Accessibility of Nilpotent Orbits in Classical Algebras

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Abstract

The work in this thesis provides a refinement in the classification of nilpotent orbits in classical algebras. Given an affine algebraic group and attached Lie algebra over an algebraically closed field κ with good characteristic, we explore the relationship between the nilpotent orbits given by taking limits along cocharacters of the group. This can be used to determine the so-called accessibility order of the nilpotent orbits in the Lie algebra. In this thesis, the accessibility of nilpotent orbits in the general linear, symplectic and orthogonal algebras are completely determined, as well as possible differences that occur when κ is no longer closed. Leaving the restriction of algebraic closure provides a topic for further research.

Contents

Introduction

The work in this thesis is based on the work in the area of Lie theory, a subject dating back to the late 19^{th} century. From the origins in “transformation groups” by Sophus Lie, the subject has developed in various directions and influenced different areas of mathematics, such as representation theory, algebraic and differential geometry, and mathematical physics. Before the 20^{th} century, semisimple complex Lie algebras were introduced, and this was quickly followed by their classification. In the 1940s, the eponymous Dynkin diagrams provided a big refinement in this classification.

The affine (and linear) algebraic groups in this thesis arise as groups which are also affine algebraic varieties. From the beginning Lie algebras and algebraic groups were closely linked. By a construction of Chevalley, an algebraic group can be constructed over any algebraically closed field from a complex semisimple Lie Algebra. This connection is made closer as every affine algebraic group has a Lie algebra attached over the same field as the group. This attachment can be considered as a linearisation of the group. The group acts naturally on its Lie algebra, which will be discussed in Chapter 1. From this action arise the so-called *nilpotent orbits*, which are the main focus of this thesis.

Understanding how an algebraic group acts on its Lie algebra helps the understanding of the group itself, this is explained in (for instance) the books about linear algebraic groups by Borel [2], Humphreys [6], and Springer [9], but it also relates to other fields of study. The notes [7] by Jantzen discuss several connections between the study of nilpotent orbits and other subjects, such as the links with representation theory.

Since the introduction of the classification of nilpotent orbits for algebraic groups, it has been refined many times. In this thesis, we will make another step in refining the classification. The original classification of Dynkin-Kostant [3] in characteristic 0, using $\mathfrak{sl}_2$-triples, is a classic result in the theory, which is still being revisited in the present day, for instance see [11]. In positive characteristic, we refer to the paper of Holt-Spaltenstein (see [5]) and the sources given there. In this thesis, the so-called *dominance order* for nilpotent orbits in classical groups is of particular relevance. The dominance order was first established by Gerstenhaber [4].

The idea for using cocharacters (or $1-$parameter subgroups) to study orbits and whether or not they are closed (which is a different approach to the Zariski-closure) finds its origin in the work of Hilbert (where the focus was on the general linear group). The Hilbert-Mumford theorem (see theorem 1.8.1) is an application, which says that if an orbit is not Zariski-closed, taking the limit of a suitable cocharacter will yield an element in a different orbit.

The first chapter of this thesis will explore the known information regarding linear algebraic groups, nilpotent elements, Lie algebras, cocharacters and the related approach of closed and open orbits, and the orthogonal and symplectic groups. The second chapter explains the results of the research in the general linear group. In the third and fourth chapter, the symplectic and orthogonal groups are covered. The final chapter summarises the developments made and gives suggestions for further research.

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Declaration

I declare that this thesis is a presentation of original work and that I am the sole author. This work has not previously been presented for an award at this or at another university. All sources used for this thesis are acknowledged as references.

Chapter 1

Algebraic groups, lie algebras and nilpotent orbits

In this chapter, we will discuss general information on algebraic groups, Lie algebras and nilpotent orbits. We start with some notation and definitions. Throughout the thesis, κ denotes an algebraically closed field with good characteristic. Given a set X and a $\kappa-$valued function $f : X \to \kappa$ we denote *evaluation* at a point $x \in X$ by ϵ_x; that is $\epsilon_x(f) := f(x)$. An *affine variety* over κ consists of:

1. a set X of *points*,

2. a finitely generated $\kappa-$algebra $\kappa[X]$ of $\kappa-$valued functions on X,

such that the evaluation map $x \mapsto \epsilon_x$ gives a bijection $X \to \mathrm{Hom}_{\kappa-\mathrm{alg}}(\kappa[X], \kappa)$. A *morphism* of affine varieties is a map $\phi : X \to Y$ of sets of points such that $f \circ \phi \in \kappa[X]$ for all $f \in \kappa[Y]$. This gives rise to the *comorphism* $\phi^\sharp : \kappa[Y] \to \kappa[X]$ defined by $\phi^\sharp(f) = f \circ \phi$.

Affine varieties carry a topology coming from the coordinate algebra: for any subset $S \subseteq \kappa[X]$, we define $V(S) \subseteq X$ to be the set $\{x \in X \mid f(x) = 0 \text{ for all } f \in S\}$. These sets form the closed sets in the *Zariski topology* on X.

The main focus of this thesis are affine algebraic groups and their actions on affine varieties. An affine algebraic group G is an affine variety, with $\kappa[G]$ its coordinate algebra, and the following are morphisms of varieties:

$$\begin{aligned} \text{multiplication } \mu &: G \times G \to G, \\ \text{inversion } \iota &: G \to G. \end{aligned}$$

Example. Let $G = \mathrm{GL}_n(\kappa)$. Then the coordinate algebra is

$$\kappa[G] = \kappa[X_{11}, X_{12}, \ldots, X_{ij}, \ldots, X_{nn}, 1/\det],$$

where the X_{ij} are the matrix coordinate functions and det is the matrix determinant.

The *multiplicative group* $\mathbb{G}_m$ can be defined as GL_1. As an abstract group it consists of the nonzero points of κ under multiplication, and its coordinate algebra can be identified with the ring of Laurent polynomials $\kappa[T, T^{-1}]$ in a single indeterminate T. To show that μ is a morphism of varieties, we need to check that the comorphism $\mu^\#$ takes an element in $\kappa[\mathrm{GL}_n]$ to $\kappa[\mathrm{GL}_n \times \mathrm{GL}_n]$:

$$
\begin{aligned}
\mu^{\#}(X_{ij}(g,h)) &= X_{ij}(\mu(g,h)) \\
&= X_{ij}(gh) \\
&= \sum_{l=1}^{n} g_{il}h_{lj} \\
&= \left(\sum_{l=1}^{n} X_{il}\otimes X_{lj}\right)(g,h).
\end{aligned}
$$

So $\mu^{\#}(X_{ij}) = \sum_{l=1}^{n} X_{il}\otimes X_{lj}$. For the inverse the argument is similar.

1.1 Algebraic groups, actions and linearisation

For further background reading, see [6], [2] and [9]. Let G be an algebraic group, and recall that the irreducible and connected components of G coincide. We let G° denote the connected component of G containing the identity; it is a normal subgroup of G of finite index, and the other components are its cosets. The group G is called connected if $G^{\circ} = G$. For the majority of this thesis, we will be concerned with the general linear group $\mathrm{GL}_n(\kappa)$, symplectic group $\mathrm{Sp}_{2n}(\kappa)$ and orthogonal group $\mathrm{O}_n(\kappa)$. The groups $\mathrm{GL}_n(\kappa)$ and $\mathrm{Sp}_{2n}(\kappa)$ are connected, and $\mathrm{O}_n(\kappa)$ has two connected components.

The aim of this thesis is to study certain actions of algebraic groups on affine varieties. We begin with some generalities. In general, let X be a set and let G be an abstract group. G acts on X if there is a map

$$
\begin{aligned}
\alpha : G\times X &\to X \\
\alpha(g,x) &= g\cdot x
\end{aligned}
$$

such that

$$
\begin{aligned}
x_1\cdot(x_2\cdot y) = (x_1x_2)\cdot y &\text{ for } x_i\in G, y\in X \\
e\cdot y = y &\text{ for all } y\in X
\end{aligned}
$$

If an algebraic group G acts on an affine variety X in such a way that the associated map

$$
\begin{aligned}
\alpha : G\times X &\to X, \\
(g,x) &\mapsto g\cdot x
\end{aligned}
$$

is a morphism of varieties, then we say G acts *morphically* on X. In this situation, we get an induced linear action on the coordinate algebra $\kappa[X]$: given $f\in\kappa[X]$ and $g\in G$, define $\tau_g(f)\in\kappa[X]$ by

$$
\tau_g(f)(x) := f(g^{-1}\cdot x).
$$

The comorphism $\alpha^{\sharp}$ takes $\kappa[X]$ to $\kappa[G\times X]$, so for any $f\in\kappa[X]$ we can write: $\alpha^{\sharp}(f) = \sum_{i=1}^{r} a_i\otimes b_i$, for some $a_i\in\kappa[G], b_i\in\kappa[X]$. But then for any $g\in G$ and $x\in X$ we have:

$$\begin{aligned}\tau_g(f)(x) &= f(g^{-1}\cdot x)\\ &= f(\alpha(g^{-1},x))\\ &= \alpha^\sharp(f)(g^{-1},x)\\ &= \sum_{i=1}^r a_i(g^{-1})b_i(x).\end{aligned}$$

Since this holds for all $x \in X$, we conclude that $\tau_g(f) = \sum_{i=1}^r a_i(g^{-1})b_i$, which is an element of $\kappa[X]$, so $\tau_g : \kappa[X] \to \kappa[X]$.

To see that this does define an action, let $g, h \in G$, $f \in \kappa[X]$, $x \in X$, and we calculate:

$$\begin{aligned}(\tau_g \circ \tau_h)(f)(x) &= \tau_g(\tau_h(f))(x)\\ &= \tau_h(f)(g^{-1}\cdot x)\\ &= f(h^{-1}\cdot(g^{-1}\cdot x))\\ &= f((h^{-1}g^{-1})\cdot x)\\ &= f((gh)^{-1}\cdot x)\\ &= \tau_{gh}(f)(x),\end{aligned}$$

so $\tau_g \circ \tau_h = \tau_{gh}$, hence τ_g defines an action.

Let G act on X. Then $G \cdot x$ is called the *orbit* of x. When the group acting is an algebraic group and the set is a variety, it is natural to ask topological questions about the orbits. In this setting, difficulties arise because not all orbits are closed, in the topological sense. Then forming a quotient (for example) is difficult.

In this setting, we have the following result (see [6, Section 8.5-8.6]), which shows that algebraic groups act "locally finitely":

Proposition 1.1.1. Let G act morphically on an affine variety X, and let F be any finite-dimensional subspace of $\kappa[X]$. Then there exists a finite-dimensional subspace E of $\kappa[X]$ such that $F \subseteq E$ and $\tau_g(E) \subseteq E$ for all $g \in G$.

This allows us to "linearise" any such action. For us, there is one particular interest of this result when we apply it to the action of an algebraic group G on itself by right multiplication, as follows:

Theorem 1.1.1. Let G be an affine algebraic group. Then G is isomorphic to a closed subgroup of $\mathrm{GL}_n(\kappa)$ for some n.

The proof of this theorem can be found in [6][Section 8.6].

1.2 Nilpotent, unipotent and semisimple elements, the Jordan decomposition

Since any affine algebraic group G can be embedded in some $\mathrm{GL}_n(\kappa)$, by Theorem 1.1.1, we will extend some familiar notions from linear algebra to algebraic groups in this section.

A particularly useful instance of this comes from nilpotent, unipotent and diagonalizable matrices. Recall that a square matrix A is called *nilpotent* if some power $A^r = 0$. A square matrix A is called *unipotent* if $A - I$ is nilpotent, where I is the identity matrix of the same size as A – equivalently, all eigenvalues are 1. A square matrix is called *diagonalizable* if there is an invertible

P such that PAP^{-1} is diagonal. Working over an algebraically closed field, one can see from the Jordan normal form for matrices (for example) that any invertible matrix A has a unique expression: $A = A_s A_u$, where A_s and A_u commute, A_s is diagonalizable and A_u is unipotent.

Now we can extend these notions to elements of an arbitrary affine algebraic group G. We call $g \in G$ *unipotent* if the image of g is a unipotent matrix in some linearisation of G; we call g *semisimple* if its image is diagonalizable in some linearisation of G. It turns out (see [6, Section 15.3]) that these notions are independent of the chosen linearisation. Further, there is a unique *Jordan decomposition* of elements of G − any $g \in G$ can be uniquely written $g = g_s g_u$ where g_s and g_u commute, g_s is semisimple and g_u is unipotent.

1.3 Borel subgroups and reductive groups

In this section, we will describe the properties of several groups that appear in the thesis, using the notation from the description in Humphreys, see [6]. Let G be an affine algebraic group and let $a, b \in G$. Recall that the commutator is defined as $(a, b) = aba^{-1}b^{-1}$, where a, b are elements of a group G. Similarly for subgroups A, B of G, then the group generated by all $(a, b), a \in A, b \in B$ will be denoted by (A, B).

G is solvable if its derived series terminates in the identity: $G = G^{(0)} \rhd (G, G) = G^{(1)} \rhd (G^{(1)}, G^{1)}) = G^{(2)} \rhd \ldots \rhd e$.

Example. Take the group of upper triangular matrices $B_n(\kappa) \subset \mathrm{GL}_n(\kappa)$. Let $x, y \in B_n$, then

$$xyx^{-1}y^{-1} = \begin{pmatrix} x_{11}y_{11}x_{11}^{-1}y_{11}^{-1} & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & x_{nn}y_{nn}x_{nn}^{-1}y_{nn}^{-1} \end{pmatrix} = \begin{pmatrix} 1 & \star & \star \\ 0 & \ddots & \star \\ 0 & 0 & 1 \end{pmatrix},$$

which is a unipotent matrix, so $(B_n, B_n) = U_n$, the group of upper triangular unipotent matrices. Continuing, one gets that $(U_n, U_n) = \{X \in U_n \mid (x_{i,i+1}) = 0\}$, the unipotent group with zero entries on the first upper diagonal. By induction, $(U_n^{(m)}, U_n^{(m)}) = I_n$, as I_n can be described as a unipotent matrix with zero entries on every upper diagonal. We conclude that the unipotent and upper triangular groups are solvable.

Definition 1.3.1. Let $R(G)$ denote the *radical*, which is defined to be the largest closed connected normal solvable subgroup of G, and let $R_u(G)$ denote the *unipotent radical*, which is the largest closed connected normal unipotent subgroup of G. Then we say that G is *semisimple* if $R(G) = 1$ and *reductive* if $R_u(G) = 1$.

Example. The matrix groups $\mathrm{GL}_n(\kappa)$ and $\mathrm{SL}_n(\kappa)$ are both reductive. $GL_n(\kappa)$ is not semisimple because the scalar matrices form a nontrivial connected normal solvable subgroup. However, $\mathrm{SL}_n(\kappa)$ is semisimple.

A Borel subgroup $B \subset G$ is a maximal connected $(G = G^0)$ solvable subgroup. The subgroup of upper triangular matrices is called the upper, or standard, Borel inside $\mathrm{GL}_n(\kappa)$. It is denoted by $B_n(\kappa)$.

A subgroup P is defined as *parabolic* if it contains a Borel subgroup B. By [6, p.134], a subgroup P is parabolic if and only if G/P is projective (a subspace of projective $n-$space, see [6][Section 1.6]). The group P is called a standard parabolic if it contains a standard Borel. In GL_n then, any parabolic is conjugate to a subgroup in upper triangular block form. For example, in GL_4, the parabolic subgroups that arise, other than G and B_4 are:

$$\left(\begin{array}{c|c} \star & \star \\ \hline 0 & \\ 0 & \star \\ 0 & \end{array}\right), \quad \left(\begin{array}{cc|c} & \star & \star \\ \hline 0 & 0 & \star \\ 0 & 0 & \end{array}\right), \quad \left(\begin{array}{ccc|c} & \star & & \star \\ \hline 0 & 0 & 0 & \star \end{array}\right).$$

One of the key points in the basic theory of affine algebraic groups is the following theorem of Borel. This theorem together with the material on Lie algebras in the next section, is the starting point for the classification of simple algebraic groups, see Section 5.

Theorem 1.3.1 (Borel's Fixed Point Theorem)**.** Let G be a connected solvable algebraic group, and let X be a (nonempty) complete variety on which G acts. Then G has a fixed point in X.

Remark (see [6][Section 6.1]): a variety X is complete if for all varieties Y, the projective map $X \times Y \to Y$ is a closed map (i.e. the map sends closed sets to closed sets).

1.4 Lie algebras

Given an affine variety X, one can form the *tangent space* to X at any point $x \in X$. The easiest way to define this is as the set of *κ-linear derivations* $T_x(X) := \mathrm{Der}_\kappa(\kappa[X], \kappa)$ – that is, linear maps from the coordinate algebra $\kappa[X]$ to κ satisfying the extra property that $D(ab) = a(x)D(b) + D(a)b(x)$ for all $a, b \in \kappa[X]$ (this is the analogue of the product rule for derivatives in this setting). It is clear that $T_x(X)$ is a vector space, and it enjoys some nice functorial properties: for example, if $\phi : X \to Y$ is a morphism of affine varieties, then there is an induced map $d_x\phi : T_x(X) \to T_{\phi(x)}(Y)$ given by $D \mapsto D \circ \phi^\sharp$, where $\phi^\sharp : \kappa[Y] \to \kappa[X]$ is the comorphism.

The following calculation shows that $D \circ \phi^\sharp$ is a derivation.

$$\begin{aligned}(D \circ \phi^\sharp)(ab) &= D(\phi^\sharp(ab)) \\ &= D(\phi^\sharp(a)\phi^\sharp(b)) \\ &= \phi^\sharp(a)(x)D(\phi^\sharp(b)) + D(\phi^\sharp)(a)\phi^\sharp(b)(x) \\ &= a(y)(D \circ \phi^\sharp)(b) + (D \circ \phi^\sharp)(a)b(y),\end{aligned}$$

for any $a, b \in \kappa[Y]$.

Example. The easiest tangent space to write down gives a good idea of how this construction works in practice. If $X = \mathbb{A}^n$ is affine n-space, with coordinate algebra $\kappa[X_1, \ldots, X_n]$, then the tangent space at a point $x = (a_1, \ldots, a_n) \in X$ is simply the linear span of the partial derivatives $(\partial/\partial X_i)|_x$: that is,

$$\left.\frac{\partial}{\partial X_i}\right|_x (f) := \frac{\partial f}{\partial X_i}(x)$$

for each $f \in \kappa[X]$.

In the case of an affine algebraic group G with identity element $1 \in G$, the comultiplication on the coordinate algebra $\kappa[G]$ can be used to give the tangent space $T_1(G)$ the additional structure of a Lie algebra. This is not straightforward to show – one approach is to identify $T_1(G)$ with the space of *left-invariant derivations* from $\kappa[G] \to \kappa[G]$, and then equip this space with the usual Lie bracket $[D_1, D_2] = D_1 \circ D_2 - D_2 \circ D_1$. The Lie algebra so defined is called the Lie algebra of G, which we will denote by $\mathfrak{g}$ (see [6, p.65])

The functoriality noted above now has some particularly nice properties: any action of G on itself which fixes the identity will differentiate to give an action on the Lie algebra. In particular, the action of G on itself by an inner automorphism differentiates to give the so-called *adjoint action* of G on $\mathfrak{g}$.

Definition 1.4.1. Let G be a linear algebraic group with Lie algebra $\mathfrak{g}$. For each $g \in G$ we have the inner automorphism

$$\begin{aligned} \mathrm{Inn}(g) : G &\to G \\ x &\mapsto gxg^{-1}. \end{aligned}$$

Then the differential $d_1(\mathrm{Inn}(g)) : \mathfrak{g} \to \mathfrak{g}$ is denoted by $\mathrm{Ad}(g)$. This gives rise to the *adjoint action* of G on $\mathfrak{g}$: for $g \in G$, $x \in \mathfrak{g}$, define $g \cdot x = \mathrm{Ad}(g)(x)$.

Fortunately, for our purposes, all of this is quite easy to write down. The general linear group is an open subset in the set of all $n \times n$ matrices, which is just affine n^2-space as a variety. Thus the tangent space at the identity of $\mathrm{GL}_n(\kappa)$ can be identified with the vector space of all $n \times n$ matrices, and it turns out that under this identification, the Lie algebra structure is the usual one for matrices: we have the Lie bracket $[x, y] = xy - yx$ for $n \times n$ matrices x and y. We write $\mathfrak{gl}_n(\kappa)$ for this Lie algebra. The adjoint action of $\mathrm{GL}_n(\kappa)$ on $\mathfrak{gl}_n(\kappa)$ is simply given by matrix conjugation. Furthermore, if H is a closed subgroup of $\mathrm{GL}_n(\kappa)$, then the Lie algebra of H can be identified as a closed subalgebra of $\mathfrak{gl}_n(\kappa)$. We describe how to do this in two specific examples in Section 1.11 below.

The above discussions give a strong motivation for a further study of the *conjugation action* of $\mathrm{GL}_n(\kappa)$ on the set $\mathrm{M}_n(\kappa)$ of all $n \times n$ matrices, since this is precisely the adjoint action of $\mathrm{GL}_n(\kappa)$ on its Lie algebra. We can see easily that this action is a morphism, by considering the conjugation action of GL_n on all matrices:

$$\begin{aligned} \mathrm{GL}_n \times M_n &\to M_n, \\ (g, A) &\mapsto gAg^{-1}. \end{aligned}$$

The mapping is a composition of the following morphisms: $\mu \circ \mu \circ \iota$,

$$\begin{aligned} \iota : g &\mapsto g^{-1}, \\ \mu : (A, g^{-1}) &\mapsto Ag^{-1}, \\ \mu : (g, Ag^{-1}) &\mapsto gAg^{-1}, \end{aligned}$$

so conjugation is a morphism.

Note that in this case, the stabilizer of $x \in X$ is defined as $G_x = \{x \in X \mid g \cdot x = x\}$. In the case that the action of G on X is conjugation, the stabilizer and centralizer $C_G(x) = \{g \in G \mid gx = xg\} = \{g \in G \mid gxg^{-1} = x\}$ are the same.

1.5 A motivating example

One of the main motivations for the work in this thesis is the paper [1], concerning the notion of cocharacter-closed orbits for algebraic groups acting on affine varieties. The technical language will be introduced later in the thesis, but this early example shows what kind of ideas appear. This is based on the example in [1, p.11].

Working over the complex numbers $\mathbb{C}$, consider the affine line $X = \mathbb{A}^1$ (which we can identify with $\mathbb{C}$ as a set) and the multiplicative group $G = \mathbb{G}_m$ (which we can identify with $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$). The algebraic group G acts on the variety X by squares:

$$\begin{aligned} &\mathbb{G}_m \times \mathbb{A}^1 \to \mathbb{A}^1, \\ &\quad (x, g) \mapsto g \cdot x := g^2 x, \quad g \in G, x \in X. \end{aligned}$$

This action has two orbits – the orbits $X \setminus \{0\}$ of nonzero points and the orbit $\{0\}$. This latter orbit is closed in the Zariski topology, and we can reach 0 from any other point in $\mathbb{A}^1$ by taking a limit – for any $x \neq 0 \in X$, we have $0 = \lim_{a \to 0} a \cdot x$. Here each of the points $a \cdot x$ lies in the first orbit, and the limit point 0 lies in the closed orbit.

Now consider the subfield $\mathbb{R} \subset \mathbb{C}$. We have the group of $\mathbb{R}$-points $\mathbb{G}_m(\mathbb{R})$ acting on the set of $\mathbb{R}$-points $X(\mathbb{R})$, but now there are three orbits – the orbit of positive real numbers $\mathbb{R}^+$, the orbit of negative real numbers $\mathbb{R}^-$, and the orbit $\{0\}$. The set $X(\mathbb{R})$ inherits a topology from the Zariski topology on X, but this topology cannot tell apart the two orbits $\mathbb{R}^+$ and $\mathbb{R}^-$ of $\mathbb{G}_m(\mathbb{R})$ because both of them are infinite, and hence dense in X (recall that a proper Zariski closed set in $\mathbb{A}^1$ is the set of zeroes of a collection of polynomials, and hence is finite).

The notion of cocharacter-closed orbits was designed in part to get round this problem: it provides a tool to topologise so-called "relative orbits" for an algebraic group which allows for a more detailed analysis than the Zariski topology inherited from the absolute (i.e., algebraically closed) setting. In this example, for the action of the real points, the closed orbit $\{0\}$ can still be obtained as a limit from either of the two other orbits, but it is not possible to jump between them using limits, so in this sense we can tell the orbits apart.

1.6 Tori and cocharacters

First, we give the definition of a torus:

Definition 1.6.1. T is called a torus if it is isomorphic to some diagonal group $D_n(\kappa) \subset \mathrm{GL}_n(\kappa)$.

From Section 1.2, recall that an element is semisimple if it is diagonalizable when represented in GL_n, so a torus T is a commutative group consisting of semisimple elements. In fact, the action of a torus in *any* representation is diagonalizable, since a set of pairwise commuting diagonalizable matrices can always be simultaneously diagonalized.

With the properties of tori, we move on to characters and weight spaces:

Definition 1.6.2. Let G be an affine algebraic group, then a character of G is defined to be a morphism of algebraic groups:

$$\alpha : G \to \mathbb{G}_m.$$

If α_1, α_2 are two characters, the product is defined as $(\alpha_1\alpha_2)(g) = \alpha_1(g)\alpha_2(g)$, which gives the structure of a commutative group to the set of all characters $X(G)$. In GL_n, the determinant of a matrix is a natural example of a character.

Let $T \subseteq \mathrm{GL}(V)$ be a torus, so it is diagonalizable in $\mathrm{GL}(V)$. For each $\alpha \in X(T)$, define the weight space V_α as follows:

$$V_\alpha = \{v \in V \mid \tau \cdot v = \alpha(\tau)v \quad \forall \tau \in T\}, \tag{1.1}$$

then α is called a weight if $V_\alpha \neq 0$.

Example. Let $G = \mathrm{GL}_2(\kappa)$ and let T be the standard diagonal torus in G. Then we can let

$$V = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}, \text{ let } T \subset \mathrm{GL}_2,$$

The characters of T correspond to pairs of integers $\alpha = (\alpha_1, \alpha_2)$ as follows: if

$$t = \begin{pmatrix} t_1 & 0 \\ 0 & t_2 \end{pmatrix}, \text{ then } \alpha(t) = t_1^{\alpha_1} t_2^{\alpha_2}.$$

Furthermore:

$$\tau \cdot e_1 = \begin{pmatrix} t_1 \\ 0 \end{pmatrix}, \text{ and } \alpha(\tau)e_1 = \begin{pmatrix} t_1^{\alpha_1} t_2^{\alpha_2} \\ 0 \end{pmatrix}.$$

It follows that if $\alpha = (1, 0)$, Equation 1.1 is satisfied. Therefore

$$V_{(1,0)} = \text{span}\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\},$$

and similarly

$$V_{(0,1)} = \text{span}\left\{\begin{pmatrix} 0 \\ 1 \end{pmatrix}\right\}.$$

In this case, it follows that $V = V_{(1,0)} \oplus V_{(0,1)}$. Now V can be denoted as the direct sum of its weight spaces: $V = \oplus_\alpha V_\alpha$. This is an instance of the general situation: for any vector space V upon which a torus acts linearly, we have that V can be decomposed as the direct sum of its T-weight spaces:

$$V = \bigoplus_\alpha V_\alpha.$$

Dual to the notion of a character is the notion of a cocharacter, which is central to the work in this thesis.

Definition 1.6.3. A **cocharacter** of G is an algebraic group homomorphism $\lambda : \mathbb{G}_m \to G$.

Let

$$\lambda(t) : \mathbb{G}_m \to \mathrm{GL}(V).$$

Let T be the standard diagonal torus in $\mathrm{GL}_n(\kappa)$. Then any cocharacter has the form:

$$\lambda : t \mapsto \begin{pmatrix} t^{m_1} & & \\ & \ddots & \\ & & t^{m_n} \end{pmatrix},$$

where the m_i are integers. For reductive G acting on affine X over κ, we define $Y_\kappa(G) = \{\text{cocharacters of } G \text{ over } \kappa\}$ (see [1, p2]).

Now suppose G acts on an affine variety X, then given any λ and $x \in X$, we can define:

$$\begin{aligned} \psi = \psi_{x,\lambda} : \mathbb{G}_m &\to X \\ a &\mapsto \lambda(a) \cdot x. \end{aligned}$$

Then $\mathrm{Im}(\psi) \subseteq G \cdot x$. So there is a point on the line for every $a \neq 0$, and $x = \lambda(1) \cdot x$. We can make the following observations:

1. If ψ extends to $\bar{\psi} : \mathbb{A}^1 \to X$ (i.e. $\bar{\psi}$ is a morphism and $\bar{\psi} = \psi \ \forall a \in \mathbb{G}_m$), then the limit $\lim_{a\to 0} \lambda(a) \cdot x$ exists.

2. If $\bar{\psi}$ exists then we write $\lim_{a\to 0} \lambda(a) \cdot x = \bar{\psi}(0)$.

We identify $\mathbb{G}_m \subseteq \mathbb{A}^1$ in the obvious way, which places the coordinate algebra of $\mathbb{A}^1$ as $\kappa[T]$ inside $\kappa[T, T^{-1}]$. Since

$$\psi : \mathbb{G}_m \to X,$$
$$\psi^{\#} : \kappa[X] \to \kappa[\mathbb{G}_m] = \kappa[T, T^{-1}],$$

it follows that $\bar{\psi}$ exists if and only if $\mathrm{Im}(\psi^{\#}) \subseteq \kappa[T]$, and we can get $\bar{\psi}^{\#} : \kappa[X] \to \kappa[T]$ by restricting the codomain, so $\bar{\psi}$ is given automatically. This gives a third observation:

3. If $\bar{\psi}$ exists then it is unique.

Since we will often be using cocharacters, we introduce the following definition.

