Random matrix models for Gram's Law

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Abstract

Gram's Law refers to the empirical observation that the zeros of the Riemann zeta function typically alternate with certain prescribed points, called Gram points. Although this pattern does not hold true for each and every zero, numerical results suggest that, as the height up the critical line increases, the proportion of zeros that obey Gram's Law converges to a finite, non-zero limit. It is also well-known that the eigenvalues of random unitary matrices provide a good statistical model for the distribution of zeros of the zeta function, so one could try to determine the value of this limit by analyzing an analogous model for Gram's Law in the framework of Random Matrix Theory. In this thesis, we will review an existing model based on random unitary matrices, for which the limit can be computed analytically, but has the wrong rate of convergence. We will then present an alternative model that uses random special unitary matrices, which gives the correct convergence rate, and discuss the large-N limit of this model. We shall conclude that at very large heights up the critical line, the local distribution of the zeta zeros is the same with respect to any sequence of points that are spaced like the Gram points. For the purpose of this thesis, we will assume throughout that all Gram points are different from zeta zeros, although this is not a proven fact.

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Acknowledgements

I would like to thank Dr. David Platt and the entire LMFDB Collaboration [62] for providing access to their numerical data on the first 100 billion non-trivial zeros of the Riemann zeta function, without which the computation of the results from Table 2.2, as well as Section 4.6, would not have been possible.

Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged in the Bibliography. The thesis includes published work [37].

Chapter 1

Gram's Law for the Riemann zeta function

1.1 The Riemann zeta function

The Riemann zeta function can be defined, in the case of complex numbers $s = \sigma + it$ with real part $\sigma > 1$, as a convergent series of the form

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s}.$$

It was first studied by Euler [27] in 1744, who remarked that in the region $\sigma > 1$ of the complex plane, $\zeta(s)$ can also be represented as a convergent infinite product over all the prime numbers (the Euler product formula)

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \prod_{p} \left(1 - \frac{1}{p^s}\right)^{-1} \qquad (\mathfrak{Re}\, s > 1).$$

And because an infinite product that converges doesn't vanish, this formula implies that $\zeta(s)$ does not have any zeros in the semi-plane $\sigma > 1$. One of the main results of Riemann's 1859 paper [86], was to show that $\zeta(s)$ can be extended by analytic continuation to a meromorphic function on the entire complex plane except at the point $s = 1$, where it has a simple pole of residue 1.

Riemann also proved that the zeta function satisfies a functional equation, which may be expressed in several equivalent ways. For example, let

$$\Gamma(s) := \int_0^{\infty} e^{-t} t^{s-1} dt$$

denote the gamma function for $\mathfrak{Re}\, s > 0$, and analytically continued elsewhere; it can be shown that $\Gamma(s)$ has simple poles at all the non-

positive integers $s = 0, -1, -2, \ldots$. Also, consider the function

$$\chi(s) := 2^s \pi^{s-1} \sin\left(\frac{\pi s}{2}\right) \Gamma(1-s) \tag{1.1}$$

$$= \pi^{s-\frac{1}{2}} \frac{\Gamma\left(\frac{1-s}{2}\right)}{\Gamma\left(\frac{s}{2}\right)}. \tag{1.2}$$

With these definitions, the functional equation for the zeta function can be represented as

$$\zeta(s) = \chi(s)\zeta(1-s). \tag{1.3}$$

Because of the presence of the sine function in (1.1), the functional equation implies that $\zeta(s)$ has simple zeros at all the negative even integers $s = -2, -4, -6, \ldots$; these are called the trivial zeros and are the only zeros of $\zeta(s)$ in the semi-plane $\sigma < 0$ (in the case when s is a positive even number, the zeros of the sine function are canceled by the corresponding poles of $\Gamma(1-s)$: $-1, -3, -5, \ldots$). Therefore, all the remaining zeros of $\zeta(s)$ are constrained to the vertical strip $0 \leq \sigma \leq 1$ in the complex plane, and are called the non-trivial zeros of the zeta function.

If we now define the entire function

$$\xi(s) := \frac{s(s-1)}{2} \pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \zeta(s), \tag{1.4}$$

one can use (1.2) to re-express the functional equation (1.3) in a way that is invariant under the substitution $s \to 1 - s$, and highlights one of the fundamental symmetries of the zeta function

$$\xi(s) = \xi(1-s). \tag{1.5}$$

We also note that in definition (1.4), the trivial zeros of the zeta function are canceled by the poles of the gamma function, which means that the zeros of $\xi(s)$ coincide with all the non-trivial zeros of $\zeta(s)$.

1.1.1 Zero-free regions

In 1896, Hadamard [38] and de la Vallée Poussin [104] extended the methods developed by Riemann, and independently proved the Prime Number Theorem (PNT)

$$\pi(x) \sim \mathrm{Li}(x),$$

where $\pi(x)$ represents the number of primes less than x, and $\mathrm{Li}(x)$ is the logarithmic integral

$$\mathrm{Li}(x) := \int_2^x \frac{1}{\log t}\, dt.$$

It can be shown that the PNT is equivalent to the statement that $\zeta(s)$ has no zeros along the line $\sigma = 1$ (and, by symmetry, neither on $\sigma = 0$); therefore, the PNT implies that all the non-trivial zeros of the zeta function are strictly inside the region $0 < \sigma < 1$, which is called the critical strip.

De la Vallée Poussin [105] also proved that $\zeta(s)$ has no zeros $\sigma + it$ in the region

$$\sigma \geq 1 - \frac{C}{\log |t|},$$

for sufficiently large $|t|$ and some constant $C > 0$. This zero-free region was subsequently extended by others, and the best estimate currently known is due to Korobov [52] and Vinogradov [106], who independently showed that $\zeta(s)$ is non-zero in

$$\sigma \geq 1 - \frac{C}{(\log |t|)^{\frac{2}{3}} (\log \log |t|)^{\frac{1}{3}}} \qquad (|t| \geq 3),$$

for some constant $C > 0$. Recently, Ford [28] has given an explicit numerical value of the constant to be $C = \frac{1}{57.54}$.

The functional equation (1.5) suggests that the non-trivial zeros are distributed symmetrically about the vertical line $\sigma = \frac{1}{2}$, which is named the critical line; in addition, it can be proved that $\zeta(\bar{s}) = \overline{\zeta(s)}$ for any $s \in \mathbb{C} \setminus \{1\}$, which means that the zeros are also symmetric with respect to the real axis $t = 0$.

In his paper [86], Riemann conjectured that all the roots of $\xi(\frac{1}{2} + it)$ are real, which is equivalent to the statement that all the non-trivial zeros of $\zeta(s)$ lie on the critical line (translation from German)

> "*it is very likely that all the roots are real. Of course, one would like to have a rigorous proof of this, but for the time being I have put aside the search for such a proof, after some fleeting vain attempts, because it is not necessary for the immediate objective of my investigation.*"

Although the Riemann Hypothesis has not been proven, it can be verified numerically up to a given height T by comparing the number of zeros in the rectangle $\{0 < \sigma < 1,\ 0 < t < T\}$ with the number of zeros on the segment $\{\frac{1}{2} + it,\ 0 < t < T\}$.

1.2 Counting zeros in the critical strip

Let $N(T)$ denote the number of zeros inside the critical strip (counted according to their multiplicity) with imaginary part between 0 and T

$$N(T) := \#\{\sigma + it \in \mathbb{C} \ : \ 0 < \sigma < 1, \ 0 < t \leq T, \ \zeta(\sigma + it) = 0\}.$$

In the case when T is the ordinate of a zeta zero, we may take

$$N(T) = \lim_{\varepsilon \to 0} \frac{1}{2}[N(T + \varepsilon) + N(T - \varepsilon)]. \tag{1.6}$$

Riemann conjectured, and in 1905 von Mangoldt [65] proved the following asymptotic formula for $N(T)$ (Riemann−von Mangoldt formula)

$$N(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \mathcal{O}(\log T). \tag{1.7}$$

This result can be obtained by considering, for $\epsilon > 0$ and $T > 0$ fixed (T different from the ordinate of a zeta zero), the following rectangle

$$R := \{s \in \mathbb{C} \mid \ -\epsilon \leq \mathfrak{Re}(s) \leq 1 + \epsilon, \ 0 \leq \mathfrak{Im}(s) \leq T\}.$$

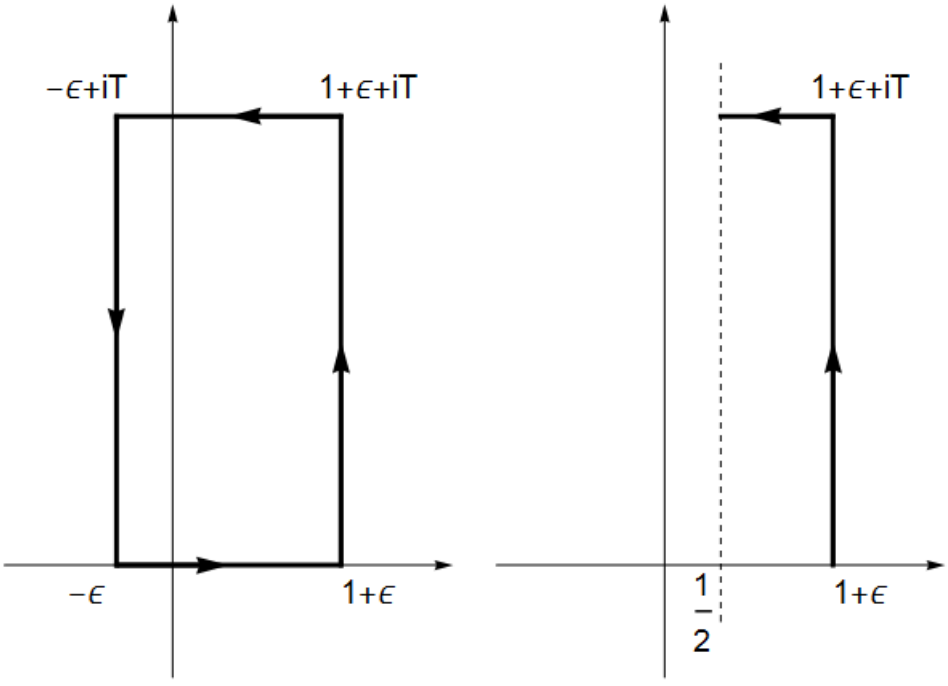

Figure 1.1: The curves ∂R (left) and C (right)

Also, let ∂R be the boundary of R oriented counterclockwise, and C the portion of ∂R that is between $1 + \epsilon$ and $\frac{1}{2} + iT$ (that is, the union of

the vertical segment from $1+\epsilon$ to $1+\epsilon+iT$, with the horizontal segment between $1+\epsilon+iT$ and $\frac{1}{2}+iT$).

As previously mentioned, the zeros of $\xi(s)$ coincide with all the zeros of $\zeta(s)$ inside the critical strip, and because the $\xi(s)$ function does not have any poles on the boundary ∂R, it will be more convenient to work with $\xi(s)$ instead of $\zeta(s)$ for the reminder of this section. Applying Cauchy's argument principle, we have that

$$N(T) = \frac{1}{2\pi i} \int_{\partial R} \frac{\xi'(s)}{\xi(s)}\, ds.$$

Now, since obviously $N(T) \in \mathbb{R}$, this becomes

$$N(T) = \frac{1}{2\pi}\, \mathfrak{Im} \int_{\partial R} \frac{\xi'(s)}{\xi(s)}\, ds.$$

Using the symmetry of the functional equation (1.5) with respect to the critical line, and with the fact that $\xi(s) \in \mathbb{R}$ along the real axis, we get

$$N(T) = \frac{1}{\pi}\, \mathfrak{Im} \int_{C} \frac{\xi'(s)}{\xi(s)}\, ds.$$

This can also be expressed in terms of a logarithmic derivative, and if we introduce the formula (1.4), we may expand it as a sum of three terms

$$\begin{aligned} N(T) =& \frac{1}{\pi}\, \mathfrak{Im} \int_{C} \frac{d}{ds} \log \xi(s)\, ds \\ =& \frac{1}{\pi}\, \mathfrak{Im} \int_{C} \frac{d}{ds} \log \left[\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \frac{s(s-1)}{2}\, \zeta(s) \right] ds \\ =& \frac{1}{\pi}\, \mathfrak{Im} \int_{C} \frac{d}{ds} \log \left[\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \right] ds + \\ &+ \frac{1}{\pi}\, \mathfrak{Im} \int_{C} \frac{d}{ds} \log[s(s-1)]\, ds + \frac{1}{\pi}\, \mathfrak{Im} \int_{C} \frac{d}{ds} \log \zeta(s)\, ds \end{aligned}$$

(the branch of the logarithm is given by continuous variation along ∂R from the value 0 at $s = 1+\epsilon$). For each term in this sum, we can apply the fundamental theorem of calculus and obtain

$$\begin{aligned} N(T) =& \frac{1}{\pi}\, \mathfrak{Im} \log \left[\pi^{-\frac{s}{2}} \Gamma\left(\frac{s}{2}\right) \right] \Big|_{s=\frac{1}{2}+iT} + \frac{1}{\pi}\, \mathfrak{Im} \log s(s-1)|_{s=\frac{1}{2}+iT} + \\ &+ \frac{1}{\pi}\, \mathfrak{Im} \log \zeta(s)|_{s=\frac{1}{2}+iT} \end{aligned}$$

$$= \frac{1}{\pi} \arg \left[\pi^{-\frac{iT}{2}} \Gamma \left(\frac{1}{4} + \frac{iT}{2} \right) \right] + \frac{1}{\pi} \arg \left(-T^2 - \frac{1}{4} \right) + \\ + \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iT \right). \tag{1.8}$$

The middle term of (1.8) can be easily computed to be

$$\frac{1}{\pi} \arg \left(-T^2 - \frac{1}{4} \right) = 1. \tag{1.9}$$

The first term of (1.8) is called the Riemann−Siegel theta function and it is denoted by

$$\theta(T) := \arg \left[\pi^{-\frac{iT}{2}} \Gamma \left(\frac{1}{4} + \frac{iT}{2} \right) \right] \\ = \mathfrak{Im} \log \Gamma \left(\frac{1}{4} + \frac{iT}{2} \right) - \frac{\log \pi}{2} T. \tag{1.10}$$

By applying Stirling's formula (1.18) for the gamma function, one can derive an asymptotic expansion for $\theta(T)$ which, although it does not converge, its first few terms give a good approximation when T is large

$$\theta(T) = \frac{T}{2} \log \frac{T}{2\pi} - \frac{T}{2} - \frac{\pi}{8} + \mathcal{O}\left(\frac{1}{T} \right). \tag{1.11}$$

The last term of (1.8) is named the $S(T)$ function

$$S(T) := \frac{1}{\pi} \arg \zeta \left(\frac{1}{2} + iT \right). \tag{1.12}$$

As in (1.6), if T coincides with the ordinate of a zeta zero, we take

$$S(T) = \lim_{\varepsilon \to 0} \frac{1}{2} [S(T + \varepsilon) + S(T - \varepsilon)].$$

It is known that $S(T)$ does not become large too fast; specifically von Mangoldt [65] proved unconditionally that

$$S(T) = \mathcal{O}(\log T), \tag{1.13}$$

and Littlewood [61] showed, assuming the Riemann Hypothesis, that

$$S(T) = \mathcal{O}\left(\frac{\log T}{\log \log T} \right).$$

Recent numerical computations made by Odlyzko [80] have confirmed that $S(T)$ does indeed grow very slow, specifically

- $|S(T)| < 1$ for $T < 280$;
- $|S(T)| < 2$ for $T < 6.8 \times 10^6$;
- largest value found so far is $|S(T)| \approx 3.3455$ at $T \approx 7.75 \times 10^{27}$ [6].

Putting everything together, we can now rewrite equation (1.8) as

$$N(T) = \frac{\theta(T)}{\pi} + 1 + S(T), \tag{1.14}$$

and by applying the estimates from (1.11) and (1.13), we recover the Riemann−von Mangoldt formula (1.7).

We may also remark that, because $N(T)$ is a step function and $\theta(T)$ is continuously increasing for $T > 6.29$, (1.14) implies that the $S(T)$ function has jump discontinuities at points equal to the zeta zeros and decreases monotonically between them.

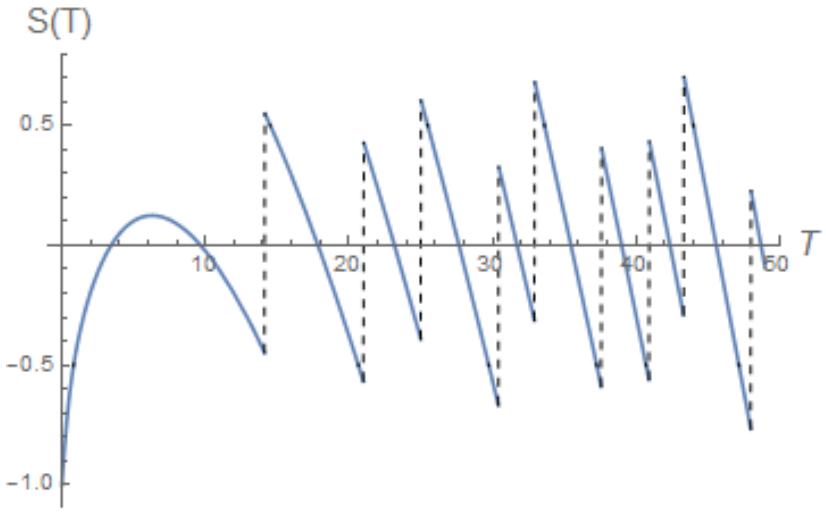

Figure 1.2: The $S(T)$ function

Backlund [4] obtained an explicit estimate for $N(T)$ when $T \geq 200$

$$\left| N(T) - \left(\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{7}{8} \right) \right| < 0.137 \log T + 0.443 \log \log T + 4.350,$$

but later developments have superseded this result. The most recent improvements were given by Trudgian and Platt [99], [100], [82] who showed that

$$|S(T)| \leq 0.110 \log T + 0.290 \log \log T + 2.290, \quad \text{for } T \geq e.$$

1.3 Counting zeros on the critical line

Let $N_0(T)$ be the number of zeta zeros (counted according to their multiplicity) located on the critical line up to height T

$$N_0(T) := \#\left\{t \in \mathbb{R} \ : \ 0 < t \leq T, \ \zeta\left(\frac{1}{2} + it\right) = 0\right\}.$$

In 1914, Hardy [39] proved that the zeta function has infinitely many zeros along the critical line. Later, Hardy and Littlewood [41] showed that there are at least $C \cdot T$ zeros on the critical line up to height T, for some positive constant C and sufficiently large T

$$N_0(T) > C \cdot T \quad (C > 0, \ T \geq T_0).$$

This result was subsequently improved by Selberg [88], who proved that a positive proportion of the zeta zeros lie on the critical line

$$N_0(T) > C \cdot N(T) \quad (C > 0, \ T \geq T_0).$$

The value of this proportion was first estimated by Min [70] to be at least $\frac{1}{14074731}$, and the best result known so far was obtained by Pratt et al. [85], who showed that at least $\frac{5}{12}$ of the zeros are on the critical line.

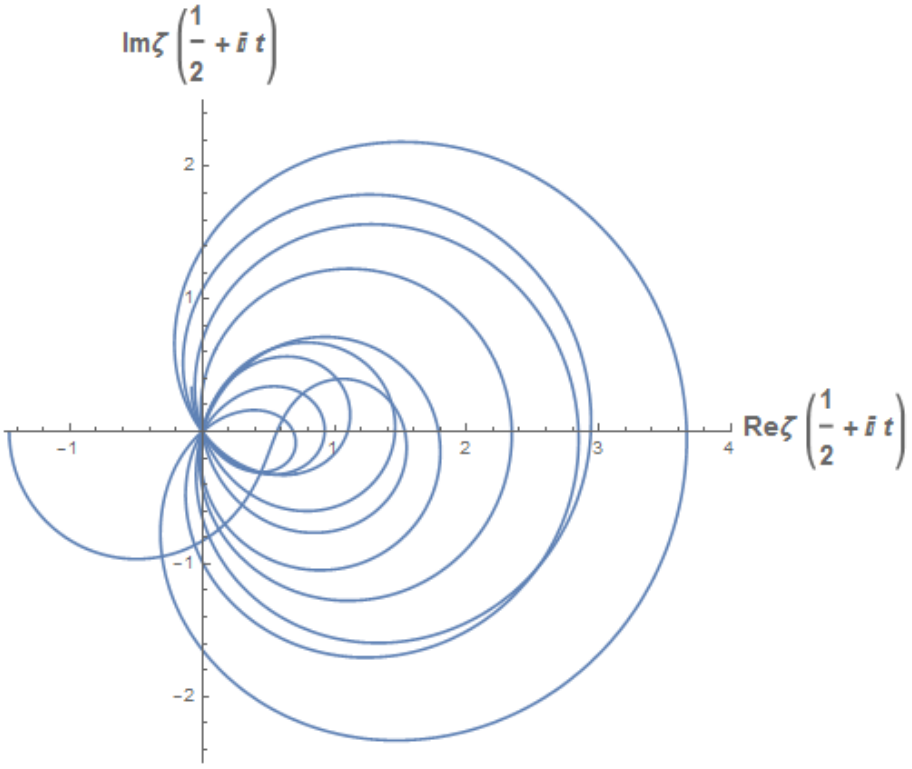

Figure 1.3: Parametric plot of $\zeta(\frac{1}{2} + it)$

In the same way in which it was more convenient to work with $\xi(s)$, rather than with $\zeta(s)$, when counting the zeros inside the critical strip, in this section we will introduce and use a new function whose zeros coincide with all the zeros of $\zeta(s)$ that are on the critical line. In order to derive this function, we begin by first noticing that the functional equation (1.5) can be reduced to

$$\xi(s) = \xi(\overline{s}), \qquad \text{if } s = \frac{1}{2} + it \quad (t \in \mathbb{R}).$$

And since it is well-known that

$$\xi(\overline{s}) = \overline{\xi(s)} \quad \text{for any } s \in \mathbb{C},$$

we get that

$$\xi(s) = \overline{\xi(s)} \quad \text{whenever } s = \frac{1}{2} + it,$$

which implies that the $\xi(s)$ function is real along the critical line. We can now apply the definition (1.4) to evaluate $\xi(s)$ at $s = \frac{1}{2} + it$, and obtain

$$\xi\left(\frac{1}{2} + it\right) = -\frac{1}{2}\left(t^2 + \frac{1}{4}\right)\Gamma\left(\frac{1}{4} + \frac{it}{2}\right)\pi^{-\frac{1}{4}-\frac{it}{2}}\zeta\left(\frac{1}{2} + it\right).$$

This expression can be further separated into two factors, as follows

$$\xi\left(\frac{1}{2} + it\right) = \underbrace{\left[-\frac{1}{2}\left(t^2 + \frac{1}{4}\right)e^{\mathfrak{Re}\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)}\pi^{-\frac{1}{4}}\right]}_{<0} \times \underbrace{\left[e^{i\left(\mathfrak{Im}\log\Gamma\left(\frac{1}{4}+\frac{it}{2}\right)-\frac{t}{2}\log\pi\right)}\zeta\left(\frac{1}{2} + it\right)\right]}_{Z(t)}.$$

Because $\xi(\frac{1}{2} + it) \in \mathbb{R}$ and the first factor above is obviously negative, this implies that the second factor is real, and has the opposite sign of $\xi(\frac{1}{2} + it)$. We also recognize that the exponent in the second factor is the theta function $\theta(t)$, defined previously in (1.10). This factor plays an important role in the theory of the zeta function, and is called the Hardy Z function

$$Z(t) := e^{i\theta(t)}\zeta\left(\frac{1}{2} + it\right). \tag{1.15}$$

As can be seen from its definition, the zeros of $Z(t)$ coincide with the imaginary parts of all the zeros of $\zeta(s)$ which lie on the critical line.

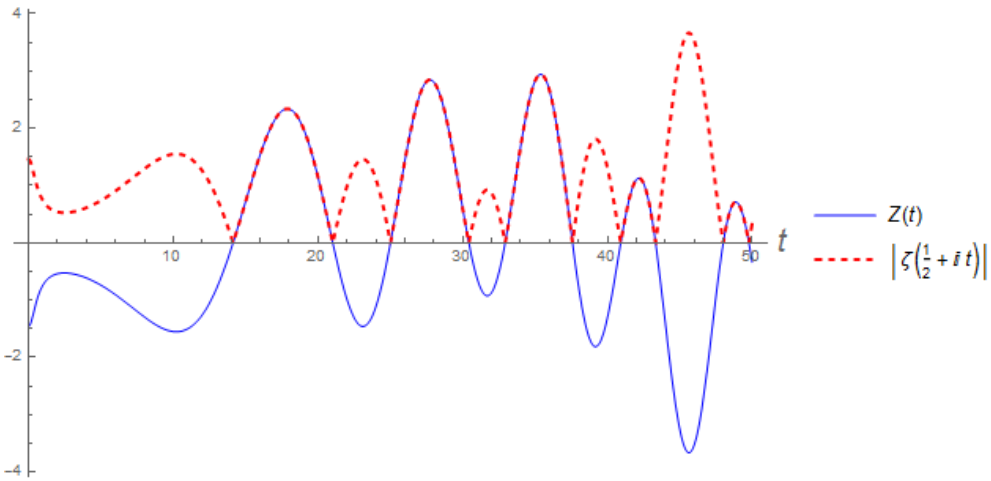

Figure 1.4: The functions $Z(t)$ and $\left|\zeta(\frac{1}{2}+it)\right|$

However, in order to numerically compute these zeros, one needs to first be able to evaluate the $Z(t)$ function, which can be done with one of the following methods.

1.3.1 The Euler−Maclaurin summation formula

The Euler−Maclaurin summation formula is a technique for approximating a sum by an integral, and provides a closed expression for the error term.

Theorem 1 (Euler−Maclaurin summation formula)**.** *If $p \in \mathbb{N}$ and $f(x)$ is a real or complex valued function that is p times continuously differentiable on the interval $[M, N]$, then we have*

$$\sum_{n=M}^{N} f(n) = \int_{M}^{N} f(x)\,dx + \frac{f(M)+f(N)}{2} + \sum_{k=1}^{[\frac{p}{2}]} T_k + R_p, \tag{1.16}$$

where

$$T_k = \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(N) - f^{(2k-1)}(M)\right),$$

and the remainder term is given by

$$R_p = \frac{(-1)^{p+1}}{p!}\int_{M}^{N} B_p(\{x\}) f^{(p)}(x)\,dx.$$

R_p has the order of magnitude of the first term omitted from the sum. Here $[x]$ represents the integer part of $x \in \mathbb{R}$, and $\{x\} = x - [x]$ the fractional part; also B_k denotes the k-th Bernoulli number, while $B_k(x)$ is the k-th Bernoulli polynomial.

Remark 1. *The Bernoulli numbers B_n are given by the coefficients in the expansion around $x = 0$ of the generating function*

$$\frac{x}{e^x - 1} = \sum_{n=0}^{\infty} \frac{B_n x^n}{n!}.$$

The odd-indexed ones B_{2k+1} (apart from the first) are all zero, and the even-indexed ones B_{2k} are related to the zeta function by

$$B_{2k} = \frac{(-1)^{k+1} 2(2k)!}{(2\pi)^{2k}} \zeta(2k).$$

The Bernoulli polynomial is defined as

$$B_p(x) = \sum_{j=0}^{p} \binom{p}{j} B_{p-j} x^j.$$

The Euler−Maclaurin summation method can be applied to derive an approximation formula for the zeta function

Theorem 2 (Euler−Maclaurin evaluation of $\zeta(s)$)**.** *For all $m \geq 0$, $N \geq 1$ and $s \in \mathbb{C}$ such that $\sigma = \mathfrak{Re}\, s > -(2m+1)$, we have*

$$\zeta(s) = \sum_{n=1}^{N} \frac{1}{n^s} + \frac{1}{(s-1)N^{s-1}} - \frac{1}{2N^s} + \sum_{k=1}^{m} T_{k,N}(s) + R_{m,N}(s), \quad (1.17)$$

where

$$T_{k,N}(s) = \frac{B_{2k}}{(2k)!} N^{1-s-2k} \prod_{j=0}^{2k-2} (s+j),$$

and

$$R_{m,N}(s) \leq \left| T_{m+1,N}(s) \frac{s+2m+1}{\sigma+2m+1} \right|.$$

It can also be used to obtain Stirling's formula [1], which is used in the asymptotic expansion of the $\theta(t)$ function

Theorem 3 (Stirling's formula)**.** *We have*

$$\log \Gamma(s) = s \log s - s + \frac{1}{2} \log \frac{2\pi}{s} + \mathcal{O}\left(\frac{1}{s}\right). \quad (1.18)$$

In principle, one can combine the formulas for $\theta(t)$ (1.11) and $\zeta(\frac{1}{2}+it)$ (1.17) to evaluate $Z(t)$ at any desired accuracy. However, there is a more efficient method that was initially discovered by Riemann and later improved by Siegel [91].

1.3.2 The Riemann–Siegel formula

In 1922, Hardy and Littlewood [42] proved the following formula for the zeta function

Theorem 4 (Approximate functional equation)**.** *Let $h > 0$ be constant, and $s = \sigma + it$ with $0 < \sigma < 1$ and $t > 0$. If $x, y \in \mathbb{R}$ such that $x, y > h > 0$ and $2\pi xy = t$, then we have*

$$\zeta(s) = \sum_{n \leq x} \frac{1}{n^s} + \chi(s) \sum_{n \leq y} \frac{1}{n^{1-s}} + \mathcal{O}(x^{-\sigma}) + \mathcal{O}(t^{\frac{1}{2}-\sigma} y^{\sigma-1}),$$

where $\chi(s)$ is the function defined in (1.1).

Now, if we set $x = y = \sqrt{\frac{t}{2\pi}}$, we obtain

$$\zeta(s) = \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^s} + \chi(s) \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{1-s}} + \mathcal{O}(t^{-\frac{\sigma}{2}}).$$

In particular, if we take $\sigma = \frac{1}{2}$, the equation becomes

$$\zeta\left(\frac{1}{2} + it\right) = \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{\frac{1}{2}+it}} + \chi\left(\frac{1}{2} + it\right) \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{\frac{1}{2}-it}} + \mathcal{O}(t^{-\frac{1}{4}}).$$

Next, we multiply by $e^{i\theta(t)}$ and use the fact that $\chi(\frac{1}{2} + it) = e^{-2i\theta(t)}$ to arrive at

$$Z(t) = e^{i\theta(t)} \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{\frac{1}{2}+it}} + e^{-i\theta(t)} \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{1}{n^{\frac{1}{2}-it}} + \mathcal{O}(t^{-\frac{1}{4}}).$$

This formula can be expressed in a more compact way as

Theorem 5 (The Riemann–Siegel formula)**.** *For any $t \in \mathbb{R}$, we have*

$$Z(t) = 2 \sum_{n \leq \sqrt{\frac{t}{2\pi}}} \frac{\cos(\theta(t) - t \log n)}{\sqrt{n}} + \mathcal{O}(t^{-\frac{1}{4}}). \tag{1.19}$$

As previously mentioned, the zeros of the $Z(t)$ function coincide with all the zeros of $\zeta(\frac{1}{2} + it)$. However, unlike $\zeta(\frac{1}{2} + it)$, which takes complex values for most $t \in \mathbb{R}$, $Z(t)$ is a real-valued function. Therefore, like any real-valued function, the zeros of $Z(t)$ can be determined by finding short intervals in which the function changes sign and applying the intermediate value theorem.

