Algebraic models and change of groups for equivariant spectra

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Abstract

In this thesis we consider the interaction between extra structures present in category of rational free G-spectra, namely the monoidal structure and the change of groups adjunctions, and the passage to the algebraic model. We prove that the category of rational cofree G-spectra admits a monoidal algebraic model in terms of L-complete modules. In order to prove this, we develop the Left Localization Principle which gives mild conditions under which a Quillen adjunction descends to a Quillen equivalence between left Bousfield localizations.

We give a model categorical argument showing that the induction, restriction and coinduction functors between categories of (co)free rational equivariant spectra correspond to functors between the algebraic models for connected compact Lie groups. In order to do this, we provide general tools to check whether Quillen functors correspond. Since the construction of algebraic models relies upon the fact that polynomial rings are strongly intrinsically formal as commutative DGAs, we pay careful attention to the interaction of model structures on commutative algebras and modules. In particular, we prove that Shipley's algebraicization theorem respects the extra compatibility between commutative algebras and modules in the flat model structure on spectra.

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Note on format

This thesis is in 'publication format'. Chapters 2, 3 and 4 are in the style of research papers. Chapter 2 consists of a paper which has been accepted for publication in the Journal of Pure and Applied Algebra and has appeared in the online version with DOI 10.1016/j.jpaa.2020.106408. Chapter 3 has been accepted for publication in Homology, Homotopy and Applications, and is available on the arXiv e-print system. Chapter 4 has been submitted for publication, and is available on the arXiv e-print system. All of the chapters are modified from the versions available on the arXiv e-print system.

The paper which comprises Chapter 2 is joint work with Luca Pol, who has agreed for it to be included in this thesis. As usual in pure mathematics, joint authorship is equal and authors are listed alphabetically.

Contents

CHAPTER 1

Introduction

1. Rational equivariant cohomology theories

The field of algebraic topology is concerned with the classification of spaces up to a weak notion of equivalence which disregards information about distance. Instead, this notion of 'homotopy equivalence' records information about holes. In order to distinguish spaces (up to homotopy equivalence), one creates rigid algebraic information from the slippery topological data, using *invariants*.

Cohomology theories are powerful invariants for topological spaces. A cohomology theory is a contravariant functor from spaces to graded abelian groups, satisfying several axioms which enable effective computation. Cohomology theories are represented by objects called *spectra*, in the sense that given a cohomology theory $E^*(-)$, there is a spectrum E such that for all spaces X,

$$E^*(X) = [X, E]_*$$

where $[-,-]_*$ denotes the graded set of homotopy classes of maps. Similarly, given a spectrum one may define a cohomology theory in this way.

We can mimic this result for spaces with the action of a (compact Lie) group G. Equivariant cohomology theories are powerful invariants which take into account the action of G. There are countless examples, such as equivariant K-theory, equivariant cobordism and Borel cohomology. As in the non-equivariant setting, G-equivariant cohomology theories are represented by objects called G-spectra.

If one seeks to find good invariants, it is important to be able to look at all invariants and choose those that are particularly well structured, or which capture the properties one cares about. Therefore, studying the collection of all cohomology theories provides crucial information about invariants. From this point of view, it is valuable to package the category of G-spectra into a rigid format which is well understood. In other words, we would like to find an *algebraic model* for G-spectra.

A priori, this goal is unreasonably difficult. Indeed, even in the non-equivariant case, the endomorphism ring of the unit object in the category of spectra is the ring of stable homotopy groups of spheres. This is a notoriously complicated ring, and therefore one must reassess the goal. Instead of seeking a full integral understanding, one might ask for a classification modulo torsion. In other words, we would aim to classify all G-equivariant cohomology theories which take values in *rational* vector spaces. This idea takes its cue from the unstable rational homotopy theory pioneered by Quillen and Sullivan, where the goal was to classify spaces (under finiteness conditions) up to rational homotopy type. Sullivan [19] proved that the rational homotopy type of a simply connected space of finite type is determined by a *Sullivan model* which is a certain type of commutative DGA.

In order to classify rational spaces, Quillen developed the framework of model categories [12]. A model category $\mathcal{C}$ is a category with a notion of a homotopy theory. It is a category with a class of maps called weak equivalences, along with additional structure and axioms which allow the construction of a homotopy category $h\mathcal{C}$. A map is an isomorphism in the homotopy category if and only if it is represented by a weak equivalence in the model category. For example, for a commutative ring R, the category of chain complexes of R-modules with weak equivalences given by quasi-isomorphisms can be made into a model category, and has homotopy category equivalent to the derived category $\mathcal{D}(R)$. A *Quillen equivalence* between model categories $\mathcal{C}$ and $\mathcal{D}$, is the correct formalism of the fact that $\mathcal{C}$ and $\mathcal{D}$ present the same homotopy theory. In particular, a Quillen equivalence $\mathcal{C} \simeq_Q \mathcal{D}$ implies that their homotopy categories $h\mathcal{C}$ and $h\mathcal{D}$ are equivalent. Modern introductions to the theory of model categories can be found in [9] and [8].

The following conjecture of Greenlees seeks to extend the programme of classifying rational homotopy type into the equivariant and stable setting.

Conjecture (Greenlees)**.** For every compact Lie group G, there is a (graded) abelian category $\mathcal{A}(G)$ and a Quillen equivalence

$$\mathrm{Sp}_G/\mathbb{Q} \simeq_Q d\mathcal{A}(G)$$

where $d\mathcal{A}(G)$ denotes objects in $\mathcal{A}(G)$ with a differential. Furthermore, the algebraic model comes equipped with an Adams spectral sequence converging to the set of equivariant stable maps $[X, Y]^G_*$.

For the rest of this introduction, everything is implicitly rationalized without comment. For example, Sp_G now denotes the category of rational G-spectra and S^0 denotes the rational sphere spectrum.

Considering the case where G is the trivial group shows that the goal of classifying rational equivariant cohomology theories algebraically is now more achievable. Indeed, Serre's calculation of the homotopy groups of spheres shows that the rational sphere spectrum has homotopy $\mathbb{Q}$, concentrated in degree zero. Combining this with Morita theory [15], one can conclude that $\mathrm{Sp} \simeq_Q \mathrm{Ch}_{\mathbb{Q}}$, so that rational cohomology theories are determined by graded $\mathbb{Q}$-modules.

This conjecture has been proved in many cases: G finite [1], $G = SO(2)$ [16], $G = O(2)$ [2], $G = SO(3)$ [10] and G a torus of any rank [7]. Throughout this thesis, we will mainly be interested in certain classes of G-spectra, rather than in certain groups. In particular, we will focus on the classes of *free* and *cofree* G-spectra, see below for definitions.

One can also strengthen this conjecture to ask for more structure. For example, one could ask that the zig-zag of Quillen equivalences is monoidal, so that it gives rise to an algebraic model for ring G-spectra and module spectra. The consideration of extra structure with the zig-zag of Quillen equivalences is a key theme in this thesis. Firstly, we will deal with monoidal considerations for equivariant cohomology theories on free G-spaces, and then we turn to studying algebraic counterparts of change of groups functors.

2. Free and cofree equivariant spectra

For a compact Lie group G, we write EG for a contractible space with a free G-action. It is characterized by the fact that the fixed point space $(EG)^H$ is contractible for $H = 1$ and is empty otherwise. Collapsing EG to a point gives rise to the *isotropy separation* cofibre sequence

$$EG_+ \to S^0 \to \widetilde{E}G.$$

A G-spectrum X is said to be *free* if the natural map $EG_+ \wedge X \to X$ is an equivalence, and is said to be *cofree* if the natural map $X \to F(EG_+, X)$ is an equivalence.

Free and cofree G-spectra are interesting for several reasons. Firstly, they represent cohomology theories on free G-spaces. Secondly, the understanding of the free case is an important step towards understanding the general case, since the algebraic models for general compact Lie groups G should be built up from information at each closed subgroup, where the model resembles that of free spectra.

Greenlees-Shipley [4, 6] have given an algebraic model for free G-spectra for G any compact Lie group. Write N for the identity component of G and $W = G/N$ for the component group. There is a zig-zag of Quillen equivalences

$$\mathrm{Sp}_G^{\mathrm{free}} \simeq_Q \mathrm{Mod}^{\mathrm{torsion}}_{H^*\widetilde{BN}[W]}$$

between the categories of rational free G-spectra and torsion dg-modules over the skewed group ring $H^*\widetilde{BN}[W]$. The category of torsion modules has no tensor unit and therefore this Quillen equivalence cannot be refined to a monoidal algebraic model. Providing a solution to the lack of monoidality is the first goal of this thesis.

The full subcategories of the homotopy category of G-spectra of free and cofree G-spectra are equivalent. This suggests an alternative approach to modelling rational equivariant cohomology theories on free G-spaces, via the category of cofree G-spectra. In order to describe the algebraic model for cofree G-spectra, we firstly need to set the algebraic backdrop. Given an ideal I in a graded commutative ring R, the I-adic completion functor is neither left nor right exact in general. We therefore regard its zeroth left derived functor L_0^I to be the correct homological notion of completion. Note that the right exactness of a functor F is not needed to construct the left derived functors L_iF. It is only used to prove that L_0F is isomorphic to F, which is *not* true in our case of interest, where F is the adic completion functor.

We say that a module is *L-complete* if the natural map $M \to L_0^I M$ is an isomorphism, and is *derived complete* if the natural map $M \to \mathbb{L}(-)^\wedge_I(M)$ is a quasi-isomorphism, where $\mathbb{L}(-)^\wedge_I$ denotes the total left derived functor of I-adic completion.

The following is the first main result of this thesis, which appears in Chapter 2 and was proved in joint work with Luca Pol [11].

Theorem. *Let G be a compact Lie group with identity component N and component group W. There is zig-zag of symmetric monoidal Quillen equivalences*

$$\mathrm{Sp}_G^{\mathrm{cofree}} \simeq_Q \mathrm{Mod}^{L\text{-complete}}_{H^*\widetilde{BN}[W]}$$

between the categories of rational cofree G-spectra and L-complete dg-modules over the skewed group ring $H^*\widetilde{BN}[W]$ with respect to the augmentation ideal.

We restrict to connected groups for simplicity in the following discussion, and write I for the augmentation ideal of H^*BG. A Quillen equivalence between rational cofree G-spectra and derived complete H^*BG-modules was already known by passing through free G-spectra in the following way:

$$\begin{tikzcd}
\text{free } G\text{-spectra} \arrow[r, leftrightarrow, "\simeq_Q"] \arrow[d, leftrightarrow, "\simeq_Q"'] & I\text{-power torsion-}H^*BG\text{-modules} \arrow[d, leftrightarrow, "\simeq_Q"] \\
\text{cofree } G\text{-spectra} \arrow[r, leftrightarrow, dashed] & \text{derived complete-}H^*BG\text{-modules.}
\end{tikzcd}$$

The horizontal Quillen equivalence is the algebraic model for free G-spectra of Greenlees-Shipley [4] and the right vertical Quillen equivalence follows from Morita theory [3] (also known as the MGM equivalence), together with the identification of I-power torsion modules as an abelian model for derived torsion modules [5, §5]. However this approach is unsatisfactory for two main reasons. Firstly, it does not give rise to a symmetric monoidal Quillen equivalence since the category of I-power torsion modules has no tensor unit. Secondly, it does not give an abelian model as desired in the conjecture of Greenlees. In light of this, our contribution is threefold: we prove the algebraic model for rational cofree G-spectra directly, we upgrade it to a symmetric monoidal Quillen equivalence, and we give an abelian model for derived complete modules.

3. Change of groups functors

Let $i\colon H \to G$ be the inclusion of a closed subgroup H into a compact Lie group G. This induces an adjoint triple of Quillen functors

$$\begin{tikzcd}
\mathrm{Sp}_G \arrow[r, "i^*" description] & \mathrm{Sp}_H \arrow[l, bend right, "i_*"'] \arrow[l, bend left, "i_!"]
\end{tikzcd}$$

where $i_* = G_+ \wedge_H -$, i^* is the forgetful functor and $i_! = F_H(G_+, -)$.

If algebraic models for G-spectra and H-spectra are known, one could ask for functors between the algebraic models corresponding to this adjoint triple. Diagrammatically, we can try to find functors

$$\begin{tikzcd} \mathrm{Sp}_G \arrow[r, leftrightarrow, "\simeq_Q"] \arrow[d, "i^*"] & d\mathcal{A}(G) \arrow[d, "?"] \\ \mathrm{Sp}_H \arrow[r, leftrightarrow, "\simeq_Q"'] \arrow[u, shift left=2, "i_*"] \arrow[u, shift right=2, "i_!"'] & d\mathcal{A}(H) \arrow[u, shift left=2, "?"] \arrow[u, shift right=2, "?"'] \end{tikzcd}$$

which model those in topology.

In this thesis we consider this question for the cases of free and cofree equivariant spectra for *connected* compact Lie groups. Recall that the algebraic model for free G-spectra (where G is connected) is torsion modules over H^*BG. The inclusion $i\colon H \to G$ induces a ring map $\theta\colon H^*BG \to H^*BH$, and hence an adjoint triple

$$\begin{tikzcd} \mathrm{Mod}_{H^*BG} \arrow[r, shift left=3, "\theta_*"] \arrow[r, shift right=3, "\theta_!"'] & \mathrm{Mod}_{H^*BH} \arrow[l, "\theta^*"'] \end{tikzcd}$$

where $\theta_* = H^*BH \otimes_{H^*BG} -$ is extension of scalars, θ^* is restriction of scalars, and $\theta_! = \mathrm{Hom}_{H^*BG}(H^*BH, -)$ is coextension of scalars. Comparing the two adjoint triples, one notices that there is a mismatch. In topology, two of the functors go from H-spectra to G-spectra, but in algebra only one functor goes in this direction. Therefore one must construct additional functors in algebra to model the functors in topology.

The following is the main result of the thesis which appears in the preprint [20]. We will not state precisely what we mean by a 'correspondence of Quillen functors' in this introduction. Instead we direct the reader to Chapter 4 for more details. Recall that the restriction functor i^* is both left and right Quillen. Therefore, there are two functors which correspond to it in algebra; one as a left Quillen functor and one as a right Quillen functor. This can be seen in the diagram below where there are four functors in algebra rather than the three in topology.

Theorem. *Let $i\colon H \to G$ be the inclusion of a connected subgroup into a connected compact Lie group. We have the following correspondence of Quillen functors*

$$\begin{tikzcd} \textit{free } G\textit{-spectra} \arrow[r, leftrightarrow, "\simeq_Q"] \arrow[d, "i^*"] & \textit{derived torsion } H^*BG\textit{-modules} \arrow[d, shift left=1, "\Sigma^{-a}\theta_!"] \arrow[d, shift left=4, "\theta_*"] \\ \textit{free } H\textit{-spectra} \arrow[r, leftrightarrow, "\simeq_Q"'] \arrow[u, shift left=2, "i_*"] \arrow[u, shift right=2, "i_!"'] & \textit{derived torsion } QH^*BH\textit{-modules} \arrow[u, shift left=7, "\theta^*"'] \arrow[u, shift right=2, "\Sigma^a\theta^*"] \end{tikzcd}$$

where $a = \dim(G/H)$ and QH^*BH is a cofibrant replacement of H^*BH as a commutative H^*BG-algebra. In other words, (i_*, i^*) corresponds to $(\Sigma^a\theta^*, \Sigma^{-a}\theta_!)$ and $(i^*, i_!)$ corresponds to (θ_*, θ^*). Similarly, when the induction, forgetful functor and coinduction functors are viewed as functors between the categories of cofree spectra, they correspond to the same functors as in the free case, now viewed as functors between the categories of derived complete modules.

If the ranks of G and H are equal there is a stronger statement. In this case, the restriction of scalars along $H^*BG \to H^*BH$ is both left and right adjoint to the extension of scalars functor.

Theorem. *Let $i\colon H \to G$ be the inclusion of a connected subgroup into a connected compact Lie group and assume that $\mathrm{rk}G = \mathrm{rk}H$. Then we have the correspondence of functors*

$$\begin{tikzcd} \textit{free } G\textit{-spectra} \arrow[r, leftrightarrow, "\simeq_Q"] \arrow[d, "i^*"] & I\textit{-power torsion } H^*BG\textit{-modules} \arrow[d, "\theta_*"] \\ \textit{free } H\textit{-spectra} \arrow[r, leftrightarrow, "\simeq_Q"'] \arrow[u, shift left=2, "i_*"] \arrow[u, shift right=2, "i_!"'] & J\textit{-power torsion } H^*BH\textit{-modules} \arrow[u, shift left=2, "\theta^*"] \arrow[u, shift right=2, "\theta^*"'] \end{tikzcd}$$

where I and J are the augmentation ideals of H^*BG and H^*BH respectively. Similarly, when the induction, forgetful functor and coinduction are viewed as functors between the categories of cofree spectra, they correspond to θ^*, θ_* and θ^* respectively, between the categories of L-complete modules.

We take a moment to discuss the model categorical underpinnings of this result. Given a monoid in a monoidal model category, we equip the category of modules with the projective model structure in which weak equivalences and fibrations are created by the forgetful functor. If we have a map of monoids $\theta\colon S \to R$ in a monoidal model category, the extension-restriction of scalars adjunction $\theta_* \dashv \theta^*$ is always a Quillen adjunction. However, the restriction-coextension of scalars adjunction $\theta^* \dashv \theta_!$ need not be a Quillen adjunction in general. If the unit of the underlying monoidal model category is cofibrant, then $\theta^* \dashv \theta_!$ is Quillen if and only if R is cofibrant as an S-module. For more details, see Proposition 3.8 in Chapter 4. Providing a context appropriate for our goal in which one can force this condition is subtle. One might expect that a cofibrant replacement of R as an S-algebra would suffice here since Schwede-Shipley [14, 4.1] have proved that if the unit of the underlying monoidal model category is cofibrant, a cofibrant S-algebra is cofibrant as an S-module. However, this breaks another step in the proof, the formality step, as we now describe.

In order to provide the necessary context to describe this issue, we recall the general procedure for constructing algebraic models. We do so in the case of free G-spectra for G a connected compact Lie group, since this is the main case of interest in this thesis. Firstly, one notices that every free G-spectrum is a module over the commutative ring G-spectrum $DEG_+ = F(EG_+, S^0)$, the G-spectrum representing Borel cohomology. Change of rings along the ring map $S^0 \to DEG_+$ yields a Quillen equivalence after a suitable cellularization of the model structures. Taking G-fixed points produces a Quillen equivalence to (a cellularization of) the category of modules over the commutative ring spectrum DBG_+. Since DBG_+ is a commutative $H\mathbb{Q}$-algebra, we can apply Shipley's algebraicization theorem [18] which gives a Quillen equivalence

$$\mathrm{Mod}_{DBG_+} \simeq_Q \mathrm{Mod}_{\Theta DBG_+}$$

where ΘDBG_+ is a commutative DGA. Moreover we know that

$$H_*(\Theta DBG_+) = \pi_* DBG_+ = H^*BG = \mathbb{Q}[x_1, \ldots, x_r]$$

where r is the rank of G. The next step is to apply a formality argument. The necessary formality statement is that polynomial rings are strongly intrinsically formal as commutative DGAs. This means that given a commutative DGA X with polynomial homology, there is a quasi-isomorphism $H_*X \xrightarrow{\sim} X$. One can see this by choosing cocycle representatives for the polynomial generators and noting that this is a quasi-isomorphism by construction. Change of rings along the quasi-isomorphism $H^*BG \xrightarrow{\sim} \Theta DBG_+$ gives a Quillen equivalence between the category of modules over ΘDBG_+ and the category of modules over H^*BG. The commutativity assumption is crucial for the formality step; polynomial rings are strongly intrinsically formal as *commutative* DGAs but not as DGAs.

Let H be a connected subgroup of G. There is a ring map $\theta\colon DBG_+ \to DBH_+$ but DBH_+ need not be cofibrant as a DBG_+-module and therefore θ^* will not be left Quillen. In order to force this, we must replace DBH_+. In light of the formality step described above, any replacements we perform must be as commutative objects. From this point of view, it seems sensible to cofibrantly replace DBH_+ as a commutative DBG_+-algebra. However, the analogue of [14, 4.1] is not true in general anymore. In particular, in the stable model structure on spectra, a cofibrant commutative algebra need not be cofibrant as a module. Shipley [17] constructed a model structure on spectra called the *flat model structure* which does satisfy this extra compatibility. This model structure is vital for our proof of the correspondence for change of groups functors. However, we therefore must also show that all the necessary Quillen equivalences still hold in the flat model structure.

4. Shipley's algebraicization theorem

As described above, the flat model structure is crucial for our approach to finding counterparts of the change of groups functors. Given a commutative $H\mathbb{Q}$-algebra S, Shipley [18] has proved that there is a commutative DGA ΘS and a zig-zag of Quillen equivalences

$$\mathrm{Mod}_S \simeq_Q \mathrm{Mod}_{\Theta S}$$

using the stable model structure on spectra. This is a key step in the zig-zag of Quillen equivalences in the construction of algebraic models. Therefore, we require the zig-zag of Quillen equivalences given by Shipley to still hold in the flat model structure.

In Chapter 3 we prove the following result, which appears in the preprint [21].

Theorem. *There is a zig-zag of symmetric monoidal Quillen equivalences*

$$\mathrm{Mod}_{H\mathbb{Q}}^{\mathrm{flat}} \simeq_Q \mathrm{Ch}_{\mathbb{Q}}$$

where the intermediate categories have the flat model structure. It follows that for A a commutative $H\mathbb{Q}$-algebra, there is a zig-zag of symmetric monoidal Quillen equivalences

$$\mathrm{Mod}_A^{\mathrm{flat}} \simeq_Q \mathrm{Mod}_{\underline{\Theta} A}$$

where $\underline{\Theta} A$ is a commutative DGA and the intermediate categories have the flat model structure.

In addition to providing the correct formal context for studying change of groups functors, the use of the flat model structure also has benefits in the algebraic model for commutative $H\mathbb{Q}$-algebras. Richter-Shipley [13] have showed that there is a zig-zag of Quillen equivalences

$$\text{commutative } HR\text{-algebras} \simeq_Q E_\infty\text{-dg-}R\text{-algebras}$$

where R is any commutative ring. In the case that $R = \mathbb{Q}$, E_∞-algebras can be rectified to strictly commutative ones, and therefore one obtains a zig-zag of six Quillen equivalences

$$\text{commutative } H\mathbb{Q}\text{-algebras} \simeq_Q \text{commutative dg-}\mathbb{Q}\text{-algebras.}$$

Using the flat model structure instead gives a zig-zag of only three Quillen equivalences between the category of commutative $H\mathbb{Q}$-algebras and the category of commutative dg-$\mathbb{Q}$-algebras. Moreover, this zig-zag of Quillen equivalences does not use the rectification step and therefore provides a more concrete approach.

5. Structure of the thesis

As described above, this thesis is in 'publication format'. Chapter 2 consists of a joint paper with Luca Pol which has been accepted for publication in the Journal of Pure and Applied Algebra [11]. Chapter 3 is a revised version of the preprint [21] and Chapter 4 comprises the preprint [20], both of which have been submitted for publication. Finally, Chapter 5 contains a discussion of possible future directions of research related to the work in this thesis.

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CHAPTER 2

The Left Localization Principle, completions and cofree G-spectra

THE LEFT LOCALIZATION PRINCIPLE, COMPLETIONS, AND COFREE G-SPECTRA

LUCA POL AND JORDAN WILLIAMSON

Abstract. We show under mild hypotheses that a Quillen adjunction between stable model categories induces another Quillen adjunction between their left localizations, and we provide conditions under which the localized adjunction is a Quillen equivalence. Moreover, we show that in many cases the induced Quillen equivalence is symmetric monoidal. Using our results we construct a symmetric monoidal algebraic model for rational cofree G-spectra. In the process, we also show that L-complete modules provide an abelian model for derived complete modules.

Contents

1. Introduction

In this paper we investigate the interplay between adjoint pairs and localizations. In homotopy theory there are two versions of localizations available: the left and right Bousfield localization. The former is ubiquitous in chromatic stable homotopy theory, while the latter has seen interesting applications in the study of torsion objects in algebraic categories, see [24, §5]. Often in the literature the right Bousfield localization is called cellularization since in the stable setting it picks out the localizing subcategory on the set of cells (if they are stable).

We now give an informal overview of our results and refer to the main body of the paper for the precise statements.

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The Cellularization Principle. Let $\mathcal{C}$ be a stable model category, and let $\mathcal{K}$ be a set of objects of $\mathcal{C}$. The Cellularization Principle of Greenlees-Shipley [24] provides conditions under which a Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ descends to a Quillen equivalence

$$F : \mathrm{Cell}_{\mathcal{K}}\mathcal{C} \rightleftarrows \mathrm{Cell}_{F\mathcal{K}}\mathcal{D} : G$$

between the cellularizations. The Cellularization Principle is a crucial ingredient in the construction of algebraic models for rational equivariant spectra, see for instance [27]. There is also a version of the Principle where the cells are passed along the right adjoint, and a variant [9, §5.1] in which symmetric monoidal structures are taken into account. The main limitation of the Cellularization Principle is that the preservation of symmetric monoidal structures is *not* automatic.

Since the symmetric monoidal structure need not be preserved by cellularization, the symmetric monoidal version of the Cellularization Principle requires stronger assumptions. For instance, when passing cells along the right adjoint, the Cellularization Principle gives a symmetric monoidal Quillen equivalence between the cellularizations if the original adjunction was *already* a symmetric monoidal Quillen equivalence [9, 5.1.7].

On the other hand, the monoidal structure is often preserved by left Bousfield localization.

The Left Localization Principle. The Left Localization Principle which we develop, gives mild conditions under which a symmetric monoidal Quillen adjunction $F\colon \mathcal{C} \rightleftarrows \mathcal{D} : G$ descends to a *symmetric monoidal* Quillen equivalence between the homological localizations. For an object E of a stable, symmetric monoidal model category $\mathcal{C}$, the homological localization $L_E\mathcal{C}$ is the localization of $\mathcal{C}$ at the class of E-equivalences, that is those morphisms that become equivalences after tensoring with E.

Theorem (3.15)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be stable, symmetric monoidal model categories, E an object of $\mathcal{C}$ and $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a symmetric monoidal Quillen adjunction. Suppose that $\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ of objects and that $\mathcal{D}$ is homotopically compactly generated by $F\mathcal{K}$. Suppose that:*

(i) *The derived unit map $K \to GFK$ is an E-equivalence for all $K \in \mathcal{K}$;*
(ii) *G sends FE-equivalences to E-equivalences.*

Then the induced Quillen adjunction

$$F : L_E\mathcal{C} \rightleftarrows L_{FE}\mathcal{D} : G$$

is a symmetric monoidal Quillen equivalence.

The major advantage of the Left Localization Principle over the Cellularization Principle is that the symmetric monoidal structure is preserved automatically. There are several variations of the Principle that we do not include in this introduction. Of particular note is the Compactly Generated Localization Principle, see Theorem 3.16. Although the assumptions of this last Principle are quite restrictive, there are interesting examples where it applies, as we show in our applications.

