Metric Diophantine Approximation on Manifolds by Algebraic Points

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PhD

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Mathematics

September 2020

Abstract

This thesis is concerned with various aspects of the metric theory of Diophantine Approximation by algebraic points. It is comprised of three introductory chapters, the presentation of our original work (Section 3.1, Chapters 4 and 5), and two appendices.

At the end of Chapter 3 we introduce a simple application of the quantitative non-divergence estimates of Kleinbock and Margulis to a problem of approximation of points on a circle in the complex plane by ratios of Gaussian integers, which is motivated by recent advances in the theory of Wireless Communications.

Then, in Chapter 4 we prove some partial results towards a Hausdorff measure description of the set of real numbers close to the zeros of polynomials of bounded degree, expanding on previous work of Hussain and Huang (see Section 1.2.1). Specifically, we use an estimate on the number of cubic polynomials with bounded discriminant due to Kaliada, Götze and Kukso and a measure bound due to Beresnevich to prove a convergence result for irreducible cubic polynomials, as well as for polynomials of arbitrary degree and large discriminant.

Finally, in Chapter 5 we apply a quantitative non-divergence estimate and the theory of ubiquitous systems to derive both a counting lower bound and a divergence Hausdorff measure result for a problem of approximation on manifolds by points with algebraic conjugate coordinates, subject to a geometric constraint.

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Acknowledgments

I would like to express my gratitude to all the people who gave me strength during the past four years, making this work possible.

First and foremost I would like to thank my supervisor, Victor Beresnevich, for his guidance, his patience, and for having faith in me when I had none. I would also like to thank Demi Allen for her friendship and her support in the more practical aspects of research work, Faustin Adiceam for pointing me in the right direction to understand the geometric condition at the heart of Chapter 5, and the rest of the Number Theory research group which over my time in York included: Sky Brewer, Sam Chow, Lifan Guan, Catalin Hanga, Christopher Huges, Alexey Kudin, Jason Levesley, Kayleigh Measures, Antoine Marnat, Erez Nesharim, David Simmons, Matthew Thompson, Niclas Technau, Sanju Velani, Ben Ward, Agamemnon "Manos" Zafeiropoulos and Evgeniy Zorin.

I would then like to thank my parents for their unconditional love, and my girlfriend, who supported me while I was writing this thesis and who kept me sane while a pandemic was raging outside.

I would also like to thank the wonderful people of the UoY Archery Club and our coaches Mick and Kath Fitzpatrick, who gave me the strength to overcome some dark times.

Finally, I would like to acknowledge the financial support of the Engineering and Physical Sciences Research Council through the International Doctoral Scholars IDS Grant number EP/N509802/1.

Author's declaration

I declare that the work presented in this thesis, except where otherwise stated, is based on my own research carried out at the University of York and has not been submitted previously for any degree at this or any other university. Sources are acknowledged by explicit references.

Chapter 4 was previously published as:
A. Pezzoni. "A Jarník-type theorem for a problem of approximation by cubic polynomials". In: *Acta Arith.* 193.3 (2020), pp. 269–281. ISSN: 0065-1036

Chapter 5 was previously published as an arXiv pre-print:
A. Pezzoni. *Quantitative non-divergence and lower bounds for points with algebraic coordinates near manifolds.* 2020. arXiv: 2006.10790 [math.NT]

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Introduction

Traditionally, Diophantine Approximation is concerned with the density of the rational numbers within the reals. In other words: given a real number α, how efficiently can it be approximated by rational numbers $\frac{p}{q}$, in terms of the size of the (reduced) integers p and q? We will discuss this problem in $[0,1]$ only for simplicity, and everything we say here can be straightforwardly extended to the whole $\mathbb{R}$. Immediately, one can see that for *every* positive q there is a p such that

$$\left|\alpha - \frac{p}{q}\right| \leq \frac{1}{2q},$$

which can be rewritten as $\langle\!\langle q\alpha \rangle\!\rangle \leq \frac{1}{2}$, where $\langle\!\langle \cdot \rangle\!\rangle$ denotes the distance from the nearest integer. Furthermore, a classical result of Dirichlet's shows that we can improve on the rate of approximation with only a slight relaxation of our requirements, i.e. by replacing *every* with *infinitely many*:

Theorem 1.0.1 (Dirichlet). *For every $\alpha \in [0,1]$ and for every $Q \in \mathbb{N}$ there are is an integer $0 < q \leq Q$ such that*

$$\langle\!\langle q\alpha \rangle\!\rangle < \frac{1}{Q}.$$

In particular, there are infinitely many positive integers q such that

$$\langle\!\langle q\alpha \rangle\!\rangle < \frac{1}{q}.$$

More broadly, we would like to determine when the inequality $\langle\!\langle q\alpha \rangle\!\rangle < \psi(q)$ admits infinitely many integer solutions q, where $\psi \colon \mathbb{R}^+ \to \mathbb{R}^+$ is called an *approximation function*; α is said to be *ψ-well approximable* if this is the case. Unfortunately, in general Dirichlet's theorem is best possible (up to a constant), as it can be shown that algebraic numbers behave particularly poorly with respect to approximation by rationals. For example, consider the following theorem due to Hurwitz, which can be proven using continued fraction methods.

Theorem 1.0.2 (Hurwitz)**.** *Let $\gamma = \frac{1+\sqrt{5}}{2}$ be the golden ratio. Then for any $0 < \varepsilon < \frac{1}{\sqrt{5}}$ the inequality*

$$\langle\!\langle q\gamma \rangle\!\rangle < \frac{\varepsilon}{q}$$

has at most finitely many solutions $q \in \mathbb{N}$.

Even though we are not able to improve on Dirichlet's theorem for every real number in $[0,1]$, it is often interesting to determine the likelihood of a random $\alpha \in [0,1]$ of being ψ-well approximable, and this is precisely where a classical result due to Khinchin comes in. The modern formulation below is a little different from Khinchin's original statement, and can be found e.g. in [19, Theorem 1.2.5]. Here $|\cdot|$ denotes the Lebesgue measure on $\mathbb{R}$, and $W(\psi)$ is the set of ψ-well approximable points in $[0,1]$.

Theorem 1.0.3 (Khinchin [65])**.** *Let ψ be an approximation function. Then*

$$|W(\psi)| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q) < \infty \\ 1 & \text{if } \psi \text{ is monotonic and } \sum_{q=1}^{\infty} \psi(q) = \infty. \end{cases}$$

The convergence part can be proved in a fairly elementary way via the following standard lemma from Probability Theory, and this argument is enough to exemplify the common approach to this sort of theorem. We will return to this Theorem later in Chapter 2, where we will show how the notion of *ubiquitous systems* can be used to prove the divergence part.

Lemma 1.0.4 (Borel-Cantelli)**.** *Let (Ω, μ) be a measure space. Further, let $\{E_q\}_{q\in\mathbb{N}}$ be a sequence of μ-measurable sets, and let E_∞ be the set of $x \in \Omega$ such that $x \in E_q$ for infinitely many $q \in \mathbb{N}$, i.e.*

$$E_\infty = \limsup_{q\to\infty} E_q = \bigcap_{q_0=1}^{\infty} \bigcup_{q=q_0}^{\infty} E_q.$$

If $\sum_{q\in\mathbb{N}} \mu(E_q) < \infty$*, then* $\mu(E_\infty) = 0$.

Proof of the convergence part of Theorem 1.0.3.
Fix $q \in \mathbb{N}$, and observe that $\alpha \in [0,1]$ satisfies $\langle\!\langle q\alpha \rangle\!\rangle < \psi(q)$ if and only if there is an integer $p \in \{0, \ldots, q\}$ such that

$$\left| \alpha - \frac{p}{q} \right| < \frac{\psi(q)}{q}.$$

Therefore α must lie in the set

$$E_q := \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap [0,1],$$

where $B(x,r)$ denotes the ball centred at x with radius r. Furthermore,

$$|E_q| \leq \sum_{p=0}^{q} \left| B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right) \cap [0,1] \right| = q\frac{\psi(q)}{q} = \psi(q).$$

Since $W(\psi) = \limsup_{q\to\infty} E_q$, a direct application of the Borel-Cantelli Lemma concludes the proof. □

Remark 1.0.5. Khinchin's original proof assumed the monotonicity of $q\psi(q)$, but this requirement was later relaxed through the notion of *regular systems* introduced by Baker and Schmidt in [7]. Furthermore, Duffin and Schaeffer [50] showed that this hypothesis is indeed necessary.

While Khinchin's Theorem is striking in its simplicity, it doesn't fully describe $W(\psi)$. For example, consider the sets $W(2)$ and $W(2020)$, where with a common abuse of notation we wrote $W(\tau)$ instead of $W(q^{-\tau})$: they both have null Lebesgue measure, but intuitively we would expect the former to be larger. Therefore, to gain a better understanding of $W(\psi)$ we need a more fine-grained way to measure it, like the Hausdorff measure $\mathcal{H}^s$ and dimension $\dim_{\mathcal{H}}$ (see Appendix A). Again, the modern formulation below of Jarník's classical result can be found in [19, Theorem 1.3.4].

Theorem 1.0.6 (Jarník [62]). *Let ψ be an approximation function. Then for any $0 < s < 1$*

$$\mathcal{H}^s(W(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{1-s}\psi(q)^s < \infty \\ \infty & \text{if } \psi \text{ is monotonic and } \sum_{q=1}^{\infty} q^{1-s}\psi(q)^s = \infty. \end{cases}$$

As for Khinchin's Theorem, here we will only prove the convergence part, and we will return to the divergence part in Chapter 2.

Proof of the convergence part of Theorem 1.0.6.
First, observe that if $F \subset \mathbb{R}$ and $\{B_i\}_{i\in\mathbb{N}}$ is a ρ-cover of F, then by definition of Hausdorff measure we have an upper bound

$$\mathcal{H}^s_\rho(F) \leq \sum_{i\in\mathbb{N}} r(B_i)^s.$$

Now let E_q be the same as in the proof of Theorem 1.0.3, and note that for every $q_0 > 0$

$$W(\psi) \subset \bigcup_{q\geq q_0} E_q \subset \bigcup_{q\geq q_0} \bigcup_{p=0}^{q} B\left(\frac{p}{q}, \frac{\psi(q)}{q}\right).$$

Then if $\rho(q_0) := \sup_{q\geq q_0} q_0^{-1}\psi(q_0)$ we have

$$\begin{aligned} \mathcal{H}^s_{\rho(q_0)}(W(\psi)) &\leq \sum_{q\geq q_0} \sum_{p=0}^{q} \left(\frac{\psi(q)}{q}\right)^s \\ &= \sum_{q\geq q_0} (q+1)\left(\frac{\psi(q)}{q}\right)^s \\ &\leq 2\sum_{q\geq q_0} q^{1-s}\psi(q)^s. \end{aligned}$$

Since the latter is the tail of a convergent series, it must tend to 0 as $q_0 \to \infty$. Furthermore, the convergence of this series also implies $\rho(q_0) \to 0$, which gives

$$\mathcal{H}^s(W(\psi)) = \lim_{q_0\to 0} \mathcal{H}^s_{\rho(q_0)}(W(\psi)) = 0. \qquad \square$$

Finally, the following theorem was proven independently by Jarník in 1928 and Besicovitch in 1932, and it can also be seen as a direct consequence of Jarník's Theorem.

Theorem 1.0.7 (Jarník-Besicovitch). *Fix $\tau > 1$. Then $\dim_{\mathcal{H}} W(\tau) = \frac{2}{1+\tau}$.*

1.1 THE THEORY IN HIGHER DIMENSIONS

There are essentially two ways to approach the problem of approximating a point $\boldsymbol{\alpha} = (\alpha_1, \ldots, \alpha_n) \in \mathbb{R}^n$ with rationals. We can treat each component separately, leading to a *simultaneous approximation* problem of the form

$$\langle\!\langle q_j \alpha_j \rangle\!\rangle < \psi_j(\|\boldsymbol{q}\|) \qquad (1 \leq j \leq n) \tag{1.1}$$

or we can treat them all at the same time by asking how far $\boldsymbol{\alpha}$ is from lying on a rational hyperplane, leading to a *dual approximation* problem of the form

$$\langle\!\langle \boldsymbol{q} \cdot \boldsymbol{\alpha} \rangle\!\rangle < \psi(\|\boldsymbol{q}\|), \tag{1.2}$$

where here and in what follows $\|\cdot\|$ denotes the sup norm, unless otherwise stated. These two types of approximation can be straightforwardly combined by taking an $m \times n$ matrix $A \in \mathbb{R}^{mn}$ and considering the problem of simultaneous approximation of the columns A_j of A:

$$\langle\!\langle \boldsymbol{q} \cdot A_j \rangle\!\rangle < \psi_j(\|\boldsymbol{q}\|) \qquad (1 \leq j \leq n). \tag{1.3}$$

We can then recover (1.1) or (1.2) by setting $m = 1$ or $n = 1$, respectively.

As a direct application of Minkowski's Theorem for linear forms, we can obtain the following equivalent of Dirichlet's Theorem for (1.3), i.e. in the special case where $\psi_j(q) = q^{-i_j m}$ for some $i_j > 0$:

Theorem 1.1.1 ([43, Chapter 1, Theorem VI]). *Let $i_1, \ldots, i_n \in [0,1]$ be real numbers such that $\sum_{j=1}^n i_j = 1$. Then for every $A \in \mathbb{R}^{mn}$ and for every $Q \in \mathbb{N}$ there is a $\boldsymbol{q} \in \mathbb{Z}^m$ such that $0 < \|\boldsymbol{q}\| \leq Q$ and*

$$\langle\!\langle \boldsymbol{q} \cdot A_j \rangle\!\rangle < Q^{-i_j m} \qquad (1 \leq j \leq n).$$

In particular, there are infinitely many $\boldsymbol{q} \in \mathbb{Z}^m$ such that

$$\langle\!\langle \boldsymbol{q} \cdot A_j \rangle\!\rangle < \|\boldsymbol{q}\|^{-i_j m} \qquad (1 \leq j \leq n). \tag{1.4}$$

Now suppose that all the approximation functions ψ_j in (1.3) coincide with the same function ψ. To lighten the notation, we will then write

$$\langle\!\langle \boldsymbol{q} A \rangle\!\rangle < \psi(\|\boldsymbol{q}\|),$$

where $\langle\!\langle \boldsymbol{q}A \rangle\!\rangle := \max_{1\le j\le n} \langle\!\langle \boldsymbol{q}\cdot A_j \rangle\!\rangle$, and we will say that A is *ψ-well approximable* if this inequality is satisfied for infinitely many $\boldsymbol{q} \in \mathbb{Z}^m$. Also let I^{mn} be the unit cube $[0,1]^{mn}$ and, like in the previous section, consider the set

$$W(m,n;\psi) := \{A \in I^{mn} : \langle\!\langle \boldsymbol{q}A \rangle\!\rangle < \psi(\|\boldsymbol{q}\|) \text{ for infinitely many } \boldsymbol{q} \in \mathbb{Z}^m\}\,.$$

The following theorem is a modern improvement on a classical result of Khinchin and Groshev [22]. Here $|\cdot|$ denotes the Lebesgue measure on I^{mn}.

Theorem 1.1.2 (Khinchin-Groshev)**.** *Let ψ be an approximation function. Then*

$$|W(m,n;\psi)| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{m-1}\psi(q)^n < \infty \\ 1 & \text{if } \psi \text{ is monotonic and } \sum_{q=1}^{\infty} q^{m-1}\psi(q)^n = \infty. \end{cases}$$

Furthermore, the monotonicity assumption can be dropped when $mn > 1$.

Finally, we ought to mention the Mass Transference Principle, a powerful result originally due to Beresnevich and Velani which allows one to obtain Hausdorff measure versions of Lebesgue measure statements about lim sup sets. Although we will not cover it in more detail in this thesis, we invite the interested reader to consult [5] for an overview of the MTP and some of the many extensions that have been proved since its inception. In particular, one such theorem due to Allen and Beresnevich allows us to derive the following Jarník-type result for $W(m,n;\psi)$ from Theorem 1.1.2 (see [4, Theorem 2]).

Theorem 1.1.3. *Let ψ be an approximation function. Also let g be a dimension function such that $r^{-mn}g(r)$ is monotonic, and define $\tilde{g}(r) := r^{-m(n-1)}g(r)$. Then*

$$\mathcal{H}^g(W(m,n;\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n+m-1}\tilde{g}\left(\frac{\psi(q)}{q}\right) < \infty \\ 1 & \text{if } \psi \text{ is monotonic and } \sum_{q=1}^{\infty} q^{n+m-1}\tilde{g}\left(\frac{\psi(q)}{q}\right) = \infty. \end{cases}$$

Furthermore, the monotonicity assumption can be dropped when $mn > 1$.

Remark 1.1.4. For the usual Hausdorff s-dimension, i.e. when $g(r) = r^s$ for some $s > 0$, the monotonicity condition on g is vacuously satisfied.

1.2 DIOPHANTINE APPROXIMATION ON MANIFOLDS

In 1932, Mahler [83] proposed a classification of the real numbers in terms of their approximation properties by algebraic numbers. More precisely, the class of a number $x \in \mathbb{R}$ depended on the supremum $w_n(x)$ of the numbers $w > 0$ such that

$$|P(x)| < \mathrm{H}(P)^{-w} \tag{1.5}$$

has infinitely many solutions in polynomials $P = a_n X^n + \cdots + a_0 \in \mathbb{Z}[X]$ of degree at most n, where $\mathrm{H}(P)$ is the *(naive) height* of P, i.e.

$$\mathrm{H}(P) := \max_{0 \le i \le n} |a_i|.$$

We should also mention that a few years later, in 1939, Koksma [73] proposed a similar classification based on the supremum $w_n^*(x)$ of the numbers $w^* > 0$ such that

$$|x - \alpha| < \mathrm{H}(\alpha)^{-w^*-1} \tag{1.6}$$

has infinitely many solutions in real algebraic numbers α of degree at most n, where $\mathrm{H}(\alpha)$ denotes the hight of the minimal polynomial of α. We will not go into more detail here, but it should be noted that there are some subtle differences between the two approaches, even though the resulting classifications ultimately coincide (see [42, Chapter 3]).

Just like in the case of Theorem 1.1.1, Minkowski's Theorem for linear forms shows that $w_n(x) \ge n$ for every $x \in \mathbb{R}$, and Mahler conjectured that $w_n(x) = n$ for Lebesgue almost every $x \in \mathbb{R}$. The case $n = 1$ follows directly from Khinchin's Theorem, while Kubilius [75] and Volkmann [105] proved the quadratic and cubic cases, respectively, before Sprindžuk settled the conjecture in its generality in 1969 [101], as well as its complex and p-adic equivalents.

Observe that (1.5) can be reinterpreted as

$$|\boldsymbol{a} \cdot \boldsymbol{x}| < \|\boldsymbol{a}\|^{-w}, \tag{1.7}$$

where $\boldsymbol{a} \in \mathbb{Z}^{n+1}$ and $\boldsymbol{x} = (1, x, \ldots, x^n) \in \mathbb{R}^{n+1}$. In other words, it can be seen as a problem of approximation of points on the Veronese curve (also known as rational normal curve) of degree n by rational hyperplanes. Thus Mahler's conjecture marked the beginning of what Sprindžuk called the theory of *approximation of dependent quantities*, i.e. of points constrained to a manifold. A key problem in this field is answering the following general question.

Question 1.2.1. *When does a given manifold $\mathcal{M} \subset \mathbb{R}^n$ inherit the same metric approximation properties of its ambient space $\mathbb{R}^n$?*

As a special case of this problem, in Sprindžuk's terminology a manifold $\mathcal{M}$ is called *extremal* if for almost all of its points (with respect to the induced Lebesgue measure on $\mathcal{M}$) the exponents in (1.4) cannot be improved.

Remark 1.2.2. Although (1.7) looks like an inhomogeneous version of the inequalities considered in the previous sections, it is not a real issue in this case. Indeed, suppose that $x \in [-\frac{1}{2}, \frac{1}{2})$ and let $\tilde{\boldsymbol{a}} = (a_1, \ldots, a_n)$ and $\tilde{\boldsymbol{x}} = (x, \ldots, x^n)$. Without loss of generality, also assume that $\|\boldsymbol{a}\| \geq 2$, so that $\|\boldsymbol{a}\|^{-w} \leq 2^{-n}$ for $w \geq n$, as well as $\tilde{\boldsymbol{a}} \neq 0$. Then

$$|a_0| \leq 2^{-n} + |\tilde{\boldsymbol{a}} \cdot \tilde{\boldsymbol{x}}| \leq 2^{-n} + \|\tilde{\boldsymbol{a}}\| \sum_{i=1}^{n} 2^{-i} < \|\tilde{\boldsymbol{a}}\|,$$

hence every solution of (1.7) is also a solution of

$$\langle\!\langle \tilde{\boldsymbol{a}} \cdot \tilde{\boldsymbol{x}} \rangle\!\rangle < \|\tilde{\boldsymbol{a}}\|^{-w}. \tag{1.8}$$

Finally, a brief argument from Section 4.3.1 allows us to lift any Hausdorff measure result from $[-\frac{1}{2}, \frac{1}{2})$ to the whole $\mathbb{R}$.

1.2.1 Veronese curves

Now, given an approximation function ψ, consider the set

$$\mathscr{L}_n(\psi) := \{x \in \mathbb{R} : |P(x)| < \psi(\mathrm{H}(P)) \text{ for i.m. } P \in \mathbb{Z}[X], \deg P \leq n\},$$

where "i.m." stands for "infinitely many". Following the blueprint of the previous sections, the first step in answering question 1.2.1 for the Veronese curves is an equivalent of Theorem 1.1.2 for $\mathscr{L}_n(\psi)$, which we obtain by combining results due to Bernik [27] and Beresnevich [11, 10].

Theorem 1.2.3. *Let ψ be an approximation function and let $I \subset \mathbb{R}$ be an interval. Then*

$$|\mathscr{L}_n(\psi) \cap I| = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} q^{n-1}\psi(q) < \infty \\ |I| & \text{if } \psi \text{ is monotonic and } \sum_{q=1}^{\infty} q^{n-1}\psi(q) = \infty. \end{cases}$$

The next step is to determine the Hausdorff dimension of $\mathscr{L}_n(\psi)$, which was done by Bernik in [25]. Here w_ψ denotes the lower order of ψ^{-1} at infinity, that is

$$w_\psi \coloneqq \liminf_{q\to\infty} -\frac{\log\psi(q)}{\log q}. \tag{1.9}$$

Theorem 1.2.4. *Let ψ be an approximation function such that $n \le w_\psi < \infty$. Then*

$$\dim_{\mathcal{H}} \mathscr{L}_n(\psi) = \frac{n+1}{w_\psi + 1}.$$

Furthermore, the equivalent of the divergence part of the analogue of Theorem 1.1.3 is a special case of a much more general result due to Beresnevich, Dickinson and Velani (see Theorem 1.2.8 below), although we should note that it could also be obtained directly from Theorem 1.2.3 via the Mass Transference Principle.

Theorem 1.2.5. *Let ψ be a decreasing approximation function. Also let g be a dimension function such that $r^{-1}g(r)$ is decreasing and $r^{-1}g(r) \to \infty$ as $r \to 0$. Then*

$$\mathcal{H}^g(\mathscr{L}_n(\psi)) = \infty \quad \text{if} \quad \sum_{q=1}^{\infty} q^n g\left(\frac{\psi(q)}{q}\right) = \infty.$$

Unfortunately, however, we are still missing a complete convergence counterpart to Theorem 1.2.5. Hussain [60] and Huang [57] independently proved the quadratic case for ψ decreasing and with an extra growth condition on g, while in [92] (reproduced in Chapter 4 below) we proved the cubic case for approximation functions of the form q^{-w}, as well as some partial results for irreducible cubic polynomials and for polynomials of arbitrary degree and large discriminant.

Remark 1.2.6. There are essentially two reasons why, unlike in Jarník's Theorem, the convergence case is actually the harder one to prove here.

The first is that, while it is possible to quantify how close a point needs to be to a root of a polynomial P to achieve a certain rate of approximation (see the estimates in Section 4.2), in general we are still missing a way to control the overlap between these intervals when two or more roots of P are close to each other, i.e. when P has small discriminant.

The second is that it is difficult to treat reducible polynomials by induction on n when ψ is not fully multiplicative, i.e. when $\psi(q_1 q_2) \neq \psi(q_1)\psi(q_2)$ (see Section 4.5 for an example of this approach).

1.2.2 More general manifolds

Most of the more general results currently available involve a class of manifolds called non-degenerate (see below), and their proofs often rely on a combination of two of the most useful tools in Diophantine Approximation: ubiquitous systems and the quantitative non-divergence estimates, which we will introduce in Chapters 2 and 3, respectively. Here we will only mention a couple of results that are particularly relevant to the discussion in later chapters; for a more detailed overview, see the introduction to [6], as well as the survey papers [18] and [19].

So let $\mathcal{M} \subset \mathbb{R}^n$ be an m-dimensional differential manifold, and consider a local chart $(\boldsymbol{f}, U)$ of $\mathcal{M}$ such that $\boldsymbol{f} \in \mathcal{C}^\ell(U)$. Then $\boldsymbol{f}$ is *ℓ-non-degenerate at $\boldsymbol{x}_0 \in U$* if there are n linearly independent derivatives of $\boldsymbol{f}$ at $\boldsymbol{x}_0$ of order up to ℓ. Furthermore, $\boldsymbol{f}$ is *non-degenerate* at $\boldsymbol{x}_0$ if it is ℓ-non-degenerate for some $\ell > 0$, and $\mathcal{M}$ is *non-degenerate* if almost every $\boldsymbol{x} \in \mathcal{M}$ (with respect to the induced Lebesgue measure) admits a non-degenerate local chart. Geometrically, this means that locally $\mathcal{M}$ diverges at least polynomially from any affine subspace of $\mathbb{R}^n$ almost everywhere (see [9, Lemma 1.c] for a more precise formulation).

Remark 1.2.7. Analytic manifolds are an important special case, since by the Wronskian criterion an analytic map $\boldsymbol{f} = (f_1, \ldots, f_n)$ is non-degenerate almost everywhere if and only if $1, f_1, \ldots, f_n$ are linearly independent functions over $\mathbb{R}$. In particular, the Veronese curves are non-degenerate 1-dimensional manifolds.

The following theorem, which was obtained by Beresnevich, Dickinson and Velani in [17], is the equivalent of the divergence part of Theorem 1.1.3 for non-degenerate manifolds in the special case of dual approximation.

Theorem 1.2.8 ([17, Theorem 18])**.** *Let $\mathcal{M} \subset \mathbb{R}^n$ be a non-degenerate m-dimensional manifold, and let ψ be a decreasing approximation function. Also let g be a dimension function such that $\tilde{g}(r) := r^{-(m-1)}g(r)$ is increasing, $q^{-m}g(r)$ is decreasing, and $r^{-m}g(r) \to \infty$ as $r \to 0$. Then*

$$\mathcal{H}^g(W(m,1;\psi) \cap \mathcal{M}) = \infty \quad \textit{if} \quad \sum_{q=1}^{\infty} q^n \tilde{g}\left(\frac{\psi(q)}{q}\right) = \infty.$$

As Remark 1.2.6 would suggest, the convergence counterpart of Theorem 1.2.8 is still an open problem. However, Dickinson and Dodson [48] proved the following lower bound for the Hausdorff dimension of $W(m,1;\psi) \cap \mathcal{M}$, which is conjectured to hold as an equality (cfr. Theorem 1.2.4). We should also mention that in [14] Beresnevich, Bernik and Dodson proved equality in the special case where $\mathcal{M}$ is a curve and w_ψ is close to n, namely $0 < w_\psi - n < n/(4n^2 + 2n - 4)$.

Theorem 1.2.9. *Let $\mathcal{M} \subset \mathbb{R}^n$ be a non-degenerate manifold. Further, let ψ be an approximation function such that $n \leq w_\psi < \infty$, where w_ψ is as in* (1.9)*. Then*

$$\dim_{\mathcal{H}}\Big(W(m,1;\psi) \cap \mathcal{M}\Big) \geq \frac{n+1}{w_\psi + 1} + \dim \mathcal{M} - 1.$$

Remark 1.2.10. As Sprindžuk notes in [102, Chapter 2, Section 12], when $\mathcal{M}$ can be factored as a product

$$\mathcal{M}_1 \times \cdots \times \mathcal{M}_m$$

of a sufficiently large number of manifolds with a relatively simple structure, "the metric theory of Diophantine approximation on $[\mathcal{M}]$ is in many ways analogous to the theory of approximation of independent quantities."

We conclude this section by mentioning that, although not much has been proven with regard to a general simultaneous approximation analogue of Theorem 1.2.8, Beresnevich, Bernik and Budarina [13] proved the following result for systems of linear forms in the spirit of Remark 1.2.10 (cfr. Theorems 5.1.6 and 5.2.16).

Fix $m, n \in \mathbb{N}$ and for each $j \in \{1, \ldots, m\}$ consider a map $\boldsymbol{f}_j : U_j \to \mathbb{R}^{n+1}$, where U_j is an open ball in $\mathbb{R}^{d_j}$. Then let

$$U := U_1 \times \cdots \times U_j \subset \mathbb{R}^d, \quad d = d_1 + \cdots + d_j,$$

and consider the set $\mathcal{F}$ of functions $F : U \to \mathbb{R}^m$ with coordinates of the form

$$F_j(\boldsymbol{x}_j) = \boldsymbol{a} \cdot \boldsymbol{f}_j(\boldsymbol{x}_j)$$

as $\boldsymbol{a}$ ranges in $\mathbb{Z}^{n+1} \setminus \{0\}$. Further, define

$$\mathrm{H}(F) := \|\boldsymbol{a}\| = \max_{0 \le i \le n} |a_i|$$

and, given an approximation function ψ, let

$$\mathscr{L}(\mathcal{F}, \psi) := \Big\{ (\boldsymbol{x}_1, \ldots, \boldsymbol{x}_m) \in U : \max_{1 \le j \le m} |F_j(\boldsymbol{x}_j)| < \psi\big(\mathrm{H}(F)\big) \text{ for i.m. } F \in \mathcal{F} \Big\}.$$

Theorem 1.2.11 ([13, Theorem 4])**.** *Suppose that, for each $j \in \{1, \ldots, m\}$, the coordinate functions of $\boldsymbol{f}_j$ are analytic and linearly independent over $\mathbb{R}$. Also, let g be a dimension function such that $\tilde{g}(r) := r^{-d+m} g(r)$ is increasing and $r^{-d} g(r)$ is non-increasing. Then*

$$\mathcal{H}^g(\mathscr{L}(\mathcal{F}, \psi)) = \mathcal{H}^g(U) \quad \text{if} \quad \sum_{q=1}^{\infty} q^n \tilde{g}\left(\frac{\psi(q)}{q}\right) = \infty.$$

Remark 1.2.12. In the same paper, Beresnevich, Bernik and Budarina also prove a convergence counterpart of Theorem 1.2.11, but only for Lebesgue measure.

