξ_j^1, ξ_j^2, ξ_j^3 and ξ_e. All these vector fields vanish at the origin. Then from Theorem 6.2.1 we have ${}_\Theta\mathcal{K}_e{-}\mathrm{cod}(h) > 0$.

Suppose that $q \geq 3$, then from Theorem 6.2.1, we get ${}_\Theta\mathcal{K}_e - -\mathrm{cod}(h) \geq 3$ and this is a contradiction because ${}_\Theta\mathcal{K}_e\ -\mathrm{cod}(h) \leq 2$. Therefore, we have $1 \leq {}_\Theta\mathcal{K}_e{-}\mathrm{cod}(h) \leq 2$ and $1 \leq q \leq 2$.

Second Step: Classification of function-germs from $(\mathbb{K}^{2d-1}, 0)$ to $(\mathbb{K}, 0)$.

We consider the 1-jet of h, i.e., its linear part:

$$j^1 h = \sum_{i=1}^{d-2} \alpha_i u_i + \sum_{i=1}^{d-1} \beta_i v_i + \gamma_1 w_1 + \gamma_2 w_2$$

for some constants α_i, β_i, $\gamma_i \in \mathbb{K}$.

For $d = 2$, Bruce and West gave the classification of function-germs as in $\mathrm{I}_{2,\mathrm{k+1}}$ and $\mathrm{II}_{2,2}$ ([BW98], theorem 3.13).

Now, we suppose that $d \geq 3$,

$$j^1 h = \sum_{i=1}^{d-2} \alpha_i u_i + \sum_{i=1}^{d-1} \beta_i v_i + \gamma_1 w_1 + \gamma_2 w_2.$$

(1) If $u_{d-2} \notin N_\Theta\mathcal{K}_e(h)$, then we have $\alpha_{d-2} \neq 0$. Since if $\alpha_{d-2} = 0$, then we can not get u_{d-2} in $T_\Theta\mathcal{K}_e(h)$. By using the 1-jets of Φ_j^3 and Φ_j^2 we can remove

$u_1, \ldots, u_{d-3}, v_1, \ldots, v_{d-2}, w_1$ and w_2.

It follows, we have

$$j^1 h = \alpha u_{d-2} + \beta_{d-1} v_{d-1}.$$

By using the matrix $M = \left[\frac{1}{\alpha_{d-2}}\right]$ (in the definition of $\mathcal{K}$-equivalence we need a diffeomorphism and a matrix)), then we fix $\alpha_{d-2} = 1$ and we get

$$j^1 h = u_{d-2} + \beta_{d-1} v_{d-1}.$$

- If $\beta_{d-1} \neq 0$, then we fix $\beta_{d-1} = \pm 1$ by using the diffeomorphism in Example 6.1.1 and we get

  $$j^1 h = u_{d-2} \pm v_{d-1}.$$

  Then, from Proposition 6.2.7, we have that h is ${}_{\Theta}\mathcal{K}$-equivalent to $u_{d-2} \pm v_{d-1}$ which is 1-${}_{\Theta}\mathcal{K}$-determined and has ${}_{\Theta}\mathcal{K}_e$-codimension 1.

- If $\beta_{d-1} = 0$, then we have $j^1 h = u_{d-2}$. Therefore, from Proposition 6.2.8, we have h is ${}_{\Theta}\mathcal{K}$-equivalent to $u_{d-2} \pm v_{d-1}^2$ which is 2-${}_{\Theta}\mathcal{K}$-determined and has ${}_{\Theta}\mathcal{K}_e$-codimension 2.

(2) If $u_{d-2} \in N_{\Theta}\mathcal{K}_e(h)$, then we have $N_{\Theta}\mathcal{K}_e(h) = \langle 1, u_{d-2} \rangle$. We can see that $\beta_{d-1} \neq 0$. (Since if $\beta_{d-1} = 0$, then $v_{d-1} \in N_{\Theta}\mathcal{K}_e(h)$).

a) If $d = 3$, then we have

$$j^1 h = \beta_2 v_2 + \gamma_1 w_1.$$

By using the 1-jets of Φ_j^1 we can remove v_1 and w_2. Then, we get

$$j^1 h = \beta_2 v_2 + \gamma_1 w_1.$$

We fix $\beta_2 = 1$ by using the matrix $M = [\frac{1}{\beta_2}]$. Then, we get $j^1 h = v_2 + \gamma_1 w_1$. If $\gamma_1 = 0$, then $w_1 \in N_{\Theta}\mathcal{K}_e(h)$. Therefore, we have $\gamma_1 \neq 0$.

By integrating the vector field

$$\xi = 2\xi_e + \frac{1}{3}\xi_1^2$$
$$= (6u_1, 3v_1, 0, 9w_1, 6w_2)^T.$$

We get a diffeomorphism

$$\Phi(u_1, v_1, v_2, w_1, w_2) = (e^{6\mu}u_1, \ldots, e^{3\mu}v_1, v_2, e^{9\mu}w_1, e^{9\mu}w_2), \quad \mu \in \mathbb{K}.$$

From this diffeomorphism we fix $\gamma_1 = \pm 1$ and we get

$$j^1 h = v_2 \pm w_1.$$

Then, from Proposition 6.2.10, we have that h is ${}_\Theta\mathcal{K}$-equivalent to $v_2 \pm w_1$ which is 2-${}_\Theta\mathcal{K}$-determined and has ${}_\Theta\mathcal{K}_e$-codimension 2.

b) If $d \geq 4$, then we have that $\alpha_{d-3} \neq 0$. (Since if $\alpha_{d-3} = 0$, then $u_{d-3} \in N_\Theta\mathcal{K}_e(h)$).

By using the 1-jets of Φ_j^2 and Φ_j^1 we can remove $u_1, \ldots, u_{d-4}, w_1, v_1, \ldots, v_{d-2}$ and w_2. Then, we get

$$j^1 h = \alpha_{d-3}u_{d-3} + \beta_{d-1}v_{d-1}.$$

We fix $\alpha_{d-3} = 1$ by using the matrix $M = [\frac{1}{\alpha_{d-3}}]$. Then, we get

$$j^1 h = u_{d-3} + \beta_{d-1}v_{d-1}.$$

By using the diffeomorphism in Example 6.1.1 then we fix $\beta_{d-1} = \pm 1$ and we get

$$j^1 h = u_{d-3} \pm v_{d-1}.$$

From Proposition 6.2.9, we have that h is ${}_\Theta\mathcal{K}$-equivalent to

$$u_{d-3} \pm v_{d-1} \pm u_{d-2}^2$$

which is 2-${}_\Theta\mathcal{K}$-determined and has ${}_\Theta\mathcal{K}_e$-codimension 2.

Third Step: Classification of map-germs from $(\mathbb{K}^{2d-1},0)$ to $(\mathbb{K}^2,0)$.

Since $q=2$, then from Theorem 6.2.1 there is not a map-germ with ${}_\Theta\mathcal{K}_e-\text{cod}(h)=1$. Therefore, we assume that ${}_\Theta\mathcal{K}_e-\text{cod}(h)=2$.

If $d\geq 5$, then according to Corollary 6.2.5, we can see that there is not a map-germ with ${}_\Theta\mathcal{K}_e-\text{cod}(h)=2$. Hence, we consider $d=2,3$ and 4 only.

Suppose that $e_1=\begin{pmatrix}1\\0\end{pmatrix}$ and $e_2=\begin{pmatrix}0\\1\end{pmatrix}$. Since every vector field in Θ vanishes at the origin, then we can see that e_1 and e_2 not in $T_\Theta\mathcal{K}_e(h)$. It follows e_1 and e_2 in $N_\Theta\mathcal{K}_e(h)$. Since ${}_\Theta\mathcal{K}_e-\text{cod}(h)=2$, then we have

$$N_\Theta\mathcal{K}_e(h)=\langle e_1,e_2\rangle \tag{6.3.1}$$

We consider the 1-jet of h, i.e., its linear part:

$$j^1h=\left(\sum_{i=1}^{d-2}\alpha_{1,i}u_i+\sum_{i=1}^{d-1}\beta_{1,i}v_i+\sum_{i=1}^{2}\gamma_{1,i}w_i,\sum_{i=1}^{d-2}\alpha_{2,i}u_i+\sum_{i=1}^{d-1}\beta_{2,i}v_i+\sum_{i=1}^{2}\gamma_{2,i}w_i\right),$$

for some constants $\alpha_{j,i}$, $\beta_{j,i}$ and $\gamma_{j,i}$ in $\mathbb{K}$.

(A) Suppose that $d=2$. We have

$$j^1h=\left(\beta_{1,1}v_1+\sum_{i=1}^{2}\gamma_{1,i}w_i,\beta_{2,1}v_1+\sum_{i=1}^{2}\gamma_{2,i}w_i\right),$$

for some constants $\beta_{j,1}$ and $\gamma_{j,i}$ in $\mathbb{K}$.

If $\beta_{j,1}=0$ for all $1\leq j\leq 2$, then one finds

$$\begin{aligned}T_\Theta\mathcal{K}(j^1h) \;=\; \Big\langle & \begin{pmatrix}2\gamma_{1,2}w_2\\2\gamma_{2,2}w_2\end{pmatrix},\begin{pmatrix}2\gamma_{1,1}w_2+_{1,2}v_1^2\\2\gamma_{2,1}w_2+_{2,2}v_1^2\end{pmatrix},\begin{pmatrix}2\gamma_{1,1}w_1\\2\gamma_{2,1}w_1\end{pmatrix},\begin{pmatrix}2\gamma_{1,2}v_1w_1\\2\gamma_{1,2}v_1w_1\end{pmatrix},\\ & \begin{pmatrix}\sum_{i=1}^{2}\gamma_{1,i}w_i\\0\end{pmatrix},\begin{pmatrix}0\\\sum_{i=1}^{2}\gamma_{1,i}w_i\end{pmatrix},\begin{pmatrix}\sum_{i=1}^{2}\gamma_{2,i}w_i\\0\end{pmatrix},\begin{pmatrix}0\\\sum_{i=1}^{2}\gamma_{2,i}w_i\end{pmatrix}\Big\rangle.\end{aligned}$$

We can see that v_1e_1 and v_1e_2 not in $T_{j^1\Theta}\mathcal{K}_e(j^1h)$. It follows from Proposition 6.2.12 we have v_1e_1 and v_1e_2 not in $T_\Theta\mathcal{K}_e(h)$ and this is a contradiction with equation (6.3.1).

Therefore, we have $\beta_{j,1} \neq 0$ for some $1 \leq j \leq 2$. We can assume that $\beta_{1,1} \neq 0$ (if $\beta_{1,1} = 0$, then we can swap). Also if $\beta_{j,1} \neq 0$ for $j = 1, 2$, then we can remove v_1 in the second coordinate by using the matrix $M_1 = \text{diagonal}(-\beta_{2,1}/\beta_{1,1}, 1)$.

Therefore, we have

$$j^1h = \left(\beta_{1,1}v_1 + \sum_{i=1}^{2}\gamma_{1,i}w_i, \sum_{i=1}^{2}\gamma_{2,i}w_i\right).$$

We fix $\beta_{1,1} = 1$ by using $M_2 = \text{diagonal}(1/\beta_{1,1}, 1)$. It follows that

$$j^1h = \left(v_1 + \sum_{i=1}^{2}\gamma_{1,i}w_i, \sum_{i=1}^{2}\gamma_{2,i}w_i\right).$$

If $\gamma_{1,2} \neq 0$, then we can we can remove w_2 from the first coordinate by using the 1-jet of the diffeomorphism Φ_1^1. Hence, we have

$$j^1h = \left(v_1 + \gamma_{1,1}w_1, \sum_{i=1}^{2}\gamma_{2,i}w_i\right).$$

Now, we consider the following cases:

(1A) If $\gamma_{1,1} \neq 0$, then from integrating the vector field $\xi = \xi_e + \xi_1^2$ we get a diffeomorphism Φ which fixes $\gamma_{1,1} = \pm 1$.

Therefore, we have

$$j^1h = \left(v_1 \pm w_1, \sum_{i=1}^{2}\gamma_{2,i}w_i\right).$$

(1A.1) If $\gamma_{2,1} \neq 0$, then we can fix $\gamma_{2,1} = 1$ by using the matrix $M_3 = \text{diagonal}(1, 1/\beta_{2,1})$. Suppose that $\gamma_{2,2} \neq 0$, then by using the 1-jet of the diffeomorphism Φ_1^3 we can remove w_2 from the second coordinate and we get

$$j^1h = (v_1 \pm w_1 + \alpha w(2), w_1) \quad \text{with } \alpha \in \mathbb{K}.$$

By using the 1-jet of the diffeomorphism Φ_1^1 we can remove w_2 from the first coordinate and we get

$$j^1h = (v_1 \pm w_1, w_1)\,.$$

Also by using the matrix

$$M_4^{\mp} = \begin{pmatrix} 1 & \mp 1 \\ 0 & 1 \end{pmatrix}$$

We get

$$j^1h = (v_1, w_1)\,.$$

Then from the `CAST` package we find that the 2-transversal is empty. Hence for all $k \geq 1$ we have an empty $(k+1)$-transversal. Therefore, any $(k+1)$-jet with k-jet (v_1, w_1) and $k \geq 1$ is ${}_{\Theta}\mathcal{K}$-equivalent to (v_1, w_1). Also, we deduce that (v_1, w_1) is 1-${}_{\Theta}\mathcal{K}$-determined with ${}_{\Theta}\mathcal{K}_e$-codimension 2.