1.4 Gram points and Gram intervals

Definition 1. *For any integer $M \geq -1$, we define the M-th Gram point g_M as the unique solution in the range $[7, \infty)$ of the equation*

$$\theta(g_M) = M\pi, \tag{1.20}$$

and we call a Gram interval any interval that lies between two consecutive Gram points g_M, g_{M+1}.

Remark 2. *We note that at the point $t_{min} \approx 6.29$, the theta function has a local minimum of $\theta(t_{min}) \approx -3.53 < -\pi$, and because of this, the equation $\theta(g_M) = M\pi$ has two distinct solutions when $M \in \{-1, 0\}$. Therefore, in order to avoid any ambiguities in the definition of Gram points, we are considering only solutions in the range $t \geq t_{min}$, where $\theta(t)$ is strictly increasing.*

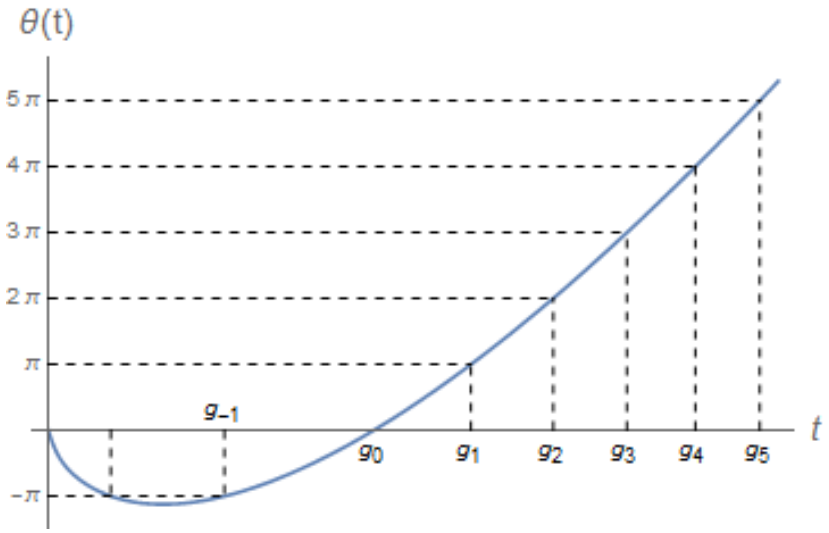

Figure 1.5: The $\theta(t)$ function

Before discussing how the Gram points are related to the zeros of the zeta function, we will first review a few of their basic properties. The most important fact about the Gram points is that they are relatively easy to compute numerically: if we truncate the expansion of the $\theta(t)$ function (1.11) after the constant term, equation (1.20) becomes

$$\frac{g_M}{2} \log \frac{g_M}{2\pi} - \frac{g_M}{2} - \frac{\pi}{8} = M\pi.$$

This can be re-expressed as

$$\frac{g_M}{2} \log \frac{g_M}{2\pi e} = M\pi + \frac{\pi}{8},$$

and by multiplying with $\frac{1}{\pi e}$, we get

$$\frac{g_M}{2\pi e}\log\frac{g_M}{2\pi e}=\frac{1}{e}\left(M+\frac{1}{8}\right).$$

Denoting

$$W=\log\frac{g_M}{2\pi e}\qquad\text{and}\qquad x=\frac{1}{e}\left(M+\frac{1}{8}\right),\tag{1.21}$$

the above equation now becomes

$$We^W=x.$$

If we try to solve this for W in terms of x, the solution $W(x)$ is called the Lambert W function [53]; it is known that $W(x)$ can not be explicitly expressed in terms of elementary functions, but it does have an asymptotic expansion, valid for large x

$$W(x)=\log x-\log\log x+\frac{\log\log x}{\log x}+\mathcal{O}\left(\left(\frac{\log\log x}{\log x}\right)^2\right).\tag{1.22}$$

Going back to (1.21), we obtain that the M-th Gram point is approximatively equal to

$$g_M\approx 2\pi e\cdot e^{W(x)}=2\pi\exp\left[1+W\left(\frac{8M+1}{8e}\right)\right].\tag{1.23}$$

In case a better precision is required, one can use this value as the starting point for repeated application of Newton's method

$$g_M^{\text{new}}=g_M^{\text{old}}-\frac{\theta(g_M^{\text{old}})-M\pi}{\theta'(g_M^{\text{old}})},$$

where

$$\theta'(t)=\frac{1}{2}\log\frac{t}{2\pi}+\mathcal{O}\left(\frac{1}{t^2}\right).\tag{1.24}$$

We can also use the expansion of the Lambert W function (1.22), together with equation (1.23), to write down an asymptotic formula for the Gram points ([47], [96])

$$g_n=\frac{2\pi n}{\log n}\left[1+\frac{1+\log\log n}{\log n}+\mathcal{O}\left(\left(\frac{\log\log n}{\log n}\right)^2\right)\right].$$

If we denote by $N_g(T)$ the number of Gram points smaller than T, it can be shown that

$$N_g(T) = \frac{T}{2\pi}\log T\left[1 - \frac{\log 2\pi e}{\log T} - \frac{\pi}{4T\log T} + \mathcal{O}\left(\frac{1}{T^2\log T}\right)\right]. \quad (1.25)$$

One could also prove that the lengths of Gram intervals decrease to 0 as $n \to \infty$

$$g_{n+1} - g_n = \frac{2\pi}{\log\frac{g_n}{2\pi} + \mathcal{O}\left(\frac{1}{g_n \log g_n}\right)}.$$

This result may be generalized to differences between non-consecutive Gram points $g_m, g_n \in [T,\, 2T]$ where, as $T \to \infty$, we have

$$g_m - g_n \sim \frac{2\pi(m-n)}{\log m} \sim \frac{2\pi(m-n)}{\log T}.$$

1.5 Gram's Law and the Weak Gram's Law

The definition of the $Z(t)$ function (1.15), together with (1.20) and Euler's identity, imply that at every Gram point g_M we have

$$\zeta\left(\frac{1}{2} + ig_M\right) = e^{-i\pi M} Z(g_M) = (-1)^M Z(g_M). \quad (1.26)$$

Keeping in mind that $Z(t)$ is a real-valued function, we obtain an alternative definition of Gram points, namely as points on the critical line at which the zeta function $\zeta\left(\frac{1}{2} + ig_M\right)$ takes real (non-zero) values.

In particular, if $\zeta\left(\frac{1}{2} + it\right)$ has the same sign at two successive Gram points $t = g_M$ and $t = g_{M+1}$, then according to (1.26) $Z(t)$ must have opposite signs at these points. This means that $Z(t)$ has at least a root between g_M and g_{M+1}, which is equivalent to $\zeta(\frac{1}{2} + it)$ having at least one zero inside the Gram interval $[g_M,\, g_{M+1})$.

This technique was initially used by Danish mathematician Gram [35] in 1903 to find the first 15 zeros of $\zeta(\frac{1}{2} + it)$ in the range $0 < t < 66$. He noticed that $\zeta\left(\frac{1}{2} + ig_M\right) > 0$ for all $-1 \leq M \leq 14$ and that each of these Gram intervals contained exactly one zero of the zeta function or, in other words, that the Gram points alternated with the zeta zeros. Gram believed that this pattern would continue beyond the first 15 intervals, but also that it would not necessarily hold true all the time. When it does hold, this phenomenon is named Gram's Law.

Definition 2. *Given two consecutive Gram points g_M and g_{M+1}, we say that Gram's Law (GL) holds true for $[g_M,\, g_{M+1})$ if this Gram interval contains exactly one zero of $\zeta(\frac{1}{2} + it)$.*

M	g_M	$\theta(g_M)$	$\zeta\left(\frac{1}{2}+ig_M\right)$
-1	9.66690...	$-\pi$	1.53182...
0	17.84559...	0	2.34018...
1	23.17028...	π	1.45743...
2	27.67018...	2π	2.84509...
3	31.71797...	3π	0.92526...
4	35.46718...	4π	2.93812...
5	38.99920...	5π	1.78672...
6	42.36355...	6π	1.09876...
7	45.59302...	7π	3.66299...
8	48.71077...	8π	0.68829...
9	51.73384...	9π	2.01121...
10	54.67523...	10π	2.91239...
11	57.54516...	11π	1.75816...
12	60.35181...	12π	0.53858...
13	63.10186...	13π	4.16439...
14	65.80088...	14π	1.05387...

n	γ_n
1	14.13472...
2	21.02203...
3	25.01085...
4	30.42487...
5	32.93506...
6	37.58617...
7	40.91871...
8	43.32707...
9	48.00515...
10	49.77383...
11	52.97032...
12	56.44624..
13	59.34704...
14	60.83177...
15	65.11254...

Table 1.1: The first Gram points and zeta zeros

Gram's Law refers only to the simple zeros of $\zeta(\frac{1}{2}+it)$, but there is another, less restrictive version called the Weak Gram's Law, which may be formulated as

Definition 3. *Given two consecutive Gram points g_M and g_{M+1}, we say that they satisfy the Weak Gram's Law (WGL) if*

$$(-1)^M Z(g_M) > 0 \quad \text{and} \quad (-1)^{M+1} Z(g_{M+1}) > 0.$$

This is equivalent to the statement that the Gram interval $[g_M, g_{M+1})$ contains an odd number of simple zeros, or a zero with odd multiplicity.

Neither Gram's Law, nor the Weak Gram's Law are concerned with potential zeros of the zeta function which might be off the critical line. The original definition of Gram's Law was proposed by Hutchinson [44] in 1925, and was given in terms of the zeros of the $Z(t)$ function

> "*Gram calculated the first fifteen roots [of $Z(t)$] and called attention to the fact that the [roots] and the [Gram points] separate each other. I will refer to this property of the roots as* Gram's Law. *Gram expressed the belief that this law is not a general one.*"

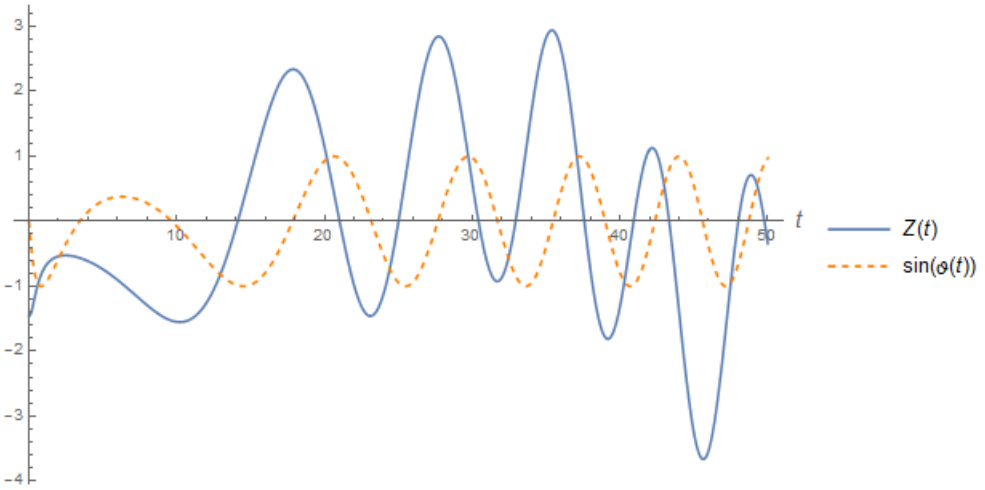

Figure 1.6: Alternation between Gram points and zeros of $Z(t)$

Hutchinson also extended Gram's computations from the first 15 to the first 138 zeros, and discovered the first instances where Gram's Law fails: the interval $[g_{125},\, g_{126})$ doesn't contain any zeros, while the next one $[g_{126},\, g_{127})$ has two. The Gram interval with the largest number of zeros known so far is the one that starts at Gram point g_M of index $M = 3,680,295,786,520$, and contains five zeros [34].

Remark 3. *Currently it is not known whether Gram's Law is true for infinitely many Gram intervals (much less for a positive proportion).*

Despite this uncertainty, extensive numerical computations at very high regions up the critical line suggest that Gram's Law does hold for a large proportion of these intervals: in a series of four papers published during the 80's, Brent, van de Lune and others [12], [13], [63], [64] have analyzed the first 1.5 billion Gram intervals and reported that approximately 72.61% of them obey Gram's Law. A summary of their results can be seen in Table 1.2 below.

Definition 4. *For $k, L, M \in \mathbb{Z}$ such that $k \geq 0$ and $0 \leq L < M$, we define $G_{L,M}(k) \in [0,1]$ to be the proportion of Gram intervals between g_L and g_M that contain exactly k zeros*

$$G_{L,M}(k) := \frac{1}{M-L}\#\{j \in [L,\, M-1] \;:\; N_0(g_{j+1}) - N_0(g_j) = k\}. \quad (1.27)$$

In particular, $G_{L,M}(1)$ represents the proportion of intervals in that range that obey Gram's Law, while $(M-L)\, G_{L,M}(k)$ will be the number of Gram intervals with exactly k zeros.

Remark 4. *In most of what follows from here on, the first zeta zero* $\gamma_1 = 14.13472\ldots$ *and the corresponding Gram interval* $[g_{-1},\, g_0)$ *are going to be ignored.*

M	$M \cdot G_{0,M}(0)$	$M \cdot G_{0,M}(1)$	$M \cdot G_{0,M}(2)$	$M \cdot G_{0,M}(3)$
100		100		
1,000	42	916	42	
10,000	808	8,390	796	6
100,000	10,330	79,427	10,157	86
1,000,000	116,055	769,179	113,477	1,289
10,000,000	1,253,556	7,507,820	1,223,692	14,932
100,000,000	13,197,331	73,771,910	12,864,188	166,570
1,000,000,003	137,078,283	727,627,708	133,509,764	1,784,225

Table 1.2: Summary of results by Brent, van de Lune et al.

Van de Lune et al. have concluded the last paper in their series with the following remarks:

> "*Our statistical material suggests that the zeros of* $Z(t)$ *are distributed among the Gram intervals according to some hitherto unknown probabilistic law.* (...) *It would be interesting to have a probabilistic model which could explain or at least support this phenomenon.*"

The main purpose of this thesis will be to make use of a conjecture from Random Matrix Theory in order to develop such a model, that describes the asymptotic limit of $G_{0,M}(k)$ for large M, as well as its rate of convergence. This will be achieved throughout the following chapters.

The remaining sections of this chapter will be devoted to reviewing other currently known results regarding the frequency with which Gram's Law and the Weak Gram's Law are valid. We will also discuss a generalization of Gram's Law, called Rosser's Rule.

1.5.1 WGL is true infinitely often

Intuitively, one could argue that the Weak Gram's Law is often true, by applying the following line of reasoning: if we evaluate the $Z(t)$ function at a Gram point $t = g_M$ using the Riemann−Siegel formula (1.19), we have that

$$
\begin{aligned}
Z(g_M) &= 2 \sum_{n \leq \sqrt{\frac{g_M}{2\pi}}} \frac{\cos(\theta(g_M) - g_M \log n)}{\sqrt{n}} + \mathcal{O}(g_M^{-\frac{1}{4}}) \\
&= 2(-1)^M \sum_{n \leq \sqrt{\frac{g_M}{2\pi}}} \frac{\cos(g_M \log n)}{\sqrt{n}} + \mathcal{O}(g_M^{-\frac{1}{4}}) \\
&= 2(-1)^M \left[1 + \frac{\cos(g_M \log 2)}{\sqrt{2}} + \frac{\cos(g_M \log 3)}{\sqrt{3}} + \ldots \right] + \mathcal{O}(g_M^{-\frac{1}{4}}).
\end{aligned}
$$

In the above sum, the first term is $+1$, while the remaining terms oscillate between positive and negative values, and decrease in absolute value. It seems reasonable to expect that for most Gram points g_M there would be enough cancellation between these terms, such that $+1$ becomes the dominant term, which then implies that $(-1)^M Z(g_M)$ is positive.

These ideas were rigorously refined by Titchmarsh [92] in 1934, when he gave two separate proofs of the fact that the Weak Gram's Law holds true infinitely many times.

His first result was to prove that

$$
\sum_{M \leq N} Z(g_{2M}) = 2N + \mathcal{O}(N^{\frac{3}{4}} \log^{\frac{3}{4}} N), \tag{1.28}
$$

$$
\sum_{M \leq N} Z(g_{2M+1}) = -2N + \mathcal{O}(N^{\frac{3}{4}} \log^{\frac{3}{4}} N). \tag{1.29}
$$

The error term in the above formulas was later improved by Ivić ([47], Theorem 6.5) to $\mathcal{O}(N^{\frac{3}{4}} \log^{\frac{1}{4}} N)$, and as recently as March 2020, Cao et al. [16] have shown that the error term can be further lowered down to $\mathcal{O}(N^{\frac{1}{4}} \log^{\frac{3}{4}} N \log\log N)$.

We note that equation (1.28) is equivalent to

$$
\frac{1}{N} \sum_{M \leq N} Z(g_{2M}) \sim 2,
$$

which means that on average, $Z(g_{2M})$ is positive, while (1.29) gives us

$$
\frac{1}{N} \sum_{M \leq N} Z(g_{2M+1}) \sim -2
$$

and therefore $Z(g_{2M+1})$ is negative on the average. Putting these two results together, we obtain that there are infinitely many Gram intervals that contain an odd number of zeros of $Z(t)$.

Within the same paper, Titchmarsh also showed that

$$\sum_{M\leq N} Z(g_M)Z(g_{M+1}) = -2N(\gamma+1) + o(N)$$

(where γ is the Euler−Mascheroni constant).

This implies that $Z(g_M)Z(g_{M+1})$ on average is negative, from where we can deduce again that for an infinity of values of M, the interval $[g_M,\ g_{M+1})$ has an odd number of zeros of $\zeta(\frac{1}{2}+it)$.

1.5.2 Discrete moments of Hardy's function

In addition to the results mentioned above, Titchmarsh also conjectured that

$$\sum_{M\leq N} Z^2(g_M)Z^2(g_{M+1}) = \mathcal{O}(N\log^A N),$$

for some positive constant A.

In a series of papers, Moser [74], [75], [76] proved several similar results concerning Hardy's function evaluated at Gram points, among which

$$\sum_{M\leq N} Z^4(g_M) = \mathcal{O}(N\log^4 N). \tag{1.30}$$

Combining this with the Cauchy−Schwarz inequality, one obtains a proof of Titchmarsh's conjecture for $A = 4$.

More recently, Kalpokas and Steuding [48] have shown that

$$\sum_{M\leq N} Z^2(g_M) = N\log N(1+o(1)).$$

Applying again the Cauchy−Schwarz inequality on this formula, we get a lower bound for Moser's result

$$\sum_{M\leq N} Z^4(g_M) \geq N\log^2 N(1+o(1)).$$

The main term in Moser's formula has been computed explicitly by Lavrik [55], who proved that

$$\sum_{M\leq N} Z^4(g_M) = \frac{1}{2\pi^2}N\log^4 N + \mathcal{O}(N\log^{\frac{7}{2}} N).$$

And finally, Conrey and Ghosh [18] have deduced a mean value theorem for the $Z(t)$ function at its relative extrema between consecutive

zeros γ_n, assuming the Riemann Hypothesis

$$\frac{1}{N}\sum_{n=1}^{N}\max_{\gamma_n<t<\gamma_{n+1}} Z^2(t) \sim \frac{e^2-5}{2}\log\frac{\gamma_N}{2\pi}$$

(where e is Euler's constant). Later, in Section 2.2, we will discuss known results and conjectures concerning the continuous moments of $|\zeta(\frac{1}{2}+it)|$.

1.5.3 WGL is true for a positive proportion

Moser's formula (1.30) can also be used to deduce that the proportion of Gram intervals up to height T that satsify the Weak Gram's Law is at least $\frac{A}{\log^3 T}$, for some positive constant A. This result was significantly improved by Trudgian in 2009, when he proved that ([96], [98])

Theorem 6 (Trudgian)**.** *There exists a $K>0$ such that, for sufficiently large T, there is a positive proportion of Gram intervals between T and $2T$ which contain at least 1 zero and not more than K zeros of $\zeta(\frac{1}{2}+it)$.*

In particular, this implies that the Weak Gram's Law is true a positive proportion of the time.

1.5.4 GL and WGL fail infinitely often

As previously mentioned, an alternative definition for Gram points is that they are points on the critical line at which the zeta function $\zeta(\frac{1}{2}+it)$ takes real (non-zero) values. Yet another definition, this time in terms of the $S(t)$ function (1.12), may be obtained in the following way: if we evaluate equation (1.14) at $T=g_M$ and use the formula (1.20), we have

$$N(g_M) = M+1+S(g_M).$$

And since obviously $N(g_M)\in\mathbb{N}$, we get that Gram points can also be considered as the points at which the $S(t)$ function takes integer values.

In 1924, Littlewood [61] proved the following result concerning $S(t)$

$$\int_0^T S(t)\,dt = \mathcal{O}(\log T). \tag{1.31}$$

It can be easily seen that this implies that the average value of $S(t)$ is zero, because

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T S(t)\,dt = 0.$$

Titchmarsh [93] later improved this observation, by showing that

Theorem 7 (Titchmarsh)**.**

The equation $S(t) = 0$ is satisfied for arbitrarily large values of t.

He then argued that, under the assumption that the $S(t)$ function is not bounded below, one can apply the above theorem to prove that Gram's Law fails infinitely many times; the same argument may also be used to reach a similar conclusion for the Weak Gram's Law.

However, at that time, the best known results concerning whether $S(t)$ is unbounded either above or below were only conditional. Specifically, Bohr and Landau [8] had proved that

Theorem 8 (Bohr−Landau)**.** *Assuming that the Riemann Hypothesis is true, then for every $\epsilon > 0$, the inequality*

$$|S(t)| > (\log t)^{\frac{1}{2}-\epsilon}$$

holds for arbitrary large values of t.

Using a theorem of Bohr [7], Landau [54] also proved a weaker version of this result: namely, that the $S(t)$ function can not be bounded below if the Riemann Hypothesis has only a finite number of exceptions.

The first unconditional result of this type was obtained by Selberg [90], who showed that $|S(t)|$ can become arbitrarily large, independently of any unproven assumptions.

Theorem 9 (Selberg)**.** *There is a positive constant C such that*

$$|S(t)| > C\frac{(\log t)^{\frac{1}{3}}}{(\log\log t)^{\frac{7}{3}}},$$

for arbitrarily large values of t.

The most recent refinements to the above theorems are due to Montgomery [73], who proved, assuming the Riemann Hypothesis, that

$$S(t) = \Omega_{\pm}\left(\left(\frac{\log t}{\log\log t}\right)^{\frac{1}{2}}\right),$$

and Tsang [102], who showed unconditionally that

$$S(t) = \Omega_{\pm}\left(\left(\frac{\log t}{\log\log t}\right)^{\frac{1}{3}}\right).$$

Therefore, together with Titchmarsh's work, this implies that Gram's Law and the Weak Gram's Law fail infinitely often, unconditionally.

1.5.5 GL and WGL fail for a positive proportion

The above results were recently strengthened by Trudgian ([96], [98]). Before presenting the main theorems, we first need to introduce some new notation: specifically, for $j = 0, 1, 2, \ldots$, let F_j be a Gram interval that contains the ordinates of exactly j zeta zeros, regardless of whether or not those zeros are on the critical line (so this implies that, in particular, an F_1 interval is one in which Gram's Law is satisfied, but the converse is not necessarily true).

Also, let $N_{F_j}(T)$ be the number of F_j intervals located between T and $2T$; and finally, let $N_G(T)$ denote the number of non-F_1 intervals between T and $2T$

$$\begin{aligned} N_G(T) &= N_{F_0}(T) + N_{F_2}(T) + N_{F_3}(T) + \ldots \\ &= N_g(2T) - N_g(T) - N_{F_1}(T) \end{aligned}$$

(where $N_g(T)$ was defined above (1.25)).

With this notation in mind, we have that

Theorem 10 (Trudgian)**.** *For sufficiently large T*

$$N_G(T) \gg T \log T.$$

This, in turn, can be used to deduce that for a positive constant A

$$\frac{N_{F_0}(T)}{N_g(2T) - N_g(T)} \geq \frac{A}{2} + \mathcal{O}\left(\frac{1}{\log T}\right),$$

which may be interpreted as

Theorem 11 (Trudgian)**.** *For sufficiently large T, there is a positive proportion of Gram intervals between T and $2T$ which do not contain a zero of $\zeta(\frac{1}{2} + it)$.*

And as an immediate consequence of this, we get that

Corollary 1 (Trudgian)**.** *For sufficiently large T there is a positive proportion of failures of the Weak Gram Law and of Gram's Law, between T and $2T$.*

1.6 Gram blocks and Rosser's Rule

We recall that one of the main results from Hutchinson's paper [44] was the discovery of a pair of consecutive Gram intervals $[g_{125},\, g_{126})$, $[g_{126},\, g_{127})$ such that one of them didn't contain any zeta zero, but the other one compensated by having two. Therefore, even though separately both these intervals contradict Gram's Law, if they are considered together, they have the same total number of zeros as a pair of consecutive Gram intervals that obey Gram's Law. This remark leads naturally to the following generalization for the concept of Gram interval

Definition 5. *Gram points that satisfy the condition* $(-1)^M Z(g_M) > 0$ *are called 'good', and those that don't are referred to as 'bad'.*

We define a Gram block of length $k \in \mathbb{N}$ *to be an interval of the form* $[g_M,\, g_{M+k})$*, where the exterior Gram points* g_M *and* g_{M+k} *are good, while the interior ones* $g_{M+1}, \ldots, g_{M+k-1}$ *are all bad Gram points.*

So, in essence, a Gram interval that obeys the Weak Gram's Law is a Gram block of length $k = 1$, and the pair of Gram intervals from Hutchinson's example form a Gram block of length $k = 2$ (g_{126} is the first bad Gram point). The above definition also implies that for $k \geq 3$, if $[g_M,\, g_{M+k})$ is a Gram block, then the exterior intervals $[g_M,\, g_{M+1})$ and $[g_{M+k-1},\, g_{M+k})$ have an even number of zeros (including no zeros), while all interior Gram intervals must contain an odd number of zeros.

Based on this definition, Rosser, Yohe and Schoenfeld [87] proposed in 1968 the following extension to Gram's Law

Definition 6. *A Gram block* $[g_M,\, g_{M+k})$ *of length* k *is said to satisfy Rosser's Rule (RR) if it contains exactly* k *zeros of* $\zeta(\frac{1}{2} + it)$*.*

In analogy with the Weak Gram's Law, we may also consider a weaker version of Rosser's Rule

Definition 7. *A Gram block* $[g_M,\, g_{M+k})$ *of length* k *is said to satisfy the Weak Rosser's Rule (WRR) if it contains at least* k *zeros of* $\zeta(\frac{1}{2} + it)$*.*

Both Rosser's Rule and the Weak Rosser's Rule do not take into account the possible existence of any zeta zeros off the critical line. In their paper, Rossser et al. numerically computed the first 3,500,000 zeros of the zeta function, and verified that all the Gram blocks in this range obey Rosser's Rule; however, they also expressed a belief that this phenomenon will not continue forever [26].

1.6.1 RR and WRR fail infinitely often

Lehman [57] adapted Titchmarsh's arguments from [93] to show that Rosser's Rule does have exceptions, and furthermore, that it has infinitely many of them.

Theorem 12 (Lehman)**.** *The Weak Rosser's Rule fails infinitely often, and therefore so does the Rosser Rule.*

As in the case of [93], Lehman's proof was based on the assumption that the $S(t)$ function is unbounded below, which at that time had already been proved by Selberg [90], so the above theorem is unconditional. Lehman also conjectured that Rosser's Rule would hold true for the first 10,000,000 zeros of the zeta function. Later Brent [12] extended the computations of zeta zeros beyond this range and found the first exception to Rosser's Rule at the 13,999,525-th Gram point, thus confirming Lehman's conjecture.

Remark 5. *As in the case of Gram's Law, it is not known if Rosser's Rule is true infinitely many times (or for a positive proportion).*
Unlike the Weak Gram's Law, it is not known if the Weak Rosser Rule is true infinitely many times (or for a positive proportion).

1.6.2 RR and WRR fail for a positive proportion

Lehman's theorem regarding the frequency of failures was substantially enhanced by Trudgian ([96], [98]), from infinitely many to a positive proportion. We will first introduce some notation that is analogous to that used in the case of the Weak Gram's Law.

For $k \in \mathbb{N}$ fixed and $j = 0, 1, 2, \ldots$, we denote B_j to be a Gram block of length k that has a total of $k + j - 2$ zeros (therefore, B_2 represents a Gram block that obeys Rosser's Rule). As previously mentioned, all the interior intervals of a Gram block must contain an odd number of zeros (so at least one zero), which means that in the case of a B_0 block, every interior interval has exactly one zero and the exterior intervals have none. It can also be seen from the definition that it is not possible to distribute an additional $2l + 1$ zeros inside a B_0 block, which implies that Gram blocks of odd index B_{2l+1} do not exist.

Now, if one denotes by $N_{GB}(T)$ the total number of Gram blocks between T and $2T$, it can be shown that

$$N_{GB}(T) \asymp T \log T.$$

Furthermore, we consider $N_{B_j}(T)$ to be the number of B_j Gram blocks between T and $2T$, and $N_B(T)$ the number of non-B_2 Gram blocks situated between T and $2T$

$$\begin{aligned} N_B(T) &= N_{B_0}(T) + N_{B_4}(T) + N_{B_6}(T) + \dots \\ &= N_{GB}(T) - N_{B_2}(T). \end{aligned}$$

For this quantity, one could prove that

Theorem 13 (Trudgian)**.** *For sufficiently large T,*

$$N_B(T) \gg T \log T.$$

As a consequence of this theorem, we have that

$$N_{B_0}(T) \gg T \log T,$$

which, combined with the above lemma, implies that

Theorem 14 (Trudgian)**.** *For sufficiently large T there is a positive proportion of Gram blocks between T and $2T$ which contain two fewer zeros of $\zeta(\frac{1}{2} + it)$ than their length.*

In particular, we may conclude the following

Corollary 2 (Trudgian)**.** *For sufficiently large T there is a positive proportion of failures of the Weak Rosser Rule, and therefore of Rosser's Rule, between T and $2T$.*

1.7 Turing's method

In 1953, Turing [103] gave a quantitative version of Littlewood's theorem regarding the integral of the $S(t)$ function (1.31), by showing that

Theorem 15 (Turing)**.** *If $t_2 > t_1 > 168\pi$, then we have*

$$\left| \int_{t_1}^{t_2} S(t)\, dt \right| \leq a \log t_2 + b, \tag{1.32}$$

where the coefficients are given by $(a,\, b) = (0.128,\, 2.07)$.

