Algebraic models and rational global spectra

Luca Pol

A thesis submitted for the degree of
Doctor of Philosophy

School of Mathematics and Statistics

The University of Sheffield

November 6, 2020

Abstract

In the first part of the thesis we define and study free global spectra: global spectra with non-trivial geometric fixed points only at the trivial group. We show that free global spectra often do not exist, and when they do, their homotopy groups satisfy strong divisibility conditions.

The second part of the thesis is dedicated to the study of the algebraic model of rational global spectra for the family of finite groups as constructed in [**81**]. We study the homological algebra of this category with a particular focus on the tensor triangulated geometry of its derived category. Along the way we make contact with the theory of representation stability and show that some algebraic invariants coming from global homotopy theory exhibit such a stability phenomenon.

Finally, in the third part of the thesis we construct a symmetric monoidal algebraic model for the category of rational cofree G-spectra for all compact Lie groups G. The key ingredient in the proof is the Left Localization principle which gives mild hypotheses under which a Quillen adjunction between stable model categories descends to a Quillen equivalence between their left localizations. This last part is joint work with Jordan Williamson.

Acknowledgements

I would like to express my gratitude to my supervisors John Greenlees and Neil Strickland for their patience, help and guidance. It has been an absolute privilege to work under their supervision and to learn so much from them.

A big thanks to the members of the Sheffield Topology group for all the reading groups, seminars and gong-shows; I have learnt so much from all of you. Particular thanks are owed to Jordan Williamson for many enlightening conversations and suggestions, and to Dimitar Kodjabachev for his guidance during the first years of my PhD studies.

A huge thanks all my friends in Sheffield that have made the last four years a truly enjoyable experience.

Finally, I would like to thank my family and Natasha for their constant support and encouragement.

Contents

General introduction

The goal of algebraic topology is to use algebraic invariants to capture topological and geometric information. Prominent examples of such invariants are the Betti numbers, the Euler characteristic and cohomology theories such as cobordism and K-theory. As algebraic objects are more rigid than topological ones, we do not expect to be able to capture all topological information via algebraic techniques. For example, a celebrated result of Schwede tells us that the category of spectra cannot be equivalent to any algebraic category [**79**, 3.1]. Nonetheless, Quillen and Sullivan's work in rational homotopy theory showed that algebraic invariants can capture a large portion of the homotopy theory of interest [**72**, **89**]. Their work reduced the topological problem of calculating the homotopy type of rational spaces to a purely algebraic one which one hope to understand better.

Algebraic models. In topology we are often interested in studying the homotopy theory of a certain category $\mathcal{C}$ which has a topological flavour. For example, this could be the category of topological spaces or spectra. Informally, we say that the category $\mathcal{C}$ admits an *algebraic model* if there exists another category $\mathcal{A}$ which models the same homotopy theory but has a much more algebraic flavour. For example $\mathcal{A}$ could be the category of chain complexes over a ring or more generally an abelian category, and in this case the homotopy category of $\mathcal{C}$ would be equivalent to the derived category of $\mathcal{A}$. An algebraic model becomes a powerful tool if the category $\mathcal{A}$ is as small and simple as possible so that calculations can easily be performed. For example, one can construct pathological objects useful for counterexamples or prove the existence of topological objects satisfying specific properties by simply constructing the algebraic counterpart in the model [**35**]. Most importantly, the existence of an algebraic model provides a bridge between topology and algebra: we can translate topological problems to algebraic ones but also import algebraic constructions to topology. Creating a dictionary between topological and algebraic constructions is the first step needed to make the algebraic model a useful and powerful tool.

The main protagonists. In this thesis we will be concerned with the study of cohomology theories on spaces with an action of a compact Lie group G. We can organize these cohomology theories in the category of G-spectra and study their homotopy theory, which is called G-equivariant stable homotopy theory. It has been noticed since the beginning of equivariant homotopy theory that there are many cohomology theories that exist in a uniform way for all compact Lie groups in a specific class $\mathcal{F}$ rather than just for a particular group. These cohomology theories are called *global* as they satisfy extra compatibility conditions as the group varies. These are represented by $\mathcal{F}$-global spectra and their study is called global stable homotopy theory. Examples of global spectra are the sphere spectrum, cobordism, topological K-theory, Borel cohomology spectra and symmetric product spectra.

The global perspective in the equivariant homotoy theory can often be illuminating and lead to powerful applications. For example the global framework was a key ingredient in Hausmann's proof of the equivariant Quillen's theorem [**47**] and in Schwede's calculation of the zeroth equivariant homotopy groups of the symmetric product spectrum [**80**]. However, the global approach introduces significant new challenges as the machinery and techniques developed when studying equivariant phenomena for a single group often do not generalize to a family of groups, and so genuinely new ideas are required to tackle this. For example, in part 2 we will develop what we call “global representation theory” to describe the algebraic behaviour of topological invariants coming from rational global homotopy theory.

The rational case. We can think of the categories of G-spectra and $\mathcal{F}$-global spectra as having two degrees of complexity: a topological one, coming from the fact that we are studying cohomology theories on spaces, and an equivariant one, coming from the action of the groups. We can reduce the topological contribution and let the equivariant phenomena stand out by considering *rational* cohomology theories. After this simplification we might hope to study these categories by purely algebraic tools.

It has been conjectured by Greenlees that the category of rational G-spectra admits a small and calculable algebraic model for all compact Lie groups G. So far this programme has been successful for a large class of groups including all finite groups [**8**] and tori [**45**]. On the other hand not much is known about the category of rational global spectra as an algebraic model has only been constructed for families of finite groups [**81**, 4.5.29].

In order to explain the contribution of this thesis to the theory of algebraic models we first need to understand the general strategy behind the construction of an algebraic model.

An algebraic example. There are two main steps in the construction of an algebraic model. We first break up the category into smaller pieces and then reassemble them in a coherent way. The splitting and the assembly procedures can be performed in different ways giving different algebraic models for the same category. In this introduction we give an informal overview on two methods for constructing algebraic models: the *Zariski* and *adelic* method. Let us consider the example of the derived category of abelian groups $\mathcal{D}(\mathbb{Z})$. The ideas described here will turn out to be helpful for more complicated cases.

In the Zariski method we think geometrically, and observe that the category of abelian groups is equivalent to the category of quasi-coherent sheaves over the Zariski spectrum $\mathrm{Spec}(\mathbb{Z})$. This suggests that we can recover the derived category as a certain sheaf over $\mathrm{Spec}(\mathbb{Z})$ whose value at a maximal ideal $(p) \in \mathrm{Spec}(\mathbb{Z})$ is given by the subcategory of complexes which are supported exactly at (p), or equivalently by the subcategory of p-local complexes. The topology of $\mathrm{Spec}(\mathbb{Z})$ and the sheaf condition tell us how to patch these categories together. Even if this method often does not produce a small and calculable algebraic model, this geometric intuition can still be very helpful. In particular, the idea of looking at subcategories of objects supported at closed points can be extremely valuable as it gives an estimate on the complexity of the algebraic model.

On the other hand, the adelic method has a more algebraic flavour and often outputs a small model. In this method we note that the category of abelian groups is equivalent to the category of $\mathbb{Z}$-modules. We then break up the category of $\mathbb{Z}$-modules using the Hasse square

$$\begin{tikzcd}
\mathbb{Z} \arrow[r] \arrow[d] & \mathbb{Q} \arrow[d] \\
\prod_p \mathbb{Z}_p \arrow[r] & (\prod_p \mathbb{Z}_p) \otimes \mathbb{Q}
\end{tikzcd}$$

which is a pullback. This suggests that we can recover the derived category of abelian groups from the data of a rational complex and a p-adic complex for each prime number p. The gluing data is encoded in the adele ring $(\prod_p \mathbb{Z}_p) \otimes \mathbb{Q}$.

It is worth noticing that both methods rely on the knowledge of the Zariski spectrum $\mathrm{Spec}(\mathbb{Z})$ as this is used to break up the derived category into smaller pieces. Therefore one can hope to apply these methods whenever a similar formalism is available.

tt-geometry. In this thesis we are interested in topological categories $\mathcal{C}$ whose homotopy categories $\mathrm{h}\mathcal{C}$ admit the structures of symmetric monoidal triangulated categories. This makes

$\mathrm{h}\mathcal{C}$ into a tensor triangulated category, which is a vast generalization of the structure present in the derived category of a ring. In fact most of the algebraic constructions and notions available in the derived category of a ring extends to this setting. For example, there is a space of prime ideals $\mathrm{spc}(\mathcal{C})$ which categorifies the Zariski spectrum of a ring [**6**]. This space is called the *Balmer spectrum* and it is related to the usual Zariski spectrum of a ring R via a natural homeomorphism $\mathrm{spc}(\mathcal{D}(R)) \simeq \mathrm{Spec}(R)$. The Balmer spectrum comes equipped with a well-defined notion of support for objects of $\mathcal{C}$ extending the usual notion of support in $\mathcal{D}(R)$. This allows us to think geometrically even in the setting of triangulated categories. Moreover for any Balmer prime $\wp \in \mathrm{spc}(\mathcal{C})$, there are functors $L_\wp, \Gamma_\wp$ and $\Lambda_\wp$ modelling the usual algebraic functors of localization, derived torsion and derived completion respectively [**5**]. For example, $\Gamma_\wp M$ and $L_\wp M$ are the universal approximation to M which are supported at and away from (the closure of) $\wp$ respectively. We note that this is the type of formalism needed to extend the two methods (Zariski and adelic) to our topological categories of interest.

The general strategy. Consider a tensor triangulated category $\mathcal{C}$ and suppose that we have a description of its Balmer spectrum $\mathrm{spc}(\mathcal{C})$. Our goal is to construct an algebraic model for $\mathcal{C}$ and so we can follow the same strategy outlined in the previous example.

The starting point of the Zariski method is to consider the full subcategory $\mathrm{h}\mathcal{C}_\mathfrak{m} \leq \mathrm{h}\mathcal{C}$ of objects supported exactly at a closed point $\mathfrak{m} \in \mathrm{spc}(\mathcal{C})$. As the support of these objects is just a point, we expect this subcategory to be easier to understand than $\mathrm{h}\mathcal{C}$. Therefore our first step is to construct an algebraic model for $\mathrm{h}\mathcal{C}_\mathfrak{m}$. In the special case that the Balmer spectrum is discrete, then there is no assembly process and the algebraic model of $\mathcal{C}$ splits as a product of the algebraic model of $\mathrm{h}\mathcal{C}_\mathfrak{m}$ for $\mathfrak{m} \in \mathrm{spc}(\mathcal{C})$. In general however, we need to take into account the topology of the Balmer spectrum which usually introduces a lot of relations.

The adelic method instead starts by considering the unit object $\mathbb{1} \in \mathcal{C}$. If the Balmer spectrum $\mathrm{spc}(\mathcal{C})$ is finite dimensional, we can break up the unit object into simpler pieces using a generalization of the Hasse square, called the *adelic approximation cube* [**5**, 8.1]. The corners of this cube consist of products of objects of the form $L_\wp \Lambda_\wp \mathbb{1}$ for $\wp \in \mathrm{spc}(\mathcal{C})$. Therefore our first step is to study the category of $L_\wp \Lambda_\wp \mathbb{1}$-modules and then use the adelic cube to recover $\mathcal{C}$.

Once again we note that both methods rely on a good description of the Balmer spectrum of $\mathcal{C}$. Now that we have highlighted the general strategy we can discuss the content of this thesis.

1. Organization of the thesis

This thesis consists of three parts each of which has its own detailed introduction where the main objects and results are discussed. In this section we give an informal overview of the thesis where we focus on explaining how each part fits in the general picture of the theory of algebraic models.

Fix a family of compact Lie groups $\mathcal{F}$ and consider a compact Lie group G. In this section we will write $\mathrm{Sp}_\mathcal{F}$ for the category of rational $\mathcal{F}$-global spectra, and Sp_G for the category of rational G-spectra. We will not need to know much about these categories, only that their homotopy categories are tensor triangulated so they fit into the framework described in the previous sections.

Part 1. In the first part of the thesis we are interested in the category $\mathrm{Sp}_\mathcal{F}$ of rational global spectra. We would like to study this category using the Zariski method therefore by looking at subcategories with small support. The main challenge that we are facing is that the Balmer spectrum of $\mathrm{hSp}_\mathcal{F}$ is not known. We can overcome this problem by using the restriction functor $\mathrm{Sp}_\mathcal{F} \to \mathrm{Sp}_G$ which is built in the framework of global homotopy theory, and the knowledge of the Balmer spectrum of $\mathrm{spc}(\mathrm{hSp}_G)$. Greenlees [**36**] identified the Balmer spectrum of hSp_G with the set of conjugacy classes of closed subgroups of G. This means that for any closed subgroup

$H \leq G$ there exists an associated Balmer prime $\wp_H \in \operatorname{spc}(\mathrm{hSp}_G)$, and that $\wp_H = \wp_K$ whenever H and K are conjugate in G.

In this part of the thesis we are interested in the Balmer prime $\wp_1$ associated to the trivial group. This is a special prime as it is always closed in $\operatorname{spc}(\mathrm{hSp}_G)$. More precisely, we are interested in the subcategory of G-spectra supported exactly at $\wp_1$ which coincides with the full subcategory of rational free G-spectra. These can also be described as the full subcategory of G-spectra X for which the canonical map $EG_+ \wedge X \to X$ is an equivalence. This a well-behaved class of equivariant spectra whose objects represents cohomology theories on free G-spaces. This category has been studied by Greenlees–Shipley who constructed an algebraic model in terms of torsion modules over the group cohomology ring [**43**]. This shows that the category of rational free G-spectra is well-understood and that it admits a simple algebraic model.

Inspired by the equivariant story we would like to proceed in a similar way for the category of rational global spectra. By pulling back the Balmer primes $\wp_1$ along the restriction functor $\mathrm{Sp}_{\mathcal{F}} \to \mathrm{Sp}_G$, we can construct a Balmer prime $\wp_1^{gl} \in \operatorname{spc}(\mathrm{hSp}_{\mathcal{F}})$. We are interested in the subcategory of global spectra supported exactly at $\wp_1^{gl}$. We call a global spectrum in this subcategory *free* since its image under the restriction functor $\mathrm{Sp}_{\mathcal{F}} \to \mathrm{Sp}_G$ is a rational free G-spectrum for all $G \in \mathcal{F}$. We hoped that the category of free global spectra would give us an insight on the algebraic model for rational global spectra. However the following result shows that the global world is drastically different from the equivariant one.

Theorem. *Let $\mathcal{F}$ be a global family containing an infinite group. Then all free $\mathcal{F}$-global spectra are contractible.*

From the Balmer spectrum point of view, this result is a consequence of the fact that the Balmer prime $\wp_1^{gl}$ is not closed in $\operatorname{spc}(\mathrm{hSp}_{\mathcal{F}})$. This is a new global phenomenon which suggests that we cannot reduce the problem of constructing an algebraic model for the category of global spectra to the equivariant case. Instead, we should study the global stable homotopy category in its own right starting from its Balmer spectrum. This bring us to the second part of the thesis.

Part 2. The goal of the second part of the thesis is to study the Balmer spectrum of $\mathrm{Sp}_{\mathcal{F}}$ for a family of finite groups. The main tool available is the algebraic model for $\mathrm{Sp}_{\mathcal{F}}$ constructed in [**81**, 4.5.29]. For this introduction it is enough to know that there exists an abelian category $\mathcal{A}$ whose derived category is equivalent to $\mathrm{hSp}_{\mathcal{F}}$. Therefore we can reduce the problem of calculating the Balmer spectrum of $\mathrm{Sp}_{\mathcal{F}}$ to a purely algebraic one. This motivates the systematic study of the abelian category $\mathcal{A}$ in the second part.

The algebraic model for rational global spectra also has connections with representation theory as it provides a good framework for studying the representation theory of the outer automorphism groups. The abelian category $\mathcal{A}$ fits into the general framework of representation stability as introduced by Church–Farb [**24**]. The main difference with Church–Farb's work is that we are no longer considering a one-parameter family of representations but rather collections of representations which are indexed by a family of groups. This introduces a new level of complexity into the story that we aim to investigate in this thesis.

Part 2 is divided into two chapters. The goal of chapter 1 is to study the abelian category $\mathcal{A}$ via representation stability techniques. In particular, we show that under a certain noetherian condition, any finitely generated object of $\mathcal{A}$ satisfies a version of Church–Farb's representation stability. We also show that the geometric fixed points of any rational compact global spectrum assemble into a finitely generated object of $\mathcal{A}$ and so manifest such a stability phenomenon.

In chapter 2 we study the derived category of $\mathcal{A}$ and its Balmer spectrum. As before we can construct Balmer primes by pulling back along the restriction functor $\mathrm{hSp}_{\mathcal{F}} \to \mathrm{hSp}_G$. It follows that for any finite group G, the full subcategory of compact rational global spectra given by

$$\wp_G = \{X \mid \Phi_*^G(X) = 0\}$$

defines a prime ideal, which we call a *group prime*. Not all prime ideals in the rational global stable homotopy category are group primes. In fact we showed that the closure properties of the family of groups considered affects the general structure of the category.

THEOREM. *Fix p a prime number. Let $\mathcal{E}$ be the family of elementary abelian p-groups, and $\mathcal{C}$ be the family of cyclic p-groups.*

(a) *The Balmer spectrum of* $\mathrm{hSp}_{\mathcal{E}}$ *consists of the group primes and the zero ideal. A basis of open sets for the topology is given by the finite sets not containing zero. Furthermore, any thick ideal is finitely generated.*
(b) *The Balmer spectrum of* $\mathrm{hSp}_{\mathcal{C}}$ *consists of group primes and the ideal*

$$\wp_{tors} = \{X \mid \pi_*(\Phi^{C_{p^n}} X) = 0 \mathit{\ for\ } n \gg 0\}.$$

The topology can be described explicitly (we omit it here) and not all thick ideals are finitely generated.

Furthermore, all the ideals are radical so we deduce a complete classification of ideals of the rational global stable homotopy category for the two families above.

Part 3. The final part of the thesis is a revised version of a paper with Jordan Williamson [**68**]. The page and environment numbering have been modified so that they run concurrently throughout the thesis. The version in this thesis is not identical to the published version. We have fixed a few imprecisions throughout the paper and removed the section on the Adams spectral sequence as it depended on an incorrect bit of algebra. We believe that the spectral sequence does exist as stated in the published paper and we are preparing an account which fills this gap.

The goal of the paper is to construct an algebraic model for the category of rational cofree G-spectra. This is a well-behaved subcategory of G-spectra whose objects represent cohomology theories on free G-spaces, and it is equivalent to the category of free G-spectra mentioned earlier.

The interest in cofree G-spectra was motivated by the result of part 1 of this thesis, and by the observation that a well-behaved category of *cofree global spectra* does exist, see Remark 0.5.6. This realization suggested a new approach to the construction of an algebraic model for rational global spectra via the theory of cosupport rather than the usual theory of support. This started our quest for an algebraic model for cofree G-spectra and the study of various type of completions in algebra: adic completion, L-completion and derived completion.

Given a commutative ring R and an ideal I, the I-adic completion of an R-module M is a universal approximation to M built from I-powers torsion modules. The adic completion functor is neither left nor right exact, so the (left) derived functor L_n^I provides a good homological replacement. These derived functors are the local homology modules of Greenlees-May [**38**]. In particular, we can consider the zeroth left derived functor L_0^I and call a module *L-complete* if the natural map $M \to L_0^I M$ is an isomorphism. This gives an abelian category of L-complete modules which is also symmetric monoidal with respect to the L-complete tensor product of modules. For convenience we will only state the main result of this part for the connected case. The general case involves considering the action of the group of components on all rings and modules considered.

THEOREM. *Let G be a connected compact Lie group and I be the augmentation ideal of $H^*(BG;\mathbb{Q})$. Then there is a symmetric monoidal Quillen equivalence*

$$\text{rational cofree } G\text{-spectra} \simeq L\text{-complete dg-}H^*(BG;\mathbb{Q})\text{-mod}.$$

A key ingredient in the proof is the Left localization principle that we develop in chapter 3, which gives mild conditions under which a Quillen adjunction descends to a Quillen equivalence after left Bousfield localization.

Part 1

Free global spectra

Introduction

For a fixed compact Lie group G, we may consider spaces with a G-action and develop their homotopy theory. It is natural to consider algebraic invariants on them which keep track of the action of the group. Important examples of such invariants are equivariant cohomology theories such as equivariant K-theory and equivariant bordism. These cohomology theories are represented by G-spectra and their study is called G-equivariant stable homotopy theory.

It has been noticed since the beginning of equivariant homotopy theory that there exist numerous examples of cohomology theories which exist in a uniform way for all groups in a specific class rather than just for a particular group. The cohomology theories exhibiting this uniform behaviour are called *global*. These cohomology theories are represented by global spectra, and their study is called global stable homotopy theory.

There has been a lot of work towards finding a good framework for the study of global homotopy theory [**40**, Section 5], [**18**] and [**58**, Chapter II]. In this thesis we will use the framework developed by Schwede [**81**]. His approach has the advantage of being very concrete as the category of global spectra is the usual category of orthogonal spectra but with a finer notion of equivalence, called global equivalence. As any orthogonal spectrum is a global spectrum, this approach comes with a good range of examples. For instance, there are global analogues of the sphere spectrum, cobordism and K-theory spectra, Borel cohomology, symmetric product spectra and many others. The key insight is that an orthogonal spectrum gives rise to an orthogonal G-spectrum for all compact Lie groups G, and the fact that all these individual equivariant objects come from one orthogonal spectrum implicitly encodes strong compatibility conditions as the group G varies. We will recall the necessary background on global homotopy theory in section 4.

In this part of the thesis we are interested in two particular classes of spectra, that of free and cofree G-spectra which we will define in section 5. These classes are interesting for several reasons. Firstly, they represent (co)homology theories on free G-spaces, the most prominent example of which is Borel cohomology. Secondly, they are the simplest equivariant objects that one can consider as any equivariant map between them which is a non-equivariant equivalence is automatically an equivalence. Informally, we can say that these classes interpolate between stable homotopy theory and equivariant stable homotopy theory. Thirdly, the (co)free functors are used to define the homotopy fixed points and homotopy orbit functors whose associated spectral sequences provide powerful tools to calculate equivariant homotopy type. Finally, the techniques employed in studying the categories of rational (co)free G-spectra are instructive for more general cases in the theory of algebraic model for rational equivariant spectra.

For all these reasons we would like to study the analogue constructions of free and cofree spectra in global homotopy theory. Let us fix a global family $\mathcal{F}$, that is a collection of compact Lie groups which is closed under isomorphism, passage to subgroups and passage to quotients. We call a $\mathcal{F}$-global spectrum free if its underlying G-spectrum is free for all compact Lie groups $G \in \mathcal{F}$. In a similar way we define cofree $\mathcal{F}$-global spectra.

The category of cofree global spectra has already been considered by Schwede. He constructed a Borel global functor $b\colon \mathrm{hSp} \to \mathrm{hSp}_{\mathcal{G}}$ from the stable homotopy category to the global stable homotopy category, which is initial amongst the functors sending global spectra to cofree global spectra. Schwede [**81**, 4.5.6] also showed that the category of cofree global spectra coincides with the essential image of the Borel functor. In particular, cofree global spectra exist and they are easy to construct.

On the other hand the story for free global spectra is surprisingly different as our first result demonstrates, which appears in the body of the thesis as Proposition 0.5.15.

Theorem A. *Let $\mathcal{F}$ be a global family of compact Lie groups containing an infinite group. Then all free $\mathcal{F}$-global spectra are contractible.*

The proof of the result uses the fact that free G-spectra are modules over a complex orientable cohomology theory which has a theory of Euler classes. We will recall the necessary background in sections 3 and 5.

The situation is substantially different if we restrict to considering global families of finite groups. In that case Proposition 0.5.11 shows that free global spectra do exist and that their homotopy groups exhibit strong divisibility conditions. We explore the behaviour of free global spectra for a family of finite groups in subsection 5.1.

As an application of our work we obtain the following result, see Theorem 0.5.16.

Theorem B. *Let $\mathcal{F}$ be any non-trivial global family. Then there exists no $\mathcal{F}$-global spectrum X whose underlying G-equivariant homotopy type is equivalent to EG_+ for all $G \in \mathcal{F}$. More informally, the universal free G-spectrum EG_+ does not admit a global refinement.*

2. Preliminaries

Convention 0.2.1. Throughout we let G be a compact Lie group; many results work more generally for topological groups but we will not need this level of generality. We will only consider *closed* subgroups to retain the homeomorphism between orbits and homogenous spaces. By a G-representation we mean a finite-dimensional real inner product space equipped with a continuous G-action by linear isometries. We will refer to the category of weak Hausdorff k-spaces and continuous maps simply as the category of topological spaces and denote it by $\mathcal{T}$. We also let $G\mathcal{T}$ be the topological category of spaces with a left G-action and continuous G-maps. Finally, we write $G\mathcal{T}_*$ and $\mathcal{T}_*$ for the corresponding categories of based spaces.

Definition 0.2.2. A *complete G-universe* is a countably infinite dimension G-representation such that every finite dimensional G-representation embeds into it.

Example 0.2.3. The regular representation ρ_G of a finite group G contains all the irreducible representations. It follows that $\mathcal{U} = \bigoplus_{n=1}^{\infty} \rho_G$ is a complete G-universe.

Definition 0.2.4. A *family* is a collection of subgroups of G closed under passage to conjugates and subgroups. Given a family $\mathcal{F}$, there exists an unbased G-CW-complex $E\mathcal{F}$ which is characterized up to equivariant equivalence by

$$(E\mathcal{F})^H \simeq \begin{cases} \emptyset & \text{if } H \notin \mathcal{F} \\ * & \text{if } H \in \mathcal{F}. \end{cases}$$

The mapping cone of the unique non-constant map $E\mathcal{F}_+ \to S^0$ is denoted by $\widetilde{E}\mathcal{F}$. By construction,

$$(\widetilde{E}\mathcal{F})^H \simeq \begin{cases} S^0 & \text{if } H \notin \mathcal{F} \\ * & \text{if } H \in \mathcal{F} \end{cases}$$

and we have cofibre sequence $E\mathcal{F}_+ \to S^0 \to \widetilde{E}\mathcal{F}$ called *isotropy separation sequence.*

Example 0.2.5.

(a) Consider the family $A\ell\ell$ of all subgroups of G. Then we have $EA\ell\ell = *$ and $\widetilde{E}A\ell\ell = S^0$.

(b) Consider the family $\{1\}$ consisting only of the trivial group. Then we have that $E\{1\} = EG$ is a universal free G-space. If there exists a nonzero G-representation V such that $V^H = 0$ for all non-trivial subgroups $H \leq G$, then an explicit model for $\widetilde{E}G$ is given by $S^{\infty V} = \operatorname{colim}_n S^{nV}$.
(c) Consider the family $\mathcal{P}$ of proper subgroups of G and a complete G-universe $\mathcal{U}_G$. An explicit model for $\widetilde{E}\mathcal{P}$ is given by $S^{\infty V(G)} = \bigcup_V S^V$ where V runs over subrepresentations $V \leq \mathcal{U}_G$ with $V^G = 0$.

3. The equivariant stable homotopy category

We briefly introduce the category of orthogonal G-spectra, define the equivariant stable homotopy category and list some useful properties that it enjoys. This is a short and dense recollection of the material, so we refer to [**81**, Chapter 3] for more details.

Definition 0.3.1.

- Write $\mathbb{L}(V, W)$ for the space of linear isometric embeddings between inner product spaces V and W. If φ is a linear isometric embedding, then there is a bijection
$$O(W)/O(W - \varphi(V)) \simeq \mathbb{L}(V, W), \quad A \cdot O(W - \varphi(V)) \mapsto A \circ \varphi.$$
We topologize $\mathbb{L}(V, W)$ so that this bijection is a homeomorphism, and one checkes that this does not depend on φ.
- Write $\mathbb{O}$ for the based topological category whose objects are inner product spaces. The morphisms space $\mathbb{O}(V, W)$ is the Thom space, i.e. the one-point compactification of the total space of the bundle given by
$$\xi(V, W) = \{(\varphi, w) \in \mathbb{L}(V, W) \times W \mid w \perp \varphi(V)\} \to \mathbb{L}(V, W), \qquad (\varphi, w) \mapsto \varphi.$$
The space $\mathbb{O}(V, W)$ comes with a canonical basepoint. Composition in $\mathbb{O}$ is defined by applying the Thom space functor to the bundle map covering the composition in $\mathbb{L}$.
- An *orthogonal G-spectrum* is a based continuous functor $\mathbb{O} \to G\mathcal{T}_*$. A morphism of orthogonal G-spectra is a natural transformation.
- We write Sp^O_G for the category of orthogonal G-spectra and G-equivariant morphisms.

Convention 0.3.2. In the special case when G is the trivial group, we will just write Sp^O for the category of orthogonal G-spectra, and we will refer to it as the category of orthogonal spectra.

Remark 0.3.3. We think of an orthogonal G-spectrum X as consisting of a collection of based G-spaces $\{X(V)\}_V$ each of which has a compatible $O(V)$-action, and equivariant structure maps for each linear isometric embedding $\varphi : V \to W$ given by
$$\sigma_{V,W} : S^{W-\varphi(V)} \wedge X(V) \xrightarrow{\varphi_* \wedge \mathrm{id}} \mathbb{O}(V, W) \wedge X(V) \xrightarrow{X} X(W)$$
where $\varphi_*(w) = (\varphi, w)$.

Construction 0.3.4. Let A be a based G-space. We define an orthogonal G-spectrum via $(\Sigma^\infty A)(V) = S^V \wedge A$, and given a linear isometric embedding $\varphi \colon V \to W$, we let $\sigma_{V,W} : S^{W-\varphi(V)} \wedge S^V \wedge A \simeq S^W \wedge A$ be the obvious homeomorphism. This defines a functor $\Sigma^\infty \colon G\mathcal{T}_* \to \mathrm{Sp}^O_G$ called the suspension spectrum functor. We will often drop the symbol Σ^∞ and just write A for the associated G-spectrum.

Construction 0.3.5. Fix a complete G-universe $\mathcal{U}_G$ and write $s(\mathcal{U}_G)$ for the poset, under inclusion, of finite dimensional G-subrepresentations of $\mathcal{U}_G$. Define the G-equivariant homotopy groups of an orthogonal G-spectrum X by
$$\pi_k^G(X) = \begin{cases} \operatorname{colim}_{V \in s(\mathcal{U}_G)} \ [S^{k+V}, X(V)]^G_{top} & \text{for } k \geq 0 \\ \operatorname{colim}_{V \in s(\mathcal{U}_G)} \ [S^{k+V}, X(\mathbb{R}^{-k} \oplus V)]^G_{top} & \text{for } k \leq 0 \end{cases}$$

where the connecting maps in the colimit system are induced by the structure maps, and $[-,-]^G_{top}$ means G-equivariant homotopy classes of based G-maps.

Remark 0.3.6. We could have also defined $\pi_k^G(X)$ as a colimit over the category of all G-representations with morphisms $\pi_0(\mathbb{L}(V,W))$ see [**81**, 3.1.14].

Remark 0.3.7. It is standard to check that the G-equivariant homotopy groups define a functor from the category of orthogonal G-spectra to graded abelian groups. However, it is a non-trivial fact that these homotopy groups come equipped with some extra structure, namely that of a G-Mackey functor [**58**, V.9]. This means that for $H \leq G$ and $g \in G$, we have restriction maps $\mathrm{res}^G_H : \pi_*^G(X) \to \pi_*^H(X)$, transfer maps $\mathrm{tr}^G_H : \pi_*^H(X) \to \pi_*^G(X)$ and conjugation maps $g_\star : \pi_*^H(X) \to \pi_*^{gHg^{-1}}(X)$ satisfying unitality, transitivity and the double coset formula. The notion of Mackey functor was first introduced by Dress [**26**] and Green [**33**].

Definition 0.3.8. A morphism of orthogonal G-spectra $f : X \to Y$ is a $\underline{\pi}_*$-*isomorphism* if the induced map $\pi_k^H(f)\colon \pi_k^H(X) \to \pi_k^H(Y)$ is an isomorphism for all integer k and all subgroups $H \leq G$. We define the G-*equivariant stable homotopy category* hSp_G as the category obtained from Sp_G^O by formally inverting the $\underline{\pi}_*$-isomorphisms.

We will refer to an object in the equivariant stable homotopy category simply as a G-spectrum.

Remark 0.3.9. We note that the orthogonal setting is important here. With symmetric spectra for example, the analogous definitions of stable homotopy groups and π_*-isomorphisms will not give the correct notions of homotopy groups and stable equivalence, see [**53**, 3.1.10].

Remark 0.3.10. The $\underline{\pi}_*$-isomorphisms are part of a cofibrantly generated, stable and topological model structure on the category of orthogonal G-spectra, see [**59**, III.4.2]. The homotopy category associated to this model category gives an explicit model for the equivariant stable homotopy category.

Notation 0.3.11. We will write $[-,-]^G$ for the set (actually an abelian group) of morphisms in hSp_G and similarly, $[-,-]$ for the morphism set in hSp_G when G is trivial.

We list some properties that the equivariant stable homotopy category hSp_G enjoys.

(1) It admits the structure of a closed symmetric monoidal category. The tensor product is denoted by $\wedge$, the unit object by S^0 and internal hom-functor by $F(-,-)$. In particular, we can talk about (commutative) algebra objects in hSp_G, which are often called homotopy (commutative) ring G-spectra. As we will only work in the homotopy category, we will drop 'homotopy' and simply call these spectra (commutative) ring G-spectra. Given a commutative ring G-spectrum R, we have a module category Mod_R and a forgetful-free adjunction
$$\begin{tikzcd} \mathrm{Mod}_R \arrow[r, shift left, "\mathrm{fgt}"'] & \mathrm{hSp}_G. \arrow[l, shift left, "R\wedge -"'] \end{tikzcd}$$
We will use the notation $[-,-]^R$ for maps in Mod_R similarly to Notation 0.3.11.

(2) For $H \leq G$, there is an adjoint triple
$$\begin{tikzcd} \mathrm{hSp}_G \arrow[r, "\mathrm{res}^G_H"] & \mathrm{hSp}_H, \arrow[l, bend right, "G_+\wedge_H(-)"'] \arrow[l, bend left, "F_H(G_+,-)"] \end{tikzcd}$$
and the Wirthmüller Isomorphism gives a natural equivalence
$$F_H(G_+, \Sigma^{L(H)}X) \simeq G_+ \wedge_H X$$
for all G-spectra X. Here $L(H)$ denotes the tangent H-representation at the identity coset of G/H.

(3) For $H \leq G$, there is a categorical fixed points functor $(-)^H \colon \mathrm{hSp}_G \to \mathrm{hSp}$ and a geometric fixed points functor $\Phi^H \colon \mathrm{hSp}_G \to \mathrm{hSp}$. The latter is determined by the fact that it is symmetric monoidal, it commutes with filtered colimits and it extends the usual fixed points functor of G-spaces in the sense that

$$\Phi^G(\Sigma^\infty X) \simeq \Sigma^\infty(X^G)$$

for all G-spaces X. We can calculate the homotopy groups of the geometric fixed points via the formula $\pi_*(\Phi^G X) = \pi_*^G(\widetilde{E}\mathcal{P} \wedge X)$ where $\mathcal{P}$ is the family of proper subgroups of G. More details can be found in [**59**, V.4].

(4) It is a triangulated category and the hom-sets have the structure of a graded abelian group via $[-,-]_n^G = [\Sigma^n -, -]^G$. Moreover, it is compactly generated, which means that any G-spectrum can be built using cones, sums and suspensions from the set of compact generators

$$\{G/H_+ = G_+ \wedge_H S^0 \mid H \leq G\},$$

which are characterized by the formula $[\Sigma^n G/H_+, X]^G = \pi_n^H(X)$.

In this thesis we will also be concerned with equivariant spectra with a good theory of Euler classes.

Definition 0.3.12. Let R be an commutative ring G-spectrum and let V be a G-representation of dimension $|V|$. An *R-orientation u_V* for V is a G-equivariant map $u_V \colon S^V \to R \wedge S^{|V|}$ such that the composite

$$u_V^R \colon R \wedge S^V \xrightarrow{R \wedge u_V} R \wedge R \wedge S^{|V|} \xrightarrow{\mu \wedge S^{|V|}} R \wedge S^{|V|}$$

is a G-equivalence. We say that R is *complex stable* if every complex representation has an R-orientation. These are subject to the following conditions:

(i) *Unitality*: u_0 represents the unit element $1 \in \pi_0^G(R)$;
(ii) *Transitivity*: Given R-orientations u_V and u_W, then $u_{V\oplus W}$ is represented by the composite

$$S^{V\oplus W} \simeq S^V \wedge S^W \xrightarrow{u_V \wedge u_W} R \wedge S^{|V|} \wedge R \wedge S^{|W|} \simeq R \wedge R \wedge S^{|V+W|} \xrightarrow{\mu \wedge 1} R \wedge S^{|V+W|}.$$

Remark 0.3.13. Given a G-equivalence $u_V^R \colon R \wedge S^V \to R \wedge S^{|V|}$ which is also an R-module map, we can precompose u_V^R with the map $\eta \wedge S^V \colon S^0 \wedge S^V \to R \wedge S^V$ to obtain an R-orientation.

Lemma 0.3.14. *Let R be a commutative ring G-spectrum which is complex stable. Consider an R-module E and a complex representation V. Then the orientation u_V^R induces an isomorphism $\pi_*^G(E \wedge S^{-V}) \cong \pi_{*+|V|}^G(E)$.*

Proof. Note that $\pi_*^G(E \wedge S^{-V}) = [R \wedge S^V, E]_*^R$. Precomposition with the orientation u_V^R gives the desired isomorphism. $\square$

Definition 0.3.15. Let V be a G-representation and consider the map $a_V : S^0 \to S^V$ given by the one point compactification of the inclusion $0 \subset V$. If R is a complex stable ring G-spectrum, then the *Euler class of* V is the element $e(V) \in \pi_{-|V|}^G(R)$ represented by the composite

$$S^0 \wedge S^0 \xrightarrow{\eta \wedge a_V} R \wedge S^V \xrightarrow{u_V^R} R \wedge S^{|V|}.$$

Remark 0.3.16. Let R be a commutative ring G-spectrum and M an R-module. Pick $r \in [R,R]^R = \pi_0^G(R)$ and $m \in [R,M]^R = \pi_0^G(M)$. Then $\pi_0^G(M)$ becomes a $\pi_0^G(R)$-module in two ways:

(1) we can define $m \cdot r$ as the composite

$$R \xrightarrow{r} R \xrightarrow{m} M$$

(2) or as the composite

$$S^0 \wedge R \xrightarrow{\eta \wedge r} R \wedge R \xrightarrow{m \wedge R} M \wedge R \xrightarrow{\text{act}} M.$$

We can see that these two actions coincide by looking at the following commutative diagram

$$\begin{tikzcd} S^0 \wedge R \arrow[r, "\eta\wedge r"] \arrow[dr, "r"'] & R \wedge R \arrow[r, "m\wedge R"] \arrow[d, "\text{act}"] & M \wedge R \arrow[d, "\text{act}"] \\ & R \arrow[r, "m"] & M \end{tikzcd}$$

where we used that m is an R-module map.

4. The global stable homotopy category

We now give a brief introduction to global stable homotopy theory following [**81**, Chapter 4].

Definition 0.4.1. A *global family* $\mathcal{F}$ is a collection of compact Lie groups that is closed under passage to subgroups, isomorphisms and passage to quotients.

We think of an $\mathcal{F}$-global spectrum as a spectrum with simultaneous and compatible actions for compact Lie groups in the global family $\mathcal{F}$. It is a crucial observation that an orthogonal spectrum exhibits such a "global" structure. In fact we can define a functor

$$i_G^* : \mathrm{Sp}^O \to \mathrm{Sp}_G^O$$

by letting G act trivially on the values of X. If we choose an action of G on V to make it a representation, this gives an action of G on $X(V)$ by functoriality. More precisely, the $O(V)$-space $X(V)$ gets a G-action by pulling-back along the map $\rho\colon G \to O(V)$ which exhibits V as an orthogonal G-representation.

Definition 0.4.2. The G-homotopy groups of an orthogonal spectrum X are defined by the formula $\pi_*^G(X) = \pi_*^G(i_G^*(X))$.

Remark 0.4.3. It is clear from the definition that the homotopy groups of an orthogonal spectrum admit the structure of a Mackey functor. In particular, we have conjugation, restriction and transfer maps as in Remark 0.3.7. Schwede [**81**, 4.2] showed that these homotopy groups admit a richer structure, that of a *global functor*. We refer the reader to [**81**, p. 373] for a complete list of axioms for a global functor and here we only mention that there are "restriction maps" along *any* continuous homomorphism $H \to G$ (not necessarily injective). Directly from the axioms one obtains the following important consequences:

(i) the conjugation action on $\pi_*(X)$ is always trivial [**81**, p. 351];
(ii) for any orthogonal spectrum X and any finite group G, we have the relation

$$\mathrm{res}_1^G \circ \mathrm{tr}_1^G = |G| \cdot \mathrm{res}_1^G \colon \pi_*(X) \to \pi_*(X)$$

see [**81**, 3.4.10].

Example 0.4.4 (Constant global functor). Let A be an abelian group and consider a global family $\mathcal{F}$ of finite groups. The constant global functor $\underline{A}$ is given by $\underline{A}(G) = A$ for all $G \in \mathcal{F}$ and all restriction maps are identities. The transfer $\mathrm{tr}_H^G\colon \underline{A}(H) \to \underline{A}(G)$ is multiplication by the index $[G : H]$.

Example 0.4.5. Consider the cyclic group $C_3 = \{1, \tau, \tau^2\}$ and let $\mathcal{G}$ be the global family consisting of C_3 and the trivial group. The generating operations of a global functor M on $\mathcal{G}$

can be displayed as follows:

$$\begin{tikzcd} M(C_3) \arrow[loop left, "\alpha^*"] \arrow[r, "\mathrm{res}_1^{C_3}"] & M(1). \arrow[l, bend right, "\mathrm{tr}_1^{C_3}"'] \arrow[l, bend left, "p^*"] \end{tikzcd}$$

Here $p\colon C_3 \to 1$ is the projection and $\alpha\colon C_3 \to C_3$ is the automorphism with $\alpha(\tau) = \tau^2$. These are subject to the following relations:

$$\mathrm{res}_1^{C_3} \circ p^* = \mathrm{id} \quad \mathrm{res}_1^{C_3} \circ \mathrm{tr}_1^{C_3} = 3 \cdot \mathrm{id} \quad \alpha^* \circ \alpha^* = \mathrm{id}$$
$$\alpha^* \circ p^* = p^* \quad \mathrm{res}_1^{C_3} \circ \alpha^* = \mathrm{res}_1^{C_3} \quad \alpha^* \circ \mathrm{tr}_1^{C_3} = \mathrm{tr}_1^{C_3}.$$

Remark 0.4.6. The underlying orthogonal G-spectrum of an orthogonal spectrum is always split, i.e., the restriction map $\mathrm{res}_e^G : \pi_*^G(X) \to \pi_*(X)$ is a split epimorphism. The splitting is induced by the group homomorphism $G \to e$.

Definition 0.4.7. A morphism $f : X \to Y$ of orthogonal spectra is an *$\mathcal{F}$-global equivalence* if the induced map $\pi_k^H(f)\colon \pi_k^H(X) \to \pi_k^H(Y)$ is an isomorphism for all integer k and all groups $H \in \mathcal{F}$. We define the *$\mathcal{F}$-global stable homotopy category* $\mathrm{hSp}_{\mathcal{F}}$ as the category obtained from Sp^O by formally inverting the $\mathcal{F}$-global equivalences.

We will refer to an object in the global stable homotopy category as an $\mathcal{F}$-global spectrum or global spectrum.

Remark 0.4.8. The $\mathcal{F}$-global equivalences are part of a topological, cofibrantly generated, stable model structure on the category of orthogonal spectra, see [**81**, 4.3.17, 4.5.28, 4.3.24]. The homotopy category associated to this model category gives an explicit model for the $\mathcal{F}$-global stable homotopy category.

Notation 0.4.9. We will write $[-,-]^{\mathcal{F}}$ for the set of morphisms in $\mathrm{hSp}_{\mathcal{F}}$.

We will list some properties that the global stable homotopy category enjoys.

(1) It admits the structure of a closed symmetric monoidal category which makes the restriction functor $i_G^*\colon \mathrm{hSp}_{\mathcal{F}} \to \mathrm{hSp}_G$ into a strong monoidal functor for all $G \in \mathcal{F}$. We will denote by $\wedge$ the tensor product and by $\mathbb{S}$ the unit. We will refer to a (commutative) algebra object in $\mathrm{hSp}_{\mathcal{F}}$ simply as a (commutative) ring global spectrum.
(2) For $G \in \mathcal{F}$, we have an adjoint triple

$$\begin{tikzcd} \mathrm{hSp}_{\mathcal{F}} \arrow[r, "i_G^*"] & \mathrm{hSp}_G. \arrow[l, bend right, "L_G"'] \arrow[l, bend left, "R_G"] \end{tikzcd}$$

There is also a similar adjoint triple if we fix a global subfamily $\mathcal{F}' \subset \mathcal{F}$.
(3) It is a triangulated category which is compactly generated by $\{B_{gl}G_+ := L_G S^0 \mid G \in \mathcal{F}\}$. If G is the trivial group, then $B_{gl}G_+ = \mathbb{S}$. These global spectra are characterized by the property $[B_{gl}G_+, X]_*^{\mathcal{F}} = \pi_*^G(X)$. It follows that $B_{gl}(G/H)_+ \simeq B_{gl}H_+$ for $H \leq G$.
(4) Given an abelian group A, there exists an Eilenberg-MacLane $\mathcal{F}$-global spectrum $H\underline{A}$ which is characterized by the property that

$$\pi_*^G(H\underline{A}) = \pi_0^G(\underline{A}) = A$$

for all $G \in \mathcal{F}$. The Mackey structure on $\underline{\pi}_0(H\underline{A})$ is as follows: the restriction and conjugations maps are identities and for finite index subgroups $H \leq G$, the transfer map tr_H^G is multiplication by the index $[G:H]$. More generally, there is an Eilenberg-MacLane $\mathcal{F}$-global spectrum $H\underline{M}$ for any global functor M characterized by the property $\pi_*^G(H\underline{M}) = \pi_0^G(H\underline{M}) = M(G)$.

5. Free G-spectra and free global spectra

We recall the notion of a free G-spectrum and then extend this definition to global homotopy theory. Finally, we investigate the existence and properties of such free global spectra.

DEFINITION 0.5.1. A G-spectrum X is said to be *free* if $\pi_*(\Phi^H X) = 0$ for all $1 \neq H \leq G$.

EXAMPLE 0.5.2. The suspension spectrum of any free G-space is free as a G-spectrum.

LEMMA 0.5.3. *A G-spectrum X is free if and only if the canonical map $X = X \wedge S^0 \to X \wedge EG_+$ is an isomorphism in the equivariant stable homotopy category.*

PROOF. Recall that EG is a contractible space with a free G-action so $EG^H = \emptyset$ for all $1 \neq H \leq G$. Using that the geometric fixed points functor is symmetric monoidal and commutes with the suspension spectrum, we calculate

$$\Phi^H(X \wedge EG_+) = \Phi^H(X) \wedge (EG)_+^H \simeq \begin{cases} X & \text{if } \ H = 1 \\ * & \text{if } \ H \neq 1. \end{cases}$$

This proves the backwards implication. The forward implication follows from the fact that geometric fixed points detect isomorphisms in the equivariant stable homotopy category [**81**, 3.3.10]. □

LEMMA 0.5.4. *Any free G-spectrum is a module over the complex stable commutative ring G-spectrum $DEG_+ := F(EG_+, S^0)$.*

PROOF. Note that DEG_+ is a commutative ring G-spectrum as we have a diagonal map $EG_+ \to EG_+ \wedge EG_+$. Using the properties of the geometric fixed points functor and that EG is free, we see that there is a G-equivalence $EG_+ \wedge S^{-V} \simeq EG_+ \wedge S^{-|V|}$ for all representations V. Then we have a chain of equivalences

$$DEG_+ \wedge S^V \simeq D(EG_+ \wedge S^{-V}) \simeq D(EG_+ \wedge S^{-|V|}) \simeq DEG_+ \wedge S^{|V|}$$

which shows that DEG_+ is complex stable. The spectrum EG_+ is a module over DEG_+ by the map

$$DEG_+ \wedge EG_+ \xrightarrow{1\wedge\delta} DEG_+ \wedge EG_+ \wedge EG_+ \xrightarrow{\mathrm{ev}\wedge 1} S^0 \wedge EG_+ = E_+.$$

The module structure on $EG_+ \wedge X$ is obtained in a straightforward way from this. This shows that any free G-spectrum is a module over DEG_+. □

DEFINITION 0.5.5. An $\mathcal{F}$-global spectrum X is said to be *free* if $\pi_*(\Phi^H X) = 0$ for all $1 \neq H \in \mathcal{F}$. Equivalently an $\mathcal{F}$-global spectrum is free if its underlying G-spectrum is free for all $G \in \mathcal{F}$.

REMARK 0.5.6. Dually we call a G-spectrum *cofree* if the canonical map $X \to F(EG_+, X)$ is an equivalence, and an $\mathcal{F}$-global spectrum *cofree* if its restriction to G-spectra is cofree for all $G \in \mathcal{F}$. By [**81**, 4.5.16], the class of cofree global spectra coincides with the class of global spectra which are right induced from the trivial family, i.e. those global spectra that are in the image of the fully faithful functor $R : \mathrm{hSp} \to \mathrm{hSp}_{\mathcal{F}}$ which is right adjoint to restriction. In particular, cofree spectra are well-understood and easy to construct. We warn the reader that the class of free global spectra does not coincide with the class of global spectra which are left induced from the trivial family, see [**81**, 4.5.8].

5.1. Finite case. In this section we will only consider global families of finite groups.

Definition 0.5.7. Let $\mathcal{F}$ be a global family of finite groups and let A be a graded abelian group. We say that A is *uniquely $\mathcal{F}$-divisible* if it is uniquely $|G|$-divisible for all $G \in \mathcal{F}$, that is multiplication by $|G|$ on A is an isomorphism for all $G \in \mathcal{F}$.

Example 0.5.8. Let $\mathcal{G}$ be the global family of finite groups. An abelian group is uniquely $\mathcal{G}$-divisible if and only if it is a $\mathbb{Q}$-vector space.

Lemma 0.5.9. *For a finite group G and a G-spectrum X, there is a conditionally convergent spectral sequence*

$$E_2^{s,t} = H_{t-s}(G;\pi_s(X)) \Rightarrow \pi_{t-s}^G(X \wedge EG_+).$$

Moreover, the spectral sequence is strongly convergent if X is non-equivariantly (-1)-connected, that is $\pi_k(X) = 0$ for $k < 0$.

Proof. This follows from applying the Adams isomorphism $(X \wedge EG_+)^G \simeq X \wedge_G EG_+$ to the Atiyah-Hirzebruch spectral sequence associated to the fibre sequence $X \to X \wedge_G EG_+ \to BG_+$. □

Lemma 0.5.10. *Let X be G-spectrum with a trivial G-action on $\pi_0(X)$, and suppose that X is non-equivariantly (-1)-connected. Then the canonical map*

$$\beta\colon \pi_0^G(X \wedge EG_+) \to \pi_0^G(X)$$

can be identified with the transfer map $\mathrm{tr}_1^G\colon \pi_0(X) \to \pi_0^G(X)$.

Proof. Recall that the transfer map is the composite

$$\mathrm{tr}_1^G\colon \pi_*(X) \simeq \pi_0^G(X \wedge G_+) \to \pi_*^G(X)$$

where we used the Wirthmüller Isomorphism and the canonical projection $G_+ \to S^0$. We can factorize this further as

$$\mathrm{tr}_1^G\colon \pi_0(X) \simeq \pi_0^G(X \wedge G_+) \xrightarrow{i_*} \pi_0^G(X \wedge EG_+) \to \pi_0^G(X)$$

by including the zero skeleton $i\colon G_+ \to EG_+$ and then projecting $EG_+ \to S^0$. Accordingly, it is enough to show that i_* is an isomorphism. If X is (-1)-connected, then the spectral sequence of Lemma 0.5.9 is concentrated in the first quadrant, so we deduce that

$$\pi_0(X) = H_0(G;\pi_0(X)) \simeq \pi_0^G(X \wedge EG_+)$$

as G-acts trivially on $\pi_0(X)$. It follows that i_* is an isomorphism. □

We now construct some examples of free global spectra.

Proposition 0.5.11. *Let $\mathcal{F}$ be a global family of finite groups and let M be a $\mathcal{F}$-global functor. Then the Eilenberg-MacLane global spectrum $H\underline{M}$ is free if and only if $M(1)$ is uniquely-$\mathcal{F}$-divisible and the transfer maps $\mathrm{tr}_1^G\colon M(1) \to M(G)$ are isomorphisms for all $G \in \mathcal{F}$. In this case, the restriction maps $\mathrm{res}_1^G\colon M(G) \to M(1)$ are also isomorphisms for all $G \in \mathcal{F}$.*

Proof. The global spectrum $H\underline{M}$ is free if and only if the canonical map

$$\beta\colon \pi_*^G(H\underline{M} \wedge EG_+) \to \pi_*^G(H\underline{M})$$

is an isomorphism for all $G \in \mathcal{F}$. The target of β is $M(G)$ concentrated in degree zero, and the source can be calculated using the homotopy orbits spectral sequence Lemma 0.5.9. We find that $\pi_*^G(H\underline{M} \wedge EG_+) = H_*(G;M(1))$ where $M(1)$ has a trivial G-action by Remark 0.4.3. Therefore we conclude that $H\underline{M}$ is free if and only if:

(i) $H_n(G;M(1)) = 0$ for all $n > 0$ and all $G \in \mathcal{F}$;
(ii) the map $\beta\colon M(1) = H_0(G,M(1)) \to M(G)$ is an isomorphism for all $G \in \mathcal{F}$.

By Lemma 0.5.10, condition (ii) is equivalent to asking that the transfer maps $\mathrm{tr}_1^G\colon M(1)\to M(G)$ are isomorphisms for all $G\in\mathcal{F}$. For condition (i), take G a cyclic group of order p, and write $p\colon M(1)\to M(1)$ for the multiplication by p map. Then we calculate that

$$H_n(G;M(1))=\begin{cases}\mathrm{cok}(p) & \text{if } n=1,3,5,\dots\\ \ker(p) & \text{if } n=2,4,6,\dots\\ M(1) & \text{if } n=0.\end{cases}$$

This shows that (i) is equivalent to $M(1)$ being uniquely p-divisible.