We now turn to the applications of the Left Localization Principle. The main motivation of the authors for developing the Left Localization Principle comes from rational equivariant stable homotopy theory.

Algebraic models. The programme of finding algebraic models for rational G-spectra was begun by Greenlees, who conjectured that for every compact Lie group G, there is an *abelian* category $\mathcal{A}(G)$, together with a Quillen equivalence between the category of rational G-spectra and the category of differential objects in $\mathcal{A}(G)$. The programme looks for abelian categories with finite homological dimension so that calculations can easily be performed, and equipped with an Adams spectral sequence to calculate homotopy classes of maps between G-spectra. This programme has so far been successful in the cases of G finite [6], $G = SO(2)$ [43, 9] $G = O(2)$ [7], $G = SO(3)$ [32], G a torus of any rank [27], the toral part of G-spectra [8], and free G-spectra for G a compact Lie group [23, 25]. One can also ask for equivalences with extra structure such as being monoidal, so that the equivalence passes to ring and module spectra.

When attempting to find algebraic models for categories of interest, there are several techniques we can apply. One approach is to use Morita theory [42] which gives an equivalence with modules over the endomorphism ring of a generator. However, the endomorphism ring need not be commutative so that formality arguments are inaccessible, and the module category often has infinite homological dimension. Another alternative is to use the Cellularization Principle to reduce the problem to checking conditions on generating cells. In this paper, we show that the Left Localization Principle is another technique that we can use. Balchin-Greenlees [4] show that stable model categories can be split into pieces determined by left localizations in an adelic fashion, by proving that the stable model category is a homotopy pullback of an 'adelic cube'. We hope that the Left Localization Principle may be applied in these situations as well, to simplify the adelic cube.

Completions. In order to verify the conjecture of Greenlees in our case of interest, we discuss some homotopical aspects of completion. We briefly recall the relevant results about the different types of completions in algebra and we refer the reader to Section 5 for a more detailed exposition and references.

Let I be a finitely generated ideal in a commutative ring R. The I-adic completion functor is a fundamental tool in algebra, but has poor homological properties as it is neither left nor right exact. Our approach is to work with its zeroth left derived functor which we denote by L_0^I. We say that an R-module M is L_0^I-complete if the canonical map $M \to L_0^I M$ is an isomorphism. The full subcategory of L_0^I-complete modules is a symmetric monoidal abelian category which supports a projective model structure under a mild condition on the ideal considered. This condition is called weak pro-regularity and holds in many cases; for example, any ideal in a Noetherian ring is weakly pro-regular.

For homotopical purposes it is often convenient to consider the derived I-completion functor. This is defined in terms of the stable Koszul complex whose filtration provides a spectral sequence making the derived completion accessible. Under the weak pro-regularity hypothesis on the ideal I, the derived I-completion functor is equivalent to the total left derived functor of I-adic completion, and therefore calculates the local homology modules, see [20, 36].

We give a proof using the language of model categories that derived I-complete modules can be modelled via the abelian category of L_0^I-complete modules, see Theorem 6.11. It follows that a dg-module is derived I-complete if and only if its homology is L_0^I-complete. This generalises a result of Dwyer-Greenlees [16, 6.15] and clarifies an observation of Porta-Shaul-Yekutieli [36, 4.33] that derived I-complete modules need not have I-adically complete homology. We note the related work of Barthel-Heard-Valenzuela who have given an ∞-categorical approach to derived completion in the general setup of comodules over Hopf algebroids [12].

Rational cofree G-spectra. The equivariant stable homotopy category contains two classes of objects of particular note: the free and cofree G-spectra. An algebraic model for rational free G-spectra was constructed by Greenlees-Shipley [23, 25] in terms of torsion modules over the group cohomology ring. However, the abelian category of torsion modules is not monoidal as it has no tensor unit and therefore the Quillen equivalence in the free case cannot be refined to a symmetric monoidal Quillen equivalence.

By exploiting the equivalence between free and cofree G-spectra, we give a symmetric monoidal algebraic model for the category of rational cofree G-spectra. For convenience, we only state the result for the connected case in this introduction. See Theorem 9.6 for the general case.

Theorem (8.4)**.** *Let G be a connected compact Lie group and I be the augmentation ideal of H^*BG. Then there is a symmetric monoidal Quillen equivalence*

$$\mathrm{Sp}_G^{\mathrm{cofree}} \simeq_Q \mathrm{Mod}^{\wedge}_{H^*BG}$$

between rational cofree G-spectra and L_0^I-complete dg-H^*BG-modules. In particular, there is a tensor-triangulated equivalence

$$\textit{cofree } G\textit{-spectra} \simeq_\triangle \mathcal{D}(L_0^I\textit{-complete } H^*BG\textit{-modules}).$$

In this application, the Left Localization Principle manifests its advantages over the Cellularization Principle. Firstly, the proof of the equivalence is formal as it only requires a few elementary iterations of the Left Localization Principle and some formality arguments in algebra. In particular we avoid any "topological" formality argument using the Adams spectral sequence. Secondly, it gives a tensor-triangulated equivalence of the homotopy categories.

Free and cofree G-spectra are interesting for three particular reasons. Firstly, they represent cohomology theories on free G-spaces, the most prominent example of which is Borel cohomology. Secondly, the techniques employed in the construction of the algebraic models for free and cofree G-spectra are instructive for more general cases, such as that of torus-equivariant spectra [27]. Finally, the algebraic models for free and cofree G-spectra fit in the general picture of a local duality context in the sense of [11]. This means that the equivalence between free and cofree G-spectra in equivariant stable homotopy theory translates to the equivalence between torsion and complete modules in algebra.

Contribution of this paper and related work. Let us restrict to connected groups for simplicity, and continue to write I for the augmentation ideal. A Quillen equivalence between rational cofree G-spectra and derived complete H^*BG-modules was already known by passing through free G-spectra in the following way:

$$\begin{tikzcd} \text{free } G\text{-spectra} \arrow[r, leftrightarrow, "\simeq_Q"] \arrow[d, leftrightarrow, "\simeq_Q"'] & I\text{-power torsion-}H^*BG\text{-modules} \arrow[d, leftrightarrow, "\simeq_Q"] \\ \text{cofree } G\text{-spectra} \arrow[r, leftrightarrow, dashed] & \text{derived } I\text{-complete-}H^*BG\text{-modules.} \end{tikzcd}$$

The horizontal Quillen equivalence is the algebraic model for free G-spectra of Greenlees-Shipley [23] and the right vertical follows from Dwyer-Greenlees' Morita theory [16] together with [24, §5]. However this is unsatisfactory for two main reasons. Firstly, it cannot be refined to a symmetric monoidal Quillen equivalence since the category of I-power torsion modules has no tensor unit. Secondly, it does not give an abelian model as desired in the conjecture of Greenlees. In light of this, our contribution is threefold: we prove the algebraic model for rational cofree G-spectra directly, we upgrade it to a symmetric monoidal Quillen equivalence, and we give an abelian model for derived complete modules. In addition, we collect several results about homotopical aspects of algebraic completions which we believe will be of independent interest.

Although our strategy is analogous to that employed by Greenlees-Shipley in the study of free G-spectra, the tools we use differ. In particular, the Left Localization Principle which we develop is a new and key ingredient in our proof.

Outline of the paper. The paper is divided into two main parts.

In the first part we give some necessary background on left Bousfield localizations and then state and prove the Left Localization Principle. We then investigate the implications in the case of homological localizations, which provide many key examples.

In the second part of the paper we focus on the applications of the Left Localization Principle. We apply the Left Localization Principle to understand completions of module categories and to construct a symmetric monoidal algebraic model for rational cofree G-spectra. We have decided to first construct the algebraic model for a connected compact Lie group and then show how to generalize our proofs to the non-connected case. In the final section, we construct a strongly convergent Adams spectral sequence to calculate homotopy classes of maps between cofree G-spectra.

Conventions. We shall follow the convention of writing the left adjoint above the right adjoint in an adjoint pair. We will use $q\colon QX \to X$ and $r\colon X \to RX$ to denote cofibrant and fibrant replacements of X respectively.

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Part 1. The Left Localization Principle

2. Left Bousfield localization of model categories

In this section we recall some necessary background on left Bousfield localizations following [28] and [10].

Definition 2.1. Let $\mathcal{C}$ be a model category and let S be a collection of maps in $\mathcal{C}$.

- An object W in $\mathcal{C}$ is *S-local* if it is fibrant in $\mathcal{C}$ and for every $s\colon A \to B$ in S, the natural map $\mathrm{Map}(B, W) \to \mathrm{Map}(A, W)$ is a weak equivalence of homotopy function complexes.
- A map $f\colon X \to Y$ in $\mathcal{C}$ is an *S-local equivalence* if for every S-local object W, the natural map $\mathrm{Map}(Y, W) \to \mathrm{Map}(X, W)$ is a weak equivalence of homotopy function complexes.

Remark 2.2. If the model category is stable, then the homotopy function complexes in the previous Definition can be replaced with the graded set of maps in the homotopy category, see [10, 4.5].

In many cases, we can endow the model category $\mathcal{C}$ with a new model structure, the *left Bousfield localization* of $\mathcal{C}$, in which the weak equivalences are the S-local equivalences, the cofibrations are unchanged, and the fibrant objects are the S-local objects. If it exists, we denote this model category by $L_S\mathcal{C}$.

Hypothesis 2.3. Throughout this paper we assume that all the required left Bousfield localizations exist.

Remark 2.4. The left Bousfield localization exists under mild conditions on the model category $\mathcal{C}$. For example, when $\mathcal{C}$ is left proper, cellular and S is a set [28, 4.1.1], or when $\mathcal{C}$ is left proper, combinatorial and S is a set [14, 4.7]. In particular, left Bousfield localizations (at sets of morphisms) exist for the stable model structure on spectra [34, 9.1], the stable model structure on equivariant spectra for any compact Lie group [33, III.4.2] and the projective model structure on dg-modules [13, 3.3].

Recall that a model category is *symmetric monoidal* if it is a closed symmetric monoidal category and it satisfies the *pushout-product axiom*: if $f\colon A \to B$ and $g\colon X \to Y$ are cofibrations, then the pushout-product map

$$f \square g\colon A \otimes Y \bigcup_{A \otimes X} B \otimes X \to B \otimes Y$$

is a cofibration, which is acyclic if either f or g is acyclic; and the *unit axiom*: the natural map $Q\mathbb{1} \otimes X \to \mathbb{1} \otimes X \cong X$ is a weak equivalence for all cofibrant X. We denote the internal hom functor by $F(-,-)$.

Definition 2.5. We say that a stable model category $\mathcal{C}$ is *homotopically compactly generated* by a set $\mathcal{K}$ of objects if its homotopy category $\mathrm{h}\mathcal{C}$ is compactly generated by $\mathcal{K}$:

- for all $K \in \mathcal{K}$ and collections $\{M_i\}$ of objects of $\mathcal{C}$, the natural map $\bigoplus \mathrm{h}\mathcal{C}(K, M_i) \to \mathrm{h}\mathcal{C}(K, \bigoplus M_i)$ is an isomorphism;
- an object X of $\mathrm{h}\mathcal{C}$ is trivial if and only if $\mathrm{h}\mathcal{C}(\Sigma^n K, X) = 0$ for all $K \in \mathcal{K}$ and $n \in \mathbb{Z}$.

Next we recall the definition of a monoidal Quillen adjunction from [41].

Definition 2.6. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction between symmetric monoidal model categories.

(1) We say that (F, G) is a *weak symmetric monoidal Quillen adjunction* if the right adjoint G is lax monoidal (which gives the left adjoint F an oplax monoidal structure) and the following two conditions hold:
   (a) for cofibrant A and B in $\mathcal{C}$, the oplax monoidal structure map $\phi\colon F(A \otimes B) \to F(A) \otimes F(B)$ is a weak equivalence in $\mathcal{D}$
   (b) for a cofibrant replacement $Q\mathbb{1}_{\mathcal{C}}$ of the unit in $\mathcal{C}$, the map $\phi_0\colon F(Q\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ is a weak equivalence in $\mathcal{D}$.
(2) If the oplax monoidal structure maps ϕ and ϕ_0 are isomorphisms, then we say that (F, G) is a *strong symmetric monoidal Quillen pair*.
(3) We say that the adjunction (F, G) is *symmetric monoidal* if it is a weak symmetric monoidal Quillen adjunction.
(4) We say that the adjucntion (F, G) is a *symmetric monoidal Quillen equivalence* if it is a symmetric monoidal adjunction and a Quillen equivalence.

Remark 2.7. A Quillen adjunction is symmetric monoidal if the left adjoint is strong monoidal and the unit object of $\mathcal{C}$ is cofibrant.

Definition 2.8. A set of morphisms S of a stable model category $\mathcal{C}$ is said to be *stable* if the collection of S-local objects is closed under (de)suspensions. We say that a stable set of cofibrations S of a stable, cellular, symmetric monoidal model category $\mathcal{C}$ is *monoidal* if $S \square I = \{s \square i \mid s \in S,\ i \in I\}$ is contained in the class of S-equivalences, where I is the set of generating cofibrations for $\mathcal{C}$.

We will need the following result.

Proposition 2.9 ([10, 5.1])**.** *Let $\mathcal{C}$ be a proper, cellular, stable, symmetric monoidal model category and let S be a stable set of cofibrations between cofibrant objects. Then the localization $L_S\mathcal{C}$ is a symmetric monoidal model category if and only if S is monoidal.*

Remark 2.10. Any map in a model category can be replaced up to weak equivalence by a cofibration between cofibrant objects: first cofibrantly replacing the source and then factoring the composite into a cofibration followed by an acyclic fibration. Since left Bousfield localization depends only on the homotopy type of the class of maps, we can assume without loss of generality that S consists of cofibrations between cofibrant objects.

Remark 2.11. Since we work with stable model categories throughout this paper, one could formulate the Bousfield localizations differently. In particular, given a collection of objects $\mathcal{A}$ in $h\mathcal{C}$ one can take $S = \{f \mid \mathrm{cofibre}(f) \in \mathcal{A}\}$. The objects of $\mathcal{A}$ then become acyclic in the localized model structure. In order to be consistent with the model categorical literature, for example [28], we will always take a set of maps S in the stable model category $\mathcal{C}$.

3. The Left Localization Principle

We are now ready to work towards the Left Localization Principle. Before we can prove an induced Quillen equivalence, we must check that the Quillen adjunction descends to the localizations. Recall that Q and R denote cofibrant and fibrant replacement in the original model structures on $\mathcal{C}$ and $\mathcal{D}$ respectively.

Proposition 3.1. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction between stable model categories satisfying Hypothesis 2.3. Let S and T be stable sets of morphisms in $\mathcal{C}$ and $\mathcal{D}$ respectively, and suppose that F sends S-equivalences between cofibrant objects to T-equivalences. Then the adjunction*

$$F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$$

is a Quillen adjunction. Furthermore, it is a symmetric monoidal Quillen adjunction if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a symmetric monoidal Quillen adjunction and S and T are monoidal.

Proof. By Hirschhorn [28, 3.3.18], to prove that $L_S\mathcal{C} \rightleftarrows L_T\mathcal{D}$ is a Quillen adjunction, it is sufficient to check that F sends S-equivalences between cofibrant objects to T-equivalences, which was our hypothesis. The claim about the monoidality follows from the fact that the cofibrations in a left Bousfield localization are the same as in the original category, and the local equivalences contain the original weak equivalences. □

Remark 3.2. If we apply the previous Proposition with $S = GRT$, then the hypothesis that F sends GRT-equivalences between cofibrant objects to T-equivalences may seem hard to verify in practice. However, we show in Lemma 3.14 that in the case of homological localization, this hypothesis can be replaced by a condition which is much easier to verify.

Remark 3.3. If S is monoidal, it often happens that FQS is also monoidal. Write $I_{\mathcal{C}}$ and $I_{\mathcal{D}}$ for the sets of generating cofibrations in $\mathcal{C}$ and $\mathcal{D}$ respectively. For instance, one can easily check that FQS is monoidal when (F, G) is a strong symmetric monoidal Quillen pair and $I_{\mathcal{D}} \subseteq F(I_{\mathcal{C}})$, or, when (F, G) is a weak symmetric monoidal Quillen pair, the domains of $I_{\mathcal{C}}$ are cofibrant and $I_{\mathcal{D}} \subseteq F(I_{\mathcal{C}})$. Note that the condition that $I_{\mathcal{D}} \subseteq F(I_{\mathcal{C}})$ is satisfied in the case when the model structure on $\mathcal{D}$ is right induced from $\mathcal{C}$.

We can now state and prove the Left Localization Principle. We note that as the cofibrations are the same in the left Bousfield localization as in the original model structure, we continue to write Q for the cofibrant replacement in the localization. However, since being fibrant in the localization is a stronger condition than being fibrant in the original model structure, we write $\overline{R}$ for the fibrant replacement in the localization.

Theorem 3.4 (Left Localization Principle)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be stable model categories satisfying Hypothesis 2.3 and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction.*

(1) *Suppose that $\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ and that $\mathcal{D}$ is homotopically compactly generated by $FQ\mathcal{K}$. Let S and T be stable sets of morphisms in $\mathcal{C}$ and $\mathcal{D}$ respectively. Suppose that the following conditions hold:*

(i) *The derived unit map* $\eta_K \colon QK \to GRFQK$ *is an* S*-equivalence for all* $K \in \mathcal{K}$*;*
(ii) G *sends* T*-equivalences between fibrant objects in* $\mathcal{D}$ *to* S*-equivalences.*
(iii) F *sends* S*-equivalences between cofibrant objects to* T*-equivalences.*
Then the induced Quillen adjunction

$$F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$$

is a Quillen equivalence. Moreover, if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *is a symmetric monoidal Quillen adjunction and* S *and* T *are monoidal, then* $F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$ *is a symmetric monoidal Quillen equivalence.*

(2) *Suppose that* $\mathcal{D}$ *is homotopically compactly generated by a set* $\mathcal{L}$ *and that* $\mathcal{C}$ *is homotopically compactly generated by* $GR\mathcal{L}$*. Let* T *be a stable set of morphisms in* $\mathcal{D}$*. Suppose that the following conditions hold:*
(i) *The derived counit map* $\epsilon_L \colon FQGRL \to RL$ *is a weak equivalence in* $\mathcal{D}$ *for all* $L \in \mathcal{L}$*;*
(ii) G *sends* T*-equivalences between fibrant objects in* $\mathcal{D}$ *to* GRT*-equivalences;*
(iii) F *sends* GRT*-equivalences between cofibrant objects to* T*-equivalences.*
Then the induced Quillen adjunction

$$F : L_{GRT}\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$$

is a Quillen equivalence. Moreover, if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ *is a symmetric monoidal Quillen adjunction and* T *and* GRT *are monoidal, then* $F : L_{GRT}\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$ *is a symmetric monoidal Quillen equivalence.*

Proof. Let us prove (1). Note that condition (iii) ensures that the Quillen adjunction descends to the localizations, see Proposition 3.1. We now show that the derived functor GR preserves sums, so that the subcategories

$$\mathcal{A} = \{X \in \mathrm{h}\mathcal{C} \mid \eta_X \colon QX \xrightarrow{\sim_S} GRFQX\} \quad \text{and} \quad \mathcal{A}' = \{Y \in \mathrm{h}\mathcal{D} \mid \epsilon_Y \colon FQGRY \xrightarrow{\sim_T} RY\}$$

are localizing. Let $(X_i)_{i\in I}$ be a collection of objects in $\mathrm{h}\mathcal{D}$. Using compactness we see that for all $K \in \mathcal{K}$

$$\mathrm{h}\mathcal{C}(K, GR(\bigoplus_{i\in I} X_i)) \cong \mathrm{h}\mathcal{D}(FQK, \bigoplus_{i\in I} X_i) \cong \bigoplus_{i\in I} \mathrm{h}\mathcal{D}(FQK, X_i) \cong \bigoplus_{i\in I} \mathrm{h}\mathcal{C}(K, GR(X_i)) \cong \mathrm{h}\mathcal{C}(K, \bigoplus_{i\in I} GR(X_i)).$$

Since $\mathcal{K}$ generates $\mathrm{h}\mathcal{C}$ we conclude that GR preserves arbitrary sums.

By assumption (i), we know that $\mathcal{K} \subset \mathcal{A}$ thus $\mathcal{A} = \mathrm{h}\mathcal{C}$ as $\mathcal{K}$ generates $h\mathcal{C}$. Note that $FQ\eta_K$ is a T-equivalence by condition (iii). Using the triangular identity of the derived adjunction

$$\begin{tikzcd} FQK \arrow[r, "FQ\eta_K"] \arrow[dr, "r"'] & FQGRFQK \arrow[d, "\epsilon_{FQK}"] \\ & RFQK \end{tikzcd}$$

and 2-out-of-3, we obtain that $FQK \in \mathcal{A}'$ and hence $\mathcal{A}' = \mathrm{h}\mathcal{D}$ as $FQ\mathcal{K}$ generates $h\mathcal{D}$.

We must prove that $\overline{\eta}_X \colon QX \to G\overline{R}FQX$ is an S-equivalence for all $X \in \mathrm{h}\mathcal{C}$ and that $\overline{\epsilon}_Y \colon FQG\overline{R}Y \to \overline{R}Y$ is a T-equivalence for all $Y \in \mathrm{h}\mathcal{D}$. Note that the canonical map $GRFQX \to G\overline{R}FQX$ is an S-equivalence by condition (ii). Therefore the derived unit

$$\overline{\eta}_X \colon QX \xrightarrow{\sim_S} GRFQX \xrightarrow{\sim_S} G\overline{R}FQX$$

is an S-equivalence. For the derived counit, note that the canonical map $GRY \to G\overline{R}Y$ is an S-equivalence and therefore $FQGRY \to FQG\overline{R}Y$ is a T-equivalence by condition (iii). By considering the diagram

$$\begin{tikzcd} FQGRY \arrow[r, "\epsilon_Y"] \arrow[d, "\sim_T"'] & RY \arrow[d, "\sim_T"] \\ FQG\overline{R}Y \arrow[r, "\overline{\epsilon}_Y"'] & \overline{R}Y \end{tikzcd}$$

we see that $\overline{\epsilon}_Y$ is a T-equivalence if and only if ϵ_Y is so. Since $\mathcal{A}' = \mathrm{h}\mathcal{D}$ the claim follows.

The proof of part (2) follows from (1) by taking $S = GRT$. $\square$

Remark 3.5. Notice that the conditions in (1) imply that the derived functor FQ preserves all compact objects. Moreover, in the proof we showed that GR preserves sums so it also follows that under the conditions in (2) the derived functor GR preserves all compact objects.

Remark 3.6. In [29, 2.3] Hovey gives criteria for when left Bousfield localization preserves Quillen equivalences. His result does not assume stability but does not treat the case where the original adjunction is not a Quillen equivalence.

In the Left Localization Principle we assumed that $\mathcal{C}$ and $\mathcal{D}$ are homotopically compactly generated whereas in the following we assume that the localizations are homotopically compactly generated. This is a stronger condition but holds in certain cases when the localization is homological, see Remark 3.18.

Theorem 3.7 (Compactly Generated Localization Principle)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be stable model categories satisfying Hypothesis 2.3 and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a Quillen adjunction. Consider stable sets S and T of morphisms in $\mathcal{C}$ and $\mathcal{D}$ respectively.*

(1) *Suppose that $L_S\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ and that $L_T\mathcal{D}$ is homotopically compactly generated by $FQ\mathcal{K}$. Suppose that the derived unit map $\overline{\eta}_K \colon QK \to G\overline{R}FQK$ is an S-equivalence for all $K \in \mathcal{K}$ and that F sends S-equivalences between cofibrant objects to T-equivalences. Then the induced Quillen adjunction*
$$F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$$
is a Quillen equivalence. Moreover, if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a symmetric monoidal Quillen adjunction and S and T are monoidal, then $F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$ is a symmetric monoidal Quillen equivalence.

(2) *Suppose that $L_T\mathcal{D}$ is homotopically compactly generated by a set $\mathcal{L}$ and that $L_S\mathcal{C}$ is homotopically compactly generated by $G\overline{R}\mathcal{L}$. Suppose that the derived counit $\overline{\epsilon}_L \colon FQG\overline{R}L \to \overline{R}L$ is a T-equivalence for all $L \in \mathcal{L}$ and that F sends S-equivalences between cofibrant objects to T-equivalences. Then the induced Quillen adjunction*
$$F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$$
is a Quillen equivalence. Moreover, if $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a symmetric monoidal Quillen adjunction and S and T are monoidal, then $F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$ is a symmetric monoidal Quillen equivalence.

Proof. Apply the Cellularization Principle [24, 2.7] to the Quillen adjunction $F : L_S\mathcal{C} \rightleftarrows L_T\mathcal{D} : G$ obtained from Proposition 3.1. □

3.1. **Homological localization.** We now rephrase the Left Localization Principle for homological Bousfield localizations. This setting provides a large family of examples in which our result simplifies.

Definition 3.8. Let $\mathcal{C}$ be a symmetric monoidal model category. We say that an object $E \in \mathcal{C}$ is *flat* if $E \otimes -$ preserves weak equivalences.

Remark 3.9. If E is a cofibrant object in a symmetric monoidal model category $\mathcal{C}$, then $E \otimes -$ preserves weak equivalences between cofibrant objects by Ken Brown's lemma. However, in many cases of interest all cofibrant objects are in fact flat:

(i) The cofibrant objects in the projective model structure on dg-modules are the dg-projective modules. Any dg-projective module P has the property that $P \otimes -$ preserves quasiisomorphisms [3, 11.1.6, 11.2.1] and so any cofibrant object is flat in the projective model structure on dg-modules.

(ii) Any cofibrant object in the stable model structure on modules over a ring spectrum is flat [34, 12.3, 12.7]. Similarly, any cofibrant object in the stable model structure on modules over a ring G-spectrum is flat [33, 7.3, 7.7].

Definition 3.10. Let $\mathcal{C}$ be a stable and symmetric monoidal model category, and let E be a flat cofibrant object of $\mathcal{C}$. We say that $f \colon X \to Y$ is an *E-equivalence* if $E \otimes f \colon E \otimes X \to E \otimes Y$ is a weak equivalence.

When it exists, localizing at the E-equivalences produces a model structure on $\mathcal{C}$ in which the weak equivalences are the E-equivalences, the cofibrations are unchanged and the fibrant objects are the E-local objects. We call this new model category the *homological localization* of $\mathcal{C}$ at E and write $L_E\mathcal{C}$. One can prove that the weak equivalences in $L_E\mathcal{C}$ are the E-equivalences as follows. Write S for the collection of E-equivalences. It is immediate that any E-equivalence is an S-equivalence, so it remains to show the converse. This follows from the universal property of $h\mathcal{C}[S^{-1}]$ applied to the functor $E \wedge -\colon h\mathcal{C} \to h\mathcal{C}$.

