Graph Granularity through Bi-intuitionistic Modal Logic

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Submitted in accordance with the requirements
for the degree of Doctor of Philosophy

The University of Leeds
School of Computing
July 2020

The candidate confirms that the work submitted is her own, except where work which has formed part of a jointly authored publication has been included. The contribution of the candidate and the other authors to this work has been explicitly indicated below. The candidate confirms that appropriate credit has been given within the thesis where reference has been made to the work of others.

Some parts of the work presented in Chapters 2, 3, 4 and 5 have been published in the following articles:

Publication 1 G. Sindoni and J. G. Stell, The logic of discrete qualitative relations, in *COSIT' 17 proceedings*, volume 86, pages 1:1-1:15. Schloss Dagstuhl - Leibniz-Zentrum für Informatik, 2017.

Publication 2 G. Sindoni, K. Sano, and J. G. Stell. Axiomatizing discrete spatial relations, in *International Conference on Relational and Algebraic Methods in Computer Science*, pages 113-130. Springer, 2018.

Publication 3 G. Sindoni, K. Sano, and J. G. Stell. Expressing discrete spatial relations under granularity. *Journal of Logical and Algebraic Methods in Program ming* (submitted).

The candidate's independent contributions to these publications, present in the thesis, have been explicitly indicated in the Section "**Contribution**". Any work included that is the result of joint work with the coauthors has been explicitly indicated in the body of the thesis.

Acknowledgements

I wish to express my sincere gratitude to my supervisor John Stell, for his expertise, advice and constant guidance throughout my PhD studies. I'm also grateful to my co-supervisor Brandon Bennett for his help and the many interesting discussions, books and conferences on logic that we shared. I wish to thank Katsuhiko Sano, whom I had the opportunity to work with, and to learn many things from.

A great thanks goes out to all the companions of this PhD adventure, Alexia, Bernhard, Claudia, Francesco, Jack, Jon, Luis, Marina, Moritz, Valerio and Wissam, for the constant support and friendship we gave each other. Thanks to my dear friend Simone, with whom I've always shared the interest for logic and the desire to learn more. And thanks to my best friends, Lorenza and Rosamaria, for always being there for me.

I owe a great deal to my family, especially to my parents Pina and Santo, my brother Giuseppe, and my amazing grandma Nicolina, for their never-ending encouragement during my academic career, and for representing that place that I will always call "home". Thanks also to my "British family", Fiona and Tony. A special thanks to Mark, for his constant presence and patience, and for pushing me to go "outside my comfort zone".

I would finally like to thank the examiners for their effort, and to acknowledge the University of Leeds for funding my research.

Abstract

This thesis concerns the use of a bi-intuitionistic modal logic, **UBiSKt**, in the field of Knowledge Representation and Reasoning. The logic is shown to be able to represent qualitative spatial relations between subgraphs at different levels of detail, or granularity. The level of detail is provided by the modal accessibility relation R defined on the set of nodes and edges. The connection between modal logic and mathematical morphology is exploited to study notions of granulation on subgraphs, namely the process of changing granularity, and to define qualitative spatial relations between these "granular" regions. In addition, a special case of graph and hypergraph granularity is analysed, namely when the accessibility relation gives rise to a partition of the underlying set of nodes and edges. Different **S5** extensions of intuitionistic modal logic are considered and compared in the thesis. It is shown that these logics, and their associated semantics, provide different ways of partitioning a graph, a hypergraph, or, more generally, a partially ordered set.

Contents

Introduction

General aim

This thesis concerns the use of a bi-intuitionistic modal logic with universal modalities, **UBiSKt**, in the field of Knowledge Representation and Reasoning. The thesis is divided into three parts, each treating a distinct but related topic. The first part concerns the use of the logic to express discrete spatial relations on graphs and hypergraphs. The second part concerns the study of the phenomenon of granularity for graphs and hypergraphs, and how spatial relations under granularity can be expressed within the logic. The third part concerns the study of graph and hypergraph partitions. In what follows, the three parts will be introduced.

Part 1. The logic **UBiSKt** is an extension of **BiSKt**, first introduced in [74]. Under certain conditions the intuitionistic frame (U, H), where U is a set and $H \subseteq U \times U$ is a partial order, can be seen as forming an undirected graph, or more generally a hypergraph, i.e. a structure made of edges and nodes where an edge is possibly incident with more than two nodes. The set U is the union of the set of edges and the set of nodes, whilst the partial order arises from the reflexive closure of the incidence relation between edges and nodes. In this work this fact is used to express discrete topological spatial relations (RCC-8 style, from [57]) between subgraphs of a (hyper)graph, within the bi-intuitionistic modal logic **UBiSKt**. Expressing spatial relations in discrete space is different from expressing spatial relations in dense space. It appears to be the case that little work has been done previously on spatial relations that apply to graphs and hypergraphs. The calculus of **UBiSKt** can be implemented, using the theorem prover generator *Mettel* [85], [71] (see [68] for the implementation). Thus it is also possible to automate spatial reasoning about regions in a network, using this tool.

Part 2. Mathematical morphology is a discipline within the field of image processing that provides a set of tools to analyse images at different levels of detail. The level of detail is parametrised by the structuring element, a shape that acts as a probe [65] through which an image is looked at. As Dougherty reports [19], according to George Matheron, who together with Jean Serra can be considered the originator of mathematical morphology, knowledge about an object is relative to the way we probe it, i.e. how we observe it. In the context of mathematical morphology for image processing, where images are usually interpreted as subsets of the grid of pixels, the basic operations are dilation and erosion by a structuring element. From these, two other fundamental operations are defined: opening, obtained by applying erosion followed by dilation, and closing, obtained by applying dilation followed by erosion. The opening can be described as the operation of fitting copies of the structuring elements within the image. The closing can be described as fitting copies of the structuring element, rotated by a half turn, on the complement of the image, and then take the complement of this [9]. This idea of describing an image by a certain shape, can be seen as a way to approximate the image, thus a way to change the level of detail. Mathematical morphology has been situated in a more general context than the pixels grid. Heijmans and Ronse [37] develop a general theory of mathematical morphology on complete lattices, where dilation and erosion are respectively join and meet preserving functions on complete lattices. It is well known that modal logic is closely connected to mathematical morphology, [6], [77]. A structuring element gives rise to a relation on the set and this relation can be interpreted as the accessibility relation of modal logic (more details about this can be found in Section 4.1). Moreover, morphological dilation corresponds to the diamond modal operator $\blacklozenge$, arising from the associate relation, whilst erosion corresponds to the box modal operator $\Box$[1]. It is well known that the theory of mathematical morphology can be extended to complex structures, such as graphs [86], [15] [14], hypergraphs, and simplicial complexes [17].

This thesis investigates how to represent graph regions at different levels of detail, or granularity. The term granularity refers to the presence of some granules, or clusters in the information. We exploit the connection between modal logic and mathematical morphology to define, within **UBiSKt**, the spatial relations between subgraphs under granularity, i.e. when the subgraphs have undergone a granulation process, which is a type of approximation. The idea is that, instead of being able to see individual nodes and edges of a subgraph, only groups of those that can be described by a structuring element can be seen. We express all the RCC-8 spatial relations between subgraphs in terms of R-dilates, i.e. copies of the structuring element. R-dilates can be seen as the new atomic parts of the space, the granules of information. In this way we can check what spatial relations occur between subgraphs, not only at the detailed level, i.e. when we consider every single node and edge as a singleton of the representation, but also when we look at subgraphs through the probe of a certain structuring element. As far as we know, the use of intuitionistic modal logic for expressing spatial relations between subgraphs at different levels of detail is new.

Part 3. In rough set theory, introduced for the first time by Pawlak [54] as an extension of set theory, elements of a set are grouped together whenever they share certain attributes. This yields an equivalence relation between the elements, that in turn gives rise to a partition of the set and to its quotient structure. The quotient structure is the set of blocks of the partition, the collection of the subsets of elements that cannot be distinguished on the basis of the available information provided by certain attributes. Looking at the quotient stucture, we get a coarser view of the initial set. It is well known that rough set theory connects with the classical modal logic **S5**, where the semantic frame is a set U plus an equivalence relation $R \subseteq U \times U$, [53], [92]. Two kinds of subset approximations are considered in rough set theory, the lower approximation $\underline{X}$, that has the same semantics of the modality **S5**-$\Box$, and the

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[1] Usually in modal logic the modalities $\Diamond$ and $\Box$ are considered. In the temporal reading of modal logic, they express possibility and necessity in the future respectively, whilst $\blacklozenge$ and $\blacksquare$ are used to express possibility and necessity in the past.

upper approximation $\overline{X}$, linked in the same way to the modality **S5-**$\blacklozenge$. Notice that usually the white diamond $\Diamond$ is considered alongside the white box $\Box$, in the modal logic **S5**. However, as the relation R is an equivalence relation and thus symmetric, white diamond and black diamond are equivalent.

As we will see, partitions of graphs and hypergraphs are important, as they give a way to look at these structures at a coarser level of detail. Equivalence relations, which are reflexive, symmetric and transitive, are well known to correspond to partitions on sets. This work considers these questions: (i) are there analogous properties of relations on hypergraphs that correspond to partitions on hypergraphs? (ii) what properties does the associated quotient function, linking the hypergraph to its quotient structure, have? An obvious possibility is considering simply equivalence relations on hypergraphs, so relations on hypergraphs that are reflexive, symmetric and transitive. But, as we will see, things are not so simple, as observed in [66], where question i is also addressed, but the properties of the quotient function are not considered. As we model a hypergraph as a poset (U, H), and thus as an intuitionistic frame, we look at different versions of the intuitionistic modal logic **S5**. These are indeed the intuitionistic analogues of classical modal logic **S5**, that, as mentioned, can be described as the logic of equivalence relations on sets. Many systems have been proposed as intuitionistic analogues of **S5**, as in [52]. Whilst there is agreement on imposing reflexivity and transitivity on the **S5** accessibility relation R, different constraints substituting symmetry have been proposed. We will consider some constraints on R substituting symmetry, appearing in [52] and [66], and we will see that each of them corresponds to a different constraint on the quotient function, linking the initial poset to its quotient structure. They all give rise to different types of partition and quotient structure of a poset. Finally, the thesis goes on to discuss a new **S5** intuitionistic logic, extending **UBiSKt**. Its semantics describes an intuitionistic frame (U, H, R), where R has all the properties that we believe are fundamental to give a satisfactory account of hypergraph partition, and to build the related quotient structure, i.e. a coarser view on the starting hypergraph. Thus, the main contribution of this part of the thesis is looking at intuitionistic analogues of **S5** under the light of a theory of hypergraph partition, or more generally poset partition. Different axioms, and corresponding properties that have been proposed in the literature, are evaluated and compared, under the light of this application.

Structure

This thesis is structured as follows.

- Chapter 1 introduces the background and analyses in detail the motivations for this work.
- Chapter 2 introduces the logic **BiSKt**, and its extension **UBiSKt**.
- Chapter 3 shows how **UBiSKt** can be used as a spatial logic, where topological spatial relations between subgraphs can be defined.
- Chapter 4 puts forward a notion of granulation for subgraphs based on mathematical morphology, and spatial relations between granular regions within **UBiSKt** are defined and analysed.

- Chapter 5 looks at a special case of hypergraph granularity, or more generally poset granularity, namely when a partition of the underlying of set is obtained.
- Chapter 6 provides a conclusion and further work.

Contribution

During her PhD studies, the author of the present work has collaborated with other researchers, thus results disseminated in this work come from these collaborations. This is why the author believes it is important to explicitly underline her independent contributions. The use of the logic **UBiSKt** to express discrete spatial relations and predicates between subgraphs, and the analysis of this, presented in Chapter 3, are contributions of the author, as well as the definitions and analysis of the granular spatial relations between granular subgraphs presented in Chapter 4. The use of the two formulae presented in Theorem 21 and 25, to express the notions of hypergraphs and graphs in **UBiSKt** is the contribution of the author. The work on the tableau-style calculus for **UBiSKt** and its implementation within the theorem-prover generator *Mettel* is the contribution of the author. The proof of completeness of a logic containing **UBiSKt** and the formula expressing the notion of graphs with respect to the graph's class of H-frames (Lemma 42 and Theorem 43) is the contribution of the author. The analysis of the different constraints yielding to different intuitionistic modal logics **S5**, under the light of a theory of poset partitions in Chapter 5 is the contribution of the author, as well as the definitions of the spatial relations in the new **S5** system for **UBiSKt**. Results achieved in collaboration are explicitly indicated in the thesis.

Chapter 1

Background and Related Work

1.1 Modal logic

Modal logic is an extension of classical logic that enables the evaluation of the truth of statements expressing necessity and possibility. Many approaches have been taken to develop the semantics of modal logic, one of the most well known being the relational semantics approach developed by Kripke in [41].

Definition 1. A *Kripke relational frame* F is a pair (U, R), where U is a set and $R \subseteq U \times U$ is a binary relation.

The syntax of classical modal logic provides propositional variables $p, q, r, \ldots$, the usual logical connectives $\vee$, $\wedge$, $\rightarrow$, $\neg$, and the modalities $\Box$ and $\Diamond$. In the context of tense logics, i.e. when the relation R models some temporal order over the elements of U, $\Box$ and $\Diamond$ are interpreted as necessity and possibility in the future. So a statement like $\Box p$ can be read as "p will definitely hold in the future" and $\Diamond p$ can be read as "p will possibly hold in the future". Thus in tense logics two further modalities have been considered: $\blacksquare$ and $\blacklozenge$, and they express necessity and possibility in the past. We also remark that, in a classical modal logic, $\Box$ and $\Diamond$ are inter-definable as $\Diamond p \leftrightarrow \neg\Box\neg p$ and $\Box p \leftrightarrow \neg\Diamond\neg p$ are theorems, an analogous relationship holds between $\blacksquare$ and $\blacklozenge$. Formulae are defined by stipulating that propositional variables are formulae, and if φ, ψ are formulae then so are $\varphi \wedge \psi$, $\varphi \vee \psi$, $\varphi \rightarrow \psi$, $\neg\varphi$, $*\varphi$, with $* \in \{\Box, \Diamond, \blacksquare, \blacklozenge\}$. The semantics for this logic allows an interpretation of atomic propositions as subsets of U, and formulae correspond to subsets constructed out of these.

Definition 2. A *Kripke relational model* M is a tuple (U, R, V) where (U, R) is a frame and $V : Prop \rightarrow \mathcal{P}(U)$, assigns each propositional variable $p \in Prop$ to a subset of U.

V can then be extended to a function $[\![\ ]\!]_M$ taking as input generic formulae. The set $[\![\varphi]\!]_M$ is usually called the *truth set* of φ, and, when a model is defined and a valuation is given, $[\![\varphi]\!]_M = \{u \in U \mid u \models \varphi\}$, where '$\models$' is the usual relation of satisfiability between elements of U and formulae, or semantic truth. We will omit the subscript M when no confusion arises.

Formulae in the language of modal logic are assigned to subsets of U, and truth and falsity in the language, denoted as usual as $\top$ and $\bot$, are interpreted respectively as U and $\varnothing$. The logical connectives $\vee$, $\wedge$, $\neg$ are classically interpreted as the set-theoretic operations of union $\cup$, intersection $\cap$, and complement $-$. Implication $\rightarrow$ is handled by defining $[\![\varphi \rightarrow \psi]\!] = -[\![\varphi]\!] \cup [\![\psi]\!]$. This means $[\![\varphi \rightarrow \psi]\!]$ holds in a given interpretation if and only if $[\![\varphi]\!] \subseteq [\![\psi]\!]$. The semantics for the modalities can be expressed as follows:

Definition 3. Given a model $M = (U, R, V)$, given a subset $X \subseteq U$, such that $X = [\![p]\!]_M$ for some propositional variable p, we define

$$\begin{array}{rcl} [\![\Box p]\!] & = & \{u \in U \mid \forall v (u\ R\ v \text{ implies } v \in [\![p]\!])\} \\ [\![\Diamond p]\!] & = & \{u \in U \mid \exists v (u\ R\ v \text{ and } v \in [\![p]\!])\} \\ [\![\blacksquare\, p]\!] & = & \{u \in U \mid \forall v (v\ R\ u \text{ implies } v \in [\![p]\!])\} \\ [\![\blacklozenge\, p]\!] & = & \{u \in U \mid \exists v (v\ R\ u \text{ and } v \in [\![p]\!])\} \end{array}$$

We also introduce the following notation about binary relations[1] that is going to be used in the thesis.

Definition 4. Given sets U, V and W and relations $R \subseteq U \times V$ and $S \subseteq V \times W$, $R\,;S \subseteq U \times W$ is the *relation composition* of R and S, that is: $\{\, (u, w) \in U \times W \mid \exists v \in V \text{ such that } u\ R\ v\ S\ w \,\}$. The *converse* of a relation $R \subseteq U \times V$ is $\breve{R} = \{\, (u, v) \in V \times U \mid (v, u) \in R \,\}$. The *relational complement* of R is $\overline{R} = \{(u, v) \in U \times V \mid (u, v) \notin R\}$. The *identity relation* $I \subseteq U \times U$ on a set U, is the set $\{\, (u, u) \mid u \in U \,\}$.

We also remind the reader about the following facts regarding relations: $R \subseteq S$ iff $\breve{R} \subseteq \breve{S}$ and $\breve{\overline{R}} = \overline{\breve{R}}$. Also $\breve{\breve{R}} = R$ and $\overline{\overline{R}} = R$. (cf. [45] page 6).

1.2 Modal Logic and Mathematical Morphology

Mathematical morphology is a discipline in the field of image processing, that has been applied to the analysis of the structure of materials in different fields, such as mineralogy, petrography, cytology and so on [37]. Mathematical morphology uses mainly concepts from set theory, and a prominent aspect of the discipline is its algebraic basis. This is explored in [37] and [60], whilst for a general introduction we refer to [65]. Modal logic and mathematical morphology are two different disciplines that are now being recognised as closely connected [6], [8],[9]. The connection between them comes from the fact that the two basic operations in mathematical morphology, dilation and erosion, have the same algebraic properties of the modalities $\blacklozenge$ and $\Box$. Both these pairs form an *adjunction*, an algebraic concept that we will shortly introduce. Let U be a set with $X \subseteq U$ and $R \subseteq U \times U$.

Definition 5. Dilation $\oplus$ and erosion $\ominus$ are operations with signature $\mathcal{P}(U), \mathcal{P}(U \times U) \mapsto \mathcal{P}(U)$ and $\mathcal{P}(U \times U), \mathcal{P}(U) \mapsto \mathcal{P}(U)$ respectively.

$$\begin{array}{rcl} X \oplus R & = & \{u \in U \mid \exists v (v\ R\ u \text{ and } v \in X\} \\ R \ominus X & = & \{u \in U \mid \forall v (u\ R\ v \text{ implies } v \in X\} \end{array}$$

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[1] In the following definition we assumethe general case of a relation $R \subseteq U \times V$ where the sets U and V are possibly different. When $U = V$ we talk about a *homogeneous relation.* When $U \neq V$, the relation R is known in the literature as a *heterogeneous relation*, see [63].

From Definition 3 it is immediate the connection that $\Box$ is associated to $X \mapsto R \ominus X$, and $\blacklozenge$ is associated to $X \mapsto X \oplus R$. We can also consider converse erosion and dilation, giving $\blacksquare$ associated to $X \mapsto \breve{R} \ominus X$, and $\Diamond$ associated to $X \mapsto X \oplus \breve{R}$.

Usually in mathematical morphology the operations of dilation and erosion are defined in terms of a *structuring element*, a small shape acting as a sort of probe, by which images, that can be interpreted as subsets of a set of pixels, can be modified. However a structuring element gives rise to relation defined on the set (details of this are given in Section 4.1), and, on this basis, the relational approach to mathematical morphology has been developed, [9] and [77]. An example of a structuring element and associated relation on the pixel grid, with operations of dilation and erosion over a subset of pixels is given in Figure 1.1

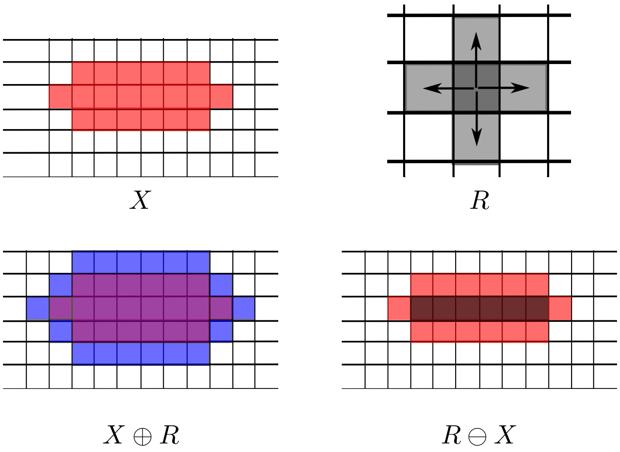

Figure 1.1: A subset of a set of pixels X with its dilation, $X \oplus R$ (blue and purple area) and erosion, $R \ominus X$ (brown area). A structuring element can be seen as a way to generate a relation. In this case, the cross-shape structuring element can be seen as (the reflexive closure of) of the 4-adjacency relation on the grid of pixels.

For a fixed relation $R \subseteq U \times U$, the operations $\_ \oplus R$ and $R \ominus \_$ with signatures $\mathcal{P}(U) \mapsto \mathcal{P}(U)$ form an adjunction on the lattice of $\mathcal{P}(U)$. The dilation by R, $\_ \oplus R$, is called the left adjoint and the erosion by R, $R \ominus \_$, is the right adjoint, in the following sense.

Definition 6. Let $(V, \leq_V)$ and $(W, \leq_W)$ be partially ordered sets. An adjunction between V and W is a pair of functions $f : V \mapsto W$ and $g : W \mapsto V$ such that $f(v) \leq_W w$ iff $v \leq_V g(w)$, for all $v \in V$ and $w \in W$. The function f is called the *left adjoint* and g is the *right adjoint*.

If (f, g) is an adjunction with f left adjoint and g right adjoint, then we have that, whenever join $\bigvee$ and meet $\bigwedge$ operations for the posets $(V, \leq_V)$ and $(W, \leq_W)$ exist, the left adjoint f preserves the join of any subset of V, and the right adjoint preserves the meet, so for any family v_i, $i \in I$, where $v_i \subseteq V$, we have that $f(\bigvee_{i \in I} v_i) = \bigvee_{i \in I} f(v_i)$, and for any family w_i, $i \in I$, where $w_i \subseteq W$, we have that $g(\bigwedge_{i \in I} w_i) = \bigwedge_{i \in I} g(w_i)$.

Then as the pair $(\_ \oplus R, R \ominus \_)$ forms an adjunction on the lattice $\mathcal{P}(U)$, they will have the following property: given $X, Y \subseteq U$, $(X \oplus R) \subseteq Y$ iff $X \subseteq (R \ominus Y)$. This translates into modal logic by the fact that given any modal frame (U, R), given any formulae φ and ψ, if the implication $\blacklozenge \varphi \to \psi$ is valid (i.e. true for any valuation V at any $u \in U$), then we can derive that $\varphi \to \Box\psi$, is also valid, and vice-versa. Analogous reasoning holds for the pair $(\_ \oplus \breve{R}, \breve{R} \ominus \_)$ and $(\Diamond, \blacksquare)$. By the property of preservation of join and meet operations, that in the case of the lattice formed by $\mathcal{P}(U)$ with the relation $\subseteq$ are union and intersection between subsets, given $X, Y \subseteq U$ we have that $(X \cup Y) \oplus R = (X \oplus R) \cup (Y \oplus R)$, and $R \ominus (X \cap Y) = (R \ominus X) \cap (R \ominus Y)$.

We remark that, in the literature, the concept of adjunction is more general than the one presented in Definition 6, and it comes from category theory, where it expresses a certain relationship between functors, i.e. mapping between categories (see [31] for an introduction to category theory).

1.3 Modal Logic + Intuitionistic Logic = ?

Intuitionistic logic is a non classical logic, based on the notion of provability more than on the notion of truth. It can be described as the logic done without the law of excluded middle, $p \vee \neg p$. Its semantics framework, developed by Kripke in [42] consists of a set U ordered by a relation of partial order $H \subseteq U \times U$, i.e. a relation that is reflexive, transitive and antisymmetric.

Classical modal logics are classical in the sense that they are built on top of classical propositional logic. Similarly, intuitionistic modal logics, *IML* for brevity, have intuitionistic logic as a base. The study of *IML* is motivated by various applications. Some examples are philosophical applications as the development of temporal intuitionistic logic by Ewald [23], and epistemic intuitionistic logic by Williamson [89]. Computational applications are mentioned by Plotkin and Stirling [56], and also studied by Wijesekera in [88], and Nishimura [50] who studies constructive variants of propositional dynamic logic. As pointed out by Simpson [67] and Kojima [39], although a notable amount of work has been done on *IML*, there is no agreement on what an *IML* should be, and there's not a unique semantic framework. Both modal and intuitionistic logic rely on a relational frame. Thus a common approach to build a semantic framework for *IML* is to put together the two Kripkeian accounts. Indeed, the common ground for the majority of approaches to *IML* found in the literature is a bi-relational semantic frame of the form (U, H, R).

The reason for such a variety of approaches comes from the fact that the classical interpretation of modalities as given in Definition 3 has to agree with the monotonicity of the valuation function proper of intuitionistic logic, according to which information about states is preserved w.r.t. H-successor. The valuation function V of an intuitionistic model $M = (U, H, V)$, defined for propositional variables, requires that for any $u, v \in U$, if $u \in V(p)$ and $u\ H\ v$ then $v \in V(p)$. This monotonicity has to extend to valuation of generic formulae, thus if $u \in [\![\varphi]\!]_M$ and $u\ H\ v$ then $v \in [\![\varphi]\!]_M$. We can say that the valuation of formulae in intuitionistic formulae is closed under H-successor, or that it assigns formulae to H-sets, where $X \subseteq U$ is an H-set if, whenever $u \in X$ and $u\ H\ v$, then $v \in X$. In Figure 1.2 we give an example showing that the classical semantic clauses for $\Box$ and $\blacklozenge$ might disagree with the above mentioned monotonicity rule.

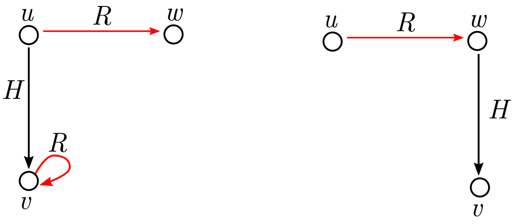

Figure 1.2: Let us consider the frame on the left. Suppose we have a model M such that $V(p) = \{ w \}$. Notice that $V(p)$ is closed under H successor, as w is the only H-successor of w. Then $[\![\Box p]\!]_M = \{ w, u \}$. By intuitionistic monotonicity, as $u \; H \; v$ and $u \in [\![\Box p]\!]$, we should have that $v \in [\![\Box p]\!]$. However $v \notin [\![\Box p]\!]$, as $v \; R \; v$ and $v \notin V(p)$. This is an example of disagreement between the classical interpretation of $\Box$ and the intuitionistic monotonicity. Let us consider the frame on the right. Suppose a model where $V(p) = \{ u \}$. Also here $V(p)$ is closed under H-successor. Thus $[\![\blacklozenge\, p]\!]_M = \{ w \}$. Then by monotonicity we should have that $v \in [\![\blacklozenge\, p]\!]$. But then we should have that for some $j \in U$, $j \; R \; v$ and $j \in V(p)$. Such a j is not present in this model. Thus, here's another case of disagreement between classical modalities and intuitionistic monotonicity.

There are different ways to solve this problem, and from this stems the variety of approaches taken towards *IML*. Roughly, we can make the following categorisation of approaches for *IML*, even if they are not mutually exclusive as, as we are going to see, both the approaches are taken at the same time by some authors:

1. The agreement between modalities and monotonicity of the valuation function is obtained by modifying the semantic clauses for the modalities.

2. Some connecting properties are imposed between the modal accessibility relation R and the intuitionistic partial order H.

The first approach is taken for example from Wijesekera in [88]. Here the author defines the semantic clauses for $\Box$ and $\Diamond$ in a different way from their classical counter-parts (he considers only these two modalities in his work):

$$[\![\Box p]\!] = \{ u \in U \mid \forall w \forall v (u \; H \; w \; R \; v \text{ implies } v \in [\![p]\!]) \} \tag{1.1}$$

$$[\![\Diamond p]\!] = \{ u \in U \mid \forall w (u \; H \; w \text{ implies } (\exists v \text{ s. t. } w \; R \; v \text{ and } v \in [\![p]\!])) \} \tag{1.2}$$

Wijesekera's semantics clause for $\Box$, equation 1.1, can be translated in terms of mathematical morphology as follows: $[\![\Box p]\!] = H \,;\, R \ominus [\![p]\!]$. It is hard to see what the "morphological translation" for equation 1.2 would be. Simpson [67] actually criticises Wijesekera's semantics for $\Diamond$ as bringing "some rather strange properties" to the system, as the fact that $\Box$ and $\Diamond$ don't seem related at all, and that even after adding the law of the excluded middle to Wijesekera's system, the two modalities are still not inter-definable. Thus the rule of excluded middle doesn't yield, in Wijesekera's system, a classical modal logic. Simpson [67] considers this feature an important requirement for *IML*, and thus criticises

Wisekejera's choice. He suggests to substitute the semantic clause for $\Diamond$ with the following one:

$$[\![\Diamond p]\!] = \{\, u \in U \mid \exists w \exists v (u \breve{H} \, w \, R \, v \text{ and } v \in [\![p]\!])) \,\} \tag{1.3}$$

This gives rise to the following morphological definition: $[\![\Diamond\varphi]\!] = [\![\varphi]\!] \oplus \breve{R} ; H$.

The second approach has been taken in the majority of cases, with some differences, as the connecting properties may vary from work to work. The difference in the chosen connecting properties depends also on which fragment of intuitionistic modal logic one chooses to consider. For example one could work on the $\Diamond$-free modal fragment, or vice-versa on the $\Box$-free modal fragment. Also one could choose whether or not to consider the modalities $\blacksquare$ and $\blacklozenge$ as well.

A first strand starts with a series of papers by Fischer Servi [24], [25] and [26]. Here the following connecting properties for a modal intuitionistic frame are introduced: $R;H \subseteq H;R$ and $\breve{R};H \subseteq H;\breve{R}$. We refer to these constraints as $FS1$ and $FS2$. Plotkin and Stirling [56] consider the FS conditions alongside two additional conditions that, as they say, "spring to mind" when looking at bi-relational frames: $H ; R \subseteq R ; H$ and $H ; \breve{R} \subseteq \breve{R} ; H$. However the authors don't discuss these latter constraints, and, for their *IML* semantics, adopt the FS conditions. They motivate their choice by the fact that $FS1$ and $FS2$ ensure the validity of two modal formulae that the authors call "very natural", and thus according to them, desirable in *IML*: $\neg\Diamond p \to \Box\neg p$ and $\Diamond p \to \neg\Box\neg p$. They say that, more generally, $FS2$ ensures that $[\![\Diamond p]\!]$ is closed under H-successor. Plotkin and Stirling also define the $\Box$ semantic clause in the intuitionistic way as in equation 1.1, to make sure that also boxed formulae are closed under H-successor. Thus they actually take a mixed approach between approach 1 and 2. Ewalds [23], Simpson [67], and Amati and Pirri [1] take the same mixed approach. A similar approach is taken also by Goré et al. in [33], with the difference that two accessibility relations R and S on U are considered, one defining the semantics for $\Box$ and $\blacklozenge$ and the other one defining $\blacksquare$ and $\Diamond$. The $FS1$ condition is imposed on R and $FS2$ is imposed on S. Moreover, the clause for $[\![\Box p]\!]$ is defined intuitionistically as in equation 1.1, and $[\![\blacksquare\, p]\!]$ is defined in the analogous way by $H ; \breve{S} \ominus [\![p]\!]$. The approach of imposing the connecting property $H ; R \subseteq R ; H$, instead of adopting equation 1.1 for $[\![\Box p]\!]$ is taken by Božić and Došen in [10] and by Došen in [18] for the semantics of their $\Diamond$-free fragment of *IML*. They then adopt $FS2$ for the semantics of the $\Box$-free fragment. The two approaches are equivalent.

Different connecting properties from the ones seen so far are considered by Nishimura [50], Wolter and Zakharyaschev [91], Kojima [39] and Stell et al. [74]. An equivalent approach is taken also by Ono [52], for his intuitionistic modal logics $S4$ and $S5$. Also Božić and Došen [10], analyse this approach to the semantics for *IML*, as an alternative to the approach mentioned above, that imposes the FS-like conditions. In this second strand of works the connecting property between R and H is $H;R;H \subseteq R$. We call this property *stability* of R, following [74]. Notice that this condition is equivalent to the conjunction of two conditions: $H ; R \subseteq R$ and $R ; H \subseteq R$. Moreover $R \subseteq H ; R ; H$ always holds by $I \subseteq H$, and thus imposing stability means imposing the identity $R = H ; R ; H$. In [91] actually two relations are introduced, one for $\Box$, $R_\Box$, and one for $\Diamond$, $R_\Diamond$. Stability is imposed on $R_\Box$, and on the converse of $R_\Diamond$, giving the constraints $H ; R_\Box ; H \subseteq R_\Box$ and $H ; \breve{R}_\Diamond ; H \subseteq \breve{R}_\Diamond$. On the other hand, in [74], only one stable relation is considered (besides the partial order H, that is trivially stable by transitivity of H). Stability ensures that $[\![\Box p]\!] = R \ominus [\![p]\!]$ and $[\![\blacklozenge p]\!] = R \oplus [\![p]\!]$ are H-sets. However, given a stable relation R, its standard converse $\breve{R}$ is not necessarily stable. Thus in [74], in order to make sure that $\Diamond$ and $\blacksquare$ give rise to H-sets, the authors consider the smallest stable relation containing $\breve{R}$, i.e. $H\,;\breve{R}\,;H$. They call it the *left converse* of R, in symbols $\circlearrowright R$, and the semantic clauses of $\Diamond$ and $\blacksquare$ are defined by using dilation and erosion by $\circlearrowright R$ respectively. As explained in [74], thanks to the use of the left converse, the intuitionistic modal logic that they consider has a feature that other intuitionistic modal logics don't have: the $\Diamond$-modality is definable in terms of $\Box$. Indeed the formula $\Diamond p \leftrightarrow \lrcorner\Box\neg p$ is a theorem in the logic. However $\Box$ is not definable in terms of $\Diamond$, and thus the two modalities are still "independent", as it is expected in *IML*, see [67]. Similarly, the modality $\blacksquare$ is definable in terms of $\blacklozenge$, as $\blacksquare p \leftrightarrow \neg\blacklozenge\lrcorner p$ is a theorem, but not viceversa.

In [74] the authors consider the relationship between the approach to *IML* with stable relations and the approach of imposing the FS-like conditions. They compare their system to the one presented by Goré et al. [33]. They notice that the $FS1$ condition is more general than stability. Indeed stability implies $FS1$: $R\,;H \subseteq I\,;R\,;H \subseteq H\,;R\,;H \subseteq R \subseteq I\,;R \subseteq H\,;R$. But not vice-versa. However it is shown that this generality is not essential, as it is possible to rephrase the semantics of modalities from [33] in terms of stable relations. Whilst the stable relation R giving rise to $\Box$ (and its adjoint $\blacklozenge$) and the stable relation S giving rise to $\Diamond$ (and its adjoint $\blacksquare$) would be unrelated in the approach from [33], the system of [74] is the special case where the relation giving rise to $\Diamond$ is not just any stable relation, but it is the left converse of the relation giving rise to $\Box$, thus a function of the first relation considered. Thus, in [74], all the four modalities can be seen as arising from a single relation, and it is for this reason that the novel definability of $\Diamond$ from $\Box$ holds there.

Also Božić and Došen [10] mention stability as an alternative to the FS conditions, and they show that their *IML* is sound and complete w.r.t. both semantics frames, with the FS-like conditions and with the stability. This is again evidence of the fact that the generality provided by the FS-like conditions is not essential, as the two types of frames compared in [10] validate the same set of formulae.

Moreover, we notice that stability on R is an "analytical" constraint, in the sense that the conclusion of the constraint is simpler than its premise. Stability says: every time we find an H ; R ; H path, we need to add an R path. On the other hand, in the case of the FS's conditions, and the related conditions $H\ ;\ R \subseteq R\ ;\ H$ and $H\ ;\ \breve{R} \subseteq \breve{R}\ ;\ H$, every time we find the path stated in the hypothesis, we have to introduce the existence of a new $u \in U$ that makes the conclusion of the constraint true. In this sense, all these rules are non-analytical. Analyticity is a very important feature when implementing some forms of proof systems for modal logic as, for example, tableau-style proof systems.

1.3.1 Bi-intuitionistic Logic

The logics used in this work, **BiSKt** and **UBiSKt**, are *bi-intuitionistic* modal logics. Propositional bi-intuitionistic logic was studied by Rauzser in [58]. Here it is called Heyting-Brouwer (H-B) logic. Bi-intuitionistic logic is obtained by adding to intuitionistic logic the dual operator of implications, called *co-*

implication $\prec$. Using this, a new negation operator can be defined, whose semantics differs from the intuitionistic standard negation $\neg$. It is usually called *co-negation* or *dual negation*, and we can indicate it with the symbol $\lrcorner$.

In the same way that Heyting algebra was introduced to formalise intuitionistic logic, bi-intuitionistic logic relates to *bi-Heyting* algebra. For formal definitions of these concepts we refer to [80]. As the author explains there, bi-Heyting algebras are both Heyting algebras and co-Heyting algebras. The study of co-Heyting algebra starts with Lawvere in [43], where it is described as a generalisation of a Boolean algebra where the conjunction of a proposition p and its negation is not necessarily a contradictory formula. Lawvere states that this concept deserves to be called the *boundary* of p. We will see how this applies to subgraphs and their spatial boundaries in Section 3.3. An analysis of modal extensions of bi-intuitionistic logic can be found in [90].

1.4 Stable Relations as Relations on Hypergraphs

In this thesis, we will take the approach of imposing the stability condition. Indeed we make use of the logic introduced in [74] called **BiSKt**, and its extension with universal modalities **UBiSKt**, introduced in [71]. The main objects of investigation of this thesis are undirected graphs (with possibly multiple edges and self-loops), or, more generally, hypergraphs. A hypergraph is a generalisation of the idea of a graph, where an edge can be incident with more than two nodes. We are going to see in Chapter 2 in more detail how a hypergraph always gives rise to a partially ordered set (U, H), and thus to an intuitionistic Kripke frame. For now we just say that we can consider $U = E \cup N$, where E is the set of edges and N is the set of nodes, and the partial-order $H \subseteq U \times U$ is the reflexive closure of the incidence relation from edges to nodes. Under this semantics, formulae in the **BiSKt** logic can be assigned to subgraphs of the hypergraph-domain (U, H), i.e. to subsets of U closed under H-successor: whenever an edge is present in a subgraph, all the nodes it is incident with are present. This is analogous to the requirement that interpretation of formulae is closed under H-successor in intuitionistic logic. Operations of dilation and erosion on subgraphs can be considered, by adding a relation $R \subseteq U \times U$ to the frame (U, H). Thus we have a modal-intuitionistic frame (U, H, R). The relation R is stable w.r.t. H.

The account of stable relations as relation on hypergraphs has been developed by Stell in [77], [78], [79] and [80]. The main motivation there was developing a theory of mathematical morphology for graphs and hypergraphs, following a fruitful strand of works [36], [15], [14], but in terms of relations on hypergraphs. Thus, an account of a relation on hypergraphs was needed. The fact that imposing a relation R on $U = E \cup N$ without any extra property is not a satisfactory account, relates to the problem of interpreting modalities, i.e. dilation and erosion, classically in an intuitionistic context, as discussed in Section 1.3. The starting point of [77], [78] to develop a satisfactory notion of relations on hypergraph, is noticing that all relations on a set U have an important property: they correspond to union-preserving functions on the lattice of all subsets of U, $\mathcal{P}(U)$.

Definition 7. A function f from $\mathcal{P}(U)$ to itself is *union preserving* if and only if for any indexed family of sets Z_i, $i \in I$, such that $Z_i \subseteq \mathcal{P}(U)$ we have

$f(\cup_{i \in I} Z_i) = \cup_{i \in I} f(Z_i)$.

Every relation R on U gives rise to a union-preserving function on U (namely the dilation by R), and vice-versa given every union-preserving function on $\mathcal{P}(U)$ defines a relation on U as follows: given $u, v \in U$, we have that $u \; R \; v$ if $v \in f(\{ u \})$. Thus there is a correspondence between relations on a set and union-preserving functions on the lattice of its subsets. Relations on a set can be characterised by this correspondence.

Let us consider the lattice of all the subgraphs of a hypergraph (U, H). It is clear that this is not the same as the lattice of all the subsets of U. Indeed, as we said, subgraphs are subsets that are closed under H-successor, thus not all subsets of U are subgraphs of (U, H). A simple example is given in Figure 1.3.

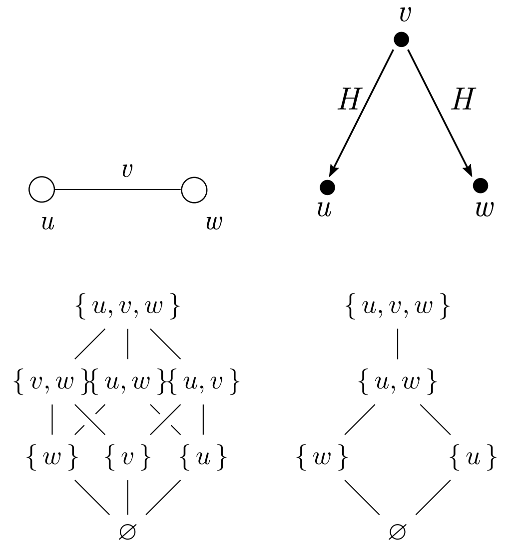

Figure 1.3: A graph with two nodes u and w, and one edge v, with its poset representation (U, H). As we can see, the lattice of all subsets of U is different from the lattice of all subgraphs of (U, H). Whilst $\{ v \}$ is a subset of U, it is not a subgraph of (U, H), as it a set made of a “naked” edge. Notice that the lattice of all subgraphs is not complemented, as there’s no element x in the lattice such that $x \vee \{ u, w \} = \{ u, v, w \}$ and $x \wedge \{ u, w \} = \varnothing$. Hence the lattice of all subgraphs of a graph is not necessarily Boolean; we remind the reader that a Boolean lattice is always complemented, and thus the associated logical system is not classical.

In [77] and [78] the correspondence between relations on a set and union-preserving functions on the lattice of its subsets is exploited, and extended to the hypergraph context. It is shown that union-preserving functions on the lattice of all subgraphs of a hypergraph correspond to those relations $R \subseteq U \times U$ that are stable: $H \; ; R \; ; H \subseteq R$. This result eventually motivates the account of stable relations as the correct account for defining relations on graphs and hypergraphs, and associated morphological operations of dilation and erosion on hypergraphs.

1.5 Qualitative approach to Spatial Representation

One contribution of this thesis is the use of the bi-intuitionistic modal logic **UBiSKt** to define qualitative spatial relations between subgraphs. As we will see, thanks to the modal part of the logic, and exploiting its connection to mathematical morphology, appropriate definitions of spatial relations at different levels of detail can also be given within this logic. The level of detail is parametrised by the modal accessibility relation that can be seen as a probe according to which the underlying graph is structured. Thus this contribution can be considered within the subfield of knowledge representation known as qualitative spatial representation and reasoning, QSR for short.

QSR is concerned with providing formal methods for encoding and reasoning about spatial knowledge. Examples of the importance of QSR in computer science are given by Bennett [2], and they are reasoning about physical systems, robot's navigation and planning, and computer vision, as this is primarly concerned with extracting information from sensor data, that are usually 2-d images.

Qualitative methods for representing space are called so as opposed to quantitative methods. It is hard to strictly define what is qualitative and what is quantitative; however, to quote Galton, [28] "the divisions of qualitative space correspond to salient discontinuities in our apprehension of quantitative space". What the author seems to mean here, is that space has measurable, quantitative properties[2], but these are often very hard to grasp and process for humans. And, for some purposes, we don't even need to have such detailed information to be able to describe space in an effective way, and make inferences about it. For example, we don't need to know the exact set of coordinates two objects occupy (in the ideal Cartesian plane of space) to know whether the two objects lie next to each other, or whether they are apart. Similarly, we don't need detailed numerical information to know that, if two objects touch each other boundaries and a third object is contained in the core part of the second one, then the first and the third object must be disconnected, apart from each other (notice that this example is more complex and involves some reasoning, however, this doesn't have to be numerical reasoning, but logical reasoning). Qualitative descriptions of space, in this sense, aim to model the way humans see and reason about space, in order to provide artificial agents with the same power, without overloading them with quantitative information that is, for many tasks, unnecessary. This type of knowledge is sometimes referred to as "common sense knowledge" and many recognize its importance in developing true intelligent artificial agents [2].

Spatial situations are usually represented in a qualitative way by specifying spatial predicates on the spatial entities involved. Extracting information from qualitative data requires logical reasoning about the objects and the relations involved, and thus the development of formal theories of spatial representation [2]. Spatial predicates can be of different kinds, depending on which aspect of space we are interested in. For example we might want to investigate the way

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[2] It is a philosophical question whether space actually *possesses* quantitative properties, and it is at the centre of the debate between platonic and constructive interpretation of space. We don't mean here to take one, or the other position. But, in any case, we can say that space has quantitative properties, leaving open whether they are human constructions or actual entities with ontological status, independent from who sees them.

spatial entities connect to each other. This kind of spatial relations is known as topological, as topology mainly concerns how things are connected. Topological properties are those properties that do not change after continuous deformation of the objects; they depend more on the way objects are put together. As opposed to topological properties, we have geometrical properties, that concern exactly the shape of the objects. Another strand is mereology, meaning, from Greek, the *theory of parts and wholes.* Mereological spatial relations are the relations of parthood, equality, or partial overlapping. Other types of spatial predicates and relations might involve the orientation of objects in the space, as as "X is at the left of Y", or "X is north in respect to Y". In this thesis we are going to focus on the so called mereotopological spatial relations. Mereotopology comes, obviously, from putting together topology with mereology. Within mereotopology, for example, the predicate of part can be enhanced by distinguishing between a peripheral, or tangential part, and a non-tangential one.

There are a few works that have been particularly important to the development of mereotopology: the work of Whitehead [87], and De Laguna [16], as they present a region-based account where the spatial relation of connection plays a central role, and the work of Clark [12] that is based on Whitehead's work, and that in turn provides the basis for the development of the Region Connection Calculus, RCC for short [57]. For a more detailed history of qualitative spatial reasoning see [76]. Nowadays, we can single out two major strands in mereotopology: the logical approach started with the RCC by Randell et al. [57], and the 9-intersection approach started by Egenhofer and Herring [22]. In this work we are interested in the logical approach.

RCC is a first-order logic theory with a primitive predicate of Connection C between regions of the space. The predicate of Parthood is defined using Connection, as follows: $P(x, y) := \forall z(C(x, z) \implies C(y, z))$. Using Parthood and Connection, a set of eight jointly-exhaustive and pairwise-disjoint spatial relations is definable. This is known as RCC-8, and it is as in Figure 1.4. As we are shortly going to see, RCC is applied to reasoning in continuous space. Although RCC is a first-order logic theory, Bennett [2] shows that propositional logic is enough. He uses classical propositional logic to represent the mereological side of the theory. Then he uses modal propositional logic to define the topological side of the RCC, using the connection between the $\Box$ operator and the interior operator of topology. Another direction in the modelling of qualitative relations in continuous space using modal logic was initiated by Bloch [7]. She exploited the connection of modal logic with the image processing techniques of mathematical morphology, mentioned earlier.

Our use of modal logic within spatial representation also builds on the link between modal operators and dilation and erosion, by which, as we will see, we can define a closure topological operator, as well as a interior topological operator. However, differently from Bloch and the RCC approach, we are interested in developing spatial relations for a kind of space that is discrete, i.e. a graph (actually, we use the more general account of a hypergraph). Thus, we present new results in the area of mereotopology for discrete space.

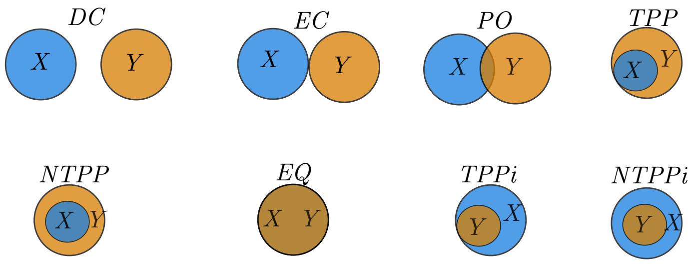

Figure 1.4: RCC-8 can express disjointness, or Disconnection (DC), and connection on the boundaries, or External Connection (EC), as well as the relation of sharing only a part, or Partial Overlapping (PO). RCC-8 can distinguish Tangential Proper Part (TPP) from Non-Tangential Proper Part ($NTPP$). Equality (EQ), and the inverses of TPP and $NTPP$ are also included in RCC-8 relations.

1.6 Discrete Space

When developing spatial theories, besides having in mind which kinds of spatial properties to model (topological, geometrical, etc...), one has also to make some choices about the nature of the space itself. One of these choices is about discreteness versus density of space. If we model regions as, for example, subsets of $\mathbb{R}^2$, we assume the space to be dense: between any two distinct points x, y, we can always find a third point z that is closer to x, than y is to x. On the other hand, discrete space, as for example the one induced by $\mathbb{Z}^2$, is such that every point has its "nearest" neighbours. In discrete space there are regions with a special status, i.e. *atomic regions*. Atomic regions do not have any proper part, they are the points at which space stops being dividable. For some purpose it is better to model space as discrete, instead of as continuous. Any kind of network (road networks, railway networks, airlines networks, cable or pipelines networks) is naturally represented by a discrete structure like a graph, where nodes are the atomic regions. Images in image processing are in the form of pixel arrays, and pixel is the atomic component of space. In Geographical Information Science many kinds of data are classified as discrete, for examples objects with distinct boundaries, like cities and districts linked by roads.

It is well known that RCC is not able to represent discrete space, as shown in [57]. Indeed a theorem in the theory is that every region has a non-tangential proper part. It is immediate to see how the assumption of an atomic region, i.e. a region without any proper part, leads to contradiction. The authors suggest that the culprit is the RCC definition of parthood, done in terms of connection. We can see that this definition cannot be applied to discrete space. Indeed, let us take a graph with two nodes u and v and one edge between them. Then each node is connected to itself, to the other node, and to the whole underlying graph. Thus, the definition of parthood from the RCC holds between u and v (and vice-versa between v and u), however, it is clear that neither node is part of the other one.

A solution to this is put forward by Galton [27], [30], and by Li and Ying, [44]. Here we focus on Galton's approach, who develops a theory named Discrete

Mereotopology, i.e. a version of the RCC for discrete space. The kind of discrete space that Galton considers is an adjacency space, so a set of primitive elements N called nodes, or sometimes cells, and a relation $A \subseteq N \times N$ reflexive and symmetric, called the adjacency relation. This approach is in turn inspired by the digital topology theory developed by Rosenfeld [61]. From this, two subsets of the set of nodes X and Y are connected iff there are $x \in X$ and $y \in Y$ and $x = y$ or $x \; A \; y$. Besides connection, Galton defines all the RCC-8 relations between regions of an adjacency space.