(1A.2) If $\gamma_{2,1} = 0$, then we have

$$j^1h = (v_1 \pm w_1, \gamma_{2,2} w_2)\,.$$

If $\gamma_{2,2} = 0$, then $j^1h = (v_1 \pm w_1, 0)$. This gives a non-submersive map-germ for h. Thus we can assume $\gamma_{2,2} \neq 0$.

We can fix $\gamma_{2,2} = 1$ by using the matrix $M_5 = \text{diagonal}(1, 1/\beta_{2,2})$ and we get

$$j^1h = (v_1 \pm w_1, w_2)\,.$$

From the `CAST` package we have $w_1e_2 \notin T_{j^1\Theta}\mathcal{K}(j^1h)$. It follows from Proposition 6.2.12 that we have $w_1e_2 \notin T_\Theta\mathcal{K}(h)$ and this again contradict with equation 5.3.1.

(2A) If $\gamma_{1,1} = 0$, then we have

$$j^1h = \left(v_1, \sum_{i=1}^{2}\gamma_{2,i}w_i\right).$$

If $\gamma_{2,1} = 0$, then w_1e_1 and w_1e_2 are not in $T_{j^1\Theta}\mathcal{K}(j^1h)$. It follows from Proposition 6.2.12 we have w_1e_1 and w_1e_2 not in $T_\Theta\mathcal{K}_e(h)$ and this again contradict with equation 5.3.1.

Thus we assume $\gamma_{2,1} \neq 0$. Then we can fix $\gamma_{2,1} = 1$ by using the matrix $M_5 = \text{diagonal}(1, 1/\beta_{2,1})$. If $\gamma_{2,2} \neq 0$, then by using the 1-jet of the diffeomorphism Φ_1^3 we can remove w_2 from the second coordinate and we get

$$j^1h = (v_1, w_1).$$

This similar to the case 1A.1.

(B) Suppose that $3 \leq d \leq 4$. We have

$$j^1h = \left(\sum_{i=1}^{d-2}\alpha_{1,i}u_i + \sum_{i=1}^{d-1}\beta_{1,i}v_i + \sum_{i=1}^{2}\gamma_{1,i}w_i, \sum_{i=1}^{d-2}\alpha_{2,i}u_i + \sum_{i=1}^{d-1}\beta_{2,i}v_i + \sum_{i=1}^{2}\gamma_{2,i}w_i\right),$$

If $\alpha_{1,d-2} = \alpha_{2,d-2} = 0$, we can see that $u_{d-2}e_1$ and $u_{d-2}e_2$ are not in $T_{j^1\Theta}\mathcal{K}(j^1h)$. It follows from Proposition 6.2.12 we have $u_{d-2}e_1$ and $u_{d-2}e_2$ not in $T_\Theta\mathcal{K}_e(h)$ and this is a contradiction with equation 5.3.1.

Therefore, we have either $\alpha_{1,d-2} \neq 0$ or $\alpha_{1,d-2} \neq 0$. We assume that $\alpha_{1,d-2} \neq 0$ (if $\alpha_{1,d-2} = 0$, then we can swap.)

We can fix $\alpha_{1,d-2} = 1$ by using the matrix $M_6 = \text{diagonal}(1/\alpha_{1,d-2}, 1)$. Also if $\alpha_{2,d-2} \neq 0$, then we can remove u_{d-2} form the second coordinate by using the matrix

$$M_7^{\mp} = \begin{pmatrix} 1 & 0 \\ -\alpha_{2,d-2} & 1 \end{pmatrix}.$$

By using the 1-jets of Φ_j^3 and Φ_j^2 we can remove $u_1, \ldots, u_{d-3}, v_1, \ldots, v_{d-2}, w_1$ and w_2 from the first coordinate. Therefore, we have

$$j^1h = \left(u_{d-2} + \beta_{1,d-1}v_{d-1}, \sum_{i=1}^{d-3} \alpha_{2,i}u_i + \sum_{i=1}^{d-1} \beta_{2,i}v_i + \sum_{i=1}^{2} \gamma_{2,i}w_i \right),$$

If $\beta_{2,d-1} = 0$, then we can see that $v_{d-1}e_2 \notin \mathrm{T}_{j^1\Theta}\mathcal{K}(j^1h)$. It follows from Proposition 6.2.12 we have $v_{d-1}e_2 \notin \mathrm{T}_\Theta\mathcal{K}_e(h)$ and this is a contradiction with equation 5.3.1.

Therefore, we assume that $\beta_{2,d-1} \neq 0$. We can fix $\beta_{2,d-1} = 1$ by using the matrix $M_8 = \text{diagonal}(1, 1/\beta_{2,d-1})$. It follows

$$j^1h = \left(u_{d-2} + \beta_{1,d-1}v_{d-1}, \sum_{i=1}^{d-3} \alpha_{2,i}u_i + \sum_{i=1}^{d-2} \beta_{2,i}v_i + v_{d-1} + \sum_{i=1}^{2} \gamma_{2,i}w_i \right),$$

If $\beta_{1,d-1} \neq 0$, then we consider j^1h as a 1-parameter family $H_{\beta_{1,d-1}}$, then we have

$$\begin{pmatrix} \sum_{i=1}^{d-2} \beta_{2,i}v_i + v_{d-1} + \sum_{i=1}^{2} \gamma_{2,i}w_i \\ 0 \end{pmatrix} = \begin{pmatrix} v_{d-1} \\ 0 \end{pmatrix} + \varphi, \quad \varphi \in \mathbf{m}^2\theta(H_{\beta_{1,d-1}}).$$

It follows that

$$\frac{\partial H_{\beta_{1,d-1}}}{\partial \beta_{1,d-1}} \in T_\Theta\mathcal{K}(H_{\beta_{1,d-1}}) + \mathbf{m}^2\theta(H_{\beta_{1,d-1}}).$$

Thus, $H_{\beta_{1,d-1}}$ is 1-${}_\Theta\mathcal{K}$-trivial. Hence

$$j^1h = \left(u_{d-2}, \sum_{i=1}^{d-3} \alpha_{2,i}u_i + \sum_{i=1}^{d-2} \beta_{2,i}v_i + v_{d-1} + \sum_{i=1}^{2} \gamma_{2,i}w_i \right).$$

By using the 1-jets of Φ_j^1 we can remove $v_1, \ldots, v_{d-2}$ and w_2 from the second coordinate. Therefore, we have

$$j^1h = \left(u_{d-2}, \sum_{i=1}^{d-3} \alpha_{2,i}u_i + v_{d-1} + \gamma_{2,1}w_1 \right).$$

Consider $d = 3$. Then, we have $j^1h = (u_1, v_2 + \gamma_{2,1}w_1)$.

a) If $\gamma_{2,1} \neq 0$, then from integrating the vector field $\xi = 2\xi_e + \frac{1}{3}\xi_1^2$ we get a diffeomorphism Φ such that

$$\Phi(u_1, v_1, v_2, w_1, w_2) = (e^{4\alpha}u_1, e^{-\alpha}v_1, v_2, e^{3\alpha}w_1, e^{6\alpha}w_2), \quad \alpha \in \mathbb{K}.$$

We fix $\gamma_{2,1} = \pm 1$ by using Φ. From the matrix $M_8 = \text{diagonal}(e^{-4\alpha}, 1)$ we have $j^1h = (u_1, v_2 \pm w_1)$.

If we have $j^kh = (u_1, v_2 \pm w_1)$, then from the `CAST` package we find that the 2-transversal is empty. Hence for all $k \geq 1$ we have an empty $(k+1)$-transversal. Therefore, any $(k+1)$-jet with k-jet $(u_1, v_2 \pm w_1)$ and $k \geq 1$ is ${}_{\Theta}\mathcal{K}$-equivalent to $(u_1, v_2 \pm w_1)$. Also, we deduce $(u_1, v_2 \pm w_1)$ is 1-${}_{\Theta}\mathcal{K}$-determined with ${}_{\Theta}\mathcal{K}_e$-codimension 2.

b) If $\gamma_{2,1} = 0$, then we have $j^1h = (u_1, v_2)$. Let $g(u_1, v_1, v_2, w_1, w_2) = (u_1, v_2)$. Then we have

$$\begin{aligned} T_{\Theta}\mathcal{K}(g) \;=\; \Big\langle & \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \\ & \begin{pmatrix} w_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ w_2 \end{pmatrix} \Big\rangle. \end{aligned}$$

It follows that for all $k \geq 1$, we have

$$\mathbf{m}_5^{k+1}\mathcal{E}_5^2 \subseteq T_{\Theta}\mathcal{K}(g) + \Big\langle \begin{pmatrix} 0 \\ w_1^{k+1} \end{pmatrix} \Big\rangle.$$

Thus, if we have a map-germ with k-jet equal to (u_1, v_2), then a $(k+1)$-transversal is $\left\{ \begin{pmatrix} 0 \\ w_2^{k+1} \end{pmatrix} \right\}$.

Let $h = (u_1, v_2 + \lambda w_1^{k+1})$ with $\lambda \neq 0$, we can fix $\lambda = \pm 1$ by using the diffeomorphism Φ as in step (i). Also, from the matrix $M_9 = \text{diagonal}(e^{-4\alpha}, 1)$ we get $h = (u_1, v_2 \pm w_1^{k+1})$. For all $k \geq 1$ we have

$$T_\Theta\mathcal{K}(h) = \left\langle \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_1 \end{pmatrix}, \begin{pmatrix} v_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \begin{pmatrix} v_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ w_1^{k+1} \end{pmatrix}, \begin{pmatrix} w_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ w_2 \end{pmatrix} \right\rangle.$$

Thus

$$\mathbf{m}_5^{k+1}\mathcal{E}_5^2 \subseteq T_\Theta\mathcal{K}(h).$$

Hence h is a $(k+1)$-${}_\Theta\mathcal{K}$-determined. Furthermore, from the description of $T_\Theta\mathcal{K}(h)$ we get ${}_\Theta\mathcal{K}_e{-}\text{cod}(h) = k+2$.

Consider $d = 4$. Then, we have $j^1h = (u_2, \alpha_{2,1}u_1 + v_3 + \gamma_{2,1}w_1)$. We find

$$\begin{aligned} T_\Theta\mathcal{K}(j^1h) = \Big\langle & \begin{pmatrix} u_1 \\ 0 \end{pmatrix}, \begin{pmatrix} u_2 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ u_2 \end{pmatrix}, \begin{pmatrix} 0 \\ v_1 \end{pmatrix}, \begin{pmatrix} 0 \\ v_2 \end{pmatrix}, \begin{pmatrix} 0 \\ v_3 \end{pmatrix}, \begin{pmatrix} w_1 \\ 0 \end{pmatrix}, \\ & \begin{pmatrix} v_2 \\ \gamma_{2,1}w_2 \end{pmatrix}, \begin{pmatrix} v_1 \\ -\alpha_{2,1}w_2 \end{pmatrix}, \begin{pmatrix} -16w_1 \\ 3\alpha_{2,1}u_1^2 \end{pmatrix}, \begin{pmatrix} -4w_2 - u_1v_3 \\ 0 \end{pmatrix}, \\ & \begin{pmatrix} v_3 + \gamma_{2,1}w_1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ -4w_2 + 9\gamma_{2,1}u_1^2 + 12\gamma_{2,1}u_1w_1 \end{pmatrix}, \\ & \begin{pmatrix} 0 \\ 3\alpha_{2,1}u_1 + 4\gamma_{2,1}w_1 \end{pmatrix}, \begin{pmatrix} 0 \\ \alpha_{2,1}u_1 + \gamma_{2,1}w_1 \end{pmatrix} \Big\rangle. \end{aligned}$$

If $\alpha_{2,1} = 0$, then we can see that $u_1 e_2 \notin \mathrm{T}_{j^1\Theta}\mathcal{K}(j^1 h)$. It follows from Proposition 6.2.12 we have $u_1 e_2 \notin \mathrm{T}_{\Theta}\mathcal{K}_e(h)$ and this is a contradict (6.3.1). Similarly, if $\gamma_{2,1} = 0$, then we can see that $w_1 e_2 \notin \mathrm{T}_{j^1\Theta}\mathcal{K}(j^1 h)$. It follows from Proposition 6.2.12 we have $w_1 e_2 \notin \mathrm{T}_{\Theta}\mathcal{K}_e(h)$ and this is a contradiction.