Remark 1.2.13. An elegant, general theorem like Theorem 1.1.2 is not actually possible in the case of simultaneous approximation on a manifold $\mathcal{M}$. The reason is that when $\psi(q)$ decreases faster than a critical threshold, the approximating rational points appear to be forced to lie on $\mathcal{M}$ itself, bringing into play its arithmetic properties (see also Remark 2.0.8).

More precisely, Rynne [97] proved that if $\mathcal{M}$ is a $\mathcal{C}^k$ manifold, then there are two manifolds "$\mathcal{C}^k$-close" to $\mathcal{M}$ with radically different approximation behaviour. Therefore the behaviour of $\mathcal{M}$ cannot be captured by its analytic properties alone. However, it is widely believed that it should be possible to prove a general theorem for a wide class of manifolds (including non-degenerate manifolds) when the rate of decay of the approximation function is above a critical threshold [16, Section 1.4].

For the current state of the art, see [20] for the divergence case, and [21, 99, 58] for the convergence case. In particular, the introduction of [58] contains a useful overview of some of the previous results. We should also mention [59], which deals with the convergence case for affine subspaces which satisfy a certain Diophantine condition.

1.3 Outline

Chapter 2 contains a brief introduction to the theory of ubiquitous systems.

Chapter 3 is devoted to the *quantitative non-divergence* estimates of Kleinbock and Margulis [71], a powerful tool at the heart of the argument in Chapter 5. After a brief introduction, we provide a simple application to a problem of approximation by ratios of Gaussian integers in the complex plane.

Chapter 4 contains some partial results towards a Hausdorff measure version of Theorem 1.2.3 which were published in [92]. Our argument follows Volkmann's proof of the cubic case of Mahler's conjecture [107] and uses some recent asymptotic estimates by Kaliada, Götze and Kukso [63] on the number of cubic polynomials with bounded discriminant.

Finally, in Chapter 5 we use a quantitative non-divergence estimate to extend some lower bounds for the number of points with algebraic conjugate coordinates close to curves and surfaces, to points close to manifolds in $\mathbb{R}^n$ which satisfy certain algebraic conditions. In the process we also prove a Hausdorff measure divergence result for approximations of the form

$$\max_{1\leq j\leq m}|P(x_j)| < \mathrm{H}(P)$$

for points $\boldsymbol{x}$ lying on one such manifold, providing a partial extension of Theorem 1.2.11. At the time of writing, this work is also available in pre-print form on arXiv [93].

— 2 —

Ubiquitous systems

Ubiquitous systems were introduced by Dodson, Rynne and Vickers in [49] as an extension of the notion of regular system of Baker and Schmidt [7], and notably they were improved upon in [17] by Beresnevich, Dickinson and Velani. Essentially, a ubiquitous system represents a family of sets $\mathcal{B}_i$ of balls with decreasing radii in a probability space Ω, such that each $\mathcal{B}_i$ covers a positive measure portion of Ω. These balls arise as neighbourhoods of points, called "resonant", which are fixed throughout. This turns out to be a good approximation of Ω in terms of its metric properties, and by comparing an approximation function ψ with the rate of decrease of the radii as i varies, one can then prove results like the divergence part of Khinchin's Theorem.

Remark 2.0.1. The theory developed in [17] allows for much more general resonant sets, but for the purposes of this thesis points will suffice.

More formally, let (Ω, d) be a compact metric space equipped with a probability measure μ for which there are constants $\delta, r_0 > 0$ such that for every $x \in \Omega$ and $r \leq r_0$

$$ar^\delta \leq \mu\Big(B(x,r)\Big) \leq br^\delta, \tag{2.1}$$

where the constants $a, b > 0$ are independent of x and r.

Remark 2.0.2. Note that (2.1) implies that $\dim_{\mathcal{H}}(\Omega) = \delta$, and it is therefore trivially satisfied when $\mu = \mathcal{H}^\delta$.

Then consider the following setting:

- J, a countable set;
- $\mathcal{R} = (R_\alpha)_{\alpha \in J}$ a family of points in Ω indexed by J, referred to as *resonant points*;
- a function $\beta\colon J \to \mathbb{R}^+, \alpha \mapsto \beta_\alpha$, which assigns a *weight* to each R_α in $\mathcal{R}$;
- a function $\rho\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{r\to\infty} \rho(r) = 0$, referred to as a *ubiquitous function*; and
- $J(t) = J_\kappa(t) \coloneqq \{\alpha \in J\colon \beta_\alpha \leq \kappa^t\}$, assumed to be finite for every $t \in \mathbb{N}$, where $\kappa > 1$ is fixed.

The pair $(\mathcal{R}, \beta)$ is then said to be a *locally μ-ubiquitous system (in Ω) with respect to ρ* if for any ball $B \in \Omega$

$$\mu\Big(\bigcup_{\alpha \in J(t)} B\Big(\alpha, \rho(\kappa^t)\Big) \cap B\Big) > c\mu(B)$$

for every t large enough, where the constant $c > 0$ is absolute.

Remark 2.0.3. The above setup is a simplified version of the one appearing in [17], but it is more than enough for our purposes.

Example 2.0.4. In the setting of Theorem 1.0.3 we have:

- $\Omega = [0, 1]$, equipped with the Euclidean metric and the Lebesgue measure;
- $J = \{(p, q) \in \mathbb{Z} \times \mathbb{N} : 0 \leq p \leq q\}$;
- $R_{(p,q)} = \frac{p}{q}$;
- $\beta_{(p,q)} = q$.

Then there is a choice of $\kappa > 0$ such that $(\mathcal{R}, \beta)$ is a locally Lebesgue-ubiquitous system with respect to $\rho(r) = cr^{-2}$, where $c > 0$ is constant (see [17, Lemma 1]).

Now, given an approximation function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$, we will also consider the limsup set

$$\Lambda_{\mathcal{R}}(\psi) := \{x \in \Omega : d(x, R_\alpha) < \psi(\beta_\alpha) \text{ for infinitely many } \alpha \in J\}.$$

The following theorems, which are again a simplified version of the main results from [17], allow us to estimate the measure of $\Lambda_{\mathcal{R}}(\psi)$ when $(\mathcal{R}, \beta)$ is a ubiquitous system. Before we can state them, though, we need one final definition: a function ψ is *κ-regular* if there is a constant $0 < c < 1$ (possibly dependent on κ) such that for all t large enough

$$\psi(\kappa^{t+1}) \leq c\psi(\kappa^t).$$

Theorem 2.0.5 ([17, Corollary 2]). *Suppose that every open set of Ω is μ-measurable. Also assume that ψ is κ-regular. If $(\mathcal{R}, \beta)$ is a locally μ-ubiquitous system (with respect to some ubiquity function ρ), then*

$$\mu\Big(\Lambda_{\mathcal{R}}(\psi)\Big) = 1 \quad \textit{if} \quad \sum_{t=0}^{\infty} \left(\frac{\psi(\kappa^t)}{\rho(\kappa^t)}\right)^{\delta} = \infty.$$

Theorem 2.0.6 ([17, Corollary 3]). *Suppose that $(\mathcal{R}, \beta)$ is a locally ubiquitous system in Ω with respect to ρ, and let g be a dimension function such that $r^{-\delta} g(r)$ is decreasing. Furthermore, suppose that there are constants $r_0, c_1, c_2 \in (0, 1)$ such that*

$$g(c_1 r) \leq c_2 g(r) \textit{ for any } r \in (0, r_0). \tag{2.2}$$

Also assume that ψ is decreasing and κ-regular. Then

$$\mathcal{H}^g\Big(\Lambda_{\mathcal{R}}(\psi)\Big) = \infty \quad \textit{if} \quad \sum_{t=0}^{\infty} \frac{g(\psi(\kappa^t))}{\rho(\kappa^t)^{\delta}} = \infty.$$

Example 2.0.7. Let ψ be an approximation function, and define $\hat{\psi}(q) := \frac{\psi(q)}{q}$. Then observe that in the setting of Example 2.0.4 we have

$$W(\psi) = \Lambda_{\mathcal{R}}(\hat{\psi}).$$

Furthermore, $\hat{\psi}$ is clearly κ-regular whenever ψ is decreasing, and condition (2.2) is trivially satisfied when $g(r) = r^s$. Therefore a direct application of Theorems 2.0.5 and 2.0.6 results in the divergence part of Theorems 1.0.3 and 1.0.6, respectively.

Remark 2.0.8. Let Ω' be a metric subspace of Ω. Then there are essentially two ways we can consider approximating the points of Ω': *intrinsic* and *extrinsic*.

Intrinsic approximation involves only points within Ω', that is, $\mathcal{R} \subset \Omega'$. In this case the theory of ubiquitous systems can be applied directly, provided we are given an appropriate probability measure μ on Ω'. However, the results can be highly unsatisfactory: for example, conics can have either infinitely many rational points or none, like the circles in $\mathbb{R}^2$ with equations $x^2 + y^2 = 1$ and $x^2 + y^2 = 3$, respectively. Even worse, a celebrated theorem by Faltings says that any algebraic curve over $\mathbb{Q}$ of genus greater than 1 has at most finitely many rational points (see e.g. [56, Theorem E.0.1]), stifling any hope of intrinsic approximation by rationals.

For extrinsic approximation, on the other hand, $\mathcal{R}$ can also contain points in $\Omega \setminus \Omega'$. Applying the theory of ubiquitous systems is often still possible in this case, but can be considerably harder: it involves projecting $\mathcal{R}$ onto Ω' in order to obtain an approximate set of resonant points, and carefully handling the resulting error to avoid affecting the overall approximation rate. We will give an example of this approach in Section 5.7.

— 3 —

Quantitative non-divergence

The quantitative non-divergence estimates of Kleinbock and Margulis for flows on homogeneous spaces are among the most powerful tools available in Diophantine Approximation, and to understand how these come into play we need to introduce the so called *Dani correspondence*. For simplicity we will present this here only in the case of the approximation problem $\langle\!\langle q\alpha \rangle\!\rangle < \psi(q)$ covered by the classical Khinchin and Jarník theorems, with the extra assumption that ψ is decreasing, and for more details we invite the interested reader to consult [70] and the survey papers [68, 18].

With a slight rewording, we would like to determine whether the system

$$\begin{cases} |q\alpha + p| < \psi(Q) \\ |q| \leq Q \end{cases} \tag{3.1}$$

admits a non-zero solution $p, q \in \mathbb{Z}$ for infinitely many integers $Q > 0$. The right-hand side can be further rewritten in matrix form as

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix}, \tag{3.2}$$

and, after denoting the matrix $\left(\begin{smallmatrix} 1 & \alpha \\ 0 & 1 \end{smallmatrix}\right)$ by U_α, it is now clear that we are looking for (non-zero) points of the lattice $\Gamma = U_\alpha \mathbb{Z}^2$ which lie within a rectangle of sides $2\psi(Q)$ and $2Q$. This is closely related to the classical *shortest vector problem* in the theory of lattices: given a lattice $\Gamma \subset \mathbb{R}^n$ and a norm on $\mathbb{R}^n$, find the non-zero vector in Γ with the smallest norm; we will denote this smallest norm by $\lambda_1(\Gamma)$. Indeed, all we need to do to reinterpret our problem in these terms is to rescale our rectangle into a square, i.e. a ball under the sup norm $\|\cdot\|$. To this end, consider the scaling matrix

$$g_t := \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix}$$

and let $t = t(Q) = \frac{1}{2}\Big(\log Q - \log\psi(Q)\Big)$, as well as $\varepsilon = \varepsilon(Q) = \sqrt{Q\psi(Q)}$. Now suppose we can find a point $\boldsymbol{v} \in g_t U_\alpha \mathbb{Z}^2$ with norm $\|\boldsymbol{v}\| < \varepsilon$. Then, by definition, there is a point $(p,q)^T \in \mathbb{Z}^2$ such that

$$\begin{cases} |q\alpha + p| < e^{-t}\varepsilon = \psi(Q) \\ |q| < e^t \varepsilon = Q. \end{cases}$$

Also note that we can recover the solutions of (3.1) with $|q| = Q$ simply by slightly increasing the value of t (which doesn't affect ε).

Thus, as t varies, $g_t U_\alpha \mathbb{Z}^2$ describes the orbit of a flow on the space of unimodular lattices $\mathcal{L}_2 = \mathrm{SL}_2(\mathbb{R})/\mathrm{SL}_2(\mathbb{Z})$, and the key to our argument is the following criterion of Mahler's which originally appeared in [80] (for the formulation below see [95, Corollary 10.9] and [44, Chapter V]). Note that this holds in higher dimensions as well.

Theorem 3.0.1 (Mahler's Compactness Criterion)**.** *Let $\mathcal{L}_2(\varepsilon) \subset \mathcal{L}_2$ be the set of lattices whose shortest non-zero vector has norm at least ε. Then $\mathcal{L}_2(\varepsilon)$ is compact for any $\varepsilon > 0$. Vice versa, every compact set of $\mathcal{L}_2$ is contained in a set of the form $\mathcal{L}_2(\varepsilon)$.*

Therefore, a number α is ψ-well approximable if and only if the orbit $g_t U_\alpha \mathbb{Z}^2$ escapes infinitely many compact sets of the form $\mathcal{L}_2(\varepsilon(Q))$, i.e. if

$$\lambda_1\Big(g_{t(Q)} U_\alpha \mathbb{Z}^2\Big) < \varepsilon(Q)$$

for infinitely many $Q > 0$. Hence the key to applying this framework to derive metric approximation results is to provide a *quantitative estimate* for the proportion of $\alpha \in [0,1]$ for which this flow *diverges* at a certain rate.

In 1998 Kleinbock and Margulis introduced this kind of estimate to prove an extremality conjecture of Sprindžuk's about a problem of multiplicative approximation [71]. However, to be able to state their theorem we need to state a couple more definitions.

First off, consider an open subset $U \subseteq \mathbb{R}^d$ and a Lebesgue-measurable function $f\colon U \to \mathbb{R}$. For any open ball $B \subset U$ and $\varepsilon > 0$, define

$$B^{f,\varepsilon} := \{x \in B : |f(x)| < \varepsilon\}. \tag{3.3}$$

Then we say that f is *(C,α)-good on U* if there are constants $C, \alpha > 0$ such that for any open ball $B \subset U$ we have

$$\left|B^{f,\varepsilon}\right| \leq C\left(\frac{\varepsilon}{\|f\|_B}\right)^{\alpha}|B| \quad \text{for all } \varepsilon > 0, \tag{3.4}$$

where $\|f\|_B := \sup_{x\in B}|f(x)|$. Note that we will assume $\frac{1}{0} = \infty$, so that (3.4) holds trivially when $\|f\|_B = 0$. We refer the reader to Sections 3.1.2 and 5.4 for some properties and examples.

Given integers $k \geq \tau > 0$, a *primitive tuple* of $\mathbb{Z}^k$ is a subset $\{\boldsymbol{w}_1, \dots, \boldsymbol{w}_\tau\}$ that can be completed to a basis of $\mathbb{Z}^k$. We will denote by $\mathcal{W}_\tau^k$ the set of elements

$$\boldsymbol{w} = \boldsymbol{w}_1 \wedge \dots \wedge \boldsymbol{w}_\tau \in \bigwedge^{\tau} \mathbb{Z}^k \quad \text{where } \{\boldsymbol{w}_1, \dots, \boldsymbol{w}_\tau\} \text{ is a primitive tuple,}$$

and by $\mathcal{W}^k$ the union of all such sets over $\tau \in \{1, \dots, k\}$. Observe that, up to a sign, the elements of $\mathcal{W}_\tau^k$ can be identified with the τ-dimensional rational subspaces of $\mathbb{R}^k$.

Here and in what follows, if $c > 0$ and $B = B(x,r)$, then cB will denote the ball $B(x,cr)$. Unless otherwise stated, $\|\cdot\|$ will denote the sup norm, although a change of norm only implies a change of constant in (3.5) below, and with a slight abuse of notation we will also denote by $\|\cdot\|$ the extension of $\|\cdot\|$ to $\bigwedge^\tau \mathbb{R}^k \simeq \mathbb{R}^{\binom{k}{\tau}}$ with respect to the standard basis of $\mathbb{R}^k$, e.g. in the case of the sup norm:

$$\|\boldsymbol{v}\| := \max_{I \in [\![k]\!]^{\tau}_{<}} |v_I| \quad \text{for every} \quad \boldsymbol{v} = \sum_{I \in [\![k]\!]^{\tau}_{<}} v_I \boldsymbol{e}_I \in \bigwedge^{\tau} \mathbb{R}^k.$$

Theorem 3.0.2. *Fix $d, k \in \mathbb{N}$, $C, \alpha > 0$, $0 < \rho \leq 1$. Let B be a ball in $\mathbb{R}^d$ and let $\tilde{B} = 3^k B$. Suppose that $\eta\colon \tilde{B} \to \mathrm{GL}_k(\mathbb{R})$ is a map such that for every $\boldsymbol{w} \in \mathcal{W}^k$:*

1. *the function $\boldsymbol{x} \mapsto \|\eta(\boldsymbol{x})\boldsymbol{w}\|$ is (C, α)-good on $\tilde{B}$; and*
2. *$\|\eta(\cdot)\boldsymbol{w}\|_B \geq \rho$.*

Then for any $0 < \varepsilon \leq \rho$ we have

$$\left|\left\{\boldsymbol{x} \in B : \lambda_1(\eta(\boldsymbol{x})\mathbb{Z}^k) < \varepsilon\right\}\right| \leq k(3^d N_d)^k C\left(\frac{\varepsilon}{\rho}\right)^\alpha |B|, \tag{3.5}$$

where N_d is a constant depending only on d.

Remark 3.0.3. The original statement of [71, Theorem 5.2] only allowed $\rho \in (0, 1/k]$, but the range can be extended to $(0, 1]$ as a special case of [34, Theorem 6.2].

Remark 3.0.4. Although the statements of Theorem 3.0.2 and its variants include the condition $\varepsilon \leq \rho$, this is not restrictive, since (3.5) is non-trivial only when $k(3^d N_d)^k C\left(\frac{\varepsilon}{\rho}\right)^\alpha < 1$.

Remark 3.0.5. [67, Theorem 3.4] shows that the factor $3^d N_d$ in (3.5) can be replaced with 2 when $d = 1$, the image of η is contained in $\mathrm{SL}_k(\mathbb{R})$, and $\|\cdot\|$ denotes the Euclidean norm.

Over the years, Theorem 3.0.2 has been improved upon in many ways. Most notably, Kleinbock, Lindenstrauss and Weiss extended it to a wider class of measures [69, Theorem 4.3], and Kleinbock and Tomanov proved a p-adic analogue [72, Theorem 9.3]. More recently, Das, Fishman, Simmons, and Urbánski extended it to even more general measures [46, Lemma 4.6], although they stated their result only in the context of the classical extremality approximation problems. Here we will only mention the following version, which is the one we will use in Chapter 5, but in order to state it we need to define the notion of (C, α)-goodness for more general metric spaces.

Let X be a metric space and ν a Radon measure on an open subset $U \subseteq X$. A ν-measurable function $f \colon U \to \mathbb{R}$ is said to be (C, α)*-good on* U *with respect to* ν if there are constants $C, \alpha > 0$ such that for every open ball $B \subset U$ centred on $\operatorname{supp} \nu$ we have

$$\nu\Big(B^{f,\varepsilon}\Big) \leq C\left(\frac{\varepsilon}{\|f\|_{\nu,B}}\right)^{\alpha} \nu(B) \quad \text{for all } \varepsilon > 0,$$

where $B^{f,\varepsilon}$ is as in (3.3) and $\|f\|_{\nu,B} \coloneqq \sup_{x \in B \cap \operatorname{supp} \nu} |f(x)|$.

Following [69], we will also say that ν is D*-Federer (or doubling) on* U for some $D > 0$ if

$$\nu\Big(3^{-1}B\Big) > D^{-1}\nu(B)$$

for any ball $B \subset U$ centred on $\operatorname{supp} \nu$. Furthermore, a Radon measure ν on X is said to be *Federer* if for ν-almost every $x \in X$ there are a neighbourhood U of x and a $D > 0$ such that ν is D-Federer on U.

Finally, again following [69], we will say that X is *Besicovitch* if there is an $N > 0$ such that, for any bounded set $A \subset X$ and any collection of balls $\mathcal{B}$ such that every $x \in A$ is in the centre of a ball in $\mathcal{B}$, there is a countable collection $\Omega \subseteq \mathcal{B}$ which covers A and such that every point $x \in A$ lies in at most N balls in Ω. It is well known that $\mathbb{R}^d$ with the Euclidean metric is Besicovitch, see e.g. [84, Theorem 2.7], and the constant N_d in Theorem 3.0.2 is precisely the Besicovitch constant of $\mathbb{R}^d$.

Theorem 3.0.6 ([66, Theorem 2.2])**.** *Fix* $k, N \in \mathbb{N}$ *and* $C, D, \alpha, \rho > 0$. *Given an* N*-Besicovitch metric space* X*, let* B *be a ball in* X *and* ν *be a measure which is* D*-Federer on* $\tilde{B} = 3^k B$. *Suppose that* $\eta \colon \tilde{B} \to \mathrm{GL}_k(\mathbb{R})$ *is a map such that for every* $1 \leq \tau \leq k$ *and for every* $\boldsymbol{w} \in \mathcal{W}_\tau^k$*:*

1. *the function* $\boldsymbol{x} \mapsto \|\eta(\boldsymbol{x})\boldsymbol{w}\|$ *is* (C, α)*-good on* $\tilde{B}$ *with respect to* ν*, and*
2. $\|\eta(\cdot)\boldsymbol{w}\|_{\nu,B} \geq \rho^\tau$.

Then for any $0 < \varepsilon \leq \rho$ *we have*

$$\nu\left(\left\{\boldsymbol{x} \in B : \lambda_1\left(\eta(\boldsymbol{x})\mathbb{Z}^k\right) < \varepsilon\right\}\right) \leq k(ND^2)^k C \left(\frac{\varepsilon}{\rho}\right)^{\alpha} \nu(B).$$

While the improvement of 2 is not often needed in applications, we note here that it is sometimes crucial for problems in Metric Diophantine

Approximation; see for example [66, Section 3] and the discussion that precedes it. Finally, we should also mention the important application of Theorem 3.0.6 by Aka, Breuillard, Rosenzweig and de Saxcé, who in 2018 used it to determine the precise geometric constraints to extremality [2].

3.1 APPROXIMATION ON CIRCLES BY GAUSSIAN RATIONALS

In recent years, results from metric Diophantine Approximation have found many fruitful applications in the field of wireless communications, where they have been used to analyse the performance of some novel communication channels based on the principle of *interference alignment* (see [23] for a mathematician-friendly introduction). Proposition 3.1.1 below is motivated by one such application: the upper bound we prove here could then be used in a manner similar to [91, Section VI.B], where an analogous bound for approximation by rationals is used to estimate the achievable rate of communication of a symmetric K-user interference channel.

Consider the following subset of the circle $S_r := \{z \in \mathbb{C} : |z| = r\}$:

$$L_r(\Delta, Q) := \{z \in S_r : |qz - p| < \Delta, |q| \leq Q \text{ for some } p, q \in \mathbb{Z}[i]\}.$$

As a simple application of Theorem 3.0.2 we will prove an effective upper bound for the Lebesgue measure of $L_r(\Delta, Q)$; that is, we will show that:

Proposition 3.1.1. *For any $\Delta, Q > 0$ we have*

$$|L_r(\Delta, Q)| < 128\pi\sqrt[4]{2}\, C \max\left\{1, r^{-1}\right\} \sqrt[4]{\Delta Q},$$

where C can be taken to be $36\sqrt{3}$.

Remark 3.1.2. Note that the components of (the numerator and denominator of) $\frac{p}{q}$ are quadratic in the components of the Gaussian integers p, q. This means that many of the usual geometry of numbers results are unavailable to us, since they tend to rely on linear forms in integers. Therefore proving an upper bound for the measure of $L_r(\Delta, Q)$ is subtly more difficult than proving a similar bound for the equivalent set of approximation by

Gaussian rationals. Cfr. [104] and [16], both of which deal with the problem of simultaneous approximation of planar curves by rational points, and in particular the proof of [16, Theorem 7].

3.1.1 Lattices

Note that the points of S_r can be parametrised by the map $\varphi\colon [-\pi,\pi) \to \mathbb{C}$ given by $x \mapsto r(\cos(x) + i\sin(x))$ and, since φ is periodic, we shall tacitly extend its domain to the whole $\mathbb{R}$ whenever needed. Also, here and for the remainder of this Chapter $\|\cdot\|$ will denote the Euclidean norm on $\mathbb{R}^4$. Then the inequalities that define $L_r(\Delta, Q)$ imply that

$$\begin{aligned}\left\| u_{\varphi(x)} \begin{pmatrix} p_1 \\ p_2 \\ q_1 \\ q_2 \end{pmatrix} \right\| &= \left\| \begin{pmatrix} q_1\cos(x) - q_2\sin(x) + p_1 \\ q_1\sin(x) + q_2\cos(x) + p_2 \\ q_1 \\ q_2 \end{pmatrix} \right\| \\ &= \sqrt{|qz-p|^2 + |q|^2} \\ &< \sqrt{\Delta^2 + Q^2}\end{aligned} \tag{3.6}$$

where $q = q_1 + iq_2$, $-p = p_1 + ip_2$, and

$$u_z = \left(\begin{array}{c|c} I_2 & A_z \\ \hline 0 & I_2 \end{array}\right), \text{ where } A_z = \begin{pmatrix} \mathrm{Re}(z) & -\mathrm{Im}(z) \\ \mathrm{Im}(z) & \mathrm{Re}(z) \end{pmatrix}.$$

However, recall that to be able to apply Theorem 3.0.2 we need to restate this in terms of a smallest vector problem. In other words, we need to rescale (3.6) so that the bound is uniform across all four components. To this end, consider the diagonal matrix

$$g_t = \begin{pmatrix} e^{t/2} & & & \\ & e^{t/2} & & \\ & & e^{-t/2} & \\ & & & e^{-t/2} \end{pmatrix} \text{ where } t \in \mathbb{R}.$$

Then if $z = \varphi(x) \in S_r$ we have that

$$\left\| g_t u_{\varphi(x)} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \right\| = \sqrt{e^t|qz-p|^2 + e^{-t}|q|^2},$$

from which, by choosing t such that $e^t\Delta^2 = e^{-t}Q^2$, we deduce the existence of a non-zero $(\mathbf{p},\mathbf{q})^T \in \mathbb{Z}^4$ such that

$$\left\| g_t u_{\varphi(x)} \begin{pmatrix} \mathbf{p} \\ \mathbf{q} \end{pmatrix} \right\| < \sqrt{2\Delta Q} =: \varepsilon.$$

3.1.2 Good functions

The following properties of (C,α)-good functions are direct consequences of the definition:

Lemma 3.1.3 ([71, Lemma 3.1],[34, Lemma 3.1])**.** *Given $V \subseteq \mathbb{R}^d$ and $C,\alpha > 0$:*

1. *A function f is (C,α)-good on V iff so is $|f|$.*
2. *A function f is (C,α)-good on V iff so is λf for any $\lambda \in \mathbb{R}$.*
3. *If the functions f_i for $i \in I$ are (C,α)-good on V, then so is $\sup_{i\in I}|f_i|$.*
4. *If f is (C,α)-good on V and $c_1 \le \frac{|f(x)|}{|g(x)|} \le c_2$ for all $x \in V$, then g is $(C(c_2/c_1)^\alpha,\alpha)$-good on V.*

For the purpose of our argument, we are interested in showing that every linear combination of $1, \sin, \cos$ is $(C,1/2)$-good on $\mathbb{R}$ for some effectively computable $C > 0$. This is a straightforward application of the following Lemma, which is a special case of [71, Lemma 3.3].

Lemma 3.1.4. *Let V be an open subset of $\mathbb{R}$ and let $f \in \mathcal{C}^k(V)$ be such that there are constants $A_1, A_2 > 0$ with*

$$\left\| f^{(i)} \right\|_V \le A_1 \text{ for all } i \le k$$

and

$$\left| f^{(k)}(x) \right| \ge A_2 \text{ for all } x \in V$$

where $f^{(i)}$ denotes the i-th derivative of f. Then for any open interval $B \subset V$ and for any $\varepsilon > 0$ we have

$$\left| \{x \in B : |f(x)| < \varepsilon\} \right| \le C \left(\frac{\varepsilon}{\|f\|_B} \right)^{\frac{1}{k}} |B|$$

where we can choose

$$C = k(k+1)\left(\frac{A_1}{A_2}(k+1)(2k^k+1)\right)^{\frac{1}{k}}.$$

Now let f be a linear combination of 1, sin and cos, say

$$f(x) = c_0 + c_1\cos(x) + c_2\sin(x).$$

Also let $V \subset \mathbb{R}$ be an open interval containing $[-\pi, \pi]$, for example $V = (-\pi - v_0, \pi + v_0)$ for some $v_0 > 0$. Then, to apply Lemma 3.1.4 it is enough to find

$$A_2 \leq \max_V\{|f'(x)|, |f''(x)|\} =: M.$$

Using Cramer's rule on the system

$$\begin{pmatrix} -\sin(x) & \cos(x) \\ -\cos(x) & -\sin(x) \end{pmatrix}\begin{pmatrix} c_1 \\ c_2 \end{pmatrix} = \begin{pmatrix} f'(x) \\ f''(x) \end{pmatrix}$$

we find that

$$c_1 = -f'(x)\sin(x) - f''(x)\cos(x) \qquad c_2 = f'(x)\cos(x) - f''(x)\sin(x).$$

Therefore

$$|c_1| \leq |\sin(x) + \cos(x)|\, M \leq \sqrt{2}M$$
$$|c_2| \leq |\cos(x) - \sin(x)|\, M \leq \sqrt{2}M,$$

so in the end we have

$$A_2 := \frac{1}{\sqrt{2}}\max\{|c_1|, |c_2|\} \leq M \leq \sqrt{2}\max\{|c_1|, |c_2|\} =: A_1$$

and $A_1/A_2 = 2$. Applying Lemma 3.1.4 we see that (on an interval $B \subset V$) when $|f'(x)| \geq |f''(x)|$ we may take $(C, \alpha) = (24, 1)$, while when $|f'(x)| < |f''(x)|$ we may take $(C, \alpha) = (18\sqrt{6}, 1/2)$. Since we may always assume that $\varepsilon/\|f\|_B < 1$, it follows that in the worst case we need $(C, \alpha) = (18\sqrt{6}, 1/2)$.