Adjacency space can be seen as graphs (undirected graphs without self-loops nor multiple edges), but there are actually differences between graph theory and adjacency theory, as Galton underlines ([30], page 6). These differences are clear when one considers substructures of one and the other. We know that a subgraph is determined by a subset of nodes, as well as a subset of edges (with the proviso that if an edge is in the subgraph, then all the nodes it is incident with are in the subgraph as well). For example given a graph with two nodes and one edge between them, the subgraph made by the union of the two nodes and the empty set of edges, is different from the subgraph made by the two node plus the edge as well. Once we determine which subset of nodes is in the subgraph we want to consider, we still need to determine which subset of edges, between these nodes, we want to consider. Different subgraphs can share the same set of nodes. In an adjacency space the story is different: its substructures are automatically determined by its subsets of nodes. In an adjacency space the edges of the graph are reduced to be just the elements of the symmetric relation A. They do not figure as spatial elements of the domain, and thus they cannot carry any additional information, beside the fact that they relate certain pairs of nodes. There's no difference between the subgraph made by the two nodes, and the subgraph made by the same nodes *and* the edge between them.

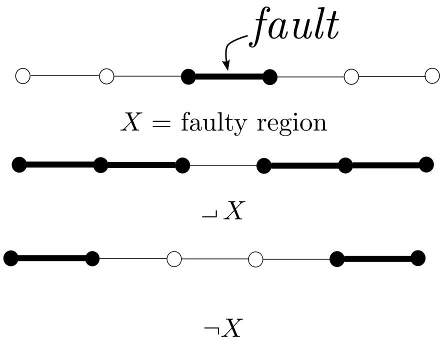

Figure 1.5: Example of a rail network with a fault occurring on one edge. If the faulty region of the network is X, the working part of the network could be $\lrcorner X$ or $\neg X$.

However, being able to distinguish between subgraphs that share the same set of nodes, but have a different set of edges, and thus giving importance to edges as well, is relevant for some purpose. For example, if we have a transport network, represented by a graph, we might have a fault occurring on some links between some stations (for example a fault on the rail tracks). The information about the fault is assigned to, say, an edge where the fault occurs (see Figure 1.5). Then we might want to single out the "working" part of the network, that is the complement of the faulty edge, thus the whole graph *without* this edge. The whole network is distinguished by the working part of the network by a single edge, having the two regions exactly the same set of nodes. On the other hand the "faulty" subregion of the network, i.e. the region of the network affected by the fault, could be the edge where the fault is plus the pair of nodes it connects, as these nodes, i.e. the stations, are affected by the fault: travelling between these two stations is not possible. This region is different from the union of the two nodes alone, as no fault occurs directly in either of the two nodes (it is still possible to travel from and to these nodes) but precisely on the edge between them. Notice that if X is the faulty region of the network, then we can distinguish between two type of complement-subgraphs of X: $(-X) \oplus H$ and $H \ominus (-X)$, where $-X$ is the set-theoretical complement of X. Notice that $-X$ is not a subgraph itself as it contains a "naked" edge. We are going to see that these two operations on the complement of a subgraph give rise to the two negations in **BiSKt**: the dual pseudo-complement $\lrcorner X$, also called, co-negation, and the pseudo-complement $\neg X$ respectively. The working part of the network could be $\lrcorner X$, as it includes all the working links, and the stations these links lead to. We have that $X \cap \lrcorner X \neq \varnothing$ as there are two nodes, that, is a sense, belong both to the faulty part of the network, as a fault occurs on one of the links they are incident with, and to the working part of it as well, as travelling from and to these nodes is still possible. On the other hand, if we want to consider as the working part of the network the subgraph that is totally "unaffected" by the fault, then we use $\neg X$. This includes indeed exactly all the stations that are not linked to any of the faulty links. We have that $X \cap \neg X = \varnothing$, and in this sense $\neg X$ is totally unaffected by the fault. Notice that these two types of complements are possible exactly because we are able to consider edges as elements of the domain, as, in this simple case, $\lrcorner$ can be seen as the complement w.r.t. the edges (take the edges in $-X$ and then complete with the nodes between them) and $\neg$ can be seen as the complement w.r.t. the nodes (take the nodes in $-X$ and then complete with the edges between them). Thus we have demonstrated the importance of edges in a discrete spatial theory, and the difference between adjacency spaces and graphs. As Galton says [28, page 93], adjacency spaces are a special case of closure spaces, a notion that we will introduce in Section 3.2, and, as we will see, any model of the logic we use **UBiSKt**, can be associated to a closure space.

The fact that edges are important in defining a graph, and might carry additional information about it, is one of the main motivations of a strand of works, aimed to generalise the theory of mathematical morphology for image processing beyond sets, and precisely to graphs. This strand was initiated by Hejmans and Vincent in [86] and [36], and had increasing attention in more recent times with the work of Najman et al. [48], Cousty et al. [14], and Najman et al. [47]. As Cousty et al. [14] say, although usually in image processing images are interpreted as subsets of $\mathbb{Z}^2$, and thus the space is essentially made of adjacent pixels, there is a growing interest in considering digital objects not only composed of points, but also composed of elements lying between points, i.e. edges, that might carry additional information about how the points are glued together. Najmans [47] also names an interesting extension of the study of mathematical morphology on graphs: extension of this to hypergraphs. Indeed

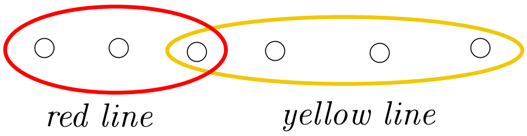

Figure 1.6: Representation of an underground network as a hypergraph: the two hyperedges are the red line and the yellow line. They connect to each other more than a pair of nodes (stations on the lines).

with hypergraphs it is possible to generalise the idea of edge-connection beyond pairs of nodes: we could have three nodes all incident with the same hyperedge, and thus all connected by the same edges. For example in an underground network, we might want to represent several stations belonging to the same line. In this case we need not just edges, but hyperedges, and thus not graphs, but hypergraphs. An example of this is given in Figure 1.6.

Another type of generalisation of graph that the idea of adjacency space cannot capture, is the one of a multi-graph, i.e. when we have possible multiple edges between the same pair of nodes. These multiple connections could be important, for example, to model two cities connected by two different roads, or two different types of transportation, or two airports connected by flights operated by different airlines. Also, a multi-graph allows the presence of self-loops, i.e. when an edge is incident with a single node. This sort of graph could represent, at another level of detail, a network that is self connected, i.e. any pair of nodes is connected by a path in the graph.

Within our approach to incidence spaces of nodes and edges as posets (U, H), we can represent all the above mentioned structures: undirected graphs, undirected multi-graphs, and hypergraphs. Thus the encoding of the spatial relations within **BiSKt** is suitable for all these types of discrete spaces, and not just for adjacency spaces. Finally, as we are going to see in Chapter 6, the structure of a partial order, by which we represent graphs and hypergraphs, can extend to incidency structures with more that two layers (not just nodes and edges): we talk, in this sense of simplicial complexes [17], and simplicial sets [51]. With these constructions we can represent not only 0 and 1-dimensional objects as nodes and edges, but also 2 dimensional cells, and 3 dimensional volumes. Even though we do not explore this direction, it is worth mentioning it as direction for future work in modelling spatial relations in a discrete setting.

1.7 Granularity

As stated in [21], every observation is subject to imprecision. *Granularity* is related to imprecision, as it refers to the presence of grains or clusters in the information: individual elements within the same grain cannot be distinguished. *Granulation* is the process of going from the information at a more detailed level to a coarser level, i.e. the result of distinct entities becoming indistinguishable. The representation of information at a coarser granularity offers *fewer details*, as things that were distinguishable from each other earlier, are no longer such. This is why we can refer to a change in granularity as a change in the level of detail. The study of change in granularity with respect to certain structures is a central topic of this work. However, it is important to stress that change in granularity is not the only type of change in the level of detail that one might adopt on some information. Stell and Worboys [82] consider another type of this change, namely a *selection* process. This consists of forgetting details of a representation. A less detailed view can be adopted by selecting some information we want to focus on, and forgetting about the rest[3]. As we are interested in graphs, in their general form of multigraphs and hypergraphs, and as we represent them as posets, we will study granularity of objects that can be seen as posets. We will explore the theme of hypergraph and poset granularity in Chapters 4 and 5.

As Galton [28] suggests, granularity is a property of the representation instead of a property of the data itself. Viewing a situation in a less detailed way is a commonplace. Here's an example. If we look at a bowl full of sugar cubes from a very close distance, we can distinguish each grain of each sugar cube. At a further distance, we distinguish the single cubes from each other, but we cannot see (or we are not interested in seeing) the sugar grains. At an even further distance we can see that the bowl contains sugar, but we cannot distinguish how the sugar is composed. This is an example of taking a gradually coarser view on some information, thus an example of granulation. The same process is known in cartography as *generalisation*: as Kraak and Ormeling [40] say (cited by Galton [28]), generalisation is "the process of reducing the amount of details in a map in a meaningful way". For example, we could have maps where the data is represented with a great amount of detail, so considering all the roads and buildings in a city, for every city of the region at issue, or we could just collapse all the details of a city into a single block, and look at each city as a unique element, a singleton of our representation, where no further information about what is within a city is revealed.

Mathematical morphology offers an approach to images where it is possible to parametrise the level of detail, thanks to the structuring element. It is indeed common to talk about the structuring element as a *probe* [65], through which the image processed. When zooming out from an image (seen as a subset of a set of pixels) we intuitively expect narrow spikes to fuse and narrow cracks and holes to disappear. This intuitive expectation can be formalised in mathematical morphology. The idea is that, instead of being able to see individual pixels, only certain groups of pixels can be seen. These groups of pixels are copies of the structuring element. Given a structuring element and the associated relation R on the set of pixels (we will see in Section 4.1 how a structuring element generates a relation on the pixel grid), and a subset $X \subseteq U$, the following operations are defined using dilation and erosion by R: the opening of X that is $(R \ominus X) \oplus R$, and the closing of X that is $R \ominus (X \oplus R)$. The opening of X consists of the subset formed by fitting copies of the structuring element within X. Only the subset of pixels of X that can be described the structuring element will be included in the opening, see Figure 1.7. The closing of X consists in overlapping copies of the structuring element (rotated by half a turn) wholly outside X, and then taking the complement of this. Thus, it will have the effect of filling the holes that are small enough that the rotated structuring element

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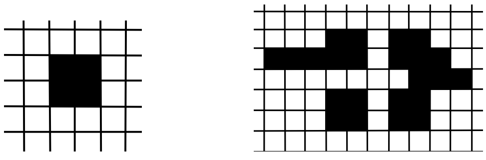
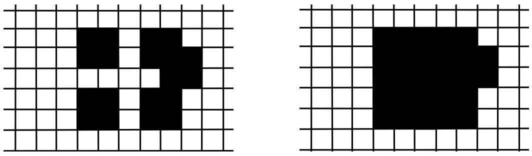
[3] Stell and Worboys [82] also consider the topic of granularity, and they call *amalgamation* what here we referred to as granulation.

doesn't fit in them (Figure 1.7). Combining these two operations, we can obtain a coarser view on the starting image. Compositions of closing and openings are known in mathematical morphology as filters [65]. The idea in Figure 1.7 is that, performing an opening and then a closing on X, instead of being able to see every single pixel, we are able to see only groups of four pixels arranged as in the structuring element (obviously, using a different structuring element, a different effect on the same image would have been obtained). Thus the composition of these two operations gives an intuitive way to visualise an image (represented by a subset of the grid of pixels) in a coarser way. We will analyse this idea in more detail in Section 4.1.



Structuring element   a subset of pixels X



opening of X closing of opening of X

Figure 1.7: Granulation of a subset X of $\mathbb{Z}^2$ by a 2×2 structuring element. The application of the opening to X will make the narrow spikes present in X disappear. The closing will fill the holes present in X (figure adapted from [71]).

In Section 4.1 we will explain in detail how the opening and the closing operations on subgraphs provide with a coarser description on them. For now it suffices to say that when we move to graphs, and to mathematical morphology on graphs, the idea of a structuring element can be generalised to the concept of an *R-dilate*, i.e. any $\{\, u \,\} \oplus R$ for any $u \in U$. In this case, the idea of the opening of a subgraph as fitting copies of R-dilates within the subgraph, remains meaningful. R-dilates can be seen as the new atomic parts of a space coarser than the starting graph. Notice that when we consider the partial order H, the H-dilates are exactly the minimal subgraphs a graph is composed of. Indeed an element of the set of all $\{\, u \,\} \oplus H$ for any $u \in U$ will be either a singleton of a node, i.e. $\{\, of \,\}$ if u is a node, or the set formed by an edge with all the nodes incident with it, if u is an edge. Thus it makes sense the idea that, at the very detailed level, the atomic parts of a graph, or hypergraph, are its H-dilates, i.e. its minimal subgraphs. In Chapter 4 we put forward some ideas for a notion of granulation of a subgraph, as the opening followed by the closing (as in Figure 1.7), or the closing followed by the opening. There's no definite answer for which sequence of morphological operations, i.e. which sequence of modalities, is the "correct" notion of granulation for subgraphs. This very much depends on the application, and on the properties of the structuring element and thus of the relation R that we wish to consider (reflexivity, symmetry and so on). Then we study spatial relations under granulation, that is how we should express for example that two granular subgraphs are connected, or disconnected, or one is part of the other and so on. Indeed, when we zoom out on subgraphs, we consider also the underlying space being made of R-dilates, and no longer H-dilates. Then we might find out that two regions that are disconnected at the detailed level become connected at another level of detail. Indeed, as noticed by Galton [29], who also studies spatial relations under granulation, some spatial attributes are sensitive to granulation. The information about the attributes holding or not depends on the level of detail. Information about certain spatial attributes and relations can be gained, or lost, whenever we change the level of detail. Thus, being able to define spatial relations for subgraphs both at a detailed level , and at "granular" level, will give us finer ways to describe space.

As with mathematical morphology, also rough set theory, first described by Pawlak [54], can be seen as an approach to information at different levels of detail. This theory has indeed been developed to deal with imprecision and vagueness in information. Also the framework of rough set theory is essentially based on a set, and a relation R imposed on that set. But differently from mathematical morphology, where no properties are imposed a priori on the structuring element and thus on the relation, rough set theory usually considers *equivalence* relations, thus reflexive, transitive and symmetric relations.

We might call the kind of granularity that rough set theory deals with *conceptual* granularity. The basic framework of rough set theory is a set with a partition, and thus with an equivalence relation. These are obtained usually as follows: a set of attributes on the elements of the set is chosen. Each attribute hold with a certain value for each element[4]. Whenever two elements share the same values for all the attributes considered, they are regarded as indistinguishable, and they are grouped within the same "granule" of information. It is clear that the relation between pairs of elements in the same granule is an equivalence relation, and also gives a partition of the set. Looking at the set of granules formed in this way, instead of looking at the initial set elements, we obtain a coarser view of the initial set. As a simple example, suppose we have a database of living things. We can classify them as mammals, birds, fishes, insects, and so on. We can then consider a new set made of these *categories* of living things. This provides a coarser description of the original database. The coarser description of the initial set, formed in this way is usually referred to as the *quotient structure*.

A central part of rough set theory is concerned with the idea of a rough set, as opposed to a crisp set. A crisp set is a subset $X \subseteq U$ in the classical sense. When an equivalence relation, and thus a partition, is imposed on U, we can build two kinds of approximations for any subset $X \subseteq U$: the lower approximation, $\underline{X}$ and the upper approximation $\overline{X}$. Rough sets arise from those $X \subseteq U$ that are not (union of collections of) blocks of the partition. Indeed these subsets contain elements that are indistinguishable from other elements that are not in X, on the basis of the equivalence relation. See Figure 1.8 for an example of a subset with its lower and upper approximations.

It is well known that rough set theory has connections with the modal logic

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[4] Usually we have "true or false" values, for qualitative attributes as color, shape as so on, but we can also have degree values as "high, medium, low" if the attributes concerns some measurable quantity, as for example the temperature of a body.

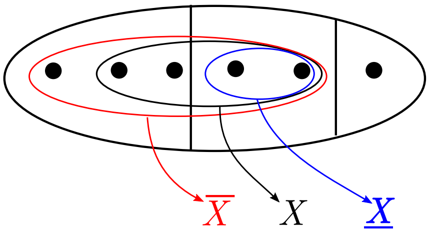
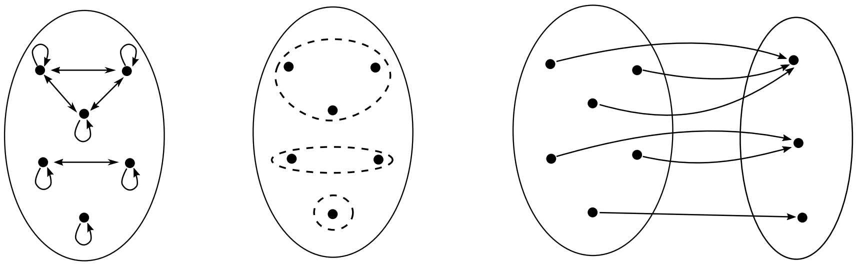


Figure 1.8: A set U and a subset X with its lower and upper approximations, $\underline{X}$ and $\overline{X}$, generated by the given partition. X contains some imprecision, or vagueness under the imposed partition, as it doesn't cover full blocks. Thus, at the coarser level, we can't "see" X, and we have two ways of building collections of blocks out of X, i.e. approximating X: completing with the missing information, $\overline{X}$, or forgetting the incomplete information, $\underline{X}$.

S5 [53], [92]. Indeed **S5** Kripke-frames are of the form (U, R) where R is an equivalence relation. We have that **S5**-$\Box$ is associated to $X \mapsto \underline{X}$ and **S5**-$\blacklozenge$ is associated to $X \mapsto \overline{X}$ (and, by symmetry of R, **S5**-$\Box$ has the same semantics of **S5**-$\blacksquare$, and analogously for **S5**-$\blacklozenge$ and **S5**-$\Diamond$).



Equivalence Relation *Partition* *Surjective quotient function*

Figure 1.9: Three equivalent ways to produce a coarser view on a set: an equivalence relation, the associated partition, and the surjective function, called the quotient function, that maps each element into its block (figure adapted from [75]).

There are three distinct forms of conceptual granulation for a set. We have already mentioned equivalence relations and partitions, but we also have surjective functions, Figure 1.9. With an equivalence relation we compare pairs of elements and we say whether they are indistinguishable under certain attributes. With a partition, we decompose the whole domain into disjoint blocks, or granules, i.e. the equivalence classes, and elements within the same block are indistinguishable. With a surjective function from the set to its quotient structure we categorise the elements of the domain. These three ways of granulation can be distinguished conceptually but they are mathematically equivalent [75]. However, it is through the surjective function, that we call the *quotient function*, that the relationship between the two levels of detail becomes evident.

1.8 Graph and Hypergraph Granularity

The approach to conceptual granularity with equivalence relations is appropriate if the kind of structures we are interested in can be seen simply as a collection of data, as the example of the database of living things given earlier. However, what if the set we are looking at carries some additional structure? This is the case of graphs and hypergraphs, where the underlying set is made of two types of objects, nodes and edges, and they are related by a partial order, i.e. the (reflexive closure of the) incidence relation from edges to nodes. What kind of relation on hypergraphs is associated to partitions on hypergraphs, and to their quotient structures? What properties does this relation have, and what properties does the associated quotient function, linking the hypergraph to its quotient structure, have? We are going to explore these questions in Chapter 5. First we motivate the importance of conceptual granularity for graphs and hypergraphs.

Looking at graphs and hypergraphs at different levels of detail is common place. Stell and Worboys [82] give many examples of looking at a transport network at different levels of detail. Two different users of a transport network might need to adopt different views on it. An engineer needs very detailed information about all the routes of a network, in order to maintain it and fix issues. But a passenger needs a much coarser view, that is its start point, its end point, and the fact that there is some route between them. If a user has to travel between two stations, they are not interested in all the stops occurring between them, but just on the fact that the two stations are connected via some link. Their point of view can be represented at a coarser level, collapsing several stations and links into a single path, see Figure 1.10 (left) for an example of this. Another example could be a granulation on a transport network or a map where we want to distinguish its north region from the south. In this case it is possible that different nodes and edges will be clustered into a single node, i.e. the region with a certain attribute, see Figure 1.10 (right). We might also want to cluster nodes, and the edges between them, depending on how close they are to each other. Nodes that are within a certain distance will be merged together into a single node, and so the edges between these nodes, see Figure 1.11. This "node-merging" action is widely used within elastic fibre network modelling [35], [38], where nodes are cross-links between the fibres, and thus a netweork of fibres generated a undirected graph.

These examples have demonstrated that nodes and edges might be clustered into a single block, and this block might play the role of either an edge (first example) or a node (second and third example) in the quotient structure, in the sense that the block a node belongs to might be an edge in the quotient structure, and vice-versa the block in which an edge is put, might be a node. Notice that it is also possible that a coarser view on a graph gives rise to a quotient structure that is a hypergraph, as in Figure 1.12.

There are many other works that use graphs and hypergraphs at different levels of detail for practical applications. We mention some of them: [46], [82], [84], [20], [83], [81]. Here graphs and hypergraphs are considered at multiple levels of detail, in contexts like way-finding, pedestrian and car navigation, geographical information systems, and ordered information systems. The theory of graph partition seems to be relevant also for ontologies [4]. An ontology is not

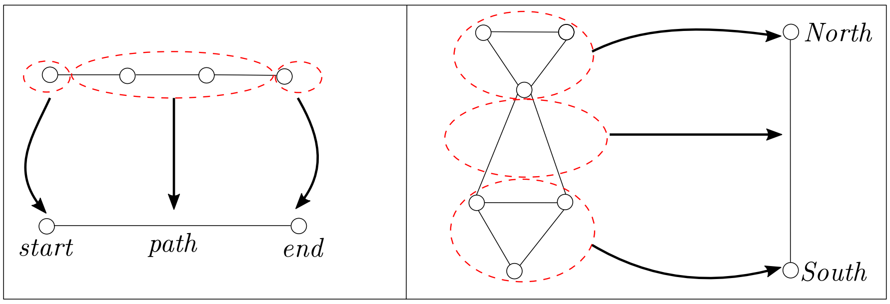


Figure 1.10: Two examples of graph granulation generated by a partition. A coarser view on the graph is obtained by collapsing different elements together. In the first case several nodes and edges are clustered into a single edge, representing the path between a start point and an end point. The start and end points are still distinguishable as single elements in the quotient structure, i.e. they do not merge with anything else. In the second case, nodes and edges get clustered together into nodes, representing the north region and the south region respectively. Also the two edges between the two regions collapse into a single edge, in this example, representing the simpler fact that the two regions are connected, instead of considering connection by each single edge.

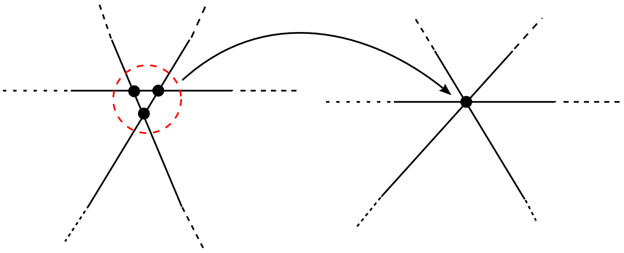


Figure 1.11: Another example of graph granulation by merging elements together, used in modelling large networks of fibres. The network is naturally represented by a graph, where the edges compose the fibres and a nodes represent the cross-links between them. In this context it is typical to merge tuples of nodes, and the edges between them, into a single node, whenever their reciprocal distance is smaller than a certain parameter. The tuple of nodes and edges in the same block will be seen as a single node.

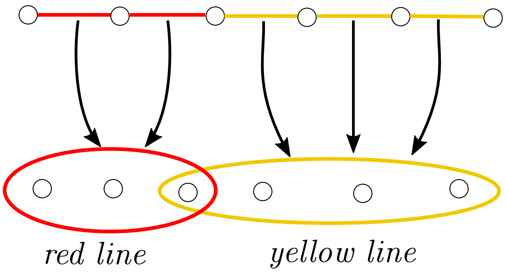


Figure 1.12: Hyper-edges might arise when moving between two levels of detail. In this example we have an underground network, and we merge together two edges whenever they belong to the same line, and thus we obtain a hypergraph quotient structure. For simplicity we don't draw the blocks of the partition in this case, but it is clear that edges are in the same block whenever they have the same colour. Each node goes in a block containing just the node itself.

just a set of data: ontologies are often represented as *knowledge graphs* [5] i.e. some data connected by certain relationships. Also in this case there is a structure more complex than a set. As we have already mentioned, generalisation of incidence structures as graphs and hypergraphs are simplicial complexes [48]. Thus, giving foundation to a theory of graph and hypergraph partition, based on the sole assumption that they are posets, as we will to do in Chapter 5, we can build a foundation of a theory of partition of objects that can be represented as posets. We will briefly look at simplicial complexes in Section6.1.1.

From the examples above, one thing is evident: when we look at a graph (or hypergraph) at another level of detail, we still expect to see an incidence structure. In all the previous examples we have gone from a graph to another graph or to a hypergraph. The quotient structure is made of blocks that are represented as edges and nodes, and they are incident in a certain way with each other. We know we can model this as a partial order. Thus we can say that the quotient structure of a poset (U, H) is of the form (U', H') where U' is the set of blocks, that are certain subsets of U, and H' is a partial order on U'. The quotient structure of a poset is a poset. The fact that the same type of structures (posets) are present at both levels can also be seen as an extension of the intuition that, when a partition is applied to a set, the resulting quotient structure is still a set.

Now that we have motivated the importance of granularity of objects like graphs and hypergraphs, and we have seen that a coarser view on a poset should give a poset as well, we can go back to the original question asked at the beginning of this section. What kind of relations on graphs and hypergraphs, or more generally on posets, are associated to a partition of the underlying set, and to their quotient structures? What kind of properties does the associated quotient function, linking (U, H) to (U', H'), have?

For the reasons discussed in Sections 1.3 and 1.4, we work with stable relations. Thus an obvious idea would be to consider stable relations that are

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[5] In this case the term *graph* is used in the sense of directed graph, as an ontology can be represented as data linked by certain relations.

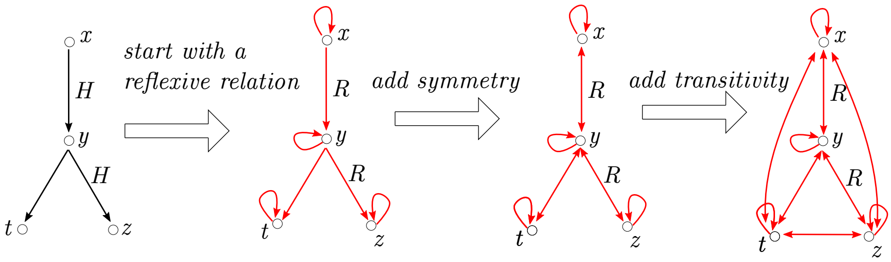

Figure 1.13: A poset (U, H) with a single component. If we require a stable relation $R \subseteq U \times U$ that is reflexive, symmetric and transitive, we end up with the universal relation $U \times U$. Notice that when R is stable, imposing reflexivity, i.e. $I \subseteq R$, is equivalent to impose $H \subseteq R$. We are going to see this in details in Chapter 5.

additionally reflexive, transitive and symmetric, i.e. equivalence relations on hypergraphs. These relations have the advantage that they are just ordinary equivalence relations that are additionally stable. After all, in the set case, it is precisely equivalence relations that correspond to partitions. However, as shown by Shaheen and Stell [66] if we consider these relations as defining partitions on hypergraphs, we will have quite a limited range of relations to choose from. As they show ([66], Theorem 6), in a hypergraph with only one connected component (i.e. every node is reachable by a path from any other node), there is only one relation with those properties: the universal relation $U \times U$. This result doesn't hold just for hypergraphs, i.e. two-levels posets, but for posets in general, where the idea of poset with a single connected component is that (U, H) is made of "one piece". In Figure 1.13 we can see that if a stable relation R is reflexive, transitive and symmetric, then it will be the relation $U \times U$. Thus, the only possible partition arising from this would be the one where everything collapses into one block, $U' = \{U\}$. It is clear that this is very restrictive, and it wouldn't allow any of the partitions we have seen earlier in Figures 1.10, 1.11 and 1.12. These partitions and quotient structures must then come from another relation, somewhat weaker than a standard equivalence on posets.

Clearly, things are not so simple when from a set U we move to the more general case of a poset (U, H). We can already see that parallelism between classical logic (framework is a set, and operations are applied to subsets) and intutionistic logic (framework is a poset, and operations are applied to subsets closed under H), and, as we are talking about an additional relation R on these structures, the parallelism is actually between classical modal logic frames (U, R), and intuitionistic modal logic frames (U, H, R). The classical Kripke-frame (U, R) where R is an equivalence relation is the base of the semantics for the system **S5**. So the question is: is there an intuitionistic version of **S5**, whose semantics is therefore based on a bi-relational frame (U, H, R)? What properties does R have here? This system, and its semantics, could help us to answer to the questions asked before. The answer is yes, there is an intuitionistic version of **S5**. But it turns out there is not just one, there are many. In just one work, [52], Ono mentions several possiblities for the intuitionistic analogous of classical **S5**. Different sets of axioms, i.e. different logics, are considered as intuitionistic equivalent of **S5**. The axioms correspond, by correspondence theorem, to different constraints on R. We are going to compare these constraints, and thus the axioms nd the logics, under the light of a specific application: R gives rise to a partition on (U, H) and to its quotient structure, i.e. to a coarser view on the inital poset. As far as we are aware, no such a thing has been done in the literature before. Intuitionistic modal logic **S5** has never been seen as the logic describing poset partitions and their quotient structures, even if the connection between partitions on sets and equivalence relations is a basic part of maths, and their connection to classical modal logic **S5** is well known [53], [92].

As we will see in Chapter 5, Section 5.3, reflexive and transitive relations on (U, H) already give an account of partitions on the underlying set U. These relations are also used in [81] to give partitions of ordered information systems, even if in a different way from us. Also other works on intuitionistic modal logic start from reflexive and transitive relations, i.e. from **S4** intuitionistic frames, and then build **S5** as an extension of this (see [52] and [18]). The other reason we keep reflexivity and transitivity, beside the fact that the same approach has been taken in the literature is that from reflexive and transitive relations we can define a partition of (U, H) and we can prove that the related quotient structure (U', H') is also a poset (Theorem 108). Then we ask the question whether any additional constraint on R is needed. We are going to look at some examples (Section 5.4) where a reflexive and transitive relation alone might not give a satisfactory account of hypergraphs partitions and their quotient structures. Thus we discuss additional constraints substituting symmetry, to impose on a stable preorder R. We talk about these constraints as "weaker forms of symmetry" as, when $H = I$ and thus we are in a classical modal logic context, all these constraints on R that substitute symmetry of R, are actually equivalent to symmetry. Thus in the case of $H = I$ all these intutionistic analogues of **S**5 collapse to classical **S**5, as expected. We consider mainly three forms of "weak" symmetry found in the literature: $R \subseteq H ; \overset{\leftrightarrow}{R}$, $R \subseteq \overset{\leftrightarrow}{R} ; H$, and $R \subseteq \overset{\leftrightarrow}{R} ; H ; \overset{\leftrightarrow}{R}$, where $\overset{\leftrightarrow}{R} = R \cap \breve{R}$, i.e. the symmetric part of R. The first two constraints are considered in Ono [52], and the first one is also considered by Dosën [18] in intuitionistic version of **S5** proposed there. The third constraint is introduced in [66] and it is known as *symmetry-generation.* Also their aim is generating partitions on hypergraphs by a stable relation. However the logic of an intuitionistic frame (U, H, R) where R is reflexive, transitive and symmetrically-generated is not investigated there, and symmetry-generation constraint is not compared with other similar constraints appearing in the literature, as the ones already mentioned from [52] and [18], and from [52]. We do both of these things in the present work. Moreover we investigate what all these weaker forms of symmetry mean in terms of the quotient function, and what effects each of them will have on the type of partition and quotient structure that can be generated after imposing them. Indeed, as we will see in Section 5.4, each of these constraints on R arises from imposing a particular condition on the quotient function, linking the starting poset (U, H) to its quotient structure (U', H'). These conditions on the quotient function, in turn, express a "dependency" of the resulting partial order H' from the original partial order H.

We will see that the two constraints proposed by Ono [52], might be too restrictive in the kind of partitions they allow. The first one, $R \subseteq H ; \overset{\leftrightarrow}{R}$, implies that a partition and a quotient structure like the one in Figure 1.10 (left) is not allowed. Indeed in this granulation we have nodes that get clustered into an element that is an edge in the quotient structure. We are going to see that this is not allowed under the above mentioned constraint (Proposition 112). On the other hand, the other constraint considered by Ono, $R \subseteq \overleftrightarrow{R};H$, will disallow the partitions in Figure 1.10 (right) and Figure 1.11, as there we have an edge that gets clustered into a block that plays the role of a node. We will see (Proposition 115) that this is not possible under the R-constraint at issue. Being able to connect these constraints on R to certain quotient function properties enabled us to see the potential restrictions with the constraints proposed by Ono. We will also see symmetry-generation constraint is equivalent to imposing a desirable property of the quotient function linking (U, H) and (U', H') (Theorem 117). This property doesn't imply any restriction on the type of elements of the quotient structure (node or edge) that nodes and edges of a hypergraph can get mapped to. Thus we will settle on stable relations that are reflexive, transitive and symmetrically-generated, as corresponding to partitions on posets and generating their quotient structures. All the partitions given in Figures 1.10, 1.11 and 1.12 arise from a relation on the graphs that has the three above mentioned properties. We will finally show that symmetry-generation corresponds to an axiom, in the sense of correspondence theorem for modal logic, expressible in the **BiSKt** logic. Thus a new intuitionistic **S5** logic, developed with the aim of a theory for hypergraphs partitions, or more in general poset partitions, is obtained.

With this work, we hope to shed light on the intricate question of what is the intuitionistic analogous of the modal logic **S5**. As mentioned above, other works suggest axioms substituting symmetry but they don't say why an axiom might be more appropriate than another one in certain situations. We evaluate **S5** axioms and corresponding constraints on R under the light of a theory of poset partition.

Chapter 2

The logic UBiSKt

> In this chapter we will introduce the logic **UBiSKt** and its proof-systems, a Hilbert-style proof system and an equivalent tableau proof system. We will also show some correspondence results concerning this logic, such as the definability of the notions of hypergraph and graph within the logic.

2.1 Graphs and Hypergraphs as Posets

The notions of graphs and hypergraphs are central concepts in this work, so let us introduce them.

Definition 8. An *edge-node hypergraph* is a triple (E, N, i). E and N are disjoint sets, E is called the set of edges, N is called the set of nodes, and $i : E \to \mathcal{P}(N)$ is a function where $\mathcal{P}(N)$ denotes the *power-set* of N. For each $e \in E$ we have that $i(e) \neq \varnothing$.

An alternative notion of hypergraph is definable, with a set U being the union set of the edges set and the nodes set, and a relation expressing the edge-node incidence:

Definition 9. A *hypergraph* (U, H) consists of a set U and a reflexive relation $H \subseteq U \times U$ such that for all $u, v, w \in U$, uHv and vHw implies $v = u$ or $w = v$. Given $u \in U$, u is an *edge* if there is some $v \in U$ such that uHv and $u \neq v$. An element $u \in U$ that is not an edge, is called a *node*.

It is clear that the relation H is transitive and anti-symmetric, thus it is a partial-order. The following result is shown in [66]:

Proposition 10. There is a bijective correspondence between edge-node hypergraphs in the sense of Definition 8 and hypergraphs in the sense of Definition 9.

Proposition 10 states that any edge-node hypergraph uniquely corresponds to a poset (U, H) where U is the set of edges and nodes together and H describes the edge-node incidence. Let us briefly explain how we can go from one construction to the other one, and vice-versa. Let $K = (E, N, i)$ be an edge-node hypergraph. We construct a hypergraph (U_K, H_K) as follows: $U_K = E \cup N$ and given two elements $u, v \in U_K$, the relation $u \; H_K \; v$ holds iff $u = v$ or $u \in E$ and $v \in N$ and $v \in i(u)$. We need to check that $u \; H_K \; v$ and $v \; H_K \; w$ implies that $u = v$ or $v = w$, so that (U_K, H_K) satisfies Definition 9. Suppose that (u, v) and (v, w) are both in H_K and that $u \neq v$. Then $u \in E$ and $v \in N$ and $v \in i(u)$ by our definition of H_K. Now if also $v \neq w$ holds, we have that $v \in E$, but this is impossible as E and N are disjoint sets. Hence we conclude that $v = w$. In the opposite direction, let $G = (U, H)$ be a hypergraph. We can construct an edge-node hypergraph (E_G, N_G, i_G) as follows. E_G is the set of all edges of G, so $E_G = \{\, u \in U \mid \exists v (u \; H \; v \text{ and } u \neq v) \,\}$, and similarly N_G is the set of all nodes of G. Then we can define the function $i_G : E_G \to \mathcal{P}(N_G)$ by: $i_G(u) = \{\, v \in U \mid u \; H \; v \text{ and } u \neq v \,\}$. We can easily check that no node in G is also an edge, by Definition 9, thus the sets E_G and N_G are disjoint, and $i_G(u)$ is always non-empty for any edge u. Thus (E_G, N_G, i_G) satisfies Definition 8, and it is an edge-node hypergraph. Finally it is possible to check that the constructions are inverse of each other, namely that if from $G = (U, H)$ we construct $K = (E_G, N_G, i_G)$, then $(U_K, H_K) = G$, and if from $K = (E, N, i)$ we construct $G = (U_K, H_K)$, then $(E_G, N_G, i_G) = K$. For a detailed proof of this fact, we refer the reader to [66] (p. 79-80).

A graph is a special case of a hypergraph, where every edge is incident with at most two nodes. By specialising Definition 9, we can define the notion of graph as follows:

Definition 11. A graph is a hypergraph (U, H) where for all $u \in U$, the set $\{v \in U \mid u \; H \; v \text{ and } u \neq v\}$ has at most cardinality 2.

As the two notions of hypergraph from Definitions 8 and 9 are equivalent, the above definition of graph is equivalent to the following one, more common in the literature.

Definition 12. An edge-node graph (E, N, i) is an edge-node hypergraph where for all $e \in E$, $1 \leq |i(e)| \leq 2$

A graph in this sense is undirected and might have multiple edges between the same pair of nodes as well as loops on nodes. These structures are also known as multi-graphs, or pseudo-graphs [34], but we will refer to them simply as graphs. In Fig. 2.1 we give an example of an edge-node graph, its representation as edge-node hypergraph, and the associated poset (U, H).

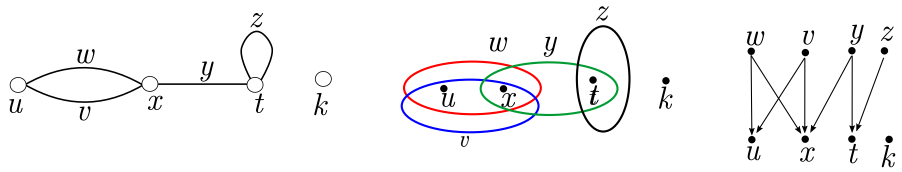

Figure 2.1: The edge-node graph on the left has four nodes, u, x, t, k, and four edges w, v, y, z. It can also be represented as an edge-node hypergraph, shown in the middle. The corresponding poset for this graph is the reflexive closure of the relation on the set $U = \{u, x, t, k, w, v, y, z\}$, shown on the right hand side (figure adapted from [72]).

A subgraph of a hypergraph as in Definition 9, and thus of a graph, can be seen a subset of U such that if an edge is present, then all its end-points are present as well. This idea can be formalised by the following definition.

Definition 13. Given a hypergraph (U, H) and a set $K \subseteq U$, K is a *subgraph* if $u \in K$ and uHv jointly imply $v \in K$, for any $u, v \in U$.

2.2 Kripke Semantics for UBiSKt

The logic **UBiSKt** is a bi-intuitionistic modal logic with universal modalities, that we introduced in [73] and we have also studied in [71]. It is an extension of the logic **BiSKt**, [74], [62]. The name of the logic stands for bi-intuitionistic stable tense logic, and the letter "K" indicates the basic normal modal system K. The semantics of **UBiSKt** is given by a set U with a preorder relation[1] $H \subseteq U \times U$. Modal operators in **UBiSKt** are interpreted with respect to a stable relation $R \subseteq U \times U$, for the reason discussed in Section 1.4. The universal modalities A and E are interpreted with respect to the universal relation $U \times U$, that is trivially stable.

Let Prop be a countable set of propositional variables. Our syntax $\mathcal{L}$ for **UBiSKt** consists of all logical connectives of bi-intuitionistic logic, i.e., two constant symbols $\bot$ and $\top$, disjunction $\vee$, conjunction $\wedge$, implication $\rightarrow$, coimplication $\prec$, and a finite set $\{\blacklozenge, \Box, \mathsf{A}, \mathsf{E}\}$ of modal operators. The set $\mathsf{Form}_{\mathcal{L}}$ of all well formed formulas in $\mathcal{L}$ is defined inductively as follows:

$$\varphi ::= \top \mid \bot \mid p \mid \varphi \wedge \varphi \mid \varphi \vee \varphi \mid \varphi \rightarrow \varphi \mid \varphi \prec \varphi \mid \blacklozenge \varphi \mid \Box\varphi \mid \mathsf{E}\,\varphi \mid \mathsf{A}\,\varphi \quad (p \in \mathsf{Prop}).$$

We define the following abbreviations (see [74]):

$$\neg\varphi := \varphi \rightarrow \bot, \quad \lrcorner\,\varphi := \top \prec \varphi, \quad \varphi \leftrightarrow \psi := (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi),$$
$$\Diamond\,\varphi := \lrcorner\,\Box\neg\varphi, \quad \blacksquare\,\varphi := \neg\blacklozenge\lrcorner\,\varphi.$$

Definition 14. Let H be a preorder on a set U. We say that $X \subseteq U$ is an *H-set* if X is closed under H-successors, i.e., $u\ H\ v$ and $u \in X$ jointly imply $v \in X$ for all elements $u,\ v \in U$. Given a preorder (U, H), a binary relation $R \subseteq U \times U$ is *stable*, if it satisfies $H \,;\, R \,;\, H \subseteq R$.

Stability is displayed diagrammatically in Figure 2.2. It is easy to see that a relation R on U is stable, if and only if, $H \,;\, R \subseteq R$ and $R \,;\, H \subseteq R$. Indeed we have that $H ; R ; H \subseteq R ; H \subseteq R$ by application of $H ; R \subseteq R$ and $R ; H \subseteq R$, and $H ; R \subseteq H ; R ; I \subseteq H ; R ; H ; R$, by reflexivity of H $(I \subseteq H)$, and application of $H ; R ; H \subseteq R$. Similarly we can deduce $R ; H \subseteq R$ from stability. When (U, H) is a graph, or a hypergraph, the two parts of stability imply the following facts: $R;H \subseteq R$ implies that every time an element $u \in U$ (edge or node) is related by R to an edge $v \in U$, then u is related by R to all the nodes that v is H-incident with. $H \,;\, R \subseteq R$ implies that every time a node $u \in U$ is related to an element $v \in U$ (edge or node), then all the edges H-incident with u are R-related to v as well.

---

[1] The relation H was originally assumed to be a preorder in the first paper where **BiSKt** appeared, [74]. However, we will often talk about the frame (U, H) as a partially ordered set, poset for short, thus with H being a preorder that is additionally antisymmetric, because our main objects of interest are graphs and hypergraphs. We have already seen that they give rise to posets. Also generalisations of these, i.e. simplicial complexes, that we will briefly look at in Chapter 6, can be modelled as posets.

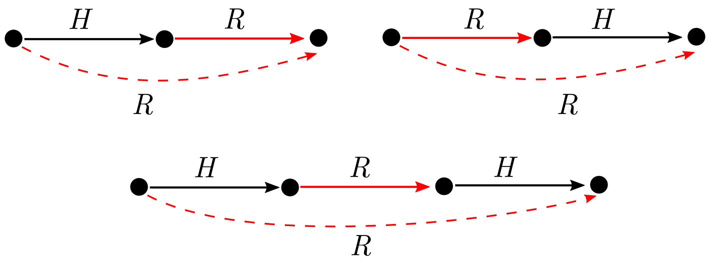

Figure 2.2: The inclusion conditions $H \,;\, R \subseteq R$ and $R \,;\, H \subseteq R$ represented by a diagram. Imposing both these conditions is equivalent to impose stability, i.e. $H \,;\, R \,;\, H \subseteq R$. The diagram has to be read as follows: whenever a pair of elements of U belong to the relational composition of the relations represented by thick arrows, then the pair belongs to the relation represented by the dashed arrow.

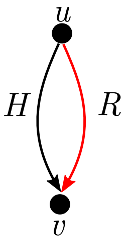

Figure 2.3: Example of a stable relation R such that its converse is not stable (reflexive loops of H are left implicit). We have that $R = H \,;\, R \,;\, H$ hence stability of R. However $(v, v) \in \breve{R} \,;\, H$ but $(v, v) \notin \breve{R}$. Thus $\breve{R} \,;\, H \not\subseteq \breve{R}$, and so $\breve{R}$ is not stable.

Even if R is a stable relation on U, its converse $\breve{R}$ may be not stable, as Figure 2.3 shows. Thus we introduce the concept of the *left converse* of a relation R, that is the smallest stable relation containing $\breve{R}$.

Definition 15 ([74])**.** The *left converse* $\circlearrowleft R$ of a stable relation R is $H \,;\, \breve{R} \,;\, H$.

Definition 16. We say that $F = (U, H, R)$ is an *H-frame* if U is a nonempty set, H is a preorder on U, and R is a *stable* binary relation on U. A *valuation* on an H-frame $F = (U, H, R)$ is a mapping V from Prop to the set of all H-sets on U. $M = (F, V)$ is an *H-model* if $F = (U, H, R)$ is an H-frame and V is a valuation. Given an H-model $M = (U, H, R, V)$, a state $u \in U$ and a formula φ, the satisfaction relation $M, u \models \varphi$ is defined inductively as follows:

$$\begin{array}{lcl}
M, u \models p & \iff & u \in V(p), \\
M, u \models \top, & & \\
M, u \not\models \bot, & & \\
M, u \models \varphi \vee \psi & \iff & M, u \models \varphi \text{ or } M, u \models \psi, \\
M, u \models \varphi \wedge \psi & \iff & M, u \models \varphi \text{ and } M, u \models \psi, \\
M, u \models \varphi \to \psi & \iff & \text{For all } v \in U \ ((uHv \text{ and } M, v \models \varphi) \text{ imply } M, v \models \psi), \\
M, u \models \varphi \mathrel{-\!\!<} \psi & \iff & \text{For some } v \in U \ (vHu \text{ and } M, v \models \varphi \text{ and } M, v \not\models \psi), \\
M, u \models \blacklozenge\, \varphi & \iff & \text{For some } v \in U \ (vRu \text{ and } M, v \models \varphi), \\
M, u \models \Box \varphi & \iff & \text{For all } v \in U \ (uRv \text{ implies } M, v \models \varphi), \\
M, u \models \mathsf{E}\, \varphi & \iff & \text{For some } v \in U \ (M, v \models \varphi), \\
M, u \models \mathsf{A}\, \varphi & \iff & \text{For all } v \in U \ (M, v \models \varphi).
\end{array}$$

The *truth set* $[\![\varphi]\!]_M$ of a formula φ in an H-model M is defined by $[\![\varphi]\!]_M := \{\, u \in U \mid M, u \models \varphi \,\}$. If the underlying model M in $[\![\varphi]\!]_M$ is clear from the context, we drop the subscript and simply write $[\![\varphi]\!]$. We say that φ is valid in M (in symbols $M \models \varphi$) when $M, u \models \varphi$ for all $u \in U$. If Γ is a set of formulas, $M \models \Gamma$ means that $M \models \gamma$ for all $\gamma \in \Gamma$.

Given any H-frame $F = (U, H, R)$, we say that a formula φ is *valid* in F (in symbols $F \models \varphi$) when $(F, V) \models \varphi$ for any valuation V and any $u \in U$.

As for the abbreviated symbols, we may derive the following satisfaction conditions:

$$\begin{array}{lcl}
M, u \models \neg \varphi & \iff & \text{For all } v \in U \ (uHv \text{ implies } M, v \not\models \varphi), \\
M, u \models \lrcorner\, \varphi & \iff & \text{For some } v \in U \ (vHu \text{ and } M, v \not\models \varphi), \\
M, u \models \Diamond\, \varphi & \iff & \text{For some } v \in U \ ((v, u) \in \curvearrowright R \text{ and } M, v \models \varphi), \\
M, u \models \blacksquare\, \varphi & \iff & \text{For all } v \in U \ ((u, v) \in \curvearrowright R \text{ implies } M, v \models \varphi).
\end{array}$$

Proposition 17. Given any H-model M, the truth set $[\![\varphi]\!]_M$ is an H-set.

Proof. By induction on φ. When φ is of the form $\blacklozenge\, \psi$, $\Box\psi$, $\Diamond\psi$ or $\blacksquare\, \psi$, we need to use $R\,;H \subseteq R$, and $H\,;R \subseteq R$, and $\curvearrowright R\,;H \subseteq \curvearrowright R$ and $H\,;\curvearrowright R \subseteq \curvearrowright R$ respectively, and all these facts hold as R and $\curvearrowright R$ are stable relations. When φ is of the form $\mathsf{E}\, \psi$, $\mathsf{A}\, \psi$, we remark that $[\![\varphi]\!]_M = U$ or $[\![\varphi]\!]_M = \varnothing$, which are both trivially H-sets. ■

The semantics of **UBiSKt** is built upon a set with a relation of preorder. We have seen how a hypergraph gives rise to a poset, that is also a preorder. So a special case of an H-model as in Definition 16 is when (U, H) is a hypergraph.

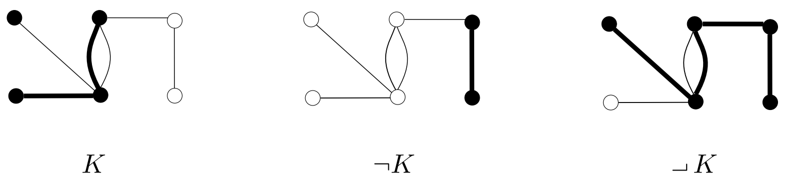

Figure 2.4: The two kinds of complement of a subgraph K (figure adapted from [71]).

By Definition 13, the subgraphs of a hypergraph (U, H) are the subsets of U that are closed under H-successor, therefore they are H-sets as in Definition 14. Since any formula φ in the logic is interpreted on the H-set $[\![\varphi]\!]_M$, formulae in the logic can be regarded as names for subgraphs of an underlying hypergraph (U, H). Similarly, operations in the logic provide operations on subgraphs, following the semantics from Definition 16 . Figure 2.4 shows a graph with a subgraph and the two operations of complement $\neg$ and $\lrcorner$, where the leftmost is a graph (U, H) with subgraph K and the remaining graphs are the subgraphs obtained by the operation $\neg$ and $\lrcorner$. We note that $\neg K$ is the largest subgraph disjoint from K, i.e. the largest subgraph such that $K \cap \neg K = \varnothing$ and $\lrcorner K$ is the smallest subgraph whose union with K gives all of the underlying graph G, i.e. the smallest subgraph such that $K \cup \lrcorner K = U$. We call $\neg K$ the *pseudo-complement* of K, and $\lrcorner K$ the *dual pseudo-complement* of K.