Definition 1.6.4. Let $x \in X$ be a matrix, and let $\lambda \in G$ be a cocharacter. Then conjugating x with a cocharacter and taking the limit $\lim_{t\to 0} \lambda(t) \cdot x$ is called *taking the limit along a cocharacter.*

We also introduce a simplified notation for taking the limit along a cocharacter. When $\lim_{a\to 0} \lambda(t) \cdot x$ exists, we denote it by $\lim_\lambda x$.

Example. We return to the vector space from the previous example, and now apply a cocharacter λ:

$$V = \mathrm{span}\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\},$$

and let $\lambda(t) : \kappa^* \to \mathrm{GL}_2$:

$$\lambda : t \mapsto \begin{pmatrix} t^2 & \\ & t \end{pmatrix}.$$

Next, let $G = \mathrm{GL}_2$ act by conjugation on $X = M_2$, the matrices of size 2. Suppose

$$x = \begin{pmatrix} p & q \\ r & s \end{pmatrix},$$

is an arbitrary matrix in M_2. Then:

$$\lambda(t) \cdot x = \begin{pmatrix} p & tq \\ t^{-1}r & s \end{pmatrix},$$

and we observe that

1. $\lim_\lambda x = x$ if $\lambda(t) \cdot x = x$, so when λ centralizes x.

2. $\lim_\lambda x$ exists if $r = 0$, this is when x is upper triangular.

So, for this choice of cocharacter, the limit exists for a matrix x if and only if x is upper triangular, and then the limit is a diagonal matrix.

This is a special case of the Hilbert-Mumford Theorem below.

Next, we determine the comorphism for $\psi = \psi_{x,\lambda}$ for a fixed choice of x as above. Recall that $\kappa[G_m] = k[T, T^{-1}]$ and we can write $\kappa[M_2] = \kappa[x_{11}, x_{12}, x_{21}, x_{22}]$, where the x_{ij} are the coordinate functions on matrices. Then $\psi^\sharp$ should be a map from $\kappa[M_2] \to \kappa[T, T^{-1}]$. Working with the coordinate functions on M_2, we get:

$$\psi^\sharp(x_{11}) = (x_{11} \circ \psi)(t) = x_{11}(\psi(t)) = p.$$

So $\psi^\sharp(x_{11}) = p$ is constant. Similarly we can calculate:

$$\begin{aligned}
\psi^\sharp(x_{12}) &= qT, \\
\psi^\sharp(x_{21}) &= rT^{-1}, \\
\psi^\sharp(x_{22}) &= s.
\end{aligned}$$

This shows that the image of $\psi^\sharp$ lies in $k[T]$ if and only if $r = 0$, and since $\lim_\lambda x$ exists if and only if $r = 0$, we conclude that $\lim_\lambda x$ exists if and only if $\mathrm{Im}(\psi^\#)$ lies in $k[T]$.

1.7 Further results on limits

In this section we will discuss further results on taking taking limits, specifically with the parabolic subgroup, Levi subgroup and unipotent radical in mind. Suppose G is a reductive group acting on an affine variety X.

Lemma 1.7.1. There exists a vector space V and embedding

$$\begin{aligned}
\rho &: G \to \mathrm{GL}(V), \\
\phi &: X \to V,
\end{aligned}$$

for all $g \in G$ and $x \in X$. In other words, the action of G on X can be linearised.

Proof. For a full proof, see [8, Section 1]. This follows in a similar fashion to the proof that any algebraic group can be linearised (see theorem 1.1.1). The action of G on X induces an action on the coordinate algebra $\kappa[X]$. We can find a $G-$stable finite dimensional subspace E of $\kappa[X]$ containing a generating set for $\kappa[X]$, and then we can identify the symmetric algebra on E with the coordinate ring of a vector space V. The inclusion $E \subset \kappa[X]$ and the action of G on E induces the required maps ϕ and ρ. $\square$

Note that if $G \cdot x$ is a closed orbit, then $\lim_\lambda x \in G \cdot x$, whenever it exists. This follows since the limit is given by a morphism, and morphisms are continuous.

For a cocharacter $\lambda \in Y(G)$ and an element $g \in G$, the cocharacter $g \cdot \lambda$ is defined by $(g \cdot \lambda)(a) = g\lambda(a)g^{-1}$ for each $a \in \mathbb{G}_m$. We begin with a general result.

Lemma 1.7.2. Let $x \in X$ and $\lambda \in Y(G)$. Suppose $y = \lim_\lambda x$ exists. Then:

(i) (The image of) λ centralizes y.

(ii) For all $g \in G$, $\lim_{g \cdot \lambda} g \cdot x = g \cdot y$.

Proof. (i) By Lemma 1.7.1 we may assume that X is a vector space and G is acting linearly. As $y = \lim_\lambda x$, we can denote $x = y + x_0$, where x_0 is the part that gets killed off: $\lim_\lambda x_0 = 0$. Then

$$y = \lim_\lambda x = \lim_\lambda (y + x_0) = \lim_\lambda y + \lim_\lambda x_0 = \lim_{t \to 0} \lambda \cdot y.$$

Hence it follows that λ must centralize y.

(ii) Note that for all $t \in \mathbb{G}_m$ we have:

$$(g \cdot \lambda)(t) \cdot (g \cdot x) = (g\lambda(t)g^{-1}) \cdot (g \cdot x) = g \cdot (\lambda(t) \cdot x).$$

Taking the limit as $t \to 0$ now gives the result. $\square$

Remark: if λ is trivial, it centralizes x, that is $\lim_\lambda x = x$.

The material above on limits has particularly interesting consequences when applied to the conjugation action of G on itself. It turns out, by [9] that if we take any cocharacter $\lambda \in Y(G)$, then:

$$P_\lambda := \{g \in G \mid \lim_\lambda g \text{ exists}\},$$

is a parabolic subgroup of G. Moreover the unipotent radical $R_u(P_\lambda)$ can be recovered as the set of elements which are sent to 1 in the limit:

$$R_u(P_\lambda) = \{g \in G \mid \lim_\lambda g = 1\},$$

and if we define

$$L_\lambda := \{g \in G \mid \lim_\lambda g = g\} = C_G(\mathrm{Im}(\lambda)),$$

then L_λ is a reductive subgroup of P_λ and $P_\lambda = L_\lambda \ltimes R_u(P_\lambda)$ gives a so-called Levi decomposition of P_λ. Moreover, for every pair (P, L) consisting of a parabolic subgroup P of G and Levi subgroup L of P, there is some $\lambda \in Y(G)$ with $P = P_\lambda$ and $L = L_\lambda$.

1.8 The Hilbert-Mumford theorem

The Hilbert-Mumford Theorem allows us to define closedness with respect to cocharacters. In [8, Section 1], the theorem is described as follows:

Theorem. Let X be an affine $G-$variety over an algebraically closed field κ. Let $x \in X$ and let S be a closed $G-$subvariety of X which meets the closure of the orbit $G \cdot x$. Then there is a λ such that $\lim_\lambda x$ exists and is contained in S.

For this thesis, we consider $S = \overline{G \cdot x} \setminus G \cdot x$. So we get:

Theorem 1.8.1. [Hilbert-Mumford Theorem] If $G \cdot x$ is **not** closed in X, then $\exists \lambda$ such that $\lim_\lambda x$ exists and $\lim_\lambda x \notin G \cdot x$.

A proof can be found in [8, Section 1]. This fundamental theorem leads us to consider the following definition, see [1]:

Definition 1.8.1. Let X be a set over κ, then $S \subset X$ is cocharacter-closed (over κ) if for all $x \in S$ and $\forall \lambda \in Y(G)$ such that $\lim_\lambda x$ exists, it follows that $\lim_\lambda x \in S$.

The cocharacter-closure of an orbit $G \cdot x$ is denoted by $\overline{G \cdot x}^c$, this gives two main ideas:

1. Use cocharacters to define a new topology on X, using the $G-$orbits in X. Letting cocharacters act on a $G-$orbit that is not closed allow us to go to another $G-$orbit.
2. With the suitable modifications, this works over any field. However, if G acts on X over a field that is not algebraically closed, there will be restrictions.

The case of interest is when $S = G \cdot x$, the orbit of $x \in X$. This is because the $\lim_\lambda x$ exists (and is $G-$conjugate to x). To show this, take any $a \in \kappa$, then:

$$(g \cdot \lambda)(a) \cdot (g \cdot x) = (g\lambda(a)g^{-1}) \cdot (g \cdot x) = g \cdot (\lambda(a) \cdot x).$$

Moreover, if $\kappa = \bar{\kappa}$, then the Hilbert-Mumford theorem tells us that cocharacter-closedness and Zariski-closedness are equivalent.

Definition 1.8.2. Suppose $x, y \in X$. Say that $G \cdot y$ is *1-accessible* from $G \cdot x$ if there exists a $\lambda \in Y(G)$ such that $\lim_\lambda x \in G \cdot y$. Say that $G \cdot y$ is $n-$*accessible* if there is a chain $x = x_0 \to x_1 \to \ldots \to x_n = y$ such that $G \cdot x_i$ is 1-accessible from $G \cdot x_{i-1}$ for $1 \leq i \leq n$. If $G \cdot y$ is $n-$accessible for some n, we say it is *accessible.*

It is immediately clear that accessibility gives a preorder on orbits; it is transitive by definition, and to show that it is reflexive, note that $\lim_\lambda x = x$ when λ is trivial. Now a formal definition of the cocharacter-closure can be given:

Definition 1.8.3. The cocharacter-closure

$$\overline{G \cdot x}^c = \bigcup G \cdot y,$$

where the union is over the orbits $G \cdot y$ that are accessible from $G \cdot x$, is the smallest cocharacter-closed set containing $G \cdot x$.

Remark: We can now use a new notation for accessibility. If $G \cdot y$ is accessible from $G \cdot x$, then we write $\overline{G \cdot y}^c \leq \overline{G \cdot x}^c$. The following questions arise the definitions:

1. Is the preorder on orbits given by cocharacter-closure a partial order?
2. Even when $\kappa = \overline{\kappa}$, are there cases when $\overline{G \cdot x}^c \neq \overline{G \cdot x}$?
3. Does accessibility imply 1-accessibility, e.g. if $G \cdot y$ is 1-accessible from $G \cdot x$ and $G \cdot z$ is 1-accessible from $G \cdot y$, does it folllow that $G \cdot z$ is 1-accessible from $G \cdot x$? To put it in diagram form:

$$\begin{tikzcd} G \cdot x \arrow[r] \arrow[dr, "?"'] & G \cdot y \arrow[d] \\ & G \cdot z. \end{tikzcd}$$

We can answer the first question, whenever κ is algebraically closed.

Lemma 1.8.1. Accessibility gives a partial order on nilpotent orbits over an algebraically closed field.

Proof. Since the accessibility order is defined by taking limits along cocharacters, it is clear that if $\overline{G \cdot y}^c \leq \overline{G \cdot x}^c$, then the orbit $G \cdot y$ must lie in the Zariski-closure of the orbit $\overline{G \cdot x}$. This means that taking limits cannot take us up the dominance order on partitions, and hence if $\overline{G \cdot y}^c < \overline{G \cdot x}^c$ but $\overline{G \cdot y}^c \neq \overline{G \cdot x}^c$, the orbit of y must be lower in the dominance order than the orbit of x. $\square$

Remark: note that this proof relies on the dominance order, so it relies on the Zariski-closure. So with this proof, we cannot say that the preorder on orbits given by cocharacter-closure is a partial order when the restriction of algebraically closed fields is lifted. Note also that we can say that if $\overline{G \cdot y}^c \leq \overline{G \cdot x}^c$, we also have that $\overline{G \cdot y} \leq \overline{G \cdot x}$, but the reverse is not necessarily true. Hence, we do not have an answer to question 2 yet.

1.9 Nilpotent elements

Recall that a square matrix x is called nilpotent if $x^n = 0$ for some $n \in \mathbb{N}$. For each integer $n \geq 1$, denote by J_n the $(n \times n)$ matrix where the $(i, i+1)$ entry equals 1, for $1 \leq i < n$, and all other entries equal 0. J_n is called a Jordan block. If x is nilpotent then there is a basis $\mathcal{B}$ such that x has a Jordan normal form: a diagonal block matrix of Jordan blocks. In this thesis we will denote Jordan normal forms with a direct sum:

$$x = J_{n_1}^{(r_1)} \oplus \cdots \oplus J_{n_p}^{(r_p)}, \tag{1.2}$$

where the (r_i) denote the multiplicity of each J_{n_i}. The matrix has a partition π, defined as follows:

Definition 1.9.1. A partition π of a natural number m is an expression $m = n_1 + \cdots + n_m$ where the n_i are positive integers. Partitions are typically denoted by listing the numbers $n_i, \ldots, n_p$ in decreasing order of size inside square brackets, with powers to indicate the number of occurrences of each number listed in the partition. So the partition π of $n = n_1 + n_1 + \cdots + n_1 + n_2 + \cdots + n_p$ is denoted as:

$$\pi = [n_1^{(r_1)}, \ldots, n_p^{(r_p)}].$$

There is a partial order on partitions, as described in [4]:

Definition 1.9.2. Let $\pi_1 = [n_1, \ldots, n_p]$ (not all n_i necessarily distinct) and let $\pi_2 = [m_1, \ldots, m_q]$ be two partitions of some n. π_1 is said to dominate π_2 if:

$$\sum_{i=1}^{j} n_i \geq \sum_{i=1}^{j} m_i \text{ for } 1 \leq j \leq p. \tag{1.3}$$

The partial ordering of nilpotent partitions is called the dominance order, which can be visualised in a diagram. As example, Figure 1.1 shows the dominance order for the partitions of 6:

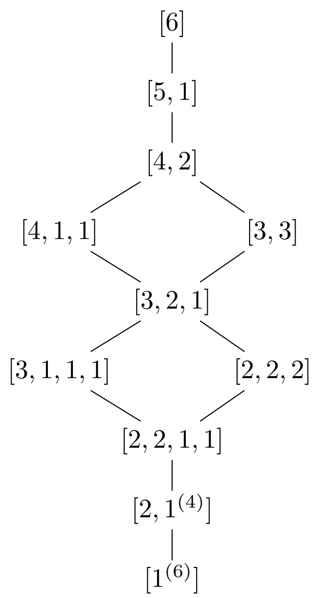

Figure 1.1: Dominance of partititions in $\mathfrak{gl}_6$.

As we show below, the orbits of interest in this thesis can be labelled by partitions, and hence the accessibility between these orbits can be described by changes between partitions, which we define as *moves*. It is useful to have some terminology to describe such moves. In a partition

$\pi_1 = [n_1, n_2]$, n_1 and n_2 are called the *parts*, and in a move to partition $\pi_2 = [n_1 - a, n_2 + a]$, a is the *piece* that is moved from part n_1 to n_2.

Let G be a reductive group and $\mathfrak{g}$ its Lie Algebra. An element $x \in \mathfrak{g}$ is called *nilpotent* if it acts nilpotently in every representation of $\mathfrak{g}$. The set of all nilpotent elements in $\mathfrak{g}$ is called the *nilpotent cone* in $\mathfrak{g}$. As seen above, G acts on $\mathfrak{g}$ via the adjoint action, and the nilpotent cone is a closed G-stable subset for this action.

The main subject of this thesis is an analysis of the orbits in the nilpotent cone for linear, symplectic and orthogonal groups, concentrating on the notion of accessibility we have defined above.

Example. For $G = \mathrm{GL}_n(\kappa)$, the Lie Algebra is the algebra of all matrices and the nilpotent elements are the nilpotent matrices described above. Hence the G-orbits in the nilpotent cone correspond to partitions.

The Jordan normal form gives a labelling of the $G-$orbits of $\mathfrak{g}$, because G acts on $\mathfrak{g}$ by conjugation. In fact, when we consider an $n \times n$ nilpotent matrix x, it has eigenvectors $\{v_1, \dots, v_n\}$. Then conjugating with a matrix V, constructed of these eigenvactors, yields the Jordan form of the orbit $\mathcal{O}(x) \in \mathfrak{g}$. We want to see if these orbits are topologically closed. By conjugating with the cocharacter $\lambda = \mathrm{diag}(t^n, t^{-1}, \dots, 1)$ and then taking the limit as $t \to 0$, any nilpotent matrix in Jordan form can be killed off. So 0 is in the closure of every orbit, and it is also the only closed orbit in the Zariski topology.

We now define distinguished partitions, which will show to be not accessible from any other partition:

Definition 1.9.3. A nilpotent element $x \in \mathfrak{g}$ is called distinguished if each torus contained in the stabilizer $C_G(x) = \{g \in G \mid g \cdot x = x\}$ is contained in the center of G.

The following lemma is simple to prove, but it turns out to be a central tool in our work below as it gives a criterion for “non-accessibility”.

Lemma 1.9.1. If x, y are nilpotent elements with y distinguished, then $y = \lim_\lambda x$ occurs only if $x = y$ and λ is central in G.

Proof. If $\lim_\lambda x = y$, then the image of λ is a torus which centralizes y. But any torus which centralizes y lies in the centre of G, by definition. Hence λ is central and this means that $y = \lim_\lambda x = x$. $\square$

For nilpotents in $\mathfrak{gl}_n$, first consider that $Z(\mathrm{GL}_n) = aI_n$, with $a \in \kappa^*$. Then x is distinguished if the aI_n are the only matrices in $\mathrm{Stab}_G(x)$. In $\mathfrak{gl}_n$, let x_1 have partition $\pi_1 = [n_1]$, then it is of the form:

$$x_1 = J_{n_1},$$

and let $t_1 = \mathrm{diag}(a_1, \dots, a_{n_1}) \in C_G(x)$, then for the equation

$$J_{n_1} = t_1 \cdot J_{n_1} = \begin{pmatrix} 0 & \frac{a_1}{a_2} & & \\ & \ddots & \ddots & \\ & & \ddots & \frac{a_{n_1-1}}{a_{n_1}} \\ & & & 0 \end{pmatrix}$$

to hold, we require $a_i = a_{i+1}$ for $1 \leq i \leq n_1 - 1$. So $t_1 = aI_{n_1}$ and x_1 is distinguished.

Next, let x_2 have partition $\pi_2 = [n_1, n_2]$, so

$$x_2 = J_{n_1} \oplus J_{n_2},$$

and let

$$t_2 = \mathrm{diag}(a_1, \ldots, a_{n_1}, a_{n_1+1}, \ldots, a_{n_1+n_2}),$$

with $a_i = a_{i+1}$ for $i \leq n_1$, $a_{n_1} \neq a_{n_1+1}$, and again $a_i = a_{i+1}$ for $n_1 + 1 \leq i \leq n_1 + n_2$. Then $t_2 \cdot x_2 = x_2$, but t_2 is not in the center of G, so x is not distinguished. The lemma follows:

Lemma 1.9.2. $x \in \mathfrak{gl}_n$ is distinguished if and only if x has partition $\pi = [n]$.

1.10 A key technical result

Suppose that G is a general linear, symplectic or orthogonal group, and X is the Lie algebra. By conjugating if necessary, we may put a cocharacter λ in any standard form we like – that is we can make λ evaluate in the standard diagonal torus, and we can ensure that the powers in λ are decreasing in size as we go down the diagonal. This means that the parabolic subgroup P_λ will be of standard upper block triangular form. Furthermore, if x is a nilpotent element such that the limit exists, we have that x is in the Lie algebra of the parabolic subgroup P_λ. Let $y = \lim_\lambda x$, then since λ fixes y, we have that y is in the Lie algebra of the Levi subgroup L_λ (which is the block diagonal subgroup).

If we take any $g \in L_\lambda$ and replace x with $g \cdot x$, replace y by $g \cdot y$ and λ by $g \cdot \lambda$, then if $\lim_\lambda x = y$, it follows that $\lim_{g \cdot \lambda} g \cdot x = g \cdot y$, by Lemma 1.7.2. By this conjugation we may further assume that y is in standard form for the Levi subgroup L_λ. But conjugating λ by an element from L_λ doesn't change λ, which means that we can put y in standard form for the Levi subgroup without changing λ.

Hence we have:

Lemma 1.10.1. When considering limits $y = \lim_\lambda x$ in the general linear, symplectic or orthogonal Lie algbras, we may assume that λ is diagonal, y is in a standard form, and x is in a corresponding upper block triangular form.

1.11 The symplectic and the orthogonal groups

When we deal with symplectic and orthogonal groups, we make the extra assumption that $\mathrm{char}(\kappa) \neq 2$. This is because the theory of bilinear forms in characteristic 2 deviates quite significantly from that in other characteristics. Let V be a finite dimensional vector space over κ and let $\phi : V \times V \to \kappa$ be a bilinear form that is non-degenerate (so for all $v \in V$, there is a $w \in V$ such that $\phi(v, w) \neq 0$). Assume ϕ is either alternating or symmetric, in other words, there is an $\varepsilon = \{\pm 1\}$ with

$$\phi(v, w) = \varepsilon \phi(w, v) \text{ for } v, w \in V.$$

If ϕ is symmetric, then $\varepsilon = 1$ and if ϕ is alternating, then $\varepsilon = -1$. Set

$$G = \{g \in \mathrm{GL}(V) \mid \phi(g(v), g(w)) = \phi(v, w) \text{ for all } v, w \in V\}.$$

This is an algebraic subgroup of $\mathrm{GL}(V)$, called the orthogonal group $\mathrm{O}(V, \phi)$ if $\varepsilon = 1$ (so ϕ is symmetric), and the symplectic group $\mathrm{Sp}(V, \phi)$ if $\phi = -1$ (ϕ is alternating). In this thesis, we will mostly be working with matrix forms of these groups, which amounts to choosing a suitable basis of the vector space V. Since all bilinear forms are conjugate, our work is independent of the choice of ϕ, so the theorems we use and prove for the symplectic and orthogonal groups and algebras hold for any bilinear form ϕ and basis $\mathcal{B}$.

If $\mathcal{B} = \{e_1, \ldots, e_n\}$ is a basis and the bilinear form ϕ is alternating, then the matrix of ϕ with respect to this basis is of the form $\Omega_{\mathcal{B}} = (\phi(e_i, e_j))$. Let $v, w \in V$, then there are column vectors:

$$\underline{v} = \begin{pmatrix} a_1 \\ \vdots \\ a_n \end{pmatrix}, \quad \underline{w} = \begin{pmatrix} b_1 \\ \vdots \\ b_n \end{pmatrix},$$

and with respect to the basis $\mathcal{B}$:

$$\begin{aligned} v &= a_1 e_1 + \cdots + a_n e_n, \\ w &= b_1 e_1 + \cdots + b_n e_n. \end{aligned}$$

Then $\phi(v, w) = \underline{v}^T \Omega_{\mathcal{B}} \underline{w}$. We want to choose a suitable basis so that $\Omega_{\mathcal{B}}$ has an nice structure. Pick a $v \in V$, then find a $w \in V$, with $\phi(v, w) \neq 0$. Note that $w \neq cv$ for any $c \in \kappa$ because $\phi(v, v) = 0$. Suppose $\phi(v, w) = a$, then if $w' = \frac{1}{a} w$ is substituted, then $\phi(v, w') = 1$ and $\phi(w', v) = -1$. Next, let $e_1 = v$, and $f_1 = w'$, and consider:

$$\langle e_1, f_1 \rangle^{\perp} = \{v \in V \mid \phi(e_1, v) = \phi(f_1, v) = 0\}.$$

There are two observations: $\dim(\langle e_1, f_1 \rangle^{\perp}) = \dim V - 2$ and $V = \langle e_1, f_1 \rangle \oplus \langle e_1, f_1 \rangle^{\perp}$. By taking $W = \langle e_1, f_1 \rangle^{\perp}$, we get a pair e_2, f_2 such that $\phi(e_2, f_2) = 1 = -\phi(e_2, f_2)$, and by iteration, we obtain a basis for V:

$$\mathcal{B} = \{e_1, \ldots, e_r, f_r, \ldots, f_1\}, \tag{1.4}$$

so $\dim(V) = 2r$, an even dimension for alternating ϕ. With respect to this basis, the form has matrix:

$$\Omega_{\mathrm{S}} = \left(\begin{array}{ccc|ccc} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ \hline & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{array}\right).$$

So

$$\mathrm{Sp}(V, \phi) \cong \mathrm{Sp}_{2n}(\kappa) = \{g \in \mathrm{GL}_{2n}(\kappa) \mid g^T \Omega_{\mathrm{S}} g = \Omega_{\mathrm{S}}\}.$$

Next, if ϕ is symmetric, the process of determining a matrix Ω_{O}, with respect to $\mathcal{B}$ is the same, but as $\phi(v, w) = \phi(w, v)$, we can have that $\phi(v, v) = 1$. So $\dim(V) = n$ can be odd or even:

$$\begin{aligned} \mathcal{B} &= \{e_1, \ldots, e_r, v, f_r, \ldots, f_1\} \text{ if } \dim(V) \text{ is odd}, \\ \mathcal{B} &= \{e_1, \ldots, e_r, f_r, \ldots, f_1\} \text{ if } \dim(V) \text{ is even}. \end{aligned}$$

In both cases:

$$\Omega_{\mathrm{O}} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix},$$

then

$$O(V, \phi) \cong \mathrm{O}_n(\kappa) = \{g \in \mathrm{GL}_n(\kappa) \mid g^T \Omega_{\mathrm{O}} g = \Omega_{\mathrm{O}}\}.$$

Next, we can determine their Lie algebras, as by [6, Chapter 9]. Given Ω as above, there is an isomorphism

$$\phi : \mathrm{GL}_n \to \mathrm{GL}_n,$$
$$\text{given by } g \mapsto \Omega^{-1}(g^T)^{-1}\Omega.$$

The subgroup G (orthogonal or symplectic) is defined by $\phi(g) = g$. If we take the differential of this map at the identity, we get an isomorphism of the Lie Algebra. First, we decompose ϕ as a sequence of maps:

$$g \mapsto g^{-1} \mapsto (g^T)^{-1} \mapsto \Omega^{-1}(g^T)\Omega,$$

then when differentiated, we get the following map on the Lie Algebra:

$$x \mapsto -x \mapsto -x^T \mapsto -\Omega^{-1}x^T\Omega,$$

so the Lie Algebra of G is the subalgebra of the fixed points of this map. Now we see that:

$$x^T = -\Omega^{-1}x^T\Omega$$

if and only if:

$$x^T\Omega + \Omega x = 0.$$

Hence the Lie algebras have the following condition:

$$\mathfrak{sp}_{2n} = \{x \in \mathfrak{gl}_{2n} \mid x^T\Omega_{\mathrm{S}} + \Omega_{\mathrm{S}}x = 0\},$$
$$\mathfrak{o}_n = \{x \in \mathfrak{gl}_n \mid x^T\Omega_{\mathrm{O}} + \Omega_{\mathrm{O}}x = 0\}.$$

To determine (1-)accessibility in the symplectic and orthogonal algebras, one first has to determine which partitions label nilpotent orbits in $\mathfrak{sp}_{2n}$ and $\mathfrak{o}_n$, respectively. The following theorem, which we take from [7, Section 1.6], provides the answer.

Theorem 1.11.1. Let $\pi = [n_1^{(r_1)}, \ldots, n_p^{(r_p)}]$ be a partition of $\dim(V)$.

1. Assume ϕ is alternating (the symplectic algebra). Then there exists a nilpotent element in $\mathfrak{g}$ with this partition if and only if r_i is even for all odd n_i.

2. Assume ϕ is symmetric (the orthogonal algebra). Then there exists a nilpotent element in $\mathfrak{g}$ with this partition if and only if r_i is even for all even n_i.

Since all matrices in an orbit are conjugate, we can determine a standard form for nilpotent symplectic and orthogonal matrices, which we can use in this thesis to determine examples of accessibility between orbits (see Chapters 3 and 4). By conjugating, we can always assume that the cocharacter is diagonal, and that $y = \lim_\lambda x$ is in standard form, so when we check the accessibility relation of x and y, most work will be in checking the orbit of x. In the following subsections, we will determine the standard forms of the symplectic and orthogonal nilpotent matrices, so we can prove Theorem 1.11.1, and determine when a nilpotent matrix is distinguished. From now on, we fix the standard bases and bilinear forms as described in these sections.

1.11.1 The symplectic algebras

In this section we will determine the standard forms of symplectic elements, and by doing so, we will prove the first part of theorem 1.11.1. First, recall the notation of Jordan forms: let $x = J_{2n}$ be a Jordan matrix form of size $2n \times 2n$, so:

$$x = J_{2n} = \begin{pmatrix} 0 & 1 & & & \\ & 0 & \ddots & & \\ & & \ddots & \ddots & \\ & & & 0 & 1 \\ & & & & 0 \end{pmatrix}.$$

We can also denote this as a diagonal Jordan block matrix, with an extra 1-entry in position $(n, n+1)$, which is in the upper right block of x. For notation purposes, we will use a shorthand notation of the subscript in the matrix, and then indicate the location. So

$$x = \left(\begin{array}{c|c} J_n & 1_{(a)} \\ \hline & J_n \end{array}\right),$$

with $(a) = (n, n+1)$. We will use this notation of block matrices often, as most upper right blocks have zeroes almost everywhere.