Therefore, we assume that $\alpha_{2,1} \neq 0$ and $\gamma_{2,1} \neq 0$. We consider the vector fields $\xi_1 = \frac{3}{4}\xi_e + \frac{1}{16}\xi_1^2$ and $\xi_2 = \frac{1}{4}\xi_e - \frac{1}{16}\xi_1^2$.

By integrating these vector fields we get diffeomorphisms Φ_1 and Φ_2 respectively such that

$$\begin{aligned} \Phi_1(\underline{u}, \underline{v}, \underline{w}) &= (e^{3\lambda_1}u_1, e^{2\lambda_1}u_2, e^{2\lambda_1}v_1, e^{\lambda_1}v_2, v_3, e^{4\lambda_1}w_1, e^{3\lambda_1}w_2) \quad \text{and} \\ \Phi_2(\underline{u}, \underline{v}, \underline{w}) &= (u_1, u_2, e^{\lambda_2}v_1, e^{\lambda_2}v_2, e^{\lambda_2}v_3, w_1, w_2), \end{aligned}$$

where $\lambda_i \in \mathbb{K}$.

We fix $\alpha_{2,1} = 1$ by using the matrix $M_{10} = \mathrm{diagonal}(1, 1/\alpha_{2,1})$. Therefore we have

$$j^1 h = \left(u_2, u_1 + \frac{1}{\alpha_{2,1}} v_3 + \frac{\gamma_{2,1}}{\alpha_{2,1}} w_1 \right).$$

We can fix $\dfrac{\gamma_{2,1}}{\alpha_{2,1}} = \pm 1$ by using Φ_1 and the matrix $M_{11} = \mathrm{diagonal}(e^{-2\lambda_1}, e^{-3\lambda_1})$. Therefore, we get

$$j^1 h = \left(u_2, u_1 + \frac{1}{\alpha_{2,1}} v_3 \pm w_1 \right).$$

By using Φ_2 we can fix $\dfrac{1}{\alpha_{2,1}} = \pm 1$ and hence we get

$$j^1 h = (u_2, u_1 \pm v_3 \pm w_1).$$

Then from the `CAST` package we find that the 2-transversal is empty. Hence for all $k \geq 1$ we have an empty $(k+1)$-transversal. Therefore, any $(k+1)$-jet with k-jet $(u_2, u_1 \pm v_3 \pm w_1)$ and $k \geq 1$ is ${}_{\Theta}\mathcal{K}$-equivalent to $(u_2, u_1 \pm v_3 \pm w_1)$. Also, we deduce that $(u_2, u_1 \pm v_3 \pm w_1)$ is 1-${}_{\Theta}\mathcal{K}$-determined with ${}_{\Theta}\mathcal{K}_e$-codimension 2.

□

6.4 The proof of corollary 6.0.21

We shall use the diffeomorphism in Proposition 6.1.2 in the proof. This diffeomorphism defined by

$$\Phi(\underline{u}, v_1, \ldots, v_{d-1}, w_1, w_2) = (\underline{u}, -v_1, \ldots, -v_{d-1}, w_1, -w_2).$$

i) For $\mathrm{I}^-_{\mathrm{d,k+1}}$, $\mathrm{III}^-_{\mathrm{d,1}}$ and $\mathrm{V}^-_{3,2}$. By using Φ and the matrix $M = [-1]$ we have that $\mathrm{I}^-_{\mathrm{d,k+1}} \sim_{\mathrm{V}\mathcal{K}} \mathrm{I}^+_{\mathrm{d,k+1}}$, $\mathrm{III}^-_{\mathrm{d,1}} \sim_{\mathrm{V}\mathcal{K}} \mathrm{III}^+_{\mathrm{d,1}}$ and $\mathrm{V}^-_{3,2} \sim_{\mathrm{V}\mathcal{K}} \mathrm{V}^+_{3,2}$.

ii) For $\mathrm{VII}^-_{3,1}$. By using Φ and the matrix $M = \mathrm{diagonal}(1, -1)$ we have that $\mathrm{VII}^-_{3,1} \sim_{\mathrm{V}\mathcal{K}} \mathrm{VII}^+_{3,1}$.

iii) For $\mathrm{VIII}^-_{4,1}$. By using Φ we have that $\mathrm{VIII}^-_{4,1} \sim_{\mathrm{V}\mathcal{K}} \mathrm{VIII}^+_{4,1}$.

□

Chapter 7

Application to right-left classification

A fair number of $\mathcal{A}$-classifications of map-germs $(\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ can be found in the literature: namely $(\mathbb{C}, 0) \to (\mathbb{C}^2, 0)$ ([BG82]), $(\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ ([Mon85] and [Rat95]), $(\mathbb{K}^n, 0) \to (\mathbb{K}^2, 0)$ ([Rie87] and [RR91]), $(\mathbb{R}^3, 0) \to (\mathbb{R}^3, 0)$ ([MT96]), $(\mathbb{R}^3, 0) \to (\mathbb{R}^4, 0)$ ([HK99]) and $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ ([Coo93]).

The classification of map-germ under $\mathcal{A}$-equivalence is hard to do. However, we can classify map-germs under ${}_V\mathcal{K}$-equivalence and by using Theorem 2.6.3 in chapter 2 we can do a sharp pullback to get $\mathcal{A}$-classification.

In the previous chapter we got the classification of map-germs under ${}_V\mathcal{K}$-equivalence, where V is the image of the minimal cross cap mapping of multiplicity $d \geq 2$. In this chapter we shall use our classification under ${}_V\mathcal{K}$-equivalence to classify corank 1 $\mathcal{A}_e$-codimension 2 map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$.

7.1 Relationship between $\mathcal{A}_e$ and ${}_V\mathcal{K}_e$-codimension

In this section we shall give the relationship between ${}_V\mathcal{K}_e$ codimension of a submersion map-germ $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ and $\mathcal{A}_e$ codimension of its a sharp pullback.

Definition 7.1.1. *Let $F : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ be a smooth map-germ and $g : (\mathbb{K}^r, 0) \to (\mathbb{K}^p, 0)$ an immersion which is transverse to F, i.e.,*

$$dF(T_0(\mathbb{K}^n, 0) + T_0(g(\mathbb{K}^r, 0)) = T_0(\mathbb{K}^p, 0)$$

where T_0 means the tangent space at 0. The **pullback** *of F by g, denoted $g^*(F)$, is the natural map from*

$$(\mathbb{K}^{r-(p-n)}, 0) \cong \{(x, y) \in (\mathbb{K}^n, 0) \times (\mathbb{K}^r, 0) : F(x) = g(y)\}$$

to $(\mathbb{K}^r, 0)$ given by projection on the second factor.

Example 7.1.2. *Let $F : (\mathbb{K}^3, 0) \to (\mathbb{K}^4, 0)$ be the trivial extension of the Whitney umbrella given by $F(x, v_1, y) = (x, v_1, y^2, v_1 y)$ and $h(x, v_1, w_1, w_2) = v_1 - p(x, w_1)$ for a function p.*

Let $g : (\mathbb{K}^3, 0) \to (\mathbb{K}^4, 0)$ be given by $g(x, w_1, w_2) = (x, p(x, w_1), w_1, w_2)$. Then the image of g is equal to $h^{-1}(0)$ and g is transverse to F.

The pullback of F by g is a map-germ of the form $f(x, y) = (x, y^2, yp(x, y^2))$. Also, we can see that the sharp pullback is a map-germ of the form $f(x, y) = (x, y^2, yp(x, y^2))$.

Remark 7.1.3. *In [Mat69], Mather shows that if $f : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ has finite $\mathcal{A}_e$-codimension then there is a stable map-germ $F : (\mathbb{K}^N, 0) \to (\mathbb{K}^P, 0)$ and an immersion map-germ $g : (\mathbb{K}^p, 0) \to (\mathbb{K}^P, 0)$ with g transverse to F such that f is obtained as a pullback in the following commutative diagram*

$$\begin{array}{ccc} (\mathbb{K}^N, 0) & \xrightarrow{F} & (\mathbb{K}^P, 0) \\ i \uparrow & & g \uparrow \\ (\mathbb{K}^n, 0) & \xrightarrow{f} & (\mathbb{K}^p, 0) \end{array}$$

In ([Dam91] and [Dam06]), Damon found the relationship between $\mathcal{A}_e$-codimension and ${}_V\mathcal{K}_e$-codimension. In fact, he showed that for a finitely $\mathcal{A}$-determined map-germ f as the pullback $g^*(F), \mathcal{A}_e{-}\mathrm{cod}(f) = \mathcal{K}_{V,e}{-}\mathrm{cod}(g)$ in [Dam91]. In [MM94], Mond and Montaldi prove that f does not necessarily have to be finitely $\mathcal{A}$-determined. In [Dam06], Damon proves $\mathcal{K}_{V,e}{-}\mathrm{cod}(g) = {}_V\mathcal{K}_e{-}\mathrm{cod}(h)$ where h is a submersion $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^{p-r}, 0)$ such that $h^{-1}(0)$ is the image of g.

Theorem 7.1.4 ([Dam91], [MM94] and [Dam06])**.** *Suppose that $F : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is a $\mathbb{K}$-analytic stable map-germ and $g : (\mathbb{K}^r, 0) \to (\mathbb{K}^p, 0)$ is an immersion transverse to F. Then,*

$$\mathcal{A}_e{-}\mathrm{cod}(g^*(F)) = {}_V\mathcal{K}_e{-}\mathrm{cod}(h)$$

where h is a submersion $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^{p-r}, 0)$ such that $h^{-1}(0)$ is the image of g and V is the $\mathbb{K}$-part of the complexification of the discriminant of F.

Now, we shall find the relationship between ${}_V\mathcal{K}_e$ codimension of a map-germ and $\mathcal{A}_e$ codimension of a sharp pullback.

Theorem 7.1.5. *Suppose that $F : (\mathbb{K}^n, 0) \to (\mathbb{K}^p, 0)$ is a smooth map-germ. Let $h : (\mathbb{K}^p, 0) \to (\mathbb{K}^q, 0)$ be a submersion transverse to F and $g : (\mathbb{K}^{p-q}, 0) \to (\mathbb{K}^p, 0)$ be an immersion such that $\mathrm{im}(g) = h^{-1}(0)$. Then,*

$$g^*(F) \sim_{\mathcal{A}} h^\sharp(F).$$

Proof. Let $U = (h \circ F)^{-1}(0)$ and $V = h^{-1}(0)$. Then $h^\sharp(F)$ is the map from U to V given by F.

Since F is transverse to $h^{-1}(0)$ we know that U and V are manifolds.

Now, let $g : (\mathbb{K}^{p-q}, 0) \to (\mathbb{K}^p, 0)$ be any immersion such that $\mathrm{im}(g) = h^{-1}(0)$. Then the pullback is defined as follows: Let $Z = \{(x, y) \in (\mathbb{K}^n, 0) \times (\mathbb{K}^{p-q}, 0) : F(x) = g(y)\}$. Since g is transverse to F, then Z is a manifold of dimension $n - q$.

We define $g^*(F)$ as the map $(Z,0) \to (\mathbb{K}^p,0)$ given by restriction to Z of the projection $pr_2 : (\mathbb{K}^n \times \mathbb{K}^{p-q},0) \to (\mathbb{K}^{p-q},0)$.

We produce a square diagram by using the other projection, $pr_1 : (\mathbb{K}^n \times \mathbb{K}^{p-q},0) \to (\mathbb{K}^n,0)$.

$$\begin{array}{ccc} (\mathbb{K}^n,0) & \xrightarrow{F} & (\mathbb{K}^p,0) \\ pr_1 \uparrow & & g \uparrow \\ (\mathbb{K}^n \times \mathbb{K}^{p-q},0) & \xrightarrow{pr_2} & (\mathbb{K}^{p-q},0) \end{array}$$

The square is commutative: Let $z \in Z$, then $z = (x,y)$ for some x and y with $F(x) = g(y)$ by definition. We have

$$F(pr_1(z)) = F(pr_1(x,y)) = F(x) = g(y) = g(pr_2(x,y)) = g(pr_2(z)).$$

Now to get $g^*(F) \sim_{\mathcal{A}} h^\sharp(F)$. We need Z to be mapped diffeomorphically to U by pr_1 and $\mathbb{K}^{p-q}$ to be mapped diffeomorphically to V by g. Since g is an immersion, then it is a diffeomorphism onto its image. Hence, we have that $\mathbb{K}^{p-q}$ and V are diffeomorphic.