Definition 18. Given a set $\Gamma \cup \{\varphi\}$ of formulas and a class $\mathbb{F}$ of H-frames, φ is a *semantic consequence* of Γ in $\mathbb{F}$ (in symbols $\Gamma \models_{\mathbb{F}} \varphi$) if, whenever $M \models \Gamma$, $M \models \varphi$ holds, for all H-models $M = (U, H, R, V)$ such that $(U, H, R) \in \mathbb{F}$. When Γ is a singleton $\{\psi\}$ formula, we simply write $\psi \models_{\mathbb{F}} \varphi$ instead of $\{ \psi \} \models_{\mathbb{F}} \varphi$. When Γ is empty, we also simply write $\models_{\mathbb{F}} \varphi$ instead of $\varnothing \models_{\mathbb{F}} \varphi$. If $\mathbb{F}$ is the class of all H-frames, we drop the subscript to write $\Gamma \models \varphi$ provided no confusion arises.

Definition 19. We say that a set Γ of formula *defines* a class $\mathbb{F}$ of H-frames if for all H-frames F, $F \in \mathbb{F}$ iff $F \models \varphi$ for all formulas $\varphi \in \Gamma$. When Γ is a singleton $\{ \varphi \}$, we simply say that φ defines a class $\mathbb{F}$.

The following frame definability result was already established in [74, Theorem 10].

Proposition 20 ([74])**.** Let $F = (U, H, R)$ be an H-frame. Let $S_i \in \{ R, \curvearrowright R \}$ for $i = 1, \dots, m$ and for each i let

$$\mathsf{B}_i = \begin{cases} \Box & \text{if } S_i = R \\ \blacksquare & \text{if } S_i = \curvearrowright R \end{cases} \quad \text{and let } \mathsf{D}_i = \begin{cases} \blacklozenge & \text{if } S_i = R \\ \Diamond & \text{if } S_i = \curvearrowright R \end{cases}$$

Let $0 \leqslant k \leqslant m$ (where the composition of a sequence of length 0 is understood as H). Then the following are equivalent: (1) $S_1 \,; \cdots ; S_k \subseteq S_{k+1} \,; \cdots ; S_m$; (2) $F \models \mathsf{D}_k \cdots \mathsf{D}_1 p \to \mathsf{D}_m \cdots \mathsf{D}_{k+1} p$; (3) $F \models \mathsf{B}_{k+1} \cdots \mathsf{B}_m \to \mathsf{B}_1 p \cdots \mathsf{B}_k p$.

2.2.1 Alternative semantics in terms of morphological operations

It is useful to see how we can express the semantics of some connectives and operators in **UBiSKt** in terms of dilation and erosion. From Definition 14 we have that given $X \subseteq U$, X is an H-set iff $X \oplus H \subseteq X$. By reflexivity of H it follows that this condition is equivalent to $X = X \oplus H$. Moreover, as $\_ \oplus H$ and $H \ominus \_$ form an adjunction in the sense of Definition 6, and by reflexivity of H again, we have that $X = X \oplus H$ iff $X = H \ominus X$. Given an H-model $M = (U, H, R, V)$, given formulae φ and ψ interpreted over some H-sets $[\![\varphi]\!]_M$ and $[\![\psi]\!]_M$, the semantics of the connectives and operators in **UBiSKt** involving a relation in their definition, can be expressed by using dilation and erosion as follows ($\mathbb{U}$ is the universal relation $U \times U$).

$$\begin{array}{ll}
\llbracket \neg\varphi \rrbracket = H \ominus (-\llbracket \varphi \rrbracket) & \llbracket \lrcorner\, \varphi \rrbracket = (-\llbracket \varphi \rrbracket) \oplus H \\
\llbracket \varphi \to \psi \rrbracket = H \ominus ((-\llbracket \varphi \rrbracket) \cup \llbracket \psi \rrbracket) & \llbracket \varphi \prec \psi \rrbracket = (\llbracket \varphi \rrbracket \cap (-\llbracket \psi \rrbracket)) \oplus H \\
\llbracket \Box\varphi \rrbracket = R \ominus \llbracket \varphi \rrbracket & \llbracket \blacklozenge\, \varphi \rrbracket = \llbracket \varphi \rrbracket \oplus R \\
\llbracket \blacksquare\, \varphi \rrbracket = \curvearrowright R \ominus \llbracket \varphi \rrbracket & \llbracket \Diamond\varphi \rrbracket = \llbracket \varphi \rrbracket \oplus \curvearrowright R \\
\llbracket \mathsf{A}\, \varphi \rrbracket = \mathbb{U} \ominus \llbracket \varphi \rrbracket & \llbracket \mathsf{E}\, \varphi \rrbracket = \llbracket \varphi \rrbracket \oplus \mathbb{U}
\end{array}$$

The remaining connectives, conjunction and disjunction, can be expressed in a similar set-theoretic fashion as $\llbracket \varphi \wedge \psi \rrbracket = \llbracket \varphi \rrbracket \cap \llbracket \psi \rrbracket$, and $\llbracket \varphi \vee \psi \rrbracket = \llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket$. Let us take the example of an implication $\varphi \to \psi$. From Definition 16 we have that given a $u \in U$, $M, u \models \varphi \to \psi$ iff for all v such that $u\ H\ v$, if $v \models \varphi$ then $v \models \psi$. Using the truth-set notation, this can be written as $u \in \llbracket \varphi \to \psi \rrbracket$ iff for all v such that $u\ H\ v$, $v \notin \llbracket \varphi \rrbracket$ or $v \in \llbracket \psi \rrbracket$, that is $v \in (-\llbracket \varphi \rrbracket) \cup \llbracket \psi \rrbracket$. So using this fact and Definition 5 of erosion we have that $\llbracket \varphi \to \psi \rrbracket = \{\, u \in U \mid \forall v (u\ H\ v \text{ implies } v \in (-\llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket)\,\} = H \ominus (-\llbracket \varphi \rrbracket \cup \llbracket \psi \rrbracket)$.

2.3 Defining the notion of Graph and Hypergraph with the logic

Hypergraphs can be regarded as posets where only two kinds of elements occur: 0-dimensional elements, i.e. nodes, H-related only to themselves by reflexivity of H, and 1-dimensional elements, i.e. edges, H-related to the nodes they are incident with, and to themselves by reflexivity. Given a poset (U, H), this property can be expressed as a constraint on H: $(H \cap \breve{H})^2 = \varnothing$. Indeed this constraint expresses the fact that the non-symmetric part of H, i.e. $H \cap \breve{H}$ is at most "one step long" (we also remind the reader that as a hypergraph is a poset and thus H is antisymmetric and reflexive, the symmetric part of H, i.e. $(H \cap \breve{H})$ is the identity relation I on U). We can reformulate the relational constraint as follows: for any u, v and $w \in U$, if $uHvHw$ holds then at least one of the following must hold: vHu or wHv. When the preorder H is additionally antisymmetric, so when the symmetric steps are just identity loops, it is clear the constraint implies that U has only layers, the edges layer H incident with the nodes layer.

In what follows we are going to show that there is a formula in **UBiSKt** that corresponds to this constraint on a preorder H, so that when we restrict our attention to H-frames where H is also antisymmetric, the formula singles out the class of H-frames that are hypergraphs. The following results are valid for any H-frame $F = (U, H, R)$, but are independent from the relation R.

Theorem 21. *Let $F = (U, H, R)$ be an H-frame. Then $F \models q \vee (q \to (p \vee \neg p))$ iff $(H \cap \breve{H})^2 = \varnothing$.*

Proof. For the right-to-left direction: let us assume that $(H \cap \breve{H})^2 = \varnothing$ and that $F \not\models q \vee (q \to (p \vee \neg p))$. We are going to derive a contradiction. If the formula $q \vee (q \to (p \vee \neg p))$ is not valid at the H-frame F, then there is a valuation V and a world u such that $F, V, u \not\models q \vee (q \to (p \vee \neg p))$. Let us fix F and u. Then $F, V, u \not\models q$ and $F, V, u \not\models q \to (p \vee \neg p)$. Then there is a $v \in U$ such that $u\ H\ v$ and $F, V, v \models q$ and $F, V, v \not\models p \vee \neg p$, so $F, V, v \not\models p$ and $F, V, v \not\models \neg p$. Thus there is a $w \in U$ such that $v\ H\ w$ and $F, V, w \models p$. Since $(H \cap \breve{H})^2 = \varnothing$ holds by assumption and $u\ H\ v\ H\ w$, we have that *i)* $v\ H\ u$, or *ii)* $w\ H\ v$. Let us assume the first case. $F, V, v \models q$ and $v\ H\ u$ jointly imply $F, V, u \models q$, by monotonicity of knowledge w.r.t. the preorder H. But this contradicts the assumption that $F, V, u \not\models q$. Then *ii)* must be the case. But $F, V, w \models p$ and $w\ H\ v$ jointly imply $F, V, v \models p$, that contradicts $F, V, v \not\models p$. As the initial assumption has been contradicted, we have proved that $(H \cap \breve{\breve{H}})^2 = \varnothing$ implies $F \models q \vee (q \to (p \vee \neg p))$ for any valuation V and world $u \in U$.

For the left-to-right direction: we prove the converse. Let us assume that $(H \cap \breve{\breve{H}})^2 \neq \varnothing$ for an H-frame $F = (U, H, R)$. Our goal is to show that $F \not\models q \vee (q \to (p \vee \neg p))$, so that it is always possible to find a valuation V and a world $u \in U$ such that $F, V, u \not\models q \vee (q \to (p \vee \neg p))$. Since $(H \cap \breve{\breve{H}})^2 \neq \varnothing$ we know that there are u, v and $w \in U$ such that $u\ H\ v$ and $v\ H\ w$ holds and both $v\ H\ u$ and $w\ H\ v$ fail. Let us define the following valuation V: $V(p) = \{\, x \in U \mid w\ H\ x \,\}$, and $V(q) = \{\, x \in U \mid v\ H\ x \,\}$. It is clear that both $V(p)$ and $V(q)$ are H-set. Now we have: $F, V, w \models p$, as, by reflexivity of H we have that $w\ H\ w$. Then, since $v\ H\ w$, $F, V, v \not\models \neg p$. But also $F, V, v \not\models p$, as $w\ H\ v$ fails. Therefore $F, V, v \not\models (p \vee \neg p)$. Moreover $F, V, v \models q$, as, by reflexivity of H we have that $v\ H\ v$. So by $u\ H\ v$, we have that $F, V, u \not\models q \to (p \vee \neg p)$. Moreover $F, V, u \not\models q$ because $v\ H\ u$ fails. Therefore $F, V, u \not\models q \vee (q \to (p \vee \neg p))$, as desired. ■

Corollary 22. Let H be a partial order, and let $F = (U, H, R)$ be an H-frame. Then $F \models q \vee (q \to (p \vee \neg p))$ iff F is a hypergraph.

Proof. The condition $(H \cap \breve{\breve{H}})^2 = \varnothing$ says that for all $u, v, w \in U$, $u\ H\ v$ and $v\ H\ w$ implies $v\ H\ u$ or $w\ H\ v$, so that for all $u, v, w \in U$, $u\ H\ v$ and $v\ H\ w$ then $u\ (H \cap \breve{H})\ v$ or $v\ (H \cap \breve{H})\ w$. When H is antisymmetric and reflexive, as for a partial order, then $H \cap \breve{H} = I$. So we have that for all $u, v, w \in U$, $u\ H\ v$ and $v\ H\ w$ then $v = u$ or $w = v$. Thus, under this condition, a poset (U, H) is a hypergraph by Definition 9. ■

It is well known that the law of excluded middle, $p \vee \neg p$, defines the property of symmetry of a preorder H, $H \subseteq \breve{H}$. When H is also anti-symmetric, the validity of this formula implies that (U, H) is a set, i.e. a hypergraph without any hyper-edges but with only nodes, namely discrete points.

Theorem 23. Let $F = (U, H, R)$ be an H-frame. Then $F \models p \vee \neg p$ iff $H \subseteq \breve{H}$.

Corollary 24. Let H be a partial order, and let $F = (U, H, R)$ be an H-frame. Then $F \models p \vee \neg p$ iff $H = I$.

Proof. The condition $H \subseteq \breve{H}$ expresses symmetry of H and it is equivalent to $H = H \cap \breve{H}$. When H is a partial-order we have that $H \cap \breve{H} = I$. Therefore for $F = (U, H)$, with H partial order, $F \models p \vee \neg p$ iff $H = H \cap \breve{H} = I$. ■

It is clear that if $H = (H \cap \breve{H})$ then the non-symmetric part of H is empty: $(H \cap \breve{\breve{H}}) = \varnothing$. Thus also $(H \cap \breve{\breve{H}})^2 = \varnothing$ holds, so if (U, H) is a poset, then under this condition, it will also be a hyperhraph, and a special type of hypergraph: one where only one type of elements are presents, i.e. nodes. Indeed, the incidence relation H is simply the identity, and from Definition 9, an element of U is an edge iff it is H-predecessor of an element distinct from itself. Thus in these cases such elements cannot exists, that means we are in the presence of a hypergraph without edges, that we can consider simply as a set of points.

The frame (U, H) is fully determined by a set U with its identity relation I. It is indeed well known that, when adding the validity of excluded middle axiom to intuitionistic logic, we obtain classical logic.

The results show that, by adding the extra assumption of antisymmetry, by the formula $q \vee (q \to (p \vee \neg p))$ or with the formula $p \vee \neg p$, we can restrict our attention to the class of H-frames that are hypergraphs and sets respectively. We note that the formulae $q \vee (q \to (p \vee \neg p))$ and $p \vee \neg p$ and their generalisation are also studied in [11], where Kripke frames for intuitionistic logic are already considered posets, and not generic pre-orderings. Here we have interpreted these frame correspondence results in terms of a preorder H and partial order H, and we have noticed that in the case of H being a partial order, the formulae at issue define the notions hypergraph and set.

A graph is a special case of a hypergraph where each edge is incident with at most two nodes (see Definition 11). We are going to show now that there is a formula in **UBiSKt** that defines the following property of an H-frame: any element u of U is H-incident with at most two distinct elements that are not in turn H-accessible from each others. This means that the width of H is bounded to a maximum size of 2. A general version of this formula and the corresponding frame property expressing the width size of an intuitionistic frame is already studied in [11]. Here we are going to see that, with the extra assumption that (U, H) is a hypergraph, imposing the validity of this formula, and thus the corresponding property, is equivalent to require that (U, H) is a graph.

Theorem 25. *Let $F = (U, H)$ be an H-frame. Then $F \models (p \to (q \vee r)) \vee (q \to (p \vee r)) \vee (r \to (p \vee q))$ iff $\forall u, v, w, t\ ((u\ H\ v \wedge u\ H\ w \wedge u\ H\ t) \to (v\ H\ w \vee w\ H\ v \vee v\ H\ t \vee t\ H\ v \vee w\ H\ t \vee t\ H\ w))$.*

Proof. For the right-to-left direction, let us assume that the frame condition is true for an H-frame $F = (U, H)$ and that the axiom is not valid at F. The goal is to show that this leads to contradiction. If $F \not\models (p \to (q \vee r)) \vee (q \to (p \vee r)) \vee (r \to (p \vee q))$ then there is a $u \in U$ and a valuation V such that $F, V, u \not\models (p \to (q \vee r))$, and $F, V, u \not\models (q \to (p \vee r))$ and $F, V, u \not\models (r \to (p \vee q))$. Then there are v, w, t such that $u\ H\ v$ and $u\ H\ w$ and $u\ H\ t$ such that: $F, V, v \models p$ and $F, V, v \not\models q$, $F, V, v \not\models r$; $F, V, w \models q$ and $F, V, w \not\models p$, $F, V, w \not\models r$; and $F, V, t \models r$ and $F, V, t \not\models p$, $F, V, t \not\models q$. By our frame condition, since $u\ H\ v$ and $u\ H\ j$ and $u\ H\ t$ hold, at least one of the following possibilities must hold: *i)* $v\ H\ w$, *ii)* $w\ H\ v$, *iii)* $v\ H\ t$, *iv)* $t\ H\ v$, *v)* $w\ H\ t$, *vi)* $t\ H\ w$. But it is easy to see that all of *i*–*vi* lead to contradiction. Suppose *i* is the case. $v\ H\ w$ and $F, V, v \models p$ jointly imply $F, V, w \models p$. But this contradicts with $F, V, w \not\models p$. Suppose *ii* is the case. $w\ H\ v$ and $F, V, w \models q$ jointly imply that $F, V, v \models q$. This contradicts with $F, V, v \not\models q$. Similar reasoning holds for the remaining case. Therefore we have shown that if the frame condition at issue holds at F, then the formula must be valid at F, as desired.

For the left-to-right direction: let us suppose that the frame condition doesn't hold. The goal is to show that the formula is not valid at such an H-frame, so that it is always possible to find a valuation V and a world u such that $F, V, u \not\models (p \to (q \vee r)) \vee (q \to (p \vee r)) \vee (r \to (p \vee q))$. Since the frame condition doesn't hold, there are $u, v, w, t \in U$ such that $u\ H\ v$ and $u\ H\ w$

and $u\ H\ t$, and $v\ \bar{H} \cap \breve{\bar{H}}\ w$, and $v\ \bar{H} \cap \breve{\bar{H}}\ t$, and $w\ \bar{H} \cap \breve{\bar{H}}\ t^2$. Let us choose the following valuation V: $V(p) = \{\, x \in U \mid v\ H\ x \,\}$, $V(q) = \{\, x \in U \mid w\ H\ x \,\}$ and $V(r) = \{\, x \in U \mid t\ H\ x \,\}$. Put $M = (F, V)$. It is clear that $V(p)$, $V(q)$ and $V(r)$ are all H-sets. Now, we have that $M, v \models p$. Also, since $w\ \bar{H}\ v$, i.e. v is not an H-successor of w, we have that $M, v \not\models q$, and since $t\ \bar{H}\ v$, we have that $M, v \not\models r$. Thus $M, v \not\models q \vee r$, that implies that $M, u \not\models p \to (q \vee r)$. Similarly, $M, w \models q$, and since $v\ \bar{H}\ w$, we have that $M, w \not\models p$. Since $t\ \bar{H}\ w$, we have that $M, w \not\models r$. Thus $M, w \not\models p \vee r$, that implies that $M, u \not\models q \to (p \vee r)$. Analogous reasoning on t leads to the conclusion that $M, u \not\models r \to (p \vee q)$. Thus $M, u \not\models (p \to (q \vee r)) \vee (q \to (p \vee r)) \vee (r \to (p \vee q))$, as desired. ■

Corollary 26. Let $F = (U, H)$ be a hypergraph. Then $F \models (p \to (q \vee r)) \vee (q \to (p \vee r)) \vee (r \to (p \vee q))$ iff F is a graph.

Proof. For the left to right direction, suppose (U, H) is a hypergraph and suppose the formula at issue holds. By Theorem 25 we know then for all $u, v, w, t \in U$, $((u\ H\ v \wedge u\ H\ w \wedge u\ H\ t) \to (v\ H\ w \vee w\ H\ v \vee v\ H\ t \vee t\ H\ v \vee w\ H\ t \vee t\ H\ w))$ holds. The goal is to show that under this conditions, (U, H) is a graph, i.e. by Definition 11, the cardinality of the set $\{\, u \in U \mid v\ H\ u \text{ and } v \neq u \,\}$ is at most 2, for any $u \in U$. So let us assume that for some $u, v, w, t \in U$, $u\ H\ v$ and $u \neq v$, $u\ H\ w$ and $u \neq w$, and $u\ H\ t$ and $u \neq t$. The goal is to show that then $v = w$ or $w = t$ or $v = t$. Since $u\ H\ v$ and $u\ H\ w$ and $u\ H\ t$ hold, then we must have one of these cases: (i) $v\ H\ w$, (ii) $w\ H\ v$, (iii) $w\ H\ t$, (iv) $t\ H\ w$, (v) $v\ H\ t$, (vi) $t\ H\ v$. Suppose (i) is the case. Then we have $u\ H\ v\ H\ w$. Then as (U, H) is a hypergraph and then $(H \cap \breve{H})^2$ holds, we have that $u = v$ or $v = w$. But $u \neq v$ by assumption. Thus it must be that $v = w$. Case (ii) implies the same. Suppose (iii) is the case. Then $u\ H\ w\ H\ t$. Thus, by the assumption of working with hypergraphs, we have that $u = w$ or $w = t$. But $u \neq w$, thus it must be that $w = t$. Case (iv) implies the same. Finally suppose (v) is the case. Then by $u\ H\ v\ H\ t$ we have that $u = v$ or $v = t$. As the first is impossible it must be the second, and the same result will be obtained analysing case (vi). Thus we have proved that $v = w$ or $w = t$ or $v = t$, as desired. Thus, given any $u \in U$, the cardinality of the set $\{\, u \in U \mid v\ H\ u \text{ and } v \neq u \,\}$ is at most 2, and thus (U, H) is a graph.

The other direction is trivial. ■

2.4 Axiomatisation and Tableau calculus for UBiSKt

2.4.1 Hilbert-style axiomatisation

HUBiSKt is an Hilbert-style axiomatization for the logic **UBiSKt**, introduced in [71]. The set of axioms and rules forming **HUBiSKt** is provided in Table 2.1. In what follows we assume that the reader is familiar with theorems and derived inference rules in intuitionistic logic (see [3] for an introduction to the topic).

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[2] The relation $x\ (\bar{H} \cap \breve{\bar{H}})\ y$ means indeed that the pair (x, y) belongs neither to H nor to $\breve{H}$.

Table 2.1: Hilbert System H**UBiSKt**

Axioms and Rules for Intuitionistic Logic			
(A0)	$p \to (q \to p)$		
(A1)	$(p \to (q \to r)) \to ((p \to q) \to (p \to r))$		
(A2)	$p \to (p \vee q)$	(A3)	$q \to (p \vee q)$
(A4)	$(p \to r) \to ((q \to r) \to (p \vee q \to r))$	(A5)	$(p \wedge q) \to p$
(A6)	$(p \wedge q) \to q$	(A7)	$(p \to (q \to p \wedge q))$
(A8)	$\bot \to p$	(A9)	$p \to \top$
(MP)	From φ and $\varphi \to \psi$, infer ψ		
(US)	From φ, infer a substitution instance φ' of φ		
Additional Axioms and Rules for Bi-intuitionistic Logic			
(A10)	$p \to (q \vee (p \prec q))$	(A11)	$((q \vee r) \prec q) \to r$
(Mon$\prec$)	From $\delta_1 \to \delta_2$, infer $(\delta_1 \prec \psi) \to (\delta_2 \prec \psi)$		
Additional Axioms and Rules for Tense Operators			
(A12)	$p \to \Box\blacklozenge p$	(A13)	$\blacklozenge\Box p \to p$
(Mon$\Box$)	From $\varphi \to \psi$, infer $\Box\varphi \to \Box\psi$	(Mon$\blacklozenge$)	From $\varphi \to \psi$, infer $\blacklozenge\varphi \to \blacklozenge\psi$
Additional Axioms and Rules for Universal Modalities			
(A14)	$p \to \mathsf{A}\,\mathsf{E}\,p$	(A15)	$\mathsf{E}\,\mathsf{A}\,p \to p$
(A16)	$\mathsf{A}\,p \to p$	(A17)	$\mathsf{A}\,p \to \mathsf{A}\,\mathsf{A}\,p$
(A18)	$\mathsf{A}\,\neg p \leftrightarrow \neg\,\mathsf{E}\,p$	(A19)	$(\mathsf{A}\,p \wedge \mathsf{E}\,q) \to \mathsf{E}(p \wedge q)$
(A20)	$\mathsf{A}\,p \to \Box p$	(A21)	$(\mathsf{A}\,p \wedge \blacklozenge q) \to \blacklozenge(p \wedge q)$
(A22)	$(\mathsf{A}\,p \wedge (q \prec r)) \to ((p \wedge q) \prec r)$		
(Mon A)	From $\varphi \to \psi$, infer $\mathsf{A}\,\varphi \to \mathsf{A}\,\psi$	(Mon E)	From $\varphi \to \psi$, infer $\mathsf{E}\,\varphi \to \mathsf{E}\,\psi$

Definition 27. Let $\varphi \in \mathsf{Form}_{\mathcal{L}}$ and $\Gamma \subseteq \mathsf{Form}_{\mathcal{L}}$. A *derivation* in **HUBiSKt** φ from Γ, in symbols $\Gamma \vdash_{\mathbf{HUBiSKt}} \varphi$, is a finite sequence of formulae $\varphi_1, \ldots, \varphi_n$ with the following properties:

1. each φ_i, $1 \leq i \leq n$ is either:
   - an axiom of **HUBiSKt** or
   - a member of Γ or
   - it follows from some previous formula φ_j, $j < i$, by application of a rule of **HUBiSKt**.
2. $\varphi_n = \varphi$.

When $\Gamma \vdash_{\mathbf{HUBiSKt}} \varphi$ we say that φ is *provable* from Γ in **HUBiSKt**. If $\Gamma \vdash_{\mathbf{HUBiSKt}} \varphi$ and $\Gamma = \varnothing$, then φ is a *theorem* in **HUBiSKt**, in symbols $\vdash_{\mathbf{HUBiSKt}} \varphi$.

The following results on **HUBiSKt** are shown in [71].

Theorem 28 (Soundness of **HUBiSKt**)**.** Given any formula φ, $\vdash_{\mathbf{HUBiSKt}} \varphi$ implies $\models \varphi$.

Proposition 29. All the following hold for **HUBiSKt**.

1. $\vdash (\psi \prec \gamma) \to \rho$ iff $\vdash \psi \to (\gamma \vee \rho)$.
2. If $\vdash \varphi \leftrightarrow \psi$ then $\vdash (\gamma \prec \varphi) \leftrightarrow (\gamma \prec \psi)$.
3. $\vdash \neg(\varphi \prec \varphi)$.
4. $\vdash \varphi \vee \lrcorner\, \varphi$.
5. $\vdash \lrcorner\, \lrcorner\, \varphi \to \varphi$.
6. $\vdash \neg\varphi \to \lrcorner\, \varphi$.
7. $\vdash \varphi \to \neg\psi$ iff $\vdash \psi \to \neg\varphi$.
8. $\vdash \lrcorner\, \varphi \to \psi$ iff $\vdash \lrcorner\, \psi \to \varphi$.
9. $\vdash \lrcorner \neg\varphi \to \psi$ iff $\vdash \varphi \to \neg \lrcorner\, \psi$.
10. $\vdash \varphi \to \lrcorner \neg\varphi$.
11. $\vdash \neg \lrcorner\, \varphi \to \varphi$.
12. If $\vdash \varphi \to \psi$ then $\vdash \lrcorner\, \psi \to \lrcorner\, \varphi$.
13. $\vdash \lrcorner(\varphi \wedge \lrcorner\, \varphi)$.
14. $\vdash \mathsf{E}\,\varphi \to \psi$ iff $\vdash \varphi \to \mathsf{A}\,\psi$.
15. $\vdash \varphi \to \mathsf{E}\,\varphi$.
16. $\vdash \mathsf{E}\,\mathsf{E}\,\varphi \to \mathsf{E}\,\varphi$.
17. $\vdash \mathsf{A}\,\mathsf{E}\,\varphi \leftrightarrow \mathsf{E}\,\varphi$.
18. $\vdash \neg\,\mathsf{A}\,\varphi \leftrightarrow \lrcorner\,\mathsf{A}\,\varphi$.
19. $\vdash \mathsf{A}\,\varphi \vee \neg\,\mathsf{A}\,\varphi$.
20. $\vdash \neg\,\mathsf{E}\,\varphi \leftrightarrow \lrcorner\,\mathsf{E}\,\varphi$.
21. $\vdash \mathsf{E}\,\varphi \vee \neg\,\mathsf{E}\,\varphi$.
22. $\vdash \mathsf{E}\,\varphi \leftrightarrow \neg\,\mathsf{A}\,\neg\varphi$.
23. $\vdash \mathsf{A}(\lrcorner\,\varphi \to \psi) \leftrightarrow \mathsf{A}(\lrcorner\,\psi \to \varphi)$.
24. $\vdash \mathsf{E}(\lrcorner \neg\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \lrcorner \neg\psi)$.

Strong completeness of **HUBiSKt** (Theorem 37) is shown in [71] and [72], using the canonical model method and it is contribution of the second author of the papers, as well as decidability of **UBiSKt** (Theorem 44). We introduce here the main definitions and lemmas needed for the canonical model construction, and thus for the completeness theorem, and we refer the reader to [71] and [72] for the full proofs. Moreover we have completeness of some extensions of **HUBiSKt**: (i) completeness of any logic *ubist* logic extended with the formula used for defining hypergraphs, introduced in Section 2.3 w.r.t the class of H-frames that are hypergraphs (Lemma 39 and Theorem 40), and (ii) completeness of any *ubist* logic extended with the formula used for defining graphs, introduced in Section 2.3 w.r.t the class of H-frames that are graphs (Lemma 42 and Theorem 43). The first point is already presented in [72] and it is contribution of the second author of the paper, whilst the second point is a novel contribution of the author of the present work.

Definition 30. A *ubist logic* is any set Λ of formulae satisfying the following conditions: i) Λ contains all the axioms of Table 2.1 and ii) Λ is closed under all the rules of Table 2.1. The term *ubist logic* is used as short for *bi-intuitionistic stable tense logic with universal modalities.* Given a *ubist* logic Λ, a pair of formulae (Γ, Δ) is Λ-*provable* if $\Gamma \vdash_\Lambda \bigvee \Delta'$, i.e. when for some finite $\Delta' \subseteq \Delta$ and some finite $\Gamma' \subseteq \Gamma$, we have that $\bigwedge \Gamma' \to \bigvee \Delta' \in \Lambda$. (Γ, Δ) is Λ-*unprovable* if it is not Λ-provable. It is *complete*, if for all formulae φ, $\varphi \in \Gamma$, or $\varphi \in \Delta$.

The following lemmas hold because Λ contains intuitionistic logic.

Lemma 31. Let Λ be a *ubist* logic, and (Γ, Δ) a complete and Λ-unprovable pair. Then:

1. For all formulas φ, $\Gamma \vdash_\Lambda \varphi$ implies $\varphi \in \Gamma$.
2. $\Lambda \subseteq \Gamma$.
3. If $\varphi \in \Gamma$ and $\varphi \to \psi$, then $\psi \in \Gamma$.
4. $\bot \notin \Gamma$.
5. $\varphi \wedge \psi \in \Gamma$ iff $\varphi \in \Gamma$ and $\psi \in \Gamma$.
6. $\varphi \vee \psi \in \Gamma$ iff $\varphi \in \Gamma$ or $\psi \in \Gamma$.

Lemma 32. Let Δ be a *ubist*-logic. Given a Λ-unprovable pair (Γ, Δ), there exists a Λ-unprovable and complete pair (Γ^+, Δ^+) such that $\Gamma \subseteq \Gamma^+$ and $\Delta \subseteq \Delta^+$.

Definition 33. Let Λ be a *ubist*-logic and (Γ, Δ) be a complete and Λ-unprovable pair. The Λ-canonical H-model $M^\Lambda_{(\Gamma,\Delta)} = (U^\Lambda, H^\Lambda, R^\Lambda, V^\Lambda)$ is defined as:

- $U^\Lambda := \{ (\Sigma, \Theta) \mid (\Sigma, \Theta) \text{ is a complete and } \Lambda - \text{unprovable pair and } (\Gamma, \Delta) S^\Lambda (\Sigma, \Theta) \}$ where the relation S^Λ is defined as:

$$(\Gamma, \Delta) S^\Lambda (\Sigma, \Theta) \quad \Longleftrightarrow \quad (\mathsf{A}\,\varphi \in \Gamma \iff \mathsf{A}\,\varphi \in \Sigma) \text{ for all formulas } \varphi.$$

- $(\Sigma_1, \Theta_1) H^\Lambda (\Sigma_2, \Theta_2)$ iff $\Sigma_1 \subseteq \Sigma_2$.
- $(\Sigma_1, \Theta_1) R^\Lambda (\Sigma_2, \Theta_2)$ iff ($\Box\varphi \in \Sigma_1$ implies $\varphi \in \Sigma_2$) for all formulas φ.
- $(\Sigma, \Theta) \in V^\Lambda(p)$ iff $p \in \Sigma$.

Let $F^\Lambda_{(\Gamma,\Delta)} = (U^\Lambda, H^\Lambda, R^\Lambda)$ be the Λ-canonical H-model.

It is clear that H^{Λ} is not just a preorder but it is also a partial order.

Lemma 34. Let $(\Sigma_1, \Theta_1), (\Sigma_2, \Theta_2) \in U^{\Lambda}$. The following are equivalent:

1. $(\Sigma_1, \Theta_1)\ R^{\Lambda}\ (\Sigma_2, \Theta_2)$
2. ($\varphi \in \Sigma_1$ implies $\blacklozenge \varphi \in \Sigma_2$) for all formulas φ.

Proof. From Definition 33 we know that $(\Sigma_1, \Theta_1) R^{\Lambda} (\Sigma_2, \Theta_2)$ iff ($\Box\varphi \in \Sigma_1$ implies $\varphi \in \Sigma_2$) for all formulas φ. Thus we need to prove that this is equivalent to item 2. For one direction, suppose that $\Box\varphi \in \Sigma_1$. The goal is to show that $\varphi \in \Sigma_2$ using item 2. If $\Box\varphi \in \Sigma_1$, then $\blacklozenge \Box\varphi \in \Sigma 2$. Then as $\blacklozenge \Box\varphi \to \varphi \in \Lambda$, and $\Lambda \subseteq \Sigma_2$, we have that $\blacklozenge \Box\varphi \to \varphi \in \Sigma_2$, and thus $\varphi \in \Sigma_2$, as desired. For the other direction, suppose that $\varphi \in \Sigma_1$, the goal is to show that $\blacklozenge \varphi \in \Sigma_2$ using the definition of R^{Λ}. Suppose by contradiction that $\varphi \in \Sigma_1$ and $\blacklozenge \varphi \notin \Sigma_2$. Then $\Box \blacklozenge \varphi \notin \Sigma_1$. But then as $\varphi \to \Box \blacklozenge \varphi \in \Lambda \subseteq \Sigma_1$, we have that $\varphi \notin \Sigma_1$. This is a contradiction as we assumed that $\varphi \in \Sigma_1$. Thus $\blacklozenge \varphi \in \Sigma_2$ as wanted. ■

Lemma 35. R^{Λ} is stable in the canonical model H-model $M^{\Lambda}_{(\Gamma,\Delta)}$.

Proof. The goal is to prove that (i) $H^{\Lambda}\ ;\ R^{\Lambda} \subseteq R^{\Lambda}$, and (ii) $R^{\Lambda}\ ;\ H^{\Lambda} \subseteq R^{\Lambda}$. For (i), suppose $(\Gamma_1, \Delta_1)\ H^{\Lambda}\ (\Gamma_2, \Delta_2)\ R^{\Lambda}\ (\Gamma_3, \Delta_3)$. Suppose $\Box\varphi \in \Gamma_1$ for any formula φ. The goal is to show that $\varphi \in \Gamma_3$. $(\Gamma_1, \Delta_1)\ H^{\Lambda}\ (\Gamma_2, \Delta_2)$ implies that $\Box\varphi \in \Gamma_2$. As $(\Gamma_2, \Delta_2)\ R^{\Lambda}\ (\Gamma_3, \Delta_3)$, we have that $\varphi \in \Gamma_3$. Similar argument proves (ii). ■

For full proof of the following truth lemma we refer to [72].

Lemma 36 (Truth Lemma)**.** Let Λ be a *ubist*-logic and (Γ, Δ) be a complete and Λ-unprovable pair. Then, for any formula φ and any complete Λ-unprovable pair (Σ, Θ), the following equivalence holds:

$$\varphi \in \Sigma \iff M^{\Lambda}_{(\Gamma,\Delta)}, (\Sigma, \Theta) \models \varphi.$$

Theorem 37 (Strong Completeness of $\mathsf{HUBiSKt}$)**.** If $\Gamma \models \varphi$ then $\Gamma \vdash_{\mathsf{HUBiSKt}} \varphi$, for every set $\Gamma \cup \{\varphi\}$ of formulas.

Proof. Put $\Lambda := \mathbf{UBiSKt}$. Fix any set $\Gamma \cup \{\, \varphi \,\}$ of formulas. We prove the contrapositive implication and so assume that $\Gamma \nvdash_{\Lambda} \varphi$. It follows that $(\Gamma, \{\, \varphi \,\})$ is Λ-unprovable. By Lemma 32, we can find a complete and Λ-unprovable pair $(\Sigma, \Theta) \in U^{\Lambda}$ such that $\Gamma \subseteq \Sigma$ and $\varphi \in \Theta$. By Lemma 36 (Truth Lemma), $M^{\Lambda}_{(\Sigma,\Theta)}, (\Sigma, \Theta) \models \gamma$ for all $\gamma \in \Gamma$ and $M^{\Lambda}_{(\Sigma,\Theta)}, (\Sigma, \Theta) \not\models \varphi$. Since M^{Λ} is an H-model by Lemma 35, we can conclude $\Gamma \not\models \varphi$, as desired. ■

Moreover, we can prove that by extending **UBiSKt** with the formula $q \vee (q \to (p \vee \neg p))$, expressing the property $(H \cap \breve{\bar{H}})^2 = \varnothing$ by Theorem 21, we get a sound and complete system w.r.t. the class of H-frames that are hypergraphs.

Definition 38. We define $\mathbb{HG}$ as the class of all H-frames (U, H, R) such that (U, H) is a hypergraph, and let $\mathbf{bd}_2$ be the formula $q \vee (q \to (p \vee \neg p))$.

Lemma 39. Given a *ubist*-logic Λ such that $\mathbf{bd}_2 \in \Lambda$, the Λ-canonical H-frame $F^{\Lambda}_{(\Gamma,\Delta)} = (U^{\Lambda}, H^{\Lambda}, R^{\Lambda})$ satisfies $(H^{\Lambda} \cap \breve{\overline{H^{\Lambda}}})^2 = \varnothing$, i.e. $F^{\Lambda}_{(\Gamma,\Delta)} \in \mathbb{HG}$.

Proof. We prove that $(\Gamma_1,\Delta_1)H^\Lambda(\Gamma_2,\Delta_2)H^\Lambda(\Gamma_3,\Delta_3)$ implies $(\Gamma_2,\Delta_2)H^\Lambda(\Gamma_1,\Delta_1)$ or $(\Gamma_3,\Delta_3)H^\Lambda(\Gamma_2,\Delta_2)$. Suppose that $(\Gamma_1,\Delta_1)H^\Lambda(\Gamma_2,\Delta_2)H^\Lambda(\Gamma_3,\Delta_3)$ and assume that $(\Gamma_2,\Delta_2)H^\Lambda(\Gamma_1,\Delta_1)$ fails, i.e., $\Gamma_2 \not\subseteq \Gamma_1$. To show that $(\Gamma_3,\Delta_3)H^\Lambda(\Gamma_2,\Delta_2)$, let us suppose that $\varphi \in \Gamma_3$. Our goal is to establish $\varphi \in \Gamma_2$. Since $\Gamma_2 \not\subseteq \Gamma_1$, there exists a formula ψ such that $\psi \in \Gamma_2$ but $\psi \notin \Gamma_1$. By $\mathbf{bd}_2 \in \Lambda$, $\psi \vee (\psi \to (\varphi \vee \neg\varphi)) \in \Gamma_1$ hence $\psi \to (\varphi \vee \neg\varphi) \in \Gamma_1$ by $\psi \notin \Gamma_1$. It follows from $\psi \in \Gamma_2$ that $\varphi \vee \neg\varphi \in \Gamma_2$. Since $\varphi \in \Gamma_3$ and $(\Gamma_2,\Delta_2)H^\Lambda(\Gamma_3,\Delta_3)$, we obtain $\neg\varphi \notin \Gamma_2$ by Lemma 31 item 4. Therefore, we deduce from $\varphi \vee \neg\varphi \in \Gamma_2$ that $\varphi \in \Gamma_2$, as desired. Because H^Λ is antisymmetric, we conclude that $F^\Lambda_{(\Gamma,\Delta)} \in \mathbb{HG}$. ■

By Theorem 21 and Lemmas 36 and 39, we can establish the following.

Theorem 40 (Soundness and Strong Completeness of $\mathsf{H}\mathbf{UBiSKtbd}_2$). The logic $\mathsf{H}\mathbf{UBiSKt}$ extended with **bd2** is sound and complete for the class $\mathbb{HG}$, i.e. $\Gamma \vdash_{\mathsf{H}\mathbf{UBiSKtbd}_2} \varphi$ iff $\Gamma \models_{\mathbb{HG}} \varphi$, for every set $\Gamma \cup \{\varphi\}$ of formulas.

We can obtain the same result for the class of H-frames that have width bounded to maximum size of 2, in the sense of Theorem 25.

Definition 41. We define $\mathbb{W}2$ as the class of H-frames (U,H,R) such that H has width of maximum size 2, i.e. $\forall u,v,w,t\ ((u\ H\ v \wedge u\ H\ w \wedge u\ H\ t) \to (v\ H\ w \vee w\ H\ v \vee v\ H\ t \vee t\ H\ v \vee w\ H\ t \vee t\ H\ w))$. Let **w2** be the formula $(p \to (q \vee r)) \vee (q \to (p \vee r)) \vee (r \to (p \vee q))$.

Lemma 42. Given a *ubist* logic Λ such that $\mathbf{w2} \in \Lambda$, the Λ-canonical H-frame $F^\Lambda_{(\Gamma,\Delta)} = (U^\Lambda, H^\Lambda, R^\Lambda)$ satisfies the fact that H^Λ has width of maximum size 2, i.e. $F^\Lambda \in \mathbb{W}2$.

Proof. Suppose $(\Gamma_1,\Delta_1)\ H^\Lambda\ (\Gamma_2,\Delta_2)$ and $(\Gamma_1,\Delta_1)\ H^\Lambda\ (\Gamma_3,\Delta_3)$, and $(\Gamma_1,\Delta_1)\ H^\Lambda\ (\Gamma_4,\Delta_4)$. We need to show that (i) $(\Gamma_2,\Delta_2)\ H^\Lambda\ (\Gamma_3,\Delta_3)$, or (ii) $(\Gamma_3,\Delta_3)\ H^\Lambda\ (\Gamma_2,\Delta_2)$, or (iii) $(\Gamma_2,\Delta_2)\ H^\Lambda\ (\Gamma_4,\Delta_4)$, or (iv) $(\Gamma_4,\Delta_4)\ H^\Lambda\ (\Gamma_2,\Delta_2)$, or (v) $(\Gamma_3,\Delta_3)\ H^\Lambda\ (\Gamma_4,\Delta_4)$, or (vi) $(\Gamma_4,\Delta_4)\ H^\Lambda\ (\Gamma_3,\Delta_3)$. Suppose that none of (i), (ii), (iii) and (v) hold. We need to show that then at least one of (iv) and (vi) holds. Thus the goal is to show that given any formula $\theta \in \Gamma_4$, $\theta \in \Gamma_2$, or $\theta \in \Gamma_3$. From the fact that neither (i) nor (iii) hold, we have that $\Gamma_2 \not\subseteq \Gamma_3$ and $\Gamma_2 \not\subseteq \Gamma_4$, thus there is a $\varphi_1 \in \Gamma_2$ such that $\varphi_1 \notin \Gamma_3$ and there is a $\varphi_2 \in \Gamma_2$ such that $\varphi_2 \notin \Gamma_4$. So by Lemma 31 item 5, we have that $\varphi_1 \wedge \varphi_2 \in \Gamma_2$ and $\varphi_1 \wedge \varphi_2 \notin \Gamma_3$ and $\varphi_1 \wedge \varphi_2 \notin \Gamma_4$. Put $\varphi = \varphi_1 \wedge \varphi_2$. From the fact that neither (ii) nor (v) hold, we have that $\Gamma_3 \not\subseteq \Gamma_2$ and $\Gamma_3 \not\subseteq \Gamma_4$, thus there is a $\psi_1 \in \Gamma_3$ such that $\psi_1 \notin \Gamma_2$ and there is a $\psi_2 \in \Gamma_3$ such that $\psi_2 \notin \Gamma_4$. So by Lemma 31 item 5, we have that $\psi_1 \wedge \psi_2 \in \Gamma_3$ and $\psi_1 \wedge \psi_2 \notin \Gamma_2$ and $\psi_1 \wedge \psi_2 \notin \Gamma_4$. Put $\psi = \psi_1 \wedge \psi_2$. As $\mathbf{w2} \in \Gamma_1$ by Lemma 31 item 2, and as any substitution instance of **w2** is in Γ_1 by Lemma 31 item 1, we have that the formula $(\varphi \to (\psi \vee \theta)) \vee (\psi \to (\varphi \vee \theta)) \vee (\theta \to (\psi \vee \theta)) \in \Gamma_1$, then by Lemma 31 item 6, at least one of the three disjuncts must be in Γ_1. Suppose that $(\varphi \to (\psi \vee \theta)) \in \Gamma_1$. Then as $(\Gamma_1,\Delta_1)\ H^\Lambda\ (\Gamma_2,\Delta_2)$ and thus $\Gamma_1 \subseteq \Gamma_2$, we have that $(\varphi \to (\psi \vee \theta)) \in \Gamma_2$. But $\varphi \in \Gamma_2$, thus $(\psi \vee \theta) \in \Gamma_2$ by Lemma 31 item 3, and thus $\psi \in \Gamma_2$ or $\theta \in \Gamma_2$. But $\psi \notin \Gamma_2$, thus $\theta \in \Gamma_2$ must be the case. Suppose that $(\psi \to (\varphi \vee \theta)) \in \Gamma_1$. Then as $(\Gamma_1,\Delta_1)\ H^\Lambda\ (\Gamma_3,\Delta_3)$ and thus $\Gamma_1 \subseteq \Gamma_3$, we have that $(\psi \to (\varphi \vee \theta)) \in \Gamma_3$. As $\psi \in \Gamma_3$, we have that $(\varphi \vee \theta) \in \Gamma_3$, i.e. $\varphi \in \Gamma_3$ or $\theta \in \Gamma_3$. But $\varphi \notin \Gamma_3$, thus $\theta \in \Gamma_3$ must be the case. Finally suppose that $(\theta \to (\psi \vee \theta)) \in \Gamma_1$. Then as $\Gamma_1 \subseteq \Gamma_4$, we have that

$(\theta \to (\psi \vee \theta)) \in \Gamma_4$, and as $\theta \in \Gamma_4$ by assumption, $\varphi \in \Gamma_4$ or $\psi \in \Gamma_4$. But this is impossible as we have already seen that, by construction, neither φ nor ψ is in Γ_4. Thus we can conclude that, given a formula $\theta \in \Gamma_4$, we have $\theta \in \Gamma_2$ or $\theta \in \Gamma_3$, and thus $\Gamma_4 \subseteq \Gamma_2$ or $\Gamma_4 \subseteq \Gamma_3$ holds, i.e. case (iv) holds or case (vi) holds, as desired. We can conclude that the canonical frame $F^{\Lambda}_{(\Gamma,\Delta)} \in \mathbb{W}2$.

Finally we notice that, as H^{Λ} is antisymmetric, the canonical frame is also in the class of $\mathbb{W}2$ H-frames that are additionally partial orders. ■

By Theorem 25 and Lemmas 36 and 42, we can establish the following.

Theorem 43 (Soundness and Strong Completeness of **HUBiSKtw**$_2$)**.** The logic **HUBiSKt** extended with **w2** is sound and complete for the class $\mathbb{W}2$, i.e. $\Gamma \vdash_{\mathbf{HUBiSKtw_2}} \varphi$ iff $\Gamma \models_{\mathbb{W}2} \varphi$, for every set $\Gamma \cup \{\varphi\}$ of formulas.

Thus, from Theorems 40 and 43, we can conclude that **UBiSKt** extended with both **bd2** and **w2** is sound and complete for the class of H-frames $\mathbb{HG} \cap \mathbb{W}2$, i.e. the class of hypergraphs that are also graphs.

Finally in [71], decidability of **UBiSKt** has been proved.

Theorem 44 (Decidability of **HUBiSKt**)**.** For every non-theorem φ of **HUBiSKt**, there is a finite frame F such that $F \not\models \varphi$. Therefore, **HUBiSKt** is decidable.

Questions about the size of the finite frames needed for Theorem 44 haven't been explored yet, and this is left as an open problem.

2.4.2 Tableau-style calculus

The calculus **TabUBiSKt** is a tableau-style calculus for **UBiSKt**. It has been implemented using the theorem-prover generator *MetTel* [85]. Our implementation of **TabUBiSKt** is available at [68]. In this section we are going to show that it is formally equivalent to **HUBiSKt**, so the tableau calculus can be seen as a computational tool for reasoning with **UBiSKt**. **TabUBiSKt** is the extension of the tableau-style calculus for **BiSKt**, as described in [74], with the rules contained in Table 2.2, that are the rules handling truth and falsity of formulae where the main operators are the universal modalities A and E (for the full tableau calculus, see Table A.1 in Appendix A.3. Expressions in **TabUBiSKt** have one of these forms:

$$s : S\varphi \qquad \bot \qquad sHt \qquad sRt \qquad s \approx t \qquad s \not\approx t$$

where S denotes a sign, either T for true or F for false, and s, t are names or labels from a fixed set Label in the tableau language whose intended meaning are elements of U.

As in ordinary tableau calculi, rules in **TabUBiSKt** are used to decompose formulae analysing their main connective. As some rules are branching, i.e. two possible conclusions are derivable from the rule premise, the tableau derivation process constructs a tree-structure. When contradiction $\bot$ is derived on a branch then the branch is said to be *closed*, and no more rules are applied to that branch. If a branch is not closed, then it is *open*. If a branch is open and no more rules can be applied to it, as all the formulae have been analysed, then the branch is said to be *fully-expanded*. A tableau derivation is *closed* when all its branches are closed, it is open otherwise. The derivation process is said to be *fully expanded*

$$\frac{s : T(\mathsf{A}\,\varphi), \quad t : S\psi}{t : T\varphi}\ (T\,\mathsf{A}) \qquad \frac{s : F(\mathsf{A}\,\varphi)}{m : F\varphi}\ (F\,\mathsf{A})\ m \text{ is fresh in the branch}$$

$$\frac{s : T(\mathsf{E}\,\varphi)}{m : T\varphi}\ (T\,\mathsf{E})\ m \text{ is fresh in the branch} \qquad \frac{s : F(\mathsf{E}\,\varphi), \quad t : S\psi}{t : F\varphi}\ (F\,\mathsf{E})$$

Table 2.2: Rules for handling truth and falsity of universal modalities in Tab**UBiSKt**.

when any the branch in the tableau derivation is either closed or open and fully expanded. An open fully expanded branch will give the information for building model for a set of tableau expressions given as derivation input. The goal of the derivation process is to produce a closed tableau, or to find at least one open fully expanded branch, given some input formulae. If for example we are interested in the validity of a formula φ, then the input to tableau derivation will be the set $\{ a : F\varphi \}$, where 'a' is a constant label which is intended to represent the initial world. If an open and fully expanded branch can be constructed, then a model for this input is found, i.e. a counter-model for the validity of φ. If on the other hand a closed tableau derivation is constructed, then the validity of φ is proved. A formula φ is a *theorem* in Tab**UBiSKt** if a tableau derivation for the input set $\{a : F\varphi\}$, gives a closed tableau derivation. A formula φ is *provable* from a finite set Γ of formulae if a tableau derivation for the input set $\{a : T\Gamma\} \cup \{a : F\varphi\}$ gives a closed tableau derived, where $a : T\Gamma$ means $(a : T\gamma)$, for all $\gamma \in \Gamma$.