To explicitly determine the standard form of a symplectic matrix x with partition π, we start with a partition of one part, then we add new parts and determine the requirements for the bigger partition to be symplectic. First, let x_1 have partition $\pi_1 = [2n]$, then if we let x_1 be of the strictly upper triangular form:

$$x_1 = \left(\begin{array}{c|c} J_n & 1_{(a)} \\ \hline & -J_n \end{array}\right),$$

with $(a) = (n, n+1)$. Then x_1 is symplectic as $x_1^T \Omega_S + \Omega_S x_1 = 0$, and we call this the standard form of x_1. For what follows it will be useful to describe the action of a nilpotent matrix on a basis by writing down a *vector sequence*, which helps read off the corresponding parts in the partition for the matrix, as well as other useful information. We denote the result of successively applying the matrix to vectors by joining the vectors with arrows. So in this example the matrix x_1 induces the following vector chain:

$$f_1 \to -f_2 \to \cdots \to (-1)^{n-1} f_n \to (-1)^n e_n \to \cdots \to (-1)^{n-1} e_1 \to 0.$$

And the vectors of the $[2n]$ part form pairs e_i, f_i such that $\phi(e_i, f_i) = -1$, so x_1 is symplectic. Next, let x_2 have partition $\pi_2 = [2n, 2m]$, then x_2 has to induce the chains:

$$
\begin{aligned}
f_1 \to -f_2 \to \cdots \to (-1)^{m-1} f_m \to & (-1)^m e_m \to \cdots \to (-1)^{m-1} e_1 \to 0,\\
h_1 \to -h_2 \to \cdots \to (-1)^{n-1} h_n \to & (-1)^n g_n \to \cdots \to (-1)^{n-1} g_1 \to 0,
\end{aligned}
$$

with vector pairings $\phi(e_i, f_i) = -1$ and $\phi(g_i, h_i) = -1$ such that both parts are symplectic. If we take:

$$
\begin{aligned}
h_1 = f_{m+1}, \ldots, & h_n = f_{m+n},\\
g_1 = e_{m+1}, \ldots, & g_n = e_{m+n}.
\end{aligned}
$$

Then we obtain the required pairings, and x_2 is of the form:

$$
x_2 = \left(\begin{array}{c|c|c|c}
J_m & & & 1_{(b)} \\ \hline
 & J_n & 1_{(a)} & \\ \hline
 & & -J_n & \\ \hline
 & & & -J_m
\end{array}\right) = \left(\begin{array}{c|c|c}
J_m & & 1_{(b)} \\ \hline
 & x_1 & \\ \hline
 & & -J_m
\end{array}\right),
$$

with $(a) = (m+n, m+n+1)$ and $(b) = (m, m+2n+1)$. As the partition $\pi_2 = [2n, 2m]$ contains the partition of x_1, we denote the partition of x_2 with a direct sum: $\pi_2 = \pi_1 \oplus [2m]$. By induction, any matrix x_p with partition $\pi_p = [2n_1, \ldots, 2n_p] = \pi_{p-1} \oplus [2n_p]$, of p distinct even parts, is symplectic in the following form:

$$
x_p = \left(\begin{array}{c|c|c}
J_{n_p} & & 1_{(a)} \\ \hline
 & x_{p-1} & \\ \hline
 & & -J_{n_p}
\end{array}\right),
$$

with $(a) = (n_p, n_p + \sum_{i=1}^{p-1}(2n_i) + 1)$. We call this the standard form of x_p. Next, suppose that x_p has a partition of p distinct parts, and x_{p+1} has a partition with a repeated part:

$$
\pi_p \oplus [n_{p+1}^{(2)}].
$$

Then x_{p+1} induces the vector chains that x_p induces, and in addition it induces the vector chains:

$$h_1 \to -h_2 \to \ldots \to (-1)^{n_{p+1}-1} h_{n_{p+1}} \to 0,$$

$$g_{n_{p+1}} \to g_{n_{p+1}-1} \to \ldots \to g_1 \to 0,$$

With pairings g_i, h_i such that $\phi(g_i, h_i) = -1$, hence the two $[n_{p+1}]$ parts are a symplectic pair; observe that n_{p+1} can be any positive integer. Then x_{p+1} is of the form:

$$x_{p+1} = \left(\begin{array}{c|c|c} J_{n_{p+1}} & & \\ \hline & x_p & \\ \hline & & -J_{n_{p+1}} \end{array} \right).$$

By induction, x_{p+q} with partition $\pi_{p+q} = \pi_p \oplus \pi_q = [2n_1, \ldots, 2n_p] \oplus [n^{(2)}_{p+1}, \ldots, n^{(2)}_{p+q}]$ is of the form:

$$x_{p+q} = \left(\begin{array}{c|c|c|c|c|c|c} J_{n_{p+q}} & & & & & & \\ \hline & \ddots & & & & & \\ \hline & & J_{n_{p+1}} & & & & \\ \hline & & & x_p & & & \\ \hline & & & & -J_{n_{p+1}} & & \\ \hline & & & & & \ddots & \\ \hline & & & & & & -J_{n_{p+q}} \end{array} \right). \tag{1.5}$$

We call this the standard form of x_{p+q}. We can now show that a nilpotent element x with a partition π does appear in the symplectic Lie algebra, if π has the form given in Theorem 1.11.1. For such a partition we can denote any odd part $[\ldots, 2n+1^{2r}, \ldots]$ occurring $2r$ times, as pairs of size $2n+1$, occurring r times: $[\ldots, 2n+1^2, \ldots 2n+1^2, \ldots]$. Hence x appears in the symplectic algebra.

We now need to show that every nilpotent element in the symplectic Lie algebra has a partition of the given form. We begin by noting that since all alternating bilinear forms are conjugate, we are free to change our basis if it helps the proof.

Now, we will can prove the following lemma:

Lemma 1.11.1. Let $x \in \mathfrak{sp}_{2n}$ have an arbitrary partition $\pi = [p_1, \ldots, p_3]$. Then any odd parts in π pair up.

The calculations in this proof are standard, but it is still worth to work them out because it shows how the pairings of odd parts occur.

Proof. Let $x \in \mathfrak{sp}_{2n}$ with partition $\pi = [p_1, \ldots, p_r]$. Then for each part p_i, we have a vector sequence induced by x:

$$e \to xe \to \cdots \to x^{p_i-1}e.$$

We may take the vectors in this chain to be basis vectors, so we obtain a decomposition of the space $V = M_1 \oplus \cdots \oplus M_r$, where each M_i is spanned by the vector chain of the i^{th} part in the partition. We have a non-degenerate form ϕ on the space V, and we can take the complement of each M_i with respect to the form. Then we have the following spaces:

$$M_i^{\perp} := \{v \in V \mid \phi(v, w) = 0 \text{ for all } w \in M_i\}.$$

Since ϕ is non-degenerate, we have that $\dim M_i + \dim M_i^{\perp} = \dim V$, and since M_i is stable under x, so is $M_i^{\perp}$. We start by considering M_1, which we may assume is the largest part of the partition π. Since M_i and $M_i^{\perp}$ are x-stable, so is $M_i \cap M_i^{\perp}$. There are now two possibilities.

1. $M_1 \cap M_1^{\perp} = \{0\}$, this means the restriction of the form to the space M_1 is also non-degenerate. So M_1 is a space carrying a non-degenerate symplectic form, hence it must be of even dimension, and the corresponding part p_1 is even.

2. $M_1 \cap M_1^{\perp}$ is non-trivial. Then there is a nonzero vector $z \in M_1 \cap M_1^{\perp}$. The basis for M_1 is $\{e, xe, \ldots, x^{p_1-1}\}$, as above. We can then write z as $z = \sum_{j=1}^{p_1-1} a_j x^j e$, with at least one nonzero a_j. The intersection $M_1 \cap M_1^{\perp}$ is x-stable, so if we let x act on z enough times, we obtain a vector ax^{p_1-1}, with $a \neq 0$, so we conclude that $x^{p_1-1} \in M_1 \cap M_1^{\perp}$. As x is symplectic, we can now find a vector e' such that $\phi(x^{p_1-1}e, e') = -1$, and we can let M_1' be spanned by $\{e', \ldots, x^{p_1-1}e'\}$. By the choice of ϕ, we have that $\phi(x^{p_1-i-1}e, x^i e') = -1$. In particular, x^{p_1-1} is nonzero, and all $x^i e'$ are independent. Hence $\dim M_1' \geq 1$, and since p_1 is the largest block size, the dimension of M_1' is exactly p_1. Since we have $\phi(x^{p_1-i-1}e, x^i e') = 1$ for each i, the restriction of ϕ to the direct sum $M_1 \oplus M_1'$ is non-degenerate.

Writing $V = M_1 \oplus M_1^{\perp}$ in the first case or $V = (M_1 \oplus M_1') \oplus (M_1 \oplus M_1')^{\perp}$ in the second case, we can reduce $\dim V$ by an even number by going to the perpendicular space in either case, so we can break V into pieces up by induction such that ϕ is non-degenerate on a piece (case 1) or pair of pieces (case 2). We have seen that in case 1, the single piece cannot be odd, so all odd parts must pair up in pieces M_i and M_i', hence odd parts in $\mathfrak{sp}_{2n}$ have even multiplicity. $\square$

We now determine the maximal tori and center of $\mathrm{Sp}_{2n}(\kappa)$. We have chosen the matrix Ω_{S} so that there is a diagonal maximal torus in each case, with restriction $g^T \Omega_{\mathrm{S}} g = \Omega_{\mathrm{S}}$. Then a diagonal matrix $g = \mathrm{diag}(g_1, g_2, \ldots, g_{2n})$ must satisfy the following property for its entries:

$$x_i = x_{2n+1-i}^{-1} \quad \forall i \leq 2n. \tag{1.6}$$

The center of GL_n is aI, with $a \in \kappa^*$, so the extra restriction in Sp_{2n} shows that $Z(\mathrm{Sp}_{2n}) = \pm I$. With this information we can now determine which nilpotent symplectic elements are distinguished. Let $x \in \mathfrak{sp}_{2n}$ and recall that it is distinguished if the only tori in $C_{Sp_{2n}}(x)$ are central. So x is distinguished if there is no non-trivial torus contained in the stabilizer. In standard form, there are two types of matrices:

1. Matrix x has distinct Jordan blocks of even size and has a partition $\pi = [2n_1, \ldots, 2n_p]$. Without loss, we may assume that of generality $\pi = [2n, 2m]$.

2. Matrix x has at least one repeated Jordan block, and has partition $\pi = [2n_1, \ldots, 2n_p, n_{p+1}^{(2)}, \ldots, n_{p+q}^{(2)}]$. Without loss of generality, we may assume that $\pi = [2n, m^{(2)}]$.

A matrix x with only distinct Jordan blocks is of the form:

$$
x = \left(\begin{array}{c|c|c|c}
J_m & & & 1_{(b)} \\
\hline
 & J_n & 1_{(a)} & \\
\hline
 & & -J_n & \\
\hline
 & & & -J_m
\end{array}\right),
$$

with $(a) = (m+n, m+n+1)$ and $(b) = (m, m+2n+1)$. Next, let $g \in T \subset \mathrm{Sp}_{2n}$, so

$$
g = \mathrm{diag}(g_1, \ldots, g_m, g_{m+1}, \ldots, g_{m+n}, g_{m+n}^{-1} \ldots, g_{m+1}^{-1}, g_m^{-1}, g_1^{-1}),
$$

and not that for $gxg^{-1} = x$, we need to preserve all Jordan blocks, so we require $g_i = g_{i+1}^{-1}$ for $1 \leq i \leq m-1$ and for $m+1 \leq i \leq m+n$. We also need to preserve the entries $1_{(a)}$ (so $g_{m+n} = g_{m+n}^{-1} = 1$) and $1_{(b)}$ (so $g_m = g_m^{-1} = 1$). This means we require all $g_i = \pm 1$, hence $g = \pm I_{2m+2n}$. We conclude that x is distinguished.

Next, let x have a repeated part, so it is of the form:

$$
x = \left(\begin{array}{c|c|c|c}
J_m & & & \\
\hline
 & J_n & 1_{(a)} & \\
\hline
 & & -J_n & \\
\hline
 & & & -J_m
\end{array}\right),
$$

with $(a) = (m+n, m+n+1)$. We again want $gxg^{-1} = x$, so all Jordan blocks need to preserved (we require $g_i = g_{i+1}^{-1}$ for $1 \leq i \leq m-1$ and $m+1 \leq i \leq m+n$), and the $1_{(a)}$ entry needs to be preserved (so $g_{m+n} = g_{m+n}^{-1}$), but there are no other entries, so g_m does not have to equal 1. Then g can be of the form $g = \mathrm{diag}(tI_m, I_{2n}, t^{-1}I_m) \neq I_{2m+2n}$, for $t \in \kappa^*$. So x is not distinguished. We conclude with the following lemma:

Lemma 1.11.2. Let $x \in \mathfrak{sp}_{2n}$, then x is distinguished if and only if its partition π has distinct even parts (and no odd parts).

1.11.2 The orthogonal algebras

Similar to the process in the symplectic algebra, we will determine the standard Jordan normal form of a nilpotent element in $\mathfrak{o}_n$. Starting with a partition of one part, let x_1 have partition $\pi_1 = [n]$, then x_1 is of the form:

$$
x_1 = \begin{pmatrix}
0 & 1 & & & & & & \\
& \ddots & \ddots & & & & & \\
& & \ddots & 1 & & & & \\
& & & \ddots & -1 & & & \\
& & & & \ddots & \ddots & & \\
& & & & & \ddots & -1 & \\
& & & & & & 0 &
\end{pmatrix}.
$$

x_1 satisfies the orthogonal equation: $x_1^T\Omega_{\mathrm{O}} + \Omega_{\mathrm{O}}x_1 = 0$ and it induces the vector chain

$$
f_1 \to -f_2 \cdots \to (-1)^{n-1}f_n \to (-1)^{n-1}v \to (-1)^{n-1}e_n \to \cdots \to (-1)^{n-1}e_1 \to 0,
$$

with orthogonal pairings: $(e_i, f_i) = 1$ for all i, and $(v, v) = 1$.

Next, let x_2 have partition $\pi_2 = [2n+1, 2m+1]$. Then x_2 has to induce vector chains:

$$
\begin{gathered}
f_1 \to -f_2 \to \cdots \to (-1)^{m-1}f_m \to (-1)^{m-1}v \to (-1)^{m-1}e_m \cdots \to (-1)^{m-1}e_1 \to 0. \\
h_1 \to -h_2 \to \cdots \to (-1)^{n-1}h_n \to (-1)^{n-1}w \to (-1)^{n-1}g_n \cdots \to (-1)^{n-1}e_1 \to 0.
\end{gathered}
$$

with orthogonal vector pairings $\phi(e_i, f_i) = 1$, $\phi(g_i, h_i) = 1$, and $\phi(v, v) = \phi(w, w) = 1$, while $\phi(v, w) = 0$. If we take

$$
\begin{aligned}
h_1 &= f_{m+1}, \ldots, h_n = f_{m+n}, \\
g_1 &= e_{m+1}, \ldots, g_n = f_{m+n},
\end{aligned}
$$

then by taking:

$$
\begin{aligned}
v &= \frac{1}{\sqrt{2}}(e_{m+n+1} + f_{m+n+1}), \\
w &= \frac{i}{\sqrt{2}}(-e_{m+n+1} + f_{m+n+1}),
\end{aligned}
$$

we get $\phi(v, v) = \phi(w, w) = 1$ and $\phi(v, w) = 0$, as required. Here, as usual, we let i denote a square root of -1 in κ. Then x_2 is of the form:

$$x_2 = \left(\begin{array}{c|c|c|c|c}
J_m & & \frac{1}{\sqrt{2}}{}_{(a)} \quad \frac{1}{\sqrt{2}}{}_{(b)} & & \\
\hline
 & J_n & \frac{i}{\sqrt{2}}{}_{(c)} \quad -\frac{i}{\sqrt{2}}{}_{(d)} & & \\
\hline
 & & & \begin{array}{c} \frac{i}{\sqrt{2}}{}_{(e)} \\ -\frac{i}{\sqrt{2}}{}_{(g)} \end{array} & \begin{array}{c} -\frac{1}{\sqrt{2}}{}_{(f)} \\ -\frac{1}{\sqrt{2}}{}_{(h)} \end{array} \\
\hline
 & & & -J_n & \\
\hline
 & & & & -J_m
\end{array}\right), \tag{1.7}$$

with

$(a) = (m, m+n+1)$	$(b) = (m, m+n+2)$
$(c) = (m+n, m+n+1)$	$(d) = (m+n, m+n+2)$
$(e) = (m+n+1, m+n+3)$	$(f) = (m+n+1, m+2n+3)$
$(g) = (m+n+2, m+n+3)$	$(h) = (m+n+2, m+2n+3)$

Note that the coefficients of v and w here involve square roots of 2 and -1 which are not always defined if κ is not closed. So this process may fail if we are not working over an algebraically closed field (e.g., if we work over $\mathbb{R}$ instead of $\mathbb{C}$), Hence it can occur that nilpotent elements of partition $[2n+1, 2m+1]$ in $\mathfrak{o}_n$ over κ cannot always be put into this form if κ is not algebraically closed.

By induction, let x_p have partition $\pi_p = [2n_1+1, \ldots, 2n_p+1]$, of all distinct parts. Then x_p is of the form:

$$x_p = \left(\begin{array}{c|c|c|c|c|c|c}
J_{n_p} & & \frac{1}{\sqrt{2}}{}_{(a)} & & \frac{1}{\sqrt{2}}{}_{(b)} & & \\
\hline
 & J_{n_p-1} & \frac{i}{\sqrt{2}}{}_{(c)} & & -\frac{i}{\sqrt{2}}{}_{(d)} & & \\
\hline
 & & & & & \frac{i}{\sqrt{2}}{}_{(e)} & \frac{1}{\sqrt{2}}{}_{(f)} \\
\hline
 & & & x_{p-2} & & & \\
\hline
 & & & & & -\frac{i}{\sqrt{2}}{}_{(g)} & \frac{1}{\sqrt{2}}{}_{(h)} \\
\hline
 & & & & & -J_{n_p-1} & \\
\hline
 & & & & & & -J_{n_p}
\end{array}\right),$$

with

$(a) = (n_{p-1}+1)$
$(b) = (n_p, n_p+n_{p-1}+\sum_{i=1}^{p-2}(2n_i+1)+2)$
$(c) = (n_p+n_{p-1}, n_p+n_{p-1}+1)$
$(d) = (n_p+n_{p-1}, n_p+n_{p-1}+\sum_{i=1}^{p-2}(2n_i+1)+2)$
$(e) = (n_p+n_{p-1}+1, n_p+n_{p-1}+\sum_{i=1}^{p-2}(2n_i+1)+3)$
$(f) = (n_p+n_{p-1}+1, n_p+\sum_{i=1}^{p-2}(2n_i+1)+2n_{p-1}+3)$
$(g) = (n_p+n_{p-1}+\sum_{i=1}^{p-2}(2n_i+1)+2, n_p+n_{p-1}+\sum_{i=1}^{p-2}(2n_i+1)+3)$
$(h) = (n_p+n_{p-1}+\sum_{i=1}^{p-2}(2n_i+1)+2, n_p+\sum_{i=1}^{p-2}(2n_i+1)+2n_{p-1}+3)$

Then x_p induces the vector chains that x_{p-2} induces, in addition to the chains:

$$\begin{aligned} f_1 \to -f_2 \to \cdots \to (-1)^{n_p-1} f_{n_p} \to (-1)^{n_p-1} v_p \\ \to (-1)^{n_p-1} e_{n_p} \to \cdots \to (-1)^{n_p-1} e_1 \to 0, \\ f_{n_{p-1}+1} \to -f_{n_{p-1}+2} \to \cdots \to (-1)^{n_{p-1}-1} f_{n_p+n_{p-1}} \to (-1)^{n_{p-1}-1} v_{p-1} \\ \to (-1)^{n_{p-1}-1} e_{n_p+n_{p-1}} \to \cdots \to (-1)^{n_{p-1}-1} e_1 \to 0. \end{aligned}$$

Where:

$$\begin{aligned} v_p &= \frac{1}{\sqrt{2}}(f_{n_p+n_{p-1}+1} + e_{n_p+n_{p-1}+1}), \\ v_{p-1} &= \frac{i}{\sqrt{2}}(-f_{n_p+n_{p-1}+1} + e_{n_p+n_{p-1}+1}). \end{aligned}$$

Then we get $\phi(v_p, v_p) = \phi(v_{p-1}, v_{p-1}) = 1$, while $\phi(v_p, v_{p-1}) = 0$, we get the standard orthogonal pairings for the e_i and f_i, with $i < n_p + n_{p-1} + 1$ and $i \neq n_p$. Next, let x_{p+1} have partition $\pi_{p+1} = [2n_1+1, \ldots, 2n_p+1, n_{p+1}^2]$, then x_{p+1} induces the vector chains of x_p, in addition to the chains:

$$\begin{aligned} f_1 \to -f_2 \to \cdots \to f_{n_p+1} \to 0, \\ e_{n_{p+1}} \to e_{n_{p+1}-1} \to \cdots \to e_1 \to 0. \end{aligned}$$

With vector pairings $\phi(e_i, f_i) = 1$ for all $i < n_{p+1}$. Then x_{p+1} is of the form:

$$x_{p+1} = \left(\begin{array}{c|c|c} J_{n_{p+1}} & & \\ \hline & x_p & \\ \hline & & -J_{n_{p+1}} \end{array}\right).$$

By induction, the standard form of x_{p+q} with partition $\pi_{p+q} = [2n_1+1, \ldots, 2n_{p+1}] \oplus [n_{p+1}^2, \ldots, n_{p+q}^2]$ is as follows:

$$x_{p+q} = \left(\begin{array}{c|c|c|c|c|c|c}
J_{n_{p+q}} & & & & & & \\ \hline
 & \ddots & & & & & \\ \hline
 & & J_{n_{p+1}} & & & & \\ \hline
 & & & x_p & & & \\ \hline
 & & & & -J_{n_{p+1}} & & \\ \hline
 & & & & & \ddots & \\ \hline
 & & & & & & -J_{n_{p+q}}
\end{array}\right).$$

Now that we have determined what a standard Jordan form for orthogonal nilpotent matrices, we will prove part 2 of Theorem 1.11.1:

Theorem. Assume ϕ is symmetric (so we consider elements in the orthogonal algebra). Then there exists a nilpotent element in $\mathfrak{g}$ with a partition $[n_1^{r_1}, \ldots, n_p^{r_p}]$ if and only if r_i is even for all even n_i.

First, it is worth noting that a basis of even dimension $2n$ can be orthogonal, as for any pair e_i and f_i, we get $\phi(e_i, f_i) = 1$ for all $i \leq n$. We will prove a matrix cannot be orthogonal if it contains an even part an odd number of times.

Proof. First, let x have partition $\pi_r = [2n_r]$, then x is of the form:

$$x = \left(\begin{array}{c|c}
J_{n_r} & 1_{(a)} \\ \hline
 & -J_{n_r}
\end{array}\right),$$

with $(a) = (n_r + 1, n_r + 1)$. Then we calculate that $x^T \Omega_{\mathrm{O}} + \Omega_{\mathrm{O}} \neq 0$ (it has an entry 2 in position $(n_r + 1, n_r + 1)$), so x is not orthogonal. We conclude that a partition consisting of a single even part cannot occur in $\mathfrak{o}_{2n}$. Next, take an arbitrary partition $\pi_{p+q+r} = [2n_1 + 1, \ldots, 2n_p + 1, n_{p+1}^{(2)}, \ldots, n_{p+q}^{(2)}, 2n_r^(2k + 1)]$. First, note that we can rewrite $[2n_r^{(2k+1)}]$ as $[2n_r^{(2k)}] \oplus [2n_r]$. Then a matrix with this partition has Jordan normal form:

$$x = J_{2n_1+1} \oplus \cdots \oplus J_{2n_p+1} \oplus J_{n_{p+1}}^{(2)} \oplus \cdots J_{n_{p+q}}^{(2)} \oplus J_{2n_r}^{(2k)} \oplus J_{2n_r}.$$

Let m be the size of matrix x. We can describe x as $\tilde{x} \oplus J_{2n_r}$, with $\tilde{x} = J_{2n_1+1} \oplus \cdots \oplus J_{2n_p+1} \oplus J_{n_{p+1}}^{(2)} \oplus \cdots \oplus J_{n_{p+q}}^{(2)}$. Then:

$$x = \left(\begin{array}{c|c|c}
J_{n_r} & & 1_{(a)} \\ \hline
 & \tilde{x} & \\ \hline
 & & -J_{n_r}
\end{array}\right),$$

with $(a) = (r, m + 1 - r)$. Then we can calculate that the entry $(m + 1 - r, m + 1 - r)$ of

$x^T\Omega_{\mathrm{O}} + \Omega_{\mathrm{O}}x \neq 0$, so x is not orthogonal. The entry (j,j) in $x^T\Omega_{\mathrm{O}} + \Omega_{\mathrm{O}}x$ is calculated as follows:

$$\begin{aligned}(x^T\Omega_{\mathrm{O}} + \Omega_{\mathrm{O}}x)_{(j,j)} &= \sum_{i=1}^{m}(x^T)_{(j,i)}\Omega_{\mathrm{O}(i,j)} + \Omega_{\mathrm{O}(j,i)}x_{(i,j)} \\ &= \sum_{i=1}^{m} x_{(i,j)}\Omega_{\mathrm{O}(i,j)} + \Omega_{\mathrm{O}(j,i)}x_{(i,j)}.\end{aligned}$$

In our case, $j = (m+1-r)$, so we get $(x^T\Omega_{\mathrm{O}} + \Omega_{\mathrm{O}}x)_{(j,j)} = \sum_{i=1}^{m} x_{(i,m+1-r)}\Omega_{\mathrm{O}(i,m+1-r)} + \Omega_{\mathrm{O}(m+1-r,i)}x_{(i,m+1-r)}$. The matrix Ω has 1-entries on the anti-diagonal, and 0-entries everywhere else, so $\Omega_{\mathrm{O}(i,m+1-r)} = 1$ if and only if $i+m+1-r = m+1$, so if and only if $i = r$. Otherwise $\Omega_{\mathrm{O}(i,m+1-r)} = 0$. Then:

$$\begin{aligned}(x^T\Omega_{\mathrm{O}} + \Omega_{\mathrm{O}}x)_{(m+1-r,m+1-r)} &= x_{(r,m+1-r)}\Omega_{\mathrm{O}(r,m+1-r)} + \Omega_{\mathrm{O}(m+1-r,r)}x_{(r,m+1-r)} \\ &= 2x_{(r,m+1-r)} \\ &= 2.\end{aligned}$$

So $x^T\Omega_{\mathrm{O}} + \Omega_{\mathrm{O}}x \neq 0$, hence x is not orthogonal. We conclude that a matrix cannot be orthogonal if an even part occurs an odd number of times. □

Finally, we determine when an orthogonal partition is distinguished. As we have chosen Ω_{O} so that there is a diagonal maximal torus in each case, again with restriction $g^T\Omega_{\mathrm{O}}g = \Omega_{\mathrm{O}}$ (restricting from GL_n). Hence a diagonal matrix $g = \mathrm{diag}(g_1, g_2, \ldots, g_n)$ must satisfy the equation:

$$x_i = x_{n-i}^{-1} \text{ for all } i \leq n,$$

similar to Equation 1.6 for the symplectic algebra. Then it is immediate that $Z(O_n) = \pm I$. Now we let $x \in \mathfrak{o}_n$, and recall that it is distinguished if each torus in $\mathrm{Stab}_{\mathrm{O}_n} = \{g \cdot x = x\}$ is contained in $Z(O_n)$. So x is distinguished if only $\pm I$ are contained in the stabilizer. The two types of matrices in standard form are:

1. Matrix x has distinct Jordan blocks of even size and has a partition $\pi = [2n_1+1, \ldots, 2n_p+1]$. Without loss of generality, we may assume that $\pi = [2n+1, 2m+1]$.
2. Matrix x has at least one repeated Jordan block, and has partition $\pi = [2n_1, \ldots, 2n_p + 1, n_{p+1}^{(2)}, \ldots, n_{p+q}^{(2)}]$. Without loss of generality, we may assume that $\pi = [2n+1, m^{(2)}]$.

Recall the standard form of a matrix with partition $\pi = [2n+1, 2m+1]$ from Equation 1.7, then it is clear that only $\pm I$ stabilize x_2, so x_2 is distinguished. By induction, any matrix with only distinct odd parts is distinguished. Conversely, a matrix with partition $\pi = [2n+1, m^{(2)}]$ can be stabilized with a matrix of the form:

$$g = \mathrm{diag}(tI_m, I_{2n+1}, t^{-1}I_m), t \in \kappa^*,$$

so a matrix with partition $[2n+1, m^{(2)}]$ is not distinguished. Hence the lemma follows:

Lemma 1.11.3. Let $x \in \mathfrak{o}_n$, then x is distinguished if and only if its partition π has distinct odd parts (and no even parts).

1.12 Symplectic and orthogonal orbits

In this section, we finish the description of the nilpotent orbits for symplectic and orthogonal groups by showing that the orbits are still labelled by partitions. Let $G = \mathrm{Sp}_{2n}$ or $G = \mathrm{O}_n$, and we look at the orbits of elements x in $\mathfrak{g}$, where $\mathfrak{g}$ is the symplectic or orthogonal algebra. The following theorem from [7, Section 1.4], helps out in the classification of nilpotent $G-$orbits in $\mathfrak{g}$:

Theorem 1.12.1. Two elements in $\mathfrak{g}$ belong to the same $G-$orbit if and only if they belong to the same $\mathrm{GL}(V)$-orbit.