If $M(1)$ is uniquely $\mathcal{F}$-divisible and the transfers are all isomorphisms, then the double coset formula tells us that $|G|=\mathrm{res}_1^G\circ\mathrm{tr}_1^G\colon M(1)\to M(1)$ which shows that the restriction maps are also isomorphisms. □

5.2. The rational case. Let us consider the category of *rational* $\mathcal{F}$-global spectra; that is the full subcategory of those global spectra whose homotopy groups are $\mathbb{Q}$-vector spaces. We claim that we have an equality

$$h\mathrm{Sp}_{\mathcal{F},\mathbb{Q}}^{\mathrm{free}}=\mathrm{hSp}_{\mathcal{F},\mathbb{Q}}^{\mathrm{cofree}}$$

between the subcategory of free and cofree rational $\mathcal{F}$-global spectra. To prove the claim it is enough to show that if G is a finite group, then a rational G-spectrum is free if and only if it is cofree. The key idea is to use the idempotents of the Burnside ring $A(G)\otimes\mathbb{Q}=[S^0_{\mathbb{Q}},S^0_{\mathbb{Q}}]^G$. As we are working rationally, we have a ring isomorphism

$$\phi\colon [S^0_{\mathbb{Q}},S^0_{\mathbb{Q}}]^G\xrightarrow{\simeq}\prod_{(H)\leq G}\mathbb{Q}\qquad [f]\mapsto((H)\mapsto\deg(\Phi^H f))$$

with H-component given by the degree of the H-geometric fixed points. Pick the idempotent $e_1\in A(G)\otimes\mathbb{Q}$ corresponding to the trivial group and let $S^0[e_1^{-1}]$ be the mapping telescope of consecutive iterations of e_1. Using the description of the ring map ϕ, we easily see that $S^0[e_1^{-1}]$ is a model for the rational spectrum EG_+. It follows that EG_+ is self-dual and small. Using these two properties we obtain canonical equivalences

$$F(EG_+,X)\simeq X\wedge DEG_+\simeq X\wedge EG_+$$

showing that X is free if and only if is cofree as claimed.

We can go even further since Remark 0.5.6 lets us identify the category of rational cofree $\mathcal{G}$-global spectra with that of rational spectra. The latter is equivalent to the derived category of $\mathbb{Q}$-vector spaces by Shipley's theorem [**86**, 2.15]. Thus we have equivalences

$$\mathrm{hSp}_{\mathcal{F},\mathbb{Q}}^{\mathrm{free}}=\mathrm{hSp}_{\mathcal{F},\mathbb{Q}}^{\mathrm{cofree}}\simeq\mathcal{D}(\mathrm{Mod}_{\mathbb{Q}}).$$

5.3. Non-finite case.

Notation 0.5.12. Let $\mathbb{T}$ denote the circle group and write $[\mathbb{T}]$ for the global family of closed subgroups of $\mathbb{T}$.

Recall the functor $L_{\mathbb{T}}$ which is left adjoint to the restriction functor $i_{\mathbb{T}}^*\colon \mathrm{hSp}_{[\mathbb{T}]}\to\mathrm{hSp}_{\mathbb{T}}$ and that by definition $B_{gl}\mathbb{T}_+=L_{\mathbb{T}}(S^0)$.

Lemma 0.5.13. *Let z be the standard $\mathbb{T}$-representation. Then we have a non-canonical splitting $B_{gl}\mathbb{T}_+=\mathbb{S}\vee L_{\mathbb{T}}(S^z)$.*

Proof. Note that there is a cofibre sequence of $\mathbb{T}$-spectra

$$\mathbb{T}_+\xrightarrow{p}S^0\xrightarrow{a_z}S^z\to\Sigma\mathbb{T}_+$$

where a_z is the map from Definition 0.3.15 and p is induced by the projection $\mathbb{T}\to\mathbb{T}/\mathbb{T}$. Apply $L_{\mathbb{T}}$ to the cofibre sequence above to get

$$\mathbb{S}\xrightarrow{L_{\mathbb{T}}(p)}B_{gl}\mathbb{T}_+\xrightarrow{L_{\mathbb{T}}(a_z)}L_{\mathbb{T}}(S^z)\to\Sigma\mathbb{S}\tag{5.3.1}$$

where we used that $\mathbb{S} = L_{\mathbb{T}}(\mathbb{T}_+)$ as they both corepresent the functor $\pi_*(-)$. We will show that there are no non-zero maps $L_{\mathbb{T}}(S^z) \to \Sigma\mathbb{S}$, or equivalently that $[S^z, S^1]^{\mathbb{T}} = 0$. Consider the following long exact sequence

$$[S^0, S^1]^{\mathbb{T}} \xleftarrow{a_z^*} [S^z, S^1]^{\mathbb{T}} \leftarrow [\mathbb{T}_+, S^0]^{\mathbb{T}} \xleftarrow{p^*} [S^0, S^0]^{\mathbb{T}}.$$

The leftmost group is zero by connectivity, and p^* is an isomorphism as it can be identified with the restriction map $\pi_0^{\mathbb{T}}(S^0) \to \pi_0(S^0)$. By the long exact sequence we deduce that $[S^z, S^1]^{\mathbb{T}} = 0$ as required. $\square$

Proposition 0.5.14. *Consider a $[\mathbb{T}]$-global spectrum X and a commutative ring $\mathbb{T}$-spectrum R which is complex stable. Then there is a split short exact sequence of the form*

$$0 \to \pi_{*+z}^{\mathbb{T}}(X) \xrightarrow{a_z^*} \pi_*^{\mathbb{T}}(X) \xrightarrow{\mathrm{res}_e^{\mathbb{T}}} \pi_*(X) \to 0.$$

If additionally $i_{\mathbb{T}}^* X$ admits an R-module structure, then the monomorphism a_z^* can be identified with multiplication by the Euler class $e(z) \in \pi_{-2}^{\mathbb{T}}(R)$.

Proof. Apply the functor $[-, X]_*^{[\mathbb{T}]}$ to the split triangle (5.3.1) to get the short exact sequence

$$0 \to \pi_{*+z}^{\mathbb{T}}(X) \xrightarrow{a_z^*} \pi_*^{\mathbb{T}}(X) \xrightarrow{\mathrm{res}_e^{\mathbb{T}}} \pi_*(X) \to 0.$$

If X admits an R-module structure, then Remark 0.3.16 shows us that the composite

$$\pi_{*+2}^{\mathbb{T}}(X) = [S^2 \wedge R, X]_*^R \cong [S^z \wedge R, X]_*^R \xrightarrow{(a_z \wedge R)^*} [R \wedge S^0, X]_*^R = \pi_*^{\mathbb{T}}(X)$$

is multiplication by $e(z)$. $\square$

Proposition 0.5.15. *Let $\mathcal{F}$ be a global family containing at least one infinite group. Then every free $\mathcal{F}$-global spectrum is contractible.*

Proof. First of all note that if $\mathcal{F}$ contains an infinite group then it must contain $[\mathbb{T}]$ since global families are closed under passage to subgroups. Accordingly, we can restrict to $[\mathbb{T}]$ and show that all free $[\mathbb{T}]$-global spectra are contractible. Suppose that there exists a nonzero free global spectrum X so by Lemma 0.5.4 its restriction $i_{\mathbb{T}}^* X$ is a $DE\mathbb{T}_+$-module. Proposition 0.5.14 tells us that the Euler class $e(z) \in \pi_{-2}^{\mathbb{T}}(DE\mathbb{T}_+)$ is regular on $\pi_*^{\mathbb{T}}(X)$. This is a contradiction since

$$0 \overset{(1)}{=} \pi_*^{\mathbb{T}}(X \wedge \widetilde{E}\mathbb{T}) \overset{(2)}{=} \pi_*^{\mathbb{T}}(X \wedge S^{\infty z}) \overset{(3)}{=} \mathrm{colim}_k\, \pi_{*-2k}^{\mathbb{T}}(X) \overset{(4)}{=} \pi_*^{\mathbb{T}}(X)[e(z)^{-1}]$$

where we used that X is free for (1), the explicit model for $\widetilde{E}\mathbb{T}$ given in Example 0.2.5 for (2) , Lemma 0.3.14 for (3) and finally Proposition 0.5.14 for (4). $\square$

Theorem 0.5.16. *Let $\mathcal{F}$ be any non-trivial global family. Then there exists no $\mathcal{F}$-global spectrum X whose underlying G-equivariant homotopy type is equivalent to EG_+ for all $G \in \mathcal{F}$. More informally, the universal free G-spectrum EG_+ does not admit a global refinement.*

Proof. By Proposition 0.5.15 we can assume that $\mathcal{F}$ consists only of finite groups. Suppose that such a global spectrum X exists, so that X is free and non-equivariantly (-1)-connected. Then by Lemma 0.5.10 the canonical map $\beta\colon \pi_0^G(X \wedge EG_+) \to \pi_0^G(X)$ can be identified with the transfer map tr_1^G. This shows that the maps $\mathrm{tr}_1^G\colon \pi_0(X) \to \pi_0^G(X)$ are all isomorphisms. Then the double coset formula gives us the relation $|G| = \mathrm{res}_1^G \circ \mathrm{tr}_1^G$. Since the restriction map res_1^G is always a split epimorphism by Remark 0.4.6, we deduce that multiplication by $|G|\colon \pi_0(X) \to \pi_0(X)$ is surjective. This is a contradiction since $\pi_0(X) = \mathbb{Z}$. $\square$

Part 2

Out-representation theory of finite groups

Introduction

In this part of the thesis we develop a framework for studying families of representations of the outer automorphism groups. A common theme in representation theory is that there is a conceptual advantage in encoding this large amount of (possibly complicated) data into a single object, which lives in a convenient abelian category. Using purely algebraic techniques we will deduce strong constraints on naturally occurring families of representations of the outer automorphism groups. We will then apply these results to study the tensor triangulated geometry of the derived category of interesting diagram categories which appear in representation theory and algebraic topology.

The main character. Fix k a field of characteristic zero and let $\mathcal{G}$ denote the category of finite groups and conjugacy classes of surjective group homomorphisms. We are interested in the category $\mathcal{A} = [\mathcal{G}^{\mathrm{op}}, \mathrm{Vect}_k]$ of contravariant functors from $\mathcal{G}$ to the category of k-vector spaces. More generally, we could restrict our attention to a replete full subcategory $\mathcal{U} \leq \mathcal{G}$ and then consider the smaller category $\mathcal{AU} = [\mathcal{U}^{\mathrm{op}}, \mathrm{Vect}_k]$.

Note that the endomorphism group of an object $G \in \mathcal{U}$ is the outer automorphism group $\mathcal{U}(G, G) = \mathrm{Out}(G)$. Therefore any object $X \in \mathcal{AU}$ gives rise to a collection of $\mathrm{Out}(G)$-representations $X(G)$ for $G \in \mathcal{U}$. The functoriality of X imposes further compatibility conditions on these representations. There are two main examples where all this data can be made very explicit.

Example. Consider the category $\mathcal{C}_2$ of cyclic 2-groups. An object $X \in \mathcal{AC}_2$ gives rise to a consistent sequence of representations of cyclic 2-groups:

$$\begin{tikzcd}
X(1) \arrow[r] \arrow[loop above, "1"] & X(C_2) \arrow[r] \arrow[loop above, "1"] & X(C_4) \arrow[r] \arrow[loop above, "C_2"] & X(C_8) \arrow[r] \arrow[loop above, "C_4"] & X(C_{16}) \arrow[r] \arrow[loop above, "C_8"] & X(C_{32}) \arrow[r] \arrow[loop above, "C_{16}"] & \dots
\end{tikzcd}$$

where the horizontal maps are induced by the canonical projections.

Example. Fix a prime number p and consider category $\mathcal{E}_p$ of elementary abelian p-groups. An object $X \in \mathcal{AE}_p$ gives rise to a consistent sequence of representations of the finite general linear groups:

$$\begin{tikzcd}
X(1) \arrow[r] \arrow[loop above, "1"] & X(C_p) \arrow[r] \arrow[loop above, "\mathrm{GL}_1(\mathbb{F}_p)"] & X(C_p^2) \arrow[r] \arrow[loop above, "\mathrm{GL}_2(\mathbb{F}_p)"] & X(C_p^3) \arrow[r] \arrow[loop above, "\mathrm{GL}_3(\mathbb{F}_p)"] & X(C_p^4) \arrow[r] \arrow[loop above, "\mathrm{GL}_4(\mathbb{F}_p)"] & X(C_p^5) \arrow[r] \arrow[loop above, "\mathrm{GL}_5(\mathbb{F}_p)"] & \dots
\end{tikzcd}$$

where the horizontal maps are induced by the projection into the first coordinates.

As we have already seen in the previous examples, it will often be convenient to restrict attention to special subcategories $\mathcal{U}$ for which certain phenomena stand out more clearly. For example:

- We might fix a prime p and restrict attention to p-groups.
- We might restrict attention to solvable, nilpotent or abelian groups.
- We might impose upper or lower bounds on the exponent, nilpotence class, order, or on the size of a minimal generating set.

- As special cases of the above, we might consider only cyclic groups, or only elementary abelian p-groups for some fixed prime p.
- For a fixed prime number p, we let $\mathcal{C}_p, \mathcal{E}_p$ and $\mathcal{P}_p$ denote the families of cyclic p-groups, elementary abelian p-groups and abelian p-groups respectively. If the fixed prime number is clear from the context, we will often omit it from the notation.

To ensure good homological properties, we will also impose additional conditions on $\mathcal{U}$ such as:

- Closure under products: If $G, H \in \mathcal{U}$, then $G \times H \in \mathcal{U}$. We say that $\mathcal{U}$ is *multiplicative*.
- Closure under passage to subgroups: If $G \in \mathcal{U}$ and $H \leq G$, then $H \in \mathcal{U}$.
- Downwards closure (i.e. closure under passage to quotients): If $G \in \mathcal{U}$ and $\mathcal{G}(G, H) \neq \emptyset$, then $H \in \mathcal{U}$.
- Upwards closure: if $H \in \mathcal{U}$ and $\mathcal{G}(G, H) \neq \emptyset$, then $G \in \mathcal{U}$.
- Convexity: if $G, K \in \mathcal{U}$ and $\mathcal{G}(G, H) \neq \emptyset$ and $\mathcal{G}(H, K) \neq \emptyset$, then $H \in \mathcal{U}$.

We will see throughout this introduction that $\mathcal{AU}$ has its best homological behaviour when $\mathcal{U}$ is a *global family*, that is replete, closed downwards and closed under passage to subgroups.

Before presenting our results we put the abelian category $\mathcal{AU}$ in the relevant context.

Representations of combinatorial categories. The abelian category $\mathcal{AU}$ is part of a larger family of categories appearing in representation theory and algebraic topology. Given a category $\mathcal{I}$ whose objects are finite sets (with possibly extra structure) and whose morphisms are functions (possibly respecting the extra structure), we can consider the associated diagram category $\mathcal{A}_{\mathcal{I}} = [\mathcal{I}, \mathrm{Vect}_k]$. Some examples of interest include:

- Let FI be the category of finite sets and injections. The associated diagram category is the category of FI-modules which appears in [**78**] in the context of stable homotopy groups of symmetric spectra, and in [**22**, **23**] in relation to the representation theory of the symmetric groups.
- Let VI be the category of finite dimensional $\mathbb{F}_p$-vector spaces and injective linear maps. The associated diagram category is the category of VI-modules which appears in [**31**, **64**] in relation to the representation theory of the finite general linear groups. This category is equivalent by Pontryagin duality to the category $\mathcal{AE}$ mentioned earlier.
- Let VA be the category of finite dimensional $\mathbb{F}_p$-vector spaces and all linear maps. The associated diagram category have been studied in relation to algebraic K-theory, rational cohomology, and the Steenrod algebra [**57**].
- Various categories encoding the reprensentation theory of different families of groups, such as wreath groups [**76**], classical Weyl groups [**95**], various linear groups [**71**] and variants of FI [**30**].

Despite the similarities with other abelian categories appearing in representation theory, there is a major difference between $\mathcal{AU}$ and all these categories. We are no longer considering a one-parameter family of representations but rather collections of representations which are indexed by a family of groups. This brings into play group-theoretic properties of the family $\mathcal{U}$ and so introduces a new level of complexity into the story which has so far not been explored.

Noetherian condition. The category $\mathcal{AU}$ is a Grothendieck abelian category with generators given by the representable functors

$$e_G = k[\mathcal{U}(-, G)] \qquad G \in \mathcal{U}.$$

Many of the familiar notions from the theory of modules carry over to this setting. For example we call an object $X \in \mathcal{AU}$:

- *finitely generated* if it admits an epimorphism $\alpha \colon P \to X$ where P is a finite sum of generators;
- *finitely presented* if in addition $\ker(\alpha)$ is finitely generated;

- *noetherian* if all its subobjects are finitely generated.

Finally, we say that $\mathcal{AU}$ is locally noetherian if the generators e_G are noetherian for all $G \in \mathcal{U}$. It is then a formal consequence of the definition that subobjects of finitely generated objects are again finitely generated, and that any finitely generated object is also finitely presented.

Work of Church–Ellenberg–Farb in the category of FI-modules showed that the noetherian condition plays a fundamental role when working with sequences of representations [**22**]. This key technical innovation allowed them to prove an asymptotic structure theorem for finitely generated FI-modules which gave an elegant explanation for the representation-theoretic patterns observed in earlier work [**24**]. Motivated by this, we investigated for which choices of $\mathcal{U}$ the category $\mathcal{AU}$ is locally noetherian.

Theorem C. *Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$ and let p be a prime number.*

(a) *If $\mathcal{U}$ is a multiplicative global family of finite abelian p-groups, then $\mathcal{AU}$ is locally noetherian.*
(b) *If $\mathcal{U}$ is the global family of cyclic p-groups, then $\mathcal{AU}$ is locally noetherian.*

If $\mathcal{U}$ contains the trivial group and infinitely many cyclic groups of prime order, then $\mathcal{AU}$ is not locally noetherian. In particular, $\mathcal{A}$ is not locally noetherian.

Proof. The result combines Proposition 1.11.3 and Theorem 1.11.14. □

There are several combinatorial criteria available in the literature to show that the category $\mathcal{AU}$ is locally noetherian. We prove part (a) using the theory of Gröbner bases developed by Sam and Snowden [**74**], and part (b) using the criterion developed in [**31**]. Our result does not aim to give a complete classification of locally noetherian categories, as this will be costly and highly non-trivial, but rather aims to give a good range of examples and counterexamples to which our theory applies.

We then turn to study homological properties of our category of interest.

Homological properties. The levelwise tensor product of k-vector spaces gives $\mathcal{AU}$ a symmetric monoidal structure in which the unit object $\mathbb{1}$ is the constant functor with value k. For all $X, Y \in \mathcal{AU}$, there exists an internal hom functor that we denote by $\underline{\mathrm{Hom}}(X, Y) \in \mathcal{AU}$.

We list a few interesting homological properties that our category enjoys.

(i) As is typical for diagram categories, the finitely generated projective objects are not strongly dualizable. In particular this means that the canonical map

$$e_G \otimes \underline{\mathrm{Hom}}(e_G, \mathbb{1}) \to \underline{\mathrm{Hom}}(e_G, e_G)$$

is in general not an isomorphism, see Remark 1.2.3. However, the finitely generated projective objects of $\mathcal{AU}$ still form a subcategory that is closed under tensor products and internal hom, see Propositions 1.2.12 and 1.2.19.
(ii) As is typical for diagram categories, any projective object is a retract of a direct sum of generators, see Lemma 1.6.1. However, under mild conditions on $\mathcal{U}$ (satisfied by $\mathcal{G}$) the projective objects of $\mathcal{AU}$ are also injective, see Proposition 1.13.3. In particular, the generators e_G are injectives.
(iii) Under mild conditions on $\mathcal{U}$ (satisfied by $\mathcal{G}$), the only objects with a finite projective resolution are the projective ones, see Proposition 1.9.5.
(iv) The abelian category $\mathcal{AU}$ is semisimple if and only if $\mathcal{U}$ is a groupoid, see Proposition 1.4.3.

Representation stability. We also study the abelian category $\mathcal{AU}$ through representation stability techniques. For example, we show that any finitely presented object in $\mathcal{AU}$ can be recovered by finite amount of data via a "stabilization recipe". This phenomenon is called *central stability* and it was first introduced by Putman [**70**] for describing certain stability phenomena of the homology groups of congruence subgroups of the general linear groups. Since then, central stability has been shown to hold for various diagram categories such as FI-modules [**23**] and complemented categories [**71**]. Before presenting our result we need to introduce a bit of notation.

Let $\mathcal{U}$ be multiplicative and closed under subgroups and let $\mathcal{U}_{\leq n}$ denote the full subcategory of groups of cardinality less than or equal to n. For any $G \in \mathcal{U}$, we consider

$$\mathcal{N}_n(G) = \bigcap_{N \triangleleft G \,:\, G/N \in \mathcal{U}_{\leq n}} N$$

and put $q_n G = G/\mathcal{N}_n(G)$. The following summarizes some of the results in section 12.

Theorem D (central stability). *Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$, and consider a finitely presented object $X \in \mathcal{AU}$. Then there exists a natural number $m \in \mathbb{N}$ such that for all $G \in \mathcal{U}$, we have*

$$X(G) = \varinjlim_{H \in N(G,m)} X(G/H)$$

where $N(G,m) = \{H \triangleleft G \mid |G/H| \leq m\}$. Furthermore, if $\mathcal{U}$ is a multiplicative and closed under passage to subgroups, then there exists $n \in \mathbb{N}$ such that for all $G \in \mathcal{U}$, we have $X(G) = X(q_n G)$.

Consider the family $\mathcal{P}_p$ of finite abelian p-groups. For all $G \in \mathcal{P}_p$, we have $q_n G = G/nG$. In this case the second part of Theorem D tells us that X is uniquely determined by its values at p-groups of exponent less than or equal to n. This illustrates the fact that the representations encoded in a finitely presented object need to satisfy strong compatibility conditions. The first part of Theorem D is a more refined version of this phenomenon. It tells us that we can recover the value $X(G)$ from a finite amount of data, namely the poset $N(G,m)$ and the representations $X(G/H)$. We note that the poset $N(G,m)$ is always finite and can be determined by purely combinatorial means. For instance, in the abelian p-group case its cardinality can be explicitly calculated using the Hall polynomials [**21**, 2.1.1].

Given an epimorphism $\alpha\colon B \to A$, we also investigate the behaviour of the structure maps $\alpha^*\colon X(A) \to X(B)$ for sufficiently large groups A and B. In this case however, we need to restrict to the locally noetherian case. Consider the following families of finite abelian p-groups:

$$\mathcal{F}_n^p = \{C_{p^n}^s \mid s \geq 0\} \quad \text{and} \quad \mathcal{C}_p = \{C_{p^s} \mid s \geq 0\}.$$

Note that $\mathcal{C}_p$ is the family of cyclic p-groups and $\mathcal{F}_1^p$ is the family of elementary abelian p-groups. The following is a reformulation of the injectivity and surjectivity conditions in the definition of representation stability due to Church–Farb [**24**, 1.1].

Definition E. Let $\mathcal{U}$ be either $\mathcal{C}_p$ or $\mathcal{F}_n^p$ for some $n \geq 1$. Consider an object $X \in \mathcal{AU}$.

- We say that X is *eventually torsion-free* if there exists $r_0 \in \mathbb{N}$ such that the induced map $\alpha^*\colon X(A) \to X(B)$ is injective for all α in $\mathcal{U}$ with $|A| \geq r_0$.
- We say that X is *stably surjective* if there exists $r_0 \in \mathbb{N}$ such that the canonical map

  $$X(A) \otimes k[\mathcal{U}(B,A)] \to X(B), \quad (x, \alpha) \mapsto \alpha^*(x)$$

  is surjective, for all $|B| \geq |A| \geq r_0$.

We are finally ready to state our second result which illustrates the fact that the structure maps of a finitely generated object need to satisfy strong compatibility conditions.

Theorem F. *Fix a prime number p. Let $\mathcal{P}$ be the family of finite abelian p-groups and consider a finitely generated object $X \in \mathcal{AP}$. Then the restriction of X to $\mathcal{C}_p$ and $\mathcal{F}_n^p$ for $n \geq 1$, is eventually torsion-free and stably surjective.*

Proof. See Theorem 1.12.6. □

Global homotopy theory. A good source of examples of finitely generated objects satisfying representation stability comes from global stable homotopy theory: the study of spectra with a uniform and compatible group action for all groups in a specific class. These are particular kind of spectra that give rise to cohomology theories on G-spaces for all groups in the chosen class. The fact that all these individual cohomology theories come from a single object imposes extra compatibility conditions as the group varies. In this thesis we will use the framework of global homotopy theory developed by Schwede [**81**]. His approach has the advantage of being very concrete as the category of global spectra is the usual category of orthogonal spectra but with a finer notion of equivalence, called global equivalence. As any orthogonal spectrum is a global spectrum, this approach comes with a good range of examples. For instance, there are global analogues of the sphere spectrum, cobordism and K-theory spectra, Borel cohomology and many others. It is a special feature of such a global spectrum X that the assignment $G \mapsto \pi_0(\Phi^G X) \otimes \mathbb{Q}$ defines an object $\underline{\Phi}_0(X) \in \mathcal{A}$, where we put $k = \mathbb{Q}$. The connection with global homotopy theory is even stronger as there is a triangulated equivalence

$$\Phi^{\mathcal{G}} \colon \mathrm{hSp}_{\mathcal{G}}^{\mathbb{Q}} \simeq_{\triangle} \mathcal{D}(\mathcal{A})$$

between the homotopy category of rational $\mathcal{G}$-global spectra and the derived category of $\mathcal{A}$ [**81**, 4.5.29]. This equivalence is compatible with geometric fixed points in the sense that $\pi_*(\Phi^G X) = H_*(\Phi^{\mathcal{G}}(X))(G)$. As an immediate consequence of Lemma 2.4.7 we obtain the following application to global homotopy theory which highlights the good behaviour of the geometric fixed points functor on the full subcategory of compact global spectra. Recall that a rational global spectrum X is said to be compact if the corepresentable functor $[X, -]^{\mathcal{G}}$ preserves arbitrary sums.

Theorem G. *Fix a prime number p. Let $\mathcal{P}$ be the family of finite abelian p-groups and let X be a rational $\mathcal{P}$-global spectrum. If X is compact, then for all $k \in \mathbb{Z}$ the geometric fixed points homotopy groups $\underline{\Phi}_k(X) \in \mathcal{AP}$ satisfy the conditions of Theorems D and F.*

An interesting source of examples is given by the rational n-th symmetric product spectra.

Example. For $n \geq 1$, we let Sp^n denote the orthogonal spectrum whose value at inner product space V is given by

$$\mathrm{Sp}^n(V) = (S^V)^{\times n}/\Sigma_n.$$

Its rationalization is a compact rational $\mathcal{P}$-global spectrum by [**46**, 2.10, 5.1]. Therefore its geometric fixed points are eventually torsion, stably surjective and they satisfy central stability.

This connection to global homotopy theory suggests a deeper study of the derived category of $\mathcal{A}$.

The derived category. An explicit model for the derived category of $\mathcal{A}$ is given by the homotopy category of complexes of projective objects $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$. We have seen that projective objects of $\mathcal{A}$ are closed under $\otimes$ and $\underline{\mathrm{Hom}}$. It follows that the tensor product and internal hom functors of $\mathcal{A}$ descend to give a closed symmetric monoidal structure on $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ which is compatible with the triangulation. Therefore $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ has the structure of a *tensor triangulated category.* In this part we will mostly focus on the full subcategory of perfect complexes $\mathrm{K}(\mathcal{A})_{\mathrm{perf}} \subset \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ which is also a tensor triangulated category. Recall that a complex is said to be perfect if it is homotopy equivalent to a bounded complex of finitely generated projective objects. We will see that the category $\mathrm{K}(\mathcal{A})_{\mathrm{perf}}$ is much more manageable than the homotopy category $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ whilst still retaining a lot of its key features. In fact we will see that the category $\mathrm{K}(\mathcal{A}_{\mathrm{perf}})$ coincides with the subcategory of compact objects of $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$.

Tensor triangulated geometry. A triangulated subcategory $\mathcal{I} \subset \mathrm{K}(\mathcal{A})_{\mathrm{perf}}$ is a *thick ideal* if it is closed under retracts and it satisfies

$$X \in \mathcal{I} \text{ and } Y \in \mathrm{K}(\mathcal{A})_{\mathrm{perf}} \Rightarrow X \otimes Y \in \mathcal{I}.$$

We say that $\mathcal{I}$ is *prime* if in addition we have: $X \otimes Y \in \mathcal{I} \Rightarrow X \in \mathcal{I}$ or $Y \in \mathcal{I}$. The second main goal of this paper is to describe the lattice of thick ideals in the subcategory of perfect complexes $\mathrm{K}(\mathcal{A})_{\mathrm{perf}}$. One possible way to obtain such a classification is by describing the Balmer spectrum: the space of prime ideals of the tensor triangulated category [**6**]. Inspired by the Zariski spectrum of a ring in commutative algebra, it allows one to do geometry in the setting of triangulated categories. The Balmer spectrum provides a unifying language for several classification results performed in many areas of mathematics: stable homotopy theory [**25**], commutative algebra [**66**], algebraic geometry [**90**] and representation theory [**17**]. In the eyes of tensor triangulated geometry these classifications are all descriptions of Balmer spectra.

Our motivation for studying the Balmer spectrum of $\mathrm{K}(\mathcal{A})_{\mathrm{perf}}$ comes from global homotopy theory. Wimmer [**96**, 3.20] showed that the triangulated equivalence $\Phi^{\mathcal{G}} \colon \mathrm{Sp}_{\mathcal{G}}^{\mathbb{Q}} \simeq \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ can be refined to a symmetric monoidal equivalence with respect to the smash product of global spectra. Therefore, we can reduce the study of the tensor triangulated geometry of the rational global stable homotopy category to that of $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$. In this part we aim to demonstrate that the tensor triangulated geometry of the rational global homotopy category is far from being obvious (despite the simplification of working over the rationals and only with finite groups) and its study often requires highly non-trivial combinatorial arguments. Our work is a first step towards a classification of the thick ideals in the global stable homotopy category which will be the global analogue of the chromatic filtration in classical stable homotopy theory due to Devinatz–Hopkins–Smith [**25**, **50**].

Rigidity. Despite sharing several formal properties with the derived category of a ring, the homotopy category $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ is not a stable homotopy category in the sense of [**52**, 1.1.4] as the generators e_G are not strongly dualizable (unless G is trivial). Recall that e_G is strongly dualizable if the canonical map

$$\underline{\mathrm{Hom}}(e_G, \mathbb{1}) \otimes X \to \underline{\mathrm{Hom}}(e_G, X)$$

is a quasi-isomorphism for all complexes X. We also say that $\mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ is *rigid* if the generators e_G are strongly dualizable for all $G \in \mathcal{U}$. It is then a formal consequence of the definition that all perfect complexes are strongly dualizable.

Theorem H. *Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$. Then $\mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ is rigid if and only if $\mathcal{U}$ is a groupoid. Furthermore, a complex is strongly dualizable if and only if it belongs to the thick subcategory generated by the unit object $\mathbb{1} = e_1$.*

Proof. See Propositions 2.5.4 and 2.5.6. ☐

Not much is known about the tensor triangulated geometry of non-rigid triangulated categories; there are only a couple of complete descriptions of such Balmer spectra in the literature, see for instance [**4**] and [**97**], and there is no general strategy. This paper provides two new complete descriptions of such Balmer spectra as well as several results on the tensor triangulated geometry of $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ which might be of independent interest. Our approach is purely algebraic and relies on a good understanding of the homology of a perfect complex.

Homology of perfect complexes. We say that an object $X \in \mathcal{A}$ is *torsion* if for all $x \in X(H)$ there exists an epimorphism $\alpha \colon G \to H$ such that $\alpha^*(x) = 0$ in $X(G)$. We say that X is *torsion-free* if the maps $\alpha^* \colon X(H) \to X(G)$ are injective for all α. It is not difficult to see that the generators e_G and hence the projective objects are torsion-free. As any perfect complex consists of torsion-free objects, it is natural to ask to what extent the homology of a perfect complex can be torsion. An answer to this questions is given by the following result.

THEOREM I. *Let $\mathcal{U}$ be a nontrivial multiplicative global family of finite groups. If a perfect complex in* $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ *has torsion homology, then it has trivial homology.*

PROOF. See Theorem 2.6.7. □

Using the equivalence between $\mathrm{K}(\mathcal{AU})$ and the rational global stable homotopy category, we deduce that the homotopy groups of the geometric fixed points of any rational compact $\mathcal{U}$-global spectrum admits a torsion-free element. This implies that the geometric fixed points homotopy groups are nonzero for infinitely many groups.

Support theory. In order to distinguish different thick ideals in the category of perfect complexes we use support theory. We define the support of a perfect complex X to be

$$\mathrm{supp}(X) = \{G \in \mathcal{U} \mid H_*(X(G)) \neq 0\}.$$

We extend this to a collection of perfect complexes $\mathcal{S}$ by $\mathrm{supp}(\mathcal{S}) = \bigcup_{X\in\mathcal{S}} \mathrm{supp}(X)$. We can reinterpret Theorem I as saying that the support of a perfect complex contains the upwards closure of a group $H \in \mathcal{U}$, namely the set $\{G \in \mathcal{U} \mid \mathcal{U}(G, H) \neq \emptyset\}$. It follows that any two given perfect complexes share a large portion of their support. Therefore it is natural to ask to what extent this notion of support detects thick ideals. We show in Remark 2.9.9 that there exist thick ideals which are not determined by their support. However, these ideals cannot be finitely generated. Recall that an ideal $\mathcal{I} \subset \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ is said to be finitely generated if there exists a finite collection of perfect complexes $\mathcal{S}$ such that $\mathcal{I} = \mathrm{thickid}(\mathcal{S})$.

THEOREM J. *Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$ and consider $X, Y \in \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$. Then we have* $\mathrm{supp}(X) \subseteq \mathrm{supp}(Y)$ *if and only if* $X \in \mathrm{thickid}(Y)$. *More generally, given thick ideals* $\mathcal{I}, \mathcal{J} \in \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ *with $\mathcal{J}$ finitely generated, then* $\mathrm{supp}(\mathcal{I}) \subseteq \mathrm{supp}(\mathcal{J})$ *if and only if* $\mathcal{I} \subseteq \mathcal{J}$.

PROOF. See Theorem 2.7.9. □

As an immediate consequence we deduce that any (not necessarily finitely generated) thick ideal of $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ is radical, see Proposition 2.7.10. We note that since $\mathrm{K}(\mathcal{AU})$ is not rigid, this is not automatic and an argument is needed. It follows that we can classify thick ideals of $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ by calculating its Balmer spectrum [**6**, 4.10].

Balmer spectra. The Balmer spectrum is the space of prime ideals

$$\mathrm{spc}(\mathcal{U}) = \{\wp \subset \mathrm{K}(\mathcal{AU})_{\mathrm{perf}} \mid \wp \text{ prime}\}$$

endowed with the Zariski topology. It is not difficult to see that for any $G \in \mathcal{U}$, the full subcategory of perfect complexes given by

$$\wp_G = \{X \mid H_*(X(G)) = 0\}$$

defines a prime ideal, which we call a *group prime*. We show that there are no containments amongst group primes (unless the groups are isomorphic) and that not all prime ideals in the category of perfect complexes are group primes. In fact, using our support theory we show that any finitely generated prime ideal $\wp$ can be written as an intersection of group primes $\bigcap_{G\in U(\wp)} \wp_G$ for a set $U(\wp) \subseteq \mathcal{U}$. This reduces the classification of finitely generated prime ideals to a purely combinatorial problem, namely determining the subset $U(\wp)$. This is still a highly non-trivial problem which relies on a good understanding of the lattice of groups of $\mathcal{U}$. Nonetheless, we obtain two general results on $U(\wp)$ which we summarize in the following theorem.

THEOREM K. *Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$. Define a preorder on $\mathcal{U}$ by $G \gg H$ if and only if $\mathcal{U}(G, H) \neq \emptyset$.*

(a) *Let $\wp$ be a finitely generated prime ideal of* $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$. *Then $U(\wp)$ has a maximal element with respect to $\gg$ if and only if $\wp$ is a group prime.*

(b) *Suppose that $\mathcal{U}$ is multiplicative global family. Then the thick ideal $\wp_{\mathcal{U}} = \bigcap_{G\in\mathcal{U}} \wp_G$ is prime in $\mathrm{K}(\mathcal{A})_{\mathrm{perf}}$, and the zero ideal is prime in $\mathrm{K}(\mathcal{A}\mathcal{U})_{\mathrm{perf}}$.*

Proof. See Corollary 2.8.5 and Lemma 2.8.8. □

At present we do not have a general strategy to tackle the classification of prime ideals of $\mathrm{K}(\mathcal{A})_{\mathrm{perf}}$. This is mainly due to the fact that objects in $\mathcal{A}$ can be torsion in different ways (see section 10). Instead we focus on two small subcategories of finite groups: elementary abelian p-groups and cyclic p-groups. These examples demonstrate how the closure properties of the family of groups considered affect the tensor triangulated geometry of the category.

Theorem L. *Fix a prime number p. Let $\mathcal{E}$ be subcategory of elementary abelian p-groups and choose a skeleton $\mathcal{E}'$ for $\mathcal{E}$. The Balmer spectrum $\mathrm{spc}(\mathcal{E})$ consists of the group primes and the zero ideal:*

$$\begin{tikzcd}[column sep=small] & & & 0 \arrow[dlll, no head] \arrow[dll, no head] \arrow[d, no head] \arrow[drr, no head] & & & \\ \wp_1 & & \wp_{C_p} & \wp_{C_p^2} & \cdots & \wp_{C_p^n} & \cdots \end{tikzcd}$$

Furthermore, there is an order preserving bijection

$$\mathcal{I}\colon \{\textit{empty or cofinite subsets of } \mathcal{E}'\} \leftrightarrow \{\textit{thick ideals of } \mathrm{K}(\mathcal{A}\mathcal{E})_{\mathrm{perf}}\}$$

given by $\mathcal{I}(V) = \{X \mid \mathrm{supp}(X) \subseteq V\}$.

Proof. See Theorem 2.9.4 and Corollary 2.9.5. □

In the previous diagram a solid line denotes a containment between the prime ideals. From a topological point of view this means that the zero ideal is the only closed point, and it is contained in the closure of any other prime ideal. In the diagram below instead, we use a dotted line to denote that there is no containment between the prime ideals, but the top prime is contained in the closure of the subset $\{\wp_{C_{p^n}} \mid n \gg 0\}$. We refer to the proof of the result for a description of the topology of the Balmer spectrum.

Theorem M. *Fix a prime number p. Let $\mathcal{C}$ be the subcategory of cyclic p-groups and choose a skeleton $\mathcal{C}'$ for $\mathcal{C}$. The Balmer spectrum $\mathrm{spc}(\mathcal{C})$ consists of the group primes and the ideal*

$$\wp_{tors} = \{X \mid H_*(X(C_{p^n})) = 0 \textit{ for } p^n \gg 0\}$$

which is not finitely generated:

$$\begin{tikzcd}[column sep=small] & & \wp_{tors} \arrow[drr, dotted, no head] \arrow[drrr, dotted, no head] & & & & \\ \wp_1 & \wp_{C_p} & \wp_{C_{p^2}} & \cdots & \wp_{C_{p^n}} & \wp_{C_{p^{n+1}}} & \cdots \end{tikzcd}$$

Put $\mathcal{S}(\mathcal{C}) = \{V \subseteq \mathcal{C}' \sqcup \{*\} \mid * \notin V$ or V cofinite$\}$. Then there is an order preserving bijection

$$\mathcal{I}\colon \mathcal{S}(\mathcal{C}) \leftrightarrow \{\textit{thick ideals of } \mathrm{K}(\mathcal{A}\mathcal{C})_{\mathrm{perf}}\}$$

where

$$\mathcal{I}(V) = \begin{cases} \{X \mid \mathrm{supp}(X) \subseteq V\} & \text{if } * \in V \\ \{X \mid \mathrm{supp}(X) \subseteq V,\ H_*(X) \text{ torsion}\} & \text{if } * \notin V. \end{cases}$$

Proof. See Theorem 2.9.8 and Corollary 2.9.10. □

Remark. We obtain a classification of thick ideals in the rational global stable homotopy category by replacing perfect complexes with compact rational global spectra, and $H_*(X(G))$ with $\pi_*(\Phi^G X)$.

Related work. Our study of the representation theory and homological algebra of $\mathcal{AU}$ is inspired by earlier work in the categories of FI-modules [**22**, **23**] and VI-modules [**32**, **64**]. Our Theorem C recovers the result that the category of VI-modules is locally noetherian, which was proved independently by Sam–Snowden [**74**, 8.3.3] and Gan–Li [**31**]. Versions of our representation stability theorems were already known to hold for the category of FI-modules [**23**], VI-modules [**32**] and complemented categories [**71**]. Finally our study of indecomposable injective objects recovers part of the classification of injective VI-modules due to Nagpal [**64**].

The tensor geometry of $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ has already been studied by Fei in the case where $\mathcal{U}$ is finite [**97**], and by Antieau–Stevenson [**2**] in the case where $\mathcal{U}^{\mathrm{op}}$ is a Dynkin quiver.

Nonetheless, to the best of our knowledge Theorems C-M are new and they generalize several known results to a wider class of examples of interest.

Organization of Part 2. This part is divided into two chapters. In the first three sections of chapter 1, we introduce the abelian category $\mathcal{A}$ and the necessary notation that we will use throughout. The main results of chapter 1 are contained in sections 11 and 12, and they rely almost exclusively on the results of sections 9 and 10.

In chapter 2 we study the derived category of $\mathcal{A}$ and its Balmer spectrum. In the first four sections we show that the homotopy category of complexes of projective objects is a model for the derived category of $\mathcal{A}$, and explicitly describe its tensor triangulated structure. We study the homology of the perfect complexes in section 6 where we use the notion of complete subcategory which is introduced in section 8 of chapter 1. The main results on the Balmer spectrum are contained in sections 7, 8 and 9 where the necessary background is also introduced.

CHAPTER 1

The abelian category $\mathcal{A}$

In this chapter we introduce the abelian category $\mathcal{A}$ and study its homological algebra. We classify all the indecomposable projective objects and obtain a partial classification of indecomposable injective objects. We define and study torsion, absolutely torsion and torsion-free objects and give various equivalent characterizations. We then turn to study finiteness conditions on the abelian category and give examples of abelian subcategories of $\mathcal{A}$ which are locally noetherian. Finally, we prove that any finitely presented object satisfies a variant of Church–Farb's representation stability. Along the way we give explicit combinatorial models for the tensor product and the internal hom functor.

1. Preliminaries

We start by introducing the main object of study of this paper, the abelian category $\mathcal{A}$.

Definition 1.1.1.

- Two homomorphisms $\varphi, \psi\colon G \to H$ between finite groups are conjugate if there exists $h \in H$ such that $h\varphi h^{-1} = \psi$. We write $[\varphi]$ for the conjugacy class of $\varphi\colon G \to H$.
- Denote by $\mathcal{G}$ the category of finite groups and conjugacy classes of epimorphisms. We will write $\mathrm{Out}(G) = \mathcal{G}(G, G)$.

Lemma 1.1.2. *Let $\alpha\colon H \to G$ be a surjective group homomorphism between finite groups. Then $[\alpha]$ is an epimorphism in $\mathcal{G}$.*

Proof. For $k \in K$, we write $c_k\colon K \to K$ for the conjugation homomorphism $h \mapsto khk^{-1}$. Consider two surjective group homomorphisms $\beta, \gamma\colon G \to K$, and suppose that $[\beta\alpha] = [\gamma\alpha]$. This means that $c_k\beta\alpha = \gamma\alpha$ for some $k \in K$. Since α is surjective we have $c_k\beta = \gamma$ which shows that $[\beta] = [\gamma]$. $\square$

Definition 1.1.3. Fix a field k of characteristic zero and set $\mathcal{A} = [\mathcal{G}^{\mathrm{op}}, \mathrm{Vect}_k]$. More generally, we put $\mathcal{AU} = [\mathcal{U}^{\mathrm{op}}, \mathrm{Vect}_k]$ for a replete full subcategory $\mathcal{U} \leq \mathcal{G}$.

Remark 1.1.4. The category $\mathcal{A}$ is abelian and admits limits and colimits for all small diagrams. These (co)limits are computed pointwise, so they are preserved by the evaluation functors $\psi^G\colon \mathcal{A} \to \mathrm{Vect}_k$.

Definition 1.1.5. Let $G \in \mathcal{G}$ and V be an $\mathrm{Out}(G)$-representation. We define the following objects of $\mathcal{A}$:

- We define e_G by $e_G(T) = k[\mathcal{G}(T, G)]$. Yoneda's Lemma tells us that $\mathcal{A}(e_G, X) = X(G)$.
- We define objects $e_{G,V}$ and $t_{G,V}$ by
$$e_{G,V}(T) = V \otimes_{k[\mathrm{Out}(G)]} k[\mathcal{G}(T, G)] \qquad t_{G,V}(T) = \mathrm{Hom}_{k[\mathrm{Out}(G)]}(k[\mathcal{G}(G, T)], V).$$
- We put $c_G = e_{G,k} = e_G^{\mathrm{Out}(G)}$ for the trivial $\mathrm{Out}(G)$-representation k. Here we used that coinvariants and invariants are isomorphic in this context.
- We define $s_{G,V}$ as the image of the canonical map $e_{G,V} \to t_{G,V}$ so that
$$s_{G,V}(T) = \begin{cases} e_{G,V}(T) & \text{if } G \simeq T \\ 0 & \text{if } G \not\simeq T. \end{cases}$$

- For a convex subcategory $\mathcal{C} \leq \mathcal{G}$, we define the "characteristic function" $\chi_{\mathcal{C}}$ by

$$\chi_{\mathcal{C}}(T) = \begin{cases} k & \text{if } T \in \mathcal{C} \\ 0 & \text{if } T \notin \mathcal{C}. \end{cases}$$

Remark 1.1.6. The abelian category $\mathcal{A}$ is Grothendieck with generators given by e_G for all $G \in \mathcal{G}$. This means that filtered colimits are exact and that any $X \in \mathcal{A}$ admits an epimorphism $P \to X$ where P is a direct sum of generators.

Lemma 1.1.7. *For $G \in \mathcal{G}$, we let $\mathcal{M}_G$ denote the category of $k[\mathrm{Out}(G)]$-modules. Then the evaluation functor*

$$\mathrm{ev}_G \colon \mathcal{A} \to \mathcal{M}_G, \quad X \mapsto X(G)$$

has a left and right adjoint which are respectively given by $e_{G,\bullet}$ and $t_{G,\bullet}$. In particular, $e_{G,V}$ is projective and $t_{G,V}$ is injective.

Proof. The unit of the adjunction $\eta_V \colon V \to e_{G,V}(G) = V$ is a natural isomorphism, and the counit is given by

$$\epsilon_X(T) \colon e_{G,X(G)}(T) \to X(T), \quad x \otimes [\alpha] \mapsto \alpha^*(x)$$

for all $T \in \mathcal{G}$. Similarly, the counit map $t_{G,V}(G) \to V$ is a natural isomorphism, and the unit is given by

$$\eta_X(T) \colon X(T) \to t_{G,X(G)}(T), \quad x \mapsto ([\beta] \mapsto \beta^*(x))$$

for all $T \in \mathcal{G}$. We leave to the reader to check that these maps are natural and that they satisfy the triangular identities. The second part of the claim follows immediately from the fact that the evaluation functor is exact as colimits are computed pointwise. □

2. Closed monoidal structure

It is convenient to add a bit of structure on $\mathcal{A}$.

Definition 1.2.1. We give $\mathcal{A}$ the symmetric monoidal structure given by $(X \otimes Y)(T) = X(T) \otimes Y(T)$. The unit object $\mathbb{1}$ is the constant functor with value k (or equivalently, $\mathbb{1} = e_1$). We also put

$$\underline{\mathrm{Hom}}(X, Y)(T) = \mathcal{A}(e_T \otimes X, Y).$$

Standard arguments show that this defines an object of $\mathcal{A}$ with

$$\mathcal{A}(W, \underline{\mathrm{Hom}}(X, Y)) \simeq \mathcal{A}(W \otimes X, Y),$$

so $\mathcal{A}$ is a closed symmetric monoidal category. We write DX for $\underline{\mathrm{Hom}}(X, \mathbb{1})$, and call this the *dual* of X.

Remark 1.2.2. Note that the tensor product is both left and right exact, so all objects are flat.

Remark 1.2.3. We warn the reader that DX is not obtained from X by taking levelwise duals, so the canonical map $X \otimes DX \to \underline{\mathrm{Hom}}(X, X)$ is usually not an isomorphism. To demonstrate this consider the case $X = e_G$ for any non-trivial group G. If we evaluate at the trivial group, we find $e_G(1) \otimes De_G(1) = 0$ and $\underline{\mathrm{Hom}}(e_G, e_G)(1) = k[\mathrm{Out}(G)]$. Therefore the map is far from being an isomorphism.

For the rest of this section we study the effect of the tensor product and internal hom functor on the generators. The main results are Propositions 1.2.12 and 1.2.19 and they both rely on the following notion.

Definition 1.2.4. Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$. A *permuted family* of groups consists of a finite group Γ, a finite Γ-set A, a family of groups $G_a \in \mathcal{U}$ for each $a \in A$, and a system of isomorphisms $\gamma_* \colon G_a \to G_{\gamma(a)}$ (for $\gamma \in \Gamma$ and $a \in A$) satisfying the functoriality conditions $1_* = 1$ and $(\delta\gamma)_* = \delta_*\gamma_*$. The system of isomorphisms gives maps $\mathrm{stab}_\Gamma(a) \to \mathrm{Aut}(G_a)$ for each

$a \in A$. We say that the family is *outer* if the image of this map contains the inner automorphism group $\mathrm{Inn}(G_a)$ for all a. Given a permuted family $\underline{G}$ which is outer, we define the set

$$\widetilde{B}(\underline{G})(T) = \{(a, \alpha) \mid a \in A,\ \alpha \in \mathrm{Epi}(T, G_a)\}.$$

The group Γ acts on $\widetilde{B}(\underline{G})(T)$ via the formula $\gamma \cdot (a, \alpha) = (\gamma(a), \gamma_* \circ \alpha)$. We define $B(\underline{G})(T) = \widetilde{B}(\underline{G})(T)/\Gamma$ and $F(\underline{G})(T) = k[B(\underline{G})(T)]$. This is contravariantly functorial in T, so $F(\underline{G}) \in \mathcal{AU}$.

Proposition 1.2.5. *For all $X \in \mathcal{AU}$ there is a natural isomorphism*

$$\mathcal{AU}(F(\underline{G}), X) = \left(\prod_{a \in A} X(G_a)\right)^{\Gamma}.$$

If we choose a subset $A_0 \subset A$ containing one element of each Γ-orbit, we get an isomorphism

$$F(\underline{G}) = \bigoplus_{a \in A_0} e_{G_a}^{\mathrm{stab}_\Gamma(a)}.$$

Thus, $F(\underline{G})$ is finitely projective.

Proof. We can reduce to the case where A is a single orbit, say $A = \Gamma a \simeq \Gamma/\Delta$, where $\Delta = \mathrm{stab}_\Gamma(a)$. We can define $\phi\colon \mathrm{Epi}(T, G_a)/\Delta \to B(\underline{G})(T)$ by $\phi[\alpha] = [a, \alpha]$. If $[b, \beta] \in B(\underline{G})(T)$ then $b = \gamma(a)$ for some a. We can then put $\alpha = \gamma_*^{-1} \circ \beta\colon T \to G_a$ and we find that $[b, \beta] = \phi[\alpha]$. On the other hand, if $\phi(\alpha) = \phi(\alpha')$ then there exists $\gamma \in \Gamma$ with $(\gamma(a), \gamma_* \circ \alpha = \alpha')$ which means that $\gamma \in \Delta$ and $[\alpha] = [\alpha']$ in $\mathrm{Epi}(T, G_a)/\Delta$. It follows that ϕ is a natural bijection. Thus, if we let Φ denote the image of Δ in $\mathrm{Out}(G_a)$, we have $F(\underline{G}) \simeq e_{G_a}^{\Phi}$. Note that the inclusion $e_{G_a}^{\Phi} \leq e_{G_a}$ is split by the map $x \to 1/|\Phi| \sum_{\phi \in \Phi} \phi \cdot x$. It follows that $e_{G_a}^{\Phi}$ is projective. $\square$

Definition 1.2.6. Let $(G_i)_{i \in I}$ be a finite family of groups in $\mathcal{U}$ with product $P = \prod_i G_i$. We say that a subgroup $W \leq P$ is *wide* if all the projections $\pi_i\colon W \to G_i$ are surjective. We say that a homomorphism $f\colon T \to P$ is *wide* if all the homorphisms $\pi_i \circ f$ are surjective, or equivalently $f(T)$ is a wide subgroup of P. For $G, H \in \mathcal{U}$, we let $\mathrm{Wide}(G, H)$ denote the set of wide subgroups of $G \times H$ which belong to $\mathcal{U}$. This set is covariantly functorial in G and H with respect to morphism in $\mathcal{U}$. Given $\varphi\colon G' \to G$ in $\mathcal{U}$ and $W' \in \mathrm{Wide}(G', H)$, we put $\varphi_* W' = (\varphi \times \mathrm{id}_H)(W')$ which is wide in $G \times H$. This comes with a map $j_\varphi\colon W' \to \varphi_* W'$ which makes the following diagram

$$\begin{tikzcd} G' \times H \arrow[r, "\varphi \times \mathrm{id}"] & G \times H \\ W' \arrow[u, hook] \arrow[r, "j_\varphi"] & \varphi_* W' \arrow[u, hook] \end{tikzcd}$$

commute. The assignment $W' \mapsto \varphi_* W'$ defines a map $\varphi_*\colon \mathrm{Wide}(G', H) \to \mathrm{Wide}(G, H)$ between the set of wide subgroups. Similar functoriality holds for H as well.

Example 1.2.7. If G_1 and G_2 are simple, then the wide subgroups of $G_1 \times G_2$ are the whole group $G_1 \times G_2$, and $\mathrm{Gr}(\alpha) = \{(g, \alpha(g)) \mid g \in G_1\}$ the graphs of isomorphisms $\alpha\colon G_1 \to G_2$. In particular, if G_1 and G_2 are not isomorphic, then the only wide subgroup is $G_1 \times G_2$. Similarly, suppose that G_1 and G_2 are not necessarily simple, but have coprime orders; we again find that the only wide subgroup is $G_1 \times G_2$ itself.

Remark 1.2.8. Suppose we have normal subgroups $N_i \lhd G_i$ for $i = 1, 2$ and an isomorphism $\alpha\colon G_1/N_1 \to G_2/N_2$. We can then define

$$H(N_1, \alpha, N_2) = \{(x_1, x_2) \in G_1 \times G_2 \mid \alpha(x_1 N_1) = x_2 N_2\} \leq G_1 \times G_2.$$

This is easily seen to be a wide subgroup, and we claim that every wide subgroup arises in this way. Indeed, suppose $K \leq G_1 \times G_2$ is wide. Put

$$N_1 = \{n_1 \in G_1 \mid (n_1, 1) \in K\},$$

and similarly for N_2. If $n_1 \in N_1$ and $g_1 \in G_1$ then wideness gives $g_2 \in G_2$ such that $(g_1, g_2) \in K$. It follows that the element $(g_1 n_1 g_1^{-1}) = (g_1, g_2)(n_1, 1)(g_1, g_2)^{-1} \in K$ and that N_1 is normal. The same argument shows that N_2 is normal in G_2 too. This means that K is the preimage in $G_1 \times G_2$ of the subgroup $\bar{K} = K/(N_1 \times N_2) \leq (G/N_1) \times (G/N_2)$. We now find that the projections $\pi_i \colon \bar{K} \to G_i/N_i$ are both isomorphisms, so we can define $\alpha \colon \pi_2 \pi_1^{-1} \colon G/N_1 \to G/N_2$. It is now easy to see that $K = H(N_1, \alpha, N_2)$, as required.