Hypothesis 3.11. From now on we assume that the required homological localizations exist.

Remark 3.12. The homological localization exists if $\mathcal{C}$ is a stable, symmetric monoidal, proper and compactly generated model category in the sense of [45, 1.2.3.4]; see [17, §VIII.1] for the special case of spectra, and [4, §6.A] for the general case.

Proposition 3.13. *Let $\mathcal{C}$ be a symmetric monoidal model category satisfying Hypothesis 3.11, and let E be a flat cofibrant object of $\mathcal{C}$. Then the homological localization $L_E\mathcal{C}$ is a symmetric monoidal model category.*

Proof. Take two cofibrations i and j. Since the cofibrations in $L_E\mathcal{C}$ are the same as in $\mathcal{C}$, the pushout-product map $i\square j$ is a cofibration since $\mathcal{C}$ satisfies the pushout-product axiom. Now suppose that i is an E-equivalence also. We have that $E\otimes(i\square j) = (E\otimes i)\square(E\otimes j)$ since $E\otimes -$ is a left adjoint. The functor $E\otimes -$ is left Quillen since E is cofibrant, so $E\otimes i$ is an acyclic cofibration and $E\otimes j$ is a cofibration. Therefore, $E\otimes(i\square j)$ is a weak equivalence by the pushout-product axiom for $\mathcal{C}$. In other words, $i\square j$ is an E-equivalence as required. The unit axiom follows immediately from the unit axiom for $\mathcal{C}$, since the cofibrations are the same in the left Bousfield localization as in the original model category. $\square$

Lemma 3.14. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a symmetric monoidal Quillen adjunction between stable symmetric monoidal model categories and let E' be a flat bifibrant object in $\mathcal{D}$. If $\epsilon_E\colon FQGE' \to E'$ is a weak equivalence in $\mathcal{D}$, then F sends QGE'-equivalences between cofibrant objects to E'-equivalences.*

Proof. Let $X \to Y$ be a QGE'-equivalence between cofibrant objects. By Ken Brown's lemma, $F(QGE'\otimes X) \to F(QGE'\otimes Y)$ is a weak equivalence. We have the commutative diagram

$$\begin{tikzcd}
F(QGE'\otimes X) \arrow[r, "\sim"] \arrow[d, "\sim"'] & FQGE'\otimes FX \arrow[r, "\sim"] \arrow[d] & E'\otimes FX \arrow[d] \\
F(QGE'\otimes Y) \arrow[r, "\sim"'] & FQGE'\otimes FY \arrow[r, "\sim"'] & E'\otimes FY
\end{tikzcd}$$

in which the first horizontal maps are equivalences by definition of a symmetric monoidal Quillen pair, and the second horizontal maps are equivalences since $\epsilon_E\colon FQGE' \to E'$ is a weak equivalence and tensoring with a cofibrant object preserves weak equivalences between cofibrants by Ken Brown's lemma. Hence by two-out-of-three, $E'\otimes FX \to E'\otimes FY$ is a weak equivalence as required. $\square$

Recall that the homological localization at an object E is a special case of left Bousfield localization which inverts the E-equivalences. Therefore we can combine this Lemma with the Left Localization Principle to obtain our version for homological localizations.

Theorem 3.15 (Left Localization Principle)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be stable, symmetric monoidal model categories satisfying Hypothesis 3.11 and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a symmetric monoidal Quillen adjunction.*

(1) *Suppose that $\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ and that $\mathcal{D}$ is homotopically compactly generated by $FQ\mathcal{K}$. Let $E \in \mathcal{C}$ be a flat cofibrant object. Suppose that the following conditions hold:*
   (i) *The derived unit map $QK \to GRFQK$ is an E-equivalence for all $K \in \mathcal{K}$;*
   (ii) *G sends FE-equivalences between fibrant objects in $\mathcal{D}$ to E-equivalences.*
   Then the induced Quillen adjunction
$$F : L_E\mathcal{C} \rightleftarrows L_{FE}\mathcal{D} : G$$
   is a symmetric monoidal Quillen equivalence.
(2) *Suppose that $\mathcal{D}$ is homotopically compactly generated by a set $\mathcal{L}$ and that $\mathcal{C}$ is homotopically compactly generated by $GR\mathcal{L}$. Let $E' \in \mathcal{D}$ be a flat bifibrant object. Suppose that the following conditions hold:*
   (i) *The derived counit map $FQGRL \to RL$ is a weak equivalence in $\mathcal{D}$ for all $L \in \mathcal{L}$;*
   (ii) *G sends E'-equivalences between fibrant objects in $\mathcal{D}$ to QGE'-equivalences.*

(iii) *The map $FQGE' \to E'$ is a weak equivalence in $\mathcal{D}$;*
Then the induced Quillen adjunction

$$F : L_{QGE'}\mathcal{C} \rightleftarrows L_{E'}\mathcal{D} : G$$

is a symmetric monoidal Quillen equivalence.

We now give a mixing of the Left Localization Principle and the Cellularization Principle. Note that we again write $\overline{R}$ for a fibrant replacement in the Bousfield localization.

Theorem 3.16 (Compactly Generated Localization Principle)**.** *Let $\mathcal{C}$ and $\mathcal{D}$ be stable, symmetric monoidal model categories satisfying Hypothesis 3.11 and let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a symmetric monoidal Quillen adjunction.*

(1) *Let E be a flat cofibrant object of $\mathcal{C}$. Suppose that $L_E\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ and that $L_{FE}\mathcal{D}$ is homotopically compactly generated by $FQ\mathcal{K}$. If the derived unit map $Q\overline{\eta}_K \colon K \to G\overline{R}FQK$ is an E-equivalence for all $K \in \mathcal{K}$ then the induced Quillen adjunction*

$$F : L_E\mathcal{C} \rightleftarrows L_{FE}\mathcal{D} : G$$

is a symmetric monoidal Quillen equivalence.

(2) *Let E' be a flat bifibrant object of $\mathcal{D}$. Suppose that $L_{E'}\mathcal{D}$ is homotopically compactly generated by a set $\mathcal{L}$ and that $L_{QGE'}\mathcal{C}$ is homotopically compactly generated by $G\overline{R}\mathcal{L}$. Suppose that the derived counit $\overline{\epsilon}_L \colon FQG\overline{R}L \to \overline{R}L$ is an E'-equivalence for all $L \in \mathcal{L}$ and that $FQGE' \to E'$ is a weak equivalence in $\mathcal{D}$. Then the induced Quillen adjunction*

$$F : L_{QGE'}\mathcal{C} \rightleftarrows L_{E'}\mathcal{D} : G$$

is a symmetric monoidal Quillen equivalence.

Remark 3.17. Barnes-Roitzheim have compared left and right Bousfield localizations of stable model categories at dualizable objects [10, 9.6]. More precisely, they proved that the identity functors

$$L_A\mathcal{C} \leftrightarrows \mathrm{Cell}_{DA}\mathcal{C}$$

give a Quillen equivalence, where $D = F(-, \mathbb{1})$ is the dual functor and A is dualizable. Accordingly, in some cases the Left Localization Principle can be replaced by the Cellularization Principle and vice versa. However, there are some subtleties that need to be considered. Firstly, the two principles are "exchangeable" only if the functors interact well with taking duals and we localize at dualizable objects. This a big disadvantage for instance in global stable homotopy theory where almost no compact objects are dualizable. This was one of the main motivations of the authors to develop the Left Localization Principle. Secondly, the two principles have quite different behaviour when we take into account the symmetric monoidal structure. While the Left Localization Principle for homological localization automatically yields a monoidal Quillen equivalence, the Cellularization Principle requires strong conditions, in particular when passing cells along the right adjoint, see [9, 5.1.7].

Remark 3.18. If we want to apply the Compactly Generated Localization Principle we need to know that the category of local objects is compactly generated. This holds for instance, when we localize at dualizable objects. More precisely, let $\mathcal{C}$ be a stable, symmetric monoidal model category, and let A be a dualizable object of $\mathcal{C}$. It is not difficult to see that if $\mathcal{C}$ is homotopically compactly generated by a set $\mathcal{K}$ then the homological localization $L_A\mathcal{C}$ is homotopically compactly generated by $DA \otimes \mathcal{K}$. Firstly, $DA \otimes K$ is A-local for all $K \in \mathcal{K}$ since if $A \otimes Z \simeq 0$, then $h\mathcal{C}(Z, DA \otimes K) = h\mathcal{C}(Z, F(A, K)) = h\mathcal{C}(Z \otimes A, K) = 0$. Compactness follows from the fact that $A \otimes - : hL_A\mathcal{C} \to h\mathcal{C}$ preserves colimits, and the generation is an immediate consequence of the duality adjunction. For more details, see for instance [35, 2.27].

4. Completion of module categories

In this section we apply the Left Localization Principle to obtain symmetric monoidal Quillen equivalences relating a ring to its completion. We provide a general statement and then give several concrete examples of interest.

Notation 4.1. Given a commutative monoid R in a symmetric monoidal model category $\mathcal{C}$, we denote by $\mathrm{Mod}_R(\mathcal{C})$ the category of R-modules equipped with the projective model structure (if it exists) in which the weak equivalences and fibrations are created by the forgetful functor $\mathrm{Mod}_R(\mathcal{C}) \to \mathcal{C}$. If the underlying category is clear, we will often write Mod_R.

Hypothesis 4.2. Throughout this paper we assume that the projective model structure on $\mathrm{Mod}_R(\mathcal{C})$ exists and that it is left proper, so that left Bousfield localizations exist.

Remark 4.3. Note that the projective model structure exists if $\mathcal{C}$ satisfies the monoid axiom [40, 4.1], and it is left proper in many cases: for instance in categories of (equivariant) spectra [34, 12.1(i)] and [33, III.7.6], and in dg-modules [13, 3.3].

Proposition 4.4. *Let $\mathcal{C}$ be a stable, symmetric monoidal model category with cofibrant monoidal unit, which is homotopically compactly generated by a set $\mathcal{K}$.*

(i) *Let S be a stable, monoidal set of maps in $\mathrm{Mod}_R(\mathcal{C})$ and $\theta\colon R \to R'$ be a map of commutative monoids in $\mathcal{C}$ which is an S-equivalence. The map θ induces an extension-restriction of scalars Quillen adjunction*

$$R' \otimes_R - : \mathrm{Mod}_R(\mathcal{C}) \rightleftarrows \mathrm{Mod}_{R'}(\mathcal{C}) : \theta^*$$

and if θ^* sends $R' \otimes_R QS$-equivalences between fibrant objects to S-equivalences, the Left Localization Principle applies to give a symmetric monoidal Quillen equivalence

$$L_S\mathrm{Mod}_R(\mathcal{C}) \simeq_Q L_{R'\otimes_R QS}\mathrm{Mod}_{R'}(\mathcal{C}).$$

(ii) *Let E be a flat cofibrant R-module and $\theta\colon R \to R'$ be a map of commutative monoids in $\mathcal{C}$ which is an E-equivalence. The map θ induces a symmetric monoidal extension-restriction of scalars Quillen adjunction*

$$R' \otimes_R - : \mathrm{Mod}_R(\mathcal{C}) \rightleftarrows \mathrm{Mod}_{R'}(\mathcal{C}) : \theta^*$$

between the categories of modules, and the Left Localization Principle applies to give a symmetric monoidal Quillen equivalence

$$L_E\mathrm{Mod}_R(\mathcal{C}) \simeq_Q L_E\mathrm{Mod}_{R'}(\mathcal{C}).$$

Remark 4.5. Note that there is an abuse of notation in the second part of the proposition above since in general there is no natural R'-module structure on E at the model category level. More precisely, on the right hand side of the Quillen equivalence above we should have localized at $R' \otimes_R E$ instead of E. However, this abuse of notation does no harm since there is a natural weak equivalence $E \xrightarrow{\sim} R' \otimes_R E$ in $\mathcal{C}$ and the class of $R' \otimes_R E$-equivalences is detected in the homotopy category of $\mathcal{C}$.

Proof. Without loss of generality we may assume that $\mathcal{K}$ consists of cofibrant objects. The set $R \otimes \mathcal{K}$ provides a set of compact generators for $h\mathrm{Mod}_R(\mathcal{C})$. The left adjoint is strong monoidal and maps compact generators to compact generators since $R' \otimes_R (R \otimes \mathcal{K}) \cong R' \otimes \mathcal{K}$.

Since S is a monoidal set, the class of S-equivalences is closed under tensor product with cofibrant objects. Let $K \in \mathcal{K}$ and consider the diagram

$$\begin{tikzcd} QR \otimes K \arrow[r] \arrow[d, "\sim_S"'] & R' \otimes_R (QR \otimes K) \arrow[d, "\sim_S"] \\ R \otimes K \arrow[r, "\sim_S"'] & R' \otimes K \end{tikzcd}$$

in which the top horizontal arrow is the derived unit. Since $QR \to R$ is an S-equivalence and K is cofibrant, the left vertical is an S-equivalence. As $R \to R'$ is an S-equivalence, the bottom horizontal is also an S-equivalence since K is cofibrant. By Ken's Brown's lemma, it follows that the right hand vertical is also an S-equivalence. Therefore, the derived unit is an S-equivalence for $K \in \mathcal{K}$. Therefore, Part (i) of the statement follows from the Left Localization Principle.

Part (ii) is a consequence of Part (i), once we show that the right adjoint θ^* automatically preserves E-equivalences between fibrant objects. Note that there is a natural map $E \otimes_R \theta^* M \to \theta^*(E \otimes_{R'} M)$ of

R-modules, which is a weak equivalence as $E \simeq R' \otimes_R E$. Now suppose that $M \to N$ is an E-equivalence between fibrant R'-modules. By considering the diagram

$$\begin{tikzcd} E \otimes_R \theta^* M \arrow[r] \arrow[d, "\sim"'] & E \otimes_R \theta^* N \arrow[d, "\sim"] \\ \theta^*(E \otimes_{R'} M) \arrow[r] & \theta^*(E \otimes_{R'} N) \end{tikzcd}$$

we see that $\theta^* M \to \theta^* N$ is an E-equivalence of R-modules. $\square$

Example 4.6. Let $\mathbb{Z}_p$ denote the p-adic integers and consider the ring map $\theta\colon \mathbb{Z} \to \mathbb{Z}_p$ which induces a symmetric monoidal Quillen adjunction between the categories of chain complexes

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} - : \mathrm{Mod}_{\mathbb{Z}} \rightleftarrows \mathrm{Mod}_{\mathbb{Z}_p} : \theta^*$$

via extension and restriction of scalars. We can apply Proposition 4.4 to obtain a symmetric monoidal Quillen equivalence

$$L_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}} \simeq_Q L_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}_p}.$$

By [21, 4.2], we can identify the homotopy categories of the two localizations with the subcategories of the derived categories consisting of derived p-complete modules which we denote $\Lambda_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}}$ and $\Lambda_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}_p}$ respectively. Putting everything together we get a tensor-triangulated equivalence

$$\Lambda_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}} \simeq_{\triangle} \Lambda_{\mathbb{Z}/p}\mathrm{Mod}_{\mathbb{Z}_p}.$$

Before stating the next example, we discuss the construction of a commutative ring orthogonal G-spectrum which presents the homotopy type of the functional dual of $E\mathcal{F}_+$, for $\mathcal{F}$ a family of subgroups of a compact Lie group G.

Construction 4.7. Let G be a compact Lie group, and $\mathcal{F}$ a family of subgroups of G. We briefly discuss the construction of an orthogonal G-spectrum modelling the homotopy theory of the functional dual $DE\mathcal{F}_+$. Since S^0 is not a fibrant G-spectrum, the point-set object $F(E\mathcal{F}_+, S^0)$ does not have the same homotopy type as the derived hom. Since we desire a model for $DE\mathcal{F}_+$ which is a commutative ring G-spectrum, the construction requires some care.

Let fS^0 be a fibrant replacement of S^0 as a commutative ring G-spectrum. This will be positive fibrant as a G-spectrum but need not be fibrant. Write $fS^0 \to \widehat{f}S^0$ for a fibrant replacement of fS^0 as a G-spectrum. We claim that if $i\colon A \to B$ is cofibration and $p\colon X \to Y$ is a positive fibration, that the pullback-product map

$$F(B, X) \to F(A, X) \times_{F(A,Y)} F(B, Y)$$

is a positive fibration. To prove this, by adjunction it is sufficient to show that the pushout-product of a cofibration with a positive cofibration is a positive cofibration. Recall that a map is a positive cofibration if and only if it is a cofibration which is an isomorphism in all levels V with $V^G = 0$, see [33, III.2.10]. Since G-spectra is a monoidal model category, it remains to check that the pushout-product of a cofibration with a positive cofibration is an isomorphism for all levels V with $V^G = 0$. This follows from the definitions of the generating cofibrations. It follows by Ken Brown's lemma that if A is cofibrant and $p\colon X \to Y$ is a weak equivalence between positively fibrant objects, then $F(A, X) \to F(A, Y)$ is a weak equivalence.

Note that $E\mathcal{F}_+$ is a cofibrant G-spectrum, and since fibrant G-spectra are also positively fibrant, this shows that the natural map $F(E\mathcal{F}_+, fS^0) \to F(E\mathcal{F}_+, \widehat{f}S^0)$ is a weak equivalence. Since $E\mathcal{F}_+$ is cofibrant and $\widehat{f}S^0$ is fibrant, the mapping spectrum $F(E\mathcal{F}_+, \widehat{f}S^0)$ has the homotopy type of the derived mapping spectrum, and hence so does $F(E\mathcal{F}_+, fS^0)$. Since fS^0 is a commutative ring G-spectrum we note that $F(E\mathcal{F}_+, fS^0)$ is a commutative ring G-spectrum via the diagonal $\triangle\colon E\mathcal{F}_+ \to E\mathcal{F}_+ \wedge E\mathcal{F}_+$.

If we work rationally, then more concretely one can take $F(E\mathcal{F}_+, \mathrm{inf} H\mathbb{Q})$ as a model for $DE\mathcal{F}_+$ since the inflated Eilenberg-MacLane spectrum is a commutative ring G-spectrum (as inflation is strong monoidal) and is fibrant (as inflation is right Quillen). This spectrum will play an important role in the second part of this paper for the family of subgroups consisting of only the trivial subgroup. In this case we write DEG_+ for this mapping spectrum.

Example 4.8. Let G be a compact Lie group and $\mathcal{F}$ a family of subgroups of G. Note that the G-spectrum $DE\mathcal{F}_+$ is a commutative ring G-spectrum as discussed in Construction 4.7. It is easy to check that the unit map $\eta\colon S^0 \to DE\mathcal{F}_+$ is a $E\mathcal{F}_+$-equivalence. We can then apply Proposition 4.4 to obtain a symmetric monoidal Quillen equivalence

$$L_{E\mathcal{F}_+}\mathrm{Sp}_G \simeq_Q L_{E\mathcal{F}_+}\mathrm{Mod}_{DE\mathcal{F}_+}(\mathrm{Sp}_G).$$

Example 4.9. Let $\mathcal{G}$ be the global family of compact Lie groups. Denote by $\mathrm{Sp}_\mathcal{G}$ the category of orthogonal spectra with the $\mathcal{G}$-global model structure which is proper [39, 4.3.17]. By [39, 4.5.21, 4.5.22(ii)], there exists a morphism of ultracommutative ring spectra $i_\mathbb{S}\colon \mathbb{S} \to \mathrm{b}\mathbb{S}$ between the global sphere spectrum and the global Borel construction which exhibits $\mathrm{b}\mathbb{S}$ as a localization of the global sphere spectrum at the class of non-equivariant equivalences. Note that the projective model structure on $\mathrm{Mod}_{\mathrm{b}\mathbb{S}}(\mathrm{Sp}_\mathcal{G})$ exists by [39, 4.3.29] and it is proper by a similar argument as in [34, 12.1(i)] so that we can perform left Bousfield localizations. Since $i_\mathbb{S}^*$ preserves non-equivariant equivalences, it follows from Proposition 4.4 that by localizing at the class 1 of non-equivariant equivalences (see Remark 4.10 for justification of its existence), we obtain a symmetric monoidal Quillen equivalence

$$L_1\mathrm{Sp}_\mathcal{G} \simeq_Q L_1\mathrm{Mod}_{\mathrm{b}\mathbb{S}}(\mathrm{Sp}_\mathcal{G}).$$

We note that this is a symmetric monoidal Quillen equivalence using Remark 3.3, since the model structure on $\mathrm{Mod}_{\mathrm{b}\mathbb{S}}(\mathrm{Sp}_\mathcal{G})$ is right induced from the $\mathcal{G}$-global model structure on $\mathrm{Sp}_\mathcal{G}$. Finally using the language of [39] we can identify the homotopy category of $L_1\mathrm{Sp}_\mathcal{G}$ with the full subcategory of the global stable homotopy category consisting of those global spectra which are right induced from the trivial family.

Remark 4.10. It is not immediate that the left Bousfield localization of $\mathrm{Sp}_\mathcal{G}$ at the *class* of non-equivariant equivalence actually exists. This localization cannot be constructed as a homological localization since in global stable homotopy theory an analogue of the free G-space EG does not exist. Instead we apply Bousfield-Friedlander localization [15, 9.3] to the natural transformation $i_X\colon X \to \mathrm{b}X$ which is a non-equivariant equivalence. By construction, the global Borel functor b has the property that for all $G \in \mathcal{G}$, the underlying G-spectrum of $\mathrm{b}X$ is cofree, see [39, 4.5.16, 4.5.22]. In particular this shows that $f\colon X \to Y$ is a non-equivariant equivalence if and only if $\mathrm{b}f\colon \mathrm{b}X \to \mathrm{b}Y$ is a global equivalence. The conditions (A1) and (A2) from [15, 9.2] easily follow from this observation. The final condition (A3) follows from the right properness of $\mathrm{Sp}_\mathcal{F}$ for the trivial family $\mathcal{F} = \{1\}$, together with the fact that any $\mathcal{G}$-global fibration is a $\mathcal{F}$-global fibration. The argument for $\mathrm{Mod}_{\mathrm{b}\mathbb{S}}(\mathrm{Sp}_\mathcal{G})$ is similar as the weak equivalences and fibrations are created by the forgetful functor $\mathrm{Mod}_{\mathrm{b}\mathbb{S}}(\mathrm{Sp}_\mathcal{G}) \to \mathrm{Sp}_\mathcal{G}$.

Part 2. Rational cofree G-spectra

We give a symmetric monoidal algebraic model for the category of rational cofree G-spectra for G a compact Lie group, in the sense of [18]. We will initially prove the result for G connected and then show how to extend our proofs to any compact Lie group. In the final section we construct a strongly convergent Adams spectral sequence calculating homotopy classes of maps between cofree G-spectra.

5. Completions in algebra

We now recall some results about complete modules following [20].

Let R be a graded commutative ring and let I be a finitely generated homogeneous ideal. The I-adic completion of a module M is defined by

$$M_I^\wedge = \lim_n M/I^nM.$$

We say that a module M is *I-adically complete* if the natural map $M \to M_I^\wedge$ is an isomorphism. A dg-module is said to be *I-adically complete* if its underlying graded module is.

Since the I-adic completion functor is neither left nor right exact in general, our approach is to consider the zeroth left derived functor L_0^I of I-adic completion as the 'correct' notion.

Definition 5.1.

- We say that a module M is *L_0^I-complete* if the natural map $M \to L_0^IM$ is an isomorphism.

- We say that a dg-module N is *L_0^I-complete* if its underlying graded module is L_0^I-complete.

We write Mod_R for the category of dg-R-modules, and $\mathrm{Mod}_R^\wedge$ for the full subcategory of L_0^I-complete dg-modules. We denote the internal hom of R-modules by $\underline{\mathrm{Hom}}_R(-,-)$.

Lemma 5.2.

(a) *The category $\mathrm{Mod}_R^\wedge$ is abelian, and the inclusion functor $i\colon \mathrm{Mod}_R^\wedge \to \mathrm{Mod}_R$ is exact. In particular, the homology of an L_0^I-complete dg-module is L_0^I-complete.*
(b) *The inclusion functor is right adjoint to the L-completion functor L_0^I.*
(c) *The category $\mathrm{Mod}_R^\wedge$ has all limits and colimits.*

Proof. The proofs of (a) and (b) can be found in [30, A.6(e), A.6(f)]. Their proofs depend only upon the fact that L_0^I is right exact and the existence of a long exact sequence of derived functors. Therefore, the restriction to local rings and regular ideals made in [30] does not affect the stated results. It follows from (b) that limits of L_0^I-complete modules are calculated in Mod_R, and that colimits of L_0^I-complete modules are calculated by L_0^I-completion of the colimit in Mod_R. □

Proposition 5.3.

(a) *If N is L_0^I-complete, then $\underline{\mathrm{Hom}}_R(M,N)$ is L_0^I-complete.*
(b) *The category $\mathrm{Mod}_R^\wedge$ is closed symmetric monoidal with tensor product $L_0^I(M\otimes_R N)$ and internal hom $\underline{\mathrm{Hom}}_R(M,N)$.*

Proof. By taking a free presentation $R^{J_1}\to R^{J_0}\to M\to 0$, we obtain an exact sequence

$$0\to \underline{\mathrm{Hom}}_R(M,N)\to \prod_{J_0} N \to \prod_{J_1} N$$

which proves (a), since L_0^I-complete modules are closed under products and kernels.

For (b) we follow the argument of Rezk [37, 6.2]. We first prove that the map $L_0^I(M\otimes_R N)\to L_0^I(L_0^I M\otimes_R N)$ induced by $\eta_M\colon M\to L_0^I M$ is an isomorphism. It is enough to check that for any L_0^I-complete module C, the map

$$\underline{\mathrm{Hom}}_R(L_0^I(L_0^I M\otimes_R N),C)\to \underline{\mathrm{Hom}}_R(L_0^I(M\otimes_R N),C)$$

is an isomorphism. By adjunction, it is an isomorphism if and only if the induced map

$$\underline{\mathrm{Hom}}_R(L_0^I M,\underline{\mathrm{Hom}}_R(N,C))\to \underline{\mathrm{Hom}}_R(M,\underline{\mathrm{Hom}}_R(N,C))$$

is. This now follows as $\underline{\mathrm{Hom}}_R(N,C)$ is L_0^I-complete by part (a). Therefore $L_0^I(M\otimes_R N)\to L_0^I(L_0^I M\otimes_R N)$ is an isomorphism. By symmetry, we also have that $L_0^I(M\otimes_R N)\to L_0^I(M\otimes_R L_0^I N)$ is an isomorphism, and therefore so is $L_0^I(M\otimes_R N)\to L_0^I(L_0^I M\otimes_R L_0^I N)$. This completes the proof of (b). □

We will also be concerned with a homotopical version of completion that we shall now recall. For any $x\in R$, we define the *unstable Koszul complex*

$$K(x)=\mathrm{fib}(\Sigma^{|x|}R\xrightarrow{\cdot x} R),$$

and the *stable Koszul complex*

$$K_\infty(x)=\mathrm{fib}(R\to R[1/x])$$

where the fibre is taken in the category of dg-modules. For an ideal $I=(x_1,\ldots,x_n)$ we put

$$K(I)=K(x_1)\otimes_R\cdots\otimes_R K(x_n)\quad\text{and}\quad K_\infty(I)=K_\infty(x_1)\otimes_R\cdots\otimes_R K_\infty(x_n).$$

If no confusion is likely to arise, we suppress notation for the ideal and write K for the unstable Koszul complex and K_∞ for the stable Koszul complex. We will also write $\mathrm{Hom}_R(-,-)$ for the derived internal hom functor. We say that a dg-module M is *derived complete* if the natural map $M\to \mathrm{Hom}_R(K_\infty,M)=:\Lambda_I M$ is a quasi-isomorphism. Then the *nth local homology* of M is defined to be $H_n^I(M)=H_n(\Lambda_I M)$.