So if x, y are symplectic (the same will hold for orthogonal elements) and are in the same Sp_{2n}-orbit, then they are in the same GL_{2n}-orbit, so for x and y we can find a $g_1 \in \mathrm{GL}_{2n}$ such that $y = g_1 x g_1^{-1}$, while g_1 is not symplectic. Then by the reverse process, there exists a $g \in \mathrm{Sp}_{2n}$ such that $y = gxg^{-1}$. The following process, which describes the proof of the theorem given in [7, Section 1.4, 1.5], will be used below to find the explicit change of basis matrices in some of our calculations.

First, we recall that:

$$\begin{aligned}\mathrm{Sp}_{2n} &= \{g \in \mathrm{GL}_{2n} \mid g^T \Omega_{\mathrm{S}} g = \Omega_{\mathrm{S}}\} \\ &= \{g \in \mathrm{GL}_{2n} \mid \Omega_{\mathrm{S}}^{-1} g^T \Omega_{\mathrm{S}} = g^{-1}\}.\end{aligned}$$

With this in mind, we define for any $g \in \mathrm{GL}_{2n}(\kappa)$ a new element $g^* = \Omega_{\mathrm{S}}^{-1} g^T \Omega_{\mathrm{S}}$. Then g is in the symplectic group if and only if $g^* = g^{-1}$. Then as

$$\begin{aligned}\mathfrak{sp}_{2n} &= \{x \in \mathfrak{gl}_{2n} \mid x^T \Omega_{\mathrm{S}} + \Omega_{\mathrm{S}} x = 0\} \\ &= \{x \in \mathfrak{gl}_{2n} \mid \Omega_{\mathrm{S}}^{-1} x^T \Omega_{\mathrm{S}} = -x\},\end{aligned}$$

we can see that for an arbitrary matrix x, it is in the symplectic algebra if and only if $x^* = -x$.

As x and y belong to the same orbit in GL_{2n}, we have that $y = g_1 x g_1^{-1}$ for some $g_1 \in \mathrm{GL}_{2n}(\kappa)$. Set $g_1^* = \Omega_{\mathrm{S}}^{-1} g_1^T \Omega_{\mathrm{S}}$, then:

$$\begin{aligned}y^* &= (g_1 x g_1^{-1})^* \\ &= \Omega_{\mathrm{S}}^{-1} (g_1 x g_1^{-1})^T \Omega_{\mathrm{S}} \\ &= \Omega_{\mathrm{S}}^{-1} (g_1^{-1})^T x^T g_1^T \Omega_{\mathrm{S}} \\ &= \Omega_{\mathrm{S}}^{-1} (g_1^{-1})^T \Omega_{\mathrm{S}} \Omega_{\mathrm{S}}^{-1} x^T \Omega_{\mathrm{S}} \Omega_{\mathrm{S}}^{-1} g_1^T \Omega_{\mathrm{S}} \\ &= (g_1^*)^{-1} x^* g_1^* \\ &= -(g_1^*)^{-1} x g_1^* = -y.\end{aligned}$$

So $g_1 x g_1^{-1} = (g_1^*)^{-1} x g_1^*$, or $g_1^* g_1 x = x g_1^* g_1$. We define $g_2 = g_1^* g_1$, then g_2 commutes with x. Next, we take the following Lemma (see [7, section 1.5]):

Lemma. [7, Section 1.5] Let $g \in \mathrm{GL}(V)$. There exists a polynomial $f(t) \in \kappa[t]$ such that $f(g)^2 = g$.

We will denote $h = f(g)^2$, and apply the Lemma to g_2. Then:

$$f(t) = \sum_{i=0}^{r} a_i t^i,$$

$$\text{such that } h = \sum_{i=0}^{r} a_i g_2^r.$$

We choose h because of three reasons:

1. Since $h^2 = g_2 \in \mathrm{GL}(V)$, and as $\det(h^2) = \det(g_2) \neq 0$, it follows that $\det(h) \neq 0$, hence $h \in \mathrm{GL}(V)$.

2. Because $xg_2 = g_2x$, it follows that:

$$\begin{aligned} xh &= x(a_0I + a_1g_2 + \ldots + a_rg^r) \\ &= (a_0I + a_1g_2 + \ldots + a_rg^r)x \\ &= hx. \end{aligned}$$

So h commutes with x.

3.

$$\begin{aligned} h^* &= (a_0I + \ldots + a_rg_2^r)^* \\ &= a_0I^* + \ldots + a_r(g_2^*)^r \\ &= h \end{aligned}$$

Then if we take $g = g_1h^{-1}$, we get

$$\begin{aligned} g^* = (g_1h^{-1})^* &= (h^*)^{-1}g_1^* \\ &= h^{-1}g_1^*g_1g_1^{-1} \\ &= h^{-1}g_2g_1^{-1} \\ &= h^{-1}h^2g_1^{-1} \\ &= hg_1^{-1} = g^{-1}. \end{aligned}$$

So as $g^* = g^{-1}$, we conclude that g is symplectic. Next,

$$\begin{aligned} gXg^{-1} &= g_1h^{-1}Xhg_1^{-1} \\ &= g_1h^{-1}hXg_1^{-1} \\ &= g_1Xg_1^{-1} \\ &= y. \end{aligned}$$

Thus we have found $g \in \mathrm{Sp}_{2n}(\kappa)$ such that $gxg^{-1} = y$, as required. We will be using this process in two examples in the symplectic (Section 3.1.1) and orthogonal (Section 4.1.1) results.

Chapter 2

Results in the general linear algebra

In this chapter, the accessibility between orbits of nilpotent elements in the general linear algebra will be discussed. We start with an illustrative example, before moving on to the general case. We first show how to move between two orbits where the partitions differ by a single move between two parts. After this, we show how to combine these moves to prove accessibility between arbitrary orbits.

2.1 An example

We start with an example with matrices in $\mathfrak{gl}_4$, showing that the nilpotent orbit of partition $[2,2]$ is one-accessible from the orbit of partition $[3,1]$. This example illustrates some of the important ideas which are needed in the general case. In particular, we see that in order to show accessibility, it is necessary to first conjugate one of the elements away from its standard form. Let $x' \in \mathfrak{gl}_4$ be

$$x' = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

then x' is in the nilpotent orbit of the partition $[3,1]$, which we will denote as $x' \in \mathcal{O}([3,1])$. This will be shown in two ways. For a direct argument, one can check that $(x')^2 \neq 0$, but $(x')^3 = 0$, so the Jordan normal form has a block of size 3, and the other block can only be of size 1. So $x' \in \mathcal{O}([3,1])$. For the general case, it will be more useful to think in terms of basis elements. Let the standard basis be $\{e_1, e_2, e_3, e_4\}$. Then the kernel of x' is spanned by e_1 and $e_2 - e_3$. Moreover, $x'e_4 = e_3$ and $x'e_3 = e_1$, so if we change basis to $\{e_1, e_3, e_4, e_2 - e_3\}$, we get that

$$x' \sim \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} = x.$$

Furthermore, the basis change corresponds to conjugating x with $g \in \mathrm{GL}_4$, with

$$g = (e_1|e_3|e_4|e_2 - e_3) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

so

$$x' = g \cdot x = gxg^{-1},$$

which finishes the proof. Next, define a cocharacter λ by $\lambda(t) = \mathrm{diag}(t, t, 1, 1)$ for $t \in \kappa^*$. Then

$$\lim_{t \to 0} \lambda(t) \cdot x' = \lim_{\lambda} x' = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

which is the standard form of matrices in the orbit $\mathcal{O}([2, 2])$, so $\mathcal{O}([2, 2])$ is 1-accessible from $\mathcal{O}([3, 1])$.

2.2 Matrices of any size

In this section, we will show that any matrix with a partition of the form $\pi = [r, s]$ is 1-accessible from a matrix with a partition of the form $\pi_2 = [r + k, s - k]$, for $k \in \mathbb{Z}_{\geq 0}$. Recall that we denote matrices with their Jordan blocks, that is, if x is a Jordan matrix of size $(n \times n)$, then $x = J_n$, and if it is a Jordan normal form of multiple Jordan blocks, we denote $x = J_{n_1} \oplus J_{n_2} \oplus \cdots \oplus J_{n_p}$.

Next, suppose that x' consists of two Jordan blocks on the diagonal (e.g two Jordan blocks of size 3), and an upper right block with one nonzero entry (e.g. in the second row and the fourth column). Then we denote x' as follows, e.g:

$$x' = \left(\begin{array}{c|c} J_3 & t_{(a)} \\ \hline & J_3 \end{array}\right) = \left(\begin{array}{ccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & t & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

where the subscript at the nonzero entry in the upper right block, in this case $(a) = (2, 4)$.

Theorem 2.2.1. Let $\pi_1 = [\ldots, r + k, s - k, \ldots]$ and $\pi_2 = [\ldots, r, s, \ldots]$ be two partitions, which only differ in the positions of the entries shown, so π_1 dominates π_2. Then $\mathcal{O}(\pi_2)$ is 1-accessible from $\mathcal{O}(\pi_1)$.

Proof. First, note that $r \geq s$. If $k = 1$, we show that the orbit $\mathcal{O}([r, s])$, is one-accessible from the orbit $\mathcal{O}([r + 1, s - 1])$ (recall that $[r + 1, s - 1]$ dominates $[r, s]$, see equation 1.3). Let $n = r + s$, and let $x \in \mathcal{O}([r + 1, s - 1])$ be the matrix in standard form, so:

$$x = J_{r+1} \oplus J_{s-1}.$$

Next, let

$$x' = \left(\begin{array}{c|c} J_s & 1_{(a)} \\ \hline & J_r \end{array}\right),$$

with $(a) = (1, s+1)$. Recall that we can describe the action of a nilpotent matrix by writing down a vector sequence, which helps read off the corresponding parts in the partition of the matrix. We compare vector sequences of x and x', so we check how the actions of x and x' compare, and that they are in the same orbit. We want to check that x and x' are in the orbit $\mathcal{O}([r+1, s-1])$, so there are two vector sequences to compare. The sequence of x and x', of size $s-1$ is as follows (starting with the first vector of the sequence):

$$\begin{aligned} xe_{r+s} &= e_{r+s-1} & x'(e_s - e_{s+s-1}) &= e_{s-1} - e_{s+s-2}, \\ xe_{r+s-1} &= e_{r+s-2} & x'(e_{s-1} - e_{s+s-2}) &= e_{s-2} - e_{s+s-3}, \\ &\vdots & &\vdots \\ xe_{r+2} &= 0 & x'(e_{s-(s-2)} - e_{s+s-1-(s-2)}) &= 0. \end{aligned}$$

Next, the sequence of size $r+1$ is as follows:

$$\begin{aligned} xe_{r+1} &= e_r & x'e_{r+s} &= e_{r+s-1}, \\ xe_r &= e_{r-1} & x'e_{r+s-1} &= e_{r+s-2}, \\ &\vdots & &\vdots \\ xe_2 &= e_1 & x'e_{s+1} &= e_1, \\ xe_1 &= 0 & x'e_1 &= 0. \end{aligned}$$

This suggests using a base change as follows:

$$\begin{aligned} e_{r+s} &\mapsto e_s - e_{s+s-1}, \\ e_{r+s-1} &\mapsto e_{s-1} - e_{s+s-2}, \\ &\vdots \\ e_{r+2} &\mapsto e_{s-(s-2)} - e_{s+s-1-(s-2)} = e_2 - e_{s+1}. \end{aligned}$$

$$\begin{aligned} e_{r+1} &\mapsto e_{r+s}, \\ e_r &\mapsto e_{r+s-1}, \\ &\vdots \\ e_2 &\mapsto e_{s+1}, \\ e_1 &\mapsto e_1. \end{aligned}$$

We describe the matrix g associated with this base change by its column vectors:

$$g = (e_1, e_{s+1}, \dots, e_{s+r}, e_2 - e_{s+1}, \dots, e_s - e_{s+s-1})$$

And $g \cdot x = gxg^{-1} = x'$, so x' is in the orbit of $x \in \mathcal{O}([r+1, s-1])$. Finally, let λ be defined by

$$\lambda = \operatorname{diag}(tI_s, I_r), \text{ for } t \in \kappa^*.$$

Then $\lim_\lambda x' \in \mathcal{O}([r,s])$, so $\mathcal{O}([r,s])$ is 1-accessible from $\mathcal{O}([r+1,s-1])$. Iterating this process shows that $\mathcal{O}([r,s])$ is accessible from $\mathcal{O}([r+k,s-k])$. We now proceed with the proof of the stronger result, that $\mathcal{O}([r,s])$ is 1-accessible from $\mathcal{O}([r+k,s-k])$. Let $x \in \mathcal{O}([r+k,s-k])$ be in the standard form, that is $x = J_{r+k} \oplus J_{s-k}$, and let

$$x' = \left(\begin{array}{c|c} J_s & 1_{(a)} \\ \hline & J_r \end{array}\right),$$

with $(a) = (k, s+1)$. Then if

$$\lambda = \operatorname{diag}(tI_s, I_r), \text{ for } t \in \kappa^*$$

it follows that $\lim_\lambda \cdot x' \in \mathcal{O}([r,s])$. Now we will prove that x and x' are conjugate: as with the case of $x, x' \in \mathcal{O}([r+1,s-1])$, we compare vector sequences. There will again be two vector sequences, one of size $s-k$:

$$\begin{aligned}
xe_{r+s} &= e_{r+s-1} & x'(e_s - e_{s+s-k}) &= e_{s-1} - e_{s+s-k-1}, \\
xe_{r+s-1} &= e_{r+s-2} & x'(e_{s-1} - e_{s+s-k-1}) &= e_{s-2} - e_{s+s-k-2}, \\
&\vdots & &\vdots \\
xe_{r+k+1} &= 0 & x'(e_{s-(s-k)+1)} - e_{s+s-k-(s-k)+1}) &= x'(e_{k+1} - e_{s+1}) = 0.
\end{aligned}$$

And a sequence of size $r+k$.

$$\begin{aligned}
xe_{r+k} &= e_{r+k-1} & x'e_{r+s} &= e_{r+s-1}, \\
xe_{r+k-1} &= e_{r-k-2} & x'e_{r+s-1} &= e_{r+s-2}, \\
&\vdots & &\vdots \\
xe_{k+1} &= e_k & x'e_{s+1} &= e_k, \\
xe_k &= e_{k-1} & x'e_k &= e_{k-1}, \\
&\vdots & &\vdots \\
xe_2 &= e_1 & x'e_2 &= e_1, \\
xe_1 &= 0 & x'e_1 &= 0.
\end{aligned}$$

Then the base change is:

$$
\begin{aligned}
e_{r+s} &\mapsto e_s - e_{s+s-k}, \\
e_{r+s-1} &\mapsto e_{s-1} - e_{s+s-k-1}, \\
&\vdots \\
e_{r+k+1} &\mapsto e_{k+1} - e_{s+1}, \\
e_{r+k} &\mapsto e_{r+s}, \\
xe_{r+k-1} &\mapsto e_{r+s-1}, \\
&\vdots \\
e_{k+1} &\mapsto e_{s+1}, \\
e_k &\mapsto e_k, \\
e_{k-1} &\mapsto e_{k-1}, \\
&\vdots \\
e_1 &\mapsto e_1.
\end{aligned}
$$

We describe the matrix g associated with this base change by writing down its column vectors:

$$g = (e_1, \dots, e_k, e_{s+1}, \dots, e_{s+r}, e_{k+1} - e_{s+1}, \dots, e_s - e_{s+s-k}).$$

Then, it can be checked that $g \cdot x = x'$, as intended. Finally, let

$$\lambda = \mathrm{diag}(tI_s, I_r),$$

so that $\lim_\lambda \lambda \cdot x' \in \mathcal{O}([r, s])$, finishing the proof. $\square$

With Theorem 2.2.1, we have shown that if x' is of the form

$$x' = \left(\begin{array}{c|c} J_s & 1_{(a)} \\ \hline & J_r \end{array}\right),$$

with $(a) = (k, s+1)$, and $k \leq s$, then x' induces a vector sequence:

$$x : e_{r+s} \mapsto e_{r+s-1} \mapsto \dots \mapsto e_{s+1} \mapsto e_k \mapsto \dots \mapsto e_1 \mapsto 0.$$

So $x \in \mathcal{O}([r+k, r-k])$. In this matrix, there are two Jordan blocks (of size J_s and J_r, in that order, with $s \leq r$) on the diagonal, an there is a nonzero entry in the upper-right block. We will now prove two lemmas that determine the orbit of x' if it has multiple nonzero entries in an arbitrary j^{th} column, with $j > s$.

Lemma 2.2.1. Let x' be of the form:

$$x' = \left(\begin{array}{c|c} J_s & \begin{matrix} 1_{(1,s+1)} \\ \vdots \\ 1_{(k,s+1)} \end{matrix} \\ \hline & J_r \end{array}\right).$$

such that it has two Jordan blocks of sizes s and r, with $r \geq s$, and that multiple 1-entries in the first s rows of the $(s+1)^{\text{th}}$ column. Then x' is in the orbit of $\mathcal{O}([r+k, s-k])$, where k is the row of the last nonzero entry in the $(s+1)^{\text{th}}$ column.

Proof. Suppose that x' has two Jordan blocks, of size J_s and J_r on the diagonal $(s \leq r)$, and two $1-$entries, in positions $(k, s+1)$ and $(k+i, s+1)$. Then x' induces a vector sequence:

$$x' : e_{r+s} \mapsto e_{r+s+1} \mapsto \ldots \mapsto e_{s+1} \mapsto e_{k+i} + e_k \mapsto \ldots e_i + e_1 \mapsto e_{i-1} + 0 \mapsto e_{i-2} \mapsto e_1 \mapsto 0.$$

At step r, the entries $1_{(k,s+1)}$ and $1_{(k+i,s+1)}$ are picked. The entry $1_{(k,s+1)}$ is killed off k steps later, but the entry $1_{(k+i,s+1)}$ is killed off $k+i$ steps later. So the vector sequence has a length $r+k+i$ and the sequence $e_k \mapsto \ldots \mapsto e_1 \mapsto 0$ is entirely contained in it. So we conclude that the last nonzero entry in the $(s+1)^{\text{th}}$ column determines the orbit of x', hence $x' \in \mathcal{O}([r+k+i, s-k-i])$. $\square$

Lemma 2.2.2. Let x'_1 be of the form:

$$x'_1 = \left(\begin{array}{c|c} J_s & a_{(k,s+1)} \\ \hline & J_r \end{array}\right),$$

such that it has two Jordan blocks of sizes s and r, with $r \geq s$, and that it has one nonzero entry in the top right block, in position $(k, s+1)$, and let x'_2 be of the form

$$x'_2 = \left(\begin{array}{c|c} J_s & \begin{matrix} v_{(1,j)} \\ \vdots \\ v_{(s,j)} \end{matrix} \\ \hline & J_r \end{array}\right),$$

such that it has Jordan blocks of sizes s and r, with $r \geq s$, and nonzero entries $v_{(1,j)}, \ldots, v_{(s,j)}$, in the first s rows of the j^{th} column. Let $v_{(t,j)}$, with $t \leq s$ be the last nonzero entry in the first s rows of the j^{th} column. Then x'_1 and x'_2 are in the same orbit if $(k, s+1) = (t-m, j-m)$ for some m.

Proof. First, we note that x'_1 and x'_2 are strictly upper triangular, and that x'_1 has one nonzero entry in the $s+1^{\text{th}}$ column. If x'_1 has more nonzero entries in this column, we apply lemma 2.2.1 and look at the last nonzero entry. Next, in matrix x'_2, we look closely at the first s rows of the j^{th} column of x'_2, which we will denote by a vector v. Then v is of the form:

$$v = \begin{pmatrix} v_{1,j} \\ \vdots \\ v_{t,j} \\ 0 \\ \vdots \end{pmatrix}.$$

$v_{(t,j)}$ is the last nonzero entry in the first s rows of the j^{th} column. As $j \geq s+1$, there another nonzero entry, in the J_r block, say in row i with $i = j-1$.

Let u be the unipotent matrix with 1s down the diagonal and $-v_t$ in position (t,i), so $u \in R_u(P_\lambda)$ and $\lim_\lambda u = 1$ (see Section 1.7). Then:

$$u(e_p) = \begin{cases} e_p - v_t e_t & \text{if } p = i, \\ e_p & \text{otherwise.} \end{cases} \tag{2.1}$$

$$u^{-1}(e_p) = \begin{cases} e_p + v_t e_t & \text{if } p = i, \\ e_p & \text{otherwise.} \end{cases} \tag{2.2}$$

Then

$$ux_2'u^{-1}(e_i) = ux_2'(e_i + v_t e_t) = ux_2'(e_i) + u(v_t e_{t-1}),$$

by the equations and the properties of x_2' (we know $x_2' : e_a \to e_{a-1}$ for $2 \leq a \leq s$, in particular $x_2'(e_t) = e_{t-1}$). Without having to detail the actions of x_2' for e_b when $b > s$, note that it is upper triangular, so $ux_2'(e_i) = x_2'(e_i)$. Therefore conjugating x_2' with u adds v_t in position $(t-1, j-1)$:

$$ux_2'u^{-1}(e_i) = X(e_i) + v_t e_{t-1},$$

and

$$ux_2'u(e_j) = ux(e_j + \sum_{l=1}^{t} v_l e_l) = u(e_i - v_t e_t + \sum_{l=1}^{t} v_l e_l) = e_i + \sum_{l=1}^{t-1} v_l e_l,$$

so the j^{th} column of $ux_2'u^{-1}$ is the j^{th} column of x_2' with the v_t entry removed (the other entries are left the same). Hence conjugating with u takes v_t from position (t,j) and puts it in position $(t-1, j-1)$, i.e. it moves up its superdiagonal. By repeating the process of conjugating with unipotent matrices, we get that for some $m \in \mathbb{N}$:

$$u_m \cdot \dots \cdot u_1 \cdot x_2'$$

has v_t in position $(t-m, j-m) = (k, s+1)$. With the same process of conjugating with unipotent matrices, we can also take other all $v_{s,j}$-entries (with $s < t$) out of the j^{th} column, an add their values to certain $(l, s+1)$ entries in the $(s+1)^{\text{th}}$ column. Since $s < t$, we also have that $l < k$, and we can apply Lemma 2.2.1 to see that so x_2' is in the same orbit as $x_1' : \mathcal{O}([r+k, s-k])$. $\square$

We can now describe in detail what orbit a matrix x' is in. Let:

$$x' = \left(\begin{array}{c|cccc} & x_{(1,1+s)} & x_{(1,2+s)} & \cdots & x_{(1,r+s)} \\ J_s & x_{(2,1+s)} & \ddots & & \vdots \\ & \vdots & & & \vdots \\ & x_{(s,1+s)} & \cdots & \cdots & x_{(s,r+s)} \\ \hline & & & & \\ & & J_r & & \\ & & & & \end{array}\right),$$

and suppose that $x_{i,j}$ is the last nonzero entry in the first s rows of column j. Then x' is in the orbit $\mathcal{O}([r+k, s-k])$, with the value of k denoted in the (i,j)-spot:

	$s+1$	$s+2$	$\dots$	$2s$	$2s+1$	$\dots$	r
1	1	0	$\dots$	0	0	$\dots$	0
2	2	1					$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\ddots$				$\vdots$
s	s	$s-1$	$\dots$	1	0	$\dots$	0

We note that block order does matter with the following corollary.

Lemma 2.2.3. Let y be of the form:

$$y' = \left(\begin{array}{c|c} J_r & \\ & y_{(r,r+s+1-k)} \\ \hline & \\ & J_s \end{array}\right)$$

for some $k \leq s$. Then $y \in \mathcal{O}([r+k, s-k])$.

Proof. The matrix y' has two Jordan blocks of size r and s on the diagonal, and a nonzero entry in the r^{th} row of the upper right block, so it starts a vector sequence of length $[r+k]$ by acting on e_{r+s} :

$$\begin{aligned} y' : e_{r+s} &\mapsto e_{r+s-1} \mapsto \cdots \mapsto e_{r+s+1-k} \\ &\mapsto e_{r+s-k} + e_r \mapsto \cdots \mapsto e_{r+1} + e_{r-(s-k)+1} \\ &\mapsto e_{r-(s-k)} \mapsto \cdots \mapsto e_1 \mapsto 0, \end{aligned}$$

hence $y' \in \mathcal{O}([r+k, s-k])$. $\square$

Remark: Similar to the case of multiple nonzero values in the $r+1^{\text{th}}$ column in x', if there are two nonzero entries in the r^{th} row of y', $y'_{r,r+s+1-k_1}$ and $y'_{r,r+s+1-k_2}$ with $k_1 > k_2$, then $y' \in \mathcal{O}([r+k_1, s-k_1])$.

To determine the orbit when the y' is in an arbitrary position, we adapt lemma 2.2.2.

Lemma 2.2.4. Let y'_1 be of the form:

$$
y_1' = \left(\begin{array}{c|c} J_r & \\ & y_{(r,r+s+1-k)} \\ \hline & J_s \end{array} \right),
$$

and let y_2' be of the form

$$
y' = \left(\begin{array}{c|c} J_r & y_{(t,j)} \\ \hline & J_s \end{array} \right),
$$

with $t \leq r$, and $j > r$. Then y_2' is in $\mathcal{O}([r+k, s-k])$ if $(t+m, j+m) = (r, r+s+1-k)$ for some $m \in \mathbb{Z} \geq \not{\vdash}$.

Proof. Again, denote by v the first r rows of column j, then $v_t = y_{t,j}$ is the last nonzero entry of v, and in column j there is exactly one nonzero entry $y_{i,j}$ with $i > t$ (here $i = j-1$). As before, unipotent matrices can remove the entry in position (t, j) in exchange for gaining one in position $(t-1, j-1)$. Here the process is reversed: let u be a unipotent matrix which is the identity with v_t in position $(t+1, j+1)$, then u^{-1} is the identity with $-v_t$ in the $(t+1, j+1)$ position and $uy'u^{-1}$ is y', but with v_t in position (t, j) replaced to position $(t+1, j+1)$. By repeating the process, we get that:

$$
u_1 \cdot u_2 \cdot \cdots \cdot u_m \cdot y'
$$

has v_t in position $(t+m, j+m) = (r, r+s+1-k)$ for some m. We conclude that y' is in the orbit of $\mathcal{O}([r+k, s-k])$. $\square$

2.3 Multiple parts

In this section, we will determine when moves of multiple parts are possible using cocharacters. In the dominance order, we can identify two moves:

1. Multiple parts give pieces to one part, e.g. $[r_1 + k_1, r_2 + k_2, s - k_1 - k_2] \to [r_1, r_2, s]$.
2. Multiple parts take pieces from one part, e.g. $[r + k_1 + k_2, s_1 - k_1, s_2 - k_2] \to [r, s_1, s_2]$.

Theorem 2.3.1. Let $\pi_1 = [\ldots, r_1+k_1, \ldots, r_p+k_p, s-\sum_{i=1}^{p} k_i, \ldots]$ and $\pi_2 = [\ldots, r_1, \ldots, r_p, s_2, \ldots]$. Then $\mathcal{O}(\pi_2)$ is 1-accessible from $\mathcal{O}(\pi_1)$.

Proof. Since the general proof for this result is technically quite involved and might be difficult to follow, we begin with the special case that $p = 2$. That is, we show that $\mathcal{O}([r_1, r_2, s])$ is one-accessible from $\mathcal{O}([r_1 + k_1, r_2 + k_2, s - k_1 - k_2])$.