Definition 1.2.9. Any wide subgroup $W \leq G \times H$ is of the form $W = H(N_1, \alpha, N_2)$ for some isomorphism $\alpha \colon G/N_1 \to H/N_2$. We call G/N_1 the *left spread* and H/N_2 the *right spread* of W. We will write $\mathrm{ls}(W)$ and $\mathrm{rs}(W)$ for the left and right spread of W respectively.

Remark 1.2.10. The left and right spread of a wide subgroup W depend on the ambient group $G \times H$.

Definition 1.2.11. Let $\underline{W}$ be the tautological family indexed by $\mathrm{W}(G, H)$, so the group indexed by $U \in \mathrm{W}(G, H)$ is U itself. Then $G \times H$ acts on $\mathrm{W}(G, H)$ by conjugation. We use this to regard $\underline{W}$ as a permuted family, and thus define a finitely projective object $F(\underline{W}) \in \mathcal{AU}$.

We now consider tensor products of generators.

Proposition 1.2.12. *Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$. Then $e_G \otimes e_H$ is naturally isomorphic to $F(\underline{W})$. In particular, $e_G \otimes e_H$ is finitely projective.*

Proof. Consider a pair $(\alpha, \beta) \in \mathrm{Epi}(T, G) \times \mathrm{Epi}(T, H)$. This gives a wide subgroup $U = \langle \alpha, \beta \rangle(T) \leq G \times H$, and we can regard (α, β) as a surjective homomorphism from T to U, so we have an element $\phi(\alpha, \beta) = (U, \langle \alpha, \beta \rangle) \in \widetilde{B}(\underline{W})(T)$. This is easily seen to give a $(G \times H)$-equivariant natural bijection

$$\phi \colon \mathrm{Epi}(T, G) \times \mathrm{Epi}(T, H) \to \widetilde{B}(\underline{W})(T).$$

It follows easily that we get an induced bijection $\mathcal{U}(T, G) \times \mathcal{U}(T, H) \to B(\underline{W})(T)$ and an isomorphism $e_G \otimes e_H \to F(\underline{W})$ as required. $\square$

Remark 1.2.13. If G and H are abelian, then $G \times H$ acts trivially on $\underline{W}$ and so $e_G \otimes e_H = \bigoplus_{U \in \mathrm{W}(G,H)} e_U$.

Remark 1.2.14. It is not true that $e_G \otimes e_H$ is always a direct sum of objects of the form e_K. In particular, this fails when $G = H = D_8$. To see this, let N be the subgroup of G isomorphic to C_4, and put $W = \{(g, h) \in G \times H \mid gN = hN\}$. This is wide, and has index 2 in $G \times H$, so it is normal in $G \times H$. The quotient $Q = (G \times H)/W$ acts on e_W by outer automorphisms, and the natural map $e_W \to e_G \otimes e_H$ factors through the coinvariants $(e_W)_Q \simeq e_W^Q$, so the corresponding summand is not of the form e_K.

Definition 1.2.15. A *virtual homomorphism* from G to H is a pair $\alpha = (A, A')$ where $A' \lhd A \leq G \times H$ and A is wide and $A' \cap (1 \times H) = 1$. We write $\mathrm{VHom}(G, H)$ for the set of virtual homomorphisms. We then let $\underline{Q}$ be the parametrised family of groups with $Q_\alpha = A/A'$ for all $\alpha = (A, A') \in \mathrm{VHom}(G, H)$. We call Q_α the *spread* of α. Note that $G \times H$ acts compatibly on $\mathrm{VHom}(G, H)$ and Q_α by conjugation. We use this to regard $\underline{Q}$ as a permuted family, and thus to define a finitely projective object $F(\underline{Q}) \in \mathcal{AU}$.

Example 1.2.16. For any surjective homomorphism $u \colon G \to H$, we can define

$$A = A' = \mathrm{graph}(u) = \{(g, u(g)) \mid g \in G\}.$$

This gives a virtual homomorphism with trivial spread. We claim that every virtual homomorphism with trivial spread arises in this way from a unique homomorphism. Indeed, let $\alpha = (A, A)$ be any such virtual homomorphism and consider the projection map $A \leq G \times H \to G$. The condition $A \cap (1 \times H) = 1$ ensures that every element $g \in G$ has a unique preimage $(g, u(g)) \in A$ under the projection. It is easy to check that the assignment $u \colon G \to H$ defines a surjective group homomorphism, and by construction $A = \mathrm{graph}(u)$.

Example 1.2.17. Consider a virtual homomorphism $\alpha = (A, A') \in \mathrm{VHom}(1, G)$. The group A must be wide in $1 \times G$, which just means that $A = 1 \times G$. The group $A' \leq 1 \times G$ must satisfy $A' \cap (1 \times G) = 1$, which means that $A' = 1$. Thus, there is a unique virtual homomorphism $\alpha = (1 \times G, 1)$, whose spread is G.

Example 1.2.18. Consider a virtual homomorphism $\alpha = (A, A') \in \mathrm{VHom}(G, 1)$. We find that A must be equal to $G \times 1$ (which we identify with G) and A' can be any normal subgroup of G.

Proposition 1.2.19. *Let $\mathcal{U}$ be a multiplicative global family of finite groups. Fix groups $G, H \in \mathcal{U}$ and let $\underline{Q}$ be the parametrised family of virtual homomorphisms from G to H. Then $\underline{\mathrm{Hom}}(e_G, e_H)$ is naturally isomorphic to $F(\underline{Q})$ (and so is a finitely generated projective object of $\mathcal{AU}$).*

Before proving the Proposition we need a little bit of preparation.

Definition 1.2.20. Fix groups $G, H \in \mathcal{U}$. We let $\mathcal{M}(T)$ denote the k-linearization of the set

$$\{(A, A', \theta) \mid (A, A') \in \mathrm{VHom}(G, H), \quad \theta \in \mathrm{Epi}(T, A/A')\}.$$

This set has an action of $T \times G \times H$ by conjugation. The functoriality in T is the obvious one given by precomposing θ with an epimorphism $T' \to T$. The space $\mathcal{M}(T)$ has a natural filtration induced by the spread of a virtual homomorphisms

$$0 = \mathcal{M}(T)_{\leq 0} \subseteq \mathcal{M}(T)_{\leq 1} \subseteq \mathcal{M}(T)_{\leq 2} \subseteq \ldots \subseteq \mathcal{M}(T)_{\leq n} \subseteq \ldots \subseteq \mathcal{M}(T)$$

where

$$\mathcal{M}(T)_{\leq n} = \{(A, A', \theta) \in \mathcal{M}(T) \mid |A/A'| \leq n\}.$$

Definition 1.2.21. Fix groups $G, H, T \in \mathcal{U}$. We let $\mathcal{N}(T)$ denote the k-linearization of the set

$$\{(W, \lambda) \mid W \in \mathrm{Wide}(T, G), \quad \lambda \in \mathrm{Epi}(W, H)\}$$

For $\varphi \colon T' \to T$ in $\mathcal{U}$, there is a map $\varphi^* \colon \mathcal{N}(T) \to \mathcal{N}(T')$ that sends a basis vector $[W, \lambda] \in N(T)$ to

$$\sum_{W = \varphi_* W'} [W', \lambda \circ j_\varphi] \in \mathcal{N}(T')$$

where the sum is over $W' \in \mathrm{Wide}(T', G)$ such that $\varphi_* W' = W$. The vector space $\mathcal{N}(T)$ has a canonical filtration

$$0 = \mathcal{N}(T)_{\leq 0} \subseteq \mathcal{N}(T)_{\leq 1} \subseteq \mathcal{N}(T)_{\leq 2} \subseteq \ldots \subseteq \mathcal{N}(T)_{\leq n} \subseteq \ldots \subseteq \mathcal{N}(T)$$

which is defined as follows. For $(W, \lambda) \in \mathcal{N}(T)$ we let $\widetilde{W}$ denote the graph subgroup of λ which is a wide subgroup of $T \times G \times H$. We can also see $\widetilde{W}$ as a wide subgroup of $T \times A$ where A is the image of $\widetilde{W}$ under the projection $\widetilde{W} \to G \times H$. We finally put

$$\mathcal{N}(T)_{\leq n} = \{(W, \lambda) \in \mathcal{N}(T) \mid |\operatorname{ls}(\widetilde{W})| = |\operatorname{rs}(\widetilde{W})| \leq n\}$$

where the left and right spread of $\widetilde{W}$ are calculated as a wide subgroup of $T \times A$ (and not of $T \times (G \times H)$).

Lemma 1.2.22. *The canonical filtration of $\mathcal{N}(T)$ is functorial with respect to epimorphisms $\varphi \colon T_1 \to T_0$. Furthermore, given $(W_0, \lambda_0) \in \mathcal{N}(T_0)$ there exists at most one pair $(W_1, \lambda_1) \in \mathcal{N}(T_1)$ such that $\varphi_* W_1 = W_0$, $\lambda_1 = \lambda_0 \circ j_\varphi$ and $|\operatorname{rs}(\widetilde{W_1})| = |\operatorname{rs}(\widetilde{W_0})|$.*

Proof. Consider pairs $(W_0, \lambda_0) \in \mathcal{N}(T_0)$ and $(W_1, \lambda_1) \in \mathcal{N}(T_1)$ such that $\varphi_* W_1 = W_0$ and $\lambda_1 = \lambda_0 \circ j_\varphi$. We let $\widetilde{W_i}$ denote the graph subgroup of λ_i which is a wide subgroup of $T_i \times (G \times H)$. The condition $\varphi_* W_1 = W_0$ ensures that we have a commutative diagram

$$\begin{tikzcd}
\widetilde{W_1} \arrow[rr] \arrow[d, hook] & & \widetilde{W_0} \arrow[d, hook] \\
T_1 \times (G \times H) \arrow[rr, "\varphi \times \mathrm{id}"] & & T_0 \times (G \times H).
\end{tikzcd}$$

This shows that the images of the two projections $\widetilde{W}_i \to G \times H$ (for $i = 0, 1$) coincide. If we denote this image by A then we can view $\widetilde{W}_i$ as a wide subgroup of $T_i \times A$. By our classification of wide subgroups, we can write

$$\widetilde{W}_i = \{(t_i, a) \in T_i \times A \mid \alpha_i(t_i) = a.A'_i\}$$

for $A'_i \triangleleft A$ and epimorphisms $\alpha_i \colon T_i \to A/A'$. Now suppose that $(W_0, \lambda_0) \in \mathcal{N}(T_0)_{\leq n}$ which means that $|A/A'_0| \leq n$. We ought to show that $|A/A'_1| \leq n$ so that $(W_1, \lambda_1) \in \mathcal{N}(T_1)_{\leq n}$. By our classification of wide subgroups, we have that

$$A'_i = \{(g, h) = a \in A \mid \lambda_i(g, 1) = h\}.$$

Using that $\lambda_1 = \lambda_0 \circ j_\varphi$ we see that we have a projection $j_\varphi \colon A'_1 \to A'_0$ showing that $|A/A'_1| \leq |A/A'_0| \leq n$ as required. For the final claim note that the condition $|\operatorname{rs}(\widetilde{W}_1)| = |\operatorname{rs}(\widetilde{W}_0)|$ ensures that the projection $A/A'_0 \to A/A'_1$ is an isomorphism. $\square$

Lemma 1.2.23. *Let $\mathcal{U}$ be a multiplicative global family of finite groups. For all $T \in \mathcal{U}$, we have bijective maps $\phi_T \colon \mathcal{M}(T) \to \mathcal{N}(T)$ which are equivariant with respect to the conjugation action of $T \times G \times H$ on both sides, and are compatible with the canonical filtrations on $\mathcal{M}(T)$ and $\mathcal{N}(T)$ (however, the maps ϕ_T are not natural in T). Furthermore, these maps induce natural isomorphisms*

$$\phi_T \colon \mathcal{M}(T)_{\leq n}/\mathcal{M}(T)_{\leq n-1} \to \mathcal{N}(T)_{\leq n}/\mathcal{N}(T)_{\leq n-1}.$$

Proof. We define a bijection $\phi_T \colon \mathcal{M}(T) \to \mathcal{N}(T)$ as follows. Given $(A, A', \theta) \in \mathcal{M}(T)$ we put

$$\widetilde{W} = \{(t, g, h) \in T \times G \times H \mid (g, h) \in A \text{ and } \theta(t) = (g, h).A'\}.$$

Note that $\widetilde{W}$ is a wide subgroup of $T \times G \times H$ so it lies in $\mathcal{U}$. We then let W denote the image of $\widetilde{W}$ in $T \times G$, which is easily seen to be wide. Using the condition $A' \cap (1 \times H) = 1$ we see that the projection $\pi \colon \widetilde{W} \to W$ is an isomorphism. We also have another projection $\pi' \colon \widetilde{W} \to H$, which is again surjective, so we can define $\lambda = \pi' \circ \pi^{-1} \in \operatorname{Epi}(W, H)$. Our map $\phi_T \colon \mathcal{M}(T) \to \mathcal{N}(T)$ is defined by $\phi_T(A, A', \theta) = (W, \lambda)$.

In the opposite direction, we can define $\psi_T \colon \mathcal{N}(T) \to \mathcal{M}(T)$ as follows. Given $(W, \lambda) \in \mathcal{N}(T)$, we put

$$\widetilde{W} = \{(t, g, h) \in T \times G \times H \mid (t, g) \in W,\ \lambda(t, g) = h\} \in \mathcal{U}.$$

We then let A be the projection of $\widetilde{W}$ in $G \times H$, so

$$A = \{(g, h) \in G \times H \mid \text{ there exists } t \in T \text{ with } (t, g) \in W \text{ and } \lambda(t, g) = h\}.$$

We also let $\widetilde{A}$ be the kernel of the projection $\widetilde{W} \to T$, which is normal in $\widetilde{W}$, and we let A' be the image of $\widetilde{A}$ in $G \times H$, which is normal in A. More explicitly, we have

$$A' = \{(g, \lambda(1, g)) \mid g \in G,\ (1, g) \in W\}.$$

From this description it is clear that $A' \cap (1 \times H) = 1$. Next, for any $t \in T$ we can choose $g \in G$ with $(t, g) \in W$ (because W is assumed to be wide). We then have $(t, g, \lambda(t, g)) \in \widetilde{W}$ and so $(g, \lambda(t, g)) \in A$. One can check that the coset $\theta(t) = (g, \lambda(t, g)).A' \in A/A'$ is independent of the choice of g, so this gives a surjective homomorphism $\theta \colon T \to A/A'$ and thus an element $\psi_T(W, \lambda) = (A, A', \theta) \in \mathcal{N}$. One can check that ψ is inverse to ϕ, and that both maps are equivariant with respect to $T \times G \times H$ and that they are compatible with the filtrations. The final claim is a consequence of the uniqueness result from Lemma 1.2.22. $\square$

We are finally ready to prove the Proposition.

Proof of Proposition 1.2.19. Fix another group $T \in \mathcal{U}$ and consider $\mathcal{M}(T)$ and $\mathcal{N}(T)$ as in Definitions 1.2.20 and 1.2.21. Both of these sets have an action of $T \times G \times H$ by conjugation. We define two objects of $\mathcal{AU}$ by $\widetilde{\mathcal{M}}(T) = \mathcal{M}(T)/(T \times G \times H)$ and $\widetilde{\mathcal{N}}(T) = \mathcal{N}(T)/(T \times G \times H)$.

For $t \in T$ and $\theta \in \mathrm{Epi}(T, A/A')$ we have $\theta \circ c_t = c_{\theta(t)} \circ \theta$. Using this we see that $\widetilde{\mathcal{M}}$ is naturally identified with $F(\underline{Q})$. We also have natural isomorphisms

$$\underline{\mathrm{Hom}}(e_G, e_H)(T) = \mathcal{AU}(e_T \otimes e_G, e_H) \simeq \left(\prod_W e_H(W)\right)^{T\times G} \simeq \left(\prod_W k[\mathrm{Epi}(W, H)]\right)^{T\times G\times H}$$

and the latter can be identified with $\widetilde{\mathcal{N}}(T)$. Since an epimorphism $\varphi\colon T' \to T$, induces a map $e_{T'} \otimes e_G \to e_T \otimes G$ which sends a wide subgroup $W' \leq T' \times G$ to the wide subgroup $\varphi_* W' \leq T \times G$, one easily checks that $\underline{\mathrm{Hom}}(e_G, e_H)(T)$ is naturally identified with $\widetilde{\mathcal{N}}(T)$. Therefore, it suffices to show that $\widetilde{\mathcal{M}}$ and $\widetilde{\mathcal{N}}$ are isomorphic in $\mathcal{AU}$.

Now consider the natural filtrations of $\mathcal{N}(T)$ and $\mathcal{M}(T)$. These induce filtrations on $\widetilde{\mathcal{N}}$ and $\widetilde{\mathcal{M}}$. Using Lemma 1.2.23 we see that the associated graded of the filtration of $\widetilde{\mathcal{N}}$ is naturally isomorphic to the associated graded of the filtration of $\widetilde{\mathcal{M}}$. The latter can be identified with $F(\underline{Q})$ which is projective. This shows that the short exact sequences

$$\widetilde{\mathcal{N}}_{<n} \to \widetilde{\mathcal{N}}_{\leq n} \to \widetilde{\mathcal{N}}_{\leq n}/\widetilde{\mathcal{N}}_{<n}$$

split for all n. By choosing splitting of these short exact sequences, we construct a map in $\mathcal{AU}$

$$F(\underline{Q}) \simeq \bigoplus_n (\widetilde{\mathcal{N}}_{\leq n}/\widetilde{\mathcal{N}}_{<n}) \to \widetilde{\mathcal{N}}$$

which is easily seen to be an isomorphism. This concludes the proof. □

Corollary 1.2.24. *Let $\mathcal{U}$ be closed downwards. Then for all $G, H \in \mathcal{U}$, the object $\underline{\mathrm{Hom}}(e_G, e_H)$ is finitely projective (see Definition 1.9.1).*

Proof. Choose a multiplicative global family $\mathcal{V}$ containing $\mathcal{U}$ and let $i\colon \mathcal{U} \to \mathcal{V}$ denote the inclusion functor. To avoid confusion, we will write $e_G \otimes_{\mathcal{U}} e_H$ and $\underline{\mathrm{Hom}}_{\mathcal{U}}(e_G, e_H)$ to mean respectively, the tensor product and internal hom calculated in $\mathcal{AU}$. We use similar notation for $\mathcal{V}$ as well. Fix $G, H \in \mathcal{U}$, and consider $i^*\underline{\mathrm{Hom}}_{\mathcal{V}}(e_G, e_H) \in \mathcal{AU}$. The restriction functor i^* preserves finitely projectives by Lemma 1.9.3(a). This together with Proposition 1.2.19 tells us that $i^*\underline{\mathrm{Hom}}_{\mathcal{V}}(e_G, e_H)$ is finitely projective. Therefore, it will suffices to show that $\underline{\mathrm{Hom}}_{\mathcal{U}}(e_G, e_H)$ is a retract of $i^*\underline{\mathrm{Hom}}_{\mathcal{V}}(e_G, e_H)$. For all $T \in \mathcal{U}$, there is a split projection map $e_G \otimes_{\mathcal{V}} e_T \to e_G \otimes_{\mathcal{U}} e_T$ that picks up the wide subgroups which lie in $\mathcal{U}$. It is also clear that this split projection is natural in T. Therefore we get an induced split projection

$$i^*\underline{\mathrm{Hom}}_{\mathcal{V}}(e_G, e_H) = \mathcal{AV}(e_G \otimes_{\mathcal{V}} e_T, e_H) \to \mathcal{AU}(e_G \otimes_{\mathcal{U}} e_T, e_H) = \underline{\mathrm{Hom}}_{\mathcal{U}}(e_G, e_H)$$

which is also natural in T. This concludes the proof. □

3. Subcategories

In this section we study the formalism that relates the abelian category $\mathcal{A}$ to its smaller subcategories $\mathcal{AU}$.

Definition 1.3.1. Let $\mathcal{U}$ be any full and replete subcategory of $\mathcal{G}$. The inclusion functor $i_{\mathcal{U}}\colon \mathcal{U} \to \mathcal{G}$ gives a pullback functor $i^*_{\mathcal{U}}\colon \mathcal{A} \to \mathcal{AU}$ which has a left and right adjoint $i^{\mathcal{U}}_!, i^{\mathcal{U}}_*\colon \mathcal{AU} \to \mathcal{A}$ (the left and right Kan extensions along $i_{\mathcal{U}}$). As $i_{\mathcal{U}}$ is the inclusion of a full subcategory, we have that $i^*_{\mathcal{U}} i^{\mathcal{U}}_! = 1$ and $i^*_{\mathcal{U}} i^{\mathcal{U}}_* = 1$, so both $i^{\mathcal{U}}_!$ and $i^{\mathcal{U}}_*$ are full and faithful embeddings.

Remark 1.3.2. We will be often interested in the case where $\mathcal{U}$ is $\mathcal{G}_n$ or $\mathcal{G}_{\leq n}$ or $\mathcal{G}_{\geq n}$, respectively the subcategories of groups of cardinality exactly n, less or equal to n, and greater or equal to n. In these cases we use the abbreviated notation like i^*_n or $i^{\leq n}_*$ or $\mathcal{A}_{\geq n}$. If $\mathcal{U}$ is clear from the context we will just write i for $i_{\mathcal{U}}$, and similarly for i^*, i_* and $i_!$.

Lemma 1.3.3. *Let $i\colon \mathcal{U} \to \mathcal{V}$ be the inclusion of a replete full subcategory.*

(a) *The (co)unit maps $i^* i_* X \to X \to i^* i_! X$ are isomorphisms, for all $X \in \mathcal{AU}$.*

(b) *The essential image of $i_!$ is $\{Y \in \mathcal{AV} \mid \epsilon_Y : i_! i^* Y \to Y$ is iso$\}$. The essential image of i_* is $\{Y \in \mathcal{AV} \mid \eta_Y : Y \to i_* i^* Y$ is iso$\}$.*

(c) *There are natural isomorphisms $i^*(\mathbb{1}) = \mathbb{1}$ and $i^*(X \otimes Y) = i^*(X) \otimes i^*(Y)$ giving a strong monoidal structure on i^*. However, the corresponding map $i^*\underline{\mathrm{Hom}}(X,Y) \to \underline{\mathrm{Hom}}(i^*X, i^*Y)$ is typically not an isomorphism.*

(d) *There are natural maps $i_!\mathbb{1} \to \mathbb{1} \to i_*\mathbb{1}$ and $i_!(X \otimes Y) \to i_!(X) \otimes i_!(Y)$ and $i_*(X) \otimes i_*(Y) \to i_*(X \otimes Y)$ giving (op)lax monoidal structures.*

(e) *In all cases $i_!$ preserves all colimits and i_* preserves all limits and i^* preserves both limits and colimits. Also $i_!$ preserves projectives and i_* preserves injectives. Both i_* and $i_!$ preserve indecomposable objects.*

(f) *If $\mathcal{U}$ is closed upwards then $i_!$ is extension by zero and so preserves all (co)limits and tensors (but not the unit). If $\mathcal{U}$ is multiplicative and closed under subgroups, then $i_!$ is strongly monoidal.*

(g) *If $\mathcal{U}$ is closed downwards then i_* is extension by zero and so preserves all (co)limits and tensors (but not the unit).*

(h) *If i has a left adjoint $q \colon \mathcal{V} \to \mathcal{U}$ then $i_! = q^*$ (and so $i_!$ preserves all (co)limits).*

(i) *Suppose that $G \in \mathcal{U}$ and $\mathcal{C} \leq \mathcal{U}$ is convex. Then we have*

$$i^*(e_{G,V}) = e_{G,V} \quad i^*(t_{G,V}) = t_{G,V} \quad i^*(s_{G,V}) = s_{G,V}$$
$$i^*(\chi_{\mathcal{C}}) = \chi_{\mathcal{C}} \quad i_!(e_{G,V}) = e_{G,V} \quad i_*(t_{G,V}) = t_{G,V}.$$

If $\mathcal{U}$ is closed upwards, we also have

$$i_!(\chi_{\mathcal{C}}) = \chi_{\mathcal{C}} \quad i_!(s_{G,V}) = s_{G,V} \quad i_!(t_{G,V}) = \chi_{\mathcal{U}} \otimes t_{G,V}.$$

If $\mathcal{U}$ is closed downwards, we also have

$$i_*(e_{G,S}) = \chi_{\mathcal{U}} \otimes e_{G,S} \quad i_!(s_{G,V}) = s_{G,V} \quad i_*(\chi_{\mathcal{C}}) = \chi_{\mathcal{C}}.$$

PROOF. Recall that the functors i_* and $i_!$ are full and faithful embeddings so for (a), we can check that $i_*i^*i_*X \to i_*X$ and $i_!(X) \to i_!i^*i_!(X)$ are isomorphisms. These now follow from the relations $i^*i_! = 1$ and $i^*i_* = 1$.

For the first part of (b), it is enough to show that if $Y = i_!(X)$ for some X, then the counit ϵ_Y is an isomorphism, as the other implication is clear. By part (a), the unit map $\eta_X : X \to i^*i_!X$ is an isomorphism, so the map $i_!(\eta_X) : i_!X \to i_!i^*i_!X$ is also invertible. The triangular identities for an adjunction tell us that the counit map $\epsilon_{i_!X}$ is a right inverse for $i_!(\eta_X)$, so it is also an isomorphism. By naturality, ϵ_Y is an isomorphism whenever Y is isomorphic to $i_!X$. This proves (b) as the argument for the second part is similar.

For all $G \in \mathcal{U}$, we have $(i^*\mathbb{1})(G) = k = \mathbb{1}(G)$ and $i^*(X \otimes Y)(G) = X(G) \otimes Y(G) = (i^*X \otimes i^*Y)(G)$ which proves that i^* is strongly monoidal. For the rest of (c), suppose that $\mathcal{U}$ consists only of the trivial group. If $G \neq 1$, then $(i^*\underline{\mathrm{Hom}}(e_G, e_G))(1) = \mathcal{AV}(e_G, e_G) = e_G(G)$ but $\underline{\mathrm{Hom}}(i^*e_G, i^*e_G)(1) = 0$ since $i^*e_G = 0$. This shows that the map $i^*\underline{\mathrm{Hom}}(X,Y) \to \underline{\mathrm{Hom}}(i^*X, i^*Y)$ adjoint to the evaluation map is not always an isomorphism.

The lax symmetric monoidal structure of i_* follows directly from (c). The map $i_*(X) \otimes i_*(Y) \to i_*(X \otimes Y)$ is adjoint to

$$i^*(i_*(X) \otimes i_*(Y)) \simeq i^*i_*(X) \otimes i^*i_*(Y) \xrightarrow{\varepsilon_X \otimes \varepsilon_Y} X \otimes Y,$$

and similarly for the oplax structure of $i_!$.

Most of part (e) is formal and follows from the properties of adjunctions. If P is indecomposable, we see that the only idempotent elements in $\mathrm{End}(P)$ are 0 and 1, and that $0 \neq 1$. As $i_!$ is full and faithful, we see that $\mathrm{End}(i_!P)$ is isomorphic to $\mathrm{End}(P)$, and so has the same idempotent structure. A similar proof works for i_* too.

Consider $Y = i_!(X)$. Using the formula for Kan extensions, we see that $Y(T)$ can be written as a colimit over the comma category $T \downarrow \mathcal{U}$. If $T \in \mathcal{U}$ then this has a terminal object, namely

$(T \xrightarrow{1} T)$ so $Y(T) = X(T)$. If $\mathcal{U}$ is closed upwards and $T \notin \mathcal{U}$, then $T \downarrow \mathcal{U}$ is empty so $Y(T) = 0$. The first part of (f) follows directly, and (g) can be proved by a very similar argument. Now suppose that i has a left adjoint q as in (h). Then the comma category $T \downarrow \mathcal{U}$ is equivalent to $qT \downarrow \mathcal{U}$ which has a terminal object $(qT \xrightarrow{1} qT)$ giving $Y(T) = X(qT)$. It follows that q^* and $i_!$ are naturally isomorphic which gives (h).

We now prove the rest of (f). Suppose that $\mathcal{U}$ is multiplicative and closed under passage to subgroups and notice that both $i_!$ and $\otimes$ commutes with colimits, so it is enough to check that the canonical map $i_!(e_G \otimes e_H) \to i_!(e_G) \otimes i_!(e_H) = e_G \otimes e_H$ is an isomorphism. This is now clear as $\mathcal{U}$ contains all the wide subgroups of $G \times H$.

That the restriction functor i^* preserves all the objects in (i) is straightforward. For any $X \in \mathcal{AV}$ we have

$$\mathcal{AV}(i_!(e_{G,V}), X) = \mathcal{AU}(e_{G,V}, i^*X) = \mathcal{M}_G(V, X(G)) = \mathcal{AV}(e_{G,V}, X)$$

where we used Lemma 1.1.7. A similar proof gives that $i_*(t_{G,V}) = t_{G,V}$. The final claims of (i) follows from (f) and (g) as the functor $i_!$ and i_* are extension by zero. □

Remark 1.3.4. The functor $i_! \colon \mathcal{AU} \to \mathcal{A}$ is not always strongly symmetric monoidal. For a counterexample take $\mathcal{U}$ the family of cyclic 2-groups. Note that the only wide subgroups of $C_4 \times C_2$ are the whole group $C_4 \times C_2$ and the graph subgroup $\mathrm{Gr}(\pi) \simeq C_4$ of the canonical projection $\pi : C_4 \to C_2$. Using Corollary 1.2.12, we see that $e_{C_2} \otimes e_{C_4} \simeq e_{C_4 \times C_2} \oplus e_{\mathrm{Gr}(\pi)}$ in $\mathcal{A}$ but $e_{C_2} \otimes e_{C_4} \simeq e_{\mathrm{Gr}(\pi)}$ in $\mathcal{AU}$. Thus, the canonical map

$$e_{\mathrm{Gr}(\pi)} = i_!(e_{\mathrm{Gr}(\pi)}) \simeq i_!(e_{C_2} \otimes e_{C_4}) \to i_!(e_{C_2}) \otimes i_!(e_{C_4}) = e_{C_2} \otimes e_{C_4} \simeq e_{C_4 \times C_2} \oplus e_{\mathrm{Gr}(\pi)}$$

is not an isomorphism in $\mathcal{A}$.

4. Simple objects

In this section we classify the simple objects and show that $\mathcal{AU}$ is semisimple if and only if $\mathcal{U}$ is a groupoid.

Definition 1.4.1. Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$.

- An object $X \in \mathcal{AU}$ is *simple* if the only subobjects are 0 and X.
- An object $X \in \mathcal{AU}$ is *semisimple* if it is a sum of finitely many simple objects.
- The abelian category $\mathcal{AU}$ is *semisimple* if every object is semisimple.

We immediately get the following result.

Lemma 1.4.2. *Any simple object of $\mathcal{AU}$ is isomorphic to $s_{G,V}$ for an irreducible representation V.*

Proof. Suppose that X is simple. Choose H of minimal order so that $X(H) \neq 0$. Then define

$$Y(G) = \begin{cases} X(G) & \text{if } |G| > |H| \\ 0 & \text{if } |G| \leq |H|. \end{cases}$$

Then Y is a proper subobject of X, therefore Y is zero. In other words, we have $X \simeq s_{G,V}$ for $V = X(G)$. If U is a nonzero irreducible subrepresentation of V we get that $s_{G,U} \leq s_{G,V}$. Thus $s_{G,U} \simeq s_{G,V}$ as required. □

Proposition 1.4.3. *The abelian category $\mathcal{AU}$ is semisimple if and only if $\mathcal{U}$ is a groupoid.*

Proof. If $\mathcal{U}$ is a groupoid then $\mathcal{AU}$ is semisimple by Proposition 1.5.3. Conversely, suppose that $\mathcal{U}$ is not a groupoid so there exists an epimorphism $\varphi \colon G \to H$ which is not an isomorphism.

Consider the canonical epimorphism $\pi\colon c_H \to s_{H,k}$. We claim that π cannot split so $e_{H,k}$ is not a direct sum of its subobjects. The commutative diagram

$$\begin{tikzcd} 0 = s_{H,k}(G) \arrow[r, "s(G)"] & c_H(G) \neq 0 \\ k = s_{H,k}(H) \arrow[u, "\varphi^*"] \arrow[r, "s(H)"'] & c_H(H) = k. \arrow[u, "\varphi^*"'] \end{tikzcd}$$

shows that any map $s\colon s_{H,k} \to c_H$ must be zero. □

5. Finite groupoids

In this section we study the abelian category $\mathcal{AU}$ in the special case that $\mathcal{U} \leq \mathcal{G}$ is a finite groupoid. For example we could take $\mathcal{U} = \{G \in \mathcal{G} \mid |G| = n\}$.

Lemma 1.5.1. *Suppose we choose a list of groups $G_1, \dots, G_r$ containing precisely one representative of each isomorphism class of groups in $\mathcal{U}$, so $\mathcal{G}(G_i, G_j) = \emptyset$ for $i \neq j$. Let $\mathcal{M}_i$ be the category of modules for the group ring $k[\mathrm{Out}(G_i)]$ and put $\mathcal{M} = \prod_{i=1}^r \mathcal{M}_i$. Then the functor $X \mapsto (X(G_i))_{i=1}^r$ gives an equivalence of categories $\mathcal{A}_n \to \mathcal{M}$.*

Proof. This is straightforward. The key point is just that every morphism in $\mathcal{U}$ is an isomorphism. □

Remark 1.5.2. Let $i\colon \mathcal{U} \to \mathcal{G}$ denote the inclusion functor. After choosing a list of groups $G_1, \dots, G_r \in \mathcal{U}$ as in Lemma 1.5.1, we have identifications

$$i_! = \bigoplus_{i=1}^r e_{G_i,\bullet} \quad \text{and} \quad i_* = \bigoplus_{i=1}^r t_{G_i,\bullet}.$$

Proposition 1.5.3. *Let $i\colon \mathcal{U} \to \mathcal{G}$ denote the inclusion.*

(a) *All monomorphisms and epimorphisms in $\mathcal{AU}$ are split.*
(b) *All objects in $\mathcal{AU}$ are both injective and projective.*
(c) *All objects in the image of $i_!$ are projective, and all objects in the image of i_* are injective.*
(d) *The functor $i_!$ preserves all limits and colimits, as does the functor i_*.*

Proof. We identify $\mathcal{AU}$ with $\mathcal{M}$ as in Lemma above. Maschke's Theorem shows that (a) and (b) hold in $\mathcal{M}_i$, and it follows that they also hold in $\mathcal{M}$ and $\mathcal{AU}$. If $X \in \mathcal{AU}$ then the functor $\mathcal{A}(i_!(X), -)$ is isomorphic to $\mathcal{AU}(X, i^*(-))$. Here i^* and $\mathcal{AU}(X, -)$ preserves epimorphisms, so $i_!X$ is projective. Similarly, we see that $i_*(X)$ is injective, which proves (c).

We next claim that i_* preserves all (co)limits. As it is a right adjoint it is enough to show that it preserves all colimits. By Remark 1.5.2, it is enough to show that the functor $t_{G_k,\bullet}$ preserves colimits for all $1 \leq k \leq r$. Choose $f_1, \dots, f_s \in \mathcal{G}(G_k, G)$, containing precisely one element from each $\mathrm{Out}(G_k)$-orbit. Let $\Delta_s \leq \mathrm{Out}(G_k)$ be the stabiliser of f_s. We find that

$$t_{G_k,V} = \mathrm{Hom}_{k[\mathrm{Out}(G_k)]}(k[\mathcal{G}(G_k, G)], V) = \prod_s V^{\Delta_s},$$

and this is easily seen to preserve all colimits as required. A similar argument shows that $i_!$ preserves all (co)limits. As before, it is enough to show that the functor $e_{G_k,\bullet}$ preserves all limits. We find that

$$e_{G,V} = k[\mathcal{G}(G, G_k)] \otimes_{k[\mathrm{Out}(G_k)]} V = V^{\oplus N}$$

where N is the number of different orbits of the free $\mathrm{Out}(G_k)$-action on $\mathcal{G}(G, G_k)$. This is easily seen to preserve all limits. □

The following results are standard.

Proposition 1.5.4.

(a) *The simple objects of $\mathcal{AU}$ are the same as the indecomposable objects.*
(b) *Every nonzero morphism to a simple object is a split epimorphism, and every nonzero morphism from a simple object is a split monomorphism.*
(c) *If S and S' are non-isomorphic simple objects in $\mathcal{AU}$, then $\mathcal{AU}(S,S')=0$.*
(d) *If S is a simple object in $\mathcal{AU}$, then $\mathrm{End}(S)$ is a division algebra of finite dimension over k.*
(e) *The category $\mathcal{AU}$ has finitely many isomorphism classes of simple objects.*
(f) *Suppose that the list $S_1,\dots,S_s$ contains precisely one simple object from each isomorphism class, and put $D_j=\mathrm{End}(S_j)$. Let $\mathcal{N}_j$ be the category of right modules over D_j, and put $\mathcal{N}=\prod_j\mathcal{N}_j$. Define functors*

$$\mathcal{AU}\xrightarrow{\phi}\mathcal{N}\xrightarrow{\psi}\mathcal{AU}$$

by $\phi(X)_j=\mathcal{AU}(S_j,X)$ and $\psi(N)=\bigoplus_j N_j\otimes_{D_j}S_j$. Then ϕ and ψ are inverse to each other, and so are equivalences.

Proof. Claim (a) is clear from the fact that all monomorphisms are split. For (b), suppose that $\alpha\colon X\to S$ is nonzero, where S is simple. Then $\mathrm{image}(\alpha)$ is a nonzero subobject of S, so it must be all of S, so α is an epimorphism, and all epimorphisms are split. This gives half of (b), and the other half is similar. Now suppose that $\alpha\colon S\to S'$, where both S and S' are simple. If $\alpha\neq 0$ then (b) tell us that α is both a split monomorphism and a split epimorphism, so it is an isomorphism. The contrapositive gives claim (c), and the special case $S'=S$, gives most of (d), apart from the finite-dimensionality statement. For that, we choose a list of groups G_i as in Lemma 1.5.1, and put $U=\bigoplus_i e_{G_i}$ which is a generator for $\mathcal{AU}$. We can decompose U as a finite direct sum of indecomposables, say $U=\bigoplus_{j=1}^s S_j^{d_j}$ with $0<d_j<\infty$ and $S_j\not\simeq S_k$ for $j\neq k$. If S is simple, there is an nonzero map $U\to S$ and so a nonzero map $S_j\to S$ for some j, that has to be an isomorphism from (b). This proves (e). We also note that S is a summand in U, so $\mathrm{End}(S)$ is a summand in $\mathrm{End}(U)$ and hence it has finite dimension over k, completing the proof of (d).

Now define ϕ and ψ as in (f). Put $T_m=\psi(S_m)\in\mathcal{N}$, so $(T_m)_m=D_m$ and $(T_m)_j=0$ for $j\neq m$. Define

$$\eta_N\colon N\to\phi\psi(N)=\mathcal{AU}(S_j,\bigoplus_k N_k\otimes_{D_k}S_k)$$

$$\epsilon_X\colon\psi\phi(X)=\bigoplus_j\mathcal{AU}(S_j,X)\otimes_{D_j}S_j\to X$$

as follows. First, any $n\in N_j$ gives a map $D_j\to N_j$ and thus a map

$$S_j=S_j\otimes_{D_j}D_j\to S_j\otimes_{D_j}N_j\leq\bigoplus_k N_k\otimes_{D_k}S_k$$

we take this to be the j-th component of η_N. Similarly, there is an evaluation morphism $\mathcal{AU}(S_j,X)\otimes S_j\to X$, which is easily seen to factor through $\mathcal{AU}(S_j,X)\otimes_{D_j}S_j$. We combine these maps to give ϵ_X.

We claim that ϵ_X is an isomorphism. Indeed, we know that the object U is a generator for $\mathcal{AU}$, so the objects S_j form a generating family. As all epimorphisms in $\mathcal{AU}$ split, we see that every object is a retract of a direct sum of objects of the form S_m. We also see that both ϕ and ψ preserve all direct sums. It will therefore suffice to check that ϵ_{S_m} is an isomorphism, and this follows easily from our description of $T_m=\psi(S_m)$.

Because every module over a division algebra is free, we also see that every object of $\mathcal{N}$ is a direct sum of objects of the form T_m. It is easy to see that η_{T_m} is an isomorphism, and it follows that η_N is an isomorphism for all N. $\square$

Remark 1.5.5. Let S be an indecomposable object of $\mathcal{AU}$. If we choose a list of groups $G_1, \dots, G_r$ as in Lemma 1.5.1, we know that $\mathcal{AU} \simeq \mathcal{M}$. Since S is indecomposable we see that $V_i = S(G_i)$ must be nonzero for exactly one index i, and so $S \simeq e_{G_i,V_i}$ where V_i is an irreducible $\mathrm{Out}(G_i)$-representation.

6. Projectives

The goal of this section is to classify the indecomposable projective objects of $\mathcal{AU}$.

Lemma 1.6.1. *Consider an object P in $\mathcal{AU}$. Then the following are equivalent:*

(a) *P is projective in $\mathcal{AU}$.*
(b) *P is isomorphic to a retract of a direct sum of objects of the form e_G with $G \in \mathcal{U}$.*
(c) *$i_!(P)$ is projective in $\mathcal{A}$.*

Proof. First suppose that (a) holds. Let $\mathcal{U}_0$ be a countable collection of objects of $\mathcal{U}$ that contains at least one representative of every isomorphism class. Put

$$FP = \bigoplus_{G \in \mathcal{U}_0} \bigoplus_{x \in P(G)} e_G \in \mathcal{AU}.$$

Each pair (G, x) defines a morphism $\epsilon_{(G,x)} \colon e_G \to P$ by Yoneda Lemma. By combining these for all pairs (G, x), we get a morphism $\epsilon \colon FP \to P$ which is an epimorphism by construction. As P is assumed to be projective this epimorphism must split, so P is a retract of FP, so (b) holds.

Next, $i_!$ preserves colimits by Lemma 1.3.3(e), so it preserves all direct sums and retracts, and it sends e_G to itself by Lemma 1.3.3(i). It follows that (b) implies (c).

Now suppose that (c) holds. We claim that (a) also holds, or equivalently that any epimorphism $f \colon X \to P$ in $\mathcal{AU}$ splits. As $i_!$ preserves all colimits, it also preserves epimorphisms. Thus $i_!(f) \colon i_!(X) \to i_!(P)$ is an epimorphism with projective target, so there exists $h \colon i_!(P) \to i_!(X)$ with $i_!(f) \circ h = 1$. We now apply i^*, recalling that $i^* i_! \simeq 1$. We find that the map $h' = i^*(h) \colon P \to X$ satisfies $f \circ h' = 1$, as required. □

Proposition 1.6.2. *Consider an object $Q \in \mathcal{A}$. Then the following are equivalent:*

(a) *$Q \simeq i_!(P)$ for some projective object $P \in \mathcal{AU}$.*
(b) *Q is a retract of $i_!(P)$ for some projective object $P \in \mathcal{AU}$.*
(c) *Q is a retract of some direct sum of objects e_G, with $G \in \mathcal{U}$.*
(d) *Q is projective, and the counit map $i_! i^* Q \to Q$ is an isomorphism.*

Moreover, if these conditions hold then $i^*(Q)$ is projective in $\mathcal{AU}$.

Proof. From what we have seen already it is clear that $(a) \Rightarrow (b) \Leftrightarrow (c)$ and that $(a) \Rightarrow (d)$. Now suppose that (b) holds, so there is a projective object $P \in \mathcal{AU}$ and an idempotent $e \colon i_! P \to i_! P$ with $Q = e.(i_! P) = \mathrm{coker}(1 - e)$. As $i_!$ is full and faithful, there is an idempotent $f \colon P \to P$ with $i_!(f) = e$. As $i_!$ preserves cokernels, it follows that $Q = i_!(f.P)$, and of course $f.P$ is projective, so (a) holds. Also, if $Q \simeq i_! P$ as in (a) holds then $i^* Q$ is isomorphic to P and so is projective.

Now all that is left is to prove that $(d) \Rightarrow (b)$. Suppose that Q is projective, and that the counit map $i_! i^* Q \to Q$ is an isomorphism. Choose a projective $P \in \mathcal{AU}$ and an epimorphism $f \colon P \to i^* Q$. As $i_!$ preserves epimorphisms, we see that $i_!(f) \colon i_! P \to i_! i^* Q \simeq Q$ is an epimorphism, but Q is projective, so Q is a retract of $i_! P$ as required. □

Construction 1.6.3. Consider an object $X \in \mathcal{AU}$. For $n \geq 0$, we let $F_{\leq n} X$ denote the smallest subobject of X containing $X(H)$ for all $H \in \mathcal{U}_{\leq n}$. Using Proposition 1.12.2 we see that $F_{\leq n} X$ can more formally be defined as the image of the counit map $i_!^{\leq n} i_{\leq n}^* X \to X$. This gives a filtration

$$0 \subseteq F_{\leq 1} X \subseteq \ldots \subseteq F_{\leq n} X \subseteq F_{\leq n+1} X \subseteq \ldots \subseteq X$$

with subquotients denoted by F_nX. Consider a map $f\colon X \to Y$ and an element $x \in (F_{\le n}X)(G)$. We can write $x = \sum_{i=1}^{s} \alpha_i^*(x_i)$ where $x_i \in X(H_i)$ with $|H_i| \le n$ and $\alpha_i \in \mathcal{U}(G, H_i)$. Note that

$$f(x) = \sum_i f\alpha_i^*(x) = \sum_i \alpha_i^* f(x) \in (F_{\le n}Y)(G),$$

so we have induced maps $f_{\le n}\colon F_{\le n}X \to F_{\le n}Y$ and $f_n\colon F_nX \to F_nY$ for all n.

EXAMPLE 1.6.4. For all $G \in \mathcal{U}$, we have

$$F_{\le n}e_G = \begin{cases} e_G & \text{if } |G| \le n \\ 0 & \text{if } |G| > n. \end{cases}$$

PROPOSITION 1.6.5. *If P is projective in $\mathcal{AU}$, then we have (not natural) isomorphism $P \simeq \bigoplus_{n\ge 0} F_nP$. Furthermore, F_nP is projective in $\mathcal{AU}_n$ and $(i_n)_!(F_nP) = F_nP$ where $i_n\colon \mathcal{U}_n \to \mathcal{U}$.*

PROOF. We know by Lemma 1.6.1 that P is a retract of an object $Q = \bigoplus_i e_{G_i}$, so $P = u.Q$ for some idempotent $u\colon Q \to Q$. Consider the filtration $F_{\le n}Q$ and note that by naturality, the induced maps $u_{\le n}\colon F_{\le n}Q \to F_{\le n}Q$ and $u_n\colon F_nQ \to F_nQ$ are still idempotents. We define $F_nP' = u_n.F_nQ$ and put $P' = \bigoplus_n F_nP' \le Q$. Let $f\colon P' \to P$ be the composite

$$P' \xrightarrow{\text{inc}} Q \xrightarrow{u} u.Q = P.$$

We claim that f induces isomorphisms $f_n\colon F_nP' \to F_nP$ and $f_{\le n}\colon F_{\le n}P' \to F_{\le n}P$ for all n. It follows that f itself is an isomorphism and that $P \simeq \bigoplus_n F_nP$ which concludes the Proposition. To prove the claim, note that by naturality of the filtration we have $F_{\le n}P = u.F_{\le n}Q$ and so $F_{\le n}P \le F_{\le n}Q$. Note also that the inclusion $P = u.Q \to Q$ induces a map $g_n\colon F_nP \to F_nQ$. As u acts as the identity of F_nP and preserves $F_{<n}P$, we see that u_n acts as the identity on the image of the map g_n, so we have a map $F_nP \to F_nP'$. This is easily seen to be inverse to $f_n\colon F_nP' \to F_nP$. An induction based on this shows that f induces isomorphisms $f_n\colon F_nP' \to F_nP$ and $f_{\le n}\colon F_{\le n}P' \to F_{\le n}P$ for all n as claimed. $\square$

COROLLARY 1.6.6. *Suppose we choose a complete system of simple objects in $\mathcal{AU}_n$ for all n, giving a sequence $(e_{G_i,S_i} \mid G_i \in \mathcal{U}_n)_n$ of indecomposable projectives in $\mathcal{AU}$. Then every projective object is a direct sum of objects of the form e_{G_i,S_i}. In particular, every indecomposable projective is isomorphic to some e_{G_i,S_i}.*

PROOF. Apply Proposition 1.6.5 and Remark 1.5.5. $\square$

PROPOSITION 1.6.7. *Any projective object P can be written as $P \simeq \prod_n F_nP$. Furthermore, products of projective objects are projective.*

PROOF. By Proposition 1.6.5 we can write $P = \bigoplus_n F_nP$ with $(i_n)!(F_nP) = F_nP$. Now note that for a fixed $G \in \mathcal{U}$, there are only finitely many indices n for which $(F_nP)(G)$ is nonzero, so $\prod_n F_nP = \bigoplus_n F_nP$. For the second claim, let (P_t) be a family of projectives, and put $P = \prod_t P_t$. We can write $P_t = \prod_k F_kP_{tk}$ as above, so $P = \prod_k Q_k$ where $Q_k = \prod_t F_kP_{tk}$. We know from Proposition 1.5.3 that $(i_k)_!$ preserves products, so Q_k is in the image of $(i_k)_!$. It follows that Q_k is projective and also that $P = \prod_k Q_k$ is the same as $\bigoplus_k Q_k$, so P is projective. $\square$

PROPOSITION 1.6.8. *Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$. Then the full subcategory of projective objects is closed under tensor products. If $\mathcal{U}$ is a multiplicative global family, it is also closed under the internal homs.*

PROOF. Consider projective objects $P, Q \in \mathcal{AU}$. Write $P = \bigoplus_n F_nP$ and $Q = \bigoplus_m F_mQ = \prod_m F_mQ$. Then we see that

$$P \otimes Q = \bigoplus_{n,m} F_nP \otimes F_mQ \quad \text{and} \quad \underline{\mathrm{Hom}}(P, Q) = \prod_{n,m} \underline{\mathrm{Hom}}(F_nP, F_mQ).$$

So it suffices to show that $e_G \otimes e_H$ and $\underline{\mathrm{Hom}}(e_G, e_H)$ are projective for all $G, H \in \mathcal{U}$. This now follows from Propositions 1.2.12 and 1.2.19. $\square$

7. An exact colimit

In this section we prove the following result that we will need later on.

THEOREM 1.7.1. *Let $\mathcal{U}$ be multiplicative and closed under passage to subgroups. Then the colimit functor*
$$\operatorname{colim}\colon \mathcal{AU} \to \mathrm{Vect}_k$$
is exact.

The proof will follow after some preliminaries.

CONSTRUCTION 1.7.2. Fix a sketeton $\mathcal{U}'$ of $\mathcal{U}$, and put $\mathcal{U}'_{\leq n} = \{G \in \mathcal{U}' \mid |G| \leq n\}$. Let F be a finitely generated group and let $\mathcal{K}_n(F)$ be the set of all normal subgroups $H \leq F$ such that $F/H \in \mathcal{U}_{\leq n}$. Each $H \in \mathcal{K}_n(F)$ occurs as the kernel of some surjective homomorphism $\alpha_H\colon F \to G_H$ with $G_H \in \mathcal{U}'_{\leq n}$, and there are only finitely many such morphisms so $\mathcal{K}_n(F)$ is finite. We define $\mathcal{N}_n(F)$ to be the interesection of all the groups in $\mathcal{K}_n(F)$, so $\mathcal{N}_n(F)$ is a characteristic group of finite index in F. We then put $q_n(F) = F/\mathcal{N}_n(F)$ which is finite. We note that the homomorphisms α_H combine to induce an embedding $q_n(F) \to \prod_H G_H$, from which it follows that $q_n(F) \in \mathcal{U}$. This construction is also functorial with respect to surjective homomorphisms $\alpha\colon G \to G'$. Using that α is surjective, we see that we have a well-defined map $\alpha\colon \mathcal{K}_n(G) \to \mathcal{K}_n(G')$ sending N to $\alpha(N)$, which is easily seen to be surjective. This shows that $\alpha(\mathcal{N}_n(G)) \subseteq \mathcal{N}_n(G')$ and so we have an induced surjective group homomorphism $q_n(\alpha)\colon q_n(G) \to q_n(G')$ as required.

EXAMPLE 1.7.3. For any finite set X of cardinality n, let FX be the free group on X. Then we put $TX = q_n(FX) \in \mathcal{U}$. This is finite and functorial for bijections of X. If G is any group in $\mathcal{U}$ with $|G| \leq n$, then we can choose a surjective map $X \to G$, and extend it to a surjective homomorphism $FX \to G$. The kernel of this homomorphism is in $\mathcal{K}_n(FX)$ and so contains $\mathcal{N}_n(FX)$, so we get an induced surjective homomorphism $TX \to G$. In particular, we can take $X = G$ and use the identity map to get a canonical epimorphism $\epsilon\colon TG \to G$.

LEMMA 1.7.4. *Let X be a finite set and consider a diagram of epimorphisms between groups in $\mathcal{U}$*
$$\begin{tikzcd} & G \arrow[d, two heads, "\alpha"] \\ TX \arrow[ur, dotted, two heads, "\mu"] \arrow[r, two heads, "\lambda"'] & H \end{tikzcd}$$
in which $|G| \leq |X|$. Then the dotted arrow can be filled in by another epimorphism.

PROOF. Put $L = \ker(\alpha)$, so $|L||H| = |G| \leq |X|$. Let $i\colon X \to TX$ be the canonical inclusion, and put $X_h = (i\lambda)^{-1}(h)$ for each $h \in H$. We then have $\sum_h |X_h| = |X| \geq |H||L|$, so we can choose h_0 with $|X_{h_0}| \geq |L|$. Let $\mu_h\colon X_h \to \alpha^{-1}\{h\}$ be choose arbitrarily, except that we choose μ_{h_0} to be surjective. By combining these maps, we get $\mu'\colon X \to G$ such that $\alpha\mu' = \lambda i$. By the defining properties of TX, we see that there is a unique homomorphism $\mu\colon TX \to G$ with $\mu i = \mu'$. This satisfies $\alpha\mu i = \lambda i$ and $i(X)$ generates TX so $\alpha\mu = \lambda$. Now note that the restriction of α to the image of μ is an epimorphism since $\alpha\mu$ is surjective. Also, the image of μ contains L as μ_{h_0} is surjective. It follows that μ is surjective as required. $\square$

LEMMA 1.7.5. *If $G \neq 1$ then $\epsilon\colon TG \to G$ is not injective, so $|TG| \geq 2|G|$.*

PROOF. Choose any nontrivial $g \in G$ and let $\tau\colon G \to G$ be the transposition that exchanges 1 and g. Let e_1 and e_g denote the corresponding generators of FG or TG. The map τ induces an automorphism α of TG which exchanges e_1 and e_g. The homomorphism $\epsilon\alpha$ sends e_1 to $g \neq 1$, so $e_1 \notin N$, so e_1 gives a nontrivial element of TG. However, this lies in the kernel of ϵ, so ϵ is not injective, and $|TG| = |G||\ker(\epsilon)| \geq 2|G|$. $\square$

REMARK 1.7.6. This lower bound is pitifully weak; in practice TG is enormously larger than G.