Theorem 45 (Soundness of Tab**UBiSKt**)**.** Given a finite set $\Gamma \cup \{ \varphi \}$ of formulae, if φ is provable from Γ in Tab**UBiSKt** then $\Gamma \models \varphi$.

Before giving a proof, we 'lift' our semantics based on an H-model for formulae to tableau expressions. As already explained in [74], the semantics of tableau expressions and the corresponding relation of satisfaction, $\Vdash$, is defined by an H-model (M, ι) with an assignment ι, where $M = (U, H, R, V)$ is an H-model and an assignment $\iota : \mathsf{Label} \to U$ is a function mapping labels of the tableau language to elements $u \in U$. Satisfaction of tableau expressions is defined as follows:

$$\begin{array}{lll} M, \iota \nVdash \bot, & & \\ M, \iota \Vdash s : T\varphi & \text{iff} & M, \iota(s) \models \varphi, \\ M, \iota \Vdash s : F\varphi & \text{iff} & M, \iota(s) \not\models \varphi, \\ M, \iota \Vdash sHt & \text{iff} & \iota(s)\ H\ \iota(t), \\ M, \iota \Vdash sRt & \text{iff} & \iota(s)\ R\ \iota(t), \\ M, \iota \Vdash s \approx t & \text{iff} & \iota(s) = \iota(t), \\ M, \iota \Vdash s \not\approx t & \text{iff} & \iota(s) \neq \iota(t). \end{array}$$

Let us say that a set of tableau expression is *satisfiable* if there exists an H-model $M = (U, H, R, V)$ and an assignment $\iota : \mathsf{Label} \to U$ such that all the tableau expressions are satisfied in the pair (M, ι).

Now we proceed to our proof of Theorem 45.

Proof. Suppose that φ is provable from Γ, that is, the input set $\{a : T\Gamma\} \cup \{a : F\varphi\}$ has a closed tableau derivation. Let us fix this closed tableau and let us suppose by contradiction that $\Gamma \not\models \varphi$. So there is an H-model $M = (U, H, R, V)$

and state $u \in U$ such tha $M, u \models \gamma$ for all $\gamma \in \Gamma$, and $M, u \not\models \varphi$. Consider an assignment function ι such that $\iota(a) = u$, then $M, \iota(a) \models \Gamma$ and $M, \iota(a) \not\models \varphi$. But then, under the assumption that the rules of the calculus preserve satisfiability, there should be at least one open branch for the set of labelled expressions $\{a : T\Gamma\} \cup \{a : F\varphi\}$. This is impossible by the initial assumption that $\{a : T\Gamma\} \cup \{a : F\varphi\}$ has a closed tableau. So the proof of soundness boils down to show that the rules of the calculus preserve satisfiability, that is, if the premise of a rule is satisfiable, so is at least one of its conclusions. Since **TabBiSKt** has already been proved sound in [74], we focus only on the new rules $T\,\mathsf{A}$, $F\,\mathsf{A}$. Our arguments for $T\,\mathsf{E}$ and $F\,\mathsf{E}$ are similarly shown.

(i) $T\,\mathsf{A}$: assume that the premise of the rule is satisfiable, so for some H-model and for some assignment ι we have $M, \iota(s) \models \mathsf{A}\,\varphi$ and $M, \iota(t) \models^{*} \psi$ where $\models^{*}$ is either $\models$ or $\not\models$. By the former, we have that $M, u \models \varphi$ for all $u \in U$. But $\iota(t) \in U$. So $M, \iota(t) \models \varphi$. So $M, \iota \Vdash t : T\varphi$ and an expanded branch with the rule's conclusion is satisfiable.

(ii) $F\,\mathsf{A}$: assume that the premise of the rule is satisfiable, so for some H-model and for some assignment ι we have $M, \iota(s) \not\models \mathsf{A}\,\varphi$. But then there is some world $v \in W$ such that $M, v \not\models \varphi$. Fix such world v. Recall that m is a fresh label in the rule as underlined in Table 2.2. We define a new label assignment ρ by $\rho(m) = v$ and $\rho(x) = \iota(x)$ for all labels $x \neq m$. Then it follows from $M, v \not\models \varphi$ that $M, \rho(m) \not\models \varphi$, that is $M, \rho \Vdash m : F\varphi$. Therefore, the tableau expression $m : F\varphi$ is satisfiable in the pair (M, ρ), and thus an expanded branch with the rule's conclusion is satisfiable. ■

Theorem 46. Given a formula φ the following are equivalent:

1. φ is a theorem in **HUBiSKt**,
2. φ is a theorem in **TabUBiSKt**,
3. φ is valid in all H-models.

Proof. The proof of equivalence of items 1 and 3 are due to the soundness and completeness results for **HUBiSKt**. So we need to show: (i) if φ is a theorem in **HUBiSKt** then φ is a theorem in **TabUBiSKt**, and (ii) if φ is a theorem in **TabUBiSKt** then φ is a theorem in **HUBiSKt**. Proof of (ii) follows from theorem 45 (soundness of **TabUBiSKt**) and theorem 37 (completeness of **HUBiSKt**). So we focus on (i).

Proof of (i): Recall that φ is a theorem **HUBiSKt** when φ follows from a set of axioms and rules given in Table 2.1. To show this direction, we reformulate our Hilbert-system into an equivalent system in the following two respects. First of all, to avoid the rule of uniform substitutions, we formulate our system in terms of axioms *schemes*, so instead of propositional variables we use generic formulae as variables of the axioms. Second, we reformulate the inference rules of Table 2.1 into an axiom as follows: for a rule of the form "from φ infer ψ", we can derive a formula $\mathsf{A}\,\varphi \to \mathsf{A}\,\psi$. Instead of showing an inference rule "from φ infer ψ" preserves theoremhood in **TabUBiSKt**, we show that $\mathsf{A}\,\varphi \to \mathsf{A}\,\psi$ is a theorem of **TabUBiSKt**. Then we can show that all the axiom schemes constructed from Table 2.1 are theorems in **TabUBiSKt** and that the above derived formula from an inference rule of Table 2.1 is a theorem in **TabUBiSKt**. We give here our proof for `Mon` ⧫ as an example. The rule states that if $(\varphi \to \psi)$ is a theorem then $(\blacklozenge\varphi \to \blacklozenge\psi)$ is a theorem. That means that the following implication must be a theorem: $\mathsf{A}(\varphi \to \psi) \to \mathsf{A}(\blacklozenge\varphi \to \blacklozenge\psi)$. For a proof of this formula using Tab**UBiSKt**, see Table 2.3.

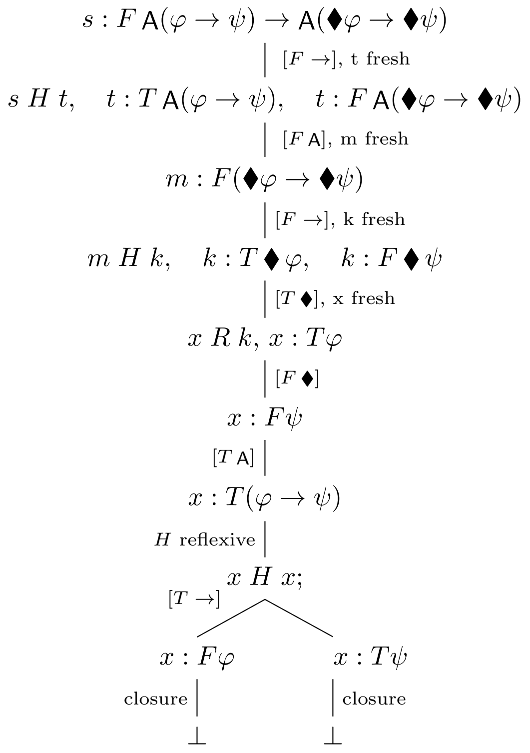

Table 2.3: Proof of the formula $\mathsf{A}(\varphi \to \psi) \to \mathsf{A}(\blacklozenge\varphi \to \blacklozenge\psi)$ in Tab**UBiSKt**.

The input set $\{s : F\,\mathsf{A}(\varphi \to \psi) \to \mathsf{A}(\blacklozenge\varphi \to \blacklozenge\psi)\}$ gives a closed tableau derivation. Therefore the formula $\mathsf{A}(\varphi \to \psi) \to \mathsf{A}(\blacklozenge\varphi \to \blacklozenge\psi)$ is a theorem in Tab**UBiSKt**.

We also note that axioms and rules of H**UBiSKt** have been proved using our implementation of Tab**UBiSKt** by the theorem prover generator *MetTel* [85], in terms of the idea above on our reformulation. This can be checked following the link at [68]. ■

Theorem 46 shows that the proof systems H**UBiSKt** and Tab**UBiSKt** capture the same set of theorems. Since H**UBiSKt** is decidable (Theorem 44), the tableau-system Tab**UBiSKt** can be seen as the specification of a concrete algorithm for deciding whether a formula $\varphi \in \mathsf{Form}_{\mathcal{L}}$ is a theorem in H**UBiSKt**.

Chapter 3

UBiSKt for Discrete Spatial Representation

> In this chapter we will show how the logic **UBiSKt** can be used to represent and reason with topological spatial relations between subgraphs, i.e. spatial relations in a discrete setting.

3.1 Mereotopological connection

In the work *Mereotopological Connection* [13] the authors aim to provide with a unifying framework for comparing mereotopological theories, so theories for representing space, but also time, based on the notion of topological connection. As the authors say, there are many of these accounts, that are not always in agreement on basic terms. The framework of [13] is a topological space. They study how spatial predicates from different mereotopological theories get interpreted when the variables these predicates apply to range over the elements of a topological space. In this section we review the approach of Cohn and Varzi [13] and show that it needs to be generalized if it is to capture discrete connection as the one defined by Galton [27].

There are many ways to define a topological space, and one of these is the definition via Kuratowski closure axioms.

Definition 47. Let U be a set and let $c : \mathcal{P}(U) \to \mathcal{P}(U)$ be a function. We say c is a *Kuratowski closure* if it satisfies the following axioms (Kuratowski axioms) for all $X, Y \in \mathcal{P}(U)$.

K1 $c(\varnothing) = \varnothing$

K2 $X \subseteq c(X)$

K3 $c(c(X)) \subseteq c(X)$

K4 $c(X \cup Y) = c(X) \cup c(Y)$

Given a set U together with an operator c satisfying $K1$-$K4$ axioms, the set $\tau = \{\, X \subseteq U \mid c(X) = X \,\}$ is the *topology* on U associated with c, and the pair (U, τ) is called a *topological space.* The elements of τ are the *closed sets* of the topological space.

It follows from axiom $K4$ that a Kuratowski closure is monotone, i.e. for any $X, Y \in \mathcal{P}(U)$, if $X \subseteq Y$ then $c(X) \subseteq c(Y)$, see [55]. A dual notion of the Kuratowski closure operator is known as *Kuratowski interior*, which is also a function from $\mathcal{P}(U)$ to itself such that is satisfies: (1): $i(U) = U$, (2): $i(X) \subseteq X$, (3): $i(X) \subseteq i(i(X))$ and (4): $i(X \cap Y) = i(X) \cap i(Y)$, [55]

Cohn and Varzi [13] give three definitions of connection which all depend on the above defined notion of topological closure.

Definition 48. Let c be a Kuratowski closure on U, and $X, Y \subseteq U$. Three binary relations of connection between subsets X, Y are defined as follows.

1. $C_1(x, y) := x \cap y \neq \varnothing$
2. $C_2(x, y) := c(x) \cap y \neq \varnothing$ or $x \cap c(y) \neq \varnothing$
3. $C_3(x, y) := c(x) \cap c(y) \neq \varnothing$

An important model of the RCC [57] consists in taking regions to be non-empty regular closed subsets of $\mathbb{R}^2$ [59], with the usual topology. A subset is called regular closed when it is equal to the closure of its interior In particular, this means that although a single point, or a line including its endpoints, is a closed set in $\mathbb{R}^2$, it is not regular closed, as its interior is empty. Therefore, such elements are not considered as regions in this context, and RCC belongs to those theories which do not allow boundary elements in their domain. In the regular-closed model of RCC, all three connections above yield the same relation between regions[1]. However, in other models of RCC and in other mereotopological systems the three notions of connection can have substantially different properties.

Although Cohn and Varzi [13, p359] aim for neutrality with respect to density of space, that is whether space can be repeatedly sub-divided *ad infinitum*, we shall see next that the use of Kuratowski closure prevents the expression of some notions of connection including one of the most straightforward examples of connection in a discrete space.

3.1.1 Example of Discrete Connection

As we have seen in Section 1.6, Galton [27] studied a notion of connection between subsets of a particular kind of discrete space, i.e. an adjacency space, defined by set N together with a symmetric and reflexive relation of adjacency $A \subseteq N \times N$. Connection, C_A, is defined for subsets $X, Y \subseteq N$ by $X \; C_A \; Y$ if there are $x \in X$ and $y \in Y$ such that $x \; A \; y$. We shall show next that there are spaces (N, A) where this connection is not expressible as any C_i, in the sense of Cohn and Varzi, for any topological closure on N. A specific example appears in Figure 3.1 where the links indicate adjacencies between distinct elements of the five element set $N = \{ m, n, p, q, r \}$.

First, C_A cannot be C_1 as two adjacent nodes give disjoint singleton subsets which are C_A connected. So suppose that $C_A = C_2$ for some closure c. If x is any node in N then $\{x\}$ is C_A connected to no singletons except those $\{y\}$ such that $x \; A \; y$. Thus, as C_A is C_2 by assumption, $c(\{x\})$ contains only nodes which are

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[1] Indeed if we can quantify only over regular closed regions X and Y, we have that $X \cap Y$ is equivalent to $c(i(X) \cap c(i(Y))$ that is in turn the same as $c(c(i(X))) \cap c(i(Y))$, equivalent to $c(c(i(X))) \cap c(c(i(Y)))$ by axioms $K2$ and $K3$, where i is the interior operator.

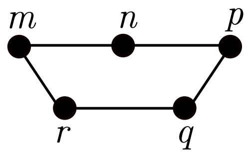

Figure 3.1: A discrete space where connection cannot be defined in terms of Kuratwoski closure (figure adapted from [73]).

adjacent to x. Hence for the specific nodes m and n we have $c(\{m\}) \subseteq \{r, m, n\}$ and $c(\{n\}) \subseteq \{m, n, p\}$. Now $\{m\}$ and $\{n\}$ are connected in the connection C_A so if they are C_2 connected we must have (i) $n \in c(\{m\})$ or (ii) $m \in c(\{n\})$. Consider first the case (i). This implies $\{n\} \subseteq c(\{m\})$ so, by monotonicity of c and axiom $K3$ we have that $c(\{n\}) \subseteq c(c(\{m\})) \subseteq c(\{m\})$. So $c(\{\, m \,\}) \subseteq \{\, m, n, p \,\}$ and $c(\{\, n \,\}) \subseteq \{\, r, m, n \,\}$. But $p \notin c(\{m\})$ and $r \notin \{\, m, n, p \,\}$, so $c(\{n\}) \subseteq \{m, n\}$. But $\{n\}$ and $\{p\}$ are connected in C_A, so $n \in c(\{p\})$, and hence $c(\{n\}) \subseteq c(\{p\})$ (it can't be that $p \in c(\{\, n \,\})$ as we have established that this is subset of $\{\, m, n \,\}$). As $m \notin c(\{p\}) \subseteq \{\, n, p, q \,\}$ we conclude $c(\{n\}) = \{n\}$ in the case that case (i) holds, i.e. $n \in c(\{m\})$. If we consider case (ii), i.e. that $m \in c(\{n\})$, we can argue in analogous way and conclude that $c(\{m\}) = \{m\}$. Thus in either case one of the sets $\{m\}$ and $\{n\}$ is a closed set, and they cannot both be closed at the same time since they need to be C_2 connected, as they are adjacent.

This applies to each pair of adjacent nodes in N; one of them is a closed set and the other is not. With an odd number of nodes in total this is a contradiction, as we will always have two adjacent nodes that are both closed, and thus they won't be connected in C_2. Hence no such topological closure, c, can generate a C_2 connection equal to C_A. There remains the possibility that C_A is of the form C_3. Suppose then that some topological closure c on N generates C_A as C_3. We must have $c(\{m\}) \cap c(\{n\}) \neq \varnothing$. For similar reasons to the C_2 case we must have $c(\{m\}) \subseteq \{r, m, n\}$ and $c(\{n\}) \subseteq \{m, n, p\}$, so to obtain the non-empty intersection either (i) $n \in c(\{m\})$ or (ii) $m \in c(\{n\})$. If (ii) is the case, from $n \in c(\{m\})$ we get that $c(\{\, n \,\}) \subseteq c(\{\, m \,\})$. Thus $p \notin c(\{n\})$ and $c(\{\, n \,\}) \subseteq \{\, m, n \,\}$. But n and p are adjacent so their singletons are connected singletons and thus $c(\{n\})$ and $c(\{p\})$ must intersect. The only possibility for this intersection is n. Thus $\{\, n \,\} \in c(\{\, p \,\})$, hence $c\{\, n \,\} \subseteq c(c(\{\, p \,\})) \subseteq c(\{\, p \,\})$. But as $m \notin c(\{\, p \,\})$ as m and p are not adjacent, thus we have that $c(\{\, n \,\})$ can only be $\{\, n \,\}$. Again case (i) implies that $\{\, n \,\}$ is closed and case (ii) implies, by analogous reasoning, that $\{\, m \,\}$ is closed. This applies again to any pair of adjacent nodes, and in the case of an odd number of node it is straightforward to continue to a contradiction as in the C_2 case.

Thus we can conclude that this type of discrete connection C_A is neither C_1, C_2 nor C_3 type.

3.2 Discrete Connection in UBiSKt

We have seen that connection in Galton's sense cannot always be expressed as one of the connections by Kuratowski closure in the framework of Cohn and Varzi. However there is a weaker notion of closure. This is known as Čech closure, and it satisfies axioms $K1$, $K2$ and $K4$ but not necessarily $K3$. As we are going to see there is an operator in **UBiSKt** that is a Čech closure, and it can be used to encode a satisfactory notion of connection in discrete space.

Definition 49. Let X be a Heyting Algebra[2] with bottom element 0 and top element 1, and let $c : X \to X$ be a function. We say the (X, c) is a *Čech closure algebra* if for all $x, y \in X$:

C1 $c(0) = 0$,

C2 $x \leq c(x)$,

C3 $c(x \cup y) = c(x) \cup c(y)$.

Let $i : X \to X$ be a function. We say that (X, i) is a Čech interior algebra if for all $x, y \in X$:

I1 $i(1) = 1$,

I2 $i(x) \leq x$,

I3 $i(x \cap y) = i(x) \cap i(y)$.

In what follows we are going to consider two functions from the set of all H-sets in an H-model M, i.e. $\{ [\![\varphi]\!]_M \mid \varphi \in \mathsf{Form} \}$, to itself.

Definition 50. Let $Hsets_M$ be the set of all H-sets in an H-model M, i.e. $\{ [\![\varphi]\!]_M \mid \varphi \in \mathsf{Form} \}$. We define the functions $\lrcorner\neg : Hsets_M \to Hsets_M$ and $\neg\lrcorner : Hsets_M \to Hsets_M$ by following mapping respectively: $[\![\varphi]\!]_M \mapsto [\![\lrcorner\neg\varphi]\!]_M$ and $[\![\varphi]\!]_M \mapsto [\![\neg\lrcorner\varphi]\!]_M$ (given a formula φ interpreted over the H-set $[\![\varphi]\!]_M$)[3].

Theorem 51. Let M be an H-model. Then the set of H-sets $HSets_M$ with $\lrcorner\neg$ defines a Čech closure algebra. $Hsets_M$ with $\neg\lrcorner$ defines a Čech interior algebra.

Proof. Since **UBiSKt** is an extension of intuitionistic logic, we have that given an H-model $M = (U, H, V)$, the set $Hsets_M = \{ [\![\varphi]\!]_M \mid \varphi \in \mathsf{Form} \}$ forms a Heyting algebra with $[\![\bot]\!]_M = \varnothing$ as the bottom element 0 and $[\![\top]\!]_M = U$ as the top element 1. Then we can show using the axiomatisation that the following holds:

1. $\vdash \lrcorner\neg\bot \leftrightarrow \bot$,

---

[2] We remind the reader that a Heyting algebra can be defined as a tuple $(A, \vee, \wedge, \to, \bot, \top)$, where $(A, \vee, \wedge, \bot, \top)$ is a lattice and for all $x, y, z \in A$ we have that $x \leq (y \to z)$ iff $(x \wedge y) \leq z$. This definition of a Heyting algebra is found in [80]. As explained in [74], given a set (U, H), the set of all its H-sets is a Heyting algebra, and thus also the set of all H-sets that are named by some formula in an H-model is a Heyting algebra.

[3] It is clear that the functions are well defined as given any formula $\varphi \in \mathsf{Form}$ such $[\![\varphi]\!]_M \in Hsets_M$ (i.e. φ is "name" of some H-set in the model) we can always build $[\![\lrcorner\neg\varphi]\!]_M$, and the H-set $[\![\lrcorner\neg\varphi]\!]_M$ must be in $Hsets_M$ as it is the H-set to which the formula $\lrcorner\neg\varphi \in \mathsf{Form}$ is assigned to. Same for $\neg\lrcorner$.

2. $\vdash p \to \lrcorner\neg p$,
3. $\vdash \lrcorner\neg(p \vee q) \leftrightarrow \lrcorner\neg p \vee \lrcorner\neg q$.
4. $\vdash \neg\lrcorner\top \leftrightarrow \top$,
5. $\vdash \neg\lrcorner p \to p$,
6. $\vdash \neg\lrcorner(p \wedge q) \leftrightarrow \neg\lrcorner p \wedge \neg\lrcorner q$.

Proof of items 1 *and* 4. By item 9 of Proposition 29 we have that $\vdash \bot \to \neg\lrcorner\bot$ iff $\vdash \lrcorner\neg\bot \to \bot$. But $\vdash \bot \to \neg\lrcorner\bot$ holds as $\bot \to \varphi$ is a theorem in intuitionistic logic. Therefore $\vdash \lrcorner\neg\bot \to \bot$ holds. The direction $\vdash \bot \to \lrcorner\neg\bot$ trivially holds. Thus we have proved item 1. Similarly for item 4 we have: $\vdash \neg\lrcorner\top \to \top$ as $\varphi \to \top$ is a theorem in intuitionistic logic. Thus we have that $\vdash \top \to \neg\lrcorner\top$ holds. The other direction trivially holds, we have proved item 1.
Proof of items 3 *and* 6. We note that item 9 of Proposition 29 implies that the pair $(\lrcorner\neg, \neg\lrcorner)$ forms an adjunction, with $\lrcorner\neg$ left adjoint and $\neg\lrcorner$ right adjoint. Thus the first preserves disjunctions and the second preserves conjunctions. This proves items 3 and 6.
Finally items 2 and 5 follow from items 10 and 11 of Proposition 29. By soundness of **HUBiSKt** (Theorem 28) it follows that $(Hsets_M, \lrcorner\neg)$ is a Čech closure algebra and $(Hsets_M, \neg\lrcorner)$ is a Čech interior algebra. ■

The following theorem shows that the operators $\lrcorner\neg$ and $\neg\lrcorner$ acting on H-sets can be also expressed respectively as dilation and erosion by a specific stable, i.e. $\curvearrowright H$.

Theorem 52. Let M be an H-model and $\llbracket\varphi\rrbracket_M \in Hset_M$. Then $\llbracket\varphi\rrbracket_M \oplus \curvearrowright H = \llbracket\lrcorner\neg\varphi\rrbracket_M$, and $\curvearrowright H \ominus \llbracket\varphi\rrbracket_M = \llbracket\neg\lrcorner\varphi\rrbracket_M$.

Proof. Theorem 4 from [74] states that given any stable relation R and H-set X we have that we have that $X \oplus \curvearrowright R = (-(R \ominus (H \ominus (-X)))) \oplus H$, and $\curvearrowright R \ominus X = H \ominus (-(((-X) \oplus H) \oplus R))$. Now suppose $R = H$ and take any H-set $\llbracket\varphi\rrbracket_M$ in an H-model M. Then, as H is stable we have that $\llbracket\varphi\rrbracket_M \oplus \curvearrowright H = (-(H \ominus (H \ominus (-\llbracket\varphi\rrbracket_M)))) \oplus H = (-(H \ominus (\llbracket\neg\varphi\rrbracket_M))) \oplus H = (-\llbracket\neg\varphi\rrbracket_M) \oplus H = \llbracket\lrcorner\neg\varphi\rrbracket_M$ by the semantics in terms of dilation and erosion given in Section 2.2.1 and because $\llbracket\varphi\rrbracket_M$ is an H-set. Similarly for the erosion, if $R = H$ we have that $\curvearrowright H \ominus \llbracket\varphi\rrbracket_M = H \ominus (-(((-\llbracket\varphi\rrbracket_M) \oplus H) \oplus H)) = H \ominus (-(\llbracket\lrcorner\varphi\rrbracket_M \oplus H)) = H \ominus (-\llbracket\lrcorner\varphi\rrbracket_M) = \llbracket\neg\lrcorner\varphi\rrbracket_M$. ■

Theorem 52 shows that the two operations can be seen in terms of dilation and erosion, and therefore they are equivalent to the two modalities ⧫ and $\Box$ when $R = \curvearrowright H$ or to $\Diamond$ and ■ when $R = H$, following the semantic definitions given in Section 2.2.1. Figure 3.2 shows the effect of the relation $\curvearrowright H$ on edges and nodes. Indeed when (U, H) is a graph or a hypergraph, thus a poset made of nodes and edges, taking the dilation by $\curvearrowright H$ of a subgraph K consists of taking its one-edge extension. In turn, the eroding a subgraph by $\curvearrowright H$ can be seen as a one-edge retraction. See Figure 3.3 for an example.

Hence, using the closure operator $\lrcorner\neg$ the spatial relation of connection between discrete regions $\llbracket\varphi\rrbracket_M$ and $\llbracket\psi\rrbracket_M$ in an H-model M can be expressed by an appropriate formula in **UBiSKt**:

$$C(\varphi, \psi) := \mathsf{E}(\lrcorner\neg\varphi \wedge \psi).$$

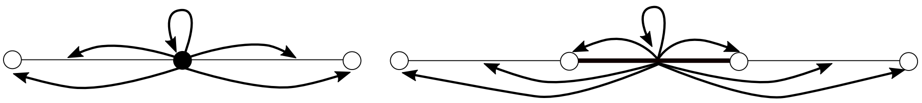

Figure 3.2: Effect of the left converse of H on one node and on one edge (figure adapted from [73]).

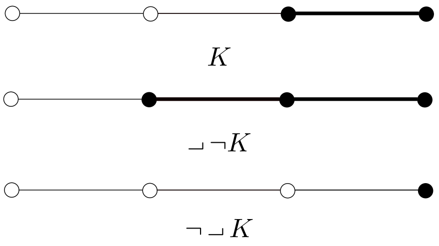

Figure 3.3: An example of a graph (U, H) with subgraph K in bold, with the Čech closure generated by $\lrcorner\neg$ and the associated Čech interior generated by $\neg\lrcorner$.

Indeed two discrete regions are connected if they are one edge apart, in the limit case. This is equivalent to require the closure of the first region, that is indeed its one-edge extension, intersect the second region. The predicate of connection is reflexive and symmetric, as expected. Reflexivity of connection is restricted to non-empty regions (see [13]), thus the theorem that we aim to prove is that every region is connected to itself whenever the region is non-empty:

Proposition 53. The following hold for H**UBiSKt**:

1. $\vdash \mathsf{E}(\varphi) \to \mathsf{E}(\varphi \wedge \lrcorner\neg\varphi)$.
2. $\vdash \mathsf{E}(\lrcorner\neg\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \lrcorner\neg\psi)$.

Proof. Item 1: $\vdash \varphi \to \varphi$ and $\vdash \varphi \to \lrcorner\neg\varphi$ both hold for H**UBiSKt** (the second one is item 2 of Theorem 51). Thus $\vdash \varphi \to (\varphi \wedge \lrcorner\neg\varphi)$. Thus by ($\mathtt{Mon}\,\mathsf{E}$) rule from Table 2.1 we have that $\vdash \mathsf{E}(\varphi) \to \mathsf{E}(\varphi \wedge \lrcorner\neg\varphi)$ as desired.
Item 2: $\vdash \mathsf{A}\,\neg(\lrcorner\neg\varphi \wedge \psi) \leftrightarrow \mathsf{A}(\lrcorner\neg\varphi \to \neg\psi)$, as $\neg(\alpha \wedge \beta) \leftrightarrow (\alpha \to \neg\beta)$ is a theorem in intuitionistic logic. $\vdash \mathsf{A}(\lrcorner\neg\varphi \to \neg\psi) \leftrightarrow \mathsf{A}(\varphi \to \neg\lrcorner\neg\psi)$, by adjunction between $\lrcorner\neg$ and $\neg\lrcorner$, from item 9 of Proposition 29. $\vdash \mathsf{A}(\varphi \to \neg\lrcorner\neg\psi) \leftrightarrow \mathsf{A}\,\neg(\varphi \wedge \lrcorner\neg\psi)$. So by concatenation $\vdash \mathsf{A}\,\neg(\lrcorner\neg\varphi \wedge \psi) \leftrightarrow \mathsf{A}\,\neg(\varphi \wedge \neg\lrcorner\psi)$, that is $\vdash \neg\,\mathsf{A}\,\neg(\lrcorner\neg\varphi \wedge \psi) \leftrightarrow \neg\,\mathsf{A}\,\neg(\varphi \wedge \lrcorner\neg\psi)$ by contraposition. Therefore $\mathsf{E}(\lrcorner\neg\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \lrcorner\neg\psi)$ by item 22 of Proposition 29[4]. ■

Item 2 of Proposition 53 implies that, given two subgraphs $[\![\varphi]\!]_M$ and $[\![\psi]\!]_M$ in an H-model M, whenever the formula $C(\varphi, \psi)$ holds in M, then $C(\varphi, \psi)$ or

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[4] Notice that here we use the adjunction rule between $\lrcorner\neg$ and $\neg\lrcorner$, that is $\vdash_{\mathbf{HUBiSKt}}$ $\lrcorner\neg\varphi \to \psi$ iff $\vdash_{\mathbf{HUBiSKt}}$ $\varphi \to \neg\lrcorner\psi$, in the form of the syntactic equivalence $\vdash_{\mathbf{HUBiSKt}}$ $\mathsf{A}(\lrcorner\neg\varphi \to \psi) \leftrightarrow \mathsf{A}(\varphi \to \neg\lrcorner\psi)$.

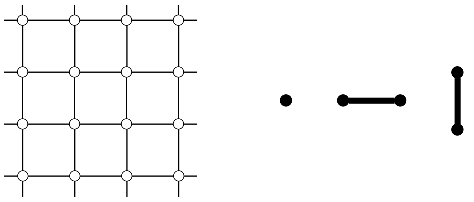
$C(\psi, \varphi)$ hold. Thus our idea of discrete connection is equivalent to the second notion of connection by closure proposed by Cohn and Varzi, C_2, with the difference that a Čech closure is considered instead of the usual notion of Kuratowski closure. The first notion of connection from Cohn and Varzi is equivalent to the idea of two subgraphs overlapping, so sharing at least a node, and the third idea of connection is expressed by the formula $\mathsf{E}(\lrcorner\neg\varphi \wedge \lrcorner\neg\psi)$ that means that the two subgraphs are, at most, two edges apart.

Lemma 54 ([74])**.** Let M be an H-model and $\llbracket\varphi\rrbracket_M$ and $\llbracket\psi\rrbracket_M \in Hset_M$. Then $\llbracket\varphi\rrbracket_M \subseteq \llbracket\psi\rrbracket_M$ iff $M \models \mathsf{A}(\varphi \to \psi)$

We now introduce the concept of an H-dilate.

Definition 55. Given a poset (U, H) and an element $u \in U$, we call *H-dilate* the set $\{\, v \in U \mid u\ H\ v \,\}$, i.e. the set $\{\, u \,\} \oplus H$.

In the case of a graph, where H-sets are the subgraphs, H-dilates can be seen as the smallest possible subgraphs contained in a graphs, the "minimal" H-sets built upon a single element of the domain. See Figure 3.4 for an example of H-dilates when the graph (U, H) is the infinite $\mathbb{Z}^2$ grid.



(U, H) is infinite grid Dilates of H

Figure 3.4: Shapes of the dilates of H when (U, H) is the graph shown (figure adapted from [71]).

Proposition 56. Given $X, Y \subseteq U$ such that X and Y are H-sets, the following are equivalent:

1. $X \subseteq Y$,
2. for all $u \in U$, $\{\, u \,\} \oplus H \subseteq X$ implies $\{\, u \,\} \oplus H \subseteq Y$

Proof. The proof of the fact that item *1* implies item *2* is trivial and it holds for any subset X and Y, as we don't need to use the assumption that they are H-sets. Indeed assume *1*, so that $X \subseteq Y$, and suppose $\{\, u \,\} \oplus H \subseteq X$ for some $u \in U$. Then, by transitivity of the subset relation $\subseteq$, we immediately have that $\{\, u \,\} \oplus H \subseteq Y$. The proof of the fact that item *2* implies item *1* is as follows. Let us assume *2*, and suppose that for some $u \in U$, $u \in X$. Then, as X is an H-set, $\{\, u \,\} \oplus H \subseteq X$. Then, by item *2*, $\{\, u \,\} \oplus H \subseteq Y$. By reflexivity of H, $u \in \{\, u \,\} \oplus H$, thus $u \in Y$, so $X \subseteq Y$, as wanted. ■

Proposition 3.2, together with Lemma 54, shows that the formula $\mathsf{A}(\varphi \to \psi)$ holding at a model M truly expresses the idea that all the "minimal" subgraphs in $\llbracket\varphi\rrbracket_M$ are also in $\llbracket\psi\rrbracket_M$, so that $\llbracket\varphi\rrbracket$ is a subgraph of $\llbracket\psi\rrbracket$.

Lemma 57. Let M be an H-model and $[\![\varphi]\!]_M$ and $[\![\psi]\!]_M \in Hset_M$. Then $[\![\varphi]\!]_M \not\subseteq [\![\psi]\!]_M$ iff $M \models \mathsf{E}(\varphi \prec \psi)$.

Proof. Let φ and ψ be formulae and $[\![\varphi]\!]_M$ and $[\![\psi]\!]_M$ the associated H-sets in a given model M. In what follows we omit the subscription M. $[\![\varphi]\!] \not\subseteq [\![\psi]\!]$ iff for some $u \in U$, $u \in [\![\varphi]\!]$ and $u \notin [\![\psi]\!]$. Since H is reflexive, uHu holds, hence there is a $v \in U$ such that vHu and $v \in [\![\varphi]\!]$ and $v \notin [\![\psi]\!]$. By definition 16 this means that $M, u \models \varphi \prec \psi$, hence $M \models \mathsf{E}(\varphi \prec \psi)$. On the other direction, $M \models \mathsf{E}(\varphi \prec \psi)$ iff for some $u \in U$ $M, u \models \varphi \prec \psi$. Hence there is a $v \in U$ such that vHu and $M, v \models \varphi$ and $M, v \not\models \psi$, that means that $v \in [\![\varphi]\!]$ and $v \notin [\![\psi]\!]$, for some $v \in U$. Therefore $[\![\varphi]\!] \not\subseteq [\![\psi]\!]$. ■

Lemmas 54 and 57 justify our choice of expressing the spatial relations of parthood and non-parthood as follows: $P(\varphi, \psi) := \mathsf{A}(\varphi \to \psi)$, and *non-*$P(\varphi, \psi) := \mathsf{E}(\varphi \prec \psi)$.

Using connection, parthood and non-parthood, and overlapping, expressed by the formula $\mathsf{E}(\varphi \wedge \psi)$ meaning that the intersection of $[\![\varphi]\!]_M$ and $[\![\psi]\!]_M$ in a given model M is non-empty, the RCC-8 spatial relations can be translated into **UBiSKt** language, and therefore applied to discrete regions as adjacency space and, more in general, graphs and hypergraphs. These spatial relations are listed in Table 3.1. Notice that we are actually using only the propositional part of **UBiSKt**, thus we can say that the discrete spatial relations are definable in a bi-intuitionistic logic with universal modalities.

Table 3.1: Spatial Relations and the corresponding formulae

Spatial Relation	Formula
$P(\varphi, \psi)$	$\mathsf{A}(\varphi \to \psi)$
$non\text{-}P(\varphi, \psi)$	$\mathsf{E}(\varphi \prec \psi)$
$O(\varphi, \psi)$	$\mathsf{E}(\varphi \wedge \psi)$
$PP(\varphi, \psi)$	$P(\varphi, \psi) \wedge non\text{-}P(\psi, \varphi)$
$NTPP(\varphi, \psi)$	$PP(\varphi, \psi) \wedge P(\lrcorner\neg\varphi, \psi)$
$TPP(\varphi, \psi)$	$PP(\varphi, \psi) \wedge non\text{-}P(\lrcorner\neg\varphi, \psi)$
$EC(\varphi, \psi)$	$C(\varphi, \psi) \wedge \neg O(\varphi, \psi)$
$DC(\varphi, \psi)$	$\neg C(\varphi, \psi)$
$PO(\varphi, \psi)$	$O(\varphi, \psi) \wedge non\text{-}P(\varphi, \psi)$ $\wedge non\text{-}P(\psi, \varphi)$
$EQ(\varphi, \psi)$	$P(\varphi, \psi) \wedge P(\psi, \varphi)$
$NTPP^i(\varphi, \psi)$	$NTPP(\psi, \varphi)$
$TPP^i(\varphi, \psi)$	$TPP(\psi, \varphi)$

Notice that classically the formula $\mathsf{A}(\varphi \to \psi)$, i.e. our definition of parthood, is equivalent to $\mathsf{A}\neg(\varphi \wedge \neg\psi)$ as $\vdash (\varphi \to \psi) \leftrightarrow \neg(\varphi \wedge \neg\psi)$ is a theorem in classical logic. Therefore the spatial relation of parthood could be expressed as $P(\varphi, \psi) := \mathsf{A}\neg(\varphi \wedge \neg\psi)$, using classical propositional logic with universal modalities. Is this a good definition for parthood between two subgraphs? Let us take the example of a simple graph (U, H), with $U = \{a, b, c\}$ and $H = I \cup \{(b, a), (b, c)\}$. Suppose we have a valuation V such that for propositional variables p and q, $V(p) = U$ and $V(q) = \{a, b\}$. Both $V(p)$ and $V(q)$ are H-sets thus $M = (U, H, V)$ is an H-model. Now $[\![\neg q]\!]_M = H \ominus (-\{a, c\}) = H \ominus \{b\} =$

$\varnothing$. So $\llbracket p \wedge \neg q \rrbracket_M = \varnothing$, and $\llbracket \neg(p \wedge \neg q) \rrbracket_M = H \ominus (-\varnothing) = H \ominus U = U$. So we have that $M \models \mathsf{A}\,\neg(p \wedge \neg q)$. Thus, if this is also our definition of parthood, we would be compelled to say that U, i.e. the whole graph, is a part of the subgraph made of the two nodes $\{\, a, b \,\}$. This is counter-intuitive. However in **UBiSKt** the implication $\neg(\varphi \wedge \neg\psi) \to (\varphi \to \psi)$ is not a theorem (as in intuitionistic logic this is not necessarily a theorem), thus we can have that $M \models \mathsf{A}\,\neg(\varphi \wedge \neg\psi)$ holds, without compelling us to accept that $\llbracket \varphi \rrbracket_M$ is part of $\llbracket \psi \rrbracket_M$, as in this case for $\llbracket p \rrbracket_M$ and $\llbracket q \rrbracket_M$. Indeed although $M \models \mathsf{A}\,\neg(p \wedge \neg q)$, it is not the case that $M \models \mathsf{A}(p \to q)$ as $b\ H\ b$ and $M, b \models p$ and $M, b \not\models q$. The fact that the two formulae are not equivalent in **UBiSKt** enables us to express that whole graph is not a part of a subgraph made of two nodes. This is a difference between the properties of spatial relations on graphs from the spatial relations on adjacency spaces and on sets.

We remark that the computation of the composition table for the discrete spatial relations defined in **UBiSKt** hasn't been considered yet, but we expect that the implementation of Tab**UBiSKt** (see [68] [69] and [70]) will facilitate this task.

3.3 Graph-Boundary

The boundary of a region is the part of that region that is adjacent to both the inside and to the outside of that region[5]. Whilst specifying what is *inside* a subgraph seems easy, as an element is inside X if it simply belongs to X, specifying what the *outside* of a subgraph is, might be more subtle. There seem to be three options. (1)We might consider an element that doesn't belong to a subgraph X, and thus belongs to its complement $-X$, as being outside of X. Or we might consider as the outside of X one of the two negations of X, i.e. (2) $\neg X$ or (3) $\lrcorner X$. All these operations on X, its complement and the two intuitionistic way of negating of X, are all equivalent in a boolean context but not in our context. Moreover we might think of the boundary of a subgraph as made of only nodes, or as made of both nodes and edges.

A first notion of boundary that can be adapted to the graph context and that can be expressed in **UBiSKt** is found already in Lawvere [43]: they call the *boundary* of any element X of a co-Heyting algebra, the meet of X and its dual pseudo-complement $\lrcorner X$. We call this set the boundary-nodes of X:

$$\beta^{\bullet}(X) := X \wedge \lrcorner X$$

This set can be described as follows: $\{\, u \in U \mid u \in X \text{ and } \exists v(v\ H\ u \text{ and } v \notin X) \,\}$. This is the set of nodes[6] of X that are immediately incident with at least one edge that does not belong to X. In this sense, these nodes lead outside X, and the idea of the *outside* of X is simply everything that is not in X. Thus the first

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[5] Notice that Galton [27], [28] says that the boundary of a region X might lie partly within and partly outside that region, as it is the set of points that are adjacent to both points within X, and to points outside X. The part of the boundary of X that lies fully within X is called there the *margin* of X. Thus the idea of idea of boundary we are describing here corresponds to what Galton defines as margin, and not to what he actually calls the boundary.

[6] It is clear $\beta^{\bullet}(X)$, if not empty, will only include nodes, and not edges of X. Indeed if u is an edge in X then asking that $u \in \lrcorner X$ means asking that there is a v such that vHu and $v \notin X$. But the only H-predecessor of edge u is u itself, as we are in a 2-tier poset. Thus as the edge u is in X, it can't be in $\lrcorner X$

idea of *outside* of X, namely $-X$, is employed here. Figure 3.5 (left and centre) shows a subgraph X with its boundary-nodes subgraph. As we are working with graphs, and they are made of both nodes and edges, an idea of boundary that inlcudes nodes only might seem unsatisfactory: it is reasonable to ask that also the edges of X that lie between any tuple of boundary-nodes are considered part of the boundary. We can then refine the notion of boundary-nodes of a subgraph X as follows, and we call this the *general boundary* of X

$$\beta(X) = \neg\neg(X \wedge \lrcorner X) \wedge X$$

Indeed the operator $\neg\neg$ on subgraphs can be seen as associating to a subgraph X its "regular" subgraph: if any two edge-adjacent nodes belong to X, but not the edge between them, then $\neg\neg X$ will be X plus these missing edges[7]. It can thus be seen as associating to each subgraph X its adjacency subspace $\neg\neg X$, as we have seen that an adjacency subspace needs to define only the nodes included in it, and the edges (the adjacency) automatically come. By intersecting $\neg\neg(X \wedge \lrcorner X)$ with X again, we will make sure that we include in the general boundary of X only edges that actually belong to X: in Figure 3.5 (right), we can see that the edge c is incident with only boundary nodes a and d, and thus it is in $\neg\neg(X \wedge \lrcorner X)$, but wasn't originally in the subgraph X. Thus intersecting with X again will get rid of this kind of edges.

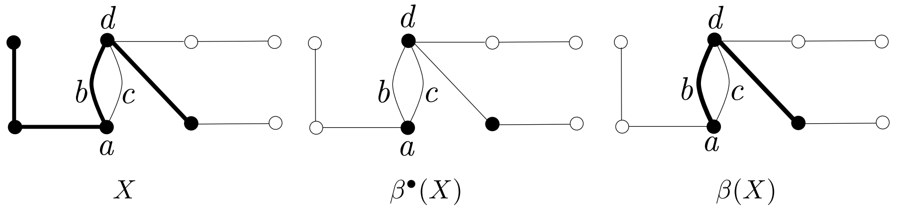

Figure 3.5: A graph X with its node boundary $\beta^\bullet(X)$, and its general boundary $\beta(X)$ (figure adapted from [73]).

Let us look at the ideas of boundary represented in Figure 3.5. The node a qualifies to be in $\beta^\bullet(X)$ as it is incident with the edge c that is not in X. However, travelling along c we still end up in nodes that belong to X, as both a and d are in X. Such an account of boundary might be unsatisfactory if we want to consider the boundary of X made of those parts that lead "properly outside" of X, where by "properly outside" we mean in the biggest region that does not share any point with X, i.e. $\neg X$. In this sense the boundary of X are the parts of X that are adjacent to $\neg X$. To single out this notion of boundary we apply the one-edge expansion to $\neg X$, namely $\lrcorner \neg\neg X$, and see where this intersects X. Thus we put forward two alternative ideas of boundary of a subgraph X, the first one referring to the nodes of X such that they are within $\lrcorner \neg\neg X$, i.e. they are adjacent to $\neg X$ (obviously this is not the case for node a in Figure 3.5), and

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[7] This is another difference between using classical propositional logic and intuitionistic logic. In **UBiSKt** we have that $\vdash p \to \neg\neg p$ but $\nvdash \neg\neg p \to p$. A formula is not equivalent to its double negation, just like a subgraph X is not equal to its "regular" version $\neg\neg X$. The latter will include all the edges between any tuple of nodes in X, whilst the first might not include all these edges.

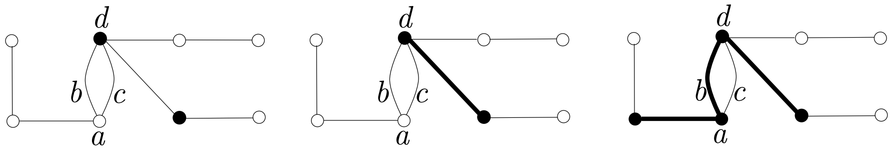
the second one adding the edges between these boundary-nodes as well. They are:

$$\beta^{\bullet}_{\neg}(X) := X \wedge \lrcorner\neg\neg X$$

and

$$\beta_{\neg}(X) := \neg\neg(X \wedge \lrcorner\neg\neg X) \wedge X$$

In this case the *outside* of X is understood as the pseudo-complement of X, $\neg X$, and the second idea of *outside* of X is employed. Examples of these ideas of boundary for a subgraph X are given in Figure 3.6.



$\beta^{\bullet}_{\neg}(X)$ $\beta_{\neg}(X)$ $\beta_{\lrcorner}(X)$

Figure 3.6: Additional notions for boundary (figure adapted from [73].

Eventually, if the outside of X was to be understood as $\lrcorner X$, and we are looking to the part of X that are adjacent to $\lrcorner X$, we would have another possible notion of boundary $\beta_{\lrcorner}(X) = (X \wedge \lrcorner\neg\lrcorner X)$, as in Figure 3.6 (right).

So we have seen that there are many possibilities for the notion of boundary of a subgraphs of a graph (U, H) (obviously the same ideas apply to the more general case of hypergraphs and their subgraphs). The choice of one or the other depends on which idea of boundary a user of the discrete spatial relations is interested in capturing.

3.4 Reasoning with Spatial Relations in UBiSKt

In this section we are going to show some entailments between spatial properties of subgraphs, that can be derived syntactically in **UBiSKt**. Indeed all the following have been proved using **HUBiSKt**. For these axiomatic proofs the reader is referred to Appendix A.1. The propositions are also mechanically verified using the implementation of **TabUBiSKt** using *Mettel* [68]. In what follows we use the term "region" to mean any H-set in an H-model M, represented by any formula φ in the language. Thus when the model we have in mind is a graph or a hypergraph, regions are its subgraphs. We also remark that the formula $\mathsf{E}\,\varphi$ holds at a model M iff $[\![\varphi]\!]_M \neq \varnothing$, $\mathsf{A}\,\varphi$ holds at M iff $[\![\varphi]\!]_M = U$, and $\mathsf{A}\,\neg\varphi$ or $\neg\,\mathsf{E}\,\varphi$ hold at M iff $[\![\varphi]\!]_M = \varnothing$.

Proposition 58. $\vdash_{\mathbf{HUBiSKt}} \beta^{\bullet}(\varphi) \leftrightarrow \beta^{\bullet}(\varphi) \wedge \lrcorner\,\beta^{\bullet}(\varphi)$. The boundary-nodes of a region are always boundary-nodes of itself.

We might want to distinguish between regions whose interior is empty and regions whose interior is non-empty. The following propositions show that requiring that the interior of a (non-empty) region is empty is the same as requiring that the region is (non-empty and) equal to its own boundary.

Definition 59. $BR(\varphi) := \mathsf{E}(\varphi) \wedge EQ(\varphi, \beta(\varphi))$. A region is a Boundary-Region in a model if it is not empty and it is equal to its own general-boundary.

In what follows we use the term "exterior" to mean the dual pseudo-complement $[\![\neg\varphi]\!]_M$ of a region $[\![\varphi]\!]_M$ in a given H-model M.

Proposition 60. (i) $\vdash_{\mathbf{HUBiSKt}} BR(\varphi) \to P(\varphi, \neg(\neg \lrcorner\, \varphi))$. If φ is a Boundary-Region then it is part of the exterior of its own interior.
(ii) $\vdash_{\mathbf{HUBiSKt}} P(\psi, \neg\delta) \to \neg\,\mathsf{E}(\psi \wedge \delta)$. If a region is part of the exterior of another region, then the two regions do not overlap.
(iii) $\vdash_{\mathbf{HUBiSKt}} \mathsf{E}(\psi) \wedge \neg\,\mathsf{E}(\psi \wedge \neg \lrcorner\, \psi) \to \mathsf{A}\,\neg(\neg \lrcorner\, \psi)$. If a region is non-empty and it does not overlap its own interior, then the interior of that region is empty.

From the results contained in Proposition 60 we can infer the following.

Proposition 61. $\vdash_{\mathbf{HUBiSKt}} BR(\varphi) \to \mathsf{A}\,\neg(\neg \lrcorner\, \varphi)$. Any boundary-region has an empty interior.

Proposition 62. $\vdash_{\mathbf{HUBiSKt}} \mathsf{A}\,\neg(\neg \lrcorner\, \varphi) \to EQ(\varphi, \beta(\varphi))$. If the interior of a region is empty, then the region is equal to its own boundary. With the extra assumption that the region is non-empty we have that if its interior is empty then it is a Boundary-region.

Propositions 61 and 62 show that our notion of Boundary-region is equivalent to the notion of non-empty region whose interior is empty:

Proposition 63. $\vdash_{\mathbf{HUBiSKt}} BR(\varphi) \leftrightarrow \mathsf{E}\,\varphi \wedge \mathsf{A}\,\neg(\neg \lrcorner\, \varphi)$

Definition 64. A region is a Substantial-region if its interior is not empty: $SR(\varphi) := \mathsf{E}\,\neg \lrcorner\, \varphi$.

We do not need to add the assumption that the Substantial region is non-empty, as this follows from the predicate itself:

Proposition 65. $\vdash_{\mathbf{HUBiSKt}} \mathsf{E}(\neg \lrcorner\, \varphi) \to \mathsf{E}(\varphi)$. If the interior of a region is non-empty, then the region is non-empty.