Let:

$$
x = J_{r_1+k_1} \oplus J_{r_2+k_2} \oplus J_{s-k_1-k_2},
$$

and let

$$x' = \left(\begin{array}{c|c|c} J_s & 1_{(a_1)} & 1_{(b_1)} \\ \hline & J_{r_1} & -1_{(b_2)} \\ \hline & & J_{r_2} \end{array}\right),$$

with $(a_1) = (k_1, s+1), (b_1) = (k_1 + k_2, s + r_1 + 1), (b_2) = (s + k_2, s + r_1 + 1)$. Then the vector sequences are as follows, starting with the part of size $r_1 + k_1$:

$$\begin{aligned}
xe_{r_1+k_1} &= e_{r_1+k_1-1} & x'e_{r_1+s} &= e_{r_1+s-1}, \\
xe_{r_1+k_1-1} &= e_{r_1-k_1-2} & x'e_{r_1+s-1} &= e_{r_1+s-2}, \\
&\vdots & &\vdots \\
xe_{k_1+1} &= e_{k_1} & x'e_{s_1+1} &= e_{k_1}, \\
xe_{k_1} &= e_{k_1-1} & x'e_{k_1} &= e_{k_1-1}, \\
&\vdots & &\vdots \\
xe_2 &= e_1 & x'e_2 &= e_1, \\
xe_1 &= 0 & x'e_1 &= 0.
\end{aligned}$$

Next, the part of size $r_2 + k_2$:

$$\begin{aligned}
xe_{r_1+k_1+r_2+k_2} &= e_{r_1+k_1+r_2+k_2-1} & x'e_{r_1+r_2+s} &= e_{r_1+r_2+s-1}, \\
xe_{r_1+r_2+s-1} &= e_{r_1+r_2+s-2} & x'e_{r_1+r_2-s-1} &= e_{r_1+r_2+s-2}, \\
&\vdots & &\vdots \\
xe_{r_1+k_1+r_2+1} &= e_{r_1+k_1+r_2} & x'e_{r_1+s+1} &= e_{k_1+k_2} - e_{s+k_2}, \\
xe_{r_1+k_1+r_2} &= e_{r_1+k_1+r_2-1} & x'(e_{k_1+k_2} - e_{s+k_2}) &= e_{k_1+k_2-1} - e_{s+k_2-1}, \\
&\vdots & &\vdots \\
xe_{r_1+k_1+1} &= 0 & x'(e_{k_1+1} - e_{s+1}) &= 0.
\end{aligned}$$

Finally, the part of size $s - k_1 - k_2$:

$$\begin{aligned} xe_{r_1+r_2+s} &= e_{r_1+r_2+s-1} & x'(e_s - e_{s+k_1} - e_{s+r_1+k_2}) &= e_{s-1} - e_{s-k_1-1} - e_{s+r_1+k_2-1}, \\ &\vdots & \vdots \\ xe_{r+1+k_1+r_2+k_2+1} &= 0 & x'(e_{s-s+k_1+k_2+1} &- e_{s+k_1-s+k_1+k_2+1} - e_{s+r_1+k_2-s+k_1+k_2+1}) \\ & & &= x'(e_{k_1+k_2+1} - e_{2k_1+k_2+1} - e_{r_1+k_1+2k_2+1}) = 0. \end{aligned}$$

So we have the following base change, for the $r_1 + k_1$ part:

$$\begin{aligned} e_1 &\mapsto e_1, \\ e_2 &\mapsto e_2, \\ &\vdots \\ e_{k_1} &\mapsto e_{k_1}, \\ e_{k_1+1} &\mapsto e_{s+1}, \\ &\vdots \\ e_{r_1+k_1} &\mapsto e_{r_1+s}. \end{aligned}$$

For the $r_2 + k_2$ part:

$$\begin{aligned} e_{r_1+k_1+1} &\mapsto e_{k_1+1} - e_{s+1}, \\ e_{r_1+k_1+2} &\mapsto e_{k_1+2} - e_{s+2}, \\ &\vdots \\ e_{r_1+k_1+k_2} &\mapsto e_{k_1+k_2} - e_{s+k_2}, \\ e_{r_1+k_1+k_2+1} &\mapsto e_{r_1+s+1}, \\ &\vdots \\ e_{r+1+k+1+r_2+k_2} &\mapsto e_{r_1+r_2+s}. \end{aligned}$$

For the $s - k_1 - k_2$ part:

$$\begin{aligned} e_{r+1+k_1+r_2+k_2+1} &\mapsto e_{k_1+k_2+1} - e_{2k_1+k_2+1} - e_{r_1+k_1+2k_2+1}, \\ &\vdots \\ e_{r_1+r_2+s} &\mapsto e_s - e_{s+k_1} - e_{s+r_1+k_2}. \end{aligned}$$

So we have shown that x' is in the orbit of x. With the appropriate λ it follows that $\lim_\lambda x = y$ with $y \in \mathcal{O}([r_1, r_2, s])$, and in the standard form. So $\mathcal{O}([r_1, r_2, s])$ is 1-accessible from $\mathcal{O}([r_1 + k_1, r_2 + k_2, s - k_1 - k_2])$.

By induction, we can now prove that $\mathcal{O}([r_1, r_2, \ldots, r_p, s])$ is 1-accessible from $\mathcal{O}([r_1 + k_1, r_2 + k_2, \ldots, r_p + k_p, s - \sum_{i=1}^{p} k_i])$.

Let

$$x = J_{r_1+k_1} \oplus J_{r_2+k_2} \oplus \cdots \oplus J_{r_p+k_p} \oplus J_{s-\sum_{i=1}^{p} k_i}.$$

And let

$$x' = \left(\begin{array}{c|c|c|c|c} J_s & 1_{(a_1)} & 1_{(b_1)} & 1_{(c_1)} & 1_{(d_1)} \\ \hline & J_{r_1} & -1_{(b_2)} & -1_{(c_2)} & -1_{(d_2)} \\ \hline & & J_{r_2} & \vdots & -1_{(d_3)} \\ \hline & & & \ddots & \vdots \\ \hline & & & & J_{r_p} \end{array}\right),$$

where

$(a_1) = (k_1, s+1)$	$(d_1) = ((\sum_{i=1}^{p} k_i), s + (\sum_{i=1}^{p-1} r_i) + 1)$
$(b_1) = (k_1 + k_2, s + r_1 + 1)$	$(d_2) = (s + (\sum_{i=2}^{p} k_i), s + (\sum_{i=1}^{p-1} r_i) + 1)$
$(b_2) = (s + k_2, s + r_1 + 1)$	$(d_3) = (s + r_1 + (\sum_{i=3}^{p} k_i), s + (\sum_{i=1}^{p-1} r_i) + 1)$
$(c_1) = ((\sum_{i=1}^{3} k_i), s + (\sum_{i=1}^{2} r_i) + 1)$	
$(c_2) = (s + r_1 + k_3, s + r_1 + r_2 + 1)$	

Then the base change is (blocks are separated by a $'|'$):

$$\begin{aligned}
&\{e_1, \ldots, e_{k_1}, e_{s+1}, \ldots, e_{s+r_1} | \\
&\ e_{k_1+1} - e_{s+1}, \ldots, e_{k_1+k_2} - e_{s+k_2}, e_{s+r_1+1}, \ldots, e_{s+r_1+r_2} | \\
&\ e_{k_1+k_2+1} - e_{s+k_2+1} - e_{s+r_1+1}, \ldots, e_{k_1+k_2+k_3} - e_{s+k_2+k_3} - e_{s+r_1+k_3}, e_{s+r_1+r_2+1}, \ldots, e_{s+r_1+r_2+r_3} | \\
&\ \vdots \\
&\ e_{(\sum_{i=1}^{p-1} k_i)+1} - e_{s+(\sum_{i=2}^{p-1} k_i)+1} - \cdots - e_{s+(\sum_{i=1}^{m-1} r_i)+(\sum_{i=m+1}^{p-1} k_i)+1} - \cdots - e_{s+(\sum_{i=1}^{p-2} r_i)+1}, \ldots, \\
&\ e_{(\sum_{i=1}^{p} k_i)} - e_{s+(\sum_{i=2}^{p} k_i)} - \cdots - e_{s+(\sum_{i=1}^{m-1} r_i)+(\sum_{i=m+1}^{p} k_i)} - \cdots - e_{s+(\sum_{i=1}^{p-2} r_i)+k_p}, \\
&\ e_{s+(\sum_{i=1}^{p-1} r_i)+1}, \ldots, e_{s+(\sum_{i=1}^{p} r_i)} | \\
&\ e_{(\sum_{i=1}^{p} k_i)+1} - e_{s+(\sum_{i=2}^{p} k_i)+1} - \cdots - e_{s+(\sum_{i=1}^{m-1} r_i)+(\sum_{i=m+1}^{p} k_i)+1} - \cdots - e_{s+(\sum_{i=1}^{p-1} r_i)+1}, \ldots, \\
&\ e_s - e_{s+(\sum_{i=2}^{p} k_i)+s-(\sum_{i=1}^{p} k_i)} - e_{s+(\sum_{i=1}^{m-1} r_i)+(\sum_{i=m+1}^{p} k_i)+s-(\sum_{i=1}^{p} k_i)} - \cdots \\
&\qquad\qquad\qquad\qquad \cdots - e_{s+(\sum_{i=1}^{p-1} r_i)+s-(\sum_{i=1}^{p} k_i)}\}
\end{aligned} \tag{2.3}$$

And, as before the matrix g consists of exactly the above base change vectors, in that order. Then $g \cdot x = x'$. Next, let

$$\lambda = \mathrm{diag}(t^p I_s, t^{p-1} I_{r_1}, \ldots, I_{r_p}),$$

so that $\lim_\lambda \cdot x' \in \mathcal{O}([r_1, r_2, \ldots, r_p, s])$, finishing the proof. $\square$

Theorem 2.3.2. Let $\pi_1 = [\ldots, r + \sum_{i=1}^{q} k_i, s_1 - k_1, \ldots, s_q - k_q, \ldots]$ and let $\pi_2 = [\ldots, r, s_1, \ldots, s_q, \ldots]$. Then $\mathcal{O}(\pi_2)$ is 1-accessible from $\mathcal{O}(\pi_1)$.

Proof. Again, we first analyse the case $q = 2$, in order to illustrate the general case with a more traceable one. Let $\pi_1 = [r + k_1 + k_2, s_1 - k_1, s_2 - k_2]$ and let $\pi_2 = [r, s_1, s_2]$. Then

$$x = J_{r+k_1+k_2} \oplus J_{s_1-k_1} \oplus J_{s_2-k_2},$$

and

$$x' = \left(\begin{array}{c|c|c} J_{s_1} & 1_{(a)} & \\ \hline & J_{s_2} & 1_{(b)} \\ \hline & & J_r \end{array}\right),$$

with $(a) = (k_1, s_1 + 1)$ and $(b) = (s_1 + k_2, s_1 + s_2 + 1)$. Again, we compare vector sequences, first the block of size $r + k_1 + k_2$.

$$\begin{aligned} xe_{r+k_1+k_2} &= e_{r+k_1+k_2-1} & x'e_{s_1+s_2+r} &= e_{s_1+s_2+r-1} \\ xe_{r+k_1+k_2-1} &= e_{r+k_1+k_2-2} & x'e_{s_1+s_2+r-1} &= e_{s_1+s_2+r-2} \\ &\vdots & &\vdots \\ xe_{k_1+k_2+1} &= e_{k_1+k_2} & x'e_{s_1+s_2+1} &= e_{s_1+k_2} \\ xe_{k_1+k_2} &= e_{k_1+k_2-1} & x'e_{s_1+k_2} &= e_{s_1+k_2-1} \\ &\vdots & &\vdots \\ xe_{k_1+1} &= e_{k_1} & x'e_{s_1+1} &= e_{k_1} \\ xe_{k_1} &= e_{k_1-1} & x'e_{k_1-1} &= e_{k-2} \\ &\vdots & &\vdots \\ xe_1 &= 0 & x'e_1 &= 0. \end{aligned}$$

Next, there is a block of size $s_2 - k_2$.

$$\begin{aligned} xe_{r+s_1+s_2} &= e_{r+s_1+s_2-1} & x'(e_{s_1+s_2} - e_{s_1+s_2+s_2-k_2}) &= e_{s_1+s_2-1} - e_{s_1+s_2+s_2-k_2-1}, \\ xe_{r+s_1+s_2-1} &= e_{r+s_1+s_2-2} & x'(e_{s_1+s_2-1} - e_{s_1+s_2+s_2-k_2-1}) &= e_{s_1+s_2-2} - e_{s_1+s_2+s_2-k_2-2}, \\ &\vdots & &\vdots \\ xe_{r+s_1+k_2+1} &= 0 & x'(e_{s_1+s_2-(s_2-k_2)+1} &- e_{s_1+s_2+s_2-k_2-(s_2-k_2)+1}), \\ & & = x'(e_{s_1+k_2+1} &- e_{s_1+s_2+1}) = 0. \end{aligned}$$

And finally there is a block of size $s_1 - k_1$.

$$\begin{aligned}
xe_{r+s_1} &= e_{r+s_1-1} & x'(e_{s_1} - e_{s_1+s_1-k_1}) &= e_{s_1-1} - e_{s_1+s_1-k_1-1} \\
xe_{r+s_1-1} &= e_{r+s_1-2} & x'(e_{s_1-1} - e_{s_1+s_1-k_1-1}) &= e_{s_1-2} - e_{s_1+s_1-k_1-2} \\
&\vdots & &\vdots \\
x = e_{r+k_1+1} &= 0 & x'(e_{s_1-(s_1-k_1)+1} &- e_{s_1+s_1-k_1-(s_1-k_1)+1}) \\
& & &= x'(e_{k_1+1} - e_{s_1+1}).
\end{aligned}$$

Which gives the following base change for the $r + k_1 + k_2$ part:

$$\begin{aligned}
e_{r+k_1+k_2} &\mapsto e_{s_1+s_2+r}, \\
&\vdots \\
e_{k_1+k_2+1} &\mapsto e_{s_1+s_2+1}, \\
e_{k_1+k_2} &\mapsto e_{s_1+k_2}, \\
&\vdots \\
e_{k_1+1} &\mapsto e_{s_1+1}, \\
e_{k_1} &\mapsto e_{s_1}, \\
&\vdots \\
e_1 &\mapsto e_1.
\end{aligned}$$

And for the $s_2 - k_2$ part:

$$\begin{aligned}
e_{r+s_1+s_2} &\mapsto e_{s_1+s_2} - e_{s_1+s_2+s_2-k_2}, \\
&\vdots \\
e_{r+s_1+k_2+1} &\mapsto e_{s_1+k_2+1} - e_{s_1+s_2+1}.
\end{aligned}$$

And for the $s_1 - k_1$ part:

$$\begin{aligned}
e_{r+s_1} &\mapsto e_{s_1} - e_{s_1+s_1-k_1}, \\
&\vdots \\
e_{r+k_1+1} &\mapsto e_{k_1+1} - e_{s_1+1}.
\end{aligned}$$

Then the base change is:

$$\begin{aligned}
\{e_1, \ldots, e_{k_1}, e_{s_1+1}, \ldots, e_{s_1+k_2}, e_{s_1+s_2+1}, \ldots, e_{s_1+s_2+r} \mid \\
e_{k_1+1} - e_{s_1+1}, \ldots, e_{s_1} - e_{s_1+s_1-k_1} \\
e_{s_1+k_2+1} - e_{s_1+s_2-k_1}, \ldots, e_{s_1+s_2} - e_{s_1+s_2+s_2-k_2}\}.
\end{aligned}$$

Let g be the matrix with the vectors of the base change, in that order, then $g \cdot x = x'$, and let:

$$\lambda = \operatorname{diag}(t^2 I_{s_1}, tI_{s_2}, I_r).$$

Finally

$$\lim_{\lambda} \lambda \cdot x' = J_{s_1} \oplus J_{s_2} \oplus J_r.$$

So $\mathcal{O}([r, s_1, s_2])$ is 1-accessible from $\mathcal{O}([r + k_1 + k_2, s_1 - k_1, s_2 - k_2])$.

Next, we can prove that $\mathcal{O}([r, s_1, s_2, \ldots, s_q])$ is 1-accessible from $\mathcal{O}([r+\sum_{i=1}^{q}, s_1, \ldots, s_1])$. First, let

$$x = J_{r+\sum_{i=1}^{q} k_i} \oplus J_{s_1-k_1} \oplus \cdots \oplus J_{s_q-k_q},$$

and let

$$x' = \left(\begin{array}{c|c|c|c|c} J_{s_1} & 1_{(a)} & & & \\ \hline & J_{s_2} & & & \\ \hline & & \ddots & & \\ \hline & & & J_{s_q} & 1_{(b)} \\ \hline & & & & J_r \end{array}\right),$$

with $(a) = (k, s_1 + 1)$ and $(b) = (\sum_{i=1}^{q-1} s_i + k, \sum_{i=1}^{q} s_i + 1)$. Then the base change is:

$$\begin{aligned}
&\{e_1, \ldots, e_{k_1}, e_{s_1+1}, \ldots, e_{s_1+k_2}, \ldots, e_{(\sum_{i=1}^{q-1} s_i)+1}, \ldots, e_{(\sum_{i=1}^{q-1} s_i)+k_q}, e_{(\sum_{i=1}^{q} +s_i)+1}, \ldots, e_{(\sum_{i=1}^{q} s_i)+r} | \\
&e_{k_1+1} - e_{s_1+1}, \ldots, e_{s_1} - e_{s_1+s_1-k_1} | \\
&e_{s_1+k_2+1} - e_{s_1+s_2+1}, \ldots, e_{s_1+s_2} - e_{s_1+s_2+s_2-k_2} | \\
&\ldots, \\
&e_{\sum_{i=1}^{q-1}(s_i+k_q)+1} - e_{\sum_{i=1}^{q-1}(s_i+s_q)+1}, \ldots, e_{\sum_{i=1}^{q} s_i} - e_{(\sum_{i=1}^{q} s_i)+s_q-k_q}\}
\end{aligned}$$

Let g be the matrix with the vectors of the base change, in that order, then $g \cdot x = x'$, and let:

$$\lambda = \operatorname{diag}(t^q I_{s_1}, t^{q-1} I_{s_2}, \ldots, t I_{s_q}, I_r).$$

Then $\lim_\lambda \lambda \cdot x \in \mathcal{O}[r, s_1, s_2, \ldots s_2]$, finishing the proof that $\mathcal{O}[r, s_1, \ldots s_k]$ is one-accessible from $\mathcal{O}[r + \sum_{i=1}^{q} k_i, s_1 - k_1, \ldots, s_q - k_q]$. $\square$

We have now shown that a move is valid if it involves moving pieces from one part, to any number of smaller parts, or if it involves moving pieces from any number of larger parts to one smaller part. Note that no larger part can receive a piece from a smaller part (see 1.8.1). Here we will show that if a move involves a part receiving pieces from a larger part, and giving pieces to a smaller part, we can instead denote it as either of the two moves described above.

Lemma 2.3.1. Let $\pi_1 = [r + k, s + l - k, t - l]$, with $k > l$, and let $\pi_2 = [r, s, t]$. Then π_2 is 1-accessible from π_1.

Proof. Denote $k = l + k'$, then $k - l = k'$, so $[r + k, s + l - k, t - l] = [r + l + k', s - k', t - l]$, which is of the form $[r + k_1, s_1 - k_1, s_2 - k_2]$, so clearly $[r, s, t]$ is 1-accessible from $[r + k, s + l - k, t - l]$.

Conversely, suppose there is a partition of the form $[r+k, s+l-k, t-l]$ with $k < l$, and one wants to show that $[r, s, t]$ is 1-accessible. Denote $k = l-l'$, then $[r+k, s+l-k, t-l] = [r+l-l', s+l', r-l]$, which is of the form $[r_1 + k_1, r_2 + k_2, s - k_1 - k_2]$.

Finally, if $k = l$, it immediately follows that $[s + k - l] = [s]$, so the problem reduces to showing $[r, \ldots, t]$ is 1-accessible from $[r + k, \ldots, t - k]$, as shown in Section 2.2. $\square$

2.4 Conclusion for $\mathfrak{gl}_n$

In this section we will use Theorems 2.3.1 and 2.3.2 to prove the following theorem:

Theorem 2.4.1. Let π_1 and π_2 be any two partitions such that π_1 dominates π_2. Then $\mathcal{O}(\pi_2)$ is one-accessible from $\mathcal{O}(\pi_1)$. Hence, for GL_n, accessibility and 1-accessibility coincide, and the partial order on orbits given by accessibility is the same as the dominance order.

Proof. π_1 dominates π_2, so letting $\pi_1 = [a_1, \ldots, a_n]$, $\pi_2 = [b_1, \ldots, b_m]$, we have $\sum_{i=1}^{n} a_i \geq \sum_{i=1}^{m} b_i$ for $1 \leq i \leq m$ (note also that $n \leq m$, or equivalently, all $a_{n+1}, \ldots, a_m$ parts are of size zero).

We can rewrite $\pi_2 = [\ldots, r_i, \ldots, s_j, \ldots, t_l, \ldots]$ and $\pi_1 = [\ldots, r_i + P_i, \ldots, s_j - Q_j, \ldots, t_l, \ldots]$, so $r_i + P_i$ parts lose pieces, and $s_j - Q_j$ parts gain pieces (t_l parts are unchanged). There are a finite number of $r_i + P_i$ parts, say p, and a finite number of $s_j - Q_j$ parts, say q.

Then we can denote each $r_i + P_i$ as $r_i + \sum_{j=1}^{q} p_{i,j}$, where $p_{i,j}$ are the pieces transferred to all $s_j - Q_j$, for $1 \leq j \leq q$, and some $p_{i,j}$ may be zero. Hence we can describe the move $\pi_1 \to \pi_2$ as p number of moves of type 1; all r_i parts lose pieces simultaneously. Similarly, all $s_j - Q_j$ parts can be denoted as $s_j - \sum_{i=1}^{p} q_{i,j}$, hence the move can be described as q moves of type 2; all s_j parts gain pieces simultaneously. Then we have described all changes to the r_i and the s_j parts in one move, so π_2 is 1-accessible from π_1. $\square$

As the dominance order and the partial order given by accessibility coincide, the example for accessibility in $\mathfrak{gl}_6$ is the same as figure 1.1. Recall that the general linear group contains as a subgroup the *special linear group* $\mathrm{SL}_n(\kappa)$ consisting of matrices of determinant 1. The corresponding Lie algebra $\mathfrak{sl}_n(\kappa)$ consists of trace zero matrices in $\mathfrak{gl}_n(\kappa)$. Since the trace of nilpotent matrices is zero, all nilpotent matrices in $\mathfrak{gl}_n$ are also in $\mathfrak{sl}_n$.

The accessibility of nilpotent orbits in the general linear group is helpful for determining the accessibility in the special linear group. In fact, the following two lemmas will show that the accessibility for $\mathfrak{sl}_n$ is the same as that for $\mathfrak{gl}_n$.

Lemma 2.4.1. The $GL_n(\kappa)$-orbits and $SL_n(\kappa)$-orbits of nilpotent matrices are identical.

Proof. Let x be a nilpotent matrix in standard form, in any orbit. Then x is strictly upper triangular, so it is immediate that all diagonal entries are zero, hence $\mathrm{Tr}x = 0$. We conclude that $x \in \mathfrak{sl}_n$. Since any $g \in \mathrm{GL}_n(\kappa)$ can be written $g = zh$ with z a scalar matrix and $h \in \mathrm{SL}_n(\kappa)$, we see that two nilpotent elements are conjugate by $\mathrm{GL}_n(\kappa)$ if and only if they are conjugate by $\mathrm{SL}_n(\kappa)$. $\square$

Furthermore, the accessibility of orbits in $\mathfrak{sl}_n$ is the same as that for the orbits in $\mathfrak{gl}_n$.

Lemma 2.4.2. Let $x_1, x_2 \in \mathfrak{gl}_n$ be nilpotent matrices corresponding to partitions π_1 and π_2, respectively. If x_2 is accessible from x_1 in $\mathfrak{gl}_n$, then it is accessible in $\mathfrak{sl}_n$.

Proof. Let x_2 be accessible from x_1 in $\mathfrak{gl}_n$, then there is a $g \in \mathrm{GL}_n$ and a cocharacter λ of $\mathrm{GL}_n(\kappa)$ such that $\lim_\lambda(g \cdot x_1) = x_2$. We can find an $h \in \mathrm{SL}_n$ and $\mu \in \kappa^*(t)$ with $\det(\mu) = 1$, such that

$\lim_{t\to 0} \mu \cdot (h \cdot x_1) = x_2$. Specifically, if $\det g = c$, then we take $h = \sqrt[n]{\frac{1}{c}}g$, and it follows that $\det(h) = \det \sqrt[n]{\frac{1}{c}}g = \frac{1}{c}\det g = 1$.

The approach to find a suitable form of λ is similar. Let x_2 have partition $[r_1, \ldots, r_p]$ of p parts, each of size r_i. Then λ is of the form:

$$\lambda = \begin{pmatrix} t^{p-1}I_{r_1} & & & \\ & t^{p-2}I_{r_2} & & \\ & & \ddots & \\ & & & I_{r_p} \end{pmatrix},$$

where each I_{r_i} is an identity block of size r_i. Then

$$\begin{aligned} \det(\lambda) &= \Pi_{i=1}^{p}\Pi_{j=1}^{r_i} t^{p-i} \\ &= \Pi_{i=1}^{p} t^{r_i(p-i)} \\ &= t^{\sum_{i=1}^{p} r_i(p-i)}. \end{aligned}$$

The effect of the cocharacter λ when limits are taken does not depend on the values of the powers of t, only on the fact that these powers form a decreasing sequence. Hence, if we like, we can replace the powers with a decreasing sequence of integers summing to 0 – that is, we replace each t^{p-i} (including when $i = p$) with powers t^{a_i} with $a_i \in \mathbb{Z}$ chosen so that $\sum_{i=1}^{p} r_i a_i = 0$. Then the new cocharacter has the same effect in the limit, but now evaluates in $\mathrm{SL}_n(\kappa)$.

By Lemma 2.4.2, x_1 and x_2 are present in the special linear group, so with μ and h, we have found a move such that $\lim_\mu x_1 = \lim_{t\to 0} \mu \cdot (h \cdot x_1) = x_2$. We conclude that x_2 is accessible from x_1 in the special linear group. $\square$

Chapter 3

Results in the symplectic algebras

In this chapter, we describe the results in the symplectic algebras. We consider the five possible moves in the dominance order of the symplectic algebras, and analyse four which are also possible with cocharacters. The last move, which doesn't occur with cocharacters, will be analysed in detail. First, recall from Equation 1.4 that the symplectic basis we choose is as follows:

$$\mathcal{B} = \{e_1, \dots, e_n, f_n, \dots, f_1\},$$

and that the matrix of the bilinear form ϕ with respect to this basis is:

$$\Omega_{\mathrm{S}} = \begin{pmatrix} & & & & & 1 \\ & & & & \iddots & \\ & & & 1 & & \\ & & -1 & & & \\ & \iddots & & & & \\ -1 & & & & & \end{pmatrix}.$$

Then the symplectic group is

$$\mathrm{Sp}_{2n} = \{g \in \mathrm{GL}_{2n} \mid g^T \Omega_{\mathrm{S}} g = \Omega_{\mathrm{S}}\},$$

and the symplectic algebra is:

$$\mathfrak{sp}_{2n} = \{x \in \mathfrak{gl}_{2n} \mid x^T \Omega_{\mathrm{S}} + \Omega_{\mathrm{S}} x = 0\}.$$

Recall that bilinear forms are conjugate, so the our analysis of the moves in this chapter are independent of the choice of the bilinear form and symplectic basis. Recall that the group and algebra are over an algebraically closed field $\kappa = \bar{\kappa}$. In the Section 5.1, we describe why the proofs do not always hold if the restriction of algebraically closed fields is lifted.

3.1 Partitions and moves

To determine (1-)accessibility in the symplectic algebra, we first recall the possible symplectic partitions, and which orbits in $\mathfrak{sp}_{2n}$ are distinguished. The symplectic partitions are given by part 1 of Theorem 1.11.1:

Theorem. Let x be a nilpotent element with partition $[r_1^{(n_1)}, \dots, r_p^{(n_p)}]$. Then x appears in $\mathfrak{sp}_{2n}$ if and only if n_i is even for all odd r_i.

And by Lemma 1.11.2, we can determine when a partition is distinguished:

Lemma. Let $x \in \mathfrak{sp}_{2n}$, then x is distinguished if and only if its partition π has distinct even parts (and no odd parts).

By considering minimal through the dominance order of the partitions corresponding to symplectic nilpotent orbits, we can identify the following moves (note that $m = 0$ can occur):

1. $\mathcal{O}([2m, 2m-2]) \to \mathcal{O}([2m-1, 2m-1])$.
2. $\mathcal{O}([2n, m, m]) \to \mathcal{O}([2n-2, m+1, m+1])$.
3. $\mathcal{O}([n, n, 2m]) \to \mathcal{O}([n-1, n-1, 2m+2])$.
4. $\mathcal{O}([n, n, m, m]) \to \mathcal{O}([n-1, n-1, m+1, m+1])$.
5. $\mathcal{O}([2n, 2m]) \to \mathcal{O}([2n-2, 2m+2])$.

It is clear that move 5 cannot occur with cocharacters if $m = 0$, and in section 3.2 we will analyse why it fails in general. First, we will analyse moves 1-4 with examples, and the generalized moves as given above. The base changes between matrices x and x' will be omitted, but can be determined with the same method as in chapter 2. While these base changes will not necessarily be symplectic, theorem 1.12.1 tells us that if x and x' are in the same $\mathrm{GL}(V)$-orbit, they will also be in the same $\mathfrak{g}$-orbit, in this case $\mathfrak{g} = \mathfrak{sp}_{2n}$. In chapter 5, we will give an example in which start with a base change that is not symplectic, and then determine one that is symplectic. This gives an interesting topic for further research, as the process involves elements that are not always defined in fields that are not algebraically closed.

3.1.1 The cocharacter realizing move 1

We begin with an example, the move, $\mathcal{O}([6,4]) \to \mathcal{O}([5,5])$ in dimension 10.

Recall that the goal is to find an $x' \in \mathcal{O}([6,4])$ and a cocharacter λ such that $\lim_\lambda x' \in \mathcal{O}([5,5])$, and a symplectic base change g, such that $gxg^{-1} = x'$, where x is the standard form of a symplectic matrix with partition $[6,4]$. That is:

$$x = \left(\begin{array}{ccc|cccc|ccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

Let

$$x' = \left(\begin{array}{ccccc|ccccc} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\lambda(t) = \operatorname{diag}(t, t, t, t, t, t^{-1}, t^{-1}, t^{-1}, t^{-1}, t^{-1}).$$

So $y = \lim_\lambda x' \in \mathcal{O}([5,5])$, and row-elimination shows that $\dim(\ker(x')) = 2$, while $(x')^5 \neq 0$ and $(x')^6 = 0$, therefore $x' \in \mathcal{O}([6,4])$. Alternatively, the information about the nonzero entries on the s^{th} superdiagonal can be used to clarify the form of x', see Lemma 2.2.2 on page 40. We conclude that $\mathcal{O}([5,5])$ is 1-accessible from $\mathcal{O}([6,4])$.