Definition 5.4. Let $I = (x_1, \ldots, x_n)$ be a finitely generated homogeneous ideal. For all $s \in \mathbb{N}$ and $x \in R$, we put

$$K_s(x) = \mathrm{fib}(\Sigma^{s|x|}R \xrightarrow{\cdot x^s} R) \quad \text{and} \quad K_s(I) = K_s(x_1) \otimes_R \cdots \otimes_R K_s(x_n).$$

We say that I is generated by the *weakly pro-regular* sequence $(x_1 \ldots, x_n)$ if the inverse system $(H_k(K_s(I)))_s$ is pro-zero for all $k \neq 0$. That is, for each $s \in \mathbb{N}$ there is $m \geq s$ such that the natural map

$$H_k(K_m(I)) \to H_k(K_s(I))$$

is zero.

Note that if R is Noetherian then any finitely generated ideal is weakly pro-regular [36, 4.34]. Indeed this is true even when weakly pro-regular is replaced by pro-regular [20].

Theorem 5.5. *Let R be a graded commutative ring and let I be a finitely generated homogeneous ideal that is generated by a weakly pro-regular sequence. Then for all dg-modules M, there is a natural quasi-isomorphism*

$$\mathrm{tel}^L_{I,M} \colon \Lambda_I(M) \xrightarrow{\sim} \mathbb{L}(-)^\wedge_I(M)$$

between the derived completion functor and the total left derived functor of I-adic completion, making the diagram

$$\begin{tikzcd} & M \arrow[dl] \arrow[dr] & \\ \Lambda_I M \arrow[rr, "\mathrm{tel}^L_{I,M}"'] & & \mathbb{L}(-)^\wedge_I(M) \end{tikzcd}$$

commute. Moreover, taking homology on both sides we get

$$H^I_*(M) \cong L^I_*(M).$$

Proof. Greenlees-May proved that if R has bounded torsion and I is pro-regular then $H^I_* M \cong L^I_* M$, see [20, 2.5]. Schenzel [38, 1.1] proved the above result for ideals generated by weakly pro-regular sequences and bounded complexes with R bounded torsion. Finally, Porta-Shaul-Yekutieli [36, 5.25] removed the hypothesis that R has bounded torsion and extended the result to unbounded complexes. □

As an application, we prove the following result which we will use in the construction of an Adams spectral sequence, see Theorem 10.8.

Proposition 5.6. *Let R be a graded commutative ring and let I be a finitely generated homogeneous ideal that is generated by a weakly pro-regular sequence.*

(a) *If M is an L^I_0-complete module and $P_\bullet \to M$ is a projective resolution of M, then $L^I_0 P_\bullet \to M$ is a projective resolution in L^I_0-complete modules.*
(b) *Write $\widehat{\mathrm{Ext}}_R$ for the Ext-groups in the abelian category of L^I_0-complete modules. Then*

$$\widehat{\mathrm{Ext}}_R(M, N) \cong \mathrm{Ext}_R(M, N)$$

for all L^I_0-complete modules M and N.

Proof. Given an L^I_0-complete module M, choose a projective resolution $P_\bullet \to M$ in R-modules. Since L^I_0 is left adjoint to the inclusion, $L^I_0 P_\bullet$ is a complex of projective L^I_0-complete modules. We now show that $L^I_0 P_\bullet \to M$ is a projective resolution in L^I_0-complete modules. Note that $\Lambda_I P_\bullet \to \Lambda_I M$ is a quasi-isomorphism. Using Theorem 5.5 and [20, 4.1] we have that $L^I_0 P_\bullet \simeq \Lambda_I P_\bullet$ and $M \simeq \Lambda_I M$. Therefore $L^I_0 P_\bullet \to M$ is a projective resolution in L^I_0-complete modules. By adjunction we deduce that for all L^I_0-complete modules M and N we have

$$\widehat{\mathrm{Ext}}_R(M, N) = H_*(\mathrm{Hom}_R(L^I_0 P_\bullet, N)) \cong H_*(\mathrm{Hom}_R(P_\bullet, N)) = \mathrm{Ext}_R(M, N).$$

□

6. An abelian model for derived completion

In this section we use the language of model categories to show that the category of L_0^I-complete modules forms an abelian model for derived complete modules, see Theorem 6.11. Our result can be thought as "dual" to the fact that I-power torsion modules forms an abelian model for derived torsion modules [24, §5]. We will be working under the following:

Hypothesis 6.1. We will assume our ideal I to be generated by a weakly pro-regular sequence and we continue to write K for its associated unstable Koszul complex.

In order to prove our main result we need to consider model structures on the categories of interest. Recall that the category of dg-modules Mod_R has a projective model structure in which the weak equivalences are the quasi-isomorphisms, the fibrations are the epimorphisms and the cofibrations are the monomorphisms which have dg-projective cokernel and are split on the underlying graded modules, see [13, 3.3] and [1, 3.15]. A dg-module M is said to be *dg-projective* if $\mathrm{Hom}_R(P, -)$ preserves surjective quasi-isomorphisms. It is important to note that any dg-projective module is (graded) projective, but the converse is not generally true. For example, the complex

$$X = \cdots \to \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \xrightarrow{2} \mathbb{Z}/4 \to \cdots$$

of free $\mathbb{Z}/4$-modules with each differential given by multiplication by 2 is not dg-projective, since the surjective quasi-isomorphism $X \to 0$ is not preserved by $\mathrm{Hom}_{\mathbb{Z}/4}(X, -)$. For more details on dg-projectivity see [3, §9].

Lemma 6.2. *If P is dg-projective, then there is a natural quasi-isomorphism $\Lambda_I P \xrightarrow{\sim} L_0^I P$.*

Proof. This is the trivial case of Theorem 5.5. □

Lemma 6.3. *Consider a dg-module M.*

(a) *If the map $M \to L_0^I M$ is a quasi-isomorphism, then $L_n^I(M) = 0$ for all $n > 0$.*
(b) *If $H_*(M)$ is L_0^I-complete, then M is derived complete.*

Proof. Part (a) follows from the fact that $L_n^I(M) = L_n^I(L_0^I M) = 0$ for $n > 0$ by [20, 4.1]. For part (b) we consider the spectral sequence [21, 3.3]

$$E^2_{p,q} = (L_p^I H_* M)_q \implies H_{p+q}(\Lambda_I M)$$

which collapses by part (a), hence showing that $M \to \Lambda_I M$ is a quasi-isomorphism. □

We will now put a projective model structure on L_0^I-complete modules following Rezk's unpublished note [37, 10.2].

Lemma 6.4.

(a) *The functor L_0^I takes cofibrations in Mod_R to morphisms which have the left lifting property with respect to surjective quasi-isomorphisms of L_0^I-complete modules.*
(b) *The functor L_0^I takes acyclic cofibrations in Mod_R to morphisms which have the left lifting property with respect to surjections of L_0^I-complete modules.*
(c) *If $M \to N$ is a cofibration in Mod_R, the homology $H_* N$ is L_0^I-complete and $M \to L_0^I M$ is a quasi-isomorphism, then $N \to L_0^I N$ is a quasi-isomorphism.*

Proof. Part (a) and (b) follow from the lifting properties in Mod_R. For part (c), note that by definition $M \to N$ is an injection with dg-projective cokernel P so we have a diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "\sim"'] & N \arrow[r] \arrow[d] & P \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r, dotted] & L_0^I M \arrow[r] & L_0^I N \arrow[r] & L_0^I P \arrow[r] & 0 \end{tikzcd}$$

in which the top row is exact and the bottom row is right exact. By Lemma 6.3, M and N are derived complete, and so it immediately follows that P is derived complete too. Therefore $P \to L_0^I P$ is a quasi-isomorphism by Lemma 6.2, and $L_n^I(P) = 0$ for $n > 0$ by Lemma 6.3. The long exact sequence of left derived functors shows that $L_0^I M \to L_0^I N$ is injective, and so by the five lemma we conclude that $N \to L_0^I N$ is a quasi-isomorphism. □

Proposition 6.5. *There is a model structure on* $\mathrm{Mod}_R^\wedge$ *in which the weak equivalences are the quasi-isomorphisms, the fibrations are the surjections, and the cofibrations are the maps with the left lifting property with respect to the acyclic fibrations. Furthermore, the adjunction*

$$L_0^I : \mathrm{Mod}_R \rightleftarrows \mathrm{Mod}_R^\wedge : i$$

is Quillen.

Proof. The only parts that need elaboration are the factorization axiom and the lifting axiom. Firstly we prove the factorization axiom.

Let $f\colon M \to N$ in $\mathrm{Mod}_R^\wedge$. Take a factorization $M \xrightarrow{i} D \xrightarrow{p} N$ in Mod_R where one of i or p is acyclic. Since L_0^I is left adjoint to the inclusion, maps $L_0^I D \to N$ are in bijection with maps $D \to N$. Therefore, there is a unique $q\colon L_0^I D \to N$ making the square

$$\begin{tikzcd} M \arrow[r] \arrow[d, "i"'] & L_0^I D \arrow[d, "q"] \\ D \arrow[ur, "\alpha"] \arrow[r, "p"'] & N \end{tikzcd}$$

commute. Note that q is a fibration since $q \cong L_0^I p$ and L_0^I preserves surjections.

If p is acyclic, Lemma 6.4(c) shows that α is a quasi-isomorphism since $H_*D \cong H_*N$, and hence by the two-of-three property, q is a weak equivalence. Lemma 6.4(a) shows that the factorization $f = q(\alpha i)$ is a factorization into a map with the left lifting property with respect to acyclic fibrations, followed by an acylic fibration. This completes the first part of the proof of the factorization axiom.

For the other part we suppose that i is a weak equivalence. Since $\alpha i \cong L_0^I(i)$, Lemma 6.4(b) shows that αi has the left lifting property with respect to fibrations in $\mathrm{Mod}_R^\wedge$. Lemma 6.4(c) shows that α is a quasi-isomorphism since $H_*D \cong H_*M$. Therefore $f = q(\alpha i)$ is a factorization into a weak equivalence with the left lifting property with respect to fibrations followed by an fibration, which completes the proof of the factorization axiom.

For the lifting axiom, we note that one part is by definition. For the other part, we use the standard method of the retract argument. Consider the square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "f"] \\ B \arrow[r] & Y \end{tikzcd}$$

in which i is an acyclic cofibration and f is a fibration. Factor i into a map with the left lifting property with respect to fibrations followed by a fibration to give $A \xrightarrow{j} C \xrightarrow{p} B$. Since j has the left lifting property with respect to fibrations, there is a lift $g\colon C \to X$.

As i and j are weak equivalences, p is an acyclic fibration. Since i has the left lifting property with respect to acyclic fibrations, there exists a lift $h\colon B \to C$. Therefore $gh\colon B \to X$ gives the required lift in the square.

It is clear that the adjunction is Quillen by the definition of the weak equivalences and fibrations. □

Remark 6.6. One might first think of attempting to prove the existence of this model structure via right inducing it from Mod_R. However, in order to be able to do this, we need to know that the inclusion $i\colon \mathrm{Mod}_R^\wedge \to \mathrm{Mod}_R$ preserves filtered colimits. This is false; take $R = \mathbb{Z}$ and $I = (p)$ and consider the direct system $\mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p \xrightarrow{p} \ldots$. Then the colimit in the category of abelian groups is $\mathbb{Q}_p$, while the colimit in the category of L_0^I-complete abelian groups is $L_0^{(p)}(\mathbb{Q}_p)$ which is zero.

Proposition 6.7. *The model structure on* $\mathrm{Mod}_R^\wedge$ *is symmetric monoidal.*

Proof. The category of L_0^I-complete modules is closed symmetric monoidal with tensor product given by $L_0^I(M \otimes N)$; see Proposition 5.3.

Let $M \to N$ and $X \to Y$ be fibrations in $\mathrm{Mod}_R^\wedge$. Since the inclusion $i\colon \mathrm{Mod}_R^\wedge \to \mathrm{Mod}_R$ preserves limits, we have that the pullback product map is

$$\underline{\mathrm{Hom}}_R(iN, iX) \to \underline{\mathrm{Hom}}_R(iM, iX) \times_{\underline{\mathrm{Hom}}_R(iM,iY)} \underline{\mathrm{Hom}}_R(iN, iY).$$

Since Mod_R is a symmetric monoidal model category and i is right Quillen, the pullback product map is a fibration. A similar proof shows that the pullback product of a fibration with an acyclic fibration is an acyclic fibration. The unit axiom is immediate since the unit in $\mathrm{Mod}_R^\wedge$ is $L_0^I R$ which is cofibrant as R is cofibrant in Mod_R. □

We need a model category modelling the homotopy theory of derived complete dg-modules. The left Bousfield localization of R-modules at the unstable Koszul complex is such a model category by the following result.

Lemma 6.8 ([21, 4.2])**.** *There is an equivalence of categories*

$$\mathrm{h}L_K\mathrm{Mod}_R \simeq \Lambda\mathrm{Mod}_R$$

where $\Lambda\mathrm{Mod}_R$ *denotes the full subcategory of the derived category of dg-modules consisting of derived complete dg-modules.*

We relate the model category of L_0^I-complete modules to derived complete modules. We will use these results to show that cofree G-spectra have an abelian model in terms of L_0^I-complete modules.

Lemma 6.9. *There is a symmetric monoidal Quillen adjunction*

$$L_0^I : L_K(\mathrm{Mod}_R) \rightleftarrows \mathrm{Mod}_R^\wedge : i.$$

Proof. The cofibrations in $L_K\mathrm{Mod}_R$ are the same as the cofibrations in Mod_R so they are preserved since $L_0^I\colon \mathrm{Mod}_R \to \mathrm{Mod}_R^\wedge$ is left Quillen. Now suppose that $f\colon M \to N$ is an acyclic cofibration in $L_K(\mathrm{Mod}_R)$ so that the cokernel C is dg-projective. In particular, $K \otimes C$ and $\mathrm{Hom}_R(K, C)$ are acyclic as K is self-dual up to suspension. We also know that K_∞ is built from K so $\Lambda_I C = \mathrm{Hom}_R(K_\infty, C)$ is acyclic as well. By Lemma 6.2, we have $\Lambda_I C \simeq L_0^I C$ and so $L_0^I M \to L_0^I N$ is a quasi-isomorphism. This is a symmetric monoidal Quillen adjunction since L_0^I is strong monoidal by Lemma 5.2, and the unit in $L_K(\mathrm{Mod}_R)$ is cofibrant. □

Before we can prove that the above Quillen adjunction is actually a Quillen equivalence, we need the following:

Lemma 6.10. *For any dg-module* M*, the natural map* $K \otimes M \to \Lambda_I(K \otimes M)$ *is a quasi-isomorphism.*

Proof. There is a fibre sequence $K_\infty \to R \to \check{C}R$ where $\check{C}R = \Sigma\ker(K_\infty \to R)$ is the Čech complex. This gives rise to another fibre sequence

$$\mathrm{Hom}_R(K_\infty, N) \leftarrow N \leftarrow \mathrm{Hom}_R(\check{C}R, N)$$

for any dg-module N. Now let $I = (x_1, \ldots, x_n)$. Note that $\check{C}R$ is finitely built from $R[\frac{1}{x_i}]$ and that the multiplication map $x_i\colon K \to K$ is null-homotopic. Thus $\mathrm{Hom}_R(\check{C}R, K \otimes M) \simeq 0$ and $K \otimes M$ is derived complete. □

We can now prove that L_0^I-complete modules are a model for derived complete modules.

Theorem 6.11. *There is a symmetric monoidal Quillen equivalence*

$$L_0^I : L_K(\mathrm{Mod}_R) \rightleftarrows \mathrm{Mod}_R^\wedge : i.$$

Proof. We now show that this Quillen adjunction is in fact a Quillen equivalence. Let P be cofibrant (i.e., dg-projective) in $L_K(\mathrm{Mod}_R)$ and M be fibrant in the category of L_0^I-complete R-modules. We must show that $L_0^I P \to M$ is a quasi-isomorphism if and only if $K \otimes P \to K \otimes M$ is a quasi-isomorphism.

Firstly, if $L_0^I P \to M$ is a quasi-isomorphism, then $K \otimes L_0^I P \to K \otimes M$ is a quasi-isomorphism since K is homotopically flat. Now note that there is a weak equivalence $K \otimes \Lambda_I P \xrightarrow{\sim} \Lambda_I(K \otimes P)$ since K is small. By Lemma 6.2, $K \otimes L_0^I P \simeq \Lambda_I(K \otimes P)$ as P is projective. Hence $K \otimes L_0^I P \simeq K \otimes P$ by Lemma 6.10. We conclude that $K \otimes P \to K \otimes M$ is a quasi-isomorphism as required.

Conversely, if $K \otimes P \to K \otimes M$ is a quasi-isomorphism then $\mathrm{Hom}_R(K, P) \to \mathrm{Hom}_R(K, M)$ is too since K is self-dual up to suspension. Since K_∞ is built from K, we also deduce $\mathrm{Hom}_R(K_\infty, P) \to \mathrm{Hom}_R(K_\infty, M)$ is a quasi-isomorphism. It follows that $\Lambda_I P \to \Lambda_I M$ is a quasi-isomorphism. By Lemma 6.2, we have $L_0^I P \simeq \Lambda_I P$ and $M \simeq \Lambda_I M$. Hence $L_0^I P \to M$ is a quasi-isomorphism. □

As a consequence we obtain the following corollary which extends [16, 6.15] to non-Noetherian rings.

Corollary 6.12. *A dg-module M is derived complete if and only if its homology H_*M is L_0^I-complete.*

Proof. Let M be derived complete. By Theorem 6.11, M is quasi-isomorphic to its L_0^I-completion $L_0^I M$. As the homology of an L_0^I-complete object is still L_0^I-complete by Lemma 5.2, we deduce that M has L_0^I-complete homology. The converse is Lemma 6.3(b). □

7. The category of rational cofree G-spectra

From now on we will be working rationally. This means that all spectra are rationalized without comment and all homology and cohomology theories will be unreduced and with rational coefficients.

Notation 7.1. Fix G a compact Lie group. We denote by Sp_G the model category of rational orthogonal G-spectra with the rational G-stable model structure, which is a compactly generated, stable, symmetric monoidal model category, see [33, III.7.6]. We write $\wedge$ for the tensor product and $F(-,-)$ for the internal hom functor. We also write hSp_G for its associated homotopy category.

Definition 7.2. A G-spectrum X is said to be *cofree* if the natural map $X \to F(EG_+, X)$ is an isomorphism in the homotopy category. We denote by $\mathrm{hSp}_G^{\mathrm{cofree}}$ the full subcategory of hSp_G of cofree G-spectra.

Lemma 7.3. *There is a natural equivalence*

$$\mathrm{h}L_{EG_+}\mathrm{Sp}_G \simeq \mathrm{hSp}_G^{\mathrm{cofree}}.$$

Furthermore, $L_{EG_+}\mathrm{Sp}_G$ is a symmetric monoidal model category.

Proof. A fibrant replacement functor in $L_{EG_+}\mathrm{Sp}_G$ is given by $F(EG_+, R(-))$ where R is the fibrant replacement in Sp_G. Therefore, the collection of bifibrant objects in $L_{EG_+}\mathrm{Sp}_G$ is equivalent to the full subcategory of cofree G-spectra. The model category $L_{EG_+}\mathrm{Sp}_G$ is symmetric monoidal by Proposition 3.13. □

8. The symmetric monoidal equivalence: connected case

In this section we fix a connected compact Lie group G. We aim to find an algebraic model for the category of rational cofree G-spectra. There are several steps needed. Recall that our model for cofree G-spectra is the homological localization $L_{EG_+}\mathrm{Sp}_G$.

Step 1. Consider the complex orientable commutative ring G-spectrum $DEG_+ = F(EG_+, \mathrm{inf} H\mathbb{Q})$, see Construction 4.7 for more details. Restriction and extension of scalars along the unit map $S^0 \to DEG_+$ induces a symmetric monoidal Quillen adjunction

$$DEG_+ \wedge - : L_{EG_+}(\mathrm{Sp}_G) \rightleftarrows L_{EG_+}(\mathrm{Mod}_{DEG_+}) : U$$

between the localizations, since $DEG_+ \wedge EG_+ \simeq EG_+$. By the Left Localization Principle this is a symmetric monoidal Quillen equivalence, since the unit is an EG_+-equivalence and U preserves non-equivariant equivalences.

Remark 8.1. This is a special case of Proposition 4.4 and Example 4.8.

Step 2. We can now take categorical fixed points to remove equivariance. As a functor from G-spectra to non-equivariant spectra, the categorical fixed points is right adjoint to the inflation functor. Using [41, §3.3] we have a symmetric monoidal Quillen adjunction

$$(-)^G : \mathrm{Mod}_{DEG_+} \leftrightarrows \mathrm{Mod}_{DBG_+} : DEG_+ \otimes_{DBG_+} -$$

between the categories of modules. Note that we suppress notation for the inflation functor. A more detailed discussion of this adjunction can be found in [26].

Since G is connected, DEG_+ generates Mod_{DEG_+} by [19, 3.1] and so the counit is an equivalence on all objects as it is an equivalence on DEG_+ and the fixed points functor preserves sums. In order to apply the Left Localization Principle, it remains to check that the fixed points functor $(-)^G$ sends non-equivariant equivalences between fibrant DEG_+-modules to BG_+-equivalences. This is equivalent to the derived functor of $(-)^G$ sending non-equivariant equivalences to BG_+-equivalences, which follows from [19, 3.3]. Therefore the Left Localization Principle applies and we get a symmetric monoidal Quillen equivalence

$$(-)^G : L_{EG_+}\mathrm{Mod}_{DEG_+} \leftrightarrows L_{BG_+}\mathrm{Mod}_{DBG_+} : DEG_+ \otimes_{DBG_+} -.$$

Step 3. We now apply Shipley's theorem [44, 2.15] (see also [46, 7.2]) which gives a symmetric monoidal Quillen equivalence

$$\Theta\colon \mathrm{Mod}_{DBG_+} \simeq_Q \mathrm{Mod}_{\Theta DBG_+}$$

where ΘDBG_+ is a commutative dga with the property that $H_*(\Theta DBG_+) = \pi_*(DBG_+) = H^*BG$. It follows that there is a symmetric monoidal Quillen equivalence

$$L_{BG_+}\mathrm{Mod}_{DBG_+} \simeq_Q L_{\Theta BG_+}\mathrm{Mod}_{\Theta DBG_+}$$

where $H_*(\Theta BG_+) \cong \pi_*(BG_+) \cong H_*BG$.

Step 4. Since H^*BG is a polynomial ring it is strongly intrinsically formal as a commutative dga. In other words, for any commutative dga R with $H_*R \cong H^*BG$, there is a quasi-isomorphism $H^*BG \to R$. Therefore, taking cycle representatives we have a quasi-isomorphism $z\colon H^*BG \to \Theta DBG_+$. We also need the following result to identify ΘBG_+.

Lemma 8.2. *There is a natural weak equivalence $\Theta BG_+ \to H_*BG$.*

Proof. Write $(-)^\vee = \mathrm{Hom}_{\mathbb{Q}}(-, \mathbb{Q})$ and note that it is exact. There is a canonical map $\Theta BG_+ \to (\Theta BG_+)^{\vee\vee}$ which is a quasi-isomorphism since the homotopy groups of BG_+ are degreewise finite. There is a natural map $\Theta DBG_+ \to (\Theta BG_+)^\vee$ obtained as the transpose of the natural composite

$$\Theta BG_+ \otimes \Theta DBG_+ \to \Theta(BG_+ \wedge DBG_+) \to \mathbb{Q}.$$

Since Θ gives a symmetric monoidal equivalence of homotopy categories, the natural map $\Theta DBG_+ \to (\Theta BG_+)^\vee$ is a weak equivalence.

Since DBG_+ is a commutative $H\mathbb{Q}$-algebra, ΘDBG_+ is a commutative dga by [44, 1.2]. As H^*BG is strongly intrinsically formal as a commutative dga, there exists a quasi-isomorphism $H^*BG \to \Theta DBG_+$. Putting all this together, we have quasi-isomorphisms

$$\Theta BG_+ \to (\Theta BG_+)^{\vee\vee} \to (H^*BG)^\vee \to H_*BG.$$

$\square$

Extension and restriction of scalars along the map $z\colon H^*BG \to \Theta DBG_+$

$$\mathrm{Mod}_{\Theta DBG_+} \underset{z^*}{\overset{\Theta DBG_+ \otimes_{H^*BG} -}{\leftrightarrows}} \mathrm{Mod}_{H^*BG}$$

is a symmetric monoidal Quillen equivalence since chain complexes satisfies Quillen invariance of modules. Therefore we have a symmetric monoidal Quillen equivalence

$$L_{H_*BG}\mathrm{Mod}_{\Theta DBG_+} \simeq_Q L_{H_*BG}\mathrm{Mod}_{H^*BG}.$$

Step 5. It remains to internalize the localization. Let I be the augmentation ideal of H^*BG and let K denote its unstable Koszul complex.

Proposition 8.3. *The homology H_*BG finitely builds K and K builds H_*BG.*

Proof. Suppose that $H^*BG = \mathbb{Q}[x_1, ..., x_n]$. There is a cofibre sequence

$$\Sigma^{|x_1|}\mathbb{Q}[x_1, ..., x_n] \xrightarrow{\cdot x_1} \mathbb{Q}[x_1, ..., x_n] \to \Sigma K(x_1)$$

and applying $\mathrm{Hom}_{\mathbb{Q}}(-, \mathbb{Q})$ gives the cofibre sequence

$$H_*BG \to \Sigma^{-|x_1|}H_*BG \to \Sigma K(x_1)^\vee.$$

Since $K(x_1)$ is self-dual up to suspension, this shows that $K(x_1)$ is finitely built from H_*BG. A repeated argument using the cofibre sequence $\Sigma^{|x_i|}K_{i-1} \to K_{i-1} \to K_i$ where $K_i = K(x_1, ..., x_i)$ and $K_0 = H^*BG$ shows that K is finitely built from H_*BG.