Lemma 1.7.7. *Suppose that $\alpha,\beta\colon G\to H$ are surjective homomorphisms in $\mathcal{U}$. Then there is an automorphism γ of TG making the following diagram commute:*

$$\begin{tikzcd} TG \arrow[rr, "\gamma"] \arrow[d, "\epsilon"] & & TG \arrow[d, "\epsilon"'] \\ G \arrow[r, "\alpha"'] & H & G \arrow[l, "\beta"] \end{tikzcd}$$

Proof. Put $m=|G|/|H|=|\ker(\alpha)|=|\ker(\beta)|$. For each $h\in H$ we have $|\alpha^{-1}\{h\}|=m=|\beta^{-1}\{h\}|$, so we can choose a bijection $\alpha^{-1}\{h\}\to\beta^{-1}\{h\}$. By combining these choices, we obtain a bijection $\sigma\colon G\to G$ with $\beta\sigma=\alpha$. This gives an automorphism $\gamma=T\sigma$ of TG. We claim that $\beta\epsilon\gamma=\alpha\epsilon\colon TG\to H$. It will suffice to check this on the generating set $G\subset TG$, and that reduces to the relation $\beta\sigma=\alpha$, which holds by construction. $\square$

Construction 1.7.8. Let G_0 be any nontrivial group in $\mathcal{U}$, and put $G_n=T^nG_0$, so we have a tower

$$G_0\xleftarrow{\epsilon}G_1\xleftarrow{\epsilon}G_2\xleftarrow{\epsilon}\cdots.$$

Given $X\in\mathcal{AU}$, we define F_nX to be the coinvariants for the action of $\mathrm{Out}(G_n)$ on $X(G_n)$. Because G_{n+1} is a functor of G_n, we get a homomorphism $\mathrm{Aut}(G_n)\to\mathrm{Aut}(G_{n+1})$, and this is compatible with ϵ by naturality. It follows that the map $\epsilon^*\colon X(G_n)\to X(G_{n+1})$ induces a map $\epsilon^*\colon F_n(X)\to F_{n+1}(X)$. We define $F_\infty(X)$ to be the colimit. Note that by definition, we have natural maps $X(G_n)\to\mathrm{colim}(X)$, that assemble to give a map $F_\infty(X)\to\mathrm{colim}(X)$.

Lemma 1.7.9. *The functor $F_\infty\colon\mathcal{AU}\to\mathrm{Vect}_k$ is exact.*

Proof. Recall that as we are working over a field of charachteristic zero, finite group coinvariants are exact on k-modules, by an averaging argument, so F_n is exact. Colimits of sequences are also exact. $\square$

Remark 1.7.10. Lemma 1.7.5 shows that $|G_n|\to\infty$. If $H\in\mathcal{U}$ and $|H|\leq|G_n|$, then H admits a surjective homomorphism from G_{n+1}. Thus, any finite group admits a surjective homomorphism from some G_n.

Construction 1.7.11. Let H be an object of $\mathcal{U}$. Choose any surjective homomorphism $\alpha\colon G_n\to H$, and let $t_\alpha\colon X(H)\to F_\infty(X)$ be the composite

$$X(H)\xrightarrow{\alpha^*}X(G_n)\to F_n(X)\to F_\infty(X).$$

Note that this is the same as

$$X(H)\xrightarrow{\alpha^*}X(G_n)\xrightarrow{(\epsilon^k)^*}X(G_{n+k})\to F_{n+k}(X)\to F_\infty(X)$$

for any $k\geq 0$. In other words, $t_\alpha=t_{\epsilon^k\alpha}$. Also, if α and β are two different surjective homomorphisms $G_n\to H$, then Lemma 1.7.7 implies that $t_{\epsilon\alpha}=t_{\epsilon\beta}$. By combining these rules, we see that the map $t_\alpha\colon X(H)\to F_\infty(X)$ is independent of α, so we can just call it t_H.

Now suppose we have a surjective homomorphism $\phi\colon H\to K$. We then have $t_H\circ\phi^*=t_\alpha\circ\phi^*=t_{\phi\alpha}=t_K$. Thus, the maps t_H fit together to give a map $t\colon\mathrm{colim}(X)\to F_\infty(X)$.

Proof of Theorem 1.7.1. Our maps between $F_\infty(X)$ and $\mathrm{colim}(X)$ are easily seen to be inverse to each other, so the claim follows from Lemma 1.7.9. $\square$

Lemma 1.7.12. *The functor F_∞ is left adjoint to the constant functor, so we have a natural equivalence*

$$\mathrm{Vect}_k(F_\infty(X),V)=\mathcal{AU}(X,V\otimes\mathbb{1})$$

for all $X\in\mathcal{AU}$ and $V\in\mathrm{Vect}_k$. In particular, we have $F_\infty(e_{G,V})=V_{\mathrm{Out}(G)}$, $F_\infty(t_{G,V})=F_\infty(s_{G,V})=0$ and $F_\infty(\chi_{\mathcal{U}})=0$ unless $\mathcal{U}$ is closed upwards.

Proof. The colim functor is left adjoint to the constant functor and we have seen that $F_\infty(X) \simeq \operatorname{colim}(X)$, so this gives the first part of the claim. By adjunctions, we have

$$\mathrm{Vect}_k(F_\infty(e_{G,V}), k) = \mathcal{AU}(e_{G,V}, \mathbb{1}) = \mathcal{M}_G(V, k) = V_{\mathrm{Out}(G)}.$$

The rest of the claim follows similarly as $\mathcal{AU}(X, \mathbb{1}) = 0$ if $X = t_{G,V}, s_{G,V}$ or $\chi_{\mathcal{U}}$. $\square$

Remark 1.7.13. It is easy to check that the canonical maps $t_K^X \otimes t_K^Y \colon X(K) \otimes Y(K) \to F_\infty(X) \otimes F_\infty(X)$ give a colimit cone so by universal property we have a map $F_\infty(X \otimes Y) \to F_\infty(X) \otimes F_\infty(Y)$. This shows that the functor F_∞ is always oplax monoidal. The following counterexample shows that F_∞ is not strong monoidal. Suppose that $G, H \in \mathcal{U}$ are abelian. Then we calculate $F_\infty(e_G) \otimes F_\infty(e_H) = k$, and using Corollary 1.2.12 we see that $F_\infty(e_G \otimes e_H) = \mathrm{Map}(\mathrm{W}(G, H), k)$.

8. Complete subcategories

In this section we introduce a well-behaved type of subcategory and present some examples.

Definition 1.8.1. Let $\mathcal{U}$ be a full subcategory of $\mathcal{G}$.

- For $T \in \mathcal{G}$, we denote by $\delta(T)$ the minimum possible size of a generating set for T.
- For $m \in \mathbb{N}$, we put $\mathcal{R}_m = \{T \in \mathcal{U} \mid \delta(T) \geq m\}$.
- We say that $\mathcal{U}$ is *expansive* if for all $G \in \mathcal{U}$ and $m \in \mathbb{N}$ there exists $T \in \mathcal{U}$ with $\delta(T) \geq m$ and $\mathcal{U}(T, G) \neq \emptyset$.
- Let $\mathcal{U}$ be expansive. For $X \in \mathcal{AU}$ and $n > 0$ we put
$$\omega_n^{\mathcal{U}}(X) = \limsup_{m \to \infty} \{\dim(X(T))/n^{\delta(T)} \mid T \in \mathcal{R}_m\} \in [0, \infty].$$
and
$$\mathcal{W}(\mathcal{U})_n = \{X \in \mathcal{AU} \mid \omega_n^{\mathcal{U}}(X) < \infty\}.$$
It is easy to see that if $\omega_n^{\mathcal{U}}(X) > 0$ then $\omega_m^{\mathcal{U}}(X) = \infty$ for $m < n$. Similarly, if $\omega_n^{\mathcal{U}}(X) < \infty$ then $\omega_m^{\mathcal{U}}(X) = 0$ for $m > n$. Thus, there is at most one n such that $0 < \omega_n^{\mathcal{U}}(X) < \infty$. If such an n exists, we call it the *order* of X.

Remark 1.8.2. We will often drop the superscript and just write $\omega_n(X)$.

Using the properties of the limsup we obtain the following result.

Lemma 1.8.3. *For any short exact sequence $X \to Y \to Z$ in $\mathcal{AU}$ we have*

$$\max(\omega_n(X), \omega_n(Z)) \leq \omega_n(Y) \leq \omega_n(X) + \omega_n(Z).$$

In particular, for any X and Z we have

$$\max(\omega_n(X), \omega_n(Z)) \leq \omega_n(X \oplus Z) \leq \omega_n(X) + \omega_n(Z).$$

$\square$

Corollary 1.8.4. *The category $\mathcal{W}(\mathcal{U})_n$ is closed under finite direct sums, subobjects, quotients, extensions and retracts. It also contains e_G for all $G \in \mathcal{U}_{\leq n}$.*

Proof. The closure properties easily follow from Lemma 1.8.3. For the second claim, note that if $A \subset T$ is a generating set for $T \in \mathcal{U}$, then the restriction map $\mathrm{Hom}(T, G) \to \mathrm{Map}(A, G)$ is injective, so $|\mathrm{Hom}(T, G)| \leq |G|^{|A|}$. It follows that

$$|\mathcal{U}(T, G)| = |\mathrm{Epi}(T, G)|/|\mathrm{Inn}(G)| \leq |\mathrm{Hom}(T, G)|/|\mathrm{Inn}(G)| \leq |G|^{\delta(T)}/|\mathrm{Inn}(G)| = |G|^{\delta(T)-1}|ZG|.$$

From this it is easy to see that $\omega_n(e_G) \leq |\mathrm{Inn}(G)|^{-1}$ if $|G| = n$, and $\omega_n(e_G) = 0$ if $|G| < n$. $\square$

We are now ready to introduce an important family of subcategories.

Definition 1.8.5. A subcategory $\mathcal{U}$ of $\mathcal{G}$ is *complete* if the following conditions are satisfied:

- $\mathcal{U}$ is expansive, i.e., for all $G \in \mathcal{U}$ and $n > 0$ there exists $T \in \mathcal{U}$ with $\delta(T) \geq n$ and $\mathcal{U}(T, G) \neq \emptyset$;
- For all $n > 0$ and $G \in \mathcal{U}_n$, we have $0 < \omega_n^{\mathcal{U}}(e_G) < \infty$. In other words, e_G has order exactly $|G|$.

EXAMPLE 1.8.6. Recall that we always have $\omega_n(e_G) \leq |\mathrm{Inn}(G)|^{-1}$ if $|G| = n$.

- Cyclic p-groups is not complete as it is not expansive.
- Elementary abelian p-groups is complete. Indeed we have

$$\omega_{p^n}(e_{C_p^n}) = \lim_{m\to\infty} \frac{|\mathrm{Epi}(C_p^m, C_p^n)|}{p^{nm}} = \lim_{m\to\infty} \frac{(p^m-1)(p^m-p)\cdots(p^m-p^{n-1})}{p^{nm}} = 1.$$

Let us produce more examples of complete subcategories.

PROPOSITION 1.8.7. *Any multiplicative full subcategory of $\mathcal{G}$ which is nontrivial and closed under passage to subgroups is complete. In particular, $\mathcal{G}$ is complete.*

PROOF. Let $\mathcal{U}$ be as above. Clearly $\mathcal{U}$ is expansive as for any $n > 0$ and $G \in \mathcal{U}$ we can take $T = G^n$. We only need to show that $\omega_{|G|}(e_G) > 0$ for all $G \in \mathcal{U}$. Without loss of generality we can assume that $G \neq 1$. For X_m a set with m elements, consider the group $TX_m \in \mathcal{U}$ as defined in Example 1.7.3. By definition, there is a natural bijection $\mathrm{Hom}(TX_m, G) = \mathrm{Hom}(FX_m, G) \simeq G^m$ for all the groups $G \in \mathcal{U}_{\leq m}$. Since by [**67**, Theorem 1] we have

$$\lim_{m\to\infty} |\mathrm{Epi}(FX_m, G)|/|G|^m = 1$$

we deduce that

$$\lim_{m\to\infty} |\mathrm{Epi}(TX_m, G)|/|G|^m = 1.$$

It only remains to notice that $\delta(TX_m) \leq m$ so

$$\omega_{|G|}(e_G) \geq \lim_{m\to\infty} \frac{|\mathcal{U}(TX_m, G)|}{|G|^m} = \lim_{m\to\infty} \frac{|\mathrm{Epi}(TX_m, G)|}{|\mathrm{Inn}(G)||G|^m} = \frac{1}{|\mathrm{Inn}(G)|} > 0.$$

□

The completeness assumption give us information on the growth of the indecomposable projectives.

LEMMA 1.8.8. *Let $\mathcal{U}$ be a complete full subcategory of $\mathcal{G}$. For $G \in \mathcal{U}$ and V an* Out(G)*-representation, we have $0 < \omega_{|G|}(e_{G,V}) < \infty$.*

PROOF. We show that $\dim(e_{G,V}(T)) = \dim(V)|\mathrm{Out}(G)|^{-1}|\dim(e_G(T))|$, and so the claim follows by completeness. It is easy to see that $\mathrm{Out}(G)$ acts freely on $\mathcal{U}(T, G)$. Choose a subset $M \subset \mathcal{U}(T, G)$ containing one representative of every orbit, so that $|M| = |\mathrm{Out}(G)|^{-1}|\mathcal{U}(T, G)|$. We also see that M is a basis for $e_G(T)$ as a module over the ring $R = k[\mathrm{Out}(G)]$, so

$$e_{G,V}(T) = V \otimes_R e_G(T) \simeq V^{|M|}.$$

This gives

$$\dim(e_{G,V}(T)) = \dim(V)|M| = \dim(V)|\mathrm{Out}(G)|^{-1}\dim(e_G(T))$$

as claimed. □

PROPOSITION 1.8.9. *Let $\mathcal{U}$ be complete full subcategory of $\mathcal{G}$. Then any monomorphism between projective objects of $\mathcal{AU}$ is split.*

PROOF. Let $u\colon P \to Q$ be a monomorphism between projective objects and consider the filtration from Construction 1.6.3. By Proposition 1.6.5, the filtrations split so we can write $P = \bigoplus_n F_nP$ and $Q = \bigoplus_n F_nQ$. We also know that u restricts to give a monomorphism $u_{\leq m}\colon P_{\leq m} \to Q_{\leq m}$. We will prove by induction on m that $u_{\leq m}$ splits. The claim is trivial if $m = 0$. Let $m > 0$ and let $s_{<m}\colon Q_{<m} \to P_{<m}$ be a splitting of $u_{<m}\colon P_{<m} \to Q_{<m}$. Now let

K_m be the kernel of the map $u_m\colon P_m \to Q_m$. As all monomorphisms in $\mathcal{AU}_m$ are split, we see that K_m is a retract of P_m. As $u_m(K_m) = 0$ and $u_{\leq m}$ is a monomorphism, we see that $u_{\leq m}$ induces a monomorphism from K_m to $Q_{<m}$. However, by completeness the order of $Q_{<m}$ is at most $m-1$, whereas if K_m is nonzero, it must have order m. It follows that K_m must actually be zero, so u_m is a monomorphism in $\mathcal{AU}_m$, so there is a splitting $v\colon Q_m \to P_m$. Let $s_{\leq m}\colon Q_{\leq m} \to P_{\leq m}$ be given by $s_{<m}$ on $Q_{<m}$, and by v on Q_m. Then $s_{\leq m}u_{\leq m}$ is the identity of $P_{<m}$, and it is the identity modulo $P_{<m}$ on P_m, so it is an automorphism of $P_{\leq m}$. It follows that $(s_{\leq m}u_{\leq m})^{-1} \circ s_{\leq m}$ is a splitting of $u_{\leq m}$, as required. By construction, the sections $s_{\leq m}$ assemble into a map $s\colon Q \to P$ satisfying $s \circ u = \mathrm{id}_P$, so u splits. □

9. Finiteness conditions

We introduce various finiteness conditions on objects of $\mathcal{A}$ and prove some implications amongst them. We refer the reader to Remarks 1.9.9 and 1.11.2 for a summary.

Definition 1.9.1. Consider an object $X \in \mathcal{A}$.

- We say that X has *finite type* if $\dim(X(G)) < \infty$ for all G.
- We say that X has *finite order* if there exists $n > 0$ such that $\omega_n(X) < \infty$.
- We say that X is *finitely projective* if it can be expressed as the direct sum of a finite family of indecomposable projectives.
- We say that X is *finitely generated* if there is an epimorphism $P_0 \to X$, where P_0 is finitely projective (or equivalently, $P_0 = \oplus_i e_{G_i}$).
- We say that X is *finitely presented* if there is a right exact sequence $P_1 \to P_0 \to X$, where P_0 and P_1 are finitely projective.
- We say that X is *finitely resolved* if there is a resolution $P_* \to X$, where each P_i is finitely projective.
- We say that X is *perfect* if there is a resolution $P_* \to X$, where each P_i is finitely projective for all i, and $P_i = 0$ for $i \gg 0$.

Remark 1.9.2. Most of the previous definitions can be made more generally in $\mathcal{AU}$. However, if we want to consider objects in $\mathcal{AU}$ with finite order we must require $\mathcal{U}$ to be expansive.

Lemma 1.9.3. *Let $i\colon \mathcal{U} \to \mathcal{V}$ be the inclusion of a replete full subcategory.*

(a) *The functor i^* always preserves objects of finite type. If $\mathcal{U}$ is closed downwards, then i^* preserves all finiteness conditions from Definition 1.9.1 excluding that of finite order.*
(b) *The functor $i_!$ always preserves finitely presented and finitely generated objects. If $\mathcal{U}$ is closed upwards (and expansive for finite order), then it preserves all finiteness conditions.*
(c) *If $\mathcal{U}$ is closed downwards, then i_* preserves objects of finite type.*

Proof. Clearly, i^* preserves objects of finite type. If $\mathcal{U}$ is closed downwards, then $i^*(e_G)$ is either e_G (if $G \in \mathcal{U}$) or 0 (if $G \notin \mathcal{U}$). It follows that i^* preserves (finitely) projectives. Since i^* is also exact by Lemma 1.3.3(e), it follows that i^* preserves all finiteness conditions in (a).

By Lemma 1.3.3(e) and (i), the functor $i_!$ preserves colimits and preserves (finitely) projective objects. It follows that $i_!$ preserves finitely presented and finitely generated objects. If $\mathcal{U}$ is closed upwards, then $i_!$ is extension by zero by Lemma 1.3.3(f) so it preserves objects of finite type and finite order (if $\mathcal{U}$ expansive). It is also exact so it preserves all the other finiteness conditions.

Finally, part (c) follows from Lemma 1.3.3(g) as i_* is extension by zero. □

It is useful to have a criterion to detect objects which are not finitely generated.

Lemma 1.9.4. *Let $\mathcal{V}$ be a full subcategory of $\mathcal{G}$ with infinitely many objects. Assume that $X \in \mathcal{A}$ has the following property: for all epimorphisms $V \to G$ with $V \in \mathcal{V}$ and $G \in \mathcal{G} - \{V\}$ we have $X(V) \neq 0$ and $X(G) = 0$. Then X cannot be finitely generated.*

PROOF. Assume that there is an epimorphism $\bigoplus_{i\in I} e_{G_i,S_i} \xrightarrow{\varphi} X$ where I is finite. Without loss of generality we can assume that for all $i \in I$, the composite

$$\alpha_i \colon e_{G_i,S_i} \to \bigoplus_{i\in I} e_{G_i,S_i} \xrightarrow{\varphi} X$$

is nonzero. For all $V \in \mathcal{V}$, we can find $i \in I$ such that $e_{G_i,S_i}(V) \neq 0$ as $X(V) \neq 0$ and φ is an epimorphism. In particular, there exists an epimorphism $V \to G_i$. As $\mathcal{V}$ has infinitely many objects and I is finite, we can assume that $V \neq G_i$. By our assumptions, we have that $X(G_i) = 0$. Hence, the map α_i must be zero. We found a contradiction. □

PROPOSITION 1.9.5. *Let $\mathcal{U}$ be a complete subcategory of $\mathcal{G}$. Then any object of $\mathcal{AU}$ with a finite projective resolution is projective. In particular any perfect complex is finitely projective.*

PROOF. Let $P_* \to X$ be a projective resolution and suppose that $P_i = 0$ for all $i > n$. If $n > 0$ it follows that the differential $d_n \colon P_n \to P_{n-1}$ must be a monomorphism, so Proposition 1.8.9 tells us that it is split. Now let Q_* be the same as P_* except that $Q_n = 0$ and $Q_{n-1} = \mathrm{coker}(d_n)$. We find that this is again a projective resolution of X. By repeating this construction, we eventually obtain a projective resolution of length one, showing that X itself is projective. □

REMARK 1.9.6. The Proposition above is not true if we drop the completeness condition. For example let $\mathcal{U}$ be the full subcategory of cyclic p-groups for a fixed prime p. Then there is a short exact sequence $0 \to c_{C_{p^2}} \to c_{C_p} \to t_{C_p,k} \to 0$ which shows that $t_{C_p,k}$ is perfect. It is not projective as it is torsion.

PROPOSITION 1.9.7. *Let $\mathcal{U}$ be a complete subcategory of $\mathcal{G}$. Then any finitely projective in $\mathcal{AU}$ has finite order.*

PROOF. The zero object has by definition finite order. For $r \geq 1$, we have

$$0 < \omega_n\left(\bigoplus_{i=1}^{r} e_{G_i,S_i}\right) < \infty \qquad \text{if } n = \max_i(|G_i|).$$

□

LEMMA 1.9.8. *If $\mathcal{U} \leq \mathcal{G}$ has only finitely many isomorphism classes, then any object of $\mathcal{AU}$ of finite type is perfect.*

PROOF. We define the support of $X \in \mathcal{AU}$ to be

$$\mathrm{supp}(X) = \{G \in \mathcal{U} \mid X(G) \neq 0\}.$$

We prove that claim by induction on the support. Choose n such that $\mathcal{U} = \mathcal{U}_{\leq n}$, this exists since $\mathcal{U}$ has only finitely many isomorphism classes. If $\mathrm{supp}(X) \subset \mathcal{U}_n$ and X is of finite type, then we can construct a map $\varphi\colon \bigoplus_{i=1}^m e_{G_i,S_i} \to X$ with $G_i \in \mathcal{U}_n$, which is an isomorphism for all the groups in $\mathcal{U}_n$. Note that $\ker(\varphi)$ is zero in $\mathcal{AU}_{\leq n}$ so X is perfect. Now suppose that $\mathrm{supp}(X) \subset \mathcal{U}_{\geq n-k}$ for $k \geq 1$. We can again construct a map $\varphi\colon \bigoplus_{i=1}^m e_{G_i,S_i} \to X$ with $G_i \in \mathcal{U}_{n-k}$, which is an isomorphism for all the groups in $\mathcal{U}_{n-k}$. Note that $\ker(\varphi)$ is supported in $\mathcal{U}_{>n-k}$ hence perfect by the induction hypothesis. It follows that X is perfect as required. By induction we conclude that any object of finite type is perfect. □

REMARK 1.9.9. So far we have the following implications:

$$\begin{array}{ccccccc}
\text{finitely resolved} & \Longrightarrow & \text{finitely presented} & \Longrightarrow & \text{finitely generated} & \Longrightarrow & \text{finite type} \\
\Uparrow & & & & & & \\
\text{perfect} & \overset{\textit{completeness}}{\Longleftrightarrow} & \text{finitely projective} & \overset{\textit{completeness}}{\Longrightarrow} & \text{finite order.} & &
\end{array}$$

10. Torsion and torsion-free objects

In this section we introduce the notions of torsion, absolutely torsion and torsion-free object. We study their formal properties and give some examples.

Definition 1.10.1. Consider an object X of $\mathcal{AU}$.

- We say that $x \in X(G)$ is *torsion* if there exists $H \in \mathcal{U}$ and $f \in \mathcal{U}(H, G)$ such that $f^*(x) = 0$.
- We say that $x \in X(G)$ is *absolutely torsion* if there exists $m \in \mathbb{N}$ such that for all $f \in \mathcal{U}(H, G)$ with $|H| \geq m$ we have $f^*(x) = 0$.
- We say that X is *torsion* (resp., *absolutely torsion*) if it consists entirely of torsion (resp., absolutely torsion) elements.
- We say that X is *torsion-free* if it does not contain any nonzero torsion element. Equivalently, X is torsion-free if and only if all the maps $\alpha^*\colon X(G) \to X(H)$ are injective.
- We write $\mathrm{tors}(X)(G)$ for the subset of torsion elements in $X(G)$.

Lemma 1.10.2. *Let $\mathcal{U}$ be closed under products and subgroups. Then* $\mathrm{tors}(X)$ *defines a subobject of X in $\mathcal{AU}$, which is the largest torsion subobject of X. The assignment* tors *is functorial in X so we have a functor* $\mathrm{tors}\colon \mathcal{AU} \to \mathcal{AU}$*. Moreover, for any finite dimensional subspace $V \leq \mathrm{tors}(X)(G)$, there is a map $\alpha\colon H \to G$ in $\mathcal{U}$ with $\alpha^*(V) = 0$.*

Proof. Suppose we have torsion elements $x_1, \ldots, x_n \in \mathrm{tors}(X)(G)$, so there are maps $\alpha_i\colon H_i \to G$ with $\alpha_i^*(x_i) = 0$. Put

$$H = \{(g, h_1, \ldots, h_n) \in G \times \prod_i H_i \mid \alpha_i(h_i) = g \text{ for all } i\}.$$

As $\mathcal{U}$ is closed under products and subgroups, we see that $H \in \mathcal{U}$. We define an epimorphism $\alpha\colon H \to G$ by $\alpha(g, h_1, \ldots, h_n) = g$. As it clearly factors through α_i, we have $\alpha^*(x_i) = 0$ for all i. Thus, if V is the span of $\{x_1, \ldots, x_n\}$, we have $\alpha^*(V) = 0$, so $V \leq \mathrm{tors}(X)(G)$. This proves in particular that $\mathrm{tors}(X)(G)$ is a vector subspace of $X(G)$.

Now suppose we have $\alpha^*(x) = 0$, and we also have another morphism $\beta\colon G' \to G$ in $\mathcal{U}$. We can then form a pullback diagram as follows:

$$\begin{tikzcd} H' \arrow[r, two heads, "\alpha'"] \arrow[d, two heads, "\beta'"'] & G' \arrow[d, two heads, "\beta"] \\ H \arrow[r, two heads, "\alpha"'] & G. \end{tikzcd}$$

Using the closure properties of $\mathcal{U}$ again, we see that the pullback $H' = G' \times_G H$ lies in $\mathcal{U}$. We have $(\alpha')^*\beta^*(x) = (\beta')^*\alpha^*(x) = 0$, so $\beta^*(x)$ is a torsion element. This shows that $\mathrm{tors}(X)$ is a subobject of X. All remaining claims are now clear. $\square$

Remark 1.10.3. The sum of two torsion-free subobjects need not be torsion-free. To see this, consider a torsion-free object Y, a nonzero torsion object Z and an epimorphism $f\colon Y \to Z$. In $Y \oplus Z$ we have a copy of Y, and the graph of f is another subobject $Y' \leq Y \oplus Z$ which is also isomorphic to Y and so is torsion-free. However, $Y + Y'$ is all $Y \oplus Z$ and so is not torsion-free.

Lemma 1.10.4. *Let $\mathcal{U}$ be closed under products and subgroups. For any object X of $\mathcal{AU}$, the quotient $X/\mathrm{tors}(X)$ is torsion-free.*

Proof. Consider an element $\overline{x} \in (X/\mathrm{tors}(X))(G)$, so $\overline{x}$ is represented by some element $x \in X(G)$. If $\overline{x}$ is a torsion element, then we have $\alpha^*(\overline{x}) = 0$ for some $\alpha \in \mathcal{U}(H, G)$, or equivalently $\alpha^*(x) \in \mathrm{tors}(X)(H)$. This means that there exists $\beta \in \mathcal{U}(K, H)$ with $(\alpha\beta)^*(x) = \beta^*(\alpha^*(x)) = 0$. Thus x is a torsion element and $\overline{x} = 0$ as required. $\square$

Recall the objects $e_{G,V}$ and $t_{G,V}$ from Definition 1.1.5.

Lemma 1.10.5. *For all $G \in \mathcal{U}$, we have that $e_{G,V}$ is torsion-free and $t_{G,V}$ is absolutely torsion. In particular, any projective object is torsion-free.*

Proof. It is clear that $t_{G,V}$ is absolutely torsion as it is zero as soon as $|K| > |G|$. It is enough to show that e_G is torsion-free as $e_{G,V}$ is a retract of a direct sum of e_G's. Thus, we need to show that for any epimorphism $\varphi\colon H \to K$ the linear map $\varphi^*\colon k[\mathcal{U}(K,G)] \to k[\mathcal{U}(H,G)]$ is injective. This is equivalent to proving that the map $\varphi^*\colon \mathcal{U}(K,G) \to \mathcal{U}(H,G)$ is injective, or in other words that φ is an epimorphism in the category $\mathcal{U}$. This is the content of Lemma 1.1.2. $\square$

We write $\mathcal{AU}_t$ and $\mathcal{AU}_f$ for the subcategories of torsion and torsion-free objects.

Lemma 1.10.6.

(a) *For an object $X \in \mathcal{AU}$, we have $X \in \mathcal{AU}_t$ if and only if $\mathcal{AU}(X,Y) = 0$ for all $Y \in \mathcal{AU}_f$.*
(b) *For an object $Y \in \mathcal{AU}$, we have $Y \in \mathcal{AU}_f$ if and only if $\mathcal{AU}(X,Y) = 0$ for all $X \in \mathcal{AU}_t$.*

Proof. If $f\colon X \to Y$ then $f(\mathrm{tors}(X)) \leq \mathrm{tors}(Y)$. If $X \in \mathcal{AU}_t$ and $Y \in \mathcal{AU}_f$ then $\mathrm{tors}(X) = X$ and $\mathrm{tors}(Y) = 0$ this becomes $f(X) = 0$ and $f = 0$. Thus, for $X \in \mathcal{AU}_t$ and $Y \in \mathcal{AU}_f$ we have $\mathcal{A}(X,Y) = 0$.

Now suppose that X is such that $\mathcal{AU}(X,Y) = 0$ for all $Y \in \mathcal{AU}_t$. In particular, this means that the projection $X \to X/\mathrm{tors}(X)$ is zero, so $\mathrm{tors}(X) = X$ and $X \in \mathcal{AU}_t$.

Suppose instead that Y is such that $\mathcal{AU}(X,Y) = 0$ for all $X \in \mathcal{AU}_t$. In particular, this means that the inclusion $\mathrm{tors}(Y) \to Y$ is zero, so $\mathrm{tors}(Y) = 0$ and $Y \in \mathcal{AU}_f$. $\square$

Lemma 1.10.7. *Consider objects $X \in \mathcal{AU}_t$ and $Y \in \mathcal{AU}_f$. Then for all $Z \in \mathcal{AU}$, we have*

(a) *$X \otimes Z$ is torsion;*
(b) *$\underline{\mathrm{Hom}}(X \otimes Z, Y) = 0$.*

Proof. Consider a homogeneous element $x \otimes z \in (X \otimes Z)(G)$. As X is torsion, there exists $\alpha\colon H \to G$ such that $\alpha^*(x) = 0$. Thus we have $\alpha^*(x \otimes z) = \alpha^*(x) \otimes \alpha^*(z) = 0$. It follows that $X \otimes Z$ is torsion. For all $G \in \mathcal{U}$, we have

$$\underline{\mathrm{Hom}}(X \otimes Z, Y)(G) = \mathcal{AU}(e_G \otimes X \otimes Z, Y) = 0$$

by part (a) and Lemma 1.10.6. $\square$

Lemma 1.10.8. *Let $\mathcal{U}$ be closed under products and subgroups. Then the subcategory $\mathcal{AU}_t$ is localizing that is, it is closed under arbitrary sums, subobjects, extensions and quotients.*

Proof. Consider an exact sequence $X \xrightarrow{i} Y \xrightarrow{p} Z$ in which X and Z are torsion objects. Consider an element $y \in Y(G)$. As Z is a torsion object, we can choose $\alpha\colon H \to G$ with $\alpha^*(p(y)) = 0$. This means that $p(\alpha^*(y)) = 0$, so $\alpha^*(y) = i(x)$ for some $x \in X(H)$. As X is a torsion object, we can choose $\beta\colon K \to H$ with $\beta^*(x) = 0$, and it follows that

$$(\alpha\beta)^*(y) = \beta^* i(x) = i(\beta^*(x)) = i(0) = 0.$$

This shows that Y is also a torsion object so $\mathcal{AU}_t$ is closed under extensions.

Now let X be a sum of torsion objects X_i and consider an element $x \in X(G)$. By definition, we can write $x = x_{i_1} + \ldots + x_{i_n}$ for elements $x_{i_k} \in X_{i_k}(G)$. As X_{i_k} is torsion, there exists $\alpha_{i_k}\colon H_{i_k} \to G$ such that $\alpha^*_{i_k}(x_{i_k}) = 0$. Put $\alpha = \alpha_{i_1} + \ldots + \alpha_{i_n}$ and $H = H_{i_1} \times \ldots \times H_{i_n}$ so that we have an epimorphism $\alpha\colon H \to G$. By construction, we have

$$\alpha^*(x) = \alpha^*_{i_1}(x_{i_1}) + \ldots \alpha^*_{i_n}(x_{i_n}) = 0$$

so x is torsion. This shows that $\mathcal{AU}_t$ is closed under arbitrary sums. All the other claims are clear. $\square$

Lemma 1.10.9. *The subcategory $\mathcal{AU}_f$ is closed under subobjects, extensions, arbitrary sums and arbitrary products.*

Proof. From Lemma 1.10.6(b) it is clear that $\mathcal{AU}_f$ is closed under products and subobjects. As products and sums are computed objectwise, we see that every sum injects in the corresponding product, so $\mathcal{AU}_f$ is also closed under coproducts. Now consider a short exact sequence as follows, in which X and Z are torsion-free

$$X \xrightarrow{f} Y \xrightarrow{g} Z.$$

If T is a torsion object, this gives a left exact sequence

$$0 = \mathcal{AU}(T,X) \xrightarrow{f_*} \mathcal{AU}(T,Y) \xrightarrow{g_*} \mathcal{AU}(T,Z) = 0,$$

proving that $\mathcal{AU}(T,Y) = 0$. It follows that $\mathcal{AU}_f$ is also closed under extensions. □

Let X be a finitely generated torsion object. It is tempting to conclude that $X(G)$ should be zero when G is sufficiently large, in some sense. However, the following example shows that this is not correct.

Example 1.10.10. Let $\theta\colon P \to Q$ be a non-split epimorphism between groups in $\mathcal{U}$. This gives a map $\theta_*\colon e_P \to e_Q$, and we define X to be the cokernel (so X is finitely presented). The obvious generator $x \in X(Q)$ satisfies $\theta^*(x) = 0$ by construction, so x is torsion. As x generates X, it follows that X is a torsion object. Note that $X(G)$ is the quotient of $k[\mathcal{U}(G,Q)]$ in which we kill every basis elements $[\alpha]$ for which the homomorphism $\alpha\colon G \to Q$ can be lifted to P. Note that no split epimorphism $\alpha\colon G \to Q$ can be lifted to P, because that would give rise to a splitting of θ. In particular, if H admits a split epimorphism to Q, then $X(H) \neq 0$. Thus, we have $X(H \times Q) \neq 0$ for all $Q \in \mathcal{U}$.

It is true, however, that if X is a finitely generated torsion object, and G is both sufficiently large and sufficiently free, then $X(G) = 0$. We now proceed to make a precise version of this statement.

Let $\mathcal{U}$ be closed under products and subgroups. For any nontrivial group G_0 in $\mathcal{U}$, we define a tower

$$G_0 \xleftarrow{\epsilon} G_1 \xleftarrow{\epsilon} G_2 \xleftarrow{\epsilon} \dots$$

with $G_k = T^k G_0$ as in Construction 1.7.8.

Lemma 1.10.11. *Consider an element $x \in X(H)$. Then the following are equivalent:*

(a) *x is torsion;*
(b) *There exists $\alpha \in \mathcal{U}(G_n,H)$ for some n such that $\alpha^*(x) = 0$ in $X(G_n)$.*
(c) *There exists n_0 such that for all $n \geq n_0$ and all $\alpha \in \mathcal{U}(G_n,H)$ we have $\alpha^*(x) = 0$ in $X(G_n)$.*

Proof. By the defining properties of G_n and Lemma 1.7.5, we see that $\mathcal{U}(G_n,H) \neq \emptyset$ for large n. It follows that $(c) \Rightarrow (b) \Rightarrow (a)$. Now suppose that (a) holds, so we can choose $\alpha \in \mathcal{U}(K,H)$ for some K with $\alpha^*(x) = 0$. Now let n_0 be least such that $|G_{n_0}| \geq |K|$. Suppose that $n \geq n_0$, so $|G_n| \geq |K|$. If $\lambda \in \mathcal{U}(G_n,H)$, then Lemma 1.7.4 gives us a morphism $\mu \in \mathcal{U}(G_n,K)$ with $\lambda = \alpha\mu$, and it follows that $\lambda^*(x) = 0$. Thus, condition (c) is satisfied. □

Definition 1.10.12. We say that an object X is *G_*-null* if $X(G_n) = 0$ for large n.

Lemma 1.10.13. *If X is G_*-null, then it is torsion. The converse holds if X is finitely generated.*

Proof. First suppose that X is G_*-null. Consider an element $x \in X(H)$. Choose n large enough that $X(G_n) = 0$ and $\mathcal{U}(G_n, H) \neq \emptyset$. Then for $\alpha \in \mathcal{U}(G_n, H)$ we have $\alpha^*(X) = 0$, as required.

Conversely, suppose that X is finitely generated, with generators $x_i \in X(H_i)$ for $i = 1, \ldots, d$ say. By Lemma 1.10.11 we can choose n_i such that $\alpha^*(x_i) = 0$ in $X(G_m)$ for all $m \geq n_i$, and all $\alpha \in \mathcal{U}(G_m, H_i)$. Put $n = \max(n_1, \ldots, n_d)$; then we find that $X(G_n) = 0$ for all $n \geq m$. $\square$

We thank Neil Strickland for the following list of examples:

Example 1.10.14. Let G be cyclic of order p, so $\mathrm{Aut}(G)$ is cyclic of order $p - 1$, and let $\psi \in \mathrm{Aut}(G)$ be a generator. Let X be the cokernel of $\psi_* - 1\colon e_G \to e_G$. By definition $X(H)$ is the quotient of $k[\mathcal{U}(H, G)]$ where we kill all the basis elements $[\alpha]$ such that $[\psi\alpha] = [\alpha]$. As we are working over a field of characteristic zero, we can identify X with c_G from Definition 1.1.5. In particular, X is projective and torsion-free. This illustrates the fact that we can introduce quite a lot of relations without creating torsion.

Example 1.10.15. Let $\mathcal{U}$ be the subcategory of finite (abelian) p-groups, and let C be cyclic of order p. Let $\lambda, \rho\colon C^2 \to C$ be the two projections, and let X be the cokernel of $\lambda_* - \rho_*\colon e_{C^2} \to e_C$. This means that $X(G) = k[T(G)]$, where TG is the coequaliser of the maps $\lambda_*, \rho_*\colon \mathcal{U}(G, C^2) \to \mathcal{U}(G, C)$. Let $Q(G)$ be the Frattini quotient of G, so $Q(G) \simeq C^{d(G)}$ for some $d(G) \geq 0$. If $d(G) = 0$ then $G = 1$ and $T(G) = \emptyset$ and $X(G) = 0$. If $d(G) = 1$ then G is cyclic and $\mathcal{U}(G, C^2) = \emptyset$ so $T(G) = \mathcal{U}(G, C) = \mathcal{U}(Q(G), C)$ (which is a set of size $p - 1$) so $X(G) \simeq k^{p-1}$. Now suppose that $d(G) \geq 2$. If α and β are epimorphisms from G to C with different kernels then the combined map $\phi = (\alpha, \beta)\colon G \to C^2$ is again surjective with $\lambda\phi = \alpha$ and $\rho\phi = \beta$ so $[\alpha] = [\beta]$ in $T(G)$. Even if α and β have the same kernel, we can choose a third epimorphism $\gamma\colon G \to C$ with different kernel (because of the fact that $d(G) \geq 2$); we then have $[\alpha] = [\gamma] = [\beta]$. From this we see that $T(G)$ is a singleton and so $X(G) = k$. To summarize

$$X(G) = k[T(G)] = \begin{cases} 0 & \text{if } d(G) = 0 \\ \mathcal{U}(G, C) \simeq k^{p-1} & \text{if } d(G) = 1 \\ k & \text{if } d(G) \geq 2. \end{cases}$$

From our discussion we also see that

$$\mathrm{tors}(X)(G) \simeq \begin{cases} k^{p-2} & \text{if } G \text{ is nontrivial and cyclic} \\ 0 & \text{otherwise} \end{cases}$$

$$(X/\mathrm{tors}(X))(G) \simeq \begin{cases} 0 & \text{if } G = 1 \\ k & \text{if } G \neq 1. \end{cases}$$

Example 1.10.16. Again take $\mathcal{U}$ as before but with $p = 2$. There are then three morphisms $\lambda, \rho, \sigma \in \mathcal{U}(C^2, C)$, and we define X to be the cokernel of $\lambda_* + \rho_* + \sigma_*\colon e_{C^2} \to e_C$. In other words, if we put $u = \lambda + \rho + \sigma \in e_C(C^2)$ then $X(G)$ is the quotient of $k[\mathcal{U}(G, C)]$ by all elements of the form $\phi^*(r)$ as ϕ runs over $\mathcal{U}(G, C^2)$. If $d(G) = 1$ then $\mathcal{U}(G, C)$ is a singleton and $\mathcal{U}(G, C^2) = \emptyset$ and $X(G) = k$. If $d(G) = 2$ then $k[\mathcal{U}(G, C)]$ has three elements, say α, β, γ, and

$$X(G) = k\{\alpha, \beta, \gamma\}/(\alpha + \beta + \gamma) \simeq k^2.$$

Now consider $X(C^3)$. This is spanned by the seven nonzero homomorphisms $C^3 \to C$. There are seven subgroups of order 4 in $\mathrm{Hom}(C^3, C) \simeq C^3$:

$$A_1 = \{0, e_1^*, e_2^*, (e_1 + e_2)^*\} \qquad A_2 = \{0, e_1^*, e_3^*, (e_1 + e_3)^*\}$$
$$A_3 = \{0, e_2^*, e_3^*, (e_2 + e_3)^*\} \qquad A_4 = \{0, e_3^*, (e_1 + e_2)^*, (e_1 + e_2 + e_3)^*\}$$
$$A_5 = \{0, e_2^*, (e_1 + e_3)^*, (e_1 + e_2 + e_3)^*\} \qquad A_6 = \{0, e_1^*, (e_2 + e_3)^*, (e_1 + e_2 + e_3)^*\}$$
$$A_7 = \{0, (e_1 + e_2)^*, (e_2 + e_3)^*, (e_1 + e_3)^*\}$$

where e_1^*, e_2^* and e_3^* denote the canonical generators. For each of these A_i we have a relation, saying that the sum of the three nonzero homomorphisms in that subgroup is zero. For example, the relation attached to A_1 tells us that $e_1^* + e_2^* + (e_1 + e_2)^* = 0$. Let u be the sum of all these relations, and let v_α be the sum of the subset that involve a particular morphism α. A calculation now shows that $(3v_\alpha - u)/6 = \alpha$. It follows that the resulting quotient $X(C^3)$ is zero. If $d(G) \geq 3$ then any $\alpha \in \mathcal{U}(G, C)$ can be factored through C^3, and it follows from this that $X(G) = 0$. This shows that X is a torsion object.

11. Noetherian abelian categories

The goal of this section is to study when the category $\mathcal{AU}$ is locally noetherian using the criteria developed in [**31**] and [**75**].

Definition 1.11.1. Let $\mathcal{U}$ be a full replete subcategory of $\mathcal{G}$.

- An object $X \in \mathcal{AU}$ is *noetherian* if every subobject of X is finitely generated.
- The category $\mathcal{AU}$ is *locally noetherian* if e_G is noetherian for all $G \in \mathcal{U}$.

Remark 1.11.2. If $\mathcal{AU}$ is locally noetherian we get the following implications

$$\text{finitely resolved} \Leftrightarrow \text{finitely presented} \Leftrightarrow \text{finitely generated}.$$

It is not difficult to find full subcategories of $\mathcal{G}$ for which $\mathcal{AU}$ is not locally noetherian.

Proposition 1.11.3. *Let $\mathcal{U}$ be a full subcategory containing the trivial group and infinitely many cyclic groups of prime order. Then $\mathcal{AU}$ is not locally noetherian.*

Proof. Let $\chi_e^c \in \mathcal{AU}$ be such that

$$\chi_e^c(T) = \begin{cases} 0 & \text{if } T = e \\ k & \text{if } T \neq e. \end{cases}$$

Note that χ_e^c is a subobject of $\mathbb{1}$. Apply Lemma 1.9.4 with $\mathcal{V} = \{C_p \mid p \text{ prime and } C_p \in \mathcal{U}\}$ to see that χ_e^c cannot be finitely generated. □

We shall now introduce the criterion for noetherianity developed in [**31**] which applies to a special type of subcategories.

Definition 1.11.4 ([**31**, 2.2]). Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$ and fix a skeleton $\mathcal{U}'$ for $\mathcal{U}$. If $G, H \in \mathcal{U}$ we write $G \gg H$ to mean that $\mathcal{U}(G, H) \neq \emptyset$. We say that $\mathcal{U}$ has *type* A_∞ if there exists an isomorphism of posets $(\mathcal{U}', \gg) \simeq (\mathbb{N}, \geq)$.

Example 1.11.5. The subcategory of all cyclic groups is not of type A_∞ as there are no epimorphisms $C_3 \to C_2$. However if we fix a prime number p, then the subcategory of cyclic p-groups has type A_∞.

For compatibility with our work, we reformulate [**31**, 3.1] for contravariant diagrams.

Definition 1.11.6. We say that the category $\mathcal{U}$ has the *transitivity property* if the action of $\operatorname{Out}(G)$ on $\mathcal{U}(G, H)$ is transitive whenever $G \gg H$.

Definition 1.11.7. Suppose that $\mathcal{U}$ has the transitivity property. For any pair (G, H) with $G \gg H$ we let $\operatorname{Out}(G)$ act diagonally on $\mathcal{U}(G, H)^2$ and put $\mathcal{U}_2(G, H) = \mathcal{U}(G, H)^2/\operatorname{Out}(G)$.

Lemma 1.11.8. *Suppose we fix $\alpha \in \mathcal{U}(G, H)$ and put $\Phi(\alpha) = \{\phi \in \operatorname{Out}(G) \mid \alpha\phi = \alpha\}$. Then there is a natural bijection $\zeta \colon \mathcal{U}(G, H)/\Phi(\alpha) \to \mathcal{U}_2(G, H)$.*

Proof. We have a map $\mathcal{U}(G,H) \to \mathcal{U}(G,H)^2$ given by $\gamma \mapsto (\alpha,\gamma)$, and this induces a map

$$\zeta\colon \mathcal{U}(G,H)/\Phi(\alpha) \to \mathcal{U}_2(G,H).$$

If $(\beta,\gamma) \in \mathcal{U}(G,H)^2$ then the transitivity property gives $\theta \in \mathrm{Out}(G)$ with $\beta\theta = \alpha$ and it follows that $[\beta,\gamma] = [\beta\theta,\gamma\theta] = \zeta(\gamma\theta)$ in $\mathcal{U}_2(G,H)$. This shows that ζ is surjective.

On the other hand, if $\zeta[\beta_0] = \zeta[\beta_1]$ then there exists $\phi \in \mathcal{U}(G)$ with $(\alpha\phi,\beta_0\phi) = (\alpha,\beta_1)$. This means that $\alpha\phi = \alpha$ (so $\phi \in \Phi(\alpha)$) and $\beta_0\phi = \beta_1$ (so $[\beta_0] = [\beta_1]$ in $\mathcal{U}(G,H)/\Phi(\alpha)$). This shows that ζ is also injective. □

Lemma 1.11.9. *Suppose that $G' \gg G$ and $u \in \mathcal{U}_2(G,H)$, so $u \subseteq \mathcal{U}(G,H)^2$. Put*

$$\lambda(u) = \lambda_{GG'}(u) = \{(\alpha\phi,\beta\phi) \mid (\alpha,\beta) \in u,\ \phi \in \mathcal{U}(G',G)\} \subseteq \mathcal{U}(G',H)^2.$$

Then $\lambda(u)$ is a $\mathrm{Out}(G')$-orbit, or in other words an element of $\mathcal{U}_2(G',H)$. The map λ can also be characterised by $\lambda[\alpha,\beta] = [\alpha\phi,\beta\phi]$ for any $\phi \in \mathcal{U}(G',G)$.

Proof. A typical element of $\lambda(u)$ has the form $x = (\alpha\phi,\beta\phi)$ with $(\alpha,\beta) \in u$ and $\phi \in \mathcal{U}(G,H)$. If $\theta \in \mathrm{Out}(G)$ then the map $\phi' = \phi\theta$ also lies in $\mathcal{U}(G,H)$ and $\theta^*x = (\alpha\phi',\beta\phi')$; this shows that $\lambda(u)$ is preserved by $\mathrm{Out}(G)$.

Now suppose we fix an element $x = (\alpha,\beta) \in u$ and a map $\phi \in \mathcal{U}(G,H)$ and put $x' = (\alpha\phi,\beta\phi) \in \lambda(u)$. Any element of u has the form $(\alpha\zeta,\beta\zeta)$ for some $\zeta \in \mathrm{Out}(G)$. Thus, any element $y \in \lambda(u)$ has the form $y = (\alpha\zeta\psi,\beta\zeta\psi)$ for some $\zeta \in \mathrm{Out}(G)$ and $\psi \in \mathcal{U}(G',G)$. By the transitivity property we can find $\xi \in \mathrm{Out}(G')$ with $\zeta\psi = \phi\xi$, so $y = (\alpha\phi\xi,\beta\phi\xi) = \xi^*(x')$. It follows that $\lambda[x] = [x']$, so in particular $\lambda[x]$ is an orbit as claimed. □

Definition 1.11.10. We say that $\mathcal{U}$ has the *bijectivity property* if for all H there exists $G \gg H$ such that for all $G' \gg G$ the map

$$\lambda\colon \mathcal{U}_2(G,H) \to \mathcal{U}_2(G',H)$$

is bijective.

Remark 1.11.11. Our bijectivity property is not the same as that of [**31**, 3.2]. However, Lemma 1.11.8 shows that they are equivalent.

We are finally ready to state the criterion.

Theorem 1.11.12 ([**31**, 3.7]). *Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$ of type A_∞. Suppose that $\mathcal{U}$ satisfies the transitivity and bijectivity properties. Then $\mathcal{AU}$ is locally noetherian.*

We now apply the criterion to our case of interest.

Theorem 1.11.13. *Fix a prime number p and let $\mathcal{C}$ be the family of cyclic p-groups. Then the category $\mathcal{AC}$ is locally noetherian.*

Proof. We have already seen that $\mathcal{C}$ has type A_∞ so it is enough to check that it satisfies the transitivity and bijectivity property.

Suppose that G is cyclic of order p^n. Then for $u \in \mathbb{Z}/p^n$ we have a well-defined multiplication $\mu_u\colon G \to G$, which is surjective iff it is bijective iff $u \in (\mathbb{Z}/p^n)^\times$. This gives an isomorphism $(\mathbb{Z}/p^n)^\times \to \mathrm{Aut}(G)$. Now let H be another cyclic group of order p^m, and suppose that $n \geq m$ so that $\mathcal{U}(G,H) \neq \emptyset$.

Note that the reduction map $\rho\colon \mathbb{Z}/p^n \to \mathbb{Z}/p^m$ induces a map $\rho\colon (\mathbb{Z}/p^n)^\times \to (\mathbb{Z}/p^m)^\times$ which is still surjective. For any $\alpha \in \mathcal{U}(G,H)$ and $u \in (\mathbb{Z}/p^n)^\times$ we have $\alpha \circ \mu_u = \mu_{\rho(u)} \circ \alpha$. Using this we find that $\mathcal{U}(G,H)$ is a single free orbit for $\mathrm{Aut}(H)$, and a single orbit for $\mathrm{Aut}(G)$. Thus $\mathcal{C}$ satisfies the transitivity condition.

If $(\alpha,\beta) \in \mathcal{U}(G,H)^2$ then there is a unique element $u \in (\mathbb{Z}/p^m)^\times$ with $\beta = \mu_u \circ \alpha$. This is unchanged if we compose α and β with any surjective homomorphism $\phi\colon G' \to G$. It follows that the rule $[\alpha,\beta]\mapsto u$ gives a well-defined bijections $\xi=\xi_{GH}\colon\mathcal{U}_2(G,H)\to(\mathbb{Z}/p^m)^\times$. This also satisfies $\xi_{G'H}\lambda=\xi_{GH}$, so all the maps λ are bijective, and so $\mathcal{C}$ satisfies the bijectivity condition. □

The rest of this section will be devoted to proving the following result.

Theorem 1.11.14. *Fix a prime number p.*

(a) *Let $\mathcal{P}$ be the subcategory of finite abelian p-groups. Then $\mathcal{AP}$ is locally noetherian.*
(b) *Let $\mathcal{U}$ be a multiplicative global subfamily of $\mathcal{P}$. Then $\mathcal{AU}$ is locally noetherian.*

We will apply a different criterion due to Sam and Snowden that we shall now recall [**75**].

Definition 1.11.15. Let $\mathcal{C}$ be a small category.

- A *sequence* in $\mathcal{C}$ means a map $u\colon\mathbb{N}\to\mathrm{obj}(\mathcal{C})$ and a *subsequence* of u is a map of the form $u\circ f$, where $f\colon\mathbb{N}\to\mathbb{N}$ is strictly increasing.
- We say that u is *good* if there exists $i<j$ such that $\mathcal{C}(u(i),u(j))\neq\emptyset$.
- We say that u is *very good* if $\mathcal{C}(u(i),u(j))\neq\emptyset$ for all $i\leq j$.
- We say that $\mathcal{C}$ is *well-quasi-ordered* (or *wqo*) if it satisfies the following conditions:
  (a) Every sequence is good.
  (b) Every sequence has a very good subsequence.
- We say that $\mathcal{C}$ is *cowqo* if $\mathcal{C}^{\mathrm{op}}$ is wqo.
- We say that $\mathcal{C}$ is *slice-wqo* if the slice category $X/\mathcal{C}$ is wqo for all objects X.

Remark 1.11.16. If P is a preordered set, we regard it as a category with $P(a,b)=1$ when $a\leq b$ and $P(a,b)=\emptyset$ when $a\not\leq b$. We say that P is wqo if it is wqo when regarded as a category in this way. Conversely, if $\mathcal{C}$ is a small category we can regard it as a preordered set with $X\leq Y$ if and only if $\mathcal{C}(X,Y)\neq\emptyset$. It is clear that this preordered set is wqo if and only the original category is wqo.