There are several errors in Turing's paper, that were corrected by Lehman [57], who got the values $(a,\, b) = (0.114,\, 1.7)$. These constants were later also improved by Trudgian [97], who obtained $(a,\, b) = (0.059,\, 2.067)$.

Brent [12] built on the ideas of Turing and Lehman to develop a method for obtaining an upper bound and a lower bound on the number of zeros at the edges of a collection of successive Gram blocks.

Theorem 16 (Brent−Lehman)**.** *If K consecutive Gram blocks with union $[g_n,\, g_p)$ obey Rosser's Rule, where*

$$\frac{b}{6\pi}\log^2 g_p + \frac{a - b\log 2\pi}{6\pi}\log g_p \leq K,$$

and a, b satisfy the condition (1.32), then we have

$$N(g_n) \leq n+1 \quad \textit{and} \quad p+1 \leq N(g_p).$$

As a simple application of this method, we reproduce the example from Lehman's paper for determining the value of $N(g_{250,098})$.

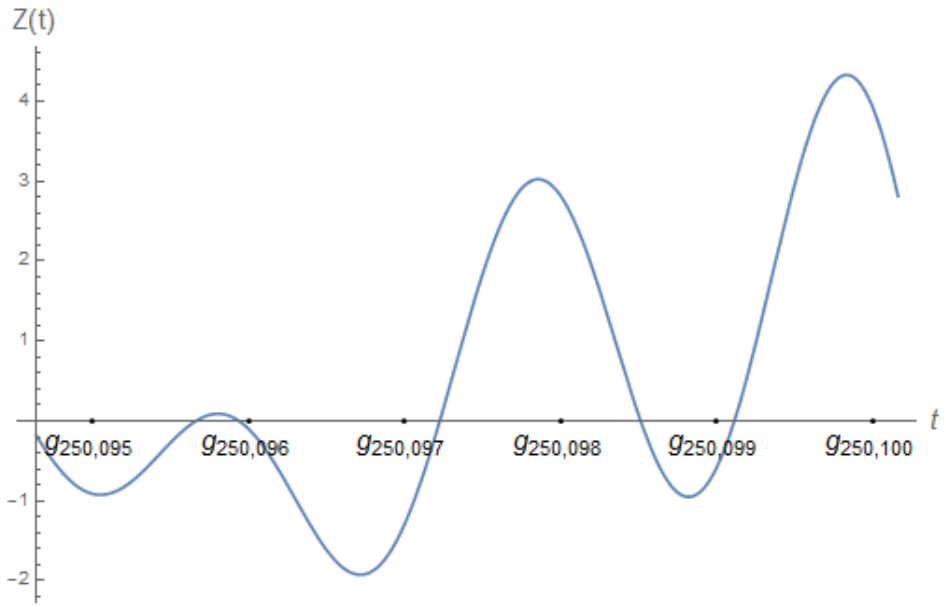

Figure 1.7: Application of the Brent−Lehman theorem

If we first use this theorem for the $K = 2$ consecutive Gram blocks $[g_{250,095},\, g_{250,097})$, $[g_{250,097},\, g_{250,098})$ that obey Rosser's Rule, we have that

$$0.0061\log^2 g_{250,098} + 0.08\log g_{250,098} \approx 1.84911 \leq 2,$$

and therefore

$$250,099 \leq N(g_{250,098}).$$

On the other hand, we may also apply the theorem for the $K = 2$

successive Gram blocks $[g_{250,098},\ g_{250,099})$, $[g_{250,099},\ g_{250,100})$ because

$$0.0061 \log^2 g_{250,100} + 0.08 \log g_{250,100} \approx 1.84912 \leq 2,$$

which implies

$$N(g_{250,098}) \leq 250,099.$$

Combining the previous two inequalities, we get

$$N(g_{250,098}) = 250,099.$$

The most recent improvement to Turing's result (1.32) is due to Trudgian [101], who showed that for $t_2 > t_1 > 10^5$, we have

$$\left| \int_{t_1}^{t_2} S(t)\, dt \right| \leq 0.049 \log t_2 + 0.183 \log \log t_2 + 1.698.$$

We now conclude this section and this chapter with a brief overview of the key milestones in the history of verifying the Riemann Hypothesis.

year	$N(T)$	T	author & reference
1859	3	26.0	Riemann (unpublished)
1903	15	66.0	Gram [35]
1914	79	200.0	Backlund [3]
1925	138	300.0	Hutchinson [44]
1935	195	390.0	Titchmarsh [93]
1936	1,041	1,468.0	Titchmarsh [94]
1953	1,104	1,540.0	Turing [103]
1956	15,000	14,041.0	Lehmer [58]
1956	25,000	21,942.6	Lehmer [59]
1958	35,337	29,751.0	Meller [66]
1966	250,000	170,571.0	Lehman [56]
1968	3,500,000	1,893,193.5	Rosser, et al. [87]
1977	40,000,000	18,114,537.6	Brent [11]
1979	81,000,001	35,018,261.5	Brent [12]
1982	200,000,001	81,702,130.0	Brent, et al [13]
1983	300,000,001	119,590,809.0	van de Lune, et al. [63]
1986	1,500,000,001	545,439,823.3	van de Lune, et al. [64]
2003	200,000,000,000		Wedeniwski [107]
2004	10,000,000,000,000		Gourdon [34]
2017	103,800,788,359	30,610,046,000.0	Platt [83]
2020	12,363,153,437,138	3,000,175,332,800.0	Platt and Trudgian [84]

Table 1.3: History of zeta zeros computation

In the case of at least one of the above authors, this computation was not necessarily motivated by proving the Riemann Hypothesis, but the contrary, as can be seen from Turing's paper [103]

> "*The calculations were done in an optimistic hope that a zero would be found off the critical line, and the calculations were directed more towards finding such zeros than proving that none existed.*"

Chapter 2

Gram's Law for random unitary matrices

2.1 GUE and the Montgomery conjecture

In this chapter, we will denote all the non-trivial zeros of the zeta function with positive imaginary parts by $\frac{1}{2} + i\gamma_n$, $n \in \mathbb{N}$. At the beginning of the 20-th century, Hilbert and Pólya [71] suggested that a possible reason for why the Riemann Hypothesis might be true would be because the γ_n's could coincide with the eigenvalues of an unbounded self-adjoint operator, which would imply that they are all real. If that were the case, one could expect to identify such an operator by comparing the distribution of its eigenvalues to the distribution of the zeta zeros. This idea became known as the Hilbert−Pólya conjecture, and it gained significant traction in the early 70's due to the work of Montgomery and Dyson.

From the Riemann−von Mangoldt formula (1.7), we know that the density of the zeta zeros is asymptotically

$$\frac{N(T)}{T} \sim \frac{1}{2\pi} \log \frac{T}{2\pi}.$$

In particular, this implies that as the height up the critical line increases, the zeros get closer together, and the average distance between consecutive zeros $\gamma_{n+1} - \gamma_n$ tends to 0. In order to study the statistical distribution of the zeros, one needs to first normalize them by their density

$$\hat{\gamma}_n = \gamma_n \times \frac{1}{2\pi} \log \frac{\gamma_n}{2\pi},$$

so that the mean spacing between normalized consecutive zeros $\hat{\gamma}_{n+1} - \hat{\gamma}_n$ converges to 1 as $n \to \infty$.

Assuming the Riemann Hypothesis, Montgomery [71] investigated the statistical distribution of the gaps between normalized zeros $\hat{\gamma}_m - \hat{\gamma}_n$, and conjectured that

Conjecture 1 (Montgomery). *For fixed $0 < \alpha < \beta < \infty$, we have*

$$\lim_{T\to\infty} \frac{1}{N(T)} \#\left\{(\gamma_m,\, \gamma_n) \,:\, 0 \le \gamma_m, \gamma_n \le T,\ \alpha \le (\gamma_m - \gamma_n)\frac{1}{2\pi}\log\frac{T}{2\pi} \le \beta\right\} = \\ = \int_\alpha^\beta \left[1 - \left(\frac{\sin \pi x}{\pi x}\right)^2\right] dx.$$

The term $1 - \left(\frac{\sin \pi x}{\pi x}\right)^2$ is called the two-point correlation function (or the pair correlation function) for the zeros of the zeta function, and it can be interpreted, intuitively, as the probability of finding a pair of normalized zeros separated by a distance x; a consequence of this is that small gaps between the zeta zeros are relatively rare. Independently of Montgomery, Dyson [25] had obtained a similar result, but in a very different context, namely that of Random Matrix Theory (RMT).

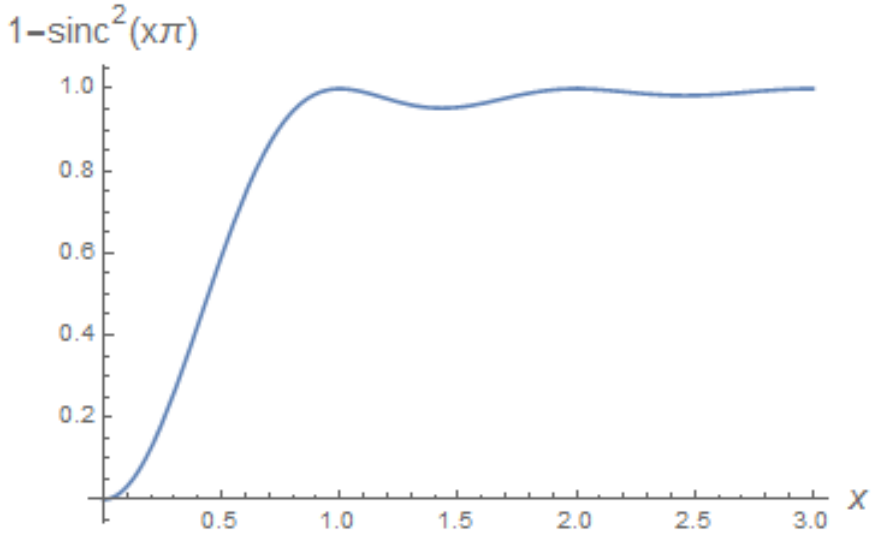

Figure 2.1: Montgomery's pair correlation function

In general, an ensemble of random matrices refers to a group of matrices with an attached probability measure. The Gaussian Unitary Ensemble (GUE) [69] contains of the group of complex Hermitian matrices

$$\mathrm{H}(N) := \{A \in \mathbb{C}^{N\times N} \ :\ A = A^\dagger\}.$$

A random element of this group is one chosen according to Haar measure.

The corresponding probability density function can be expressed in

terms of the eigenvalues $x_1, \ldots, x_N$ of the matrix [69]

$$\mathcal{P}_{\mathrm{H}(N)}(x_1, \ldots, x_N) := c \prod_{1 \le j < k \le N} |x_j - x_k|^2 e^{-(x_1^2 + \ldots + x_N^2)},$$

where

$$c = \pi^{\frac{N}{2}} 2^{-\frac{N(N-1)}{2}} \prod_{j=1}^{N} j!$$

is a normalization constant (in other words, $\mathcal{P}_{\mathrm{H}(N)}(x_1, \ldots, x_N)\, dx_1 \ldots dx_N$ represents the probability of a random $\mathrm{H}(N)$ matrix having an eigenvalue in each of $[x_1,\, x_1 + dx_1]$, $[x_2,\, x_2 + dx_2]$ and so on). It can be shown [69] that the probability density function can also be expressed as an $N \times N$ determinant

$$\mathcal{P}_{\mathrm{H}(N)}(x_1, \ldots, x_N) = \frac{1}{N!} \begin{vmatrix} K_N(x_1,\, x_1) & K_N(x_1,\, x_2) & \cdots & K_N(x_1,\, x_N) \\ K_N(x_2,\, x_1) & K_N(x_2,\, x_2) & \cdots & K_N(x_2,\, x_N) \\ \vdots & \vdots & \ddots & \vdots \\ K_N(x_N,\, x_1) & K_N(x_N,\, x_2) & \cdots & K_N(x_N,\, x_N) \end{vmatrix}.$$

Here, $K_N(x_j,\, x_k)$ is called the $\mathrm{H}(N)$ kernel function and is given by

$$K_N(x_j,\, x_k) := \sum_{m=0}^{N-1} \varphi_m(x_j) \varphi_m(x_k),$$

where

$$\varphi_m(x) := (2^m m! \sqrt{\pi})^{-\frac{1}{2}} e^{-\frac{x^2}{2}} H_m(x),$$

and $H_m(x)$ are the Hermite polynomials

$$H_m(x) := (-1)^m e^{x^2} \frac{d^m}{dx^m} e^{-x^2}.$$

The notion of pair correlation function may be defined for the eigenvalues of a random Hermitian matrix; it can also be extended to the more general case of $\mathrm{H}(N)$ n-point correlation function (or n-level density), given by

$$\begin{aligned} R^n_{\mathrm{H}(N)}(x_1, \ldots, x_n) &:= \frac{N!}{(N-n)!} \int_{\mathbb{R}^{N-n}} \mathcal{P}_{\mathrm{H}(N)}(x_1, \ldots, x_N)\, dx_{n+1} \ldots dx_N \\ &= \frac{1}{(N-n)!} \int_{\mathbb{R}^{N-n}} \det_{N \times N} K_N(x_j,\, x_k)\, dx_{n+1} \ldots dx_N, \end{aligned}$$

where $n = 1, \ldots, N$. It represents the probability of finding an eigenvalue (regardless of labeling) around each of the points $x_1, \ldots, x_n$, and with all

the other eigenvalues being integrated out. The following lemma [23] greatly simplifies the computation of n-point correlation functions in the cases where the kernel function is known.

Lemma 1 (Gaudin)**.** *Let f be a function defined on a measurable set J such that*

$$\int_J f(x,\, y) f(y,\, z)\, dy = C f(x,\, z),$$

and

$$\int_J f(x,\, x)\, dx = D,$$

where $C = C(f,\, J)$ and $D = D(f,\, J)$ are constants. Then, we have

$$\int_J \det_{N\times N} f(x_j,\, x_k)\, dx_N = (D - (N-1)C) \det_{(N-1)\times(N-1)} f(x_j,\, x_k).$$

It can be verified that the kernel function $K_N(x_j,\, x_k)$ satisfies the conditions of the above lemma with $D = N$ and $C = 1$, and through repeated applications, we obtain that

$$\int_{\mathbb{R}^m} \det_{N\times N} K_N(x_j,\, x_k)\, dx_{N-m+1} \ldots dx_N = m! \det_{(N-m)\times(N-m)} K_N(x_j,\, x_k).$$

Therefore, if we chose $m = N - n$, the n-point correlation function may be represented as an $n \times n$ determinant

$$R^n_{\mathrm{H}(N)}(x_1, \ldots, x_n) = \det_{n\times n} K_N(x_j,\, x_k). \tag{2.1}$$

Now, Wigner [109], [110] showed that the bulk of the eigenvalues of GUE matrices lie within an interval of the real line, and their density function has the shape of a semicircle

$$\sigma(x) := \begin{cases} \dfrac{1}{\pi}\sqrt{2N - x^2} & , \quad |x| < \sqrt{2N} \\ \\ 0 & , \quad |x| > \sqrt{2N}. \end{cases}$$

As a consequence of this, the mean density at the origin is given by

$$\sigma(0) = \frac{\sqrt{2N}}{\pi}.$$

And, if we rescale the eigenvalues by this factor

$$y_j = \frac{\sqrt{2N}}{\pi} x_j,$$

in order to have unit average spacing, then relation (2.1) becomes

$$R^n_{\mathrm{H}(N)}\left(\frac{\pi}{\sqrt{2N}}y_1,\ldots,\frac{\pi}{\sqrt{2N}}y_n\right)=\det_{n\times n}\left[\frac{\pi}{\sqrt{2N}}K_N\left(y_j\frac{\pi}{\sqrt{2N}},\ y_k\frac{\pi}{\sqrt{2N}}\right)\right].$$

The asymptotic value of the $\mathrm{H}(N)$ n-point correlation function can be computed with the help of the following formula [69]

$$\lim_{N\to\infty}\frac{\pi}{\sqrt{2N}}K_N\left(y_j\frac{\pi}{\sqrt{2N}},\ y_k\frac{\pi}{\sqrt{2N}}\right)=\frac{\sin\pi(y_j-y_k)}{\pi(y_j-y_k)}. \tag{2.2}$$

Using the Weyl integration formula [108], one can define the average over GUE of a function that is symmetric in all its variables

$$\begin{aligned}\int_{\mathrm{H}(N)}f(x_1,\ldots,x_N)\,d\mu_{\mathrm{H}(N)}&=\\ &=\int_{\mathbb{R}^N}f(x_1,\ldots,x_N)\mathcal{P}_{\mathrm{H}(N)}(x_1,\ldots,x_N)\,dx_1\ldots x_N.\end{aligned}$$

With this notation, we can express Dyson's formula for the pair correlation of the rescaled eigenvalues of a random Hermitian matrix in the following way

Theorem 17 (Dyson (GUE)). *For fixed $0<\alpha<\beta<\infty$, we have*

$$\begin{aligned}\lim_{N\to\infty}\int_{\mathrm{H}(N)}\frac{1}{N}\#\left\{(x_m,\,x_n)\,:\,\alpha\le(x_m-x_n)\frac{\sqrt{2N}}{\pi}\le\beta\right\}d\mu_{\mathrm{H}(N)}=\\ =\int_\alpha^\beta\left[1-\left(\frac{\sin\pi x}{\pi x}\right)^2\right]dx.\end{aligned}$$

Montgomery's pair correlation conjecture was extensively verified by Odlyzko [78] during the 80's, when he performed unprecedented large-scale, high-accuracy numerical computations on the first 10^5 zeta zeros and for zeros number $10^{12}+1$ up to $10^{12}+10^5$. Odlyzko's results provided "*a satisfying amount of agreement between the experimental data and the GUE predictions*", and led to the proposal of a more general conjecture [49] relating the zeros with the eigenvalues

Conjecture 2 (Montgomery−Odlyzko Law (GUE)). *The distribution of spacings between non-trivial zeros of the Riemann zeta function is statistically identical to the distribution of eigenvalue spacings in a Gaussian Unitary Ensemble.*

2.2 CUE and the zeta moments conjecture

Another significant result obtained by Dyson was to prove that the results from the GUE can be reproduced in the framework of a completely different matrix ensemble. The Circular Unitary Ensemble (CUE) [25] consists of all complex unitary matrices

$$\mathrm{U}(N) := \{A \in \mathbb{C}^{N\times N} \ : \ AA^{\dagger} = I_N\},$$

with the associated probability density function given by [108]

$$\mathcal{P}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N) := \frac{1}{N!(2\pi)^N} \prod_{1\leq j<k\leq N} |e^{i\theta_j} - e^{i\theta_k}|^2. \tag{2.3}$$

For any matrix $A \in \mathrm{U}(N)$, we denote its eigenvalues by $e^{i\theta_1}, \ldots, e^{i\theta_N}$, where $\theta_1, \ldots, \theta_N \in [-\pi, \pi)$. As in the case of the GUE, the probability density function can be represented in determinant form

$$\mathcal{P}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N) = \frac{1}{N!} \begin{vmatrix} S_N(\theta_1, \theta_1) & S_N(\theta_1, \theta_2) & \cdots & S_N(\theta_1, \theta_N) \\ S_N(\theta_2, \theta_1) & S_N(\theta_2, \theta_2) & \cdots & S_N(\theta_2, \theta_N) \\ \vdots & \vdots & \ddots & \vdots \\ S_N(\theta_N, \theta_1) & S_N(\theta_N, \theta_2) & \cdots & S_N(\theta_N, \theta_N) \end{vmatrix},$$

where the $\mathrm{U}(N)$ kernel function is

$$S_N(\theta_j, \theta_k) := \begin{cases} \dfrac{1}{2\pi} \dfrac{\sin \dfrac{N(\theta_j - \theta_k)}{2}}{\sin \dfrac{\theta_j - \theta_k}{2}} & , \quad j \neq k \\ \\ \dfrac{N}{2\pi} & , \quad j = k. \end{cases}$$

In order to show that the eigenangles of a random unitary matrix have a similar distribution to that of the eigenvalues of a random Hermitian matrix, we need to study the asymptotic behavior of the level densities.

Definition 8. *The* $\mathrm{U}(N)$ *n-level density (also called the n-point correlation function) for the eigenangles of a unitary matrix is defined as*

$$\begin{aligned} R^n_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_n) &:= \frac{N!}{(N-n)!} \int_{[-\pi,\pi)^{N-n}} \mathcal{P}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N)\, d\theta_{n+1} \ldots d\theta_N \\ &= \frac{1}{(N-n)!} \int_{[-\pi,\pi)^{N-n}} \det_{N\times N} S_N(\theta_j, \theta_k)\, d\theta_{n+1} \ldots d\theta_N, \end{aligned} \tag{2.4}$$

where $n = 1, \dots, N$. It can be viewed as the probability of finding an eigenangle (regardless of labeling) around each of the points $\theta_1, \dots, \theta_n$, and with all the other eigenangles being integrated out.

The $\mathrm{U}(N)$ kernel function $S_N(\theta_j, \theta_k)$ verifies the conditions of Gaudin's lemma, therefore we can express the n-level density in a more elegant way

$$R^n_{\mathrm{U}(N)}(\theta_1, \dots, \theta_n) = \det_{n\times n} S_N(\theta_j, \theta_k).$$

The one-level density thus becomes

$$\begin{aligned} R^1_{\mathrm{U}(N)}(\theta_1) &= S_N(\theta_1, \theta_1) \\ &= \frac{N}{2\pi}, \end{aligned}$$

which agrees with the intuitive notion that the density should be the total number of eigenvalues divided by the total length of the unit circle.

This also tells us that in the $\mathrm{U}(N)$ case, the local density always coincides with the mean density (in other words, there are no places on the circle that are more popular with the eigenangales than others). This is in contrast with the other classical groups, where the local density around the symmetry point changes.

By rescaling the eigenangles with this value

$$\phi_j = \frac{N}{2\pi}\theta_j, \tag{2.5}$$

we get that

$$R^n_{\mathrm{U}(N)}\left(\frac{2\pi}{N}\phi_1, \dots, \frac{2\pi}{N}\phi_n\right) = \det_{n\times n}\left[\frac{2\pi}{N} S_N\left(\phi_j \frac{2\pi}{N},\ \phi_k \frac{2\pi}{N}\right)\right].$$

Finally, because the limit

$$\lim_{N\to\infty} \frac{2\pi}{N} S_N\left(\frac{2\pi}{N}\phi_j,\ \frac{2\pi}{N}\phi_k\right) = \frac{\sin \pi(\phi_j - \phi_k)}{\pi(\phi_j - \phi_k)} \tag{2.6}$$

is the same as (2.2), we can conclude that for any $n \in \mathbb{N}$, the $\mathrm{U}(N)$ n-level density coincides asymptotically with the $\mathrm{H}(N)$ n-level density. This remark allows us to re-express the main results from the previous section in terms of unitary matrices.

Theorem 18 (Dyson (CUE))**.** *For fixed $0 < \alpha < \beta < \infty$, we have*

$$\lim_{N\to\infty}\int_{\mathrm{U}(N)}\frac{1}{N}\#\left\{(\theta_m,\,\theta_n)\,:\,\alpha\le(\theta_m-\theta_n)\frac{N}{2\pi}\le\beta\right\}d\mu_{\mathrm{U}(N)}=$$
$$=\int_\alpha^\beta\left[1-\left(\frac{\sin\pi x}{\pi x}\right)^2\right]dx,$$

where the analogous Weyl integration formula is

$$\int_{\mathrm{U}(N)}f(\theta_1,\ldots,\theta_N)\,d\mu_{\mathrm{U}(N)}= \tag{2.7}$$
$$=\int_{[-\pi,\pi)^N}f(\theta_1,\ldots,\theta_N)\mathcal{P}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N)\,d\theta_1\ldots\theta_N.$$

Conjecture 3 (Montgomery−Odlyzko Law (CUE))**.** *The zeros of the Riemann zeta function at height T on the critical line are statistically distributed like the eigenvalues of an $N\times N$ random unitary matrix around the unit circle, where the height T and the matrix size N are related by*

$$N\approx\log\frac{T}{2\pi}. \tag{2.8}$$

Remark 6. *The link between N and T is deduced by identifying the mean density of eigenangles $\frac{N}{2\pi}$ with the mean density of the zeta zeros $\frac{1}{2\pi}\log\frac{T}{2\pi}$.*

This connection was further used during the 90's to gain insight into a long-standing unsolved problem in number theory. It had been conjectured that the 2λ-th moment of the zeta function averaged along the critical line has asymptotically the following factorization

$$\lim_{T\to\infty}\frac{1}{(\log\frac{T}{2\pi})^{\lambda^2}}\frac{1}{T}\int_0^T\left|\zeta\left(\frac{1}{2}+it\right)\right|^{2\lambda}dt=a(\lambda)f(\lambda).$$

Here, the $a(\lambda)$ factor is given by

$$a(\lambda)=\prod_p\left\{\left(1-\frac{1}{p}\right)^{\lambda^2}\left[\sum_{m=0}^\infty\left(\frac{\Gamma(\lambda+m)}{m!\Gamma(\lambda)}\right)^2\frac{1}{p^m}\right]\right\},$$

and arises naturally from the problem of [17], while $f(\lambda)$ satisfies the constraint

$$f(\lambda)=\frac{g_\lambda}{(\lambda^2)!}\ ,\quad\text{with } g_\lambda\in\mathbb{N}.$$

Hardy and Littlewood [40] proved that $f(1)=1$, and Ingham [45] that $f(2)=\frac{2}{4!}$. On the basis of number-theoretical arguments, Conrey

and Ghosh [20], [21] conjectured that $f(3) = \frac{42}{9!}$, and later Conrey and Gonek [22] that $f(4) = \frac{24024}{16!}$.

Since the eigenvalues of a random $\mathrm{U}(N)$ matrix are the RMT analogues of the zeta zeros, and the unitary circle represents the analogue of the critical line, it makes sense to consider the characteristic polynomial of a unitary matrix as a RMT model for the Riemann zeta function; for $A \in \mathrm{U}(N)$, this is defined as

$$\begin{aligned} \Lambda_A(\theta) &:= \det(I_N - Ae^{-i\theta}) \\ &= \prod_{j=1}^{N}(1 - e^{i(\theta_j - \theta)}). \end{aligned} \tag{2.9}$$

Starting from here, Keating and Snaith [51] have proposed that the moments of the characteristic polynomial can be obtained by averaging over the CUE. Therefore, the RMT analogue of the $f(\lambda)$ factor can be considered to be

$$f_{\mathrm{CUE}}(\lambda) := \lim_{N\to\infty} \frac{1}{N^{\lambda^2}} \int_{\mathrm{U}(N)} |\Lambda_A(\theta)|^{2\lambda}\, d\mu_{\mathrm{U}(N)}.$$

Using an integral computed by Selberg [89], Keating and Snaith showed

Theorem 19 (Keating–Snaith). *For $\mathfrak{Re}\,\lambda > -\frac{1}{2}$, we have*

$$f_{\mathrm{CUE}}(\lambda) = \frac{G^2(1+\lambda)}{G(1+2\lambda)}.$$

In the above, $G(\lambda)$ denotes the Barnes G–function [5], which satisfies the functional equation

$$G(\lambda + 1) = \Gamma(\lambda)G(\lambda).$$

In particular, if $\lambda \in \mathbb{N}$, then $f_{\mathrm{CUE}}(\lambda)$ simplifies to

$$f_{\mathrm{CUE}}(\lambda) = \prod_{j=0}^{\lambda-1} \frac{j!}{(j+\lambda)!}.$$

It can be checked that for $\lambda = 1, 2, 3, 4$, $f_{\mathrm{CUE}}(\lambda)$ coincides with the corresponding known or assumed vales of $f(\lambda)$, which led to the conjecture that this agreement should hold true more generally

Conjecture 4 (Keating–Snaith). *For $\mathfrak{Re}\,\lambda > -\frac{1}{2}$, we have*

$$f_{\mathrm{CUE}}(\lambda) = f(\lambda).$$

2.3 Fujii's conjecture for Gram's Law

Because the eigenvalues of a random unitary matrix provide a good statistical model for the zeros of the zeta function, it is natural to ask if there could also exist a RMT model for Gram's Law. The first such model was proposed by Fujii [31] in 1987, who made a conjecture that is equivalent to the following statement

Conjecture 5 (Fujii (GUE)). *For any $k \in \mathbb{N} \cup \{0\}$ and $s \in \mathbb{N}$, we have*

$$E(k,s) = \lim_{M\to\infty} G_{0,M}(k,s). \tag{2.10}$$

Here, $G_{L,M}(k,s)$ is a generalization of definition (1.27), and it represents the proportion of s consecutive Gram intervals in the range $[L,\ M]$ that together contain a total of k zeros.

$$G_{L,M}(k,s) := \frac{1}{M-L}\#\{j \in [L,\ M-s] \ :\ N_0(g_{j+s}) - N_0(g_j) = k\}.$$

In the particular case when $s = 1$, it reduces to the definition from the previous chapter

$$G_{L,M}(k,1) = G_{L,M}(k).$$

The quantity $E(k,s)$ was studied by Metha [67], and denotes the asymptotic probability of finding k scaled eigenvalues of a random Hermitian matrix in an interval of length s. It is obtained by integrating the $\mathrm{H}(N)$ probability density function such that k scaled eigenvalues lie in $[0,\ s)$, while the other ones are outside, and then taking the limit $N \to \infty$

$$E(k,s) := \lim_{N\to\infty} \binom{N}{k} \left(\frac{\pi}{\sqrt{2N}}\right)^N \times \\ \times \int_{[0,s)^k} \int_{(\mathbb{R}\setminus[0,s))^{N-k}} \mathcal{P}_{\mathrm{H}(N)}\left(\frac{\pi}{\sqrt{2N}}y_1, \ldots, \frac{\pi}{\sqrt{2N}}y_N\right) dy_1 \ldots dy_N$$

(the binomial factor is to account for the different possible combinations).

Because we are interested in Gram's Law, we will focus on computing the above probability for intervals of length $s = 1$

$$E(k,1) = \lim_{N\to\infty} \binom{N}{k} \left(\frac{\pi}{\sqrt{2N}}\right)^N \times \\ \times \int_{[0,1)^k} \int_{(\mathbb{R}\setminus[0,1))^{N-k}} \mathcal{P}_{\mathrm{H}(N)}\left(\frac{\pi}{\sqrt{2N}}y_1, \ldots, \frac{\pi}{\sqrt{2N}}y_N\right) dy_1 \ldots dy_N.$$

Using the limit formulas (2.2) and (2.6), we can infer that the prob-

ability of having in an interval of unit length k eigenvalues of the GUE scaled by their mean density $\frac{\sqrt{2N}}{\pi}$ is asymptotically equal to the probability of finding in an interval of the same length k eigenangles of the CUE scaled by their mean density $\frac{N}{2\pi}$

$$E(k,1) = \lim_{N\to\infty} \binom{N}{k} \left(\frac{2\pi}{N}\right)^N \times \\ \times \int_{[0,1)^k} \int_{([0,N)\setminus[0,1))^{N-k}} \mathcal{P}_{\mathrm{U}(N)}\left(\frac{2\pi}{N}\phi_1, \ldots, \frac{2\pi}{N}\phi_N\right) d\phi_1 \ldots d\phi_N.$$

Applying the change of variables (2.5), we have that the probability of having k scaled eigenangles of the CUE in an interval of the length 1 coincides with the probability of finding k unscaled eigenangles of the CUE in an interval of length $\frac{2\pi}{N}$

$$E(k,1) = \lim_{N\to\infty} \binom{N}{k} \int_{[0,\frac{2\pi}{N})^k} \int_{([0,2\pi)\setminus[0,\frac{2\pi}{N}))^{N-k}} \mathcal{P}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N)\, d\theta_1 \ldots d\theta_N.$$

Note that by the rotation invariance of Haar measure, the above integral is insensitive to the actual starting position of the interval, only to its length. Because of this, it will be convenient to define for finite N a quantity that depends on a generic interval of fixed length.