Proposition 66. $\vdash_{\mathbf{HUBiSKt}} SR(\varphi) \to$ not-$P(\varphi, \beta(\varphi))$. If a region has a non-empty interior then it is not part of its own boundary. Therefore it is not a Boundary-region.

Proposition 67. $\nvdash_{\mathbf{HUBiSKt}} \mathsf{E}(\neg \lrcorner\, \varphi) \wedge \mathsf{E}(\lrcorner\, \varphi) \to \mathsf{E}\,\beta(\varphi)$. If a region has a non-empty interior and it is not the whole graph, it does not necessarily have a boundary.

Proof. A counter-H-model for this formula is the following. Suppose $U = \{\, u, v \,\}$ and $H = I$. Suppose a valuation V, such that $V(p) = \{\, u \,\}$. In the model $M = (U, H, V)$, $[\![p]\!]_M = \{\, u \,\}$. Then $[\![\neg \lrcorner\, p]\!]_M = [\![p]\!]_M$ and $[\![\neg\neg(p \wedge \lrcorner\, p) \wedge p]\!]_M = \varnothing$: since the region is a node not connected to anything else except itself, its interior is the region itself and, for the same reason, its boundary is empty. ■

Thanks to the distinction between Boundary regions and Substantial Regions, we are able to refine the spatial relations given above, by limiting the domain of the regions considered. As an example, the relation of Partial Overlapping PO can be defined only between regions with non-empty interior, and be distinguished into three different relations.

Let φ, ψ represent substantial regions:

$PO_1(\varphi,\psi) := \mathsf{E}(\varphi \wedge \psi) \wedge \textit{not-P}(\varphi,\psi) \wedge \textit{not-P}(\psi,\varphi) \wedge \mathsf{E}(\beta(\varphi) \wedge \beta(\psi))$,

$PO_2(\varphi,\psi) := \mathsf{E}(\varphi \wedge \psi)\textit{not-P}(\varphi,\psi) \wedge \textit{not-P}(\psi,\varphi) \wedge \mathsf{E}(\neg \lrcorner \varphi \wedge \neg \lrcorner \psi) \wedge BR(\neg \lrcorner \varphi \wedge \neg \lrcorner \psi)$,

$PO_3(\varphi,\psi) := \mathsf{E}(\varphi \wedge \psi)\textit{not-P}(\varphi,\psi) \wedge \textit{not-P}(\psi,\varphi) \wedge \mathsf{E}(\neg \lrcorner \varphi \wedge \neg \lrcorner \psi) \wedge SR(\neg \lrcorner \varphi \wedge \neg \lrcorner \psi)$.

where $PO_1(\varphi,\psi)$ expresses the idea that the two regions overlap in their boundaries, $PO_2(\varphi,\psi)$ expresses that two regions overlap in their interior and region where they overlap is a Boundary-region, $PO_3(\varphi,\psi)$ expresses that two regions overlap in their interior, and the region where they overlap is in turn a Substantial-region.

Proposition 68. $\vdash_{\mathbf{HUBiSKt}} \neg SR(\beta^\bullet(\varphi) \wedge \beta^\bullet(\psi))$ The intersection of the Node-boundaries of two regions is not a Substantial-Region.

Proposition 69. $\vdash_{\mathbf{HUBiSKt}} EQ(\neg\beta^\bullet(\varphi), \neg\beta(\varphi))$. The exterior of the Node-Boundary is equal to the exterior of the General-Boundary.

Proposition 70. $\vdash_{\mathbf{HUBiSKt}} SR(\varphi) \wedge \mathsf{A}\,\neg\beta^\bullet(\varphi) \leftrightarrow \mathsf{E}\,\varphi \wedge P(\varphi, \neg \lrcorner \varphi)$ A region is a Substantial region and its Node-Boundary is empty iff the region is non-empty and it is part of its own interior.

By the results in Proposition 69, we can generalise the results obtained in Propositions 70 to the general-Boundary of a region.

Proposition 71. $\vdash_{\mathbf{HUBiSKt}} SR(\varphi) \wedge \mathsf{A}\,\neg\beta(\varphi) \leftrightarrow \mathsf{E}\,\varphi \wedge P(\varphi, \neg \lrcorner \varphi)$.

Chapter 4

Spatial Relations on Graphs under Granularity

Change in the level of granularity according to which a scene is visualised is a type of qualitative spatial change. In this chapter we will see how mathematical morphology can help to represent granularity on graphs and we are going to see how the spatial relations between subgraphs under granularity should be expressed.

4.1 Evolution of Mathematical Morphology

Mathematical morphology is usually presented as a discipline in the field of image processing. In this context, a two dimensional image can be understood as a subset of the pixel grid. The structuring element is a certain arrangement of pixels with a specified origin or centre, that placed on the chosen image generates a new one, by the basic operations of dilation and erosion, by the more complex ones of opening, closing, and by different compositions of these, known as morphological filters. The operation of opening is often described as a way to delete narrow portions of an image, and the closing as a way of filling small holes in an image, see for example [37]. Thus we can say that mathematical morphology was conceived with the intent to analyse an image at different levels of detail, where the level of detail is provided by the structuring element. For example Hejmans and Ronse in [37] explain that an image "contains an unstructured wealth of information, most of which is of no use to us", and using the tools from mathematical morphology we can "extract what interests us, obtaining thus a structure which is in fact a simplified sketch, a caricature, of the original image". The use of terms like *sketch*, or *caricature*, to describe the result of applying morphological operations, shows that these operations have always been thought of as a way to approximate images, and thus it connects to the level of detail according to which an image is visualised.

Although it was first conceived as a discipline to analyse images, and these are usually interpreted as subsets of the set of the pixel grid, mathematical morphology can be placed in a more general context. The relational approach to mathematical morphology and its connection to modal logic [7], [77], is based on the abstraction of morphological operations from a set with some additional structure (as the pixel grid) to a generic set. A structuring element on the pixel grid gives rise to a relation on the set, and this relation can be seen as the modal accessibility relation. We can go from structuring element to the associated relation as follows. A structuring element E is an certain arrangement of pixels with a designated origin. We can see the structuring element E as a subset of $\mathbb{Z}^2$, with its origin as the origin of $\mathbb{Z}^2$, not necessarily belonging to E. Then, given any $x \in \mathbb{Z}^2$, consider the translations of the form $x+E = \{\, x+e \mid e \in E \,\}$. Each set of the form $x+E$ places the structuring element with its origin at x, and the pixels in E lie over some pixels in $\mathbb{Z}^2$. Then dilation and erosion of a subset $X \subseteq \mathbb{Z}^2$ can be defined as follows: the dilation of X by E is $X \oplus E = \bigcup\{\, x+E \mid x \in X \,\} = \{\, y \in \mathbb{Z}^2 \mid \exists x(y \in (x+E) \text{ and } x \in X) \,\}$. The erosion of X by E is $E \ominus X = \{\, y \in \mathbb{Z}^2 \mid \forall x(x \in (y+E) \text{ implies } x \in X) \,\}$. We can now define a binary relation over $\mathbb{Z}^2$ from E as follows: for any $u, v \in \mathbb{Z}^2$, $u \; R_E \; v$ iff $v \in u+E$. Then it is clear that given a structuring element E and its associated relation R_E, $X \oplus E = X \oplus R_E$, and $E \ominus X = R_E \ominus X$, where $X \oplus R_E$ and $R_E \ominus X$ are dilation and erosion by a binary relaion on a set, as in Definition 5. Notice that relations are more general than structuring elements, as it is clear that not every relation on $\mathbb{Z}^2$ can be generated from a structuring element. Finally, we can abstract from the pixel grid and generalise morphological operations of dilation and erosion to any set U and relation $R \subseteq U \times U$, where given any subset $X \subseteq U$, dilation and erosion by R are defined as in Definition 5. As we have seen in Section 1.2, dilation and erosion by R have the same semantics of the classical modalities $\blacklozenge$ and $\Box$ arising from R when this is the accessibility relation of a modal frame (U, R), whilst the converse relation $\breve{R}$ gives rise to corresponding converse dilation $\Diamond$ and converse erosion $\blacksquare$.

Furthermore, morphological operations can be placed in a more general context than operations on subsets of a set. One of the main contributions of Heijmans and Ronse in [37] and [60], is indeed to show that most of the theory can be based on the idea of complete lattices, and that dilation and erosion are respectively join and meet preserving functions on complete lattices. A complete lattice is a set $\mathcal{L}$ with a relation of partial order, such that any non-empty subset X of $\mathcal{L}$ has a supremum in $\mathcal{L}$, also called least upper bound or join, and an infimum in $\mathcal{L}$, also called greatest lower bound or meet. In the case of the lattice formed by the powerset of a set U, namely $\mathcal{P}(U)$, and the partial order $\subseteq$ between its elements, the supremum of any collection of subsets of U is the operation of union, and the infimum is the intersection. Dilations and erosions here are operations on $\mathcal{P}(U)$ that preserve union (join) and intersection (meet) respectively. For every dilation, or join preserving function on a complete lattice, there is a unique associated erosion, or meet preserving function [49]. Dilation and erosion are indeed linked by the concept of adjunction (see Definition 6). The set of all subgraphs of a graph, or of a hypergraph, is a complete lattice with the join operation being the union between subgraphs and the meet being intersection[1]. Mathematical morphology for graphs and hypergraphs has been studied in a variety of works as [86] (which actually considers the complete lattice formed by the set of all nodes of a graph only), [15], [47], [14]. The relational approach to mathematical morphology on graphs and hypergraphs, based on the idea of stable relations, as well as its connection to intuitionistic

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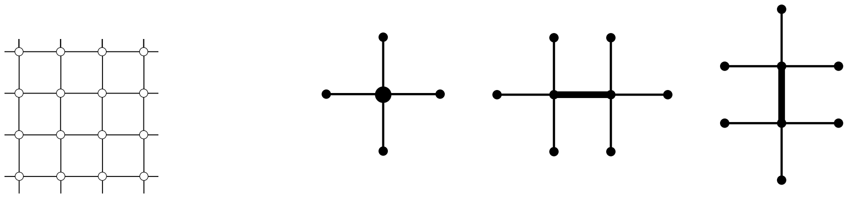
[1] More generally, the set of all H-sets of an H-frame (U, H) with the partial order $\subseteq$ forms a complete lattice.

modal logic, has been studied in [74], where, however the use of morphological operations to represent granularity has not been explored.

We have already seen the concept of H-dilates in Definition 55. If we consider a generic stable relation R instead of H, we obtain the more general idea of R-dilate.

Definition 72. Given an H-frame (U, H, R) and an element $u \in U$ we call an *R-dilate* the set $\{\, v \in U \mid u \; R \; v \,\}$, namely the set $\{u\} \oplus R$.

As R is a stable relation, the R-dilates are always H-sets, so when (U, H) is a graph or s hypergraph, R-dilates are subgraphs, and not just generic subsets, of U. In the context of mathematical morphology on the pixel grid $\mathbb{Z}^2$, where the relation comes from a structuring element, R-dilates can be seen as its copies, i.e. as the translations of the form $x + E$ introduced earlier[2]. To give concrete examples of an R-dilate on a graph, let (U, H) be the graph with $\mathbb{Z}^2$ for nodes and two nodes are connected by an edge if exactly one of their coordinates differs by 1. We refer to this as the graph $\mathbb{Z}^2$, visualized as in Fig 4.1. The dilates by $\circlearrowright H$ of a node, a horizontal edge, and a vertical edge are also shown in Figure 4.1. The significance is that any node in the graph (U, H) represented in the figure is linked to itself, to the four adjacent nodes, and to all the edges between them, by the relation $\circlearrowright H = H \mathbin{;} \breve{H} \mathbin{;} H$, and any horizontal or vertical edge is linked to itself, to the nodes it is H-incident to, and finally to their four perpendicularly adjacent nodes and to all the edges between them.



(U, H) is infinite grid, $\mathbb{Z}^2$

Shapes of $\circlearrowright H$-dilates

Figure 4.1: Shapes of dilates when $R = \circlearrowright H$ and when (U, H) is the graph shown. We can see the shape of $\{\, u \,\} \oplus \circlearrowright H$ when u is a node (in bold) in (U, H), when u is a horizontal edge (in bold) in (U, H), and when u is a vertical edge (in bold) in (U, H) (figure adapted from [71]).

We have already mentioned the operations of opening and closing on subsets in Section 1.7. The opening of a subset X is $(R \ominus X) \oplus R$. The closing of X is $R \ominus (X \oplus R)$. The same operations can be defined for any H-set of (U, H) and stable relation R. Let us denote the opening of X as $X \circ R$ and the closing of X as $X \bullet R$. It is clear that from a modal logic point of view the opening is associate to the sequence of modalities $\blacklozenge \Box$, and the closing is associated to the sequence of modalities $\Box \blacklozenge$. Thus given an H-model M where $[\![\varphi]\!]_M = X$ for some formula φ and H-set X, we have that $X \circ R = [\![\blacklozenge \Box \varphi]\!]_M$ and $X \bullet R$ $= [\![\Box \blacklozenge \varphi]\!]_M$. The opening of a set X, and thus also of an H-set X, can be expressed in terms of R-dilates, as the following lemma shows:

Lemma 73. $X \circ R = \bigcup \{\, \{\, x \,\} \oplus R \mid \{\, x \,\} \oplus R \subseteq X \,\}$

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[2] Indeed for $x, y \in \mathbb{Z}^2$ pixel grid, $y \in x + E$ iff $x R_E y$ iff $y \in \{\, x \,\} \oplus R_E$

Proof. For the following proofs we remind the reader of the definitions of dilation and erosion by a relation R as in Definition 5. Given any $x \in U$

$$\begin{aligned}
&\bigcup \{ \{ x \} \oplus R \mid \{ x \} \oplus R \subseteq X \}, \\
&\quad = \{ u \in U \mid \exists x (x \; R \; u \text{ and } \{ x \} \oplus R \subseteq X) \}, \\
&\quad = \{ u \in U \mid \exists x (x \; R \; u \text{ and } \forall y (x \; R \; y \text{ implies } y \in X)) \}, \\
&\quad = \{ u \in U \mid \exists x (x \; R \; u \text{ and } x \in R \ominus X \}, \\
&\quad = (R \ominus X) \oplus R.
\end{aligned}$$

■

In Section 1.7 we have described the opening of a subset X of a set of pixels as it is common in mathematical morphology for image processing, as providing a description of X not using individual pixels but by fitting copies of structuring elements within X. It is clear from Lemma 73 that, taking the more general idea of R-dilates instead of copies of the structuring element, this description of opening of an H-set X of (U, H) still holds: the Lemma proves that the opening of X is the union set of all the R-dilates lying within X. Thus we can say that the opening of a subgraph X of (U, H) provides a *coarse* description of X, as it describes X not in terms of its H-dilates but in terms of its R-dilates. What we can see of X will be its R-dilates only. Parts of X that can't be described in this way will be "forgotten", left out, performing an opening on X. We can see an example of opening of two H-sets by the relation $\looparrowright H$ on a specific graph (U, H), in Figure 4.3 (middle).

On the other hand the operation closing was described, in the pixel-based context, as fitting copies of the structuring element, rotated by a half turn, outside X, i.e. performing an opening by the rotated structuring element on the complement of X, and then take the complement of this. In this way the closing will have the effect of "filling holes" on the background of X where the rotated structuring element doesn't fit. This description of the closing still holds in the context of a set U with a relation R, i.e. when we calculate the closing of any $X \subseteq U$ by R. Generalising from structuring elements to R-dilates, the action of rotating a structuring element by a half turn corresponds to the idea of using the converse relation $\breve{R}$. Then the closing of a subset $X \subseteq U$ can be described as before, as taking the complement of X, performing an opening by $\breve{R}$, i.e. an erosion by $\breve{R}$ followed by a dilation by $\breve{R}$, and then take the complement of this. This is formally proved by the following lemma.

Lemma 74. $X \bullet R = -(-X \circ \breve{R})$

Proof. Let $x \in -(-X \circ \breve{R})$.

$$\begin{aligned}
&\Leftrightarrow x \notin (-X \circ \breve{R}), \\
&\Leftrightarrow x \notin ((\breve{R} \ominus (-X)) \oplus \breve{R}), \\
&\Leftrightarrow \forall y (y \; \breve{R} \; x \text{ implies } y \notin \breve{R} \ominus (-X)), \\
&\Leftrightarrow \forall y (y \; \breve{R} \; x \text{ implies } \exists z (y \; \breve{R} \; z \text{ and } z \notin -X)), \\
&\Leftrightarrow \forall y (x \; R \; y \text{ implies } \exists z (z \; R \; y \text{ and } z \in X)), \\
&\Leftrightarrow \forall y (x \; R \; y \text{ implies } y \in X \oplus R)), \\
&\Leftrightarrow x \in R \ominus (X \oplus R).
\end{aligned}$$

$\blacksquare$

This connection between the closing of a subset and the opening by the converse relation can also be derived in classical modal logic. Indeed in classical modal logic the formula $\Box \blacklozenge \varphi \leftrightarrow \neg\Diamond \blacksquare \neg\varphi$ is a theorem. Thus for any Kripke relational model M as in Definition 2 and any formula φ, we have that $\llbracket \Box \blacklozenge \varphi \rrbracket_M = \llbracket \neg\Diamond \blacksquare \neg\varphi \rrbracket_M$, which expresses the fact that the closing of any subset $\llbracket \varphi \rrbracket_M$ is equal to the complement of the opening by $\breve{R}$ of the complement of $\llbracket \varphi \rrbracket_M$: as we have seen in Section 1.2, dilation by $\breve{R}$ is associated to $\Diamond$ and erosion by $\breve{R}$ is associated to $\blacksquare$. This equivalence is derivable using the fact that, in classical modal logic, the pairs of modalities $(\Box, \Diamond)$ and $(\blacksquare, \blacklozenge)$ are inter-definable: $\Box\varphi \leftrightarrow \neg\Diamond\neg\varphi$, and $\Diamond\varphi \leftrightarrow \neg\Box\neg\varphi$ hold for any formula φ, and analogous property holds for the modalities in the pair $(\blacksquare, \blacklozenge)$.

When we move to a graph, or a hypergraph (U, H), where subgraphs are the H-sets, it is still correct to say that the closing of an H-set X by a stable relation R, $X \bullet R$, is equivalent to the idea of taking the complement of X, performing an opening by $\breve{R}$, and then take the complement of this. Indeed the result from Lemma 74 holds for any set $X \subseteq U$ and any relation R on U, thus it still holds for any H-set of (U, H), as these are still subsets of U, and any stable relation on U. So we can say that the closing of an H-set X gives a coarser description of X as it corresponds to the idea of "filling holes" in the background of X, namely $-X$, where $\breve{R}$-dilates do not fit. Notice that it makes perfect sense to analyse the complement of an H-set X by the converse of a stable relation $\breve{R}$. Indeed, if X is an H-set then its complement $-X$ is an $\breve{H}$-set (a set closed under $\breve{H}$-successor), i.e. if $u \in -X$ and $u \; \breve{H} \; v$ then $v \in -X$. Thus we will analyse this set by a relation closed under $\breve{H}$. Given a stable relation $R = H \,; R \,; H$, then its converse is $\breve{R} = \breve{H} \,; \breve{R} \,; \breve{H}$. The idea is that holes visible in an H-set X are the $\breve{H}$-dilates of the background of X, that is $-X$, and thus they are $\breve{H}$-sets. Thus we need to analyse these holes by a relation that sends $\breve{H}$-sets to $\breve{H}$-set, namely $\breve{R}$. Applying the opening by $\breve{R}$ to $-X$ we will have this effect: if the dilates by $\breve{R}$ fit in a certain hole, then it means that this hole is "big enough" to be visible after granulation, so applying the complement operation will not fill this hole– the area is not part of the granulation of X, as there is still a hole there. If the converse dilates do not fit in a certain hole, then it is small enough to be forgotten after the applying granulation, and thus the complement operation will fill its area: in the granulation of X these parts of $-X$ are no longer holes.

However, we notice that in this context it is no longer the case that $\llbracket \Box \blacklozenge \varphi \rrbracket_M = \llbracket \neg\Diamond \blacksquare \neg\varphi \rrbracket_M$, when M is an H-model and $\llbracket \varphi \rrbracket_M$ is an H-set in M. Indeed, first of all the complement of an H-set is not necessarily its negation $\neg X$. Moreover, when we are looking for the right notion of *converse opening* of an H-set, this is not just applying the opening by $\breve{R}$. Indeed, as already explained in Section 2.2, the converse $\breve{R}$ of a stable relation R is not necessarily stable, thus operations by this relation might not map H-sets to H-sets. The right notion of *converse opening* for H-sets is provided by the composition of dilation and erosion by the stable relation $\curvearrowright R$, i.e. the smallest stable relation containing $\breve{R}$. Indeed for any H-model M we have that $\llbracket \Diamond \blacksquare \varphi \rrbracket_M = (\curvearrowright R \ominus (\llbracket \varphi \rrbracket_M)) \oplus \curvearrowright R = \llbracket \varphi \rrbracket_M \circ \curvearrowright R$ (see Section 2.2.1). In **UBiSKt**, being an intuitionistic modal logic, we don't have the inter-definability of $\Box$ and $\Diamond$, and of $\blacksquare$ and $\blacklozenge$, as in classical modal logic. Thus the equality $\llbracket \Box \blacklozenge \varphi \rrbracket_M = \llbracket \neg\Diamond \blacksquare \neg\varphi \rrbracket_M$ doesn't necessarily hold, when M is an H-model and $\llbracket \varphi \rrbracket_M$ is an H-set in M A counter-model to the equality of these two formulae is given in Figure 4.2, where $R = H$, and thus $\curvearrowright R = \curvearrowright H$. This shows that morphological operations on graphs and hypergraphs are quite different from morphological operations on sets.

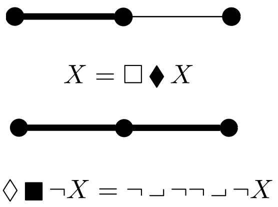
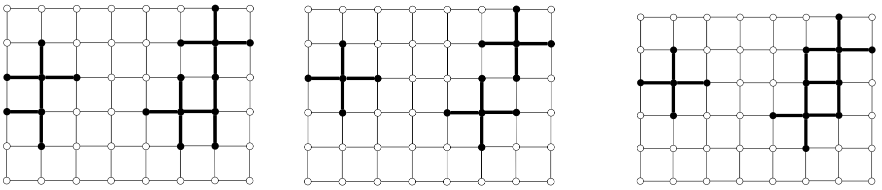


Figure 4.2: A model showing that the equality $\Box\blacklozenge X = \neg\Diamond\blacksquare\neg X$ doesn't hold intuitionistically. In this model we consider $R = H$, $X = \Box\blacklozenge X$, and $\neg\Diamond\blacksquare\neg X = \neg\lrcorner\neg\neg\lrcorner\neg X$ as the modal operator $\Diamond$ is the dilation by $\curvearrowright H$, that maps X to $\lrcorner\neg X$, as we have seen in Theorem 52 and similarly $\blacksquare$ is erosion by $\curvearrowright H$, that maps X to $\neg\lrcorner X$.

In conclusion, we can think of $(X \circ R) \bullet R$ as a granular version of X in which we cannot 'see' arbitrary H-sets, but only ones that can be described in terms of the R-dilates, in the way specified by Lemmas 73 and 74. We give an example of two H-sets and their granular version obtained by applying $(\_ \circ R) \bullet R$ in Figure 4.3, where $R = \curvearrowright H$. As we have seen, opening and closing correspond to specific sequences of modalities in the logic. So, given an H-model $M = (U, H, R)$ and an H-set $X = [\![\varphi]\!]_M$ for some formula φ in the language, we can capture its granular version by a formula in the logic.



Two H-sets
in the infinite grid $\mathbb{Z}^2$

The opening by
by $\curvearrowright H$

The closing of the opening
by $\curvearrowright H$

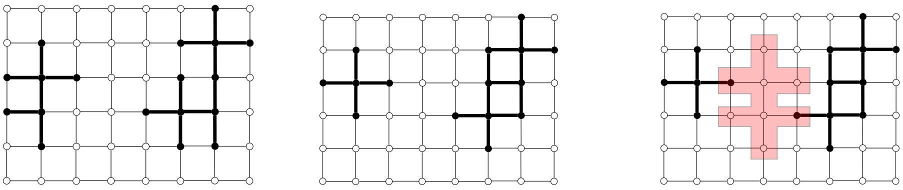
Figure 4.3: Granular view provided by the relation $\curvearrowright H$. On the left, two H-sets in the given graph (U, H). In the middle, the operation of opening of the two H-sets by the stable relation $\curvearrowright H$. Only $\curvearrowright H$-dilates lying in the H-sets will be selected by this operation. On the right, the closing by $\curvearrowright H$ applied after the opening by $\curvearrowright H$. Small gaps within the H-sets are filled by the closing.

Definition 75. The formula 'coarsely φ' is defined by $\mathsf{G}\varphi := \Box\blacklozenge\blacklozenge\Box\varphi$.

This definition of granulation is just one of many possible ones, and it comes from generalising ideas from mathematical morphology in the set based context, to graphs and hypergraphs. Alternative notions of granulation of a subgraph X could be applying the same operations in the reverse order, namely applying a closing first and then an opening, or simply applying an opening, thus considering the union of all the R-dilates present in the subgraph X as adopting a granular view on X.

4.2 Granular Connection

The operation of opening followed by closing on H-sets, and thus on subgraphs when (U, H) is a graph or a hypergraph, gives a possible way of zooming-out for a subgraph region. But how should we define connection between coarse regions, i.e. between regions that have undergone a granulation process? The issue is that the space underlying the regions should become coarser – regions disconnected may become connected for example. In the same way that coarse regions are described in terms of R-dilates using opening and closing, a coarse version of connection can be formulated using R-dilates. To motivate this, consider Fig 4.4 which shows the idea that coarse regions are coarsely connected if there is a dilate intersecting both, or visually and informally that the gap between can be bridged by an R-dilate. Requiring an R-dilate joining two regions seems a suitable notion of coarse connection, as it extends the intuition of connection at the detailed level given in Section 3.2. Indeed two subgraphs X and Y are connected at the detailed level if the gap between them can be bridged by an H-dilate, so if they are an edge apart, in the limit case (see Figure 3.4 for an example of H-dilates on a graph). Going to the granular level, single H-dilates are no longer "visible", and the space has coarser atomic parts: R-dilates.



Two H-sets

Their approximations by by $\looparrowright H$

The approximations can be joined by an $\looparrowright H$-dilate

Figure 4.4: Example of granular connection provided by Relation $\looparrowright H$. On the left, two H-sets in the given graph (U, H). In the middle, their approximation by $\looparrowright H$. On the right we can see that the resulting H-sets are coarsely connected: there is indeed an $\looparrowright H$-dilate intersecting the two H-sets. Notice that they are not connected in the sense of connection given in Table 3.1. However, the new atomic parts of the representation are now $\looparrowright H$-dilates, and the gap between the H-sets can be bridged by this. Thus, at this level of detail, the subgraphs are connected. We can say that the subgraphs are "one edge apart" where the new idea of edge is the $\looparrowright H$-dilate displayed in the Figure (figure adapted from [71]).

Definition 76. An R-dilate, D, joins H-sets X and Y if $X \cap D \neq \varnothing$ and $Y \cap D \neq \varnothing$.

It is easy to see that requiring the existence of an R-dilate that joins X and Y amounts to require that, given the union of the R-dilates intersecting X, at least one of them intersects Y. Before showing how to formally encode this idea, we need to introduce the following lemma, containing some facts about relational dilation on sets. They obviously extend to H-sets and stable relations.

Lemma 77. Given R and S relations on a set U and $X, Y \subseteq U$, the following holds:

1. $X \subseteq Y$ implies $X \oplus R \subseteq Y \oplus R$ (monotonicity of dilation).
2. $X \oplus (R; S) = (X \oplus R) \oplus S$.
3. $R \subseteq S$ implies $X \oplus R \subseteq X \oplus S$.

Proof. The proof of item 1 is easy: suppose $u \in X \oplus R$. Then there is a $v \in U$ such that $v \; R \; u$ and $v \in X$. Then $v \in Y$, as $X \subseteq Y$. Thus $u \in Y \oplus R$.
The proof of item 2 is as follows:

$$\begin{aligned}
&(X \oplus R) \oplus S,\\
&= \{\, u \in U \mid \exists y (y \; S \; u \text{ and } y \in X \oplus R) \,\},\\
&= \{\, u \in U \mid \exists y (y \; S \; u \text{ and } \exists z (z \; R \; y \text{ and } z \in X)) \,\},\\
&= \{\, u \in U \mid \exists z \exists y (z \; R \; y \; S \; u \text{ and } z \in X) \,\},\\
&= \{\, u \in U \mid \exists z (z \; R \,;\, S \; u \text{ and } z \in X) \,\},\\
&= X \oplus R \,;\, S.
\end{aligned}$$

The proof of item 3 is easy: suppose $u \in X \oplus R$. Then there is a $v \in U$ such that $v \; R \; u$ and $v \in X$. As $R \subseteq S$, we have that $v \; S \; u$, and thus $u \in X \oplus S$. ■

Now we can show that, given an H-set X, the union of all the R-dilates intersecting X can be expressed as a function of X, using dilation by $\circlearrowleft R$ and by R.

Lemma 78. Let X be an H-set and R a stable relation. The union of the R-dilates intersecting X is $X \oplus (\circlearrowleft R; R)$.

Proof. First we show that the union of the R-dilate intersecting X is $X \oplus \breve{R} \oplus R$. If $\{u\} \oplus R$ intersects X, for some $u \in U$, then there is a $x \in X$ such that $\{u\} \subseteq \{x\} \oplus \breve{R}$. Hence $\{u\} \oplus R \subseteq \{x\} \oplus \breve{R} \oplus R \subseteq X \oplus \breve{R} \oplus R$ by Lemma 77 item 1. In the other direction, if $y \in X \oplus \breve{R} \oplus R$, then there is some $u \in U$ and $x \in X$ such that uRy and uRx, so that $y \in \{u\} \oplus R$ with $\{u\} \oplus R$ intersecting X. Now, since $\breve{R} \subseteq \circlearrowleft R$ (see Definition 15), $X \oplus \breve{R} \oplus R \subseteq X \oplus \circlearrowleft R \oplus R = X \oplus \circlearrowleft R; R$ by Lemma 77 item 3, item 1 and item 2. Also $X \oplus \circlearrowleft R; R = X \oplus H; \breve{R}; H; R = X \oplus \breve{R}; H; R \subseteq X \oplus \breve{R}; R = X \oplus \breve{R} \oplus R$ because X is an H-set and R is stable. So $X \oplus \breve{R} \oplus R = X \oplus \circlearrowleft R; R$. ■

Finally, we can formally encode the idea of the existence of an R-dilate intersecting both the H-sets X and Y, and then we can express this idea as the validity of a **UBiSKt** formula in an H-model[3].

Proposition 79. There is an R-dilate joining H-sets X and Y iff $(X \oplus (\circlearrowleft R\,;R)) \cap Y \neq \varnothing$.

Proof. The union of R-dilates intersecting X is $X \oplus (\circlearrowleft R; R)$ from Lemma 78. This intersects Y iff $(X \oplus (\circlearrowleft R; R)) \cap Y \neq \varnothing$. ■

The above discussion provides a semantic justification for the following definition.

Definition 80 (Coarse connection). $C_G(\varphi, \psi) := \mathsf{E}(\blacklozenge\Diamond\mathsf{G}\varphi \wedge \mathsf{G}\psi)$.

Note that in an H-model M where $R = H$, we have that $[\![\mathsf{G}\varphi]\!]_M = [\![\varphi]\!]_M$. Indeed from the semantics given in Section 2.2.1, we know that $[\![\Box\varphi]\!]_M = R \ominus [\![\varphi]\!]_M$ that is $H \ominus [\![\varphi]\!]_M$ if $R = H$. Similarly $[\![\blacklozenge\,\varphi]\!]_M = [\![\varphi]\!]_M \oplus H$. As $[\![\varphi]\!]_M$ is always an H-set for any formula φ in the language, we have that $H \ominus [\![\varphi]\!]_M = [\![\varphi]\!]_M = [\![\varphi]\!]_M \oplus H$, because, as we have already noticed in Section 2.2.1, an H-set is always equal to both its dilation and its erosion by the relation H. Thus, in the special case of an H-model where $R = H$, $[\![\mathsf{G}\varphi]\!]_M$ is simply equivalent to $[\![\varphi]\!]_M$. Moreover in this special case we have that for any $[\![\varphi]\!]_M$, $[\![\blacklozenge\,\Diamond\varphi]\!]_M = ([\![\varphi]\!]_M \oplus \circlearrowleft H) \oplus H = [\![\varphi]\!]_M \oplus \circlearrowleft H = [\![⌐\neg\varphi]\!]_M$, using the result proved in Theorem 52 and the fact that $[\![\varphi]\!]_M$ is an H-set. Thus we can conclude that in an H-model M where $R = H$, thus where we are adopting the view of H-dilates, two regions are coarsely connected if and only if they are connected in the sense of connection from Table 3.1, as $C_G(\varphi, \psi)$ is equivalent to $C(\varphi, \psi)$, as expected. This supports the definition of coarse connection given above, as we expect that when the granular view adopted on a graph is given H, i.e. the smallest stable relation containing the identity I, zooming-out by H is simply the action of "staying there", and the idea of coarse connection becomes the standard idea of connection from Table 3.1.

The spatial relation of connection is always assumed to be symmetric. Our notion of coarse connection is symmetric as follows.

Proposition 81. $\vdash_{\mathbf{HUBiSKt}} \mathsf{E}(\blacklozenge\,\Diamond\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \blacklozenge\,\Diamond\psi)$.

Proof. We have the following derivation in **HUBiSKt**. $\vdash \neg(\blacklozenge\,\Diamond\varphi \wedge \psi) \leftrightarrow (\blacklozenge\,\Diamond\varphi \to \neg\psi)$ because $\neg(\alpha \wedge \beta) \leftrightarrow (\alpha \to \neg\beta)$ is a theorem in intuitionistic logic. Thus by $(\mathtt{Mon}\,\mathsf{A})$ we have that $\vdash \mathsf{A}\,\neg(\blacklozenge\,\Diamond\,\varphi \wedge \psi) \leftrightarrow \mathsf{A}(\blacklozenge\,\Diamond\,\varphi \to \neg\psi)$.
Now $\vdash \mathsf{A}(\blacklozenge\,\Diamond\,\varphi \to \neg\psi) \leftrightarrow \mathsf{A}(\Diamond\,\varphi \to \Box\neg\psi)$ and $\vdash \mathsf{A}(\Diamond\,\varphi \to \Box\neg\psi) \leftrightarrow \mathsf{A}(\varphi \to \blacksquare\,\Box\neg\psi)$ by adjunction between $\blacklozenge$ and $\Box$, and between $\Diamond$ and $\blacksquare$.
Then $\vdash \mathsf{A}(\varphi \to \blacksquare\,\Box\neg\psi) \leftrightarrow \mathsf{A}(\varphi \to \neg\,\blacklozenge\,⌐\,\Box\neg\psi)$ and $\mathsf{A}(\varphi \to \neg\,\blacklozenge\,⌐\,\Box\neg\psi) \leftrightarrow \mathsf{A}(\varphi \to \neg\,\blacklozenge\,\Diamond\,\psi)$ because $\blacksquare\,\alpha \leftrightarrow \neg\,\blacklozenge\,⌐\,\alpha$ and $\Diamond\,\alpha \leftrightarrow ⌐\,\Box\neg\alpha$ are both abbreviations in the syntax. $\vdash (\varphi \to \neg\,\blacklozenge\,\Diamond\,\psi) \leftrightarrow \neg(\varphi \wedge \blacklozenge\,\Diamond\,\psi)$ and thus by $(\mathtt{Mon}\,\mathsf{A})$ we have that $\vdash \mathsf{A}(\varphi \to \neg\,\blacklozenge\,\Diamond\,\psi) \leftrightarrow \mathsf{A}\,\neg(\varphi \wedge \blacklozenge\,\Diamond\,\psi)$, and then $\vdash \mathsf{A}\,\neg(\blacklozenge\,\Diamond\,\varphi \wedge \psi) \leftrightarrow \mathsf{A}\,\neg(\varphi \wedge \blacklozenge\,\Diamond\,\psi)$ by concatenation. Therefore $\vdash \neg\,\mathsf{A}\,\neg(\blacklozenge\,\Diamond\,\varphi \wedge \psi) \leftrightarrow \neg\,\mathsf{A}\,\neg(\varphi \wedge \blacklozenge\,\Diamond\,\psi)$ that is $\vdash \mathsf{E}(\blacklozenge\,\Diamond\,\varphi \wedge \psi) \leftrightarrow \mathsf{E}(\varphi \wedge \blacklozenge\,\Diamond\,\psi)$ by item 22 of Proposition 29. ■

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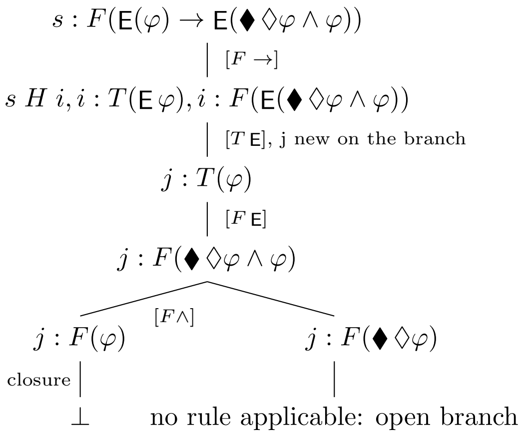
[3] Notice that the formal definition of the idea of an R-dilate intersecting X and Y, given in Proposition 79, is worked out for two *generic* H-sets X and Y. Then, as we expect to apply the predicate of coarse connection to regions resulting from some granulation process, so to coarse regions, we can apply this predicate to coarse regions only. So we can say that our predicate of coarse connection, as well as all the other coarse spatial relations introduced in this Chapter, is actually independent from the notion of granulation used, as they have been worked out for any pair of regions.

The proof of symmetry of granular connection is shown using generic formulae φ and ψ, so it will also hold for granular regions $\mathsf{G}\varphi$ and $\mathsf{G}\varphi$. It will also hold applying *any* other notion of granulation– thus not necessarily the opening followed by the closing as proposed in Definition 75, but also using another combination of these operations for example.

It is usually assumed that the spatial relation of connection is reflexive, namely that every region is connected to itself. This property of connection is actually restricted to non-empty regions, as clearly the empty region won't be connected to any other region, and thus it won't be connected to itself. It has been shown in Section 3.2, Proposition 53 item 1, that spatial relation of connection between subgraphs is reflexive when we consider non-empty regions. Thus the same question arises for the spatial relation of coarse connection presented in Definition 80. The formula that we want to test in this case is $\mathsf{E}(\varphi) \to \mathsf{E}(\blacklozenge \lozenge\varphi \wedge \varphi)$. The hypothesis of the implication stands for the assumption that self-connection is restricted to non-empty regions. As for symmetry, proved in Proposition 81, we first try to prove the property for a generic formula φ, and thus without assuming any specific notion of granulation. We are going to see next the formula is not a theorem. As we want to exhibit a counter-model, we are going to use the tableau system **TabUBiSKt** instead of the axiomatic system. The full tableau calculus can be found in Table A.1 the Appendix A.3.

Proposition 82. $\nvdash_{\mathsf{TabUBiSKt}} (\mathsf{E}\,\varphi) \to \mathsf{E}(\blacklozenge \lozenge\varphi \wedge \varphi)$

Proof.



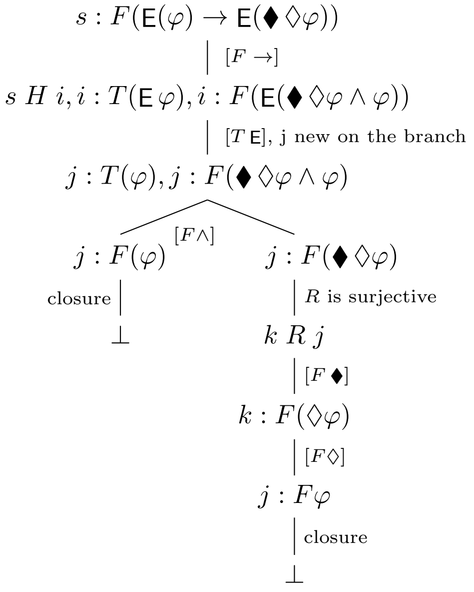
As we can see from the tableau proof, there is an open branch for the input formula $s : F(\mathsf{E}(\varphi) \to \mathsf{E}(\blacklozenge \lozenge\varphi))$. Thus the formula is not a theorem in the calculus. From the open branch we can extract the information needed to build a counter-model M for $\mathsf{E}(\varphi) \to \mathsf{E}(\blacklozenge \lozenge\varphi)$. We have a set $U = \{ s, i, j \}$, $H = I \cup \{ (s, i) \}$, and $[\![\varphi]\!]_M = \{ j \}$ therefore $[\![\varphi]\!]_M$ is not empty. However in this model $R = \varnothing$, thus $[\![\blacklozenge \lozenge\varphi]\!]_M = \varnothing$ and then $[\![\varphi]\!]_M \cap [\![\blacklozenge \lozenge\varphi]\!]_M = \varnothing$. Thus $M, s \models \mathsf{E}(\varphi)$, $s\ H\ i$, and $M, i \nvDash E(\varphi \wedge \blacklozenge \lozenge\varphi)$, i.e. $M, s \nvDash E(\varphi) \to E(\varphi \wedge \blacklozenge \lozenge\varphi)$ (same holds for worlds i and j). If we look carefully at the tableau proof we notice that once we get to $j : F(\blacklozenge \lozenge\varphi)$ no more rule is applicable. Indeed the only possibility is to apply the rule $[F \blacklozenge]$. However this rule has also a second premise, namely that j has R-predecessor (see rule $[F\blacklozenge]$ in Table A.1, Appendix A.3). This premise is missing on the tableau, as $R = \varnothing$. The world j is a dead-end world w.r.t. $\breve{R}$: it doesn't have any R-predecessor. Thus j will automatically make false all $\blacklozenge\varphi$, for any formula φ, namely $j \notin \llbracket\blacklozenge\varphi\rrbracket_M$ for any $\llbracket\varphi\rrbracket_M$. This hints that, in order to proceed towards a closed tableau, we need to be able to create an R predecessor for j. The fact that any world as an R-predecessor is a specific property that we might impose on the accessibility relation R, named *surjectivity* (see [74]). We can express this property in first-order logic as $\forall u \exists v(v\ R\ u)$, or as inclusion of relations as $I \subseteq \breve{R}\,;R$. When R is stable, surjectivity is equivalent to the inclusion $H \subseteq \circlearrowright R\,;R$.

Proposition 83. If R is stable, then $I \subseteq \breve{R}\,;R$ iff $H \subseteq \circlearrowright R\,;R$.

Proof. Assume that $I \subseteq \breve{R}\,;R$. We have the following chain of inclusion: $H \subseteq I;H \subseteq \breve{R};R;H$ by our assumption. $\breve{R};R;H \subseteq \breve{R};R \subseteq \circlearrowright R;R$ by stability of R and by the fact that $\breve{R} \subseteq \circlearrowright R$. Thus $H \subseteq \circlearrowright R\,;R$, under the assumption that $I \subseteq \breve{R}\,;R$ and R is stable. For the other direction, assume that $H \subseteq \circlearrowright R\,;R$. Since $I \subseteq H$, then $I \subseteq \circlearrowright R\,;R$. Therefore for any $u \in U$ there are elements x, y, z such that $u\ H\ x\ \breve{R}\ y\ H\ z\ R\ u$. Thus for all $u \in U$ there is some z such that $z\ R\,u$. So $I \subseteq \breve{R}\,;R$. ■

Proposition 84. Assume that R is surjective. Then $\vdash_{\mathsf{TabUBiSKt}} (\mathsf{E}\,\varphi) \rightarrow \mathsf{E}(\blacklozenge\Diamond\varphi \wedge \varphi)$

Proof.



Notice that in this tableau proof we implicitly use the fact that $\breve{R} \subseteq \circlearrowright R$. Indeed, since $k\ R\ j$ we have that $j\breve{R}k$, and thus $j \circlearrowright Rk$. This, and the fact that $k : F\Diamond\varphi$ provides the two premises for the application of $[F\Diamond]$ rule, whose conclusion is $j : F\varphi$ (see rule $[F\Diamond]$ in Table A.1).

Proposition 84 shows that imposing surjectivity on R is sufficient in order to prove that the relation of coarse connection is reflexive for any non-empty region $[\![\varphi]\!]$. Thus the following question arises: is surjectivity both sufficient *and* necessary in order to have reflexivity of coarse connection? It turns out that this is not the case. Given an H-frame $F = (U, H, R)$, there is property of R, that is the is necessary and sufficient condition for having reflexivity of coarse connection. We call this property *weak surjectivity* as surjectivity implies this condition but not the other way around.

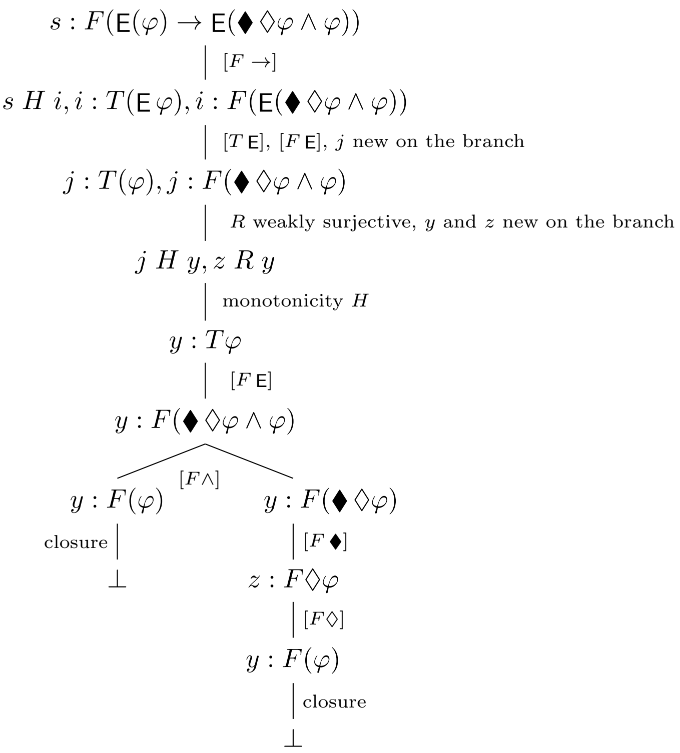
Definition 85. Given an H-frame $F = (U, H, R)$, R is weakly surjective if $\forall u \exists v \exists w (u \; H \; v \text{ and } w \; R \; v)$.

The intuition behind weak surjectivity is that every element in U has at least one H-successor that in turn has at least one R-predecessor. We notice the following facts about the weak surjectivity constraint: (i) when $H = I$, thus in the context of classical modal logic, weak surjectivity and surjectivity are equivalent, (ii) since H is reflexive, namely $I \subseteq H$, surjectivity implies weak surjectivity, (iii) the other direction doesn't hold, as we can have frames where R is weakly surjective but not surjective, for example if $U = \{ x, y \}$, $H = I \cup \{ (x, y) \}$ and $R = \{ (y, y), (x, y) \}$, and (iv) the property can also be expressed as follows: for all $u \in U$, $\{ u \} \oplus H \, ; \breve{R} \neq \varnothing$.

Now we are going to prove that, if weak surjectivity of R is assumed, then a tableau proof of $(\mathsf{E}\,\varphi) \to \mathsf{E}(\blacklozenge \lozenge \varphi \wedge \varphi)$ can be accomplished, as we did in Proposition 84 using surjectivity of R. Given the nature of the tableau calculus, attempting to build a counter-model for a given formula, we can say that this means that whenever an H-frame F is weakly reflexive, then $F \models \mathsf{E}(\varphi) \to \mathsf{E}(\blacklozenge \lozenge \varphi \wedge \varphi)$, as the closed tableau proof shows that we can't find any model M based on F where this formula is not valid.

Proposition 86. Assume that R is weakly surjective. Then $\vdash_{\mathsf{Tab}\mathbf{UBiSKt}} (\mathsf{E}\,\varphi) \to \mathsf{E}(\blacklozenge \lozenge \varphi \wedge \varphi)$

Proof.



Finally we show that if the spatial relation of coarse connection is valid in a frame F, then R in F is weakly surjective.

Proposition 87. Given an H-frame $F = (U, H, R)$, if $F \models (\mathsf{E}(\varphi) \to \mathsf{E}(\blacklozenge \Diamond\varphi \wedge \varphi))$ then R is weakly surjective.

Proof. We are going to show the contrapositive, namely that if R in F is not weakly surjective than there is a model $M = (F, V)$, such that $M \not\models (\mathsf{E}(\varphi) \to \mathsf{E}(\blacklozenge \Diamond\varphi \wedge \varphi))$, and thus $F \not\models (\mathsf{E}(\varphi) \to \mathsf{E}(\blacklozenge \Diamond\varphi \wedge \varphi))$. If R is not weakly surjective then there is an $x \in U$ such that $\{\, x \,\} \oplus H \,;\, \breve{R} = \varnothing$. Let us fix this x. Suppose a model M based on this frame F where $[\![\varphi]\!]_M = \{\, x \,\} \oplus H$ (notice that this is an H-set). Now we prove that $[\![\Diamond\varphi]\!]_M = [\![\varphi]\!]_M \oplus \curvearrowright R = \varnothing$, and hence $[\![\blacklozenge \Diamond\varphi]\!]_M = ([\![\varphi]\!]_M \oplus \curvearrowright R) \oplus R = \varnothing$. This is shown as follows:

$$\begin{aligned}
&[\![\Diamond\varphi]\!]_M = [\![\varphi]\!]_M \oplus \curvearrowright R,\\
&= (\{\, x \,\} \oplus H) \oplus H \,;\, \breve{R} \,;\, H,\\
&= \{\, x \,\} \oplus H \,;\, \breve{R} \,;\, H,\\
&= (\{\, x \,\} \oplus H \,;\, \breve{R}) \oplus H,\\
&= \varnothing \oplus H = \varnothing.
\end{aligned}$$

The key point is that $\{\, x \,\} \oplus H \,;\, \breve{R} = \varnothing$, that, as already noticed, is implied by the assumption that R is not weakly surjective. Hence we have that $[\![\blacklozenge \Diamond\varphi]\!] = ([\![\varphi]\!] \oplus \curvearrowright R) \oplus R = \varnothing$. So we can conclude that $M, x \models \varphi$, as $x\ H\ x$ and then $x \in [\![\varphi]\!]_M$, and thus $M, x \models \mathsf{E}\,\varphi$. Also $M, x, \not\models E(\blacklozenge \Diamond\varphi \wedge \varphi)$, as $[\![\blacklozenge \Diamond\varphi \wedge \varphi]\!]_M =$

$\llbracket\blacklozenge\Diamond\varphi\rrbracket \cap \llbracket\varphi\rrbracket_M = \varnothing \cap \llbracket\varphi\rrbracket_M = \varnothing$. Thus $M, x \nvDash E(\varphi) \to E(\blacklozenge\Diamond\varphi \wedge \varphi)$, and thus $F \nvDash E(\varphi) \to E(\blacklozenge\Diamond\varphi \wedge \varphi)$, under the assumption that R in F is not weakly surjective. ■

In the above discussion we have taken into consideration a generic non-empty region $\llbracket\varphi\rrbracket_M$, and we have shown that the formula expressing reflexivity of coarse connection for non-empty regions is not a theorem in the logic. Hence, by completeness, that there are models where the formula does not hold. We have seen a simple counter-model of the formula. However it is possible that if the notion of granulation G proposed in Definition 75 is considered, then the notion of connection for non-empty granular regions is reflexive. We wanted to show reflexivity for a generic non-empty region $\llbracket\varphi\rrbracket_M$, so that the result could apply to non-empty granular regions, where *any* notion of granulation could be specified, as done in the proof of symmetry of coarse connection. But maybe if we consider a specific notion of granulation, for instance the one introduced in Definition 75, then the reflexivity of coarse connection, for these kinds of non-empty regions, could be proved. Hence the question is: is the formula $\mathsf{E}(\mathsf{G}\varphi) \to \mathsf{E}(\blacklozenge\Diamond\mathsf{G}\varphi \wedge \mathsf{G}\varphi)$ a theorem in Tab**UBiSKt**? It is easy to see, without even attempting a tableau proof, that this is not the case. Given the frame $F = (U, H, R)$ where $U = \{\, u \,\}$, $H = I$, and $R = \varnothing$, any model M based on F works as a counter-model for the formula.