Conversely, since H_*BG is torsion it is built by K as K generates torsion modules [23, 8.7]. □

Therefore, a map is a H_*BG-equivalence if and only if it is a K-equivalence. It follows that

$$L_{H_*BG}\mathrm{Mod}_{H^*BG} = L_K\mathrm{Mod}_{H^*BG}.$$

Combining all the statements of this section with Theorem 6.11 gives the following result.

Theorem 8.4. *Let G be a connected compact Lie group and I be the augmentation ideal of H^*BG. Then there is a symmetric monoidal Quillen equivalence*

$$L_{EG_+}\mathrm{Sp}_G \simeq_Q \mathrm{Mod}^\wedge_{H^*BG}$$

between rational cofree G-spectra and L_0^I-complete dg-H^*BG-modules. In particular, there is a tensor-triangulated equivalence

$$\text{cofree } G\text{-spectra} \simeq_\triangle \mathcal{D}(L_0^I\text{-complete } H^*BG\text{-modules}).$$

9. The symmetric monoidal equivalence: non-connected case

In this section we extend the algebraic model for cofree G-spectra from connected G to any compact Lie group. The blueprint is the same as for the connected case, however some extra care is required which arises from taking categorical fixed points. We fix a compact Lie group G with identity component N and component group $W = G/N$, and write r for the rank of G.

9.1. **Skewed Model Categories.** We recall some results about model categories with a action of a finite group W from [31, §5.2] and [5, §7]. For any cofibrantly generated model category $\mathcal{C}$, we denote by $\mathcal{C}[W] = \mathrm{Fun}(BW, \mathcal{C})$ the category of objects of $\mathcal{C}$ with a W-action. We endow $\mathcal{C}[W]$ with the projective model structure where the weak equivalence and fibrations are created by the forgetful functor $\mathcal{C}[W] \to \mathcal{C}$. We will need the following result:

Lemma 9.1 ([31, 5.3])**.** *There is a symmetric monoidal Quillen equivalence $L_{EW_+}\mathrm{Sp}_W \simeq_Q \mathrm{Sp}[W]$.*

More generally, we can consider the category $\mathbb{E}W$ with objects the elements of W and a unique morphism connecting each pair of objects. Let $\mathcal{C}$ be a category with a W-action, that is, with functors $w_*\colon \mathcal{C} \to \mathcal{C}$ for each $w \in W$ satisfying $(ww')_* = w_*w'_*$ and $e_* = 1$. The category of objects of $\mathcal{C}$ with a *skewed W-action* is the category of equivariant functors $\mathbb{E}W \to \mathcal{C}$ and equivariant natural transformations, which we denote by $\mathcal{C}[\widetilde{W}]$. Note that if the W-action on $\mathcal{C}$ is trivial, then $\mathcal{C}[\widetilde{W}]$ is equivalent to $\mathcal{C}[W]$. We say that an adjunction between categories with a W-action is a *W-adjunction* if both the functors are W-equivariant and the unit and counit are W-equivariant natural transformations. We say that a model category $\mathcal{C}$ with a W-action is *skewable* if $w_* : \mathcal{C} \to \mathcal{C}$ is left Quillen for each $w \in W$. Note that $w_* : \mathcal{C} \to \mathcal{C}$ is left adjoint to w_*^{-1}, so equivalently, we could ask for w_* to be right Quillen for all $w \in W$.

Lemma 9.2.

(a) *Let $\mathcal{C}$ be a skewable, symmetric monoidal, cofibrantly generated model category with a W-action. Then $\mathcal{C}[\widetilde{W}]$ admits a closed symmetric monoidal structure and a projective model structure making it into a symmetric monoidal model category.*
(b) *Let $\mathcal{C}$ and $\mathcal{D}$ be skewable, symmetric monoidal model categories. Suppose that $\mathcal{C} \rightleftarrows \mathcal{D}$ is a W-adjunction which is a symmetric monoidal Quillen equivalence. Then we have a symmetric monoidal Quillen equivalence*

$$\mathcal{C}[\widetilde{W}] \simeq_Q \mathcal{D}[\widetilde{W}].$$

Proof. One can check that $\mathcal{C}[\widetilde{W}]$ is a symmetric monoidal model category in which the weak equivalences and fibrations are determined levelwise, and that Quillen equivalences extend to the skewed model category; see [5, §7.3] for the case $W = C_2$. $\square$

9.2. **The algebraic model.** The component group W acts on N by conjugation and hence on its cohomology H^*BN. We write $H^*\widetilde{BN}$ for the polynomial ring H^*BN equipped with this W-action. Accordingly, the model category $\mathrm{Mod}_{H^*\widetilde{BN}}$ inherits a W-action as follows. For $w \in W$ and a $H^*\widetilde{BN}$-module M, we define w_*M to be the same underlying abelian group as M but with module structure now defined by $r \cdot m := (wr)m$ for $r \in H^*\widetilde{BN}$ and $m \in M$. This model category is skewable since the action preserves weak equivalences and fibrations. Therefore, we can consider the model category $\mathrm{Mod}_{H^*\widetilde{BN}}[\widetilde{W}]$ of modules with a skewed W-action. More explicitly, we can identify this category with the category of modules over the skewed ring $H^*\widetilde{BN}[W]$, that is, the ring whose elements are formal linear sums $\sum_{w\in W} x_w w$ where $x_w \in H^*\widetilde{BN}$, with pointwise addition and multiplication defined by

$$(xw)\cdot(x'w') = (x(w\cdot x'))(ww') \quad \text{for } w, w' \in W \text{ and } \ x, x' \in H^*\widetilde{BN}.$$

We now turn to define a suitable notion of L_0^I-completion for a module over the skewed ring.

Definition 9.3. Let I denote the augmentation ideal of H^*BN. We say that a dg-$H^*\widetilde{BN}[W]$-module M is *L_0^I-complete* if M is L_0^I-complete as a H^*BN-module. We denote by $\mathrm{Mod}^{\wedge}_{H^*\widetilde{BN}[W]}$ the category of L_0^I-complete dg modules over the skewed ring.

Lemma 9.4.

(a) *The category of left $H^*\widetilde{BN}[W]$-modules admits a closed symmetric monoidal structure and a projective model structure making it into a symmetric monoidal model category.*
(b) *The category of L_0^I-complete left $H^*\widetilde{BN}[W]$-modules is abelian and is a symmetric monoidal model category with the projective model structure.*

Proof. The results follow from the previous sections and Lemma 9.2 by noticing that the category of (L_0^I-complete) $H^*\widetilde{BN}[W]$-modules is equivalent to $\mathcal{C}[\widetilde{W}]$ for $\mathcal{C}$ the category of (L_0^I-complete) $H^*\widetilde{BN}$-modules. $\square$

Lemma 9.5. *(Eilenberg-Moore) Consider the family $[\subseteq N] = \{H \le G \mid H \subseteq N\}$ and the Quillen adjunction*

$$(-)^N : \mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \leftrightarrows \mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W) : DEG_+ \otimes_{D\widetilde{BN}_+} -$$

where we set $D\widetilde{BN}_+ = (DEG_+)^N$. Then for all DEG_+-modules Y, the counit map

$$\epsilon_Y : DEG_+ \otimes_{D\widetilde{BN}_+} Y^N \to Y$$

is a $E[\subseteq N]_+$-equivalence.

Proof. A map of G-spectra is an $E[\subseteq N]_+$-equivalence if and only if its restriction to N-spectra is a weak equivalence. Therefore, it is sufficient to check that $DEN_+ \otimes_{DBN_+} Y^N \to Y$ is a weak equivalence. The full subcategory of DEN_+-module spectra Y for which ϵ_Y is a weak equivalence is localizing and clearly contains DEN_+. Since DEN_+ generates Mod_{DEN_+} by [19, 3.1] the claim follows. $\square$

We now ready to prove our main result.

Theorem 9.6. *Let G be a compact Lie group with identity component N and component group $W = G/N$. Let I be the augmentation ideal of H^*BN. Then there is a symmetric monoidal Quillen equivalence*

$$L_{EG_+}(\mathrm{Sp}_G) \simeq_Q \mathrm{Mod}^{\wedge}_{H^*\widetilde{BN}[W]}$$

between rational cofree G-spectra and L_0^I-complete dg-$H^*\widetilde{BN}[W]$-modules. In particular, there is a tensor-triangulated equivalence

$$\textit{cofree } G\textit{-spectra} \simeq_\triangle \mathcal{D}(L_0^I\textit{-complete } H^*\widetilde{BN}[W]\textit{-modules}).$$

Proof. We will prove the theorem using the Compactly Generated Localization Principle 3.16. To have a better control on the compact generators of the localized categories, it is convenient to change our model for cofree G-spectra. Thus we note that

$$L_{EG_+}\mathrm{Sp}_G = L_{G_+}\mathrm{Sp}_G$$

since the EG_+-equivalences are the same as the G_+-equivalences. Using Proposition 4.4 we have a symmetric monoidal Quillen equivalence $L_{G_+}(\mathrm{Sp}_G) \simeq_Q L_{G_+}(\mathrm{Mod}_{DEG_+})$.

Taking categorical G-fixed points loses too much information since Mod_{DEG_+} is no longer generated by DEG_+. Instead we slightly modify the model structure and then take N-fixed points. Consider the family $[\subseteq N] = \{H \leq G \mid H \subseteq N\}$. There is a symmetric monoidal Quillen equivalence

$$L_{G_+}\mathrm{Mod}_{DEG_+} \rightleftarrows L_{G_+}L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}$$

since $G_+ \wedge E[\subseteq N]_+ \to G_+$ is a weak equivalence.

We now take categorical N-fixed points to remove equivariance. We use the tilde in $D\widetilde{BN}_+ = (DEG_+)^N$ to emphasize that it may have a non-trivial W-action. We apply the Compactly Generated Localization Principle to the symmetric monoidal Quillen adjunction

$$(-)^N : L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \leftrightarrows \mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W) : DEG_+ \otimes_{D\widetilde{BN}_+} -$$

to obtain a symmetric monoidal Quillen equivalence after localization. There are several conditions that need to be checked. Firstly, we claim that $L_{G_+}L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}$ is compactly generated by $DG_+ \simeq DG_+ \wedge DEG_+$. It is clear that it generates so we only show that it is compact. By definition of sum in the localized category, we have to show that

$$\mathrm{hMod}_{DEG_+}(DG_+, F(EG_+, \bigvee_i Y_i)) \simeq \bigoplus_i \mathrm{hMod}_{DEG_+}(DG_+, Y_i) \tag{1}$$

where Y_i is cofree for all i. This is now clear since DG_+ is small and $DG_+ \wedge EG_+ \simeq DG_+$. We also claim that $(DG_+)^N \simeq W_+$ compactly generates $L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W)$. Since W_+ has a trivial $D\widetilde{BN}_+$-action, it builds $D\widetilde{BN}_+ \wedge W_+$ in $\mathrm{Mod}_{D\widetilde{BN}_+}$ and hence it generates $L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W)$. It is also compact by a similar argument to (1). By the Compactly Generated Localization Principle it remains to check that the derived counit is a G_+-equivalence on DG_+, and that the derived counit is an $E[\subseteq N]_+$-equivalence for G_+. These are true by the Eilenberg-Moore Lemma. Hence we have a symmetric monoidal Quillen equivalence

$$L_{G_+}L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+} \simeq_Q L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W).$$

Note that have an equality of model categories

$$L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W) = L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(L_{EW_+}\mathrm{Sp}_W)$$

since $EW_+ \wedge W_+ \simeq W_+$.

We can rewrite the target category as $L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}[W])$ and apply Shipley's theorem [44] to obtain symmetric monoidal Quillen equivalences

$$L_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}[W]) \simeq_Q L_{\Theta(W_+)}\mathrm{Mod}_{\Theta D\widetilde{BN}_+}(\mathrm{Mod}_{\mathbb{Q}[W]}) \simeq_Q L_{\Theta(W_+)}\mathrm{Mod}_{\Theta D\widetilde{BN}_+[W]}.$$

One can construct a $\mathbb{Q}[W]$-module map $H^*\widetilde{BN} \to \Theta D\widetilde{BN}_+$ which is a quasi-isomorphism as in [25, §7]. Since the map is compatible with the W-action, there is a symmetric monoidal Quillen equivalence

$$\mathrm{Mod}_{\Theta D\widetilde{BN}_+[W]} \simeq_Q \mathrm{Mod}_{H^*\widetilde{BN}[W]}.$$

Note that $H_*(\Theta(W_+)) = H_0(\Theta(W_+)) = \mathbb{Q}[W]$ and hence $\Theta(W_+)$ is formal as a $H^*\widetilde{BN}[W]$-module since we have a zig-zag of quasi-isomorphisms $\Theta(W_+) \leftarrow \tau_{\geq 0}(\Theta(W_+)) \to H_0(\Theta(W_+))$ where $\tau_{\geq 0}$ denotes the connective cover functor. Putting all this together, we deduce a zig-zag of symmetric monoidal Quillen equivalences

$$L_{EG_+}(\mathrm{Sp}_G) \simeq_Q L_{\mathbb{Q}[W]}\mathrm{Mod}_{H^*\widetilde{BN}[W]}.$$

We now claim that $L_{\mathbb{Q}[W]}\mathrm{Mod}_{H^*\widetilde{BN}[W]} = (L_{\mathbb{Q}}\mathrm{Mod}_{H^*BN})[\widetilde{W}]$. As the underlying categories are equal and the acyclic fibrations are easily seen to be the same, we only need to argue that the model categories have the same weak equivalences. This is clear since

$$\mathbb{Q}[W] \otimes_{H^*\widetilde{BN}[W]} M \cong \mathbb{Q} \otimes_{H^*BN} M$$

for all $H^*\widetilde{BN}[W]$-modules M. Hence the two model categories are equal.

Finally, using Lemma 9.2 and Theorem 6.11, we conclude that there are symmetric monoidal Quillen equivalences

$$(L_{\mathbb{Q}}\mathrm{Mod}_{H^*BN})[\widetilde{W}] \simeq_Q \mathrm{Mod}^{\wedge}_{H^*BN}[\widetilde{W}] \simeq_Q \mathrm{Mod}^{\wedge}_{H^*\widetilde{BN}[W]}.$$

□

Remark 9.7. Our proof bridges a gap in [25]. In the cited paper it is stated that there is a Quillen equivalence

$$(-)^N : \mathrm{Cell}_{G_+}\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \leftrightarrows \mathrm{Cell}_{W_+}\mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W) : DEG_+ \otimes_{D\widetilde{BN}_+} -$$

obtained by the Cellularization Principle. The claim as it is stated it is not correct. Indeed, if we want to apply the Cellularization Principle we need to check that the counit $DEG_+ \otimes_{D\widetilde{BN}_+} (G_+)^N \to G_+$ is a weak equivalence of G-spectra, which in general is false. Nonetheless, we can modify the argument as follows. Firstly there is a Quillen equivalence

$$\mathrm{Cell}_{G_+}\mathrm{Mod}_{DEG_+} \rightleftarrows \mathrm{Cell}_{G_+}L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}.$$

We note that the localization $\mathrm{Cell}_{G_+}L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}$ exists, since left Bousfield localizations of right proper, stable model categories are right proper by [10, 4.7]. We can then apply the Cellularization Principle to the Quillen adjunction

$$(-)^N : L_{E[\subseteq N]_+}\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \leftrightarrows \mathrm{Mod}_{D\widetilde{BN}_+}(\mathrm{Sp}_W) : DEG_+ \otimes_{D\widetilde{BN}_+} -$$

and the Eilenberg-Moore Lemma to show that this is a Quillen equivalence after cellularization.

10. Adams spectral sequence

In this section, we construct an Adams spectral sequence for cofree G-spectra. It provides a tool for calculating the space of maps between two cofree G-spectra in terms of L_0^I-complete modules, and furthermore gives intuition for the Quillen equivalence given in the previous section.

We describe the construction of an Adams spectral sequence based on projective resolutions as in [2].

Let $\mathcal{T}$ be a triangulated category and let $\mathcal{A}$ be a $\mathbb{Z}$-graded abelian category with enough projectives. Note that $\mathcal{T}(X,Y)$ is a $\mathbb{Z}$-graded abelian group via $\mathcal{T}(X,Y)_n = \mathcal{T}(\Sigma^n X, Y)$. Assume that we are given a $\mathbb{Z}$-graded exact functor $\pi_*^{\mathcal{A}} \colon \mathcal{T} \to \mathcal{A}$. We aim to construct a conditionally convergent Adams-type spectral sequence

$$E_2^{s,t} = \mathrm{Ext}_{\mathcal{A}}^{s,t}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \Rightarrow \mathcal{T}(X,Y)_{t-s}$$

for all $X, Y \in \mathcal{T}$. We list the steps needed.

Step 0: Choose a projective resolution of $\pi_*^{\mathcal{A}}(X)$ in $\mathcal{A}$

$$0 \leftarrow \pi_*^{\mathcal{A}}(X) \leftarrow P_0 \leftarrow P_1 \leftarrow P_2 \leftarrow \dots .$$

Step 1: Realize the projectives, i.e., find $\mathbb{P}_j \in \mathcal{T}$ so that $\pi_*^{\mathcal{A}}(\mathbb{P}_j) = P_j$.

Step 2: Let $X \in \mathcal{T}$ and $\mathbb{P}_j$ as above. Show that the functor $\pi_*^{\mathcal{A}}$ induces an isomorphism

$$\mathcal{T}(\mathbb{P}_j, X) \xrightarrow{\cong} \mathrm{Hom}_{\mathcal{A}}(P_j, \pi_*^{\mathcal{A}}(X)).$$

Step 3: Using Step 0 and Step 1, we can formally produce a tower

$$\begin{tikzcd}
\mathbb{P}_0 \arrow[d] & \Sigma^1\mathbb{P}_1 \arrow[d] & \Sigma^2\mathbb{P}_2 \arrow[d] & \Sigma^3\mathbb{P}_3 \arrow[d] & \\
X = X_0 \arrow[r] & X_1 \arrow[r] & X_2 \arrow[r] & X_3 \arrow[r] & \cdots
\end{tikzcd}$$

Step 4: Apply $\mathcal{T}(-, Y)$ to get a spectral sequence with E_1-page:

$$E_1^{s,*} = \mathcal{T}(\mathbb{P}_s, Y) = \mathrm{Hom}_{\mathcal{A}}(P_s, \pi_*^{\mathcal{A}}(Y)).$$

By construction, we will have a conditionally convergent spectral sequence

$$E_2^{*,*} = \mathrm{Ext}_{\mathcal{A}}^{*,*}(\pi_*^{\mathcal{A}}(X), \pi_*^{\mathcal{A}}(Y)) \Rightarrow \mathcal{T}(\overline{X}, Y)_*$$

where $\overline{X}$ is the fibre of the canonical map $X \to \mathrm{hocolim}_s X_s$.

Step 5: Show that $\mathrm{hocolim}_s X_s = 0$.

We apply the recipe above in the following setting. Fix a compact Lie group G with identity component N and component group W, and fix I to be the augmentation ideal of H^*BN. We consider the abelian category $\mathrm{Mod}^{\wedge}_{H^*\widetilde{BN}[W]}$ of graded L_0^I-complete modules over the skewed group ring $H^*\widetilde{BN}[W]$, and the homotopy category of rational cofree G-spectra. Before we give the exact functor, we need a preliminary result.

Lemma 10.1. *Any cofree G-spectrum is (weakly equivalent to) a DEG_+-module.*

Proof. Recall that we write DEG_+ to mean the mapping spectrum $F(EG_+, \mathrm{inf}H\mathbb{Q})$. Throughout this proof we will simply write $\mathbb{Q}$ for the inflated Eilenberg-MacLane spectrum.

Let X be a cofree G-spectrum and without loss of generality suppose that X is bifibrant. Firstly, $EG_+ \wedge \mathbb{Q}$ is a DEG_+-module with action map defined by

$$F(EG_+, \mathbb{Q}) \wedge \mathbb{Q} \wedge EG_+ \xrightarrow{1\wedge 1\wedge \triangle} F(EG_+, \mathbb{Q}) \wedge \mathbb{Q} \wedge EG_+ \wedge EG_+ \xrightarrow{\mathrm{ev}} \mathbb{Q} \wedge \mathbb{Q} \wedge EG_+ \xrightarrow{\mu} \mathbb{Q} \wedge EG_+.$$

We have a composite

$$X \xrightarrow{\sim} F(EG_+, X) \xrightarrow{\sim} F(EG_+, \mathbb{Q} \wedge X) = \mathrm{Hom}_{\mathbb{Q}}(EG_+ \wedge \mathbb{Q}, \mathbb{Q} \wedge X)$$

which is a weak equivalence as X is cofree and $S^0 \to \mathbb{Q}$ is a (rational) weak equivalence. The DEG_+-module structure on $EG_+ \wedge \mathbb{Q}$ passes to give a DEG_+-module structure on $\mathrm{Hom}_{\mathbb{Q}}(EG_+ \wedge \mathbb{Q}, \mathbb{Q} \wedge X)$ which completes the proof. $\square$

Remark 10.2. We recall a spectral sequence relating local homology to equivariant homotopy groups, see [22]. Let R be a ring G-spectrum and M an R-module. For $J = (x_1, ..., x_r)$ a finitely generated ideal in $\pi_*^G R$ define

$$M_J^{\wedge} = F(K(J), M)$$

where $K(J) = \mathrm{fib}(R \to R[1/x_1]) \otimes_R \cdots \otimes_R \mathrm{fib}(R \to R[1/x_r])$ is the Koszul spectrum. Then there is a convergent spectral sequence

$$E_2^{*,*} = L_*^J(\pi_*^G R; \pi_*^G M) \implies \pi_*^G(M_J^{\wedge}).$$

In the special case that R has Thom isomorphisms and J is the augmentation ideal of $\pi_*^G R$, there is an equivalence $M_J^{\wedge} \xrightarrow{\sim} F(EG_+, M)$ by [22, 2.5].

Lemma 10.3. *Let X be a cofree G-spectrum. Then $\pi_*^N X$ is L_0^I-complete.*

Proof. For any DEN_+-module M, there is a convergent spectral sequence

$$E_2^{*,*} = L_*^I(H^*BN; \pi_*^N M) \implies \pi_*^N(F(EG_+, M))$$

by Remark 10.2. If in addition M is cofree, we also have that $M_I^\wedge \simeq M$.

Now let X be a cofree G-spectrum. The restriction of X to an N-spectrum is still cofree, and so by Lemma 10.1, the discussion above tells us that we have a convergent spectral sequence

$$E_2^{*,*} = L_*^I(H^*BN; \pi_*^N X) \implies \pi_*^N X.$$

Since the E_2-page of the spectral sequence consists of L_0^I-complete modules by [20, 4.1], and the kernel and cokernel of a map of L_0^I-complete modules is L_0^I-complete, we have that $\pi_*^N X$ is L_0^I-complete.

Finally, note that W acts on $\pi_*^N(X)$ by conjugation, making it naturally a module over $H^*\widetilde{BN}[W]$. □

Therefore we may use the exact functor

$$\pi_*^N : \mathrm{hSp}_G^{\mathrm{cofree}} \to \mathrm{Mod}^\wedge_{H^*\widetilde{BN}[W]}$$

for the construction of the Adams spectral sequence.

Lemma 10.4 (Step 1)**.** *The abelian category* $\mathrm{Mod}^\wedge_{H^*\widetilde{BN}[W]}$ *has enough projectives. Moreover, the projectives are realized, that is*

$$\pi_*^N(F(EG_+, \bigvee \Sigma^{n_i} S^0) \wedge W_+) \cong L_0^I(\bigoplus \Sigma^{n_i} H^*\widetilde{BN})[W].$$

Proof. Using that L_0^I is right exact and left adjoint to the inclusion, we see that $L_0^I(\bigoplus \Sigma^{n_i} H^*\widetilde{BN})[W]$ is projective in $\mathrm{Mod}^\wedge_{H^*\widetilde{BN}[W]}$ and that there are enough projectives. It is left to show that the projectives are realized. Note that

$$\pi_*^N(F(EG_+, \bigvee \Sigma^{n_i} S^0) \wedge W_+) \cong \pi_*^N(F(EG_+, \bigvee \Sigma^{n_i} S^0))[W]$$

so it is enough to show that

$$\pi_*^N(F(EN_+, \bigvee \Sigma^{n_i} S^0) \cong L_0^I(\oplus \Sigma^{n_i} H^*BN).$$

By isotropy separation,

$$F(EN_+, \bigvee \Sigma^{n_i} S^0) \xrightarrow{\sim} F(EN_+, \bigvee \Sigma^{n_i} DEN_+).$$

There is a spectral sequence

$$E_2^{*,*} = L_*^I(\pi_*^N M) \implies [EN_+, M]_*^N = \pi_*^N(F(EN_+, M))$$

and when $M = \bigvee \Sigma^{n_i} DEN_+$ the E_2-page has the form

$$L_*^I \pi_*^N(\bigvee \Sigma^{n_i} DEN_+) \cong L_*^I(\oplus \Sigma^{n_i} H^*BN) = L_0^I(\oplus \Sigma^{n_i} H^*BN)$$

by Proposition 6.2. Since the E_2-page is concentrated in one line, the spectral sequence collapses so that

$$L_0^I(\oplus \Sigma^{n_i} H^*BN) \cong \pi_*^N(F(EN_+, \bigvee \Sigma^{n_i} DEN_+)).$$

□

Before we can prove Step 2 we need to recall some terminology and give a preliminary result. Let T be a triangulated category and let X be an object of T; for us, T will be the (homotopy) category of rational cofree N-spectra. A full replete subcategory of T which closed under retracts is said to be thick. If in addition it is closed under arbitrary coproducts (resp. products) it is said to be localizing (resp. colocalizing). The thick (resp. localizing, resp. colocalizing) subcategory of T generated by X is the smallest thick (resp. localizing, resp. colocalizing) subcategory of T which contains X. We then say that an object Y of T is:

- *finitely built* from X if Y is in the thick subcategory generated by X;
- *built* from X if Y is in the localizing subcategory generated by X;
- *cobuilt* from X if Y is in the colocalizing subcategory generated by X;

Lemma 10.5. *Let N be a connected compact Lie group.*

(a) *Rationally, EN_+ finitely builds N_+.*
(b) *The rational spectrum $\bigoplus_i \Sigma^{n_i} N_+$ is a retract of $\prod_i \Sigma^{n_i} N_+$.*
(c) *The category of rational cofree N-spectra coincides with the colocalizing subcategory generated by DEN_+. The same holds for the rational spectrum DN_+.*

Proof. Since N is connected, we know that $H^*(BN;\mathbb{Q}) = \mathbb{Q}[x_1, \ldots, x_r]$ with generators in even degrees. Note that $[EN_+, EN_+]^G_* = H^*(BN;\mathbb{Q})$ so we can view each generator x_i as an endomorphism of EN_+. By the Wirthmüller isomorphism, we have that $\pi^N_*(N_+) = \Sigma^d\mathbb{Q}$ where d is the dimension of N. This suggests that we can build N_+ from EN_+ via a Kozsul resolution. We put

$$EN_+//x_1 = \mathrm{fib}(\Sigma^{-|x_1|}EN_+ \xrightarrow{x_1} EN_+)$$

and

$$EN_+//(x_1, \ldots x_j) = \mathrm{fib}(\Sigma^{-|x_j|}EN_+//(x_1, \ldots, x_{j-1}) \xrightarrow{x_j} EN_+//(x_1, \ldots, x_{j-1}))$$

for $1 < j \leq r$. By construction, EN_+ finitely builds $EN_+//(x_1, \ldots, x_r)$. The latter spectrum can be identified with the bottom cell of EN_+ which is $\Sigma^{-d}N_+$. This concludes the proof of (a).