Definition 9. *Let $J \subset [-\pi, \pi)$ be an interval of fixed length $\frac{2\pi}{N}$, and arbitrarily positioned on the unit circle. The probability that J contains exactly k unscaled eigenvalues of a random $\mathrm{U}(N)$ matrix is given by*

$$E_{\mathrm{U}(N)}(k, J) := \binom{N}{k} \int_{J^k} \int_{([-\pi,\pi)\setminus J)^{N-k}} \mathcal{P}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N)\, d\theta_1 \ldots d\theta_N. \tag{2.11}$$

In Chapter 3 we will present a more efficient formula for computing this quantity. Putting everything together, we can re-express Fujii's conjecture for Gram's Law (2.10) with $s = 1$ in the framework of the Circular Unitary Ensemble as

Conjecture 6 (Fujii (CUE))**.** *For any $k \in \mathbb{N} \cup \{0\}$, we have*

$$\lim_{N\to\infty} E_{\mathrm{U}(N)}(k, J) = \lim_{M\to\infty} G_{0,M}(k) \tag{2.12}$$

(where J is an interval of fixed length $\frac{2\pi}{N}$, and arbitrarily positioned on the unit circle).

Metha [67] provided a power series expansion of $E(k, s)$ for $k = 0, 1, 2$ that is valid around the point $s = 0$

$$E(0,s) = 1 - s + \frac{\pi^2}{36}s^4 - \frac{\pi^4}{675}s^6 + \frac{\pi^6}{17640}s^8 + \dots$$
$$E(1,s) = s - \frac{\pi^2}{18}s^4 + \frac{2\pi^4}{675}s^6 + \dots$$
$$E(2,s) = \frac{\pi^2}{36}s^4 - \frac{\pi^4}{675}s^6 + \dots$$

In order to compute the numerical values of $E(k,1)$, Fujii took the limit $s \to 1$ of the above expressions and obtained

$$E(0,1) \approx 0.18434 \qquad E(1,1) \approx 0.74031 \qquad E(2,1) \approx 0.12984. \quad (2.13)$$

Although these numbers add up to more than 100%, Fujii claimed that these values can be compared to the corresponding proportions for Gram's Law obtained by van de Lune et al. [64] in the case of the first $M = 1.5 \times 10^9$ Gram intervals

$$G_{0,M}(0) \approx 0.13784 \qquad G_{0,M}(1) \approx 0.72611 \qquad G_{0,M}(2) \approx 0.13424.$$

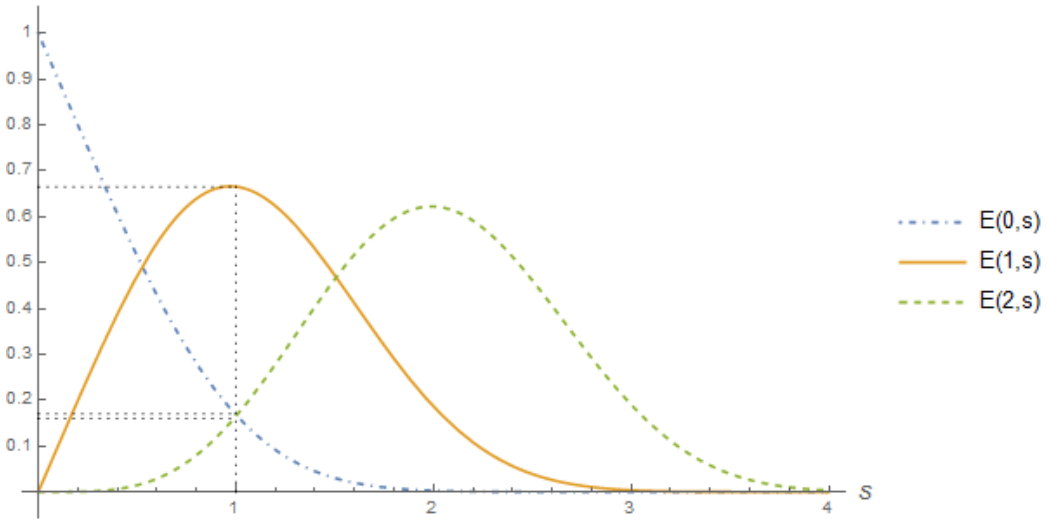

Figure 2.2: $E(k,s)$ for $k = 0, 1, 2$

However, as can be seen from the above plot, the numbers computed by Fujii (2.13) do not provide a good approximation. The correct values for these probabilities were reported later by Odlyzko [79] to be

$$E(0,1) \approx 0.17022 \qquad E(1,1) \approx 0.66143 \qquad E(2,1) \approx 0.16649. \quad (2.14)$$

2.3.1 Rate of convergence

One way of verifying whether the two limits from Fujii's conjecture (2.12) do indeed coincide is to compare their rates of convergence and see how similar they are. Using formula (2.11) we may compute the values of $E_{\mathrm{U}(N)}(k, J)$ for small N, which are given in Table 2.1. We remark that on each column, $E_{\mathrm{U}(N)}(k, J)$ converges very fast to its corresponding limit (2.14), and that on every row, they add up to almost 100%.

N	$E_{\mathrm{U}(N)}(0, J)$	$E_{\mathrm{U}(N)}(1, J)$	$E_{\mathrm{U}(N)}(2, J)$
2	0.148679	0.702642	0.148679
3	0.161362	0.678268	0.159378
4	0.165362	0.670641	0.162630
5	0.167146	0.667251	0.164060
6	0.168098	0.665445	0.164817
7	0.168666	0.664367	0.165268
8	0.169032	0.663673	0.165558
9	0.169283	0.663199	0.165755
10	0.169461	0.662860	0.165896
11	0.169593	0.662611	0.166000
12	0.169693	0.662421	0.166079
13	0.169771	0.662274	0.166140
14	0.169833	0.662157	0.166188
15	0.169882	0.662063	0.166227
16	0.169923	0.661986	0.166259
17	0.169957	0.661922	0.166286
18	0.169985	0.661869	0.166308
19	0.170009	0.661824	0.166327
20	0.170029	0.661785	0.166343
21	0.170047	0.661752	0.166357

Table 2.1: $E_{\mathrm{U}(N)}(k, J)$ for $k = 0, 1, 2$ and $N = 2, \ldots, 21$

However, the values of $G_{0,M}(k)$ that can be deduced from Table 1.2 are not directly comparable with the entries from Table 2.1, and need to be recomputed for different indices M in the following way: we know that the matrix size N is related to the height up the critical line T according to formula (2.8), which is equivalent to

$$T \approx 2\pi e^N.$$

And in our case, the height is given by the Gram points that we are interested in ($T = g_M$), so we can define M_N to be the index of the Gram point g_{M_N} that lies at the height on the critical line that corresponds to unitary matrices of size $N \times N$.

From the definition of the Gram points (1.20) we have that

$$M_N = \frac{1}{\pi}\theta(g_{M_N}),$$

and from the asymptotic formula for the theta function (1.11) we get

$$\frac{1}{\pi}\theta(T) \approx \frac{T}{2\pi}\log\frac{T}{2\pi} - \frac{T}{2\pi}.$$

Combining the previous three equations, we can derive an approximate formula for the index M_N in terms of the matrix size N

$$M_N \approx e^N(N-1).$$

The Gram points with these indices split the critical line into increasingly large segments of type $[g_{M_N},\, g_{M_{N+1}})$, and for each segment we can recompute the proportion of Gram intervals that contain exactly k zeros, $G_{M_N,M_{N+1}}(k)$; these are presented in Table 2.2 and represent the values that can be compared with the corresponding results from Table 2.1.

N	M_N	$G_{M_N,M_{N+1}}(0)$	$G_{M_N,M_{N+1}}(1)$	$G_{M_N,M_{N+1}}(2)$
2	7		1.000000	
3	40	0.016129	0.967741	0.016129
4	164	0.039351	0.921296	0.039351
5	594	0.069669	0.860661	0.069669
6	2,017	0.083059	0.834538	0.081744
7	6,580	0.095051	0.810527	0.093791
8	20,867	0.105168	0.790572	0.103348
9	64,825	0.111233	0.778687	0.108924
10	198,238	0.116361	0.768576	0.113764
11	598,741	0.121410	0.758585	0.118597
12	1,790,303	0.125309	0.750841	0.122389
13	5,308,961	0.128694	0.744212	0.125490
14	15,633,856	0.131542	0.738581	0.128210
15	45,766,243	0.134146	0.733422	0.130716
16	133,291,658	0.136422	0.728930	0.132871
17	386,479,244	0.138428	0.724956	0.134802
18	1,116,219,475	0.140223	0.721401	0.136526
19	3,212,681,417	0.141825	0.718223	0.138077
20	9,218,138,713	0.143277	0.715342	0.139481
21	26,376,314,690	0.144590	0.712736	0.140756
22	75,283,169,769	...	...	...

Table 2.2: $G_{M_N,M_{N+1}}(k)$ for $k = 0, 1, 2$ and $N = 2, \ldots, 21$

Now, according to the Montgomery−Odlyzko Law (CUE), for each finite N, $E_{\mathrm{U}(N)}(k, J)$ should provide a good approximation to $G_{M_N, M_{N+1}}(k)$, but we can see that it is not the case. This does not necessarily imply that their asymptotic limits do not coincide, but what is clear from this data is that $E_{\mathrm{U}(N)}(k, J)$ and $G_{M_N, M_{N+1}}(k)$ have very different rates of convergence.

We claim that this apparent contradiction with Fujii's conjecture originates from the use of an incorrect RMT analogy for Gram points and intervals. Recall that J was chosen to be an arbitrary fixed interval on the unit circle and as a consequence, there is nothing inherently special about its endpoints. However, these endpoints should represent the RMT analogues of Gram points and, as we have mentioned, the Gram points are special, in the sense that they are points on the critical line at which the zeta function takes real (non-zero) values.

In the following sections, we will consider and analyze an alternative RMT model for Gram's Law, in which the analogue Gram points are not fixed on the unit circle, but instead depend on the corresponding unitary matrix and are related to its characteristic polynomial in the same manner in which the actual Gram points relate to the Riemann zeta function.

2.4 $\mathrm{U}(N)$ Gram points and intervals

In order to motivate our RMT equivalent of Gram points, we recall from the previous chapter the way in which the zeta function is related to the $Z(t)$ function

$$\zeta\left(\frac{1}{2}+it\right) = Z(t)e^{-i\theta(t)} = Z(t)\cos\theta(t) - iZ(t)\sin\theta(t).$$

If we want to find the points on the critical line at which the zeta function is real, we have to impose the condition that its imaginary part should be zero, from which we get

$$\mathfrak{Im}\,\zeta\left(\frac{1}{2}+it\right) = 0 \quad \Leftrightarrow \quad Z(t)\sin\theta(t) = 0.$$

The last condition is equivalent to the following two possibilities

- $Z(t) = 0$, which also gives all the zeros of $\zeta(\frac{1}{2}+it)$;
- $\sin\theta(t) = 0 \Leftrightarrow \theta(t) = M\pi$ for $M \in \mathbb{Z}$, from which we get the Gram points g_M.

Now, the characteristic polynomial (2.9) of a random unitary matrix $A \in \mathrm{U}(N)$ can be re-expressed in terms of the eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_N}$ as

$$\begin{aligned}
\Lambda_A(\theta) &= \prod_{j=1}^{N}(1 - e^{i(\theta_j - \theta)}) \\
&= \prod_{j=1}^{N} \exp\frac{i(\theta_j - \theta)}{2}\left[\exp\left(-\frac{i(\theta_j - \theta)}{2}\right) - \exp\left(\frac{i(\theta_j - \theta)}{2}\right)\right] \\
&= (-2i)^N \prod_{j=1}^{N}\left[\exp\left(\frac{i(\theta_j - \theta)}{2}\right)\sin\left(\frac{\theta_j - \theta}{2}\right)\right] \\
&= 2^N \exp\left(-i\frac{N\pi}{2}\right)\exp\left(i\sum_{j=1}^{N}\frac{\theta_j - \theta}{2}\right)\prod_{j=1}^{N}\sin\left(\frac{\theta_j - \theta}{2}\right) \\
&= 2^N \exp\left[i\left(-\frac{N\pi}{2} + \frac{\theta_1 + \ldots + \theta_N}{2} - \frac{N\theta}{2}\right)\right]\prod_{j=1}^{N}\sin\left(\frac{\theta_j - \theta}{2}\right).
\end{aligned}$$

We continue the above analogy by searching for the points $\theta \in [-\pi, \pi)$ on the unit circle at which the characteristic polynomial is real

$$\mathfrak{Im}\,\Lambda_A(\theta) = 0 \;\Leftrightarrow\; 2^N \sin\left(-\frac{N\pi}{2} + \frac{\theta_1 + \ldots + \theta_N}{2} - \frac{N\theta}{2}\right)\prod_{j=1}^{N}\sin\left(\frac{\theta_j - \theta}{2}\right) = 0.$$

As before, this leads to two possible cases

- $2^N \prod_{j=1}^{N}\sin\left(\frac{\theta_j - \theta}{2}\right) = 0;$
- $\sin\left(-\frac{N\pi}{2} + \frac{\theta_1 + \ldots + \theta_N}{2} - \frac{N\theta}{2}\right) = 0 \quad \Leftrightarrow$
  $\Leftrightarrow \quad -\frac{N\pi}{2} + \frac{\theta_1 + \ldots + \theta_N}{2} - \frac{N\theta}{2} = m\pi$, for some $m \in \mathbb{Z}$.

From the first condition we recover the N eigenangles $\theta \in \{\theta_1, \ldots, \theta_N\}$ (which are the $\mathrm{U}(N)$ analogues of the zeta zeros). From the second condition, we obtain another set of points, given by

$$\theta \in \left\{\frac{\theta_1 + \ldots + \theta_N}{N} - \pi - \frac{2m\pi}{N}\;,\; m \in \mathbb{Z}\right\}.$$

We note that only N elements of this set are distinct modulo 2π, and because they represent the points on the unit circle at which the characteristic polynomial of a $\mathrm{U}(N)$ matrix is real (but not necessarily zero), we will consider them to be the analogous $\mathrm{U}(N)$ Gram points.

Definition 10. *If A is a $\mathrm{U}(N)$ matrix with eigenvalues $e^{i\theta_1}, \ldots, e^{i\theta_N}$, we define the corresponding $\mathrm{U}(N)$ Gram points as*

$$\psi_m^{\mathrm{U}(N)} := \frac{\theta_1 + \ldots + \theta_N}{N} - \pi + \frac{2m\pi}{N}, \quad m = 0, 1, \ldots, N-1.$$

We also define a $\mathrm{U}(N)$ Gram interval as any interval on the unit circle between two consecutive $\mathrm{U}(N)$ Gram points.

We remark that the $\mathrm{U}(N)$ Gram points are placed along the unit circle at equal distance from each other in steps of $\frac{2\pi}{N}$, rather than being distributed arbitrarily. Furthermore, they are not fixed on the unit circle, and are not the same for all $A \in \mathrm{U}(N)$ matrices, but instead depend on

$$\arg(\det A) = \theta_1 + \ldots + \theta_N \pmod{2\pi}.$$

With these definitions in mind, we can analyze the probability of having k eigenvalues of a random $\mathrm{U}(N)$ matrix inside one of these $\mathrm{U}(N)$ Gram intervals, in order to understand if and how it differs from $E_{\mathrm{U}(N)}(k, J)$.

2.4.1 The U(2) case

We begin by studying the simplest case, that of $N = 2$, which can be solved using just elementary logic, without any computations. According to the formula given above, if A is a unitary matrix of size 2×2, then its eigenangles θ_1, θ_2 are related to its $\mathrm{U}(2)$ Gram points ψ_1, ψ_2 by

$$\psi_1 = \frac{\theta_1 + \theta_2}{2} - \pi \quad \text{and} \quad \psi_2 = \frac{\theta_1 + \theta_2}{2}.$$

Now, since ψ_2 is the arithmetic average of θ_1 and θ_2, this means that it will always be located between them on the unit circle (regardless of where they are). On the other hand, ψ_1 is diametrically opposed to ψ_2, so it will also lie between θ_1 and θ_2, but on the other side of the circle. This is equivalent to having the two θ_j's positioned between the two ψ_j's, each one on a different arc. In particular, this implies that the probability of finding exactly $k = 1$ eigenvalue of a random $\mathrm{U}(2)$ matrix inside a $\mathrm{U}(2)$ Gram interval will always be 100%; it also means that the probability is zero for having an empty $\mathrm{U}(2)$ Gram interval ($k = 0$) or of having both eigenvalues in the same interval ($k = 2$). These results are not only very different from the values on row $N = 2$ of Table 2.1 but, more importantly, are in perfect agreement with the entries on row $N = 2$ of Table 2.2 .

This hints at the more general fact that the probability of finding exactly k eigenvalues of a random $\mathrm{U}(N)$ matrix in a $\mathrm{U}(N)$ Gram interval gives a much better model for $G_{M_N,M_{N+1}}(k)$ than $E_{\mathrm{U}(N)}(k,J)$. However, it becomes increasingly difficult to compute this quantity in a direct way for $N \geq 3$ (the problem comes from the fact that this probability is essentially an integral over the eigenangles $\theta_1,\ldots,\theta_N$ and each $\mathrm{U}(N)$ Gram point depends on all of them). In order to overcome this difficulty, we will relate this quantity to the corresponding probability for a particular kind of $\mathrm{U}(N)$ matrices, namely the special unitary matrices, and then focus on computing that probability.

2.5 $\mathrm{SU}(N)$ Gram points and intervals

If we define the $\mathrm{U}(N)$ Dyson product to be

$$\mathcal{D}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N) := \prod_{1\leq j<k\leq N} |e^{i\theta_j} - e^{i\theta_k}|^2, \tag{2.15}$$

then the $\mathrm{U}(N)$ probability density function (2.3) may be written as

$$\mathcal{P}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N) = \frac{1}{N!(2\pi)^N}\mathcal{D}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N). \tag{2.16}$$

A $\mathrm{SU}(N)$ matrix is a unitary matrix with determinant equal to 1. It has a Haar measure which effectively comes from the Haar measure for $\mathrm{U}(N)$; the only difference is that, in this case, one eigenangle is forced to take the value which makes the sum of all the N eigenangles congruent to $0 \pmod{2\pi}$, since that would make the determinant equal to 1.

That is, the probability density function for the N eigenangles of a Haar distributed $\mathrm{SU}(N)$ matrix is [43]

$$\mathcal{P}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_N) := \frac{1}{N!(2\pi)^{N-1}}\mathcal{D}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N)\delta(\theta_1+\ldots+\theta_N \pmod{2\pi}) \tag{2.17}$$

where $\delta(x)$ represents the Dirac delta function. If we integrate it over one of the variables, we have

$$\int_{[-\pi,\pi)} \mathcal{P}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_N)\,d\theta_N = \frac{1}{N!(2\pi)^{N-1}}\mathcal{D}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_{N-1}),$$

where $\mathcal{D}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_{N-1})$ denotes the $\mathrm{SU}(N)$ Dyson product, and is

given by

$$\begin{aligned}\mathcal{D}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_{N-1}) &= \\ &= \int_{[-\pi,\pi)} \mathcal{D}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N)\delta(\theta_1+\ldots+\theta_N \pmod{2\pi})\,d\theta_N \\ &= \mathcal{D}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_{N-1},-\theta_1-\ldots-\theta_{N-1}) \qquad (2.18) \\ &= \prod_{1\le j<k\le N-1} |e^{i\theta_j}-e^{i\theta_k}|^2 \prod_{k=1}^{N-1} |e^{i\theta_k}-e^{-i(\theta_1+\ldots+\theta_{N-1})}|^2.\end{aligned}$$

Now, if A is a $\mathrm{SU}(N)$ matrix then, by definition,

$$\arg(\det A) = \theta_1+\ldots+\theta_N = 0 \pmod{2\pi},$$

and the $\mathrm{U}(N)$ Gram points from the previous section are reduced to, what we will call, the $\mathrm{SU}(N)$ Gram points.

Definition 11. *We define the* $\mathrm{SU}(N)$ *Gram points as*

$$\psi_m^{\mathrm{SU}(N)} := -\pi + \frac{2m\pi}{N}, \quad m = 0,1,\ldots,N-1.$$

We also define a $\mathrm{SU}(N)$ *Gram interval as any interval along the unit circle between two consecutive* $\mathrm{SU}(N)$ *Gram points.*

Similar to the $\mathrm{U}(N)$ case, these represent the points on the unit circle at which the characteristic polynomial of a $\mathrm{SU}(N)$ matrix is real (but not necessarily zero), and they are distributed equidistantly in steps of $\frac{2\pi}{N}$. However, unlike the $\mathrm{U}(N)$ case, the $\mathrm{SU}(N)$ Gram points do not depend in any way on the eigenangles, which implies that they are the same for all $\mathrm{SU}(N)$ matrices, and are also fixed on the unit circle.

As we will later see, this makes it easier to compute the probability of having exactly k eigenvalues of a random $\mathrm{SU}(N)$ matrix inside a $\mathrm{SU}(N)$ Gram interval. For now, we will prove the following result, which relates this quantity with the corresponding probability from the previous subsection:

Lemma 2. *For any* $k = 0,\ldots,N$, *we have*

$$\begin{aligned}&\mathbb{P}[k \text{ eigenvalues of a } \mathrm{U}(N) \text{ matrix lie in a } \mathrm{U}(N) \text{ Gram interval}] = \\ &= \mathbb{P}[k \text{ eigenvalues of a } \mathrm{SU}(N) \text{ matrix lie in a } \mathrm{SU}(N) \text{ Gram interval}],\end{aligned}$$

where the first probability is over Haar measure for $\mathrm{U}(N)$ *and the second probability is over Haar measure for* $\mathrm{SU}(N)$.

Proof. Let $e^{i\theta_1}, \ldots, e^{i\theta_N}$ be the eigenvalues of a U(N) matrix. For simplicity, we will use

$$\mathcal{I} = \left[\frac{\theta_1 + \ldots + \theta_N}{N} - \pi,\ \frac{\theta_1 + \ldots + \theta_N}{N} - \pi + \frac{2\pi}{N}\right) \pmod{2\pi}$$

as a generic U(N) Gram interval, and denote its complement by

$$[-\pi, \pi) \setminus \mathcal{I} = \left[\frac{\theta_1 + \ldots + \theta_N}{N} - \pi + \frac{2\pi}{N},\ \frac{\theta_1 + \ldots + \theta_N}{N} + \pi\right) \pmod{2\pi}.$$

Because the U(N) probability density is a symmetric function in all eigenangles, it can be shown that the probability of having k eigenvalues of a U(N) matrix in a U(N) Gram interval is the same, for any U(N) Gram interval and for any k eigenvalues. Starting with this fact, we have

$$\begin{aligned}
&\mathbb{P}[k \text{ eigenvalues of a U}(N) \text{ matrix lie in a U}(N) \text{ Gram interval}] = \\
&\quad = \binom{N}{k} \mathbb{P}[\theta_1, \ldots, \theta_k \in \mathcal{I} \text{ and } \theta_{k+1}, \ldots, \theta_N \in [-\pi, \pi) \setminus \mathcal{I}] \\
&\quad = \binom{N}{k} \int \ldots \int_{\mathcal{R}} \mathcal{P}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N)\, d\theta_1 \ldots d\theta_N \\
&\quad = \binom{N}{k} \frac{1}{N!(2\pi)^N} \int \ldots \int_{\mathcal{R}} \mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N)\, d\theta_1 \ldots d\theta_N, \qquad (2.19)
\end{aligned}$$

where the N-dimensional integral is over a region $\mathcal{R}$ described by the restrictions

$$\mathcal{R} : \begin{cases} \theta_1, \ldots, \theta_k \in \mathcal{I} \\ \theta_{k+1}, \ldots, \theta_N \in [-\pi, \pi) \setminus \mathcal{I}. \end{cases}$$

This can be written more explicitly as

$$\mathcal{R} : \begin{cases} \dfrac{\theta_1 + \ldots + \theta_N}{N} - \pi \le \theta_n < \dfrac{\theta_1 + \ldots + \theta_N}{N} - \pi + \dfrac{2\pi}{N} & (n = 1, \ldots, k) \\ \dfrac{\theta_1 + \ldots + \theta_N}{N} - \pi + \dfrac{2\pi}{N} \le \theta_n < \dfrac{\theta_1 + \ldots + \theta_N}{N} + \pi & (n = k+1, \ldots, N) \end{cases}$$

which becomes

$$\mathcal{R} : \begin{cases} -\pi \le \theta_n - \dfrac{\theta_1 + \ldots + \theta_N}{N} < -\pi + \dfrac{2\pi}{N} & (n = 1, \ldots, k) \\ -\pi + \dfrac{2\pi}{N} \le \theta_n - \dfrac{\theta_1 + \ldots + \theta_N}{N} < \pi & (n = k+1, \ldots, N) \end{cases}.$$

We now perform the following change of variables

$$\lambda_n = \theta_n - \frac{\theta_1 + \ldots + \theta_N}{N} \quad (n = 1, \ldots, N-1),$$

and

$$\lambda_N = N\theta_N.$$

The determinant of the Jacobian matrix is 1. Note that modulo 2π the restriction on θ_N is lost when it comes to considering λ_N. Furthermore, with this change of variables, we have that

$$\theta_N - \frac{\theta_1 + \ldots + \theta_N}{N} = -\lambda_1 - \ldots - \lambda_{N-1},$$

and the previous region of integration $\mathcal{R}$ is now described by the conditions

$$\mathcal{R}' : \begin{cases} -\pi \leq \lambda_n < -\pi + \dfrac{2\pi}{N} \quad (n = 1, \ldots, k) \\ -\pi + \dfrac{2\pi}{N} \leq \lambda_n < \pi \quad (n = k+1, \ldots, N-1) \\ -\pi + \dfrac{2\pi}{N} \leq -\lambda_1 - \ldots - \lambda_{N-1} < \pi. \end{cases}$$

If we denote a generic SU(N) Gram interval by

$$\mathcal{J} = \left[-\pi,\ -\pi + \frac{2\pi}{N}\right),$$

and its complement as

$$[-\pi, \pi) \setminus \mathcal{J} = \left[-\pi + \frac{2\pi}{N},\ \pi\right),$$

then $\mathcal{R}'$ becomes

$$\mathcal{R}' : \begin{cases} \lambda_1, \ldots, \lambda_n \in \mathcal{J} \\ \lambda_{n+1}, \ldots, \lambda_{N-1} \in [-\pi,\ \pi) \setminus \mathcal{J} \\ -\lambda_1 - \ldots - \lambda_{N-1} \in [-\pi,\ \pi) \setminus \mathcal{J}. \end{cases}$$

Since there is no restriction imposed on λ_N, we may take $\lambda_N \in [-\pi,\ \pi)$. The old variables θ_n can be expressed in terms of the new variables λ_n as

$$\begin{aligned} \theta_n &= \lambda_n + (\lambda_1 + \ldots + \lambda_{N-1}) + \frac{\lambda_N}{N} \qquad (n = 1, \ldots, N-1), \\ \theta_N &= \frac{\lambda_N}{N}. \end{aligned}$$

We note that

$$\theta_m - \theta_n = \lambda_m - \lambda_n \quad \text{for} \quad m, n = 1, \ldots, N-1,$$

and

$$\theta_n - \theta_N = \lambda_n + (\lambda_1 + \ldots + \lambda_{N-1}) \quad \text{for} \quad n = 1, \ldots, N-1,$$

which implies that

$$\begin{aligned}
&\mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N) = \\
&= \prod_{1 \le m < n \le N} |e^{i\theta_m} - e^{i\theta_n}|^2 \\
&= 2^{N(N-1)} \prod_{1 \le m < n \le N} \left(\sin \frac{\theta_m - \theta_n}{2} \right)^2 \\
&= 2^{N(N-1)} \prod_{1 \le m < n \le N-1} \left(\sin \frac{\theta_m - \theta_n}{2} \right)^2 \prod_{n=1}^{N-1} \left(\sin \frac{\theta_n - \theta_N}{2} \right)^2 \\
&= 2^{N(N-1)} \prod_{1 \le m < n \le N-1} \left(\sin \frac{\lambda_m - \lambda_n}{2} \right)^2 \prod_{n=1}^{N-1} \left(\sin \frac{\lambda_n + (\lambda_1 + \ldots + \lambda_{N-1})}{2} \right)^2 \\
&= \prod_{1 \le m < n \le N-1} |e^{i\lambda_m} - e^{i\lambda_n}|^2 \prod_{n=1}^{N-1} |e^{i\lambda_n} - e^{-i(\lambda_1 + \ldots + \lambda_{N-1})}|^2 \\
&= \mathcal{D}_{\mathrm{SU}(N)}(\lambda_1, \ldots, \lambda_{N-1}).
\end{aligned}$$

Putting everything together, we obtain that the initial integral (2.19) can be expressed in the new system of variables as

$$\begin{aligned}
&\binom{N}{k} \frac{1}{N!(2\pi)^N} \int \ldots \int_{\mathcal{R}} \mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N)\, d\theta_1 \ldots d\theta_N \\
&= \binom{N}{k} \frac{1}{N!(2\pi)^N} \int_{-\pi}^{\pi} \left[\int \ldots \int_{\mathcal{R}'} \mathcal{D}_{\mathrm{SU}(N)}(\lambda_1, \ldots, \lambda_{N-1})\, d\lambda_1 \ldots d\lambda_{N-1} \right] d\lambda_N \\
&= \binom{N}{k} \frac{1}{N!(2\pi)^{N-1}} \int \ldots \int_{\mathcal{R}'} \mathcal{D}_{\mathrm{SU}(N)}(\lambda_1, \ldots, \lambda_{N-1})\, d\lambda_1 \ldots d\lambda_{N-1}.
\end{aligned}$$

If we write down explicitly the conditions of $\mathcal{R}'$ into the integral, it becomes

$$\begin{aligned}
\binom{N}{k} \frac{1}{N!(2\pi)^{N-1}} \int_{\mathcal{J}^k} \int_{([-\pi,\pi) \setminus \mathcal{J})^{N-1-k}} &\mathcal{D}_{\mathrm{SU}(N)}(\lambda_1, \ldots, \lambda_{N-1}) \times \\
&\times \chi_{[-\pi,\pi) \setminus \mathcal{J}}(-\lambda_1 - \ldots - \lambda_{N-1} \pmod{2\pi})\, d\lambda_1 \ldots d\lambda_{N-1},
\end{aligned}$$

where $\chi_I(x)$ denotes the characteristic function of the interval I. Now, because

$$\chi_I(x) = \int_I \delta(y - x)\, dy,$$

we can re-introduce into our integral the variable λ_N (that was previously

integrated out) and obtain

$$\begin{aligned}
&\binom{N}{k}\frac{1}{N!(2\pi)^{N-1}}\int_{\mathcal{J}^k}\int_{([-\pi,\pi)\setminus\mathcal{J})^{N-k}}\mathcal{D}_{\mathrm{U}(N)}(\lambda_1,\ldots,\lambda_N)\times\\
&\quad\times\delta(\lambda_1+\ldots+\lambda_N\ (\mathrm{mod}\ 2\pi))\,d\lambda_1\ldots d\lambda_N=\\
&=\binom{N}{k}\int_{\mathcal{J}^k}\int_{([-\pi,\pi)\setminus\mathcal{J})^{N-k}}\mathcal{P}_{\mathrm{SU}(N)}(\lambda_1,\ldots,\lambda_N)\,d\lambda_1\ldots d\lambda_N\\
&=\binom{N}{k}\mathbb{P}[\lambda_1,\ldots,\lambda_k\in\mathcal{J}\ \text{ and}\\
&\quad\lambda_{k+1},\ldots,\lambda_N\in[-\pi,\pi)\setminus\mathcal{J}\text{ and }\lambda_1+\ldots+\lambda_N=0\ (\mathrm{mod}\ 2\pi)]\\
&=\mathbb{P}[k\text{ eigenvalues of a SU}(N)\text{ matrix lie in a SU}(N)\text{ Gram interval}],
\end{aligned}$$

as required. In the last step we have used, as in the beginning, the fact that for any k the probability of having k eigenvalues of a $\mathrm{SU}(N)$ matrix in a $\mathrm{SU}(N)$ Gram interval is the same, for any $\mathrm{SU}(N)$ Gram interval and any k eigenvalues.