We now generalize to the move $\mathcal{O}([2m, 2m-2]) \to \mathcal{O}([2m-1, 2m-1])$. For notation purposes, let

$$\tilde{J}_{2n} = \left(\begin{array}{c|c} J_n & 1_{(a)} \\ \hline & -J_n \end{array}\right),$$

with $(a) = (n, n+1)$. Then, the standard forms of x and the form of x' are as follows:

$$x = \left(\begin{array}{c|c|c} J_m & & 1_{(a)} \\ \hline & \tilde{J}_{2m-2} & \\ \hline & & -J_m \end{array}\right),$$

where $(a) = (m, 3m-1)$. Next,

$$x' = \left(\begin{array}{c|c} J_{2m-1} & 1_{(a)} \quad 1_{(b)} \\ \hline & -J_{2m-1} \end{array}\right),$$

with $(a) = (1, 2m)$ and $(b) = (2m-1, 4m-2)$. Then x induces the vector chains:

$$f_1 \to -f_2 \to \ldots \to (-1)^{m-1} f_m \to (-1)^m e_m \to (-1)^m e_{m-1} \to \ldots \to (-1)^m e_1 \to 0.$$
$$f_{m+1} \to -f_{m+2} \to \ldots \to (-1)^{m-2} f_{2m-1} \to (-1)^{m-1} e_{2m-1} \to \ldots \to (-1)^{m-1} e_{m+1} \to 0.$$

And x' induces the vector chains:

$$f_1 \to -f_2 - e_n \to f_3 - e_{n-1} \to \ldots \to (-1)^{2m-2} f_{2m-1} - e_2 \to -2e_2 \to 0,$$
$$f_2 - e_n \to -f_3 - e_{n-1} \to \ldots \to (-1)^{2m-3} f_{2m-1} - e_2 \to 0,$$

so x' does indeed have partition $[2m, 2m-2]$. Let

$$\lambda = \begin{pmatrix} tI_{2m-1} & \\ & -t^{-1} I_{2m-2} \end{pmatrix}, \text{ then } y = \lim_{\lambda} x' = \begin{pmatrix} tJ_{2m-1} & \\ & -t^{-1} J_{2m-1} \end{pmatrix},$$

which has partition $[2m-1, 2m-1]$, hence $\mathcal{O}([2m-1, 2m-1])$ is 1-accessible from $\mathcal{O}([2m, 2m-2])$.

3.1.2 The cocharacter realizing move 2

The second move is: $\mathcal{O}([2n, m, m]) \to \mathcal{O}([2n-2, m+1, m+1])$. We start with example $\mathcal{O}([6, 2, 2]) \to \mathcal{O}([4, 3, 3])$. The standard symplectic form is:

$$x = J_2 \oplus \tilde{J}_6 \oplus J_2.$$

Let

$$x' = \left(\begin{array}{ccc|cccc|ccc} 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

$$\lambda(t) = \mathrm{diag}(t, t, t, 1, 1, 1, 1, t^{-1}, t^{-1}, t^{-1}).$$

So $y = \lim_{\lambda} x' \in \mathcal{O}([6, 2, 2])$.

Generalizing to the move $\mathcal{O}([2n, m, m]) \to \mathcal{O}([2n-2, m+1, m+1])$, let

$$x = J_m \oplus \tilde{J}_{2n} \oplus (-J_m)$$

and let

$$
x' = \left(\begin{array}{c|c|c}
J_{m+1} & 1_{(a)} & \\
\hline
 & \tilde{J}_{2n-2} & -1_{(b)} \\
\hline
 & & -J_{m+1}
\end{array}\right),
$$

with $(a) = (1, m+2)$ and $(m+2n-1, 2m+2n)$. Then x induces the vector chains:

$$
\begin{aligned}
f_1 \to -f_2 \to \cdots \to (-1)^{m-1} f_m &\to 0, \\
f_{m+1} \to -f_{m+2} \to \cdots \to (-1)^{n-1} f_{m+n} &\to (-1)^n e_{n+m} \to \cdots \to (-1)^n e_1 \to 0, \\
e_m \to \cdots \to e_1 &\to 0.
\end{aligned}
$$

and x' induces the vector chains, of length $2n, m, m$, respectively:

$$
\begin{aligned}
f_1 \to -(f_2 + f_{2+m}) \to \cdots \to (-1)^m (f_{1+m} + f_{1+2m}) &\to (-1)^{m+1} f_{2+2m} \to \\
\cdots \to (-1)^{n-3+m} f_{n+m} &\to (-1)^{n-3+m} e_{n+m} \to \\
\cdots \to (-1)^{n-3+m} e_{2+m} &\to (-1)^{n-3+m} e_1 \to 0,
\end{aligned}
$$

$$
\begin{aligned}
f_2 \to -f_3 \to \cdots \to (-1)^{m-1} f_{m+1} &\to 0, \\
e_n - e_{n+m} \to \cdots \to e_{n+1-m} - e_{n+1} &\to 0.
\end{aligned}
$$

So x' does indeed have partition $[2n, m, m]$. Let

$$
\lambda = \begin{pmatrix} tI_{m+1} & & \\ & \tilde{I}_{2n-2} & \\ & & t^{-1} I_{m+1} \end{pmatrix}, \text{ then } \lim_{\lambda} x' = \begin{pmatrix} J_{m+1} & & \\ & -\tilde{J}_{2n-2} & \\ & & -J_{m+1} \end{pmatrix},
$$

which has partition $[2n-2, m+1, m+1]$, hence $\mathcal{O}[2n-2, m+1, m+1]$ is 1-accessible from $\mathcal{O}[2n, m, m]$.

3.1.3 The cocharacter realizing move 3

The third move is: $\mathcal{O}([n, n, 2m]) \to \mathcal{O}([n-1, n-1, 2m+2])$. We start with example $\mathcal{O}([5, 5, 2]) \to \mathcal{O}([4, 4, 4])$. The standard symplectic form is:

$$x = J_5 \oplus J_2 \oplus (-J_5).$$

Let

$$
x' = \left(\begin{array}{cccc|cccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),
$$

$$
\lambda(t) = \operatorname{diag}(t, t, t, t, 1, 1, 1, 1, t^{-1}, t^{-1}, t^{-1}, t^{-1}).
$$

So $y = \lim_\lambda x' \in \mathcal{O}([4, 4, 4])$, therefore $\mathcal{O}([4, 4, 4])$ is 1-accessible from $\mathcal{O}([5, 5, 2])$. Note the position of the ones in x' compared to the position in x' in the example for move 2.

Generalizing to the move $\mathcal{O}([n, n, 2m]) \to \mathcal{O}([n-1, n-1, 2m+2])$, let

$$
x = J_n \oplus \tilde{J}_{2m} \oplus (-J_n),
$$

and let

$$
x' = \left(\begin{array}{c|c|c}
J_{n-1} & \qquad 1_{(a)} & \\
\hline
 & \tilde{J}_{2m+2} & 1_{(b)} \qquad \\
\hline
 & & -J_{n-1}
\end{array}\right),
$$

with $(a) = (n-1, n+2m+1)$ and $(b) = (n, n+2m+2)$. Then x induces the following vector chains:

$$
\begin{aligned}
&f_1 \to -f_2 \to \cdots \to (-1)^{n-1} f_n \to 0, \\
&e_n \to e_{n-1} \to \cdots \to e_1 \to 0, \\
&f_{n+1} \to -f_{n+2} \to \cdots \to (-1)^{m-1} f_n \to (-1)^m e_m \to \cdots \to (-1)^m e_1 \to 0.
\end{aligned}
$$

And x' induces the following chains of length $n, n, 2m$ respectively:

$$
\begin{aligned}
f_1 \to -f_2 \to \cdots &\to (-1)^{n-2} f_{n-1} \to (-1)^{n-1} e_n \to 0, \\
f_n \to -f_{n+1} - e_{n-1} \to \cdots &\to (-1)^m f_{n+m} - e_{n-m} \to (-1)^{m+1} e_{n+m} - e_{n-m-1} \to \\
\cdots &\to (-1)^{m+1} e_n - e_{n-2m-1} \to -e_{n-2m-1} \to \\
\cdots &\to -e_1 \to 0, \\
f_{n+1} + f_{n-2m} \to \cdots &\to (-1)^{m-1}(f_{n+m} + f_{n-m-1}) \to (-1)^m e_{n+m} + (-1)^m f_{n-m} \to \\
\cdots &\to (-1)^m e_{n+1} + (-1)^{2m-1} f_{n-1} \to 0.
\end{aligned}
$$

So x' does indeed have partition $[n, n, 2m]$. Let

$$\lambda = \begin{pmatrix} tI_{n-1} & & \\ & \tilde{I}_{2m+2} & \\ & & -t^{-1}I_{n-1} \end{pmatrix}, \text{ then } y = \lim_{\lambda} x' = \begin{pmatrix} J_{n-1} & & \\ & \tilde{J}_{2m+2} & \\ & & -J_{n-1} \end{pmatrix},$$

which has partition $[n-1, n-1, 2m+2]$, hence $\mathcal{O}([n-1, n-1, 2m+2])$ is 1-accessible from $\mathcal{O}([n, n, 2m])$.

3.1.4 The cocharacter realizing move 4

The fourth move is: $\mathcal{O}([n, n, m, m]) \to \mathcal{O}([n-1, n-1, m+1, m+1])$. We start with an example: $\mathcal{O}([5, 5, 2, 2]) \to \mathcal{O}([4, 4, 3, 3])$. In standard form

$$x = J_2 \oplus J_5 \oplus (-J_5) \oplus (-J_2),$$

let

$$x' = \left(\begin{array}{ccc|cccc|cccc|ccc}
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right),$$

$$\lambda = \operatorname{diag}(t, t, t, 1, 1, 1, 1, 1, 1, 1, 1, t^{-1}, t^{-1}, t^{-1}).$$

So $y = \lim_{\lambda} x' \in \mathcal{O}([4, 4, 3, 3])$, therefore $\mathcal{O}([4, 4, 3, 3])$ is 1-accessible from $\mathcal{O}([5, 5, 2, 2])$.

Generalizing to the move $\mathcal{O}([n, n, m, m]) \to \mathcal{O}([n-1, n-1, m+1, m+1])$, let

$$x = J_m \oplus J_n \oplus (-J_n) \oplus (-J_m),$$

and let

$$x' = \left(\begin{array}{c|c|c|c} J_{m+1} & 1_{(a)} & & \\ \hline & J_{n-1} & & \\ \hline & & -J_{n-1} & 1_{(b)} \\ \hline & & & -J_{m+1} \end{array}\right),$$

with $(a) = (1, m+2)$ and $(b) = (m+2n-1, 2m+2n)$. Then x induces the following vector chains, of sizes m, n, n, m respectively:

$$\begin{aligned} f_1 &\to \cdots \to (-1)^{m-1} f_m, \\ f_{m+1} &\to \cdots \to (-1)^{n-1} f_{m+n}, \\ e_{n+m} &\to \cdots \to e_{m+1}, \\ e_m &\to \cdots \to e_1. \end{aligned}$$

And x' induces the following chains, of sizes m, n, n, m respectively:

$$\begin{aligned} f_2 &\to -f_3 \to \cdots \to (-1)^{m-1} f_{1+m} \to 0, \\ f_1 &\to -(f_2 + f_{2+m}) \to \cdots \to (-1)^m (f_{1+m} + f_{1+2m}) \to (-1)^{m+1}(f_{2+2m}) \to \cdots \to (-1)^n (f_{n+m}) \to 0, \\ e_{n+m} &\to e_{n+m-1} \to \cdots \to e_{2+m} \to e_1 \to 0, \\ e_{1+m} - e_{1+2m} &\to \cdots \to e_2 - e_{2+m} \to 0. \end{aligned}$$

So x' has partition $[n, n, m, m]$. Finally, let $\lambda = \mathrm{diag}(tI_{m+1}, I_{n-1}, I_{n-1}, t^{-1} I_{m+1})$, then

$$y = \lim_{\lambda} x' = \begin{pmatrix} J_{m+1} & & & \\ & J_{n-1} & & \\ & & -J_{n-1} & \\ & & & -J_{m+1} \end{pmatrix},$$

which has partition $[n-1, n-1, m+1, m+1]$, hence $\mathcal{O}([n-1, n-1, m+1, m+1])$ is 1-accessible from $\mathcal{O}([n, n, m, m])$.

3.2 A non-move

This section will analyse Move 5: $\mathcal{O}([2n, 2m]) \to \mathcal{O}([2n-2, 2m+2])$. First, consider that if $2n - 2 = 2m + 2$, then this move is actually a generalized move 1, of the form $\mathcal{O}([2m, 2m-2k]) \to \mathcal{O}([2m-k, 2m-k])$, with $k = 2$. We should now consider the case where $2n - 2 \neq 2m + 2$.

As example, consider the move $\mathcal{O}([6]) \to \mathcal{O}([4, 2])$ (so $m = 0$). Let $x \in \mathcal{O}([6])$ be in standard form, then

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Next, we want to find a matrix x' such that $y = \lim_\lambda x' \in \mathcal{O}([4,2])$, for a suitable λ. Then λ has to be in the centralizer of y, so we start by determining the forms of y and λ.

$$y = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and $\lambda = \mathrm{diag}(t_1, t_2, t_3, t_4, t_5, t_6)$. As $\lim_\lambda x' = y$, we require $\lim_{t\to 0} t_1 x'_{(1,6)} t_6^{-1} = 1$, so $t_1 = t_6$, therefore all t_i must be equal, so the only λ in $\mathrm{Stab}_G(y)$ is $\pm I$, which is in the centre of y. So y is distinguished. By Lemma 1.9.1, $x' = y$, hence y is not accessible from x. This is a special case of a general phenomenom: if the target element y is distinguished, then there cannot be any non-trivial cocharacter with $\lim_\lambda x' = y$.

Now that it is shown that $\mathcal{O}([2n, 2m]) \to \mathcal{O}([2n-2, 2m+2])$ is not a valid move if $2n-2 \neq 2m+2$, we can consider the implications to the general accessibility of orbits. The following diagram is the diagram of 1-accessibility in $\mathfrak{sp}_6$, to compare to the diagram for $\mathfrak{gl}_6$ (see Figure 1.1).

$$\begin{tikzcd}
{[6]} \arrow[d, dash] \arrow[drr, dash] & & {[4,2]} \arrow[d, dash] \arrow[dll, dash] \\
{[3,3]} \arrow[dr, dash] & & {[4,1,1]} \arrow[dl, dash] \\
& {[2,2,2]} \arrow[d, dash] & \\
& {[2,2,1,1]} \arrow[d, dash] & \\
& {[2,1^{(4)}]} \arrow[d, dash] & \\
& {[1^{(6)}]} &
\end{tikzcd}$$

Figure 3.1: Accessibility in $\mathfrak{sp}_6$.

The observation that $\mathcal{O}([4,2])$ is not accessible from $\mathcal{O}([6])$ gives an answer to the second question in 1.8: When $\kappa = \overline{\kappa}$, are there cases when $\overline{G(\kappa)\cdot x}^c \neq \overline{G(\kappa)\cdot x}$? Yes, as $\mathcal{O}([4,2])$ is not in the cocharacter-closure of $\mathcal{O}([6])$. In general, however, this observation does not give us the complete answer, since the presence of other parts in the partition can complicate matters. For example, even though $\mathcal{O}([4,2])$ is not accessible from $\mathcal{O}([6])$, we can show that $\mathcal{O}([4,2,2])$ is accessible from $\mathcal{O}([6,2])$. The presence of the extra 2 in the partition makes a material difference to the outcome, which is not obvious immediately. Note in this case that the two partitions are no longer adjacent in the dominance order – the accessibility diagram for this case can be split as follows:

$$\begin{array}{ccc} [6,2] & \longrightarrow & [6,1,1] \\ & \searrow & \downarrow \\ & & [4,2,2]. \end{array}$$

Consecutively applying move 1 to the [2] part of $[6,2]$ ($[2m, 2m-2] \to [2m-1, 2m-1]$ for $m = 1$) and move 2 to the partition $[6,1,1]$ ($[2n, m, m] \to [2n-2, m+1, m+1]$ for $n = 3$, $m = 1$), shows that $\mathcal{O}([4,2,2])$ is accessible from $\mathcal{O}([6,2])$. The following matrix shows that $\mathcal{O}([4,2,2])$ is also 1-accessible. Let x' be of the form:

$$x' = \left(\begin{array}{cc|cccc|cc} 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

we can compute that $(x')^5 \neq 0$, and row-elimination shows that $\text{rank}(x') = 6$, so x' must have two parts to the partition, and hence x' is in the orbit of $\mathcal{O}([6,2])$. Taking

$$\lambda = \text{diag}(t, t, 1, 1, 1, 1, t^{-1}, t^{-1}),$$

then we get:

$$\lambda \cdot x' = \left(\begin{array}{cc|cccc|cc} 0 & 1 & t^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & t^4 & 0 \\ \hline 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -t^2 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right),$$

so $\lim_\lambda x' \in \mathcal{O}([4,2,2])$, so $\mathcal{O}([4,2,2])$ is 1-accessible from $\mathcal{O}([6,2])$. We now proceed to show precisely when the move of type 5 cannot occur.

3.2.1 The shrinking operation

We define the shrinking operation as follows: given a $2n \times 2n$ matrix A, matrix $S(A)$ is the $(2n-2) \times (2n-2)$ matrix formed by deleting rows and columns on the outside of matrix A.

For example, let $A = \begin{pmatrix} a_{(1,1)} & a_{(1,2)} & a_{(1,3)} & a_{(1,4)} \\ a_{(2,1)} & a_{(2,2)} & a_{(2,3)} & a_{(2,4)} \\ a_{(3,1)} & a_{(3,2)} & a_{(3,3)} & a_{(3,4)} \\ a_{(4,1)} & a_{(4,2)} & a_{(4,3)} & a_{(4,4)} \end{pmatrix}$, then $S(A) = \begin{pmatrix} a_{(2,2)} & a_{(2,3)} \\ a_{(3,2)} & a_{(3,3)} \end{pmatrix}$.

Denote by $S^d(A)$ the $(2n-2d) \times (2n-2d)$ matrix formed repeating the shrinking operation d times. The shrinking operation is well-behaved with respect to the transpose $S^d(A^T) = (S^d(A))^T$. To show that S is also well-behaved with respect to being in the symplectic Lie algebra, observe that when Ω_{S} is the defining matrix for the form in dimension $2n$, then $S^d(\Omega_{\text{S}})$ is the defining matrix in dimension $2n-2d$. Then if $A^T\Omega_S + \Omega_S A = 0$, it follows that $S^d(A^T)S^d(\Omega_S) + S^d(\Omega_S)S^d(A) = 0$.

Next, suppose y is in standard form and $\dim(y) = 2n$. If x is such that $\lim_\lambda x = y$ and λ is in standard form (as in 1.6), then $\lim_{S^d(\lambda)} S^d(x) = S^d(y)$. There are now two possibilities:

1. $S^d(x)$ is conjugate to $S^d(y)$, i.e. no orbit change is made.
2. $S^d(x)$ is strictly higher than $S^d(y)$ in the dominance order.

With these observations in hand, we now make some further reductions.

3.2.2 The setup

We have observed that distinguished partitions are not accessible from any partition higher in the dominance order. The question is: are non-distinguished partitions accessible through a 'move' of type 5? This is the case of interest, so we may assume that the following hold:

(i) y is in standard form with a repeated part $[\ldots, d, d, \ldots]$ appearing on the outside of y (so y is not distinguished).

(ii) x is another nilpotent element.

(iii) λ is a cocharacter in standard form such that $\lim_\lambda x = y$.

Under these hypotheses, we can denote x as $x = y + x_0$, with $\lim_\lambda x_0 = 0$. Since λ is in standard form, x_0 is strictly upper triangular. Furthermore, λ must centralize y, so λ has constant weight a on the first d basis vectors, and constant weight $-a$ on the last d vectors, i.e. for $1 \le i \le d$, we get $\lambda(t)e_i = t^a e_i$, $\lambda(t)f_i = t^{-a} f_i$. Let λ_0 be the cocharacter formed by having weight a on the first d vectors, $-a$ on the last d vectors, and weight zero elsewhere. Then λ_0 and λ are identical on the outside d vectors, and λ_0 fixes the other vectors.

Then $\lim_{\lambda_0} y = y$ and $x' := \lim_{\lambda_0} x$ exists. Since $\lim_\lambda x = y$, $\lim_\lambda x' = y$ also, hence x' lies between x and y in the order of dominance. Hence, if x and y are adjacent in the order, there are two possibilities:

1. $x' = y$,
2. x' is conjugate to x.

In the case that $x' = y$, the only difference between x and y lies in the outside d rows and columns, i.e. $S^d(x) = S^d(y)$. In the case that x' is conjugate with x, we may replace x with x' and assume x has the same repeated $[d, d]$ blocks as y on the outside.

3.2.3 The move of type 5

In the previous subsections, we have setup the requirements to analyse move 5, the move

$$\mathcal{O}([\ldots, 2r, 2s, \ldots]) \to \mathcal{O}([\ldots, 2r-2, 2s+2]).$$

We can now prove the following lemma:

Lemma 3.2.1. Let $x \in \mathcal{O}([\ldots, 2r, 2s, \ldots])$ and let $y \in \mathcal{O}([\ldots, 2r-2, 2s+2, \ldots])$. Then y is not accessible from x.

Proof. We proceed by induction on n, where $2n$ is the matrix size. If $n = 1$, there are no r, s with $2r \geq 6$. For the inductive step, suppose that for all matrix sizes smaller than $2n$ the claim holds, we now finish the proof with a combination of contradiction and some direct calculation.

Suppose there is a $\lambda \in Y(G)$ such that $\lim_\lambda x = y$. First note that y is not distinguished, by Lemma 1.9.1. So y must contain a repeated part $[\ldots, d, d, \ldots]$. With the assumption that y and λ are in standard form, we may assume that this repeated part appears on the outside of the matrix for y, so Section 3.2.2 can be used. Since x and y are adjacent, the two cases of that section apply. First, suppose that x' is conjugate to x, then x and y share the repeated $[\ldots, d, d \ldots]$ part. With a suitable conjugation, we may assume that the repeated $d-$parts in x are on the outside as well. Then the shrinking operation applied d times removes that part from both x and y. The move between $S^d(x)$ and $S^d(y)$ is of the same form as the move between x and y, i.e. it consists of the move $[\ldots, 2r, 2s, \ldots]$ to $[\ldots, 2r-2, 2s+2, \ldots]$ between adjacent parts, with $2r \geq 2s+6$. Since the matrix sizes have been decreased, this is impossible, by induction. So we get a contradiction and conclude that the second case cannot occur.

So we may assume we are in case 1 of Section 3.2.2. Replacing x with x' and λ with λ_0, we are left with the case of $S^d(x) = S^d(y)$. Note that the first d rows and last d columns of x are related because x is symplectic; if there is an entry a in position (i, j) then x also has an entry $-a$ in position $(2n+1-j, 2n+1-i)$. The main idea is to conjugate x by a suitable symplectic unipotent matrix to kill off most entries in the first d rows which have a further nonzero entry in the column below them, say in row i for $i > d$.

So we suppose that x has at least one nonzero entry in column j, with $d \leq j \leq 2n-d$, and we suppose further that in this column there is another nonzero entry in position (i, j) for some $i > d$. Then, since x looks the same as y away from the first d rows and last d columns, we can conclude that:

(i) This other entry is the only other nonzero entry in the j^{th} column, because $S^d(x) = S^d(y)$ is in standard form.

(ii) We have $d < i < j$ because x is strictly upper triangular.

(iii) The entry is a 1 if $i \leq n$ and a -1 if $i > n$.

(iv) It is the only nonzero entry in the i^{th} row, except possibly in the last d columns - in particular all entries in row i before the j^{th} column are 0.

In this situation, let v denote the first d rows in the j^{th} column, viewed as a column vector of length d. Let v_s denote every position s of vector v. We now split into the following subcases to cover all possibilities:

1. $j \leq n$, and $i \leq n$,
2. $j > n$, and $i \leq n$,
3. $j > n$, and $i > n$.

Matrices y and x are of the following form:

$$
y = \left(\begin{array}{c|c|c|c|c|c|c}
J_d & & & & & & \\ \hline
 & \ddots & & & & & \\ \hline
 & & J_{r-1} & & 1_{(a)} & & \\ \hline
 & & & \tilde{J}_{2s+2} & & & \\ \hline
 & & & & -J_{r-1} & & \\ \hline
 & & & & & \ddots & \\ \hline
 & & & & & & -J_d
\end{array}\right),
$$

$$
x = \left(\begin{array}{c|c|c|c|c|c|c}
J_d & & \begin{matrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{matrix} & \begin{matrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{matrix} & \begin{matrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{matrix} & & \\ \hline
 & \ddots & & & & & \\ \hline
 & & J_{r-1} & & 1_{(a)} & & \begin{matrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{matrix} \\ \hline
 & & & \tilde{J}_{2s+2} & & & \begin{matrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{matrix} \\ \hline
 & & & & -J_{r-1} & & \begin{matrix} \star & \star & \star \\ \star & \star & \star \\ \star & \star & \star \end{matrix} \\ \hline
 & & & & & \ddots & \\ \hline
 & & & & & & -J_d
\end{array}\right).
$$

Here the entry $1_{(a)}$ is the entry in the bottom-left location of its respective block, and the stars indicate arbitrary entries at any location in their blocks.

Subcase 1: if $j \leq n$, and $i \leq n$, then the case is almost identical to determining the orbit of a nilpotent matrix in $\mathfrak{gl}_n$, see Lemma 2.2.2 in Section 2.2. Let v_t be the last nonzero entry in the $j^{\text{th}}-$column, more precisely in position (t, j), and the symplectic property requires that $-v_t$ is in position $(2n+1-j, 2n+1-t)$. Let u be a unipotent symplectic matrix with 1s down the diagonal, $-v_t$ in position (t, i), and (since u is symplectic) v_t in position $(2n+1-i, 2n+1-t)$. Recall the action of the unipotent matrix on the basis vectors (Equation 2.1):

$$u(e_p) = \begin{cases} e_p - v_t e_t & \text{if } p = i, \\ e_p & \text{otherwise.} \end{cases}$$

$$u^{-1}(e_p) = \begin{cases} e_p + v_t e_t & \text{if } p = i, \\ e_p & \text{otherwise.} \end{cases}$$

Then

$$uxu^{-1}(e_i) = ux(e_i + v_t e_t) = ux(e_i) + u(v_t e_{t-1}),$$

so a v_t-value is added to the entry in position $(t-1, i)$, but no other entries before the j^{th} column are altered. And

$$uxu(e_j) = ux(e_j + \sum_{l=1}^{t} v_l e_l) = u(e_i - v_t e_t + \sum_{l=1}^{t} v_l e_l) = e_i + \sum_{l=1}^{t-1} v_l e_l.$$

So the v_t-entry in position (t, j) is killed off. Hence by conjugation with u, the v_t-value in position (t, j) moves up and to the left.

Subcase 2: if $j > n$, and $i \leq n$ then the other nonzero entry in the j^{th} column is found on the anti-diagonal. As before, we will determine a unipotent matrix to conjugate x with, but we calculate what entries can be found in uxu^{-1} for each column, with an example for further clarity. Recall that the second half of the symplectic basis is indexed backwards, f_1 is the last basis vector, f_2 the second to last, etc, with f_n being the $n+1^{\text{th}}$ basis vector, f_{n-1} being the $n+2^{th}$, etc. As an example, let $n = 7, j = 9, i = 6, t = 3$, so:

$$x = \left(\begin{array}{cccc|cccccc|cccc}
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & v_3 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & v_3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{array}\right).$$

Note that $x(f_i) = -e_i - v_t e_t$, and $x(f_t) = -f_{t+1} - v_t e_i$. In this example, $x(f_6) = -e_6 - v_t e_3$, and $x(f_3) = f_4 - v_3 e_6$.

Let u be the unipotent matrix with $-v_t$ in position (t, i) and v_t in position $(2n+1-i, 2n+1-t)$, then by Equation 2.1, we know that the first n columns of uxu^{-1} are the same as those in x, with a v_t removed from position (t, j) and added in position $(t-1, i)$. The calculation of interest is: $uxu^{-1}(e_i) = e_{i-1} + v_t e_{t-1}$. By the symplectic property, we also get that the v_t entry in position $(2n+1-j, 2n+1-t)$ is removed, and a $-v_t$-value in position $(2n+1-i, 2n+2-t)$ is added. To check this, we calculate the conjugation $u \cdot x$ for the last n columns:

$$u(f_p) = \begin{cases} f_p + v_t f_i & \text{if } p = t, \\ f_p & \text{otherwise.} \end{cases}$$

$$u^{-1}(f_p) = \begin{cases} f_p - v_t e_i & \text{if } p = t, \\ f_p & \text{otherwise.} \end{cases}$$

Checking for the columns p with $n < p < j$, we note that this corresponds to the basis vectors $f_{2n+1-n}, f_{2n+1-(n-1)}, \dots, f_{2n+1-(j-1)}$. As $n < j < 2n - d$, we get $f_{2n+1-n} > f_{2n+1-p} > f_{2n+1-j}$ and with $t \leq d$, we get $f_{2n+1-j} > f_t$, so $f_{2n+1-p} \neq f_t$ for $n < p < j$. For these f_{2n+1-p} it must follow that $uxu^{-1}(f_{2n+1-p}) = ux(f_{2n+1-p})$. Furthermore, as x is upper triangular, $x(f_{2n-1+p})$ does not involve f_t, so $ux(f_{2n+1-p}) = x(f_{2n+1-p})$. Next, for $p = j$, we repeat the calculation:

$$uxu^{-1}(f_{2n+1-j}) = ux(f_{2n+1-j}) = u(-e_i + \sum_{s=1}^{t} v_s e_s) = -e_i - v_t e_t + \sum_{s=1}^{t} v_s e_s = -e_i + \sum_{s=1}^{t-1} v_s e_s.$$

In the example,

$$uxu^{-1}(f_6) = ux(f_6) = u(-e_6 + \sum_{s=1}^{3} v_s e_s) = -e_6 - v_3 e_3 + \sum_{s=1}^{3} v_s e_s = -e_i + \sum_{s=1}^{2} v_s e_s = -e_6.$$

For $p > j, p \neq t-1$ and $p \neq t-1$, we again get $uxu^{-1}(f_p) = x(f_p)$. For $p = t-1$, we have that $uxu^{-1}(f_{t-1}) = -f_t - v_t f_i$, hence uxu^{-1} has a $-v_t$-entry in position $(2n+1-i, 2n+2-t)$, as u is symplectic. Finally, if $p = t$, we get that $uxu^{-1}(f_t) = ux(f_t - v_t f_i) = u(-f_{t+1} + v_t^2 e_t) = -f_{t+1} + v_t^2 e_t$, so the v_t-entry in position $(2n+1-j, 2n+1-t)$ is removed (the v_t^2-entry added in uxu^1, in position $(t, 2n+1-t)$ is not of interest).