Another possible notion of granulation for regions could be formalised by applying the closing followed by the opening. So let $\mathsf{G}'\varphi := \blacklozenge\Box\Box\blacklozenge\varphi$. Is it the case that, for this notion of granulation, that $\mathsf{E}(\mathsf{G}'\varphi) \to \mathsf{E}(\blacklozenge\Diamond\mathsf{G}'\varphi \wedge \mathsf{G}'\varphi)$ a theorem in **UBiSKt**? It turns out to be so.

Proposition 88. Let $\mathsf{G}'\varphi := \blacklozenge\Box\Box\blacklozenge\varphi$. Then $\vdash_{\mathsf{Tab}\mathbf{UBiSKt}} \mathsf{E}(\mathsf{G}'\varphi) \to \mathsf{E}(\blacklozenge\Diamond\mathsf{G}'\varphi \wedge \mathsf{G}\varphi)$.

Proof.

$$\begin{array}{c}
s : F(\mathsf{E}(\mathsf{G}'\varphi) \to \mathsf{E}(\blacklozenge\Diamond\mathsf{G}'\varphi \wedge \mathsf{G}'\varphi)) \\
\Big| \; [F \to] \\
s\ H\ i, i : T\,\mathsf{E}(\mathsf{G}'\varphi)), i : F(\mathsf{E}(\blacklozenge\Diamond\mathsf{G}'\varphi \wedge \mathsf{G}'\varphi)) \\
\Big| \; [T\,\mathsf{E}], \text{ j new on the branch} \\
j : T(\mathsf{G}'\varphi), j : F(\blacklozenge\Diamond\mathsf{G}'\varphi \wedge \mathsf{G}'\varphi) \\
[F\wedge] \\
\begin{array}{cc}
j : F(\mathsf{G}'\varphi) & j : F(\blacklozenge\Diamond\mathsf{G}'\varphi) \\
\text{closure}\;\Big| & \Big| \; T\,\blacklozenge, k \text{ new on the branch} \\
\bot & k\ R\ j, k : T\Box\Box\blacklozenge\varphi \\
 & \Big| \; [F\,\blacklozenge] \\
 & k : F\Diamond\mathsf{G}'\varphi \\
 & \Big| \; [F\Diamond] \\
 & j : F\mathsf{G}'\varphi \\
 & \Big| \; \text{closure} \\
 & \bot
\end{array}
\end{array}$$

■

Notice here the key-point of the proof: we will always be able to find an R-predecessor for any element belonging to a non-empty granular region $\mathsf{G}'\varphi$. Indeed in this case the first prefix of the new granularity predicate is $\blacklozenge$. Thus the rule $[T \blacklozenge]$ will be applied, and this will create an R-predecessor to any point belonging to the granular region $\mathsf{G}'\varphi$. This will allow the accomplishment of the proof. So, using this alternative notion of granulation, we can prove that all non-empty granular regions are coarsely connected to themselves. Finally, we notice that if the notion of granulation assumed is the opening of a region, then a proof for reflexivity of coarse connection similar to the one of Proposition 88 presented above can be obtained.

The above discussion shows that the reflexivity property of coarse connection is not independent from the notion of granulation chosen. If we want to guarantee reflexivity of coarse connection for *any* region, and thus independently from the notion of granulation chosen, some property on the accessibility relation R has to be imposed. Imposing weak surjectivity on R is a necessary and sufficient condition for this to hold.

4.3 Beyond Granular Connection

In the previous section we have considered a way to look at a subgraph at a different level of detail and we have seen how the function mapping a region to its granular version can be expressed in modal logic, giving a syntactic notion of granulation (we have seen that that is not the only possible notion of granulation). We have then proposed a compatible idea of granular connection, and we have justified our choice, as it generalises the idea of detailed connection between subgraphs in terms of H-dilates, to R-dilates. We can extend the same reasoning to all the spatial relations between subgraphs defined in Section 3.2. We will start from defining a notion of coarse parthood in terms of R-dilates. The standard notion of parthood at the detailed level (Table 3.1) says that, given H-sets X and Y, X is part of Y if and only if all the H-dilates in X lie in Y (see Proposition 56). A suitable notion of coarse parthood will require that X is coarse part of Y if and only if all the R-dilates in X lie also in Y.

Proposition 89. Let X and Y be H-sets, and R a stable relation. The following are equivalent: 1) all the R-dilates in X lie in Y and 2) $R \ominus (X) \subseteq R \ominus (Y)$.

Proof. The union of all the R-dilates in X is the opening of X: $X \circ R = (R \ominus X) \oplus R$. Hence, requiring the all the R-dilates in X lie in Y amounts to require that $(R \ominus X) \oplus R \subseteq Y$. By properties of adjunction this is equivalent to $R \ominus X \subseteq R \ominus Y$. ■

The above reasoning together with Lemma 54 provides a semantic justification for the following definition of coarse parthood between coarse regions.

Definition 90 (Coarse parthood). $P_G(\varphi, \psi) := \mathsf{A}(\Box \mathsf{G}\varphi \to \Box \mathsf{G}\psi)$.

It is easy to see that the notion of coarse parthood is reflexive: $\vdash_{\mathbf{UBiSKt}}$ $\mathsf{A}(\Box \mathsf{G}\varphi \to \Box \mathsf{G}\varphi)$ (the same will clearly holds choosing a different notion of granulation). The following proposition shows that coarse parthood is also transitive, as expected.

Proposition 91. $\vdash_{\mathsf{Tab}\mathbf{UBiSKt}} \mathsf{A}(\Box \mathsf{G}\varphi \to \Box \mathsf{G}\psi) \wedge \mathsf{A}(\Box \mathsf{G}\psi \to \Box \mathsf{G}\delta) \to \mathsf{A}(\Box \mathsf{G}\varphi \to \Box \mathsf{G}\delta)$.

Proof. See Appendix A.2. ■

The negation of the notion of coarse parthood will give a notion of coarse non-parthood: this requires that there is at least an R-dilate in X such that it is not in Y. From Proposition 89, we know that this is equivalent to $R \ominus X \not\subseteq R \ominus Y$. Because of Lemma 57 we propose the following definition.

Definition 92 (Coarse non-parthood)**.** non-$P_G(\varphi, \psi) := \mathsf{E}(\Box \mathsf{G}\varphi \mathrel{-\!\!<} \Box \mathsf{G}\psi)$.

We now analyse how to extend the spatial relation of overlapping to the granular level. Two H-sets X and Y overlap at the detailed level if and only if there is an H-dilate that lies both in X and Y. Indeed the intersection of two H-sets X and Y is always an H-set, thus if $X \cap Y \neq \varnothing$, then there is at least $u \in U$ such that $\{u\} \oplus H \in X$ and $\{u\} \oplus H \in Y$ (notice that H-dilates are always non-empty by reflexivity of H: $u \in \{u\} \oplus H$ for any $u \in U$). Following this idea, a suitable notion of coarse overlapping requires the existence of a non-empty R-dilate that lies both in X and Y.

Proposition 93. Let X and Y be H-sets and R a stable relation. The following are equivalent: 1) there is a non-empty R-dilate that lies both in X and in Y and 2) $(X \cap Y) \circ R \neq \varnothing$.

Proof. $(X \cap Y) \circ R$ is the opening of $X \cap Y$, so the union of all R-dilates both in X and in Y. Hence requiring that there is a non empty R-dilate that lies both in X and in Y amounts to require that the opening of $X \cap Y$ is non-empty: $(X \cap Y) \circ R \neq \varnothing$. ■

Thus we define coarse overlapping between coarse regions as follows.

Definition 94 (Coarse overlapping)**.** $O_G(\varphi, \psi) := \mathsf{E}(\blacklozenge \Box (\mathsf{G}\varphi \wedge \mathsf{G}\psi))$.

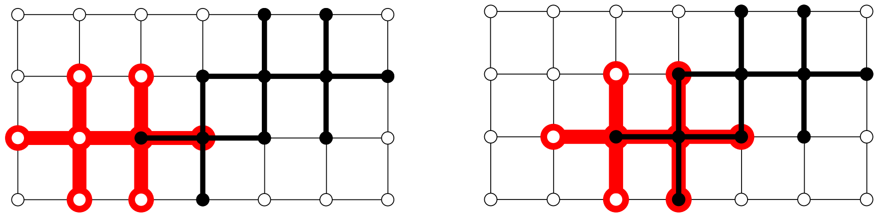
As an example, in Fig. 4.5 on the left we show two H-sets (red and black) that intersect, but an R-dilate will not fit inside the region of intersection (in this example we take $R = \curvearrowright H$). Therefore the spatial relation O_G does not hold. If the region of intersection is at least as big as an R-dilate, as it happens on the right of the figure, then the relation O_G does hold.

Given H-sets X and Y, X is non-tangential part of Y at the detailed level if X is part of Y and the closure of X, $\lrcorner \neg X$, is still part of Y. This means that all the H-dilates that intersect X lie in Y (indeed now we can see from Lemma 78 that the union of all H-dilates intersecting X is $X \oplus \curvearrowright H \oplus H$ i.e. $X \oplus \curvearrowright H$ that is $\lrcorner \neg X$ by theorem 52). Hence, a suitable notion of coarse non-tangential part between H-sets X and Y is obtained by requiring that X is coarse part of Y and all the R-dilates intersecting X lie in Y.

Proposition 95. Let X and Y be H-sets and R a stable relation. The following are equivalent: 1) all the R-dilates overlapping X lie in Y, and 2) $X \oplus \curvearrowright R \subseteq R \ominus Y$.

Proof. Requiring that the union of the R-dilates overlapping X lie in Y is $(X \oplus \curvearrowright R \oplus R) \subseteq Y$ by Lemma 78. This is equivalent to $X \oplus \curvearrowright R \subseteq R \ominus Y$ by properties of adjunctions. ■

The above reasoning provides a semantic justification for the following definition.



Two H-sets not sharing a whole R-dilate   Two H-sets sharing a whole R-dilate

Figure 4.5: Cases of coarse non-overlapping and of coarse overlapping, where R is $\curvearrowleft H$ (figure adapted from [71]).

Definition 96 (Coarse non-tangential part)**.** $NTP_G(\varphi, \psi) := \mathsf{A}(\Box\mathsf{G}\varphi \rightarrow \Box\mathsf{G}\psi) \wedge \mathsf{A}(\Diamond\mathsf{G}\varphi \rightarrow \Box\mathsf{G}\psi)$.

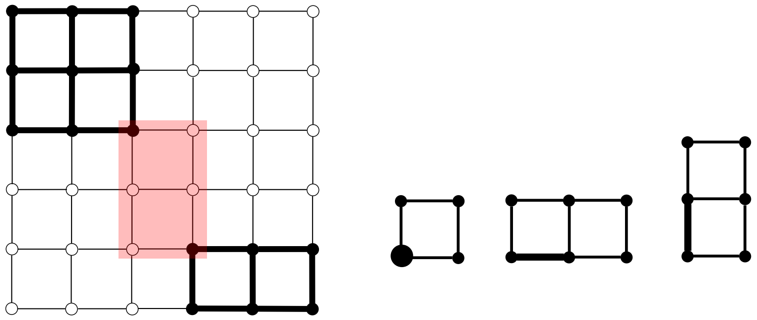
Finally, we analyse the notion of coarse tangential part. At the detailed level, an H-set X is tangential part of Y if X is part of Y and there is at least an H-dilate intersecting X that does not lie in Y. This is obtained by requiring that $\lrcorner\neg X$, that is the union of all H-dilates intersecting X, is not part of Y. Hence, at the granular level we will require that the union of all R-dilates intersecting X is not part of Y. This means that we have to negate the requirement for NTP_G: by Proposition 95 this is $X \oplus \curvearrowleft R \not\subseteq R \ominus Y$. Because of this and Lemma 57 we propose the following.

Definition 97 (Coarse tangential part)**.** $TP_G(\varphi, \psi) := \mathsf{A}(\Box\mathsf{G}\varphi \rightarrow \Box\mathsf{G}\psi) \wedge \mathsf{E}(\Diamond\mathsf{G}\varphi \prec \Box\mathsf{G}\psi)$.

Using the predicates C_G, P_G, non-P_G, O_G, NTP_G and TP_G, a set of RCC-8 style coarse spatial relations between coarse subgraphs can be obtained in the obvious way, as in Table 4.1. For example the coarse spatial relation of external connection $EC_G(\varphi, \psi)$ will be defined as $C_G(\varphi, \psi) \wedge \neg O_G(\varphi, \psi)$.

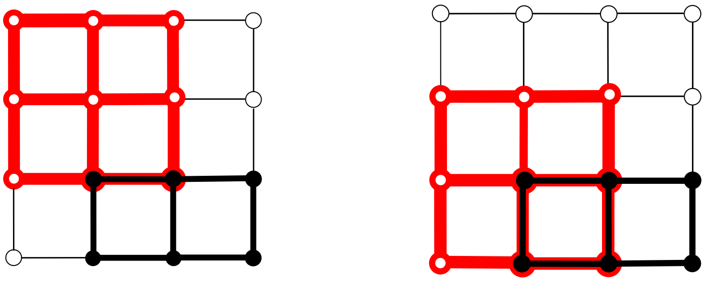
Notice that all the definitions of the coarse spatial relations above have been worked out for generic X and Y. they will be applied to regions that have undergone a granulation process, as for example the opening followed by the closing as proposed in Definition 75, but the definition of the coarse spatial relations are independent from this choice of granulation, thus they will apply also when another notion of granulation (as for example the closing followed by the opening, or just the opening) is chosen.

To conclude this chapter, we present in Figure 4.6 and Figure 4.7 additional examples of coarse connection and coarse overlapping between H-sets, based on a new relation R defined on the graph (U, H).



Two H-sets coarsely connected by an R-dilate Shapes of R-dilates

Figure 4.6: An additional example of coarse connection between two H-sets (on the left), based on new shapes of R-dilates $\{ u \} \oplus R$ (on the right). The relation R on the graph (U, H) associated to these shapes can be constructed as follows: every node is related to all the elements belonging to the R-dilate when the node is the origin, and analogous reasoning holds for horizontal and vertical edges. The possible origins (a node, an horizontal edge, and a vertical edge) are highlighted in the figure (right). We notice that the two H-sets are not connected in the sense of connection presented in Section 3.2. However, there is an R-dilate intersecting the two H-sets, hence they are connected at the level of detail provided by this choice of R.



Two H-sets not sharing a whole R-dilate Two H-sets sharing an R-dilate

Figure 4.7: Another example of coarse non-overlapping and of coarse overlapping, based on the R-dilates from Figure 4.6.

Table 4.1: Granular Relations and corresponding formulae

Granular Relation	Formula
$C_G(\varphi, \psi)$	$\mathsf{E}(\blacklozenge\Diamond\mathsf{G}\varphi \wedge \mathsf{G}\psi)$
$O_G(\varphi, \psi)$	$\mathsf{E}(\blacklozenge\Box(\mathsf{G}\varphi, \mathsf{G}\psi))$
$P_G(\varphi, \psi)$	$\mathsf{A}(\Box\mathsf{G}\varphi \rightarrow \Box\mathsf{G}\psi)$
$DC_G(\varphi, \psi)$	$\neg C_G(\varphi, \psi)$
$non\text{-}P_G(\varphi, \psi)$	$\mathsf{E}(\Box\mathsf{G}\varphi \prec \Box\mathsf{G}\psi)$
$PO_G(\varphi, \psi)$	$O_G(\varphi, \psi) \wedge non\text{-}P_G(\varphi, \psi) \wedge non\text{-}P_G(\psi, \varphi)$
$PP_G(\varphi, \psi)$	$P_G(\varphi, \psi) \wedge not\text{-}P_G(\psi, \varphi)$
$EQ_G(\varphi, \psi)$	$P_G(\varphi, \psi) \wedge P_G(\psi, \varphi)$
$NTPP_G(\varphi, \psi)$	$PP_G(\varphi, \psi) \wedge \mathsf{A}(\Diamond\mathsf{G}\varphi \rightarrow \Box\mathsf{G}\psi)$
$NTPP^i_G(\varphi, \psi)$	$NTPP_G(\psi, \varphi)$
$TPP_G(\varphi, \psi)$	$PP_G(\varphi, \psi) \wedge \mathsf{E}(\Diamond\mathsf{G}\varphi \prec \Box\mathsf{G}\psi)$
$TPP^i_G(\varphi, \psi)$	$TPP_G(\psi, \varphi)$
$EC_G(\varphi, \psi)$	$C_G(\varphi, \psi)\neg O_G(\varphi, \psi)$

Chapter 5

Modal Logic for Hypergraph Partitions

In this chapter we will present and discuss different possibilities for an **S5** version of **UBiSKt**. On the semantics side, this corresponds to the logic of H-frames where the stable relation R on (U, H) provides with a partition on (U, H).

5.1 Introduction

In the previous section we explored the idea of looking at subgraphs at a different level of detail. The intuition is that instead of being able to see all H-dilates of a graph – single nodes and single edges with their end-points nodes – only groups of these that can be described by R-dilates are "visible". The modal accessibility relation R on the graph, provides a granular view on it. The notion of R-dilate comes from generalising the notion of a structuring element on the pixel grip, known from mathematical morphology. No special constraint on R has been imposed, except for stability as we are working with relations on graphs and hypergraphs (see Section 1.4).

A specific way of grouping elements of information together is when they share certain attributes. In rough set theory [54], attributes defined on a set U provide an equivalence relation on the set, and thus a partition of the set. Indistinguishable elements, namely elements that cannot be distinguished on the basis of the available attributes, coalesce into "granules", the blocks of the partition. This process gives a coarser view of the initial set. Then, given any subset $X \subseteq U$ two kinds of approximation can be considered in rough set theory: $\underline{X}$ or the lower approximation of X, and $\overline{X}$ or the upper approximation of X. The first one can be informally described as taking those clusters containing some elements in the initial set. The latter can be described as taking those clusters that contains only elements from the initial set. It is well known that rough set theory has connections with the modal logic **S5**, where indeed R is an equivalence relation, with **S5**-$\Box$ associated to $X \mapsto \underline{X}$ and **S5**-$\blacklozenge$ associated to $X \mapsto \overline{X}$, [92], [53] (notice that R being an equivalence relation, and thus symmetric, the box modalities $\Box$ and $\blacksquare$ are equivalent, and the same holds for the diamonds $\blacklozenge$ and $\Diamond$). Thus an equivalence relation on a set provides with specific way to take a coarser look on that set, and on its subsets, i.e. a specific form of a granulation.

As already explained in Section 1.7, a partition on a set is always associated to an equivalence relation as well as to a function that maps each element to its equivalence class, or block in the partition. This is known as the *quotient function.*

In [66] the correspondence between partitions and relations on hypergraphs is studied. It is shown that equivalence relations on hypergraphs are too restrictive in the kind of partitions they give rise to. We have already discussed this in Section 1.8. An alternative approach is proposed in [66]. Whilst the relations considered are still reflexive and transitive (as well as stable, as the approach of stable relations as relations on graphs and hypergraphs is taken also in this work), a constraint weaker than symmetry is imposed on R, referred as "symmetric generation".

In [52] the goal is to study some **S5** extension of intuitionistic modal logic. The author starts from an H-frame (U, H, R), where R is additionally reflexive and transitive. Besides reflexivity and transitivity, other constraints substituting symmetry are considered, and they yield what the author describes as different intuitionistic analogues of the modal logic **S5**.

In this chapter we are going to discuss and compare these different constraints on R, that can be seen as substitutes for symmetry[1]. We will see that all the constraints on R under consideration arise from imposing specific conditions on the quotient function f that links (U, H) to its quotient structure. We call this type of condition *back conditions* on the quotient function. The goal of the chapter is to study graph and hypergraph partitions and the associated relations. However most of the theory presented doesn't need the assumption that (U, H) is specifically a hypergraph, but the fact that it is a poset is enough. Specific examples will usually be two-levels posets, namely hypergraphs, as these are the objects of interest. The chapter is structured as follows. Section 5.2 presents the **S4** extension of **UBiSKt**, that is the modal logic that captures the class of reflexive and transitive H-frames. In Section 5.3 we show that a reflexive and transitive relation R on a poset (U, H) already gives rise to a partition of the underlying set U, hence in the special case of (U, H) being a hypergraph a graph, reflexive and transitive relations can be used to obtain a partition of the set of nodes and edges U. The quotient structure is a partial order (Theorem 108). Hence, given a poset (U, H), we have a way to take a "coarser look" at it, using a preorder R, and obtaining a quotient structure (U', H') that is still a poset. However, in Section 5.4 we are going to look at some cases of poset partitions that hint that reflexive and transitive relations might not give a fully satisfactory account of poset partitions. We will focus on the missing link not discussed in [66], i.e. the quotient function between a poset and its quotient structure. We will show that the constraint of symmetry-generation considered in [66] corresponds to imposing an important property on the quotient function f (Theorem 117). We call this property the *weak-zag* constraint. It will make

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[1] It should be mentioned that Ono in [52] considers an additional constraint as a substitute for symmetry, i.e. $R \subseteq H \mathbin{;} \breve{R}$. We don't analyse in this constraint in this work, as it is not of a certain form (it doesn't come from one of the *back conditions* on f w.r.t. H and H'). However we can already notice that this constraint will cause the same restriction on the type of allowed partitions $R \subseteq H \mathbin{;} \overset{\leftrightarrow}{R}$, that we do analyse, i.e. a node in a hypergraph can only go be assigned to a node in the quotient structure (Proposition 112).

sure that an important intuition from the theory of set partitions is preserved in a more general theory of poset partitions, i.e. that the quotient structure of a set is still a set. More generally, as we are going to see in Section 5.4, this property will impose a dependency between the resulting partial order H' from the initial partial order H. We will also look at other constraints to impose on the quotient function that also imply a dependency of H' from H. We show that they are equivalent to some constraints on R considered by Ono in [52], (Propositions 113 and 116). However imposing the constraints proposed by Ono, and thus considering the intuitionistic **S5** systems presented in [52], might be too restrictive in the kind of hypergraph partitions, and more generally posets partitions, that would be allowed there (Propositions 112 and 115). We remark that Ono's intuitionistic frames, for the modal logic **S**4, on the top of which the different **S**5 systems are built, are H-frames where R is a preorder, thus they are equivalent to the **UBiSKt** frames for the **S**4 extension of **UBiSKt**, introduced in section 5.2. Indeed in [52, page 695], the author considers frames $F = (U, H, R)$ where H is a partial order and R is a reflexive and transitive relation such that $H \subseteq R$. This implies stability of R by $H \,;\, R \,;\, H \subseteq R \,;\, R \,;\, R \subseteq R$. Ultimately in this section we show that the two-tierness constraint on R from [66] is equivalent to imposing that the quotient structure is not just a poset, but it is indeed a hypergraph (Theorem 119). Indeed the construction presented in Section 5.3 holds not just for hypergraphs, but more generally for posets. But if we want to make sure that, when we start from a hypergraph, its quotient structure is also a hypergraph, then we need an extra constraint on the preorder R from which the partition arises.

Finally, in Section 5.5, we are going to show that there is an extension of **UBiSKt** that captures the semantics of the class of H-frames such that R has all the above mentioned properties, i.e. it is a symmetrically generated, two-tier preorder. And in Section 5.6 we work out what the new definitions of the coarse spatial relations (from Sections 4.2 and 4.3) in this extension of **UBiSKt**.

We introduce a lemma that we are going to use throughout the chapter.

Lemma 98. Let V and W be sets and $P \subseteq V \times W$. Let I_V and I_W be the identity relations on V and W respectively. Then $I_V \,;\, P = P$ and $P \,;\, I_W = P$. Moreover we say that P is *functional* iff $\breve{P} \,;\, P \subseteq I_W$, and that P is *total* iff $I_V \subseteq P \,;\, \breve{P}$, and that P is *surjective* iff $I_W \subseteq \breve{P} \,;\, P$, and finally that P is *injective* iff $P \,;\, \breve{P} \subseteq I_V$.

We have already seen the property of surjectivity of a (stable) relation $R \subseteq U \times U$ in Chapter 4. Here the definition is generalised to when a relation is not necessarily defined on the same set (an *homogeneous* relation) but on two possibly different sets (an *heterogeneous* relation).

We also remind the reader that the following properties of relations holds: given a set W and relations $S \subseteq W \times W$ and $P \subseteq W \times W$, $S \subseteq P$ implies $\breve{S} \subseteq \breve{P}$ and $\breve{\breve{S}} = S$ (see [45]).

Next we will consider the logic of the class of H-frames where R is reflexive and transitive, and then we will build upon this to obtain the desired system.

5.2 System S4 for UBiSKt

Reflexivity of an arbitrary relation R can be expressed as the relational inclusion $I \subseteq R$. Transitivity can be expressed as $R\,;R \subseteq R$. When, additionally, R is a stable relation, as in the context of this work, we have the following fact, that can be found already in [74].

Proposition 99. If R is stable then $I \subseteq R$ iff $H \subseteq R$.

Proof. Let us assume that $H \subseteq R$. Since $I \subseteq H$ by reflexivity of H, we have that $I \subseteq R$. On the other direction, let us assume that $I \subseteq R$. We have the following chain of inclusion: $H \subseteq I\,;H \subseteq R\,;H \subseteq R$ by stability of R and Lemma 98. Thus $H \subseteq R$. ■

So the relations that are reflexive and transitive in the classical sense, and that are additionally stable, are those relations that satisfy $H \subseteq R$ and the normal condition for transitivity. It is also useful to note that, if R is transitive and $H \subseteq R$, then $H;R;H \subseteq R;R;R \subseteq R$, which means that R is stable, from Definition 14).

From the correspondence theorem in [74] we have that the properties of reflexivity and transitivity of a stable relation R can be expressed as a formula in the logic **BiSKt**, and therefore in **UBiSKt**, as this latter is an extension of the former. We will refer to the formula $p \to \blacklozenge p$ (or to its equivalent box-form $\Box p \to p$) as the *reflexivity axiom*, as this formula is valid in a H-frames $F = (U, H, R)$ iff $H \subseteq R$ holds, i.e. F is a reflexive H-frame. We will refer to the formula $\blacklozenge\blacklozenge p \to \blacklozenge p$ (or to its equivalent box-form $\Box p \to \Box\Box p$) as the *transitivity axiom*, as this formula is valid in a H-frames $F = (U, H, R)$ iff $R; R \subseteq R$ holds.

Definition 100. We define **S4** be the set $\{p \to \blacklozenge p, \blacklozenge\blacklozenge p \to \blacklozenge p\}$ and $\mathbb{S}4$ be the class of H-frames (U, H, R) such that R is reflexive (i.e., $H \subseteq R$) and R is transitive.

Theorem 101. Let Λ be a *ubist*-logic such that $\mathbf{S4} \subseteq \Lambda$. Then all the following hold in Λ.

1. $\blacklozenge\Box p \leftrightarrow \Box p$
2. $\Box\blacklozenge p \leftrightarrow \blacklozenge p$
3. $\Box\blacklozenge\blacklozenge\Box p \leftrightarrow \Box p$
4. $\blacklozenge\Box\Box\blacklozenge \leftrightarrow \blacklozenge p$

Proof. Proof of item 1.: for the left to right direction: $\Box p \to \Box\Box p$ is the transitivity axiom. By adjunction between $\blacklozenge$ and $\Box$ this is equivalent to $\blacklozenge\Box p \to \Box p$. For the right to left direction: $\Box p \to \blacklozenge\Box p$ is an instance of the reflexivity axiom. Proof of item 2: for the left to right direction: $\blacklozenge\blacklozenge p \to \blacklozenge p$ is the transitivity axiom. By adjunction between $\blacklozenge$ and $\Box$ this is equivalent to $\blacklozenge p \to \Box\blacklozenge p$. For the right to left direction: $\Box\blacklozenge p \to \blacklozenge p$ is an instantiation of reflexivity axiom. Proof of item 3: $\Box\blacklozenge\blacklozenge\Box p \leftrightarrow \blacklozenge\blacklozenge\Box p \leftrightarrow \blacklozenge\Box p \leftrightarrow \Box p$ by item 2, reflexivity and transitivity axioms and item 1. Proof of item 4: $\blacklozenge\Box\Box\blacklozenge p \leftrightarrow \Box\Box\blacklozenge p \leftrightarrow \Box\blacklozenge p \leftrightarrow \blacklozenge p$ by item 1, reflexivity and transitivity axioms and item 2. ■

Theorem 101 shows that when we assume **S4**-axioms, the notion of granulation presented in Section 4.2 corresponds to the lower approximation as in

Rough Set Theory, and the other order of composition of closing and opening corresponds to upper approximation. In terms of mathemathical morphology, we can say that, given a structuring element on a graph (U, H) and the associated stable relation, if this is transitive and reflexive, then the operation opening followed by closing will give the same results of simply performing an opening, which in turn is equivalent to take the erosion of the subgraph. Same link occurs between the operation of closing followed by opening, the closing, and the dilation.

The **S4** system for **UBiSKt** is sound and complete w.r.t. the class $\mathbb{S}4$. These results are contained in [72] and they were achieved in collaboration with Katsuhiko Sano.

Lemma 102. Given a *ubist*-logic Λ such that $\mathbf{S4} \subseteq \Lambda$, the Λ-canonical H-frame $F^{\Lambda}_{(\Gamma,\Delta)} = (U^{\Lambda}, H^{\Lambda}, R^{\Lambda})$ satisfies both $H^{\Lambda} \subseteq R^{\Lambda}$ and $R^{\Lambda}; R^{\Lambda} \subseteq R^{\Lambda}$, i.e., $F^{\Lambda}_{(\Gamma,\Delta)} \in \mathbb{S}4$.

Proof. First we establish $H^{\Lambda} \subseteq R^{\Lambda}$. Suppose that $(\Sigma_1, \Theta_1)H^{\Lambda}(\Sigma_2, \Theta_2)$, i.e., $\Sigma_1 \subseteq \Sigma_2$. Assume that $\varphi \in \Sigma_1$. Our goal is to show $\blacklozenge\varphi \in \Sigma_2$, as by Lemma 34 this is equivalent to $(\Sigma_1, \Theta_1)R^{\Lambda}(\Sigma_2, \Theta_2)$. But this is clear as $\varphi \to \blacklozenge\varphi \in \Sigma1$ by $\mathbf{S4} \subseteq \Lambda$, and thus $\blacklozenge\varphi \in \Sigma1$, hence $\blacklozenge\varphi \in \Sigma2$, as $\Sigma1 \subseteq \Sigma2$. Second, we prove $R^{\Lambda}; R^{\Lambda} \subseteq R^{\Lambda}$. Assume that $(\Sigma_1, \Theta_1)R^{\Lambda}(\Sigma_2, \Theta_2)$ and $(\Sigma_2, \Theta_2)R^{\Lambda}(\Sigma_3, \Theta_3)$. To show that $(\Sigma_1, \Theta_1)R^{\Lambda}(\Sigma_3, \Theta_3)$, suppose that $\varphi \in \Sigma_1$. We show that $\blacklozenge\varphi \in \Sigma_3$. By assumption, we have $\blacklozenge\blacklozenge\varphi \in \Sigma_3$, which implies $\blacklozenge\varphi \in \Sigma_3$ by $\blacklozenge\blacklozenge\varphi \to \blacklozenge\varphi \in \Sigma_3$ (by $\mathbf{S4} \subseteq \Lambda$). ■

By Lemmas 36, 39 and 102, Theorem 21 and Proposition 20 we can estabilish the following.

Theorem 103.

1. **UBiSKt** extended with **S4** is sound and strongly complete for the class $\mathbb{S}4$, i.e., $\Gamma \models_{\mathbb{S}4} \varphi$ iff $\Gamma \vdash_{\mathbf{UBiSKtS4}} \varphi$ for every set $\Gamma \cup \{\varphi\}$ of formulas.

2. **UBiSKt** extended with **S4** and $\mathbf{bd}_2$ is sound and strongly complete for the class of H-frames $(U, H, R) \in \mathbb{S}4$ where (U, H) is a hypergraph, i.e., $\Gamma \models_{\mathbb{HG}\cap\mathbb{S}4} \varphi$ iff $\Gamma \vdash_{\mathbf{UBiSKtS4bd_2}} \varphi$ for every set $\Gamma \cup \{\varphi\}$ of formulas.

5.3 Hypergraph Partitions and Quotient Function

Given a set U, it is well known that a partition of it can be obtained by a reflexive and transitive relation R on U, by considering the symmetric part of R, namely $(R \cap \breve{R})$ (see for example [32] Chapter 8). This is indeed an equivalence relation in the classical sense, and it will have an associated partition. Thus, given an hypergraph (U, H) and a stable relation R on U that is additionally reflexive and transitive, we can obtain a partition of the underlying set U of edges and node. Although we are interested in "two-level" posets, namely hypergraphs, in what follows we just need the assumption that (U, H) is a poset. The corresponding quotient structure, arising from $(R \cap \breve{R})$ can be defined as follows (notice that, as we work with stable relations R, reflexivity of R is equivalent to $H \subseteq R$ as already noticed in Section 5.2, Proposition 99):

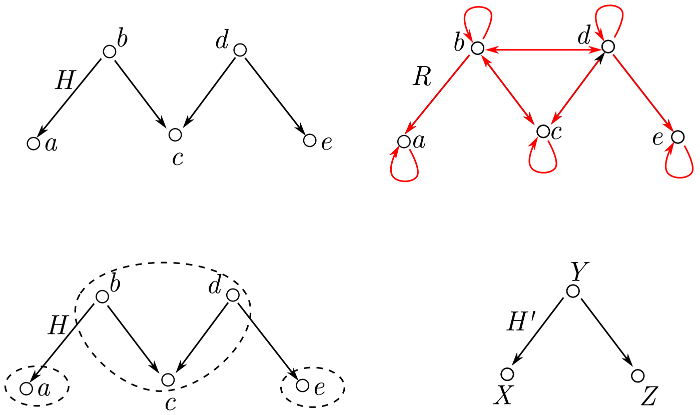

Figure 5.1: Top-left: a hypergraph (U, H), (reflexive loops of H are left implicit). Top-right: a reflexive and transitive relation $R \subseteq U \times U$ (transitive arrows are left implicit). The symmetric part of R, $\overleftrightarrow{R} = I \cup \{(b,c),(c,b),(b,d),(d,b),(d,c),(c,d)\}$ generates a partition of U (bottom-left). Bottom-right: the quotient structure (U', H'). The elements of U' are $X = \{a\}$, $Y = \{b,c,d\}$ and $Z = \{e\}$. H', that is the reflexive closure of the relation on the set U'.

Definition 104. Given a poset (U, H) and a reflexive and transitive relation $R \subseteq U \times U$, the quotient structure (U', H') is obtained by taking as elements of U' those $X \subseteq U$ such that there is an $u \in U$ such that $X = \{u\} \oplus (R \cap \breve{R})$. Given $X, Y \in U'$, $XH'Y$ iff there are $x \in X$ and $y \in Y$ such that xRy.

An example of a poset partition and associated quotient structure generated by a transitive and reflexive relation is given in Fig. 5.1.

In what follows we will use the following abbreviation: $(R \cap \breve{R}) = \overleftrightarrow{R}$.

Every element $u \in U$ can be associated to the block of the partition it contributes to form by the assignment $u \mapsto \{u\} \oplus \overleftrightarrow{R}$. We can look at the function $\{\_\} \oplus \overleftrightarrow{R} : U \to U'$ as the *quotient function*. Blocks are equivalence classes in the classical sense [2]. For sake of simplicity, we will indicate the quotient function $\{\_\} \oplus \overleftrightarrow{R}$ simply as f. It is clear from Definition 104 that f will be surjective, so for any element $X \in U'$, we can find a $u \in U$ such that $X = f(u)$. It will not be injective, except in the trivial case in which each element in U is mapped onto a block containing only the element itself. Being a function, f will be both functional, i.e. every element in U is uniquely mapped into an element in U', and total, i.e. all the elements in U get mapped into some element in U'. We

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[2] Notice that the quotient function differs slightly from the standard dilation function $\_ \oplus \overleftrightarrow{R} : \mathcal{P}(U) \to \mathcal{P}(U)$, as introduced in Section 1.2. Whilst the quotient function takes as argument an element of $u \in U$, the standard dilation takes as argument subsets of $X \subseteq U$. When a singleton-subset $\{u\}$ for $u \in U$ is considered, then the two functions will return the same value $\{u\} \oplus \overleftrightarrow{R}$, where the quotient function takes u as argument, and the dilation function takes $\{u\}$ as argument.

know how to express all these properties of f as inclusion of relations, from Lemma 98.

For any element $u \in U$ and block $X \in U'$, if $u \in X$ then u can be taken as representative of X: $X = \{ u \} \oplus \overset{\leftrightarrow}{R} = f(u)$. As blocks, namely elements of U', are equivalence classes in the classical sense, we have the following lemma.

Lemma 105. The following statements hold and they express the same fact, i.e. that two elements are in the same block iff they are related by the equivalence relation $\overset{\leftrightarrow}{R}$:

1. for any $x, y \in U$: $f(x) = f(y)$ iff $x \overset{\leftrightarrow}{R} y$.
2. $f\, ; f^{-1} = \overset{\leftrightarrow}{R}$.

The relation $f^{-1} \subseteq U' \times U$ is the converse of f, thus it relates any block of $X \in U'$ to all the elements $x \in U$ that coalesce into that block. Notice that this is just a relation, and not a function, as f is not necessarily injective, as already noticed, and thus a block $X \in U'$ is possibly related by f^{-1} to many different elements in U. By properties of dilation by symmetry of $\overset{\leftrightarrow}{R}$, we also know that $x\overset{\leftrightarrow}{R}y$ iff $y \in f(x)$ iff $x \in f(y)$, for any element x and y.

The quotient function can be seen as a link between the structures (U, R) and (U', H'), but also between (U, H) and (U, H'), i.e. the initial poset and its quotient structure. Now we are going to introduce some properties concerning relations between structures, and we are going to investigate whether the quotient function f has these properties, with respect to the pair of structures $((U, R), (U', H'))$, and $((U, H), (U', H'))$. We are going to use some of these properties to show that the quotient structure (U', H') arising from (U, H, R) as in Definition 104 is a poset as well, i.e. that H' built in that way is a partial order. This extends the intuition of the theory of rough sets, where the quotient structure of a set is also a set, to a theory of rough posets, where then we would expect that the quotient structure of a poset is a poset as well.

Definition 106. Given sets W and V and relations P, S and Q such that $P \subseteq W \times W$, $S \subseteq V \times V$, and $Q \subseteq W \times V$, Q is order-preserving with respect to P and S if $\breve{Q}; P; Q \subseteq S$, and Q is order-reflecting with respect to P and S if $Q; S; \breve{Q} \subseteq P$.

See Fig. 5.2 for a diagrammatic representation of order-preserving and order-reflecting conditions. Notice that usually the adjectives "order-preserving" and "order-reflecting" are used to describe mappings between posets (see [64]), but we will use this terminology in this more general setting, where (W, P) and (V, S) are just a pair of a set and a relation defined on the set.

From Definition 104 it immediately follows that f is order-preserving with respect to R and H', that is $f^{-1}\, ; R\, ; f \subseteq H'$. Notice that f is order-preserving also w.r.t. H and H'. This is an easy consequence of the fact that R is reflexive, and thus $H \subseteq R$, and f is order-preserving w.r.t. R and H', as already noticed: $f^{-1}\, ; H\, ; f \subseteq f^{-1}\, ; R\, ; f \subseteq H'$. Moreover we have the following.

Proposition 107. f is order-reflecting with respect to R and H'.

Proof. From Definition 104 it follows that $H' \subseteq f^{-1}\, ; R\, ; f$. So $f\, ; H'\, ; f^{-1} \subseteq f\, ; f^{-1}\, ; R\, ; f\, ; f^{-1} \subseteq \overset{\leftrightarrow}{R}; R; \overset{\leftrightarrow}{R}$ by Lemma 105, item 2, and $\overset{\leftrightarrow}{R}; R; \overset{\leftrightarrow}{R} \subseteq R$ by transitivity of R. Therefore $f\, ; H'\, ; f^{-1} \subseteq R$, as wanted. ■

$$\begin{tikzcd} w \arrow[r, "Q"] \arrow[d, "P"'] & v \arrow[d, dashed, "S"] \\ u \arrow[r, "Q"'] & z \end{tikzcd} \qquad \begin{tikzcd} w \arrow[r, "Q"] \arrow[d, dashed, "P"'] & v \arrow[d, "S"] \\ u \arrow[r, "Q"'] & z \end{tikzcd}$$

Figure 5.2: On the left, the order-preserving condition represented by a diagram. On the right the order-reflecting condition represented by a diagram. Read it as: $v\breve{Q}wPuQz$ implies vSz, and $wQvSz\breve{Q}u$ implies wPu.

We are now going to use order-reflecting and order-reflecting properties of f w.r.t. R and H', to show that the quotient structure (U', H') is a poset.

Theorem 108. Let (U, H) be a hypergraph and $R \subseteq U \times U$ a reflexive and transitive relation. Let (U', H') be the quotient structure generated by R. Then:

1. H' is reflexive
2. H' is transitive.
3. H' is anti-symmetric.

Proof. Proof of item 1: let us indicate with I' the identity relation on the set U'. Then we have $I' \subseteq f^{-1} ; f \subseteq f^{-1} ; I ; f \subseteq f^{-1} ; R ; f \subseteq H'$, by surjectivity of f, reflexivity of R and Definition 104.

Proof of item 2: we need to show that $H';H' \subseteq H'$. It follows from Definition 104 that $H' \subseteq f^{-1} ; R ; f$. So $H';H' \subseteq f^{-1} ; R ; f ; f^{-1} ; R ; f \subseteq$ $f^{-1} ; R ; \overset{\leftrightarrow}{R} ; R ; f \subseteq f^{-1} ; R ; f$, by item 2 of Lemma 105 and transitivity of R. Finally we have that $f^{-1};R;f \subseteq H'$ as the quotient functions is order-preserving with respect to R and H', as we have already noticed. Therefore $H';H' \subseteq H'$, as wanted.

Proof of item 3: suppose for some X and Y in U', $XH'Y$ and $YH'X$ holds. We need to show that $X = Y$. By surjectivity of f and Proposition 107 we have that there are $x, y, x', y' \in U$ such that $f(x) = X$ and $f(y) = Y$ and xRy, and $f(y') = Y$ and $f(x') = X$ and $y'Rx'$. Thus $x'\overset{\leftrightarrow}{R}x$ by Lemma 105 item 1, and $y'\overset{\leftrightarrow}{R}y$. Thus we have that $y\overset{\leftrightarrow}{R}y'Rx'\overset{\leftrightarrow}{R}x$ that by transitivity of R implies that yRx. Therefore $x\overset{\leftrightarrow}{R}y$ that implies that $X = f(x) = f(y) = Y$ by item 1 of Lemma 105. Therefore $X = Y$ as wanted. ■

As H' is reflexive, transitive and antisymmetric, (U', H') is a poset.

5.4 Is a Preorder Enough? Weaker Forms of Symmetry

All the properties seen so far follow from the only assumption that the quotient structure is built from a transitive and reflexive relation R. Given a poset (U, H)

– and thus given a hypergraph (U, H) as hypergraphs are posets – we can take a "coarser look" at it by using a preorder R. The resulting structure (U', H') will also be a poset. Let us now look at the situation depicted in Figure 5.3.a.

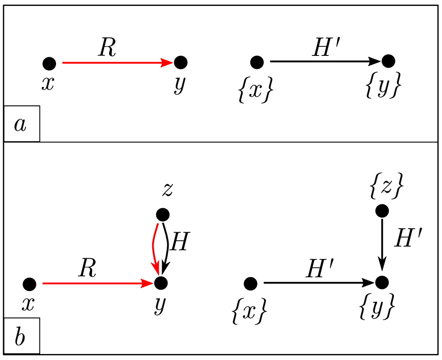

Figure 5.3: On the top: a poset (U, H) where $H = I$ (i.e. a set) is associated to a quotient structure that is a proper graph ($H \neq I$), by a preorder R. On the bottom, a simple generalisation of the situation above: even if the initial poset is not a set, an extra H' step appears in the quotient structure (reflexive loops of R, H and H' are left implicit).

The relation R is a preorder (reflexive loops of R, H and H' are left implicit). The starting poset (U, H) is just a set, i.e. $H = I$. However, its quotient structure is a two-level poset, where $\{ x \}$ is an edge and $\{ y \}$ is a node: the quotient structure is a hypergraph. The aim is to extend ideas of rough set theory, to hypergraphs and more generally to posets. The situation depicted in Figure 5.3.a seems counter-intuitive from the point of view of rough set theory. Indeed in rough set theory, or more generally in the theory of set partitions, the quotient structure of set, providing a coarser description of the initial set, is always a set itself. By modular reasoning, if the situation as in Figure 5.3.a is considered counter-intuitive, so it should be the one depicted in Figure 5.3.b (all we have done is we added an edge incident with one of the two nodes). After all, a set is a hypergraph, and more generally is poset, so a theory of hypergraph partitions, or even more generally a theory of poset partitions, should be conservative with respect to the theory of set partitions. By "conservative" we mean that what holds true in the first theory should still hold true in the new, extended theory. If we look carefully at the two examples in Figure 5.3, we can see that a relation H' between two elements in the quotient structure might arise, without it corresponding, via quotient function, to any H incidence in U. There is an H' step that has come "out of nowhere", as a corresponding H step is not present in the initial posets. This idea can be formalised by saying that there is the *lack* of a *back condition* on the quotient function f w.r.t. the two partial orders H and H'. We can think of a back condition on f relating (U, H) and (U', H') as follows: for any H' step in $U' \times U'$ there must be a corresponding H step in $U \times U$. The correspondence must happen via the quotient function, as this is the link between (U, H) and (U', H'). Thus the possible back conditions to impose on f w.r.t. H and H' are built as follows: an H' relation is always assumed (thick arrow in the Figure

5.4), and an H relation must be inferred (dotted arrow in the Figure 5.4). The two quotient function steps (there are two because we need to link two pairs, one in $U' \times U'$ and one in $U \times U$), might or might not be assumed (inferred). This gives us an exhaustive set of four possible back conditions on f w.r.t. H and H', drawn in Figure 5.4. The idea of a back condition has probably never been formalised in the literature, but it is not new. Examples of back conditions on relations between structures are the order-reflecting property on a mapping between posets (order-preserving being a *forth condition* in turn) [64], or the zag-condition on a bisimulation between modal frames (zig-condition being a forth condition), see [5].

In what follows, we are going to analyse all the possible back conditions to impose on f w.r.t. H and H' from Figure 5.4. We will see that they all correspond to specific constraints on R. Whilst the first condition is too restrictive to be considered, the following two conditions on f correspond to constraints on R that have been proposed by Ono in [52]. Here, the author's goal is to study intuitionistic analogues of the modal logic **S5**. Each of two constraints on R at issue substitutes symmetry of R, and they give rise respectively to two different intuitionistic analogues of the modal logic **S5**. The last constraint on f w.r.t. H and H' that we will analyse, corresponds in turn to a constraint on R that has been proposed in [66], called symmetry-generation, as we will show. In [66] the aim was to study relations on hypergraphs corresponding to partitions. However, the role of the quotient function, mapping the initial hypergraph (U, H) to its quotient structure (U', H'), is not analysed there, and symmetry-generation is not compared to other similar constraints on R, as those analysed by Ono [52]. Our analysis will make this comparison and we will also underline the different effects that each of these constraints will imply, in the type of partitions that can be generated from a poset (U, H).

$$\begin{tikzcd}
w \arrow[r, "f"] \arrow[d, dashed, "H"'] & z \arrow[d, "H'"] \\
u \arrow[r, "f"'] & v
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, "f"] \arrow[d, dashed, "H"'] & z \arrow[d, "H'"] \\
u \arrow[r, dashed, "f"'] & v
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, dashed, "f"] \arrow[d, dashed, "H"'] & z \arrow[d, "H'"] \\
u \arrow[r, "f"'] & v
\end{tikzcd}
\qquad
\begin{tikzcd}
w \arrow[r, dashed, "f"] \arrow[d, dashed, "H"'] & z \arrow[d, "H'"] \\
u \arrow[r, dashed, "f"'] & v
\end{tikzcd}$$

Figure 5.4: The four back-conditions on f w.r.t. H and H'. They can be written as inclusion of relations, as follows: *i)* $f \,;\, H' \,;\, f^{-1} \subseteq H$. *ii)* $f \,;\, H' \subseteq H \,;\, f$. *iii)* $H' \,;\, f^{-1} \subseteq f^{-1} \,;\, H$. *iv)* $H' \subseteq f^{-1} \,;\, H \,;\, f$.

Let us look at the first back condition from Figure 5.4. It is the order-reflecting condition on f w.r.t. H and H' (from Definition 106). It is obvious from Figure 5.3, that this condition does not hold already. Is this the right back condition to impose on f w.r.t. H and H'? As we are going to see, imposing this constraint is equivalent to requiring that $R \subseteq H$ will follow (and vice-versa). As $H \subseteq R$ already holds by assumption of reflexivity of R, we will have that $R = H$. To prove this fact, we need an auxiliary lemma.

Lemma 109. Let P, S and Q be relations, and W, V be sets, such that $P \subseteq W \times W$, $S \subseteq V \times V$ and $Q \subseteq W \times V$, and let Q be a *i)* total relation, so $I_W \subseteq Q \,;\, \breve{Q}$, and *ii)* order-preserving w.r.t. P and S. Then $P \subseteq Q \,;\, S \,;\, \breve{Q}$

Proof. $P \subseteq I_W ; P ; I_W \subseteq Q ; \breve{Q} ; P ; Q ; \breve{Q} \subseteq Q ; S ; \breve{Q}$. This holds by Lemma 98, by totality of Q, and by Q being order-preserving w.r.t. P and S. ■

Now, as f is a function, and thus is a total relation between U and U' and as it follows from Definition 104 that it is order-preserving w.r.t. R and H', we have that $R \subseteq f ; H' ; f^{-1}$ holds.