For part (b), we first note that in the category of $\mathbb{Q}$-vector spaces the direct sum is a retract of the product. It follows that the rational spectrum $\bigoplus S^{n_i}$ is a retract of $\prod S^{n_i}$. The claim then follows by applying the functor $N_+ \wedge -$ to this splitting together with the identification

$$N_+ \wedge (\prod S^{n_i}) \simeq \Sigma^d F(N_+, \prod S^{n_i}) \simeq \prod \Sigma^{n_i} N_+$$

using that $N_+ \simeq \Sigma^d DN_+$ where d is the dimension of N by the Wirthmülller isomorphism.

We now prove part (c). Note that N_+ builds EN_+ as it is free. It follows that DN_+ cobuilds DEN_+, and so to prove part (c) it is sufficient to show that DEN_+ cobuilds any rational cofree N-spectrum Y. We have seen in Lemma 10.4 that any free resolution of $\pi^N_*(Y)$ can be realized in rational cofree N-spectra by $\widehat{\mathbb{F}}_i = F(EN_+, \bigvee \Sigma^{n_i} DEN_+)$. Since the homological dimension of L^I_0-complete H^*BN-modules is $r = \mathrm{rank}(N)$, it follows that Y is finitely built from the collection of $\widehat{\mathbb{F}}_i$. Therefore it suffices to show that each $\widehat{\mathbb{F}}_i$ is cobuilt from DEN_+.

We write $\mathbb{F} = \bigvee \Sigma^{n_i} DEN_+$ and $\widehat{\mathbb{F}} = F(EN_+, \bigvee \Sigma^{n_i} DEN_+)$ for its cofree-ification. Since EN_+ finitely builds N_+, applying the dual functor shows that DEN_+ finitely builds N_+ by the Wirthmüller isomorphism. Note that $DEN_+ \wedge N_+ \simeq N_+$, and therefore DEN_+ cobuilds $\prod(\Sigma^{n_i} DEN_+ \wedge N_+)$. Hence DEN_+ cobuilds $\mathbb{F} \wedge N_+ = \bigvee \Sigma^{n_i} DEN_+ \wedge N_+$ since it is a retract of the product.

As $EN_+^{(n)}$ is a finite free N-CW-complex, $\mathbb{F} \wedge N_+$ finitely builds $\mathbb{F} \wedge DEN_+^{(n)}$. Therefore DEN_+ cobuilds $\mathbb{F} \wedge DEN_+^{(n)}$. Finally, we note that $\widehat{\mathbb{F}} = \mathrm{holim}(\mathbb{F} \wedge DEN_+^{(n)})$ since $EN_+^{(n)}$ is finite. This completes the proof of (c). $\square$

We also need to realize the maps.

Lemma 10.6 (Step 2)**.** *Taking homotopy groups gives an isomorphism*

$$\pi^N_* \colon [F(EG_+, \bigvee \Sigma^{n_i} S^0) \wedge W_+, Y]^G_* \xrightarrow{\cong} \mathrm{Hom}_{H^*\widetilde{BN}[W]}(L^I_0(\oplus \Sigma^{n_i} H^*\widetilde{BN})[W], \pi^N_*(Y)).$$

Proof. We apply the change of groups adjunctions on both sides to reduce to showing that

$$\pi^N_* \colon [F(EN_+, \bigvee \Sigma^{n_i} S^0), Y]^N_* \xrightarrow{\cong} \mathrm{Hom}_{H^*BN}(\oplus \Sigma^{n_i} H^*BN, \pi^N_*(Y)).$$

Since there is a weak equivalence $EN_+ \wedge X \xrightarrow{\sim} EN_+ \wedge F(EN_+, X)$ for any X, we see that

$$[F(EN_+, \bigvee \Sigma^{n_i} S^0), F(EN_+, Y)]^N \cong [\bigvee F(EN_+, \Sigma^{n_i} S^0), F(EN_+, Y)]^N.$$

Accordingly, it is enough to show that

$$\pi^N_* \colon [DEN_+, Y]^N_* \xrightarrow{\cong} \mathrm{Hom}_{H^*BN}(H^*BN, \pi^N_*(Y))$$

for all Y cofree N-spectra. We note that the collection of Y for which the above map is an isomorphism is colocalizing. As every rational cofree N-spectrum is cobuilt from DN_+ by Lemma 10.5 it is enough to prove the claim for $Y = DN_+$. It is easy to see that both sides evaluated at DN_+ give $\mathbb{Q}$. Thus we only have to argue that the natural transformation π_*^N is non-zero. Observe that a nontrivial N-equivariant map $f\colon DEN_+ \to DN_+$ corresponds to a nontrivial N-equivariant map $\widetilde{f}\colon N_+ \to EN_+$ which gives a map $\widetilde{f}/N_+\colon S^0 \to BN_+$ which is nontrivial in reduced H_0. It remains to note that for a free N-spectrum we have $\pi_*^N(X) = H_*(X/N)$ up to an integer shift. $\square$

Since π_*^N maps homotopy direct limits to direct limits, it is left to show the following:

Lemma 10.7 (Step 5)**.** *Let X be a cofree G-spectrum with $\pi_*^N(X) = 0$. Then $X \simeq 0$.*

Proof. We first prove the claim for the connected case and then we show how to extend it to all compact Lie groups. Note that there is an equivalence $EN_+ \simeq EN_+ \wedge DEN_+$ so that $EN_+ \in \mathrm{Mod}_{DEN_+}$. We claim that $N_+ \in \mathrm{Loc}(EN_+)$; that is N_+ is in the localizing subcategory generated by EN_+. Note that DN_+ is cofree and hence a DEN_+-module. Since DEN_+ generates the category Mod_{DEN_+} by [19, 3.1], we get that $DN_+ \in \mathrm{Loc}_{\mathrm{Mod}_{DEN_+}}(DEN_+)$. Since the forgetful functor $\mathrm{Mod}_{DEN_+} \to \mathrm{Sp}_N$ and $EN_+ \wedge -\colon \mathrm{Sp}_N \to \mathrm{Sp}_N$ preserve colimits we get $EN_+ \wedge DN_+ \in \mathrm{Loc}(EN_+ \wedge DEN_+)$. By the Wirthmüller Isomorphism, we see that $DN_+ \simeq \Sigma^{-d} N_+$ where d is the dimension of N. Putting all this together, $N_+ \in \mathrm{Loc}(EN_+)$ as required. Let us now prove that for a cofree N-spectrum X with $\pi_*^N(X) = 0$, then $\pi_*(X) = 0$ and hence $X \simeq 0$. By hypothesis, we have
$$0 = \pi_*^N(X) = [EN_+, X]^N.$$
By a localizing subcategory argument we get $\pi_*(X) = [N_+, X]^N = 0$ as required. Finally, let G be any compact Lie group and let X be a cofree G-spectrum with $\pi_*^N(X) = 0$. By the previous paragraph, we know that X is N-equivariantly contractible, that is $F(W_+, X) \simeq 0$ and hence $F(EW_+, X) \simeq 0$. Therefore
$$X \simeq F(\widetilde{E}W, X) \simeq F(\widetilde{E}W \wedge EG_+, X) \simeq 0$$
since X is cofree. $\square$

Finally, we have our Adams spectral sequence:

Theorem 10.8. *For X and Y cofree G-spectra, there is a strongly convergent Adams spectral sequence*
$$E_2^{*,*} = \mathrm{Ext}^{*,*}_{H^*\widetilde{BN}[W]}(\pi_*^N X, \pi_*^N Y) \implies [X, Y]_*^G.$$

Proof. Combining the results of this section with Proposition 5.6, we have constructed a conditionally convergent spectral sequence as above. Note that $H^*\widetilde{BN}[W]$ has global homological dimension smaller or equal to $r = \mathrm{rank}(N)$ since the projectives are induced from the category of H^*BN-modules, see Lemma 10.4. It follows that the spectral sequence is concentrated in rows 0 to r, and hence is strongly convergent. $\square$

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CHAPTER 3

Flatness and Shipley's algebraicization theorem

FLATNESS AND SHIPLEY'S ALGEBRAICIZATION THEOREM

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Abstract. We provide an enhancement of Shipley's algebraicization theorem which behaves better in the context of commutative algebras. This involves defining flat model structures as in Shipley and Pavlov-Scholbach, and showing that the functors still provide Quillen equivalences in this refined context. The use of flat model structures allows one to identify the algebraic counterparts of change of groups functors, as demonstrated in forthcoming work of the author.

Contents

1. Introduction

Many concepts and constructions in algebra can be understood in a homotopy invariant sense, and the derived category of a ring is the universal category in which to study these. In turn, these homotopy invariant algebraic notions can be translated into stable homotopy theory [13] and this translation to *spectral algebra* has led to a powerful new point of view on many areas such as modular representation theory [12, 14]. Robinson [31] showed that the category of spectra contains 'extraordinary' derived categories generalizing the derived category of a ring. Shipley [39] gave a more precise and general version of Robinson's result in terms of a zig-zag of Quillen equivalences. This paper is a contribution to the understanding of the relationship between spectral and homological algebra.

Passing between the worlds of spectral algebra and homological algebra is a valuable technique. It allows the reduction of topological questions to algebraic questions, and conversely, allows the importation of algebraic methods to the realm of spectra. Associated to any ring R there is an Eilenberg-MacLane spectrum HR, and the homological algebra of R is equivalent to the spectral algebra of HR. This relation is particularly striking in the case that $R = \mathbb{Q}$, as the rational sphere spectrum is equivalent to $H\mathbb{Q}$.

Let R be a commutative ring. It was shown by Shipley [39] that there is a zig-zag of Quillen equivalences between HR-module spectra and chain complexes of R-modules. Moreover, this is a zig-zag of symmetric monoidal Quillen equivalences, so that it gives a zig-zag of Quillen equivalences between HR-algebra spectra and differential graded R-algebras. Shipley's algebraicization theorem shows that spectral algebra is a vast generalization of homological algebra.

Moreover, it provides a bridge between the worlds of topology and algebra. This bridge has been widely used in the construction of algebraic models for rational equivariant cohomology theories by Barnes, Greenlees, Kędziorek and Shipley, see [1, 2, 5, 15, 16, 17, 23, 29, 36].

By Shipley's algebraicization theorem, an HR-algebra X corresponds to a differential graded R-algebra ΘX and there is a Quillen equivalence $\mathrm{Mod}_X \simeq_Q \mathrm{Mod}_{\Theta X}$. However, if X is in addition a commutative HR-algebra, it does not correspond to a commutative differential graded R-algebra, but rather to a differential graded E_∞-R-algebra, see [30].

When $R = \mathbb{Q}$, more is true. A commutative $H\mathbb{Q}$-algebra X does correspond to a commutative differential graded $\mathbb{Q}$-algebra by [39, 1.2]. More precisely, there is zig-zag of natural weak equivalences $\Theta X \simeq \Theta' X$ where $\Theta' X$ is a commutative DGA. However, despite the fact that the categories of modules have symmetric monoidal structures, the Quillen equivalence $\mathrm{Mod}_X \simeq_Q \mathrm{Mod}_{\Theta' X}$ is *not* a symmetric monoidal Quillen equivalence. This is because the upgrading of a Quillen equivalence to the categories of modules involves cofibrant replacement of monoids [34, 3.12(1)] which will destroy commutativity and hence the symmetric monoidal structure.

The stable model structure on spectra does not behave well with respect to commutative algebras, in the sense that for a commutative ring spectrum S, cofibrant commutative S-algebras are not cofibrant as S-modules in general. Shipley [37] constructed the flat model structure (also called the S-model structure) on symmetric spectra, which does satisfy the property that cofibrant commutative algebras are cofibrant as modules. Pavlov-Scholbach [28] extended this to the case of symmetric spectra in general model categories. This extra compatibility between commutative algebras and modules provides several useful tools that would otherwise not be valid. For a concrete example of where this compatibility can be useful, see the next section of the introduction.

In light of these considerations, the goal of this paper is threefold. Firstly, we show that the zig-zag of Quillen equivalences in Shipley's algebraicization theorem still holds in flat model structures which satisfy the extra compatibility between commutative algebras and modules discussed above. See Section 5 for a precise statement of the zig-zag of symmetric monoidal Quillen equivalences in this first theorem.

Theorem 1.1. *There is a zig-zag of symmetric monoidal Quillen equivalences*

$$\mathrm{Mod}_{H\mathbb{Q}}^{\mathrm{flat}} \simeq_Q \mathrm{Ch}_{\mathbb{Q}}$$

where the intermediate categories have the flat model structure.

In fact we show that the flat model structures on the intermediate categories are the same as the stable model structures used by Shipley [39], see Corollary 4.13.

Secondly, we use this theorem to give a new proof of the following theorem, which appears in the body of the paper as Theorem 6.6. In particular, our approach does not pass through the category of E_∞-algebras as in the proof given by Richter-Shipley [30].

Theorem 1.2. *There is a zig-zag of Quillen equivalences between the category of commutative $H\mathbb{Q}$-algebras and the category of commutative rational DGAs.*

Finally, we prove the following theorem which appears as Theorem 7.2 in the main body of the paper.

Theorem 1.3. *For a commutative $H\mathbb{Q}$-algebra X there is a zig-zag of weak symmetric monoidal Quillen equivalences $\mathrm{Mod}_X \simeq_Q \mathrm{Mod}_{\underline{\Theta} X}$ where $\underline{\Theta} X$ is a commutative DGA.*

Motivation and related work. The author's main motivation comes from the study of algebraic models for rational equivariant cohomology theories. A key step in the construction of algebraic models is the passage from modules over a commutative $H\mathbb{Q}$-algebra to modules over a commutative DGA via Shipley's algebraicization theorem. Therefore, a deep understanding of Shipley's algebraicization theorem provides key insights into the understanding of algebraic models for rational equivariant cohomology theories.

Working in the flat model structure provides valuable techniques which are not valid in the stable model structure. In forthcoming work [44], the author considers the correspondence of the change of groups functors in rational equivariant stable homotopy theory with functors between the algebraic models. In particular, this includes studying how the extension-restriction-coextension of scalars adjoint triple along a map of commutative $H\mathbb{Q}$-algebras $\theta\colon S \to R$ behaves with respect to the Quillen equivalences in Shipley's algebraicization theorem.

The restriction of scalars functor along a map of commutative monoids $\theta\colon S \to R$ in a symmetric monoidal model category is always right Quillen in the model structure right lifted from the underlying category, but it is not left Quillen in general. If the monoidal unit of the underlying category is cofibrant, then restriction of scalars is left Quillen if and only if R is cofibrant as an S-module. Since a key step in the proof of algebraic models is a formality argument based on the fact that polynomial rings are formal as commutative DGAs, one needs to be able to replace R in such a way that it is still a commutative S-algebra, and is cofibrant as an S-module. This replacement is possible in the flat model structure, but not in the stable model structure on spectra. Therefore, Theorem 1.1 provides the necessary setup in which to attack the correspondence of functors along the bridge which Shipley's algebraicization theorem provides between topology and algebra.

The use of the flat model structure allows the extension of the result to commutative algebra objects, so that we prove a Quillen equivalence between the category of commutative $H\mathbb{Q}$-algebras and the category of commutative rational DGAs. Richter-Shipley [30] prove that the category of commutative HR-algebras is Quillen equivalent to the category of differential graded E_∞-R-algebras for any commutative ring R. Since E_∞-algebras in chain complexes of $\mathbb{Q}$-modules can be rectified to strictly commutative objects, see for example [25, §7.1.4], as a corollary [30, 8.4] of Richter and Shipley's result one obtains that the category of commutative $H\mathbb{Q}$-algebras is Quillen equivalent to the category of commutative rational DGAs. We give a concrete zig-zag of Quillen equivalences which lands naturally in commutative DGAs, bypassing the need for the rectification step. We expect that this direct approach will enable a better understanding of algebraic models for naive-commutative rational G-spectra as studied by Barnes-Greenlees-Kędziorek [3, 4]. White-Yau [43] give an alternative approach to this zig-zag of Quillen equivalences by using the stable model structure and their theory of lifting Quillen equivalences to categories of coloured operads. The generality of their theory leads to more stringent hypotheses than our approach, see for example [43, 3.27]. Our approach exploits the fact that in the flat model structure, cofibrant commutative algebras forget to cofibrant modules.

Finally we give a concrete zig-zag of *symmetric monoidal* Quillen equivalences between the category of modules over a commutative $H\mathbb{Q}$-algebra and the category of modules over a commutative DGA. The result is assumed without proof in the literature, see for example [6, 3.4.4]. Due to the importance of this result in the construction of algebraic models, we believe it is valuable to make the proof explicit. Shipley proved that there is a Quillen equivalence [39, 2.15] between modules over an HR-algebra X and modules over a DGA ΘX for any ring R. In the case that $R = \mathbb{Q}$, Shipley furthermore proves that ΘX is naturally weakly equivalent to a *commutative* DGA $\Theta' X$ [39, 1.2]. A dual of a result of Schwede-Shipley [34, 3.12(2)] allows one to conclude moreover that there is a commutative DGA $\underline{\Theta} X$ and a zig-zag of *symmetric monoidal* Quillen equivalences $\mathrm{Mod}_X \simeq_Q \mathrm{Mod}_{\underline{\Theta} X}$. The fact that this is a symmetric monoidal Quillen equivalence has been a vital ingredient in the construction of symmetric monoidal algebraic models, see [6, 3.4.4] and [29, 9.6].

Outline of the paper. We recall the key background on model categories in Section 2, and on symmetric spectra in general model categories in Section 3. In Section 4, we recall results from Pavlov-Scholbach [28] which enable the construction of flat model structures on symmetric spectra in general model categories, and apply these results to our cases of interest. Section 5 is dedicated to the proof of Theorem 1.1. In Section 6 we extend our results to show that the category of commutative $H\mathbb{Q}$-algebras is Quillen equivalent to the category of commutative rational DGAs. Finally, in Section 7 we consider the extension to modules over commutative $H\mathbb{Q}$-algebras.

Conventions. We write the left adjoint above the right adjoint in an adjoint pair displayed horizontally, and on the left in an adjoint pair displayed vertically.

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2. Model categorical preliminaries

In this section we recall the necessary background on model categories which we require for the paper.

2.1. Bousfield localization.

Firstly we recall the definitions and key properties of left Bousfield localizations from [18].

Definition 2.1. Let $\mathcal{C}$ be a model category and let S be a collection of maps in $\mathcal{C}$.

- An object W in $\mathcal{C}$ is *S-local* if it is fibrant in $\mathcal{C}$ and for every $s\colon A \to B$ in S, the natural map $\mathrm{map}(B, W) \to \mathrm{map}(A, W)$ is a weak equivalence of homotopy function complexes.
- A map $f\colon X \to Y$ in $\mathcal{C}$ is an *S-local equivalence* if for every S-local object W, the natural map $\mathrm{map}(Y, W) \to \mathrm{map}(X, W)$ is a weak equivalence of homotopy function complexes.

The *left Bousfield localization* of $\mathcal{C}$ at S (if it exists), denoted $L_S\mathcal{C}$, is the model structure on $\mathcal{C}$ in which the weak equivalences are the S-local equivalences and the cofibrations are the same as in $\mathcal{C}$. The fibrant objects are the S-local objects. We call the fibrations the S-local fibrations.

The left Bousfield localization of $\mathcal{C}$ at S exists if S is a set of maps and $\mathcal{C}$ is left proper and cellular [18, 4.1.1], or if S is a set of maps and $\mathcal{C}$ is left proper and combinatorial [7, 4.7]. Any weak equivalence in $\mathcal{C}$ is an S-local equivalence, so it follows that the identity functors give a Quillen adjunction $\mathcal{C} \rightleftarrows L_S\mathcal{C}$.

Proposition 2.2 ([18, 3.3.16], [22, 7.21])**.** *Let $\mathcal{C}$ be a model category and S a set of maps in $\mathcal{C}$.*

(1) If f is an S-local equivalence between S-local objects, then f is a weak equivalence in $\mathcal{C}$.
(2) If f is a fibration between S-local objects, then f is an S-local fibration.

We now recall a result of Dugger [11, A.2], which when used in conjunction with Proposition 2.2 simplifies the process of proving a Quillen adjunction between left Bousfield localizations.

Proposition 2.3. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be an adjunction, where $\mathcal{C}$ and $\mathcal{D}$ are model categories. Then G is right Quillen if and only if G preserves fibrations between fibrant objects and all acyclic fibrations.*

2.2. Algebras and modules.

We next recall the theory of (commutative) monoids, (commutative) algebras and modules in symmetric monoidal model categories due to Schwede-Shipley [33] and White [42].

Recall that a model category is said to be *symmetric monoidal* if it has a closed symmetric monoidal structure and satisfies the following two conditions:

(1) *pushout-product axiom*: if $f\colon A \to B$ and $g\colon X \to Y$ are cofibrations, then the pushout-product map
$$f \square g\colon A \otimes Y \bigcup_{A \otimes X} B \otimes X \to B \otimes Y$$
is a cofibration, which is acyclic if either f or g is acyclic;

(2) *unit axiom*: for $c\mathbb{1} \to \mathbb{1}$ a cofibrant replacement of the unit, the natural map $c\mathbb{1} \otimes X \to \mathbb{1} \otimes X \cong X$ is a weak equivalence for all cofibrant X.

Definition 2.4. Suppose that $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a Quillen adjunction between symmetric monoidal model categories. We say that (F, U) is a *weak symmetric monoidal Quillen adjunction* if the right adjoint U is lax symmetric monoidal (which gives the left adjoint F an oplax symmetric monoidal structure) and the following conditions hold:

(1) for cofibrant A and B in $\mathcal{C}$, the oplax monoidal structure map $\varphi\colon F(A \otimes B) \to FA \otimes FB$ is a weak equivalence in $\mathcal{D}$;

(2) for a cofibrant replacement $c\mathbb{1}_{\mathcal{C}}$ of the unit in $\mathcal{C}$, the map $F(c\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ is a weak equivalence in $\mathcal{D}$.

If the oplax monoidal structure maps are isomorphisms, then we say that (F, U) is a *strong symmetric monoidal Quillen adjunction*. We say that (F, U) is a *weak (resp. strong) symmetric monoidal Quillen equivalence* if (F, U) is a weak (resp. strong) symmetric monoidal Quillen adjunction which is also a Quillen equivalence. Note that if F is strong monoidal and the unit of $\mathcal{C}$ is cofibrant, then the Quillen pair (F, U) is a strong symmetric monoidal Quillen pair.

In this paper, we will be particularly interested in the interaction of model structures and Quillen functors with categories of modules and (commutative) algebras. Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a symmetric monoidal model category. For a monoid S in $\mathcal{C}$, we denote the category of (left) S-modules by $\mathrm{Mod}_S(\mathcal{C})$. If the underlying category is clear, we will instead write Mod_S.

The categories of modules and algebras often inherit a model structure from the underlying category in the following way. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be an adjunction in which $\mathcal{C}$ is a model category and $\mathcal{D}$ is a bicomplete category. Kan's lifting theorem [18, 11.3.2] provides conditions under which $\mathcal{D}$ inherits a model structure in which a map f in $\mathcal{D}$ is a weak equivalence (resp. fibration) if and only if Uf is a weak equivalence (resp. fibration) in $\mathcal{C}$. We call such a model structure *right lifted*.

Under mild hypotheses, the categories of modules and (commutative) algebras obtain right lifted model structures. We refer the reader to [33, 2.4] for the precise smallness condition in the following theorem, and instead note that it is satisfied if $\mathcal{C}$ is locally presentable. Similarly, we refer the reader to [33, 3.3] and [42, 3.1] for the definitions of the monoid axiom and commutative monoid axiom respectively.

Theorem 2.5 ([33, 4.1], [42, 3.2])**.** *Let $\mathcal{C}$ be a cofibrantly generated, symmetric monoidal model category (with some smallness condition) and let S be a commutative monoid in $\mathcal{C}$.*

(1) If $\mathcal{C}$ satisfies the monoid axiom then the categories of S-modules and S-algebras have right lifted model structures in which a map is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. fibration) in $\mathcal{C}$.

(2) If $\mathcal{C}$ satisfies the commutative monoid axiom and the monoid axiom, then the category of commutative S-algebras has a right lifted model structure in which a map is a weak equivalence (resp. fibration) if and only if it is a weak equivalence (resp. fibration) in $\mathcal{C}$.

We say that a symmetric monoidal model category $\mathcal{C}$ satisfies *Quillen invariance of modules* if for any weak equivalence $\theta\colon S \to R$ of monoids in $\mathcal{C}$, the extension-restriction of scalars adjunction

$$\mathrm{Mod}_S \underset{\theta^*}{\overset{R \otimes_S -}{\rightleftarrows}} \mathrm{Mod}_R$$

is a Quillen equivalence, see [33, 4.3]. Throughout we write $\theta_* = R \otimes_S -$ for the left adjoint of the restriction of scalars functor θ^*.

2.3. **Cofibrations of modules and (commutative) algebras.** In general there is not an explicit description of the cofibrations in a right lifted model structure, but in many situations they have desirable properties.

Theorem 2.6 ([33, 4.1])**.** *Let $\mathcal{C}$ be a symmetric monoidal model category and let S be a commutative monoid in $\mathcal{C}$. Every cofibration of S-algebras whose source is cofibrant as an S-module is also a cofibration of S-modules. In particular, if the unit of $\mathcal{C}$ is cofibrant, then every cofibrant S-algebra is a cofibrant S-module.*

The case of commutative algebras is more subtle. White [42, 3.5, 3.6] has given an answer to this question in general, but it requires stronger assumptions that just the existence of the model structure on commutative algebras. We recall some relevant examples.

Example 2.7. If S a commutative DGA over a field of characteristic zero and R is a cofibrant commutative S-algebra, then R is cofibrant (i.e., dg-projective) as an S-module, see for instance [42, §5.1]. Note that it fails in non-zero characteristic since Maschke's theorem does not apply.