Lemma 1.11.17. *Conditions (a) and (b) in Definition 1.11.15 are equivalent.*

Proof. Let u be a sequence in $\mathcal{C}$. If (b) holds, then we can find $f\colon\mathbb{N}\to\mathbb{N}$ strictly increasing such that $u\circ f$ is very good. This means that for all $i\leq j$ we have $f(i)<f(j)$ and $u(f(i))\leq u(f(j))$. So u is good. Conversely, suppose that (a) holds. Consider the subset $S=\{i\in\mathbb{N}\mid u(i)\not\leq u(j),\ \ \forall\, j>i\}$. If S is infinite, then we can find an ordering preserving bijection $f\colon\mathbb{N}\to S$. The composite $u_{|S}\circ f$ is a sequence in $\mathcal{C}$ which is good by (a). This contradicts the definition of the set S. Therefore S must be finite. Then choose a strictly increasing $f\colon\mathbb{N}\to\mathbb{N}$ such that $f(0)>s$ for all $s\in S$. Then $u\circ f$ is very good as required. □

Definition 1.11.18. Let $\mathcal{C}$ be a small category.

- We say that $\mathcal{C}$ is *rigid* if every endomorphism is an identity.
- A *hom-ordering* on $\mathcal{C}$ consists of a system of well-orderings of the hom sets $\mathcal{C}(X,Y)$ such that for all $\alpha\colon Y\to Z$, the induced map $\alpha_*\colon\mathcal{C}(X,Y)\to\mathcal{C}(X,Z)$ is monotone.

Definition 1.11.19. Let $\mathcal{C}$ be a small category and let $\mathcal{D}$ be essentially small.

- We say that $\mathcal{C}$ is *Gröbner* if it is rigid, slice-wqo and it admits a hom-ordering.
- We say that $\mathcal{D}$ is *quasi-Gröbner* if there is a Gröbner category $\mathcal{C}$ and an essentially surjective functor $M\colon\mathcal{C}\to\mathcal{D}$ satisfying the following property:
  (F) Given any object $x\in\mathcal{D}$ there exist finitely many objects $y_1,\dots,y_n\in\mathcal{C}$ and morphisms $f_i\colon x\to M(y_i)$ in $\mathcal{D}$ such that for any object $y\in\mathcal{C}$ and any morphism $f\colon x\to M(y)$ in $\mathcal{D}$, there exists a morphism $g\colon y_i\to y$ in $\mathcal{C}$ such that $f=M(g)\circ f_i$.

We are finally ready to state the criterion.

Theorem 1.11.20. [**75**, 4.3.2] *Let $\mathcal{C}$ be a quasi-Gröbner category. Then the category $[\mathcal{C},\mathrm{Vect}_k]$ is locally noetherian.*

Remark 1.11.21. Our terminology differs from that of [**75**]. We use the term "rigid" instead of "direct" for a category, and "wqo" instead of "noetherian" for a preordered set. Furthermore, their conditions (G1) and (G2) correspond to our notion of hom-ordering and slice-wqo respectively.

Before proving Theorem 1.11.14 we need to introduce more notation and prove some technical results.

Well-quasi orders.

Remark 1.11.22. To deal with some set-theoretic issues, we let $\mathcal{X}$ denote the set of hereditarily finite sets, so $\mathcal{X}$ is countable and closed under taking subsets, products and quotients, and contains sets of all finite orders. When we discuss categories of finite sets with extra structure, we will implicitly assume that the underlying sets are in $\mathcal{X}$, so that the category will be small.

Definition 1.11.23. Let $\mathcal{C}$ and $\mathcal{D}$ be preordered sets, and let $f\colon \mathcal{C} \to \mathcal{D}$ be a function.

(a) We say that f is *monotone* if $p \leq p'$ implies $f(p) \leq f(p')$.
(b) We say that f is *comonotone* if $f(p) \leq f(p')$ implies $p \leq p'$.

Remark 1.11.24. Here $\mathcal{C}$ and $\mathcal{D}$ might be small categories, regarded as preordered sets as in Remark 1.11.16. In that case, any functor $f\colon \mathcal{C} \to \mathcal{D}$ gives a monotone map.

Proposition 1.11.25. *If $f\colon \mathcal{C} \to \mathcal{D}$ is comonotone and $\mathcal{D}$ is wqo then $\mathcal{C}$ is wqo.*

Proof. If $u\colon \mathbb{N} \to \mathcal{C}$ is a sequence, then $f \circ u$ must be good, so there exists $i \leq j$ with $fu(i) \leq fu(j)$, but that implies $u(i) \leq u(j)$ by the comonotone property. □

Proposition 1.11.26. *Any finite product of wqo preordered sets is again wqo.*

Proof. It suffices to show that if P and Q are wqo, then so is $P \times Q$. Let $u\colon \mathbb{N} \to P \times Q$ be a sequence. As P is wqo, we can find a subsequence v such that $\pi_P \circ v$ is nondecreasing. As Q is wqo, we can then find a subsequence w of v such that $\pi_Q \circ w$ is nondecreasing. Now w is nondecreasing subsequence of u. □

Definition 1.11.27. Let P be a preordered set. We say that a finite list $u \in P^n$ is *bad* if there is no pair (i, j) with $0 \leq i < j < n$ and $u(i) \leq u(j)$. We say that such a finite list u is *very bad* if there is an infinite bad sequence extending it. If so, the set

$$E(u) = \{u' \in P \mid (u(0), \ldots, u(n-1), u') \text{ is very bad}\}$$

is nonempty. Now suppose we have a well-ordered set W and a function $\lambda\colon P \to W$. Following Nash-Williams' theory of minimal bad sequences [**65**], we put

$$EM(u) = \{u' \in E(u) \mid \lambda(u') = \min(\lambda(E(u)))\} \neq \emptyset.$$

We say that a very bad list $u \in P^n$ is *λ-minimal* if for all $k < n$ we have $u(k) \in EM(u_{<k})$. We say that a bad sequence u is *λ-minimal* if every initial segment $u_{<k}$ is λ-minimal.

Lemma 1.11.28. *If P is not wqo, then it has a λ-minimal bad sequence.*

Proof. Start with the empty sequence, which is very bad by the assumption that P is not wqo. Then choose recursively $u(k) \in EM(u_{<k})$ for all $k \geq 0$. □

Proposition 1.11.29. *Let P and λ be as above. Let P_0 be a subset of P, and let $\chi\colon P_0 \to P$ be a map such that*

(a) *For all $x \in P_0$ we have $\chi(x) \leq x$ and $\lambda(\chi(x)) < \lambda(x)$.*
(b) *Every bad sequence $u\colon \mathbb{N} \to P$ has a subsequence v contained in P_0 with the following property: if $i < j$ with $\chi(v(i)) \leq \chi(v(j))$, then $v(i) \leq v(j)$.*

Then P is wqo.

Proof. Suppose not, so there exists a minimal bad sequence u. Let v be a subsequence as in (b), so $v(n) = u(f(n))$ for some strictly increasing map $f\colon \mathbb{N} \to \mathbb{N}$. Define $w(n) = u(n)$ for $n < f(0)$ and $w(f(0)+k) = \chi(v(k))$. We claim that w is bad. If not, we have $i < j$ with $w(i) \leq w(j)$. If $j < f(0)$ this gives $u(i) \leq u(j)$, contradicting the badness of u. Suppose instead that $i < f(0) \leq j$, so $w(i) = u(i)$ and $w(j) = \chi(v(j')) = \chi(u(j''))$ for some $j' \geq 0$ and $j'' \geq f(0)$. We now have $u(i) \leq \chi(u(j'')) \leq u(j'')$, again contradicting the badness of u. This just leaves the possibility that $f(0) \leq i < j$, so $w(i) = \chi(v(i')) = \chi(u(i''))$ and $w(j) = \chi(v(j')) = \chi(u(j''))$ for some i', j', i'', j'' with $i' < j'$ and $i'' < j''$. We now have $\chi(v(i')) \leq \chi(v(j'))$ so $v(i') \leq v(j')$ b y condition (b), so $u(i'') \leq u(j'')$, yet again contradicting the badness of u. It follows that w must be bad after all. However, this contradicts the λ-minimality of $u(f(0))$ in $E(u_{<f(0)})$. $\square$

Definition 1.11.30. Let $\mathcal{C}$ be a wqo category. We define $\mathcal{SC}$ to be the category of pairs (X,p), where X is a finite, totally ordered set, and $p\colon X \to \mathcal{C}$. A morphism from (X,p) to (Y,q) consists of a strictly monotone map $\overline{\phi}\colon X \to Y$ together with a family of morphisms $\phi_x\colon p(x) \to q(\overline{\phi}(x))$ for each $x \in X$. These are composed in the obvious way. We put $\lambda(X,p) = |X|$.

Remark 1.11.31. If $\mathcal{C}$ is just a preordered set, then a morphism from (X,p) to (Y,q) is just a strictly monotone map $\overline{\phi}\colon X \to Y$ such that $p(x) \leq q(\overline{\phi}(x))$ for all x.

Proposition 1.11.32 (Higman's Lemma). *$\mathcal{SC}$ is wqo.*

Proof. For (X,p) with $X \neq \emptyset$ we define $x_0 = \min(X)$ and $\epsilon(X,p) = p(x_0) \in \mathcal{C}$ and $\chi(X,p) = (X',p')$, where $X' = X \setminus \{x_0\}$ and $p' = p|_{X'}$. This clearly satisfies condition (a) of Proposition 1.11.29. If $u\colon \mathbb{N} \to \mathcal{SC}$ is bad then $u(n)$ can never be empty (otherwise we would have $u(n) \leq u(n+1)$), so we have a sequence $u_1 = \epsilon \circ u\colon \mathbb{N} \to \mathcal{C}$. As $\mathcal{C}$ is wqo, we can choose a strictly increasing map $f\colon \mathbb{N} \to \mathbb{N}$ such that $u_1 \circ f\colon \mathbb{N} \to P$ is very good. Now put $v = u \circ f$. If $i < j$ and $\chi(v(i)) \leq \chi(v(j))$ then we also have $\epsilon(v(i)) \leq \epsilon(v(j))$ and it follows easily that $v(i) \leq v(j)$. Using Proposition 1.11.29 we can now see that $\mathcal{SC}$ is wqo. $\square$

Definition 1.11.33. Let X and Y be nonempty finite totally ordered sets. Let $\phi\colon X \to Y$ be a surjective map, which need not preserve the order. We define an $\phi^\dagger\colon Y \to X$ by $\phi^\dagger(y) = \min(\phi^{-1}\{y\})$. We say that ϕ is $\dagger$-*monotone* if $\phi^\dagger$ is monotone.

Lemma 1.11.34. *For any ϕ we have $\phi\phi^\dagger(y) = y$ for all $y \in Y$, and $\phi^\dagger\phi(x) \leq x$ for all $x \in X$. If ϕ is $\dagger$-monotone then we have $\phi(x) < y$ whenever $x < \phi^\dagger(y)$. In particular, if x_0 and y_0 are the smallest elements of X and Y, then $\phi(x_0) = y_0$ and $\phi^\dagger(y_0) = x_0$.*

Proof. It is clear by definition that $\phi\phi^\dagger(y) = y$. Next, if $x \in X$ then x is a preimage of $\phi(x)$, whereas $\phi^\dagger\phi(x)$ is the smallest preimage, so $\phi^\dagger\phi(x) \leq x$. Now suppose that ϕ is $\dagger$-monotone. If $y \leq \phi(x)$ then $\phi^\dagger(y) \leq \phi^\dagger\phi(x) \leq x$. By the contrapositive, if $x < \phi^\dagger(y)$ we must have $\phi(x) < y$, as claimed. We now claim that $x_0 = \phi^\dagger(y_0)$. Indeed, if not then $x_0 < \phi^\dagger(y_0)$ so $\phi(x_0) < y_0$, contradicting the definition of y_0. We must therefore have $x_0 = \phi^\dagger(y_0)$ after all, and it follows that $\phi(x_0) = \phi\phi^\dagger(y_0) = y_0$. $\square$

Corollary 1.11.35. *Suppose we have $\dagger$-monotone maps*

$$X \xrightarrow{\phi} Y \xrightarrow{\psi} Z.$$

Then $(\psi\phi)^\dagger = \phi^\dagger\psi^\dagger$, and so $\psi\phi$ is also $\dagger$-monotone.

Proof. Given $z \in Z$ put $y = \psi^\dagger(z)$ and $x = \phi^\dagger(y) = \phi^\dagger\psi^\dagger(z)$. Using the Lemma we get $\psi\phi(x) = z$. We also see that if $x' < x = \phi^\dagger(y)$ then $\phi(x') < y = \psi^\dagger(z)$ and thus $\psi(\phi(x')) < z$. This means that x has the defining property of $(\psi\phi)^\dagger(z)$. We therefore have $(\psi\phi)^\dagger = \phi^\dagger\psi^\dagger$. This is the composite of two increasing maps, so it is again increasing, so $\psi\phi$ is $\dagger$-monotone. $\square$

Definition 1.11.36. We define a category $\mathcal{L}_\dagger$ as follows. The objects are finite nonempty sets equipped with a map $e_X\colon X \to \mathbb{N}$, together with a total order on X with respect to which e_X is monotone. The morphisms from X to Y are †-monotone surjective maps $\phi\colon X \to Y$ such that $e_Y(\phi(x)) \leq e_X(x)$ for all $x \in X$.

Definition 1.11.37. We define $\alpha, \beta\colon \mathcal{L}_\dagger \to \mathbb{N}$ by $\alpha(X) = e_X(\min(X))$ and $\beta(X) = \min(e_X(X))$. Next, for $x \in X \setminus \{\min(X)\}$ we define

$$e'_X(x) = \min\{e_X(x') \mid x' < x\} \in \mathbb{N},$$

and $e^*_X(x) = (e_X(x), e'_X(x)) \in \mathbb{N}^2$. The set $X \setminus \{\min(X)\}$ together with the map e^*_X define an object $\gamma(X) \in \mathcal{S}(\mathbb{N}^2)$.

Proposition 1.11.38. *The map $(\alpha, \beta, \gamma)\colon \mathcal{L}_\dagger^{\mathrm{op}} \to \mathbb{N}^2 \times \mathcal{S}(\mathbb{N}^2)$ is comonotone, so $\mathcal{L}_\dagger$ is cowqo.*

Proof. Suppose that $\alpha(X) \leq \alpha(Y)$ and $\beta(X) \leq \beta(Y)$ and $\gamma(X) \leq \gamma(Y)$; we need to construct a morphism from Y to X. As $\beta(X) \leq \beta(Y)$, we can choose a strictly increasing map $\psi\colon X \setminus \{\min(X)\} \to Y \setminus \{\min(Y)\}$ with $e_X(x) \leq e_Y(\psi(x))$ and $e'_X(x) \leq e'_X(\psi(x))$ for all x. We extend ψ over all of X by putting $\psi(\min(X)) = \min(Y)$, and note that the relation $e_X(x) \leq e_Y(\psi(x))$ remains true. We define $\phi\colon \psi(X) \to X$ by $\phi(\psi(x)) = x$. Now consider an element $y \in Y \setminus \psi(X)$, so $y \neq \min(Y)$. If $y > \max(\psi(X))$ we choose x with $e_X(x) = \beta(X)$ and define $\phi(y) = x$, noting that $e_Y(y) \geq \beta(Y) \geq \beta(X) = e_X(x)$. Otherwise, we let x' be least such that $\psi(x') > y$, then choose $x < x'$ with $e_X(x) = e'_X(x')$. This gives

$$e_Y(y) \geq e'_Y(\psi(x')) \geq e'_X(x') = e_X(x),$$

and we define $\phi(y) = x$. We now have a surjective map $\phi\colon Y \to X$ with $e_Y(y) \geq e_X(\phi(y))$ for all y. We also have $\phi(\psi(x)) = x$, and $\phi(y) < x$ whenever $y < \psi(x)$, so that $\psi = \phi^\dagger$. This means that ϕ is a morphism in $\mathcal{L}^\dagger$, as required. □

Corollary 1.11.39. *$\mathcal{L}_\dagger$ is slice-cowqo*

Proof. The construction $(X \xleftarrow{p} U) \mapsto (p^{-1}\{x\})_{x \in X}$ gives a full and faithful embedding $\mathcal{L}_\dagger/X \to \prod_{x \in X} \mathcal{L}_\dagger$. Finally apply Proposition 1.11.26. □

Hom-orderings.

Remark 1.11.40. A hom-ordering of $\mathcal{C}^{op}$ consists of a system of well-orderings of the hom sets $\mathcal{C}(X, Y)$ such that for all $\beta\colon W \to X$, the induced map $\beta^*\colon \mathcal{C}(X, Y) \to \mathcal{C}(W, Y)$ is monotone.

Remark 1.11.41. If $F\colon \mathcal{C} \to \mathcal{D}$ is a faithful functor and we have a hom-ordering on $\mathcal{D}$ then we can define a hom-ordering on $\mathcal{C}$ by declaring that $\phi \leq \psi$ if and only if $F\phi \leq F\psi$.

Definition 1.11.42. Let $\mathcal{F}_\dagger$ be the category of finite totally ordered sets and †-monotone surjections. We order $\mathcal{F}_\dagger(X, Y)$ lexicographically, so $\phi < \psi$ iff there exists $x_0 \in X$ with $\phi(x_0) < \psi(x_0)$ and $\phi(x) = \psi(x)$ for all $x < x_0$.

Proposition 1.11.43. *This gives a hom-ordering on $\mathcal{F}_\dagger^{\mathrm{op}}$.*

Proof. It is standard and easy that the above rule gives a total order on the finite set of surjections from X to Y. Now suppose we have $\theta\colon W \to X$ and $\phi, \psi\colon X \to Y$ with $\phi \leq \psi$; we must show that $\phi\theta \leq \psi\theta$. By assumption there exists $x_0 \in X$ with $\phi(x_0) < \psi(x_0)$ and $\phi(x) = \psi(x)$ for all $x < x_0$. Put $w_0 = \theta^\dagger(x_0) = \min(\theta^{-1}\{x_0\})$. Then $(\phi\theta)(w_0) = \phi(x_0) < \psi(x_0) = (\psi\theta)(w_0)$. On the other hand, if $w < w_0$ then Lemma 1.11.34 tells us that $\theta(w) < x_0$ and so $(\phi\theta)(w) = (\psi\theta)(w)$. □

Corollary 1.11.44. *The faithful forgetful functor $\mathcal{L}_\dagger^{\mathrm{op}} \to \mathcal{F}_\dagger^{\mathrm{op}}$ gives a hom-ordering to $\mathcal{L}_\dagger^{\mathrm{op}}$.* □

Proof of Theorem 1.11.14. For the duration of this proof we put $C[k] = C_{p^k} \in \mathcal{P}$. If $k \geq m$, we write π for the standard surjective homomorphism $C[k] \to C[m]$. For $A \in \mathcal{P}$ and $a \in A$, we let η_a be the natural number such that a has order p^{η_a}

By combining Corollaries 1.11.39 and 1.11.44, we see that $\mathcal{L}_\dagger^{\mathrm{op}}$ is Gröbner.

We define an essentially surjective functor $M\colon \mathcal{L}_\dagger^{\mathrm{op}} \to \mathcal{P}^{\mathrm{op}}$ as follows. For an object $X \in \mathcal{L}_\dagger$, we set $MX = \prod_{x \in X} C[e_X(x)]$. Given a morphism $\phi\colon X \to Y$ in $\mathcal{L}_\dagger$, we define $\phi_*\colon MX \to MY$ by

$$(\phi_* m)_y = \prod_{\phi(x)=y} \pi(m_x).$$

Let us introduce some terminology before proceeding with the proof. A *framing* of $A \in \mathcal{P}$ is a surjective homomorphism $MX \to A$ for some $X \in \mathcal{L}_\dagger$. This corresponds to a map $\alpha_0\colon X \to A$ such that $\eta(\alpha_0(x)) \leq e_X(x)$ for all x, and $\alpha_0(X)$ generates A. We say that the framing is *tautological* if X is a subset of A and α_0 is just the inclusion and

$$e_X(x) = \max\{\eta(w) \mid w \in X,\ w \leq x\}.$$

It is clear from the definition that there are only finitely many tautological framings. Unravelling the definitions, we see that M satisfies condition (F) if any framing $\alpha_0\colon X \to A$ factors as

$$X \to \overline{X} \to A$$

where the first arrow is in $\mathcal{L}_\dagger$ and the second one is a tautological framing. So if $\alpha\colon X \to A$ is an arbitrary framing, we define $\overline{X} = \alpha_0(X) \subset A$ and $e_{\overline{X}} = \eta|_{\overline{X}}$ and set $\overline{\alpha}_0\colon \overline{X} \to A$ to be the inclusion. We also define $\alpha_0^\dagger\colon A \to X$ by $\alpha_0^\dagger(a) = \min(\alpha_0^{-1}(a))$ and order $\overline{X}$ by declaring that $a < b$ iff $\alpha_0^\dagger(a) < \alpha_0^\dagger(b)$. This makes $\overline{\alpha}_0$ into a tautological framing and gives the required factorization. Therefore $\mathcal{P}^{\mathrm{op}}$ is quasi-Gröbner and so part (a) holds.

For part (b), we put

$$\Omega = \{\eta_a \mid\ A \in \mathcal{U},\ a \in A\} \subset \mathbb{N}.$$

Define $\mathcal{L}_\dagger^{\mathcal{U}}$ to be the full subcategory of $\mathcal{L}_\dagger$ consisting of objects X with $\mathrm{Image}(e_X) \subset \Omega$. This is still Gröbner by [**75**, 4.4.2]. It is now easy to check that the functor $M\colon (\mathcal{L}_\dagger^{\mathcal{U}})^{\mathrm{op}} \to \mathcal{U}^{\mathrm{op}}$ defined as above is essentially surjective and satisfies property (F). Thus $\mathcal{U}^{\mathrm{op}}$ is quasi-Gröbner and $\mathcal{A}\mathcal{U}$ is locally noetherian.

12. Representation stability

In this section we show that any finitely presented object can be recovered by a finite amount of data via a stabilization recipe. This phenomenon is called central stability and it was first introduced by Putman [**70**]. We also show that under the noetherian assumption, any finitely generated object satisfies the analogue of the injectivity and surjectivity conditions in the definition of representation stability due to Church–Farb [**24**, 1.1].

Definition 1.12.1. Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$. For $X \in \mathcal{A}\mathcal{U}$, we put

$$\tau_n(X) = i_!^{\leq n} i_{\leq n}^*(X) \in \mathcal{A}\mathcal{U},$$

and note that there is a counit map $\tau_n(X) \to X$. We also define natural maps $\tau_n(X) \to \tau_{n+1}(X)$ as follows. Let j denote the inclusion $\mathcal{U}_{\leq n} \to \mathcal{U}_{\leq (n+1)}$, so we have a counit map $j_! j^*(Y) \to Y$ for all $Y \in \mathcal{A}\mathcal{U}_{\leq (n+1)}$. Taking $Y = i_{\leq (n+1)}^*(X)$ for some $X \in \mathcal{A}\mathcal{U}$, we get a map $j_! i_{\leq (n+1)}^* X \to i_{\leq (n+1)}^* X$. Applying the functor $i_!^{\leq (n+1)}$ to this gives the required map $\tau_n(X) \to \tau_{n+1}(X)$.

We list a few important properties of the truncation functor.

Proposition 1.12.2. *Consider an object $X \in \mathcal{A}\mathcal{U}$.*

(a) *Then X is the colimit of the objects $\tau_n(X)$.*
(b) *We have $\tau_n(e_G) = e_G$ if $G \in \mathcal{U}_{\leq n}$ and $\tau_n(e_G) = 0$ otherwise.*

(c) *For all $G \in \mathcal{U}$ and $n \geq 0$, we have*

$$\tau_n(X)(G) = \varinjlim_{H \in N(G,n)} X(G/H)$$

where $N(G,n) = \{H \triangleleft G \mid |G/H| \leq n\}$.

Proof. For part (a) it is enough to notice that $\tau_n(X)(G) = X(G)$ for $|G| \leq n$. Part (b) follows from Lemma 1.3.3(i)

Using the formula for Kan extensions, we see that $\tau_n(X)(G)$ can be written as a colimit over the comma category $(G \downarrow \mathcal{U}_{\leq n})$. Suppose we have objects $(G \xrightarrow{\alpha} A)$ and $(G \xrightarrow{\beta} B)$ in the comma category so $A, B \in \mathcal{U}_{\leq n}$. As α and β are surjective, we find that there is a unique morphism from α to β if $\ker(\alpha) \leq \ker(\beta)$, and no morphisms otherwise. This shows that the comma category is equivalent to the poset $N(G,n)$ so part (c) follows. □

The following is a characterization of finitely generated and finite presented objects.

Proposition 1.12.3. *Consider an object $X \in \mathcal{AU}$.*

(a) *X is finitely generated if and only if X has finite type and there exists $N \in \mathbb{N}$ such that the canonical map $\tau_n(X) \to X$ is an epimorphism for all $n \geq N$.*
(b) *X is finitely presented if and only if X has finite type and there exists $N \in \mathbb{N}$ such that the canonical map $\tau_n(X) \to X$ is an isomorphism for all $n \geq N$.*

Proof. For part (a), assume that the map $\tau_n(X) \to X$ is an epimorphism for all $n \geq N$. Note that we can construct an epimorphism

$$\bigoplus_{G \in \mathcal{G}_{\leq n}} \dim(X(G))\, e_G \to i^*_{\leq n}(X)$$

as X has finite type. We apply $i_!^{\leq n}$ to get an epimorphism

$$\bigoplus_{G \in \mathcal{G}_{\leq n}} \dim(X(G))\, e_G \to \tau_n(X)$$

since $i_!^{\leq n}$ preserves all colimits by Lemma 1.3.3(f). Post-composition with $\tau_n(X) \to X$ gives the desired epimorphism. Conversely, assume that X is finitely generated so that we have a short exact sequence $0 \to K \to P \to X \to 0$ with P finitely projective. Note that by Proposition 1.12.2(b), there must exist $N \in \mathbb{N}$ such that $\tau_n(P) \simeq P$ for all $n \geq N$. The commutativity of the diagram

$$\begin{tikzcd} P \arrow[r] & X \arrow[r] & 0 \\ \tau_n(P) \arrow[u, "\simeq"] \arrow[r] & \tau_n(X) \arrow[u] & \end{tikzcd}$$

implies that the map $\tau_n(X) \to X$ is an epimorphism for all $n \geq N$.

For part (b), assume that X is finitely presented. Then there exists a short exact sequence $0 \to K \to P \to X \to 0$ with P finitely projective and K finitely generated. By Part (a), it is enough to show that the canonical map $\tau_n(X) \to X$ is eventually monic. Note that for large n, we have a diagram

$$\begin{tikzcd}
 & \ker(i^n_K) \arrow[d] & 0 \arrow[d] & \ker(i^n_X) \arrow[d] & \\
 & \tau_n(K) \arrow[r] \arrow[d, "i^n_K"] & \tau_n(P) \arrow[r] \arrow[d, "\simeq"] & \tau_n(X) \arrow[r] \arrow[d, "i^n_X"] & 0 \\
0 \arrow[r] & K \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & X \arrow[r] \arrow[d] & 0 \\
 & \operatorname{coker}(i^n_K) & 0 & \operatorname{coker}(i^n_X) &
\end{tikzcd}$$

where the bottom row is exact and the top is only right exact. By assumption both K and X are finitely generated, so the maps i^n_K and i^n_X are epimorphisms by part (a). Thus, the Snake Lemma tell us that $\ker(i^n_X) = 0$. Conversely, assume that the natural map is an isomorphism. By part (a), X is finitely generated so we have a short exact sequence $0 \to K \to P \to X \to 0$ with P finitely projective. By applying the Snake Lemma to the diagram above, we see that $\operatorname{coker}(i^n_K) = 0$ for large n, so K is finitely generated and X is finitely presented. □

Recall the functor q_n from Construction 1.7.2.

LEMMA 1.12.4. *Let $\mathcal{U}$ be multiplicative and closed under passage to subgroups. For $n \geq 0$, we put*

$$\mathcal{U}^*_{\leq n} = \{G \in \mathcal{U} \mid q_n(G) = G\}$$

so that the inclusion $i_{\leq n}$ factors as $\mathcal{U}_{\leq n} \xrightarrow{j} \mathcal{U}^*_{\leq n} \xrightarrow{k} \mathcal{U}$. Then we have $(i_{\leq n})_! = q^*_n \circ j_!$ as functors $\mathcal{AU}_{\leq n} \to \mathcal{AU}$.

PROOF. The functor q_n is left adjoint to k so the claim follows by Lemma 1.3.3(h). □

PROPOSITION 1.12.5. *Let $\mathcal{U}$ be multiplicative and closed under passage to subgroups, and consider a finitely presented object $X \in \mathcal{AU}$. Then there exists $n \in \mathbb{N}$ such that $X(G) = X(q_n G)$ for all $G \in \mathcal{U}$.*

PROOF. Choose a finite presentation

$$\bigoplus_{i=1}^{r} e_{G_i} \xrightarrow{f} \bigoplus_{j=1}^{s} e_{H_j} \to X \to 0.$$

Choose n large enough so that $G_i, H_j \in \mathcal{U}^*_{\leq n}$ for all i and j. Let Y be cokernel of f in $\mathcal{AU}^*_{\leq n}$. We claim that $X = q^*_n(Y)$. As the functor q^*_n preserves all colimits it is enough to show that $q^*_n e_G = e_G$ for all $G \in \mathcal{U}^*_{\leq n}$. Using that q_n is left adjoint to the inclusion $k\colon \mathcal{U}^*_{\leq n} \to \mathcal{U}$ we see that

$$(q^*_n e_G)(H) = k[\mathcal{U}(q_n H, G)] = k[\mathcal{U}(H, G)] = e_G(H)$$

which concludes the proof.

□

We now restrict to the locally noetherian case. Recall the definition of eventually torsion-free and stably surjective object from the introduction, see Definition E.

THEOREM 1.12.6. *Fix a prime number p. Let $\mathcal{P}$ be the family of finite abelian p-groups and consider a finitely generated object $X \in \mathcal{AP}$. Then the restriction of X to $\mathcal{AC}_p$ and $\mathcal{AF}^p_n$ for $n \geq 1$, is eventually torsion-free and stably surjective.*

PROOF. The restriction of X to $\mathcal{AC}_p$ is eventually torsion-free and stably surjective by [**32**, 5.1, 5.2].

Let $\mathcal{P}_n$ denote the subfamily of abelian p-groups of exponent less than or equal to p^n. Write Y_n for the restriction of X to $\mathcal{AP}_n$ and note that this is still finitely generated by Lemma **??**. Put $q = p^n$. Note that $\mathrm{tors}(Y_n)$ is finitely generated so G_*-null by Lemma 1.10.13. This means that $\mathrm{tors}(Y_n)(C_q^r) = 0$ for $r \gg 0$. Note that any elements in the kernel of $\alpha^* \colon Y_n(A) \to Y_n(B)$ lies in $\mathrm{tors}(Y_n)$. This shows that the restriction of Y_n to $\mathcal{F}_n^p$ is eventually torsion-free.

For the surjectivity condition, choose an epimorphism $P \twoheadrightarrow Y_n$ in $\mathcal{AP}_n$ from a finitely projective object. By the commutativity of the diagram

$$\begin{tikzcd} P(C_q^r) \otimes k[\mathcal{P}(C_q^{r+1}, C_q^r)] \arrow[r, "\theta_P"] \arrow[d] & P(C_q^{r+1}) \arrow[d] \\ Y_n(C_q^r) \otimes k[\mathcal{P}(C_q^{r+1}, C_q^r)] \arrow[r, "\theta_{Y_n}"] & Y_n(C_q^{r+1}) \end{tikzcd}$$

it is enough to show that θ_P is surjective. Equivalently, we need to show that the map

$$\mathcal{P}(C_q^r, A) \times \mathcal{P}(C_q^{r+1}, C_q^r) \to \mathcal{P}(C_q^{r+1}, A), \quad (\alpha, \beta) \to \beta \circ \alpha$$

is surjective, for $A \in \mathcal{P}_n$ and $r \gg 0$. This now follows from Lemma 1.7.4. $\square$

13. Injectives

We now turn to study the injective objects of $\mathcal{AU}$. Unlike in the projective case, a complete classification of the indecomposable injective objects seems at the moment far out of reach. The main difficulty arises from the fact that any projective object is necessarily torsion-free whereas an injective object can be torsion, absolutely torsion or torsion-free.

The following structural result, classically due to Matlis [**63**], suggests that we can restrict our attention to indecomposable injectives.

THEOREM 1.13.1 ([**29**, Chaper IV]). *Any injective object in a locally noetherian abelian category is a sum of indecomposable injectives.*

Let us produce some examples of injective objects.

LEMMA 1.13.2. *Let $\mathcal{U}$ be multiplicative and closed under passage to subgroups, and consider $X \in \mathcal{AU}$. Then DX is injective. However, if X is torsion then $DX = 0$.*

PROOF. Note that $\mathbb{1}$ is injective since $\mathcal{AU}(-, \mathbb{1}) \simeq \mathrm{Vect}_k(\mathrm{colim}(-), k)$ and colim is exact by Theorem 1.7.1. It follows that $\mathcal{A}(-, DX) \simeq \mathcal{A}(X \otimes -, \mathbb{1})$ is also exact, and so DX is injective. Finally note that if X is torsion, then $D(X)(G) = \mathcal{A}(e_G \otimes X, \mathbb{1}) = 0$ by Lemma 1.10.7. $\square$

PROPOSITION 1.13.3. *Let $\mathcal{U}$ be multiplicative global family. Then any projective object of $\mathcal{AU}$ is injective.*

PROOF. Consider a projective object P. We can write this as $\prod_n F_n P$ by Proposition 1.6.7, so it will suffice to show that $F_n P$ is injective. We have $F_n P = (i_n)_!(P_n)$ for some projective object $P_n \in \mathcal{AU}_n$. We can write P_n as a retract of an object $Q = \bigoplus_t e_{G_t}$ with $G_t \in \mathcal{U}_n$. This embeds in the product $R = \prod_t e_{G_t}$, and all monomorphisms in $\mathcal{AU}_n$ are split, so P_n is a retract of R. We know that $(i_n)_!$ preserves products by Lemma 1.5.3, so $(i_n)_!(P_n)$ is a retract of $\prod_t (i_n)_!(e_{G_t}) = \prod_t e_{G_t}$. Therefore, it is enough to show that e_{G_t} is injective. This now follows from the fact that De_{G_t} is injective and that e_{G_t} is a summand of De_{G_t} by Proposition 1.2.19. $\square$

REMARK 1.13.4. Let $\mathcal{U}$ be the family of cyclic 2-groups. Then we have a short exact sequence

$$0 \to e_{C_2} \to \mathbb{1} \to t_{1,k} \to 0$$

that cannot split as $\mathbb{1}$ is torsion-free and $t_{1,k}$ is torsion. Hence e_{C_2} is not injective in $\mathcal{AU}$.

LEMMA 1.13.5. *Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$ and let $I \in \mathcal{AU}$ be injective. Then I is a retract of a product of objects $t_{G,V}$ with $G \in \mathcal{U}$. If in addition I is absolutely torsion, then it is a retract of a sum of objects $t_{G,V}$ with $G \in \mathcal{U}$.*

PROOF. Fix a skeleton $\mathcal{U}'$ of $\mathcal{U}$ and consider the morphism $I \to t_{G,I(G)}$ adjoint to the identity. We can combine all these maps into a well-defined morphism

$$\mathrm{env}\colon I \to \prod_{G \in \mathcal{U}'} t_{G,I(G)} =: E(I)$$

that we claim to be a monomorphism. By definition, the composite

$$I(K) \xrightarrow{\mathrm{env}(K)} E(I)(K) \xrightarrow{\mathrm{proj}} t_{K,I(K)}(K) = I(K)$$

is the identity, so $\mathrm{env}(K)$ is injective for all $K \in \mathcal{U}$. By injectivity of I, the map env splits and so I is a retract of $E(I)$. If in addition I is absolutely torsion, then the image of any element of I under env is nonzero only for finitely many $G \in \mathcal{U}'$, so the morphism env factors through the direct sum. □

LEMMA 1.13.6. *Let $\mathcal{U}$ be multiplicative global family of $\mathcal{V}$.*

(a) *For any $G \in \mathcal{V}$ and V irreducible* $\mathrm{Out}(G)$*-representation, the object $t_{G,V}$ is indecomposable and injective in $\mathcal{AV}$. Furthermore, $t_{G,V}$ is the injective envelope of $s_{G,V}$.*
(b) *For any $G \in \mathcal{U}$ and V irreducible* $\mathrm{Out}(G)$*-representation, the object $\chi_{\mathcal{U}} \otimes e_{G,V}$ is indecomposable and injective in $\mathcal{AV}$.*

PROOF. We have seen that $t_{G,V}$ is injective and it is indecomposable by Lemma 1.3.3(e). If $\mathcal{U}$ is a multiplicative global family, then $e_{G,V}$ is injective and so combining part (e) and (i) of Lemma 1.3.3 we see that $i_*(e_{G,V}) = \chi_{\mathcal{U}} \otimes e_{G,V}$ is an indecomposable injective. Finally note that there is a canonical monomorphism $s_{G,V} \to t_{G,V}$, so the injective hull of $s_{G,V}$ is a direct summand of $t_{G,V}$ so the claim follows by indecomposability □

CHAPTER 2

The derived category $\mathcal{D}(\mathcal{A})$

In this chapter we study the tensor triangulated geometry of the derived category of $\mathcal{A}$. We start by giving an explicit model for the derived category: the homotopy category of complexes of projective objects. We show that the compact objects coincide with the perfect complexes and that not all the compact objects are strongly dualizable. In particular, we show that the derived category is almost never rigid and we measure this failure by classifying all the strongly dualizable objects. Finally, we show that the derived category fits into a recollement (or six-functors calculus) which plays a similar role as the change of subcategory functors in the abelian level. Using this formalism we prove that under some completeness assumption on the chosen family, the homology of a perfect complex cannot be torsion. We then turn to study the Balmer spectrum of the category of perfect complexes over $\mathcal{A}$. We show that finitely generated thick ideals are completely determined by their supports and can be written as an intersection of small primes, that we call *group primes*. We also construct prime ideals that are not finitely generated and hence not determined by their support. Finally, we completely describe the Balmer spectrum in the special cases where $\mathcal{U}$ is truncated $\mathcal{U}_{\leq n}$, the family of elementary abelian p-groups and cyclic p-groups, and hence obtain a complete classification of thick ideals for these cases.

1. Preliminaries

We denote by $\mathrm{Ch}(k)$ the category of unbounded chain complexes of k-vector spaces, and by $\mathrm{Ch}(\mathcal{A})$ the category of chain complexes over $\mathcal{A}$. The homotopy category $\mathrm{K}(\mathcal{A})$ is the category whose objects are chain complexes over $\mathcal{A}$, and whose morphisms are homotopy classes of chain maps.

CONSTRUCTION 2.1.1. Given $X, Y \in \mathrm{Ch}(\mathcal{A})$, we have a tensor product chain complex $X \otimes Y \in \mathrm{Ch}(\mathcal{A})$ whose n-th term is given by

$$(X \otimes Y)_n = \bigoplus_{p+q=n} X_p \otimes Y_q \in \mathcal{A},$$

and differential defined by $d(x \otimes y) = d(x) \otimes y + (-1)^p x \otimes d(y)$ for a homogeneous element $x \otimes y \in X_p \otimes Y_q$. For $i \in \mathbb{Z}$, we have a k-vector space of maps of degree i

$$\mathrm{Hom}^i(X, Y) = \prod_{q \in \mathbb{Z}} \mathcal{A}(X_q, Y_{q+i}).$$

Let $\mathrm{Hom}(X, Y)$ denote the chain complex of k-vector spaces whose i-th term is $\mathrm{Hom}^i(X, Y)$, and differential given by $df = d_Y \circ f - (-1)^i f \circ d_X$ for $f \in \mathrm{Hom}^i(X, Y)$. One can check that the zero cycles in $\mathrm{Hom}(X, Y)$ are the chain maps, and the zero boundaries are the null homotopies. It follows that

$$H_0(\mathrm{Hom}(X, Y)) = \mathrm{Hom}_{\mathrm{K}(A)}(X, Y). \tag{1.0.1}$$

There is also an internal hom functor $\underline{\mathrm{Hom}}(X, Y) \in \mathrm{Ch}(\mathcal{A})$ defined by

$$\underline{\mathrm{Hom}}(X, Y)(G) = \mathrm{Hom}(e_G \otimes X, Y).$$

It is standard to check that the tensor product and internal hom as defined above give a closed symmetric monoidal structure on the abelian category $\mathrm{Ch}(\mathcal{A})$. For X, Y and $Z \in \mathrm{Ch}(\mathcal{A})$, we have a natural isomorphism of chain complexes of k-vector spaces

$$\mathrm{Hom}(X \otimes Y, Z) \simeq \mathrm{Hom}(X, \underline{\mathrm{Hom}}(Y, Z))$$

which shows that the closed symmetric monoidal structure is compatible with the enrichment over $\mathrm{Ch}(k)$. As all the functors considered are additive, we have an induced closed symmetric monoidal structure on $\mathrm{K}(\mathcal{A})$ that we will not distinguish from that of $\mathrm{Ch}(\mathcal{A})$.

Definition 2.1.2.

- A chain map $f\colon X \to Y$ is said to be a *quasi-isomorphism* if the induced map in homology $H_*(f)\colon H_*(X) \to H_*(Y)$ is an isomorphism.
- The derived category $\mathcal{D}(\mathcal{A})$ is obtained from the homotopy category $\mathrm{K}(\mathcal{A})$ by formally inverting quasi-isomorphisms. The objects of the derived category are chain complexes over $\mathcal{A}$, and a morphism from X to Y is an equivalence class of spans $X \xleftarrow{f} Z \xrightarrow{g} Y$ where f is a quasi-isomorphism. Compositions of spans is given by pullback. There is a canonical localization functor $Q\colon \mathrm{K}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ which is the identity on objects, and on a morphism $f\colon X \to Y$ is given by $Q(f) = (X \xleftarrow{1} X \xrightarrow{f} Y)$. The universal property of the derived category says that the functor Q is initial amongst those functors sending quasi-isomorphisms to isomorphisms.

Remark 2.1.3. The categories $\mathrm{K}(\mathcal{A})$ and $\mathcal{D}(\mathcal{A})$ admit the structure of triangulated categories. In particular, they are additive categories, there is a shift functor given by $(\Sigma X)_i = X_{i-1}$, and they have a distinguished collection of triangles. The distinguished triangles in $\mathrm{K}(\mathcal{A})$ are isomorphic to term-wise split exact sequences of complexes. The triangles in $\mathcal{D}(\mathcal{A})$ are isomorphic to sequences $A \xrightarrow{f} B \to \mathrm{cone}(f) \to \Sigma A$. These data have to satisfy a list of axioms that we will not include here, instead we refer the reader to [**92**].

Remark 2.1.4. A triangulated functor between triangulated categories is an additive functor sending distinguished triangles to distinguished triangles. We will mostly consider triangulated functors which are constructed using the universal property of the derived category. For example, we will use that an additive functor $F\colon \mathcal{A} \to \mathcal{A}$ induces a triangulated functor $\mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{A})$ since it preserves split exact sequences. If in addition F is exact, it further descends to a functor $\mathcal{D}(\mathcal{A}) \to \mathcal{D}(\mathcal{A})$ by the universal property of the derived category.

2. An explicit model

In this section we show that the homotopy category of complexes of projective objects of $\mathcal{A}$ is a model for the derived category of $\mathcal{A}$.

Lemma 2.2.1. *Consider two complexes $X, N \in \mathrm{Ch}(\mathcal{A})$ with N acyclic. Then the following are equivalent:*

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X, N) = 0 \;\Leftrightarrow\; \mathrm{Hom}(X, N) \text{ is acyclic} \;\Leftrightarrow\; \underline{\mathrm{Hom}}(X, N) \text{ is acyclic.}$$

Proof. Using Equation 1.0.1 we see that

$$H_i(\mathrm{Hom}(X, N)) = H_0(\mathrm{Hom}(\Sigma^i X, N)) = \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\Sigma^i X, N)$$

so the first two conditions are equivalent. If $\underline{\mathrm{Hom}}(X, N)$ is acyclic then so is $\underline{\mathrm{Hom}}(X, N)(1) = \mathrm{Hom}(X, N)$. Conversely, suppose that $\mathrm{Hom}(X, N)$ is acyclic. We have to show that $\mathrm{Hom}(e_G \otimes X, N)$ is acyclic for all $G \in \mathcal{G}$. Note that $\underline{\mathrm{Hom}}(e_G, -)$ is exact as e_G is projective. In particular, $\underline{\mathrm{Hom}}(e_G, N)$ is acyclic and so $\mathrm{Hom}(e_G \otimes X, N) \simeq \mathrm{Hom}(X, \underline{\mathrm{Hom}}(e_G, N))$ is acyclic as required. ☐

Definition 2.2.2. Let $\mathcal{K}$ denote the full subcategory of complexes $X \in \mathrm{K}(\mathcal{A})$ that satisfy the equivalent conditions of Lemma 2.2.1, for all acyclic complexes N.

Lemma 2.2.3. *The full subcategory $\mathcal{K}$ is closed under suspensions, triangles and arbitrary sums, and it contains all projective objects of $\mathcal{A}$. In particular, $\mathcal{K}$ is a triangulated subcategory of* $\mathrm{K}(\mathcal{A})$.

Proof. It is clear that all projective objects belong to $\mathcal{K}$ and that $\mathcal{K}$ is closed under suspensions. Suppose we are given a triangle $X \to Y \to Z \to \Sigma X$ in $\mathrm{K}(\mathcal{A})$ with two vertices in $\mathcal{K}$. By rotating the triangle if necessary, we can assume that X and Z belong to $\mathcal{K}$. For any acyclic complex N, we have an exact sequence

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(Z,N) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(Y,N) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X,N)$$

By assumption, the outer terms of the exact sequence are zero. It follows that the middle term is zero and so $Z \in \mathcal{K}$. Finally, let $P = \bigoplus_i P_i$ be a sum of objects in $\mathcal{K}$. Then for any acyclic complex N, we have

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(P,N) = \prod_i \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(P_i,N) = 0$$

since $P_i \in \mathcal{K}$. Therefore $P \in \mathcal{K}$ as required. $\square$

Construction 2.2.4. Let X be a complex of projective objects and consider the filtration from Construction 1.6.3, so that $F_{\leq i}X \leq X$ is the subcomplex consisting of indecomposable projective $e_{G,S}$ with $|G| \leq i$. By construction, we have an exhaustive filtration $0 = F_0 X \subset F_{\leq 1}X \subset F_{\leq 2}X \subset \ldots \subset X$ whose subquotients $F_{i+1}X$ consist of indecomposable projective $e_{G,S}$ with $|G| = i+1$. Consider the vector space of all possible maps

$$\mathcal{A}(e_{G,S}, e_{G',S'}) = \mathrm{Hom}_{k[\mathrm{Out}(G)]}(S, e_{G',S'}(G))$$

between two indecomposable projective objects $e_{G,S}$ and $e_{G',S'}$ with $|G| = |G'| = i+1$. If $G \not\simeq G'$, then there are no nonzero maps, and if $G \simeq G'$, Schur's Lemma tells us that all nonzero maps are necessarily isomorphisms. From this it follows that the differentials of $F_{i+1}X$ are either zero or isomorphisms. Thus we can decompose $F_{i+1}X$ as a sum $P_{i+1}(X) \oplus C_{i+1}(X)$ where $P_{i+1}(X)$ is a complex with zero differentials and $C_{i+1}(X)$ is contractible.

Proposition 2.2.5. *Any complex of projective objects of $\mathcal{A}$ belongs to $\mathcal{K}$.*

Proof. Let X be a complex of projective objects and consider the filtration $F_{\leq i}X$ from Construction 2.2.4. We claim that for all $i \geq 0$, the complex $F_{\leq i}X$ is in $\mathcal{K}$. We prove the claim by induction on i. If $i = 0$, we have $F_0X = 0$ which is in $\mathcal{K}$. We assume by induction that $F_{\leq j-1}X \in \mathcal{K}$, and we consider the short exact sequence of complexes $0 \to F_{\leq j-1}X \to F_{\leq j}X \to F_jX \to 0$ which gives a triangle in $\mathrm{K}(\mathcal{A})$. For any acyclic complex N, we have an exact sequence

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(F_jX,N) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(F_{\leq j}X,N) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(F_{\leq j-1}X,N). \tag{2.0.1}$$

Note that $F_jX \in \mathcal{K}$ since it decomposes as $C_j(X) \oplus P_j(X)$ where $C_j(X)$ is contractible and $P_j(X)$ is a sum of shifts of projective objects of $\mathcal{A}$. In particular, the left term of the sequence (2.0.1) is zero, and so is the right term by the induction hypothesis. It follows that $F_{\leq j}X$ belongs to $\mathcal{K}$ as claimed.

We now prove that $X \in \mathcal{K}$. For any acyclic complex N, we have the Milnor short exact sequence

$$0 \to \varprojlim{}^1 \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(\Sigma F_{\leq i}X,N) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X,N) \to \varprojlim \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(F_{\leq i}X,N) \to 0.$$

By the previous paragraph, $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(F_{\leq i}X,N) = 0$ so the right term is zero. Note that the maps in the direct inverse system are surjective by (2.0.1), so the Mittag-Leffler argument shows that the left term is zero too. It follows that $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(X,N)$ is zero and that $X \in \mathcal{K}$. $\square$

We can also consider projective resolutions.

Construction 2.2.6. For all $X \in \mathcal{A}$, we put

$$TX = \bigoplus_{k \geq 1} (i_k)_! i_k^*(X)$$

where $i_k\colon \mathcal{G}_k \to \mathcal{G}$ denotes the inclusion. Note that the counit map $TX \to X$ is an epimorphism and that TX is projective by Proposition 1.5.3. We therefore have a functorial projective resolution of X with terms $P_0X = TX$ and $P_1X = T(\ker(TX \to X))$ and $P_iX = T(\ker(P_{i-1}X \to P_{i-2}X))$ for $i \geq 2$. Because T is an additive functor we can apply this construction to a chain complex X and get an upper half plane double complex $C(X)$ as in the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\dots & P_1X_{-1} \arrow[l] \arrow[d] & P_1X_0 \arrow[l] \arrow[d] & P_1X_1 \arrow[l] \arrow[d] & \dots \arrow[l] \\
\dots & P_0X_{-1} \arrow[l] & P_0X_0 \arrow[l] & P_0X_1 \arrow[l] & \dots \arrow[l]
\end{tikzcd}$$

Write PX for the product total complex of $C(X)$; this is a complex of projectives by Proposition 1.6.7. There is an augmentation map $\epsilon_X\colon PX \to X$ which we claim is a quasi-isomorphism. To see this, we filter the total complex by the columns of $C(X)$ and consider the associated spectral sequence

$$E^2_{p,q} = H^h_p H^v_q(C(X)) \Rightarrow H_{p+q}(PX).$$

The E^2-page of the spectral sequence is $H_*(X)$ in the line $q = 0$, and zero everywhere else. If the spectral sequence converges, this shows that the map $\epsilon_X\colon PX \to X$ is a quasi-isomorphism. By [**93**, Section 5.6], the spectral sequence is weakly convergent to $H_*(PX)$ and there is an Eilenberg-Moore filtration sequence

$$0 \to \varprojlim{}^1 H_{s+1}(\mathrm{Tot}(C_{>n}X)) \to H_s(PX) \to \varprojlim H_s(\mathrm{Tot}(C_{>n}X)) \to 0.$$

Here $C_{>n}X$ is the double complex obtained from $C(X)$ by killing all the columns $r \leq n$. Using the double complex spectral sequence for $C_{>n}X$, which is convergent since the double complex is bounded, we see that $H_s(C_{>n}X) = H_s(X)$ if $n + 1 < s$. It follows that the maps $H_s(C_{>n-1}X) \to H_s(C_{>n}X)$ are eventually isomorphism so the leftmost term in the short exact sequence is zero by the Mittag-Leffler argument. It follows that the spectral sequence converges to $H_*(PX)$ and so $\epsilon_X\colon PX \to X$ is a functorial surjective quasi-isomorphism.

Lemma 2.2.7. *Let P be a complex of projective objects. For every diagram in* $\mathrm{K}(\mathcal{A})$

$$\begin{tikzcd}
 & M \arrow[d, "s"] \\
P \arrow[r, "f"] & N
\end{tikzcd}$$

with s a quasi-isomorphism, there exists unique morphism $g\colon P \to M$ such that $s \circ g = f$ in $\mathrm{K}(\mathcal{A})$*.*

Proof. Let C be the cone of s which is acyclic. Then $\mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(P, C) = 0$ since $P \in \mathcal{K}$. It follows that the map $s_*\colon \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(P, M) \to \mathrm{Hom}_{\mathrm{K}(\mathcal{A})}(P, N)$ is an isomorphism. In particular, there exists an unique $g\colon P \to M$ such that $s \circ g = f$ in $\mathrm{K}(\mathcal{A})$. □

Corollary 2.2.8. *Any quasi-isomorphism between complexes of projective objects is a chain homotopy equivalence.*

Proof. Let $f\colon P \to Q$ be a quasi-isomorphism between complexes of projective objects. By Lemma 2.2.7, there exists $g\colon Q \to P$ such that $f \circ g = \mathrm{id}_Q$. Note that g is a quasi-isomorphism by the 2-out-of-3 property. Again by Lemma 2.2.7 there exists $f'\colon P \to Q$ such that $f' \circ g = \mathrm{id}_P$. Then $f = (f' \circ g) \circ f = f' \circ (g \circ f) = f'$ so g is an homotopy inverse of f and f is a chain homotopy equivalence. □

Definition 2.2.9. Let $\mathrm{K}(\mathcal{A}_{\mathrm{prj}}) \subset \mathrm{K}(\mathcal{A})$ be the full subcategory of complexes of projective objects in $\mathcal{A}$. More generally, we will write $\mathrm{K}(\mathcal{A}\mathcal{U}_{\mathrm{prj}})$ for a replete full subcategory $\mathcal{U}$ of $\mathcal{G}$.

It is now formal that the derived category $\mathcal{D}(\mathcal{A})$ is equivalent to $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$. We briefly recall the argument.

Corollary 2.2.10. *The projective resolution functor $P\colon \mathrm{K}(\mathcal{A}) \to \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ from Construction 2.2.6 is universal among functors that invert quasi-isomorphisms. In particular, $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ is a model for $\mathcal{D}(\mathcal{A})$.*

Proof. Note that the functor P sends quasi-isomorphisms to isomorphisms by Corollary 2.2.8. This functor comes with a natural transformation $\epsilon\colon P \Rightarrow \mathrm{Id}$ which is objectwise a surjective quasi-isomorphism. If F is another functor sending quasi-isomorphisms to isomorphisms, then $F\epsilon\colon F \circ P \Rightarrow F$ is a natural isomorphism that shows that F factors through P. It is easy to check that this factorization is unique. $\square$

Remark 2.2.11. Note that we have not used any property of the category $\mathcal{G}$ so all the results of this section extend verbatim to a full and replete subcategory $\mathcal{U}$ of $\mathcal{G}$.

Remark 2.2.12. Any exact functor $F\colon \mathcal{A} \to \mathcal{A}$ descends to a well-defined functor $PF\colon \mathrm{K}(\mathcal{A}_{\mathrm{prj}}) \to \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ by the universal property of the derived category. If in addition F preserves projectives, then we can simply write $F\colon \mathrm{K}(\mathcal{A}_{\mathrm{prj}}) \to \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$.

3. Tensor triangulated structure

In this section we study the symmetric monoidal structure on the derived category of $\mathcal{A}$.

We first need some preliminary results. The category of chain complexes of vector spaces is a closed symmetric monoidal abelian category with tensor product and internal hom functors defined in a similar way as in Construction 2.1.1.