3.1.3 Proof of Proposition 3.1.1

In the statement of Theorem 3.0.2 take $k = 4$, $d = 1$, $B = [-\pi, \pi]$ and $\eta(x) = g_t u_{\varphi(x)}$, where g, u are defined as in Section 3.1.1. Also let $\|\cdot\|_\infty$ denote the sup norm and define $\psi_{\boldsymbol{w}}(x) := \|\eta(x)\boldsymbol{w}\|_\infty$. By direct computation we can show that for every $\boldsymbol{w} \in \mathcal{W}^k$ the function $\psi_{\boldsymbol{w}}(x)$ is an integer linear combination of

$$1, r\sin(x), r\cos(x), \text{ and } r^2\sin^2(x) + r^2\cos^2(x) = r^2.$$

Therefore, by Section 3.1.2 it follows that $\psi_{\boldsymbol{w}}$ is $(18\sqrt{6}, 1/2)$-good on $\mathbb{R}$. Since $\|y\|_\infty \le \|y\| \le 2\|y\|_\infty$ for any $y \in \mathbb{R}^4$, Lemma 3.1.3 implies that $x \mapsto \|\eta(x)\boldsymbol{w}\|$ is $(36\sqrt{3}, 1/2)$-good on $\mathbb{R}$. Furthermore, if

$$\|\eta(x)\boldsymbol{w}\| = d_0 + d_1 r^2 + d_2 r\sin(x) + d_3 r\cos(x),$$

then

$$\sup_B \|\eta(x)\boldsymbol{w}\| \ge \left|d_0 + d_1 r^2\right| + r\max\{|d_2|, |d_3|\} \ge \min\{1, r^2\},$$

since the d_j are all integers and at least one must be non-zero. Therefore, by Section 3.1.1 and Theorem 3.0.2, as well as Remark 3.0.5, we conclude that for every $\Delta, Q > 0$ we have

$$\begin{aligned} |L_r(\Delta, Q)| &\le \left|\left\{x \in [-\pi,\pi] : \lambda_1(g_t u_{\varphi(x)}\mathbb{Z}^4) < \sqrt{\Delta Q}\right\}\right| \\ &< |B| k 2^k C \sqrt{\frac{\varepsilon}{\min\{1, r^2\}}} \\ &= 2\pi \cdot 4 \cdot 2^4 \cdot 36\sqrt{3} \cdot \sqrt[4]{2} \frac{\sqrt[4]{\Delta Q}}{\min\{1, r\}} \\ &= 128\pi\sqrt[4]{2} \cdot 36\sqrt{3} \max\left\{1, r^{-1}\right\} \sqrt[4]{\Delta Q}. \end{aligned}$$

3.1.4 Further remarks

It seems that this bound is far from being sharp: a more direct computation of C, as well as computer simulations, suggests that it should be possible to take $C = 4$. Unfortunately, as of this writing we have been unable to prove that our analysis is exhaustive.

Furthermore, we should point out that the above proof can be carried through for any curve in the complex plane, parametrised as $\varphi(x) = \varphi_1(x) + i\varphi_2(x)$, as long as one can find $C, \alpha > 0$ such that every integer linear combination of 1, φ_1, φ_1 and $\varphi_1^2 + \varphi_2^2$ is (C, α)-good. In particular, this is the case when φ_1, φ_2 are analytic and linearly independent.

— 4 —

A Jarník-type theorem for a problem of Mahler's [1]

4.1 INTRODUCTION

Recall from Chapter 1 that classical Diophantine Approximation studies the density of the rational numbers in the set of real numbers, starting with Dirichlet's theorem, which states that for any real number x there are infinitely many pairs $(p, q) \in \mathbb{Z} \times \mathbb{Z} \setminus \{0\}$ such that

$$|qx - p| < \frac{1}{q}.$$

This is in a sense optimal, since by Hurwitz's theorem

$$|q\varphi - p| < \frac{1}{cq}$$

has at most finitely many solutions (p, q) as above when $c > \sqrt{5}$ and $\varphi = (\sqrt{5} - 1)/2$ (see [55, Theorem 194] for a proof). However, Khinchin's and Jarník's theorems tell us — in a precise way — how likely it is that a randomly chosen real number can be approximated by rationals up to a certain accuracy, which is given in terms of a decreasing function of the denominators.

More precisely, let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be such a function, called an *approximation function*, and define

$$W_n(\psi) \coloneqq \{x \in \mathbb{R}^n : |q \cdot x - p| < \psi(|q|) \text{ for i.m. } (p, q) \in \mathbb{Z} \times \mathbb{Z}^n \setminus \{0\}\}$$

[1]The content of this chapter was previously published in [92].

where $|q| = \max|q_i|$ and "i.m." is shorthand for "infinitely many". Here and in what follows $\mathcal{H}^s$ denotes the usual s-dimensional Hausdorff measure, which we recall in the next section.

Theorem 4.1.1 (Jarník). *In the above setting, for any $s \geq 0$ we have*

$$\mathcal{H}^s(W_1(\psi)) = \begin{cases} 0 & \text{if } \sum_{q=1}^{\infty} \psi(q)^s q^{1-s} < \infty \\ \infty & \text{otherwise.} \end{cases}$$

Furthermore, with some extra hypotheses (detailed in Corollary 4.1.3 below) Jarník's theorem can be generalised to Hausdorff g-measures $\mathcal{H}^g$, and this gives a pretty accurate description of the geometry of the set $W_1(\psi)$. A more general approximation problem consists of looking at the set $\mathscr{L}_n(w)$ of real numbers x such that

$$|P(x)| < \mathrm{H}(P)^{-w}$$

for infinitely many integer polynomials P with degree bounded above by n, where $w > 0$ is given and $\mathrm{H}(P)$ denotes the *height* of P, i.e. the maximum absolute value of its coefficients. This is related to, but subtly different from, the problem of approximating x by algebraic numbers of bounded degree; we refer the interested reader to [42, Chapter 3], in particular to the last part of Section 3.4.

The study of this problem dates back to Mahler. With Minkowski's linear forms theorem one can prove that $\mathscr{L}_n(w)$ has full Lebesgue measure for any $w \leq n$, and in 1932 Mahler conjectured that $\mathscr{L}_n(w)$ has measure 0 for every $w > n$ [82]. In 1969 Sprindžuk proved this in full generality [101], although the cases $n = 2$ and $n = 3$ had already been settled by Kubilyus, Kasch, and Volkmann (see [105] for more details).

The picture for Hausdorff measures, on the other hand, is a bit less clear. In 1983 Bernik proved that $\mathscr{L}_n(w)$ has Hausdorff dimension $\frac{n+1}{w+1}$ [25], and in 2006 Beresnevich, Dickinson, and Velani proved [17, Theorem 18], which specialises to the divergence part of a Jarník-type theorem for Mahler's problem. Interestingly, though, the convergence case is not quite as straightforward as for Jarník's theorem and the only results so far in this direction are for $n = 2$; these were the work of Hussain [60] and Huang [57], who gave a more general proof for the case of non-degenerate $\mathcal{C}^2$ plane curves.

4.1.1 Hausdorff measures and dimension

Let X be a subset of $\mathbb{R}^n$ and let $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a *dimension function*, i.e. a continuous, increasing function s.t. $g(r) \to 0$ as $r \to 0$. Given $\rho > 0$, a *ρ-cover* of X is a (possibly countable) collection $\{B_i\}$ of balls of $\mathbb{R}^n$ s.t. the radius $r(B_i)$ of each ball B_i lies in $(0, \rho]$ and $X \subseteq \bigcup B_i$. Now define

$$\mathcal{H}^g_\rho(X) \coloneqq \inf \left\{ \sum g\left(r(B_i)\right) : \{B_i\} \text{ is a } \rho\text{-cover of } X \right\}$$

and note that this is increasing as $\rho \to 0$. Therefore the limit

$$\mathcal{H}^g(X) \coloneqq \lim_{\rho \to 0^+} \mathcal{H}^g_\rho(X) = \sup_{\rho > 0} \mathcal{H}^g_\rho(X)$$

exists, and is called the *Hausdorff g-measure* of X. When $g(r) = r^s$ for some $s \geq 0$, it is customary to write $\mathcal{H}^s(X)$ for $\mathcal{H}^g(X)$, which is then called the *s-dimensional Hausdorff measure* of X. Moreover, if s is an integer, then $\mathcal{H}^s$ is just a constant multiple of the Lebesgue measure on $\mathbb{R}^s$.

If h, g are two dimension functions, a straightforward standard argument shows that if $h(r)/g(r) \to 0$ when $r \to 0$, then

$$\mathcal{H}^h(X) = 0 \text{ whenever } \mathcal{H}^g(X) < \infty.$$

In particular, this implies that if $s > t \geq 0$, then $\mathcal{H}^s(X) = 0$ when $\mathcal{H}^t(X) < \infty$. We can then define the *Hausdorff dimension* of X as

$$\dim_{\mathcal{H}}(X) \coloneqq \inf\left\{s \geq 0 : \mathcal{H}^s(X) = 0\right\}.$$

Finally, note that if $\dim_{\mathcal{H}}$ is an integer, then it coincides with the usual "naive" notion of dimension.

4.1.2 Our setting

From here on we will make heavy use of Vinogradov's notation $\ll$ and $\asymp$, where $a \ll b$ means that $a \leq cb$ for some constant $c > 0$, while $a \asymp b$ means that both $a \ll b$ and $b \ll a$. We will also use subscripts to emphasise the dependence of the implied constant on certain quantities; for example, $a \ll_n b$ means that the implied constant c depends on n. For later convenience, define

$$\mathcal{P}_n \coloneqq \{P \in \mathbb{Z}[X] : \deg(P) \leq n\}$$

and

$$\mathcal{A}_n(\psi) \coloneqq \{x \in \mathbb{R} : |P(x)| \leq \psi(\mathrm{H}(P)) \text{ for i.m. } P \in \mathcal{P}_n\}.$$

Theorem 4.1.2 (Beresnevich, Dickinson, Velani [17, Theorem 18])**.** *Let $\mathcal{M}$ be a non-degenerate submanifold of $\mathbb{R}^n$ of dimension m. Let ψ be an approximation function, and let g be a dimension function such that $q^{-m}g(q)$ is decreasing and $q^{-m}g(q) \to \infty$ as $q \to 0$. Furthermore, suppose that $q^{1-m}g(q)$ is increasing. Then*

$$\mathcal{H}^g(W_n(\psi) \cap \mathcal{M}) = \infty \quad \textit{if} \quad \sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) \psi(q)^{1-m} q^{m+n-1} = \infty.$$

Corollary 4.1.3. *Let ψ be an approximation function and consider an increasing dimension function g such that $q^{-1}g(q)$ is decreasing and $q^{-1}g(q) \to \infty$ as $q \to 0$. Then*

$$\mathcal{H}^g(\mathcal{A}_n(\psi)) = \infty \quad \textit{if} \quad \sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) q^n = \infty.$$

Recall that the *discriminant* of an integer polynomial P of degree n is defined as

$$\mathrm{D}(P) \coloneqq a_n^{2n-2} \prod_{1 \leq i < j \leq n} (\alpha_i - \alpha_j)^2$$

where $\alpha_1, \ldots, \alpha_n$ are the (possibly complex) roots of P, counted with multiplicity, and a_n is the leading coefficient of P. Furthermore, it can be shown that $\mathrm{D}(P)$ is a homogeneous polynomial of degree $2n-2$ in the coefficients of P, and its value is bounded above by $c_n \,\mathrm{H}(P)^{2n-2}$, for some constant c_n that depends only on n.

Now, for any given $0 < \lambda \leq n-1$ and fixed $0 < \tau \leq c_n$, consider

$$\mathcal{P}_n^\lambda := \left\{ P \in \mathcal{P}_n : |\mathrm{D}(P)| \geq \tau H(P)^{2(n-1-\lambda)} \right\}$$

and

$$\mathcal{A}_n^\lambda(\psi) := \left\{ x \in \mathbb{R} : |P(x)| \leq \psi(\mathrm{H}(P)) \text{ for i.m. } P \in \mathcal{P}_n^\lambda \right\}.$$

In this chapter we will examine the case $n = 3$ of the convergence equivalent of Corollary 4.1.3 and provide a partial result for general n; moreover, our conclusions do not depend on the choice of τ. Namely, we will prove the following:

Theorem 4.1.4. *Let ψ and g as in Corollary 4.1.3. Then for any $0 < \lambda < 1$ we have that*

$$\mathcal{H}^g\left(\mathcal{A}_n^\lambda(\psi)\right) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) q^n < \infty.$$

For the counterpart, set the notation

$$\mathcal{P}_{n,\lambda} := \left\{ P \in \mathcal{P}_n : |\mathrm{D}(P)| < \tau H(P)^{2(n-1-\lambda)} \right\}$$

and

$$\mathcal{A}_{n,\lambda}(\psi) := \left\{ x \in \mathbb{R} : |P(x)| \leq \psi(\mathrm{H}(P)) \text{ for i.m. } P \in \mathcal{P}_{n,\lambda} \right\}.$$

Theorem 4.1.5. *Consider ψ and g as in Corollary 4.1.3. Let $\mathcal{P}_{3,\lambda}^*$ be the set of irreducible polynomials in $\mathcal{P}_{3,\lambda}$ and let $\mathcal{A}_{3,\lambda}^*(\psi)$ be the corresponding* lim sup *set. Further assume that $0 \leq \lambda < 9/20$. Then*

$$\mathcal{H}^g\left(\mathcal{A}_{3,\lambda}^*(\psi)\right) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) q^{3-2\lambda/3} < \infty.$$

Corollary 4.1.6. *Suppose that $\psi(q) = q^{-w}$ for some $w > 0$ and that $0 \leq \lambda < 9/20$. As customary, write $\mathcal{A}_{3,\lambda}(w)$ for $\mathcal{A}_{3,\lambda}(\psi)$. Then*

$$\mathcal{H}^g(\mathcal{A}_{3,\lambda}(w)) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} g\left(q^{-w-1}\right) q^{3-2\lambda/3} < \infty.$$

Note that the condition $0 \leq \lambda < 9/20$ stems from the fact that our proof is based on the discriminant estimate from [63, Corollary 2], much like Volkmann's proof of the cubic case of Mahler's conjecture [105] relied on a similar estimate by Davenport [47]. As a special case, for $\lambda = 0$ we recover Bernik's result for $n = 3$, namely that the Hausdorff dimension of $\mathscr{L}_3(w) = \mathcal{A}_{3,0}(w)$ is $\frac{4}{w+1}$.

4.2 A FEW LEMMAS ON POLYNOMIALS

In this section we will collect some lemmas that we will use later in the chapter. Some we prove here, while others are taken from [101], often restated in a slightly simpler way that is enough for our purpose.

Lemma 4.2.1. *[107, Hilfssatz 3] Let $P_1, \dots, P_k$ be integer polynomials. Then*

$$\mathrm{H}(P_1 \cdots P_k) \asymp \mathrm{H}(P_1) \cdots \mathrm{H}(P_k)$$

where the implied constants depend only on the degrees of the polynomials.

Proof. Recall that the Mahler measure of a polynomial P of degree d is defined as

$$\mathrm{M}(P) \coloneqq |a_d| \prod_{i=1}^{d} \max\{1, |\alpha_i|\}$$

where a_d and α_i are the leading coefficient and roots of P, respectively. Now, Mahler [81] showed that $\mathrm{M}(P)$ satisfies

$$\binom{d}{\lfloor d/2 \rfloor}^{-1} \mathrm{H}(P) \leq \mathrm{M}(P) \leq \sqrt{d+1}\, \mathrm{H}(P).$$

Hence the result follows by noting that the Mahler measure is multiplicative, which can easily be seen from its definition. □

Lemma 4.2.2. *[64, Lemma 3] Let P be an integer polynomial of degree at most $n \geq 2$ and with non-zero discriminant. If α is a root of P, then*

$$|P'(\alpha)| \gg |\mathrm{D}(P)|^{\frac{1}{2}}\, \mathrm{H}(P)^{-n+2}$$

where the implied constant depends only on n.

Lemma 4.2.3. *[64, Lemma 4] Let P be as in Lemma 4.2.2 and consider some $x \in \mathbb{C}$. If α is the closest root of P to x, then*

$$|x - \alpha| \ll \mathrm{H}(P)^{n-2} |\mathrm{D}(P)|^{-\frac{1}{2}} |P(x)|$$

where the implied constant depends only on n.

Lemma 4.2.4. *In the setting of Lemma 4.2.3 write H for $\mathrm{H}(P)$. Furthermore, assume that $x \in \left[-\frac{1}{2}, \frac{1}{2}\right)$ and that $|P(x)| < \psi(H)$ for some approximation function ψ. If $|\mathrm{D}(P)| \gg_n H^{2(n-1-\lambda)}$ for some $0 < 2\lambda < 1 - \log_H \psi(H)$, then we have $|P'(x)| \asymp_n |P'(\alpha)|$ for sufficiently large H.*

Proof. First, observe that by Lemma 4.2.2 we have

$$|P'(\alpha)| \gg_n H^{n-1-\lambda} H^{-n+2} = H^{1-\lambda}.$$

Then note that by Lemma 4.2.3 we have

$$|x - \alpha| \ll_n H^{n-2} H^{\lambda+1-n} \psi(H) = H^{\lambda-1} \psi(H) < H^{-1/2} \psi(H)^{1/2}.$$

Hence we can assume $|\alpha| < 1$, since this is less than $1/2$ for H large enough. Now, by the mean value theorem we can find some z between x and α, thus with $|z| < 1$, such that

$$|P'(x) - P'(\alpha)| = |P''(z)||x - \alpha| \ll_n H H^{\lambda-1} \psi(H) = H^{\lambda} \psi(H).$$

Finally, the hypothesis on λ implies that $H^{\lambda}\psi(H) < H^{1-\lambda}$, therefore up to choosing H large enough we have

$$|P'(x) - P'(\alpha)| < \frac{1}{2} |P'(\alpha)|,$$

from which it follows that $|P'(x)| \asymp_n |P'(\alpha)|$, as required. □

Lemma 4.2.5. *Fix $P \in \mathbb{C}[X]$ and $m \in \mathbb{C}$. If $P(X) = a_n X^n + \cdots + a_1 X + a_0$, then the coefficients of $P(X + m) = b_n X^n + \cdots + b_n X + b_0$ are*

$$b_k = \sum_{j=k}^{n} \binom{j}{k} a_j m^{j-k} \quad \textit{for each} \quad 0 \leq k \leq n.$$

Proof. We proceed by induction on n. If $n = 1$ then $P(X + m) = a_1 X + a_0 + ma_1$, which agrees with the above formula. Now assume the lemma is true for $n - 1$. Since we can write $P(X) = a_n X^n + Q(X)$, where $Q(X) = a_{n-1}X^{n-1} + \cdots + a_0$, we have that

$$P(X+m) = Q(X+m) + a_n(X+m)^n = Q(X+m) + a_n \sum_{i=0}^{n} \binom{n}{i} X^i m^{n-i}.$$

Thus $b_n = a_n$ and, by the induction hypothesis, for $0 \le k \le n-1$

$$b_k = \binom{n}{k} a_n m^{n-k} + \sum_{j=k}^{n-1} \binom{j}{k} a_j m^{j-k} = \sum_{j=k}^{n} \binom{j}{k} a_j m^{j-k}. \qquad \square$$

Note. While we chose to state the lemma over $\mathbb{C}$ for simplicity, there is nothing specific to it in the proof, which carries over as-is for any other commutative ring with unity.

Corollary 4.2.6. *In the setting of Lemma 4.2.5 we have*

$$\mathrm{H}(P(X+m)) \le (1+|m|)^n \, \mathrm{H}(P).$$

Proof. By Lemma 4.2.5, for each $0 \le k \le n$ we have

$$|b_k| \le \sum_{j=k}^{n} \binom{j}{k} |a_j||m|^{j-k} \le \mathrm{H}(P) \sum_{j=k}^{n} \binom{j}{k} |m|^{j-k}.$$

Since $(1+|m|)^n = \sum_{s=0}^{n} \binom{n}{s} |m|^s$, it is enough to prove that

$$\binom{n}{s} \ge \binom{s+k}{k} = \binom{s+k}{s}$$

for any $0 \le k \le n$ and $0 \le s \le n-k$. On the other hand, $\binom{t}{s}$ is monotonic in t for $t \ge s$, which can be readily seen from

$$\binom{t+1}{s} = \frac{t+1}{t+1-s} \frac{t!}{(t-s)!s!} \ge \binom{t}{s}.$$

Therefore the observation that $n \ge j = s + k$ completes the proof. $\square$

4.3 PROOF OF THEOREM 4.1.4

Let $I := \left[-\frac{1}{2}, \frac{1}{2}\right)$. We prove the result for $\mathcal{A}_n^\lambda(\psi) \cap I$, and then extend it to the whole $\mathcal{A}_n^\lambda(\psi)$. Our first goal is to estimate how much each polynomial in $\mathcal{P}_n^\lambda$ can contribute towards $\mathcal{A}_n^\lambda(\psi)$. To do so, consider some $\varepsilon > 0$ and $Q \in \mathbb{N}$. For a polynomial $P \in \mathcal{P}_n^\lambda$ with $\mathrm{H}(P) \leq Q$ define

$$\sigma_\varepsilon(P) := \{x \in I : |P(x)| \leq \varepsilon, |P'(x)| \geq 2\}.$$

Then let $B_n(Q, \varepsilon)$ be the union of $\sigma_\varepsilon(P)$ over all such polynomials. We will rely on the following specialisation of [11, Proposition 1]:

Lemma 4.3.1. *For any $Q > 4n^2$ and any $\varepsilon < n^{-1}2^{-n-2}Q^{-n}$ we have*

$$|B_n(Q, \varepsilon)| \leq n2^{n+2}\varepsilon Q^n,$$

where $|B_n(Q, \varepsilon)|$ denotes the Lebesgue measure of $B_n(Q, \varepsilon)$.

Now, partition $\mathcal{P}_n^\lambda$ into sets

$$\mathcal{P}_n^\lambda(t) := \left\{P \in \mathcal{P}_n^\lambda : 2^t \leq \mathrm{H}(P) < 2^{t+1}\right\}$$

and observe that

$$\mathcal{A}_n^\lambda(\psi) \cap I = \limsup \gamma_\psi(P) = \bigcap_{t_0=1}^{\infty} \bigcup_{t=t_0}^{\infty} \bigcup_{P \in \mathcal{P}_n^\lambda(t)} \gamma_\psi(P)$$

where $\gamma_\psi(P) := \{x \in I : |P(x)| \leq \psi(\mathrm{H}(P))\}$. Then, for t large enough and for any $P \in \mathcal{P}_n^\lambda(t)$, letting $\varepsilon = \psi(2^t)$ we have that $\gamma_\psi(P) \subseteq \sigma_\varepsilon(P)$, so that the sets $\sigma_\varepsilon(P)$ form a cover of $\mathcal{A}_n^\lambda(\psi) \cap I$. Indeed, $\psi(\mathrm{H}(P)) \leq \varepsilon$ since ψ is assumed to be decreasing. Furthermore, if α is the root of P closest to x, then up to choosing t_0 large enough Lemma 4.2.4 ensures that $|P'(x)|$ is comparable to $|P'(\alpha)|$, hence

$$|P'(x)| \gg_n \mathrm{H}(P)^{1-\lambda} \geq 2^{t(1-\lambda)}$$

so $|P'(x)| > 2$, again up to choosing t_0 large enough.

Note that each $\sigma_\varepsilon(P)$ is a union of finitely many intervals, the number of which is bounded above by a constant that depends only on n. We can't use this directly to obtain an upper bound for the Hausdorff dimension of $\mathcal{A}_n^\lambda$, though, because those intervals can be arbitrarily small, and we also don't know how many polynomials there are in each $\mathcal{P}_n^\lambda(t)$. To fix this, consider the sets

$$\tilde{\sigma}_\varepsilon(P) := \bigcup_{x\in\sigma_\varepsilon(P)} \left\{ y \in I : |y-x| < 2^{-t}\varepsilon \right\}.$$

Clearly $\sigma_\varepsilon(P) \subseteq \tilde{\sigma}_\varepsilon(P)$. Furthermore, by the mean value theorem, for each $y \in \tilde{\sigma}_\varepsilon(P)$ there is a $z \in I$ which lies between y and the corresponding $x \in \sigma_\varepsilon(P)$ such that

$$|P(y) - P(x)| = |P'(z)||y-x|.$$

Since $|z| < 1$ we have $|P'(z)| \ll_n \mathrm{H}(P) < 2^{t+1}$, thus

$$|P(y)| \leq |P(x)| + |P'(z)||y-x| \ll_n |P(x)| + 2\varepsilon \ll \varepsilon.$$

Now, let c be the constant implied in the above inequality, so that $\sigma_\varepsilon(P)$ is covered by intervals in $\sigma_{c\varepsilon}(P)$ of length at least $\ell = 2^{1-t}\varepsilon$. From this we can obtain a cover made up of intervals of length exactly ℓ, splitting up the larger intervals and allowing some overlap at the edges as necessary, and by Lemma 4.3.1 the polynomials in $\mathcal{P}_n^\lambda(t)$ contribute at most

$$\frac{|B_n(2^{t+1}, c\varepsilon)|}{\ell} \ll_n 2^{t(n+1)} =: N$$

of these intervals. To conclude, it follows that

$$\begin{aligned}
\mathcal{H}^g\left(\mathcal{A}_n^\lambda \cap I\right) &\ll_n \lim_{t_0\to\infty} \sum_{t\geq t_0} g(\ell)N \\
&= \lim_{t_0\to\infty} \sum_{t\geq t_0} g\left(\frac{\psi(2^t)}{2^{t-1}}\right) 2^{t(n+1)} \\
&\leq \lim_{t_0\to\infty} \sum_{t\geq t_0} g\left(\frac{\psi(2^t)}{2^{t}}\right) 2^{t(n+1)} \\
&= 0
\end{aligned}$$

because g is assumed to be increasing, ψ is decreasing, and by Cauchy's condensation test we know that

$$\sum_{t\geq 0} g\left(\frac{\psi(2^t)}{2^t}\right) 2^{t(n+1)} < \infty \quad \text{iff} \quad \sum_{q\geq 1} g\left(\frac{\psi(q)}{q}\right) q^n < \infty.$$

4.3.1 Extending the argument

Fix $m \in \mathbb{Z}$ and consider $x \in \left[m - \frac{1}{2}, m + \frac{1}{2}\right)$. Then suppose that $P \in \mathcal{P}_n^\lambda$ is such that $|P(x)| \le \psi(\mathrm{H}(P))$. Now, note that $y = x - m \in I$ and let $Q(X) = P(X + m)$, so that $Q(y) = P(x)$. Furthermore, by Lemma 4.2.6 we know that $c\,\mathrm{H}(Q) \le \mathrm{H}(P)$, where $c = (1 + |m|)^{-n}$ is independent of P. Therefore $Q \in \mathcal{P}_n^\lambda$ and

$$|Q(y)| \le \psi\left(\mathrm{H}(P)\right) \le \psi\left(c\,\mathrm{H}(Q)\right).$$

Hence the following lemma, together with the previous argument, is enough to complete the proof.