Example 2.8. In categories of spectra the situation is more complicated. It is well known by Lewis' obstruction [24] that the stable model structure on (symmetric) spectra cannot be right lifted to a model structure on commutative algebra spectra as the sphere spectrum is cofibrant. Indeed, a fibrant replacement of the sphere spectrum as a commutative ring spectrum would be an Ω-spectrum whose zeroth space is a commutative topological monoid which is weakly equivalent to $\mathrm{colim}(\Omega^n\Sigma^n S^0)$. This implies that $\mathrm{colim}(\Omega^n\Sigma^n S^0)$ is weakly equivalent to a product of Eilenberg-MacLane spaces which is false. Instead, one must consider the *positive* stable model structure in which the sphere spectrum is not cofibrant. This model structure can be right lifted to give a model structure on commutative algebras, however, a cofibrant commutative algebra in the positive stable model structure on spectra need not be cofibrant as a module. Nonetheless there is a model structure on spectra called the flat model structure, for which this property is true, see Corollary 4.11.

3. Symmetric spectra in general model categories

In this section we recall the definition of the category of symmetric spectra in general model categories and its properties and stable model structure as in [20]; see also [30, §2].

Let $(\mathcal{C}, \otimes, \mathbb{1})$ be a bicomplete, closed symmetric monoidal category and $K \in \mathcal{C}$. Let Σ be the category whose objects are the finite sets $\underline{n} = \{1, \ldots, n\}$ for $n \geq 0$ where $\underline{0} = \emptyset$, and whose morphisms are the bijections of $\underline{n}$. The category of symmetric sequences in $\mathcal{C}$ is the functor category $\mathcal{C}^\Sigma$. The category $\mathcal{C}^\Sigma$ inherits a closed symmetric monoidal structure from $\mathcal{C}$ via the Day convolution, with tensor product given by

$$(A \odot B)(n) = \coprod_{p+q=n} \Sigma_n \times_{\Sigma_p \times \Sigma_q} A(p) \otimes B(q).$$

The category of *symmetric spectra* $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is the category of modules over $\mathrm{Sym}(K)$ in $\mathcal{C}^\Sigma$, where $\mathrm{Sym}(K) = (\mathbb{1}, K, K^{\otimes 2}, \cdots)$ is the free commutative monoid on K. Therefore, $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ inherits a closed symmetric monoidal structure with tensor product defined by the coequalizer

$$X \wedge Y = \mathrm{coeq}\,(X \odot \mathrm{Sym}(K) \odot Y \rightrightarrows X \odot Y)$$

of the actions of $\mathrm{Sym}(K)$ on X and Y. More explicitly, an object X of $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is a collection of Σ_n-objects $X(n) \in \mathcal{C}$ with Σ_n-equivariant maps

$$K \otimes X(n) \to X(n+1)$$

for all $n \geq 0$, such that the composite

$$K^{\otimes m} \otimes X(n) \to X(n+m)$$

is $\Sigma_m \times \Sigma_n$-equivariant for all $m, n \geq 0$. Note that taking $\mathcal{C} = \mathrm{sSet}_*$ and $K = S^1$ recovers the usual notion of symmetric spectra as defined and studied by Hovey-Shipley-Smith [21].

We now sketch the construction of the stable model structure on $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ due to Hovey [20]. If $\mathcal{C}$ is a left proper and cellular model category, one can equip $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ with a level model structure in which the weak equivalences and fibrations are levelwise weak equivalences and levelwise fibrations in $\mathcal{C}$ respectively [20, 8.2]. One can then left Bousfield localize this level model structure to obtain the stable model structure [20, 8.7]. We call the weak equivalences in this model structure the stable equivalences and the fibrations the stable fibrations.

There is also a positive stable model structure, which allows the construction of right lifted model structures on commutative algebras, see for instance [26, §14]. However, these model structures do not have the property that cofibrant commutative algebras are cofibrant modules. In order to rectify this, we turn to the flat model structure in the next section.

Notation 3.1. We set notation for the categories of symmetric spectra of interest.

- We write $\mathrm{Sp}^\Sigma = \mathrm{Sp}^\Sigma(\mathrm{sSet}_*, S^1)$ for the category of symmetric spectra in simplicial sets.
- We write $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ for the category $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod}, \widetilde{\mathbb{Q}}S^1)$ where $\mathrm{s}\mathbb{Q}$-mod is the category of simplicial $\mathbb{Q}$-modules and $\widetilde{\mathbb{Q}}\colon \mathrm{sSet}_* \to \mathrm{s}\mathbb{Q}\text{-mod}$ is the functor which takes the levelwise free $\mathbb{Q}$-module on the non-basepoint simplices.
- We write $\mathrm{Sp}^\Sigma(\mathrm{Ch}_\mathbb{Q}^+)$ for the category $\mathrm{Sp}^\Sigma(\mathrm{Ch}_\mathbb{Q}^+, \mathbb{Q}[1])$ where $\mathrm{Ch}_\mathbb{Q}^+$ is the category of non-negatively graded chain complexes of $\mathbb{Q}$-modules and $\mathbb{Q}[1]$ is the chain complex which contains a single copy of $\mathbb{Q}$ concentrated in degree 1.

4. Flat model structures

In this section we show that the categories used in Shipley's algebraicization theorem support a flat model structure. Recall from Example 2.8 that a cofibrant commutative algebra need not be a cofibrant module in the stable model structure on spectra. To rectify this, Shipley [37] constructs a flat (and a positive flat) model structure on symmetric spectra in simplicial sets in which this property holds. Pavlov-Scholbach [28] extended these flat model structures to symmetric spectra in general model categories. The flat model structure has the same weak equivalences as the stable model structure on spectra (i.e., the stable equivalences), but has more cofibrations. In particular, the identity functor from the stable model structure to the flat model structure is a left Quillen equivalence.

4.1. **Equivariant model structures.** The stable model structure on symmetric spectra disregards the actions of the symmetric groups on each level. Instead, the flat model structure proceeds by remembering this equivariance and building it into the model structure. There are two extreme cases: the naive case is where no equivariance is recorded and the genuine case is when all equivariance is recorded. The flat model structure on $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ (when it exists) is built from the blended model structure on G-objects in $\mathcal{C}$ which is intermediate between the naive and genuine structures. Note that some authors refer to this model structure as the mixed model structure, but we do not since it is not mixed in the sense of Cole mixing [10].

From now on, we assume that $\mathcal{C}$ is a pretty small model category [27, 2.1]. We note that this condition is satisfied for simplicial sets, simplicial $\mathbb{Q}$-modules and non-negatively graded chain complexes of $\mathbb{Q}$-modules.

We now recall the conditions needed for the genuine and blended model structures to exist, see for instance [40]. Let G be a finite group. We write $G\mathcal{C}$ for the category of G-objects in $\mathcal{C}$; that is, the functor category $[BG, \mathcal{C}]$ where BG is the one-object category whose morphisms are elements of G.

Definition 4.1. We say that $\mathcal{C}$ satisfies the *weak cellularity conditions for G* if the following are true for all subgroups $H, K \leq G$:

(1) $(-)^H$ preserves directed colimits of diagrams in $G\mathcal{C}$ where each underlying arrow in $\mathcal{C}$ is a cofibration,
(2) $(-)^H$ preserves pushouts of diagrams where one leg is of the form $G/K \otimes f$ for f a cofibration in $\mathcal{C}$,
(3) $(G/K \otimes -)^H$ takes generating cofibrations to cofibrations and generating acyclic cofibrations to acyclic cofibrations.

We say that it satisfies the *strong cellularity conditions for G* if (1) and (2) from above hold, and for any $H, K \leq G$ and any $X \in \mathcal{C}$,

$$(G/H \otimes X)^K \cong (G/H)^K \otimes X.$$

Definition 4.2. We say that a map $f\colon X \to Y$ in $G\mathcal{C}$ is:

- a *naive weak equivalence* if the underlying morphism is a weak equivalence in $\mathcal{C}$;
- a *naive fibration* if the underlying morphism is a fibration in $\mathcal{C}$;
- a *naive cofibration* if it has the left lifting property with respect to the naive acyclic fibrations;
- a *genuine weak equivalence* if for every subgroup H of G, the map $f^H\colon X^H \to Y^H$ is a weak equivalence in $\mathcal{C}$;
- a *genuine fibration* if for every subgroup H of G, the map $f^H\colon X^H \to Y^H$ is a fibration in $\mathcal{C}$;
- a *genuine cofibration* if it has the left lifting property with respect to all genuine acyclic fibrations.
- a *blended fibration* if it has the right lifting property with respect to maps which are both naive weak equivalences and genuine cofibrations.

The cellularity conditions control when the genuine model structure on $G\mathcal{C}$ exists.

Proposition 4.3. *If the weak cellularity conditions hold for $\mathcal{C}$ then the genuine weak equivalences, genuine cofibrations and genuine fibrations give a cofibrantly generated, model structure on $G\mathcal{C}$ called the genuine model structure. Furthermore, if $\mathcal{C}$ is proper, then so is the genuine model structure on $G\mathcal{C}$, and if $\mathcal{C}$ is a monoidal model category with cofibrant unit, then so is the genuine model structure on $G\mathcal{C}$.*

Proof. The claim that the genuine model structure exists and is cofibrantly generated is due to Stephan [40, 2.6]. The generating cofibrations and generating acyclic cofibrations are given by $\cup_{H\leq G}\{G/H \otimes i \mid i \in I\}$ and $\cup_{H\leq G}\{G/H \otimes j \mid j \in J\}$ respectively, where I and J are the sets of generating cofibrations and acyclic cofibrations for $\mathcal{C}$ respectively.

We now prove that the genuine model structure is left proper. It suffices to prove that in a diagram of pushouts of the form

$$\begin{tikzcd} G/H \otimes A \arrow[r] \arrow[d, hook] & C \arrow[r, "\sim"] \arrow[d] & X \arrow[d] \\ G/H \otimes B \arrow[r] & D \arrow[r] \arrow[ul, phantom, "\ulcorner", very near start] & Y \arrow[ul, phantom, "\ulcorner", very near start] \end{tikzcd}$$

where $A \to B$ is a generating cofibration for $\mathcal{C}$, the map $D \to Y$ is a genuine weak equivalence. This is because as $\mathcal{C}$ is pretty small, weak equivalences are closed under transfinite composition [27, 2.2], and therefore the class of maps for which pushing out along them preserves weak equivalences is closed under retracts, pushouts and transfinite compositions. By the second cellularity condition, after taking K fixed points, the left hand square and the outer rectangle are still pushouts. It follows that the right hand square is also still a pushout.

By the third cellularity condition, the left most vertical map is still a cofibration after taking K fixed points. Since cofibrations are stable under pushout, the map $C^K \to D^K$ is a cofibration, and since $\mathcal{C}$ is left proper, we have that $D^K \to Y^K$ is a weak equivalence for all K. Hence the genuine model structure is left proper. The fact that the model structure is right proper follows immediately from the fact that fixed points determine fibrations and weak equivalences.

We now prove that the genuine model structure is monoidal. Firstly we must show that the pushout-product of two genuine cofibrations is a genuine cofibration. We use the description of the generating cofibrations $\cup_{H \leq G}\{G/H \otimes i \mid i \in I\}$ for the genuine model structure where I is the set of generating cofibrations for $\mathcal{C}$. Take generating cofibrations

$$G/H \otimes i \colon G/H \otimes A \to G/H \otimes B \quad \text{and} \quad G/K \otimes i' \colon G/K \otimes X \to G/K \otimes Y$$

for the genuine model structure. Since $G/H \otimes -$ and $G/K \otimes -$ are left adjoints, the pushout product map $(G/H \otimes i)\square(G/K \otimes i')$ can be identified with the map $(G/H \otimes G/K) \otimes (i \square i')$, which in turn can be identified with

$$\coprod_{x \in [H\backslash G/K]} G/(H \cap xKx^{-1}) \otimes (i \square i')$$

by the double coset formula. Since $\mathcal{C}$ is monoidal, $i\square i'$ is a cofibration in $\mathcal{C}$ and hence the pushout product map $(G/H \otimes i)\square(G/K \otimes i')$ is a genuine cofibration as required. It follows by a similar argument that the pushout product of a genuine cofibration with a genuine acyclic cofibration is a genuine acyclic cofibration.

For the unit axiom, note that the monoidal unit in $G\mathcal{C}$ is the unit of $\mathcal{C}$ equipped with the trivial G-action. The functor which equips an object with the trivial G-action is left adjoint to the G-fixed points functor, and hence is left Quillen. It then follows that since the unit of $\mathcal{C}$ is cofibrant, the unit in $G\mathcal{C}$ is genuine cofibrant. $\square$

We can then localize the genuine model structure to give the blended model structure.

Theorem 4.4. *Let $\mathcal{C}$ be a simplicial, proper model category which satisfies the weak cellularity conditions. Then the naive weak equivalences, genuine cofibrations and blended fibrations give a proper, cofibrantly generated model structure on $G\mathcal{C}$ which we call the blended model structure.*

Proof. We apply Bousfield-Friedlander localization [9, 9.3] to the genuine model structure on $G\mathcal{C}$, with $QX = \mathrm{map}(EG, \widehat{f}X)$ where $\widehat{f}$ is a genuine fibrant replacement functor and map denotes the simplicial cotensor. We must verify that the conditions (A1), (A2) and (A3) from [9, 9.3] are satisfied. Note that a map $f \colon X \to Y$ in $G\mathcal{C}$ is a naive weak equivalence if and only if $Qf \colon QX \to QY$ is a genuine weak equivalence. To see this, if $f \colon X \to Y$ is a naive weak equivalence, then $\mathrm{map}(G, f)$ is a genuine weak equivalence. Therefore, $\mathrm{map}(Z, f)$ is a genuine weak equivalence if Z is built from free cells. Conversely, since $EG \to *$ is a naive weak equivalence, if $\mathrm{map}(EG, f)$ is a genuine weak equivalence then f is a naive weak equivalence. The conditions (A1) and (A2) follow from this observation. Since Q preserves fibrations and pullbacks, condition (A3) follows from the right properness of the genuine model structure on $G\mathcal{C}$. $\square$

Note that [9, 9.3] also gives an explicit description of the blended fibrations as those maps $X \to Y$ which are genuine fibrations and have the property that

$$(\star) \qquad \begin{tikzcd} X \arrow[r] \arrow[d] & \mathrm{map}(EG, \widehat{f}X) \arrow[d] \\ Y \arrow[r] & \mathrm{map}(EG, \widehat{f}Y) \end{tikzcd}$$

is a homotopy pullback square. The genuine fibrant replacement ensures that this is equivalent to the square being a homotopy pullback after taking H-fixed points for all $H \leq G$.

Proposition 4.5. *The blended model structure exists on $G\mathcal{C}$ for $\mathcal{C} = \mathrm{sSet}_*$, $\mathrm{s}\mathbb{Q}\text{-mod}$ and $\mathrm{Ch}^+_{\mathbb{Q}}$.*

Proof. The categories of based simplicial sets and simplicial $\mathbb{Q}$-modules satisfy the strong cellularity conditions by [40, 2.14]. The category of non-negatively graded rational chain complexes satisfies the weak cellularity conditions by [40, 2.19]. Therefore the result follows from Theorem 4.4. □

Finally we note that in these cases, the blended model structure can be identified with the injective model structure in which the weak equivalences and cofibrations are both underlying.

Proposition 4.6.

(i) *A map f in G-sSet_* is an underlying cofibration if and only if it is a genuine cofibration.*
(ii) *For $\mathcal{C} = \mathrm{s}\mathbb{Q}\text{-mod}$ and $\mathrm{Ch}^+_{\mathbb{Q}}$, a map f in $G\mathcal{C}$ is an underlying cofibration if and only if it is a naive cofibration if and only if it is a genuine cofibration.*

Proof. Part (i) is well known; see for example [37, 1.2] or [40, 2.16].

For part (ii), let $\mathcal{C} = \mathrm{Ch}^+_{\mathbb{Q}}$ or $\mathrm{s}\mathbb{Q}$-mod and note that we can give the same style of proof since they are both $\mathbb{Q}$-additive. From the description of the generating cofibrations of the genuine model structure given in the proof of Proposition 4.3, it is clear that any genuine cofibration is an underlying cofibration. Since any naive cofibration is also a genuine cofibration it follows that any naive cofibration is an underlying cofibration.

We now turn to proving the forward implication. Since any naive cofibration is a genuine cofibration, it suffices to show that if f is an underlying cofibration then it is a naive cofibration. Let $f\colon X \to Y$ be an underlying cofibration in $G\mathcal{C}$. In order to prove that f is a naive cofibration we must show that it has the left lifting property with respect to the naive acyclic fibrations. Consider a commutative square

$$\begin{tikzcd} X \arrow[r, "\alpha"] \arrow[d, "f"'] & A \arrow[d, "h"] \\ Y \arrow[r, "\beta"'] & B \end{tikzcd}$$

in $G\mathcal{C}$, in which h is a naive acyclic fibration. Since f is an underlying cofibration, there is a lift $\theta\colon Y \to A$ making the diagram commute, but this need not be an equivariant map. Define $\varphi\colon Y \to A$ by

$$\varphi(y) = \frac{1}{|G|} \sum_{g \in G} g\theta(g^{-1}y).$$

This is an equivariant map, so it remains to check that it is indeed a lift.

Since f and α are equivariant maps,

$$\varphi(f(x)) = \frac{1}{|G|} \sum_{g \in G} g\theta(f(g^{-1}x)) = \frac{1}{|G|} \sum_{g \in G} g\alpha(g^{-1}x) = \alpha(x).$$

In a similar way, one can show that $h\varphi = \beta$. Therefore φ is a lift, and the map f is a naive cofibration and hence also a genuine cofibration. □

Corollary 4.7. *For $\mathcal{C} = \mathrm{s}\mathbb{Q}\text{-mod}$ and $\mathrm{Ch}^+_{\mathbb{Q}}$, the blended model structure, injective model structure and the naive model structure on $G\mathcal{C}$ are the same.*

Proof. The weak equivalences in all three model structures are the naive weak equivalences. The cofibrations in each coincide by Proposition 4.6. □

We emphasize that in the case of $\mathcal{C} = \mathrm{sSet}_*$, the blended model structure is the same as the injective model structure on $G\mathcal{C}$, but is *not* the same as the naive model structure.

Corollary 4.8. *For* $\mathcal{C} = \mathrm{sSet}_*$, $\mathrm{s}\mathbb{Q}\text{-mod}$ *and* $\mathrm{Ch}^+_{\mathbb{Q}}$*, the blended model structure on* $G\mathcal{C}$ *is monoidal.*

Proof. Note that in each case, $\mathcal{C}$ is monoidal and has cofibrant unit. Since the blended model structure is the same as the injective model structure by Proposition 4.6, it is immediate that the pushout-product axiom holds. The unit axiom holds by the same argument as in Proposition 4.3. □

4.2. **The flat model structure.** We can equip Sp^Σ, $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ and $\mathrm{Sp}^\Sigma(\mathrm{Ch}^+_{\mathbb{Q}})$ with the *level flat model structure*, in which the weak equivalences (resp. fibrations) are the levelwise naive weak equivalences (resp. levelwise blended fibrations) [28, 3.1.3]. The cofibrations in the level flat model structure are the *flat cofibrations*; that is, the maps which have the left lifting property with respect to maps which are both levelwise naive weak equivalences and levelwise blended fibrations. In a similar manner, Sp^Σ, $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ and $\mathrm{Sp}^\Sigma(\mathrm{Ch}^+_{\mathbb{Q}})$ can be given the *positive level flat model structure* in which the weak equivalences (resp. fibrations) are the maps which are naive weak equivalences (resp. blended fibrations) for each level $n > 0$.

A left Bousfield localization of the level flat model structure yields the *flat model structure*. The weak equivalences in the flat model structure are the stable equivalences, and the cofibrations are the flat cofibrations. We call the fibrations in the flat model structure the *flat fibrations*. Similarly, a left Bousfield localization of the positive level flat model structure gives the *positive flat model structure* in which the weak equivalences are also the stable equivalences.

Theorem 4.9. *The flat and positive flat model structures on* Sp^Σ*,* $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ *and* $\mathrm{Sp}^\Sigma(\mathrm{Ch}^+_{\mathbb{Q}})$ *(and on modules over monoids in these categories) exist. Furthermore, they satisfy Quillen invariance of modules, and are stable, left proper, symmetric monoidal and combinatorial model structures.*

Proof. Since the genuine cofibrations are the same as the underlying cofibrations by Proposition 4.6, the blended model structure coincides with the injective model structure. The injective model structure is strongly admissible by [28, 2.3.7] and therefore the flat model structure exists by [28, 3.2.1]. Quillen invariance holds by [28, 3.3.9], monoidality follows as it is defined to be a monoidal left Bousfield localization, stability by [28, 3.4.1] and left properness and combinatoriality follows from [28, 3.4.2]. □

We now record some key properties of the flat model structure which we will use throughout this paper.

Proposition 4.10.

(i) *A map is an acyclic flat fibration if and only if it is a levelwise acyclic flat fibration.*
(ii) *A map between flat fibrant objects is a flat fibration if and only if it is a levelwise flat fibration.*
(iii) *The identity functor is a left Quillen equivalence from the stable model structure to the flat model structure.*

Proof. Part (i) follows from the fact that left Bousfield localization does not change the acyclic fibrations and part (ii) follows from Proposition 2.2. For part (iii), since the stable model structure and the flat model structure have the same weak equivalences, it suffices to show that any stable cofibration is a flat cofibration. A map is a stable cofibration if and only if it has the left lifting property with respect to maps which are levelwise naive acyclic fibrations, and a map is a flat cofibration if and only if it has the left lifting property with respect to maps which are both levelwise naive weak equivalences and blended fibrations. Any blended fibration is a naive fibration, and therefore a stable cofibration is also a flat cofibration. □

Corollary 4.11. *Let S be a commutative monoid in Sp^Σ, $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ or $\mathrm{Sp}^\Sigma(\mathrm{Ch}_\mathbb{Q}^+)$. The positive flat model structure can be right lifted to give a model structure on commutative S-algebras. Moreover, a positively flat cofibrant commutative S-algebra is also flat cofibrant as an S-module.*

Proof. Since the blended model structure coincides with the injective model structure in these cases by Proposition 4.6, and the injective model structure is strongly admissible [28, 2.3.7], this is a consequence of [28, 4.1, 4.4]. □

The flat model structure is a left Bousfield localization of the level flat model structure where weak equivalences and fibrations are determined levelwise in the blended model structure. We can give a characterization of the fibrant objects in the flat model structure.

Proposition 4.12 ([28, 3.2.1])**.** *An object X of $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is flat fibrant if and only if X is level flat fibrant and $X_n \to \underline{\mathrm{Hom}}(K, X_{n+1})$ is a naive weak equivalence where $\underline{\mathrm{Hom}}(K, -)$ is the right adjoint to $K \otimes -$.*

The following corollary shows that stable model structures on $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ and $\mathrm{Sp}^\Sigma(\mathrm{Ch}_\mathbb{Q}^+)$ satisfy extra compatibility between commutative algebras and modules, unlike the stable model structure on Sp^Σ.

Corollary 4.13. *The flat (resp. positive flat) model structure on $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ and $\mathrm{Sp}^\Sigma(\mathrm{Ch}_\mathbb{Q}^+)$ is the same as the stable (resp. positive stable) model structure.*

Proof. The weak equivalences in both the flat and stable model structure are the stable equivalences. Therefore it suffices to show that they have the same acyclic fibrations. A map is an acyclic fibration in the stable model structure if and only if it is a levelwise naive acyclic fibration. By Corollary 4.7, this is the case if and only if it is a levelwise acyclic fibration in the blended model structure, i.e., an acyclic flat fibration. □

In light of the previous corollary, we could call the model structure which we use on $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ and $\mathrm{Sp}^\Sigma(\mathrm{Ch}_\mathbb{Q}^+)$ either flat or stable. However, we will often refer to it as the flat model structure to remind the reader of the extra compatibility between commutative algebras and modules given by Corollary 4.11, which we will use throughout the paper.

Remark 4.14. One can give explicit characterizations of the stable and flat cofibrations in terms of latching objects. Define an object $\overline{\mathrm{Sym}}(K)$ of $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ to be 0 in level 0 and $K^{\otimes n}$ in level n, with the evident structure maps. The nth latching space of $X \in \mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is defined by $L_n X = \mathrm{Ev}_n(X \wedge \overline{\mathrm{Sym}}(K))$. This is an object in $\Sigma_n\mathcal{C}$, and the natural map $\overline{\mathrm{Sym}}(K) \to \mathrm{Sym}(K)$ induces a Σ_n-equivariant map $i_n \colon L_n X \to X$. Recall from [20, 8.5] that a map $f \colon X \to Y$ in $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is a stable cofibration if and only if the pushout-product map $X_n \cup_{L_n X} L_n Y \to Y_n$ is a naive cofibration in $\Sigma_n\mathcal{C}$ for all $n \geq 0$. In a similar way, one can show that a map $f : X \to Y$ is a flat cofibration if and only if $X_n \cup_{L_n X} L_n Y \to Y_n$ is a genuine cofibration in $\Sigma_n\mathcal{C}$ for all $n \geq 0$. This observation together with Proposition 4.6 gives another proof of Corollary 4.13.

Remark 4.15. Recall that in Sp^Σ every π_*-isomorphism is a stable equivalence [21, 3.1.11], but the converse fails [21, 3.1.10]. However, since we work rationally throughout this paper, the situation can be somewhat simplified. Shipley [35] constructs a detection functor $D \colon \mathrm{Sp}^\Sigma \to \mathrm{Sp}^\Sigma$ with the property that a map f is a stable equivalence if and only if Df is a π_*-isomorphism. The functor D is defined in terms of a homotopy colimit over the injection category. Since colimits over the injection category are exact rationally, it follows that in the rational setting a map is a stable equivalence if and only if it is a π_*-isomorphism. Moreover, $\pi_*(DX) \cong \pi_*(LX)$ where L is a stable fibrant replacement functor [35, 2.1.3], so that rationally the genuine stable homotopy groups (i.e., maps from the sphere in the homotopy category) coincide with the naive homotopy groups. For more details, also see [32, 8.49].