Lemma 2.3.1. *The following hold in the category $\mathrm{Ch}(k)$:*

(a) *Quasi-isomorphisms are chain homotopy equivalences.*
(b) *The functors $\otimes, \underline{\mathrm{Hom}}\colon \mathrm{Ch}(k) \times \mathrm{Ch}(k) \to \mathrm{Ch}(k)$ preserve quasi-isomorphisms.*
(c) *The canonical map $H_*(X) \otimes H_*(Y) \to H_*(X \otimes Y)$ is an isomorphism, for all $X, Y \in \mathrm{Ch}(k)$.*

Proof. Consider a complex $X \in \mathrm{Ch}(k)$ and write $Z(X) \leq X$ for the subcomplex of cycles of X. Choose a complement T for $Z(X)$ in X, and a complement H for dT in $Z(X)$. This gives a splitting $X = H \oplus (dT \oplus T)$ where H has zero differential and $(dT \oplus T)$ is contractible. It follows that the inclusion map $H \to X$ is a chain homotopy equivalence, and that (a) holds. The functors $\otimes$ and $\underline{\mathrm{Hom}}$ preserve homotopies and so chain homotopy equivalences since they are additive. Part (b) then follows from (a). Using the splitting constructed above, we see that the canonical map in (c) is always an isomorphism. $\square$

Lemma 2.3.2. *For all $X, Y \in \mathrm{Ch}(\mathcal{A})$, the canonical map $H_*(X) \otimes H_*(Y) \to H_*(X \otimes Y)$ is an isomorphism. In particular, the tensor product preserves quasi-isomorphisms.*

Proof. Using that evaluation at $G \in \mathcal{G}$ is an exact functor and Lemma 2.3.1(c), we see that

$$H_*(X \otimes Y)(G) = H_*(X(G) \otimes Y(G)) \simeq H_*(X(G)) \otimes H_*(Y(G)) = H_*(X)(G) \otimes H_*(Y)(G).$$

The equivalences are natural in G, so we have $H_*(X) \otimes H_*(Y) \simeq H_*(X \otimes Y)$ as objects of $\mathcal{A}$. $\square$

Recall the tensor product and internal hom functors as defined in Construction 2.1.1.

Proposition 2.3.3. *We have well-defined triangulated functors*

$$- \otimes - \colon \mathrm{K}(\mathcal{A}_{\mathrm{prj}}) \times \mathrm{K}(\mathcal{A}_{\mathrm{prj}}) \to \mathrm{K}(\mathcal{A}_{\mathrm{prj}}) \quad \textit{and} \quad \underline{\mathrm{Hom}}(-,-)\colon \mathrm{K}(\mathcal{A}_{\mathrm{prj}}) \times \mathrm{K}(\mathcal{A}_{\mathrm{prj}}) \to \mathrm{K}(\mathcal{A}_{\mathrm{prj}}).$$

These functors give $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ *a closed symmetric monoidal structure.*

Proof. The tensor product and internal hom functors preserve projective objects by Proposition 1.6.8. As the functors are additive, the tensor-hom adjunction at the abelian level descends to a well-defined adjuction at the level of the homotopy category. □

Remark 2.3.4. The tensor product preserves projective objects in $\mathcal{AU}$ for any replete full subcategory $\mathcal{U}$ by Proposition 1.6.8. As it is additive, it descends to $\mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$. If $\mathcal{U}$ is closed downwards, then the internal hom functor preserves projectives by Corollary 1.2.24 and so it descends to $\mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$. In complete generality however, the internal hom might not preserve projectives. Instead one needs to consider $\mathrm{P}\underline{\mathrm{Hom}}(X,Y)$ a projective functorial resolution of $\underline{\mathrm{Hom}}(X,Y)$. It is not difficult to see that this projective version is the internal hom in $\mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$. As we will mostly be interested in the homology of the internal hom functor, there is no harm in considering $\underline{\mathrm{Hom}}(X,Y)$ instead of its projective version.

4. Compact objects and perfect complexes

In this section we show that $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ is a compactly generated tensor triangulated category, and that the subcategory of compact objects coincides with that of perfect complexes.

We start off by recalling some useful definitions.

Definition 2.4.1. Consider complexes $X \in \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$.

- We say that X is *perfect* if it is homotopy equivalent to a bounded complex of finitely projective objects. We write $\mathrm{K}(\mathcal{A})_{\mathrm{perf}}$ for the full subcategory of perfect complexes.
- We say that X is *compact* if the functor $\mathrm{Hom}_{\mathrm{K}(\mathcal{A}_{\mathrm{prj}})}(X,-)$ preserves arbitrary direct sums. We denote by $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})^{\omega}$ the full subcategory of compact objects of $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$.
- We say that $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ is *compactly generated* if there exists a set of compact generators, i.e., there exists a set $\mathcal{S}$ of compact objects of $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ such that an object $X \in \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ is zero if and only if $\mathrm{Hom}_{\mathrm{K}(\mathcal{A}_{\mathrm{prj}})}(S, \Sigma^i X) = 0$ for all $S \in \mathcal{S}$ and $i \in \mathbb{Z}$.
- A full triangulated subcategory $\mathcal{T} \subset \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ is *thick* if it contains 0 and it is closed under retracts. We write $\mathrm{thick}(\mathcal{S})$ for the thick subcategory generated by a set of objects $\mathcal{S}$ of $\mathcal{T}$.
- A full triangulated subcategory $\mathcal{L} \subset \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ is *localizing* if it is thick and closed under arbitrary small sums.

Remark 2.4.2.

- A standard argument shows that $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})^{\omega}$ is thick.
- Let $\mathcal{S}$ be a set of compact generators of $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ in the sense of the previous definition. Then the smallest localizing subcategory containing $\mathcal{S}$ is $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ itself, see [**84**, 2.2.1]. Equivalently, any object $X \in \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ can be built from objects in $\mathcal{S}$ using triangles, retracts and sums.

Remark 2.4.3. Consider the functor $P\colon \mathcal{A} \to K(\mathcal{A}_{\mathrm{prj}})$ sending an object X to its projective resolution. Then $X \in \mathcal{A}$ is perfect in the sense of Definition 1.9.1 if and only if PX is perfect in the sense of above.

Lemma 2.4.4. *Let* $\mathcal{G}'$ *be a skeleton for* $\mathcal{G}$*. Then* $\{e_G \mid G \in \mathcal{G}'\}$ *is a set of compact generators for* $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$*. Therefore, the homotopy category* $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ *is a compactly generated tensor triangulated category.*

Proof. By Yoneda Lemma we have

$$\mathrm{Hom}_{\mathrm{K}(\mathcal{A}_{\mathrm{prj}})}(e_G, \Sigma^n X) = H_{-n}(X(G))$$

for all $X \in \mathrm{K}(\mathcal{A}_{\mathrm{prj}})$. Using this fact and that colimits in $\mathcal{A}$ are computed levelwise, we easily see that e_G is compact. The same formula above tells us that $\{e_G \,|G \in \mathcal{G}'\}$ is a set of generators. $\square$

We recall the following fundamental result.

Proposition 2.4.5. *The following equalities hold:*

$$\mathrm{thick}(e_G \mid G \in \mathcal{G}) = \mathrm{K}(\mathcal{A}_{\mathrm{prj}})^{\omega} = \mathrm{K}(\mathcal{A})_{\mathrm{perf}}.$$

Proof. The first claim is [**52**, 2.3.12]. Note that the cited result assumes the triangulated category to be rigid, but this is nowhere used in the proof. For the second claim, we use that by [**87**] any compact object is a retract in the derived category of a perfect complex. Then a similar argument as in [**19**, 3.4] shows that the full subcategory of perfect complexes is idempotent complete. Thus a retract in the derived category of a perfect complex is again perfect. It remains only to prove that any perfect complex is compact which is standard, see for example [**88**]. $\square$

Remark 2.4.6. Suppose that we want to show that for all perfect complexes of $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ a certain property P is satisfied. Then, by the previous Proposition it is enough to check that the full subcategory of objects of $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ satisfying P is thick and contains e_G for all $G \in \mathcal{G}$.

Lemma 2.4.7. *Suppose that $\mathcal{AU}$ is locally noetherian and let $X \in \mathrm{K}(\mathcal{AU})$ be a perfect complex. Then $H_*(X)$ is a finitely generated object in $\mathcal{AU}$.*

Proof. Without loss of generality we can assume that X is a bounded complex of finitely projective objects. Since X has only finitely many nonzero entries, we see that there are only finitely many nonzero homology groups. Note that there is a canonical quotient map $Z_n(X) \twoheadrightarrow H_n(X)$ and that $Z_n(X) \leq X_n$ is finitely generated by the noetherian condition. $\square$

5. Rigidity

In this section we classify strongly dualizable objects and study when the derived category is rigid.

Recall that for all $X, Y \in \mathrm{K}(\mathcal{AU})$, there is an internal hom functor given by

$$\underline{\mathrm{Hom}}(X, Y)(G) = \underline{\mathrm{Hom}}(e_G \otimes X, Y).$$

We put $DX = \underline{\mathrm{Hom}}(X, \mathbb{1})$, and call this the *dual* of X.

Definition 2.5.1. Let $\mathcal{U}$ be a full replete subcategory of $\mathcal{G}$ and let $X \in \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$.

- We say that X is *strongly dualizable* if the natural map $DX \otimes Y \to \underline{\mathrm{Hom}}(X, Y)$ is a quasi-isomorphism for all Y.
- We say that $\mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ is *rigid* if e_G is strongly dualizable for all $G \in \mathcal{U}$.

Remark 2.5.2. In the definition of strongly dualizable object we required the map to be a quasi-isomorphism instead of a homotopy equivalence. This is because in complete generality DX and $\underline{\mathrm{Hom}}(X, Y)$ might not be projective. See also the discussion in Remark 2.3.4.

The following result shows that the rigidity condition can be checked at the abelian level.

Lemma 2.5.3. *The derived category $\mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ is rigid if and only if the canonical map*

$$De_G \otimes e_H \to \underline{\mathrm{Hom}}(e_G, e_H)$$

in an isomorphism for all $G, H \in \mathcal{U}$.

Proof. It is enough to prove the backward implication. Consider the full subcategory

$$\mathcal{L} = \{X \in \mathrm{K}(\mathcal{AU}_{\mathrm{prj}}) \mid De_G \otimes X \xrightarrow{\sim} \underline{\mathrm{Hom}}(e_G, X) \ \ \forall\, G \in \mathcal{U}\}.$$

By assumption $e_H \in \mathcal{L}$ for all $H \in \mathcal{U}$, and it is easily seen that $\mathcal{L}$ is localizing. As the objects e_H generate, we deduce that $\mathcal{L} = \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ as required. □

The main result of this section is the following.

Proposition 2.5.4. *Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$. Then the homotopy category $\mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ is rigid if and only if $\mathcal{U}$ is a groupoid.*

Proof. Suppose that $\mathcal{U}$ is a groupoid and let $\mathcal{U}'$ be a skeleton for $\mathcal{U}$. Write $\mathcal{M}_G$ for the category of $k[\mathrm{Out}(G)]$-modules so that $\mathcal{AU} \simeq \prod_{G\in\mathcal{U}'} \mathcal{M}_G$. Accordingly, it is enough to check that $\mathcal{M}_G$ satisfies the condition of Lemma 2.5.3. In other words, we need to check that the canonical map

$$\mathrm{Hom}_k(k[\mathrm{Out}(G)], k) \otimes_k k[\mathrm{Out}(G)] \to \mathrm{Hom}_k(k[\mathrm{Out}(G)], k[\mathrm{Out}(G)])$$

is an isomorphism. This is clear since $k[\mathrm{Out}(G)]$ is finite dimensional over k.

Conversely, suppose that $\mathcal{U}$ is not a groupoid so there exists $\varphi : G \to H$ in $\mathcal{U}$ which is not an isomorphism. We will show that the canonical map $De_G \otimes e_G \to \underline{\mathrm{Hom}}(e_G, e_G)$ is not an isomorphism. If we evaluate at H, we have $De_G(H) \otimes e_G(H) = 0$ since H is not isomorphic to G. On the other hand, we claim that $\underline{\mathrm{Hom}}(e_G, e_G)(H) = \mathcal{AU}(e_G \otimes e_H, e_G) \neq 0$. Let G' be the graph of φ and note that $G' \simeq G \in \mathcal{U}$. Write W for the Weyl group of G' in $G \times H$. By Corollary 1.2.12, the tensor product $e_G \otimes e_H$ contains $e_{G'}^W$ as a direct summand. It follows that $\mathcal{AU}(e_G \otimes e_H, e_G)$ contains $\mathcal{AU}(e_{G'}^W, e_G) = e_G(G')/W \neq 0$ as a summand which proves the claim. □

We finish this section by classifying all strongly dualizable objects.

Lemma 2.5.5. *Let $X \in \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ be strongly dualizable, then X is compact. Furthermore, if X is nonzero then $X(1)$ is nonzero.*

Proof. By definition we have a natural quasi-isomorphism $\underline{\mathrm{Hom}}(X, Y) \simeq DX \otimes Y$ so the functor $\underline{\mathrm{Hom}}(X, -)$ preserves arbitrary sums. Since

$$H_0(\underline{\mathrm{Hom}}(X, \Sigma^n Y)(1)) = \mathrm{Hom}_{\mathrm{K}(\mathcal{A}_{\mathrm{prj}})}(X, \Sigma^n Y)$$

we deduce that X is compact. For the second claim note that if X is nonzero then $\underline{\mathrm{Hom}}(X, X)(1)$ is nonzero since the identity map $X \to X$ is nonzero. Thus $X(1) \otimes DX(1)$ and $X(1)$ are nonzero. □

Proposition 2.5.6. *A complex $X \in \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ is strongly dualizable if and only if $X \in \mathrm{thick}(\mathbb{1})$.*

Proof. First of all note that the full subcategory of strongly dualizable objects is thick and contains $\mathbb{1}$. It follows that if $X \in \mathrm{thick}(\mathbb{1})$, then X is strongly dualizable. Conversely, suppose that X is nonzero and strongly dualizable, so that X is compact and $X(1) \neq 0$. We can construct a nonzero morphism $u\colon \bigoplus_{i=1}^m \Sigma^{n_i} \mathbb{1} \to X$ that is an isomorphism after evaluating at 1. Set $Y = \mathrm{cof}(u)$ which is again strongly dualizable since the subcategory of strongly dualizable objects is closed under cofibre. By construction, $Y(1) \simeq 0$ so $Y \simeq 0$ and u is a quasi-isomorphism. □

6. The homology of perfect complexes

We show that the derived category $\mathrm{K}(\mathcal{A}_{\mathrm{prj}})$ can be reconstructed from $\mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ and its complement $\mathrm{K}(\mathcal{AU}^c_{\mathrm{prj}})$ via a recollement. We use this formalism to construct a truncation functor $\tau_{\leq n}\colon \mathrm{K}(\mathcal{AU}_{\mathrm{prj}}) \to \mathrm{K}(\mathcal{AU}^{\leq n}_{\mathrm{prj}})$ that we use to show that the homology of a perfect complex cannot be torsion.

Proposition 2.6.1. *Let $\mathcal{U}$ be a closed downwards full subcategory of $\mathcal{G}$, and let $\mathcal{U}^c$ be the complement that is closed upwards. Write $j\colon \mathcal{U}^c \to \mathcal{G}$ and $i\colon \mathcal{U} \to \mathcal{G}$ for the inclusion functors. Then there exists a recollement*

$$\begin{tikzcd}
\mathrm{K}(\mathcal{A}\mathcal{U}^c_{\mathrm{prj}}) \arrow[r, "j_!"] & \mathrm{K}(\mathcal{A}_{\mathrm{prj}}) \arrow[l, bend right, "j^!"'] \arrow[l, bend left, "Pj^*"] \arrow[r, "i^*"] & \mathrm{K}(\mathcal{A}\mathcal{U}_{\mathrm{prj}}) \arrow[l, bend right, "i_!"'] \arrow[l, bend left, "Pi_*"]
\end{tikzcd}$$

that is

(i) *both $(i_!, i^*, Pi_*)$ and $(Pj^*, j_!, j^!)$ are adjoint triples,*
(ii) *$i^*j_! = 0$,*
(iii) *the functors $i_!$, Pi_* and $j_!$ are fully faithful,*
(iv) *for each complex X there are exact triangles*
  (a) $i_!i^*(X) \to X \to j_!j^!(X) \to \Sigma i_!i^*(X)$
  (b) $j_!Pj^*(X) \to X \to Pi_*i^*(X) \to \Sigma j_!Pj^*(X)$
  where the maps to X are counit maps, and the maps out of X are unit maps.

Furthermore, the functor i^* is strong symmetric monoidal, Pi_* is lax symmetric monoidal and $i_!$ is oplax symmetric monoidal. If in addition $\mathcal{U}$ is a multiplicative global family, then $i_!$ is strong symmetric monoidal.

Proof. Note that the functors i^* and j^* are exact and that $i_!$ and $j_!$ preserve projective objects by Lemma 1.3.3(e). Similarly, parts (f) and (g) of the same Lemma show that the functor i_* is exact and that $j_!$ admits a left adjoint $j^!$ by Brown representability. It follows from the exactness of i_* that i^* preserves projective objects. Using Remark 2.2.12, it is easy to see that (i), (ii) and (iii) hold. We prove part (iv)(b) as (a) follows from (b) by taking opposite categories. Let $\varphi(X)$ be the cofibre of the counit $j_!Pj^*(X) \to X$ so that we have a triangle

$$j_!Pj^*(X) \to X \to \varphi(X) \to \Sigma j_!Pj^*(X).$$

By applying i^* to the triangle above we see that $i^*(X) \simeq i^*(\varphi(X))$. If we apply Pj^* instead we see that $\varphi(X) \in \ker(Pj^*)$. Lemma 1.3.3(e) tell us that $\ker(Pj^*) = \mathrm{essim}(Pi_*)$ so $\varphi(X) = Pi_*(Y)$ for some Y. Assembling we get $i^*(\varphi(X)) \simeq i^*(Pi_*(Y)) \simeq Y$ thus $\varphi(X) \simeq Pi_*i^*(X)$.

We now prove the second part of the claim. By Lemma 1.3.3(c), the functor i^* is strong symmetric monoidal so it extends naturally to a strong symmetric monoidal functor between the derived categories. It is then formal to show that Pi_* is lax and $i_!$ is oplax. If $\mathcal{U}$ is a multiplicative global family, then $i_!$ is symmetric monoidal by Lemma 1.3.3(f). □

Remark 2.6.2. From the axioms of a recollement one can see that the triangles (a) and (b) above are natural and essentially unique, see [**48**, 1.5].

A similar proof gives us the following.

Corollary 2.6.3. *For all full and replete subcategories $\mathcal{U}$ and natural numbers n, there exists a recollement*

$$\begin{tikzcd}
\mathrm{K}(\mathcal{A}\mathcal{U}^{>n}_{\mathrm{prj}}) \arrow[r, "j_!^{>n}"] & \mathrm{K}(\mathcal{A}\mathcal{U}_{\mathrm{prj}}) \arrow[l, bend right, "j^!_{>n}"'] \arrow[l, bend left, "Pj^*_{>n}"] \arrow[r, "i^*_{\leq n}"] & \mathrm{K}(\mathcal{A}\mathcal{U}^{\leq n}_{\mathrm{prj}}) \arrow[l, bend right, "i_!^{\leq n}"'] \arrow[l, bend left, "Pi_*^{\leq n}"]
\end{tikzcd}$$

with an associated natural triangle

$$\tau_{\leq n}(X) \to X \to \tau_{>n}(X) \to \Sigma\tau_{\leq n}(X)$$

where $\tau_{>n}(X) = j_!^{>n}j^!_{>n}(X)$ and $\tau_{\leq n}(X) = i_!^{\leq n}i^*_{\leq n}(X)$.

We now describe the truncation functor $\tau_{\leq n}$ explicitly. Consider a complex $X \in \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ so that each entry of X is a sum of indecomposable projective objects $e_{G,S}$ for some G and S. Using Lemma 1.3.3(i) we calculate

$$\tau_{\leq n}(e_{G,S}) = i_!^{\leq n} i_{\leq n}^*(e_{G,S}) = \begin{cases} 0 & \text{if } |G| > n \\ e_{G,S} & \text{if } |G| \leq n. \end{cases} \tag{6.0.1}$$

In particular this shows that $\tau_{\leq n}(X)$ is the subcomplex of X whose entries are sums of indecomposable projective $e_{G,S}$ with $|G| \leq n$. From this description it is clear that we have inclusion maps $\tau_{\leq n}(X) \to \tau_{\leq n+1}(X)$ and that the colimit is X. In other words, the complexes $\tau_{\leq n}(X)$ define an exhaustive filtration of X; this is the filtration of Construction 2.2.4. In particular, we have a decomposition of the quotients

$$\tau_{\leq n+1}(X)/\tau_{\leq n}(X) = P_{n+1}(X) \oplus C_{n+1}(X) \tag{6.0.2}$$

where $P_{n+1}(X)$ is a complex of projective objects with zero differential and $C_{n+1}(X)$ is contractible. In the case that X is perfect, we can a say a bit more.

Lemma 2.6.4. *Let $X \in \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ be perfect.*

(a) *The truncation $\tau_{\leq n}(X)$ is a perfect complex for all $n \geq 0$.*
(b) *The counit map $\tau_{\leq n}(X) \to X$ is an isomorphism for large n.*

Proof. Parts (a) and (b) follow from the description of the truncation functor as given above. □

Lemma 2.6.5. *For all perfect complexes $X, Y \in \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$, the canonical map*

$$\tau_{\leq n}(X \otimes Y) \to \tau_{\leq n}(X) \otimes \tau_{\leq n}(Y)$$

is an isomorphism for large n.

Proof. Fix $G \in \mathcal{U}$ and let $\mathcal{T}_G$ denote the full subcategory of perfect complexes X for which the canonical map $\tau_{\leq n}(X \otimes e_G) \to \tau_{\leq n}(X) \otimes \tau_{\leq n}(e_G)$ is an isomorphism for n large enough. Combining Equation (6.0.1) with Corollary 1.2.12, and the extra care of taking only those wide subgroups which lie in $\mathcal{U}$, we see that $e_H \in \mathcal{T}_G$ for all $H \in \mathcal{U}$. As $\mathcal{T}_G$ is also thick we deduce $\mathrm{K}(\mathcal{AU}_{\mathrm{perf}}) \subset \mathcal{T}_G$ by Proposition 2.4.5.
Now let $\mathcal{T}'$ be the full subcategory of perfect complexes Y such that for any other perfect complex X there exists n large enough so that the canonical map $\tau_{\leq n}(X \otimes Y) \to \tau_{\leq n}(X) \otimes \tau_{\leq n}(Y)$ is an isomorphism. Again $\mathcal{T}'$ is thick and contains e_G for all $G \in \mathcal{U}$ by the previous paragraph. Therefore $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}} \subset \mathcal{T}'$ as required. □

Recall the subcategory $\mathcal{W}(\mathcal{U})_n \subset \mathcal{AU}$ from Definition 1.8.1.

Lemma 2.6.6. *Let $\mathcal{U}$ be expansive and let $X \in \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ be a perfect complex. Then $H_*(\tau_{\leq n}(X)) \in \mathcal{W}(\mathcal{U})_n$.*

Proof. Note that by definition we have $\tau_{\leq n}(X) \in \mathrm{thick}(e_G \mid G \in \mathcal{U}_{\leq n})$. Consider the subcategory

$$\mathcal{T} = \{X \in \mathrm{K}(\mathcal{AU})_{\mathrm{perf}} \mid H_*(X) \in \mathcal{W}(\mathcal{U})_n\}.$$

Note that to prove that $X \in \mathcal{T}$ it is enough to show that $\mathcal{T}$ is thick and that it contains e_G for all $G \in \mathcal{U}_{\leq n}$. It follows from Corollary 1.8.4 that $\mathcal{T}$ is closed under finite sums and that $e_G \in \mathcal{T}$ for all $G \in \mathcal{U}_{\leq n}$. It is only left to show that $\mathcal{T}$ is closed under triangles. Consider a triangle $X \to Y \to Z \to \Sigma X$ and its associated long exact sequence in homology

$$\ldots \to H_*(X) \xrightarrow{d_X} H_*(Y) \xrightarrow{d_Y} H_*(Z) \xrightarrow{d_Z} H_{*-1}(X) \to \ldots.$$

As we can always rotate the triangle, we can assume that $X, Y \in \mathcal{T}$ and show that Z is too. Consider the exact diagram

$$\begin{tikzcd}
 & H_*(Y) \arrow[d, "d_Y"] & & 0 \arrow[d] & \\
0 \arrow[r] & \ker(d_Z) \arrow[r, "\mathrm{incl}"] \arrow[d] & H_*(Z) \arrow[r, "\mathrm{d_Z}"] & \mathrm{image}(d_Z) \arrow[r] \arrow[d, "\mathrm{incl}"] & 0 \, . \\
 & 0 & & H_{*-1}(X) &
\end{tikzcd}$$

Using that $\mathcal{W}(\mathcal{U})_n$ is closed under subobjects, quotients and extensions, we see that $H_*(Z) \in \mathcal{W}(\mathcal{U})_n$. □

THEOREM 2.6.7. *Let $\mathcal{U}$ be a complete subcategory of $\mathcal{G}$ and let $X \in \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ be a nonzero perfect complex. Then $H_*(X)$ cannot be torsion.*

PROOF. Let $\kappa + 1$ be the smallest natural number for which the counit map $\tau_{\leq \kappa+1}(X) \to X$ is an isomorphism, see Lemma 2.6.4. Then we have a natural triangle of perfect complexes

$$\tau_{\leq\kappa}(X) \to X \to \tau_{>\kappa}(X) \to \Sigma\tau_{\leq\kappa}(X)$$

by Corollary 2.6.3 and Lemma 2.6.4. Equation 6.0.2 shows that $\tau_{>\kappa}(X)$ is homotopy equivalent to $P_{\kappa+1}(X)$ which has torsion-free homology. If $\tau_{\leq\kappa}(X) = 0$ then we are done. Suppose that this is not the case and note that by completeness, the homology of $\tau_{>\kappa}(X)$ has order exactly $\kappa + 1$. We also have that $H_*(\Sigma\tau_{\leq\kappa}(X)) \in \mathcal{W}(\mathcal{U})_\kappa$ by Lemma 2.6.6. We conclude that the map $\tau_{>\kappa}(X) \to \Sigma\tau_{\leq\kappa}(X)$ cannot be injective in homology. Denote by K the kernel of the natural map $H_*(\tau_{>\kappa}X) \to H_*(\Sigma\tau_{\leq\kappa}X)$, and note that K is nonzero and torsion-free by Lemma 1.10.9. The long exact sequence in homology gives us an epimorphism $H_*(X) \to K$ that implies that $H_*(X)$ cannot contain only torsion elements. □

7. Support

In this section we define a notion of support and show that any finitely generated thick ideal is uniquely determined by its support. We also exhibit a counterexample for localizing subcategories.

Unless otherwise stated $\mathcal{U}$ will denote a replete full subcategory of $\mathcal{G}$.

DEFINITION 2.7.1. A *thick ideal* $\mathcal{I} \subset \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ is a full triangulated subcategory which is closed under retracts, it contains 0 and it satisfies the ideal condition:

$$X \in \mathcal{I} \text{ and } Y \in \mathrm{K}(\mathcal{AU})_{\mathrm{perf}} \Rightarrow X \otimes Y \in \mathcal{I}$$

Given a set $\mathcal{S}$ of perfect complexes, we write $\mathrm{thickid}(\mathcal{S})$ for the smallest thick ideal containing $\mathcal{S}$. A thick ideal $\mathcal{I}$ is said to be *finitely generated* if there exists a finite collection of perfect complexes $\mathcal{S}$ such that $\mathcal{I} = \mathrm{thickid}(\mathcal{S})$.

We introduce our definition of support.

DEFINITION 2.7.2. For $X \in \mathrm{K}(\mathcal{AU})$, we define the *support* of X to be the set

$$\mathrm{supp}(X) = \{G \in \mathcal{U} \mid H_*(X)(G) \neq 0\}.$$

More generally, for any subset $\mathcal{S} \subset \mathrm{K}(\mathcal{AU})$ we set

$$\mathrm{supp}(\mathcal{S}) = \{G \in \mathcal{U} \mid \exists X \in \mathcal{S} \text{ such that } H_*(X)(G) \neq 0\}.$$

We summarize some properties of the support, the proofs are straightforward and left to the reader.

PROPOSITION 2.7.3. *The support enjoys the following properties:*

(a) $\operatorname{supp}(0) = \emptyset$ *and* $\operatorname{supp}(\mathbb{1}) = \mathcal{U}$*;*
(b) $\operatorname{supp}(X \oplus Y) = \operatorname{supp}(X) \cup \operatorname{supp}(Y)$*;*
(c) $\operatorname{supp}(\Sigma X) = \operatorname{supp}(X)$*;*
(d) $\operatorname{supp}(Y) \subset \operatorname{supp}(X) \cup \operatorname{supp}(Z)$ *for any exact triangle* $X \to Y \to Z \to \Sigma X$*;*
(e) $\operatorname{supp}(X \otimes Y) = \operatorname{supp}(X) \cap \operatorname{supp}(Y)$.

□

Our strategy is to first show that the support classifies thick ideals in $\mathrm{K}(\mathcal{AU}_{\leq n})_{\mathrm{perf}}$ and then extend the result to $K(\mathcal{AU})_{\mathrm{perf}}$ using Lemma 2.6.4. Before we prove the result, we need a little bit of preparation.

Recall the simple objects $s_{G,V}$ from Definition 1.1.5. As these objects will play an important role in this chapter, we introduce the following notation.

Definition 2.7.4. We denote by $\rho_{G,V}$ a projective resolution of $s_{G,V}$ so that $\rho_{G,V} \in \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$.

Remark 2.7.5. Note that if V is finite dimensional then there exists n such that $\rho_{G,V} \in \mathrm{K}(\mathcal{AU}_{\leq n})_{\mathrm{perf}}$ by Lemma 1.9.8.

Lemma 2.7.6. *For any finite dimensional* $\mathrm{Out}(G)$*-representation* V*, we have* $\rho_{G,k} \in \operatorname{thickid}(\rho_{G,V})$.

Proof. Note that $\rho_{G,V} \otimes \rho_{G,W} = \rho_{G,V\otimes W}$ so it is enough to show that there exists a positive integer m such that $V^{\otimes m}$ contains the trivial representation. Put $d = \dim_k(V)$ and $m = d \cdot |\mathrm{Out}(G)|$. Then $\Lambda^d V$ is a one dimensional subrepresentation of $V^{\otimes d}$. By character theory, we see that $(\Lambda^d V)^{\otimes m}$ contains the trivial representation and so $V^{\otimes m}$ does too. □

Construction 2.7.7. Consider a complex $Z \in \mathrm{K}(\mathcal{AU})$. For all $k \geq 1$, we define $F_k Z \in \mathrm{K}(\mathcal{AU})$ by

$$(F_k Z)(G) = \begin{cases} Z(G) & \text{if } |G| \geq k \\ 0 & \text{if } |G| < k. \end{cases}$$

By construction, we have a filtration

$$\ldots \to F_n Z \to F_{n-1} Z \to \ldots \to F_2 Z \to F_1 Z = Z$$

with subquotients given by

$$(F_k Z / F_{k+1} Z)_n = \bigoplus_{G \in \mathcal{U}'_k} s_{G, Z_n(G)}$$

where $\mathcal{U}'$ is a fixed skeleton for $\mathcal{U}$. If we apply a projective resolution functor $P\colon \mathrm{K}(\mathcal{AU}) \to \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ to this filtration, we obtain a collection of triangles

$$PF_{k+1} Z \to PF_k Z \to \bigoplus_{G \in \mathcal{U}'_k} \rho_{G, H_*(Z)(G)} \to \Sigma PF_{k+1} Z$$

for all $k \geq 1$.

Proposition 2.7.8. *For all* $Z \in \mathrm{K}(\mathcal{AU}_{\leq n})_{\mathrm{perf}}$*, we have*

$$\operatorname{thickid}(Z) = \operatorname{thickid}(\rho_{G,k} \mid G \in \operatorname{supp}(Z)).$$

Furthermore, given thick ideals $\mathcal{I}, \mathcal{J} \subset \mathrm{K}(\mathcal{AU}_{\leq n})_{\mathrm{perf}}$ *we have* $\operatorname{supp}(\mathcal{I}) \subseteq \operatorname{supp}(\mathcal{J})$ *if and only if* $\mathcal{I} \subseteq \mathcal{J}$*.*

Proof. Note that if $G \in \operatorname{supp}(Z)$ then $Z \otimes \rho_{G,k} = \bigoplus_n \Sigma^n \rho_{G, H_n(Z(G))}$ and so $\rho_{G,H_n(Z(G))} \in \operatorname{thickid}(Z)$. By Lemma 2.7.6, we know that $\rho_{G,k} \in \operatorname{thickid}(Z)$ and so we have the first inclusion. For the other inclusion, note that the filtration of Construction 2.7.7 is finite since $F_{n+1} Z = 0$. As Z is perfect, the homology $H_*(Z)$ is of finite type and so the mapping cones of $PF_k Z \to PF_{k+1} Z$ are semisimple. It follows that $\operatorname{thickid}(Z) = \operatorname{thickid}(\rho_{G,k} \mid G \in \operatorname{supp}(Z))$ as claimed. The second claim follows easily from the first one. □

We are ready to prove the main result of this section.

THEOREM 2.7.9. *For all* $X, Y \in \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$*, we have* $\mathrm{supp}(X) \subseteq \mathrm{supp}(Y)$ *if and only if* $X \in \mathrm{thickid}(Y)$*. More generally, given thick ideals* $\mathcal{I}, \mathcal{J} \subseteq \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ *with* $\mathcal{J}$ *finitely generated, then* $\mathrm{supp}(\mathcal{I}) \subseteq \mathrm{supp}(\mathcal{J})$ *if and only if* $\mathcal{I} \subseteq \mathcal{J}$*.*

PROOF. Let us first prove that if $X \in \mathrm{thickid}(Y)$ then $\mathrm{supp}(X) \subseteq \mathrm{supp}(Y)$. By assumption we can build X from Y using triangles, retracts and the tensor products. Proposition 2.7.3 shows that $\mathrm{supp}(X) \subseteq \mathrm{supp}(Y)$ as claimed. For the other implication note that $\mathrm{supp}(i^*_{\leq n}X) \subseteq \mathrm{supp}(i^*_{\leq n}Y)$ if $\mathrm{supp}(X) \subseteq \mathrm{supp}(Y)$. Thus $i^*_{\leq n}(X) \in \mathrm{thickid}(i^*_{\leq n}(Y))$ by Proposition 2.7.8. That is, there exists a filtration in $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$

$$0 = F_0 \leq F_1 \leq F_2 \leq \ldots \leq F_k = i^*_{\leq n}X$$

with cones F_i/F_{i-1} which are retracts of $Q_i \otimes i^*_{\leq n}Y$. Apply $i^{\leq n}_!$ to this filtration to produce cofibre sequences

$$i^{\leq n}_! F_{i-1} \to i^{\leq n}_! F_i \to i^{\leq n}_!(Q_i \otimes i^*_{\leq n}Y) \qquad \text{for } 1 \leq i \leq k$$

which build $\tau_{\leq n}X$. Choose n large enough so that the canonical maps

$$i^{\leq n}_!(Q_i \otimes i^*_{\leq n}Y) \to i^{\leq n}_!(Q_i) \otimes \tau_{\leq n}(Y) \qquad \text{for } 1 \leq i \leq k$$

and

$$\tau_{\leq n}Y \to Y \qquad \tau_{\leq n}X \to X$$

are isomorphisms, see Lemma 2.6.5 and Lemma 2.6.4. For such n, we have $\tau_{\leq n}(X) \in \mathrm{thickid}(\tau_{\leq n}(Y))$ and therefore $X \in \mathrm{thickid}(Y)$ as required.

For the second claim note that if $\mathcal{J}$ is generated by $\{S_1, \ldots, S_n\}$ then $\mathcal{J} = \mathrm{thickid}(\bigoplus_i S_i)$. For all $X \in \mathcal{I}$, we have $\mathrm{supp}(X) \subseteq \mathrm{supp}(\bigoplus_i S_i)$ and therefore $X \in \mathcal{J}$ as claimed. $\square$

Recall that the radical $\sqrt{\mathcal{I}}$ of a thick ideal $\mathcal{I}$ is given by

$$\sqrt{\mathcal{I}} = \{X \mid \exists\, n \geq 1 \text{ such that } X^{\otimes n} \in \mathcal{I}\},$$

and that a thick ideal is radical if $\mathcal{I} = \sqrt{\mathcal{I}}$.

PROPOSITION 2.7.10. *Any thick ideal of* $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ *is radical.*

PROOF. By [**6**, 4.4], it is enough to show that $X \in \mathrm{thickid}(X \otimes X)$ for all $X \in \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$. Note that if $X(G) \neq 0$, then $X(G) \otimes X(G) \neq 0$. This shows that $\mathrm{supp}(X) \subset \mathrm{supp}(X \otimes X)$ and hence $X \in \mathrm{thickid}(X \otimes X)$ as required. $\square$

The following example shows that localizing subcategories are not determined by their support.

EXAMPLE 2.7.11. Let $\mathcal{U}$ be multiplicative and closed under subgroups, and fix a skeleton $\mathcal{U}'$ for $\mathcal{U}$. Consider the object $X = \bigoplus_{G \in \mathcal{U}'} \rho_{G,k}$. We obviously have $\mathrm{supp}(\mathbb{1}) = \mathrm{supp}(X)$. However $\mathbb{1}$ does not belong to the localizing subcategory generated by X as it does not have torsion homology. Thus localizing ideals and subcategories are not determined by their support.

8. Prime ideals

In this section we study the prime ideals of the category of perfect complexes. Firstly, we recall the following definitions from [**6**].

DEFINITION 2.8.1. Let $\mathcal{U}$ be a replete full subcategory of $\mathcal{G}$.

- A *prime* of $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ is a proper thick ideal $\wp$ satisfying:
$$X \otimes Y \in \wp \Rightarrow X \in \wp \text{ or } Y \in \wp.$$
- The *Balmer spectrum* $\mathrm{spc}(\mathcal{U})$ is the set of primes of $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$.

We will need few general facts on the Balmer spectrum, details can be found in [**6**].

(1) The Balmer spectrum can be upgraded to a topological space by endowing it with a Zariski type topology. A basis of closed subsets is given by
$$Z(\{X\}) = \{\wp \in \mathrm{spc}(\mathcal{U}) \mid X \notin \wp\}, \quad \forall X \in \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}.$$
(2) The closure of a prime $\wp \in \mathrm{spc}(\mathcal{U})$ is given by
$$\overline{\wp} = \{\wp' \mid \wp' \subset \wp\}.$$
(3) The Balmer spectrum is functorial with respect to exact symmetric monoidal functors. Given any such a functor $F\colon \mathrm{K}(\mathcal{AU})_{\mathrm{perf}} \to \mathrm{K}(\mathcal{AU}')_{\mathrm{perf}}$, then the map
$$\mathrm{spc}(F)\colon \mathrm{spc}(\mathcal{U}') \to \mathrm{spc}(\mathcal{U}), \quad \wp \mapsto F^{-1}(\wp)$$
is well-defined and continuous. We note that the functor F needs to preserve perfect complexes.
(4) Consider a thick ideal $\mathcal{I} \subset \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$. Then there is an exact sequence of tensor triangulated categories
$$\mathcal{I} \to \mathrm{K}(\mathcal{AU})_{\mathrm{perf}} \xrightarrow{q} \mathcal{Q}.$$
Then $\mathrm{spc}(q)$ induces an homeomorphism between $\mathrm{spc}(\mathcal{Q})$ and the subspace $\{\wp \in \mathrm{spc}(\mathcal{U}) \mid \mathcal{I} \subset \wp\}$.

We now construct some primes.

Definition 2.8.2. For $G \in \mathcal{U}$, we define the *group prime*
$$\wp_G = \{X \in \mathrm{K}(\mathcal{AU})_{\mathrm{perf}} \mid H_*(X)(G) = 0\}.$$
More generally for a subcategory $\mathcal{E}$ of $\mathcal{U}$, we define the thick ideal
$$\wp_{\mathcal{E}} = \{X \in \mathrm{K}(\mathcal{AU})_{\mathrm{perf}} \mid H_*(X)(G) = 0 \ \ \forall G \in \mathcal{E}\}.$$

Lemma 2.8.3. *Let $\mathcal{U}$ be a full replete subcategory of $\mathcal{G}$.*

(a) *For all $G \in \mathcal{U}$, the thick ideal $\wp_G$ is prime.*
(b) *There are no containments amongst the group primes unless the groups are isomorphic.*

Proof. For part (a), note that $\wp_G$ is a thick ideal as homology sends exact triangles to long exact sequences and commutes with direct sums. It is prime by Lemma 2.3.2.

For part (b), suppose that we have $\wp_H \subset \wp_G$ with H and G not isomorphic. Then there are two possibilities:

- There are no epimorphisms $H \to G$. In this case we have $e_G \in \wp_H$ and hence $e_G \in \wp_G$ which is a contradiction.
- There exists an epimorphism $H \to G$. In this case, let X be the cofibre of the canonical map $c_H \to \mathbb{1}$. By construction, we have $X \in \wp_H$ but $X \notin \wp_G$. We have found a contradiction.

□

Recall the definition of complete subcategory and the list of examples from Section 8.

Proposition 2.8.4. *Let $\mathcal{U}$ be a complete subcategory. Then the zero ideal is prime in $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$. In particular, the zero ideal is prime in $\mathrm{K}(\mathcal{A})_{\mathrm{perf}}$.*

Proof. We need to show that if X and Y are nonzero then so is $X \otimes Y$. By Theorem 2.6.7, we know that there exists $G, K \in \mathcal{U}$ and torsion-free elements $x_G \in H_i(X)(G)$ and $y_K \in H_j(Y)(K)$. Let $p_G\colon G \times K \to G$ be the projection onto G, and similarly for p_K. Put $x = H_i(X(p_G))(x_G) \neq 0$ and $y = H_j(Y(p_K))(y_K) \neq 0$. Thus the element $x \otimes y \in H_{i+j}(X \otimes Y)(G \times K)$ is nonzero and $X \otimes Y \not\simeq 0$. □

Corollary 2.8.5. *Let $\mathcal{U}$ be a multiplicative global family. Then $\wp_{\mathcal{U}}$ is prime in $\mathrm{K}(\mathcal{A})_{\mathrm{perf}}$.*

Proof. We have exact strong monoidal functors $i^*\colon \mathrm{K}(\mathcal{A})_{\mathrm{perf}} \to \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ and an induced functor $\mathrm{spc}(i^*)$ between Balmer spectra, see Fact (3). By definition, we have

$$\mathrm{spc}(i^*)(0) = \{X \mid i^*(X) \simeq 0\} = \wp_{\mathcal{U}}$$

so the claim follows. $\square$

Remark 2.8.6. The restriction functor $i^*\colon \mathrm{K}(\mathcal{A}_{\mathrm{prj}}) \to \mathrm{K}(\mathcal{AU}_{\mathrm{prj}})$ does not preserve perfect complexes in general. However, it does when $\mathcal{U}$ is closed downwards by Lemma **??**.

Proposition 2.8.7. *Any finitely generated thick ideal $\mathcal{I} \subset \mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$ can be written as*

$$\mathcal{I} = \bigcap_{G \in U(\mathcal{I})} \wp_G$$

where $U(\mathcal{I}) = \{G \in \mathcal{U} \mid H_*(X)(G) = 0, \ \forall X \in \mathcal{I}\}$.

Proof. It is easy to check that $\mathrm{supp}(\mathcal{I}) = \mathrm{supp}(\cap_{G\in U(\mathcal{I})}\wp_G)$, then apply Theorem 2.7.9. $\square$

We define a preorder on $\mathcal{U}$ by $G \gg H$ if and only if $\mathcal{U}(G,H) \neq \emptyset$.

Lemma 2.8.8. *Let $\wp$ be a finitely generated prime ideal of $\mathrm{K}(\mathcal{AU})_{\mathrm{perf}}$. Then $U(\wp)$ has a maximal element with respect to $\gg$ if and only if $\wp = \wp_G$ for some $G \in \mathcal{U}$.*

Proof. It is easy to see that $U(\wp_G) = \{H \in \mathcal{U} \mid G \simeq H\}$. Conversely, suppose that $U(\wp)$ has a maximal element G and that it is not a group prime. Then $U(\wp)$ contains at least two non-isomorphic groups so we can write

$$\wp = \wp_G \cap \bigcap_{H \in U(\wp) - U(\wp_G)} \wp_H.$$

Note that by Lemma 2.8.3, there must exist $X \in \wp_G$ such that $X \notin \bigcap_{H\in U(\wp)-U(\wp_G)} \wp_H$. Then we have $X \notin \wp$ and $e_G \notin \wp$ but $X \otimes e_G \in \wp$ which is a contradiction. Therefore $\wp$ must be a group prime. $\square$

9. Examples

In this section we calculate some examples of Balmer spectra and hence obtain a complete classification of thick ideals in the category of perfect complexes. In particular, we show that the closure properties of the chosen family affect the general structure of the Balmer spectrum.

9.1. The Balmer spectrum of $\mathcal{AU}_{\leq n}$. Consider a full replete subcategory $\mathcal{U}$ of $\mathcal{G}$ and fix a skeleton $\mathcal{U}'$ for $\mathcal{U}$. We will be interested in the full subcategory $\mathcal{U}_{\leq n}$ of groups of cardinality less than or equal to n.

Theorem 2.9.1. *The Balmer spectrum $\mathrm{spc}(\mathcal{U}_{\leq n})$ consists only of group primes $\wp_G$ for $G \in \mathcal{U}'_{\leq n}$. Furthermore, the topology is discrete.*

Proof. Note that any thick ideal is finitely generated by Proposition 2.7.8 together with the fact that there are finitely many groups in $\mathcal{U}_{\leq n}$. Consider a prime $\wp$ and suppose that is not a group prime. Then $\wp = \bigcap_{G\in\mathcal{U}(\wp)} \wp_G$ and there are $G, H \in U(\wp)$ not isomorphic. We have $0 \simeq \rho_{G,k} \otimes \rho_{H,k} \in \wp$ but $\rho_{G,k} \notin \wp$ and $\rho_{H,k} \notin \wp$ which is a contradiction as $\wp$ was assumed to be prime. Hence $U(\wp)$ consists only of one group and $\wp$ is a group prime. The topology is discrete by Fact (2) together with Lemma 2.8.3 $\square$

Corollary 2.9.2. *Fix a skeleton $\mathcal{U}'$ for $\mathcal{U}$. Then there is an order preserving bijection*

$$\mathcal{I}\colon \{\textit{subsets of } \mathcal{U}'_{\leq n}\} \leftrightarrow \{\textit{thick ideals of } \mathrm{K}(\mathcal{A}\mathcal{U}_{\leq n})_{\mathrm{perf}}\}$$

given by

$$\mathcal{I}(V) = \{X \mid \mathrm{supp}(X) \subset V\}.$$

Proof. This is [**6**, 4.10] together with Proposition 2.7.10. ☐

9.2. Elementary abelian p-groups. Fix a prime number p and let $\mathcal{E}$ denote the subcategory of elementary abelian p-groups. Throughout we will use the abbreviated notation $\underline{n}$ for the elementary abelian p-group of order p^n.

Lemma 2.9.3. *Any thick ideal of $\mathrm{K}(\mathcal{A}\mathcal{E})_{\mathrm{perf}}$ is finitely generated. Furthermore, the support of any nonzero thick ideal is cofinite in $\mathcal{E}$.*

Proof. Consider a thick ideal $\mathcal{I}$. It suffices to find $Y \in \mathcal{I}$ such that $\mathrm{supp}(Y) = \mathrm{supp}(\mathcal{I})$ and apply Theorem 2.7.9 to deduce that $\mathcal{I} = \mathrm{thickid}(Y)$. If $\mathcal{I}$ is zero we can choose $Y = 0$, so let us assume that I is nonzero. By Theorem 2.6.7, the homology of $X \in \mathcal{I}$ contains at least one torsion-free element, and it follows that $\mathrm{supp}(X)$ and $\mathrm{supp}(\mathcal{I})$ are cofinite in $\mathcal{E}$. In particular, there exists $n > 0$ such that $e_{\underline{n}} \in \mathrm{thickid}(X) \subseteq \mathcal{I}$ since $\mathrm{supp}(e_{\underline{n}}) \subseteq \mathrm{supp}(X)$. For all $m < n$ such that $\underline{m} \in \mathrm{supp}(\mathcal{I})$, we can choose $X_m \in \mathcal{I}$ such that $\underline{m} \in \mathrm{supp}(X)$. This exists, otherwise we would have $\underline{m} \in U(\mathcal{I})$. Finally put $Y = e_{\underline{n}} \oplus \bigoplus_{m<n} X_m \in \mathcal{I}$ and note that by construction $\mathrm{supp}(Y) = \mathrm{supp}(\mathcal{I})$. ☐

Theorem 2.9.4. *The Balmer spectrum $\mathrm{spc}(\mathcal{E})$ consists of the zero ideal and the group primes $\wp_{\underline{n}}$ for $n \geq 0$. Furthermore, a basis of closed subsets for the topology is given by the empty set together with the cofinite subsets containing 0. In particular, the zero ideal is the only closed point. The Balmer spectrum can be depicted in the following way:*

$$\begin{tikzcd}[column sep=small]
 & & & 0 \arrow[dlll, dash] \arrow[dll, dash] \arrow[dl, dash] \arrow[dr, dash] & & \\
\wp_{\underline{0}} & \wp_{\underline{1}} & \wp_{\underline{2}} & \cdots & \wp_{\underline{n}} & \cdots
\end{tikzcd}$$

Proof. Note that 0 and $\wp_{\underline{n}}$ for $n \geq 0$, are prime by Proposition 2.8.4 and Lemma 2.8.3. If $\wp$ is any nonzero prime ideal, then by Lemma 2.9.3 we can write $\wp$ as a finite intersection of group primes, see Proposition 2.8.7. In particular, $U(\wp)$ has a maximal element so by Lemma 2.8.8 we conclude that $\wp$ is a group prime. The fact that the zero ideal is the only closed point follows from Lemma 2.8.3 and Fact (2). By Fact (1), a basis of closed subsets is given by the empty set and $Z(\{X\}) = \{0\} \cup \mathrm{supp}(X)$ for all nonzero X. ☐

As a consequence we obtain a complete classification of the thick ideals.

Corollary 2.9.5. *Fix a skeleton $\mathcal{E}'$ for $\mathcal{E}$. Then there is an order preserving bijection*

$$\mathcal{I}\colon \{\textit{empty or cofinite subsets of } \mathcal{E}'\} \leftrightarrow \{\textit{thick ideals of } \mathrm{K}(\mathcal{A}\mathcal{E})_{\mathrm{perf}}\}$$

given by

$$\mathcal{I}(V) = \{X \mid \mathrm{supp}(X) \subseteq V\}.$$

Proof. This is [**6**, Theorem 4.10] together with Proposition 2.7.10. ☐

9.3. Cyclic p-groups. Fix a prime number p and let $\mathcal{C}$ denote the subcategory of cyclic p-groups. Throughout we will use the abbreviated notation $[n]$ for the cylic group of order p^n.

Lemma 2.9.6. *The support of any perfect complex is either finite or cofinite.*

Proof. Consider a perfect complex $X \in \mathrm{K}(\mathcal{AC})_{\mathrm{perf}}$. Recall that the canonical map $\tau_{\leq n}X \to X$ is an isomorphism for large n, see Lemma 2.6.4. Without loss of generality we can assume that n is a power of p. By Proposition 2.7.8 we know that

$$X \in \mathrm{thickid}(\tau_{\leq n}(\rho_{[m],k}) \mid [m] \in \mathrm{supp}(X) \cap \mathcal{C}_{\leq n}).$$

Using the short exact sequence $0 \to e_{[m+1],k} \to e_{[m],k} \to \chi_{[m],k} \to 0$ we see that

$$\tau_{\leq n}(\rho_{[m],k}) = \begin{cases} \rho_{[m],k} & \text{if } p^m < n \\ e_{[n],k} & \text{if } p^m = n \\ 0 & \text{if } p^m > n. \end{cases}$$

From this it is easy to see that the support of X is either finite or cofinite. $\square$

Lemma 2.9.7. *The thick ideal*

$$\wp_{tors} = \{X \mid H_*(X)([n]) = 0, \ \forall\, n \gg 0\}$$

is prime in $\mathrm{K}(\mathcal{AC})_{\mathrm{perf}}$.

Proof. Suppose that $X \otimes Y \in \wp_{tors}$ which means that $\mathrm{supp}(X \otimes Y) = \mathrm{supp}(X) \cap \mathrm{supp}(Y)$ is finite. It follows that at least one between X and Y has finite support, i.e., at least one is in $\wp_{tors}$. $\square$

Theorem 2.9.8. *The Balmer spectrum* $\mathrm{spc}(\mathcal{C})$ *consists of the group primes* $\wp_{[n]}$ *for* $n \geq 0$*, and the prime* $\wp_{tors}$ *which is not finitely generated. A basis of closed subsets is given by the following collections:*

- *finite subsets not containing* $\wp_{tors}$*;*
- *closed upwards subsets containing* $\wp_{tors}$ *with respect to* $\wp_{[i]} \leq \wp_{[j]}$ *if and only if* $i \leq j$.

The Balmer spectrum can be depicted in the following way:

$$\begin{tikzcd} & & \wp_{tors} \arrow[drr, dotted, no head] \arrow[drrr, dotted, no head] & & & & \\ \wp_{[0]} & \wp_{[1]} & \wp_{[2]} & \cdots & \wp_{[n]} & \wp_{[n+1]} & \cdots \end{tikzcd}$$

Proof. The ideals $\wp_{[n]}$ are prime by Lemma 2.8.3 and $\wp_{tors}$ is prime by Lemma 2.9.7. Consider a prime ideal $\wp$ and recall that the support of $X \in \wp$ can only be finite or cofinite by Lemma 2.9.6. We claim that if there exists $X \in \wp$ with cofinite support, then $\wp$ is a group prime. Suppose that such X exists so that by Theorem 2.7.9 we have $e_{[n+1]} \in \mathrm{thickid}(X) \subset \wp$ for some n. In particular, we see that $\wp \supset \mathcal{I}_{>n} = \mathrm{thickid}(e_{[n+1]})$. Note that there is an exact sequence of tensor triangulated categories

$$\mathcal{I}_{>n} \to \mathrm{K}(\mathcal{AC})_{\mathrm{perf}} \xrightarrow{i^*_{\leq n}} \mathrm{K}(\mathcal{AC}_{\leq n})_{\mathrm{perf}}.$$

By Fact (4), the prime $\wp$ corresponds to a prime in $\mathrm{spc}(\mathcal{C}_{\leq n})$ under $\mathrm{spc}(i^*_{\leq n})$. As the Balmer spectrum of $\mathcal{C}_{\leq n}$ consists only of group primes by Theorem 2.9.1, we conclude that $\wp$ must be a group prime as claimed.