□

In analogy with the quantity from (2.11), we will denote the latter probability of Lemma 6 by $E_{\mathrm{SU}(N)}(k,\mathcal{J})$.

Definition 12. *We define the probability of having exactly k unscaled eigenvalues of a random Haar-distributed* $\mathrm{SU}(N)$ *matrix inside a* $\mathrm{SU}(N)$ *Gram interval $\mathcal{J}$ as*

$$E_{\mathrm{SU}(N)}(k,\mathcal{J}):=\binom{N}{k}\int_{\mathcal{J}^k}\int_{([-\pi,\pi)\setminus\mathcal{J})^{N-k}}\mathcal{P}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_N)\,d\theta_1\ldots d\theta_N. \tag{2.20}$$

As we have just proved, $E_{\mathrm{SU}(N)}(k,\mathcal{J})$ also represents the probability of finding precisely k eigenvalues of a random $\mathrm{U}(N)$ matrix in a $\mathrm{U}(N)$ Gram interval, and by using the above formula, we can compute it numerically for the same values of k and N as in Table 2.1; these are presented in Table 2.3.

On the one hand, if we compare those values with the corresponding values from Table 2.1, we see that $E_{\mathrm{SU}(N)}(k,\mathcal{J})$ has a different rate of convergence, in the sense that it doesn't converge to its large N limit (2.14) as fast as $E_{\mathrm{U}(N)}(k,J)$. On the other hand, if we look at the entries in Table 2.2, we notice that for each k and N, $E_{\mathrm{SU}(N)}(k,\mathcal{J})$ does provide a good approximation for $G_{M_N,M_{N+1}}(k)$, in accordance with the Montgomery−Odlyzko Law (CUE).

N	$E_{\mathrm{SU}(N)}(0,\mathcal{J})$	$E_{\mathrm{SU}(N)}(1,\mathcal{J})$	$E_{\mathrm{SU}(N)}(2,\mathcal{J})$
2		1.000000	
3	0.023074	0.954844	0.021090
4	0.040362	0.920641	0.037630
5	0.067146	0.867251	0.064059
6	0.084764	0.832111	0.081483
7	0.097237	0.807224	0.093839
8	0.106532	0.788673	0.103058
9	0.113727	0.774310	0.110200
10	0.119461	0.762860	0.115896
11	0.124138	0.753520	0.120545
12	0.128026	0.745755	0.124412
13	0.131309	0.739197	0.127678
14	0.134118	0.733586	0.130474
15	0.136549	0.728730	0.132894
16	0.138673	0.724486	0.135009
17	0.140545	0.720746	0.136874
18	0.142207	0.717424	0.138530
19	0.143693	0.714455	0.140011
20	0.145029	0.711785	0.141343
21	0.146237	0.709371	0.142547

Table 2.3: $E_{\mathrm{SU}(N)}(k,\mathcal{J})$ for $k=0,1,2$ and $N=2,\ldots,21$

The observation that these two quantities appear to have the same rate of convergence hints at the possibility that they should also have the same asymptotic limit, which leads us to put forward the following alternative to Fujii's conjecture (CUE)

Conjecture 7. *For any $k \in \mathbb{N} \cup \{0\}$, we have*

$$\lim_{N\to\infty} E_{\mathrm{SU}(N)}(k,\mathcal{J}) = \lim_{M\to\infty} G_{0,M}(k)$$

(where $\mathcal{J}$ is a $\mathrm{SU}(N)$ Gram interval).

Finally, we remark that although $E_{\mathrm{SU}(N)}(k,\mathcal{J})$ converges at a slower rate compared to $E_{\mathrm{U}(N)}(k,J)$, this does not necessarily imply that they don't tend to the same limit; in order to clarify whether this is or not the case, we have to obtain a more explicit formula that describes how $E_{\mathrm{SU}(N)}(k,\mathcal{J})$ depends on the matrix size for finite, but arbitrarily large N. This will be the topic of the next chapter, and we close the current chapter with a well-known quote by the great statistician Box [10]

"*All models are wrong, but some are useful.*"

Chapter 3

SU(N) theory

3.1 The standard method

The traditional technique for computing probabilities like $E_{\mathrm{U}(N)}(k, J)$ is presented in detail in [23], [29] and [32]. One of the main quantities of interest needed when studying the eigenvalue distribution of random matrices is the generating function, defined in the case of CUE as follows

Definition 13. *Let $J \subset [-\pi, \pi)$ be an interval of any length, and arbitrarily positioned on the unit circle. If $E_{\mathrm{U}(N)}(k, J)$ is defined as in (2.11), the $\mathrm{U}(N)$ generating function is given by*

$$\mathcal{E}_{\mathrm{U}(N)}(z, J) := \sum_{k=0}^{N} (1+z)^k E_{\mathrm{U}(N)}(k, J).$$

If the generating function is known, one can immediately recover the desired probabilities through repeated differentiation

$$E_{\mathrm{U}(N)}(k, J) = \frac{1}{k!} \left(\frac{d^k}{dz^k} \mathcal{E}_{\mathrm{U}(N)}(z, J) \right) \Bigg|_{z=-1}. \tag{3.1}$$

It can be shown [23] that it is also possible to express the generating function in terms of all the n-level densities (2.4), as a sum

$$\mathcal{E}_{\mathrm{U}(N)}(z, J) = 1 + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J^n} R^n_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n. \tag{3.2}$$

We recall that, because the CUE has a kernel function, we may use Gaudin's lemma to represent each n-level density as an $n \times n$ determinant

$$R^n_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_n) = \det_{n \times n} S_N(\theta_j, \theta_k). \tag{3.3}$$

This, in turn, allows us to apply an identity that is due to Gram [23] to show that the sum (3.2) is equal to an $N \times N$ determinant

Lemma 3 (Gram's identity)**.** *For an interval $J \in [-\pi, \pi)$, we have*

$$1 + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J^n} R^n_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n = \\ = \det_{N\times N} \left[I_N + \frac{z}{2\pi} \left(\int_J e^{i(j-k)\theta} d\theta \right)_{1\le j,k\le N} \right].$$

Putting together the above equations, we obtain an analytic formula for each $E_{\mathrm{U}(N)}(k, J)$, as the k-th order derivative of this determinant

$$E_{\mathrm{U}(N)}(k, J) = \frac{1}{k!} \frac{d^k}{dz^k} \left\{ \det_{N\times N} \left[I_N + \frac{z}{2\pi} \left(\int_J e^{i(j-k)\theta} d\theta \right)_{1\le j,k\le N} \right] \right\} \Bigg|_{z=-1}.$$

The notions of n-level density and generating function may also be extended to the case of $\mathrm{SU}(N)$ matrices

Definition 14. *We define the $\mathrm{SU}(N)$ n-level density of a special unitary matrix as*

$$R^n_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_n) := \frac{N!}{(N-n)!} \int_{[-\pi,\pi)^{N-n}} \mathcal{P}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_N)\, d\theta_{n+1} \ldots d\theta_N. \tag{3.4}$$

Here, we can only have $n = 1, \ldots, N-1$, because if the values of $N-1$ eigenangles are given, then the N-th one is already determined by the restriction $\theta_1 + \ldots + \theta_N = 0 \pmod{2\pi}$, so it can not be assigned to an arbitrary value.

Definition 15. *Let $J \subset [-\pi, \pi)$ be an interval of any length, and arbitrarily positioned on the unit circle. If $E_{\mathrm{SU}(N)}(k, J)$ is defined as in (2.20), the $\mathrm{SU}(N)$ generating function is given by*

$$\mathcal{E}_{\mathrm{SU}(N)}(z, J) := \sum_{k=0}^{N} (1+z)^k E_{\mathrm{SU}(N)}(k, J).$$

As before, the quantity of interest $E_{\mathrm{SU}(N)}(k, J)$ may be expressed in terms of the $\mathrm{SU}(N)$ generating function

$$E_{\mathrm{SU}(N)}(k, J) = \frac{1}{k!} \left(\frac{d^k}{dz^k} \mathcal{E}_{\mathrm{SU}(N)}(z, J) \right) \Bigg|_{z=-1}. \tag{3.5}$$

However, it appears that for $\mathrm{SU}(N)$ matrices there is no known kernel function that would allow one to use Gaudin's lemma and express the n-level density $R_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_n)$ as an $n \times n$ determinant, similar to (3.3). Because of this, it is also not possible to apply Gram's identity and continue the analogy with the CUE. So, instead of trying to compute $E_{\mathrm{SU}(N)}(k, J)$ in analogy with $E_{\mathrm{U}(N)}(k, J)$, in the following sections we will develop a method for computing $E_{\mathrm{SU}(N)}(k, J)$ in terms of $E_{\mathrm{U}(N)}(k, J)$.

3.2 $\mathrm{SU}(N)$ n-level density

In this section, we are going to deduce a formula that expresses the $\mathrm{SU}(N)$ n-level density in terms of the corresponding $\mathrm{U}(N)$ n-level density. We first note that the $\mathrm{U}(N)$ Dyson product (2.15) can be expanded in the form of a trigonometric Fourier series as

$$\begin{aligned}
&\mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N) = \\
&\quad = \prod_{1 \le j < k \le N} |e^{i\theta_j} - e^{i\theta_k}|^2 \\
&\quad = \prod_{1 \le j < k \le N} [2 - 2\cos(\theta_j - \theta_k)] \\
&\quad = \sum_{\substack{p=0 \\ p \neq 1}}^{N} \sum_{\substack{j_1 + \ldots + j_p = 0 \\ j_1, \ldots, j_p \neq 0 \\ |j_l| < N}} b_{j_1 \ldots j_p} \left[\sum_{\{h_1, \ldots, h_p\} \subset \{1, \ldots, N\}} 2\cos(j_1\theta_{h_1} + \ldots + j_p\theta_{h_p}) \right],
\end{aligned}$$

where $b_{j_1 \ldots j_p}$ are called the Dyson coefficients. We have that $b_{j_1, \ldots, j_N} \neq 0$ if and only if the corresponding $j_1, \ldots, j_N \in \mathbb{Z}$ satisfy the conditions $j_1 + \ldots + j_N = 0$ and $|j_l| < N$ for all $l = 1, \ldots, N$. The terms are grouped by the number of distinct variables on which they depend. The term with $p = 0$ represents the constant term (which doesn't depend on any variable), and there is no group with $p = 1$, because it is not possible for a cosine term to depend on only a single variable.

Combining formula (2.4) with (2.16), we can write the $\mathrm{U}(N)$ n-level density in terms of the $\mathrm{U}(N)$ Dyson product as

$$R^n_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_n) = \frac{1}{(N-n)!(2\pi)^N} \int_{[-\pi, \pi)^{N-n}} \mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N) d\theta_{n+1} \ldots d\theta_N. \tag{3.6}$$

If we now insert the expansion of the Dyson product, the n-level

density becomes

$$R^n_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_n) = \frac{1}{(N-n)!(2\pi)^N}\sum_{\substack{p=0\\p\neq 1}}^{N}\sum_{\substack{j_1+\ldots+j_p=0\\j_1,\ldots,j_p\neq 0\\|j_l|<N}} b_{j_1\ldots j_p}\times$$

$$\times\int_{[-\pi,\pi)^{N-n}}\left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,N\}} 2\cos(j_1\theta_{h_1}+\ldots+j_p\theta_{h_p})\right]d\theta_{n+1}\ldots d\theta_N.$$

The above expression can be simplified with the following observation: for each cosine term in the interior sum, if any of the coefficients j_l corresponding to the integration variable θ_{h_l} $(h_l = n+1,\ldots,N)$ is non-zero, then the integral over that term is zero, since the cosine is being integrated over a full period. This implies that the only terms in the sum that give non-zero contributions to the n-level density are those terms for which $j_{n+1} = \ldots = j_N = 0$ or, in other words, the terms that depend only on the eigenangles $\theta_1,\ldots,\theta_n$, plus the constant term.

$$R^n_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_n) = \frac{1}{(N-n)!(2\pi)^N}\sum_{\substack{p=0\\p\neq 1}}^{n}\sum_{\substack{j_1+\ldots+j_p=0\\j_1,\ldots,j_p\neq 0\\|j_l|<N}} b_{j_1\ldots j_p}\times$$

$$\times\left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,n\}} 2\cos(j_1\theta_{h_1}+\ldots+j_p\theta_{h_p})\right]\int_{[-\pi,\pi)^{N-n}} d\theta_{n+1}\ldots d\theta_N.$$

The integral at the end is obviously just $(2\pi)^{N-n}$, which means that

$$R^n_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_n) = \frac{1}{(N-n)!(2\pi)^n}\sum_{\substack{p=0\\p\neq 1}}^{n}\sum_{\substack{j_1+\ldots+j_p=0\\j_1,\ldots,j_p\neq 0\\|j_l|<N}} b_{j_1\ldots j_p}\times$$

$$\times\left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,n\}} 2\cos(j_1\theta_{h_1}+\ldots+j_p\theta_{h_p})\right].$$

If we combine equations (2.17), (2.18) and (3.4), we can relate the $\mathrm{SU}(N)$ n-level density with the $\mathrm{SU}(N)$ Dyson product as

$$R^n_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_n) = \frac{1}{(N-n)!(2\pi)^{N-1}}\times$$

$$\times\int_{[-\pi,\pi)^{N-n-1}}\mathcal{D}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_{N-1})\,d\theta_{n+1}\ldots d\theta_{N-1}. \tag{3.7}$$

We are now going prove the following statement

Theorem 20. *For $n = 1, \ldots, N-1$ fixed, we have*

$$R^n_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_n) = R^n_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_n) + X^n(\theta_1, \ldots, \theta_n), \tag{3.8}$$

where

$$\begin{aligned} X^n(\theta_1, \ldots, \theta_n) :=& \frac{2}{(N-n)!(2\pi)^n} \sum_{p=1}^{n} \sum_{m=1}^{p} \sum_{\substack{k_1+\ldots+k_p=mN \\ k_1,\ldots,k_p\neq 0 \\ -N+m<k_i<N+m}} c_{k_1\ldots k_p} \times \\ & \times \left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,n\}} \cos(k_1\theta_{h_1} + \ldots + k_p\theta_{h_p}) \right] \end{aligned} \tag{3.9}$$

(and the meaning of the $c_{k_1,\ldots,k_p}$ coefficients will be explained throughout the course of the proof).

Proof. We begin by separating the trigonometric Fourier expansion of the $\mathrm{U}(N)$ Dyson product into two parts: a part that contains the terms that depend on θ_N, and a part with the terms that don't

$$\begin{aligned} &\mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N) = \\ &= \sum_{\substack{p=0 \\ p\neq 1}}^{N} \sum_{\substack{j_1+\ldots+j_p=0 \\ j_1,\ldots,j_p\neq 0 \\ |j_l|<N}} b_{j_1\ldots j_p} \left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,N\}} 2\cos(j_1\theta_{h_1} + \ldots + j_p\theta_{h_p}) \right] \\ &= \sum_{\substack{p=0 \\ p\neq 1}}^{N-1} \sum_{\substack{j_1+\ldots+j_p=0 \\ j_1,\ldots,j_p\neq 0 \\ -N<j_i<N}} b_{j_1,\ldots j_p} \left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,N-1\}} 2\cos(j_1\theta_{h_1} + \ldots + j_p\theta_{h_p}) \right] + \\ &\quad + \sum_{\substack{j_1+\ldots+j_{N-1}+j_N=0 \\ j_N\neq 0 \\ -N<j_i<N}} b'_{j_1\ldots j_N} 2\cos(j_1\theta_1 + \ldots + j_{N-1}\theta_{N-1} + j_N\theta_N). \end{aligned}$$

In the first part, the terms are still grouped by the number of non-zero variables on which they depend, while in the second part they are not (due to this, the Dyson coefficients for the second part had to be re-labeled).

The second part will instead be grouped by the values taken by j_N, and since cosine is an even function, we can write it as

$$\sum_{\substack{j_1+\ldots+j_{N-1}+j_N=0 \\ j_N\neq 0 \\ -N<j_i<N}} b'_{j_1\ldots j_N} 2\cos(j_1\theta_1 + \ldots + j_{N-1}\theta_{N-1} + j_N\theta_N) =$$

$$= \sum_{j_N=1}^{N-1} \sum_{\substack{j_1+\ldots+j_{N-1}-j_N=0 \\ -N<j_1,\ldots,j_{N-1}<N}} b'_{j_1\ldots j_N} 2\cos(j_1\theta_1 + \ldots + j_{N-1}\theta_{N-1} - j_N\theta_N).$$

If we perform the substitution $\theta_N = -\theta_1 - \ldots - \theta_{N-1}$, this becomes

$$= \sum_{j_N=1}^{N-1} \sum_{\substack{j_1+\ldots+j_{N-1}-j_N=0 \\ -N<j_1,\ldots,j_{N-1}<N}} b'_{j_1\ldots j_N} 2\cos((j_1+j_N)\theta_1 + \ldots + (j_{N-1}+j_N)\theta_{N-1}).$$

We now group the terms by the number of distinct variables on which they depend and re-label again the coefficients in a more convenient way

$$= \sum_{p=1}^{N-1} \sum_{j_N=1}^{p} \sum_{\substack{j_1+\ldots+j_p=j_N(N-p) \\ j_1,\ldots,j_p\neq -j_N \\ -N<j_i<N}} c_{j_1+j_N,\ldots j_p+j_N} \times$$
$$\times \left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,N-1\}} 2\cos((j_1+j_N)\theta_{h_1} + \ldots + (j_p+j_N)\theta_{h_p}) \right].$$

The above restriction

$$j_1 + \ldots + j_p = j_N(N-p) \tag{3.10}$$

appears in the sum for the following reason: we started with the condition

$$j_1 + \ldots + j_{N-1} - j_N = 0,$$

which is equivalent to

$$j_1 + \ldots + j_p + j_{p+1} + \ldots + j_{N-1} = j_N. \tag{3.11}$$

The condition that only $j_1, \ldots, j_p \neq -j_N$ implies that $j_{p+1} = \ldots = j_{N-1} = -j_N$; thus, (3.11) becomes

$$j_1 + \ldots + j_p + (-j_N)[(N-1) - (p+1) + 1] = j_N,$$

and this leads to requirement (3.10). We note that in this case, there is a group with $p = 1$ because we can have cosine terms that depend on a single variable.

If we now join the two parts back together, we have obtained the

Fourier expansion for the $\mathrm{SU}(N)$ Dyson product (we re-label $j_N = m$)

$$\mathcal{D}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_{N-1}) = \mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_{N-1}, -\theta_1 - \ldots - \theta_{N-1})$$

$$= \sum_{\substack{p=0 \\ p\neq 1}}^{N-1} \sum_{\substack{j_1+\ldots+j_p=0 \\ j_1,\ldots,j_p\neq 0 \\ -N<j_i<N}} b_{j_1,\ldots j_p} \left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,N-1\}} 2\cos(j_1\theta_{h_1} + \ldots + j_p\theta_{h_p}) \right] +$$

$$+ \sum_{p=1}^{N-1} \sum_{m=1}^{p} \sum_{\substack{j_1+\ldots+j_p=m(N-p) \\ j_1,\ldots,j_p\neq -m \\ -N<j_i<N}} c_{j_1+m,\ldots j_p+m} \times$$

$$\times \left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,N-1\}} 2\cos((j_1+m)\theta_{h_1} + \ldots + (j_p+m)\theta_{h_p}) \right].$$

In order to obtain the $\mathrm{SU}(N)$ n-level density, we need to compute the integral of the $\mathrm{SU}(N)$ Dyson product

$$R^n_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_n) = \frac{1}{(N-n)!(2\pi)^{N-1}} \times$$

$$\times \int_{[-\pi,\pi)^{N-n-1}} \mathcal{D}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_{N-1})\, d\theta_{n+1} \ldots d\theta_{N-1}.$$

From the integral of the first part of the $\mathrm{SU}(N)$ Dyson product expansion we recover the $\mathrm{U}(N)$ n-level density

$$\frac{1}{(N-n)!(2\pi)^{N-1}} \sum_{\substack{p=0 \\ p\neq 1}}^{N-1} \sum_{\substack{j_1+\ldots+j_p=0 \\ j_1,\ldots,j_p\neq 0 \\ -N<j_i<N}} b_{j_1,\ldots j_p} \int_{[-\pi,\pi)^{N-n-1}}$$

$$\left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,N-1\}} 2\cos(j_1\theta_{h_1} + \ldots + j_p\theta_{h_p}) \right] d\theta_{n+1} \ldots d\theta_{N-1}$$

$$= \frac{1}{(N-n)!(2\pi)^{n}} \sum_{\substack{p=0 \\ p\neq 1}}^{n} \sum_{\substack{j_1+\ldots+j_p=0 \\ j_1,\ldots,j_p\neq 0 \\ -N<j_i<N}} b_{j_1,\ldots j_p} \times$$

$$\times \left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,n\}} 2\cos(j_1\theta_{h_1} + \ldots + j_p\theta_{h_p}) \right]$$

$$= R^n_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_n).$$

And in a similar way, by integrating the second part of the SU(N) Dyson product, we get the expression that we have denoted as $X^n(\theta_1,\ldots,\theta_n)$

$$\frac{1}{(N-n)!(2\pi)^{N-1}}\sum_{p=1}^{N-1}\sum_{m=1}^{p}\sum_{\substack{j_1+\ldots+j_p=m(N-p)\\ j_1,\ldots,j_p\neq -m\\ -N<j_i<N}} c_{j_1+m,\ldots j_p+m}\int_{[-\pi,\pi)^{N-n-1}}$$

$$\left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,N-1\}} 2\cos((j_1+m)\theta_{h_1}+\ldots+(j_p+m)\theta_{h_p})\right]d\theta_{n+1}\ldots d\theta_{N-1}$$

$$=\frac{1}{(N-n)!(2\pi)^{n}}\sum_{p=1}^{n}\sum_{m=1}^{p}\sum_{\substack{j_1+\ldots+j_p=m(N-p)\\ j_1,\ldots,j_p\neq -m\\ -N<j_i<N}} c_{j_1+m,\ldots j_p+m}\times$$

$$\times\left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,n\}} 2\cos((j_1+m)\theta_{h_1}+\ldots+(j_p+m)\theta_{h_p})\right].$$

The indices of the Dyson coefficients may be re-labeled as

$$k_i=j_i+m, \tag{3.12}$$

and the above expression becomes

$$=\frac{1}{(N-n)!(2\pi)^{n}}\sum_{p=1}^{n}\sum_{m=1}^{p}\sum_{\substack{k_1+\ldots+k_p=mN\\ k_1,\ldots,k_p\neq 0\\ -N+m<k_i<N+m}} c_{k_1\ldots k_p}\times$$

$$\times\left[\sum_{\{h_1,\ldots,h_p\}\subset\{1,\ldots,n\}} 2\cos(k_1\theta_{h_1}+\ldots+k_p\theta_{h_p})\right]$$

$$=X^n(\theta_1,\ldots,\theta_n).$$

□

The main idea of the above proof can be summarized in the following way: all the terms in $\mathcal{D}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N)$ that depend only on $\theta_1,\ldots,\theta_n$ also appear unchanged in $\mathcal{D}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_{N-1})$ (with the same coefficients) and this, in turn, implies that all the terms of the $R^n_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_n)$ are always included among the terms of the $R^n_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_n)$. Furthermore, for $m=1,\ldots,n$ there are other terms in $\mathcal{D}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N)$ that depend on all the eigenangles, such as

$$\cos(j_1\theta_1+\ldots+j_n\theta_n-m\theta_{n+1}-\ldots-m\theta_N), \quad \text{with} \quad j_1+\ldots+j_n = m(N-n),$$

which, after the substitution $\theta_N = -\theta_1 - \ldots - \theta_{N-1}$, depend only on $\theta_1, \ldots, \theta_n$

$$\cos((j_1+m)\theta_1 + \ldots + (j_n+m)\theta_n).$$

Therefore, these are terms that do not appear in the U(N) n-level density, but contribute to the SU(N) n-level density; all these extra terms are collectively denoted by $X^n(\theta_1, \ldots, \theta_n)$.

For later notational simplicity, we will extend the definition of $X^n(\theta_1, \ldots, \theta_n)$ to the cases $n = 0$ and $n = N$

$$X^n(\theta_1, \ldots, \theta_n) = \tag{3.13}$$
$$= \begin{cases} 0, & n = 0 \\ R^n_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_n) - R^n_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_n), & n = 1, \ldots, N-1 \\ N!\mathcal{P}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_N) - R^N_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N), & n = N \end{cases}$$

where $\mathcal{P}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_{N-1})$ was defined in (2.17).

3.3 SU(N) generating function

We will now use the fact that each SU(N) n-level density depends on the U(N) n-level density to prove that the SU(N) generating function can also be expressed in terms of the corresponding U(N) generating function.

Theorem 21. *Let $J \subset [-\pi, \pi)$ be an interval of any length, and arbitrarily positioned on the unit circle. We have*

$$\mathcal{E}_{\mathrm{SU}(N)}(z, J) = \mathcal{E}_{\mathrm{U}(N)}(z, J) + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J^n} X^n(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n. \tag{3.14}$$

Proof. The Weyl integration formula for the case of SU(N) matrices can be stated as

$$\int_{\mathrm{SU}(N)} f(\theta_1, \ldots, \theta_N)\, d\mu_{\mathrm{SU}(N)} =$$
$$= \int_{[-\pi,\pi)^N} f(\theta_1, \ldots, \theta_N)\mathcal{P}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_N)\, d\theta_1 \ldots d\theta_N.$$

We will begin by proving the following equality

$$\int_{\mathrm{SU}(N)} (1 + z\chi_J(\theta_1)) \dots (1 + z\chi_J(\theta_N))\, d\mu_{\mathrm{SU}(N)} = \sum_{k=0}^{N} (1+z)^k E_{\mathrm{SU}(N)}(k, J)$$

(where, again, $\chi_J(\theta)$ is the characteristic function). For $k = 0, \dots, N$, let $P_k \subset \mathrm{SU}(N)$ be the subset of all $\mathrm{SU}(N)$ matrices with exactly k eigenangles in J. Obviously, these sets are all pairwise disjoint and their union is $\mathrm{SU}(N)$, so they form a partition of $\mathrm{SU}(N)$. This implies that

$$\begin{aligned}
&\int_{\mathrm{SU}(N)} (1 + z\chi_J(\theta_1)) \dots (1 + z\chi_J(\theta_N))\, d\mu_{\mathrm{SU}(N)} = \\
&= \int_{P_0 \cup \dots \cup P_N} (1 + z\chi_J(\theta_1)) \dots (1 + z\chi_J(\theta_N))\, d\mu_{\mathrm{SU}(N)} \\
&= \sum_{k=0}^{N} \int_{P_k} (1 + z\chi_J(\theta_1)) \dots (1 + z\chi_J(\theta_N))\, d\mu_{\mathrm{SU}(N)} \\
&= \int_{P_0} d\mu_{\mathrm{SU}(N)} + (1+z) \int_{P_1} d\mu_{\mathrm{SU}(N)} + \dots + (1+z)^N \int_{P_N} d\mu_{\mathrm{SU}(N)}
\end{aligned}$$

and, by definition, $E_{\mathrm{SU}(N)}(k, J)$ represents the measure of the set of all $\mathrm{SU}(N)$ matrices which have precisely k eigenangles in J, so

$$E_{\mathrm{SU}(N)}(k, J) = \int_{P_k} d\mu_{\mathrm{SU}(N)}.$$

Therefore, we are able to write the $\mathrm{SU}(N)$ generating function as

$$\begin{aligned}
&\mathcal{E}_{\mathrm{SU}(N)}(z, J) = \\
&= \sum_{k=0}^{N} (1+z)^k E_{\mathrm{SU}(N)}(k, J) \\
&= \int_{\mathrm{SU}(N)} (1 + z\chi_J(\theta_1)) \dots (1 + z\chi_J(\theta_N))\, d\mu_{\mathrm{SU}(N)} \\
&= \int_{[-\pi.\pi)^N} (1 + z\chi_J(\theta_1)) \dots (1 + z\chi_J(\theta_N)) \mathcal{P}_{\mathrm{SU}(N)}(\theta_1, \dots, \theta_N)\, d\theta_1 \dots d\theta_N.
\end{aligned}$$

After we open all the brackets, this becomes

$$\begin{aligned}
&\mathcal{E}_{\mathrm{SU}(N)}(z, J) = \\
&= 1 + \sum_{n=1}^{N} z^n \int_{[-\pi,\pi)^N} h_n(\chi_J(\theta_1), \dots, \chi_J(\theta_N)) \mathcal{P}_{\mathrm{SU}(N)}(\theta_1, \dots, \theta_N)\, d\theta_1 \dots d\theta_N,
\end{aligned}$$

where the h_n's are elementary symmetric polynomials in $\chi_J(\theta_1), \dots, \chi_J(\theta_N)$

$$\begin{cases} h_1 = \chi_J(\theta_1) + \ldots + \chi_J(\theta_N) \\ h_2 = \sum_{1\leq i<j\leq N} \chi_J(\theta_i)\chi_J(\theta_j) \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ h_N = \chi_J(\theta_1)\ldots\chi_J(\theta_N). \end{cases}$$

Because each h_n has $\binom{N}{n}$ terms, and $\mathcal{P}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_N)$ is invariant under the permutation of any of its arguments, the $\mathrm{SU}(N)$ generating function becomes

$$\begin{aligned} &\mathcal{E}_{\mathrm{SU}(N)}(z,J) = \\ &= 1 + \sum_{n=1}^{N} z^n \frac{N!}{n!(N-n)!} \int_{[-\pi,\pi)^N} \chi_J(\theta_1)\ldots\chi_J(\theta_n)\times \\ &\quad \times \mathcal{P}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_N)\, d\theta_1\ldots d\theta_N \\ &= 1 + \sum_{n=1}^{N} \frac{z^n}{n!} \frac{N!}{(N-n)!} \int_{J^n} \int_{[-\pi,\pi)^{N-n}} \mathcal{P}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_N)\, d\theta_1\ldots d\theta_N \\ &= 1 + \sum_{n=1}^{N} \frac{z^n}{n!} \int_{J^n} \left[\frac{N!}{(N-n)!} \times \right. \\ &\quad \left. \times \int_{[-\pi,\pi)^{N-n}} \mathcal{P}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_N) d\theta_{n+1}\ldots d\theta_N \right] d\theta_1\ldots d\theta_n \\ &= 1 + \sum_{n=1}^{N-1} \frac{z^n}{n!} \int_{J^n} R^n_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_n)\, d\theta_1\ldots d\theta_n + \\ &\quad + \frac{z^N}{N!} \int_{J^N} N! \mathcal{P}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_N)\, d\theta_1\ldots d\theta_N, \end{aligned}$$

where in the last line we used (3.4) to obtain $R^n_{\mathrm{SU}(N)}$ for $n = 1,\ldots,N-1$. Now, if we apply (3.13), we find that $\mathcal{E}_{\mathrm{SU}(N)}(z,J)$ is

$$\begin{aligned} &1 + \sum_{n=1}^{N-1} \frac{z^n}{n!} \int_{J^n} [R^n_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_n) + X^n(\theta_1,\ldots,\theta_n)]\, d\theta_1\ldots d\theta_n + \\ &\quad + \frac{z^N}{N!} \int_{J^N} [R^N_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N) + X^N(\theta_1,\ldots,\theta_N)]\, d\theta_1\ldots d\theta_N. \end{aligned}$$

And by re-grouping the terms, this equals

$$1+\sum_{n=1}^{N}\frac{z^n}{n!}\int_{J^n}R^n_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_n)\,d\theta_1\ldots d\theta_n+$$
$$+\sum_{n=1}^{N}\frac{z^n}{n!}\int_{J^n}X^n(\theta_1,\ldots,\theta_n)\,d\theta_1\ldots d\theta_n.$$

Noting that the first term is the $\mathrm{U}(N)$ generating function (3.2), we have shown that

$$\mathcal{E}_{\mathrm{SU}(N)}(z,J)=\mathcal{E}_{\mathrm{U}(N)}(z,J)+\sum_{n=1}^{N}\frac{z^n}{n!}\int_{J^n}X^n(\theta_1,\ldots,\theta_n)\,d\theta_1\ldots d\theta_n,$$

which was our desired result. □

The above formula can be used, together with (3.1) and (3.5), to derive a similar relation between $E_{\mathrm{SU}(N)}(k,J)$ and $E_{\mathrm{U}(N)}(k,J)$.