Lemma 4.3.2. *Let* $0 < c_1 < c_2$. *Then*

$$\sum_{q=1}^{\infty} g\left(\frac{\psi(c_1 q)}{q}\right) q^n < \infty \quad \textit{iff} \quad \sum_{q=1}^{\infty} g\left(\frac{\psi(c_2 q)}{q}\right) q^n < \infty.$$

Proof. To begin with, assume that the series with c_1 converges. Since ψ is decreasing we have $\psi(c_1 q) \ge \psi(c_2 q)$, and since g is increasing it follows that

$$\sum_{q=1}^{\infty} g\left(\frac{\psi(c_2 q)}{q}\right) q^n \le \sum_{q=1}^{\infty} g\left(\frac{\psi(c_1 q)}{q}\right) q^n < \infty.$$

For the other implication, note that $c = c_2 c_1^{-1} > 1$ and consider the following

$$\begin{aligned}
\sum_{q \ge c} g\left(\frac{\psi(c_1 q)}{q}\right) q^n &= \sum_{r=1}^{\infty} \sum_{cr \le q < c(r+1)} g\left(\frac{\psi(c_1 q)}{q}\right) q^n \\
&\le \sum_{r=1}^{\infty} \sum_{cr \le q < c(r+1)} g\left(\frac{\psi(c c_1 r)}{cr}\right) c^n (r+1)^n \\
&\le 2^n c^n \sum_{r=1}^{\infty} \left(\sum_{cr \le q < c(r+1)} 1\right) g\left(\frac{\psi(c_2 r)}{r}\right) r^n \\
&\le 2^n c^{n+1} \sum_{r=1}^{\infty} g\left(\frac{\psi(c_2 r)}{r}\right) r^n
\end{aligned}$$

where the first two inequalities are again due to the fact that g is increasing and ψ is decreasing. Therefore the first series converges when the second does, as required. □

4.4 PROOF OF THEOREM 4.1.5

Just like in the proof of Theorem 4.1.4 we will focus on $\mathcal{A}^*_{n,\lambda}(\psi) \cap I$, after which the result immediately extends to the whole $\mathcal{A}^*_{n,\lambda}(\psi)$. Similarly to what we did there, define

$$\mathcal{P}^*_{3,\lambda}(t) := \left\{ P \in \mathcal{P}^*_{3,\lambda} : 2^t \leq \mathrm{H}(P) < 2^{t+1} \right\}.$$

Now suppose that $P \in \mathcal{P}^*_{3,\lambda}$ and let $\sigma(P)$ be the set of $x \in I$ such that $|P(x)| \leq \psi(\mathrm{H}(P))$. Furthermore, let $\sigma(t)$ be the union of $\sigma(P)$ over all P in $\mathcal{P}^*_{3,\lambda}(t)$. Then, by Lemma 4.2.3, we know that

$$|x - \alpha| \leq c\, \mathrm{H}(P)\, |\mathrm{D}(P)|^{-1/2} \psi(\mathrm{H}(P)) =: r(P, \psi)$$

where α is the root of P closest to x and where the constant $c > 0$ is independent of P and x. Hence $\sigma(P)$ is covered by at most three intervals of radius $r(P, \psi)$ centred at the roots of P. Then

$$\mathcal{A}^*_{3,\lambda}(\psi) \cap I \subseteq \limsup \sigma(t) = \bigcap_{t_0=0}^{\infty} \bigcup_{t=t_0}^{\infty} \sigma(t)$$

and

$$\begin{aligned}
|\sigma(t)| &\leq \sum_{P \in \mathcal{P}^*_{3,\lambda}(t)} |\sigma(P)| \\
&\ll \sum_{P \in \mathcal{P}^*_{3,\lambda}(t)} \mathrm{H}(P)\, |\mathrm{D}(P)|^{-1/2} \psi(\mathrm{H}(P)) \\
&\ll 2^t \psi(2^t) \sum_{P \in \mathcal{P}^*_{3,\lambda}(t)} |\mathrm{D}(P)|^{-1/2} \\
&\ll 2^{t(3-2\lambda/3)} \psi(2^t)
\end{aligned}$$

because from [63, Corollary 2] it follows immediately that

$$\sum_{P \in \mathcal{P}^*_{3,\lambda}(t)} |\mathrm{D}(P)|^{-1/2} \asymp 2^{t(2-2\lambda/3)}$$

where the implied constants are absolute. Just like we did in the proof of Theorem 4.1.4, consider a slight enlargement of $\sigma(P)$

$$\tilde{\sigma}(P) = \bigcup_{x \in \sigma(P)} \left\{ y \in \mathbb{R} : |y - x| < \psi(2^t)/2^t \right\}$$

so that for any $y \in \tilde{\sigma}(P)$ we have

$$|P(y)| \leq |P(x)| + |P'(z)||x-y| \ll \psi(\mathrm{H}(P)).$$

Thus $\sigma(P) \subseteq \tilde{\sigma}(P)$ and $|\tilde{\sigma}(t)| \asymp |\sigma(t)|$. It follows that we can cover $\sigma(t)$ with at most

$$N := \frac{|\tilde{\sigma}(t)|}{\ell} \ll 2^{t(3-2\lambda/3)}\psi(2^t)\frac{2^t}{\psi(2^t)} = 2^{t(4-2\lambda/3)}$$

intervals of length $\ell := \psi(2^t)/2^t$. Finally, this implies that

$$\mathcal{H}^g(\mathcal{A}_{3,\lambda}(\psi) \cap I) \ll \lim_{t_0\to\infty} \sum_{t=t_0}^{\infty} g\left(\frac{\psi(2^t)}{2^t}\right) 2^{t(4-2\lambda/3)} = 0$$

since by Cauchy's condensation test we know that

$$\sum_{t=0}^{\infty} g\left(\frac{\psi(2^t)}{2^t}\right) 2^{t(4-2\lambda/3)} < \infty \quad \text{iff} \quad \sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) q^{3-2\lambda/3} < \infty.$$

4.5 PROOF OF COROLLARY 4.1.6

By Theorem 4.1.5 it is enough to focus on reducible polynomials, i.e. on $\mathcal{B} := \mathcal{A}_{3,\lambda}(w) \setminus \mathcal{A}^*_{3,\lambda}(w)$. Now consider $x \in \mathbb{R}$ such that $|P(x)| \leq \mathrm{H}(P)^{-w}$ for infinitely many reducible cubic polynomials P and write $P = P_1P_2$, with $\deg(P_i) = i$. Then note that if, say, $|P_1(x)| \leq \mathrm{H}(P_1)^{-w}$ for at most finitely many P_1, then $|P_1(x)| \gg \mathrm{H}(P_1)^{-w}$ for all P_1 and by Lemma 4.2.1 we have

$$\mathrm{H}(P_1)^{-w}|P_2(x)| \ll |P_1(x)P_2(x)| \leq \mathrm{H}(P)^{-w} \ll \mathrm{H}(P_1)^{-w}\,\mathrm{H}(P_2)^{-w}.$$

It follows that for at least one $i \in \{1,2\}$ we can find a constant $c_i > 0$ such that $|P_i(x)| \leq c_i\,\mathrm{H}(P_i)^{-w}$ for infinitely many P_i. In other words, we have that

$$\mathcal{B} \subseteq \mathcal{A}_1(c_1q^{-w}) \cup \mathcal{A}_2(c_2q^{-w}).$$

Similarly, by noticing that a quadratic polynomial is either irreducible or a product or two linear polynomials, we can also find constants $c'_i > 0$ such that

$$\mathcal{A}_2(c_2q^{-2}) \subseteq \mathcal{A}_1(c'_1q^{-w}) \cup \mathcal{A}^*_2(c'_2q^{-w}),$$

where $\mathcal{A}_2^* = \mathcal{A}_{2,0}^*$. Furthermore, without loss of generality we may assume that $c_1 \geq c_1'$, so that

$$\mathcal{B} \subseteq \mathcal{A}_1(c_1 q^{-w}) \cup \mathcal{A}_2^*(c_2' q^{-w}).$$

Then Jarník's theorem implies that

$$\mathcal{H}^g\Big(\mathcal{A}_1(c_1 q^{-w})\Big) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} g(c_1 q^{-w-1}) q < \infty$$

and the proof of case II of [60] implies that

$$\mathcal{H}^g\Big(\mathcal{A}_2^*(c_2' q^{-w})\Big) = 0 \quad \text{if} \quad \sum_{q=1}^{\infty} g(c_2' q^{-w-1}) q^2 < \infty.$$

Finally, by the comparison test and by Lemma 4.3.2, we note that those two series converge when

$$\sum_{q=1}^{\infty} g(q^{-w-1}) q^2 < \infty.$$

This is enough complete the proof of Corollary 4.1.6, since $0 \leq \lambda < 9/20$ means that $3 - 2\lambda/3 > 2$.

4.6 CONCLUSIONS

The main issue with proving a convergence result in the case of reducible polynomials for more general approximation functions, similar to what we did for Corollary 4.1.6, lies in the decoupling of the resulting inequalities

$$|P_1(x)||P_2(x)| \leq \psi(\mathrm{H}(P_1 P_2)).$$

Our proof carries through as-is for any other ψ that is multiplicative, but this is by no means the general case. For the case of quadratic polynomials, Hussain [60] and Huang [57] resorted to imposing a fairly restrictive condition on the dimension function and, while this looks artificial, in private correspondence Hussain confirmed that the techniques used in those papers don't allow for its removal. Furthermore, in a recent preprint Hussain, Schleischitz and Simmons [61] showed that this decoupling can be achieved in the general case when, for all q large enough, $\log\psi(q)/\log q$ is monotonically non-increasing and for every $c_1 > 1$ there is a $c_2 > 0$ such that $\psi(q/c_1) \leq c_2\psi(q)$. Obviously these are satisfied for $\psi(q) = q^{-w}$ and multiplicative functions satisfy the latter condition, but it is not immediately clear whether multiplicative approximation functions need to satisfy the former condition.

It would also be interesting to look into an equivalent version of Theorem 4.1.5 for higher degrees, which would lead to a complete treatment of the case of approximation functions of the form q^{-w}. In other words, we propose the following conjecture, which for $\lambda = 0$ is a special case of the far reaching [18, Problem 3].

Conjecture 4.6.1. *Consider an approximation function ψ and a dimension function g such that $q^{-1}g(q)$ is decreasing and $q^{-1}g(q) \to \infty$ as $q \to 0$. Let $\mathcal{P}^*_{n,\lambda}$ be the set of irreducible polynomials in $\mathcal{P}_{n,\lambda}$ and let $\mathcal{A}^*_{n,\lambda}(\psi)$ be the corresponding* lim sup *set. Then there are positive constants λ_m and c, possibly dependent on n, such that for every $0 \leq \lambda < \lambda_m$*

$$\mathcal{H}^g\Big(\mathcal{A}^*_{n,\lambda}(\psi)\Big) = 0 \quad \textit{if} \quad \sum_{q=1}^{\infty} g\left(\frac{\psi(q)}{q}\right) q^{n-c\lambda} < \infty.$$

Many estimates for the number of polynomials of given degree and bounded discriminant have appeared recently, which could potentially allow to extend our argument to higher degrees through an analogue of the discriminant estimate from [63]. Unfortunately, obtaining good upper bounds is a difficult problem, and the only ones available at the moment are either not sharp (e.g. [76]), or cover only special classes of polynomials (e.g. [38] or [41]).

Just like Sprindžuk's solution to Mahler's conjecture required techniques different from those Volkmann used in his treatment of the cubic case, another potential avenue to establishing Conjecture 4.6.1 would be to adapt the techniques developed by Bernik in [25]. This could likely lead to the development of a whole new methodology, but it appears to be a considerably complex task.

— 5 —

Algebraic points near manifolds [1]

5.1 INTRODUCTION

In the course of developing his classification of real numbers, Mahler conjectured that for every $\varepsilon > 0$ and Lebesgue almost every $x \in \mathbb{R}$ the inequality

$$|P(x)| < \mathrm{H}(P)^{-n-\varepsilon} \tag{5.1}$$

has at most finitely many solutions $P \in \mathbb{Z}[X]$ with $\deg(P) \leq n$, where $\mathrm{H}(P)$ denotes the *(naive) height* of P, i.e. the maximum of its coefficients in absolute value. This was later proved by Sprindžuk [101], and it marked the beginning of the theory of Diophantine Approximation of dependent quantities, i.e. the study of the Diophantine properties of points bound to a given manifold.

It is then natural to wonder about the Diophantine properties of the solutions to a system of simultaneous equations of type (5.1) in multiple independent variables $x_0, \dots, x_m \in \mathbb{R}$, i.e.

$$\max_{0\leq k\leq m} |P(x_k)| < \psi(\mathrm{H}(P)) \tag{5.2}$$

for some function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$, with solutions in integer polynomials P of degree between $m+1$ and n. Indeed, in [100, Problem C] Sprindžuk conjectured that the maximum $v > 0$ for which (5.2) with $\psi(Q) = Q^{-v}$ has infinitely many solutions for all x in a set of positive measure is

$$v = \frac{n+1}{m+1} - 1, \tag{5.3}$$

[1] The content of this chapter is also available as a pre-print [93].

and this was later proved by Bernik in [26].

The statement associated with (5.2) was then considered for arbitrary ψ and $m = 1$ in [28], as well as for the case where the variables x_k can also take complex or p-adic values in [29, 31] ($m = 2$) and [40, 39] (arbitrary m). In particular, the following result is contained in the preprint [13], which deals with the more general case of systems of linear forms in dependent variables, i.e.

$$\max_{0\le k\le m} |\boldsymbol{a} \cdot \boldsymbol{f}_k(\boldsymbol{x}_k)| < \psi(\|\boldsymbol{a}\|) \tag{5.4}$$

with solutions in $\boldsymbol{a} \in \mathbb{Z}^{n+1}$, where $\boldsymbol{f_k}\colon U_k \to \mathbb{R}^n$ are sufficiently regular maps defined on open balls $U_k \subset \mathbb{R}^{d_k}$ and $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ as before. Here and throughout this chapter $\|\cdot\|$ will denote the sup norm $\mathbb{R}^n$ unless otherwise specified, although note that most of the results presented here still hold with minor modification for any other choice of norm.

Theorem 5.1.1 ([13, Theorem 1])**.** *Consider integers $n > m \ge 0$, a function $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$, and a ball $B \subset \mathbb{R}^{m+1}$. Let $|\cdot|$ be the Lebesgue measure on $\mathbb{R}^{m+1}$. Then*

$$|\mathscr{L}_{n,m+1}(\psi) \cap B| = \begin{cases} 0 & \text{if } S_{n,m+1}(\psi) < \infty \\ |B| & \text{if } \psi \text{ is monotonic and } S_{n,m+1}(\psi) = \infty \end{cases}$$

where $\mathscr{L}_{n,m+1}(\psi)$ denotes the set of $(x_0, \ldots, x_m) \in \mathbb{R}^{m+1}$ which satisfy (5.2) *for infinitely many polynomials P of degree up to n, and where*

$$S_{n,m+1}(\psi) \coloneqq \sum_{Q=1}^{\infty} Q^{n-m-1}\psi^{m+1}(Q).$$

Note. Like many other Khinchin-Groshev-type theorems, this kind of result has already found applications in communication engineering, specifically in the field of *interference alignment*; see for example [90, Appendix B] and [79, Section IV], or [54, 88] for examples which require results of approximation on manifolds. The interested reader may also find a more accessible description of how Khinchin-like theorems come into play in the theory of interference alignment in [1, Appendix A].

Finally, one might consider what changes after introducing a dependency among the variables $x_0, \dots, x_m$ of (5.2) (or $\boldsymbol{x}_0, \dots, \boldsymbol{x}_m$ in (5.4)), i.e. when they are parametrised by a sufficiently regular map $\boldsymbol{f}\colon \mathscr{B} \subset \mathbb{R}^d \to \mathbb{R}^{m+1}$, and this is the subject of the present chapter.

Clearly, if $P(x)$ is small, then x must be close to at least one of the roots of P. In particular, this means that if P is irreducible and $|P(f_0(\boldsymbol{x}))|, \dots, |P(f_m(\boldsymbol{x}))|$ are all small, then there must be a point $\boldsymbol{\alpha} \in \mathbb{R}^{m+1}$ close to $\boldsymbol{f}(\boldsymbol{x})$, where the coordinates of $\boldsymbol{\alpha}$ are algebraic and conjugate. Note, however, that there are subtle differences among these two types of approximation, as evinced by the difference between the classifications of numbers of Mahler and Koksma [42, Section 3.4].

Nonetheless, a good first step towards establishing a result like Theorem 5.1.1 is to provide an estimate for the number of such points $\boldsymbol{\alpha}$ which are sufficiently close to the manifold $\mathcal{M}$ parametrised by $\boldsymbol{f}$ (see e.g. [102, Section 2.6]). Furthermore, the techniques used to derive such estimates can be of interest in and of themselves; for example, in the case of rational points they have been adapted to derive an efficient algorithm to compute the rational points with bounded denominator on a given manifold, see [52], or [86, Section 11] for a nice overview. This problem was first considered for planar curves by Bernik, Götze and Kukso in [33]. In other words, let $\mathscr{B} \subset \mathbb{R}$ be a bounded open interval and let $f_1\colon \mathscr{B} \to \mathbb{R}$ be a $\mathcal{C}^1$ function; also, define the sets

$$\begin{aligned} \mathbb{A}_n^2(Q) &\coloneqq \{\boldsymbol{\alpha} \in \mathbb{R}^2 : \boldsymbol{\alpha} \text{ is algebraic }, \deg(\boldsymbol{\alpha}) \le n, \mathrm{H}(\boldsymbol{\alpha}) \le Q\} \\ M_{f_1}^n(Q, \gamma, \mathscr{B}) &\coloneqq \{(\alpha_0, \alpha_1) \in \mathbb{A}_n^2(Q) : \alpha_0 \in \mathscr{B}, |f_1(\alpha_0) - \alpha_1| < c_0 Q^{-\gamma}\}, \end{aligned} \tag{5.5}$$

where $c_0 > 0$ is fixed. Here by $\boldsymbol{\alpha} \in \mathbb{R}^{m+1}$ *algebraic* we mean that its coordinates are algebraic conjugate real numbers, and by $\mathrm{H}(\boldsymbol{\alpha})$ we denote the height of their minimal polynomial.

A lower bound for $\# M_{f_1}^n(Q, \gamma, \mathscr{B})$ was provided in [33] for $0 < \gamma < \frac{1}{2}$. This was soon extended in [32], where Bernik, Götze and Gusakova also provided an upper bound. We also note that recently Bernik, Budarina and Dickinson provided an analogous lower bound for surfaces in $\mathbb{R}^3$ [30].

Theorem 5.1.2 ([32, Theorem 1]). *Suppose that both* $\#\{x \in \mathscr{B} : f_1(x) = x\}$ *and* $\sup_{\mathscr{B}}|f_1'|$ *are bounded. If* c_0 *is sufficiently large, then*

$$\#M^n_{f_1}(Q, \gamma, \mathscr{B}) \asymp Q^{n+1-\gamma}$$

for every Q *large enough and* $0 < \gamma < 1$*.*

Note. Here and throughout this chapter we will make extended use of Vinogradov's notation. Namely, we will write $a \ll b$ if there is a constant $c > 0$ such that $a < cb$, as well as $b \gg a$ if $a \ll b$, and $a \asymp b$ when $a \ll b$ and $b \ll a$ simultaneously, in which case we say that a is *comparable* to b. Occasionally we will make dependencies of the implied constant explicit via a subscript, e.g. $a \ll_\varepsilon b$, and c is generally assumed to be independent of the variables that a and b depend on, although it could depend on the other parameters involved. We also extend this notation to vectors in the natural way: if $\boldsymbol{a} = (a_1, \ldots, a_r)$ and $\boldsymbol{b} = (b_1, \ldots, b_r)$, then $\boldsymbol{a} \ll \boldsymbol{b}$ means that $a_i \ll b_i$ for every $1 \le i \le r$, and similarly for $\gg$ and $\asymp$.

In the present chapter we will extend the lower bound in Theorem 5.1.2 to sufficiently regular manifolds in arbitrary dimension. While the characterisation of these manifolds is quite technical, as a special case our results hold true when $\boldsymbol{f}$ is analytic with algebraically independent components. In particular, the following is a special case of Theorem 5.2.13, which extends the range of γ to the best possible — here $M^n_{\boldsymbol{f}}$ is the higher dimensional analogue of $M^n_{f_1}$, see (5.18) for details.

Theorem 5.1.3. *Let* $\mathscr{B} \subset \mathbb{R}^d$ *be a bounded open set, and let*

$$\boldsymbol{f}(\boldsymbol{x}) = (x_0, \ldots, x_{d-1}, f_d(\boldsymbol{x}), \ldots, f_m(\boldsymbol{x}))$$

be an analytic function $\mathscr{B} \to \mathbb{R}^{m+1}$ *with algebraically independent components. Then for* $c_0 > 0$ *fixed and for every*

$$0 < \gamma \le \frac{n+1}{m+1}$$

we have

$$\#M_{\boldsymbol{f}}^{n}(Q,\gamma,\mathscr{B}) \gg Q^{n+1-\gamma(m+1-d)}$$

for every Q sufficiently large, where the implied constant does not depend on Q.

Remark 5.1.4. For $m = 1$ we are in the case of planar curves and the upper bound for γ becomes $\frac{n+1}{2}$, which when $n > 1$ is considerably larger than the bound in Theorem 5.1.2.

Remark 5.1.5. As mentioned in Section 1.2.2, most results in the theory of Diophantine Approximation on manifolds rely on the notion of non-degeneracy, and the algebraic independence condition on $\boldsymbol{f}$ is a natural extension of this. Indeed, a (non-constant) analytic map $\boldsymbol{f}$ is non-degenerate if and only if it is not the root of a *real* polynomial of degree 1. On the other hand, the components of $\boldsymbol{f}$ are algebraically independent if and only if $\boldsymbol{f}$ is not the root of a *rational* polynomial of arbitrary degree.

Examples of maps that satisfy this condition can be readily obtained from the theory of Mahler functions (see [89, Chapter 3]) or from the Lindemann-Weierstrass Theorem [8, Theorem 1.4], like the following: if $\alpha_1,\dots,\alpha_m$ are irrational algebraic numbers linearly independent over $\mathbb{Q}$, then the functions $x, e^{\alpha_1 x},\dots,e^{\alpha_m x}$ are algebraically independent.

In the process of proving Theorem 5.1.3, we will also be able to extend the divergence part of Theorem 5.1.1 as follows; here $\mathcal{H}^s$ denotes the usual Hausdorff s-measure (see Definition 5.7.2).

Theorem 5.1.6. *Let $\mathscr{B}$, $\boldsymbol{f}$ be as in Theorem 5.1.3, and let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing function such that $\psi(Q) \gg Q^{\frac{m-n}{m+1}}$. Further, denote by $\mathscr{L}_{n,\boldsymbol{f}}(\psi)$ the set of $\boldsymbol{x} \in \mathscr{B}$ such that $\boldsymbol{f}(\boldsymbol{x}) \in \mathscr{L}_{n,m+1}(\psi)$. Then for any $0 < s \le d$ we have*

$$\mathcal{H}^s(\mathscr{L}_{n,\boldsymbol{f}}(\psi)) = \mathcal{H}^s(\mathscr{B}) \quad \text{if} \quad \sum_{Q=1}^{\infty} Q^{n-m-1}\psi(Q)^{m+1}\left(\frac{\psi(Q)}{Q}\right)^{s-d} = \infty.$$

Corollary 5.1.7 (Cfr. [13, Corollary 1]). *Let* $\dim_{\mathcal{H}}$ *denote Hausdorff dimension. In the same setting of Theorem 5.1.6, we have that*

$$\dim_{\mathcal{H}}(\mathscr{L}_{n,\boldsymbol{f}}(\psi)) \geq \min\left\{d, \frac{n+1}{\tau_\psi + 1} + d - m - 1\right\},$$

where

$$\tau_\psi := \liminf_{Q\to\infty} -\frac{\log\psi(Q)}{\log Q}$$

is the lower order of ψ^{-1} *at infinity.*

Remark 5.1.8. The condition $\psi(Q) \gg Q^{\frac{m-n}{m+1}}$ of Theorem 5.1.6 implies that $\tau_\psi \leq \frac{n-m}{m+1}$, which is precisely the situation where

$$\frac{n+1}{\tau_\psi + 1} + d - m - 1 \geq d.$$

Therefore in this setting we actually have $\dim_{\mathcal{H}}(\mathscr{L}_{n,\boldsymbol{f}}(\psi)) = d$. On the other hand, Theorem 5.2.16 shows that the picture is more interesting with two separate approximation functions on $\mathbb{R}^d$ and $\mathbb{R}^{m+1-d}$.

Our proof exploits the powerful quantitative non-divergence bounds introduced in Chapter 3, and it has a similar flavour to [13]. This chapter is structured as follows:

- in the next section we will describe our setting and main results;
- in sections 5.3 and 5.4 we will discuss the regularity conditions that these depend on and provide some examples of functions that satisfy them;
- the next sections are devoted to the proofs, in this order: sections 5.5 and 5.6 for the extension of Theorem 5.1.2, section 5.7 for the extension of Theorem 5.1.1, and section 5.8 for the proof of Theorem 5.2.8, which underpins the whole argument; and
- the last section contains some final remarks about possible directions in which this work could be extended.

5.2 THE MAIN RESULT

Let X be a metric space. If $\kappa > 0$ and $B \subset X$ is a ball centred at x and with radius r, throughout this chapter κB will denote the dilation of B by κ, i.e. the ball with centre x and radius κr.

Definition 5.2.1. Let $N > 0$. As in Chapter 3, a metric space X is called *N-Besicovitch* if, for any bounded set $A \subset X$ and any collection of balls $\mathcal{B}$ such that every $x \in A$ is in the centre of a ball in $\mathcal{B}$, there is a countable collection $\Omega \subseteq \mathcal{B}$ which covers A and such that every point $x \in A$ lies in at most finitely many balls in Ω. We will also say that X is *Besicovitch* if it is N-Besicovitch for some $N > 0$.

Example 5.2.2. It is well known that $\mathbb{R}^n$ with the Euclidean metric is Besicovitch, see e.g. [84, Theorem 2.7].

Recall from Chapter 3 that a Radon measure ν on an open subset $U \subset X$ is *D-Federer on U* for some $D > 0$ if

$$\nu\Big(3^{-1}B\Big) > D^{-1}\nu(B)$$

for any ball $B \subset U$ centred on $\operatorname{supp}\nu$, and it is called *Federer* if for ν almost every $x \in X$ there are a neighbourhood U of x and a $D > 0$ such that ν is D-Federer on U. We will also introduce the following definitions (cfr. [69]).

Definition 5.2.3. Let $U \subset X$ be an open subset and let ν be a Radon measure on U. We will say that ν is:

- *(Rationally) non-planar* if $X = \mathbb{R}^d$ and $\nu(\mathcal{L}) = 0$ for every (rational) affine hyperplane of $\mathbb{R}^d$.
- (C, α)-*decaying on U* for some $C, \alpha > 0$ if $X = \mathbb{R}^d$ and for any ball B centred on $\operatorname{supp}\nu$, any affine hyperplane $\mathcal{L} \subset \mathbb{R}^d$, and any $\varepsilon > 0$ we have

$$\nu\Big(B \cap \mathcal{L}^{(\varepsilon)}\Big) \leq C\left(\frac{\varepsilon}{\|d_{\mathcal{L}}\|_{\nu,B}}\right)^{\alpha}\nu(B), \tag{5.6}$$

where $\mathcal{L}^{(\varepsilon)}$ is the ε-neighbourhood of $\mathcal{L}$, $d_{\mathcal{L}}$ is the Euclidean distance from $\mathcal{L}$, and $\|d_{\mathcal{L}}\|_{\nu,B} = \sup_{\boldsymbol{x}\in B\cap \operatorname{supp}\nu} d_{\mathcal{L}}(\boldsymbol{x})$. C.f. Definition 5.4.1.

- *Absolutely* (C,α)*-decaying on* U if (5.6) holds with the radius of B in place of $\|d_{\mathcal{L}}\|_{\nu,B}$.
- *(Absolutely) decaying* if for ν almost every $x \in X$ there are a neighbourhood U of x and constants $C, \alpha > 0$ such that ν is (absolutely) (C,α)-decaying on U.

Remark 5.2.4. Both classes of Federer and absolutely decaying measures are closed under restriction to open subsets $U \subset X$. Furthermore, they are also closed with respect to taking finite products [69, Theorem 2.4].

Example 5.2.5. Examples of measures that are Federer and absolutely decaying on $\mathbb{R}^d$ include the Lebesgue measure and measures supported on certain self-similar sets (see e.g. [69] and [85]).

Now consider a d-dimensional manifold $\mathcal{M}$ in $\mathbb{R}^{m+1}$, parametrised over a bounded open subset $\mathscr{B} \subset X$ by a continuous map

$$\boldsymbol{f}(\boldsymbol{x}) = (f_0(\boldsymbol{x}), \dots, f_m(\boldsymbol{x})).$$

Without loss of generality, if $X = \mathbb{R}^d$ we will assume that $f_i(\boldsymbol{x}) = x_i$ for $0 \le i < d$ and write $\tilde{\boldsymbol{f}}$ for $(f_d, \dots, f_m)$. Then, for $n \ge m+1$ fixed, define the vectors in $\mathbb{R}^{n+1}$

$$\begin{aligned} \boldsymbol{v}_i = \boldsymbol{v}_i(\boldsymbol{x}) &:= \begin{pmatrix} 1 & f_i(\boldsymbol{x}) & f_i(\boldsymbol{x})^2 & \cdots & f_i(\boldsymbol{x})^n \end{pmatrix} \\ \boldsymbol{v}'_i = \boldsymbol{v}'_i(\boldsymbol{x}) &:= \begin{pmatrix} 0 & 1 & 2f_i(\boldsymbol{x}) & \cdots & nf_i(\boldsymbol{x})^{n-1} \end{pmatrix} \end{aligned} \qquad 0 \le i \le m$$

and the $(n+1) \times (n+1)$ matrices

$$M_{\boldsymbol{f}} := \begin{pmatrix} \boldsymbol{v}_0 \\ \vdots \\ \boldsymbol{v}_m \\ \hline \mathbf{0} \;\big|\; \mathbf{I}_{n-m} \end{pmatrix} \tag{5.7}$$

$$U^h_{\boldsymbol{f}} \coloneqq \left(\begin{array}{c|c} \multicolumn{2}{c}{\boldsymbol{v}_0} \\ \multicolumn{2}{c}{\vdots} \\ \multicolumn{2}{c}{\boldsymbol{v}_m} \\ \multicolumn{2}{c}{\boldsymbol{v}'_h} \\ \hline \mathbf{0} & \mathbf{I}_{n-m-1} \end{array}\right). \tag{5.8}$$

Remark 5.2.6. The determinant of $U^h_{\boldsymbol{f}}$ is the same as the determinant of the submatrix $\tilde{U}^h_{\boldsymbol{f}}$ formed by its first $n+1$ rows and columns. The latter is an example of what in the literature is known as a *confluent Vandermonde matrix*, and a theorem of Schendel's (see e.g. [74, Theorem 20]) shows that

$$\left|\det \tilde{U}^h_{\boldsymbol{f}}\right| = \prod_{0 \le i < j \le m} |f_i(\boldsymbol{x}) - f_j(\boldsymbol{x})|^{e_i e_j},$$

where e_i is 2 if $i = h$ and 1 otherwise. In particular, $\det U^h_{\boldsymbol{f}} \neq 0$ if and only if the Vandermonde polynomial $\mathrm{V}(\boldsymbol{f})$ is non-zero (see (5.10) below).

For ease of notation, we will write $[\![n]\!]$ instead of $\{0, \ldots, n\}$, as well as $[\![n]\!]^\tau_<$ for the set of $I = (i_1, \ldots, i_\tau) \in [\![n]\!]^\tau$ such that $i_1 < i_2 < \cdots < i_\tau$, where $1 \le \tau \le n+1$. Given an $(n+1) \times (n+1)$ matrix A, we will also write $A_{I,J}$ for the submatrix of A with rows indexed by $I \subseteq [\![n]\!]^\tau_<$ and columns indexed by $J \subseteq [\![n]\!]^\tau_<$, and $|A|_{I,J}$ for its determinant.

Then, for every $1 \le \tau \le n+1$ and for every $I \in [\![n]\!]^\tau_<$, define the map from the set $\mathrm{M}_{n+1,n+1}$ of $(n+1) \times (n+1)$ matrices to $\bigwedge^\tau \mathbb{R}^{n+1} \simeq \mathbb{R}^{\binom{n+1}{\tau}}$ given by

$$\mathcal{G}_I \colon A \mapsto \left(|A|_{I,J}\right)_{J \in [\![n]\!]^\tau_<}. \tag{5.9}$$

In other words, $\mathcal{G}_I(A)$ is the image under the Plücker embedding of the linear subspace of $\mathbb{R}^{n+1}$ spanned by the rows of A indexed by I. Furthermore, in Section 5.3 we will see that for $I \in [\![m]\!]^\tau_<$ and $1 \le \tau \le m+1$ we have

$$\mathcal{G}_I(M_{\boldsymbol{f}}) = \left(\mathrm{V}(\boldsymbol{f}_I) s_\lambda(\boldsymbol{f})\right)_{\substack{|\lambda| \le n+1-\tau \\ \ell(\lambda) \le \tau}}$$

where $\mathrm{V}(\boldsymbol{f}_I)$ is the Vandermonde polynomial of $\boldsymbol{f}_I = (f_i)_{i \in I}$, i.e.

$$\mathrm{V}(\boldsymbol{f}_I) \coloneqq \prod_{\substack{i,j \in I \\ i<j}} (f_j - f_i), \tag{5.10}$$

and s_λ is the *Schur polynomial* in τ indeterminates corresponding to the partition λ of the integer $|\lambda|$ with $\ell(\lambda)$ parts (see Definition 5.3.3). Therefore, we also define

$$\mathcal{S}_{n,\tau}\colon \boldsymbol{T} = (T_1, \dots, T_\tau) \mapsto \Big(s_\lambda(\boldsymbol{T})\Big)_{\substack{|\lambda| \le n+1-\tau \\ \ell(\lambda) \le \tau}} \tag{5.11}$$

and with a slight abuse of notation we will write $\mathcal{S}_\tau(\boldsymbol{T})$ or $\mathcal{S}(\boldsymbol{T})$ instead of $\mathcal{S}_{n,\tau}(\boldsymbol{T})$ whenever n or τ are clear from the context. Finally, observe that if $\mathrm{V}(\boldsymbol{f})$ is bounded on $\mathscr{B}$, then $\mathcal{G}_I(M_{\boldsymbol{f}}) \asymp \mathcal{S}(\boldsymbol{f}_I)$.