On the other hand, we claim that if the support of all $X \in \wp$ is finite then $\wp = \wp_{tors}$. In this case we must have $\wp = \mathrm{thickid}(\rho_{[s],k} \mid s \in S)$ for a set $S \subset \mathbb{N}$. If $S \neq \mathbb{N}$, choose $s_0 \in \mathbb{N} - S$. Then $\rho_{[s_0],k} \notin \wp$ and $e_{[s_0+1]} \notin \wp$ as it has cofinite support, but $0 = \rho_{[s_0],k} \otimes e_{[s_0+1]} \in \wp$. As $\wp$ was assumed to be prime we conclude that $S = \mathbb{N}$, or equivalently that $\wp = \wp_{tors}$.

Finally, using Fact (1) we see that a basis of closed subsets is given by $Z(\{e_{[n]}\})$ and $Z(\{\rho_{[n],k}\})$ which are the closed upwards subsets containing $\wp_{tors}$ and the finite subsets not containing $\wp_{tors}$, respectively.

$\square$

REMARK 2.9.9. We showed that the ideal $\wp_{tors}$ is not finitely generated and that its support is all of $\mathcal{C}$. We note that $\wp_{tors}$ is not all of $\mathrm{K}(\mathcal{AC})_{\mathrm{perf}}$ since any $X \in \wp_{tors}$ has torsion homology. In particular, this shows that there exist non-finitely generated thick ideals which are not determined by the notion of support introduce in Definition 2.7.2.

As a consequence we obtain a complete classification of the thick ideals.

COROLLARY 2.9.10. *Fix a skeleton $\mathcal{C}'$ for $\mathcal{C}$. Put $\mathcal{S}(\mathcal{C}) = \{V \subseteq \mathcal{C}' \sqcup \{*\} \mid * \notin V$ or V cofinite$\}$. Then there is an order preserving bijection*

$$\mathcal{I}\colon \mathcal{S}(\mathcal{C}) \leftrightarrow \{\textit{thick ideals of } \mathrm{K}(\mathcal{AC})_{\mathrm{perf}}\}$$

where

$$\mathcal{I}(V) = \begin{cases} \{X \mid \mathrm{supp}(X) \subseteq V\} & \text{if } * \in V \\ \{X \mid \mathrm{supp}(X) \subseteq V,\ H_*(X) \text{ torsion}\} & \text{if } * \notin V. \end{cases}$$

PROOF. We can identify the Balmer spectrum with the set $\mathcal{C}' \sqcup \{*\}$ where $\wp_{tors}$ corresponds to the the singleton $\{*\}$. Then apply [**6**, Theorem 4.10] and Proposition 2.7.10. $\square$

Part 3

The Left Localization Principle, completions, and cofree G-spectra

Introduction

In this paper we investigate the interplay between adjoint pairs and localizations. In homotopy theory there are two versions of localizations available: the left and right Bousfield localization. The former is ubiquitous in chromatic stable homotopy theory, while the latter has seen interesting applications in the study of torsion objects in algebraic categories, see [**42**, Section 5]. Often in the literature the right Bousfield localization is called cellularization since in the stable setting it picks out the localizing subcategory on the set of cells (if they are stable and compact).

We now give an informal overview of our results and refer to the main body of the paper for the precise statements.

The Cellularization Principle. Let $\mathcal{C}$ be a stable model category, and let $\mathcal{K}$ be a set of objects of $\mathcal{C}$. The Cellularization Principle of Greenlees-Shipley [**42**] provides conditions under which a Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ descends to a Quillen equivalence

$$F : \mathrm{Cell}_{\mathrm{K}}\mathcal{C} \rightleftarrows \mathrm{Cell}_{F\mathcal{K}}\mathcal{D} : G$$

between the cellularizations. The Cellularization Principle is a crucial ingredient in the construction of algebraic models for rational equivariant spectra, see for instance [**45**]. There is also a version of the Principle where the cells are passed along the right adjoint, and a variant [**11**, Section 5.1] in which symmetric monoidal structures are taken into account. The main limitation of the Cellularization Principle is that the preservation of symmetric monoidal structures is *not* automatic.

Since the symmetric monoidal structure need not be preserved by cellularization, the symmetric monoidal version of the Cellularization Principle requires stronger assumptions. For instance, when passing cells along the right adjoint, the Cellularization Principle gives a symmetric monoidal Quillen equivalence between the cellularizations if the original adjunction was *already* a symmetric monoidal Quillen equivalence [**11**, 5.1.7].

On the other hand, the monoidal structure is often preserved by left Bousfield localization.

The Left Localization Principle. The Left Localization Principle which we develop, gives mild conditions under which a symmetric monoidal Quillen adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ descends to a *symmetric monoidal* Quillen equivalence between the homological localizations. For an object E of a stable, symmetric monoidal model category $\mathcal{C}$, the homological localization $L_E\mathcal{C}$ is the localization of $\mathcal{C}$ at the class of E-equivalences, that is those morphisms that become equivalences after tensoring with E.

Theorem N (3.2.15). *Let $\mathcal{C}$ and $\mathcal{D}$ be stable, symmetric monoidal model categories, E an object of $\mathcal{C}$ and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a symmetric monoidal Quillen adjunction. Suppose that $\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ of objects and that $\mathcal{D}$ is homotopically compactly generated by $F\mathcal{K}$. Suppose that:*

(i) *The derived unit map $K \to GFK$ is an E-equivalence for all $K \in \mathcal{K}$;*
(ii) *G sends FE-equivalences to E-equivalences.*

Then the induced Quillen adjunction

$$F : L_E\mathcal{C} \rightleftarrows L_{FE}\mathcal{D} : G$$

is a symmetric monoidal Quillen equivalence.

The major advantage of the Left Localization Principle over the Cellularization Principle is that the symmetric monoidal structure is preserved automatically. There are several variations of the Principle that we do not include in this introduction. Of particular note is the Compactly Generated Localization Principle, see Theorem 3.2.16. Although the assumptions of this last Principle are quite restrictive, there are interesting examples where it applies, as we show in our applications.

We now turn to the applications of the Left Localization Principle. The main motivation of the authors for developing the Left Localization Principle comes from rational equivariant stable homotopy theory.

Algebraic models. The programme of finding algebraic models for rational G-spectra was begun by Greenlees, who conjectured that for every compact Lie group G, there is an *abelian* category $\mathcal{A}(G)$, together with a Quillen equivalence between the category of rational G-spectra and the category of differential objects in $\mathcal{A}(G)$. The programme looks for abelian categories with finite homological dimension so that calculations can easily be performed, and equipped with an Adams spectral sequence to calculate homotopy classes of maps between G-spectra. This programme has so far been successful in the cases of G finite [**8**], $G = SO(2)$ [**11, 85**] $G = O(2)$ [**9**], $G = SO(3)$ [**56**], G a torus of any rank [**45**], the toral part of G-spectra [**10**], and free G-spectra for G a compact Lie group [**41, 43**]. One can also ask for equivalences with extra structure such as being monoidal, so that the equivalence passes to ring and module spectra.

When attempting to find algebraic models for categories of interest, there are several techniques we can apply. One approach is to use Morita theory [**84**] which gives an equivalence with modules over the endomorphism ring of a generator. However, the endomorphism ring need not be commutative so that formality arguments are inaccessible, and the module category often has infinite homological dimension. Another valuable technique is to use the Cellularization Principle to reduce the problem to checking conditions on generating cells. In this paper, we show that the Left Localization Principle is another technique that we can use. Balchin-Greenlees [**5**] show that stable model categories can be split into pieces determined by left localizations in an adelic fashion, by proving that the stable model category is a homotopy pullback of an 'adelic cube'. We hope that the Left Localization Principle may be applied in these situations as well, to simplify the adelic cube.

Completions. In order to verify the conjecture of Greenlees in our case of interest, we discuss some homotopical aspects of completion. We briefly recall the relevant results about the different types of completions in algebra and we refer the reader to Section 1 for a more detailed exposition and references.

Let I be a finitely generated ideal in a commutative ring R. The I-adic completion functor is a fundamental tool in algebra, but has poor homological properties as it is neither left nor right exact. Our approach is to work with its zeroth left derived functor which we denote by L_0^I. We say that an R-module M is L_0^I-complete if the canonical map $M \to L_0^I M$ is an isomorphism. The full subcategory of L_0^I-complete modules is a symmetric monoidal abelian category which supports a projective model structure under a mild condition on the ideal considered. This condition is called weak pro-regularity and holds in many cases; for example, any ideal in a Noetherian ring is weakly pro-regular.

For homotopical purposes it is often convenient to consider the derived I-completion functor. This is defined in terms of the stable Koszul complex whose filtration provides a spectral sequence making the derived completion accessible. Under the weak pro-regularity hypothesis on the ideal I, the derived I-completion functor is equivalent to the total left derived functor of I-adic completion, and therefore calculates the local homology modules, see [**38, 69**].

We give a proof using the language of model categories that derived I-complete modules can be modelled via the abelian category of L_0^I-complete modules, see Theorem 4.2.11. It follows that a dg-module is derived I-complete if and only if its homology is L_0^I-complete. This generalises a result of Dwyer-Greenlees [**27**, 6.15] and clarifies an observation of Porta-Shaul-Yekutieli [**69**, 4.33] that derived I-complete modules need not have I-adically complete homology. We note the related work of Barthel-Heard-Valenzuela who have given an ∞-categorical approach to derived completion in the general setup of comodules over Hopf algebroids [**14**].

Rational cofree G-spectra. The equivariant stable homotopy category contains two classes of objects of particular note: the free and cofree G-spectra. An algebraic model for rational free G-spectra was constructed by Greenlees-Shipley [**41**, **43**] in terms of torsion modules over the group cohomology ring. However, the abelian category of torsion modules is not monoidal as it has no tensor unit and therefore the Quillen equivalence in the free case cannot be refined to a symmetric monoidal Quillen equivalence.

By exploiting the equivalence between free and cofree G-spectra, we give a symmetric monoidal algebraic model for the category of rational cofree G-spectra. For convenience, we only state the result for the connected case in this introduction. See Theorem 4.5.6 for the general case.

Theorem O (4.4.4). *Let G be a connected compact Lie group and I be the augmentation ideal of H^*BG. Then there is a symmetric monoidal Quillen equivalence*

$$\mathrm{Sp}_G^{\mathrm{cofree}} \simeq_Q \mathrm{Mod}_{H^*BG}^{\wedge}$$

between rational cofree G-spectra and L_0^I-complete dg-H^*BG-modules. In particular, there is a tensor-triangulated equivalence

$$\textit{cofree } G\textit{-spectra} \simeq_\triangle \mathcal{D}(L_0^I\textit{-complete } H^*BG\textit{-modules}).$$

In this application, the Left Localization Principle manifests its advantages over the Cellularization Principle. Firstly, the proof of the equivalence is formal as it only requires a few elementary iterations of the Left Localization Principle and some formality arguments in algebra. In particular we avoid any "topological" formality argument using the Adams spectral sequence. Secondly, it gives a tensor-triangulated equivalence of the homotopy categories.

Free and cofree G-spectra are interesting for three particular reasons. Firstly, they represent cohomology theories on free G-spaces, the most prominent example of which is Borel cohomology. Secondly, the techniques employed in the construction of the algebraic models for free and cofree G-spectra are instructive for more general cases, such as that of torus-equivariant spectra [**45**]. Finally, the algebraic models for free and cofree G-spectra fit in the general picture of a local duality context in the sense of [**13**]. This means that the equivalence between free and cofree G-spectra in equivariant stable homotopy theory translates to the equivalence between torsion and complete modules in algebra.

Contribution of this paper and related work. Let us restrict to connected groups for simplicity, and continue to write I for the augmentation ideal. A Quillen equivalence between rational cofree G-spectra and derived complete H^*BG-modules was already known by passing through free G-spectra in the following way:

$$\begin{tikzcd}
\text{free } G\text{-spectra} \arrow[r, leftrightarrow, "\simeq_Q"] \arrow[d, leftrightarrow, "\simeq_Q"'] & I\text{-power torsion-}H^*BG\text{-modules} \arrow[d, leftrightarrow, "\simeq_Q"] \\
\text{cofree } G\text{-spectra} \arrow[r, leftrightarrow, dashed] & \text{derived } I\text{-complete-}H^*BG\text{-modules.}
\end{tikzcd}$$

The horizontal Quillen equivalence is the algebraic model for free G-spectra of Greenlees-Shipley [**41**] and the right vertical follows from Dwyer-Greenlees' Morita theory [**27**] together with [**42**, Section 5]. However this is unsatisfactory for two main reasons. Firstly, it cannot be refined to a symmetric monoidal Quillen equivalence since the category of I-power torsion modules has no tensor unit. Secondly, it does not give an abelian model as desired in the conjecture of Greenlees. In light of this, our contribution is threefold: we prove the algebraic model for rational cofree G-spectra directly, we upgrade it to a symmetric monoidal Quillen equivalence, and we give an abelian model for derived complete modules. In addition, we collect several results about homotopical aspects of algebraic completions which we believe will be of independent interest.

Although our strategy is analogous to that employed by Greenlees-Shipley in the study of free G-spectra, the tools we use differ. In particular, the Left Localization Principle which we develop is a new and key ingredient in our proof.

Outline of the paper. The paper is divided into two main parts.

In the first part we give some necessary background on left Bousfield localizations and then state and prove the Left Localization Principle. We then investigate the implications in the case of homological localizations, which provide many key examples.

In the second part of the paper we focus on the applications of the Left Localization Principle. We apply the Left Localization Principle to understand completions of module categories and to construct a symmetric monoidal algebraic model for rational cofree G-spectra. We have decided to first construct the algebraic model for a connected compact Lie group and then show how to generalize our proofs to the non-connected case.

Conventions. We shall follow the convention of writing the left adjoint above the right adjoint in an adjoint pair. We will use $q\colon QX \to X$ and $r\colon X \to RX$ to denote cofibrant and fibrant replacements of X respectively. In addition we shall assume that these replacements are functorial. For instance this always holds if the factorization of the model category is obtained via the small object argument.

Acknowledgements. We are extremely grateful to John Greenlees for many helpful discussions and suggestions. We would also like to thank Scott Balchin, Magdalena Kędziorek and Gabriel Valenzuela for their interest and comments.

CHAPTER 3

The Left Localization Principle

In this chapter we developed the Left Localization Principle which gives mild conditions under which a Quillen adjunction between stable model categories descends to a Quillen equivalence between the left Bousfield localizations. Finally we present one application of this principle to completion of module categories.

1. Left Bousfield localization of model categories

In this section we recall some necessary background on left Bousfield localizations following [**49**] and [**12**].

DEFINITION 3.1.1. Let $\mathcal{C}$ be a model category and let S be a collection of maps in $\mathcal{C}$.

- An object W in $\mathcal{C}$ is *S-local* if it is fibrant in $\mathcal{C}$ and for every $s\colon A \to B$ in S, the natural map $\mathrm{Map}(B, W) \to \mathrm{Map}(A, W)$ is a weak equivalence of homotopy function complexes.
- A map $f\colon X \to Y$ in $\mathcal{C}$ is an *S-local equivalence* if for every S-local object W, the natural map $\mathrm{Map}(Y, W) \to \mathrm{Map}(X, W)$ is a weak equivalence of homotopy function complexes.

REMARK 3.1.2. If the model category is stable, then the homotopy function complexes in the previous Definition can be replaced with the graded set of maps in the homotopy category, see [**12**, 4.5].

In many cases, we can endow the model category $\mathcal{C}$ with a new model structure, the *left Bousfield localization* of $\mathcal{C}$, in which the weak equivalences are the S-local equivalences, the cofibrations are unchanged, and the fibrant objects are the S-local objects. If it exists, we denote this model category by $L_S\mathcal{C}$.

HYPOTHESIS 3.1.3. Throughout this paper we assume that all the required left Bousfield localizations exist.

REMARK 3.1.4. The left Bousfield localization exists under mild conditions on the model category $\mathcal{C}$. For example, when $\mathcal{C}$ is left proper, cellular and S is a set [**49**, 4.1.1], or when $\mathcal{C}$ is left proper, combinatorial and S is a set [**16**, 4.7]. In particular, left Bousfield localizations (at sets of morphisms) exist for the stable model structure on spectra [**61**, 9.1], the stable model structure on equivariant spectra for any compact Lie group [**60**, III.4.2] and the projective model structure on dg-modules [**15**, 3.3].

Recall that a model category is *symmetric monoidal* if it is a closed symmetric monoidal category and it satisfies the *pushout-product axiom*: if $f\colon A \to B$ and $g\colon X \to Y$ are cofibrations, then the pushout-product map

$$f\square g\colon A\otimes Y \bigcup_{A\otimes X} B\otimes X \to B\otimes Y$$

is a cofibration, which is acyclic if either f or g is acyclic; and the *unit axiom*: the natural map $Q\mathbb{1}\otimes X \to \mathbb{1}\otimes X \cong X$ is a weak equivalence for all cofibrant X. We denote the internal hom functor by $F(-,-)$.

DEFINITION 3.1.5. We say that a stable model category $\mathcal{C}$ is *homotopically compactly generated* by a set $\mathcal{K}$ of objects if its homotopy category $\mathrm{h}\mathcal{C}$ is compactly generated by $\mathcal{K}$:

- for all $K \in \mathcal{K}$ and collections $\{M_i\}$ of objects of $\mathcal{C}$, the natural map $\bigoplus \mathrm{h}\mathcal{C}(K, M_i) \to \mathrm{h}\mathcal{C}(K, \bigoplus M_i)$ is an isomorphism;
- an object X of $\mathrm{h}\mathcal{C}$ is trivial if and only if $\mathrm{h}\mathcal{C}(\Sigma^n K, X) = 0$ for all $K \in \mathcal{K}$ and $n \in \mathbb{Z}$.

Next we recall the definition of a monoidal Quillen adjunction from [**83**].

Definition 3.1.6. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction between symmetric monoidal model categories.

(1) We say that (F, G) is a *weak symmetric monoidal Quillen adjunction* if the right adjoint G is lax monoidal (which gives the left adjoint F an oplax monoidal structure) and the following two conditions hold:
 (a) for cofibrant A and B in $\mathcal{C}$, the oplax monoidal structure map $\phi\colon F(A \otimes B) \to F(A) \otimes F(B)$ is a weak equivalence in $\mathcal{D}$
 (b) for a cofibrant replacement $Q\mathbb{1}_{\mathcal{C}}$ of the unit in $\mathcal{C}$, the map $\phi_0\colon F(Q\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ is a weak equivalence in $\mathcal{D}$.
(2) If the oplax monoidal structure maps ϕ and ϕ_0 are isomorphisms, then we say that (F, G) is a *strong symmetric monoidal Quillen pair.*
(3) We say that the adjunction (F, G) is *symmetric monoidal* if it is a weak symmetric monoidal Quillen adjunction.
(4) We say that the adjucntion (F, G) is a *symmetric monoidal Quillen equivalence* if it is a symmetric monoidal adjunction and a Quillen equivalence.

Remark 3.1.7. A Quillen adjunction is symmetric monoidal if the left adjoint is strong monoidal and the unit object of $\mathcal{C}$ is cofibrant.

Definition 3.1.8. A set of morphisms S of a stable model category $\mathcal{C}$ is said to be *stable* if the collection of S-local objects is closed under (de)suspensions. We say that a stable set of cofibrations S of a stable, cellular, symmetric monoidal model category $\mathcal{C}$ is *monoidal* if $S \square I = \{s \square i \mid s \in S,\ i \in I\}$ is contained in the class of S-equivalences, where I is the set of generating cofibrations for $\mathcal{C}$.

We will need the following result.

Proposition 3.1.9 ([**12**, 5.1]). *Let $\mathcal{C}$ be a proper, cellular, stable, symmetric monoidal model category and let S be a stable set of cofibrations between cofibrant objects. Then the localization $L_S\mathcal{C}$ is a symmetric monoidal model category if and only if S is monoidal.*

Remark 3.1.10. Any map in a model category can be replaced up to weak equivalence by a cofibration between cofibrant objects: first cofibrantly replacing the source and then factoring the composite into a cofibration followed by an acyclic fibration. Since left Bousfield localization depends only on the class of maps up to equivalence, we can assume without loss of generality that S consists of cofibrations between cofibrant objects.

2. The Left Localization Principle

We are now ready to work towards the Left Localization Principle. Before we can prove an induced Quillen equivalence, we must check that the Quillen adjunction descends to the localizations. Recall that Q and R denote cofibrant and fibrant replacement in the original model structures on $\mathcal{C}$ and $\mathcal{D}$ respectively.

Proposition 3.2.1. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction between stable model categories satisfying Hypothesis 3.1.3. Let S and T be stable sets of morphisms in $\mathcal{C}$ and $\mathcal{D}$ respectively, and suppose that F sends S-equivalences between cofibrant objects to T-equivalences. Then the adjunction*
$$F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$$
is a Quillen adjunction. Furthermore, it is a symmetric monoidal Quillen adjunction if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a symmetric monoidal Quillen adjunction and S and T are monoidal.

Proof. By Hirschhorn [**49**, 3.3.18], to prove that $L_S\mathcal{C} \rightleftarrows L_T\mathcal{D}$ is a Quillen adjunction, it is sufficient to check that F sends S-equivalences between cofibrant objects to T-equivalences, which was our hypothesis. The claim about the monoidality follows from the fact that the cofibrations in a left Bousfield localization are the same as in the original category, and the local equivalences contain the original weak equivalences. □

Remark 3.2.2. If we apply the previous Proposition with $S = GRT$, then the hypothesis that F sends GRT-equivalences between cofibrant objects to T-equivalences may seem hard to verify in practice. However, we show in Lemma 3.2.14 that in the case of homological localization, this hypothesis can be replaced by a condition which is much easier to verify.

Remark 3.2.3. If S is monoidal, it often happens that FQS is also monoidal. Write $I_{\mathcal{C}}$ and $I_{\mathcal{D}}$ for the sets of generating cofibrations in $\mathcal{C}$ and $\mathcal{D}$ respectively. For instance, one can easily check that FQS is monoidal when (F, G) is a strong symmetric monoidal Quillen pair and $I_{\mathcal{D}} \subseteq F(I_{\mathcal{C}})$, or, when (F, G) is a weak symmetric monoidal Quillen pair, the domains of $I_{\mathcal{C}}$ are cofibrant and $I_{\mathcal{D}} \subseteq F(I_{\mathcal{C}})$. Note that the condition that $I_{\mathcal{D}} \subseteq F(I_{\mathcal{C}})$ is satisfied in the case when the model structure on $\mathcal{D}$ is right induced from $\mathcal{C}$.

We can now state and prove the Left Localization Principle. We note that as the cofibrations are the same in the left Bousfield localization as in the original model structure, we continue to write Q for the cofibrant replacement in the localization. However, since being fibrant in the localization is a stronger condition than being fibrant in the original model structure, we write $\overline{R}$ for the fibrant replacement in the localization.

Theorem 3.2.4 (Left Localization Principle). *Let $\mathcal{C}$ and $\mathcal{D}$ be stable model categories satisfying Hypothesis 3.1.3 and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction.*

(1) *Suppose that $\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ and that $\mathcal{D}$ is homotopically compactly generated by $FQ\mathcal{K}$. Let S and T be stable sets of morphisms in $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose that the following conditions hold:*
   (i) *The derived unit map $\eta_K \colon QK \to GRFQK$ is an S-equivalence for all $K \in \mathcal{K}$.*
   (ii) *G sends T-equivalences between fibrant objects in $\mathcal{D}$ to S-equivalences.*
   (iii) *F sends S-equivalences between cofibrant objects to T-equivalences.*
   Then the induced Quillen adjunction
   $$F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$$
   is a Quillen equivalence. Moreover, if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a symmetric monoidal Quillen adjunction and S and T are monoidal, then $F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$ is a symmetric monoidal Quillen equivalence.
(2) *Suppose that $\mathcal{D}$ is homotopically compactly generated by a set $\mathcal{L}$ and that $\mathcal{C}$ is homotopically compactly generated by $GR\mathcal{L}$. Let T be a stable set of morphisms in $\mathcal{D}$. Suppose that the following conditions hold:*
   (i) *The derived counit map $\epsilon_L \colon FQGRL \to RL$ is a weak equivalence in $\mathcal{D}$ for all $L \in \mathcal{L}$;*
   (ii) *G sends T-equivalences between fibrant objects in $\mathcal{D}$ to GRT-equivalences;*
   (iii) *F sends GRT-equivalences between cofibrant objects to T-equivalences.*
   Then the induced Quillen adjunction
   $$F : L_{GRT}\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$$
   is a Quillen equivalence. Moreover, if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a symmetric monoidal Quillen adjunction and T and GRT are monoidal, then $F : L_{GRT}\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$ is a symmetric monoidal Quillen equivalence.

Proof. Let us prove (1). Note that condition (iii) ensures that the Quillen adjunction descends to the localizations, see Proposition 3.2.1. We now show that the derived functor GR

preserves sums, so that the subcategories

$$\mathcal{A} = \{X \in \mathrm{h}\mathcal{C} \mid \eta_X \colon QX \xrightarrow{\sim_S} GRFQX\} \quad \text{and} \quad \mathcal{A}' = \{Y \in \mathrm{h}\mathcal{D} \mid \epsilon_Y \colon FQGRY \xrightarrow{\sim_T} RY\}$$

are localizing. Let $(X_i)_{i\in I}$ be a collection of objects in $\mathrm{h}\mathcal{D}$. Using compactness we see that for all $K \in \mathcal{K}$

$$\begin{aligned} \mathrm{h}\mathcal{C}(K, GR(\bigoplus_{i\in I} X_i)) \cong \mathrm{h}\mathcal{D}(FQK, \bigoplus_{i\in I} X_i) \cong \bigoplus_{i\in I} \mathrm{h}\mathcal{D}(FQK, X_i) \cong \\ \cong \bigoplus_{i\in I} \mathrm{h}\mathcal{C}(K, GR(X_i)) \cong \mathrm{h}\mathcal{C}(K, \bigoplus_{i\in I} GR(X_i)). \end{aligned}$$

Since $\mathcal{K}$ generates $\mathrm{h}\mathcal{C}$ we conclude that GR preserves arbitrary sums.

By assumption (i), we know that $\mathcal{K} \subset \mathcal{A}$ thus $\mathcal{A} = \mathrm{h}\mathcal{C}$ as $\mathcal{K}$ generates $h\mathcal{C}$. Note that $FQ\eta_K$ is a T-equivalence by condition (iii). Using the triangular identity of the derived adjunction

$$\begin{tikzcd} FQK \arrow[r, "FQ\eta_K"] \arrow[dr, "r"'] & FQGRFQK \arrow[d, "\epsilon_{FQK}"] \\ & RFQK \end{tikzcd}$$

and 2-out-of-3, we obtain that $FQK \in \mathcal{A}'$ and hence $\mathcal{A}' = \mathrm{h}\mathcal{D}$ as $FQ\mathcal{K}$ generates $h\mathcal{D}$.

We must prove that $\overline{\eta}_X \colon QX \to G\overline{R}FQX$ is an S-equivalence for all $X \in \mathrm{h}\mathcal{C}$ and that $\overline{\epsilon}_Y \colon FQG\overline{R}Y \to \overline{R}Y$ is a T-equivalence for all $Y \in \mathrm{h}\mathcal{D}$. Note that the canonical map $GRFQX \to G\overline{R}FQX$ is an S-equivalence by condition (ii). Therefore the derived unit

$$\overline{\eta}_X \colon QX \xrightarrow{\sim_S} GRFQX \xrightarrow{\sim_S} G\overline{R}FQX$$

is an S-equivalence. For the derived counit, note that the canonical map $GRY \to G\overline{R}Y$ is an S-equivalence and therefore $FQGRY \to FQG\overline{R}Y$ is a T-equivalence by condition (iii). By considering the diagram

$$\begin{tikzcd} FQGRY \arrow[r, "\epsilon_Y"] \arrow[d, "\sim_T"'] & RY \arrow[d, "\sim_T"] \\ FQG\overline{R}Y \arrow[r, "\overline{\epsilon}_Y"'] & \overline{R}Y \end{tikzcd}$$

we see that $\overline{\epsilon}_Y$ is a T-equivalence if and only if ϵ_Y is so. Since $\mathcal{A}' = \mathrm{h}\mathcal{D}$ the claim follows.

The proof of part (2) follows from (1) by taking $S = GRT$. $\square$

Remark 3.2.5. Notice that the conditions in (1) imply that the derived functor FQ preserves all compact objects. Moreover, in the proof we showed that GR preserves sums so it also follows that under the conditions in (2) the derived functor GR preserves all compact objects.

Remark 3.2.6. In [**51**, 2.3] Hovey gives criteria for when left Bousfield localization preserves Quillen equivalences. His result does not assume stability but does not treat the case where the original adjunction is not a Quillen equivalence.

In the Left Localization Principle we assumed that $\mathcal{C}$ and $\mathcal{D}$ are homotopically compactly generated whereas in the following we assume that the localizations are homotopically compactly generated. This is a stronger condition but holds in certain cases when the localization is homological, see Remark 3.2.18.

Theorem 3.2.7 (Compactly Generated Localization Principle). *Let $\mathcal{C}$ and $\mathcal{D}$ be stable model categories satisfying Hypothesis 3.1.3 and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction. Consider stable sets S and T of morphisms in $\mathcal{C}$ and $\mathcal{D}$ respectively.*

(1) *Suppose that $L_S\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ and that $L_T\mathcal{D}$ is homotopically compactly generated by $FQ\mathcal{K}$. Suppose that the derived unit map $\overline{\eta}_K\colon QK \to G\overline{R}FQK$ is an S-equivalence for all $K \in \mathcal{K}$ and that F sends S-equivalences between cofibrant objects to T-equivalences. Then the induced Quillen adjunction*

$$F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$$

is a Quillen equivalence. Moreover, if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a symmetric monoidal Quillen adjunction and S and T are monoidal, then $F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$ is a symmetric monoidal Quillen equivalence.

(2) *Suppose that $L_T\mathcal{D}$ is homotopically compactly generated by a set $\mathcal{L}$ and that $L_S\mathcal{C}$ is homotopically compactly generated by $G\overline{R}\mathcal{L}$. Suppose that the derived counit $\overline{\epsilon}_L\colon FQG\overline{R}L \to \overline{R}L$ is a T-equivalence for all $L \in \mathcal{L}$ and that F sends S-equivalences between cofibrant objects to T-equivalences. Then the induced Quillen adjunction*

$$F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$$

is a Quillen equivalence. Moreover, if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a symmetric monoidal Quillen adjunction and S and T are monoidal, then $F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$ is a symmetric monoidal Quillen equivalence.

Proof. Apply the Cellularization Principle [**42**, 2.7] to the Quillen adjunction $F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$ obtained from Proposition 3.2.1. □

2.1. Homological localization. We now rephrase the Left Localization Principle for homological Bousfield localizations. This setting provides a large family of examples in which our result simplifies.

Definition 3.2.8. Let $\mathcal{C}$ be a symmetric monoidal model category. We say that an object $E \in \mathcal{C}$ is *flat* if $E \otimes -$ preserves weak equivalences.

Remark 3.2.9. If E is a cofibrant object in a symmetric monoidal model category $\mathcal{C}$, then $E \otimes -$ preserves weak equivalences between cofibrant objects by Ken Brown's lemma. However, in many cases of interest all cofibrant objects are in fact flat:

(i) The cofibrant objects in the projective model structure on dg-modules are the dg-projective modules. Any dg-projective module P has the property that $P \otimes -$ preserves quasiisomorphisms [**3**, 11.1.6, 11.2.1] and so any cofibrant object is flat in the projective model structure on dg-modules.

(ii) Any cofibrant object in the stable model structure on modules over a ring spectrum is flat [**61**, 12.3, 12.7]. Similarly, any cofibrant object in the stable model structure on modules over a ring G-spectrum is flat [**60**, 7.3, 7.7].

Definition 3.2.10. Let $\mathcal{C}$ be a stable and symmetric monoidal model category, and let E be a flat cofibrant object of $\mathcal{C}$. We say that $f\colon X \to Y$ is an *E-equivalence* if $E \otimes f\colon E \otimes X \to E \otimes Y$ is a weak equivalence.

When it exists, localizing at the E-equivalences produces a model structure on $\mathcal{C}$ in which the weak equivalences are the E-equivalences, the cofibrations are unchanged and the fibrant objects are the E-local objects. We call this new model category the *homological localization* of $\mathcal{C}$ at E and write $L_E\mathcal{C}$.

Hypothesis 3.2.11. From now on we assume that the required homological localizations exist.

Remark 3.2.12. The homological localization exists if $\mathcal{C}$ is a stable, symmetric monoidal, proper and compactly generated model category in the sense of [**91**, 1.2.3.4]; see [**28**, Section VIII.1] for the special case of spectra, and [**5**, 6.A] for the general case.

Proposition 3.2.13. *Let $\mathcal{C}$ be a symmetric monoidal model category satisfying Hypothesis 3.2.11, and let E be a flat cofibrant object of $\mathcal{C}$. Then the homological localization $L_E\mathcal{C}$ is a symmetric monoidal model category.*

Proof. Take two cofibrations i and j. Since the cofibrations in $L_E\mathcal{C}$ are the same as in $\mathcal{C}$, the pushout-product map $i\Box j$ is a cofibration since $\mathcal{C}$ satisfies the pushout-product axiom. Now suppose that i is an E-equivalence also. We have that $E \otimes (i\Box j) = (E \otimes i)\Box(E \otimes j)$ since $E \otimes -$ is a left adjoint. The functor $E \otimes -$ is left Quillen since E is cofibrant, so $E \otimes i$ is an acyclic cofibration and $E \otimes j$ is a cofibration. Therefore, $E \otimes (i\Box j)$ is a weak equivalence by the pushout-product axiom for $\mathcal{C}$. In other words, $i\Box j$ is an E-equivalence as required. The unit axiom follows immediately from the unit axiom for $\mathcal{C}$, since the cofibrations are the same in the left Bousfield localization as in the original model category. □

Lemma 3.2.14. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a symmetric monoidal Quillen adjunction between stable symmetric monoidal model categories and let E' be a flat bifibrant object in $\mathcal{D}$. If $\epsilon_E \colon FQGE' \to E'$ is a weak equivalence in $\mathcal{D}$, then F sends QGE'-equivalences between cofibrant objects to E'-equivalences.*

Proof. Let $X \to Y$ be a QGE'-equivalence between cofibrant objects. By Ken Brown's lemma, $F(QGE' \otimes X) \to F(QGE' \otimes Y)$ is a weak equivalence. We have the commutative diagram

$$\begin{tikzcd}
F(QGE' \otimes X) \arrow[r, "\sim"] \arrow[d, "\sim"'] & FQGE' \otimes FX \arrow[r, "\sim"] \arrow[d] & E' \otimes FX \arrow[d] \\
F(QGE' \otimes Y) \arrow[r, "\sim"'] & FQGE' \otimes FY \arrow[r, "\sim"'] & E' \otimes FY
\end{tikzcd}$$

in which the first horizontal maps are equivalences by definition of a symmetric monoidal Quillen pair, and the second horizontal maps are equivalences since $\epsilon_E \colon FQGE' \to E'$ is a weak equivalence and tensoring with a cofibrant object preserves weak equivalences between cofibrants by Ken Brown's lemma. Hence by two-out-of-three, $E' \otimes FX \to E' \otimes FY$ is a weak equivalence as required. □

Recall that the homological localization at an object E is a special case of left Bousfield localization which inverts the E-equivalences. Therefore we can combine this Lemma with the Left Localization Principle to obtain our version for homological localizations.

Theorem 3.2.15 (Left Localization Principle). *Let $\mathcal{C}$ and $\mathcal{D}$ be stable, symmetric monoidal model categories satisfying Hypothesis 3.2.11 and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a symmetric monoidal Quillen adjunction.*

(1) *Suppose that $\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ and that $\mathcal{D}$ is homotopically compactly generated by $FQ\mathcal{K}$. Let $E \in \mathcal{C}$ be a flat cofibrant. Suppose that the following conditions hold:*
   (i) *The derived unit map $QK \to GRFQK$ is an E-equivalence for all $K \in \mathcal{K}$;*
   (ii) *G sends FE-equivalences between fibrant objects in $\mathcal{D}$ to E-equivalences.*
   Then the induced Quillen adjunction
   $$F : L_E\mathcal{C} \rightleftarrows L_{FE}\mathcal{D} : G$$
   is a symmetric monoidal Quillen equivalence.
(2) *Suppose that $\mathcal{D}$ is homotopically compactly generated by a set $\mathcal{L}$ and that $\mathcal{C}$ is homotopically compactly generated by $GR\mathcal{L}$. Let $E' \in \mathcal{D}$ be a flat bifibrant object. Suppose that the following conditions hold:*
   (i) *The derived counit map $FQGRL \to RL$ is a weak equivalence in $\mathcal{D}$ for all $L \in \mathcal{L}$;*
   (ii) *G sends E'-equivalences between fibrant objects in $\mathcal{D}$ to QGE'-equivalences.*

(iii) *The map $FQGE' \to E'$ is a weak equivalence in $\mathcal{D}$;*
Then the induced Quillen adjunction

$$F : L_{QGE'}\mathcal{C} \rightleftarrows L_{E'}\mathcal{D} : G$$

is a symmetric monoidal Quillen equivalence.

We now give a mixing of the Left Localization Principle and the Cellularization Principle. Note that we again write $\overline{R}$ for a fibrant replacement in the Bousfield localization.

Theorem 3.2.16 (Compactly Generated Localization Principle). *Let $\mathcal{C}$ and $\mathcal{D}$ be stable, symmetric monoidal model categories satisfying Hypothesis 3.2.11 and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a symmetric monoidal Quillen adjunction.*

(1) *Let E be a flat cofibrant object of $\mathcal{C}$. Suppose that $L_E\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ and that $L_{FE}\mathcal{D}$ is homotopically compactly generated by $FQ\mathcal{K}$. If the derived unit map $Q\overline{\eta}_K : K \to G\overline{R}FQK$ is an E-equivalence for all $K \in \mathcal{K}$ then the induced Quillen adjunction*

$$F : L_E\mathcal{C} \rightleftarrows L_{FE}\mathcal{D} : G$$

is a symmetric monoidal Quillen equivalence.

(2) *Let E' be a flat bifibrant object of $\mathcal{D}$. Suppose that $L_{E'}\mathcal{D}$ is homotopically compactly generated by a set $\mathcal{L}$ and that $L_{QGE'}\mathcal{C}$ is homotopically compactly generated by $G\overline{R}\mathcal{L}$. Suppose that the derived counit $\overline{\epsilon}_L : FQG\overline{R}L \to \overline{R}L$ is an E'-equivalence for all $L \in \mathcal{L}$ and that $FQGE' \to E'$ is a weak equivalence in $\mathcal{D}$. Then the induced Quillen adjunction*

$$F : L_{QGE'}\mathcal{C} \rightleftarrows L_{E'}\mathcal{D} : G$$

is a symmetric monoidal Quillen equivalence.

Remark 3.2.17. Barnes-Roitzheim have compared left and right Bousfield localizations of stable model categories at dualizable objects [**12**, 9.6]. More precisely, they proved that the identity functors

$$L_A\mathcal{C} \leftrightarrows \mathrm{Cell}_{DA}\mathcal{C}$$

give a Quillen equivalence, where $D = F(-, \mathbb{1})$ is the dual functor and A is dualizable. Accordingly, in some cases the Left Localization Principle can be replaced by the Cellularization Principle and vice versa. However, there are some subtleties that need to be considered. Firstly, the two principles are "exchangeable" only if the functors interact well with taking duals and we localize at dualizable objects. This is a big disadvantage for instance in global stable homotopy theory where almost no compact objects are dualizable. This was one of the main motivations of the authors to develop the Left Localization Principle. Secondly, the two principles have quite different behaviour when we take into account the symmetric monoidal structure. While the Left Localization Principle for homological localization automatically yields a monoidal Quillen equivalence, the Cellularization Principle requires strong conditions, in particular when passing cells along the right adjoint, see [**11**, 5.1.7].

Remark 3.2.18. If we want to apply the Compactly Generated Localization Principle we need to know that the category of local objects is compactly generated. This holds for instance, when we localize at dualizable objects. More precisely, let $\mathcal{C}$ be a stable, symmetric monoidal model category, and let A be a dualizable object of $\mathcal{C}$. It is not difficult to see that if $\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ then the homological localization $L_A\mathcal{C}$ is homotopically compactly generated by $DA \otimes \mathcal{K}$. Firstly, $DA \otimes K$ is A-local for all $K \in \mathcal{K}$ since if $A \otimes Z \simeq 0$, then $h\mathcal{C}(Z, DA \otimes K) = h\mathcal{C}(Z, F(A, K)) = h\mathcal{C}(Z \otimes A, K) = 0$. Compactness follows from the fact that $A \otimes - : hL_A\mathcal{C} \to h\mathcal{C}$ preserves colimits, and the generation is an immediate consequence of the duality adjunction. For more details, see for instance [**62**, 2.27].

3. Completion of module categories

In this section we apply the Left Localization Principle to obtain symmetric monoidal Quillen equivalences relating a ring to its completion. We provide a general statement and then give several concrete examples of interest.

NOTATION 3.3.1. Given a commutative monoid R in a symmetric monoidal model category $\mathcal{C}$, we denote by $\mathrm{Mod}_R(\mathcal{C})$ the category of R-modules equipped with the projective model structure (if it exists) in which the weak equivalences and fibrations are created by the forgetful functor $\mathrm{Mod}_R(\mathcal{C}) \to \mathcal{C}$. If the underlying category is clear, we will often write Mod_R.

HYPOTHESIS 3.3.2. Throughout this paper we assume that the projective model structure on $\mathrm{Mod}_R(\mathcal{C})$ exists and that it is left proper, so that left Bousfield localizations exist.

REMARK 3.3.3. Note that the projective model structure exists if $\mathcal{C}$ satisfies the monoid axiom [**82**, 4.1], and it is left proper in many cases: for instance in categories of (equivariant) spectra [**61**, 12.1(i)] and [**60**, III.7.6], and in dg-modules [**15**, 3.3].

PROPOSITION 3.3.4. *Let $\mathcal{C}$ be a stable, symmetric monoidal model category, homotopically compactly generated by a set $\mathcal{K}$. Let E be a flat cofibrant R-module, $\theta\colon R \to S$ be a map of commutative monoids in $\mathcal{C}$, and suppose that $\theta\colon R \to S$ is an E-equivalence. The map θ induces a symmetric monoidal extension-restriction of scalars Quillen adjunction*

$$S \otimes_R - : \mathrm{Mod}_R(\mathcal{C}) \rightleftarrows \mathrm{Mod}_S(\mathcal{C}) : \theta^*$$

between the categories of modules. Then the Left Localization Principle applies and gives a symmetric monoidal Quillen equivalence

$$L_E\mathrm{Mod}_R(\mathcal{C}) \simeq_Q L_E\mathrm{Mod}_S(\mathcal{C}).$$

REMARK 3.3.5. Note that there is an abuse of notation in the Proposition above since in general there is no natural S-module structure on E at the model category level. More precisely, on the right hand side of the Quillen equivalence above we should have localized at $S \otimes_R E$ instead of E. However, this abuse of notation does no harm since there is a natural weak equivalence $E \xrightarrow{\sim} S \otimes_R E$ in $\mathcal{C}$ and the class of $S \otimes_R E$-equivalences is detected in the homotopy category of $\mathcal{C}$.

PROOF. Without loss of generality we may assume that $\mathcal{K}$ consists of cofibrant objects. The set $R \otimes \mathcal{K}$ provides a set of compact generators for $h\mathrm{Mod}_R(\mathcal{C})$. The left adjoint is strong monoidal and maps compact generators to compact generators since $S \otimes_R (R \otimes \mathcal{K}) \cong S \otimes \mathcal{K}$.

As $R \to S$ is an E-equivalence, we obtain a weak equivalence $E \xrightarrow{\sim} S \otimes_R E$ by tensoring with E. Therefore, the derived unit $R \otimes K \to S \otimes_R (R \otimes K) = S \otimes K$ is an E-equivalence for all $K \in \mathcal{K}$.

Finally we must show that the right adjoint θ^* preserves E-equivalences between fibrant objects. Note that there is a natural map $E \otimes_R \theta^* M \to \theta^*(E \otimes_S M)$ of R-modules, which is a weak equivalence as $E \simeq S \otimes_R E$. Now suppose that $M \to N$ is an E-equivalence between fibrant S-modules. By considering the diagram

$$\begin{tikzcd} E \otimes_R \theta^* M \arrow[r] \arrow[d, "\sim"'] & E \otimes_R \theta^* N \arrow[d, "\sim"] \\ \theta^*(E \otimes_S M) \arrow[r] & \theta^*(E \otimes_S N) \end{tikzcd}$$

we see that $\theta^* M \to \theta^* N$ is an E-equivalence of R-modules. $\square$

EXAMPLE 3.3.6. Let $\mathbb{Z}_p$ denote the p-adic integers and consider the ring map $\theta\colon \mathbb{Z} \to \mathbb{Z}_p$ which induces a symmetric monoidal Quillen adjunction between the categories of chain complexes

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} - : \mathrm{Mod}_{\mathbb{Z}} \rightleftarrows \mathrm{Mod}_{\mathbb{Z}_p} : \theta^*$$

via extension and restriction of scalars. We can apply Proposition 3.3.4 to obtain a symmetric monoidal Quillen equivalence

$$L_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}} \simeq_Q L_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}_p}.$$

By [**39**, 4.2], we can identify the homotopy categories of the two localizations with the subcategories of the derived categories consisting of derived p-complete modules which we denote $\Lambda_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}}$ and $\Lambda_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}_p}$ respectively. Putting everything together we get a tensor-triangulated equivalence

$$\Lambda_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}} \simeq_\triangle \Lambda_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}_p}.$$

EXAMPLE 3.3.7. Let G be a compact Lie group and $\mathcal{F}$ a family of subgroups of G. Note that the G-spectrum $DE\mathcal{F}_+$ is a commutative ring G-spectrum via the diagonal map $\Delta\colon E\mathcal{F}_+ \to E\mathcal{F}_+ \wedge E\mathcal{F}_+$. It is easy to check that the unit map $\eta\colon S^0 \to DE\mathcal{F}_+$ is a $E\mathcal{F}_+$-equivalence. We can then apply Proposition 3.3.4 to obtain a symmetric monoidal Quillen equivalence

$$L_{E\mathcal{F}_+}\mathrm{Sp}_G \simeq_Q L_{E\mathcal{F}_+}\mathrm{Mod}_{DE\mathcal{F}_+}(\mathrm{Sp}_G).$$

Note that the proof of Proposition 3.3.4 works more generally for localizations at a set of maps S.

EXAMPLE 3.3.8. Let $\mathcal{G}$ be the global family of compact Lie groups. Denote by $\mathrm{Sp}_{\mathcal{G}}$ the category of orthogonal spectra with the $\mathcal{G}$-global model structure which is proper [**81**, 4.3.17]. By [**81**, 4.5.21, 4.5.22(ii)], there exists a morphism of ultracommutative ring spectra $i_{\mathbb{S}}\colon \mathbb{S} \to \mathrm{b}\mathbb{S}$ between the global sphere spectrum and the global Borel construction which exhibits $\mathrm{b}\mathbb{S}$ as a localization of the global sphere spectrum at the class of non-equivariant equivalences. Note that the projective model structure on $\mathrm{Mod}_{\mathrm{b}\mathbb{S}}(\mathrm{Sp}_{\mathcal{G}})$ exists by [**81**, 4.3.29] and it is proper by a similar argument as in [**61**, 12.1(i)] so that we can perform left Bousfield localizations. Following the proof of Proposition 3.3.4 and localizing at the class 1 of non-equivariant equivalences (see Remark 3.3.9 for justification of its existence), we obtain a symmetric monoidal Quillen equivalence

$$L_1\mathrm{Sp}_{\mathcal{G}} \simeq_Q L_1\mathrm{Mod}_{\mathrm{b}\mathbb{S}}(\mathrm{Sp}_{\mathcal{G}}).$$

We note that this is a symmetric monoidal Quillen equivalence using Remark 3.2.3, since the model structure on $\mathrm{Mod}_{\mathrm{b}\mathbb{S}}(\mathrm{Sp}_{\mathcal{G}})$ is right induced from the $\mathcal{G}$-global model structure on $\mathrm{Sp}_{\mathcal{G}}$. Finally using the language of [**81**] we can identify the homotopy category of $L_1\mathrm{Sp}_{\mathcal{G}}$ with the full subcategory of the global stable homotopy category consisting of those global spectra which are right induced from the trivial family.

REMARK 3.3.9. It is not immediate that the left Bousfield localization of $\mathrm{Sp}_{\mathcal{G}}$ at the *class* of non-equivariant equivalences actually exists. This localization cannot be constructed as a homological localization since in global stable homotopy theory an analogue of the free G-space EG does not exist. Instead we apply Bousfield-Friedlander localization [**20**, 9.3] to the natural transformation $i_X\colon X \to \mathrm{b}X$ which is a non-equivariant equivalence. By construction, the global Borel functor b has the property that for all $G \in \mathcal{G}$, the underlying G-spectrum of $\mathrm{b}X$ is cofree, see [**81**, 4.5.16, 4.5.22]. In particular this shows that $f\colon X \to Y$ is a non-equivariant equivalence if and only if $\mathrm{b}f\colon \mathrm{b}X \to \mathrm{b}Y$ is a global equivalence. The conditions (A1) and (A2) from [**20**, 9.2] easily follow from this observation. The final condition (A3) follows from the right properness of $\mathrm{Sp}_{\mathcal{F}}$ for the trivial family $\mathcal{F} = \{1\}$, together with the fact that any $\mathcal{G}$-global fibration is a $\mathcal{F}$-global fibration. The argument for $\mathrm{Mod}_{\mathrm{b}\mathbb{S}}(\mathrm{Sp}_{\mathcal{G}})$ is similar as the weak equivalences and fibrations are created by the forgetful functor $\mathrm{Mod}_{\mathrm{b}\mathbb{S}}(\mathrm{Sp}_{\mathcal{G}}) \to \mathrm{Sp}_{\mathcal{G}}$.

CHAPTER 4

Rational cofree G-spectra

We give a symmetric monoidal algebraic model for the category of rational cofree G-spectra for G a compact Lie group, in the sense of [**34**]. We will initially prove the result for G connected and then show how to extend our proofs to any compact Lie group.

1. Completions in algebra

We now recall some results about complete modules following [**38**].

Let R be a graded commutative ring and let I be a finitely generated homogeneous ideal. The I-adic completion of a module M is defined by

$$M_I^\wedge = \lim_n M/I^n M.$$

We say that a module M is *I-adically complete* if the natural map $M \to M_I^\wedge$ is an isomorphism. A dg-module is said to be *I-adically complete* if its underlying graded module is.

Since the I-adic completion functor is neither left nor right exact in general, our approach is to consider the zeroth left derived functor L_0^I of I-adic completion as the 'correct' notion.

Definition 4.1.1.

- We say that a module M is *L_0^I-complete* if the natural map $M \to L_0^I M$ is an isomorphism.
- We say that a dg-module N is *L_0^I-complete* if its underlying graded module is L_0^I-complete.

We write Mod_R for the category of dg-R-modules, and $\mathrm{Mod}_R^\wedge$ for the full subcategory of L_0^I-complete dg-modules. We denote the internal hom of dg-R-modules by $\underline{\mathrm{Hom}}_R(-,-)$.

Lemma 4.1.2.

(a) *The category* $\mathrm{Mod}_R^\wedge$ *is abelian, and the inclusion functor* $i\colon \mathrm{Mod}_R^\wedge \to \mathrm{Mod}_R$ *is exact. In particular, the homology of an L_0^I-complete dg-module is L_0^I-complete.*
(b) *The inclusion functor is right adjoint to the L-completion functor L_0^I.*
(c) *The category* $\mathrm{Mod}_R^\wedge$ *has all limits and colimits.*

Proof. The proofs of (a) and (b) can be found in [**54**, A.6(e), A.6(f)]. Their proofs depend only upon the fact that L_0^I is right exact and the existence of a long exact sequence of derived functors. Therefore, the restriction to local rings and regular ideals made in [**54**] does not affect the stated results. It follows from (b) that limits of L_0^I-complete modules are calculated in Mod_R, and that colimits of L_0^I-complete modules are calculated by L_0^I-completion of the colimit in Mod_R. □

Proposition 4.1.3.

(a) *If N is L_0^I-complete, then $\underline{\mathrm{Hom}}_R(M,N)$ is L_0^I-complete.*
(b) *The category* $\mathrm{Mod}_R^\wedge$ *is closed symmetric monoidal with tensor product $L_0^I(M \otimes_R N)$ and internal hom $\underline{\mathrm{Hom}}_R(M,N)$.*

Proof. By taking a free presentation $R^{J_1} \to R^{J_0} \to M \to 0$, we obtain an exact sequence

$$0 \to \underline{\mathrm{Hom}}_R(M,N) \to \prod_{J_0} N \to \prod_{J_1} N$$

which proves (a), since L_0^I-complete modules are closed under products and kernels.

For (b) we follow the argument of Rezk [**73**, 6.2]. We first prove that the map $L_0^I(M \otimes_R N) \to L_0^I(L_0^I M \otimes_R N)$ induced by $\eta_M\colon M \to L_0^I M$ is an isomorphism. It is enough to check that for any L_0^I-complete module C, the map

$$\underline{\mathrm{Hom}}_R(L_0^I(L_0^I M \otimes_R N), C) \to \underline{\mathrm{Hom}}_R(L_0^I(M \otimes_R N), C)$$

is an isomorphism. By adjunction, it is an isomorphism if and only if the induced map

$$\underline{\mathrm{Hom}}_R(L_0^I M, \underline{\mathrm{Hom}}_R(N,C)) \to \underline{\mathrm{Hom}}_R(M, \underline{\mathrm{Hom}}_R(N,C))$$

is. This now follows as $\underline{\mathrm{Hom}}_R(N,C)$ is L_0^I-complete by part (a). Therefore $L_0^I(M \otimes_R N) \to L_0^I(L_0^I M \otimes_R N)$ is an isomorphism. By symmetry, we also have that $L_0^I(M \otimes_R N) \to L_0^I(M \otimes_R L_0^I N)$ is an isomorphism, and therefore so is $L_0^I(M \otimes_R N) \to L_0^I(L_0^I M \otimes_R L_0^I N)$. This completes the proof of (b). □

We will also be concerned with a homotopical version of completion that we shall now recall. For any $x \in R$, we define the *unstable Koszul complex*

$$K(x) = \mathrm{fib}(\Sigma^{|x|} R \xrightarrow{\cdot x} R),$$

and the *stable Koszul complex*

$$K_\infty(x) = \mathrm{fib}(R \to R[1/x])$$

where the fibre is taken in the category of dg-modules. For an ideal $I = (x_1, \ldots, x_n)$ we put

$$K(I) = K(x_1) \otimes_R \cdots \otimes_R K(x_n) \quad \text{and} \quad K_\infty(I) = K_\infty(x_1) \otimes_R \cdots \otimes_R K_\infty(x_n).$$

If no confusion is likely to arise, we suppress notation for the ideal and write K for the unstable Koszul complex and K_∞ for the stable Koszul complex. We will also write $\mathrm{Hom}_R(-,-)$ for the derived internal hom functor. We say that a dg-module M is *derived complete* if the natural map $M \to \mathrm{Hom}_R(K_\infty, M) =: \Lambda_I M$ is a quasi-isomorphism. Then the *nth local homology* of M is defined to be $H_n^I(M) = H_n(\Lambda_I M)$.