Proposition 110. f is order-reflecting w.r.t. H and H' iff $R \subseteq H$

Proof. First we prove that if f is order-reflecting w.r.t. H and H' then $R \subseteq H$. We have that: $R \subseteq f ; H' ; f^{-1}$ by application of Lemma 109 to R, H' and f, and $f ; H' ; f^{-1} \subseteq H$ is the assumption of order-reflecting of f w.r.t. H and H'. Thus $R \subseteq H$.
For the other direction, suppose that $R \subseteq H$. Then $R = H$, by reflexivity of R. Then we can substitute H to R in the result stated in Proposition 107 (this holds for any preorder relation, so also for H). Thus we have that $f ; H' ; f^{-1} \subseteq H$, as wanted (notice that here f is dilation by $(H \cap \breve{H})$, i.e. dilation by I, as we are assuming that $R = H$). ■

Thus, imposing this back condition on f w.r.t. H and H' seems to have a big drawback. We would be restricted to a unique and quite trivial choice of preorder R for poset partitions, namely H. Every element will be mapped by the quotient function f into a block containing only the element itself, as f will be given by the dilation by $H \cap \breve{H} = I$, by antisymmetry of H.

Let us consider the next back condition as in Figure 5.4. We will call this constraint the *zag* constraint as it is equivalent to imposing on f the zag-condition of a bounded morphism, w.r.t. modal frames (U, H) and (U', H') (see [5] p. 17). It can be written as the following inclusion of relations: $f ; H' \subseteq H ; f$.

Definition 111. Let (U, H) be a poset. An element $u \in U$ is *minimal* if for every $v \in U$, $u\ H\ v$ implies $v = u$.

It is clear that, in poset (U, H) that is a hypergraph, minimal elements are the nodes, as in Definition 9.

Now, if we impose the zag condition on f w.r.t. H and H', we have the following effect.

Proposition 112. Let f satisfy the zag condition, and let u be minimal in the poset (U, H). Then $f(u)$ is minimal in the poset (U', H').

Proof. Suppose the zag condition holds and u is minimal. Suppose $f(u)H'Y$ for some $Y \in U$. Then the zag condition gives that there is a $v \in U$ such that $f(v) = Y$ and $u\ H\ v$. As u is minimal, $v = u$, and hence $Y = f(u)$. Thus $f(u)$ is minimal in (U', H') ■

As shown from Proposition 112, imposing the zag-constraint might also be quite restrictive. Under this condition, we would have that nodes of a hypergraph (U, H) can only get mapped into blocks of the quotient structure (U', H') that are nodes as well. Partitions of a hypergraph like the one presented in Figure 5.1 would not be allowed. Indeed there the node c of the initial hypergraph (U, H) is assigned to a block of U' that plays the roles of an edge in (U', H'), namely Y. We have a minimal element, c, that gets mapped onto a non-minimal element, Y, by f. Thus the zag condition is not respected. If we want to allow these kinds of partitions, then the zag has to be ruled out as a candidate for a back condition to impose on f w.r.t. H and H'. We have seen in Section 1.8, Figure 1.10 (left) a practical example of graph granulation, showing that a node might get clustered, with other elements, into a block that plays the role of an edge in the coarser description provided by the quotient structure. In that case indeed we had a graph representing a certain journey with a start point and an end point. Then, we wanted to get a coarser description of this just by looking at all the middle points between the start and the end, simply as a *path* between them.

We are now going to show that the zag condition corresponds to a constraint on R. This constraint on R is also found in [52], where the author proposes it alongside with transitivity and reflexivity of a stable relation R, to define the semantic frame of an intuitionistic analogue of the modal logic **S5**.

Proposition 113. Imposing the zag condition is equivalent to imposing the following constraint on R: $R \subseteq H \,;\, \overset{\leftrightarrow}{R}$.

Proof. First we prove that $R \subseteq H \,;\, \overset{\leftrightarrow}{R}$ implies that $f \,;\, H' \subseteq H \,;\, f$. We have the following: $f\,;H' \subseteq f\,;H'\,;I' \subseteq f\,;H'\,;f^{-1}\,;f \subseteq R\,;f \subseteq H\,;\overset{\leftrightarrow}{R}\,;f \subseteq H\,;f\,;f^{-1}f \subseteq H\,;f\,;I' = H\,;f$ by Lemma 98 being f surjective, Proposition 107, R constraint $R \subseteq H\,;\overset{\leftrightarrow}{R}$, Lemma 105, Lemma 98 being f functional, and by Lemma 98 again. Thus $f \,;\, H' \subseteq H \,;\, f$, under the assumption that $R \subseteq H \,;\, \overset{\leftrightarrow}{R}$ holds.
Now we prove the other direction. We have that $R \subseteq I;R;I \subseteq f;f^{-1};R;f;f^{-1} \subseteq f\,;H'\,;f^{-1} \subseteq H\,;f\,;f^{-1} \subseteq H\,;\overset{\leftrightarrow}{R}$. This is by Lemma 98, being f total, by f being order-preserving w.r.t. R and H', by zag-constraint, and finally by Lemma 105. Thus $R \subseteq H \,;\, \overset{\leftrightarrow}{R}$, under the assumption that $f \,;\, H' \subseteq H \,;\, f$ holds. ■

So we can identify the zag condition on f as a constraint on R. When $R = H$, $R \subseteq H \,;\, \overset{\leftrightarrow}{R}$ holds, as we have that $H \subseteq H \,;\, I \subseteq H \,;\, (H \cap \breve{H})$. Therefore, by Proposition 110 and Proposition 113 we can establish that the order-reflecting constraint implies the zag-constraint. In the other direction, the implication does not hold. A very simple example is a graph with only two nodes x and y with $H = I$, where they both get clustered into a unique block $X = \{\, x, y \,\}$. We can therefore establish a strict order between the first two back-conditions that we have analysed so far.

The third candidate for a back-condition on f w.r.t. H and H' (Figure 5.4) can be expressed as follows: $H' \,;\, f^{-1} \subseteq f^{-1} \,;\, H$. We are going to refer to this condition as *co-zag*-condition.

Definition 114. Let (U, H) be a poset. An element $u \in U$ is *maximal* if for every $v \in U$, $v \; H \; u$ implies $v = u$.

It is clear that, in a poset (U, H) that is a hypergraph, the maximal elements are edges and isolated nodes (i.e. those nodes such that there's no edge that is H-incident with them).

If f satisfies the co-zag condition, then we have the following.

Proposition 115. Let f satisfy the co-zag condition, and let u be a maximal element in the poset (U, H). Then $f(u)$ is maximal in the poset (U', H')

Proof. Suppose the co-zag condition holds and u is maximal. Suppose $Y \; H' \; f(u)$ for some $Y \in U'$. Then the co-zag condition gives that there is a $v \in U$ such that $f(v) = Y$ and $v \; H \; u$. As u is maximal, $v = u$, and hence $Y = f(u)$. Thus $f(u)$ is maximal in (U, H'). ■

The co-zag puts a restriction on the kind of allowed partitions: in a hypergraph (U, H), an edge, that is a maximal element in (U, H), can only get clustered to an element of the quotient structure that is an edge, or, at most, isolated nodes, i.e. to maximal elements of the quotient structure. An edge can never get mapped to a node like in Figure 5.5, making such a partition impossible to obtain. Hence, if we want to allow the kind of partitions of Figure 5.5, the co-zag has to be ruled out as a candidate for a back condition on f w.r.t H and H'. We have seen in Section 1.8, Figure 1.10 (right) and Figure 1.11, other practical examples of edges of a graph that get clustered, together with other elements, in a block that plays the role of a node in the coarser view of the starting graph.

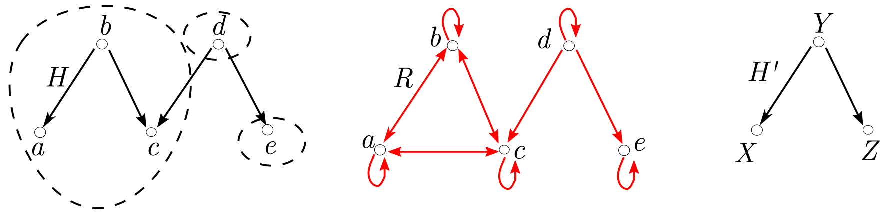

Figure 5.5: Example of a partition on hypergraph arising from a preorder R that does not respect the co-zag condition. We have the edge b, i.e. a maximal element in the poset (U, H), that gets mapped by f to the node $X = \{\, a, b, c \,\}$ that is not maximal in (U', H'). This type of partition is not allowed if we impose the co-zag condition on f.

Next, we are going to show that imposing the cozag condition on f w.r.t. H and H' is equivalent to imposing a constraint on R. Also this constraint is proposed in [52], and it contributes to define the semantic frame of an intuitionistic analogue of **S5**.

Proposition 116. Imposing the co-zag is equivalent to imposing the following constraint on R: $R \subseteq \overset{\leftrightarrow}{R} \,; H$.

Proof. First we prove that if $R \subseteq \overset{\leftrightarrow}{R} \,; H$ holds, then co-zag holds. We have the following chain of inclusions: $H' \,; f^{-1} \subseteq I' \,; H' \,; f^{-1} \subseteq f^{-1} \,; f \,; H' \,; f^{-1} \subseteq f^{-1} \,; R \subseteq f^{-1} \,; \overset{\leftrightarrow}{R} \,; H \subseteq f^{-1} \,; f \,; f^{-1} \,; H \subseteq I' \,; f^{-1} \,; H = f^{-1} \,; H$, by Lemma 98 since f is surjective, by Proposition 107, by $R \subseteq \overset{\leftrightarrow}{R} \,; H$, Lemma 105, by Lemma 98 f being functional.
For the other direction, let us assume that the co-zag holds. We have the following chain of inclusion: $R \subseteq I \,; R \,; I \subseteq f \,; f^{-1} \,; R \,; f \,; f^{-1} \subseteq f \,; H' \,; f^{-1} \subseteq f \,; f^{-1} \,; H \subseteq \overset{\leftrightarrow}{R} \,; H$ by Lemma 105, by Lemma 98 because of totality of f, by f being order-preserving w.r.t R and H', by the assumption of co-zag, and by Lemma 105. ■

It is easy to see that if $R = H$ then $R \subseteq \overleftrightarrow{R} \,; H$ holds (by reflexivity and anti-symmetry of H), but not the other way around (an example is again a set with only two nodes, where they both get clustered into a single node). Thus, in terms of the quotient function, we can say that if f is order-reflecting w.r.t. H and H', then f respects the co-zag condition, but not the other way around. Also in Figure 5.6 we have situations where $R \subseteq H \,; \overleftrightarrow{R}$ holds but $R \subseteq \overleftrightarrow{R} \,; H$ does not (top), and vice versa (bottom). Thus we can deduce that the zag and the co-zag condition are not comparable, neither of them implies the other one.

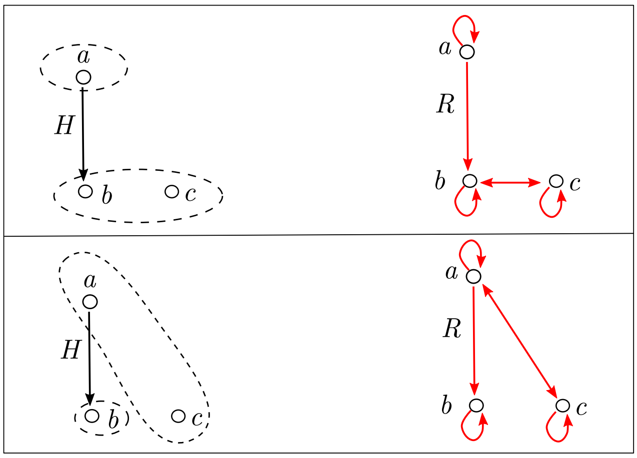

Figure 5.6: In the situation depicted on the top we have that $R \subseteq H \,; \overleftrightarrow{R}$, but $R \nsubseteq \overleftrightarrow{R} \,; H$. Indeed $a \; R \; b \; R \; c$ implies $a \; R \; c$ by transitivity of R (left implicit in the figure), but there's no x such that $a\overleftrightarrow{R}xHc$. So $R \subseteq H \,; \overleftrightarrow{R}$ does not imply $R \subseteq \overleftrightarrow{R} \,; H$. In the situation depicted on the bottom we have that $R \subseteq \overleftrightarrow{R} \,; H$, and $R \nsubseteq H \,; \overleftrightarrow{R}$. Thus $R \subseteq \overleftrightarrow{R} \,; H$ does not imply $R \subseteq H \,; \overleftrightarrow{R}$

The last candidate for a back-condition on f w.r.t. H and H' is the following (see Figure 5.4): $H' \subseteq f^{-1} \,; H \,; f$. Let us call it *weak-zag* condition. The weak-zag won't impose any restriction on what type of elements of (U', H') each element of a hypergraph (U, H) gets mapped to. Indeed we can see from Figure 5.1, where the weak-zag is respected, that it is possible for a node to become an edge in the quotient structure, and from Figure 5.5, where again the weak-zag is satisfied, that an edge becomes a node in the quotient structure. More generally, the weak-zag won't impose the same restrictions as the zag and the co-zag conditions, implying that minimal and maximal elements "stay" minimal and maximal, respectively, in the quotient structure. Still, imposing this condition on f will create a dependency of H' from the initial partial order H, so that cases like the ones presented in Figure 5.3, won't be allowed (the weak-zag is indeed not respected there).

Now we are going to prove that imposing the weak-zag condition on f is equivalent to imposing a constraint on R called symmetry generation, presented in [66].

Theorem 117. Imposing the weak-zag on f w.r.t. H and H' is equivalent to imposing that $R \subseteq \overset{\leftrightarrow}{R} ; H\overset{\leftrightarrow}{R}$.

Proof. First we prove that symmetry generation of R implies that f respects the weak-zag w.r.t. H and H'. We have the following chain of inclusions: $H' \subseteq f^{-1};R;f \subseteq f^{-1};\overset{\leftrightarrow}{R};H\overset{\leftrightarrow}{R};f \subseteq f^{-1};f;f^{-1};H;f;f^{-1};f \subseteq I';f^{-1};H;f;I' \subseteq f^{-1} ; H ; f$ by Definition 104, symmetry-generation of R, Lemma 105, Lemma 98 by functionality of f, and by Lemma 98 again.
For the other direction, we have the following chain of inclusion: $R \subseteq f ; H' ; f^{-1} \subseteq f ; f^{-1} ; H ; f ; f^{-1} \subseteq \overset{\leftrightarrow}{R} ; H\overset{\leftrightarrow}{R}$ by Lemma 109 applied to R, H' and f, weak-zag, and by Lemma 105. ■

It is easy to see that if either $R \subseteq \overset{\leftrightarrow}{R} ; H$ or $R \subseteq H ; \overset{\leftrightarrow}{R}$ holds, then R is symmetrically generated. This follows by the fact that $I \subseteq \overset{\leftrightarrow}{R}$. Both these implications do not hold in the other direction. For example in Figure 5.1 and Figure 5.5, we have that the symmetry-generation constraint holds but $R \subseteq H;\overset{\leftrightarrow}{R}$ and $R \subseteq \overset{\leftrightarrow}{R} ; H$ do not, respectively. We can draw the following diagram, to frame the implications between these constraints on preorder R.

$$\begin{tikzcd}
& R \subseteq H \arrow[dl, dash] \arrow[dr, dash] & \\
R \subseteq \overset{\leftrightarrow}{R} ; H \arrow[dr, dash] & & R \subseteq H ; \overset{\leftrightarrow}{R} \arrow[dl, dash] \\
& R \subseteq \overset{\leftrightarrow}{R} ; H ; \overset{\leftrightarrow}{R} &
\end{tikzcd}$$

Summing up, we have analysed all the possibilities for a back-condition on f w.r.t. H and H'. We have seen that imposing order-reflecting is very restrictive, as the only relation that satisfies this is H, and thus the only possible partition would be the one generated by $\_ \oplus I$, where each element is clustered with itself only. The two middle possibilities are not as restrictive but they still rule out some kinds of partitions that one might want to allow. We have seen that these two conditions on f correspond to R-constraints, and to modal axioms, already occuring in the literature, in Ono's work [52]. However, for the effects we have seen in Proposition 112 and Proposition 115, the **S5** intuitionistic logics arising from considering these constraints on R alongside reflexivity and transitivity, might be too restrictive to represent hypergraph and poset partitions. If one wishes to impose a back-condition on f w.r.t. H and H', that is some dependency of partial order H' from H, via f, we need to impose the weak-zag condition. Notice that as $f^{-1};H;f \subseteq H'$ already holds, then imposing weak-zag is equivalent to imposing the identity $H' = f^{-1};H;f$, and thus the partial order H' will be a "function" of both R (as the quotient function f is defined from R) *and* the initial partial order H. We have seen that weak-zag corresponds to a constraint on R, already appearing in [66], known as symmetry-generation of R. Thus if we want to obtain a poset partition from a stable relation R with this property, then we need R to be a symmetrically generated preorder. In Table 5.1 we can see all the back conditions of f w.r.t. H and H', with the corresponding constraint on R. By Proposition 20, we can establish that $R \subseteq H$ corresponds to the validity of the axiom $p \to \Box p$ in an H-frame. For what concerns the middle constraints $R \subseteq \overleftrightarrow{R}\,; H$ and $R \subseteq H\,; \overleftrightarrow{R}$, the corresponding axioms are presented in [52]. And in Section 5.5 we are going to see that the symmetry-generation constraint, and thus the weak-zag, corresponds to the axiom shown in Table 5.1.

Table 5.1: The back conditions on f w.r.t. H and H', with the corresponding constraint on R, and the corresponding axiom in intuitionistic modal logic.

Back condition on f	**Constraint on R**	**Modal axiom**
$f\,; H'\,; f \subseteq H$	$R \subseteq H$	$p \to \Box p$
$f\,; H' \subseteq H\,; f$	$R \subseteq H\,; \overleftrightarrow{R}$	$(\Box p \to \Box q) \to \Box(\Box p \to \Box q)$
$H'\,; f^{-1} \subseteq f^{-1}\,; H$	$R \subseteq \overleftrightarrow{R}\,; H$	$\Box(\Box p \vee q) \to (\Box p \vee \Box q)$
$H' \subseteq f^{-1}\,; H\,; f$	$R \subseteq \overleftrightarrow{R}\,; H\,; \overleftrightarrow{R}$	$(\Box(\Box p \vee (\Box q \to \Box r))) \to$ $(\Box p \vee \Box(\Box q \to \Box r))$

We can consider these conditions on the preorder R as weaker forms of symmetry in the following sense. First of all, symmetry of R implies any of these properties (except for the first one $R \subseteq H$, but we have seen that imposing this constraint on R is highly restrictive), but not the other way around. As an example, if R is symmetric, i.e. $R \subseteq \breve{R}$, then we have that $R \subseteq \overleftrightarrow{R}$, and thus $R \subseteq I\,; \overleftrightarrow{R} \subseteq H\,; \overleftrightarrow{R}$. There is another sense in which the constraints, substituting symmetry, can be seen as weaker forms of symmetry: when $H = I$, thus in the context of classical modal logic, all these constraints imply symmetry (they are actually equivalent to symmetry, except for $R \subseteq H$). As an example, if $H = I$ then $R \subseteq H\,; \overleftrightarrow{R}$ that is $R \subseteq \overleftrightarrow{R}$ that is symmetry of R.

The fact that (U, H) is not just a poset, but it is indeed a hypergraph, doesn't play a role in the construction presented in Definition 104. So we can say that we have a method to partition a poset in a way that its quotient structure is also a poset, as proved in Theorem 108. However, we might wish to ensure that, if we start from a poset (U, H) that is also a hypergraph, then the arising quotient structure is not just a poset, but it is a hypergraph. This is what happens when partitioning a set: its quotient structure is still a set. In [66], an extra condition on R is considered. It is two-tierness of R: $(R \cap \breve{\breve{R}})^2 = \varnothing$. We are going to show that this property ensures that the quotient structure (U', H') is not just a generic poset, but it is indeed a hypergraph. We introduce first the following Lemma. For sake of simplicity, we indicated the non-symmetric part of the partial order H', namely $(H' \cap \breve{\breve{H}}')$ simply as J'.

Lemma 118. Let (U, H) be a a poset and R a stable preorder on U. Let (U', H') be the associated quotient structure as in Definition 104. Then we have the following:

1. $J' \subseteq f^{-1} ; (R \cap \breve{R}) ; f$.
2. $(R \cap \breve{R}) \subseteq f ; J' ; f^{-1}$.

Proof. Item 1: From Definition 104, we have that $H' \subseteq f^{-1} ; R ; f$. So $J' \subseteq H' \subseteq f^{-1} ; R ; f$. So it must be that $J' \subseteq f^{-1} ; \overleftrightarrow{R} ; f$ or $J' \subseteq f^{-1} ; (R \cap \breve{R}) ; f$. Suppose it is the first. Then $J' \subseteq f^{-1} ; \overleftrightarrow{R} ; f \subseteq f^{-1} ; f ; f^{-1} ; f \subseteq I' ; I' \subseteq I'$, by Lemma 105 and by functionality of f. But $J' \subseteq I'$ iff $J' = \varnothing$, as J' is the non-symmetric part of H and I' is its symmetric part. So $J' \subseteq f^{-1} ; \overleftrightarrow{R} ; f$ iff $J' = \varnothing$. But when $J' = \varnothing$, we have that $J' \subseteq f^{-1} ; (R \cap \breve{R}) ; f$ holds. Thus $J' \subseteq f^{-1} ; (R \cap \breve{R}) ; f$ holds.
Item 2 : Very similar to proof of item 1. We know that $(R \cap \breve{R}) \subseteq R \subseteq f ; H' ; f^{-1}$, by Proposition 109. Then it must be $(R \cap \breve{R}) \subseteq f ; I' ; f^{-1}$ or $(R \cap \breve{R}) \subseteq f ; J' ; f^{-1}$, as $H' = I' \cup J'$. If we assume that the first inclusion holds, we get $(R \cap \breve{R}) \subseteq \overleftrightarrow{R}$, which is only possible when $(R \cap \breve{R}) = \varnothing$. Then if the first inclusion holds, the second one will hold. So, in any case, $(R \cap \breve{R}) \subseteq f ; J' ; f^{-1}$ holds. ■

Theorem 119. Let (U, H) be a poset and R a stable preorder on U. Let (U', H') be its quotient structure associated by the quotient function f, defined in the usual way. $(R \cap \breve{R})^2 = \varnothing$ iff $J'^2 = \varnothing$.

Proof. This follows from Lemma 118. In order to prove $(R \cap \breve{R})^2 = \varnothing$ implies $J'^2 = \varnothing$, we prove the contrapositive, i.e. we suppose that $J'^2 \neq \varnothing$ and we show that $(R \cap \breve{R})^2 \neq \varnothing$. If $J'^2 \neq \varnothing$, then, by Lemma 118 item 1, $f^{-1} ; (R \cap \breve{R}) ; f ; f^{-1} ; (R \cap \breve{R}) ; f \neq \varnothing$. So, by Lemma 105, $f^{-1} ; (R \cap \breve{R}) ; \overleftrightarrow{R} ; (R \cap \breve{R}) ; f \neq \varnothing$, and by transitivity of R we have that $f^{-1} ; (R \cap \breve{R})^2 ; f \neq \varnothing$. But then there are $X, Y \in U'$, and $x, y \in U$, such that $X f^{-1} x (R \cap \breve{R})^2 y f Y$, so $(R \cap \breve{R})^2 \neq \varnothing$, as wanted.
The proof of the other direction is analogous. We can prove the contrapositive form, i.e. assuming that $(R \cap \breve{R})^2 \neq \varnothing$, it will follow that $J'^2 \neq \varnothing$. To do so we employ Lemma 118 item 2, and the fact that f is functional. ■

We have seen in Section 2.3, Corollary 22, that a 2-tier partial order on a set defines a hypergraph as in Definition 9. We have already proved that H' is a partial-order on U' (Theorem 108). Then, with the proviso that R is 2-tier, we get that (U', H') is a hypergraph.

5.5 System S5 for UBiSKt

In this section we will see that, given an H-frame F such that $F \in \mathbb{S}4$, there are two formulas corresponding to the property of R in F being two-tier and symmetrically generated. These results are presented in [72], and they were achieved in collaboration with Katsuhiko Sano.

Theorem 120. Let $F = (U, H, R)$ be an H-frame where R is reflexive and transitive. Then $F \models \Box q \lor \Box(\Box q \to (\Box p \lor \Box\neg\Box p))$ iff $(R \cap \breve{\overline{R}})^2 = \varnothing$.

We note that the following equivalence holds: $(R \cap \breve{\overline{R}})^2 = \varnothing$ iff xRy and yRz jointly imply yRx or zRy for all x, y, $z \in U$.

Proof. **Right-to-left direction**: Assume that $(R \cap \breve{\overline{R}})^2 = \varnothing$. Fix any valuation V and any $w \in U$. Put $M = (F, V)$ and assume that $M, w \not\models \Box q$. To show that $M, w \models \Box(\Box q \to (\Box p \lor \Box\neg\Box p))$, fix any v such that wRv and $M, v \models \Box q$. We show that $M, v \models \Box p \lor \Box\neg\Box p$. So suppose that $M, v \not\models \Box p$ and let us show that $M, v \models \Box\neg\Box p$. Let us fix any u such that $v\ R\ u$.To show that $M, u \models \neg\Box p$, let us fix any $u\ H\ i$. Our goal is to show that $M, i \not\models \Box p$. By $v\ R\,;H\ i$, we have $v\ R\ i$. Since $w\ R\ v$ and $v\ R\ i$, two-tierness implies $v\ R\ w$ or $i\ R\ v$. If $v\ R\ w$, we should have $M, w \models \Box q$ by transitivity of R and $M, v \models \Box q$. But this is a contradiction with $M, w \not\models \Box q$. So we have iRv. By $M, v \not\models \Box p$, we can find a state $x \in U$ such that vRx and $M, x \not\models p$. By transitivity of R and iRv, we have $M, i \not\models \Box p$.
Left-to-right direction: Suppose that $F \models \Box q \lor \Box(\Box q \to (\Box p \lor \Box\neg\Box p))$. To show the two-tierness of R, let us suppose that wRv and vRu and that vRw fails. Our goal is to show that uRv. Define $V(p) = \{\, u \,\} \oplus R$ and $V(q) = \{\, v \,\} \oplus R$, where we note that both sets are H-sets by stability of R. Let us write $M = (F, V)$. We have $M, w \not\models q$ and wRw hence $M, w \not\models \Box q$. By the initial supposition (the validity of the formula), we obtain $M, w \models \Box(\Box q \to (\Box p \lor \Box\neg\Box p))$. Because wRv and $M, v \models \Box q$ by our definition of V, we obtain $M, v \models \Box p \lor \Box\neg\Box p)$. Let us establish $M, v \not\models \Box\neg\Box p$. It suffices to show $M, u \not\models \neg\Box p$. This is clear from uHu and $M, u \models \Box p$ by our definition of V. It follows from $M, v \models \Box p \lor \Box\neg\Box p$ that $M, v \models \Box p$, which implies $\{\, v \,\} \oplus R \subseteq \{\, u \,\} \oplus R$. Since vRv, and thus $v \in \{\, v \,\} \oplus R$ we have that $v \in \{\, u \,\} \oplus R$, and then we conclude uRv, as required. ■

Definition 121. We use t_2 to mean the formula $\Box q \lor \Box(\Box q \to (\Box p \lor \Box\neg\Box p))$. Let $\mathbb{T}_2$ be be the class of H-frames $F = (U, H, R)$ such that R is a preorder, i.e. $F \in \mathbb{S}4$, and $(R \cap \breve{\overline{R}})^2 = \varnothing$.

Lemma 122. Given a *ubist*-logic Λ such that $\mathbf{S4} \cup \{\, \mathbf{t}_2 \,\} \subseteq \Lambda$, the Λ-canonical H-frame $F^{\Lambda}_{(\Gamma,\Delta)} = (U^{\Lambda}, H^{\Lambda}, R^{\Lambda})$ is a preorder and it also satisfies two-tierness, i.e., $((R \cap \breve{\overline{R}})^{\Lambda})^2 = \varnothing$.

Proof. It suffices to prove the two-tierness of R^{Λ} alone. Suppose that $(\Sigma_1, \Theta_1)R^{\Lambda}(\Sigma_2, \Theta_2)$ and $(\Sigma_2, \Theta_2)R^{\Lambda}(\Sigma_3, \Theta_3)$. We need to prove that $(\Sigma_2, \Theta_2)R^{\Lambda}(\Sigma_1, \Theta_1)$ or $(\Sigma_3, \Theta_3)R^{\Lambda}(\Sigma_2, \Theta_2)$. Suppose that $(\Sigma_2, \Theta_2)R^{\Lambda}(\Sigma_1, \Theta_1)$ fails, i.e., we can find a formula ψ such that $\Box\psi \in \Sigma_2$ and $\psi \notin \Sigma_1$. To show $(\Sigma_3, \Theta_3)R^{\Lambda}(\Sigma_2, \Theta_2)$, fix any $\Box\varphi \in \Sigma_3$. Our goal is to establish $\varphi \in \Sigma_2$. Since $\Box\psi \in \Sigma_2$ and $\Box\psi \to \psi$, then $\psi \in \Sigma_2$ (because $\mathbf{S4} \subseteq \Lambda \subseteq \Sigma_2$ by item 2 of Lemma 31). Since $\psi \notin \Sigma_1$, we get $\Box\psi \notin \Sigma_1$. Because any substitution instance of $\mathbf{t}_2$ is in Σ_1, we deduce from $\Box\psi \notin \Sigma_1$ that $\Box(\Box\psi \to (\Box\varphi \lor \Box\neg\Box\varphi)) \in \Sigma_1$. By $(\Sigma_1, \Theta_1)R^{\Lambda}(\Sigma_2, \Theta_2)$, we obtain $\Box\psi \to (\Box\varphi \lor \Box\neg\Box\varphi) \in \Sigma_2$. Since $\Box\psi \in \Sigma_2$, we have $\Box\varphi \lor \Box\neg\Box\varphi \in \Sigma_2$. Recall that our goal is to show $\varphi \in \Sigma_2$. By our assumption of $\Box\varphi \in \Sigma_3$ we have that $\neg\Box\varphi \notin \Sigma_3$, by item 4 of Lemma 31. Since $(\Sigma_2, \Theta_2)R^{\Lambda}(\Sigma_3, \Theta_3)$, we get $\Box\neg\Box\varphi \notin \Sigma_2$. It follows from $\Box\varphi \lor \Box\neg\Box\varphi \in \Sigma_2$ that $\Box\varphi \in \Sigma_2$, and then $\varphi \in \Sigma_2$, because $\mathbf{S4} \subseteq \Lambda$, as desired. ■

We can establish the following strong completeness result by Lemmas 36, 122 and 102, Theorem 120 and Proposition 20.

Theorem 123. The logic $\mathsf{HUBiSKt}$ extended with **S4** and $\mathbf{t}_2$ is sound and strongly complete for the class $\mathbb{T}_2$, i.e., $\Gamma \models_{\mathbb{T}_2} \varphi$ iff $\Gamma \vdash_{\mathbf{HUBiSKtS4t_2}} \varphi$ for every set $\Gamma \cup \{\varphi\}$ of formulas.

The proof of the following lemma comes from the general fact that the complement of a lower set is an upset[3].

Lemma 124. Given $u \in U$, the set $-(\{ u \} \oplus \breve{H})$, namely the complement of an $\breve{H}$-dilate, is an H-set.

Proof. By Definition 14 of H-set, we need to show that $(-(\{ u \} \oplus \breve{H})) \oplus H \subseteq -(\{ u \} \oplus \breve{H})$. Suppose $v \in (-(\{ u \} \oplus \breve{H})) \oplus H$. Then there is $w \in U$ such that $w\ H\ v$ and $w \in -(\{ u \} \oplus \breve{H})$, namely $w \notin \{ u \} \oplus \breve{H}$. Hence it is not the case that $u\ \breve{H}\ w$, i.e. it is not the case that $w\ H\ u$. Therefore, as $w\ H\ v$ and by transitivity of H we have that also $v \notin \{ u \} \oplus \breve{H}$, hence $v \in -(\{ u \} \oplus \breve{H})$, as wanted. ■

Theorem 125. Let $F = (U, H, R)$ be an H-frame where R is reflexive and transitive. Then the following equivalence holds:

$$F \models (\Box(\Box p \vee (\Box q \to \Box r))) \to (\Box p \vee \Box(\Box q \to \Box r)) \text{ iff } R \subseteq \overleftrightarrow{R}; H; \overleftrightarrow{R}.$$

Proof. **Right-to-left direction**: Let us fix any valuation V. Let $M = (F, V)$. Let us fix $u \in U$. We need to show that $M, u \models (\Box(\Box p \vee (\Box q \to \Box r))) \to (\Box p \vee \Box(\Box q \to \Box r))$. Let us take any $v \in U$ such that $u\ H\ v$ and $M, v \models \Box(\Box p \vee (\Box q \to \Box r))$. We need to show that $M, v \models \Box p \vee \Box(\Box q \to \Box r)$. Let us assume that $M, v \not\models \Box p$, i.e., we can find a $w \in U$ such that $v\ R\ w$ and $M, w \not\models p$. Now we need to show that $M, v \models \Box(\Box q \to \Box r)$. Let us fix any $x \in U$ such that $v\ R\ x$. We show that $M, x \models \Box q \to \Box r$. Fix any $a \in U$ such that $x\ H\ a$ and $M, a \models \Box q$. To show that $M, a \models \Box r$, fix any $b \in U$ such that $a\ R\ b$. Our goal is to establish $M, b \models r$. By R being transitive and reflexive, we know that vRa. By R being symmetrically generated, we also know that there are $z, t \in U$ such that $v\ \overleftrightarrow{R}\ z\ H\ t\ \overleftrightarrow{R}\ a$. See the model in Fig. 5.7.

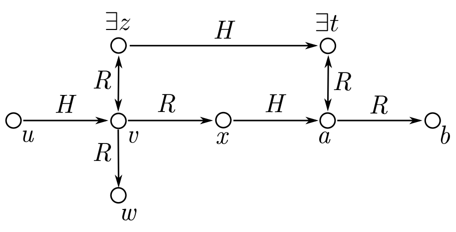

Figure 5.7: Model constructed for proving the right-to-left direction of Theorem 125.

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[3] We haven't used the terms *upper set* and *lower set* so far, but it is well known that upper sets are the subsets of a partially ordered set closed under partial-order successor, thus in (U, H) they are our H-sets. Lower sets are the dual notions, i.e. X is a lower set if $X \oplus \breve{H} \subseteq X$

Since $v\ R\ z$ and $M, v \models \Box(\Box p \vee (\Box q \to \Box r))$, we have that $M, z \models \Box p \vee (\Box q \to \Box r)$. Since $M, v \not\models \Box p$ and $z\ R\ v$, we know that $M, z \not\models \Box\Box p$. Then, by transitivity of R, $M, z \not\models \Box p$. Thus it must be the case that $M, z \models \Box q \to \Box r$. Then, as $z\ H\ t$, we have if $M, t \models \Box q$ we can deduce that $M, t \models \Box r$. Since $a\ R\ t$ and $M, a \models \Box q$ by assumption, we deduce from transitivity of R that $M, t \models \Box q$. Therefore $M, t \models \Box r$. By transitivity $t\ R\ b$ holds from $t\ R\ a$ and $a\ R\ b$, and then $M, b \models r$, as wanted.
Left-to-right direction: Assume the validity of the formula in $F = (U, H, R)$. Suppose that xRy. We need to show that $x\ \overleftrightarrow{R}\,;H\,;\overleftrightarrow{R}\ y$. Consider the following valuation: $V(p) = -(\{\,x\,\} \oplus \breve{H})$, $V(q) = \{\,y\,\} \oplus R$ and $V(r) = -(\{\,y\,\} \oplus \breve{H})$. These sets are H-sets by Lemma 124 and by stability of R. Let us write $M = (F, V)$. By our assumption, we get $M, x \models (\Box(\Box p \vee (\Box q \to \Box r))) \to (\Box p \vee \Box(\Box q \to \Box r))$. By $x \notin V(p)$ and xRx, we have $M, x \not\models \Box p$. Moreover we can prove that $M, x \not\models \Box(\Box q \to \Box r)$ as follows: by xRy and yHy, it suffices to show that $M, y \models \Box q$ and $M, y \not\models \Box r$. By our definition of V, the former is easy and the latter holds by yRy and $M, y \not\models r$. This finishes showing $M, x \not\models \Box(\Box q \to \Box r)$. Therefore, $M, x \not\models \Box p \vee \Box(\Box q \to \Box r)$ hence $M, x \not\models \Box(\Box p \vee (\Box q \to \Box r))$. So we can find a state $z \in U$ such that $x\ R\ z$ and $M, z \not\models \Box p$ and $M, z \not\models \Box q \to \Box r$. It follows from $M, z \not\models \Box p$ that there is a k such that $z\ R\ k$ and $M, k \not\models p$, hence $k \in \{\,x\,\} \oplus \breve{H}$, that is $k\ H\ x$. Then we have that $z\ R\,;H\ x$ and so $z\ R\ x$ by stability of R. Thus we have $x\overleftrightarrow{R}z$. On the other hand, from $M, z \not\models \Box q \to \Box r$ we can find a state w such that $z\ H\ w$ and $M, w \models \Box q$ and $M, w \not\models \Box r$. By $M, w \not\models \Box r$, we know that $w\ R\,;H\ y$ and so $w\ R\ y$. But it follows from $M, w \models \Box q$ that $\{\,w\,\} \oplus R \subseteq \{\,y\,\} \oplus R$. As $w\ R\ w$, we have that $w \in \{\,y\,\} \oplus R$ that is $y\ R\ w$. Hence $w\ \overleftrightarrow{R}\ y$. Therefore, we have $x\ \overleftrightarrow{R}\ z$, $z\ H\ w$ and $w\ \overleftrightarrow{R}\ y$, i.e., $x\ \overleftrightarrow{R}\,;H\,;\overleftrightarrow{R}\ y$, as desired. ■

Definition 126. We use **sg** to mean the formula $(\Box(\Box p \vee (\Box q \to \Box r))) \to (\Box p \vee \Box(\Box q \to \Box r))$. We also use **S5** to mean the set $\mathbf{S4} \cup \{\,\mathbf{sg}\,\}$. Let $\mathbb{S}5$ be the class of all H-frames $F = (U, H, R)$ such that R is a symmetry generated preorder, i.e., $F \in \mathbb{S}4$, and F is symmetry generated. We also use $\mathbb{HGT}_2\mathbb{S}5$ to mean $\mathbb{HG} \cap \mathbb{T}_2 \cap \mathbb{S}5$, where recall that $\mathbb{T}_2 \subseteq \mathbb{S}4$.

By Proposition 20, Corollary 22 and Theorems 120 and 125, we obtain the following.

Theorem 127. The logic $\mathsf{HUBiSKt}$ extended with **bd2**, $\mathbf{t}_2$ and with **S5** is sound for the class $\mathbb{HGS}5$.

Conjecture 128. Given a *ubist*-logic Λ such that $\mathbf{S5} \subseteq \Lambda$, the Λ-canonical H-frame $F^{\Lambda}_{(\Gamma,\Delta)} = (U^{\Lambda}, H^{\Lambda}, R^{\Lambda})$ is a preorder and it also satisfies symmetrical generation, i.e., $R^{\Lambda} \subseteq \overleftrightarrow{R}^{\Lambda}\,; H^{\Lambda}\,; \overleftrightarrow{R}^{\Lambda}$.

In order to accomplish proof of Conjecture 128 we need to show that, given pairs (Σ_1, Θ_1) and (Σ_2, Θ_2) in the canonical frame F^{Λ}, if $(\Sigma_1, \Theta_1)\ R^{\Lambda}\ (\Sigma_2, \Theta_2)$, then there are two elements of the canonical frame (Σ_3, Θ_3) and (Σ_4, Θ_4) such that $(\Sigma_1, \Theta_1)\ \overleftrightarrow{R}^{\Lambda}\ (\Sigma_3, \Theta_3)$, and $(\Sigma_3, \Theta_3)\ H^{\Lambda}\ (\Sigma_4, \Theta_4)$, and $(\Sigma_4, \Theta_4)\ \overleftrightarrow{R}^{\Lambda}\ (\Sigma_2, \Theta_2)$. The challenge of this proof is that the constraint of R at issue, namely symmetry generation, is *non-analytical*. We call a constraint on a relation R *analytical*, when its conclusion is simpler than its assumption. An example of an analytical constraint is transitivity of R, where from the assumption that $x\ R\ y\ R\ z$ holds for some elements x, y and z, we can infer that $x\ R\ z$ holds. On the other hand, for symmetry generation, from the assumption that $x\ R\ y$ holds for some elements x and y of the domain, we then need to prove the existence of new elements z and t, that relate to each other and to x and y in a certain way. There is not much literature on canonical model proofs and related completeness proofs, where the constraints at issue are non-analytical, and where, additionally, the modal logic has intuitionistic logic as base. An exception to this is Ono's work [52] (p. 698-700). As we know he considers two constraints related to symmetry generation, namely $R \subseteq H\ ;\overleftrightarrow{R}$ and $R \subseteq \overleftrightarrow{R}\ ; H$. Thus a proof of Conjecture 128 could be inspired by Ono's method, although symmetry generation has a higher degree of non-analyticity than Ono's constraints, as we need to prove existence of not just one, but two elements of the canonical frame that relate to existing elements *and* to each other in the expected way. Thus the proof of Conjecture 128 seems more challenging and it is left as future work, together with the possibility of exploring new methods for proving completeness of intuitionistic modal logics with non-analytical constraints.

Hence, if Conjecture 128 is proved, together with Lemma 39, 122 and the truth lemma 36, we would have the following completeness theorem.

Conjecture 129. The logic **HUBiSKt** extended with **bd2**, $\mathbf{t}_2$ and **S5** is strongly complete for the class $\mathbb{HGT}_2\mathbb{S}5$ i.e. for the class of H-frames where (U, H) is a hypergraph and where R is a two-tier, symmetrically generated preorder.

In this logic we can represent and reason about hypergraph partitions and related quotient structures. If the constraints **bd2**, making (U, H) a hypergraph, and the related constraint $\mathbf{t}_2$, making the quotient structure a hyprgraph, are dropped, then we have a new intuitionistic version of the modal logic **S5**, where poset partitions can be represented. We have seen in Section 5.4 how this logic relates to other intuitionistic analogues of **S5**, namely the ones studied by Ono in [52].

5.6 Expressing Coarse Spatial Relations

In Chapter 4 we have seen how connection and other spatial relations can be expressed when we look at regions of a graph at another level of detail, induced by a stable relation R. But what happens when the relation R has the properties discussed in Section 5.5? This section is going to answer this question. We have already seen in Section 5.2 that when R is reflexive and transitive the notion of granulation as it was presented in Section 4.2, Definition 75 is equivalent to taking the lower approximation, or erosion, of the subgraph at issue. So 'coarsely p' will be equivalent to $\Box p$ in the logic **HUBiSKt** extended with **S5** as well as in **HUBiSKt** extended with **S5**, **bd2** and **t2**, namely when both (U, H) and (U', H') are hypergraphs. This leads to the following notion of coarse connection:

Proposition 130 (**S5** Coarse Connection)**.** $C_G(\varphi, \psi) \leftrightarrow \mathsf{E}(\blacklozenge \lozenge \Box\varphi \wedge \Box\psi)$ is a theorem of **HUBiSKtS5**. Therefore we can define coarse connection in this system as follows: $C_{G\mathbf{S5}}(\varphi, \psi) := \mathsf{E}(\blacklozenge \lozenge \Box\varphi \wedge \Box\psi)$.

Proof. $\mathsf{E}(\blacklozenge \Diamond G\varphi \wedge G\psi) \leftrightarrow \mathsf{E}(\blacklozenge \Diamond \Box\varphi \wedge \Box\psi)$ is a theorem in **HUBiSKtS5**, by Theorem 101, item 3. ■

A remark is important. This predicate of coarse connection gives more evidence of the fact that imposing the classical notion of equivalence relation as corresponding to a partition on a poset is too restrictive, as argued in [66]. Indeed symmetry of R in an H-frame corresponds to the property $R = \circlearrowright R$, that corresponds in turn to the validity of the formula $\blacklozenge p \leftrightarrow \Diamond p$ in all symmetric H-frames (see correspondence results from [74]). Therefore $\mathsf{E}(\blacklozenge \Diamond \Box\varphi \wedge \Box\psi) \leftrightarrow \mathsf{E}(\blacklozenge \blacklozenge \Box\varphi \wedge \Box\psi) \leftrightarrow \mathsf{E}(\blacklozenge \Box\varphi \wedge \Box\psi) \leftrightarrow \mathsf{E}(\Box\varphi \wedge \Box\psi)$ is a theorem in the extension of **HUBiStS4** with the symmetry axiom $\blacklozenge p \leftrightarrow \Diamond p$. The notion of coarse connection collapses with a notion of overlapping, since the formula $\mathsf{E}(\Box\varphi \wedge \Box\psi)$ means that the granulation of φ overlaps with the granulation of ψ. Cases of external-connection, i.e. edge-connection in a hypergraph, will no longer happen in such a setting.

To give an example of **S5**-connection, let us look at the graph partition in Fig. 5.1. R is transitive, reflexive and symmetrically generated, but not symmetric, and (U, H, R) is an H-frame, let us call it F. Let us impose a valuation V such that $V(p) = \{\, a \,\}$ and $V(q) = \{\, e \,\}$ for propositional variables p, q in the language. We remark that these sets are H-sets. Let $M = (F, V)$. Then the granulation of the subgraph $\{a\}$ is $[\![\Box p]\!]_M = R \ominus [\![p]\!]_M = R \ominus \{a\} = \{a\}$ and the granulation of subgraph $\{e\}$ is $[\![\Box q]\!]_M = R \ominus [\![p]\!]_M = R \ominus \{e\} = \{e\}$. Also $[\![\blacklozenge \Diamond \Box p]\!]_M = U$, (notice that here we have evidence of the fact that the formula $\blacklozenge \Diamond \Box p \leftrightarrow \Box p$ is not valid in **HUBiStS5**, therefore connection can be distinguished from overlapping) and $M \models \mathsf{E}(\blacklozenge \Diamond \Box p \wedge \Box q)$. Indeed the granulations of $\{a\}$ and $\{e\}$ are connected by the edge $Y = \{b, c, d\}$ in the quotient structure. Notice also that when $H = I$, so when (U, H) is a set, the symmetry generation property of R is equivalent to symmetry of R, as $R \subseteq \overset{\leftrightarrow}{R} \,;\, H \,;\, \overset{\leftrightarrow}{R}$ iff $R \subseteq \breve{R}$ iff $R = \breve{R}$ iff $R = \circlearrowright R$ since $\breve{R} = \circlearrowright R$ when $H = I$. Indeed in this case it is correct to assume that coarse connection collapses to a form of overlapping, as no elements other than nodes, i.e. discrete points, are present in a set, so the only possible form of connection between two sets is when they overlap.

Now we look at the notion of coarse parthood in this system.

Proposition 131 (**S5** Coarse Parthood). $P_G(\varphi, \psi) \leftrightarrow \mathsf{A}(\Box\varphi \rightarrow \Box\psi)$ is a theorem in **HUBiSKtS5**. Therefore $P_{G\mathbf{S5}}(\varphi, \psi) := \mathsf{A}(\Box\varphi \rightarrow \Box\psi)$.

Proof. $\mathsf{A}(\Box G\varphi \rightarrow \Box G\psi) \leftrightarrow \mathsf{A}(\Box\Box\varphi \rightarrow \Box\Box\psi) \leftrightarrow \mathsf{A}(\Box\varphi \rightarrow \Box\psi)$ is a theorem in **HUBiSKtS5** by reflexivity and transitivity axioms and by Theorem 101. ■

Proposition 132 (**S5** Coarse Overlapping). $O_G(\varphi, \psi) \leftrightarrow \mathsf{E}(\Box\varphi \wedge \Box\psi)$ is a theorem in **HUBiSKtS5**. Therefore $O_{G\mathbf{S5}}(\varphi, \psi) := \mathsf{E}(\Box\varphi \wedge \Box\psi)$

Proof. $\mathsf{E}(\blacklozenge \Box(G\varphi \wedge G\psi)) \leftrightarrow \mathsf{E}(\blacklozenge \Box(\Box\varphi \wedge \Box\psi)) \leftrightarrow \mathsf{E}(\Box(\Box\varphi \wedge \Box\psi)) \leftrightarrow \mathsf{E}(\Box\Box\varphi \wedge \Box\Box\psi) \leftrightarrow \mathsf{E}(\Box\varphi \wedge \Box\psi)$ is a theorem in **HUBiSKtS5** by transitivity and reflexivity axioms and by Theorem 101. ■

Proposition 133 (**S5** Coarse Non-tangential part). $NTP_G(\varphi, \psi) \leftrightarrow \mathsf{A}(\Box\varphi \rightarrow \Box\psi) \wedge \mathsf{A}(\Diamond \Box\varphi \rightarrow \Box\psi)$ is a theorem in **HUBiSKtS5**. Therefore $NTP_{G\mathbf{S5}}(\varphi, \psi) := \mathsf{A}(\Box\varphi \rightarrow \Box\psi) \wedge \mathsf{A}(\Diamond \Box\varphi \rightarrow \Box\psi)$

Proof. The first part of the conjunction is the predicate of parthood, that has already been shown. $\mathsf{A}(\Diamond G\varphi \to \Box G\psi) \leftrightarrow \mathsf{A}(\Diamond \Box\varphi \to \Box\Box\psi) \leftrightarrow \mathsf{A}(\Diamond \Box\varphi \to \Box\psi)$ is a theorem in **HUBiSKtS5** by Theorem 101 item 3 and by reflexivity and transitivity axioms. ■

Also here a remark is important, as for coarse connection. $\mathsf{A}(\Diamond \Box\varphi \to \Box\psi) \leftrightarrow \mathsf{A}(\blacklozenge \Box\varphi \to \Box\varphi) \leftrightarrow \mathsf{A}(\Box\varphi \to \Box\psi)$ is a theorem in the extension of **HUBiSKtS4** with the symmetry axiom $\blacklozenge \varphi \leftrightarrow \Diamond \varphi$ and by item 1 of Theorem 101. In this extension the predicate of non-tangential parthood collapses with the predicate of parthood, showing that, if we choose to impose symmetry on R instead of a weaker property as symmetry generation, then we wouldn't be able to distinguish a generic spatial relation of parthood from the more specific notion of non-tangential parthood. When $H = I$, so when (U, H) is a set, thus when symmetrically generated property of R is equivalent to symmetry of R, it is correct to say that non-tangential parthood is just parthood: a set has no elements other than nodes, thus it has no edges and therefore it has no boundary-nodes. Hence any of its subsets is a non-tangential part.

Finally, let us look at the predicate of tangential-part in **HUBiSKtS5**.

Proposition 134. (**S5** Coarse Tangential part.) $TP_G(\varphi, \psi) \leftrightarrow \mathsf{A}(\Box\varphi \to \Box\psi) \wedge \mathsf{E}(\Diamond \Box\varphi \prec \Box\psi)$ is a theorem in **HUBiSKtS5**. Therefore $NTP_{G\mathbf{S5}}(\varphi, \psi) := \mathsf{A}(\Box\varphi \to \Box\psi) \wedge \mathsf{E}(\Diamond \Box\varphi \prec \Box\psi)$

Proof. The first conjunct is parthood. Then we have $\mathsf{E}(\Diamond\mathsf{G}\varphi \prec \Box\mathsf{G}\psi) \leftrightarrow \mathsf{E}(\Diamond\Box\varphi \prec \Box\Box\psi) \leftrightarrow \mathsf{E}(\Diamond\Box\varphi \prec \Box\psi)$ is a theorem in **HUBiSKtS5** by Theorem 101 item 3 and transitivity and reflexivity axioms. ■

When R is reflexive, transitive and additionally symmetric, the predicate of tangential part becomes $\mathsf{A}(\Box\varphi \to \Box\psi) \wedge \mathsf{E}(\Box\varphi \prec \Box\psi)$, and this leads to contradiction, as $\Box\varphi$ would be a part and a non-part of $\Box\psi$ at the same time. The spatial relation of tangential parthood wouldn't occur, indeed, as we have already seen that all spatial relations of parthood are non-tangential, if R is a preorder and additionally symmetric. When (U, H) is a set, it is correct to assume that the predicate of tangential part is contradictory as only non-tangential parts of a set exist.