Definition 5.2.7. Let X be a measure space and ν a measure on X. Fix $\tau \ge 1$ and let Λ_τ^k be the space of rational symmetric polynomials of degree up to k. Given a map $\boldsymbol{f}\colon \mathscr{B} \subseteq X \to \mathbb{R}^\tau$, the pair $(\boldsymbol{f}, \nu)$ is called:

- *Non-symmetric (of degree k) at x* if for every neighbourhood $B \ni x$ and $s \in \Lambda_\tau^k$ we have that $\boldsymbol{f}(B \cap \operatorname{supp}\nu)$ is not contained in the zero locus of s in $\mathbb{R}^\tau$. Cf. the definitions of non-planarity from [66, 72].
- *Non-symmetric (of degree k) on $\mathscr{B}$* if it is non-symmetric of degree k at every $x \in \mathscr{B} \cap \operatorname{supp}\nu$.
- *Symmetrically good (of degree k) on $\mathscr{B}$* if it is non-symmetric (of degree k) on $\mathscr{B}$ and there are constants $C, \alpha > 0$ such that $s(\boldsymbol{f})$ is (C, α)-good on B with respect to ν for every $s \in \Lambda_\tau^k$ (see Definition 5.4.1).

Note. See Corollary 5.3.5 for an equivalent characterisation of $(\boldsymbol{f}, \nu)$ being non-symmetric of degree k in terms of the components of an appropriate $\mathcal{S}(\boldsymbol{f})$.

Now take functions $\psi_0, \dots, \psi_m, \varphi_{m+1}, \dots, \varphi_n\colon \mathbb{R}^+ \to \mathbb{R}^+$, and for $Q > 0$ consider the system of inequalities

$$\begin{cases} |P(f_k(\boldsymbol{x}))| < \psi_k(Q) & \text{for } 0 \le k \le m \\ \max_i |P'(f_i(\boldsymbol{x}))| \le \varphi_{m+1}(Q) & \\ |a_k| \le \varphi_k(Q) & \text{for } m+1 < k \le n \end{cases} \tag{5.12}$$

with solutions in integer polynomials $P = a_n X^n + \ldots + a_0$ of degree at most n. Our main result concerns the set

$$\mathcal{D}^n_{\boldsymbol{f}}(Q, \mathscr{B}) = \mathcal{D}^n_{\boldsymbol{f}}(Q, \mathscr{B}; \psi_0, \ldots, \psi_m, \varphi_{m+1}, \ldots, \varphi_n)$$

of points $x \in \mathscr{B}$ for which (5.12) admits a solution. For ease of notation, given $I \in [\![0, m]\!]^{\tau_1}_{<}$ and $J \in [\![m+1, n]\!]^{\tau_2}_{<}$, let

$$\boldsymbol{\psi}_I := \prod_{i \in I} \psi_i, \quad \boldsymbol{\varphi}_J := \prod_{j \in J} \varphi_j,$$

as well as $\boldsymbol{\psi} := \boldsymbol{\psi}_{[\![0,m]\!]}$ and $\boldsymbol{\varphi} := \boldsymbol{\varphi}_{[\![m+1,n]\!]}$.

Theorem 5.2.8. *Let X be an N-Besicovitch space, $\mathscr{B} \subset X$ a bounded open subset, and let ν be a D-Federer measure on $\mathscr{B}$. Let $\boldsymbol{f} \colon \mathscr{B} \to \mathbb{R}^{m+1}$ be a continuous function such that $c_1 \geq \mathrm{V}(\boldsymbol{f}) \geq c_2$ on $\mathscr{B} \cap \operatorname{supp} \nu$, where $\mathrm{V}(\boldsymbol{f})$ is the Vandermonde polynomial of $\boldsymbol{f}$. Furthermore, let $\boldsymbol{\psi}, \boldsymbol{\varphi}$ be as above, and suppose that for some $n > 0$*

$$\varphi_{m+1}(Q)^{n+1} \gg \boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q) \tag{5.13}$$

and that for every $1 \leq \tau \leq m+1$ there is a choice of $I \in [\![m]\!]^{\tau}_{<}$ such that

$$\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q) \gg \psi_I(Q)^{\frac{n+1}{\tau}}, \text{ and} \tag{5.14}$$

$$(\boldsymbol{f}_I, \nu) \text{ is symmetrically } (\tilde{C}, \alpha)\text{-good of degree } n+1-\tau \text{ on } \mathscr{B} \tag{5.15}$$

for some $\tilde{C}, \alpha > 0$. Then for any $0 < \theta < 1$ and for Q large enough we may find a subset $\mathscr{B}_\theta \subset \mathscr{B}$ with measure $\nu(\mathscr{B}_\theta) > \theta \nu(\mathscr{B})$, as well as $C = C(\tilde{C}, c_1, c_2, n, N, D) > 0$ and $\rho = \rho(\boldsymbol{f}, n, \mathscr{B}_\theta) > 0$, such that

$$\nu\Big(\mathcal{D}^n_{\boldsymbol{f}}(Q, \mathscr{B}_\theta)\Big) \leq C \left(\frac{\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q)}{\rho^{n+1}} \right)^{\frac{\alpha}{n+1}} \nu(\mathscr{B}_\theta).$$

Remark 5.2.9. The set $\mathscr{B}_\theta$ can be chosen to be either compact or a union of finitely many open balls (which form a cover for this compact set).

Note. Corollaries 5.3.10 and 5.4.6 below show that it is relatively straightforward to check condition (5.15) when $X = \mathbb{R}^d$ and $\boldsymbol{f}$ is analytic.

Corollary 5.2.10. *Let $\mathcal{D}^n_{\boldsymbol{f}}(\mathscr{B}) = \limsup_Q \mathcal{D}^n_{\boldsymbol{f}}(Q, \mathscr{B})$. Under the hypothesis of Theorem 5.2.8, further assume that $X = \mathbb{R}^d$, $\boldsymbol{f}$ is analytic and ν is the Lebesgue measure on $\mathscr{B}$. Then α can be chosen to be one of:*

- $1/d(N-1)$, *or*
- $1/k \deg(\boldsymbol{f})$ *if $\boldsymbol{f}$ is a polynomial map.*

Moreover,

$$\sum_{Q=1}^{\infty} \Big(\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q)\Big)^{\frac{\alpha}{n+1}} < \infty \qquad \textit{implies} \qquad \nu\Big(\mathcal{D}^n_{\boldsymbol{f}}(\mathscr{B})\Big) = 0. \tag{5.16}$$

Corollary 5.2.11. *Consider $\psi_0, \dots, \psi_m$, $\boldsymbol{f}$ as in Theorem 5.2.8 such that condition (5.14) holds with $\varphi_{m+1}(Q) = \cdots = \varphi_n(Q) = Q$. Furthermore, assume that for every $0 \le i \le m$ and for Q large enough $\psi_i(Q)Q^{-1}$ is decreasing, and that there are constants $c_3, c_4 > 0$ such that*

$$c_3 \le \boldsymbol{\psi}(Q)Q^{n-m} \le c_4.$$

Then for every $0 < \theta < 1$ there are a constant $c > 0$ and a subset $\mathscr{B}_\theta$ of $\mathscr{B}$, independent of Q, such that $\nu(\mathscr{B}_\theta) > \theta\nu(\mathscr{B})$ and every $\boldsymbol{x} \in \mathscr{B}_\theta$ admits $n+1$ distinct points $(\alpha_0, \dots, \alpha_m) \in \mathbb{R}^{m+1}$ with algebraic conjugate coordinates of height $\mathrm{H}(\alpha_k) \ll Q$ which satisfy

$$|f_k(\boldsymbol{x}) - \alpha_k| < c\frac{\psi_k(Q)}{Q} \qquad \textit{for } 0 \le k \le m \tag{5.17}$$

whenever $Q > 0$ is sufficiently large.

Remark 5.2.12. Here c can be chosen to be $\frac{c_5}{c_6}$, where the constants c_5, c_6 are the same as in Corollary 5.5.1. In particular, it depends on c_3 and c_4 but not on the functions ψ_i themselves.

Now suppose that $\mathscr{B} \subset \mathbb{R}^d$ and without loss of generality assume that $f_i(\boldsymbol{x}) = x_i$ for each $0 \le i < d$. Then for a given $c_0 > 0$, let (c.f. (5.5))

$$\begin{aligned} \mathbb{A}^{m+1}_n(Q) &:= \{\boldsymbol{\alpha} \in \mathbb{R}^{m+1} : \boldsymbol{\alpha} \text{ is algebraic }, \deg(\boldsymbol{\alpha}) \le n, \mathrm{H}(\boldsymbol{\alpha}) \le Q\} \\ M^n_{\boldsymbol{f}}(Q, \gamma, \mathscr{B}) &:= \Big\{(\alpha_0, \dots, \alpha_m) \in \mathbb{A}^{m+1}_n(Q) : \\ &\qquad \boldsymbol{\alpha} = (\alpha_0, \dots, \alpha_{d-1}) \in \mathscr{B}, \max_{d \le j \le m} |f_j(\boldsymbol{\alpha}) - \alpha_j| < c_0 Q^{-\gamma}\Big\} \end{aligned} \tag{5.18}$$

where and $\gamma > 0$. Then we are able to extend the lower bound from Theorem 5.1.2 as follows:

Theorem 5.2.13. *Let $\boldsymbol{f}\colon \mathscr{B} \to \mathbb{R}^{m+1}$ be a $\mathcal{C}^1$ map as above such that* $\mathrm{V}(\boldsymbol{f}) \neq 0$*, and assume that, up to reordering $f_d, \ldots, f_m$,*

$$(\boldsymbol{x}, f_d, \ldots, f_\tau) \text{ is symmetrically good of degree } n-\tau \text{ on } \mathscr{B} \tag{5.19}$$

for every $d \leq \tau \leq m$. Then for $c_0 > 0$ fixed and for every

$$0 < \gamma \leq \frac{n+1}{m+1} \tag{5.20}$$

we have

$$\# M_{\boldsymbol{f}}^n(Q, \gamma, \mathscr{B}) \gg Q^{n+1-\gamma(m+1-d)}$$

for every Q sufficiently large, where the implied constant does not depend on Q.

Remark 5.2.14. Theorem 5.2.13 is proved by taking $\psi_k = Q^{1-\gamma}$ when $d \leq k \leq m$ in Corollary 5.2.11. In particular, Sprindžuk's conjecture (5.3) shows that the upper bound $\gamma \leq \frac{n+1}{m+1}$ is in general the best possible.

Remark 5.2.15. If we could show that the lower bound for Q and the implied constant in Theorem 5.2.13 can be chosen independently of translations of $\mathcal{M}$, then we would also have an upper bound for $\# M_{\boldsymbol{f}}^n(Q, \gamma, \mathscr{B})$. Indeed, without loss of generality we may assume that $\boldsymbol{f}$ is bounded on $\mathscr{B}$, hence $\mathcal{M}$ is contained in an open set K of volume comparable to $\mathrm{vol}(\mathscr{B})$. Now let $\mathcal{M}_\gamma$ be the γ-neighbourhood of $\mathcal{M}$, i.e. the set

$$\mathcal{M}_\gamma := \left\{ (\boldsymbol{x}, \boldsymbol{y}) \in \mathscr{B} \times \mathbb{R}^{m+1-d} : \|\boldsymbol{f}(\boldsymbol{x}) - \boldsymbol{y}\| < Q^{-\gamma} \right\}$$

and note that $\mathrm{vol}(\mathcal{M}_\gamma) \asymp \mathrm{vol}(\mathscr{B}) Q^{-\gamma(m+1-d)}$. In particular, up to replacing K with a slightly bigger open set, we may assume that K contains a union of disjoint translated copies $\{\mathcal{M}_\gamma^j\}_{j \in J}$ of $\mathcal{M}_\gamma$, with

$$\# J \asymp \mathrm{vol}(K) / \mathrm{vol}(\mathcal{M}_\gamma) \asymp Q^{\gamma(m+1-d)}.$$

If the implied constant in Theorem 5.2.13 can be chosen to be in a translation invariant way, then we may find $c, Q_0 > 0$ such that for every $Q > Q_0$ and for every $j \in J$ we have

$$\#\Big(\mathcal{M}_\gamma^j \cap \mathbb{A}_n^{m+1}(Q)\Big) > cQ^{n+1-\gamma(m+1-d)}.$$

It follows that

$$\#\Big(K \cap \mathbb{A}_n^{m+1}(Q)\Big) \gg Q^{n+1-\gamma(m+1-d)}\#J \gg Q^{n+1}.$$

However, since there are only Q^{n+1} polynomials of degree at most n and height at most Q, we can conclude that

$$\#M_{\boldsymbol{f}}^n(Q,\gamma,\mathscr{B}) \ll Q^{n+1-\gamma(m+1-d)}$$

as well, matching the lower bound.

We conclude this section by stating our extension of Theorem 5.1.1. Define $\mathscr{L}_{n,m+1}(\psi,\Psi;d)$ to be the set of $\boldsymbol{x} \in \mathbb{R}^{m+1}$ such that

$$\max_{0\le k<d}|P(x_k)| < \psi(\mathrm{H}(P)) \quad \text{and} \quad \max_{d\le k\le m}|P(x_k)| < \Psi(\mathrm{H}(P))$$

for infinitely many $P \in \mathbb{Z}[X]$ with $\deg(P) \le n$, and note that the set defined in Theorem 5.1.1 can be seen as $\mathscr{L}_{n,m+1}(\psi,\psi;d) = \mathscr{L}_{n,m+1}(\psi)$. When $\boldsymbol{f}$ parametrises a d-dimensional manifold $\mathcal{M}$ and $f_k(\boldsymbol{x}) = x_k$ for every $0 \le k < d$, will also write $\mathscr{L}_{n,\boldsymbol{f}}(\psi,\Psi)$ for the set of $\boldsymbol{x} \in \mathscr{B}$ such that $\boldsymbol{f}(\boldsymbol{x}) \in \mathscr{L}_{n,m+1}(\psi,\Psi;d)$.

Theorem 5.2.16. *Let $\psi, \Psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be decreasing functions such that*

$$\Psi(Q) \gg \max\Big\{Q^{\frac{m-n}{m+1}}, \psi(Q)\Big\}, \tag{5.21}$$

and let g be a dimension function such that $r^{-d}g(r)$ is non-increasing. Also assume that there are constants $r_0, c_7, c_8 \in (0,1)$ such that

$$g(c_7 r) \le c_8 g(r) \text{ for any } r \in (0, r_0). \tag{5.22}$$

Further suppose that $\boldsymbol{f}$ is Lipschitz continuous, that $\mathrm{V}(\boldsymbol{f}) \neq 0$, and that $\boldsymbol{f}$ is symmetrically good of degree $n+1-d$ on $\mathscr{B}$. Then

$$\mathcal{H}^g(\mathscr{L}_{n,\boldsymbol{f}}(\psi,\Psi)) = \mathcal{H}^g(\mathscr{B}) \quad \textit{if} \quad \sum_{Q=1}^{\infty} Q^{n-m-1+d}\Psi(Q)^{m+1-d} g\left(\frac{\psi(Q)}{Q}\right) = \infty.$$

Moreover,

$$|\mathscr{L}_{n,\boldsymbol{f}}(\psi,\Psi)| = 0 \quad \textit{if} \quad \sum_{Q=1}^{\infty} Q^{n-m-1}\psi(Q)^d\Psi(Q)^{m+1-d} < \infty.$$

Note. The generalised Hausdorff measure $\mathcal{H}^g$ will be introduced in Definition 5.7.1. For the moment observe that when $g(r) = r^d$ we have that $\mathcal{H}^g$ is a constant multiple of the Lebesgue measure on $\mathbb{R}^d$, thus we recover a version of Theorem 5.1.1 for symmetrically good manifolds.

Note. Condition (5.22) is not particularly restrictive, and in particular it is trivially satisfied for the usual Hausdorff s-measures, i.e. when $g(r) = r^s$ for some real $s > 0$.

5.3 SCHUR POLYNOMIALS

Throughout this section, we will denote by $\Lambda_\tau = \mathbb{Q}[T_0, \dots, T_{\tau-1}]^{S_\tau}$ the space of symmetric polynomials in τ variables, and define

$$\Lambda_\tau^k := \{s \in \Lambda_\tau : \deg(s) \leq k\}.$$

Definition 5.3.1. Let $f_0, \dots, f_{\tau-1}$ be a collection of τ real valued functions. The *order of symmetric independence of* $f_0, \dots, f_{\tau-1}$, denoted by $\mathfrak{s}(f_0, \dots, f_{\tau-1})$, is either

$$\max\{k : s(f_0, \dots, f_{\tau-1}) \neq 0 \text{ for every } s \in \Lambda_\tau^k\},$$

or ∞ when $f_0, \dots, f_{\tau-1}$ are algebraically independent over $\mathbb{Q}$.

Note. The functions $f_0, \dots, f_{\tau-1}$ are algebraically independent over $\mathbb{Q}$ if and only if there is no symmetric polynomial $S \in \Lambda_\tau$ with $S(f_0, \dots, f_{\tau-1}) = 0$.

Indeed, observe that if $P(f_0, \dots, f_{\tau-1}) = 0$ for some rational polynomial P in τ variables, then

$$S(T_0, \dots, T_{\tau-1}) \coloneqq P^{S_\tau} = \prod_{\sigma \in S_\tau} P(T_{\sigma(0)}, \dots, T_{\sigma(\tau-1)})$$

is a symmetric polynomial such that $S(f_0, \dots, f_{\tau-1}) = 0$.

Note. Comparing with Definition 5.2.7, we see that $(\boldsymbol{f}, \nu)$ is non-symmetric of degree k at $\boldsymbol{x}$ if and only if for every ball B containing $\boldsymbol{x}$ we have $\mathfrak{s}\left(\boldsymbol{f}|_{B \cap \operatorname{supp} \nu}\right) \geq k$.

Now consider $\lambda = (\lambda_0, \dots, \lambda_{\tau-1})$ with $\lambda_0 \geq \lambda_1 \geq \dots \geq \lambda_{\tau-1} \geq 0$, i.e. a partition of the integer $|\lambda| \coloneqq \lambda_0 + \dots + \lambda_{\tau-1}$ with $\ell(\lambda) \leq \tau$ parts. Given two such partitions λ^1 and λ^2, we will define their sum component by component, i.e. $\lambda^1 + \lambda^2 = (\lambda^1_0 + \lambda^2_0, \dots, \lambda^1_{\tau-1} + \lambda^2_{\tau-1})$. Also, let $\mu \coloneqq (\tau-2, \tau-3, \dots, 0)$ be the minimal such partition with distinct parts. Then, the *alternating polynomial* corresponding to λ is

$$a_{\lambda+\mu}(T_0, \dots, T_{\tau-1}) \coloneqq \det\left(T_i^{\lambda_j + \mu_j}\right) = \begin{vmatrix} T_0^{\lambda_0+\mu_0} & \cdots & T_0^{\lambda_{\tau-1}+\mu_{\tau-1}} \\ \vdots & & \vdots \\ T_{\tau-1}^{\lambda_0+\mu_0} & \cdots & T_{\tau-1}^{\lambda_{\tau-1}+\mu_{\tau-1}} \end{vmatrix}.$$

Example 5.3.2. The alternating polynomial corresponding to $(0, \dots, 0)$ is the Vandermonde polynomial on $\boldsymbol{T}$, i.e. $a_\mu = \mathrm{V}(\boldsymbol{T})$.

Definition 5.3.3. By Cauchy's Bi-Alternant Formula we know that a_μ divides $a_{\lambda+\mu}$ for every partition λ (see [103] for a concise proof). Further, the quotient is a symmetric polynomial, and we define the *Schur polynomial* in τ variables corresponding to λ as

$$s_\lambda \coloneqq \frac{a_{\lambda+\mu}}{a_\mu}.$$

We can also extend this to $\ell(\lambda) > \tau$ by setting $s_\lambda = 0$, and we will denote by $\mathcal{S}^k_\tau(\boldsymbol{T})$ the collection of all the $s_\lambda(\boldsymbol{T})$ with $|\lambda| \leq k$ and $\ell(\lambda) \leq \tau$ (c.f. the definition of $\mathcal{S}_{n,\tau}(T)$ at (5.11)). Note that $\#\mathcal{S}^k_\tau(\boldsymbol{T}) = \binom{k+\tau-1}{\tau}$.

One can show that s_λ is symmetric and homogeneous of degree $|\lambda|$, which makes it straightforward to see that

$$s_\lambda(T_0, \dots, T_{\ell(\lambda)}, 0, \dots, 0) = s_\lambda(T_0, \dots, T_{\ell(\lambda)})$$

when $\ell(\lambda) < \tau$. There is a wealth of literature about Schur polynomials, and the interested reader is invited to consult either I. G. Macdonald's book [78], or [51] for a more gentle introduction. In particular, we will need the following result.

Proposition 5.3.4 ([78, (3.3), p. 41])**.** *The Schur polynomials in $\mathcal{S}_\tau^k(\boldsymbol{T})$ form a basis for Λ_τ^k as a module over $\mathbb{Q}$.*

Corollary 5.3.5. *The following are equivalent:*

- *$(\boldsymbol{f}, \nu)$ is non-symmetric of degree k on $\mathscr{B}$;*
- *for every $x \in \mathscr{B} \cap \operatorname{supp} \nu$, every neighbourhood $B \ni x$, and every partition λ with $|\lambda| \leq k$ and $\ell(\lambda) \leq \tau$, the restrictions of $s_\lambda(\boldsymbol{f})$ to $B \cap \operatorname{supp} \nu$ are linearly independent over $\mathbb{Q}$.*
- *$(\mathcal{S}^k \circ \boldsymbol{f})_* \nu$ is rationally non-planar.*

Remark 5.3.6. It follows that $(\boldsymbol{f}, \nu)$ is symmetrically good of degree k if and only if $(\mathcal{S}^k \circ \boldsymbol{f})_* \nu$ is decaying and rationally non-planar. C.f. the notion of *friendly* measure from [69], i.e. a measure that is Federer, decaying and non-planar.

We conclude this section with some criteria to estimate $\mathfrak{s}(\boldsymbol{f})$.

Proposition 5.3.7. *Let $\boldsymbol{f} = (f_0, \dots, f_{\tau-1})$. Then for every $2 \leq t \leq \tau$ and for every $I \in [\![\tau]\!]_<^t$ we have*

$$\mathfrak{s}(\boldsymbol{f}) < \frac{n!}{t!}(\mathfrak{s}(\boldsymbol{f}_I) + 1).$$

Proof. Fix t, I, and let $s \in \Lambda_t$ be a polynomial of degree $\mathfrak{s}(\boldsymbol{f}_I) + 1$ such that $s(\boldsymbol{f}_I) = 0$. Since s is symmetric we know that S_t fixes s, thus there is a well defined action of $G_t := S_\tau/S_t$ on the image of s under the inclusion $\mathbb{Q}[T_0,\dots,T_{t-1}] \subset \mathbb{Q}[T_0,\dots,T_{\tau-1}]$. It follows that s^G is a symmetric polynomial in τ variables of degree $\deg(s)\#G$ such that $s^G(\boldsymbol{f}) = 0$. □

Now let d, N be positive integers, and for every $0 \le s \le N-1$ let Δ_s be a differential operator of the form

$$\Delta_s = \left(\frac{\partial}{\partial x_0}\right)^{j_0} \cdots \left(\frac{\partial}{\partial x_{d-1}}\right)^{j_{d-1}} \quad \text{where } j_0 + \cdots + j_{d-1} \le s. \tag{5.23}$$

Given a $\mathcal{C}^{N-1}$ map $\boldsymbol{g}(x_0,\dots,x_{d-1})$ with $N-1$ components, define the *generalised Wronskian of* $\boldsymbol{g}$ *associated with* $\Delta_0,\dots,\Delta_{N-2}$ to be the determinant

$$\det(\Delta_i(g_j)) = \begin{vmatrix} \Delta_0(g_0) & \dots & \Delta_0(g_{N-1}) \\ \vdots & & \vdots \\ \Delta_{N-1}(g_0) & \dots & \Delta_{N-1}(g_{N-1}) \end{vmatrix}.$$

This definition can also be extended to the case where $g_0,\dots,g_{N-1}$ are formal power series with coefficients in a field K. Furthermore, note that if the components of $\boldsymbol{g}$ are linearly dependent, then all of its generalised Wronskians vanish. In [36], Bostan and Dumas proved the following partial converse.

Theorem 5.3.8 ([36, Theorem 3]). *Let $g_0,\dots,g_{N-1}$ be formal power series with coefficients in a field K of characteristic 0. If they are linearly independent over K, then at least one of their generalised Wronskians is non-zero.*

Corollary 5.3.9 (Wronskian Criterion). *Let $\boldsymbol{g} = (g_1,\dots,g_{N-1})$ be a $\mathcal{C}^{N-1}$ real valued map. If at least one of the generalised Wronskians of $\boldsymbol{g}$ is non-zero, then $g_1,\dots,g_N$ are linearly independent over $\mathbb{R}$, and the converse holds when $\boldsymbol{g}$ is analytic.*

Corollary 5.3.10. *Let k,τ be positive integers and let $N = \binom{k+\tau-1}{\tau}$. If $\boldsymbol{f}$ is a $\mathcal{C}^{N-1}$ real valued map with τ components and at least one of the generalised Wronskians of $\mathcal{S}_\tau^k(\boldsymbol{f})$ is non-zero, then $\mathfrak{s}(f) \ge k$.*

To simplify the proof of the final result we will rely on another special kind of symmetric polynomial, the *monomial symmetric polynomials* m_λ. Let λ be a partition of integers with at most τ parts; then m_λ is defined as

$$m_\lambda := \sum_\sigma T_0^{\sigma(\lambda_0)} \cdots T_{\tau-1}^{\sigma(\lambda_{\tau-1})}$$

where σ runs over the distinct permutations of $\lambda_0, \ldots, \lambda_{\tau-1}$. Again, it can be shown that the collection of monomial symmetric polynomials corresponding to λ with $|\lambda| \leq k$ and $\ell(\lambda) = \tau$ forms a basis for Λ_τ^k as a module over $\mathbb{Q}$.

Proposition 5.3.11. *Let $\boldsymbol{p} = (p_0, p_1)$ be a polynomial map such that $\deg p_0 > \deg p_1$. Then $\mathfrak{s}(\boldsymbol{p}) \geq \frac{\deg p_0}{\deg p_1}$.*

Proof. Let $d_i := \deg p_i$, and note that if $\lambda = (\lambda_0, \lambda_1)$ is a partition with $k \geq \lambda_0 \geq \lambda_1 \geq 0$, then $\deg m_\lambda(\boldsymbol{p}) = d_0\lambda_0 + d_1\lambda_1$. We will show that the map $\lambda \mapsto \deg m_\lambda(\boldsymbol{p})$ is injective for $k \leq \frac{d_0}{d_1}$, which immediately gives a lower bound for $\mathfrak{s}(\boldsymbol{p})$.

Suppose that $d_0\lambda_0^1 + d_1\lambda_1^1 = d_0\lambda_0^2 + d_1\lambda_1^2$ for some $\lambda^1 \neq \lambda^2$, and without loss of generality assume $\lambda_1^1 > \lambda_1^2$. Then $d_1(\lambda_1^1 - \lambda_1^2) = d_0(\lambda_0^2 - \lambda_0^1)$, which results in

$$\begin{aligned} k &\geq \lambda_0^2 \\ &= \lambda_0^1 + \frac{d_1}{d_0}(\lambda_1^1 - \lambda_1^2) \\ &\geq 1 + \frac{d_1}{d_0} \end{aligned}$$

□

Example 5.3.12. At least when $\boldsymbol{p}(\boldsymbol{x}) = (p_0(\boldsymbol{x}), \ldots, p_{\tau-1}(\boldsymbol{x}))$ is a polynomial map with rational coefficients, we can compute $\mathfrak{s}(\boldsymbol{p})$ with relative efficiency using variable elimination via Gröbner bases. Even more, it is possible to describe all the symmetric polynomials that vanish on $\boldsymbol{p}$. Indeed, let $e_1, \ldots, e_\tau$ be the elementary symmetric polynomials in τ variables, that is

$$e_k(T_0, \ldots, T_{\tau-1}) := \sum_{I \in [\![\tau-1]\!]_<^k} T_{i_1} \cdots T_{i_k}.$$

Then it is well known that $\Lambda_\tau = \mathbb{Q}[e_1(\boldsymbol{T}), \dots, e_\tau(\boldsymbol{T})]$; in other words, every symmetric polynomial in $\boldsymbol{T}$ can be written as a polynomial in $e_1, \dots, e_\tau$. Now let $\boldsymbol{Y} = (Y_1, \dots, Y_\tau)$ and consider the ideal $\mathcal{I} \subset \mathbb{Q}[\boldsymbol{x}, \boldsymbol{Y}]$ generated by the polynomials

$$Y_k - e_k(\boldsymbol{p}) \qquad \text{for } 1 \le k \le \tau.$$

It is possible to compute a Gröbner basis G for the ideal $\tilde{\mathcal{I}} := \mathcal{I} \cap \mathbb{Q}[\boldsymbol{Y}]$ through standard algorithms, and we can see that every symmetric polynomial in Λ_τ which vanishes on $\boldsymbol{p}$ is of the form $h(e_1, \dots, e_\tau)$ for some $h \in \tilde{\mathcal{I}}$. In particular,

$$\mathfrak{s}(\boldsymbol{p}) = \min_{g \in G} \deg(g(e_1, \dots, e_\tau)).$$

As an example, these are the orders of symmetric independence for the Veronese curves of degree τ between 2 and 10, i.e. for $\boldsymbol{p}(x) = (x, x^2, \dots, x^\tau)$:

τ	2	3	4	5	6	7	8	9	10
$\mathfrak{s}(\boldsymbol{p})$	4	5	5	6	6	7	7	7	7

5.4 GOOD FUNCTIONS

Definition 5.4.1. Let X be a metric space and ν a Radon measure on X. Also consider an open subset $U \subseteq X$ and a ν-measurable function $f \colon U \to \mathbb{R}$. For any open ball $B \subset U$ and $\varepsilon > 0$, define

$$B^{f,\varepsilon} := \{x \in B : |f(x)| < \varepsilon\}.$$

As in Chapter 3, we say that f is *(C, α)-good on U with respect to ν* if there are constants $C, \alpha > 0$ such that for any open ball $B \subset U$ centred on $\operatorname{supp} \nu$ we have

$$\nu\left(B^{f,\varepsilon}\right) \le C \left(\frac{\varepsilon}{\|f\|_{\nu,B}}\right)^{\alpha} \nu(B) \quad \text{for all } \varepsilon > 0, \tag{5.24}$$

where $\|f\|_{\nu,B} := \sup_{x \in B \cap \operatorname{supp} \nu} |f(x)|$. Also, when $X = \mathbb{R}^d$ and ν is the corresponding Lebesgue measure we will write $\|f\|_B$ for $\|f\|_{\nu,B}$, and we say that f is *absolutely (C, α)-good on U with respect to ν* if (5.24) holds with $\|f\|_B$ in place of $\|f\|_{\nu,B}$.