Theorem 22. *Let $J\subset[-\pi,\pi)$ be an interval of any length, and arbitrarily positioned on the unit circle. For any $k=0,\ldots,N$, we have*

$$E_{\mathrm{SU}(N)}(k,J)=E_{\mathrm{U}(N)}(k,J)+\sum_{n=k}^{N}\frac{(-1)^{n-k}}{k!(n-k)!}\int_{J^n}X^n(\theta_1,\ldots,\theta_n)\,d\theta_1\ldots d\theta_n. \tag{3.15}$$

Proof. We have

$$\begin{aligned}E_{\mathrm{SU}(N)}(k,J)&=\frac{1}{k!}\left(\frac{d^k}{dz^k}\mathcal{E}_{\mathrm{SU}(N)}(z,J)\right)\Big|_{z=-1}\\&=\frac{1}{k!}\left(\frac{d^k}{dz^k}\mathcal{E}_{\mathrm{U}(N)}(z,J)\right)\Big|_{z=-1}+\\&\quad+\frac{1}{k!}\left(\frac{d^k}{dz^k}\sum_{n=1}^{N}\frac{z^n}{n!}\int_{J^n}X^n(\theta_1,\ldots,\theta_n)\,d\theta_1\ldots d\theta_n\right)\Big|_{z=-1}\\&=E_{\mathrm{U}(N)}(k,J)+\sum_{n=k}^{N}\frac{(-1)^{n-k}}{k!(n-k)!}\int_{J^n}X^n(\theta_1,\ldots,\theta_n)\,d\theta_1\ldots d\theta_n.\end{aligned}$$

□

Since the quantity $E_{\mathrm{U}(N)}(k,J)$ is already very well understood, the last important unknown is the magnitude of terms such as

$$\int_{J^n}X^n(\theta_1,\ldots,\theta_n)\,d\theta_1\ldots d\theta_n.$$

In order to obtain this, we need to have an estimate for $c_{k_1\ldots k_p}$ in (3.9).

3.4 Upper bound on the Dyson coefficients

If we set $z_j = e^{i\theta_j}$, it can be seen that

$$\begin{aligned} |e^{i\theta_j} - e^{i\theta_k}|^2 &= |z_j - z_k|^2 \\ &= (z_j - z_k)(z_j^{-1} - z_k^{-1}) \\ &= 2 - \frac{z_j}{z_k} - \frac{z_k}{z_j}. \end{aligned}$$

With this notation, the $\mathrm{U}(N)$ Dyson product (2.15) becomes

$$\mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N) = \prod_{1 \le j < k \le N} \left(2 - \frac{z_j}{z_k} - \frac{z_k}{z_j}\right). \tag{3.16}$$

In general, it is a non-trivial task to obtain an exact analytic formula for the coefficients of the Dyson product. Numerical results suggest that they have the following upper bound

Conjecture 8. *Let $1 \le m \le p < N$ be fixed. If $j_1, \ldots, j_p \in \mathbb{Z}$ satisfy the conditions*

$$-N < j_i < N, \quad j_1 + \ldots + j_p = m(N-p) \quad \textit{and} \quad j_1, \ldots, j_p \neq -m,$$

then the terms in the expansion of the $\mathrm{U}(N)$ Dyson product (3.16) that take the form

$$z_1^{j_1} \ldots z_p^{j_p} \frac{1}{z_{p+1}^m \ldots z_N^m}$$

have coefficients whose absolute value is at most

$$p!(N-p)!.$$

In this notation, the substitution

$$\theta_N = -\theta_1 - \ldots - \theta_{N-1}$$

is equivalent to

$$z_N = \frac{1}{z_1 \ldots z_{N-1}},$$

so the above conjecture also claims that the terms in the expansion of the $\mathrm{SU}(N)$ Dyson product that take the form

$$z_1^{j_1+m} \ldots z_p^{j_p+m}$$

have coefficients whose absolute value is at most

$$p!(N-p)!.$$

Due to the way the labeling of the indices was chosen in (3.12), this inequality is assumed true, in particular, for the coefficients in the definition of the $X^n(\theta_1, \ldots, \theta_n)$ term (3.9), therefore

$$|c_{k_1 \ldots k_p}| \leq p!(N-p)!. \tag{3.17}$$

3.5 Magnitude of the additional terms

Up to this point, $J \subset [-\pi, \pi)$ was allowed to be an interval of any length, and arbitrarily positioned on the unit circle; from here on, we will focus specifically on $\mathrm{SU}(N)$ Gram intervals, such as $\mathcal{J} = \left[-\pi, -\pi + \frac{2\pi}{N}\right)$

Theorem 23. *Let $\mathcal{J}$ be a $\mathrm{SU}(N)$ Gram interval. Under the assumption (3.17), we have*

$$\int_{\mathcal{J}^n} X^n(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n = \mathcal{O}\left(\frac{n!}{N}\frac{n \log n}{\pi^n}\right), \tag{3.18}$$

uniformly for $2 \leq n \leq N$.

Proof. We know from Section 3.2 that

$$X^n(\theta_1, \ldots, \theta_n) = \frac{2}{(N-n)!(2\pi)^n} \sum_{p=1}^{n} \sum_{m=1}^{p} \sum_{\substack{k_1+\ldots+k_p = mN \\ k_1, \ldots, k_p \neq 0 \\ -N+m < k_i < N+m}} c_{k_1 \ldots k_p} \times$$
$$\times \left[\sum_{\{h_1, \ldots, h_p\} \subset \{1, \ldots, n\}} \cos(k_1 \theta_{h_1} + \ldots + k_p \theta_{h_p}) \right].$$

It can be shown that for $\{h_1, \ldots, h_p\} \subset \{1, \ldots, n\}$ and $k_1, \ldots, k_p \neq 0$

$$\int_{\mathcal{J}^n} \cos(k_1 \theta_{h_1} + \ldots + k_p \theta_{h_p})\, d\theta_1 \ldots d\theta_n =$$
$$= \left(\frac{2\pi}{N}\right)^{n-p} \frac{2^p}{k_1 \ldots k_p} \sin\left(\frac{k_1 \pi}{N}\right) \ldots \sin\left(\frac{k_p \pi}{N}\right) \times$$
$$\times \cos\left((k_1 + \ldots + k_p)\left(-\pi + \frac{\pi}{N}\right)\right).$$

If, in addition, we also have $k_1 + \ldots + k_p = mN$, then this becomes

$$\int_{\mathcal{J}^n} \cos(k_1\theta_{h_1} + \ldots + k_p\theta_{h_p})\, d\theta_1 \ldots d\theta_n =$$
$$= \left(\frac{2\pi}{N}\right)^{n-p} \frac{2^p}{k_1 \ldots k_p} \sin\left(\frac{k_1\pi}{N}\right) \ldots \sin\left(\frac{k_p\pi}{N}\right) (-1)^{m(N-1)}.$$

We can apply this formula to compute the integral of $X^n(\theta_1, \ldots, \theta_n)$, term by term, and obtain

$$\int_{\mathcal{J}^n} X^n(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n =$$

$$= \frac{2}{(N-n)!(2\pi)^n} \sum_{p=2}^{n} \sum_{m=1}^{p} \sum_{\substack{k_1+\ldots+k_p=mN \\ k_1,\ldots,k_p\neq 0 \\ -N+m<k_i<N+m}} c_{k_1\ldots k_p} \times$$
$$\times \binom{n}{p} \left(\frac{2\pi}{N}\right)^{n-p} \frac{2^p}{k_1 \ldots k_p} \sin\left(\frac{k_1\pi}{N}\right) \ldots \sin\left(\frac{k_p\pi}{N}\right) (-1)^{m(N-1)}$$

$$= \frac{2}{(N-n)!} \sum_{p=2}^{n} \sum_{m=1}^{p} \sum_{\substack{k_1+\ldots+k_p=mN \\ k_1,\ldots,k_p\neq 0 \\ -N+m<k_i<N+m}} c_{k_1\ldots k_p} \times$$
$$\times \binom{n}{p} \left(\frac{1}{N}\right)^{n-p} \left(\frac{1}{\pi}\right)^{p} \frac{1}{k_1 \ldots k_p} \sin\left(\frac{k_1\pi}{N}\right) \ldots \sin\left(\frac{k_p\pi}{N}\right) (-1)^{m(N-1)}.$$

Using the triangle inequality, we have

$$\left| \int_{\mathcal{J}^n} X^n(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n \right| \leq$$
$$\leq 2 \sum_{p=2}^{n} \sum_{m=1}^{p} \sum_{\substack{k_1+\ldots+k_p=mN \\ k_1,\ldots,k_p\neq 0 \\ -N+m<k_i<N+m}} |c_{k_1\ldots k_p}| \frac{1}{(N-n)!} \binom{n}{p} \left(\frac{1}{N}\right)^{n-p} \left(\frac{1}{\pi}\right)^{p} \times$$
$$\times \left| \frac{1}{k_1 \ldots k_p} \sin\left(\frac{k_1\pi}{N}\right) \ldots \sin\left(\frac{k_p\pi}{N}\right) \right|.$$

This is where we apply the upper bound (3.17) on the Dyson coefficients

$$|c_{k_1\ldots k_p}| \leq p!(N-p)!,$$

to obtain

$$\left|\int_{\mathcal{J}^n} X^n(\theta_1,\ldots,\theta_n)\,d\theta_1\ldots d\theta_n\right| \leq$$
$$\leq 2\sum_{p=2}^{n}\sum_{m=1}^{p}\sum_{\substack{k_1+\ldots+k_p=mN\\ k_1,\ldots,k_p\neq 0\\ -N+m<k_i<N+m}} \frac{(N-p)!}{(N-n)!}\binom{n}{p}\left(\frac{1}{N}\right)^{n-p}\frac{p!}{\pi^p}\times$$
$$\times\left|\frac{1}{k_1\ldots k_p}\sin\left(\frac{k_1\pi}{N}\right)\ldots\sin\left(\frac{k_p\pi}{N}\right)\right|.$$

Next, it can be shown that

$$\frac{(N-p)!}{(N-n)!}\left(\frac{1}{N}\right)^{n-p}\leq 1 \quad \text{for} \quad p=2,\ldots,n,$$

which implies

$$\left|\int_{\mathcal{J}^n} X^n(\theta_1,\ldots,\theta_n)\,d\theta_1\ldots d\theta_n\right| \leq 2\sum_{p=2}^{n}\frac{n!}{(n-p)!}\times \tag{3.19}$$
$$\times\frac{1}{\pi^p}\sum_{m=1}^{p}\sum_{\substack{k_1+\ldots+k_p=mN\\ k_1,\ldots,k_p\neq 0\\ -N+m<k_i<N+m}}\left|\frac{1}{k_1\ldots k_p}\sin\left(\frac{k_1\pi}{N}\right)\ldots\sin\left(\frac{k_p\pi}{N}\right)\right|.$$

Now, we will focus on estimating the interior sum of (3.19). Using the properties of the sinc function it can be shown by induction over p that

$$\sum_{\substack{k_1+\ldots+k_p=mN\\ k_1,\ldots,k_p\neq 0\\ -N+m<k_i<N+m}}\left|\frac{1}{k_1\ldots k_p}\sin\left(\frac{k_1\pi}{N}\right)\ldots\sin\left(\frac{k_p\pi}{N}\right)\right| =$$
$$=\sum_{\substack{k_1+\ldots+k_p=mN\\ -N+m<k_i<N+m}}\left(\frac{\pi}{N}\right)^p\left|\operatorname{sinc}\left(\frac{k_1\pi}{N}\right)\ldots\operatorname{sinc}\left(\frac{k_p\pi}{N}\right)\right|+\mathcal{O}\left(\frac{1}{N^2}\right).$$

This was done because it is more convenient to work with the sum over the entire range of integers $-N+m<k_i<N+m$, without the restriction $k_1,\ldots,k_p\neq 0$. For this sum, we have

$$\sum_{\substack{k_1+\ldots+k_p=mN\\ -N+m<k_i<N+m}}\left(\frac{\pi}{N}\right)^p\left|\operatorname{sinc}\left(\frac{k_1\pi}{N}\right)\ldots\operatorname{sinc}\left(\frac{k_p\pi}{N}\right)\right| =$$
$$=\sum_{\substack{k_1+\ldots+k_p=mN\\ -N+m<k_i<N+m}}\left(\frac{\pi}{N}\right)^p\left|\frac{k_1+\ldots+k_p}{mN}\operatorname{sinc}\left(\frac{k_1\pi}{N}\right)\ldots\operatorname{sinc}\left(\frac{k_p\pi}{N}\right)\right|.$$

We may apply again the triangle inequality for the above modulus

$$\leq \left(\frac{\pi}{N}\right)^p \frac{1}{mN} \left[\sum_{\substack{k_1+\ldots+k_p=mN \\ -N+m<k_i<N+m}} \left| k_1 \operatorname{sinc}\left(\frac{k_1\pi}{N}\right) \ldots \operatorname{sinc}\left(\frac{k_p\pi}{N}\right) \right| + \ldots \right.$$

$$\left. \ldots + \sum_{\substack{k_1+\ldots+k_p=mN \\ -N+m<k_i<N+m}} \left| k_p \operatorname{sinc}\left(\frac{k_1\pi}{N}\right) \ldots \operatorname{sinc}\left(\frac{k_p\pi}{N}\right) \right| \right].$$

Because $k_1, \ldots, k_p$ range over the same values, the p terms in the bracket above are all equal to each other, so we can keep only one of them

$$= \left(\frac{\pi}{N}\right)^p \frac{p}{mN} \sum_{\substack{k_1+\ldots+k_p=mN \\ -N+m<k_i<N+m}} \left| k_p \operatorname{sinc}\left(\frac{k_1\pi}{N}\right) \ldots \operatorname{sinc}\left(\frac{k_{p-1}\pi}{N}\right) \operatorname{sinc}\left(\frac{k_p\pi}{N}\right) \right|$$

$$= \left(\frac{\pi}{N}\right)^{p-1} \frac{p}{mN} \sum_{\substack{k_1+\ldots+k_p=mN \\ -N+m<k_i<N+m}} \left| \operatorname{sinc}\left(\frac{k_1\pi}{N}\right) \ldots \operatorname{sinc}\left(\frac{k_{p-1}\pi}{N}\right) \sin\left(\frac{k_p\pi}{N}\right) \right|.$$

We now substitute $k_p = mN - k_1 - \ldots - k_{p-1}$ in order to eliminate it from the summation

$$= \frac{p}{mN} \sum_{\substack{k_1,\ldots,k_{p-1}<N+m \\ mN-N-m<k_1+\ldots+k_{p-1}}} \left(\frac{\pi}{N}\right)^{p-1} \times$$

$$\times \left| \operatorname{sinc}\left(\frac{k_1\pi}{N}\right) \ldots \operatorname{sinc}\left(\frac{k_{p-1}\pi}{N}\right) \sin\left(\frac{(k_1+\ldots+k_{p-1})\pi}{N}\right) \right|.$$

This expression can be written as an iterated sum, and through repeated application of the Euler−Maclaurin formula (1.16), this iterated sum can be approximated by a multiple integral, that is bounded by a constant

$$\sum_{\substack{k_1,\ldots,k_{p-1}<N+m \\ mN-N-m<k_1+\ldots+k_{p-1}}} \left(\frac{\pi}{N}\right)^{p-1} \left| \operatorname{sinc}\left(\frac{k_1\pi}{N}\right) \ldots \operatorname{sinc}\left(\frac{k_{p-1}\pi}{N}\right) \times \right.$$

$$\left. \times \sin\left(\frac{(k_1+\ldots+k_{p-1})\pi}{N}\right) \right| \approx$$

$$\approx \int_{\substack{x_1,\ldots,x_{p-1}<\pi(1+\frac{m}{N}) \\ \pi(m-1-\frac{m}{N})<x_1+\ldots+x_{p-1}}} |\operatorname{sinc}(x_1) \ldots \operatorname{sinc}(x_{p-1}) \times$$

$$\times \sin(x_1+\ldots+x_{p-1})|\, dx_1 \ldots dx_{p-1} = \mathcal{O}(1).$$

Putting together all of the arguments above, we get the following upper

bound for the interior sum of (3.19)

$$\sum_{\substack{k_1+\ldots+k_p=mN \\ k_1,\ldots,k_p\neq 0 \\ -N+m<k_i<N+m}} \left| \frac{1}{k_1\ldots k_p} \sin\left(\frac{k_1\pi}{N}\right)\ldots\sin\left(\frac{k_p\pi}{N}\right)\right| < \frac{p}{mN}\,\mathcal{C}$$

(where $\mathcal{C} > 0$ is a constant, and will not necessarily denote the same value at every subsequent occurrence). Summing over m and using the well-known formula

$$\sum_{m=1}^{p} \frac{1}{m} = \mathcal{O}(\log p),$$

we get an upper bound for the middle sum of (3.19)

$$\sum_{m=1}^{p} \sum_{\substack{k_1+\ldots+k_p=mN \\ k_1,\ldots,k_p\neq 0 \\ -N+m<k_i<N+m}} \left| \frac{1}{k_1\ldots k_p} \sin\left(\frac{k_1\pi}{N}\right)\ldots\sin\left(\frac{k_p\pi}{N}\right)\right| < \frac{p\log p}{N}\,\mathcal{C}.$$

Finally, we may estimate the exterior sum of (3.19) if we sum over p

$$\sum_{p=2}^{n} \frac{n!}{(n-p)!}\frac{1}{\pi^p} \sum_{m=1}^{p} \sum_{\substack{k_1+\ldots+k_p=mN \\ k_1,\ldots,k_p\neq 0 \\ -N+m<k_i<N+m}} \left| \frac{1}{k_1\ldots k_p} \sin\left(\frac{k_1\pi}{N}\right)\ldots\sin\left(\frac{k_p\pi}{N}\right)\right| <$$

$$< \sum_{p=2}^{n} \frac{n!}{(n-p)!}\frac{1}{\pi^p}\frac{p\log p}{N}\,\mathcal{C}$$

$$= \frac{n!}{N}\left[\sum_{p=2}^{n} \frac{p\log p}{(n-p)!\pi^p}\right]\mathcal{C}.$$

It can be shown that the last term in the above sum is the dominant term

$$\sum_{p=2}^{n} \frac{p\log p}{(n-p)!\pi^p} = \mathcal{O}\left(\frac{n\log n}{\pi^n}\right),$$

which gives us the result that was stated at the beginning

$$\left|\int_{\mathcal{J}^n} X^n(\theta_1,\ldots,\theta_n)\,d\theta_1\ldots d\theta_n\right| < \frac{n!}{N}\,\frac{n\log n}{\pi^n}\,\mathcal{C}.$$

□

We now have all the necessary elements to describe how $E_{\mathrm{SU}(N)}(k,\mathcal{J})$

and $E_{\mathrm{U}(N)}(k,\mathcal{J})$ are related asymptotically.

Theorem 24. *Let $\mathcal{J}$ be a $\mathrm{SU}(N)$ Gram interval. For any $k=0,\dots,N$, under the assumption (3.17), we have*

$$E_{\mathrm{SU}(N)}(k,\mathcal{J}) = E_{\mathrm{U}(N)}(k,\mathcal{J}) + \mathcal{O}\left(\frac{1}{N}\right).$$

Proof. We know from (3.15) that

$$E_{\mathrm{SU}(N)}(k,\mathcal{J}) = E_{\mathrm{U}(N)}(k,\mathcal{J}) + \sum_{n=k}^{N} \frac{(-1)^{n-k}}{k!(n-k)!} \int_{\mathcal{J}^n} X^n(\theta_1,\dots,\theta_n)\, d\theta_1 \dots d\theta_n.$$

By applying the triangle inequality for the distance between $E_{\mathrm{SU}(N)}(k,\mathcal{J})$ and $E_{\mathrm{U}(N)}(k,\mathcal{J})$, together with formula (3.18), we get

$$\begin{aligned}
\left|E_{\mathrm{SU}(N)}(k,\mathcal{J}) - E_{\mathrm{U}(N)}(k,\mathcal{J})\right| &= \left|\sum_{n=k}^{N} \frac{(-1)^{n-k}}{k!(n-k)!} \int_{J^n} X^n(\theta_1,\dots,\theta_n)\, d\theta_1 \dots d\theta_n\right| \\
&\leq \sum_{n=k}^{N} \frac{1}{k!(n-k)!} \left|\int_{\mathcal{J}^n} X^n(\theta_1,\dots,\theta_n)\, d\theta_1 \dots d\theta_n\right| \\
&< \sum_{n=k}^{N} \frac{1}{k!(n-k)!} \frac{n!}{N} \frac{n\log n}{\pi^n} \mathcal{C} \\
&= \frac{1}{N}\left[\sum_{n=k}^{N} \binom{n}{k} \frac{n\log n}{\pi^n}\right] \mathcal{C}.
\end{aligned}$$

For k fixed, if we consider the series

$$S = \sum_{n=k}^{\infty} a_n,$$

where

$$a_n = \binom{n}{k} \frac{n\log n}{\pi^n},$$

it can be shown that

$$\begin{aligned}
L &= \lim_{n\to\infty} n\left(\frac{a_n}{a_{n+1}} - 1\right) \\
&= \lim_{n\to\infty} n\left[\pi \frac{n(n+1-k)\log n}{(n+1)^2 \log(n+1)} - 1\right] \\
&= \infty.
\end{aligned}$$

And because $L > 1$, according to the Raabe−Duhamel test for convergence, this implies that the series S is convergent. In particular, it implies

that the partial sum of S is bounded by a constant

$$\sum_{n=k}^{N} \binom{n}{k} \frac{n \log n}{\pi^n} = \mathcal{O}(1),$$

which gives us the main result

$$\left| E_{\mathrm{SU}(N)}(k, \mathcal{J}) - E_{\mathrm{U}(N)}(k, \mathcal{J}) \right| < \frac{1}{N} \mathcal{C}.$$

□

3.6 The X^1 and X^2 terms

Now that we know the magnitude of $\int_{\mathcal{J}^n} X^n(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n$, we will focus on computing this quantity explicitly for $n = 1, 2$

Theorem 25. *We have*

$$a) \quad \int_{\mathcal{J}} X^1(\theta_1)\, d\theta_1 = 0,$$

$$b) \quad \int_{\mathcal{J}^2} X^2(\theta_1, \theta_2)\, d\theta_1 d\theta_2 = -\frac{1}{N} \frac{4[\gamma + \log 2\pi - \mathrm{Ci}(2\pi)]}{\pi^2} + \mathcal{O}\left(\frac{1}{N^3}\right),$$

where $\mathrm{Ci}(x)$ *is the cosine integral function.*

Proof. a) As previously mentioned, the only terms from the Fourier expansion of the Dyson product that make non-zero contributions to the one-level density are the constant term and the cosine terms that depend only on θ_1. In the case of the $\mathrm{U}(N)$ Dyson product, there is no cosine term that depends only on θ_1, and the constant term is known to be $N!$ ([33], [36], [111], [112])

$$\mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N) = N! + \ldots.$$

By applying (3.6), we recover the known $\mathrm{U}(N)$ one-level density

$$R^1_{\mathrm{U}(N)}(\theta_1) = \frac{N}{2\pi}.$$

In order to compute the $\mathrm{SU}(N)$ one-level density, we return to the notation $z_j = e^{i\theta_j}$ from Section 3.4, where we had

$$\mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N) = \prod_{1 \le j < k \le N} \left(2 - \frac{z_j}{z_k} - \frac{z_k}{z_j}\right). \tag{3.20}$$

We separate the factors with z_1 from the rest of the product

$$\mathcal{D}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N) = \left(2-\frac{z_1}{z_2}-\frac{z_2}{z_1}\right)\ldots\left(2-\frac{z_1}{z_N}-\frac{z_N}{z_1}\right)\times \\ \times \prod_{2\le j<k\le N}\left(2-\frac{z_j}{z_k}-\frac{z_k}{z_j}\right).$$

When we open all the brackets on the first row, one of the terms that we get is

$$\left(2-\frac{z_1}{z_2}-\frac{z_2}{z_1}\right)\ldots\left(2-\frac{z_1}{z_N}-\frac{z_N}{z_1}\right) = (-1)^{N-1}\frac{z_1^{N-1}}{z_2\ldots z_N} + \text{ other terms.}$$

This term is obtained simply by taking the fraction with z_1 in the numerator from each of the above factors. And since the second row

$$\prod_{2\le j<k\le N}\left(2-\frac{z_j}{z_k}-\frac{z_k}{z_j}\right) = \mathcal{D}_{\mathrm{U}(N-1)}(\theta_2,\ldots,\theta_N)$$

doesn't contain any z_1 and its constant term is $(N-1)!$, this implies that the coefficient of

$$\frac{z_1^{N-1}}{z_2\ldots z_N}$$

in the expansion of (3.20) is $(-1)^{N-1}(N-1)!$

Similarly, the coefficient of

$$\frac{z_2\ldots z_N}{z_1^{N-1}}$$

is also $(-1)^{N-1}(N-1)!$.