We need to show that Z and U are diffeomorphic. We define $\varphi : U \to Z$ through the following. Let $x \in U$, then $F(x) \in V$ by definition. But then, there exists a unique y in K^{p-q} such that $F(x) = g(y)$. Thus let $\varphi(x) = (x,y)$. Obviously, φ is the inverse of the map $pr_1 : Z \to \operatorname{im}(pr_1)$. Since the smoothness of map is also preserved by restriction to subsets, then φ and $pr_1 : Z \to \operatorname{im}(pr_1)$ are smooth. By looking at the Jacobians of the two possible compositions of these (they will be the identity) we get that they must be local diffeomorphisms. □

Corollary 7.1.6. *Suppose that* $F : (\mathbb{K}^n,0) \to (\mathbb{K}^p,0)$ *is a* $\mathbb{K}$*-analytic stable map-germ and* $h : (\mathbb{K}^p,0) \to (\mathbb{K}^q,0)$ *is a submersion transverse to* F*. Then,*

$$\mathcal{A}_e\text{−cod}(h^\sharp(F)) = {}_V\mathcal{K}_e\text{−cod}(h)$$

Proof. Follows from the theorem and Theorem 7.1.4. □

Now, we need to show that how one can determine the $\mathcal{A}$-classification of map-germs $(\mathbb{K}^n, 0) \to (\mathbb{K}^{n+1}, 0)$ from ${}_V\mathcal{K}$-classification.

We begin with the direct sum of a smooth map-germs. This gives a process producing new map-germs from old ones.

Definition 7.1.7. *([AGV88], [Wal82]) Let $h : (\mathbb{K}^{p_1}, 0) \to (\mathbb{K}^q, 0)$ and $g : (\mathbb{K}^{p_2}, 0) \to (\mathbb{K}^q, 0)$ be smooth map-germs. We define the* **direct sum** $h \oplus g : (\mathbb{K}^{p_1+p_2}, 0) \to (\mathbb{K}^q, 0)$ *by*

$$h \oplus g(z, x) = h(z) + g(x).$$

Remark 7.1.8. *Augmentations of map-germs have the property above, see [Hou98].*

Definition 7.1.9. *Let Θ_i be a set of smooth vector fields on $(\mathbb{K}^{p_i}, 0)$ $i = 1, 2$. $\Theta_i = \{\xi_{i,j}\}_{j=1}^{r_i}$. Then the* **product of** Θ_1 *and* Θ_2*, denoted $\Theta_1 \times \Theta_2$, is the set of vector fields on $(\mathbb{K}^{p_1} \times \mathbb{K}^{p_2}, 0 \times 0)$ defined by*

$$\Theta_1 \times \Theta_2 = \{\xi_{1,1}, \ldots, \xi_{1,r_1}, \xi_{2,1}, \ldots, \xi_{2,r_2}\}.$$

If we use the coordinates $(Z_1, \ldots, Z_{p_1})$ on $(\mathbb{K}^{p_1}, 0)$ and $(X_1, \ldots, X_{p_2})$ on $(\mathbb{K}^{p_2}, 0)$, then any vector field $\xi = \sum_{i=1}^{p_1} \alpha_i \frac{\partial}{\partial Z_i}$ on $(\mathbb{K}^{p_1}, 0)$ can be extended naturally to $\xi = \sum_{i=1}^{p_1} \alpha_i \frac{\partial}{\partial Z_i} + \sum_{i=1}^{p_2} \beta_i \frac{\partial}{\partial X_i}$ on $(\mathbb{K}^{p_1} \times \mathbb{K}^{p_2}, 0 \times 0)$ where $\beta_i = 0$ for all $1 \leq i \leq p_2$.

Theorem 7.1.10. *Let $h, \widetilde{h} : (\mathbb{K}^{p_1}, 0) \to (\mathbb{K}^q, 0)$ and $g, \widetilde{g} : (\mathbb{K}^{p_2}, 0) \to (\mathbb{K}^q, 0)$ be smooth map-germs. Let Θ_i be a finitely generated $\mathcal{E}_i$-module of smooth vector fields on $(\mathbb{K}^{p_i}, 0)$ with $i = 1, 2$. If h is ${}_{\Theta_1}\mathcal{R}$-equivalent to $\widetilde{h}$ and g is ${}_{\Theta_2}\mathcal{R}$-equivalent to $\widetilde{g}$, then $h + g$ is ${}_{\Theta_1 \times \Theta_2}\mathcal{R}$-equivalent to $\widetilde{h} + \widetilde{g}$.*

Proof. Since h and $\widetilde{h}$ are ${}_{\Theta 1}\mathcal{R}$-equivalent, then there exists a vector field $\xi_1 \in \Theta_2$ that can be integrated to give a diffeomorphism φ_1 such that $\widetilde{h}(z) = h \circ \varphi_1(z)$.

Similarly, since g and $\widetilde{g}$ are ${}_{\Theta 2}\mathcal{R}$-equivalent, then there exists a vector field $\xi_2 \in \Theta_2$ that can be integrated to give a diffeomorphism φ_2 such that $\widetilde{g}(x) = g \circ \varphi_2(x)$.

We take the Cartesian product of the diffeomorphisms φ_1 and φ_2, i.e., $\varphi = (\varphi_1, \varphi_1)$. Then we have

$$\begin{aligned} h \oplus g\left(\varphi(z,x)\right) &= h \oplus g\left(\varphi_1(z), \varphi_2(x)\right) \\ &= h\left(\varphi_1(z)\right) + g\left(\varphi_2(x)\right) \\ &= \widetilde{h}(z) + \widetilde{g}(x). \end{aligned}$$

□

The direct sum in Theorem 7.1.10 is not compatible with ${}_{\Theta}\mathcal{K}$-equivalence. For example we have the following

Example 7.1.11. *[Wal82] Let $h(x,y) = (x^2, y^2)$ and $g(x,y) = (x^2 + y^2, x^2 - y^2$. We can see that h and g differ only by a linear coordinate change in the target. However $h \oplus h$ is not finitely $\mathcal{K}$-determined whereas $h \oplus g$ is 2-$\mathcal{K}$-determined.*

Definition 7.1.12. *Let $g : (\mathbb{K}^p, 0) \to (\mathbb{K}, 0)$ be a smooth function-germ.*

i) The Milnor algebra of g is given by

$$M_g = \frac{\mathcal{E}_p}{\langle \frac{\partial g}{\partial X_1}, \ldots, \frac{\partial g}{\partial X_p} \rangle \mathcal{E}_p}.$$

ii) The Tjurina algebra of g is given by

$$T_g = \frac{\mathcal{E}_p}{\langle g, \frac{\partial g}{\partial X_1}, \ldots, \frac{\partial g}{\partial X_p} \rangle \mathcal{E}_p}.$$

iii) The numbers

$$\mu(g) := \dim_{\mathbb{K}} M_g \quad \text{and} \quad \tau(g) := \dim_{\mathbb{K}} T_g$$

are called the Milnor and Tjurina number of g, respectively.

The Milnor and the Tjurina number play an important role in the study of isolated hypersurface singularities.

Theorem 7.1.13. *Let $h : (\mathbb{K}^{p_1}, 0) \to (\mathbb{K}, 0)$ and $g : (\mathbb{K}^{p_2}, 0) \to (\mathbb{K}, 0)$ be smooth map-germs. Let Θ_i be a finitely generated $\mathcal{E}_{p_i}$-module of smooth vector fields on $(\mathbb{K}^{p_i}, 0)$ with $i = 1, 2$. Then*

$$_{\Theta_1\times\Theta_2}\mathcal{K}_e\text{−cod}(h \oplus g) \geq \Big({}_{\Theta_1}\mathcal{K}_e\text{−cod}(h)\Big)\Big({}_{\Theta_2}\mathcal{K}_e\text{−cod}(g)\Big).$$

If Θ_2 is the whole module of vector fields on $(\mathbb{K}^{p_2}, 0)$ and g is quasihomogeneous, then

$$_{\Theta_1\times\Theta_2}\mathcal{K}_e\text{−cod}(h \oplus g) = {}_{\Theta_1}\mathcal{K}_e\text{−cod}(h)\tau(g).$$

Proof. We have,

$$\begin{aligned}
{}_{\Theta_1\times\Theta_2}\mathcal{K}_e\text{−cod}(h \oplus g) &= \dim_{\mathbb{K}} \frac{\mathcal{E}_{p_1+p_2}}{T_{\Theta_1\times\Theta_2}\mathcal{K}_e(h \oplus g)} \\
&= \dim_{\mathbb{K}} \frac{\mathcal{E}_{p_1+p_2}}{\langle \xi(h \oplus g) \,|\, \xi \in \Theta_1 \times \Theta_2\rangle + \langle h \oplus g\rangle \mathcal{E}_{p_1+p_2}} \\
&= \dim_{\mathbb{K}} \frac{\mathcal{E}_{p_1+p_2}}{\langle \xi_{1,1}(h), \dots, \xi_{1,r_1}(h), \xi_{2,1}(g), \dots, \xi_{2,r_2}(g)\rangle + \langle h + g\rangle \mathcal{E}_{p_1+p_2}} \\
&\geq \dim_{\mathbb{K}} \frac{\mathcal{E}_{p_1+p_2}}{\langle \xi_{1,i}(h)\rangle_{i=1}^{r_1} + \langle h\rangle \mathcal{E}_{p_1} + \langle \xi_{2,i}(g)\rangle_{i=1}^{r_2} + \langle g\rangle \mathcal{E}_{p_2}} \\
&= \dim_{\mathbb{K}} \left(\frac{\mathcal{E}_{p_1}}{\langle \xi_{1,i}(h)\rangle_{i=1}^{r_1} + \langle h\rangle \mathcal{E}_{p_1}} \otimes_{\mathbb{K}} \frac{\mathcal{E}_{p_2}}{\langle \xi_{2,i}(g)\rangle_{i=1}^{r_2} + \langle g\rangle \mathcal{E}_{p_2}} \right) \\
&= \dim_{\mathbb{K}} \frac{\mathcal{E}_{p_1}}{\langle \xi_{1,i}(h)\rangle_{i=1}^{r_1} + \langle h\rangle \mathcal{E}_{p_1}} \times \dim_{\mathbb{K}} \frac{\mathcal{E}_{p_2}}{\langle \xi_{2,i}(g)\rangle_{i=1}^{r_2} + \langle g\rangle \mathcal{E}_{p_2}} \\
&= \Big({}_{\Theta_1}\mathcal{K}_e\text{−cod}(h)\Big)\Big({}_{\Theta_2}\mathcal{K}_e\text{−cod}(g)\Big).
\end{aligned}$$

If g is quasihomogeneous, then $g \in \left\langle \frac{\partial g}{\partial X_1}, \dots, \frac{\partial g}{\partial X_{p_2}} \right\rangle$ and so from above we have

$$\begin{aligned}
{}_{\Theta_1 \times \Theta_2}\mathcal{K}_e\text{−cod}(h \oplus g) &= \dim_{\mathbb{K}} \frac{\mathcal{E}_{p_1+p_2}}{\langle \xi_{1,1}(h), \dots, \xi_{1,r_1}(h), \xi_{2,1}(g), \dots, \xi_{2,p_2}(g)\rangle + \langle h+g\rangle \mathcal{E}_{p_1+p_2}} \\
&= \dim_{\mathbb{K}} \frac{\mathcal{E}_{p_1+p_2}}{\langle \xi_{1,1}(h), \dots, \xi_{1,r_1}(h), \xi_{2,1}(g), \dots, \xi_{2,p_2}(g), h\rangle \mathcal{E}_{p_1+p_2}} \\
&= \dim_{\mathbb{K}} \frac{\mathcal{E}_{p_1}}{\langle \xi_{1,i}(h)\rangle_{i=1}^{r_1} + \langle h \rangle \mathcal{E}_{p_1}} \times \dim_{\mathbb{K}} \frac{\mathcal{E}_{p_2}}{\langle \frac{\partial g}{\partial X_1}, \dots, \frac{\partial g}{\partial X_{p_2}}\rangle \mathcal{E}_{p_2}} \\
&= {}_{\Theta_1}\mathcal{K}_e\text{−cod}(h)\tau(g).
\end{aligned}$$

□

We can find new classification from other classifications by using the following theorem.

Theorem 7.1.14. *Let Θ be a finitely generated $\mathcal{E}_p$-module of smooth vector fields on $(\mathbb{K}^p, 0)$ and $h : (\mathbb{K}^p, 0) \to (\mathbb{K}, 0)$ be a smooth function-germ with ${}_\Theta\mathcal{K}$-codimension at most 4. Suppose that $\dim_{\mathbb{K}} \{\xi(0) | \xi \in \Theta\} = r$. Then,*

(A) there exist coordinates $(Z_1, \dots, Z_{p-r}, X_1, \dots, X_r)$ on $(\mathbb{K}^p, 0)$ such that

$$\Theta \cong \widetilde{\Theta} \times \langle \partial/\partial X_1, \dots, \partial/\partial X_r \rangle$$

where $\langle \partial/\partial X_1, \dots, \partial/\partial X_r \rangle$ means the $\mathcal{E}_p$-module generated by these vector fields and all vector fields in $\widetilde{\Theta}$ vanish at 0.