Definition 4.1.4. Let $I = (x_1, \ldots, x_n)$ be a finitely generated homogeneous ideal. For all $s \in \mathbb{N}$ and $x \in R$, we put

$$K_s(x) = \mathrm{fib}(\Sigma^{s|x|} R \xrightarrow{\cdot x^s} R) \quad \text{and} \quad K_s(I) = K_s(x_1) \otimes_R \cdots \otimes_R K_s(x_n).$$

We say that I is generated by the *weakly pro-regular* sequence $(x_1, \ldots, x_n)$ if the inverse system $(H_k(K_s(I)))_s$ is pro-zero for all $k \neq 0$. That is, for each $s \in \mathbb{N}$ there is $m \geq s$ such that the natural map

$$H_k(K_m(I)) \to H_k(K_s(I))$$

is zero.

Note that if R is Noetherian then any finitely generated ideal is weakly pro-regular [**69**, 4.34]. Indeed this is true even when weakly pro-regular is replaced by pro-regular [**38**].

Theorem 4.1.5. *Let R be a graded commutative ring and let I be a finitely generated homogeneous ideal that is generated by a weakly pro-regular sequence. Then for all dg-modules M, there is a natural quasi-isomorphism*

$$\mathrm{tel}_{I,M}^L\colon \Lambda_I(M) \xrightarrow{\sim} \mathbb{L}(-)_I^\wedge(M)$$

between the derived completion functor and the total left derived functor of I-adic completion (calculated using dg-projective resolutions), making the diagram

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ \Lambda_I M \arrow[rr, "\mathrm{tel}^L_{I,M}"'] & & \mathbb{L}(-)^\wedge_I(M) \end{tikzcd}$$

commute.

Proof. Greenlees-May proved that if R has bounded torsion and I is pro-regular then $H^I_* M \cong L^I_* M$, see [**38**, 2.5]. Schenzel [**77**, 1.1] proved the above result for ideals generated by weakly pro-regular sequences and bounded complexes with R bounded torsion. Finally, Porta-Shaul-Yekutieli [**69**, 5.25] removed the hypothesis that R has bounded torsion and extended the result to unbounded complexes. □

2. An abelian model for derived completion

In this section we use the language of model categories to show that the category of L^I_0-complete modules forms an abelian model for derived complete modules, see Theorem 4.2.11. Our result can be thought as "dual" to the fact that I-power torsion modules forms an abelian model for derived torsion modules [**42**, Secion 5]. We will be working under the following:

Hypothesis 4.2.1. We will assume our ideal I to be generated by a weakly pro-regular sequence and we continue to write K for its associated unstable Koszul complex.

Remark 4.2.2. For a dg-module M, we write $L^I_n M$ for the dg-module obtained from M by applying levelwise the functor L^I_n. This is in line with Definition 4.1.1. This should not be confused with the n-left derived functor of I-adic completion which will not play any role in this paper.

In order to prove our main result we need to consider model structures on the categories of interest. Recall that the category of dg-modules Mod_R has a projective model structure in which the weak equivalences are the quasi-isomorphisms, the fibrations are the epimorphisms and the cofibrations are the monomorphisms which have dg-projective cokernel and are split on the underlying graded modules, see [**15**, 3.3] and [**1**, 3.15]. A dg-module M is said to be *dg-projective* if $\mathrm{Hom}_R(P, -)$ preserves surjective quasi-isomorphisms. It is important to note that any dg-projective module is (graded) projective, but the converse is not generally true, see [**3**, 9.6.1].

Lemma 4.2.3. *If P is dg-projective, then $L^I_n P = 0$ for all $n \geq 1$. Moreover, there is a natural quasi-isomorphism $\Lambda_I P \xrightarrow{\sim} L^I_0 P = P^\wedge_I$.*

Proof. This is the trivial case of Theorem 4.1.5. □

We will now put a projective model structure on L^I_0-complete modules following Rezk's unpublished note [**73**, 10.2].

Lemma 4.2.4.

(a) *The functor L^I_0 takes cofibrations in Mod_R to morphisms which have the left lifting property with respect to surjective quasi-isomorphisms of L^I_0-complete modules.*
(b) *The functor L^I_0 takes acyclic cofibrations in Mod_R to morphisms which have the left lifting property with respect to surjections of L^I_0-complete modules.*
(c) *If $M \to N$ is a cofibration in Mod_R, the homology $H_* N$ is L^I_0-complete and $M \to L^I_0 M$ is a quasi-isomorphism, then $N \to L^I_0 N$ is a quasi-isomorphism.*

Proof. Part (a) and (b) follow from the lifting properties in Mod_R. For part (c), note that by definition $M \to N$ is an injection with dg-projective cokernel P so we have a diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "\sim"'] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & L_0^I M \arrow[r] & L_0^I N \arrow[r] & L_0^I P \arrow[r] & 0
\end{tikzcd}$$

in which the top row is exact. By Lemma 4.2.3 we have that $L_i^I P = 0$ for $i \geq 1$ so the long exact sequence of derived functors collapses to a short exact sequence. Therefore, the bottom row is exact too. Since $L_0^I M$ is L_0^I-complete, the homology $H_*M \cong H_*L_0^I M$ is L_0^I-complete by Lemma 4.1.2(a), and so H_*P is L_0^I-complete too. Now consider the spectral sequence [**39**, 3.3]

$$E^2_{p,q} = (L_p^I(H_*P))_q \implies H_{p+q}(\Lambda_I P).$$

If the homology groups are L_0^I-complete, then the spectral sequence collapses by [**38**, 4.1] to give a quasi-isomorphism $P \to \Lambda_I P$. Therefore $P \to L_0^I P$ is a quasi-isomorphism by Lemma 4.2.3. Hence $N \to L_0^I N$ is a quasi-isomorphism as required. $\square$

Proposition 4.2.5. *There is a model structure on* $\mathrm{Mod}_R^\wedge$ *in which the weak equivalences are the quasi-isomorphisms, the fibrations are the surjections, and the cofibrations are the maps with the left lifting property with respect to the acyclic fibrations. Furthermore, the adjunction*

$$L_0^I : \mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_R^\wedge : i$$

is Quillen.

Proof. The only parts that need elaboration are the factorization axiom and the lifting axiom. Firstly we prove the factorization axiom.

Let $f\colon M \to N$ in $\mathrm{Mod}_R^\wedge$. Take a factorization $M \xrightarrow{i} D \xrightarrow{p} N$ in Mod_R where one of i or p is acyclic. Since L_0^I is left adjoint to the inclusion, maps $L_0^I D \to N$ are in bijection with maps $D \to N$. Therefore, there is a unique $q\colon L_0^I D \to N$ making the square

$$\begin{tikzcd}
M \arrow[r] \arrow[d, "i"'] & L_0^I D \arrow[d, "q"] \\
D \arrow[r, "p"'] \arrow[ur, "\alpha"] & N
\end{tikzcd}$$

commute. Note that q is a fibration since $q \cong L_0^I p$ and L_0^I preserves surjections.

If p is acyclic, Lemma 4.2.4(c) shows that α is a quasi-isomorphism since $H_*D \cong H_*N$, and hence by the two-of-three property, q is a weak equivalence. Lemma 4.2.4(a) shows that the factorization $f = q(\alpha i)$ is a factorization into a map with the left lifting property with respect to acyclic fibrations, followed by an acylic fibration. This completes the first part of the proof of the factorization axiom.

For the other part we suppose that i is a weak equivalence. Since $\alpha i \cong L_0^I(i)$, Lemma 4.2.4(b) shows that αi has the left lifting property with respect to fibrations in $\mathrm{Mod}_R^\wedge$. Lemma 4.2.4(c) shows that α is a quasi-isomorphism since $H_*D \cong H_*M$. Therefore $f = q(\alpha i)$ is a factorization into a weak equivalence with the left lifting property with respect to fibrations followed by an fibration, which completes the proof of the factorization axiom.

For the lifting axiom, we note that one part is by definition. For the other part, we use the standard method of the retract argument. Consider the square

$$\begin{array}{ccc} A & \longrightarrow & X \\ {\scriptstyle i}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ B & \longrightarrow & Y \end{array}$$

in which i is an acyclic cofibration and f is a fibration. Factor i into a map with the left lifting property with respect to fibrations followed by a fibration to give $A \xrightarrow{j} C \xrightarrow{p} B$. Since j has the left lifting property with respect to fibrations, there is a lift $g\colon C \to X$.

As i and j are weak equivalences, p is an acyclic fibration. Since i has the left lifting property with respect to acyclic fibrations, there exists a lift $h\colon B \to C$. Therefore $gh\colon B \to X$ gives the required lift in the square.

It is clear that the adjunction is Quillen by the definition of the weak equivalences and fibrations. □

Remark 4.2.6. One might first think of attempting to prove the existence of this model structure via right inducing it from Mod_R. However, in order to be able to do this, we need to know that the inclusion $i\colon \mathrm{Mod}_R^\wedge \to \mathrm{Mod}_R$ preserves filtered colimits. This is false; take $R = \mathbb{Z}$ and $I = (p)$ and consider the direct system $\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \dots$. Then the colimit in the category of abelian groups is $\mathbb{Q}_p$, while the colimit in the category of L_0^I-complete abelian groups is $L_0^{(p)}(\mathbb{Q}_p)$ which is zero.

Proposition 4.2.7. *The model structure on* $\mathrm{Mod}_R^\wedge$ *is symmetric monoidal.*

Proof. The category of L_0^I-complete modules is closed symmetric monoidal with tensor product given by $L_0^I(M \otimes N)$; see Proposition 4.1.3.

Let $M \to N$ and $X \to Y$ be fibrations in $\mathrm{Mod}_R^\wedge$. Since the inclusion $i\colon \mathrm{Mod}_R^\wedge \to \mathrm{Mod}_R$ preserves limits, we have that the pullback product map is

$$\underline{\mathrm{Hom}}_R(iN, iX) \to \underline{\mathrm{Hom}}_R(iM, iX) \times_{\underline{\mathrm{Hom}}_R(iM,iY)} \underline{\mathrm{Hom}}_R(iN, iY).$$

Since Mod_R is a symmetric monoidal model category and i is right Quillen, the pullback product map is a fibration. A similar proof shows that the pullback product of a fibration with an acyclic fibration is an acyclic fibration. The unit axiom is immediate since the unit in $\mathrm{Mod}_R^\wedge$ is $L_0^I R$ which is cofibrant as R is cofibrant in Mod_R. □

We need a model category modelling the homotopy theory of derived complete dg-modules. The left Bousfield localization of R-modules at the unstable Koszul complex is such a model category by the following result.

Lemma 4.2.8 ([**39**, 4.2]). *There is an equivalence of categories*

$$\mathrm{h}L_K\mathrm{Mod}_R \simeq \Lambda\mathrm{Mod}_R$$

where $\Lambda\mathrm{Mod}_R$ *denotes the full subcategory of the derived category of dg-modules consisting of derived complete dg-modules.*

We relate the model category of L_0^I-complete modules to derived complete modules. We will use these results to show that cofree G-spectra have an abelian model in terms of L_0^I-complete modules.

Lemma 4.2.9. *There is a symmetric monoidal Quillen adjunction*

$$L_0^I : L_K(\mathrm{Mod}_R) \rightleftarrows \mathrm{Mod}_R^\wedge : i.$$

Proof. The cofibrations in $L_K\mathrm{Mod}_R$ are the same as the cofibrations in Mod_R so they are preserved since $L_0^I\colon \mathrm{Mod}_R \to \mathrm{Mod}_R^\wedge$ is left Quillen. Now suppose that $f\colon M \to N$ is an acyclic cofibration in $L_K(\mathrm{Mod}_R)$ so that the cokernel C is dg-projective. In particular, $K \otimes C$ and $\mathrm{Hom}_R(K, C)$ are acyclic as K is self-dual up to suspension. We also know that K_∞ is built from

K so $\Lambda_I C = \mathrm{Hom}_R(K_\infty, C)$ is acyclic as well. By Lemma 4.2.3, we have $\Lambda_I C \simeq L_0^I C$ and so $L_0^I M \to L_0^I N$ is a quasi-isomorphism. This is a symmetric monoidal Quillen adjunction since L_0^I is strong monoidal by Lemma 4.1.2, and the unit in $L_K(\mathrm{Mod}_R)$ is cofibrant. $\square$

Before we can prove that the above Quillen adjunction is actually a Quillen equivalence, we need the following:

LEMMA 4.2.10. *For any dg-module M, the natural map $K \otimes M \to \Lambda_I(K \otimes M)$ is a quasi-isomorphism.*

PROOF. There is a fibre sequence $K_\infty \to R \to \check{C}R$ where $\check{C}R = \Sigma\ker(K_\infty \to R)$ is the Čech complex. This gives rise to another fibre sequence

$$\mathrm{Hom}_R(K_\infty, N) \leftarrow N \leftarrow \mathrm{Hom}_R(\check{C}R, N)$$

for any dg-module N. Now let $I = (x_1, \ldots, x_n)$. Note that $\check{C}R$ is finitely built from $R[\frac{1}{x_i}]$ and that the multiplication map $x_i \colon K \to K$ is null-homotopic. Thus $\mathrm{Hom}_R(\check{C}R, K \otimes M) \simeq 0$ and $K \otimes M$ is derived complete. $\square$

We can now prove that L_0^I-complete modules are a model for derived complete modules.

THEOREM 4.2.11. *There is a symmetric monoidal Quillen equivalence*

$$L_0^I : L_K(\mathrm{Mod}_R) \rightleftarrows \mathrm{Mod}_R^\wedge : i.$$

PROOF. We now show that this Quillen adjunction is in fact a Quillen equivalence. Let P be cofibrant (i.e., dg-projective) in $L_K(\mathrm{Mod}_R)$ and M be fibrant in the category of L_0^I-complete R-modules. We must show that $L_0^I P \to M$ is a quasi-isomorphism if and only if $K \otimes P \to K \otimes M$ is a quasi-isomorphism.

Firstly, if $L_0^I P \to M$ is a quasi-isomorphism, then $K \otimes L_0^I P \to K \otimes M$ is a quasi-isomorphism since K is homotopically flat. Now note that there is a weak equivalence $K \otimes \Lambda_I P \xrightarrow{\sim} \Lambda_I(K \otimes P)$ since K is small. By Lemma 4.2.3, $K \otimes L_0^I P \simeq \Lambda_I(K \otimes P)$ as P is projective. Hence $K \otimes L_0^I P \simeq K \otimes P$ by Lemma 4.2.10. We conclude that $K \otimes P \to K \otimes M$ is a quasi-isomorphism as required.

Conversely, if $K \otimes P \to K \otimes M$ is a quasi-isomorphism then $\mathrm{Hom}_R(K, P) \to \mathrm{Hom}_R(K, M)$ is too since K is self-dual up to suspension. Since K_∞ is built from K, we also deduce $\mathrm{Hom}_R(K_\infty, P) \to \mathrm{Hom}_R(K_\infty, M)$ is a quasi-isomorphism. It follows that $\Lambda_I P \to \Lambda_I M$ is a quasi-isomorphism. By Lemma 4.2.3, we have $L_0^I P \simeq \Lambda_I P$ and $M \simeq \Lambda_I M$. Hence $L_0^I P \to M$ is a quasi-isomorphism. $\square$

As a consequence we obtain the following corollary which extends [**27**, 6.15] to non-Noetherian rings.

COROLLARY A. *A dg-module M is derived complete if and only if its homology H_*M is L_0^I-complete.*

PROOF. Let M be derived complete. By Theorem 4.2.11, M is quasi-isomorphic to its L_0^I-completion $L_0^I M$. As the homology of an L_0^I-complete object is still L_0^I-complete by Lemma 4.1.2, we deduce that M has L_0^I-complete homology. Conversely, suppose that M is a module with L_0^I-complete homology. The spectral sequence [**39**, 3.3]

$$E^2_{p,q} = (L_p^I H_* M)_q \implies H_{p+q}(\Lambda_I M)$$

collapses by [**38**, 4.1], showing that $M \to \Lambda_I M$ is a quasi-isomorphism. $\square$

3. The category of rational cofree G-spectra

From now on we will be working rationally. This means that all spectra are rationalized without comment and all homology and cohomology theories will be unreduced and with rational coefficients.

Notation 4.3.1. Fix G a compact Lie group. We denote by Sp_G the model category of rational orthogonal G-spectra with the rational G-stable model structure, which is a compactly generated, stable, symmetric monoidal model category, see [**60**, III.7.6]. We write $\wedge$ for the tensor product and $F(-,-)$ for the internal hom functor. We also write hSp_G for its associated homotopy category.

Definition 4.3.2. A G-spectrum X is said to be *cofree* if the natural map $X \to F(EG_+, X)$ is an isomorphism in the homotopy category. We denote by $\mathrm{hSp}_G^{\mathrm{cofree}}$ the full subcategory of hSp_G of cofree G-spectra.

Lemma 4.3.3. *There is a natural equivalence*

$$\mathrm{h}L_{EG_+}\mathrm{Sp}_G \simeq \mathrm{hSp}_G^{\mathrm{cofree}}.$$

Furthermore, $L_{EG_+}\mathrm{Sp}_G$ is a symmetric monoidal model category.

Proof. A fibrant replacement functor in $L_{EG_+}\mathrm{Sp}_G$ is given by $F(EG_+, R(-))$ where R is the fibrant replacement in Sp_G. Therefore, the collection of bifibrant objects in $L_{EG_+}\mathrm{Sp}_G$ is equivalent to the full subcategory of cofree G-spectra. The model category $L_{EG_+}\mathrm{Sp}_G$ is symmetric monoidal by Proposition 3.2.13. $\square$

4. The symmetric monoidal equivalence: connected case

In this section we fix a connected compact Lie group G. We aim to find an algebraic model for the category of rational cofree G-spectra. There are several steps needed. Recall that our model for cofree G-spectra is the homological localization $L_{EG_+}\mathrm{Sp}_G$.

Step 1. Consider the complex stable commutative ring G-spectrum $DEG_+ = F(EG_+, S^0)$, see Definition 0.3.12. Restriction and extension of scalars along the unit map $S^0 \to DEG_+$ induces a symmetric monoidal Quillen adjunction

$$DEG_+ \wedge - : L_{EG_+}(\mathrm{Sp}_G) \rightleftarrows L_{EG_+}(\mathrm{Mod}_{DEG_+}) : U$$

between the localizations, since $DEG_+ \wedge EG_+ \simeq EG_+$. By the Left Localization Principle this is a symmetric monoidal Quillen equivalence, since the unit is an EG_+-equivalence and U preserves non-equivariant equivalences.

Remark 4.4.1. This is a special case of Proposition 3.3.4 and Example 3.3.7.

Step 2. We can now take categorical fixed points to remove equivariance. As a functor from G-spectra to non-equivariant spectra, the categorical fixed points is right adjoint to the inflation functor. Using [**83**, 3.3] we have a symmetric monoidal Quillen adjunction

$$(-)^G : \mathrm{Mod}_{DEG_+} \leftrightarrows \mathrm{Mod}_{DBG_+} : DEG_+ \otimes_{DBG_+} -$$

between the categories of modules. Note that we suppress notation for the inflation functor. A more detailed discussion of this adjunction can be found in [**44**].

Since G is connected, DEG_+ generates Mod_{DEG_+} by [**37**, 3.1] and so the counit is an equivalence on all objects as it is an equivalence on DEG_+ and the fixed points functor preserves sums. By [**37**, 3.3], the fixed points functor sends non-equivariant equivalences to BG_+-equivalences, so the Left Localization Principle applies and we get a symmetric monoidal Quillen equivalence

$$(-)^G : L_{EG_+}\mathrm{Mod}_{DEG_+} \leftrightarrows L_{BG_+}\mathrm{Mod}_{DBG_+} : DEG_+ \otimes_{DBG_+} -.$$

Step 3. We now apply Shipley's theorem [**86**, 2.15] (see also [**94**, 7.2]) which gives a symmetric monoidal Quillen equivalence

$$\Theta\colon \mathrm{Mod}_{DBG_+} \simeq_Q \mathrm{Mod}_{\Theta DBG_+}$$

where ΘDBG_+ is a commutative dga with the property that $H_*(\Theta DBG_+) = \pi_*(DBG_+) = H^*BG$. It follows that there is a symmetric monoidal Quillen equivalence

$$L_{BG_+}\mathrm{Mod}_{DBG_+} \simeq_Q L_{\Theta BG_+}\mathrm{Mod}_{\Theta DBG_+}$$

where $H_*(\Theta BG_+) \cong \pi_*(BG_+) \cong H_*BG$.

Step 4. Since H^*BG is a polynomial ring it is strongly intrinsically formal as a commutative dga. In other words, for any commutative dga R with $H_*R \cong H^*BG$, there is a quasi-isomorphism $H^*BG \to R$. Therefore, taking cycle representatives we have a quasi-isomorphism $z\colon H^*BG \to \Theta DBG_+$. We also need the following result to identify ΘBG_+.

Lemma 4.4.2. *There is a natural weak equivalence $\Theta BG_+ \to H_*BG$.*

Proof. Write $(-)^\vee = \mathrm{Hom}_{\mathbb{Q}}(-,\mathbb{Q})$ and note that it is exact. There is a canonical map $\Theta BG_+ \to (\Theta BG_+)^{\vee\vee}$ which is a quasi-isomorphism since the homotopy groups of BG_+ are degreewise finite. There is a natural map $\Theta DBG_+ \to (\Theta BG_+)^\vee$ obtained as the transpose of the natural composite

$$\Theta BG_+ \otimes \Theta DBG_+ \to \Theta(BG_+ \wedge DBG_+) \to \mathbb{Q}.$$

Since Θ gives a symmetric monoidal equivalence of homotopy categories, the natural map $\Theta DBG_+ \to (\Theta BG_+)^\vee$ is a weak equivalence.

Since DBG_+ is a commutative $H\mathbb{Q}$-algebra, ΘDBG_+ is a commutative dga by [**86**, 1.2]. As H^*BG is strongly intrinsically formal as a commutative dga, there exists a quasi-isomorphism $H^*BG \to \Theta DBG_+$. Putting all this together, we have quasi-isomorphisms

$$\Theta BG_+ \to (\Theta BG_+)^{\vee\vee} \to (H^*BG)^\vee \to H_*BG.$$

$\square$

Extension and restriction of scalars along the map $z\colon H^*BG \to \Theta DBG_+$

$$\mathrm{Mod}_{\Theta DBG_+} \underset{z^*}{\overset{\Theta DBG_+\otimes_{H^*BG}-}{\rightleftarrows}} \mathrm{Mod}_{H^*BG}$$

is a symmetric monoidal Quillen equivalence since chain complexes satisfies Quillen invariance of modules. Therefore we have a symmetric monoidal Quillen equivalence

$$L_{H_*BG}\mathrm{Mod}_{\Theta DBG_+} \simeq_Q L_{H_*BG}\mathrm{Mod}_{H^*BG}.$$

Step 5. It remains to internalize the localization. Let I be the augmentation ideal of H^*BG and let K denote its unstable Koszul complex.

Proposition 4.4.3. *The homology H_*BG finitely builds K and K builds H_*BG.*

Proof. Suppose that $H^*BG = \mathbb{Q}[x_1, ..., x_n]$. There is a cofibre sequence

$$\Sigma^{|x_1|}\mathbb{Q}[x_1, ..., x_n] \xrightarrow{\cdot x_1} \mathbb{Q}[x_1, ..., x_n] \to \Sigma K(x_1)$$

and applying $\mathrm{Hom}_{\mathbb{Q}}(-,\mathbb{Q})$ gives the cofibre sequence

$$H_*BG \to \Sigma^{-|x_1|}H_*BG \to \Sigma K(x_1)^\vee.$$

Since $K(x_1)$ is self-dual up to suspension, this shows that $K(x_1)$ is finitely built from H_*BG. A repeated argument using the cofibre sequence $\Sigma^{|x_i|}K_{i-1} \to K_{i-1} \to K_i$ where $K_i = K(x_1, ..., x_i)$ and $K_0 = H^*BG$ shows that K is finitely built from H_*BG.

Conversely, since H_*BG is torsion it is built by K as K generates torsion modules [**41**, 8.7]. $\square$

Therefore, a map is a H_*BG-equivalence if and only if it is a K-equivalence. It follows that

$$L_{H_*BG}\mathrm{Mod}_{H^*BG} = L_K\mathrm{Mod}_{H^*BG}.$$

Combining all the statements of this section with Theorem 4.2.11 gives the following result.

THEOREM 4.4.4. *Let G be a connected compact Lie group and I be the augmentation ideal of H^*BG. Then there is a symmetric monoidal Quillen equivalence*

$$L_{EG_+}\mathrm{Sp}_G \simeq_Q \mathrm{Mod}^{\wedge}_{H^*BG}$$

between rational cofree G-spectra and L_0^I-complete dg-H^*BG-modules. In particular, there is a tensor-triangulated equivalence

$$\text{cofree } G\text{-spectra} \simeq_{\triangle} \mathcal{D}(L_0^I\text{-complete } H^*BG\text{-modules}).$$

5. The symmetric monoidal equivalence: non-connected case

In this section we extend the algebraic model for cofree G-spectra from connected G to any compact Lie group. The blueprint is the same as for the connected case, however some extra care is required which arises from taking categorical fixed points. We fix a compact Lie group G with identity component N and component group $W = G/N$, and write r for the rank of G.

5.1. Skewed Model Categories.

We recall some results about model categories with a action of a finite group W from [**55**, 5.2] and [**7**, Section 7]. For any cofibrantly generated model category $\mathcal{C}$, we denote by $\mathcal{C}[W] = \mathrm{Fun}(BW, \mathcal{C})$ the category of objects of $\mathcal{C}$ with a W-action. We endow $\mathcal{C}[W]$ with the projective model structure where the weak equivalence and fibrations are created by the forgetful functor $\mathcal{C}[W] \to \mathcal{C}$. We will need the following result:

LEMMA 4.5.1 ([**55**, 5.3]). *There is a symmetric monoidal Quillen equivalence $L_{EW_+}\mathrm{Sp}_W \simeq_Q \mathrm{Sp}[W]$.*

More generally, we can consider the category $\mathbb{E}W$ with objects the elements of W and a unique morphism connecting each pair of objects. Let $\mathcal{C}$ be a category with a W-action, that is, with functors $w_*\colon \mathcal{C} \to \mathcal{C}$ for each $w \in W$ satisfying $(ww')_* = w_*w'_*$ and $e_* = 1$. The category of objects of $\mathcal{C}$ with a *skewed W-action* is the category of equivariant functors $\mathbb{E}W \to \mathcal{C}$ and equivariant natural transformations, which we denote by $\mathcal{C}[\widetilde{W}]$. Note that if the W-action on $\mathcal{C}$ is trivial, then $\mathcal{C}[\widetilde{W}]$ is equivalent to $\mathcal{C}[W]$. We say that an adjunction between categories with a W-action is a *W-adjunction* if both the functors are W-equivariant and the unit and counit are W-equivariant natural transformations. We say that a model category $\mathcal{C}$ with a W-action is *skewable* if $w_* : \mathcal{C} \to \mathcal{C}$ is left Quillen for each $w \in W$. Note that $w_* : \mathcal{C} \to \mathcal{C}$ is left adjoint to w_*^{-1}, so equivalently, we could ask for w_* to be right Quillen for all $w \in W$.

LEMMA 4.5.2.

(a) *Let $\mathcal{C}$ be a skewable, symmetric monoidal, cofibrantly generated model category with a W-action. Then $\mathcal{C}[\widetilde{W}]$ admits a closed symmetric monoidal structure and a projective model structure making it into a symmetric monoidal model category.*
(b) *Let $\mathcal{C}$ and $\mathcal{D}$ be skewable, symmetric monoidal model categories. Suppose that $\mathcal{C} \rightleftarrows \mathcal{D}$ is a W-adjunction which is a symmetric monoidal Quillen equivalence. Then we have a symmetric monoidal Quillen equivalence*

$$\mathcal{C}[\widetilde{W}] \simeq_Q \mathcal{D}[\widetilde{W}].$$

PROOF. One can check that $\mathcal{C}[\widetilde{W}]$ is a symmetric monoidal model category in which the weak equivalences and fibrations are determined levelwise, and that Quillen equivalences extend to the skewed model category; see [**7**, 7.3] for the case $W = C_2$. $\square$

5.2. The algebraic model. The component group W acts on N by conjugation and hence on its cohomology H^*BN. We write $H^*\widetilde{BN}$ for the polynomial ring H^*BN equipped with this W-action. Accordingly, the model category $\mathrm{Mod}_{H^*\widetilde{BN}}$ inherits a W-action as follows. For $w \in W$ and a $H^*\widetilde{BN}$-module M, we define w_*M to be the same underlying abelian group as M but with module structure now defined by $r \cdot m := (wr)m$ for $r \in H^*\widetilde{BN}$ and $m \in M$. This model category is skewable since the action preserves weak equivalences and fibrations. Therefore, we can consider the model category $\mathrm{Mod}_{H^*\widetilde{BN}}[\widetilde{W}]$ of modules with a skewed W-action. More explicitly, we can identify this category with the category of modules over the skewed ring $H^*\widetilde{BN}[W]$, that is, the ring whose elements are formal linear sums $\sum_{w\in W} x_w w$ where $x_w \in H^*\widetilde{BN}$, with pointwise addition and multiplication defined by

$$(xw)\cdot(x'w') = (x(w\cdot x'))(ww') \quad \text{for } w, w' \in W \text{ and } \ x, x' \in H^*\widetilde{BN}.$$

We now turn to define a suitable notion of L_0^I-completion for a module over the skewed ring.

Definition 4.5.3. Let I denote the augmentation ideal of H^*BN. We say that a dg-$H^*\widetilde{BN}[W]$-module M is *L_0^I-complete* if M is L_0^I-complete as a H^*BN-module. We denote by $\mathrm{Mod}^\wedge_{H^*\widetilde{BN}[W]}$ the category of L_0^I-complete dg modules over the skewed ring.

Lemma 4.5.4.

(a) *The category of left $H^*\widetilde{BN}[W]$-modules admits a closed symmetric monoidal structure and a projective model structure making it into a symmetric monoidal model category.*
(b) *The category of L_0^I-complete left $H^*\widetilde{BN}[W]$-modules is abelian and is a symmetric monoidal model category with the projective model structure.*

Proof. The results follow from the previous sections and Lemma 4.5.2 by noticing that the category of (L_0^I-complete) $H^*\widetilde{BN}[W]$-modules is equivalent to $\mathcal{C}[\widetilde{W}]$ for $\mathcal{C}$ the category of (L_0^I-complete) $H^*\widetilde{BN}$-modules. □

Lemma 4.5.5. *(Eilenberg-Moore) Consider the family $[\subseteq N] = \{H \leq G \mid H \subseteq N\}$ and the Quillen adjunction*

$$(-)^N : \mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \leftrightarrows \mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W) : DEG_+ \otimes_{D\widetilde{BN}_+} -$$

where we set $D\widetilde{BN}_+ = (DEG_+)^N$. Then for all DEG_+-modules Y, the counit map

$$\epsilon_Y : DEG_+ \otimes_{D\widetilde{BN}_+} Y^N \to Y$$

is a $E[\subseteq N]_+$-equivalence.

Proof. A map of G-spectra is an $E[\subseteq N]_+$-equivalence if and only if its restriction to N-spectra is a weak equivalence. Therefore, it is sufficient to check that $DEN_+ \otimes_{DBN_+} Y^N \to Y$ is a weak equivalence. The full subcategory of DEN_+-module spectra Y for which ϵ_Y is a weak equivalence is localizing and clearly contains DEN_+. Since DEN_+ generates Mod_{DEN_+} by [**37**, 3.1] the claim follows. □

We now ready to prove our main result.

Theorem 4.5.6. *Let G be a compact Lie group with identity component N and component group $W = G/N$. Let I be the augmentation ideal of H^*BN. Then there is a symmetric monoidal Quillen equivalence*

$$L_{EG_+}(\mathrm{Sp}_G) \simeq_Q \mathrm{Mod}^\wedge_{H^*\widetilde{BN}[W]}$$

between rational cofree G-spectra and L_0^I-complete dg-$H^*\widetilde{BN}[W]$-modules. In particular, there is a tensor-triangulated equivalence

$$\textit{cofree } G\textit{-spectra} \simeq_\triangle \mathcal{D}(L_0^I\textit{-complete } H^*\widetilde{BN}[W]\textit{-modules}).$$

Proof. We will prove the theorem using the Compactly Generated Localization Principle 3.2.16. To have a better control on the compact generators of the localized categories, it is convenient to change our model for cofree G-spectra. Thus we note that

$$L_{EG_+}\mathrm{Sp}_G = L_{G_+}\mathrm{Sp}_G$$

since the EG_+-equivalences are the same as the G_+-equivalences. Using Proposition 3.3.4 we have a symmetric monoidal Quillen equivalence $L_{G_+}(\mathrm{Sp}_G) \simeq_Q L_{G_+}(\mathrm{Mod}_{DEG_+})$.

Taking categorical G-fixed points loses too much information since Mod_{DEG_+} is no longer generated by DEG_+. Instead we slightly modify the model structure and then take N-fixed points. Consider the family $[\subseteq N] = \{H \leq G \mid H \subseteq N\}$. There is a symmetric monoidal Quillen equivalence

$$L_{G_+}\mathrm{Mod}_{DEG_+} \rightleftarrows L_{G_+}L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}$$

since $G_+ \wedge E[\subseteq N]_+ \to G_+$ is a weak equivalence.

We now take categorical N-fixed points to remove equivariance. We use the tilde in $D\widetilde{BN}_+ = (DEG_+)^N$ to emphasize that it may have a non-trivial W-action. We apply the Compactly Generated Localization Principle to the symmetric monoidal Quillen adjunction

$$(-)^N : L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \leftrightarrows \mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W) : DEG_+ \otimes_{D\widetilde{BN}_+} -$$

to obtain a symmetric monoidal Quillen equivalence after localization. There are several conditions that need to be checked. Firstly, we claim that $L_{G_+}L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}$ is compactly generated by $DG_+ \simeq DG_+ \wedge DEG_+$. It is clear that it generates so we only show that it is compact. By definition of sum in the localized category, we have to show that

$$\mathrm{hMod}_{DEG_+}(DG_+, F(EG_+, \bigvee_i Y_i)) \simeq \bigoplus_i \mathrm{hMod}_{DEG_+}(DG_+, Y_i) \tag{5.2.1}$$

where Y_i is cofree for all i. This is now clear since DG_+ is small and $DG_+ \wedge EG_+ \simeq DG_+$. We also claim that $(DG_+)^N \simeq W_+$ compactly generates $L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W)$. Since W_+ has a trivial $D\widetilde{BN}_+$-action, it builds $D\widetilde{BN}_+ \wedge W_+$ in $\mathrm{Mod}_{D\widetilde{BN}_+}$ and hence it generates $L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W)$. It is also compact by a similar argument to (5.2.1). By the Compactly Generated Localization Principle it remains to check that the derived counit is a G_+-equivalence on DG_+, and that the derived counit is an $E[\subseteq N]_+$-equivalence for G_+. These are true by the Eilenberg-Moore Lemma. Hence we have a symmetric monoidal Quillen equivalence

$$L_{G_+}L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+} \simeq_Q L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W).$$

Note that have an equality of model categories

$$L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W) = L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(L_{EW_+}\mathrm{Sp}_W)$$

since $EW_+ \wedge W_+ \simeq W_+$.

We can rewrite the target category as $L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}[W])$ and apply Shipley's theorem [**86**] to obtain symmetric monoidal Quillen equivalences

$$L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}[W]) \simeq_Q L_{\Theta(W_+)}\mathrm{Mod}_{\Theta D\widetilde{BN}_+}(\mathrm{Mod}_{\mathbb{Q}[W]}) \simeq_Q L_{\Theta(W_+)}\mathrm{Mod}_{\Theta D\widetilde{BN}_+[W]}.$$

One can construct a $\mathbb{Q}[W]$-module map $H^*\widetilde{BN} \to \Theta D\widetilde{BN}_+$ which is a quasi-isomorphism as in [**43**, Section 7]. Since the map is compatible with the W-action, there is a symmetric monoidal Quillen equivalence

$$\mathrm{Mod}_{\Theta D\widetilde{BN}_+[W]} \simeq_Q \mathrm{Mod}_{H^*\widetilde{BN}[W]}.$$

Note that $H_*(\Theta(W_+)) = H_0(\Theta(W_+)) = \mathbb{Q}[W]$ and hence $\Theta(W_+)$ is formal as a $H^*\widetilde{BN}[W]$-module since we have a zig-zag of quasi-isomorphisms $\Theta(W_+) \leftarrow \tau_{\geq 0}(\Theta(W_+)) \to H_0(\Theta(W_+))$

where $\tau_{\geq 0}$ denotes the connective cover functor. Putting all this together, we deduce a zig-zag of symmetric monoidal Quillen equivalences

$$L_{EG_+}(\mathrm{Sp}_G) \simeq_Q L_{\mathbb{Q}[W]}\mathrm{Mod}_{H^*\widetilde{BN}[W]}.$$

We now claim that $L_{\mathbb{Q}[W]}\mathrm{Mod}_{H^*\widetilde{BN}[W]} = (L_{\mathbb{Q}}\mathrm{Mod}_{H^*BN})[\widetilde{W}]$. As the underlying categories are equal and the acyclic fibrations are easily seen to be the same, we only need to argue that the model categories have the same weak equivalences. This is clear since

$$\mathbb{Q}[W] \otimes_{H^*\widetilde{BN}[W]} M \cong \mathbb{Q} \otimes_{H^*BN} M$$

for all $H^*\widetilde{BN}[W]$-modules M. Hence the two model categories are equal.

Finally, using Lemma 4.5.2 and Theorem 4.2.11, we conclude that there are symmetric monoidal Quillen equivalences

$$(L_{\mathbb{Q}}\mathrm{Mod}_{H^*BN})[\widetilde{W}] \simeq_Q \mathrm{Mod}^{\wedge}_{H^*BN}[\widetilde{W}] \simeq_Q \mathrm{Mod}^{\wedge}_{H^*\widetilde{BN}[W]}.$$

$\square$

Remark 4.5.7. Our proof bridges a gap in [**43**]. In the cited paper it is stated that there is a Quillen equivalence

$$(-)^N : \mathrm{Cell}_{G_+}\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \leftrightarrows \mathrm{Cell}_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W) : DEG_+ \otimes_{D\widetilde{BN}_+} -$$

obtained by the Cellularization Principle. The claim as it is stated it is not correct. Indeed, if we want to apply the Cellularization Principle we need to check that the counit $DEG_+ \otimes_{D\widetilde{BN}_+} (G_+)^N \to G_+$ is a weak equivalence of G-spectra, which in general is false. Nonetheless, we can modify the argument as follows. Firstly there is a Quillen equivalence

$$\mathrm{Cell}_{G_+}\mathrm{Mod}_{DEG_+} \rightleftarrows \mathrm{Cell}_{G_+}L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}.$$

We note that the localization $\mathrm{Cell}_{G_+}L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}$ exists, since left Bousfield localizations of right proper, stable model categories are right proper by [**12**, 4.7]. We can then apply the Cellularization Principle to the Quillen adjunction

$$(-)^N : L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \leftrightarrows \mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W) : DEG_+ \otimes_{D\widetilde{BN}_+} -$$

and the Eilenberg-Moore Lemma to show that this is a Quillen equivalence after cellularization.

Bibliography

[1] Mohammad M. Abbasirad, *Homotopy theory of differential graded modules and adjoints of restriction of scalars*, 2014. University of Sheffield PhD thesis.

[2] Benjamin Antieau and Greg Stevenson, *Derived categories of representations of small categories over commutative Noetherian rings*, Pacific J. Math. **283** (2016), no. 1, 21–42. MR3513838

[3] H.H. Avramov, H-B. Foxby, and S. Halperin, *Differential graded homological algebra*. Preprint, 2003.

[4] Shawn Baland, Alexandru Chirvasitu, and Greg Stevenson, *The prime spectra of relative stable module categories*, Trans. Amer. Math. Soc. **371** (2019), no. 1, 489–503. MR3885152

[5] Scott Balchin and J. P. C. Greenlees, *Adelic models of tensor-triangulated categories*, arXiv e-prints (2019Mar), arXiv:1903.02669, available at `1903.02669`.

[6] Paul Balmer, *The spectrum of prime ideals in tensor triangulated categories*, J. Reine Angew. Math. **588** (2005), 149–168. MR2196732

[7] David Barnes, *Rational equivariant spectra*, 2008. University of Sheffield PhD thesis.

[8] ———, *Classifying rational G-spectra for finite G*, Homology Homotopy Appl. **11** (2009), no. 1, 141–170. MR2506130

[9] ———, *Rational $O(2)$-equivariant spectra*, Homology Homotopy Appl. **19** (2017), no. 1, 225–252. MR3633720

[10] David Barnes, John Greenlees, and Magdalena Kędziorek, *An algebraic model for rational toral G-spectra*, Algebr. Geom. Topol. **19** (2019), no. 7, 3541–3599. MR4045360

[11] David Barnes, John Greenlees, Magdalena Kędziorek, and Brooke Shipley, *Rational $SO(2)$-equivariant spectra*, Algebr. Geom. Topol. **17** (2017), no. 2, 983–1020. MR3623679

[12] David Barnes and Constanze Roitzheim, *Stable left and right Bousfield localisations*, Glasg. Math. J. **56** (2014), no. 1, 13–42. MR3137847

[13] Tobias Barthel, Drew Heard, and Gabriel Valenzuela, *Local duality in algebra and topology*, Adv. Math. **335** (2018), 563–663. MR3836674

[14] ———, *Derived completion for comodules*, Manuscripta Math. **161** (2020), no. 3-4, 409–438. MR4060488

[15] Tobias Barthel, J. P. May, and Emily Riehl, *Six model structures for DG-modules over DGAs: model category theory in homological action*, New York J. Math. **20** (2014), 1077–1159. MR3291613

[16] Clark Barwick, *On left and right model categories and left and right Bousfield localizations*, Homology Homotopy Appl. **12** (2010), no. 2, 245–320. MR2771591

[17] David J. Benson, Jon F. Carlson, and Jeremy Rickard, *Thick subcategories of the stable module category*, Fund. Math. **153** (1997), no. 1, 59–80. MR1450996

[18] Anna Marie Bohmann, *Global orthogonal spectra*, Homology Homotopy Appl. **16** (2014), no. 1, 313–332. MR3217308

[19] Marcel Bökstedt and Amnon Neeman, *Homotopy limits in triangulated categories*, Compositio Math. **86** (1993), no. 2, 209–234. MR1214458

[20] A. K. Bousfield, *On the telescopic homotopy theory of spaces*, Trans. Amer. Math. Soc. **353** (2001), no. 6, 2391–2426. MR1814075

[21] Lynne M. Butler, *Subgroup lattices and symmetric functions*, Mem. Amer. Math. Soc. **112** (1994), no. 539, vi+160. MR1223236

[22] Thomas Church, Jordan S. Ellenberg, and Benson Farb, *FI-modules and stability for representations of symmetric groups*, Duke Math. J. **164** (2015), no. 9, 1833–1910. MR3357185

[23] Thomas Church, Jordan S. Ellenberg, Benson Farb, and Rohit Nagpal, *FI-modules over Noetherian rings*, Geom. Topol. **18** (2014), no. 5, 2951–2984. MR3285226

[24] Thomas Church and Benson Farb, *Representation theory and homological stability*, Adv. Math. **245** (2013), 250–314. MR3084430

[25] Ethan S. Devinatz, Michael J. Hopkins, and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. I*, Ann. of Math. (2) **128** (1988), no. 2, 207–241. MR960945

[26] Andreas W. M. Dress, *Notes on the theory of representations of finite groups. Part I: The Burnside ring of a finite group and some AGN-applications*, Universität Bielefeld, Fakultät für Mathematik, Bielefeld, 1971. With the aid of lecture notes, taken by Manfred Küchler. MR0360771

[27] W. G. Dwyer and J. P. C. Greenlees, *Complete modules and torsion modules*, Amer. J. Math. **124** (2002), no. 1, 199–220. MR1879003

[28] A. D. Elmendorf, I. Kriz, M. A. Mandell, and J. P. May, *Rings, modules, and algebras in stable homotopy theory*, Mathematical Surveys and Monographs, vol. 47, American Mathematical Society, Providence, RI, 1997. With an appendix by M. Cole. MR1417719

[29] Pierre Gabriel, *Des catégories abéliennes*, Bulletin de la Société Mathématique de France **90** (1962), 323–448 (fr). MR38#1144

[30] Nir Gadish, *Categories of FI type: a unified approach to generalizing representation stability and character polynomials*, J. Algebra **480** (2017), 450–486. MR3633316

[31] Wee Liang Gan and Liping Li, *Noetherian property of infinite EI categories*, New York J. Math. **21** (2015), 369–382. MR3358549

[32] Wee Liang Gan and John Watterlond, *A representation stability theorem for VI-modules*, Algebr. Represent. Theory **21** (2018), no. 1, 47–60. MR3748353

[33] James A. Green, *Axiomatic representation theory for finite groups*, J. Pure Appl. Algebra **1** (1971), no. 1, 41–77. MR0279208

[34] J. P. C. Greenlees, *Triangulated categories of rational equivariant cohomology theories.* Oberwolfach Reports 8/2006, 480-488.

[35] ———, *Rational S^1-equivariant elliptic cohomology*, Topology **44** (2005), no. 6, 1213–1279. MR2168575

[36] ———, *The Balmer spectrum of rational equivariant cohomology theories*, J. Pure Appl. Algebra **223** (2019), no. 7, 2845–2871. MR3912951

[37] ———, *Borel cohomology and the relative Gorenstein condition for classifying spaces of compact Lie groups*, J. Pure Appl. Algebra **224** (2020), no. 2, 806–818. MR3987977

[38] J. P. C. Greenlees and J. P. May, *Derived functors of I-adic completion and local homology*, J. Algebra **149** (1992), no. 2, 438–453. MR1172439

[39] ———, *Completions in algebra and topology*, Handbook of algebraic topology, 1995, pp. 255–276. MR1361892

[40] ———, *Localization and completion theorems for MU-module spectra*, Ann. of Math. (2) **146** (1997), no. 3, 509–544. MR1491447

[41] J. P. C. Greenlees and Brooke Shipley, *An algebraic model for free rational G-spectra for connected compact Lie groups G*, Math. Z. **269** (2011), no. 1-2, 373–400. MR2836075

[42] ———, *The cellularization principle for Quillen adjunctions*, Homology Homotopy Appl. **15** (2013), no. 2, 173–184. MR3138375

[43] ———, *An algebraic model for free rational G-spectra*, Bull. Lond. Math. Soc. **46** (2014), no. 1, 133–142. MR3161769

[44] ———, *Fixed point adjunctions for equivariant module spectra*, Algebr. Geom. Topol. **14** (2014), no. 3, 1779–1799. MR3212584

[45] ———, *An algebraic model for rational torus-equivariant spectra*, J. Topol. **11** (2018), no. 3, 666–719. MR3830880

[46] Markus Hausmann, *Symmetric products and subgroup lattices*, Geom. Topol. **22** (2018), no. 3, 1547–1591. MR3780441

[47] ———, *Global group laws and equivariant bordism rings*, arXiv e-prints (December 2019), arXiv:1912.07583, available at `1912.07583`.

[48] Andreas Heider, *Two results from Morita theory of stable model categories*, arXiv e-prints (July 2007), arXiv:0707.0707, available at `0707.0707`.

[49] Philip S. Hirschhorn, *Model categories and their localizations*, Mathematical Surveys and Monographs, vol. 99, American Mathematical Society, Providence, RI, 2003. MR1944041

[50] Michael J. Hopkins and Jeffrey H. Smith, *Nilpotence and stable homotopy theory. II*, Ann. of Math. (2) **148** (1998), no. 1, 1–49. MR1652975

[51] Mark Hovey, *Spectra and symmetric spectra in general model categories*, J. Pure Appl. Algebra **165** (2001), no. 1, 63–127. MR1860878

[52] Mark Hovey, John H. Palmieri, and Neil P. Strickland, *Axiomatic stable homotopy theory*, Mem. Amer. Math. Soc. **128** (1997), no. 610, x+114. MR1388895

[53] Mark Hovey, Brooke Shipley, and Jeff Smith, *Symmetric spectra*, J. Amer. Math. Soc. **13** (2000), no. 1, 149–208. MR1695653

[54] Mark Hovey and Neil P. Strickland, *Morava K-theories and localisation*, Mem. Amer. Math. Soc. **139** (1999), no. 666, viii+100. MR1601906

[55] Magdalena Kędziorek, *An algebraic model for rational G-spectra over an exceptional subgroup*, Homology Homotopy Appl. **19** (2017), no. 2, 289–312. MR3731480

[56] ———, *An algebraic model for rational $SO(3)$-spectra*, Algebr. Geom. Topol. **17** (2017), no. 5, 3095–3136. MR3704254

[57] Nicholas J. Kuhn, *The generic representation theory of finite fields: a survey of basic structure*, Infinite length modules (Bielefeld, 1998), 2000, pp. 193–212. MR1789216

[58] L. G. Lewis Jr., J. P. May, M. Steinberger, and J. E. McClure, *Equivariant stable homotopy theory*, Lecture Notes in Mathematics, vol. 1213, Springer-Verlag, Berlin, 1986. With contributions by J. E. McClure. MR866482

[59] M. A. Mandell and J. P. May, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. **159** (2002), no. 755, x+108. MR1922205

[60] ———, *Equivariant orthogonal spectra and S-modules*, Mem. Amer. Math. Soc. **159** (2002), no. 755, x+108. MR1922205

[61] M. A. Mandell, J. P. May, S. Schwede, and B. Shipley, *Model categories of diagram spectra*, Proc. London Math. Soc. (3) **82** (2001), no. 2, 441–512. MR1806878

[62] Akhil Mathew, Niko Naumann, and Justin Noel, *Nilpotence and descent in equivariant stable homotopy theory*, Adv. Math. **305** (2017), 994–1084. MR3570153

[63] Eben Matlis, *Injective modules over Noetherian rings*, Pacific J. Math. **8** (1958), 511–528. MR99360

[64] Rohit Nagpal, *VI-modules in nondescribing characteristic, part I*, Algebra Number Theory **13** (2019), no. 9, 2151–2189. MR4039499

[65] C. St. J. A. Nash-Williams, *On well-quasi-ordering finite trees*, Proc. Cambridge Philos. Soc. **59** (1963), 833–835. MR153601

[66] Amnon Neeman, *The chromatic tower for $D(R)$*, Topology **31** (1992), no. 3, 519–532. With an appendix by Marcel Bökstedt. MR1174255

[67] Igor Pak and Sergey Bratus, *On sampling generating sets of finite groups and product replacement algorithm (extended abstract)*, Proceedings of the 1999 International Symposium on Symbolic and Algebraic Computation (Vancouver, BC), 1999, pp. 91–96. MR1802071

[68] Luca Pol and Jordan Williamson, *The Left Localization Principle, completions, and cofree G-spectra*, J. Pure Appl. Algebra **224** (2020), no. 11, 106408. MR4104494

[69] Marco Porta, Liran Shaul, and Amnon Yekutieli, *On the homology of completion and torsion*, Algebr. Represent. Theory **17** (2014), no. 1, 31–67. MR3160712

[70] Andrew Putman, *Stability in the homology of congruence subgroups*, Invent. Math. **202** (2015), no. 3, 987–1027. MR3425385

[71] Andrew Putman and Steven V. Sam, *Representation stability and finite linear groups*, Duke Math. J. **166** (2017), no. 13, 2521–2598. MR3703435

[72] Daniel Quillen, *Rational homotopy theory*, Ann. of Math. (2) **90** (1969), 205–295. MR258031

[73] Charles Rezk, *Analytic completion.* Avaliable from the author's webpage at `https://faculty.math.illinois.edu/~rezk/analytic-paper.pdf`.

[74] Steven V. Sam and Andrew Snowden, *GL-equivariant modules over polynomial rings in infinitely many variables*, Trans. Amer. Math. Soc. **368** (2016), no. 2, 1097–1158. MR3430359

[75] ———, *Gröbner methods for representations of combinatorial categories*, J. Amer. Math. Soc. **30** (2017), no. 1, 159–203. MR3556290

[76] ———, *Representations of categories of G-maps*, J. Reine Angew. Math. **750** (2019), 197–226. MR3943321

[77] Peter Schenzel, *Proregular sequences, local cohomology, and completion*, Math. Scand. **92** (2003), no. 2, 161–180. MR1973941

[78] Stefan Schwede, *On the homotopy groups of symmetric spectra*, Geom. Topol. **12** (2008), no. 3, 1313–1344. MR2421129

[79] ———, *The n-order of algebraic triangulated categories*, J. Topol. **6** (2013), no. 4, 857–867. MR3145142

[80] ———, *Equivariant properties of symmetric products*, J. Amer. Math. Soc. **30** (2017), no. 3, 673–711. MR3630085

[81] ———, *Global homotopy theory*, New Mathematical Monographs, vol. 34, Cambridge University Press, Cambridge, 2018. MR3838307

[82] Stefan Schwede and Brooke Shipley, *Algebras and modules in monoidal model categories*, Proc. London Math. Soc. (3) **80** (2000), no. 2, 491–511. MR1734325

[83] ———, *Equivalences of monoidal model categories*, Algebr. Geom. Topol. **3** (2003), 287–334. MR1997322

[84] ———, *Stable model categories are categories of modules*, Topology **42** (2003), no. 1, 103–153. MR1928647

[85] Brooke Shipley, *An algebraic model for rational S^1-equivariant stable homotopy theory*, Q. J. Math. **53** (2002), no. 1, 87–110. MR1887672

[86] ———, *$H\mathbb{Z}$-algebra spectra are differential graded algebras*, Amer. J. Math. **129** (2007), no. 2, 351–379. MR2306038

[87] The Stacks project authors, *The stacks project* (2019), available at `https://stacks.math.columbia.edu/tag/0948`.

[88] ———, *The stacks project* (2020), available at `https://stacks.math.columbia.edu/tag/07LT`.

[89] Dennis Sullivan, *Infinitesimal computations in topology*, Inst. Hautes Études Sci. Publ. Math. **47** (1977), 269–331 (1978). MR646078

[90] Robert W. Thomason, *The classification of triangulated subcategories*, Compositio Math. **105** (1997), no. 1, 1–27. MR1436741

[91] Bertrand Toën and Gabriele Vezzosi, *Homotopical algebraic geometry. II. Geometric stacks and applications*, Mem. Amer. Math. Soc. **193** (2008), no. 902, x+224. MR2394633

[92] Jean-Louis Verdier, *Des catégories dérivées des catégories abéliennes*, Astérisque **239** (1996), xii+253 pp. (1997). With a preface by Luc Illusie, Edited and with a note by Georges Maltsiniotis. MR1453167

[93] Charles A. Weibel, *An introduction to homological algebra*, Cambridge Studies in Advanced Mathematics, vol. 38, Cambridge University Press, Cambridge, 1994. MR1269324

[94] Jordan Williamson, *Flatness and Shipley's algebraicization theorem*, arXiv e-prints (2020Jan), arXiv:2001.06694, available at `2001.06694`.

[95] Jennifer C. H. Wilson, $\mathrm{FI}_{\mathcal{W}}$*-modules and stability criteria for representations of classical Weyl groups*, J. Algebra **420** (2014), 269–332. MR3261463

[96] Christian Wimmer, *Rational global homotopy theory and geometric fixed points*, Ph.D. Thesis, 2017.

[97] Fei Xu, *Spectra of tensor triangulated categories over category algebras*, Arch. Math. (Basel) **103** (2014), no. 3, 235–253. MR3266367