Now, because

$$\frac{z_1^{N-1}}{z_2\ldots z_N}+\frac{z_2\ldots z_N}{z_1^{N-1}} = e^{i((N-1)\theta_1-\theta_2-\ldots-\theta_N)}+e^{-i((N-1)\theta_1-\theta_2-\ldots-\theta_N)} \\ = 2\cos((N-1)\theta_1-\theta_2-\ldots-\theta_N),$$

this means that in the Fourier expansion of $\mathcal{D}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N)$ the coefficient of

$$\cos((N-1)\theta_1-\theta_2-\ldots-\theta_N)$$

will be $2(-1)^{N-1}(N-1)!$.

$$\mathcal{D}_{\mathrm{U}(N)}(\theta_1,\ldots,\theta_N) = N!+2(-1)^{N-1}(N-1)!\cos((N-1)\theta_1-\theta_2-\ldots-\theta_N)+\ldots$$

We know that by setting $\theta_N = -\theta_1 - \ldots - \theta_{N-1}$, $\mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N)$ becomes $\mathcal{D}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_{N-1})$, and the term $\cos((N-1)\theta_1 - \theta_2 - \ldots - \theta_N)$ becomes $\cos(N\theta_1)$. Obviously, the constant term of $\mathcal{D}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_{N-1})$ is still $N!$, so we have that

$$\mathcal{D}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_{N-1}) = N! + 2(-1)^{N-1}(N-1)!\cos(N\theta_1) + \ldots.$$

We see that, unlike $\mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N)$, the $\mathcal{D}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_{N-1})$ does contain a cosine term that depends only on θ_1. We now use (3.7) to get the $\mathrm{SU}(N)$ one-level density

$$\begin{aligned} R^1_{\mathrm{SU}(N)}(\theta_1) &= \frac{N}{2\pi} + \frac{2(-1)^{N-1}\cos(N\theta_1)}{2\pi} \\ &= R^1_{\mathrm{U}(N)}(\theta_1) + \frac{2(-1)^{N-1}\cos(N\theta_1)}{2\pi}. \end{aligned}$$

By definition, $X^1(\theta_1)$ is the difference between $R^1_{\mathrm{SU}(N)}(\theta_1)$ and $R^1_{\mathrm{U}(N)}(\theta_1)$,

$$X^1(\theta_1) = \frac{2(-1)^{N-1}\cos(N\theta_1)}{2\pi}.$$

It is now straightforward to see that if we integrate $X^1(\theta_1)$ over a $\mathrm{SU}(N)$ Gram interval like

$$\mathcal{J} = \left[-\pi,\ -\pi + \frac{2\pi}{N}\right),$$

we get

$$\int_{\mathcal{J}} X^1(\theta_1)\, d\theta_1 = 0.$$

b) For the $\mathrm{U}(N)$ two-level density, we get a contribution from the constant term, together with the cosine terms that depend only on θ_1, θ_2, which are

$$\mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N) = N! - (N-2)!\, 2\sum_{a=1}^{N-1}(N-a)\cos(a\theta_1 - a\theta_2) + \ldots$$

and, as a consequence of (3.6)

$$R^2_{\mathrm{U}(N)}(\theta_1, \theta_2) = \frac{1}{(2\pi)^2}\left[(N-1)N - 2\sum_{a=1}^{N-1}(N-a)\cos(a\theta_1 - a\theta_2)\right].$$

Now, in order to compute the $\mathrm{SU}(N)$ two-level density, we recall that $\mathcal{D}_{\mathrm{SU}(N)}(\theta_1, \ldots, \theta_{N-1})$ contains all the terms in $\mathcal{D}_{\mathrm{U}(N)}(\theta_1, \ldots, \theta_N)$ that depended only on θ_1, θ_2 plus several additional ones, which are

$$\mathcal{D}_{\mathrm{SU}(N)}(\theta_1,\ldots,\theta_{N-1}) = \left[N! - (N-2)!\sum_{a=1}^{N-1} 2(N-a)\,\cos(a\theta_1 - a\theta_2)\right] +$$
$$+ 2(-1)^{N-1}(N-1)![\cos(N\theta_1)+\cos(N\theta_2)] + 4(N-2)!\cos(N\theta_1+N\theta_2) -$$
$$- 2(N-2)![\cos((N+1)\theta_1+(N-1)\theta_2)+\cos((N-1)\theta_1+(N+1)\theta_2)] +$$
$$+ 4(-1)^{N-2}(N-2)!\sum_{\substack{k_1=1\\k_1+k_2=N}}^{N-1}\sum_{k_2=1}^{N-1}\cos(k_1\theta_1+k_2\theta_2) + \ldots.$$

Applying (3.7), this leads to the fact that the SU(N) two-level density can be written in terms of the U(N) two-level density plus an additional contribution

$$\begin{aligned} R^2_{\mathrm{SU}(N)}(\theta_1,\theta_2) &= \\ &= R^2_{\mathrm{U}(N)}(\theta_1,\theta_2) + \frac{2}{(2\pi)^2}[(-1)^{N-1}(N-1)[\cos(N\theta_1)+\cos(N\theta_2)] + \\ &\quad + 2(-1)^{N-2}\sum_{\substack{k_1=1\\k_1+k_2=N}}^{N-1}\sum_{k_2=1}^{N-1}\cos(k_1\theta_1+k_2\theta_2) + 2\cos(N\theta_1+N\theta_2) - \\ &\quad - \cos((N+1)\theta_1+(N-1)\theta_2) - \cos((N-1)\theta_1+(N+1)\theta_2)]. \end{aligned}$$

As mentioned, this additional contribution represents the $X^2(\theta_1,\theta_2)$ term

$$\begin{aligned} X^2(\theta_1,\theta_2) &= \\ &= \frac{2}{(2\pi)^2}[(-1)^{N-1}(N-1)[\cos(N\theta_1)+\cos(N\theta_2)] + \\ &\quad + 2(-1)^{N-2}\sum_{\substack{k_1=1\\k_1+k_2=N}}^{N-1}\sum_{k_2=1}^{N-1}\cos(k_1\theta_1+k_2\theta_2) + 2\cos(N\theta_1+N\theta_2) - \\ &\quad - \cos((N+1)\theta_1+(N-1)\theta_2) - \cos((N-1)\theta_1+(N+1)\theta_2)]. \end{aligned}$$

This can be used to show that its integral over a pair of SU(N) Gram intervals is given by

$$\begin{aligned} &\int_{\mathcal{J}^2} X^2(\theta_1,\theta_2)\,d\theta_1 d\theta_2 = \\ &\quad = \frac{4}{\pi^2}\left[\frac{1}{N^2-1}\left(\sin\frac{\pi}{N}\right)^2 - \sum_{\substack{k_1=1\\k_1+k_2=N}}^{N-1}\sum_{k_2=1}^{N-1}\frac{1}{k_1\,k_2}\sin\left(\frac{k_1\pi}{N}\right)\sin\left(\frac{k_2\pi}{N}\right)\right]. \end{aligned}$$

We remark that the double sum can be re-expressed as a single sum

$$\begin{aligned}
&\sum_{\substack{k_1=1\\k_1+k_2=N}}^{N-1}\sum_{k_2=1}^{N-1}\frac{1}{k_1\,k_2}\sin\left(\frac{k_1\pi}{N}\right)\sin\left(\frac{k_2\pi}{N}\right)=\\
&=\sum_{k=1}^{N-1}\frac{1}{k(N-k)}\sin\left(\frac{k\pi}{N}\right)\sin\left(\pi-\frac{k\pi}{N}\right)\\
&=\frac{2}{N}\sum_{k=1}^{N}\frac{1}{k}\left(\sin\frac{k\pi}{N}\right)^2,
\end{aligned}$$

and using Euler−Maclaurin summation (1.16), this sum can be approximated by an integral,

$$\begin{aligned}
&\sum_{k=1}^{N}\frac{1}{k}\left(\sin\frac{k\pi}{N}\right)^2=\\
&=\int_1^N\frac{1}{x}\left(\sin\frac{x\pi}{N}\right)^2dx+\mathcal{O}\left(\frac{1}{N^2}\right)\\
&=\int_{\frac{1}{N}}^1\frac{(\sin y\pi)^2}{y}\,dy+\mathcal{O}\left(\frac{1}{N^2}\right).
\end{aligned}$$

The integral, in turn, can be expressed in terms of the cosine integral function $\mathrm{Ci}(x)$

$$\int_{\frac{1}{N}}^1\frac{(\sin y\pi)^2}{y}\,dy=\frac{1}{2}\left[\mathrm{Ci}\left(\frac{2\pi}{N}\right)-\mathrm{Ci}(2\pi)-\log\left(\frac{1}{N}\right)\right],$$

where $\mathrm{Ci}(x)$ is defined as

$$\begin{aligned}
\mathrm{Ci}(x)&:=-\int_x^\infty\frac{\cos t}{t}\,dt\\
&=\gamma+\log x+\sum_{n=1}^{\infty}\frac{(-1)^n x^{2n}}{2n(2n)!}.
\end{aligned}$$

This implies that the above sum is equal to

$$\sum_{k=1}^{N}\frac{1}{k}\left(\sin\frac{k\pi}{N}\right)^2=\frac{1}{2}[\gamma+\log(2\pi)-\mathrm{Ci}(2\pi)]+\mathcal{O}\left(\frac{1}{N^2}\right),$$

while the double sum becomes

$$\sum_{\substack{k_1=1\\k_1+k_2=N}}^{N-1}\sum_{k_2=1}^{N-1}\frac{1}{k_1\,k_2}\sin\left(\frac{k_1\pi}{N}\right)\sin\left(\frac{k_2\pi}{N}\right)=\frac{1}{N}[\gamma+\log(2\pi)-\mathrm{Ci}(2\pi)]+\mathcal{O}\left(\frac{1}{N^3}\right).$$

In the case of the first term in the integral of $X^2(\theta_1, \theta_2)$, as N increases, it has the order of magnitude

$$\frac{1}{N^2-1}\left(\sin\frac{\pi}{N}\right)^2 = \mathcal{O}\left(\frac{1}{N^4}\right).$$

Putting the previous results back together, we obtain that the contribution coming from the integral of $X^2(\theta_1, \theta_2)$ is

$$\int_{\mathcal{J}^2} X^2(\theta_1, \theta_2)\, d\theta_1 d\theta_2 = -\frac{\alpha}{N} + \mathcal{O}\left(\frac{1}{N^3}\right),$$

where

$$\begin{aligned}\alpha &:= \frac{4[\gamma + \log(2\pi) - \mathrm{Ci}(2\pi)]}{\pi^2} \\ &\approx 0.987944\ldots.\end{aligned}$$

□

Numerical results suggest that the value of the integral of $X^n(\theta_1, \ldots, \theta_n)$ over $\mathcal{J}^n$ decreases as $n = 2, \ldots, N$ increases, which implies that the integral of the $X^2(\theta_1, \theta_2)$ term gives the main error to $E_{\mathrm{SU}(N)}(k, \mathcal{J})$. However, according to (3.15), $X^2(\theta_1, \theta_2)$ appears in $E_{\mathrm{SU}(N)}(k, \mathcal{J})$ only for $k = 0, 1, 2$. Keeping also in mind that in this case, the term $n = 1$ of the sum is zero, we get the following approximations for these probabilities

$$\begin{aligned}E_{\mathrm{SU}(N)}(0, \mathcal{J}) &= E_{\mathrm{U}(N)}(0, \mathcal{J}) + \sum_{n=2}^{N} \frac{(-1)^n}{0!n!} \int_{\mathcal{J}^n} X^n(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n \\ &= E_{\mathrm{U}(N)}(0, \mathcal{J}) + \frac{1}{2}\int_{\mathcal{J}^2} X^2(\theta_1, \theta_2)\, d\theta_1 d\theta_2 + \ldots \\ &\approx E_{\mathrm{U}(N)}(0, \mathcal{J}) - \frac{\alpha}{2N},\end{aligned}$$

$$\begin{aligned}E_{\mathrm{SU}(N)}(1, \mathcal{J}) &= E_{\mathrm{U}(N)}(1, \mathcal{J}) + \sum_{n=2}^{N} \frac{(-1)^{n-1}}{1!(n-1)!} \int_{\mathcal{J}^n} X^n(\theta_1, \ldots, \theta_n)\, d\theta_1 \ldots d\theta_n \\ &= E_{\mathrm{U}(N)}(1, \mathcal{J}) - \int_{\mathcal{J}^2} X^2(\theta_1, \theta_2)\, d\theta_1 d\theta_2 + \ldots \\ &\approx E_{\mathrm{U}(N)}(1, \mathcal{J}) + \frac{\alpha}{N},\end{aligned}$$

$$
\begin{aligned}
E_{\mathrm{SU}(N)}(2,\mathcal{J}) &= E_{\mathrm{U}(N)}(2,\mathcal{J}) + \sum_{n=2}^{N} \frac{(-1)^{n-2}}{2!(n-2)!} \int_{\mathcal{J}^n} X^n(\theta_1,\ldots,\theta_n)\, d\theta_1 \ldots d\theta_n \\
&= E_{\mathrm{U}(N)}(2,\mathcal{J}) + \frac{1}{2}\int_{\mathcal{J}^2} X^2(\theta_1,\theta_2)\, d\theta_1 d\theta_2 + \ldots \\
&\approx E_{\mathrm{U}(N)}(2,\mathcal{J}) - \frac{\alpha}{2N}.
\end{aligned}
$$

In conclusion, we note that in the large N limit, the $\mathrm{SU}(N)$-probability of finding 0, 1, or 2 eigenangles in a $\mathrm{SU}(N)$ Gram interval converges to the $\mathrm{U}(N)$-probability of finding 0, 1, or 2 eigenangles in an arbitrary interval of length $\frac{2\pi}{N}$, and we have estimated the rate of convergence. This confirms a prediction previously made by Odlyzko [79], who claimed that

> "*it seems reasonable to expect that at large heights the local distribution of the zeros will be independent of Gram points, which leads to the above assumption* (2.10)*. In other words, the expectation is that at large heights, any grid of points spaced like the Gram points would exhibit similar behavior with respect to location of zeros.*"

Chapter 4

Numerical analysis

4.1 Description of data

In order to further verify the results of the previous chapter, I have developed a computer code in C++ that analyzed Gram's Law for the zeta zeros of the LMFDB database [62], [81], [83]. This database contains the imaginary parts of the first 103,800,788,359 non-trivial zeros, all stored at $\pm 2^{-102}$ absolute precision, or an accuracy of almost 30 decimals.

The zeros are located in 14,580 binary files, which together occupy approximatively 1.3 terabytes. Each file is named `zeros_<t>.dat`, where `<t>` represents the starting point up the critical line covered by this file. The first file is `zeros_14.dat` (because 14 is the integer part of the smallest zeta zero), and the final file is `zeros_29536946000.dat`, which covers the range from `T0 = 30,607,946,000` to `T1 = 30,610,046,000`. For most files, we have `T1 - T0 = 2,100,000`, except for the first 4 files, which span a shorter range

- `zeros_14.dat` covers $[14,\ 5,000]$;
- `zeros_5000.dat` covers $[5,000,\ 26,000]$;
- `zeros_26000.dat` covers $[26,000,\ 236,000]$;
- `zeros_236000.dat` covers $[236,000,\ 446,000]$.

For each file, the zeros are grouped in a certain number of blocks (most files contain 1,000 blocks, with each covering a range of 2,100). Also, every file `zeros_<t>.dat` has the following structure

- `B`, a 64 bit unsigned integer that represents the exact number of blocks in the current file;

- for each block
  - `t0`, a 64 bit double, giving the height at which this block starts;
  - `t1`, a 64 bit double, giving the height at which this block ends;
  - `N0`, a 64 bit unsigned integer, that is the number of zeros with imaginary part between 0 and `t0`;
  - `N1`, a 64 bit unsigned integer, that is the number of zeros with imaginary part between 0 and `t1`;
  - $y_1, \ldots, y_{N1-N0}$, are 104 bit unsigned integers, which contain the information necessary to reconstruct the zeros.

Remark 7. *These blocks are not related in any way to the concept of Gram blocks, introduced in Chapter 1.*

4.2 The IEEE double-precision floating-point format

For each block, the values which represent the heights `t0`, `t1` are stored in the IEEE 754 double-precision binary floating-point format. For example, in the case of the `zeros_14.dat` file, the first 8 bytes (or 64 bits), expressed in hexadecimal, are

```
01 00 00 00 00 00 00 00.
```

This implies that the file has `B = 1` blocks. The next 8 bytes are

```
00 00 00 00 00 00 2C 40.
```

If we reverse the order, this becomes

```
40 2C 00 00 00 00 00 00.
```

Converting from hexadecimal to binary, we get a string of 64 bits

```
01000000 00101100 00000000 00000000
00000000 00000000 00000000 00000000
```

The first bit represents the sign

```
s = 0.
```

The next 11 bits give the exponent,

`10000000010`

which, converted to decimal, is

`e = 1026.`

The remaining 52 bits form the mantissa

$$b_{51} \dots b_0 = 1100 \dots 00.$$

The general formula for converting from IEEE 754 double-precision binary floating-point format is

$$(-1)^s \, 2^{e-1023} \left(1 + \sum_{i=1}^{52} b_{52-i} 2^{-i} \right).$$

Applying this in the above example, we get

$$(-1)^0 \, 2^{1026-1023} (1 + 2^{-1} + 2^{-2}) = 2^3 (1 + 2^{-1} + 2^{-2}) = 14.0,$$

which is the height `t0` at which the file `zeros_14.dat` begins.

4.3 Reconstruction of zeta zeros

Each block covers a certain range `[t0, t1]` on the critical line, and contains the information necessary to reconstruct the zeros in that range. This information is essentially a list of gaps between the consecutive `N1-N0` zeros within that block. Each gap y_m is stored as 13 bytes (104 bits), and the n-th zero of that block may be computed from the first n gaps, using the formula

$$z_n = t_0 + 2^{-101} \sum_{m=1}^{n} y_m.$$

The n-th zero can also be computed in terms of the previous zero and y_n

$$z_n = z_{n-1} + 2^{-101} y_n.$$

For example, going back to the file `zeros_14.dat`, the first gap is

given by

```
04 4F AB 19 B6 62 18 6D 73 DE 65 80 DA.
```

Converting from hexadecimal to decimal, this becomes

$$y_1 = 341568813571635725508154851546.$$

Therefore, the value of the first zero is

$$\begin{aligned} z_1 &= t_0 + 2^{-101} y_1 \\ &= 14.134725141734693790457251983 5625. \end{aligned}$$

Similarly, for the second gap, we have

```
DC 64 E1 5F 41 47 BE 5F DE 59 F2 29 77
```

which can be covered to

$$y_2 = 17461416712258725981254736554359,$$

and thus we get that the second zero is

$$\begin{aligned} z_2 &= t_0 + 2^{-101}(y_1 + y_2) \\ &= 21.022039638771554992628479593 8969. \end{aligned}$$

With these remarks, it is straightforward to implement a function that computes the $(i+1)$-th zero inside a block, where $i \geq 0$. For simplicity and clarity, all the functions presented in this chapter are written in pseudo-code, rather than a particular programming language (all integer variables will implicitly be assumed as unsigned integers).

```
function compute_ith_zero( &zero, i, start, *memblock )
{
  float y = HexToDec(&memblock[start + 40 + i*13], 13)
  zero = zero + pow(2.0, -101) * y
}
```

Here, `memblock` is a pointer to the location of the memory address where the content of the entire current file is stored; `start` represents the number of bytes from the beginning of the file up to the current block. In order to reach the gap for the $(i+1)$-th zero, we have to pass the first `start` bytes of `memblock`, plus another $8 \times 5 = 40$ bytes (from `B`,

`t0`, `t1`, `N0`, `N1` of the current block), plus the gaps for the first i zeros, each of which has 13 bytes. Once this position is reached, the 13 bytes corresponding to the $(i+1)$-th gap are converted from hexadecimal to decimal and stored in the variable `y`. The variable `zero` initially contains the value of the i-th zero, which is subsequently replaced with the $(i+1)$-th zero.

4.4 Computation of Gram points

Unlike the zeta zeros, the first 100,000,000,000 Gram points are not stored in memory; instead they are computed directly during run-time. The function `g(n, error)` takes as input an integer `n` and a real number `error`. It computes the `n`-th Gram point at a precision of at least `error` (set to 10^{-30}), using the technique described in Section 1.4. The final result is stored in the real variable `res`.

The following auxiliary functions are also required

- `lambert_w0(x)` is a C++ implementation for the Lambert W function (1.22) (which can be found in the Boost library [9]);
- `theta(x)` is an implementation of the theta function, using the first terms of the asymptotic series expansion (1.11);
- `theta_prime(x)` represents the first order derivative of the theta function (1.24);
- `abs(x)` returns the absolute value of `x`.

```
function g(n, error)
{
  float e  = 2.71828182845904523536028747135 2
  float pi = 3.14159265358979323846264338327 9

  float res = 2 * pi * exp(1 + lambert_w0((8*n + 1) /
     (8*e)))

  // newton's method for f(x) = 0
  // f(x) = theta(x) - n*pi
  // f'(x) = theta'(x)
  // x = x - f(x)/f'(x)

  float f = theta(res) - n * pi
  float abs_f = abs(f)
```

```
  while( abs_f > error )
    {
      float f_prime = theta_prime(res)
      res = res - f / f_prime
      f = theta(res) - n * pi
      abs_f = abs(f)
    }
  return res
}
```

4.5 Gram's Law within a block

In general, we will be interested in verifying Gram's Law between two Gram points denoted by `g_n1` and `g_n2`. The input variables are the indices `n1`, `n2`, and the Gram points are computed with the function described in the previous section

```
float g_n1 = g(n1, error)
float g_n2 = g(n2, error)
```

The simplest case for checking Gram's Law is that in which both `g_n1` and `g_n2` are located in the same block, of the same file. The function `grams_law_in_block_from_upto()` treats this case. Among its arguments, `g_n1`, `g_n2`, `n1`, `start`, `t0`, `t1`, `memblock` and `error` have already been presented; `nr_of_zeros` represents the total number of zeros in that block. `GL[]` is a vector (or array) with 5 integer elements, that are at first all initialized to 0. At run-time, the zeros between `t0` and `g_n1` are ignored. Then, for each Gram interval between `g_n1` and `g_n2`, the variable `counter` counts the number of zeros in that interval, and `GL[counter]` is incremented by 1. At the end, `GL[0]` has the number of Gram intervals with no zeros, `GL[1]` gives the number of Gram intervals that have exactly 1 zero, and so on.

The case when `g_n1` and `g_n2` are in different blocks requires 3 separate functions; however, each of these is just a variation of the function presented above. The function `grams_law_in_block_from()` processes the block that contains `g_n1`, the function `grams_law_in_block_upto()` processes the block that has `g_n2`, and `grams_law_in_block_whole()` analyses every intermediate block between them.

The function `grams_law_in_block_from()` ignores all the zeros between `t0` and `g_n1` and analyses in the usual way all the Gram intervals between `g_n1`, and the last Gram point before `t1`; finally, it counts the zeros between the last Gram point and `t1`, it stores the number in the

transfer_counter variable, which is sent to the function that handles the next block, together with the vector GL[]

The function grams_law_in_block_upto() receives transfer_counter and GL[] from the function that handled the previous block; it counts the zeros between t0 and the first Gram point, it adds the number to transfer_counter, and GL[transfer_counter] is increased by 1. The Gram intervals, first Gram point and g_n2 are analyzed in the usual way.

The function grams_law_in_block_whole() is essentially a combination of the two functions described above.

```
function grams_law_in_block_from_upto( g_n1, g_n2, n1,
   nr_of_zeros, start, t0, t1, *memblock, GL[], error )
{
  // initialization
  int i = 0
  float zero = t0
  compute_ith_zero( zero, i, start, memblock )

  while( (zero < g_n1) & (i < nr_of_zeros) )
    {
      i++
      compute_ith_zero( zero, i, start, memblock )
    }

  int counter
  int n = n1 + 1    // index of the next gram point
  float gp_old = g(n-1)
  float gp_new = g(n)  // intermediate gram point

  while( gp_new <= g_n2 )
    {
      counter = 0

      while( (zero < gp_new) & (i < nr_of_zeros) )
        {
          counter++
          i++
          compute_ith_zero( zero, i, start, memblock )
        }
      GL[counter]++
      n++
      gp_old = gp_new
      gp_new = g(n)
    }
}
```

```
function grams_law_in_block_from( g_n1, n1, nr_of_zeros
   , start, t0, t1, *memblock, GL[], &transfer_counter,
    &transfer_n, error )
{
  // initialization
  int i = 0
  float = t0
  compute_ith_zero( zero, i, start, memblock )

  // ignore zeros below g_n1
  while( (zero < g_n1) & (i < nr_of_zeros) )
    {
      i++
      compute_ith_zero( zero, i, start, memblock )
    }

  int counter
  int n = n1 + 1
  float gp_old = g(n-1)
  float gp_new = g(n)

  // verifify GL between g_n1 and the last gram point
     smaller than t1
  while( gp_new <= t1 )
    {
      counter = 0
      while( (zero < gp_new) & (i < nr_of_zeros) )
        {
          counter++
          i++;
          compute_ith_zero( zero, i, start, memblock )
        }
      GL[counter]++
      n++
      gp_old = gp_new
      gp_new = g(n)
    }

  // count the zeros between the last gram point
     smaller than t1, and t1
  transfer_counter = nr_of_zeros - i
  transfer_n = n

  if (transfer_counter > 0)
      for ( int j = i+1; j < nr_of_zeros; j++ )
          compute_ith_zero( zero, j, start, memblock )
}
```

```
function grams_law_in_block_upto( g_n2, nr_of_zeros,
   start, t0, t1, *memblock, GL[], &transfer_counter, &
   transfer_n, error )
{
  // initialization
  int i = 0
  float zero = t0
  compute_ith_zero( zero, i, start, memblock )

  int counter = transfer_counter
  int n = transfer_n
  float gp_old = g(n-1)
  float gp_new = g(n)

  // take care of the gram interval in the gap between
     blocks
  while( (zero < gp_new) & (i < nr_of_zeros) )
    {
      counter++
      i++
      compute_ith_zero( zero, i, start, memblock )
    }

  GL[counter]++
  n++

  gp_old = gp_new
  gp_new = g(n)

  // take care of the rest of the gram intervals up to
     g_n2
  while( gp_new <= g_n2 )
    {
      counter = 0

      while( (zero < gp_new) & (i < nr_of_zeros) )
        {
          counter++
          i++
          compute_ith_zero( zero, i, start, memblock )
        }
      GL[counter]++
      n++
      gp_old = gp_new
      gp_new = g(n)
    }
}
```

```
function grams_law_in_block_whole( nr_of_zeros, start,
   t0, t1, *memblock, GL[], &transfer_counter, &
   transfer_n, error )
{
  // initialization
  int i = 0
  float zero = t0
  compute_ith_zero( zero, i, start, memblock )

  int counter = transfer_counter
  int n = transfer_n
  float gp_old = g(n-1)
  float gp_new = g(n)

  // take care of the gram interval in the gap between
     blocks
  while( (zero < gp_new) & (i < nr_of_zeros) )
    {
      counter++
      i++
      compute_ith_zero( zero, i, start, memblock )
    }

  GL[counter]++
  n++

  gp_old = gp_new
  gp_new = g(n)

  // verifify GL between g_n1 and the last gram point
     smaller than t1
  while( gp_new <= t1 )
    {
      counter = 0

      while( (zero < gp_new) & (i < nr_of_zeros) )
        {
          counter++
          i++
          compute_ith_zero( zero, i, start, memblock )
        }

      GL[counter]++
      n++
      gp_old = gp_new
      gp_new = g(n)
    }
```

```
    // count the zeros between the last gram point
       smaller than t1, and t1
    transfer_counter = nr_of_zeros - i
    transfer_n = n

    if (transfer_counter > 0)
        for ( int j = i+1; j < nr_of_zeros; j++ )
            compute_ith_zero( zero, j, start, memblock )
}
```

It is straightforward to see how the above functions can be combined to treat the case when `g_n1` and `g_n2` are in different blocks, of different files (this was the case for all the results presented in the next section).

4.6 Gram's Law for the first 10^{11} intervals

The following table displays the number (not the proportion) of Gram intervals that contain exactly k zeros, for the first 100 billion Gram intervals, in groups of 1 billion intervals per row (in other words, we have $10^9\, G_{M,M+10^9}(k)$, for $M = 0, 10^9, \ldots, 99 \times 10^9$).

M	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
0	137078280	727627711	133509760	1784226	23
1,000,000,000	139668886	722499776	135993815	1837499	24
2,000,000,000	140511345	720832311	136801377	1854932	35
3,000,000,000	141037402	719785380	137317062	1860129	27
4,000,000,000	141417109	719033978	137680746	1868138	29
5,000,000,000	141705730	718460220	137962408	1871603	39
6,000,000,000	141947538	717979999	138197436	1874980	47
7,000,000,000	142167980	717544668	138406770	1880536	46
8,000,000,000	142338445	717206257	138572188	1883073	37
9,000,000,000	142482740	716919638	138712555	1885016	51
10,000,000,000	142634785	716619586	138856535	1889032	62
11,000,000,000	142761621	716366072	138983042	1889216	49
12,000,000,000	142870614	716151258	139085699	1892373	56
13,000,000,000	142971851	715949722	139185047	1893335	45
14,000,000,000	143066090	715761605	139278578	1893670	57
15,000,000,000	143153458	715590466	139358746	1897278	52
16,000,000,000	143239730	715418877	139443120	1898208	65
17,000,000,000	143310496	715275867	139516832	1896751	54
18,000,000,000	143392098	715116064	139591643	1900130	65
19,000,000,000	143451241	714998796	139648761	1901126	76

Table 4.1: $10^9\, G_{M,M+10^9}(k)$ for $M = 0, 10^9, \ldots, 19 \times 10^9$

M	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
20,000,000,000	143522107	714858356	139717026	1902452	59
21,000,000,000	143590287	714722152	139784903	1902590	68
22,000,000,000	143637865	714625660	139835144	1901272	59
23,000,000,000	143693181	714516592	139887332	1902836	59
24,000,000,000	143756289	714395768	139939663	1908213	67
25,000,000,000	143795064	714316431	139982011	1906430	64
26,000,000,000	143847858	714210151	140036191	1905733	67
27,000,000,000	143889000	714129704	140073650	1907589	57
28,000,000,000	143932799	714042955	140115747	1908444	55
29,000,000,000	143985994	713937651	140166782	1909507	66
30,000,000,000	144024610	713859117	140207998	1908213	62
31,000,000,000	144057847	713793432	140239652	1909011	58
32,000,000,000	144105138	713701490	140281672	1911634	66
33,000,000,000	144139426	713630359	140321078	1909063	74
34,000,000,000	144176147	713559787	140352050	1911953	63
35,000,000,000	144209054	713494545	140383813	1912522	66
36,000,000,000	144237921	713435636	140415021	1911366	56
37,000,000,000	144273180	713366219	140448090	1912443	68
38,000,000,000	144308785	713293899	140485920	1911323	73
39,000,000,000	144340647	713233335	140511464	1914479	75
40,000,000,000	144376060	713163245	140545404	1915218	73
41,000,000,000	144401710	713110714	140573517	1913983	76
42,000,000,000	144434488	713045395	140605823	1914218	76
43,000,000,000	144462344	712991960	140629124	1916495	77
44,000,000,000	144478251	712957862	140649578	1914254	55
45,000,000,000	144512422	712891763	140679292	1916439	84
46,000,000,000	144550011	712816411	140717205	1916312	61
47,000,000,000	144556732	712804567	140720742	1917888	71
48,000,000,000	144594118	712730015	140757705	1918073	89
49,000,000,000	144608245	712699958	140775412	1916322	63
50,000,000,000	144632777	712651920	140797900	1917332	71
51,000,000,000	144662888	712592768	140825880	1918384	80
52,000,000,000	144687093	712545048	140848694	1919096	69
53,000,000,000	144706592	712505135	140870038	1918151	84
54,000,000,000	144722747	712474556	140882730	1919884	83
55,000,000,000	144749487	712421085	140909435	1919927	66
56,000,000,000	144768133	712384525	140926634	1920625	83
57,000,000,000	144790088	712339643	140950519	1919681	69
58,000,000,000	144819333	712282985	140976101	1921511	70
59,000,000,000	144836167	712247734	140996104	1919922	73

Table 4.2: $10^9\, G_{M,M+10^9}(k)$ for $M = 20 \times 10^9, \ldots, 59 \times 10^9$

M	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
60,000,000,000	144849912	712221917	141006489	1921623	59
61,000,000,000	144871888	712176703	141031008	1920323	78
62,000,000,000	144879388	712161738	141038434	1920366	74
63,000,000,000	144905074	712110758	141063336	1920759	73
64,000,000,000	144923535	712074213	141081032	1921157	63
65,000,000,000	144940978	712040285	141096571	1922090	76
66,000,000,000	144959021	712005111	141112786	1923011	71
67,000,000,000	144966619	711988810	141122582	1921930	59
68,000,000,000	144993784	711933665	141151392	1921085	74
69,000,000,000	145015351	711893640	141166742	1924192	75
70,000,000,000	145034919	711853720	141187872	1923420	69
71,000,000,000	145048712	711826374	141201176	1923678	60
72,000,000,000	145060118	711804086	141211541	1924188	67
73,000,000,000	145081017	711763036	141230950	1924924	73
74,000,000,000	145092325	711738945	141245204	1923457	69
75,000,000,000	145104401	711715891	141255087	1924549	72
76,000,000,000	145120237	711684575	141270215	1924897	76
77,000,000,000	145123475	711678524	141272607	1925314	80
78,000,000,000	145155927	711614343	141303606	1926051	73
79,000,000,000	145166484	711590898	141318828	1923713	77
80,000,000,000	145181893	711562560	141329279	1926191	77
81,000,000,000	145192199	711539830	141343824	1924066	81
82,000,000,000	145205330	711514203	141355678	1924715	74
83,000,000,000	145213980	711497481	141363166	1925305	68
84,000,000,000	145227407	711470386	141377083	1925048	76
85,000,000,000	145252566	711421485	141399412	1926458	79
86,000,000,000	145256363	711414062	141402858	1926645	72
87,000,000,000	145281048	711365703	141425510	1927679	60
88,000,000,000	145287264	711351123	141436034	1925507	72
89,000,000,000	145301306	711322414	141451335	1924864	81
90,000,000,000	145315454	711296405	141460910	1927149	82
91,000,000,000	145332364	711265217	141472548	1929797	74
92,000,000,000	145335930	711256553	141479191	1928239	87
93,000,000,000	145346214	711233625	141494159	1925951	51
94,000,000,000	145350374	711225992	141496971	1926586	77
95,000,000,000	145372098	711182814	141518147	1926872	69
96,000,000,000	145393373	711140846	141538271	1927428	82
97,000,000,000	145395439	711136346	141541068	1927070	77
98,000,000,000	145401383	711126476	141542966	1929108	67
99,000,000,000	145434630	711060311	141575571	1929405	83

Table 4.3: $10^9\, G_{M,M+10^9}(k)$ for $M = 60 \times 10^9, \ldots, 99 \times 10^9$

Summing up the results in all the above columns, we obtain that out the entire range that was considered, the number of Gram intervals with precisely k zeros is

$k = 0$	$k = 1$	$k = 2$	$k = 3$	$k = 4$
14,419,450,104	71,352,179,843	14,037,296,603	191,066,849	6,601

In particular, we remark that the total number of zeros in the first 100,000,000,000 Gram intervals is exactly

$$71,352,179,843 + 2 \times 14,037,296,603 + 3 \times 191,066,849 + 4 \times 6,601 = \\ = 100,000,000,000$$

(as mentioned in Chapter 1, $[g_{-1}, g_0)$ and γ_1 are not counted).