(B) The function-germ h is ${}_\Theta\mathcal{K}$-equivalent to one of the following function-germs:- ($\varepsilon_i = \pm 1$ when $\mathbb{K} = \mathbb{R}$ and $\varepsilon = 1$ when $\mathbb{K} = \mathbb{C}$)

(B_1) X_l,

(B_2) $\widetilde{h}(\underline{Z}) + g(\underline{X})$, where $\widetilde{h} : (\mathbb{K}^{p-r}, 0) \to (\mathbb{K}, 0)$ has ${}_{\widetilde{\Theta}}\mathcal{K}$-codimension at most 4 and $g : (\mathbb{K}^r, 0) \to (\mathbb{K}, 0)$ is $\mathcal{K}$-equivalent to a function-germ of $\tau \leq 4$.

(B_3) $\widetilde{h}(\underline{Z}) + \alpha Z_\alpha X_l + X_l^m + \mathbf{Q}$ with $m = 3$ or 4,

(B_4) $\widetilde{h}(\underline{Z}) + (\alpha_1 Z_{\alpha_1} + \alpha_2 Z_{\alpha_2})X_l + X_l^3 + \mathbf{Q}$,

where

$$\begin{aligned} \alpha_1.\alpha_2 &= 0 \\ \mathbf{Q} &= \sum_{\substack{j=1 \\ j\neq l}}^{r} X_j^2 \end{aligned}$$

and the function-germs in (B_3) and (B_4) have ${}_\Theta\mathcal{K}$-codimension ≥ 3.

Proof. We need the following notation in the proof. Let $\alpha = (\alpha_1, \ldots, \alpha_n)$ be an n-tuple of non-negative integers and e_j^i be an r-tuple with a 1 in the jth position and zeros elsewhere. We use the coordinate $(Z_1, \ldots, Z_{p-r}, X_1, \ldots, X_r)$ on $\mathbb{K}^p = \mathbb{K}^{p-r} \times \mathbb{K}^r$. We define

$$\begin{aligned} |\alpha| &:= \sum_{i=1}^{p-r} \alpha_i, \\ |\beta| &:= \sum_{i=1}^{r} \beta_i, \\ Z^\alpha &:= Z_1^{\alpha_1}.Z_2^{\alpha_2} \ldots Z_{p-r}^{\alpha_{p-r}}, \\ X^\alpha &:= X_1^{\alpha_1}.X_2^{\alpha_2} \ldots X_r^{\alpha_r}. \end{aligned}$$

(A) We have $\dim_{\mathbb{K}} \{\xi(0) | \xi \in \Theta\} = r$, i.e., there exist vector fields $\xi_1, \ldots, \xi_r$ in Θ such that these vector fields do not vanish at 0 and the vectors $\xi_1(0), \ldots, \xi_r(0)$ are linearly independent. Therefore, a change of coordinates in $(\mathbb{K}^p, 0)$ allows to assume $\xi_1 = \dfrac{\partial}{\partial X_1}, \ldots, \xi_r = \dfrac{\partial}{\partial X_r}$. In this case we have $\mathbb{K}^p = \mathbb{K}^{p-r} \times \mathbb{K}^r$ such that the coordinates $(Z_1, \ldots, Z_{p-r})$ on $(\mathbb{K}^{p-r}, 0)$ and $\widetilde{\Theta}$ will denote the module of vector fields on $(\mathbb{K}^{p-r}, 0)$ with all these vector fields vanishing at 0.

(B) Since h has ${}_\Theta\mathcal{K}$-codimension at most 4. Then, we consider

$$j^5 h = \sum_{|\alpha|+|\beta| \leq 5} C_{\alpha,\beta} Z^\alpha X^\beta.$$

If $C_{0,e_j^1} \neq 0$ for some $1 \leq j \leq r$ (say for $j = l$), then by using the coordinate changes $X_j \mapsto X_j + \alpha_i Z_i$, $X_j \mapsto X_j + \beta_i X_i$ and the matrix $M = [\frac{1}{C_{0,e_l^1}}]$ such that $1 \leq j \leq r$ and $1 \leq i \leq p - r$ we get h is ${}_\Theta\mathcal{K}$-equivalent to X_1.

Now, we suppose that $C_{0,e_j^1} = 0$ for all $1 \leq j \leq r$.

a) If $C_{0,e_j^2} \neq 0$ for all $1 \leq j \leq r$, then by using the coordinate changes $X_j \mapsto X_j + \alpha_i Z_i$ for $1 \leq j \leq r$ and $1 \leq i \leq p - r$ and $X_j \mapsto X_j + \beta_i X_i$ for $1 \leq i, j \leq r$ and $i \neq j$ we can remove $Z_i X_j$ and $X_i X_j$ from $j^5 h$.

   Therefore, we have

$$j^5 h = \sum_{\substack{|\alpha|+|\beta| \leq 5 \\ \alpha \neq 1 \\ \beta \neq 1}} C_{\alpha,\beta} Z^\alpha X^\beta.$$

   By using the diffeomorphism $\Phi(\underline{Z}, \underline{X}) = (\underline{Z}, X_1, \ldots, e^{\lambda_j} X_j, X_{j+1}, \ldots, X_r)$ we fix $C_{0,e_j^2} = \pm 1$ for all $1 \leq j \leq r$. Also we have

$$\partial j^5 h / \partial X_j = \pm 2X_j + G, \quad \text{with } G \in \mathbf{m}_p^2.$$

   If we consider $j^5 h$ as a 1-parameter unfolding for all α, then the terms of the form $Z^\alpha X^\beta$ are 2-trivial.

   It follows,

$$j^5 h = \widetilde{h}(\underline{Z}) + \sum_{j=1}^{r} \varepsilon_j X_j^2.$$

   such that $\varepsilon_i = \pm 1$ when $\mathbb{K} = \mathbb{R}$ and $\varepsilon = 1$ when $\mathbb{K} = \mathbb{C}$.

b) Suppose that there exists l such that $C_{0,e_l^2} = 0$ and $C_{0,e_j^2} \neq 0$ for all $1 \leq j \leq r$ with $j \neq l$.

   (b_1) If the coefficient of $X_l X_j$ does not equal 0 for some $1 \leq j \leq r$, then by the same way above we have h is ${}_\Theta\mathcal{K}$-equivalent to a function-germ of the form $\widetilde{h}(\underline{Z}) + \sum_{j=1}^{r} \varepsilon_j X_j^2$.

(b$_2$) If the coefficient of X_lX_j equals 0 for all $1 \leq j \leq r$, then we can see that 1 and x_l are not in $T_\Theta\mathcal{K}_e(h)$ because all vector fields in $\widetilde{\Theta}$ vanish at 0.

We consider the following cases:

(1) If $C_{0,e_l^3} \neq 0$, then we can fix $C_{0,e_l^3} = 1$, then by the same way above we have

$$j^5h = \widetilde{h}(\underline{Z}) \pm X_l^3 + \sum_{\substack{j=1 \\ j\neq l}}^{r} \varepsilon_j X_j^2 + \sum_{|\alpha|\leq 4} C_{\alpha,e_l^1} Z^\alpha X_l.$$

i. If ${}_\Theta\mathcal{K}-\text{cod}(h) = 2$, then we have $N_\Theta\mathcal{K}_e(h) = \langle 1, X_l\rangle$ and

$$\mathbf{m}_p - \{X_l\} \subseteq \mathbf{m}_p T_\Theta\mathcal{K}_e(h) \subseteq T_\Theta\mathcal{K}(h).$$

If we consider j^5h as a 1-parameter unfolding for all C_{α,e_l^1}, then all terms in $\sum_{|\alpha|\leq 4} C_{\alpha,e_l^1} Z^\alpha X_l$ are trivial and hence h is ${}_\Theta\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) \pm X_l^3 + \sum_{\substack{j=1 \\ j\neq l}}^{r} \varepsilon_j X_j^2.$$

ii. If ${}_\Theta\mathcal{K}-\text{cod}(h) = 3$. We suppose that $N_\Theta\mathcal{K}_e(h) = \langle 1, X_l, U\rangle$.

ii.1 If $U \neq Z_tX_l$ for all $1 \leq t \leq p-r$, then we have

$$\mathbf{m}_p - \{X_l, U\} \subseteq \mathbf{m}_p T_\Theta\mathcal{K}_e(h) \subseteq T_\Theta\mathcal{K}(h).$$

Similarly, we have h is ${}_\Theta\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) \pm X_l^3 + \sum_{\substack{j=1 \\ j\neq l}}^{r} \varepsilon_j X_j^2.$$

ii.2 If $U = Z_tX_l$ for single t, then we have h is ${}_\Theta\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) + \alpha_t Z_t X_l \pm X_l^3 + \sum_{\substack{j=1 \\ j\neq l}}^{r} \varepsilon_j X_j^2.$$

iii. If ${}_{\Theta}\mathcal{K}-\text{cod}(h) = 4$. We suppose that $N_{\Theta}\mathcal{K}_e(h) = \langle 1, X_l, U, V\rangle$. Similarly,

iii.1 If $U \neq Z_{t_1}X_l$ and $V \neq Z_{t_2}X_l$ for all $1 \leq t_1, t_2 \leq p-r$, then we have $Z_{t_1}X_l$ and $Z_{t_1}X_l$ in $T_{\Theta}\mathcal{K}_e(h)$.

Therefore, we have h is ${}_{\Theta}\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) \pm X_l^3 + \sum_{\substack{j=1\\ j\neq l}}^{r} \varepsilon_j X_j^2.$$

iii.2 If $U = Z_{t_1}X_l$ and $V \neq Z_{t_2}X_l$, then we have h is ${}_{\Theta}\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) + \alpha_{t_1}Z_{t_1}X_l \pm X_l^3 + \sum_{\substack{j=1\\ j\neq l}}^{r} \varepsilon_j X_j^2.$$

iii.3 If $U \neq Z_{t_1}X_l$ and $V = Z_{t_2}X_l$, then we have h is ${}_{\Theta}\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) + \alpha_{t_2}Z_{t_2}X_l \pm X_l^3 + \sum_{\substack{j=1\\ j\neq l}}^{r} \varepsilon_j X_j^2.$$

iii.4 If $U = Z_{t_1}X_l$ and $V = Z_{t_2}X_l$, then we have h is ${}_{\Theta}\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) + \alpha_{t_1}Z_{t_1}X_l + \alpha_{t_2}Z_{t_2}X_l \pm X_l^3 + \sum_{\substack{j=1\\ j\neq l}}^{r} \varepsilon_j X_j^2.$$

(2) If $C_{0,e_l^3} = 0$ and $C_{0,e_l^4} \neq 0$, then we have 1, X_l and X_l^2 are not in $T_{\Theta}\mathcal{K}(h)$.

i. If ${}_{\Theta}\mathcal{K}-\text{cod}(h) = 3$, then we have $N_{\Theta}\mathcal{K}_e(h) = \langle 1, X_l, X_l^2\rangle$. We can see that

$$\mathbf{m}_p - \left\{X_l, X_l^2\right\} \subseteq \mathbf{m}_p T_{\Theta}\mathcal{K}_e(h) \subseteq T_{\Theta}\mathcal{K}(h).$$

Hence, h is ${}_{\Theta}\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) \pm X_l^4 + \sum_{\substack{j=1\\ j\neq l}}^{r} \varepsilon_j X_j^2.$$

ii. If ${}_{\Theta}\mathcal{K}-\text{cod}(h) = 4$. We suppose that $N_{\Theta}\mathcal{K}_e(h) = \langle 1, X_l, X_l^2, U\rangle$.

ii.1 If $U \neq Z_t X_l$ for all $1 \leq t \leq p-r$, then we have $Z_t X_l \in T_\Theta \mathcal{K}_e(h)$. Therefore, we have h is ${}_\Theta\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) \pm X_l^4 + \sum_{\substack{j=1 \\ j \neq l}}^{r} \varepsilon_j X_j^2.$$

ii.2 If $U = Z_t X_l$ for single t, then we have h is ${}_\Theta\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) + \alpha_t Z_t X_l \pm X_l^4 + \sum_{\substack{j=1 \\ j \neq l}}^{r} \varepsilon_j X_j^2.$$

(3) If $C_{0,e_l^3} = C_{0,e_l^4} = 0$, then we have $N_\Theta \mathcal{K}_e(h) = \langle 1, X_l, X_l^2, X_l^3 \rangle$. In this case $C_{0,e_l^2} \neq 0$. Therefore, we have h is ${}_\Theta\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) \pm X_l^5 + \sum_{\substack{j=1 \\ j \neq l}}^{r} \varepsilon_j X_j^2.$$

(c) Suppose that there exist l_1 and l_2 such that $C_{0,e_{l_1}^2} = C_{0,e_{l_2}^2} = 0$ and $C_{0,e_j^2} \neq 0$ for all $1 \leq j \leq r$ with $j \neq l_1$ and $j \neq l_2$. Since all vector fields in $\widetilde{\Theta}$ vanish at 0. Then 1, X_{l_1} and X_{l_2} are not in $T_\Theta \mathcal{K}_e(h)$.