Note that absolute (C,α)-goodness implies (C,α)-goodness, while the converse holds for measures with full support. With some minor adjustments, the properties outlined in Lemma 3.1.3 still hold in the context of measures ν other than Lebesgue. In particular, we have the following.

Lemma 5.4.2 ([72, Lemma 3.1], [34, Lemma 3.1]).

1. *If f is (C,α)-good on U wrt ν, then it is also (C',α')-good on U' wrt ν for every $C' \geq C$, $\alpha' \leq \alpha$ and $U' \subseteq U$.*
2. *If $\{f_i\}_{i\in I}$ is a collection of (C,α)-good functions on U wrt ν and the function $f \coloneqq \sup_{i\in I}|f_i|$ is Borel measurable, then f is also (C,α)-good on U wrt ν.*
3. *If f is (C,α)-good on U wrt ν and $c_9 \leq \frac{|f(x)|}{|g(x)|} \leq c_{10}$ for every $x \in U \cap \operatorname{supp}\nu$, then g is $(C(c_{10}/c_9)^\alpha, \alpha)$-good on U wrt ν.*

Proof of 3. Note that if $\varepsilon > |g(x)| \geq \frac{|f(x)|}{c_{10}}$ on $U \cap \operatorname{supp}\nu$, then

$$B^{g,\varepsilon} \cap \operatorname{supp}\nu \subseteq B^{f,c_{10}\varepsilon} \cap \operatorname{supp}\nu$$

for every ball $B \subseteq U$. Furthermore,

$$c_9\|g(x)\|_{\nu,B} = \sup_{x\in B\cap\operatorname{supp}\nu} c_9|g(x)| \leq \sup_{x\in B\cap\operatorname{supp}\nu} |f(x)| = \|f(x)\|_{\nu,B}.$$

Therefore

$$\begin{aligned}
\nu\left(B^{g,\varepsilon}\right) &\leq \nu\left(B^{f,c_{10}\varepsilon}\right) \\
&< C\left(\frac{c_{10}\varepsilon}{\|f\|_{\nu,B}}\right)^\alpha \nu(B) \\
&\leq C\left(\frac{c_{10}}{c_9}\right)^\alpha \left(\frac{\varepsilon}{\|g\|_{\nu,B}}\right)^\alpha \nu(B).
\end{aligned}$$ □

The papers [71] and [34] include various examples of real valued functions which are (C,α)-good with respect to Lebesgue measure. Moreover, [72] extends those examples to functions with values in non-Archimedean fields which satisfy a condition equivalent to (5.24). For the purposes of the present chapter we are mainly interested in the following propositions.

Proposition 5.4.3 ([3, Proposition 2.8]). *Fix $d, m, k \in \mathbb{Z}_{>0}$ and let $\boldsymbol{g} = (g_1, \ldots, g_N) \colon \mathbb{R}^d \to \mathbb{R}^N$ be a polynomial map of degree at most k. Then for any convex subset $B \subset \mathbb{R}^d$ we have*

$$|\{x \in B : \|\boldsymbol{g}(\boldsymbol{x})\| < \varepsilon\}| \leq 4d \left(\frac{\varepsilon}{\|\boldsymbol{g}\|_B}\right)^{\frac{1}{k}} |B|,$$

where $\|\boldsymbol{g}\|_B = \sup_B \|\boldsymbol{g}(\boldsymbol{x})\|$ and $\|\boldsymbol{g}(\boldsymbol{x})\| = \max_j |g_j(\boldsymbol{x})|$.

Note. This immediately implies that polynomial functions on $\mathbb{R}^d$ of degree k are $(4d, 1/k)$-good with respect to Lebesgue measure.

Now, suppose that $U \subset \mathbb{R}^d$ is open and that $\boldsymbol{g} = (g_1, \ldots, g_N) \colon U \to \mathbb{R}^N$ is a $\mathcal{C}^\ell$ map. For a given $\boldsymbol{x} \in U$, we say that $\boldsymbol{g}$ is *ℓ-non-degenerate at $\boldsymbol{x}$* if the partial derivatives of $\boldsymbol{g}$ at $\boldsymbol{x}$ of order up to ℓ span $\mathbb{R}^N$. In [71] Kleinbock and Margulis proved the following result on the (C, α)-goodness with respect to Lebesgue measure of ℓ-non-degenerate functions, which was later extended in [69] to a wider class of measures.

Proposition 5.4.4 ([71, Proposition 3.4]). *Let $\boldsymbol{g} = (g_1, \ldots, g_N) \colon U \subseteq \mathbb{R}^d \to \mathbb{R}^N$ be a $\mathcal{C}^\ell$ map, U open. If $\boldsymbol{g}$ is ℓ-non-degenerate at $\boldsymbol{x} \in U$, then there are a neighbourhood $V \subset U$ of $\boldsymbol{x}$ and a $C > 0$ such that any linear combination of $1, g_1, \ldots, g_N$ is $(C, 1/d\ell)$-good on V with respect to Lebesgue measure.*

Recall from Definition 5.2.3 that a measure ν on X is called *Federer* if for ν almost every $x \in X$ there are a neighbourhood U of x and a constant $D > 0$ such that $\nu(3^{-1}B) > \nu(B)/D$ for any ball $B \subset U$ centred on $\operatorname{supp} \nu$. Furthermore, observe that ν is absolutely (C, α)-decaying according to (5.6) precisely when every linear function is absolutely (C, α)-good with respect to ν.

Proposition 5.4.5 ([69, Proposition 7.3]). *Let $\boldsymbol{g} = (g_1, \ldots, g_N) \colon U \subseteq \mathbb{R}^d \to \mathbb{R}^N$ be a $\mathcal{C}^{\ell+1}$ map, U open. Further, let ν be a measure which is Federer and absolutely $(\tilde{C}, \alpha)$-decaying on U for some $\tilde{C}, \alpha > 0$. If $\boldsymbol{g}$ is ℓ-non-degenerate at $\boldsymbol{x} \in U$, then there are a neighbourhood $V \subset U$ of $\boldsymbol{x}$ and a $C > 0$ such that* *any linear combination of* $1, g_1, \dots, g_N$ *is absolutely* $\big(C, \alpha/(2^{\ell+1}-2)\big)$*-good on* V *with respect to* ν*.*

Note. Consider the Lebesgue measure as ν. Then Proposition 5.4.3 shows that for $d = 1$ the exponent $1/\ell$ in Proposition 5.4.4 is likely to be optimal, while $1/(2^{\ell+1} - 2)$ is much worse. However, the latter is independent of d. Unfortunately, according to [69], finding the optimal exponent seems to be a challenging open problem.

Corollary 5.4.6. *Let* k *be a positive integer,* $\boldsymbol{f} = (f_0, \dots, f_{\tau-1})$ *be an analytic map on* $U \subset \mathbb{R}^d$*, and let* ν *be a measure on* U *which is Federer and absolutely* $(\tilde{C}, \tilde{\alpha})$*-decaying on* U*. Then for every* $\boldsymbol{x} \in U \setminus Z_{\boldsymbol{f}}$ *there are a neighbourhood* $V \ni \boldsymbol{x}$ *and constants* $C_{\boldsymbol{x}}, \alpha > 0$ *such that* $s(\boldsymbol{f})$ *is* $(C_{\boldsymbol{x}}, \alpha)$*-good on* V *for every symmetric polynomial* s *of degree up to* k*, where* $Z_{\boldsymbol{f}}$ *is the zero set of a real analytic function. Furthermore, if* $N = \binom{k+\tau-1}{\tau}$*, then* α *can be chosen to be:*

- $\tilde{\alpha}/(2^N - 2)$*;*
- $1/d(N-1)$ *if* ν *is the Lebesgue measure;*
- $1/k\deg(\boldsymbol{f})$ *if* $\boldsymbol{f}$ *is a polynomial map and* ν *is the Lebesgue measure.*

Proof. Let Σ be a basis for the linear span $\langle \mathcal{S}_\tau^k(\boldsymbol{f}) \rangle_{\mathbb{R}}$, and note that we may always assume that $1 \in \Sigma$, since $1 \in \mathcal{S}_\tau^k(\boldsymbol{f})$ for every $k > 0$. Furthermore, $\#\Sigma \leq N$ and all the elements of Σ are analytic because so is $\boldsymbol{f}$.

Therefore by the Wronskian Criterion (Corollary 5.3.9) we know that at least one of the generalised Wronskians of Σ, say W, is not identically zero. Hence Σ is non-degenerate outside of the zero set $Z_{\boldsymbol{f}}$ of W, and the statement follows from Propositions 5.4.3, 5.4.4 and 5.4.5. □

Remark 5.4.7. As a consequence of the Implicit Function Theorem, one can show that the Hausdorff dimension of $Z_{\boldsymbol{f}}$ is at most $d-1$ [87]. In particular, if ν is either the Lebesgue measure on $\mathbb{R}^d$ or the natural measure supported on a sufficiently regular IFS of dimension $s > d-1$ (see Example 5.2.5), then ν satisfies the hypotheses of Corollary 5.4.6 and $\nu(Z_{\boldsymbol{f}}) = 0$.

5.5 POINTS WITH CONJUGATE COORDINATES

Proof of Theorem 5.2.13 assuming Corollary 5.2.11.
First, we are now going to check that the hypotheses of Corollary 5.2.11 are satisfied for ψ and Ψ defined by

$$\begin{cases} \psi_k^d(Q) = \psi^d(Q) := Q^{d-n-1+\gamma(m+1-d)} & \text{for } 0 \le k < d \\ \psi_k(Q) = \Psi(Q) := Q^{1-\gamma} & \text{for } d \le k \le m. \end{cases}$$

Clearly $\gamma > 0$ implies that $\psi_k(Q)Q^{-1} = Q^{-\gamma}$ is decreasing, and observe that

$$\boldsymbol{\psi}(Q)Q^{n-m} = \psi(Q)^d \Psi(Q)^{m+1-d} Q^{n-m} = 1,$$

hence condition (5.14) of Theorem 5.2.8 is satisfied for every $1 \le \tau \le m+1$ by choosing $I_\tau = (0, \dots, \tau - 1)$, since our choice of ψ and Ψ, together with the fact that $\boldsymbol{\psi}$ is decreasing, implies that $\boldsymbol{\psi}_{I_\tau}$ is decreasing as well.

Furthermore, observe that $\mathfrak{s}(\boldsymbol{x}) = \infty$ because the coordinate functions $x_0, \dots, x_{d-1}$ are algebraically independent over $\mathbb{R}$. Therefore (5.19) is enough to guarantee that condition (5.15) is satisfied as well, since the ordering of ψ_k, hence of f_k, is irrelevant for $d \le k \le m$.

Thus we can apply Corollary 5.2.11 taking the Lebesgue measure vol_d on $\mathbb{R}^d$ as ν, and through it we find $\mathscr{B}_\theta \subseteq \mathscr{B}$ with $\mathrm{vol}_d(\mathscr{B}_\theta) \gg \mathrm{vol}_d(\mathscr{B})$. Moreover, for every $x \in \mathscr{B}_\theta$ we have points $(\alpha_0, \dots, \alpha_m)$ with algebraic conjugate coordinates and $\mathrm{H}(\alpha_k) \ll Q$ such that

$$\begin{cases} |x_k - \alpha_k| \ll \frac{\psi(Q)}{Q} & \text{for } 0 \le k < d \\ |f_k(\boldsymbol{x}) - \alpha_k| \ll \frac{\Psi(Q)}{Q} & \text{for } d \le k \le m. \end{cases} \tag{5.25}$$

However, given that ψ and Ψ are multiplicative, we may assume that $\mathrm{H}(\alpha_k) \le Q$ by rescaling Q and changing the implied constants in (5.25) accordingly.

Now choose a compact subset $K \subseteq \mathscr{B}_\theta$ such that $\mathrm{vol}_d(K) \gg \mathrm{vol}_d(\mathscr{B})$, which we can always do since $\mathscr{B}$ is assumed to be bounded. Then note that the partial derivatives of $\boldsymbol{f}$ are all bounded on K, therefore the Mean Value Theorem implies that for each $d \le k \le m$ we have

$$|f_k(\boldsymbol{\alpha}) - f_k(\boldsymbol{x})| \ll_{K,d,f_k} \max_{0 \le i < d} |x_i - \alpha_i| \ll \frac{\psi(Q)}{Q}.$$

It follows that

$$|f_k(\boldsymbol{\alpha}) - \alpha_k| \leq |f_k(\boldsymbol{\alpha}) - f_k(\boldsymbol{x})| + |f_k(\boldsymbol{x}) - \alpha_k| \ll \frac{\psi(Q)}{Q} + \frac{\Psi(Q)}{Q}.$$

Since $\Psi(Q) \geq \psi(Q)$ precisely when $\gamma \leq \frac{n+1}{m+1}$, we have that

$$|f_k(\boldsymbol{\alpha}) - \alpha_k| \ll \frac{\Psi(Q)}{Q} = Q^{-\gamma}, \tag{5.26}$$

hence if c_0 is greater than the implied constant, then $(\alpha_0, \dots, \alpha_m)$ lies in $M^n_{\boldsymbol{f}}(Q, \gamma, \mathscr{B})$. Therefore we conclude that

$$\begin{aligned} \# M^n_{\boldsymbol{f}}(Q, \gamma, \mathscr{B}) &\gg \operatorname{vol}_d(K) \left(\frac{Q}{\psi(Q)} \right)^d \\ &\gg \operatorname{vol}_d(\mathscr{B}) Q^{n+1-\gamma(m+1-d)}. \end{aligned}$$ □

In section 5.6 we shall prove the following Corollary of Theorem 5.2.8, from which Corollary 5.2.11 follows immediately.

Corollary 5.5.1. *Suppose that condition* (5.14) *holds for some* $\psi_0, \dots, \psi_m$, $\varphi_{m+2}, \dots, \varphi_n$, $\boldsymbol{f}$ *as in Theorem 5.2.8 with*

$$\varphi(Q) = \varphi_{m+1}(Q) = \max\{Q, \varphi_{m+2}(Q), \dots, \varphi_n(Q)\}.$$

Furthermore, assume that there are constants $c_3, c_4 > 0$ *such that*

$$c_3 \leq \boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q) \leq c_4. \tag{5.27}$$

Then for every $0 < \theta < 1$ *there are constants* $c_5, c_6 > 0$ *and a subset* $\mathscr{B}_\theta$ *of* $\mathscr{B}$, *independent of* Q, *such that* $\nu(\mathscr{B}_\theta) > \theta\nu(\mathscr{B})$ *and every* $\boldsymbol{x} \in \mathscr{B}_\theta$ *admits* $n+1$ *linearly independent irreducible polynomials* $P = a_0 + a_1X + \cdots + a_nX^n \in \mathbb{Z}[X]$ *of degree bounded by* n *such that*

$$\begin{cases} |P(f_k(\boldsymbol{x}))| < c_5\psi_k(Q) & \\ |P'(f_k(\boldsymbol{x}))| > c_6\varphi(Q) & \text{for } 0 \leq k \leq m \\ |a_k| \leq c_5\varphi_k(Q) & \text{for } m < k \leq n \end{cases} \tag{5.28}$$

whenever Q *is sufficiently large. In particular* $\mathrm{H}(P) \ll \varphi(Q)$.

Proof of Corollary 5.2.11. Let P be as in the statement of Corollary 5.5.1, let $\varphi_{m+1}(Q) = \cdots = \varphi_n(Q) = Q$, so that $\varphi(Q) = Q$ as well, and note that by remark 5.2.9 we may choose $\mathscr{B}_\theta$ to be compact. To simplify the notation, let $y_k = f_k(\boldsymbol{x})$ for $0 \leq k \leq m$. Then observe that, since P' is continuous, $\mathscr{B}_\theta$ is compact, and $\psi_k(Q)Q^{-1}$ is decreasing, we may choose an open set U with $\mathscr{B}_\theta \subset U \subseteq \mathscr{B}$ and a constant $Q_0 > 0$ such that for every $Q > Q_0$ every interval of the form

$$I_{y_k} := \left[y_k - \kappa \frac{\psi_k(Q)}{Q}, y_k + \kappa \frac{\psi_k(Q)}{Q} \right]$$

is contained in U, where $\kappa := \frac{c_5}{c_6}$, and such that $|P'(z)| > c_6 Q$ for every $z \in U$. Furthermore, by the Mean Value Theorem we know that for every $\tilde{y}_k \in I_{y_k}$ there is a $z_k \in I_{y_k}$ such that

$$P(\tilde{y}_k) = P(y_k) + P'(z_k)(\tilde{y}_k - y_k).$$

Now note that $\mathrm{H}(P) \ll Q$, again because $\psi_k(Q)Q^{-1}$ is decreasing for every $0 \leq k \leq m$. As $\mathscr{B}$ is bounded, it follows that $|P'(z_k)|$ is bounded above by Q, up to a constant that depends on n, $\boldsymbol{f}$ and $\mathscr{B}$. Furthermore, $|P'(z_k)| > c_6 Q$ implies that for $\tilde{y}_k = y_k \pm \kappa \frac{\psi_k(Q)}{Q}$ we have

$$|P'(z_k)(\tilde{y}_k - y_k)| > c_6 \kappa \psi_k(Q) = c_5 \psi_k(Q),$$

therefore

$$P\left(y_k - \kappa \frac{\psi_k(Q)}{Q} \right) P\left(y_k + \kappa \frac{\psi_k(Q)}{Q} \right) < 0.$$

Applying once more the Mean Value Theorem we obtain, for every $0 \leq k \leq m$, a root α_k of P such that

$$|y_k - \alpha_k| < \kappa \frac{\psi_k(Q)}{Q}.$$

Finally, note that Corollary 5.5.1 gives us $n+1$ distinct irreducible polynomials, from which we obtain $n+1$ distinct points $(\alpha_0, \ldots, \alpha_m)$. □

Note. The numbers y_k are pairwise distinct on $\mathscr{B} \cap \operatorname{supp} \nu$, since $\det U_{\boldsymbol{f}}^h$ is non-zero by remark 5.2.6. By taking Q large enough if necessary, it follows that we can guarantee that the sets I_{y_k} are pairwise disjoint, hence the roots α_k are pairwise distinct. In particular, observe that the constant κ does not depend on Q and may be chosen uniformly on $\mathscr{B}$.

5.6 TAILORED POLYNOMIALS

Similarly to what Beresnevich, Bernik and Götze did in [15], we call a *tailored polynomial* an irreducible polynomial which satisfies (5.12). Our construction follows closely the argument of [15, Section 3], and it is based on Theorem 5.2.8, which we will then prove in Section 5.8 using the quantitative non-divergence method of Kleinbock and Margulis.

Now fix $\boldsymbol{x} \in \mathscr{B}$ and observe that solving for $P \in \mathbb{Z}[X]$ the system of inequalities

$$\begin{cases} |P(f_k(\boldsymbol{x}))| < \psi_k(Q) & \text{for } 0 \leq k \leq m \\ |a_k| \leq \varphi_k(Q) & \text{for } m < k \leq n \end{cases} \tag{5.29}$$

is equivalent to looking for points of the lattice $L \coloneqq M\mathbb{Z}^{n+1}$ which lie in the convex body $\mathscr{C}$, where $M = M_{\boldsymbol{f}}(\boldsymbol{x})$ is the matrix defined in (5.7) and where

$$\mathscr{C} \coloneqq \left\{ \boldsymbol{y} \in \mathbb{R}^{n+1} : \begin{array}{ll} |y_k| < \psi_k(Q) & \text{for } 0 \leq k \leq m \\ |y_k| \leq \varphi_k(Q) & \text{for } m < k \leq n \end{array} \right\}.$$

Note that $\det M \neq 0$ on $\mathscr{B} \cap \operatorname{supp} \nu$, since $\mathrm{V}(\boldsymbol{f}) \neq 0$ implies $\det U^h_{\boldsymbol{f}} \neq 0$ by remark 5.2.6. Furthermore, since $\det M$ is continuous in $\boldsymbol{x}$ we may assume without loss of generality that is bounded away from 0 on $\mathscr{B}$, up to replacing $\mathscr{B}$ with the interior of a compact subset with measure arbitrarily close to $\nu(\mathscr{B})$ (which we can always find since ν is Radon). Then Minkowski's second convex body theorem tells us that the successive minima $\lambda_0 \leq \ldots \leq \lambda_n$ of $\mathscr{C}$ with respect to L satisfy

$$\frac{2^{n+1}}{(n+1)!} \det M \leq \lambda_0 \cdots \lambda_n \operatorname{vol}(\mathscr{C}) \leq 2^{n+1} \det M$$

where $\operatorname{vol}(\mathscr{C}) = 2^{n+1} \boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q)$ is the volume of $\mathscr{C}$. Therefore we have

$$\lambda_n \leq \frac{\det(M)}{c_3 \lambda_0^n},$$

since $\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q) \geq c_3$ by condition (5.27).

Now note that if $P = a_0 + a_1X + \cdots + a_nX^n$ is such that $M\boldsymbol{a} \in \lambda_0\mathscr{C}$ where $\boldsymbol{a} = (a_0, \ldots, a_n)^T \neq (0, \ldots, 0)^T$, then $\mathrm{H}(P) \ll \lambda_0\varphi(Q)$ as long as $\det(M)$ is uniformly bounded away from 0. Indeed, there is a $\boldsymbol{b} \in \lambda_0\mathscr{C}$ such that $M\boldsymbol{a} = \boldsymbol{b}$, thus for Q large enough

$$\mathrm{H}(P) = \|\boldsymbol{a}\|_\infty \leq \left\|M^{-1}\right\|_\infty \|\boldsymbol{b}\|_\infty \leq \lambda_0\varphi(Q)\frac{\|\mathrm{adj}(M)\|_\infty}{|\det(M)|}$$

where $\mathrm{adj}(M)$ is the adjugate matrix of M, whose norm depends only on n, $\boldsymbol{x}$ and $\boldsymbol{f}(\boldsymbol{x})$, and thus can be bounded above by a constant depending on n, $\boldsymbol{f}$, and $\mathscr{B}$. Since $\mathscr{B}$ is bounded, it follows that there is a constant $c_m > 0$ such that

$$\max_{0\leq i\leq m} |P'(f_i(\boldsymbol{x}))| \leq c_m\lambda_0\varphi(Q).$$

Therefore Theorem 5.2.8 implies that for any given $\delta_0 > 0$ the set of $\boldsymbol{x} \in \mathscr{B}$ for which $\lambda_0 = \lambda_0(\boldsymbol{x}) \leq \delta_0$ is bounded above by

$$\delta_0^\alpha \nu(\mathscr{B})$$

up to a constant, since condition (5.27) implies that $\mathrm{vol}(\lambda_0\mathscr{C}) \leq 2^{n+1}c_4\lambda_0^{n+1}$. In particular, we may choose δ_0 depending only on θ, n, $\boldsymbol{f}$ and $\mathscr{B}$ such that for every $\boldsymbol{x}$ in a subset $B(\delta_0)$ of measure at least $\sqrt{\theta}\nu(\mathscr{B})$ we have $\lambda_0 > \delta_0$.

Now, let $\delta_n := \frac{\det(M)}{c_3\delta_0^n}$. Then for any $\boldsymbol{x} \in B(\delta_0)$ we may find $n+1$ linearly independent polynomials P_i whose vectors of coefficients $\boldsymbol{a}_i$ satisfy $M\boldsymbol{a}_i \in \delta_n\mathscr{C}$. If A is the matrix with columns $\boldsymbol{a}_i$, $0 \leq i \leq n$, then

$$1 \leq |\det(A)| \leq \mathrm{vol}(\delta_n\mathscr{C}) \leq 2^{n+1}c_4\delta_n^{n+1} := c'$$

and by Bertrand's postulate we may find a prime p such that

$$c' < p < 2c'.$$

In particular, this implies that $\det(A) \neq 0 \pmod{p}$, hence the system

$$A\boldsymbol{t} \equiv \boldsymbol{b}$$

has a unique solution $\boldsymbol{t} \in \mathbb{F}_p^{n+1}$, where $\boldsymbol{b} = (0, \ldots, 0, 1)^T$. Now, for ℓ in $\{0, \ldots, n\}$ define $\boldsymbol{r}_\ell = (1, \ldots, 1, 0, \ldots, 0)^T \in \mathbb{F}_p^{n+1}$, where ℓ denotes the number of zeroes. Then write $A\boldsymbol{t} - \boldsymbol{b} = p\boldsymbol{w}$ after choosing representatives for $\boldsymbol{t}$ in $\{0, \dots, p-1\}$, let $\boldsymbol{\gamma}_\ell \in \mathbb{F}_p^{n+1}$ be the unique solution to

$$A\boldsymbol{\gamma}_\ell \equiv -\boldsymbol{w} + \boldsymbol{r}_\ell$$

modulo p, and define $\boldsymbol{\eta}_\ell = \boldsymbol{t} + p\boldsymbol{\gamma}_\ell$. For each $\ell \in \{0, \dots, n\}$ let

$$\widetilde{P}_\ell := \sum_{i=0}^{n} \eta_{\ell i} P_i$$

and note that the linear independence of the vectors $\boldsymbol{r}_\ell$ implies the linear independence of the polynomials $\widetilde{P}_\ell$.

Since $A\boldsymbol{\eta}_\ell = \boldsymbol{s}$ is the vector of coefficients of $\widetilde{P}_\ell$ and since $\boldsymbol{\eta}_\ell \equiv \boldsymbol{t} \pmod{p}$, it follows that $s_n \equiv 1 \pmod{p}$ and $s_i \equiv 0 \pmod{p}$ for $0 \leq i \leq n-1$. Furthermore, the definition of $\boldsymbol{\gamma}_\ell$ implies that

$$A\boldsymbol{\eta}_\ell = \boldsymbol{b} + p\boldsymbol{r}_\ell,$$

thus $s_0 \equiv p \pmod{p^2}$. Therefore, by Eisenstein's criterion it follows that $\widetilde{P}_\ell$ is irreducible. Finally, observe that taking representatives for $\boldsymbol{t}$ and $\boldsymbol{\gamma}_\ell$ in $\{0, \dots, p-1\}$ we have $|\eta|_{\ell i} \leq p^2$, thus $\widetilde{P}_\ell$ satisfies

$$\begin{cases} |P(f_k(\boldsymbol{x}))| < c_5 \psi_k(Q) & \text{for } 0 \leq k \leq m \\ |a_k| \leq c_5 \varphi_k(Q) & \text{for } m < k \leq n, \end{cases} \tag{5.30}$$

where

$$\begin{aligned} c_5 &= 4(n+1)\delta_n c'^2 \\ &= 2^{2n+4}(n+1)c_4^2 \delta_n^{2n+3} \\ &= 2^{2n+4}(n+1)c_4^2 \left(\frac{\det(M)}{c_3 \delta_0^n} \right)^{2n+3}. \end{aligned} \tag{5.31}$$

Then, Theorem 5.2.8 implies that the measure of the set of $\boldsymbol{x} \in \mathscr{B}$ which admit a solution P to (5.30) such that $\max|P'(f_i(\boldsymbol{x}))| \leq c_6\varphi(Q)$ is bounded above by

$$c_6^{\frac{\alpha}{n+1}} c_5^{\frac{\alpha n}{n+1}} \nu(\mathscr{B})$$

up to a constant. In particular, we may choose $c_6 > 0$, depending only on θ, n, $\boldsymbol{f}$ and $\mathscr{B}$, such that for every $\boldsymbol{x}$ in a subset $\mathscr{B}_\theta = B(\delta_0, c_6) \subseteq B(\delta_0)$ of measure at least $\sqrt{\theta}\nu(B(\delta_0)) \geq \theta\nu(\mathscr{B})$ we have $\min|P'(f_k(\boldsymbol{x}))| > c_6\varphi(Q)$.

5.7 UBIQUITY

Definition 5.7.1. A *dimension function* $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is a continuous increasing function such that $g(r) \to 0$ as $r \to 0$. Now suppose that F is a non-empty subset of a metric space Ω. For $\rho > 0$, a *ρ-cover* of F is a countable collection $\{B_i\}$ of balls in Ω of radii $r(B_i) \leq \rho$ whose union contains F. Define

$$\mathcal{H}^g_\rho(F) \coloneqq \inf\Big\{ \sum_i g\big(r(B_i)\big) : \{B_i\} \text{ is a } \rho\text{-cover of } F\Big\}.$$

The *(generalised) Hausdorff measure* $\mathcal{H}^g(F)$ of F with respect to the dimension function g is defined as

$$\mathcal{H}^g(F) \coloneqq \lim_{\rho\to 0} \mathcal{H}^g_\rho(F) = \sup_{\rho>0} \mathcal{H}^g_\rho(F).$$

See [84, Chapter 4] for more details.

Example 5.7.2. Given $s > 0$, the usual Hausdorff s-measure $\mathcal{H}^s$ coincides with $\mathcal{H}^g$ where $g(r) = r^s$. In particular, when s is an integer $\mathcal{H}^s$ is a constant multiple of the s-dimensional Lebesgue measure.