To summarize, conjugation by u kills off the v_t-entry in position (t, j) at the expense of adding a v_t-value to the entry in position $(t-1, i)$, possibly making it nonzero. The other entries of uxu^{-1} before column j are unchanged, hence the v_t-value in position (t, j) is moved up and to the left. In particular for Subcase 2, because $t - 1 \leq d < n$, and $i \leq n$, the entry where a v_t-value is added, is in position $(t-1, i)$. If the entry in this position needs to be removed, subcase 1 can be applied to do so.

Subcase 3: in the last subcase, $i, j > n$. As before, the first d rows of column j form vector v, with v_t its last nonzero entry in position (t, j). The only nonzero entry in column j, after v, is -1 in position (i, j). With v_t in position (t, j), there is an entry v_t in position $(2n+1-j, 2n+1-t)$. Let u be the unipotent matrix with ones on the diagonal, and a v_t entry in positions (t, i) and $(2n+1-i, 2n+1-t)$.

Then for any e_p, we get $u(e_p) = e_p$ and $u^{-1}(e_p) = e_p$, so u and u^{-1} do nothing to the first half of the basis, hence the first n columns of x and uxu^{-1} are the same. Next, with $n+1 \leq i < j \leq 2n-d$, we let p range as follows ($n + 1 \leq p \leq 2n$):

$$\{n+1, \dots, i-1, i, i+1, \dots, 2n+1-t, 2n-t, \dots, 2n\}.$$

Then we get the following sequence of basis vectors f_{2n+1-p}:

$$\{f_n, \dots, f_{2n+1-(i-1)}, f_{2n+1-i}, f_{2n+1-(i+1)}, \dots, f_{t+1}, f_t, f_{t-1}, \dots f_1\},$$

and the action of u yields:

$$u(f_{2n+1-p}) = \begin{cases} f_{2n+1-p} - v_t e_t & \text{if } p = i, \\ f_{2n+1-p} - v_t e_{2n+1-i} & \text{if } p = 2n+1-t, \\ f_{2n+1-p} & \text{otherwise.} \end{cases}$$

$$u^{-1}(f_{2n+1-p}) = \begin{cases} f_{2n+1-p} + v_t e_t & \text{if } p = i, \\ f_{2n+1-p} + v_t e_{2n+1-i} & \text{if } p = 2n+1-t, \\ f_{2n+1-p} & \text{otherwise.} \end{cases}$$

The sequence of $u(f_{2n+1-p})$ is as follows:

$$\{f_n, \ldots, f_{2n+1-(i-1)}, f_{2n+1-i} - v_t e_t, f_{2n+1-(i+1)}, \ldots, f_{t+1}, f_t - v_t e_{2n+1-i}, f_{t-1}, \ldots f_1\},$$

by our assumptions, we have that $t \leq d$ and $j \leq 2n - d$. So when we look at what happens with the vectors f_{2n+1-p} for $n+1 \leq p < j$, with $p \neq i$, this never includes vector f_t.

First, for $n+1 \leq p < j$, and $p \neq i$, we get $uxu^{-1}(f_{2n+1-p}) = ux(f_{2n+1-p})$. The only nonzero entries in row i occur in column j (as $S^d(x)$ is in standard form) or later (the last d columns). So applying x to f_{2n+1-p} does not yield basis vector f_{2n+1-i}, and since x is strictly upper triangular, applying x to f_{2n+1-p} does not yield f_t. So $ux(f_{2n+1-p}) = x(f_{2n+1-p})$ when $n+1 \leq p < j$, and $p \neq i$. Hence for $n+1 \leq p < j$, and $p \neq i$, we have that the p^{th} column of uxu^{-1} is the same as the p^{th} column of x.

Second, if $p = i$, then $uxu^{-1}(f_{2n+1-i}) = ux(f_{2n+1-i} + v_t e_t) = ux(f_{2n+1-i}) + ux(v_t e_t)$. Since x is upper triangular, $x(f_{2n+1-i})$ yields basis vectors f_p, with $p > 2n+1-i$ and vectors e_p with $1 \leq p \leq n$, which are all fixed under u. Furthermore $ux(v_t e_t) = u(v_t e_{t-1}) = v_t e_{t-1}$, since we know the action of x on $v_t e_t$ explicitly, and all e_p vectors are fixed under u $(1 \leq p \leq n)$.

So the i^{th} column of uxu^{-1} is the same as the i^{th} column of x, except a v_t-value is added to the entry in position $(t-1, i)$.

Next, let $p = j$, then:

$$\begin{aligned} uxu^{-1}(f_{2n+1-j}) &= u(f_{2n+1-i} + \sum_{s=1}^{t} v_s e_s) \\ &= f_{2n+1-i} - v_t e_t + \sum_{s=1}^{t} v_s e_s \\ &= f_{2n+1-i} + \sum_{s=1}^{t-1} v_s e_s. \end{aligned}$$

So conjugating with u removes the v_t value in position (t, j).

Considering the action on the i^{th} column and the j^{th} column, conjugation by u kills off the v_t-entry in position (t, j), at the expense of adding a v_t-value to the entry in position $(t-1, i)$, which may make it nonzero. Except for the change of this value in position $(t-1, i)$, no other entry before column j is altered by the conjugation.

With these subcases in hand, we can remove nonzero entries v_t in position (t, j) in columns where a further nonzero entry in row i is present. Specifically, by conjugating, the v_t-value of the entry in position (t, j) moves up and to the left. By systematically iterating these conjugations, we can kill off all the v_t-entries in the first d rows (starting with the entry in position $(d, 2n+1-d)$; the last row of the rightmost column), in the columns where a further nonzero entry is present, until the only nonzero entries left in the first d rows are in the columns which have zero entries everywhere else. Since each of the conjugating elements is sent to the identity in the limit, we have that the limit along the "new" x is still equal to y. So we may assume that x is of the following form:

$$x = \left(\begin{array}{c|c|c|c|c|c|c}
J_d & & \begin{matrix}\star\\ \star\\ \star\end{matrix} & \begin{matrix}\star\\ \star\\ \star\end{matrix} & & & \\ \hline
 & \ddots & & & & & \\ \hline
 & & J_{r-1} & & 1_{(a)} & & \\ \hline
 & & & \tilde{J}_{2s+2} & & & \star \quad \star \quad \star \\ \hline
 & & & & -J_{r-1} & & \star \quad \star \quad \star \\ \hline
 & & & & & \ddots & \\ \hline
 & & & & & & -J_d
\end{array}\right),$$

where $1_{(a)}$ is the entry in the bottom-left of its respective block, and the stars indicate arbitrary entries in the first column or last row of their respective blocks (zero entries are omitted as usual). First, we consider the behaviour and orbit of x in an abstract way, then we analyse in detail what change of orbits occurs. While y induces vector chains of size $2r-2, 2s+2$ and two chains of size d, matrix x induces different vector chains. Consider the effect of x on a basis vector f_i for $1 \leq i \leq d$. If $x(f_i) \neq 0$, then f_i is in the vector sequence starting with f_1, and the value of f_i is added to a vector from another vector sequence in the center of the matrix. In other words, a vector sequence initiated in the $[d]$ part at the bottom of the matrix will be continued by the J_{2r-2} Jordan blocks or the J_{2s+2} Jordan block in the central portion of the matrix, where in the matrix y the vector chains of parts $[2r-2]$ and $[2s+2]$ are initiated. By the symplectic property, the vector chains of lengths $2r-2$ and $2s+2$ that are terminated in y are picked up by the J_d Jordan block at the top left of the matrix. In detail, the following moves occur, in two cases:

Case 1: if $s > d$, then x induces vector chains of size $s+2a$, $r+2b$ and two chains of size $d-a-b$, with at least one of a and b nonzero. So taking $\lim_\lambda x$ is a move of type 2, in fact two moves of type 2 occur simultaneously:

$$\mathcal{O}([r+2a, s+2b, d-a-b, d-a-b]) \to \mathcal{O}([r, s, d, d]),$$

is a combination of

$$\mathcal{O}([r+2a, s+2b, d-a-b, d-a-b]) \to \mathcal{O}([r, s+2b, d-b, d-b]),$$
$$\mathcal{O}([r, s+2b, d-b, d-b]) \to \mathcal{O}([r, s, d, d]).$$

Case 2: if $d > r$, then we analyse the matrix x separately for even d and for odd d. Without loss of generality, we can assume that the nonzero entry is in the first column after the $[d]$ block, so in the otherwise zero column of the J_{2r-2} block. If d is even, let $d = 2d'$, and denote top left $2d' \times (2d'+1)$ block of matrix x as follows:

$$\left(\begin{array}{c|c} & x_{1,2d'+1} \\ & \vdots \\ & x_{k,2d'+1} \\ & x_{k+1,2d'+1} \\ J_d & \vdots \\ & x_{k+i+,2d'+1} \\ & x_{k+i+1,2d'+1} \\ & \vdots \\ & x_{k+i+j,2d'+1} \\ \hline \end{array}\right),$$

here we let k, i and j take their maximum value, and we now relate the orbit of x to the position of the nonzero entry, and determine in what values k, i and j range.

If we consider a single nonzero entry, in a row $k+i+j$, we start by considering the last row; row $d = 2d'$ Then x has a Jordan block of size $2d'+2d'+2r-2$ (and a Jordan block of size $2s+2$, which is not involved). Then, if we consider the nonzero entry to be in one row higher, this part loses two pieces to two different parts, so x has parts of size $2d'+2d'+2r-4$, and there are now two parts of size 1. For every row higher, the largest part loses two additional pieces to the two smaller parts, until these are of size $2r-2$, hence x is in the orbit of $[2d'+2d'-(2r-2), 2r-2, 2r-2, 2s+2]$. Hence, if the nonzero entry is in one of these last $2r-2$ rows, taking the limit $\lim_\lambda x$ yields the move:

$$\mathcal{O}([2d'+2d'+(2r-2)-2j, j, j, 2s+2]) \to \mathcal{O}([2d', 2d', 2r-2, 2s+2]),$$

which is a move of type 2. Furthermore, we have determined that $1 \leq j \leq 2r-2$.

Next, if we consider the nonzero entry to be another row higher, the first part loses two pieces and the second part gains two pieces, hence taking $\lim_\lambda x$ gives the following move: $\mathcal{O}([2d'+2d'-(2r-2)-2, 2r-2+2, 2r-2, 2s+2]) \to \mathcal{O}([2d', 2d', 2r-2, 2s+2])$. Then, for every row we go up, the first part will be two additional pieces smaller, and the second part will be two additional pieces bigger, hence if the nonzero entry is in row $2d'-(2r-2)-i$, x is in the orbit of $[2d'+2d'-(2r-2)-2i, 2r-2+2i, 2r-2, 2s+2]$, and taking the limit $\lim_\lambda x$ yields the move:

$$\mathcal{O}([2d'+2d'-(2r-2)-2i, 2r-2+2i, 2r-2, 2s+2]) \to \mathcal{O}([2d', 2d', 2r-2, 2s+2]),$$

which is a move of type 1. This is possible until $2i = 2d'-(2r-2)$, because when this equality holds, x is in the orbit of $[2d', 2d', 2r-2, 2s+2]$, which is the same orbit as $y = \lim_\lambda x$, and no move occurs. Hence we have determined that $1 \leq i \leq d'-(r-1)$, and then for the maximum values of i and j, we determine that the range of k is as follows: $1 \leq k \leq 2d'-i-j = d'-(r-1)$. We can now make the following conclusion:

(i) If the nonzero entry is in position k, for $1 \leq k \leq 2d'-i-j = d'-(r-1)$, then x is in the orbit of $[2d', 2d', 2r-2, 2s+2]$. Taking $\lim_\lambda x$ will not change the orbit.

(ii) If the nonzero entry is in position $k+i$ for the maximum value of k and $1 \leq i \leq d-(r-1)$, then x is in the orbit of $[2d'+2i, 2d'-2i, 2r-2, 2s+2]$. Taking $\lim_\lambda x$ is a move of type 1, where the $[2r-2]$ and $[2s+2]$ pieces are unchanged.

(iii) If the nonzero entry is in position $k+i+j$ for maximum values of k, i, and $1 \leq j \leq 2r-2$, then x is in the orbit of $[2d'+2i+2j, 2d'-di-j, 2r-2-j]$. Taking $\lim_\lambda x$ is a combination of move 1 and move 2: $\mathcal{O}([2d'+2i+2j, 2d'-di-j, 2r-2-j]) \to \mathcal{O}([2d'+2i, 2d'-di, 2r-2]) \to \mathcal{O}([2d, 2d, 2r-2, 2s+2])$.

If d is odd, we denote $d = 2d'+1$, and this introduces a row where the nonzero entry places x in the orbit of $[2d'+1, 2d'-1, r-2, s+2]$. Then the conclusion changes only by the size of the blocks:

(i) If the nonzero entry is in position k, for $k \leq d' - (r-1)$, then x is in the orbit of $[2d'+1, 2d'+1, 2r-2, 2s+2]$. Taking $\lim_\lambda x$ will not change the orbit.

(ii) If the nonzero entry is in position $k+i$ for the maximum value of k and $1 \leq i \leq d-(r-1)+1$, then x is in the orbit of $[2d'+1+(2(i-1)+1), 2d'-(2(i-1)-1), 2r-2, 2s+2]$. Taking $\lim_\lambda x$ is a move of type 1, where the $[2r-2]$ and $[2s+2]$ blocks are unchanged.

(iii) If the nonzero entry is in position $k+i+j$ for maximum values of k, i, and $1 \leq j \leq 2r-2$, then x is in the orbit of $[2d'+(2(i-1)+1)+2j, 2d'-(2(i-1)+1)-j, 2r-2-j]$. Taking $\lim_\lambda x$ is a combination of move 1 and move 2: $\mathcal{O}([2d'+(2(i-1)+1)+2j, 2d'-(2(i-1)+1)-j, 2r-2-j]) \to \mathcal{O}([2d'+2i, 2d'-di, 2r-2]) \to \mathcal{O}([2d, 2d, 2r-2, 2s+2])$.

We now have determined the move for a nonzero entrey in the $d+1^{\text{th}}$ column: a nonzero entry in the first d rows of the first column above the J_{r-1} Jordan block gives rise to a move involving the parts that are of size $[d]$ and $[2r-2]$ after taking the limit, and we have also established that it does not involve the part of size $[2s+2]$. Similarly, a nonzero entry in the first column above the $\tilde{J}_{2s+2}$ Jordan block gives rise to a move involving the parts that are of size $[d]$ and $[2s+2]$ after taking the limit, but it doesn't involve the part of size $[2r-2]$.

In the case there are nonzero entries in the first columns above both Jordan blocks, taking the limit will give rise to two moves, that separately involve the parts which (after taking the limit) are of size $[2s+2]$ and $[d]$, and the parts which are of size $[2r-2]$ and $[d]$. However, the nonzero entries do not allow for a move between the parts that are of size $[2r-2]$ and $[2s+2]$, after taking the limit. Hence in this special case, the required move does not occur.

So for both case 1 and case 2, the nonzero entries described are only in the first d rows of the matrix so the move only involves the $[2r-2]$ and $[d]$ parts when the nonzero entries are above the J_{r-1} bock. In case the nonzero entries are above the J_{2s+2} block, the move involves the $[2s+2]$ and $[d]$ blocks.

We will now generalize as we go down the diagonal, where the J_{d_i} blocks are present, for $1 \leq i \leq m$. The nonzero entries in the first column above the J_{r-1} blocks are next to the J_{d_i} blocks, so we obtain a move involving the $[2r-2]$ and $[d_i]$ parts (after taking the limit). Regardless of the details of the move, the $[2s+2]$ part is not involved.

Vice versa, the nonzero entries in the first column above the J_{2s+2} Jordan block, in any row of one of the J_d or J_{d_i} Jordan blocks give rise to a move involving the parts $[2s+2], [d], [d_1], \ldots, [d_m]$ (after taking the limit), but the $[2r-2]$ part is not involved.

Hence, regardless of the location of the nonzero entries above the J_{r-1} and $\tilde{J}_{2s+2}$ Jordan block, a move of type $[2r, 2s] \to [2r-2, s2+2]$ does not occur.

This shows that the move $\mathcal{O}([\ldots, r, s, \ldots]) \to \mathcal{O}([\ldots, r-2, s+2, \ldots])$ does not occur. We conclude that the orbit $\mathcal{O}([\ldots, r-2, s+2, \ldots])$ is not accessible from the orbit $\mathcal{O}([\ldots, r, s, \ldots])$. $\square$

3.3 Conclusion for $\mathfrak{sp}_{2n}$

In this chapter, we considered the five minimal moves in the dominance order of partitions corresponding to symplectic nilpotent orbits (see 3.1), and determined that four are valid moves for accessibility of these orbits. We can now describe the order of the orbits in the symplectic algebras. In Section 2.4, we drew a conclusion for accessibility of nilpotent orbits in the general linear group: the partial order on orbits is the same as the dominance order. This is not the case for the symplectic algebra, as we have found in Section 3.2 that distinguished partitions are not accessible from partitions higher up in the dominance order (e.g. in $\mathfrak{sp}_8$, $\pi_1 = [6,2]$ is not accessible from $\pi_2 = [8]$). Moreover, if we consider two partitions $\pi_1 = [\ldots, r, s, \ldots]$ and $\pi_2 = [\ldots, r-2, s+2, \ldots]$, with $r \geq s+6$, then π_1 dominates π_2, but π_2 is not accessible from π_1. In Section 3.1, we have seen the moves that are valid in the symplectic algebra, so we can determine an accessibility diagram with the following theorem:

Theorem 3.3.1. The partial order on nilpotent symplectic orbits given by the accessibility relation is determined by a combination of the moves of type 1 – 4.

In sections 3.1 and 3.2, we have seen that $\mathcal{O}(y)$ is accessible from $\mathcal{O}(x)$ if its partition change is a move of type 1 – 4. But $\mathcal{O}(y)$ is not accessible from $\mathcal{O}(x)$ if the partition change can only be obtained through a move of type 5. With the proof of Lemma 1.8.1 (which tells us that the accessibility order is a partial order over algebraically closed κ), we can also determine the accessibility order with respect to the dominance order:

Lemma 3.3.1. Let x, y be two nilpotent symplectic matrices. Then $\overline{\mathcal{O}(y)}^c \leq \overline{\mathcal{O}(y)}^c$ if $\overline{\mathcal{O}(y)} \leq \overline{\mathcal{O}(x)}$, unless the move from $\mathcal{O}(x)$ to $\mathcal{O}(x)$ is of type 5.

With this theorem we can answer Questions 1 and 2 in Section 1.8.

1 The preorder on orbits given by cocharacter closure is a partial order.

2 Over a closed field $\overline{\kappa}$, the cocharacter-closure $\overline{G \cdot x}^c$ does not coincide with the Zariski-closure if the Zariski-closure contains a distinguished orbit, or if the Zariski-closure contains an orbit $G \cdot y$, where the move between $\mathcal{O}(x)$ and $\mathcal{O}(y)$ is of type 5.

We can now describe the difference between the dominance order, and the accessibility order, of orbits. Let O_1 and O_2 be two partitions with $\pi_1 = [\ldots, r_1, s_1, \ldots]$ and $\pi_2 = [\ldots, r_2, s_2, \ldots]$. Then $O_1 \geq O_2$, while O_2 is *not* accessible from O_1 if $[\ldots, r_1, s_1, \ldots] = [\ldots, r_2 + 2k, s_2 - 2k, \ldots]$ for $k \in \mathbb{Z}_{>0}$, and with $r_1 \geq s_1 + 6$ and $r_2 - 2k \neq s_2 + 2k$.

Example. The following figure shows the accessibility diagram and the dominance order of nilpotent orbits in $\mathfrak{sp}_8$.

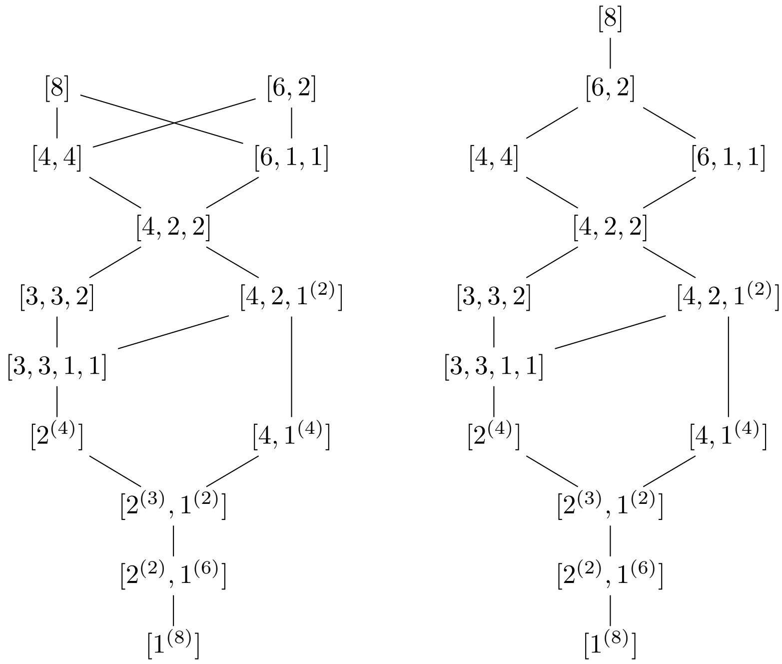

Figure 3.2: Accessibility and Dominance diagrams of the nilpotent orbits in $\mathfrak{sp}_8$

Example. The following figure shows the accessibility diagraom of the nilpotent orbits in $\mathfrak{sp}_{12}$. Note that $\mathcal{O}([6,4,1^2])$ is not accessible from $\mathcal{O}([8,2,1^2])$, because the move of type 5 does not occur in the symplectic algebra.

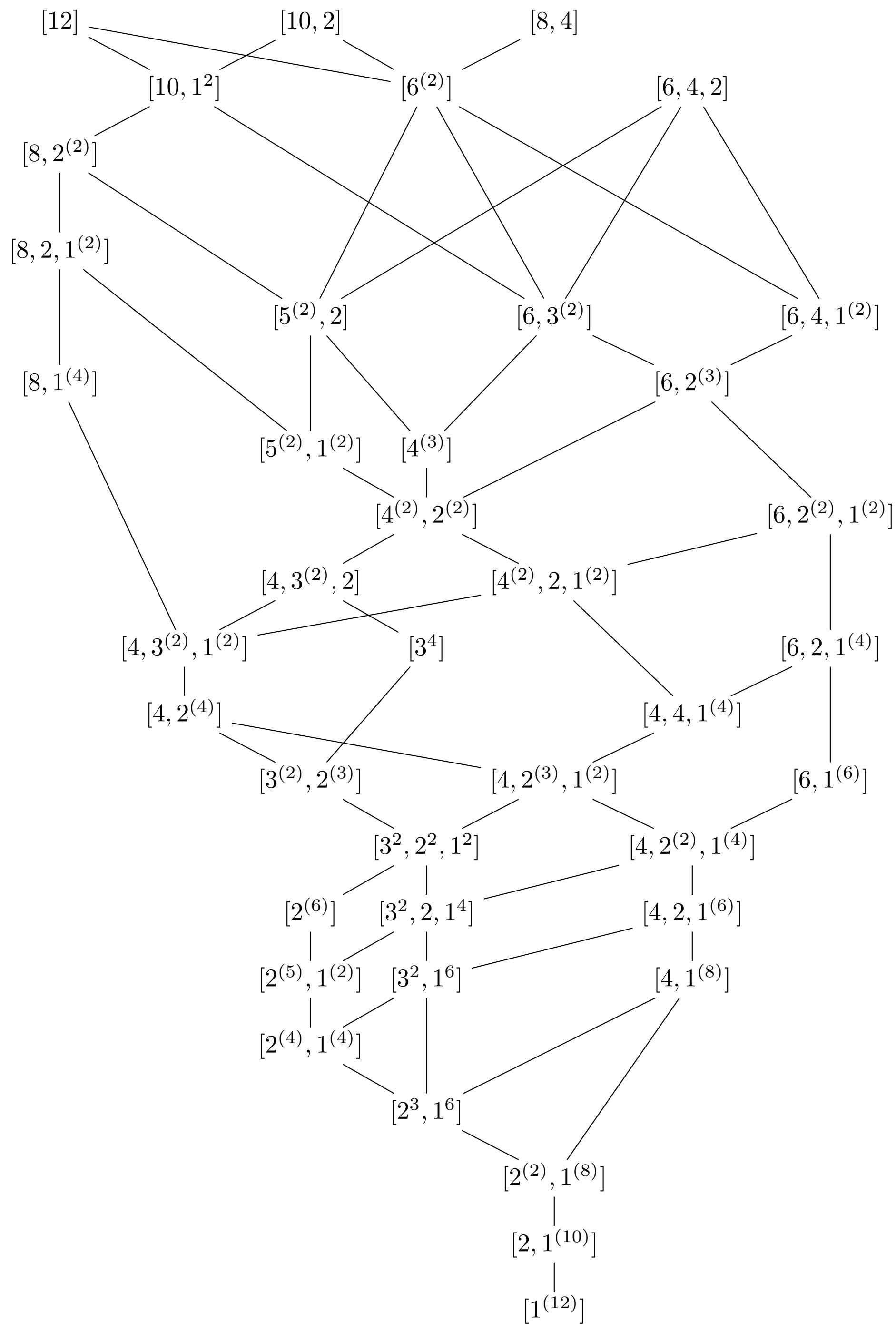

Figure 3.3: Accessibility and diagram of the nilpotent orbits in $\mathfrak{sp}_{12}$

Chapter 4

Results in the orthogonal algebra

4.1 Partitions and moves

In this section, we describe some results in the orthogonal algebra. It will show that the matrix forms of nilpotent elements are similar to those in the symplectic algebra, and that the orthogonal moves share similarities with the symplectic moves.

We recall from Section 1.11.2 that the basis of $\mathfrak{o}_n$ is as follows:

$$\mathcal{B} = \{e_1, \ldots, e_n, f_n, \ldots, f_1\} \text{ if } \operatorname{Dim}(\mathcal{B}) = 2n,$$
$$\mathcal{B} = \{e_1, \ldots, e_n, v, f_n, \ldots, f_1\} \text{ if } \operatorname{Dim}(\mathcal{B}) = 2n+1,$$

and the matrix of the bilinear form with respect to $\mathcal{B}$ is:

$$\Omega_{\mathrm{O}} = \begin{pmatrix} & & 1 \\ & \iddots & \\ 1 & & \end{pmatrix}.$$

Then:

$$O_n = \{g \in \mathrm{GL}_n \mid g^T \Omega_{\mathrm{O}} g = \Omega_{\mathrm{O}}\},$$
$$\mathfrak{o}_n = \{x \in \mathfrak{gl}_n \mid x^T \Omega_{\mathrm{O}} + \Omega_{\mathrm{O}} x = 0\}.$$

Recall the orthogonal case of Theorem 1.11.1:

Theorem. Assume ϕ is symmetric (the orthogonal algebra). Then there exists a nilpotent element in $\mathfrak{g}$ with this partition if and only if r_i is even for all even n_i.

Then from Lemma 1.11.3, we know that:

Lemma. Let $x \in \mathfrak{o}_n$, then x is distinguished if and only if its partition π has distinct odd parts (and no even parts).

Like we did in section 3.1, we consider the minimal moves through the dominance order of the partitions corresponding to orthogonal nilpotent orbits. Then we have the following moves:

1. $\mathcal{O}([2m+1, 2m-1]) \to \mathcal{O}([2m, 2m])$.
2. $\mathcal{O}([2n+1, m, m]) \to \mathcal{O}([2n-1, m+1, m+1])$.

3. $\mathcal{O}([n, n, 2m-1]) \to \mathcal{O}([n-1, n-1, 2m+1])$.

4. $\mathcal{O}([n, n, m, m]) \to \mathcal{O}([n-1, n-1, m+1, m+1])$.

5. $\mathcal{O}([2n+1, 2m+1]) \to \mathcal{O}([2n-1, 2m+3])$.