(c$_1$) If the coefficient of $X_{l_1} X_{l_2}$ not equal 0, then by using the coordinate changes $X_j \mapsto X_j + \alpha_i Z_i$ for $1 \leq j \leq r$ and $1 \leq i \leq p-r$ and $X_j \mapsto X_j + \beta_i X_i$ for $1 \leq i, j \leq r$ and $i \neq j$ we have h is ${}_\Theta\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) + \sum_{j=1}^{r} \varepsilon_j X_j^2.$$

(c$_2$) If the coefficient of $X_{l_1} X_{l_2}$ is 0, then we have 1, X_{l_1} and X_{l_2}, $X_{l_1} X_{l_2}$ are not in $T_\Theta \mathcal{K}(h)$. We assume that $N_\Theta \mathcal{K}_e(h) = \langle 1, X_{l_1}, X_{l_2}, X_{l_1} X_{l_2} \rangle$. Therefore, we have $C_{0,e_{l_1}^3} \neq 0$ and $C_{0,e_{l_2}^3} \neq 0$. Similarly of above we have h is ${}_\Theta\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) \pm X_{l_1}^3 \pm X_{l_2}^3 + \sum_{\substack{j=1 \\ j \neq l_1, l_2}}^{r} \varepsilon_j X_j^2.$$

By using the coordinate changes $X_{l_1} \mapsto X_{l_1} + X_{l_2}$ and $X_{l_1} \mapsto X_{l_1} - X_{l_2}$, we have h is ${}_{\Theta}\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) + X_{l_1}^2 X_{l_2} \pm X_{l_2}^3 + \sum_{\substack{j=1 \\ j \neq l_1, l_2}}^{r} \varepsilon_j X_j^2.$$

(d) Suppose that $C_{0,e_{l_i}^2} = 0$ for $1 \leq i \leq 3$ and $C_{0,e_j^2} \neq 0$ for all $1 \leq j \leq r$ with $j \neq l_1, l_2, l_3$.

(d_1) If the coefficients of $X_{l_1}X_{l_2}$, $X_{l_1}X_{l_3}$ and $X_{l_2}X_{l_3}$ are not equal 0, then by using the coordinate changes $X_i \mapsto X_i + X_j$ and $X_j \mapsto X_j - X_i$ such that $l_1 \leq i, j \leq l_2$ and $i \neq$ we have h is ${}_{\Theta}\mathcal{K}$-equivalent to

$$\widetilde{h}(\underline{Z}) + \sum_{j=1}^{r} \varepsilon_j X_j^2.$$

(d_2) If there is at least one of the coefficients of $X_{l_1}X_{l_2}$, $X_{l_1}X_{l_3}$ and $X_{l_2}X_{l_3}$ is 0 and other not, then we have similar of (c_2).

(d_3) If there are at least two of the coefficients of $X_{l_1}X_{l_2}$, $X_{l_1}X_{l_3}$ and $X_{l_2}X_{l_3}$ are 0 and other not, then we have ${}_{\Theta}\mathcal{K}_e{-}\mathrm{cod}(h) \geq 5$.

□

In the following examples we shall show that how we can use the theorem above in our classification and we shall see our results coincide with the results of Mond on $\mathcal{A}$-classification of map-germs $(\mathbb{K}^2, 0) \to (\mathbb{K}^3, 0)$ (see [Mon85], theorem 1.1) and Houston and Kirk on $\mathcal{A}$-classification of map-germs $(\mathbb{K}^3, 0) \to (\mathbb{K}^4, 0)$ (see [HK99], theorem 1.1).

Example 7.1.15. *Let $V = \widetilde{V} \times \mathbb{C}$ where $\widetilde{V}$ is the image of $\varphi_2 : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ given by $\varphi_2(v_1, y) = (v_1, y^2, v_1 y)$ and let $F = \varphi_2 \times \mathrm{Id}_1$. Let $g(x) = x^5$, then we have $\tau(g) = 4$.*

i) From Corollary 6.0.21 we have $\widetilde{h}(v_1, w_1, w_2) = v_1 - w_1$ has ${}_{\widetilde{V}}\mathcal{K}_e{-}\mathrm{cod}(\widetilde{h}) = 1$.

If $h = \widetilde{h} \oplus g$, then from Theorem 7.1.13 we have ${}_V\mathcal{K}_e{-}\mathrm{cod}(h) = 4$. Thus $h \circ F(v_1, y, x) = 0$ gives $v_1 = y^2 + x^5$. Hence $h^\sharp(F)(x, y) = (v_1, y^2, y^3 + x^5y)$ and this is the S_4 singularity of Mond.

*ii) Suppose that $\widetilde{h} = v_1 - xw_1$ and $h = \widetilde{h} \oplus g$, then from the **CAST** package we have ${}_V\mathcal{K}_e{-}\mathrm{cod}(h) = 5$.*

We can find $h^\sharp(F)(x, y) = (x, y^2, xy^3 + x^5y)$ and this is the C_5 singularity of Mond.

Example 7.1.16. *Let $V = \widetilde{V} \times \mathbb{C}^2$ where $\widetilde{V}$ is the image of $\varphi_2 : (\mathbb{C}^2, 0) \to (\mathbb{C}^3, 0)$ given by $\varphi_2(v_1, y) = (v_1, y^2, v_1y)$ and let $F = \varphi_2 \times \mathrm{Id}_2$. Let $g(x, y) = x^2 + z^3$, then we have $\tau(g) = 2$.*

i) From Corollary 6.0.21 we have $\widetilde{h}(v_1, w_1, w_2) = v_1 - w_1$ has ${}_{\widetilde{V}}\mathcal{K}_e{-}\mathrm{cod}(\widetilde{h}) = 1$.

If $h = \widetilde{h} \oplus g$, then from Theorem 7.1.13 we have ${}_V\mathcal{K}_e{-}\mathrm{cod}(h) = 4$. Thus $h \circ F(v_1, y, x, z) = 0$ gives $v_1 = y^2 + x^2 + z^3$. Hence $h^\sharp(F)(x, z, y) = (x, z, y^2, y^3 + x^2y + z^3y)$ and this is the A_2 singularity of Houston and Kirk.

*ii) Suppose that $\widetilde{h} = v_1 - zw_1$ and $h = \widetilde{h} \oplus g$, then from the **CAST** package we have ${}_V\mathcal{K}_e{-}\mathrm{cod}(h) = 3$.*

We can find $h^\sharp(F)(x, z, y) = (x, z, y^2, zy^3 + x^2y + z^3y)$ and this is the C_3 singularity of Houston and Kirk.

7.2 Map-germs of $\mathcal{A}_e$-codimension two

In this section, we shall give the classification of a corank 1 $\mathcal{A}_e$-codimension 2 map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. Before that we need the following theorem.

Theorem 7.2.1. *Let $\widetilde{\Theta}$ be a finitely generated $\mathcal{E}_p$-module of smooth vector fields on $(\mathbb{K}^p, 0)$ and $\Theta = \widetilde{\Theta} \times \{\partial/\partial X_j\}_{j=1}^r$. Let $h : (\mathbb{K}^p \times \mathbb{K}^r, 0) \to (\mathbb{K}^q, 0)$ be a submersion map-germ with ${}_{\Theta}\mathcal{K}_e$-codimension 2 and $q \geq 2$. Then h is ${}_{\Theta}\mathcal{K}$-equivalent to the map-germ of the form $(X_1, X_2, \ldots, X_{r-1}, H)$ where H is ${}_{\Theta}\mathcal{K}$-equivalent to one of the following function-germs*

(1) $\widetilde{h} + \sum_{j=1}^{r} X_j^2$ *such that* ${}_{\widetilde{\Theta}}\mathcal{K}_e\text{−cod}(\widetilde{h}) = 2$,

(2) $\widetilde{h} + X_l^3 + \sum_{\substack{j=1 \\ j \neq l}}^{r} X_j^2$ *such that* ${}_{\widetilde{\Theta}}\mathcal{K}_e\text{−cod}(\widetilde{h}) = 1$.

Proof. Suppose that $h = (h_1, \ldots, h_q)$. We are looking for only terms on $X_1, \ldots, X_r$ in h.

If h has no linear terms in $X_1, \ldots, X_r$, then we have $e_j = (0, 0, \ldots, 0, 1, 0, \ldots, 0)^T \notin T_{\Theta}\mathcal{K}(h)$ for all $j = 1, 2, \ldots, q$ because every vector field in $\widetilde{\Theta}$ vanishes at the origin. It follows that ${}_{\Theta}\mathcal{K}_e\,\text{−cod}(h) \geq q$. If $q \geq 3$, then we get a contradiction since ${}_{\Theta}\mathcal{K}_e\text{−cod}(h) = 2$.

Assume that $q = 2$. Then we have

$$T_{\Theta}\mathcal{K}_e(h) = \mathbf{m}_{p+r} \langle e_1, e_2 \rangle .$$

Then, on the RHS of above we require $2r$ distinct generators (we are looking for only $X_1, \ldots, X_r$) and on the LHS we have at most r generators and this is a contradiction.

Therefore, we can assume h_1 has a linear term and the coefficient of X_1 is not zero (if the coefficient of X_1 is zero, then we can swap). We can remove all other terms in h_1 and we get $h_1 = X_1$, i.e., we have $h = (X_1, h_2, \ldots, h_q)$.

Now, for all $2 \leq i \leq r$ and $2 \leq j \leq q$ if the coefficient $\alpha_{i,j}$ of X_i in h_j is not zero, then we can remove X_i in h_j by using the matrix $M_{10} = \text{diagonal}(-\alpha_{i,j}, 1, \ldots, 1)$.

Therefore, we have $h = (X_1, h_2, \ldots, h_q)$ such that h_j has no x_1 in its linear part for all $2 \leq j \leq q$.

By the same way above we can get $h = (X_1, X_2, \ldots, X_{r-1}, h_q)$ such that the linear part of h_q has not $X_1, \ldots, X_{r-1}$. If the coefficient of X_r in h_q is not zero, then we can remove all terms in h_q and we get $h = (X_1, X_2, \ldots, X_{r-1}, X_r)$, i.e., ${}_{\Theta}\mathcal{K}_e{-}\mathrm{cod}(h) = 0$ and this is a contradiction since ${}_{\Theta}\mathcal{K}_e{-}\mathrm{cod}(h) = 2$.

Therefore, we assume that the coefficient of X_r in h_q is zero. Hence, h_q has no linear part.

Now, we can see that the sharp pullback of h is equal to the sharp pullback of h_q and hence they have the same $\mathcal{A}_e$-codimension. From Corollary 7.1.6 we get ${}_{\Theta}\mathcal{K}_e{-}\mathrm{cod}(h_q) = 2$ and Theorem 7.1.14 we can see that h_q is one of the following function-germs

(1) $\widetilde{h} + \sum\limits_{j=1}^{r} X_j^2$ such that ${}_{\widetilde{\Theta}}\mathcal{K}_e{-}\mathrm{cod}(\widetilde{h}) = 2$,

(2) $\widetilde{h} + X_l^3 + \sum\limits_{\substack{j=1 \\ j \neq l}}^{r} X_j^2$ such that ${}_{\widetilde{\Theta}}\mathcal{K}_e{-}\mathrm{cod}(\widetilde{h}) = 1$.

□

In the following theorem we give the classification of corank 1 $\mathcal{A}_e$-codimension 2 map-germs $(\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$. We shall use the coordinate $(\underline{U}, \underline{V}, \underline{X}, Y)$ on the source.

Theorem 7.2.2. *Suppose that $f : (\mathbb{C}^n, 0) \to (\mathbb{C}^{n+1}, 0)$ is a corank 1 $\mathcal{A}_e$-codimension 2 map-germ, then f is $\mathcal{A}$-equivalent to one of the map-germs in the following:*

(a_1) (Y^2, Y^5),

(a_2) $\left(\underline{U}, \underline{V}, Y^d + \sum\limits_{i=1}^{d-3} U_i Y^i + V_{d-1}^2 Y^{d-2}, \sum\limits_{i=1}^{d-1} V_i Y^i\right)$.

(a_3) $\left(\underline{U}, \underline{V}, Y^d + \sum\limits_{i=1}^{d-4} U_i Y^i + \big(V_{d-1} + U_{d-2}^2\big) Y^{d-3} + U_{d-2} Y^{d-2}, \sum\limits_{i=1}^{d-1} V_i Y^i\right)$.

(a_4) $(U_1, V_1, Y^3 + U_1Y, V_1Y + U_1Y^3 + Y^5)$.

(a_5) $(V_1, Y^3, V_1Y + Y^5)$.

(a_6) $(U_1, V_1, V_2, Y^4 + U_1Y, V_1Y + V_2Y^2 + U_1Y^3 + U_1Y^4 + Y^7)$.