Define $\mathscr{L}^*_{n,m+1}(\psi, \Psi; d)$ to be the set of $\boldsymbol{x} \in \mathbb{R}^{m+1}$ such that

$$\max_{0\leq k<d} |x_k - \alpha_k| < \frac{\psi(\mathrm{H}(\boldsymbol{\alpha}))}{\mathrm{H}(\boldsymbol{\alpha})} \quad \text{and} \quad \max_{d\leq k\leq m} |x_k - \alpha_k| < \frac{\Psi(\mathrm{H}(\boldsymbol{\alpha}))}{\mathrm{H}(\boldsymbol{\alpha})}$$

for infinitely many $\boldsymbol{\alpha} \in \mathbb{A}^{m+1}_n$. When $\boldsymbol{f}$ parametrises a d-dimensional manifold $\mathcal{M}$ and $f_k(\boldsymbol{x}) = x_k$ for every $0 \leq k < d$, will also write $\mathscr{L}^*_{n,\boldsymbol{f}}(\psi, \Psi)$ for the set of $\boldsymbol{x} \in \mathscr{B}$ such that $\boldsymbol{f}(\boldsymbol{x}) \in \mathscr{L}^*_{n,m+1}(\psi, \Psi; d)$. This section is devoted to the proof of the following Proposition, of which Theorem 5.2.16 is a direct consequence.

Proposition 5.7.3. *Let $\psi, \Psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be decreasing functions which satisfy* (5.21)*, and let g be a dimension function such that $r^{-d}g(r)$ is non-increasing. Also assume that $r^{-\gamma}g(r)$ is increasing for some $\gamma > 0$, and* *that it satisfies* (5.22)*. Further suppose that* $\boldsymbol{f}$ *is Lipschitz continuous, that* $\mathrm{V}(\boldsymbol{f}) \neq 0$*, and that* $\boldsymbol{f}$ *satisfies condition* (5.15) *on* $\mathscr{B}$*. Then*

$$\mathcal{H}^g\Big(\mathscr{L}^*_{n,\boldsymbol{f}}(\psi)\Big) = \mathcal{H}^g(\mathscr{B}) \quad \text{if} \quad \sum_{Q=1}^{\infty} Q^{n-m-1+d}\Psi(Q)^{m+1-d} g\left(\frac{\psi(Q)}{Q}\right) = \infty.$$

Remark 5.7.4. There is a constant $c_{11} > 0$, depending only on n and $\mathcal{M}$, such that

$$\mathscr{L}^*_{n,\boldsymbol{f}}(\psi, \Psi) \subseteq \mathscr{L}_{n,\boldsymbol{f}}(c_{11}\psi, c_{11}\Psi).$$

Indeed, suppose that $\boldsymbol{y} \in \mathscr{L}^*_{n,\boldsymbol{f}}(\psi)$, and let $\boldsymbol{\alpha} \in \mathbb{A}_n^{m+1}$ be such that $\|\boldsymbol{y} - \boldsymbol{\alpha}\| < \frac{\psi(\mathrm{H}(\boldsymbol{\alpha}))}{\mathrm{H}(\boldsymbol{\alpha})}$. If P is the minimum polynomial of α_0 (and hence of α_k for every $0 \leq k \leq m+1$), then by the Mean Value Theorem we have

$$\begin{aligned} |P(y_k)| &= |P(y_k) - P(\alpha_k)| \\ &\leq |y_k - \alpha_k| \sup_{\boldsymbol{z} \in \mathcal{M}} |P'(z_k)| \\ &< c_{11} \frac{\psi_k(\mathrm{H}(\boldsymbol{\alpha}))}{\mathrm{H}(\boldsymbol{\alpha})} \mathrm{H}(P) \\ &= c_{11}\psi_k(\mathrm{H}(\boldsymbol{\alpha})) \end{aligned} \qquad \text{with} \qquad \psi_k = \begin{cases} \psi & \text{for } 0 \leq k < d \\ \Psi & \text{for } d \leq k \leq m \end{cases},$$

since $\mathscr{B}$ bounded implies that P' is bounded above on $\mathcal{M}$, and of course $\mathrm{H}(P) = \mathrm{H}(\boldsymbol{\alpha})$. Thus it follows that $\boldsymbol{y} \in \mathscr{L}_{n,\boldsymbol{f}}(c_{11}\psi, c_{11}\Psi)$, as required.

Therefore the convergence part of Theorem 5.1.1 immediately gives the following partial counterpart of Proposition 5.7.3. Here, as before, $|U|$ denotes the Lebesgue measure of a measurable set $U \subset \mathbb{R}^{m+1}$.

Lemma 5.7.5. *For any function* $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ *we have*

$$\Big|\mathscr{L}^*_{n,\boldsymbol{f}}(\psi, \Psi)\Big| = 0 \quad \text{if} \quad \sum_{Q=1}^{\infty} Q^{n-m-1}\psi(Q)^d \Psi(Q)^{m+1-d} < \infty.$$

Our proof relies on a powerful tool of Diophantine Approximation, *ubiquitous systems*, adapted to the case of approximation of dependent quantities like in [12]. Consider the following setting:

- Ω, a compact subset of $\mathbb{R}^d$;

- J, a countable set;
- $\mathcal{R} = (R_{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in J}$ a family of points in Ω indexed by J, referred to as *resonant points*;
- a function $\beta\colon J \to \mathbb{R}^+, \boldsymbol{\alpha} \mapsto \beta_{\boldsymbol{\alpha}}$, which assigns a *weight* to each $R_{\boldsymbol{\alpha}}$ in $\mathcal{R}$;
- a function $\rho\colon \mathbb{R}^+ \to \mathbb{R}^+$ such that $\lim_{r\to\infty} \rho(r) = 0$, referred to as a *ubiquitous function*; and
- $J(t) = J_\kappa(t) \coloneqq \{\boldsymbol{\alpha} \in J\colon \beta_{\boldsymbol{\alpha}} \le \kappa^t\}$, assumed to be finite for every $t \in \mathbb{N}$, where $\kappa > 1$ is fixed.

Furthermore, $B(\boldsymbol{x}, r)$ will denote a ball in Ω with respect to the sup norm, and for a given function $\hat{\psi}\colon \mathbb{R}^+ \to \mathbb{R}^+$ we will also consider the limsup set

$$\Lambda_{\mathcal{R}}(\hat{\psi}) \coloneqq \{\boldsymbol{x} \in \Omega : \|\boldsymbol{x} - R_{\boldsymbol{\alpha}}\| < \hat{\psi}(\beta_{\boldsymbol{\alpha}}) \text{ for infinitely many } \boldsymbol{\alpha} \in J\}.$$

Definition 5.7.6. The pair $(\mathcal{R}, \beta)$ is a *locally ubiquitous system in Ω with respect to ρ* if for any ball $B \in \Omega$

$$\left|\bigcup_{\boldsymbol{\alpha}\in J(t)} B(\boldsymbol{\alpha}, \rho(\kappa^t)) \cap B\right| \gg |B|$$

for every t large enough, where the implied constant is absolute.

Like with [24, Theorem 1], the following statement can be readily obtained by combining Theorems 2.0.5 and 2.0.6.

Theorem 5.7.7. *In the above setting, suppose that $(\mathcal{R}, \beta)$ is a locally ubiquitous system in Ω with respect to ρ, and let g be a dimension function such that $r^{-d}g(r)$ is non-increasing. Furthermore, suppose that there are constants $r_0, c_7, c_8 \in (0, 1)$ such that*

$$g(c_7 r) \le c_8 g(r) \text{ for any } r \in (0, r_0).$$

Also assume that $\hat{\psi}$ is decreasing and that

$$\limsup_{t\to\infty} \frac{\hat{\psi}(\kappa^{t+1})}{\hat{\psi}(\kappa^t)} < 1. \tag{5.32}$$

Then

$$\mathcal{H}^g\Big(\Lambda_{\mathcal{R}}(\hat{\psi})\Big) = \mathcal{H}^g(\Omega) \quad \textit{if} \quad \sum_{t=0}^{\infty} \frac{g(\hat{\psi}(\kappa^t))}{\rho(\kappa^t)^d} = \infty.$$

Now let $\boldsymbol{f}$, ψ and Ψ be as in Proposition 5.7.3. Since $\mathscr{B}$ is assumed to be bounded, for every integer $q \geq 2$ we may find a compact subset $\mathscr{B}_q \subset \mathscr{B}$ such that $|\mathscr{B}_q| \geq (1 - \frac{1}{q})|\mathscr{B}|$. It follows that $\mathcal{H}^g(\mathscr{B}) = \lim_{q\to\infty} \mathcal{H}^g(\mathscr{B}_q)$, so it suffices to prove the proposition with $\mathscr{B}_q$ in place of $\mathscr{B}$ for any fixed q. For ease of notation, given $\boldsymbol{y} = (y_0, \ldots, y_m) \in \mathbb{R}^{m+1}$ we will write $\hat{\boldsymbol{y}}$ for $(y_0, \ldots, y_{d-1})$. Then let $\Omega \coloneqq \mathscr{B}_q$ and define

$$J \coloneqq \left\{\boldsymbol{\alpha} \in \mathbb{A}_n^{m+1} : \hat{\boldsymbol{\alpha}} \in \Omega \text{ and } \max_{d\leq k\leq m} |f_k(\hat{\boldsymbol{\alpha}}) - \alpha_k| < \frac{1}{2}\frac{\Psi(\mathrm{H}(\alpha))}{\mathrm{H}(\alpha)}\right\}$$

$$\mathcal{R} \coloneqq (\hat{\boldsymbol{\alpha}})_{\boldsymbol{\alpha}\in J} \qquad \beta_{\boldsymbol{\alpha}} \coloneqq \mathrm{H}(\boldsymbol{\alpha}).$$

Also let

$$\rho(Q) = \rho_0\Big(Q^{n-m+d}\Psi(Q)^{m+1-d}\Big)^{-\frac{1}{d}}$$

for some constant $\rho_0 > 0$ to be determined later, and observe that (5.21) implies

$$\begin{aligned}\rho(Q) &\ll \Big(Q^{n-m+d}Q^{\frac{m-n}{m+1}(m+1-d)}\Big)^{-\frac{1}{d}} \\ &= \Big(Q^{n-m+d}Q^{m-n-\frac{d}{m+1}}\Big)^{-\frac{1}{d}} \\ &= Q^{-1+\frac{1}{m+1}},\end{aligned}$$

which shows that $\rho(Q) \to 0$ as $Q \to \infty$.

Lemma 5.7.8. *Suppose that $f_d, \ldots, f_m$ are Lipschitz continuous with constant bounded above by $c_{\boldsymbol{f}}$. If $\boldsymbol{y} \in \mathbb{R}^{m+1}$ is such that $\hat{\boldsymbol{y}} \in \mathscr{B}$,*

$$\max_{0\leq k<d} |x_k - y_k| < \Theta_{\boldsymbol{x}} \quad \textit{and} \quad \max_{d\leq k\leq m} |f_k(\boldsymbol{x}) - y_k| < \Theta_{\boldsymbol{f}}$$

for some $\Theta_{\boldsymbol{x}}, \Theta_{\boldsymbol{f}} > 0$, then

$$\max_{d\leq k\leq m} |f_k(\hat{\boldsymbol{y}}) - y_k| < \Theta_{\boldsymbol{f}}\left(1 + c_{\boldsymbol{f}}\frac{\Theta_{\boldsymbol{x}}}{\Theta_{\boldsymbol{f}}}\right).$$

Proof. Simply observe that, by the triangle inequality,

$$\begin{aligned}|f_k(\hat{\boldsymbol{y}}) - y_k| &\leq |f_k(\hat{\boldsymbol{y}}) - f_k(\boldsymbol{x})| + |f_k(\boldsymbol{x}) - y_k| \\ &< c_{\boldsymbol{f}}\|\boldsymbol{x} - \hat{\boldsymbol{y}}\| + |f_k(\boldsymbol{x}) - y_k| \\ &< c_{\boldsymbol{f}}\Theta_{\boldsymbol{x}} + \Theta_{\boldsymbol{f}} \\ &= \Theta_{\boldsymbol{f}}\left(1 + c_{\boldsymbol{f}}\frac{\Theta_{\boldsymbol{x}}}{\Theta_{\boldsymbol{f}}}\right). \end{aligned}$$ □

Lemma 5.7.9. *Let $J, \mathcal{R}, \beta, \rho$ be as above, and suppose that $\boldsymbol{f}$ is Lipschitz continuous. Then there is a choice of $\rho_0 > 0$ such that $(\mathcal{R}, \beta)$ is a locally ubiquitous system in Ω with respect to ρ.*

Proof. Fix a ball $B \subset \Omega$ and let

$$\begin{cases} \psi_k(Q) = Q\rho(Q) & \text{for } 0 \leq k < d \\ \psi_k(Q) = \Psi(Q) & \text{for } d \leq k \leq m \\ \varphi_k(Q) = Q & \text{for } m < k \leq n. \end{cases}$$

Then observe that $\frac{\psi_k(Q)}{Q}$ is decreasing for every $0 \leq k \leq m$, and that

$$\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q) = \rho(Q)^d\Psi(Q)^{m+1-d}Q^{n-m+d} = \rho_0^d.$$

Furthermore, by condition (5.21) we know that $\Psi(Q) \gg Q^{\frac{m-n}{m+1}}$, which implies $\Psi(Q) \gg Q\rho(Q)$. Therefore for every $1 \leq \tau \leq m+1$ and every choice of $I \in [\![m]\!]^{\tau}_{<}$ we have that

$$\psi_I(Q) \ll \Psi(Q)^{\tau} \ll 1$$

for every Q large enough, thus we may apply Corollary 5.2.11 with $\nu(\cdot) = |\cdot|$ and B in place of $\mathscr{B}$. Hence for any fixed $0 < \theta < 1$ we find a set $B_\theta \subseteq B$ with $|B_\theta| > \theta|B|$ and a constant $c > 0$ such that, for t large enough, every $\boldsymbol{x} \in B_\theta$ admits $n+1$ points $\boldsymbol{\alpha} \in \mathbb{A}_n^{m+1}(c\kappa^t)$ with

$$\max_{0\leq k<d}|x_k - \alpha_k| < c\rho(\kappa^t) \quad \text{and} \quad \max_{d\leq k\leq m}|f_k(\boldsymbol{x}) - \alpha_k| < c\frac{\Psi(\kappa^t)}{\kappa^t}.$$

Now, again because of $\Psi(Q) \gg Q\rho(Q)$, by Lemma 5.7.8 it follows that

$$\max_{d\leq k\leq m}|f_k(\hat{\boldsymbol{\alpha}}) - \alpha_k| < c\hat{c}\frac{\Psi(\kappa^t)}{\kappa^t}$$

for some $\hat{c} > 1$. Finally, observe that Remark 5.2.12 and equation (5.31) show that we can choose c by manipulating the value of ρ_0. In particular, we can ensure that $c < \hat{c}^{-1}$, thus $\boldsymbol{\alpha} \in J(t)$ and

$$\left| \bigcup_{\boldsymbol{\alpha} \in J(t)} B(\boldsymbol{\alpha}, \rho(\kappa^t)) \cap B \right| \geq |B_\theta| > \theta |B|. \qquad \square$$

Note. Condition (5.14) is actually satisfied even in the absence of (5.21). Indeed, using the fact that Ψ is decreasing and that $\boldsymbol{\psi\varphi}$ is constant, one can show that ψ_I is decreasing for every $I = (m+1-\tau, \dots, m)$ where $1 \leq \tau \leq m+1$.

Proof of Proposition 5.7.3. Note that since $\psi \in O(\Psi)$, there is a $c_{12} > 0$ such that $\Psi(Q) > c_{12}\psi(Q)$ for any integer $Q > 0$. Then let $\hat{\psi}(Q) = \frac{c_{12}}{2c_{\boldsymbol{f}}} \frac{\psi(Q)}{Q}$, where $c_{\boldsymbol{f}}$ is as in Lemma 5.7.8. It is clear that this choice of $\hat{\psi}$ satisfies (5.32): indeed,

$$\limsup_{t \to \infty} \frac{\hat{\psi}(\kappa^{t+1})}{\hat{\psi}(\kappa^t)} = \frac{1}{\kappa} \limsup_{t \to \infty} \frac{\psi(\kappa^{t+1})}{\psi(\kappa^t)} < \frac{1}{\kappa}$$

since ψ is assumed to be decreasing. The proposition will follow as an immediate consequence of Theorem 5.7.7 once we've shown that $\Lambda_{\mathcal{R}}(\hat{\psi}) \subseteq \mathscr{L}^*_{n,\boldsymbol{f}}(\psi, \Psi)$.

If $\boldsymbol{x} \in \Lambda_{\mathcal{R}}(\hat{\psi})$, then there are infinitely many $\boldsymbol{\alpha} \in \mathbb{A}_n^{m+1}$ such that

$$\max_{0 \leq k < d} |x_k - \alpha_k| < \frac{c_{12}}{2c_{\boldsymbol{f}}} \frac{\psi(\mathrm{H}(\boldsymbol{\alpha}))}{\mathrm{H}(\boldsymbol{\alpha})} \quad \text{and} \quad \max_{d \leq k \leq m} |f_k(\hat{\boldsymbol{\alpha}}) - \alpha_k| < \frac{1}{2} \frac{\Psi(\mathrm{H}(\boldsymbol{\alpha}))}{\mathrm{H}(\boldsymbol{\alpha})}.$$

Therefore the same argument of Lemma 5.7.8 gives

$$\max_{d \leq k \leq m} |f_k(\boldsymbol{x}) - \alpha_k| < \frac{\Psi(\mathrm{H}(\boldsymbol{\alpha}))}{\mathrm{H}(\boldsymbol{\alpha})}.$$

It follows that $\boldsymbol{x} \in \mathscr{L}^*_{n,\boldsymbol{f}}(\psi, \Psi)$, since we may assume without loss of generality that $c_{\boldsymbol{f}} \geq \frac{1}{2}$. The proof is concluded by observing that by Cauchy's Condensation Test

$$\sum_{t=0}^{\infty} \frac{g(\hat{\psi}(\kappa^t))}{\rho(\kappa^t)^d} = \rho_0^{-d} \sum_{Q=1}^{\infty} \kappa^{t(n-m+d)} \Psi(\kappa^t)^{m+1-d} g\left(\frac{c_{12}}{2c_{\boldsymbol{f}}} \frac{\psi(\kappa^t)}{\kappa^t} \right) = \infty$$

if and only if

$$S_1 \coloneqq \sum_{Q=1}^{\infty} Q^{n-m-1+d}\Psi(Q)^{m+1-d} g\left(\frac{c_{12}}{2c_{\boldsymbol{f}}}\frac{\psi(Q)}{Q}\right) = \infty,$$

and that the same argument of Lemma 4.3.2 shows that the latter happens if and only if

$$S_2 \coloneqq \sum_{Q=1}^{\infty} Q^{n-m-1+d}\Psi(Q)^{m+1-d} g\left(\frac{\psi(Q)}{Q}\right) = \infty$$

when g is increasing and ψ is decreasing. Indeed, note that without loss of generality we may assume that $2c_{\boldsymbol{f}} > c_{12}$, and let $c \coloneqq \frac{2c_{\boldsymbol{f}}}{c_{12}} > 1$. Furthermore, for ease of notation let

$$\sigma(z, Q) \coloneqq Q^{n-m-1+d}\Psi(Q)^{m+1-d} g\left(z\frac{\psi(Q)}{Q}\right).$$

Now, on one hand g increasing immediately implies that $S_1 \leq S_2$. On the other hand,

$$\begin{aligned} S_2 &= \sum_{Q=1}^{c-1} \sigma(1, Q) + \sum_{q=1}^{\infty} \sum_{cq \leq Q < c(q+1)} \sigma(1, Q) \\ &\ll \sum_{q=1}^{\infty} \sigma(1.cq) \\ &\ll \sum_{q=1}^{\infty} \sigma(c^{-1}.q) \\ &= S_1, \end{aligned}$$

where the last inequality is due to the fact that ψ is decreasing and g is increasing. □

5.8 PROOFS OF THEOREM 5.2.8 AND COROLLARY 5.2.10

We will first prove the following local version of Theorem 5.2.8, which can then be extended via a compactness argument.

Theorem 5.8.1. *Under the hypotheses of Theorem 5.2.8, fix a point* $\boldsymbol{x} \in \mathscr{B} \cap \operatorname{supp} \nu$ *and let* $B \ni \boldsymbol{x}$ *be a ball such that* $\tilde{B} = 3^{n+1}B \subset \mathscr{B}$. *Then for*

Q large enough we may find constants $C, \rho > 0$, the latter dependant on B, such that

$$\nu\Big(\mathcal{D}^n_{\boldsymbol{f}}(Q, B)\Big) \leq C\left(\frac{\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q)}{\rho^{n+1}}\right)^{\frac{\alpha}{n+1}} \nu(B).$$

Proof of Theorem 5.2.8 given Theorem 5.8.1. Let $\mathscr{B}_\theta \subset \mathscr{B}$ be a compact subset such that $\nu(\mathscr{B}_\theta) \geq \nu(\mathscr{B})$, which exists because $\mathscr{B}$ is bounded and ν is Radon. Then note that, since $\mathscr{B} \cap \operatorname{supp} \nu$ is contained in the interior of $\mathscr{B}$ by hypothesis, for every $\boldsymbol{x} \in \mathscr{B}_\theta$ we may find a ball $B_{\boldsymbol{x}} \ni \boldsymbol{x}$ as in Theorem 5.8.1, as well as the respective constants $C_{\boldsymbol{x}}$ and $\rho_{\boldsymbol{x}}$. Hence by compactness there is a finite subset $\{x_k\}_{k\in K} \subset \mathscr{B}_\theta$ such that $\{B_{\boldsymbol{x}_k}\}_{k\in K}$ is an open cover of $\mathscr{B}_\theta$. Therefore the result follows by observing that

$$\nu\Big(\mathcal{D}^n_{\boldsymbol{f}}(Q, \mathscr{B}_\theta)\Big) \leq \sum_{k\in K} \nu\Big(\mathcal{D}^n_{\boldsymbol{f}}(Q, B_{\boldsymbol{x}_k})\Big)$$

and by taking $C = \max_K C_{\boldsymbol{x}_k}$ and $\rho = \min_K \rho_{\boldsymbol{x}_k}$. □

Through the Dani-Kleinbock-Margulis correspondence between Diophantine Approximation and flows on homogeneous spaces [45, 71], we will reinterpret the problem of finding points $x \in \mathscr{B}$ for which (5.12) has a solution as a shortest vector problem. First, we expand (5.12) into the $m+1$ systems of inequalities

$$\begin{cases} |P(f_k(\boldsymbol{x}))| < \psi_k(Q) & \text{for } 0 \leq k \leq m \\ P'(f_h(\boldsymbol{x})) \leq \varphi_{m+1}(Q) & \\ |a_k| \leq \varphi_k(Q) & \text{for } m+1 < k \leq n \end{cases} \qquad 0 \leq h \leq m, \tag{5.33}$$

and observe that these can be rewritten in matrix form using the matrices $U^h_{\boldsymbol{f}}$.

However, to be able to view this as a smallest vector problem we also need to rescale the inequalities. Consider the scaling matrix

$$g_{\boldsymbol{t}} \coloneqq \operatorname{diag}\Big(e^{t_0}, \ldots, e^{t_m}, e^{-t_{m+1}}, \ldots, e^{-t_n}\Big), \tag{5.34}$$

where $\boldsymbol{t} = (t_0, \ldots, t_n) \in \mathbb{R}^{n+1}$ is such that

$$t_{[\![0,m]\!]} = t_{[\![m+1,n]\!]},$$

and where for every $1 \leq \tau \leq n+1$ and $I \in [\![n]\!]^\tau_<$ we defined

$$t_I = \sum_{i\in I} t_i. \tag{5.35}$$

Then we need $\delta = \delta(Q) > 0$ such that

$$\begin{cases} \delta = e^{t_k}\psi_k(Q) & \text{for } 0 \leq k \leq m \\ \delta = e^{-t_k}\varphi_k(Q) & \text{for } m < k \leq n, \end{cases} \tag{5.36}$$

and multiplying those $n+1$ equations together we see that

$$\delta^{n+1} = \boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q). \tag{5.37}$$

Therefore, taking logarithms we may rewrite t_k in terms of ψ_k and φ_k, as

$$\begin{cases} t_k = \log\delta - \log\psi_k(Q) & \text{for } 0 \leq k \leq m \\ t_k = \log\varphi_k(Q) - \log\delta & \text{for } m < k \leq n. \end{cases} \tag{5.38}$$

We can now see that (5.12) has a solution for a given $\boldsymbol{x} \in \mathscr{B}$ if and only if for every $0 \leq h \leq m$ the lattice $g_{\boldsymbol{t}} U^h_{\boldsymbol{f}}(\boldsymbol{x})\mathbb{Z}^{n+1}$ has a non-zero vector with sup-norm at most δ, thus we have indeed reduced to a shortest vector problem. In other words,

$$\mathcal{D}^n_{\boldsymbol{f}}(Q,B) = \bigcap_{h=0}^{m} \left\{ \boldsymbol{x} \in B : \lambda\Big(g_{\boldsymbol{t}} U^h_{\boldsymbol{f}}(\boldsymbol{x})\mathbb{Z}^{n+1}\Big) < \Big(\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q)\Big)^{\frac{1}{n+1}} \right\},$$

where $\lambda(\Gamma) = \inf_{v\in\Gamma\setminus\{0\}}\|v\|$ denotes the length of the shortest vector in a discrete subgroup $\Gamma \subset \mathbb{R}^{n+1}$.

Remark 5.8.2. Condition (5.14) of Theorem 5.2.8 is equivalent to asking that for every $1 \leq \tau \leq m+1$ there is a choice of $I \in [\![m]\!]^\tau_<$ such that $t_I = t_I(Q)$ is bounded below. Indeed, by (5.38)

$$\begin{aligned} t_I &= \tau\log\delta - \sum_{i\in I}\log\psi_i(Q) \\ &= \frac{\tau}{n+1}\log(\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q)) - \log\psi_I(Q) \\ &\geq c \end{aligned}$$

precisely when $\psi(Q)\varphi(Q) \geq e^c\psi_I(Q)^{\frac{n+1}{\tau}}$.

Example 5.8.3. Let $d = m = 1$, as in the context of Theorem 5.1.2. Furthermore, let

$$\psi_0(Q) = \psi_1(Q) = Q^{-\frac{n-1}{2}}, \ \ \varphi_2(Q) = \varepsilon^{n+1}Q$$
$$\text{and } \varphi_3(Q) = \cdots = \varphi_n(Q) = Q.$$

Then by (5.37) we have $\delta = \varepsilon$. Moreover, the equations (5.38) become

$$\begin{cases} t_k = \log\varepsilon + \frac{n-1}{2}\log Q & \text{for } 0 \le k \le 1 \\ t_2 = n\log\varepsilon + \log Q & \\ t_k = \log Q - \log\varepsilon & \text{for } 3 \le k \le n. \end{cases} \tag{5.39}$$

Therefore $t_{[\![1]\!]} = 2\log\varepsilon + (n-1)\log Q$, which in particular gives that $t_{[\![1]\!]} \ge c$ for

$$\log\varepsilon \ge \frac{\log c}{2} - \frac{n-1}{2}\log Q,$$

i.e. $\varphi_2(Q) \gg Q^{\frac{3-n^2}{2}}$. Then $\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q) \gg Q^{\frac{1-n^2}{2}}$ and we easily see that condition (5.14) of Theorem 5.2.8 is satisfied for all choices of I, since

$$\boldsymbol{\psi}(Q)^{\frac{n+1}{2}} = \psi_0(Q)^{n+1} = \psi_1(Q)^{n+1} = Q^{\frac{1-n^2}{2}}.$$

The main tool in our proof will be the following Theorem from [66]. Here $\mathcal{W}_\tau$ denotes the set of elements

$$\boldsymbol{w} = \boldsymbol{w}_1 \wedge \cdots \wedge \boldsymbol{w}_\tau \in \bigwedge^{\tau} \mathbb{Z}^{n+1}$$

where $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_\tau\}$ is a *primitive τ-tuple*, i.e. it can be completed to a basis of $\mathbb{Z}^{n+1}$. Furthermore, $\|\cdot\|$ will denote both the sup-norm and the norm it induces on $\bigwedge \mathbb{R}^{n+1}$.

Note. The elements of $\mathcal{W}_\tau$ can be identified with the primitive subgroups of $\mathbb{Z}^{n+1}$ of rank τ, i.e. those non-zero subgroups $\Gamma \subseteq \mathbb{Z}^{n+1}$ of rank τ such that $\Gamma = \Gamma_{\mathbb{R}} \cap \mathbb{Z}^{n+1}$, where $\Gamma_{\mathbb{R}}$ denotes the linear subspace generated by Γ in $\mathbb{R}^{n+1}$. Therefore, up to a sign they can also be identified with the rational τ-dimensional subspaces of $\mathbb{R}^{n+1}$.

Theorem 5.8.4 ([66, Theorem 2.2]). *Fix $n, N \in \mathbb{N}$ and $\tilde{C}, D, \alpha, \rho > 0$. Given an N-Besicovitch metric space X, let B be a ball in X and ν be a measure which is D-Federer on $\tilde{B} = 3^{n+1}B$. Suppose that $\eta \colon \tilde{B} \to \mathrm{GL}_{n+1}(\mathbb{R})$ is a map such that for every $1 \leq \tau \leq n+1$ and for every $\boldsymbol{w} \in \mathcal{W}_\tau$:*

1. *the function $\boldsymbol{x} \mapsto \|\eta(\boldsymbol{x})\boldsymbol{w}\|$ is $(\tilde{C}, \alpha)$-good on $\tilde{B}$ with respect to ν, and*
2. *$\|\eta(\cdot)\boldsymbol{w}\|_{\nu,B} \geq \rho^\tau$.*

Then for any $0 < \delta \leq \rho$ we have

$$\nu\left(\left\{\boldsymbol{x} \in B : \lambda\left(\eta(\boldsymbol{x})\mathbb{Z}^{n+1}\right) < \delta\right\}\right) \leq C\left(\frac{\delta}{\rho}\right)^\alpha \nu(B)$$

with $C = (n+1)\tilde{C}(ND^2)^{n+1}$.

Note. In light of Lemma 5.4.2, we may extend this to $\delta > \rho$ as well, since we may always exchange $\tilde{C}$ with $\max\{\tilde{C}, (n+1)^{-1}(ND^2)^{-n-1}\}$, so that $C \geq 1$.

For our purposes we would like to take $\eta(\boldsymbol{x}) = g_{\boldsymbol{t}} U^h_{\boldsymbol{f}}(\boldsymbol{x})$, and to show that it satisfies hypotheses 1 and 2 we will need the following Lemma. Here, for each $I \subseteq [\![n]\!]^\tau_<$ we will denote by $\boldsymbol{e}_I$ the standard basis element $\boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_\tau}$ of $\bigwedge^\tau \mathbb{R}^{n+1}$.