As mentioned, all the Gram points and zeta zeros were considered at an accuracy of almost 10^{-30} (for this purpose, the variables in the code were implemented using the MPFR format [77]). This precision is considered to be enough, because the closest zero and Gram point in this whole range were found to be the $4,565,968,685$-th Gram point

$$g_{4,565,968,685} = 1,564,857,893.112865537745330361035860865125$$

and the $4,565,968,686$-th zero

$$\gamma_{4,565,968,686} = 1,564,857,893.112865537745413923236496208804.$$

The distance between them is significantly less than 10^{-30}

$$\gamma_{4,565,968,686} - g_{4,565,968,685} = 0.000000000000083562200635343679.$$

4.7 Comparison with previous results

As an additional way to check the veracity of the code, I also tried to use it to reproduce the previously known results of van de Lune et al, reported in Table 5 on page 672 of [64]; that table contains an analysis of Gram's Law for the first 1.5 billion Gram intervals, in steps of 100 million per row. The corresponding values generated by my algorithm are displayed here in Table 4.4. If we compare this table with the one from [64], we remark that the only minor differences appear on three of the fifteen rows; specifically, rows 1, 6 and 14 of van de Lune's table give the following numbers

L	M	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
0	100000000	13197331	7377191**0**	1286418**8**	16657**0**	1
500000001	600000000	1379503**3**	7258984**2**	1343521**5**	179909	0
1300000000	1400000000	1395273**5**	7227836**8**	1358506**0**	183836	1

As can be seen, the discrepancies (marked in bold) affect only the last digit, and since the computations in this thesis were performed at a much higher accuracy than those in [64], we claim that the values presented in Table 4.4 are the correct ones.

L	M	$k=0$	$k=1$	$k=2$	$k=3$	$k=4$
0	100000000	13197331	7377191**1**	1286418**6**	16657**1**	1
100000000	200000000	13534327	73106626	13183768	175278	1
200000000	300000000	13641172	72895112	13286261	177454	1
300000000	400000000	13711578	72755501	13354267	178651	3
400000000	500000001	13756913	72666325	13396614	180148	1
500000001	600000000	1379503**1**	7258984**6**	1343521**3**	179909	0
600000000	700000000	13826622	72528206	13463727	181440	5
700000000	800000000	13849738	72481722	13487344	181194	2
800000000	900000000	13871424	72439079	13507575	181917	5
900000000	1000000003	13894145	72393385	13530805	181664	4
1000000003	1100000001	13908465	72365865	13542874	182791	3
1100000001	1200000000	13924978	72332789	13559487	182744	1
1200000000	1300000000	13940406	72302046	13574692	182854	2
1300000000	1400000000	1395273**4**	7227837**0**	1358505**9**	183836	1
1400000000	1500000000	13964556	72254377	13597581	183483	3

Table 4.4: $10^8 G_{L,M}(k)$ for $k = 0, 1, 2, 3, 4$

Chapter 5

Other remarks and further work

5.1 Averages over $\mathrm{U}(N)$

Lemma 4. *If $f(\theta)$ is any 2π-periodic (integrable) function, and $p \in \mathbb{Z}$ is an arbitrary constant, we have that*

$$\int_{\mathrm{U}(N)} \prod_{n=1}^{N} f(\lambda_n)\, d\mu_{\mathrm{U}(N)} = \int_{\mathrm{U}(N)} \prod_{n=1}^{N} f(\lambda_n + p(\lambda_1 + \ldots + \lambda_N))\, d\mu_{\mathrm{U}(N)}, \quad (5.1)$$

where the average over $\mathrm{U}(N)$ matrices was defined in (2.7) to be

$$\int_{\mathrm{U}(N)} g(\lambda_1, \ldots, \lambda_N)\, d\mu_{\mathrm{U}(N)} = \\ = \int_{[-\pi,\pi)^N} g(\lambda_1, \ldots, \lambda_N) \mathcal{P}_{\mathrm{U}(N)}(\lambda_1, \ldots, \lambda_N)\, d\lambda_1 \ldots d\lambda_N.$$

Proof. First, we recall that the complex Fourier series expansion of the Dyson product

$$\begin{aligned} \mathcal{D}_{\mathrm{U}(N)}(\lambda_1, \ldots, \lambda_N) &= \prod_{1 \le j < k \le N} |e^{i\lambda_j} - e^{i\lambda_k}|^2 \\ &= \prod_{1 \le j < k \le N} (e^{i\lambda_j} - e^{i\lambda_k})(e^{-i\lambda_j} - e^{-i\lambda_k}) \\ &= \prod_{1 \le j < k \le N} \left(2 - \frac{e^{i\lambda_j}}{e^{i\lambda_k}} - \frac{e^{i\lambda_k}}{e^{i\lambda_j}}\right) \\ &= \sum_{\substack{(j_1,\ldots,j_N) \in \mathbb{Z}^N \\ j_1 + \ldots + j_N = 0 \\ |j_l| < N}} c_{j_1,\ldots,j_N} e^{i(j_1\lambda_1 + \ldots + j_N\lambda_N)} \end{aligned}$$

has the property that its coefficients $c_{j_1,\ldots,j_N}$ are non-zero if and only if their indices satisfy $j_1 + \ldots + j_N = 0$ and $|j_l| < N$ for all $l = 1, \ldots, N$ (for the purpose of this proof, the actual values of the coefficients $c_{j_1,\ldots,j_N}$ will not be relevant, only that they vanish if $j_1 + \ldots + j_N \neq 0$).

This means that the LHS of (5.1) can be written as

$$\int_{[-\pi,\pi)^N} \prod_{n=1}^{N} f(\lambda_n) \mathcal{P}_{\mathrm{U}(N)}(\lambda_1, \ldots, \lambda_N)\, d\lambda_1 \ldots d\lambda_N =$$

$$= \frac{1}{N!(2\pi)^N} \int_{[-\pi,\pi)^N} f(\lambda_1) \ldots f(\lambda_N) \prod_{1 \leq j < k \leq N} |e^{i\lambda_j} - e^{i\lambda_k}|^2\, d\lambda_1 \ldots d\lambda_N$$

$$= \frac{1}{N!(2\pi)^N} \int_{[-\pi,\pi)^N} f(\lambda_1) \ldots f(\lambda_N) \times$$
$$\times \sum_{\substack{(j_1,\ldots,j_N) \in \mathbb{Z}^N \\ j_1+\ldots+j_N=0 \\ |j_l|<N}} c_{j_1,\ldots,j_N} e^{ij_1\lambda_1} \ldots e^{ij_N\lambda_N}\, d\lambda_1 \ldots d\lambda_N$$

$$= \frac{1}{N!} \sum_{\substack{(j_1,\ldots,j_N) \in \mathbb{Z}^N \\ j_1+\ldots+j_N=0 \\ |j_l|<N}} c_{j_1,\ldots,j_N} \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda_1) e^{ij_1\lambda_1} d\lambda_1 \right) \ldots$$
$$\ldots \left(\frac{1}{2\pi} \int_{-\pi}^{\pi} f(\lambda_N) e^{ij_N\lambda_N} d\lambda_N \right)$$

$$= \frac{1}{N!} \sum_{\substack{(j_1,\ldots,j_N) \in \mathbb{Z}^N \\ j_1+\ldots+j_N=0 \\ |j_l|<N}} c_{j_1,\ldots,j_N} \widehat{f}_{j_1} \ldots \widehat{f}_{j_n},$$

where

$$\widehat{f}_j = \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{ij\theta} f(\theta) d\theta$$

are the coefficients in the Fourier expansion of f

$$f(\theta) = \sum_{j=-\infty}^{\infty} \widehat{f}_j e^{-ij\theta}.$$

By applying the same Fourier expansion to the integrand on the RHS of (5.1), we get

$$\prod_{n=1}^{N} f(\lambda_n + p(\lambda_1 + \ldots + \lambda_N)) =$$

$$= \prod_{n=1}^{N} f\left((p+1)\lambda_n + p\sum_{\substack{m=1\\m\neq n}}^{N}\lambda_m\right)$$

$$= \prod_{n=1}^{N}\left[\sum_{k_n\in\mathbb{Z}} \widehat{f}_{k_n} e^{-ik_n\left((p+1)\lambda_n + p\sum_{m\neq n}\lambda_m\right)}\right].$$

Expanding out the product and then re-arranging the exponents such that we collect all the λ_n terms into a single exponent, we have

$$= \sum_{k_1\in\mathbb{Z}} \dots \sum_{k_N\in\mathbb{Z}}\left[\widehat{f}_{k_1}\dots\widehat{f}_{k_N}\prod_{n=1}^{N} e^{-i\lambda_n\left((p+1)k_n + p\sum_{m\neq n}k_m\right)}\right].$$

With this, the RHS of (5.1) now becomes

$$\int_{[-\pi,\pi)^N}\prod_{n=1}^{N} f(\lambda_n + p(\lambda_1+\ldots+\lambda_N))\mathcal{P}_{\mathrm{U}(N)}(\lambda_1,\dots,\lambda_N)\,d\lambda_1\dots d\lambda_N =$$

$$= \frac{1}{N!(2\pi)^N}\int_{[-\pi,\pi)^N}\sum_{(k_1,\dots,k_N)\in\mathbb{Z}^N}\left[\widehat{f}_{k_1}\dots\widehat{f}_{k_N}\prod_{n=1}^{N} e^{-i\lambda_n\left((p+1)k_n + p\sum_{m\neq n}k_m\right)}\right]\times$$
$$\times \sum_{\substack{(j_1,\dots,j_N)\in\mathbb{Z}^N\\ j_1+\ldots+j_N=0\\ |j_i|<N}} c_{j_1,\dots,j_N} e^{ij_1\lambda_1}\dots e^{ij_N\lambda_N}\,d\lambda_1\dots d\lambda_N =$$

$$= \frac{1}{N!(2\pi)^N}\sum_{(k_1,\dots,k_N)\in\mathbb{Z}^N}\sum_{\substack{(j_1,\dots,j_N)\in\mathbb{Z}^N\\ j_1+\ldots+j_N=0\\ |j_i|<N}} c_{j_1,\dots,j_N}\widehat{f}_{k_1}\dots\widehat{f}_{k_N}\times$$
$$\times\left(\int_{-\pi}^{\pi} e^{i\lambda_1\left(j_1-(p+1)k_1-p\sum_{m\neq 1}k_m\right)}d\lambda_1\right)\dots$$
$$\dots\left(\int_{-\pi}^{\pi} e^{i\lambda_N\left(j_N-(p+1)k_N-p\sum_{m\neq N}k_m\right)}d\lambda_N\right).$$

We separate this sum into two parts

- For those N-tuples $(k_1,\dots,k_N)\in\mathbb{Z}^N$ and $(j_1,\dots,j_N)\in\mathbb{Z}^N$ for which

$$j_1 = (p+1)k_1 + p\sum_{\substack{m=1\\m\neq 1}}^{N} k_m = k_1 + p(k_1+\ldots+k_N) \tag{5.2}$$
$$\dots\dots\dots\dots\dots$$

$$j_N = (p+1)k_N + p\sum_{\substack{m=1\\ m\neq N}}^{N} k_m = k_N + p(k_1 + \ldots + k_N)$$

(at the same time) all the exponents are zero, and the sum becomes in this case

$$\frac{1}{N!(2\pi)^N} \sum_{(k_1,\ldots,k_N)\in\mathbb{Z}^N} \sum_{\substack{(j_1,\ldots,j_N)\in\mathbb{Z}^N\\ j_1+\ldots+j_N=0\\ |j_l|<N}} c_{j_1,\ldots,j_N} \widehat{f}_{k_1}\ldots\widehat{f}_{k_N} \left(\int_{-\pi}^{\pi} d\lambda_1\right)\ldots\left(\int_{-\pi}^{\pi} d\lambda_N\right) =$$

$$= \frac{1}{N!} \sum_{(k_1,\ldots,k_N)\in\mathbb{Z}^N} \sum_{\substack{(j_1,\ldots,j_N)\in\mathbb{Z}^N\\ j_1+\ldots+j_N=0\\ |j_l|<N}} c_{j_1,\ldots,j_N} \widehat{f}_{k_1}\ldots\widehat{f}_{k_N}.$$

Also, the restriction $j_1 + \ldots + j_N = 0$ combined with all the conditions in (5.2) implies $k_1 + \ldots + k_N = 0$, which again, combined with all the conditions in (5.2) gives us that $j_1 = k_1$, $\ldots$, $j_N = k_N$, so the above sum reduces to

$$= \frac{1}{N!} \sum_{\substack{(j_1,\ldots,j_N)\in\mathbb{Z}^N\\ j_1+\ldots+j_N=0\\ |j_l|<N}} c_{j_1,\ldots,j_N} \widehat{f}_{j_1}\ldots\widehat{f}_{j_N},$$

which has been shown to be equal to the LHS of (5.1);

- In the case of those N-tuples $(k_1,\ldots,k_N)\in\mathbb{Z}^N$ and $(j_1,\ldots,j_N)\in\mathbb{Z}^N$ for which at least one of the conditions in (5.2) is not true, that is

$$j_n \neq (p+1)k_n + p\sum_{\substack{m=1\\ m\neq n}}^{N} k_m$$

for at least one $n = 1,\ldots,N$, we have that the corresponding integral

$$\int_{-\pi}^{\pi} e^{i\lambda_n\left(j_n-(p+1)k_n-p\sum_{m\neq n}k_m\right)} d\lambda_n = 2\frac{\sin\pi\left(j_n-(p+1)k_n-p\sum\limits_{m\neq n}k_m\right)}{j_n-(p+1)k_n-p\sum\limits_{m\neq n}k_m}$$

$$= 0,$$

because p was assumed to be an integer. This means that all the remaining terms in the sum that were not considered in the first case are zero, and the proof is complete. □

5.2 SU(N) Haar measure has mass 1

Let A be a matrix with eigenvalues $e^{i\lambda_1}, \ldots, e^{i\lambda_N}$ and its characteristic polynomial defined as in Section 2.2

$$\begin{aligned}\Lambda_A(\theta) &= \det(I - Ae^{-i\theta}) \\ &= \prod_{n=1}^{N}(1 - e^{i(\lambda_n - \theta)}).\end{aligned}$$

If we take the function f from the previous result to be $f(\theta) = |1 - e^{i\theta}|^s$ where $s \in \mathbb{C}$ with $\mathfrak{Re}\ s > -1$, then the integrand of the LHS of (5.1) becomes

$$\begin{aligned}\prod_{n=1}^{N} f(\lambda_n) &= \left|\prod_{n=1}^{N}(1 - e^{i\lambda_n})\right|^s \\ &= |\Lambda_A(0)|^s,\end{aligned}$$

while on the RHS we get

$$\begin{aligned}\prod_{n=1}^{N} f(\lambda_n + p(\lambda_1 + \ldots + \lambda_N)) &= \left|\prod_{n=1}^{N}(1 - e^{i(\lambda_n + p(\lambda_1 + \ldots + \lambda_N))})\right|^s \\ &= |\Lambda_A(-p(\lambda_1 + \ldots + \lambda_N))|^s.\end{aligned}$$

Putting these together, the previous result implies that for any $p \in \mathbb{Z}$

$$\int_{\mathrm{U}(N)} |\Lambda_A(0)|^s \, d\mu_{\mathrm{U}(N)} = \int_{\mathrm{U}(N)} |\Lambda_A(-p(\lambda_1 + \ldots + \lambda_N))|^s \, d\mu_{\mathrm{U}(N)}. \quad (5.3)$$

It is also well-know from [51] that

$$\int_{\mathrm{U}(N)} |\Lambda_A(0)|^s \, d\mu_{\mathrm{U}(N)} = \prod_{j=1}^{N} \frac{\Gamma(j)\Gamma(j+s)}{\left[\Gamma\left(j + \frac{s}{2}\right)\right]^2}.$$

In particular, by taking $p = 1$ and $s = 2$, we can use these to prove that the Haar probability measure of the SU(N) group has total mass 1.

$$\begin{aligned}\int_{\mathrm{SU}(N+1)} d\mu_{\mathrm{SU}(N+1)} = \frac{1}{(N+1)!(2\pi)^N} \int_{[-\pi,\pi)^N} &\prod_{1 \le j < k \le N} |e^{i\lambda_j} - e^{i\lambda_k}|^2 \times \\ &\times \prod_{1 \le j \le N} |e^{i\lambda_j} - e^{-i(\lambda_1 + \ldots + \lambda_N)}|^2 \, d\lambda_1 \ldots d\lambda_N\end{aligned}$$

$$= \frac{1}{N+1}\,\frac{1}{N!(2\pi)^N}\int_{[-\pi,\pi)^N}\prod_{1\le j<k\le N}|e^{i\lambda_j}-e^{i\lambda_k}|^2\times$$
$$\times\left|\prod_{1\le j\le N}(1-e^{i(\lambda_j+\lambda_1+\ldots+\lambda_N)})\right|^2 d\lambda_1\ldots d\lambda_N$$

$$= \frac{1}{N+1}\int_{\mathrm{U}(N)}|\Lambda_A(-(\lambda_1+\ldots+\lambda_N))|^2\,d\mu_{\mathrm{U}(N)}$$

$$= \frac{1}{N+1}\int_{\mathrm{U}(N)}|\Lambda_A(0)|^2\,d\mu_{\mathrm{U}(N)}$$

$$= \frac{1}{N+1}\prod_{j=1}^{N}\frac{\Gamma(j)\Gamma(j+2)}{[\Gamma(j+1)]^2}=1.$$

5.3 Generalizing a Selberg-type integral

Start by considering the following quantity

$$\int_{\mathrm{U}(N)}|\Lambda_A(-(\lambda_1+\ldots+\lambda_N))|^s\,d\mu_{\mathrm{U}(N)}=$$

$$= \frac{1}{N!(2\pi)^N}\int_{[-\pi,\pi)^N}|\Lambda_A(-(\lambda_1+\ldots+\lambda_N))|^s\times$$
$$\times\prod_{1\le j<k\le N}|e^{i\lambda_j}-e^{i\lambda_k}|^2\,d\lambda_1\ldots d\lambda_N$$

$$= \frac{1}{N!(2\pi)^N}\int_{[-\pi,\pi)^N}\left|\prod_{1\le j\le N}(1-e^{i(\lambda_j+\lambda_1+\ldots+\lambda_N)})\right|^s\times$$
$$\times\prod_{1\le j<k\le N}|e^{i\lambda_j}-e^{i\lambda_k}|^2\,d\lambda_1\ldots d\lambda_N$$

$$= \frac{2^{sN}\,2^{N(N-1)}}{N!(2\pi)^N}\int_{[-\pi,\pi)^N}\prod_{1\le j\le N}\left|\sin\left(\frac{\lambda_j+\lambda_1+\ldots+\lambda_N}{2}\right)\right|^s\times$$
$$\times\prod_{1\le j<k\le N}\left|\sin\left(\frac{\lambda_j-\lambda_k}{2}\right)\right|^2 d\lambda_1\ldots d\lambda_N.$$

We apply the first change of variables

$$\frac{\lambda_j}{2} \to \lambda_j, \qquad 1 \le j \le N$$

with Jacobian 2^N; the above quantity becomes

$$= \frac{2^{sN}\, 2^{N(N-1)}\, 2^N}{N!(2\pi)^N} \int_{[-\pi,\pi)^N} \prod_{1\le j\le N} |\sin(\lambda_j + \lambda_1 + \ldots + \lambda_N)|^s \times$$
$$\times \prod_{1\le j<k\le N} |\sin(\lambda_j - \lambda_k)|^2 \, d\lambda_1 \ldots d\lambda_N =$$

$$= \frac{2^{N^2+sN}}{N!(2\pi)^N} \int_{[-\pi,\pi)^N} \prod_{1\le j\le N} |\sin\lambda_j \, \cos(\lambda_1 + \ldots + \lambda_N) +$$
$$+ \cos\lambda_j \, \sin(\lambda_1 + \ldots + \lambda_N)|^s \times$$
$$\times \prod_{1\le j<k\le N} |\sin\lambda_j \cos\lambda_k - \sin\lambda_k \cos\lambda_j|^2 \, d\lambda_1 \ldots d\lambda_N.$$

We perform the second change of variables

$$x_j = \tan\lambda_j \quad \Leftrightarrow \quad \lambda_j = \operatorname{atan}\, x_j. \tag{5.4}$$

Because

$$\frac{\partial}{\partial x_j} \operatorname{atan}\, x_j = \frac{1}{1+x_j^2},$$

the corresponding Jacobian is

$$\prod_{j=1}^{N} \frac{1}{1+x_j^2}.$$

We will also need the fact that

$$\cos^2\lambda_j = \frac{1}{1+\tan^2\lambda_j} = \frac{1}{1+x_j^2}.$$

We now apply this change of variables separately for the two products in the above integrand

- $$\prod_{1\le j<k\le N} |\sin\lambda_j \cos\lambda_k - \sin\lambda_k \cos\lambda_j|^2 =$$

$$= \prod_{1\le j<k\le N} (\cos\lambda_j \cos\lambda_k)^2 \prod_{1\le j<k\le N} |\tan\lambda_j - \tan\lambda_k|^2$$

$$= \prod_{j=1}^{N} (\cos^2 \lambda_j)^{N-1} \prod_{1 \le j < k \le N} |\tan \lambda_j - \tan \lambda_k|^2$$

$$= \left(\prod_{j=1}^{N} \frac{1}{1+x_j^2} \right)^{N-1} \prod_{1 \le j < k \le N} |x_j - x_k|^2.$$

- If we define the elementary symmetric polynomials in $x_1, \ldots, x_N$

$$\begin{cases} e_0 = 1 \\ e_1 = x_1 + \ldots + x_N = \tan \lambda_1 + \ldots + \tan \lambda_N \\ e_2 = \sum_{j<k} x_j x_k = \sum_{j<k} \tan \lambda_j \tan \lambda_k \\ \ldots\ldots\ldots\ldots \\ e_N = x_1 \ldots x_N = \tan \lambda_1 \ldots \tan \lambda_N, \end{cases}$$

then it can be proved (by induction) that

$$\sin \left(\sum_{k=1}^{N} \lambda_k \right) = \left(\prod_{k=1}^{N} \cos \lambda_k \right) (e_1 - e_3 + e_5 - \ldots),$$

and

$$\cos \left(\sum_{k=1}^{N} \lambda_k \right) = \left(\prod_{k=1}^{N} \cos \lambda_k \right) (e_0 - e_2 + e_4 - \ldots).$$

Also, if we denote by

$$S_0 = e_0 - e_2 + e_4 - \ldots$$
$$S_1 = e_1 - e_3 + e_5 - \ldots,$$

then it can be shown that

$$\arctan x_1 + \ldots + \arctan x_N = \arctan \frac{S_1}{S_0},$$

and

$$(1 + x_1^2) \ldots (1 + x_N^2) = S_0^2 + S_1^2.$$

We now have everything we need to apply the change of variables (5.4) to the other product in the integrand

$$\prod_{j=1}^{N} |\sin\lambda_j\, \cos(\lambda_1+\ldots+\lambda_N)+\cos\lambda_j\, \sin(\lambda_1+\ldots+\lambda_N)|^s =$$

$$= \prod_{j=1}^{N} \left| \sin\lambda_j \left(\prod_{k=1}^{N}\cos\lambda_k\right)(e_0-e_2+e_4-\ldots)+ \right.$$
$$\left. +\cos\lambda_j \left(\prod_{k=1}^{N}\cos\lambda_k\right)(e_1-e_3+e_5-\ldots)\right|^s$$

$$= \left(\prod_{k=1}^{N}\cos\lambda_k\right)^{Ns} \times$$
$$\times \prod_{j=1}^{N} |\sin\lambda_j(e_0-e_2+e_4-\ldots)+\cos\lambda_j(e_1-e_3+e_5-\ldots)|^s$$

$$= \left(\prod_{k=1}^{N}\cos\lambda_k\right)^{Ns} \left(\prod_{k=1}^{N}\cos\lambda_k\right)^{s} \times$$
$$\times \prod_{j=1}^{N} |\tan\lambda_j(e_0-e_2+e_4-\ldots)+(e_1-e_3+e_5-\ldots)|^s$$

$$= \left(\prod_{j=1}^{N}\frac{1}{1+x_j^2}\right)^{\frac{Ns}{2}+\frac{s}{2}} \prod_{j=1}^{N} |x_j\, S_0+S_1|^s\,.$$

Putting everything together we finally get

$$\int_{\mathrm{U}(N)} |\Lambda_A(-(\lambda_1+\ldots+\lambda_N))|^s\, d\mu_{\mathrm{U}(N)} =$$

$$= \frac{2^{N^2+sN}}{N!(2\pi)^N} \int_{[-\pi,\pi)^N} \prod_{1\le j\le N} |\sin\lambda_j\, \cos(\lambda_1+\ldots+\lambda_N)+$$
$$+\cos\lambda_j\, \sin(\lambda_1+\ldots+\lambda_N)|^s \times$$
$$\times \prod_{1\le j<k\le N} |\sin\lambda_j\cos\lambda_k-\sin\lambda_k\cos\lambda_j|^2\, d\lambda_1\ldots d\lambda_N$$

$$= \frac{2^{N^2+sN}}{N!(2\pi)^N} \int_{\mathbb{R}^N} \left(\prod_{j=1}^N \frac{1}{1+x_j^2} \right)^{\frac{Ns}{2}+\frac{s}{2}} \prod_{j=1}^N |x_j\, S_0 + S_1|^s \times$$
$$\times \left(\prod_{j=1}^N \frac{1}{1+x_j^2} \right)^{N-1} \prod_{1\le j<k\le N} |x_j - x_k|^2 \left(\prod_{j=1}^N \frac{1}{1+x_j^2} \right) dx_1 \dots dx_N$$

$$= \frac{2^{N^2+Ns}}{N!(2\pi)^N} \int_{\mathbb{R}^N} \left(\prod_{j=1}^N \frac{1}{1+x_j^2} \right)^{\frac{Ns}{2}+\frac{s}{2}+N} \prod_{j=1}^N |x_j\, S_0 + S_1|^s \times$$
$$\times \prod_{1\le j<k\le N} |x_j - x_k|^2\, dx_1 \dots dx_N.$$

One of the main results of [51] was to prove that

$$\int_{\mathrm{U}(N)} |\Lambda_A(0)|^s\, d\mu_{\mathrm{U}(N)} =$$

$$= \frac{2^{N^2+Ns}}{N!(2\pi)^N} \int_{\mathbb{R}^N} \left(\prod_{j=1}^N \frac{1}{1+x_j^2} \right)^{\frac{s}{2}+N} \prod_{1\le j<k\le N} |x_j - x_k|^2\, dx_1 \dots dx_N$$

so, combining this with formula (5.3) from the previous corollary, we have

$$\frac{2^{N^2+Ns}}{N!(2\pi)^N} \int_{\mathbb{R}^N} \left(\prod_{j=1}^N \frac{1}{1+x_j^2} \right)^{\frac{Ns}{2}+\frac{s}{2}+N} \prod_{j=1}^N |x_j\, S_0 + S_1|^s \times$$
$$\times \prod_{1\le j<k\le N} |x_j - x_k|^2\, dx_1 \dots dx_N =$$

$$= \frac{2^{N^2+Ns}}{N!(2\pi)^N} \int_{\mathbb{R}^N} \left(\prod_{j=1}^N \frac{1}{1+x_j^2} \right)^{\frac{s}{2}+N} \prod_{1\le j<k\le N} |x_j - x_k|^2\, dx_1 \dots dx_N$$

$$= \prod_{j=1}^N \frac{\Gamma(j)\Gamma(j+s)}{\left[\Gamma\left(j+\frac{s}{2}\right)\right]^2}$$

(it would also be interesting if one could find a more direct proof of the above equality).

5.4 Further work

The results discussed in previous chapters could be further extended in several new directions.

It is very likely that the methods and techniques developed through Chapters 2 – 3 could also be applied to create a RMT analogue of Gram blocks, as defined in Section 1.6. This could then be used to obtain a probabilistic model that quantifies the frequency with which Rosser's Rule holds true. And in order to verify the validity of this model, one could implement an algorithm that analyzes Rosser's Rule for the first 100 billion zeta zeros from the LMFDB database.

Throughout this thesis, we have focused mostly on improving Fujii's conjecture, as stated in (2.12) (for a single Gram interval); the conclusion was that the rates of convergence differ, but the limits coincide. However, it is important to keep in mind that in Fujii's paper [31], his original conjecture was defined in a more general sense, as in (2.10) (for several consecutive Gram intervals). It would be interesting to apply the ideas of this thesis to Fujii's initial conjecture, and see if the same conclusion holds (presumably it does). It is possible that this analysis would overlap to a certain extent with the one mentioned in the previous paragraph.

The final results of Chapter 3 are dependent on conjecture (3.17) regarding the upper bound on the Dyson coefficients. It would be very useful if one could find a combinatorial argument to prove this conjecture, so that the results of Chapter 3 would hold unconditionally. Ideally, one would like to have an explicit formula for as many of these coefficients as possible, but this is known to be a very difficult problem.

Another interesting question would be if it is possible to extend the results of this thesis to the case of Dirichlet L-functions. In order to do that, one would have to first find a way to define Gram points for L-functions. This could be either the points at which the L-function is real, or at which the smooth part of the zero counting function is an integer.

Following the discussion from Section 2.4, it is clear that the RMT analogue of the Hardy Z-function can be considered to be

$$z_A(\theta) := 2^N \prod_{j=1}^{N} \sin\left(\frac{\theta_j - \theta}{2}\right).$$

A final area of research would be to use this function in order to obtain RMT analogues for the moments of the Hardy Z function evaluated at Gram points, which were reviewed in subsections 1.5.1 and 1.5.2.

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