(b_1) $$\left(Y^2, Y^3 + X_l^3 Y + \Big(\sum_{\substack{j=1\\ j\neq l}}^{r} X_j^2\Big)Y, X_1, \dots, X_r\right).$$

(b_2) $$\left(Y^2, Y^5 + \Big(\sum_{j=1}^{r} X_j^2\Big)Y, X_1, \dots, X_r\right).$$

(b_3) $$\left(\underline{U}, \underline{V}, Y^d + \sum_{i=1}^{d-3} U_iY^i + \Big(V_{d-1} + X_l^3 + \sum_{\substack{j=1\\ j\neq l}}^{r} X_j^2)\Big)Y^{d-2}, \sum_{i=1}^{d-1} V_iY^i, \underline{X}\right),$$

(b_4) $$\left(\underline{U}, \underline{V}, Y^d + \sum_{i=1}^{d-3} U_iY^i + \Big(V_{d-1}^2 + \sum_{j=1}^{r} X_j^2\Big)Y^{d-2}, \sum_{i=1}^{d-1} V_iY^i, \underline{X}\right),$$

(b_5) $$\left(\underline{U}, \underline{V}, Y^d + \sum_{i=1}^{d-4} U_iY^i + \Big(V_{d-1} + U_{d-2}^2 + \sum_{j=1}^{r} X_j^2\Big)Y^{d-3} + U_{d-2}Y^{d-2}, \sum_{i=1}^{d-1} V_iY^i, \underline{X}\right),$$

(b_6) $$\left(U_1, V_1, Y^3 + U_1Y, V_1Y + \Big(\sum_{j=1}^{r} X_j^2\Big)Y^2 + U_1Y^3 + Y^5, X_1, \dots, X_r\right),$$

(b_7) $$\left(V_1, Y^3, V_1Y + \Big(\sum_{j=1}^{r} X_j^2\Big)Y^2 + Y^5, X_1, \dots, X_r\right),$$

(b_8) $$\left(U_1, V_1, V_2, Y^4 + U_1Y, V_1Y + V_2Y^2 + \Big(U_1 + \sum_{j=1}^{r} X_j^2\Big)Y^3 + U_1Y^4 + Y^7, X_1, \dots, X_r\right).$$

Proof. From Remark 7.1.3 and Theorem 2.3.9 we can get the following diagram

$$\begin{array}{ccccc}
(\mathbb{C}^{2d-2} \times \mathbb{C}^r, 0) & \xrightarrow{\varphi_d \times \mathrm{Id}_r} & (\mathbb{C}^{2d-1} \times \mathbb{C}^r, 0) & \xrightarrow{h} & (\mathbb{C}^q, 0) \\
\Phi \uparrow & & \Psi \uparrow & & \\
(\mathbb{C}^{n'}, 0) & \xrightarrow{F} & (\mathbb{C}^{n'+1}, 0) & & \\
i \uparrow & & g \uparrow & & \\
(\mathbb{C}^n, 0) & \xrightarrow{f} & (\mathbb{C}^{n+1}, 0) & &
\end{array}$$

such that φ_d is the minimal cross cap mapping of multiplicity $d \geq 2$, i.e.,

$$\varphi_d(\underline{u}, \underline{v}, y) = \left(\underline{u}, \underline{v}, y^d + \sum_{i=1}^{d-2} u_i y^i, \sum_{i=1}^{d-1} v_i y^i\right).$$

We consider the following cases

i) If $r = 0$, then from Corollary 6.0.21 we have h.

(a$_1$) Suppose that $h(v_1, w_1, w_2) = v_1 - w_1^2$. Thus $h \circ \varphi_2(v_1, y) = 0$ gives $v_1 = y^2$. Using coordinate $Y = y$ on $(h \circ \varphi_2)^{-1}(0)$, we see that the map $\varphi_2|(h \circ \varphi_2)^{-1}(0), 0) \to (h^{-1}(0), 0)$ becomes

$$Y \mapsto (Y^2, Y^5).$$

(a$_2$) Suppose that $h(\underline{u}, \underline{v}, w_1, w_2) = u_{d-2} - v_{d-1}^2$. Thus $h \circ \varphi_d(\underline{u}, \underline{v}, y) = 0$ gives $u_{d-2} = v_{d-1}^2$.

Using coordinates $U_1 = u_1, \ldots, U_{d-3} = u_{d-3}$, $V_1 = v_1, \ldots, V_{d-1} = v_{d-1}$ and $Y = y$, we see that the map $\varphi_d|(h \circ \varphi_d)^{-1}(0), 0) \to (h^{-1}(0), 0)$ becomes

$$(\underline{U}, \underline{V}, Y) \mapsto \left(\underline{U}, \underline{V}, Y^d + \sum_{i=1}^{d-3} U_i Y^i + V_{d-1}^2 Y^{d-2}, \sum_{i=1}^{d-1} V_i Y^i\right).$$

(a$_3$) Suppose that $h(, w_1, w_2) = u_{d-3} - v_{d-1}^2 - u_{d-2}^2$. Thus $h \circ \varphi_d(\underline{u}, \underline{v}, y) = 0$ gives $u_{d-3} = v_{d-1} + u_{d-2}^2$.

Using coordinates $U_1 = u_1, \ldots, U_{d-4} = u_{d-4}$, $U_{d-3} = u_{d-2}$, $V_1 = v_1$, $\ldots$, $V_{d-1} = v_{d-1}$ and $Y = y$, we see that the map $\varphi_d|(h \circ \varphi_d)^{-1}(0), 0) \to (h^{-1}(0), 0)$ becomes

$$(\underline{U}, \underline{V}, Y) \mapsto \left(\underline{U}, \underline{V}, Y^d + \sum_{i=1}^{d-4} U_i Y^i + \left(V_{d-1} + U_{d-2}^2\right) Y^{d-3} + U_{d-2} Y^{d-2}, \sum_{i=1}^{d-1} V_i Y^i\right).$$

(a$_4$) Suppose that $h(v_1, w_1, w_2) = v_2 - w_1$. Thus $h \circ \varphi_3(u_1, v_1, v_2, y) = 0$ gives $v_2 = y^3 + u_1 y$.

Using coordinates $U_1 = u_1$, $V_1 = v_1$ and $Y = y$ on $(h \circ \varphi_3)^{-1}(0)$, we see that the map $\varphi_3|(h \circ \varphi_3)^{-1}(0), 0) \to (h^{-1}(0), 0)$ becomes

$$(U_1, V_1, Y) \mapsto (U_1, V_1, Y^3 + U_1Y, V_1Y + U_1Y^3 + Y^5).$$

(a$_5$) Suppose that $h(u_1, v_1, v_2, w_1, w_2) = (u_1, v_2 - w_1)$. Thus $h \circ \varphi_3(u_1, v_1, v_2, y) = 0$ gives $u_1 = 0$ and $v_2 = y^3$.

Using coordinates $V_1 = v_1$ and $Y = y$ on $(h \circ \varphi_3)^{-1}(0)$, we see that the map $\varphi_3|(h \circ \varphi_3)^{-1}(0), 0) \to (h^{-1}(0), 0)$ becomes

$$(V_1, Y) \mapsto (V_1, Y^3, V_1Y + Y^5).$$

(a$_6$) Suppose that $h(u_1, u_2, v_1, v_2, v_3, w_1, w_2) = (u_2, u_1 - v_2 - w_1)$. Thus $h \circ \varphi_4(u_1, v_1, v_2, y) = 0$ gives $u_2 = 0$ and $v_3 = u_1 + u_1y + y^4$.

Using coordinates $U_1 = u_1$, $V_1 = v_1$, $V_2 = v_2$ and $Y = y$ on $(h \circ \varphi_3)^{-1}(0)$, we see that the map $\varphi_4|(h \circ \varphi_4)^{-1}(0), 0) \to (h^{-1}(0), 0)$ becomes

$$(U_1, V_1, V_2, Y) \mapsto (U_1, V_1, V_2, Y^4 + U_1Y, V_1Y + V_2Y^2 + U_1Y^3 + U_1Y^4 + Y^7).$$

If $h = w_1 - v_1^2$, then we have that $(h \circ \varphi_2)^{-1}(0)$ is a singular, i.e., φ_2 does not transversal to $h^{-1}(0)$. Hence we have not the sharp pullback in this case. In the same way for $h = (v_1, w_1)$.

ii) If $r \geq 1$, then from Theorem 7.1.14, Theorem 7.2.1 and Corollary 6.0.21 we have the results $b_1, \ldots, b_8$ in the same way above.

□

We can see that there is some similarity between the germs labelled (a_i) and the germs labelled (b_i) in Theorem 7.2.2. For example, in (a_1) we have (Y^2, Y^5) and in (b_2) we have $\left(Y^2, Y^5 + \left(\sum_{j=1}^{r} X_j^2\right)Y, X_1, \ldots, X_r\right)$. Similarly, between (a_2) and (b_4), (a_3) and (b_5), (a_4) and (b_6), (a_5) and (b_7), (a_6) and (b_8).

Chapter 8

Ideas for Future Work

It is clear that the work with ${}_V\mathcal{K}$-equivalence is much easier than $\mathcal{A}$-equivalence. For example, $\mathcal{A}$-equivalence classifications are hard to do but, armed with the liftable vector fields, classifications of map-germs on discriminant under ${}_V\mathcal{K}$-equivalence are much easier as they are similar to $\mathcal{K}$-equivalence. More importantly, $\mathcal{A}$-classification and ${}_V\mathcal{K}$-classification are intimately related.

It would be interesting to study the geometry of the versal deformations of the $\mathcal{A}_e$-codimension 2 map-germs listed. In particular it would be good to know if the Mond conjecture holds for them: that the $\mathcal{A}_e$-codimension is less than or equal to the image Milnor number, with equality in the quasihomogeneous cases. It should not be too hard to check this, though it is striking that most of the germs in the list are not quasihomogeneous, since in most classifications, quasihomogeneity predominates in low codimension.

A related question is whether in fact one can find the image Milnor number easily using the apparatus of ${}_\Theta\mathcal{K}$-equivalence. Perhaps $\mu_I(f)$ is equal to the ${}_\Theta\mathcal{K}$-codimension of the map-germ h for which $h^\sharp(\varphi_d) = f$, where Θ is the module of vector fields which annihilate the equation of the image of φ_d.

From Chapter 6 and Chapter 7 some more work could be done on the classification of map-germs with codimension ≤ 4. In fact, we have primary results on $\mathcal{A}$-classification and ${}_V\mathcal{K}$-classification of map-germs with codimension at most 4. We hope to find the final results of these classifications.

In Corollary 7.1.6 in Chapter 7 we have a relationship between $\mathcal{A}_e$ and ${}_V\mathcal{K}_e$ codimension. We hope to find a relationship between $\mathcal{A}_e$ and ${}_V\mathcal{K}_e$ determinacy.

In ([BW98], section 3.3), Bruce and West discussed the geometry of the function-germs on the cross cap. We could try to study the geometry of the map-germs on the generalized cross cap.

Finally, there are attempts to write new packages by using new versions of Maple, Matlab and C++ programming.