5.7 Conclusions

The problem of representing graph and hypergraph partitions by a relation is addressed in [66]. There, the authors show that an equivalence relation in the classical sense, so a reflexive, transitive and symmetric relation, is too strong to generate a partition on hypergraphs. The constraint of symmetry is substituted by a weaker constraint, referred to as symmetry-generation of R.

In this chapter we have addressed the issue of hypergraph partitions, or more generally poset partitions, by looking at the quotient function. Although given a poset (U, H), a partition of U can be obtained by simply imposing a stable preorder R on U, we have seen that certain properties of the quotient function f, linking the resulting quotient structure (U', H'), to the initial poset (U, H), are missing when R is simply a preorder, as shown from the examples in Figure 5.3. We have called this type of property *back-conditions* on f w.r.t. H and H'. We have seen that all the back conditions on f correspond to some constraint on R, that can be seen as a weak form of symmetry. The first one is trivial as it is imposing the identity $R = H$. The middle two correspond to constraints on R that have been considered in the work of Ono [52], where the author considers several intuitionistic analogues of the modal logic **S5**. Imposing either of these constraints puts some significant restrictions on the type of partition and quotient structure arising from a poset (U, H) as with the first constraint at issue minimal elements of (U, H) can be assigned just to minimal elements of (U', H') by f (Propositions 112), and similarly for maximal elements of (U, H), when imposing the latter constraint (Proposition 115). The last back condition on f w.r.t. H and H', that we named weak-zag, corresponds to the property of symmetry generation on R proposed in [66]. If a back condition on f, and thus a dependency of the partial order H' from the initial partial order H, is desirable the weak-zag constraint on f, namely symmetry generation on R, is the best choice, as it imposes the link between H' and H and it does not cause any restriction on what type of elements of quotient structure (U', H'), every element of the initial poset (U, H) can be assigned to, by the quotient function f, as it happens for the constraints on R, and associated property on f, considered by [52]. We have also shown that two-tierness on R will imply that the quotient structure, which is already a poset by reflexivity and transitivity of R, is actually a hypergraph. Thus one might wish to impose this extra condition on R if, when starting from a poset (U, H) that is a hypergraph, the desired quotient structure (U', H') needs to be a hypergraph.

We have seen that there are formulas in **UBiSKt** that correspond to the property of an H-frame to be symmetry generated and two-tiers. Thus a logic where hypergraph or, more generally, poset partitions can be represented is obtainable. Finally we have introduced the definitions of hypergraph granulation and coarse spatial relations, developed in a more general context in Chapter 4, in the new **S5** extension of **UBiSKt**.

Chapter 6

Further Work and Conclusions

6.1 Further Work

6.1.1 Simplicial Complex

The main objects of investigation of the present work are graphs and hypergraphs, represented as a posets. We considered one-tier structures made only of 0-dimensional elements, i.e. nodes, and two-tier structures where also edges, i.e. 1-dimensional elements incident with the nodes, occur. However, there's no reason why we should stop there. There are interesting kinds of structure with more than two tiers, that can be represented by a poset.

A first example are *simplicial complexes* (see [17] for a concise introduction to simplicial complexes). To understand the idea of a simplicial complex, or simply a complex, we need to introduce its building block, i.e. a simplex. A simplex is any finite non-empty set. It has a dimension, that is given by its cardinality minus one. A simplex of dimension n, is called an n-simplex. A complex is any set U of simplices such that, for any $u \in U$, any non-empty subset of u is also in U. So for example the set $\{\{a\},\{b\},\{a,b\}\}$ is a complex, $\{\{a\},\{a,b\}\}$ is not a complex, and $\{\{a\},\{b\},\{b,c\}\}$ is not a complex. Also a simplicial complex has a dimension, that is the highest dimension of its simplices. A complex of dimension n is called an n-complex. With complexes, we can represent not just points and lines, but also higher dimensional spatial objects, as cells. In Figure 6.1, an example of a cell represented as a 2-d complex. Given a complex

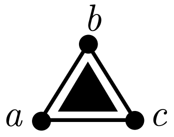

Figure 6.1: A 2-d cell represented as a 2-d simplicial complex. It is composed by three 0-d simplices ($\{a\}$, $\{b\}$, and $\{c\}$), three 1-d simplices ($\{a,b\}$, $\{b,c\}$, and $\{a,b\}$), and one 2-d simplex ($\{a,b,c\}$).

$\mathcal{C}$, a subset of $\mathcal{C}$ that is also a complex is called a subcomplex.

It is clear that a complex $\mathcal{C}$ can be seen as an H-frame (U, H), where U is the set of all simplices forming $\mathcal{C}$, and $H \subseteq U \times U$ is a partial order defined as follows: given $u, v \in U$, $u \; H \; v$ iff $u \supseteq v$. Given $X \subseteq \mathcal{C}$, X is a subcomplex of $\mathcal{C}$ if it is a complex itself. That means all its elements are such that any of their subsets is in X: for all $x \in X$, for all y such that xHy, $y \in X$. Hence subcomplexes of $\mathcal{C}$ are exactly its H-sets. It is easy to see the analogy between the idea of H as reflexive closure of the incidence relation between edges and nodes, introduced in Section 2.1, and this idea of H as the $\supseteq$ relation on the set of simplices. To clear this up, let us represent a graph (without multiple-edges and loops), both as a set with its incidence relation, and as a complex, in the way just described. Say we want to represent a graph with two nodes and one edge connecting them. Then, in the first case we have $U = \{ a, b, c \}$ and $H = I \cup \{ (c, a), (c, b) \}$. The element c is the edge incident with nodes a and b. If we represent the same structure as a complex, we need two 0-d simplices, and one 1-d simplex, so we have: $U = \{ \{ a \}, \{ b \}, \{ a, b \} \}$, and $H = I \cup \{ (\{ a, b \}, \{ a \}), (\{ a, b \}, \{ b \}) \}$. We can see that the two structures represent the same thing, with the assignments $a \mapsto \{ a \}$, $b \mapsto \{ b \}$, and $c \mapsto \{ a, b \}$. Notice that the representation as a poset (U, H) where H is the incidence relation is more general than the representation as a complex, as it allows also to have hyperedges, thus hypergraphs, multiple edges between two nodes and self loops. In the context of complexes, 1-dimension complexes are undirected graphs without multiple edges and loops. They are made of 0-dimensional elements, i.e. nodes, and 1-dimensional elements, i.e. edges. However, there is a notion that generalises, i.e. the notion of *simplicial set*. Using this, also graphs with multiple edges and hypergraphs can be represented, and these still give rise to a poset (U, H). See [51] for an introduction to simplicial sets.

So any complex $\mathcal{C}$ can be represented by an H-frame, and subcomplexes of $\mathcal{C}$ are H-sets. Hence the topological spatial relations presented in Chapter 3, as well as the coarse spatial relations presented in Chapter 4 and the various notions of granulation, could be applied to simplicial complexes of any dimensions, and not just to graphs and hypergraphs. Indeed mathematical morphology has been extended beyond sets and graphs, to simplicial complex spaces in [17]. We have seen how mathematical morphology relates to modal logic, and how the modal operators on graphs and hypergraphs are dilations and erosion by a stable relations. Dilation and erosion by a stable relation R map H-sets of the hypergraph (U, H) to H-sets. So in the case of simplicial complexes they will map subcomplexes to subcomplexes. It seems that the logic **UBiSKt** has the right semantics to represent and reason with simplicial complex spaces, and morphological operators on them correspond to **UBiSKt** modalities. Moreover, we could extend the theory of graph and hypergraph partitions and associated relations to simplicial complexes. Indeed we have seen that most of the theory works for any poset (U, H), and it is not restricted to posets with only two types of elements. Being able to take a coarser view on simplicial complexes seems to be important, as with these structure we can represent not only 0-dimensional and 1-dimensional elements as nodes and edges, but also objects of higher dimensions as surfaces and volumes. An example of a 2-d simplicial complex partition with the associated quotient structure, providing a coarser description of the initial complex, is given in Figure 6.2.

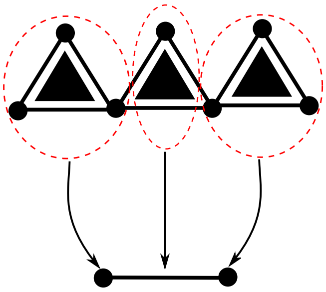

Figure 6.2: Example of a simplicial complex partition and the associated quotient structure providing a coarser description of initial complex.

6.1.2 Dependency-graphs

Another interesting type of structure that give rise to a poset is a dependency graph[1]. A dependency graph is usually defined as a finite set U plus a transitive relation $D \subseteq U \times U$. D is understood as a dependency relation, so given $u, v \in U$, $u \; D \; v$ is read as "an evaluation of u needs an evaluation of v first", or, more simply "u depends on v". The dependency relation can also be seen as reflexive (any item trivially depends on itself). From a dependency graph it is possible to derive an evaluation order, or the absence of an evaluation order, that respects the dependencies. An evaluation order within a dependency graph becomes impossible if there are circular dependencies in (U, D). We can think of circular dependencies as a cycles of the dependency relation D, other than a self-loops, so when for $x, y \in U$ such that $x \neq y$ we have that $x \; D \; y$ and $y \; D \; x$: the two elements are dependent from each other. These circularities make any evaluation orders impossible, as none of the items in the circular dependency can be evaluated first. It is clear that requiring the absence of circular dependencies makes the dependency relation also anti-symmetric. Thus, if we consider dependency graphs without circularity, then dependency relation is a partial order, being transitive, reflexive and anti-symmetric.

There are many objects that can be abstractly seen as dependency graphs. Here are some examples. Collections of events ordered in time and their influence on each others. Genealogical trees, with family members as vertices and the ancestor-successor relation between them. Sets of functions and their dependencies on each other: vertices are the functions, and dependency edges $f_1 \; D \; f_2$ are present every time a function f_1 needs f_2 to be defined. Spreadsheets, where the vertices are cells, and $c_1 \; D \; c_2$ iff c_1 uses the values from c_2. Hierarchies of concepts of a theory: the vertices are the definitions of the concepts, and the dependency relation holds between c_1 and c_2 if an understanding of concept c_2 is needed to grasp concepts c_1.

As dependency graphs are represented by a set plus a relation of partial order, they can be represented by a **UBiSKt** frame[2]. The H-sets of a dependency

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[1] Notice that here we use the term "graph", but it doesn't have to be confused with a undirected multigraph investigated in the body of this work. In this case by graph we mean a set with a relation, specifically a dependency relation, i.e. a *directed* graph.

[2] Notice that, even if there are circular dependencies in a dependency graph, and thus the

graph, i.e. the interpretation of formulae, will be the subsets of the dependency graph that are dependency graphs themselves. Thus naturally arises the question about the significance of the spatial relations applied to dependency graphs. Let us give some examples. Suppose we have a dependency graph $U = \{\{\alpha\}, \{\beta\}, \{\gamma\}, \{\delta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \delta\}\}$, and $x\ H\ y$ iff $x \supseteq y$. This could be interpreted as a concepts graph, visualising the concepts occurring in a certain theory, and the way their definitions depend on each other. The singletons are basic concepts, like axioms, and the 1-level concepts are built upon the basic concepts. H-sets are fully defined concepts. Then let us look at some of the **UBiSKt** operations and spatial relation analysed in Chapter 3. Let us calculate the closure of the basic concept $\{\alpha\}$, i.e. $\lrcorner\neg\{\alpha\}$ (with a slight abuse of notation, as the right way to write this would be $\{\alpha\} \oplus \curvearrowright H$, by Theorem 52). That is $\{\{\alpha\}, \{\beta\}, \{\gamma\}, \{\delta\}, \{\alpha, \beta, \gamma\}, \{\alpha, \delta\}\}$. Thus we can see the closure of an H-set as the complete set of concepts it contributes to define. Thus, in this situation, we have the formula $C(\{\alpha\}, \{\delta\})$ holds, as $\lrcorner\neg\{\alpha\} \cap \{\delta\} \neq \varnothing$. This assertion can be seen as the fact, even if $\{\alpha\}$ and $\{\delta\}$ are two different concepts, they both contribute to the definition of (at least) one common concept, in this case $\{\alpha, \delta\}$. Let us look at the spatial relation of Tangential-part.What are the tangential parts of the full-concept $X = \{\{\alpha\}, \{\beta\}, \{\gamma\}, \{\alpha, \beta, \gamma\}\}$? They are the subsets Y of U such that $Y \subseteq X$ and $\lrcorner\neg Y \nsubseteq X$. So, for example, $\{\alpha\}$ is a tangential part of X; the meaning of this assertion is that $\{\alpha\}$ contributes to form X, being its part, but it contributes to form concepts that are "outside" X, i.e. different from it. To evaluate X we need to evaluate α first, but α builds bits of information that are not X, as well. One last example: we have that the spatial relation of disconnection DC holds between $\{\beta\}$ and $\{\delta\}$. Indeed the formula $\neg\,\mathsf{A}(\lrcorner\neg\{\beta\} \wedge \{\delta\})$ holds in the model. This means that none of the concepts that $\{\beta\}$ contributes to define, need $\{\delta\}$ to be defined.

These are just some examples of the meaning that spatial relations on dependency graphs might have. Exploring more of these cases could be a direction for future work, as well as applying the ideas developed in Chapter 4 and thus applying the coarse spatial relations to subgraphs of dependency graphs that have been approximated by means of some stable relation R defined on the dependency graph. As a special case of this, the theory of poset partition developed in Chapter 5 could be applied to dependency graphs. We give an example that shows that, when we impose a partition on a dependency graph, the symmetry generation constraint on the associated preorder is important.

Example 135. Partitioning a set of functions dependent on each others, Figure 6.3. In Figure 6.3 there are some examples of dependency graphs with functions depending on each others in different ways. H is the dependency relation. For example in the case on the left we have $f_x\ H\ f_y$ meaning that the function f_x depends on f_y to be defined. Let us suppose that we need to impose a partition on the dependency graphs, because we need to split the work of implementing each function across a group of people, as in a group project. Say there are three members in the group, A, B and C. Every function will be assigned to one and only one member of the group. The reflexive and transitive closure of the relation $R = \{(f_x, f_y), (f_y, f_z), (f_z, f_y), (f_z, f_k)\}$ gives rise

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dependency relation is not anti-symmetric, the graph is still an H-frame as these need to be preordered sets, and not necessarily posets, although we focused on posets because graphs and hypergraphs are (two-tier) posets.

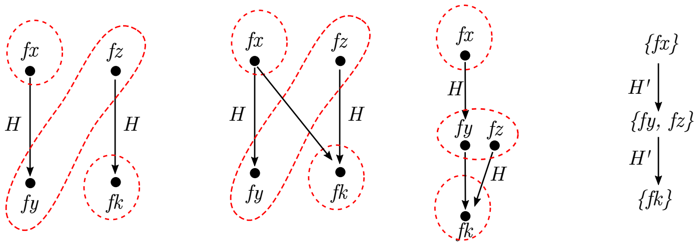


Figure 6.3: Example of three dependency graphs representing functions and their dependencies. The underlying set U is the same but the partial order H, i.e. the dependency relation, varies. The reflexive and transitive closure of $R = \{\, (f_x, f_y), (f_y, f_z), (f_z, f_y), (f_z, f_k) \,\}$ produces the partition shown and the quotient structure shown. The quotient structure can be seen as expressing how the work has been split among the members of the group, and H' how each of them depends on others.

to the partitions shown in the figure (this R is the smallest relation that gives rise to this partition). Notice we are imposing the same preorder on the three dependency graphs, indeed they give rise to the same quotient structure. The quotient structure represents how the work has been split across the members of the group (U'), and how the work of each member is dependent on the work of everybody else (H'). Now notice what happens in the first case: we have that the work of A is dependent on the work of C, however there's no function assigned to A that depends on a function assigned to C. There is a "false" H' dependency in (U', H') in the sense that it is not justified by any H dependency in (U, H). Indeed we have that R_1 is not symmetrically generated w.r.t. the partial order H_1, as $f_x\ R\ f_k$, but there are no f_i and f_j such that $f_x \overleftrightarrow{R} f_i\ H\ f_j \overleftrightarrow{R} f_k$. This tells us that the chosen partition might not the best one to split the work among the members, as in this way the work of A will be "unnecessarily" dependent on the work of C. In the remaining cases, all the H' dependencies between members' work are justified by the existence of some functions in each partition that are actually H dependent on each other. Thus we can conclude that the symmetry generation constraint is relevant, also in a theory of dependency graphs partitions.

6.1.3 Automated Reasoning

As already mentioned, the tableau calculus for **UBiSKt** has been implemented using the theorem prover generator *Mettel* [85]. The implementation can be found at [68]. This is very useful as it makes it possible to automate spatial reasoning using **UBiSKt**.

A special type of rule can be added to the *Mettel* implementation, known as *blocking rules*. They can be found already in the implementation of the tableau calculus for **BiSKt** in *Mettel*, presented in [74]. These are two examples of blocking rules:

$$\frac{s\ H\ t}{s \approx t \mid s \not\approx t}\ H \text{ blocking} \qquad \frac{s\ R\ t}{s \approx t \mid s \not\approx t}\ R \text{ blocking}$$

The idea is that every time a new element of U is added during the tableau construction, for example by the $[T\blacklozenge]$ rule that adds an R-predecessor for some $v \in U$, or by $[F\neg]$ that adds an H-successor from some $v \in U$, the blocking rules split the branch into two different branches. In the first branch one checks whether the new element that needs to be added can instead be unified with an existing element. If this is not possible, and thus this branch gets closed, the second branch is explored, where an element that is actually new is added. When a counter-model for a formula exists, blocking rules are useful as they help to build "minimal counter-models", in the sense that they try first to avoid assuming the existence of new elements, action that will increase the cardinality of U. For example, if we input the formula $s : F\neg p$, the tableau rule $[F\neg]$ is applied and a new t such that $s\ H\ t$ and $t : Tp$ is added. The H blocking rule will then unify s with t. This doesn't cause any contradiction as s can carry the information that should be carried by t, i.e. that $s : Tp$ and $s\ H\ s$. Hence the model will be made of a set $U = \{\, s \,\}$, instead of having $U = \{\, s, t \,\}$. Thus, blocking rules are a useful mechanism.

However the blocking rules seem to be problematic with the *Mettel* implementation of **UBiSKt**. Indeed we have experienced cases where, for a formula φ that we know is a theorem, when the blocking rules are not in the implementation, then input $s : F(\varphi)$ correctly produces a closed tableau. But when the blocking rules are added to the calculus the input formula $s : F(\varphi)$ produces a model, thus a counter-model for φ. This counter-model has a contradiction somewhere that is not detected, as φ is a theorem. This is the case for example with the formula $\mathsf{E}(\lrcorner\neg\varphi \wedge \psi) \to \mathsf{E}(\varphi \wedge \lrcorner\neg\psi)$. We have already seen that this formula is a theorem in H**UBiSKt** (53 item 2), and a tableau proof, thus a closed tableau for $s : F(\mathsf{E}(\lrcorner\neg\varphi \wedge \psi) \to \mathsf{E}(\varphi \wedge \lrcorner\neg\psi))$, can be produced with Tab**UBiSKt** by hand as well. If the blocking rules are not added within the *Mettel* implementation of Tab**UBiSKt**, this works correctly and produces a closed tableau. However, when the blocking rules for H and R are added, a counter-model appears to be produced (this will be a false counter-model as contradiction must be found somewhere). Thus this is evidence that the blocking rules, whilst useful, are problematic with the *Mettel* implementation of Tab**UBiSKt**.

Motivated by this, as a joint work between the author of this thesis and Brandon Bennett, Tab**UBiSKt** has been implemented using the language Prolog. Initial work has been presented in [70]. The implementation can be found at [69]. Initial tests have shown that the implementation can prove, or disprove, **UBiSKt** formulae in the expected way. The goal of a working implementation of the blocking rules, in order to ensure minimal models construction, is still to achieve and it is left as future work.

The proof procedure is implemented by a recursive algorithm. Rules will be applied until no "active" formula with a logical connective or operator remains. For some of the rules, we adopt a non-destructive-tableau approach: once a formula (or group of formulae) that matches the premise of a rule has been analysed by that rule and the corresponding conclusion has been added to the branch, the formula-premise is kept in the branch. However the formula will not be analysed again by the same rule, as the conclusion is already present in the branch. A non-destructive approach is preferable for the rules handling truth of a box, and falsity of a diamond. For example, consider the rule handling truth of the universal box A. It needs to be applied every time a new label for a new

$$\frac{w : T(\varphi), \quad w : F\varphi}{\bot}\ (\bot)$$

```
refute( Formulae, [tf_close] ) :-
   select( W:(Phi=t), Formulae, Rest ),
   member( W:(Phi=f),  Rest ), !.
```

$$\text{v new on the branch}\ \frac{w : F(\neg\varphi)}{w\ H\ v, v : T\varphi}\ (F\neg)$$

```
refute(Formulae, [f_nneg| Rules] ) :-
   select(W:(nneg(Phi)=f),
           Formulae, Rest ), !,
    V = @nneg(Phi, W),
   refute( [h(W,V), h(V,V),
                V:(Phi=t) | Rest],
                 Rules )).
```

$$\frac{w : T(\mathsf{A}\,\varphi), \quad v : S\psi}{v : T\varphi}\ (T\,\mathsf{A})$$

```
refute(Formulae, [t_ubox| Rules] ) :-
   select( _W:(ubox(Phi)=t),
           Formulae, Rest ),
   member( V:(_), Formulae ),
   \+(member( V:(Phi=t), Rest) ),!,
   refute( [V:(Phi=t) | Formulae], Rules).
```

Table 6.1: The top row shows the branch closing rule used to derive contradiction. The middle row shows the rule for the falsity of intuitionistic negation $\neg$. The bottom row shows the rule handling the truth of the universal box A.

world is added to the tree. The fact the rule will be blocked if the conclusion of the rule is already present in the branch, will stop the program from applying the rule over again, as this might cause the program to loop. Table 6.1 shows some examples of tableau rules from Tab**UBiSKt** and the way they have been implemented in Prolog.

Our initial work has indicated many possibilities for enhancing tableau-based reasoning in this kind of modal calculus, by constraining the ordering of rule applications and by special handling of formulae relating to the relational structure of possible worlds. A further challenge is to automate a theorem-prover for the **S**5 extension of **UBiSKt**, where graphs and hypergraphs partitions can be represented. Blocking rules would also be useful in the implementation of **S5**. Indeed the rule for expressing symmetry generation of R will have a premise $(x\ R\ y)$, more complex than the conclusion $(x\ \overset{\leftrightarrow}{R}\ ; H\ ; \overset{\leftrightarrow}{R}\ y)$, as in the conclusion we have to infer the existence of one or more element of U, related to x and y in a certain way. Avoiding the addition of unnecessary new elements during the model construction, using a blocking mechanism, would enhance the calculus.

Moreover this seems an interesting direction of future work as implementation of calculi for intuitionistic modal logics seems a relatively unexplored field.

6.2 Conclusions

This thesis has explored the use of a bi-intuitionistic modal logic with universal modalities, **UBiSKt**, within the field of qualitative spatial representation. A special case of an H-frame is a graph, or more generally, a hypergraph. Topological spatial relations that apply to these kinds of discrete structures have been expressed within the logic. As the logic comes with an axiomatic calculus and an equivalent tableau calculus, it is possible to prove properties about the spatial relations, and thus reasoning about spatial relations on graphs and hypergraphs. The tableau calculus has been implemented using *Mettel* (implementation available at [68]), thus the spatial reasoning can also be automated. An enhanced implementation of the calculus within Prolog, is a promising direction of future work.

The topic of graph and hypergraph granularity has been explored. Granularity refers to the presence of granules in the information. The process of changing level of granularity, or level of detail according to which information is visualised, is a type of approximation, that we called *granulation*. Thanks to the connection between modal logic and mathematical morphology, morphological operations have been used to put forward some notions for subgraphs granulation, in order to visualise subgraphs at a different granularity. The level of granularity on subgraphs is provided by a stable relation R on the set U of nodes and edges. Moreover, topological spatial relations under granularity, i.e. occurring between regions that have undergone a granulation process, have been defined and the definitions have been justified by a parallelism with the spatial relations at a "detailed" level. The detailed level is the one provided by the smallest stable relation containing the identity relation I, namely the (reflexive closure of) the incidence relation H. In this case the spatial relations are computed in terms of H-dilates, i.e. every single node, and every edge with all its end-points, as these are the smallest subgraphs a graph is composed of. If we take a granular view provided by some stable relation R, then the "atomic" parts of the graph are R-dilates, that can be seen as an abstraction of the morphological structuring element. The spatial relations can be expressed in terms of R-dilates, instead of in terms of H-dilates. A set of *coarse* spatial relations between coarse subgraphs has been defined in this way. Coarse spatial relations have been analysed; in particular it has been proved that coarse connection is always symmetric, as expected. For what concerns reflexivity of coarse connection, we have seen that this doesn't hold for any R, but there is a necessary and sufficient condition imposing which coarse connection will be reflexive. In classical modal logic this condition is the surjectivity of R, whilst in an intuitionistic modal logic like **UBiSKt**, it is a more subtle condition, called weak surjectivity as the latter is implied, but does not imply, the former. This is evidence of the fact that spatial relations on sets and spatial relations on graphs and hypergraphs are different.

Finally, a special case of hypergraph granularity has been considered, i.e. when a partition over the set of nodes and edges is obtained. The quotient structure is the coarser description of the initial hypergraph. As explained in [66], a stable preorder R on U that is additionally symmetric is too restrictive in the kinds of partitions it generates. In the present work, different constraints, substituting symmetry, have been considered. They arise from imposing some back condition on the quotient function f, that is the link between the initial poset (U, H) and its quotient structure (U', H'). Some of the constraints analysed are equivalent to properties on R studied in [52], where intuitionistic analogues of the modal logic **S5** are studied. These properties on R, or equivalently on f, rule out some poset partitions that one might wish to consider.

Indeed with the first constraint considered in [52], minimal elements of (U, H) can be assigned by f just to minimal elements of (U', H'), and with the second constraint maximal elements of (U, H) can be assigned just to maximal elements of (U', H'). The last back condition on f, the weak-zag, corresponds to the symmetry generation constraint on R introduced in [66]. This imposes a dependency of H' from H the weak-zag on f, but doesn't put the same restrictions as the constraints adopted in [52], thus minimal and maximal elements of (U, H) can be assigned to both minimal and maximal elements of (U', H'). Thus we settle for a stable preorder on U that is additionally symmetrically generated, as giving rise to partitions on a poset (U, H), and to its quotient structure (U', H'), with the right properties. This motivates the choices made in [66]. Moreover we have seen that there is a formula in **UBiSKt** corresponding to symmetry-generation. This formula is valid in all and only the H-frames that are symmetrically generated, namely the formula singles out the class of frames where R is symmetrically generated (and additionally a preorder). This gives a new intuitionistic analogue of **S5**, where poset partitions can be represented, that has never been considered before. With this work the author hopes to have given an evaluation of different intuitionistic modal systems, that can all be considered analogues of classical modal logic **S5**, under the light of a theory of poset partitions.

As the objects of interest were 2-tier posets, namely graphs and hypergraphs, we have considered as additional constraint on R, also presented in [66], the 2-tierness of R. Imposing this constraint ensures that the quotient structure of a hypergraph is a hypergraph too. Then we have see that there is a formula in **UBiSKt** whose validity singles out the 2-tier H-frame, i.e. R in F is 2-tier preorder. Finally, we have seen how the coarse spatial relations can be expressed in the **S**5 extension of **UBiSKt**.

With this work the author hopes to have contributed to the foundation of a theory of *rough things*, where the objects of study are not necessarily simple sets. Sets carrying additional structure are used everywhere. Taxonomies, knowledge graphs and ontologies, are examples of data linked in a certain way. In this work we have focused on the case where the link can be modelled as a partial order. We have seen how graphs and hypergraphs can be represented in this way. However, a theory of granularity and partition of relational structures different from a poset, providing a coarser description of the initial structure, could be an interesting direction of research. Modal logic, given its long history, could be used as a guide to develop it. These objects can be seen as modal frames, and an additional relation, providing the coarser view of the underlying set, would be considered.

Appendix A

A.1 Proof of Propositions 58 - 71 of Section 3.4

Proposition 58 :$\beta^\bullet(\varphi) \leftrightarrow \beta^\bullet(\varphi) \wedge \lrcorner\, \beta^\bullet(\varphi) := (\varphi \wedge \lrcorner\, \varphi) \leftrightarrow (\varphi \wedge \lrcorner\, \varphi) \wedge \lrcorner(\varphi \wedge \lrcorner\, \varphi)$.

1. $\vdash_{\mathbf{HUBiSKt}} \lrcorner(\varphi \wedge \lrcorner\, \varphi)$ by item 13 of Proposition 29.

2. $\vdash \lrcorner(\varphi \wedge \lrcorner\, \varphi) \to ((\varphi \wedge \lrcorner\, \varphi) \to \lrcorner(\varphi \wedge \lrcorner\, \varphi))$ by axiom `A0`.

3. $\vdash (\varphi \wedge \lrcorner\, \varphi) \to \lrcorner(\varphi \wedge \lrcorner\, \varphi)$ by `MP`.

4. $\vdash (\varphi \wedge \lrcorner\, \varphi) \to (\varphi \wedge \lrcorner\, \varphi) \wedge \lrcorner(\varphi \wedge \lrcorner\, \varphi)$.

5. $\vdash (\varphi \wedge \lrcorner\, \varphi) \wedge \lrcorner(\varphi \wedge \lrcorner\, \varphi) \to (\varphi \wedge \lrcorner\, \varphi)$ follows as an instance of `A5`

6. $\vdash (\varphi \wedge \lrcorner\, \varphi) \leftrightarrow (\varphi \wedge \lrcorner\, \varphi) \wedge \lrcorner(\varphi \wedge \lrcorner\, \varphi)$ by lines 4 and 5.

Proposition 60 item *i*: $\vdash_{\mathbf{HUBiSKt}} BR(\varphi) \to P(\varphi, \neg\neg \lrcorner\, \varphi)$

i) $BR(\varphi) \to P(\varphi, \neg\neg \lrcorner\, \varphi) := \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\, \varphi))) \to \mathsf{A}(\varphi \to \neg\neg \lrcorner\, \varphi)$

1. $\vdash \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\, \varphi) \wedge \varphi) \to (\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\, \varphi) \wedge \varphi)$ by `A16`.

2. $\vdash (\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\, \varphi) \wedge \varphi) \to (\varphi \to \neg\neg(\varphi \wedge \lrcorner\, \varphi))$

3. $\vdash (\varphi \to \neg\neg(\varphi \wedge \lrcorner\, \varphi)) \to (\varphi \to \neg\neg\varphi \wedge \neg\neg \lrcorner\, \varphi)$, as $\neg\neg(\alpha \wedge \beta) \leftrightarrow \neg\neg\alpha \wedge \neg\neg\beta$ is a theorem in intuitionistic logic.

4. $\vdash (\varphi \to \neg\neg\varphi \wedge \neg\neg \lrcorner\, \varphi) \to (\varphi \to \neg\neg \lrcorner\, \varphi)$

5. $\vdash \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\, \varphi) \wedge \varphi) \to (\varphi \to \neg\neg \lrcorner\, \varphi)$ by concatenating 1-4.

6. $\vdash \mathsf{A}(\mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\, \varphi) \wedge \varphi)) \to \mathsf{A}(\varphi \to \neg\neg \lrcorner\, \varphi)$ by `Mon` A rule.

So we have shown $\mathsf{A}(BR(\varphi)) \to P(\varphi, \neg\neg \lrcorner\, \varphi)$. But:

7. $\vdash \mathsf{A}(\mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\, \varphi) \wedge \varphi))) \leftrightarrow (\mathsf{A}\,\mathsf{E}\,\varphi \wedge \mathsf{A}\,\mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\, \varphi) \wedge \varphi)))$ because $\vdash \mathsf{A}(\alpha \wedge \beta) \leftrightarrow \mathsf{A}(\alpha) \wedge \mathsf{A}(\beta)$ due to the adjunction "$\mathsf{E} \dashv \mathsf{A}$" by item 14 of Proposition 29.

8. $\vdash (\mathsf{A}\,\mathsf{E}\,\varphi \wedge \mathsf{A}\,\mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\, \varphi) \wedge \varphi))) \leftrightarrow \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\, \varphi) \wedge \varphi))$ by $\vdash \mathsf{A}\,\mathsf{A}\,\alpha \leftrightarrow \mathsf{A}\,\alpha$ (due to `A16` and `A17`) and $\vdash \mathsf{A}\,\mathsf{E}\,\alpha \leftrightarrow \mathsf{E}\,\alpha$ (due to item 17 of Proposition 29).

9. $\vdash \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\, \varphi) \wedge \varphi) \to \mathsf{A}(\varphi \to \neg\neg \lrcorner\, \varphi)$ by lines 6, 7 and 8.

Proposition 60 item ***ii*:** $\vdash_{\mathbf{HUBiSKt}} \mathsf{A}(\psi \to \neg\delta) \to \neg\,\mathsf{E}(\psi \wedge \delta)$.

1. $\vdash \mathsf{A}(\psi \to \neg\delta) \to \mathsf{A}\,\neg(\psi \wedge \delta)$ because $(\alpha \to \neg\beta) \leftrightarrow \neg(\alpha \wedge \beta)$ is a theorem of intuitionistic logic.

2. $\vdash \mathsf{A}\,\neg(\psi \wedge \delta) \to \neg\,\mathsf{E}(\psi \wedge \delta)$ by `A18`.

3. $\vdash \mathsf{A}(\psi \to \neg\delta) \to \neg\,\mathsf{E}(\psi \wedge \delta)$ by concatenating lines 1 and 2.

Proposition 60 item ***iii*:** it suffices to show $\vdash_{\mathbf{HUBiSKt}} \neg\,\mathsf{E}(\varphi \wedge \neg \lrcorner\, \varphi) \to \mathsf{A}\,\neg\neg \lrcorner\, \varphi$.

1. $\vdash \neg \lrcorner\, \varphi \to (\varphi \wedge \neg \lrcorner\, \varphi)$ by item 11 of Proposition 29.

2. $\vdash \neg(\varphi \wedge \neg \lrcorner\, \varphi) \to \neg\neg \lrcorner\, \varphi$ by intuitionistic logic from line 1.

3. $\vdash \mathsf{A}\,\neg(\varphi \wedge \neg \lrcorner\, \varphi) \to \mathsf{A}\,\neg\neg \lrcorner\, \varphi$ by `Mon A`.

4. $\vdash \neg\,\mathsf{E}(\varphi \wedge \neg \lrcorner\, \varphi) \to \mathsf{A}\,\neg\neg \lrcorner\, \varphi$. by `A18`.

Proposition 62: $\vdash_{\textbf{HUBiSKt}} \mathsf{A}\neg(\neg\lrcorner\varphi) \to EQ(\varphi, \partial(\varphi)) := \mathsf{A}\neg(\neg\lrcorner\varphi) \to \mathsf{A}(\varphi \leftrightarrow (\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi))$.

1. $\vdash \neg\neg\lrcorner\varphi \to (\varphi \to \neg\neg\lrcorner\varphi)$ by `A0`.

2. $\vdash \mathsf{A}(\neg\neg\lrcorner\varphi) \to \mathsf{A}(\varphi \to \neg\neg\lrcorner\varphi)$ by `Mon A`.

3. $\vdash \mathsf{A}(\neg\neg\lrcorner\varphi) \to \mathsf{A}(\varphi \to \neg\neg\varphi)$ because $\alpha \to \neg\neg\alpha$ is a theorem in intuitionistic logic.

4. $\vdash \mathsf{A}(\neg\neg\lrcorner\varphi) \to \mathsf{A}(\varphi \to \neg\neg\lrcorner\varphi) \wedge \mathsf{A}(\varphi \to \neg\neg\varphi)$ from lines 3 and 4.

5. $\vdash \mathsf{A}(\neg\neg\lrcorner\varphi) \to \mathsf{A}(\varphi \to \neg\neg\varphi \wedge \neg\neg\lrcorner\varphi)$ because A preserves conjunction by adjunction between E and A, Proposition 29 item 14.

6. $\vdash \mathsf{A}(\neg\neg\lrcorner\varphi) \to \mathsf{A}(\varphi \to \neg\neg(\varphi \wedge \lrcorner\varphi))$, because $\neg\neg(\alpha \wedge \beta) \leftrightarrow \neg\neg\alpha \wedge \neg\neg\beta$ is a theorem in intuitionistic logic.

7. $\vdash \mathsf{A}(\neg\neg\lrcorner\varphi) \to \mathsf{A}(\varphi \to \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)$

8.$\vdash \mathsf{A}(\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi \to \varphi)$

9. $\vdash \mathsf{A}(\neg\neg\lrcorner\varphi) \to \mathsf{A}(\varphi \leftrightarrow \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)$ from lines 7 and 8.

Proposition 65 $\vdash_{\textbf{HUBiSKt}} \mathsf{E}(\neg\lrcorner\varphi) \to \mathsf{E}\varphi$:

1. $\vdash (\neg\lrcorner\varphi) \to \varphi$ item 11 of Proposition 29.

2. $\vdash \mathsf{E}(\neg\lrcorner\varphi) \to \mathsf{E}\varphi$ by Mon E rule.

Proposition 66: $SR(\varphi) \to$ not-$P(\varphi, \beta(\varphi)) := \mathsf{E}(\neg\lrcorner\varphi) \to \mathsf{E}(\varphi \prec (\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi))$.

First we show that

$\vdash (\alpha \wedge \neg\beta) \to (\alpha \prec \beta)$:

1. $\vdash \alpha \to (\beta \vee (\alpha \prec \beta))$ by axiom `A10`.

2. $\vdash (\alpha \wedge \neg\beta) \to (\alpha \prec \beta)$ by intuitionistic logic.

By putting

$\alpha := \varphi, \quad \beta := \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)$

we obtain:

3. $\vdash (\varphi \wedge \neg(\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)) \to (\varphi \prec \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)$

4. $\vdash \neg\lrcorner\varphi \to (\varphi \wedge \neg(\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi))$.

5. $\vdash \neg\lrcorner\varphi \to \varphi$ by item 11 of Proposition 29.

6. $\vdash \neg\lrcorner\varphi \wedge \lrcorner\varphi \to \bot$

7. $\vdash \neg\lrcorner\varphi \wedge (\varphi \wedge \lrcorner\varphi) \wedge \varphi \to \bot$

8. $\vdash \neg\lrcorner\varphi \wedge \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi \to \bot$

9. $\vdash \neg\lrcorner\varphi \to \neg(\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)$

10. $\vdash \neg\lrcorner\varphi \to (\varphi \wedge \neg(\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi))$ by lines 5 and 9

11. $\vdash \neg\lrcorner\varphi \to (\varphi \prec \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)$ by concatenating lines 10 and 3.

12. $\vdash \mathsf{E}(\neg\lrcorner\varphi) \to \mathsf{E}((\varphi \prec \neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi))$ by `Mon-E` rule

Proposition 68: $\neg SR(\beta^{\bullet}\varphi \wedge \beta^{\bullet}(\psi)) := \neg\mathsf{E}\neg\lrcorner((\varphi \wedge \lrcorner\varphi) \wedge (\psi \wedge \lrcorner\psi))$

1. $\vdash \lrcorner(\psi \wedge \lrcorner\psi)$ from item 13 of Proposition 29.

2. $\vdash \neg\neg\lrcorner(\psi \wedge \lrcorner\psi)$ by intuitionistic logic.

3. $\vdash \mathsf{A}\neg\neg\lrcorner(\psi \wedge \lrcorner\psi)$ by necessitation rule.

4. $\vdash ((\psi \wedge \lrcorner\psi) \wedge (\varphi \wedge \lrcorner\varphi)) \to (\psi \wedge \lrcorner\psi)$ by `A6`

5. $\vdash \lrcorner(\psi \wedge \lrcorner\psi) \to \lrcorner((\varphi \wedge \lrcorner\varphi) \wedge (\psi \wedge \lrcorner\psi))$ by item 12 of Proposition 29

6. $\vdash \mathsf{A}\neg\neg\lrcorner(\psi \wedge \lrcorner\psi) \to \mathsf{A}\neg\neg\lrcorner((\varphi \wedge \lrcorner\varphi) \wedge (\psi \wedge \lrcorner\psi))$ by intuitionistic logic and `Mon A`

7. $\vdash \mathsf{A}\neg\neg\lrcorner((\varphi \wedge \lrcorner\varphi) \wedge (\psi \wedge \lrcorner\psi))$ by concatenation of lines 3 and 6.

Proposition 69: $EQ(\neg\beta^{\bullet}(\varphi), \neg\beta(\varphi) := \mathsf{A}((\neg(\varphi \wedge \lrcorner\varphi)) \leftrightarrow (\neg((\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi))))$

1. $\vdash (\neg(\varphi \wedge \lrcorner\varphi) \wedge (\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi)) \to \bot$ because $\neg\alpha \wedge \neg\neg\alpha \to \bot$ is a theorem in intuitionistic logic.

2. $\vdash \neg(\neg(\varphi \wedge \lrcorner\varphi) \wedge (\neg\neg(\varphi \wedge \lrcorner\varphi) \wedge \varphi))$ from line 1.

3. $\vdash \neg(\varphi \wedge \lrcorner\,\varphi) \to (\neg(\neg\neg(\varphi \wedge \lrcorner\,\varphi) \wedge \varphi))$ because $\neg(\alpha \wedge \beta) \leftrightarrow (\alpha \to \neg\beta)$ is a theorem in intuitionistic logic.

4. $\vdash (\varphi \wedge \lrcorner\,\varphi) \to (\neg\neg(\varphi \wedge \lrcorner\,\varphi) \wedge \varphi)$ because $\vdash \alpha \to \neg\neg\alpha$.

5. $\vdash (\neg(\neg\neg(\varphi \wedge \lrcorner\,\varphi) \wedge \varphi)) \to \neg(\varphi \wedge \lrcorner\,\varphi)$ by line 4.

6. $\vdash \neg(\varphi \wedge \lrcorner\,\varphi) \leftrightarrow (\neg(\neg\neg(\varphi \wedge \lrcorner\,\varphi) \wedge \varphi))$ by lines 3 and 5.

7. $\vdash \mathsf{A}(\neg(\varphi \wedge \lrcorner\,\varphi) \leftrightarrow (\neg(\neg\neg(\varphi \wedge \lrcorner\,\varphi) \wedge \varphi)))$ by necessitation rule.

Proposition 70: we divide the proof of this formula in the two $\to$ directions, (i) and (ii).

i) $\vdash_{\mathbf{HUBiSKt}} SR(\varphi) \wedge \mathsf{A}\,\neg\partial^N(\varphi) \to \mathsf{E}\,\varphi \wedge P(\varphi, \neg\lrcorner\,\varphi) := \mathsf{E}\,\neg\lrcorner\,\varphi \wedge \mathsf{A}\,\neg(\varphi \wedge \lrcorner\,\varphi) \to \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \to \neg\lrcorner\,\varphi)$.

1. $\vdash \mathsf{E}\,\neg\lrcorner\,\varphi \to \mathsf{E}\,\varphi$ by Proposition 10.

2. $\vdash \mathsf{A}\,\neg(\varphi \wedge \lrcorner\,\varphi) \to \mathsf{A}(\varphi \to \neg\lrcorner\,\varphi)$ because $\neg(\alpha \wedge \beta) \to (\alpha \to \neg\beta)$ is a theorem in intuitionistic logic.

3. $\vdash \mathsf{E}\,\neg\lrcorner\,\varphi \wedge \mathsf{A}\,\neg(\varphi \wedge \lrcorner\,\varphi) \to \mathsf{E}\,\varphi \wedge \mathsf{A}(\varphi \to \neg\lrcorner\,\varphi)$. from 1 and 2.

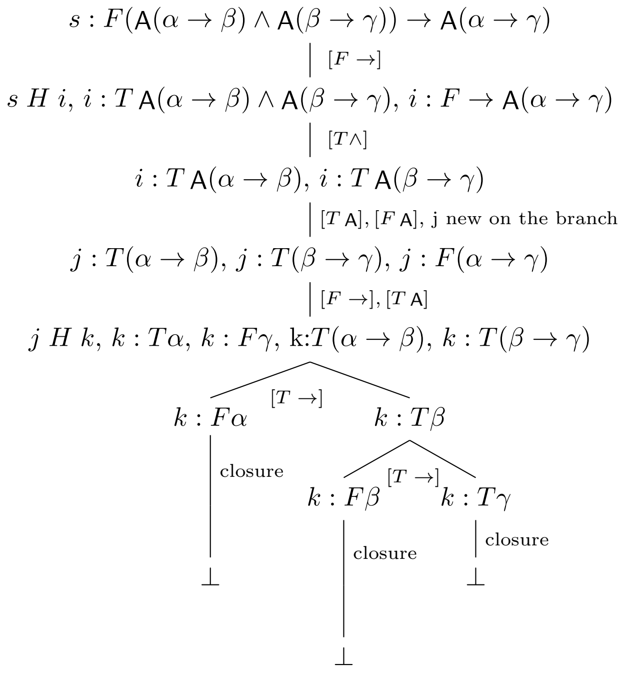
ii) $\vdash_{\mathbf{HUBiSKt}} P(\varphi, \neg\lrcorner\,\varphi) \wedge \mathsf{E}\,\varphi \to SR(\varphi) \wedge \mathsf{A}\neg\partial^N(\varphi) := \mathsf{A}(\varphi \to \neg\lrcorner\,\varphi) \wedge \mathsf{E}\,\varphi \to \mathsf{E}(\neg\lrcorner\,\varphi) \wedge \mathsf{A}\,\neg(\varphi \wedge \lrcorner\,\varphi)$.

1. $\vdash \mathsf{A}(\varphi \to \neg\lrcorner\,\varphi) \wedge \mathsf{E}\,\varphi \to \mathsf{E}((\varphi \to \neg\lrcorner\,\varphi) \wedge \varphi) \to \mathsf{E}(\neg\lrcorner\,\varphi)$ by `A19` and `MP`.

2. $\vdash \mathsf{A}(\varphi \to \neg\lrcorner\,\varphi) \wedge \mathsf{E}\,\varphi \to \mathsf{E}(\neg\lrcorner\,\varphi)$ from 1.

3. $\vdash (\mathsf{A}(\varphi \to \neg\lrcorner\,\varphi) \wedge \mathsf{E}\,\varphi) \to (\mathsf{A}\,\neg(\varphi \wedge \lrcorner\,\varphi) \wedge (\mathsf{E}\,\neg\lrcorner\,\varphi))$ because $(\alpha \to \neg\beta) \leftrightarrow \neg(\alpha \wedge \beta)$ is a theorem in intuitionistic logic.

A.2 Proof of Proposition 91



A.3 Tableau Calculus

In Table A.1 the full set of rules composing **TabUBiSKt** is introduced. This is obtained by adding the rules for universal modalities to the tableau calculus for

BiSKt, presented in [74].

Table A.1: Tableau calculus Tab**UBiSKt**

$$\frac{s : T\varphi \quad s : F\varphi}{\bot}\ \text{closure}$$

$$\frac{s : T\varphi \wedge \psi}{s : T\varphi \quad s : T\psi}\ (T\wedge)$$

$$\frac{s : F\varphi \wedge \psi}{s : F\varphi \quad s : F\psi}\ (F\vee)$$

$$\frac{s : T\neg\varphi, s\ H\ t}{t : F\varphi}\ (T\neg)$$

$$\frac{s : F \lrcorner\, \varphi \quad t\ H\ s}{t : T\varphi}\ (F \lrcorner)$$

$$\frac{s : T\varphi \rightarrow \psi \quad s\ H\ t}{t : F\varphi \mid t : T\psi}\ (T \rightarrow)$$

$$\frac{s : F\varphi \prec \psi \quad t\ H\ s}{t : F\varphi \mid t : T\psi}\ (F\prec)$$

$$\frac{s : T\Box\varphi \quad s\ R\ t}{t : T\varphi}\ (T\Box)$$

$$\frac{s : F \blacklozenge \varphi \quad t\ R\ s}{t : F\varphi}\ (F \blacklozenge)$$

$$\frac{s : F\Diamond\varphi \quad t\ H\ m \quad n\ R\ m \quad n\ H\ s}{t : F\varphi}\ (F\Diamond)$$

$$\frac{s : T \blacksquare \varphi \quad s\ H\ m \quad n\ R\ m \quad n\ H\ t}{t : T\varphi}\ (T \blacksquare)$$

$$\frac{s : T\,\mathsf{A}\,\varphi, \quad t : S\psi}{t : T\varphi}\ (T\,\mathsf{A})$$

$$\frac{s : F\,\mathsf{E}\,\varphi \quad t : S\psi}{t : F\varphi}\ F\,\mathsf{E}$$

$$\frac{s : S\varphi}{s\ H\ s}\ \texttt{refl-H}$$

$$\frac{s : T\varphi \quad s\ H\ t}{t : T\varphi}\ \texttt{mon-H}$$

$$\frac{s : T(\bot)}{\bot}\ (T\bot)$$

$$\frac{s : F(\varphi \wedge \psi)}{s : F\varphi \mid s : F\psi}\ (F\wedge)$$

$$\frac{s : T(\varphi \vee \psi)}{s : T\varphi \mid s : T\psi}\ (T\vee)$$

$$\dagger\ \frac{s : F\neg\varphi}{s\ H\ m \quad m : T\varphi}\ (F\neg)$$

$$s : T \lrcorner\, \varphi$$

$$\dagger\ \frac{s : F\varphi \rightarrow \psi}{s\ H\ m \quad m : T\varphi \quad m : F\psi}\ (F \rightarrow)$$

$$\dagger\ \frac{s : T\varphi \prec \psi}{m\ H\ s \quad m : T\varphi \quad m : F\psi}\ (T\prec)$$

$$\dagger\ \frac{s : F\Box\varphi}{s\ R\ m \quad m : F\varphi}\ (F\Box)$$

$$\dagger\ \frac{s : T \blacklozenge \varphi}{m\ R\ s \quad m : T\varphi}\ (T \blacklozenge)$$

$$\ddagger\ \frac{s : T\Diamond\varphi}{m\ H\ n \quad t\ R\ n \quad t\ H\ s \quad m : T\varphi}\ (T\Diamond)$$

$$\ddagger\ \frac{s : F \blacksquare \varphi}{s\ H\ n \quad t\ R\ n \quad t\ H\ m \quad m : F\varphi}\ (F \blacksquare)$$

$$\dagger\ \frac{s : F\,\mathsf{A}\,\varphi}{m : F\varphi}\ F\,\mathsf{A}$$

$$\dagger\ \frac{s : T\,\mathsf{E}\,\varphi}{m : T\varphi}\ (T\,\mathsf{E})$$

$$\frac{s\ H\ t \quad t\ H\ j}{s\ H\ j}\ \texttt{trans-H}$$

$$\frac{s\ H\ t \quad t\ R\ j \quad j\ H\ k}{s\ R\ k}\ \texttt{stab-R}$$

† m is fresh on the branch.

‡ m, n and t are fresh on the branch.

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