Lemma 5.8.5. *Let $\boldsymbol{w} = \boldsymbol{w}_1 \wedge \cdots \wedge \boldsymbol{w}_\tau \in \mathcal{W}_\tau$ and let A be an $(n+1) \times (n+1)$ matrix. Then, for every $I \subseteq [\![n]\!]^\tau_<$, the component of $A\boldsymbol{w}$ corresponding to $\boldsymbol{e}_I$ is an integer linear combination of the minors $|A|_{I,J}$, where J runs through $[\![n]\!]^\tau_<$. Furthermore, the coefficients are independent from I and not all zero.*

Proof. Let $W = (\boldsymbol{w}_1 | \cdots | \boldsymbol{w}_\tau)$ be the matrix obtained by juxtaposition of the vectors $\boldsymbol{w}_1, \ldots, \boldsymbol{w}_\tau$, and recall the well-known fact that the $\boldsymbol{e}_I$ component of $\boldsymbol{w}$ is just the $\tau \times \tau$ minor $|W|_{I,[\![\tau]\!]^*}$ of W (see e.g. [98, Chapter 10, Section 3]), where $[\![\tau]\!]^* = \{1, \ldots, \tau\}$. Now observe that

$$A\boldsymbol{w} = (A\boldsymbol{w}_1) \wedge \cdots \wedge (A\boldsymbol{w}_\tau) = AW(\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_\tau)$$

and that $(A\boldsymbol{w}_1 | \cdots | A\boldsymbol{w}_\tau) = AW$. Finally, the statement follows by the Cauchy-Binet formula (see e.g. [37, Cauchy-Binet Corollary, p. 214] or [98, Example 10.31]), i.e.

$$|AW|_{I,[\![\tau]\!]^*} = \sum_{J\in[\![n]\!]^\tau_<} |A|_{I,J}|W|_{J,[\![\tau]\!]^*}. \qquad \square$$

It follows that the component of $g_{\boldsymbol{t}}U^h_{\boldsymbol{f}}(\boldsymbol{x})\boldsymbol{w}$ corresponding to $\boldsymbol{e}_I$ is of the form

$$e^{t_I} \sum_{J\in[\![n]\!]^\tau_<} c_J \left|U^h_{\boldsymbol{f}}\right|_{I,J}$$

with $c_J \in \mathbb{Z}$ not all zero and independent from I. Therefore we have that

$$\left\|g_{\boldsymbol{t}}U^h_{\boldsymbol{f}}(\boldsymbol{x})\boldsymbol{w}\right\| \gg \left|\sum_{J\in[\![n]\!]^\tau_<} c_J \left|U^h_{\boldsymbol{f}}\right|_{I,J}\right|$$

as long as e^{t_I} is bounded below, which we know from Remark 5.8.2 to be guaranteed by condition (5.14) of Theorem 5.2.8. We can now prove that for every $1 \le \tau \le n+1$ the norm of $g_{\boldsymbol{t}}U^h_{\boldsymbol{f}}(\boldsymbol{x})\boldsymbol{w}$ is bounded below uniformly in $w \in \mathcal{W}_\tau$.

This is straightforward for $\tau = n+1$, since in that case

$$\left\|g_{\boldsymbol{t}}U^h_{\boldsymbol{f}}(\boldsymbol{x})\boldsymbol{w}\right\| = \left|c_{[\![n]\!]} \det U^h_{\boldsymbol{f}}\right| \ge \left|\det U^h_{\boldsymbol{f}}\right| > 0$$

by remark 5.2.6. When $\tau \le n$, on the other hand, condition (5.13) of Theorem 5.2.8 guarantees that

$$t_{m+1} = \sum_{i=0}^{m} t_i - \sum_{i=m+2}^{n} t_i$$

is bounded below, hence we may always find an index set $I \in [\![n]\!]^\tau_<$ such that $m+1 \notin I$ and e^{t_I} is bounded below. But then

$$\mathcal{G}_I\left(U^h_{\boldsymbol{f}}\right) = \mathcal{G}_I(M_{\boldsymbol{f}}) = \mathcal{G}_{\tilde{I}}(M_{\boldsymbol{f}})$$

for some $\tilde{I} \in [\![m]\!]^{\tilde{\tau}}_<$ and $1 \le \tilde{\tau} \le m+1$. Therefore it is enough to check that for every such $\tilde{I}$ and $\tilde{\tau}$

$$\left\|\boldsymbol{c}\cdot\mathcal{G}_I\left(M_{\boldsymbol{f}(\boldsymbol{x})}\right)\right\|_{\nu,B} = \sup_{\boldsymbol{x}\in B\cap\operatorname{supp}\nu}\left|\boldsymbol{c}\cdot\mathcal{G}_I\left(M_{\boldsymbol{f}(\boldsymbol{x})}\right)\right| \gg 1 \tag{5.40}$$

uniformly in non-zero integer vectors $\boldsymbol{c}$. By Lemma 5.8.7, this can be guaranteed by requiring that $\boldsymbol{f}_I$ is non-symmetric of degree $n+1-\tau$, because when $\mathrm{V}(\boldsymbol{f})$ is bounded we have

$$\left|\boldsymbol{c}\cdot\mathcal{G}_I\Big(M_{\boldsymbol{f}(\boldsymbol{x})}\Big)\right| \gg |\boldsymbol{c}\cdot\mathcal{S}_{n,\tau}(\boldsymbol{f}_I(\boldsymbol{x}))| \tag{5.41}$$

and from Proposition 5.3.4 we know that the components of $\mathcal{S}_{n,\tau}(\boldsymbol{T}) = \mathcal{S}_\tau^{n+1-\tau}(\boldsymbol{T})$ form a basis for the module of symmetric polynomials in τ variables and degree bounded by $n+1-\tau$.

Lemma 5.8.6. *Given a continuous map $\boldsymbol{g} = (g_0,\dots,g_r)\colon \mathscr{B}\to\mathbb{R}^r$, let $\tilde{\boldsymbol{g}} = (\tilde{g}_0,\dots,\tilde{g}_{\tilde{r}})$ be a basis for the linear span $\langle g_0,\dots,g_r\rangle_{\mathbb{R}}$ and let R be the real matrix such that $\tilde{\boldsymbol{g}}R = \boldsymbol{g}$. Then $\ker(R)\cap\mathbb{Q}^{r+1} = \{\boldsymbol{0}\}$ if and only if $g_0,\dots,g_r$ are linearly independent over $\mathbb{Q}$.*

Proof. The components of $\boldsymbol{g}$ are linearly dependent over $\mathbb{Q}$ if and only if there is a non-zero $\boldsymbol{q}\in\mathbb{Q}^{r+1}$ such that

$$0 = \boldsymbol{g}\cdot\boldsymbol{q} = \tilde{\boldsymbol{g}}R\boldsymbol{q},$$

but then it must be that $\boldsymbol{q}\in\ker(R)$, since by hypothesis the components of $\tilde{\boldsymbol{g}}$ are linearly independent over $\mathbb{R}$. □

Lemma 5.8.7. *Let $\boldsymbol{g} = (g_1,\dots,g_r)\colon\mathscr{B}\to\mathbb{R}^r$ be a continuous map with components linearly independent over $\mathbb{Q}$. Then there is a $\rho>0$ such that $\|g\|_{\nu,B}\geq\rho$ for every integer linear combination g of the components of $\boldsymbol{g}$.*

Proof. Let $\tilde{\boldsymbol{g}}$ be a basis for the linear span $\langle g_0,\dots,g_r\rangle_{\mathbb{R}}$. Further, let $\mathbb{S}^r$ be the unit sphere in $\mathbb{R}^{r+1}$ and note that if $\boldsymbol{b}\in\mathbb{Z}^r\setminus\{0\}$, then $\tilde{\boldsymbol{b}} := \frac{\boldsymbol{b}}{\|\boldsymbol{b}\|}\in\mathbb{S}^r\cap\mathbb{Q}^{r+1}$. Therefore

$$\min_{\boldsymbol{b}\in\mathbb{Z}^r\setminus\{0\}}\|\boldsymbol{g}\cdot\boldsymbol{b}\|_{\nu,B} \geq \min_{\tilde{\boldsymbol{b}}\in\mathbb{S}^r}\|\boldsymbol{g}\cdot\tilde{\boldsymbol{b}}\|_{\nu,B} = \min_{\tilde{\boldsymbol{b}}\in\mathbb{S}^r}\|\tilde{\boldsymbol{g}}\cdot(R\tilde{\boldsymbol{b}})\|_{\nu,B} =: \rho,$$

which is well defined since $\tilde{\boldsymbol{g}}R\tilde{\boldsymbol{b}}$ is continuous in $\tilde{\boldsymbol{b}}$ and $\mathbb{S}^r$ is compact. Finally, Lemma 5.8.6 implies that $\rho>0$. □

Having shown that $\eta(\boldsymbol{x}) = g_{\boldsymbol{t}} U_{\boldsymbol{f}}^h(\boldsymbol{x})$ satisfies condition 2 of Theorem 5.8.4, we note that (5.41) implies that η satisfies condition 1 as well. Indeed, write ϖ for $\left\| g_{\boldsymbol{t}} U_{\boldsymbol{f}}^h(\boldsymbol{x})\boldsymbol{w} \right\|_{\nu,B}$ and ϖ_I for the component of $g_{\boldsymbol{t}} U_{\boldsymbol{f}}^h(\boldsymbol{x})\boldsymbol{w}$ corresponding to $\boldsymbol{e}_I$. Since B is bounded and ϖ, ϖ_I are continuous, (5.40) implies that $\varpi \leq c\varpi_I$ on $B \cap \operatorname{supp}\nu$ for some $c > 0$. Furthermore, (5.41) shows that ϖ_I is (C,α)-good on B with respect to ν, since $(\mathcal{S}(\boldsymbol{f}_I), \nu)$ is (C,α)-good by hypothesis. Therefore by Lemma 3 we have that ϖ is $(c^\alpha C, \alpha)$-good on B with respect to ν.

This concludes the proof of Theorem 5.8.1.

Proof of Corollary 5.2.10. Note that the first part is just a special case of Corollary 5.4.6. Then for each integer $k > 1$ apply Theorem 5.2.8 with $\theta_k = 1 - \frac{1}{k}$, resulting in a sequence of subsets $\mathscr{B}_k \subset \mathscr{B}$ with $\nu(\mathscr{B}_k) > \theta_k \nu(\mathscr{B})$ and such that

$$\nu\Big(\mathcal{D}_{\boldsymbol{f}}^n(Q, \mathscr{B}_k)\Big) \ll_k (\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q))^{\frac{\alpha}{n+1}}\, \nu(\mathscr{B}_k)$$

for Q large enough, where the implied constant is independent of Q. Therefore by condition (5.16) we have

$$\sum_{Q=1}^{\infty} \nu\Big(\mathcal{D}_{\boldsymbol{f}}^n(Q, \mathscr{B}_k)\Big) \ll \nu(\mathscr{B}_k) \sum_{Q=1}^{\infty} \Big(\boldsymbol{\psi}(Q)\boldsymbol{\varphi}(Q)\Big)^{\frac{\alpha}{n+1}} < \infty$$

and by the Borel-Cantelli Lemma this implies that $\nu\Big(\mathcal{D}_{\boldsymbol{f}}^n(\mathscr{B}_k)\Big) = 0$.

Now observe that for every $k, Q > 1$ we have $\mathcal{D}_{\boldsymbol{f}}^n(Q, \mathscr{B}_k) \subseteq \mathcal{D}_{\boldsymbol{f}}^n(Q, \mathscr{B})$, hence $\mathcal{D}_{\boldsymbol{f}}^n(\mathscr{B}_k) \subseteq \mathcal{D}_{\boldsymbol{f}}^n(\mathscr{B})$. Thus

$$\begin{aligned}
\nu\Big(\mathcal{D}_{\boldsymbol{f}}^n(\mathscr{B})\Big) &\leq \nu\Big(\mathcal{D}_{\boldsymbol{f}}^n(\mathscr{B}) \setminus \mathcal{D}_{\boldsymbol{f}}^n(\mathscr{B}_k)\Big) + \nu\Big(\mathcal{D}_{\boldsymbol{f}}^n(\mathscr{B}_k)\Big) \\
&= \nu\Big(\mathcal{D}_{\boldsymbol{f}}^n(\mathscr{B}) \setminus \mathcal{D}_{\boldsymbol{f}}^n(\mathscr{B}_k)\Big) \\
&\leq \nu(\mathscr{B} \setminus \mathscr{B}_k) \\
&= \nu(\mathscr{B}) - \nu(\mathscr{B}_k) \\
&\leq \frac{1}{k}\nu(\mathscr{B}) \to 0
\end{aligned}$$

as $k \to \infty$, and we conclude that $\nu\Big(\mathcal{D}_{\boldsymbol{f}}^n(\mathscr{B})\Big) = 0$, as required. $\square$

5.9 FINAL REMARKS

There is a notable gap between the hypotheses of Theorem 5.1.2 and those of Theorem 5.2.13. For example, when $\boldsymbol{f}$ is a polynomial map our theorem only applies to at most finitely many values of n. It would therefore be interesting to explore the limit of the techniques presented in this chapter, and a possible approach would be to adapt the work of Aka, Breuillard, Rosenzweig, and de Saxcé [2] to determine the precise obstruction to the applicability of Theorem 5.8.4 to the present problem.

We also note that Theorem 5.2.8 suggests that the volume of the approximation targets plays a greater role than the length of their sides in determining whether a certain rate of approximation is achievable or not. In other words, we conjecture the following improvement of Proposition 5.7.3 for the set $\mathscr{L}^*_{n,\boldsymbol{f}}(\psi_0,\dots,\psi_m)$ of points $\boldsymbol{x} \in \mathscr{B}$ such that $\boldsymbol{f}(\boldsymbol{x}) \in \mathscr{L}^*_{n,m+1}(\psi_0,\dots,\psi_m)$, where the latter is the set of $\boldsymbol{x} \in \mathbb{R}^{m+1}$ such that

$$|x_k - \alpha_k| < \frac{\psi_k(\mathrm{H}(\boldsymbol{\alpha}))}{\mathrm{H}(\boldsymbol{\alpha})}$$

for infinitely many $\boldsymbol{\alpha} \in \mathbb{A}_n^{m+1}$.

Conjecture 5.9.1. *Let $\psi_0,\dots\psi_m\colon \mathbb{R}^+ \to \mathbb{R}^+$ be decreasing functions such that $\psi_i \in O(\psi_j)$ for every $0 \le i < d$ and $d \le j \le m$, and suppose that there is a $\kappa > 0$ such that*

$$\kappa^{n-m+d} > \lim_{t\to\infty} \frac{\psi_d(\kappa^t)\cdots\psi_m(\kappa^t)}{\psi_d(\kappa^{t+1})\cdots\psi_m(\kappa^{t+1})}.$$

Further, let g be a dimension function such that $r^{-d}g(r)$ is non-increasing, and assume that $\boldsymbol{f}$ is Lipschitz continuous, that $\mathrm{V}(\boldsymbol{f}) \neq 0$, and that $\boldsymbol{f}$ satisfies condition (5.15) on $\mathscr{B}$. Then

$$\mathcal{H}^g\Big(\mathscr{L}^*_{n,\boldsymbol{f}}(\psi_0,\dots,\psi_m)\Big) = \begin{cases} 0 & \text{if } S^g_{n,d}(\psi_0,\dots,\psi_m) < \infty \\ \mathcal{H}^g(\mathscr{B}) & \text{if } S^g_{n,d}(\psi_0,\dots,\psi_m) = \infty \end{cases}$$

where

$$S^g_{n,d}(\psi_0,\dots,\psi_m) \coloneqq \sum_{Q=1}^{\infty} Q^n \frac{\psi_d(Q)\cdots\psi_m(Q)}{Q^{m+1-d}}\, g\left(\frac{\psi_0(Q)\cdots\psi_{d-1}(Q)}{Q^d}\right).$$

Furthermore, observe that a version of [46, Lemma 4.6] for flows $\mathscr{B} \to \mathrm{GL}_{n+1}(\mathbb{R})$ would allow us to extend Theorem 5.2.8 to more general measures. In this spirit and motivated by [69, 106, 94], as well as recent work by Khalil and Luethi, we propose the following:

Conjecture 5.9.2. *Let $\boldsymbol{f}\colon \mathscr{B} \subseteq \mathbb{R}^d \to \mathbb{R}^{m+1}$ be a continuous map, and let ν be a measure on $\mathscr{B}$ such that $(\mathcal{S}^{n-d} \circ \boldsymbol{f})_*\nu$ is Federer, decaying, and rationally non-planar. Also let $\psi\colon \mathbb{R}^+ \to \mathbb{R}^+$ be a decreasing function. Then for any ball $B \subseteq \mathscr{B}$*

$$\nu\Big(\mathscr{L}^*_{n,\boldsymbol{f}}(\psi) \cap B\Big) = \nu(B) \quad \textit{if} \quad \sum_{Q=1}^{\infty} Q^{n-m-1}\psi^{m+1}(Q) = \infty.$$

— A —

Hausdorff measures and dimension

Most of the following discussion follows [84, Chapter 4], and we invite the interested reader to consult this book for more information, as well as [53, Chapter 3] and [96].

To begin with, a function $g\colon \mathbb{R}^+ \to \mathbb{R}^+$ is said to be a *dimension function* if it is increasing and $g(r) \to 0$ as $r \to 0$. Then consider a metric space X and a non-empty subset $F \subset X$. For any fixed $\rho > 0$, a collection $\{B_i\}$ of balls in X is said to be a *ρ-cover* of F if the radii $r(B_i)$ are bounded above by ρ and if F is contained in the union of $\{B_i\}$. We also define

$$\mathcal{H}^g_\rho(F) \coloneqq \inf\Big\{ \sum_i g(r(B_i)) : \{B_i\} \text{ is a } \rho\text{-cover of } F \Big\}.$$

Then the *(generalised) Hausdorff g-measure* of F is the limit

$$\mathcal{H}^g(F) \coloneqq \lim_{\rho \to 0} \mathcal{H}^g_\rho(F) = \sup_{\rho > 0} \mathcal{H}^g_\rho(F).$$

The more classical Hausdorff s-measures $\mathcal{H}^s$ can then be obtained with $g(r) = r^s$ for $s > 0$.

Theorem A.0.1 ([96, Theorem 27])**.** *The measure $\mathcal{H}^g$ is Borel regular.*

Note that $\mathcal{H}^g$ is not Radon because in general it is not locally finite. However, by Theorem A.0.1 we see that the restriction of $\mathcal{H}^g$ to a set of finite g-measure is a Radon measure.

Proposition A.0.2 ([96, Theorem 40])**.** *Let g, h be dimension functions such that $h \in o(g)$ for $r \to 0$. Then $\mathcal{H}^h(F) = 0$ whenever $\mathcal{H}^g(F) < \infty$.*

In particular, this means that for $t > s > 0$ we have that $\mathcal{H}^t(F) = 0$ if $\mathcal{H}^s(F) < \infty$. Therefore the following notion of *Hausdorff dimension* is well defined:

$$\dim_{\mathcal{H}}(F) \coloneqq \inf\{s > 0 : \mathcal{H}^s(F) = 0\}.$$

Remark A.0.3. When $n > 0$ is an integer, $\mathcal{H}^n$ is a constant multiple of Lebesgue measure. In particular, this means that Hausdorff dimension generalises the usual naive notion of dimension.

Remark A.0.4. While the s-measures are usually sufficient for most applications, the g-measures allow for a finer description of the geometry of a set. For example, it can be shown that almost surely a Brownian path in $\mathbb{R}^3$ has Hausdorff dimension 2 and 2-measure 0; however, it also has positive and finite g-measure with $g(r) = r^2 \log\log(1/t)$. For more details, see [53, Section 3.6] and references within.

— B —

Exterior products

Exterior products have applications throughout mathematics, most notably in Differential Geometry, where they are used as the basis for a theory of integration. However, here we are mainly interested in their connection with the *Grassmannian manifolds*, i.e. the moduli spaces $\mathrm{Gr}(k, V)$ of k-dimensional linear subspaces of a vector space V. Our exposition will follow [98, Chapter 10] and [77, Chapter XIX].

While exterior products can be defined in the general context of moduli over a commutative ring, here we will only be concerned with the simpler case of vector spaces. So let $V_1, \ldots, V_k, W$ be vector spaces over a field $\mathbb{K}$, and recall that a map

$$F\colon V_1 \times \cdots \times V_k \to W$$

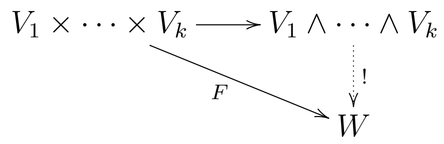
is said to be *multilinear* if it is linear in all of its components, i.e. if

$$\begin{aligned} &F(v_1, \ldots, \alpha v_{i,1} + \beta v_{i,2}, \ldots, v_k) \\ &\qquad = \alpha F(v_1, \ldots, v_{i,1}, \ldots, v_k) + \beta F(v_1, \ldots, v_{i,2}, \ldots, v_k) \end{aligned}$$

for every $i \in \{1, \ldots, n\}$, $\alpha, \beta \in \mathbb{K}$, $v_{i,1}, v_{i,2} \in V_i$ and $v_j \in V_j$ $(j \neq i)$. Furthermore, F is said to be *alternating* if

$$F(v_1, \ldots, v_k) = 0 \text{ whenever } v_i = v_j \text{ (and } i \neq j).$$

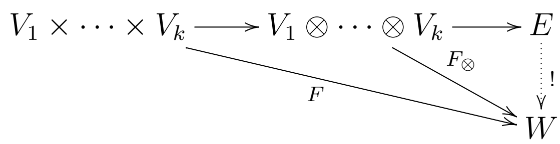
The *exterior product* of $V_1, \ldots, V_k$, denoted by $V_1 \wedge \cdots \wedge V_k$, is the vector space characterised by the following universal property: for every alternating multilinear map $F\colon V_1 \times \cdots \times V_k \to W$, there is a unique linear map $V_1 \wedge \cdots \wedge V_k \to W$ through which F factors; with a slight abuse of notation, in what follows we will also denote this map by F. Described as a commutative diagram:



Observe that a realisation of $V_1 \wedge \cdots \wedge V_k$ can be readily obtained as a quotient of the tensor product $V_1 \otimes \cdots \otimes V_k$. Indeed, let $\mathfrak{a}$ be the ideal of $V_1 \otimes \cdots \otimes V_k$ generated by the tensors of the form

$$v_1 \otimes \cdots \otimes v_k \text{ such that } v_i = v_j \text{ for some } i \neq j,$$

and let E be the quotient $(V_1 \otimes \cdots \otimes V_k)/\mathfrak{a}$. Now recall that, given a multilinear map $F\colon V_1 \times \cdots \times V_k \to W$, by the universal property of the tensor product we have a unique map $F_{\otimes}\colon V_1 \otimes \cdots \otimes V_k \to W$ through which F factors. Clearly $\mathfrak{a}$ lies in the kernel of any alternating map $F_{\otimes}$, thus $F_{\otimes}$ factors through the quotient and we obtain a commutative diagram



Therefore E satisfies the universal property of the exterior product and $E \simeq V_1 \wedge \cdots \wedge V_k$. A more elementary construction traditional in Differential Geometry can be found in [35].

Remark B.0.1. Let V be a vector space. Then for every $v_1, v_2 \in V$ we have

$$v_2 \wedge v_1 = -v_1 \wedge v_2$$

in $\bigwedge^2 V$, which can be readily deduced from $(v_1 + v_2) \wedge (v_1 + v_2) = 0$. More in general, for every $v_1, \ldots, v_k \in V$ and for every permutation σ on $\{1, \ldots, k\}$ we have

$$v_{\sigma(1)} \wedge \cdots \wedge v_{\sigma(k)} = \operatorname{sgn}(\sigma) v_1 \wedge \cdots \wedge v_k,$$

where $\operatorname{sgn}(\sigma)$ denotes the *signum* of σ, i.e. $(-1)^{\#\sigma}$, with $\#\sigma$ the number of transpositions in the decomposition of σ.

For the remainder of this section we will only consider the case where $V_1, \ldots, V_k$ all coincide with the same n-dimensional vector space V. We will also denote by $[\![n]\!]$ the set of integers $\{1, \ldots, n\}$, and by $[\![n]\!]^k_<$ the set of $(i_1, \ldots, i_k) \in [\![n]\!]^k$ such that $i_1 < \ldots < i_k$.

Proposition B.0.2 ([77, Proposition XIX.1.1])**.** *Let V be an n-dimensional vector space and let $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ be a basis. Then $\bigwedge^k V = \{0\}$ if $k > n$, while if $1 \leq k \leq n$ the elements*

$$\boldsymbol{e}_I \coloneqq \boldsymbol{e}_{i_1} \wedge \cdots \wedge \boldsymbol{e}_{i_k} \quad I \in [\![n]\!]^k_<$$

form a basis of $\bigwedge^k V$. In particular

$$\dim \bigwedge^k V = \binom{n}{k}.$$

Now observe that every linear transformation $T\colon V \to W$ naturally induces a map $\bigwedge^k V \to \bigwedge^k W$, which with a slight abuse of notation we will still denote by T, given by

$$T(\boldsymbol{v}_1 \wedge \ldots \wedge \boldsymbol{v}_k) \mapsto T\boldsymbol{v}_1 \wedge \ldots \wedge T\boldsymbol{v}_k$$

Example B.0.3. The exterior product can be used to characterise the determinant of a linear transformation $T\colon V \to W$ in a coordinate-free manner. Indeed, since $\bigwedge^n V$ is 1-dimensional by Proposition B.0.2, the map induced on it by T can only amount to the multiplication by a scalar, which turns out to be precisely $\det(T)$. In other words, if $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ is a basis of V, then

$$T(\boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n) = \det(T)\, \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n.$$

This example is directly related to the following, more general fact, whose proof can be found for example in [98, Chapter 10, Section 3].

Proposition B.0.4. *Let V be an n-dimensional vector space, fix $1 \leq k \leq n$, and let $\boldsymbol{v}_1, \ldots, \boldsymbol{v}_k \in V$. Further, let M be the $n \times k$ matrix $(\boldsymbol{v}_1 | \cdots | \boldsymbol{v}_k)$, and for every $I \in [\![n]\!]^k_<$ let M_I be the $k \times k$ submatrix formed by the lines $i_1, \ldots, i_k$ of M. Then the component of $\boldsymbol{v}_1 \wedge \cdots \wedge \boldsymbol{v}_k$ corresponding to $\boldsymbol{e}_I$ is* $\det(M_I)$.

Finally, we are now able to highlight the connection between the exterior product $\bigwedge^k V$ and the Grassmannian $\mathrm{Gr}(k, V)$, as promised. The simplest example is given by the hyperplanes of V, i.e. by the case $k = n - 1$: a hyperplane H can be described with a single equation of the form

$$\boldsymbol{h} \cdot \boldsymbol{x} = 0$$

for some $\boldsymbol{h} \in V \simeq \bigwedge^{n-1} V$. Since $\boldsymbol{h}$ is uniquely defined up to a constant, there is a bijection between the hyperplanes of V and the points of the projective space $\mathbb{P}\left(\bigwedge^{n-1} V\right)$.

Remark B.0.5. The reason we chose $\bigwedge^{n-1} V$ here is because (after fixing a basis $\{\boldsymbol{e}_1, \ldots, \boldsymbol{e}_n\}$ of V) there is an isomorphism between $\bigwedge^{n-1} V$ and the dual space V^* which assigns to $\boldsymbol{z} \in \bigwedge^{n-1} V$ the function $f \colon V \to \mathbb{K}$ such that

$$\boldsymbol{z} \wedge \boldsymbol{x} = f(\boldsymbol{x}) \boldsymbol{e}_1 \wedge \cdots \wedge \boldsymbol{e}_n$$

for every $\boldsymbol{x} \in V$. We then obtain an isomorphism $\bigwedge^{n-1} V \simeq V$ by composing this with the canonical isomorphism that assigns to each $f \in V^*$ an $\boldsymbol{h} \in V$ such that $f(\boldsymbol{x}) = \boldsymbol{h} \cdot \boldsymbol{x}$ (see [98, Section 10.5]).

Now let $W \subset V$ be a k-dimensional subspace, that is, an element of $\mathrm{Gr}(k, V)$. Further, let $\{\boldsymbol{w}_1, \ldots, \boldsymbol{w}_k\}$ be a basis of W, and let $M^W = (\boldsymbol{w}_1 | \cdots | \boldsymbol{w}_k)$. It can then be readily seen that the numbers

$$\det\left(M^W_I\right), \quad I \in [\![n]\!]^k_<,$$

called the *Plücker coordinates* of W, uniquely determine W up to a non-zero constant [98, Theorem 10.2], since M^W is uniquely determined up to multiplication by an invertible matrix. Therefore by Proposition B.0.4 we see that W corresponds a point $[\boldsymbol{w}] \in \mathbb{P}\left(\bigwedge^k V\right)$. In fact, the map

$$\mathcal{G}\colon W \mapsto [\boldsymbol{w}] \coloneqq [\boldsymbol{w}_1 \wedge \cdots \wedge \boldsymbol{w}_k]$$

is injective, and is called the *Plücker embedding*.

Example B.0.6. It is readily seen that set of projective lines in $\mathbb{P}^3(\mathbb{R})$ coincides with $\mathrm{Gr}(2, \mathbb{R}^4)$, which can be identified with the quadric in $\mathbb{P}^5(\mathbb{R})$ with equation

$$w_{12}w_{34} - w_{13}w_{24} + w_{14}w_{23} = 0,$$

where w_{ij} is the coordinate of $\boldsymbol{w}_1 \wedge \boldsymbol{w}_2$ corresponding to $\boldsymbol{e}_i \wedge \boldsymbol{e}_j$ (see [98, Example 10.3]).

Symbols

$\langle\!\langle \cdot \rangle\!\rangle$	Distance from the nearest integer
$\mathbb{N}$	The set of positive integers, also denoted by $\mathbb{Z}_{>0}$
$\lvert \cdot \rvert$	Lebesgue measure
$\mathcal{H}^s$	Hausdorff s-measure
$\dim_{\mathcal{H}}$	Hausdorff dimension
$\lVert\cdot\rVert$	The supremum norm, unless otherwise stated
$\lambda_1(\Gamma)$	Length of the shortest non-zero vector of a lattice Γ, with respect to $\lVert\cdot\rVert$

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