Comparing with the moves in Section 3.1, it shows that the moves in the orthogonal algebra are similar to those in the symplectic algebra; only the sizes of the partitions that occur once are of different size. In moves 2, 3, and 4 there is at most one distinct partition, so there are very similar matrix forms (see Sections 1.11.1 and 1.11.2 where we determined the standard forms of matrices). Therefore we will omit Moves 2, 3, and 4. In the next section, we will analyse Move 1 in more detail, because the standard form of a matrix with partition $[2m+1, 2m-1]$ is quite different from the standard form of a matrix with partition $[2m, 2m-2]$, and it is worth viewing the differences in more detail. Move 5 is impossible to realise with cocharacters, which we will analyse in section 4.2.

4.1.1 The cocharacter realizing move 1

In first move for the orthogonal algebras is: $\mathcal{O}([2m+1, 2m-1]) \to \mathcal{O}([2m, 2m])$. We start with example $\mathcal{O}([7,5]) \to \mathcal{O}([6,6])$. As established in Section 1.11.2, the standard form of an element in $\mathcal{O}([7,5])$ is:

$$x = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

and we have also determined that this type of nilpotent matrix may not occur in $\mathfrak{o}_n$ over κ if κ is not algebraically closed, since $\sqrt{-1}$ and $\sqrt{2}$ may not be defined. The matrix x induces the following vector chains (for the vector chains of the general case, see Section 1.11.2):

$$f_1 \to -f_2 \to f_3 \to \frac{1}{\sqrt{2}}(-f_6 - e_6) \to -e_3 \to -e_2 \to -e_1 \to 0,$$
$$f_4 \to -f_5 \to \frac{i}{\sqrt{2}}(f_6 - e_6) \to e_5 \to e_4 \to 0.$$

Next, let x' be of the form:

$$
x' = \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix},
$$

then it induces the vector chains:

$$
\begin{gathered}
f_1 \to -f_2 - e_6 \to f_3 - e_5 \to -f_4 - e_4 \to f_5 - e_3 \to -f_6 - e_2 \to -2e_1 \to 0. \\
f_2 - e_6 \to -f_3 - e_5 \to f_4 - e_4 \to f_5 - e_3 \to f_6 - e_2 \to 0.
\end{gathered}
$$

As some entries in x involve $i = \sqrt{-1}$, we can expect the base change to show parts containing i as well, and indeed we obtain the following base change matrix:

$$
g_1 = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1-i}{\sqrt{2}} & \frac{1+i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & \frac{1+i}{\sqrt{2}} & \frac{1-i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
$$

As some of the entries in g_1 contain i as well, it is clear that this base change does not work over any field κ, since $\sqrt{-1}$ may not be defined in some fields that are not algebraically closed. Matrix g_1 is not orthogonal, as $g_1^T \Omega_{\mathrm{O}} g_1 \neq \Omega_{\mathrm{O}}$. Further calculation, as described in Section 1.12, gives the following base change matrix:

$$g = \begin{pmatrix} \sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} \end{pmatrix},$$

which is orthogonal.

Here we see that the orthogonal base change g to obtain x', when x is given, may not occur over any field κ, since $\sqrt{-1}$ and $\sqrt{2}$ are not always defined. Earlier, we have also seen that x is not always defined when the restriction of algebraically closed fields is lifted, since x also has entries containing $\sqrt{-1}$ and $\sqrt{2}$. Hence, for the orthogonal algebras, both the orbits of nilpotent elements and the accessibility diagrams may change when the field κ is not closed.

4.2 A non-move for $\mathfrak{o}_n$

We have seen with Lemma 1.9.1 that distinguished partitions are not accessible. In $\mathfrak{gl}_n$, the only distinguished partition is $\pi = [n]$. In the symplectic algebra, distinguished partitions are of the form $[2r_1, \ldots, 2r_p]$, where each r_i is distinct. In Section 3.2, we have seen that the orbit of the non-distinguished partition $\mathcal{O}([\ldots, 2r-2, 2s+2, \ldots])$ is not accessible from $\mathcal{O}([\ldots, 2r, 2s, \ldots])$, if $2r \geq 2s+6$, and $2r$ and $2s$ are adjacent parts in the partition.

It is now a natural question to ask if a similar case occurs in the orthogonal algebra, for two odd parts in a partition. We recall the method to prove that the move of type 5 cannot occur in the symplectic algebra, and use it for the move in the orthogonal algebra.

4.2.1 The setup

First, suppose that y - of size $2n$ - is in standard form, and x is of a form such that $\lim_\lambda x = y$, with λ also in standard form. Then $\lim S^d(\lambda)S^d(x) = S^d(y)$ and we have two possibilities:

(i) $S^d(x)$ is conjugate to $S^d(y)$ so no orbit change occurs.

(ii) $S^d(x)$ is strictly higher than $S^d(y)$ in the dominance order.

As in 3.2.2, we may assume that:

(i) y is in standard form with a repeated $[\ldots, d, d, \ldots]$ part on the outside.

(ii) x is another nilpotent element.

(iii) λ is a standard-form cocharacter such that $\lim_\lambda x = y$.

Next, we denote $x = y = x_0$, then there is a λ_0 such that $\lim_{\lambda_0} x_0 = 0$. x_0 is strictly upper triangular because λ is in standard form. Then $\lim_{\lambda_0} y = y$ and $x' := \lim_{\lambda_0} x$ exists. Since $\lim_\lambda x = y$, $\lim_\lambda x' = y$ as well, hence x' lies between x and y in the dominance order. Then if x and y are adjacent, we get that one of the following two possibilities occurs:

1. $x' = y$.

2. x' is conjugate to x.

In the case that $x' = y$, the only difference between x and y lies in the outside d rows and columns; $S^d(x) = S^d(y)$. If x' is conjugate with x, we may replace x with x' and assume that x has the same repeated $[d, d]$ part on the outside.

4.2.2 The orthogonal non-move

The move of type 5 is of the form $[\ldots, 2r+1, 2s+1, \ldots] \to [\ldots, 2r-1, 2s+3, \ldots]$ between adjacent parts, with $2r - 1 > 2s + 3$. As both parts need to be odd (since they occur once in the partition), it follows that $2r - 1 \geq 2s + 3 + 2$, or $2r \geq 2s + 6$, as is the case in the symplectic move of type 5.

Now we apply Section 3.2.3 to this orthogonal case. If x' is conjugate to x, then x, y share a $[\ldots, d, d, \ldots]$ part which we may assume is on the outside. Then we may apply the shrinking operation to remove the $d-$parts. So here too the move between $S^d(x)$ and $S^d(y)$ is the same as between x and y, for the two adjacent $[\ldots, 2r+1, 2s+1, \ldots]$ parts. By induction on the shrinking operation, this is equivalent to the move $[2r+1, 2s+1] \to [2r-1, 2s+3]$ between distinguished partitions, which cannot occur.

So we are left with the first case: $x' = y$. By replacing x with x', and λ with λ_0, this is the situation of $S^d(x) = S^d(y)$. The first d rows and last d columns are related in $\mathfrak{o}_n$. By using the suitable conjugations we will again kill off nonzero entries in the first d rows (after the first d columns) and last the d columns (before the last d rows). We let v denote the first d rows of the j^{th} column, viewed as a column vector of length d. Let v_s denote the s^{th} entry of v, and let v_t denote the last nonzero entry of v. The difference between the proof in the orthogonal case and the symplectic case, is that in the orthogonal case several columns have two nonzero entries in two rows below row d, which we will denote by i_α, i_β. We can then make the following observations:

1. There is one other nonzero entry, or there are two in the j^{th} column, as $S^d(x) = S^d(y)$ is in standard form.

2. x is strictly upper triangular, so $d < i < j$.

3a. If there is one other nonzero entry, this entry is a 1 if $i \leq n$, and -1 if $i > n$.

3b. If there are two other nonzeroes, we denote these by α, β for the entries in position (i_α, j) and (i_β, j) respectively. The values α and β take are $\frac{1}{\sqrt{2}}$, $-\frac{1}{\sqrt{2}}$, $\frac{i}{\sqrt{2}}$, or $-\frac{i}{\sqrt{2}}$.

4a. If one other nonzero entry is present in column j, then this is the only nonzero entry in row i, with $i > d$.

4b. If two other nonzero entries are present in column j, then there are two nonzero entries in row i_α, and two nonzero entries in row i_β, in accordance with the “block” in the centre of the matrix.

So x is of the following standard form (see Section 1.11.2), with repeated blocks between J_d and J_{r-1} left out for symplicity:

$$
x = \left(\begin{array}{c|c|c|c|c|c|c}
J_d & \star \quad \star \quad \star & \star \quad \star \quad \star & \star \qquad \star & \star \quad \star \; \star & \star \quad \star \; \star & \\
\hline
 & J_{r-1} & & \frac{1}{\sqrt{2}}_{(a)} \qquad \frac{1}{\sqrt{2}}_{(b)} & & & \begin{matrix} \star \\ \star \\ \star \end{matrix} \\
\hline
 & & J_{s+1} & \frac{i}{\sqrt{2}}_{(c)} \qquad -\frac{i}{\sqrt{2}}_{(d)} & & & \begin{matrix} \star \\ \star \\ \star \end{matrix} \\
\hline
 & & & & \begin{matrix} \frac{i}{\sqrt{2}}_{(e)} \\ -\frac{i}{\sqrt{2}}_{(g)} \end{matrix} & \begin{matrix} -\frac{1}{\sqrt{2}}_{(f)} \\ -\frac{1}{\sqrt{2}}_{(h)} \end{matrix} & \begin{matrix} \star \\ \star \end{matrix} \\
\hline
 & & & & -J_{s+1} & & \begin{matrix} \star \\ \star \\ \star \end{matrix} \\
\hline
 & & & & & -J_{r-1} & \begin{matrix} \star \\ \star \\ \star \end{matrix} \\
\hline
 & & & & & & -J_d
\end{array}\right),
$$

where the stars indicate nonzero entries at any location in their respective blocks, and:

$(a) = (d + r - 1, d + r + s + 1)$	$(b) = (d + r - 1, d + r + s + 2)$
$(c) = (d + r + s, d + r + s + 1)$	$(d) = (d + r + s, d + r + s + 2)$
$(e) = (d + r + s + 1, d + r + s + 3)$	$(f) = (d + r + s + 1, d + r + 2s + 4)$
$(g) = (d + r + s + 2, d + r + s + 3)$	$(h) = (d + r + s + 2, d + r + 2s + 4)$

We now split into the following subcases, where subcases 1 and 3 are identical to subcases 1 and 3 in the symplectic case, see Section 3.2.3. Subcase 2 is different, and we split that up further, into four parts.

1. $j \leq n$, and $i \leq n$,

2a. $j = n - 1$, and $i_\alpha, i_\beta \leq n$,

2b. $j = n$, and $i_\alpha, i_\beta \leq n$,

2c. $j = n + 1$, with $i_\alpha = n$, and $i_\beta = n + 1$.

2d. $j > n + 1$, with $i_\alpha = n$, and $i_\beta = n + 1$

3. $j > n$, and $i > n$.

First, as subcases 1 and 3 in the orthogonal case are the same as subcases 1 and 3 in the symplectic case, a suitable unipotent matrix u will move the nonzero entry v_t in position (t, j) up and to the left, matched by a $-v_t$ entry in position $(2n + 1 - j, 2n + 1 - t)$ that moves down and to the right. By repeatedly conjugating with the desired unipotent matrices u, all entries in these cases will either be killed off, at the cost of changing the entries in the first d rows of otherwise zero columns in x, or changing the entries in the first d rows in columns j that have two nonzero entries further down the column; we kill those off by applying any of the second subcases.

In the subcases 2a through 2d, the entry v_t is in position (t, j), with $t \leq d$, and there are two other nonzero entries further down column j: α in position (i_α, j), β in position (i_β, j), with both $i_\alpha, i_\beta > d$. Then let u be a unipotent matrix with 1s down the diagonal, and entries

$\hat{\alpha}$ in position (t, i_α), $\hat{\beta}$ in position (t, i_β), and (as the matrices are orthogonal) $-\hat{\alpha}$ in position $(2n+1-i_\alpha, 2n+1-t)$, $-\hat{\beta}$ in position $(2n+1-i_\beta, 2n+1-t)$. In the following table, the values of $\hat{\alpha}$ and $\hat{\beta}$ are set out for each part of subcase 2.

	$\hat{\alpha}v_t$	$\hat{\beta}v_t$
$2a$	$-\alpha v_t = -\frac{1}{\sqrt{2}}v_t$	$-\beta v_t = -\frac{i}{\sqrt{2}}v_t$
$2b$	$-\alpha v_t = -\frac{1}{\sqrt{2}}v_t$	$\beta v_t = -\frac{i}{\sqrt{2}}v_t$
$2c$	$\alpha v_t = \frac{i}{\sqrt{2}}v_t$	$\beta v_t = -\frac{i}{\sqrt{2}}v_t$
$2d$	$-\alpha v_t = \frac{1}{\sqrt{2}}v_t$	$-\beta v_t = -\frac{1}{\sqrt{2}}v_t$

Then uxu^{-1} kills off the v_t-entry in position (t, j), at the expense of adding $-\hat{\alpha}v_t$ and $-\hat{\beta}v_t$ values to the entries in positions $(t-1, i_\alpha)$ and $(t-1, \beta)$, respectively. The conjugation also kills off the $-v_t$-entry in position $(2n+1-j, 2n+1-t)$, while adding $\hat{\alpha}v_t$ and $\hat{\beta}v_t$ values to the entries in positions $(2n+1-i_\beta, 2n+1-(t-1))$ and $(2n+1-i_\alpha, 2n+1-(t-1))$.

Hence we can conclude that in these subcases, we can conjugate by a suitable unipotent matrix to move a nonzero entry v_t in column j, in the first d rows of x, up and to the left while there are two nonzero entries further down in column j. Then by conjugating with suitable unipotent matrices for all subcases 1 through 3, we can obtain a new matrix x with the nonzero entries in the first d rows only in the columns which are zero columns in $S^d(x)$:

$$x = \left(\begin{array}{c|c|c|c|c|c|c}
J_d & \star & \star & & & & \\ \hline
 & J_{r-1} & & \frac{1}{\sqrt{2}}_{(a)} \quad \frac{1}{\sqrt{2}}_{(b)} & & & \\ \hline
 & & J_{s+1} & \frac{i}{\sqrt{2}}_{(c)} \quad -\frac{i}{\sqrt{2}}_{(d)} & & & \\ \hline
 & & & & \begin{array}{c} \frac{i}{\sqrt{2}}_{(e)} \\ -\frac{i}{\sqrt{2}}_{(g)} \end{array} & \begin{array}{c} -\frac{1}{\sqrt{2}}_{(f)} \\ -\frac{1}{\sqrt{2}}_{(h)} \end{array} & \\ \hline
 & & & & -J_{s+1} & & \star \\ \hline
 & & & & & -J_{r-1} & \star \\ \hline
 & & & & & & -J_d
\end{array}\right),$$

where the stars indicate nonzero entries in the first column or last row of their respective blocks, and:

$(a) = (d+r-1, d+r+s+1)$	$(b) = (d+r-1, d+r+s+2)$
$(c) = (d+r+s, d+r+s+1)$	$(d) = (d+r+s, d+r+s+2)$
$(e) = (d+r+s+1, d+r+s+3)$	$(f) = (d+r+s+1, d+r+2s+4)$
$(g) = (d+r+s+2, d+r+s+3)$	$(h) = (d+r+s+2, d+r+2s+4)$

Now we again analyse the vector chains induced by x, and compare it to our assumption. Recall that y induces vector chains of size $2r-1, 2s+3$, and two of size d. To determine the vector chains of x, we can apply the analysis from Section 3.2.3, to conclude that the vector chains are not of size $2r+1, 2s+1$, and d (with multiplicity 2). Hence the move made by taking the limit $\lim_\lambda x$ to obtain y is not of the form $[\ldots, 2r+1, 2s+1, \ldots] \to [\ldots, 2r-1, 2s+3, \ldots]$. Hence the move of type 5 doesn't occur in the orthogonal algebra.

4.3 Conclusion for $\mathfrak{o}_n$

As we have observed the many similarities between accessibility of nilpotent orbits in the symplectic algebra, and the accessibility of nilpotent orbits in the orthogonal algebra, we can draw the same conclusion. Theorem 3.3.1 in Section 3.3 applies to the orthogonal algebra as well:

Theorem 4.3.1. The partial order on nilpotent orthogonal orbits given by the accessibility relation is determined by a combination of the moves of type $1-4$.

Proof. To show that moves $1-4$ apply, while Move 5 doesn't, we have explicitly shown that move 1 is a valid move between orthogonal nilpotent orbits. The moves $2, 3, 4$ are constructed in the same way as they are for the symplectic algebra, in Sections 3.1.2, 3.1.3, 3.1.4. To determine that move 5 is invalid, the theory of move 5 in the symplectic case can be applied again (see 3.2.3), though with the addition for the subcases where a nonzero entry v_t in the first d rows is in a column with 2 nonzeroes in rows i_α, i_β for $i_\alpha, i_\beta > d$. That the ordering in the orthogonal algebra is reflexive, transitive and antisymmetric is covered by the proof of Theorem 3.3.1. Hence the accessibility relation gives a partial order on nilpotent orbits in the orthogonal groups. □

Then the answers to Questions 1 and 2 in Section 1.8 are the same for the orthogonal algebra and the symplectic algebra:

1 The preorder on orbits given by cocharacter closure is a partial order.

2 Over a closed field $\overline{\kappa}$, the cocharacter-closure $\overline{G \cdot x}^c$ does not coincide with the Zariski-closure if the Zariski-closure contains a distinguished orbit, or if the Zariski-closure contains an orbit $G \cdot y$, where the move between $\mathcal{O}(x)$ and $\mathcal{O}(y)$ is of type 5.

Example. The following figure shows the accessibility diagram and the dominance order of nilpotent orbits in $\mathfrak{o}_8$.

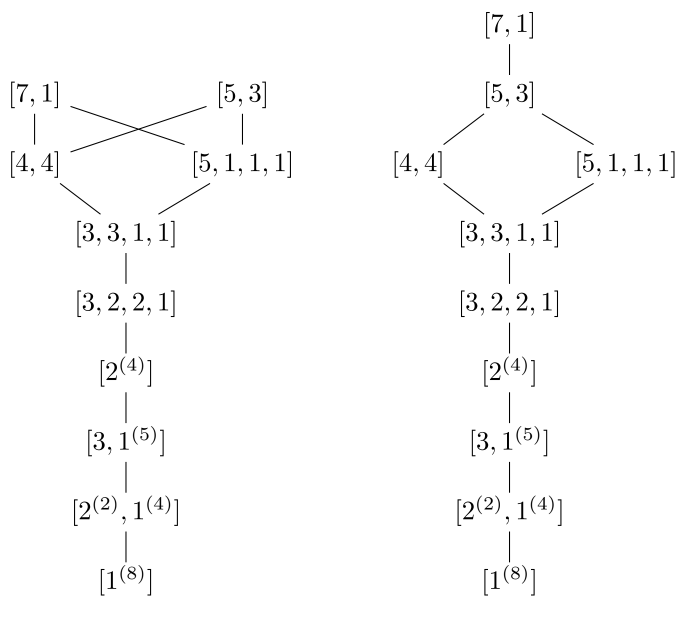

Figure 4.1: Accessibility and Dominance diagrams of the nilpotent orbits in $\mathfrak{o}_8$.

Chapter 5

Further research

5.1 Non-algebraically closed fields

In this thesis we have focused on the case where the group G acting on a set X, is over an algebraically closed field κ. In this chapter, we will explore reasons for that, and to what extent it might be possible to relax the assumption of an algebraically closed field. This is important, since one of the motivations for the introduction of cocharacter closure is to provide a formalism to work over an arbitrary field: recall that studying Zariski-closed sets may fail to pick up interesting behaviour, see also in the example of Section 1.5. We begin with a positive result: let κ be any field, then although the Jordan normal form for matrices may not work over κ, it is still the case that nilpotent $n \times n$ matrices have a standard form. This is because for nilpotent matrices the eigenvalues are all zero, and the calculations needed to put a matrix in standard form can all be done over κ.

Thus $\mathrm{GL}_n(\kappa)$-orbits of nilpotent matrices over κ still correspond to partitions as described in Chapter 2. We then have the following theorem:

Theorem 5.1.1. The accessibility for nilpotent orbits of $\mathrm{GL}_n(\kappa)$ is identical to the accessibility for nilpotent orbits of $\mathrm{GL}_n(\overline{\kappa})$.

Proof. Suppose that x_1 and x_2 are nilpotent matrices in $\mathfrak{gl}_n(\overline{\kappa})$, with partitions $\pi_1 = [r_1 + k_1, \ldots, r_p + k_p, s_1 - l_1, \ldots, s_q - l_q]$ and $\pi_2 = [r_1, \ldots, r_p, s_1, \ldots, s_q]$ respectively, such that $\mathcal{O}(\pi_2)$ is accessible from $\mathcal{O}(\pi_1)$. In Section 2.4, we showed that the accessibility $\mathcal{O}(\pi_1) \to \mathcal{O}(\pi_2)$ is a combination of two moves:

$$\mathcal{O}([r + \sum_{i=1}^{p} k_i, s_1 - k_1, \ldots, s_p - k_p]) \to \mathcal{O}([r, s_1, \ldots, s_p]),$$

$$\mathcal{O}([r_1 + k_1, \ldots, r_q + k_q, s - \sum_{i=1}^{q} k_i]) \to ([r_1, \ldots, r_q, s]).$$

So it suffices to show that these orbits and moves are defined over any field κ.

Starting with the orbits, recall that any nilpotent matrix in $\mathfrak{gl}_n$ is a Jordan normal form in its standard form:

$$x_1 = \begin{pmatrix} J_{r_1} & & & & & \\ & \ddots & & & & \\ & & J_{r_p} & & & \\ & & & J_{s_1} & & \\ & & & & \ddots & \\ & & & & & J_{s_q} \end{pmatrix},$$

where each J_i has ones on the upper diagonal, and zeroes everywhere else. So the Jordan blocks are defined over any field, hence the nilpotent matrices are also defined over any field κ.

Next, we check that the base changes are valid. Recall that a base change gives rise to a matrix g such that $x_1' = g \cdot x$. In Section 2.3, we determined the base changes for the two moves described above. In Equation 2.3, the base change for the move $\mathcal{O}([r_1 + k_1, \ldots, r_q + k_q, s - \sum_{i=1}^{q} k_i]) \to \mathcal{O}([r_1, \ldots, r_q, s])$ has no coefficients other than 1, so it is defined over any field κ, as is the base change matrix g. Then x_1' is also defined over κ, as it is the conjugation of x_1 with g. Similarly, the base change for the move $\mathcal{O}([r + \sum_{i=1}^{p} k_i, s_1 - k_1, \ldots, s_p - k_p]) \to \mathcal{O}([r, s_1, \ldots, s_p])$ is defined over κ. We can then combine the base changes, to obtain a 'final' base change that is still defined over any field κ. Hence the nilpotent gl_n-orbits are also defined over any field κ.

Finally, the cocharacters λ are diagonal matrices in their standard form, and the nonzero entries are all powers of t, that are non-increasing as they go down the diagonal. These powers of t are defined over $\kappa(t)$, for any field κ, hence the cocharacter is defined for any $\kappa(t)$. In 1.6, conjugating with a cocharacter with non-increasing powers of t showed that the limit $\lim_\lambda g \cdot x$ is defined over $\overline{\kappa}$ for suitable x and g. Since x, g and λ are defined over any field κ, the limit is still valid if we remove the restriction of algebraically closed fields. $\square$

Corollary 5.1.1. The dominance order on nilpotent orbits for $\mathrm{GL}_n(\kappa)$ describes the order from cocharacter closure over an arbitrary field.

We now discuss the symplectic and orthogonal groups, where the situation is less transparent. The main problem for our approach is that one of the key results - Theorem 1.12.1 - fails over fields that are not algebraically closed. In Section 3.1.1, we considered the orbits $\mathcal{O}([5,5])$ and $\mathcal{O}([6,4])$ in the symplectic algebra, and found a matrix $x \in \mathfrak{sp}_{10}$ in the orbit $\mathcal{O}([6,4])$, and x' in the same orbit that had its limit $y = \lim_\lambda x' \in \mathcal{O}([5,5])$ for a suitable cocharacter λ. Here, we will determine the base change by first finding a base change $g_1 \in \mathrm{GL}_{10}$, and then making it symplectic by applying Theorem 1.12.1.

For the move $\mathcal{O}([6,4]) \to \mathcal{O}([5,5])$, recall that:

$$x = \left(\begin{array}{ccc|cccc|ccc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right), x' = \left(\begin{array}{ccccc|ccccc} 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right).$$

Then we obtain a base change matrix g_1 of the form:

$$g_1 = \begin{pmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

which is not symplectic, as $g_1^T \Omega_{\mathrm{S}} g \neq \Omega_{\mathrm{S}}$. By the process described in Section 1.12, we obtain a symplectic base change g, of the following form:

$$g = \begin{pmatrix} -i\sqrt{2} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{i}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -\frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & \frac{i}{\sqrt{2}} & 0 \\ 0 & -\frac{i}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{i}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \frac{1}{\sqrt{2}} & 0 & -\frac{i}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{\sqrt{2}} \end{pmatrix},$$

which has entries containing $\sqrt{-1}$ and $\sqrt{2}$, so a symplectic base change is not possible over fields in which these values are not defined. Hence we can say that x and x' are $\mathrm{Sp}_{2n}(\bar{\kappa})-$conjugate, but not necessarily $\mathrm{Sp}_{2n}(\kappa)-$conjugate.

This means that we cannot analyse nilpotent orbits over an arbitrary field without additional tools at our disposal. However, there are some options if we restrict our attention to certain classes of fields.

As an example, further research could be fruitful in the direction of finite fields. Let G be the symplectic or orthogonal group defined over $\kappa = \overline{\mathbb{F}}_q$, the closure of the finite field with q elements. Then $\mathbb{F}_q$ can be realised as the fixed points of the Frobenius endomorphism $x \mapsto x^q$, which extends to a homomorphism σ which raises each matrix entry to the q^{th} power:

$$\sigma : G \to G,$$
$$\text{given by } a_{ij} \mapsto a_{ij}^q.$$

It is clear that the subgroup G_σ of σ-fixed points in G is the finite symplectic or orthogonal group consisting of matrices satisfying the defining conditions for G, but with all entries in $\mathbb{F}_q$. We can now extend the Frobenius endomorphism to a map on the Lie algebra $\mathfrak{g}$ of G and consider how the $G-$orbits of $\mathfrak{g}$ relate to the $G_\sigma-$orbits of $\mathfrak{g}_\sigma$.

In particular, if $x \in \mathfrak{g}$ is such that $\sigma(x) = x$, how does $(G \cdot x)_\sigma := G \cdot x \cap \mathfrak{g}_\sigma$ decompose into $G_\sigma-$orbits? The theorem of Springer-Steinberg (see [10, p.172-173]) gives the answer:

Theorem. Let $C = C_G(x)$ be the centralizer of x in G. Then the $G_\sigma-$orbits in $(G \cdot x)_\sigma$ are parametrised by the elements of the cohomology group $H^1(\sigma, C)$. In the special case that C is connected, there is a single orbit.

Here $H^1(\sigma, C)$ denotes C modulo the equivalence relation $a \sim b$ if and only if there exists $c \in C$ with $a = cb\sigma(c)^{-1}$. These are the orbits under 'twisted' conjugation. With this result and the knowledge of the centralizers of nilpotent elements from [7, Section 3], in principle we can work out the nilpotent orbits for these finite groups and then begin to analyse accessibility relations between them. In practice, this is likely to be a complex process, but it is possible that we could find a general theorem analogous to theorem 3.3.1 , which deals with accessibility of nilpotent symplectic orbits over a closed field (or the likely similar theorem for nilpotent orthogonal orbits).

5.2 Non-classical groups

Simple algebraic groups over an algebraically closed field have a classification by root data (see [6, p.229]), as in common in Lie Theory. This means that for a fixed algebraically closed field κ there are four infinite families of simple algebraic group – types A, B, C and D – and five exceptional families – types E_6, E_7, E_8, F_4 and G_2. There is in general more than one simple group of each type, but they are all closely related by isogeny – there is a so-called *simply connected* group of each type and an *adjoint* group of each type, and all the others lie somewhere in between. In any case, the structure of the nilpotent orbits is insensitive to this change of group within each type, so we are free to choose a representative of each type and study that. The groups in this thesis cover the four infinite families: type A simple groups are represented by the special linear groups SL_n, and this is covered by our work in Chapter 2, type C is symplectic groups (Chapter 3), and types B and D are orthogonal groups (Chapter 4). This leaves the exceptional groups, which we now briefly discuss.

Typically, when dealing with the exceptional groups it is rather difficult to produce uniform arguments, like we have been able to provide above for the classical groups, and one has to proceed in a more ad hoc manner, case by case. Type G_2 was explored in some depth in [1]. The next case to consisder would be F_4. In attacking these groups, there is some hope that a method similar to the method's laid out in [1] might bear fruit.

The nilpotent orbits in exceptional groups can be classified by combinatorial data, based on the Dynkin diagram of the root system, and the G_2 calculations in [1] suggest that for many of the orbits the accessibility question is relatively easy to settle by writing down cocharacters based on the coroots. As above, what is likely to be more difficult is to settle cases where orbits are *not* accessible from higher up. By Lemmas 1.9.2, 1.11.2, 1.11.3, we know that there will be non-accessible orbits, corresponding to distinguished nilpotent elements. However, it is quite likely that within the larger exceptional groups, there are more complicated relationships between the orbits, as discovered in the symplectic and orthogonal groups in Chapters 3 and 4.

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