Appendix A

Singular Library 'CAST.lib'

This appendix details the collection of the CAST codes.

```
////////////////////////////////////////////////////////////////////
//////////
version="$Id$";
category="Singularities";
info="
LIBRARY:  CAST.lib   Computational Aspects of Singularity
                     Theory
OVERVIEW:
   Computational Aspects of Singularity Theory
PROCEDURES:
 setphi(d);
 phivfs(d);
 phivfs0(d);
 def_eq(d);
 derlogV(h);
```

```
 tthe(theta, h, G) calculates the extended $_\Theta \GG$-tangent space
 of a map;
 tth(theta, h, G)  calculates the $_\Theta \GG$-tangent space
 of a map;
 nthe(theta, h, G) calculates the extended $_\Theta \GG$ normal space
 of a map;
 codthe(theta, h, G) calculates the extended $_\Theta \GG $-codimension
  of a map;
guessdet (theta, h, G) gives an estimate for the determinacy of a map;
 ct(tangent, k) calculates a complete transversal of a map;
 trivunf(ct, tangent) checks whether an unfolding is trivial or
 not;
";

LIB "ring.lib";

//////////////////////////////////////////////////////////////////////
//////////
proc setphi(int d)
{
if (d==2)
{
ring phiring = 0,(v(1),w(1),w(2)),ds;
keepring(phiring);
}
else
```

```
{
ring phiring = (0,a(1..d-2),a1(1..d-2),b(1..d-1),b1(1..d-1),c(1..2),
c1(1..2)),(u(1..d-2),v(1..d-1),w(1),w(2)),ds;
keepring(phiring);
}
}

//////////////////////////////////////////////////////////////////////
//////////
/* This procedure computes the module of liftable vector fields over
the minimal cross cap mapping of multiplicity d
*/
proc phivfs(int d)
{
if (d==2)
{
module derlog=[w(2),0,v(1)*w(1)],[-v(1),2*w(1),0],[0,2*w(2),v(1)^2],
[v(1),2*w(1),2*w(2)];
return(derlog);
}
else
{
module derlog;
int i,j,m,m1,m2;
int n1=d-2;
int n2=d-1;
```

```
int n3=2*d-1;
matrix A1[n1][n2];
matrix B1[n2][n2];
matrix C1[2][n2];
matrix F1[n3][n2];
matrix A2[n1][n2];
matrix B2[n2][n2];
matrix C2[2][n2];
matrix F2[n3][n2];
matrix A3[n1][n2];
matrix B3[n2][n2];
matrix C3[2][n2];
matrix F3[n3][n2];
matrix F[n3][n2+n3];
poly sum1,sum2,sum3;
def u(d-1)=0;
def v(d)=0;
def u(d)=1;
for(i=d+1;i<=2*d;i++)
{
def u(i)=0;
def v(i)=0;
}
for(i=-2*d;i<=0;i++)
{
def u(i)=0;
```

```
def v(i)=0;
}
for(j=1;j<=d-1;j++)
{
for(i=1;i<=d-2;i++)
{
A1[i,j]=(d-i)*(d-j)*u(i)*u(j);
}
C1[1,j]=d*(d-j)*u(j)*w(1);
C1[2,j]=-d*v(j)*w(1)+(d-j)*u(j)*w(2);
}
for(j=1;j<=(d-1);j++)
{
for(i=1;i<=(d-1);i++)
{
sum1=0;
for(m1=1;m1<=(i-1);m1++)
{
sum1=sum1+u(i+j-m1)*v(m1);
}
sum2=0;
for(m2=1;m2<=i;m2++)
{
sum2=sum2+u(m2)*v(i+j-m2);
}
B1[i,j]=d*sum1-d*sum2-(i-1)*(d-j)*u(j)*v(i)+d*v(i+j)*w(1)-d*u(i+j)*w(2);
```

```
}
}
for(j=1;j<=d-1;j++)
{
for(i=1;i<=2*d-3;i++)
{
if(i<=d-2)
{
F1[i,j]=A1[i,j];
}
if(i>d-2)
{
F1[i,j]=B1[i-d+2,j];
}
}
F1[2*d-2,j]=C1[1,j];
F1[2*d-1,j]=C1[2,j];
}
for(j=1;j<=d-1;j++)
{
for(i=1;i<=d-2;i++)
{
sum1=0;
for(m1=1;m1<=i;m1++)
{
sum1=sum1+(d+i-j-2*m1+1)*u(m1)*u(d+i-j-m1+1);
```

```
}
A2[i,j]=-d*(d+i-j+1)*u(d+i-j+1)*w(1)+d*sum1-j*(i+1)*u(i+1)*u(d-j);
}
C2[1,j]=d*(d-j+1)*u(d-j+1)*w(1)+j*u(1)*u(d-j);
C2[2,j]=d*(d-j+1)*v(d-j+1)*w(1)+j*v(1)*u(d-j);
}
for(j=1;j<=d-1;j++)
{
for(i=1;i<=d-1;i++)
{
sum2=0;
sum3=0;
for(m=1;m<=i;m++)
{
sum2=sum2+(d+i-j-m+1)*u(m)*v(d+i-j-m+1);
sum3=sum3+m*u(d+i-j-m+1)*v(m);
}
B2[i,j]=-d*(d+i-j+1)*v(d+i-j+1)*w(1)+d*sum2-d*sum3-j*(i+1)*u(d-j)*v(i+1);
}
}
for(j=1;j<=d-1;j++)
{
for(i=1;i<=2*d-3;i++)
{
if(i<=d-2)
{
```

```
F2[i,j]=A2[i,j];
}
if(i>d-2)
{
F2[i,j]=B2[i-d+2,j];
}
}
F2[2*d-2,j]=C2[1,j];
F2[2*d-1,j]=C2[2,j];
}
for(j=1;j<=d-1;j++)
{
for(i=1;i<=d-2;i++)
{
sum1=0;
sum2=0;
for(m=1;m<=i;m++)
{
sum1=sum1+(d+i-j-m+1)*u(d+i-j-m+1)*v(m);
sum2=sum2+m*u(m)*v(d+i-j-m+1);
}
A3[i,j]=-d*(d+i-j+1)*u(d+i-j+1)*w(2)+d*sum1-d*sum2-d*(i+1)*u(i+1)*v(d-j);
}
}
for(j=1;j<=d-1;j++)
{
```

```
for(i=1;i<=d-1;i++)
{
sum3=0;
for(m1=1;m1<=i;m1++)
{
sum3=sum3+(d+i-j-2*m1+1)*v(m1)*v(d+i-j-m1+1);
}
B3[i,j]=-d*(d+i-j+1)*v(d+i-j+1)*w(2)+d*sum3-d*(i+1)*v(i+1)*v(d-j);
}
C3[1,j]=d*(d-j+1)*u(d-j+1)*w(2)+d*u(1)*v(d-j);
C3[2,j]=d*(d-j+1)*v(d-j+1)*w(2)+d*v(1)*v(d-j);
}
for(j=1;j<=d-1;j++)
{
for(i=1;i<=2*d-3;i++)
{
if(i<=d-2)
{
F3[i,j]=A3[i,j];
}
if(i>d-2)
{
F3[i,j]=B3[i-d+2,j];
}
}
F3[2*d-2,j]=C3[1,j];
```

```
F3[2*d-1,j]=C3[2,j];
}
for(j=1;j<=d-1;j++)
{
for(i=1;i<=2*d-1;i++)
{
F[i,j]=F1[i,j];
}
}
for(j=d;j<=2*d-2;j++)
{
for(i=1;i<=2*d-1;i++)
{
F[i,j]=F2[i,j-d+1];
}
}
for(j=2*d-1;j<=3*d-3;j++)
{
for(i=1;i<=2*d-1;i++)
{
F[i,j]=F3[i,j-2*d+2];
}
}
for(i=1;i<d-1;i++)
{
F[i,3*d-2]=(d-i)*u(i);
```

```
}
for(i=1;i<d;i++)
{
F[i+d-2,3*d-2]=(d-i)*v(i);
}
F[2*d-2,3*d-2]=d*w(1);
F[2*d-1,3*d-2]=d*w(2);
derlog=F;
return(derlog);
}
}

/////////////////////////////////////////////////////////////////////
//////////
/* This procedure computes the module of liftable vector fields over
the minimal cross cap mapping of multiplicity d without the Euler
vector field
*/
proc phivfs0(int d)
{
module derlog=phivfs(d);
module derlog0;
derlog0[1..3*d-3]=derlog[1..3*d-3];
return(derlog0);
}
```

```
//////////////////////////////////////////////////////////////////////
//////////
/*This Procedure Computes the define equation of the image of
the minimal cross cap mapping of multiplicity d
*/
proc def_eq(int d)
{
if (d==2)
{
return(w(2)^2-v(1)^2*w(1));
}
else
{
def current = basering;
ring r = 0,(u(1..d-2),v(1..d-1),w(1),w(2),a(1..d-2),b(1..d-1),y),ds;
matrix A[1][d-2];
matrix B[1][d-1];
poly sum1,sum2,prod1,prod2;
prod1=1;
prod2=1;
int i;
for(i=1;i<=(d-2);i++)
{
sum1=sum1+u(i)*y^i;
prod1=prod1*a(i);
}
```

```
for(i=1;i<=(d-1);i++)
{
sum2=sum2+v(i)*y^i;
prod2=prod2*b(i);
}
for(i=1;i<=d-2;i++)
{
A[1,i]=u(i)-a(i);
}
for(i=1;i<=d-1;i++)
{
B[1,i]=v(i)-b(i);
}
ideal J=A,B,w(1)-(y^d+sum1),w(2)-sum2;
ideal Q=eliminate(J,prod1*prod2*y);
poly H=Q[1];
setring(current);
poly H2 = fetch(r,H);
return(H2);
}
}

//////////////////////////////////////////////////////////////////////
/////////
/* Calculate the extended _Theta\GG-tangent space of map-germ with
 respect to a module of vector fields
```

```
*/
proc tthe (module theta, ideal h, string G)
"
USAGE:  tthe( theta, h, string G); theta module, h ideal, G string
PURPOSE: Calculate the extended _Theta\GG-tangent space of h with
respect to a module of vector fields
RETURN:  Returns T_Theta\GG_{e}(h)
"
{
module dh = jacob (h);
module Ch = freemodule(ncols(h))*h;
module TVE;
def EQ=G[1];
if (EQ=="R")
{
 TVE = dh*theta;
}
if (EQ=="K")
{
 TVE = dh*theta+Ch;
}
return(TVE);
}

//////////////////////////////////////////////////////////////////////
/////////
```

```
/* Calculate the _Theta\GG-tangent space of map-germ with respect to
a module of vector fields
*/
proc tth (module theta, ideal h, string G)
"
USAGE:  tth( theta, h, string G); theta module, h ideal, G string
PURPOSE: Calculate the _Theta\GG-tangent space of h with respect to
a module of vector fields
RETURN:  Returns T_Theta\GG (h)
"
{
def EQ=G[1];
module theta1 = intersect(theta, maxideal(1)*freemodule(nrows(theta)));
module TV;
if (EQ=="R")
{
 TV = tthe(theta1,h, "R");
}
if (EQ=="K")
{
 TV= tthe(theta1,h, "K");
}
return(TV);
}

/////////////////////////////////////////////////////////////////////
```

```
/////////
/* Calculate the extended _Theta\GG_e-normal space of map-germ with
respect to a module of vector fields
*/
proc nthe (module theta, ideal h, string G)
"
USAGE:  nthe( theta, h, G); theta module, h ideal, G string
PURPOSE: Calculate the extended _Theta\GG_e-normal space of h with
respect to a module of vector fields
RETURN:  Returns N_Theta\GG_{e}(h)
"
{
def EQ=G[1];
module NTV;
if (EQ=="R")
{
 NTV =kbase(std(tthe(theta,h, "R")));
}
if (EQ=="K")
{
 NTV =kbase(std(tthe(theta,h, "K")));
}
return(NTV);
} ////////////////////////////////////////////////////////////////////
/////////
/* Calculate the extended _Theta\GG_e-codimension of map-germ with
```

```
respect to a module of vector fields
*/
proc codthe (module theta, ideal h, string G)
"
USAGE:  codthe( theta, h, G); theta module, h ideal, G string
PURPOSE: Calculate the extended _Theta\GG_e-codimension of h with
respect to a module of vector fields
RETURN:  Returns _Theta\GG_{e}\cod(h)
"
{
def EQ=G[1];
int COD;
if (EQ=="R")
{
COD = vdim(std(tthe(theta,h, "R")));
}
if (EQ=="K")
{
COD = vdim(std(tthe(theta,h, "K")));
}
return(COD);
}

/////////////////////////////////////////////////////////////////////
///////
/* Guess the k-_Theta\GG-determinacy of map-germ with respect to
```

```
 a module of vector fields. For high corner see p530 of 2nd edition
 of "A Singular introduction ..."
*/
proc guessdet (module theta, ideal h, string G)
"
USAGE:  guessdet( theta, h, G); theta module, h ideal, G string
PURPOSE: Guess the k-_Theta\GG-determinacy of h with respect to
a module of vector fields
RETURN:  Returns k-_Theta\GG-determinacy
"
{
def EQ=G[1];
vector hc;
if (EQ=="R")
{
hc = highcorner(std(tth(theta,h, "R")));
}
if (EQ=="K")
{
hc = highcorner(std(tth(theta,h, "K")));
}
return(deg(hc)+1);
}
 ////////////////////////////////////////////////////////////////////
 //////////
/* Compute a complete k-transversal. The module is usually related to
```

```
a tangent space module, eg, mT_VK. However, it can be any module,
doesn't have to be a tangent space. Returns a set of monomials of
degree k which form the k-transversal
*/
proc ct (module tangent, int k)
"
USAGE:  ct( tangent, k); tangent module, k integer. The module is
usually related to a tangent space module, eg, \M T_\Theta \GG.
However, it can be any module, doesn't have to be a tangent space
PURPOSE: Compute a complete k-transversal
RETURN:  Returns a set of monomials of degree k which
         form the k-transversal
"
{
module Ch1 = freemodule(nrows(tangent))*maxideal(k+1);
module comp = std(tangent+Ch1);
return(kbase(comp,k));
}
/////////////////////////////////////////////////////////////////////
////////
/* Check to see if an unfolding is trivial
*/
proc trivunf (module ct, module tangent)
"
USAGE:  trivunf(ct, tangent); ct module, tangent module
PURPOSE: when an unfolding is trivial
```

```
RETURN:  Returns an element equal to the input element in ct if
the unfolding is not trivial and is zero if the unfolding is trivial.
"
{
module NTV=reduce(ct,std(tangent));
return(NTV);
}
```

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