In this section we show that the chain of Quillen equivalences given by Shipley [39] for the stable model structure are still Quillen equivalences in the flat model structure, for the rational case. The identity functor from the stable model structure to the flat model structure is a left Quillen equivalence by Proposition 4.10. Therefore by the 2-out-of-3 property of Quillen equivalences, it is sufficient to check that we get Quillen adjunctions in the flat model structure. In fact, since the stable and flat model structure are the same on $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ and $\mathrm{Sp}^\Sigma(\mathrm{Ch}^+_\mathbb{Q})$ by Corollary 4.13, this reduces to just checking that the first adjunction is a Quillen adjunction. The following diagram summarises all of the adjunctions between $H\mathbb{Q}$-modules and chain complexes of $\mathbb{Q}$-modules.

$$(\dagger)\qquad \begin{tikzcd}
\mathrm{Mod}^{\mathrm{flat}}_{H\mathbb{Q}} \arrow[r, shift left, "Z"] \arrow[d, shift left, "1"] & \mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})_{\mathrm{flat}} \arrow[l, shift left, "U"] \arrow[r, shift right, "\varphi^* N"'] \arrow[d, shift left, "1"] & \mathrm{Sp}^\Sigma(\mathrm{Ch}^+_\mathbb{Q})_{\mathrm{flat}} \arrow[l, shift right, "L"'] \arrow[r, shift left, "D"] \arrow[d, shift left, "1"] & \mathrm{Ch}_\mathbb{Q} \arrow[l, shift left, "R"] \arrow[d, shift left, "1"] \\
\mathrm{Mod}^{\mathrm{stable}}_{H\mathbb{Q}} \arrow[u, shift left, "1"] \arrow[u, phantom, "\simeq_Q"] \arrow[r, shift left, "Z"] \arrow[r, phantom, "\simeq_Q"] & \mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})_{\mathrm{stable}} \arrow[u, shift left, "1"] \arrow[u, phantom, "="] \arrow[l, shift left, "U"] \arrow[r, shift right, "\varphi^* N"'] \arrow[r, phantom, "\simeq_Q"] & \mathrm{Sp}^\Sigma(\mathrm{Ch}^+_\mathbb{Q})_{\mathrm{stable}} \arrow[u, shift left, "1"] \arrow[u, phantom, "="] \arrow[l, shift right, "L"'] \arrow[r, shift left, "D"] \arrow[r, phantom, "\simeq_Q"] & \mathrm{Ch}_\mathbb{Q} \arrow[u, shift left, "1"] \arrow[u, phantom, "="] \arrow[l, shift left, "R"]
\end{tikzcd}$$

The functors will be defined throughout the rest of the section.

The model structure on simplicial $\mathbb{Q}$-modules is right lifted from simplicial sets along the forgetful functor $\mathrm{s}\mathbb{Q}\text{-mod} \to \mathrm{sSet}_*$. Applying this functor levelwise gives a forgetful functor $\widetilde{U}\colon \mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod}) \to \mathrm{Sp}^\Sigma$. Note that $\widetilde{U}\mathrm{Sym}(\widetilde{\mathbb{Q}}S^1) = (\mathbb{Q}, \widetilde{\mathbb{Q}}S^1, \widetilde{\mathbb{Q}}S^2, ...)$ which is $H\mathbb{Q}$ [21, 1.2.5]. Therefore the forgetful functor $\widetilde{U}$ can be viewed as a functor $U\colon \mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod}) \to \mathrm{Mod}_{H\mathbb{Q}}$.

Firstly, we show that the forgetful functor $\widetilde{U}$ is right Quillen when Sp^Σ is equipped with the flat model structure. Even though the flat model structure and the stable model structure on $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ are the same by Corollary 4.13, in order to prove the following it is actually convenient to work with the description of the acyclic fibrations in the flat model structure.

Lemma 5.1. *The forgetful functor*

$$\widetilde{U}\colon \mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})_{\mathrm{flat}} \to \mathrm{Sp}^\Sigma_{\mathrm{flat}}$$

preserves fibrations, and preserves and detects weak equivalences.

Proof. The forgetful functor preserves and detects weak equivalences by [39, Proof of 4.1]. We now show that it preserves the fibrations. By Proposition 2.3 it is sufficient to show that $\widetilde{U}$ preserves the acyclic flat fibrations and the flat fibrations between flat fibrant objects.

A map is an acyclic flat fibration if and only if it is a levelwise acyclic flat fibration, so it suffices to show that the forgetful functor $\mathrm{s}\mathbb{Q}\text{-mod} \to \mathrm{sSet}_*$ preserves naive weak equivalences and blended fibrations. Since the model structure on $\mathrm{s}\mathbb{Q}$-mod is right lifted from sSet_*, the forgetful functor preserves naive weak equivalences and genuine fibrations. It remains to check the homotopy pullback condition $(\star)$, which is an immediate consequence of the fact that the forgetful functor preserves homotopy pullbacks.

A flat fibration between flat fibrant objects is a levelwise flat fibration and hence $\widetilde{U}$ sends it to a levelwise flat fibration by the previous paragraph. Therefore, it remains to show that $\widetilde{U}$ preserves flat fibrant objects. Let X be a flat fibrant object in $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})_{\mathrm{flat}}$. By Proposition 4.12, X is level flat fibrant and $X_n \to \underline{\mathrm{Hom}}(\widetilde{\mathbb{Q}}S^1, X_{n+1})$ is a naive weak equivalence. It follows that $\widetilde{U}X$ is level flat fibrant, and since $\widetilde{U}$ preserves naive weak equivalences, we also have that $\widetilde{U}X_n \to \widetilde{U}\underline{\mathrm{Hom}}(\widetilde{\mathbb{Q}}S^1, X_{n+1})$ is a naive weak equivalence. By the $\widetilde{\mathbb{Q}} \dashv U$ adjunction, it follows that

$$\widetilde{U}X_n \to \underline{\mathrm{Hom}}(S^1, \widetilde{U}X_{n+1})$$

is a naive weak equivalence, and hence by Proposition 4.12, $\widetilde{U}X$ is flat fibrant. Therefore, $\widetilde{U}$ preserves flat fibrant objects. $\square$

Corollary 5.2. *The forgetful functor*

$$U\colon \mathrm{Sp}^{\Sigma}(\mathrm{s}\mathbb{Q}\text{-mod})_{\mathrm{flat}} \to \mathrm{Mod}_{H\mathbb{Q}}^{\mathrm{flat}}$$

preserves fibrations, and preserves and detects weak equivalences.

Recall from [39, 4.3] that the forgetful functor $U\colon \mathrm{Sp}^{\Sigma}(\mathrm{s}\mathbb{Q}\text{-mod}) \to \mathrm{Mod}_{H\mathbb{Q}}$ has a left adjoint Z defined by

$$Z(X) = H\mathbb{Q} \otimes_{\widetilde{\mathbb{Q}}H\mathbb{Q}} \widetilde{\mathbb{Q}}X$$

where $H\mathbb{Q}$ is viewed as a $\widetilde{\mathbb{Q}}H\mathbb{Q}$-module via the ring map $\beta\colon \widetilde{\mathbb{Q}}H\mathbb{Q} \to H\mathbb{Q}$ given by the monad structure on $\widetilde{\mathbb{Q}}$.

Proposition 5.3. *The adjunction*

$$\mathrm{Mod}_{H\mathbb{Q}}^{\mathrm{flat}} \underset{U}{\overset{Z}{\rightleftarrows}} \mathrm{Sp}^{\Sigma}(\mathrm{s}\mathbb{Q}\text{-mod})_{\mathrm{flat}}$$

is a strong symmetric monoidal Quillen equivalence with the respect to the flat model structures.

Proof. The forgetful functor U preserves weak equivalences and fibrations in the flat model structure by Corollary 5.2. Therefore, $Z \dashv U$ is a Quillen adjunction and hence by the 2-out-of-3 property of Quillen equivalences, is a Quillen equivalence; see Diagram (†). It is a strong symmetric monoidal Quillen equivalence as Z is strong symmetric monoidal and the unit $H\mathbb{Q}$ is a cofibrant $H\mathbb{Q}$-module. $\square$

Applying the normalization functor $N\colon \mathrm{s}\mathbb{Q}\text{-mod} \to \mathrm{Ch}_{\mathbb{Q}}^{+}$ levelwise yields a functor

$$N\colon \mathrm{Sp}^{\Sigma}(\mathrm{s}\mathbb{Q}\text{-mod}) \to \mathrm{Mod}_{\mathcal{N}}\left((\mathrm{Ch}_{\mathbb{Q}}^{+})^{\Sigma}\right)$$

where $\mathcal{N} = N(\mathrm{Sym}(\widetilde{\mathbb{Q}}S^1))$. There is a ring map $\varphi\colon \mathrm{Sym}(\mathbb{Q}[1]) \to \mathcal{N}$ induced levelwise by the lax symmetric monoidal structure on N, and therefore composing N and φ^* gives a functor $\varphi^*N\colon \mathrm{Sp}^{\Sigma}(\mathrm{s}\mathbb{Q}\text{-mod}) \to \mathrm{Sp}^{\Sigma}(\mathrm{Ch}_{\mathbb{Q}}^{+})$. This functor has a left adjoint denoted L by [34, §3.3]. It is important to note that the left adjoint is not just the composite of the left adjoints of N and φ^*. Shipley [39, 4.4] shows that

$$\mathrm{Sp}^{\Sigma}(\mathrm{s}\mathbb{Q}\text{-mod}) \underset{\varphi^*N}{\overset{L}{\leftrightarrows}} \mathrm{Sp}^{\Sigma}(\mathrm{Ch}_{\mathbb{Q}}^{+})$$

is a weak symmetric monoidal Quillen equivalence.

The final step is the passage from symmetric spectra in non-negatively graded chain complexes to unbounded chain complexes. The inclusion $\mathrm{Ch}_{\mathbb{Q}}^{+} \to \mathrm{Ch}_{\mathbb{Q}}$ of non-negatively graded chain complexes into unbounded complexes has a right adjoint C_0 called the connective cover. This is defined by $(C_0X)_n = X_n$ for $n \geq 1$ and $(C_0X)_0 = \mathrm{cycles}(X_0)$. Using the connective cover, one defines a functor $R\colon \mathrm{Ch}_{\mathbb{Q}} \to \mathrm{Sp}^{\Sigma}(\mathrm{Ch}_{\mathbb{Q}}^{+})$ by $(RY)_n = C_0(Y \otimes \mathbb{Q}[n])$. Recall from [39] that this functor has a left adjoint D. Moreover, D is strong symmetric monoidal as proved by Strickland [41]. Note that this fact has been subject to some confusion, see [38]. Shipley [39, 4.7] shows that

$$\mathrm{Sp}^{\Sigma}(\mathrm{Ch}_{\mathbb{Q}}^{+}) \underset{R}{\overset{D}{\rightleftarrows}} \mathrm{Ch}_{\mathbb{Q}}$$

is a strong symmetric monoidal Quillen equivalence where $\mathrm{Ch}_{\mathbb{Q}}$ is equipped with the projective model structure.

Combining the results of this section gives a proof of Theorem 1.1.

6. Extension to commutative algebras

Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a weak symmetric monoidal Quillen pair. As G is lax symmetric monoidal, it preserves commutative monoids and therefore gives rise to a functor $G\colon \mathrm{CMon}(\mathcal{D}) \to \mathrm{CMon}(\mathcal{C})$. If the Quillen pair is a *strong* symmetric monoidal Quillen pair, then F also lifts to a functor on commutative monoids. However, when F is only oplax symmetric monoidal, it will not necessarily preserve commutative monoids.

We always equip the category of commutative monoids with the model structure right lifted along the forgetful functor, see Theorem 2.5. The forgetful functor $U\colon \mathrm{CMon}(\mathcal{C}) \to \mathcal{C}$ has a left adjoint given by

$$\mathbb{P}_{\mathcal{C}}(X) = \bigvee_{n\geq 0} X^{\wedge n}/\Sigma_n.$$

The adjoint lifting theorem [8, 4.5.6] implies that the lift of G to the categories of commutative monoids has a left adjoint $\widetilde{F}$ defined by the coequalizer diagram

$$\mathbb{P}_{\mathcal{D}}F\mathbb{P}_{\mathcal{C}}X \rightrightarrows \mathbb{P}_{\mathcal{D}}FX \longrightarrow \widetilde{F}X.$$

One of the maps is obtained from the counit of the $\mathbb{P}_{\mathcal{C}} \dashv U$ adjunction, and the other map is adjunct to the natural map

$$F\mathbb{P}_{\mathcal{C}}X \cong \bigvee_{n\geq 0} F(X^{\wedge n})/\Sigma_n \to \bigvee_{n\geq 0} (FX)^{\wedge n}/\Sigma_n \cong \mathbb{P}_{\mathcal{D}}FX$$

obtained from the oplax structure on F. Since G preserves commutative monoids, there is a natural isomorphism $UG \cong GU$ and by adjunction there is a natural isomorphism

$$\mathbb{P}_{\mathcal{D}}F \cong \widetilde{F}\mathbb{P}_{\mathcal{C}}.$$

Before we can state a theorem about lifting weak symmetric monoidal Quillen equivalences to Quillen equivalences on commutative monoids, we need to impose a hypothesis.

Hypothesis 6.1. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a weak symmetric monoidal Quillen equivalence. For any cofibrant object X of $\mathcal{C}$, the natural map

$$F(X^{\wedge n})/\Sigma_n \to (FX)^{\wedge n}/\Sigma_n$$

is a weak equivalence in $\mathcal{D}$.

Lemma 6.2. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a weak symmetric monoidal Quillen equivalence. This satisfies Hypothesis 6.1 if either of the following conditions hold:*

(i) *$F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ is a strong symmetric monoidal Quillen equivalence;*
(ii) *underlying cofibrant objects in $\Sigma_n\mathcal{D}$ are naive cofibrant.*

Proof. The first part follows immediately from the definition. For the second part, let X be cofibrant in $\mathcal{C}$. By definition of a weak symmetric monoidal Quillen pair, the natural map $F(X^{\wedge n}) \to (FX)^{\wedge n}$ is a weak equivalence between cofibrant objects in the injective model structure on $\Sigma_n\mathcal{D}$. By hypothesis, this is moreover a naive weak equivalence between naive cofibrant objects. From the description of the generating (acyclic) cofibrations given in Proposition 4.3 one can see that the orbits functor $(-)/\Sigma_n\colon \Sigma_n\mathcal{D} \to \mathcal{D}$ is left Quillen when $\Sigma_n\mathcal{D}$ is equipped with the genuine model structure. Since the identity is a left Quillen functor from the naive model structure on $\Sigma_n\mathcal{D}$ to the genuine model structure, it follows that $(-)/\Sigma_n\colon \Sigma_n\mathcal{D} \to \mathcal{D}$ is left Quillen when $\Sigma_n\mathcal{D}$ is equipped with the naive model structure. By Ken Brown's lemma, it then follows that $F(X^{\wedge n})/\Sigma_n \to (FX)^{\wedge n}/\Sigma_n$ is a weak equivalence in $\mathcal{D}$. □

We now state when weak symmetric monoidal Quillen equivalences lift to Quillen equivalences between the categories of commutative monoids. This result is closely related to work of Schwede-Shipley, White and White-Yau. Schwede-Shipley [34, 3.12(3)] consider the related question on associative monoids without the commutativity assumption, White [42, 4.19] provides hypotheses under which *strong* monoidal Quillen equivalences lift to the categories of commutative monoids and White-Yau [43, 5.8] provide hypotheses under which *weak* monoidal Quillen equivalences lift. The most general of the statements is that of White-Yau where the result follows from a more general result about lifting Quillen equivalences to categories of coloured operads.

For orientation in the following statement and proof, the reader might like to consider $\mathcal{C}$ being the positive flat model structure on spectra and $\widetilde{\mathcal{C}}$ being the flat model structure on spectra. The hypotheses are designed in such a way that this example fits into the framework. We note that we write left adjoint functors on the left in an adjoint pair displayed vertically.

Theorem 6.3. *Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : G$ be a weak symmetric monoidal Quillen equivalence between cofibrantly generated model categories which satisfy the commutative monoid axiom and the monoid axiom. Suppose that the underlying categories of $\mathcal{C}$ and $\mathcal{D}$ support other model structures denoted $\widetilde{\mathcal{C}}$ and $\widetilde{\mathcal{D}}$ respectively, with the same weak equivalences, such that*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] \arrow[d, shift right, "1"'] & \mathcal{D} \arrow[l, shift left, "G"] \arrow[d, shift right, "1"'] \\ \widetilde{\mathcal{C}} \arrow[r, shift left, "F"] \arrow[u, shift right, "1"'] & \widetilde{\mathcal{D}} \arrow[l, shift left, "G"] \arrow[u, shift right, "1"'] \end{tikzcd}$$

are all Quillen adjunctions. Suppose that cofibrant commutative monoids in $\mathcal{C}$ (resp. $\mathcal{D}$) are cofibrant in $\widetilde{\mathcal{C}}$ (resp. $\widetilde{\mathcal{D}}$), the generating cofibrations I of $\mathcal{C}$ have cofibrant source (and hence target), the monoidal unit $\mathbb{1}_{\mathcal{C}}$ of $\mathcal{C}$ is cofibrant and that Hypothesis 6.1 is satisfied. Then there is a Quillen equivalence

$$\widetilde{F} : \mathrm{CMon}(\mathcal{C}) \rightleftarrows \mathrm{CMon}(\mathcal{D}) : G.$$

Proof. Since the model structures are right lifted, G preserves fibrations and acyclic fibrations and therefore is right Quillen as a functor $\mathrm{CMon}(\mathcal{D}) \to \mathrm{CMon}(\mathcal{C})$. Let A be a cofibrant commutative monoid in $\mathcal{C}$ and B be a fibrant commutative monoid in $\mathcal{D}$. We must show that the map $A \to GB$ is a weak equivalence in $\mathcal{C}$ if and only if $\widetilde{F}A \to B$ is a weak equivalence in $\mathcal{D}$.

The adjunction unit of $\widetilde{F} \dashv G$ gives rise to a map $UA \to UG\widetilde{F}A \cong GU\widetilde{F}A$ and hence by adjunction there is a natural map $FA \to \widetilde{F}A$ where we neglect to write the forgetful functors. The composite $FA \to \widetilde{F}A \to B$ is adjunct to the map $A \to GB$ in $\mathcal{C}$.

Let $X \in \mathcal{D}$. Write fX for a fibrant replacement of X in $\mathcal{D}$ and $\widetilde{f}X$ for a fibrant replacement of X in $\widetilde{\mathcal{D}}$. Consider the square

$$\begin{tikzcd} X \arrow[r] \arrow[d] & \widetilde{f}X \arrow[d] \\ fX \arrow[r] & * \end{tikzcd}$$

in which the left vertical arrow is an acyclic cofibration in $\mathcal{D}$, and the right vertical is a fibration in $\widetilde{\mathcal{D}}$ and hence in $\mathcal{D}$. By lifting properties, we obtain a map $fX \to \widetilde{f}X$ which is a weak equivalence.

We must show that the map $A \to GB$ is a weak equivalence in $\mathcal{C}$ if and only if $\widetilde{F}A \to B$ is a weak equivalence in $\mathcal{D}$, where A is cofibrant in $\mathrm{CMon}(\mathcal{C})$ and B is fibrant in $\mathrm{CMon}(\mathcal{D})$. By the previous paragraph, we have a weak equivalence $B \to \widetilde{f}B$ where $\widetilde{f}B$ is fibrant in $\widetilde{\mathcal{D}}$ and hence in $\mathcal{D}$. By Ken Brown's lemma, $GB \to G\widetilde{f}B$ is a weak equivalence, and therefore $A \to GB$ is a weak equivalence if and only if $A \to G\widetilde{f}B$ is a weak equivalence.

Note that since $\mathcal{C}$ and $\widetilde{\mathcal{C}}$ have the same weak equivalences, the identity functor $\mathcal{C} \to \widetilde{\mathcal{C}}$ is a left Quillen equivalence, and similarly for $\mathcal{D}$. Therefore, by the 2-out-of-3 property of Quillen equivalences, $F : \widetilde{\mathcal{C}} \rightleftarrows \widetilde{\mathcal{D}} : G$ is a Quillen equivalence. Since A is cofibrant in $\mathrm{CMon}(\mathcal{C})$ and hence in $\widetilde{\mathcal{C}}$, and $\tilde{f}B$ is fibrant in $\widetilde{\mathcal{D}}$, $A \to G\tilde{f}B$ is a weak equivalence if and only if $FA \to \tilde{f}B$ is a weak equivalence. Since $B \to \tilde{f}B$ is a weak equivalence, $FA \to \tilde{f}B$ is a weak equivalence if and only if $FA \to B$ is a weak equivalence. Since the composite $FA \to \widetilde{F}A \to B$ is adjunct to the map $A \to GB$ in $\mathcal{C}$, it follows that it is enough to show that $\lambda_A\colon FA \to \widetilde{F}A$ is a weak equivalence.

As $\mathcal{C}$ is cofibrantly generated, A is a retract of a $\mathbb{P}_{\mathcal{C}}(I)$-cell complex where I is the set of generating cofibrations for $\mathcal{C}$, i.e., $\emptyset \to A$ is a retract of a transfinite composition of pushouts of maps in $\mathbb{P}_{\mathcal{C}}(I)$. We proceed by transfinite induction on the transfinite composition which defines a cofibrant object. The base case is the claim that $F(\mathbb{1}_{\mathcal{C}}) \to \widetilde{F}(\mathbb{1}_{\mathcal{C}})$ is a weak equivalence. The left adjoint $\widetilde{F}$ takes the initial object $\mathbb{1}_{\mathcal{C}}$ of $\mathrm{CMon}(\mathcal{C})$ to the initial object $\mathbb{1}_{\mathcal{D}}$ of $\mathrm{CMon}(\mathcal{D})$. Since $\mathbb{1}_{\mathcal{C}}$ is cofibrant, $F(\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ is a weak equivalence by the unit axiom of the weak monoidal Quillen adjunction $F \dashv G$. Therefore the base case holds.

Write $\mathbb{P}^n X = X^{\wedge n}/\Sigma_n$, so that $\mathbb{P}X = \vee_{n\geq 0}\mathbb{P}^n X$. By Hypothesis 6.1, if X is a cofibrant object of $\mathcal{C}$, $F(\mathbb{P}^n_{\mathcal{C}}X) = F(X^{\wedge n})/\Sigma_n \to (FX)^{\wedge n}/\Sigma_n = \mathbb{P}^n_{\mathcal{D}}(FX)$ is a weak equivalence. Since X is cofibrant in $\mathcal{C}$, $\mathbb{P}^n_{\mathcal{C}}X$ is cofibrant in $\widetilde{\mathcal{C}}$ and therefore $F(\mathbb{P}^n_{\mathcal{C}}X)$ is cofibrant in $\widetilde{\mathcal{D}}$. In a similar way, one sees that $\mathbb{P}^n_{\mathcal{D}}(FX)$ is cofibrant in $\widetilde{\mathcal{D}}$. Therefore $F(X^{\wedge n})/\Sigma_n \to (FX)^{\wedge n}/\Sigma_n$ is a weak equivalence between cofibrant objects in $\widetilde{\mathcal{D}}$ and Ken Brown's lemma shows that taking coproducts preserves this weak equivalence. Therefore,

$$F\mathbb{P}_{\mathcal{C}}X = F\bigvee_{n\geq 0} X^{\wedge n}/\Sigma_n \cong \bigvee_{n\geq 0} F(X^{\wedge n})/\Sigma_n \xrightarrow{\sim} \bigvee_{n\geq 0} (FX)^{\wedge n}/\Sigma_n = \mathbb{P}_{\mathcal{D}}FX$$

is a weak equivalence. Hence using the isomorphism $\mathbb{P}_{\mathcal{D}}FX \cong \widetilde{F}\mathbb{P}_{\mathcal{C}}X$, if X is cofibrant in $\mathcal{C}$, one sees that both $F\mathbb{P}_{\mathcal{C}}X \to \widetilde{F}\mathbb{P}_{\mathcal{C}}X$ and $F\mathbb{P}^n_{\mathcal{C}}X \to \widetilde{F}\mathbb{P}^n_{\mathcal{C}}X$ are weak equivalences.

We now prove that if

$$\begin{tikzcd} \mathbb{P}_{\mathcal{C}}X \arrow[r] \arrow[d] & Y \arrow[d, "f"] \\ \mathbb{P}_{\mathcal{C}}X' \arrow[r] & P \end{tikzcd}$$

is a pushout square in $\mathrm{CMon}(\mathcal{C})$, and $FY \to \widetilde{F}Y$ is a weak equivalence where Y is cofibrant, then $FP \to \widetilde{F}P$ is a weak equivalence. Since I consists of cofibrations with cofibrant source, we may assume that X and X' are cofibrant in $\mathcal{C}$. By [42, B.2], $f\colon Y \to P$ has a filtration

$$Y = P_0 \to P_1 \to \cdots$$

where $P_{n-1} \to P_n$ is defined by the pushout

$$\begin{tikzcd} Y \wedge Q_n(f)/\Sigma_n \arrow[r] \arrow[d] & P_{n-1} \arrow[d] \\ Y \wedge \mathbb{P}^n_{\mathcal{C}}X' \arrow[r] & P_n \end{tikzcd}$$

in $\mathcal{C}$. For our purposes, it is not important precisely what $Q_n(f)$ is, apart from the fact that it is a colimit of a punctured n-dimensional cube whose vertices are given by tensor products of X, X' and Y. It follows from the commutative monoid axiom that $Q_n(f)/\Sigma_n$ is cofibrant in $\mathcal{C}$, see [42, Proof of 4.17] for details.

Since F sends pushouts in $\mathcal{C}$ to pushouts in $\mathcal{D}$,

$$\begin{tikzcd} F(Y \wedge Q_n(f)/\Sigma_n) \arrow[r] \arrow[d] & FP_{n-1} \arrow[d] \\ F(Y \wedge \mathbb{P}^n_{\mathcal{C}}X') \arrow[r] & FP_n \end{tikzcd}$$

is a pushout in $\mathcal{D}$. Since $\widetilde{F}$ preserves pushouts of commutative monoids,

$$\begin{tikzcd}
\mathbb{P}_{\mathcal{D}}FX \arrow[r] \arrow[d] & \widetilde{F}Y \arrow[d, "\widetilde{F}f"] \\
\mathbb{P}_{\mathcal{D}}FX' \arrow[r] & \widetilde{F}P
\end{tikzcd}$$

is a pushout in $\mathrm{CMon}(\mathcal{D})$ using the isomorphism $\mathbb{P}_{\mathcal{D}}F \cong \widetilde{F}\mathbb{P}_{\mathcal{C}}$. Applying [42, B.2] again, we obtain a filtration $\widetilde{F}Y = R_0 \to R_1 \to \cdots$ of $\widetilde{F}f\colon \widetilde{F}Y \to \widetilde{F}P$ where $R_{n-1} \to R_n$ is defined by the pushout

$$\begin{tikzcd}
\widetilde{F}Y \wedge Q_n(\widetilde{F}f)/\Sigma_n \arrow[r] \arrow[d] & R_{n-1} \arrow[d] \\
\widetilde{F}Y \wedge \mathbb{P}^n_{\mathcal{D}}FX' \arrow[r] & R_n
\end{tikzcd}$$

in $\mathcal{D}$. This filtration is compatible with the filtration of $Y \to P$ and therefore λ_Y sends FP_n to R_n. By applying F to the pushout square shown Diagram 1 in [42, Proof of A.1] and using that $F \dashv G$ is a weak monoidal Quillen pair, one argues by induction that there is a natural weak equivalence $FQ_n(f) \xrightarrow{\sim} Q_n(\widetilde{F}f)$. Similarly, since taking orbits commutes with taking pushouts, there is a natural weak equivalence $FQ_n(f)/\Sigma_n \xrightarrow{\sim} Q_n(\widetilde{F}f)/\Sigma_n$ by an inductive argument on Diagram 6 in [42, Proof of A.3].

We now show by induction that $\lambda_{P_n}\colon FP_n \to R_n$ is a weak equivalence. The base case holds since $\lambda_{P_0} = \lambda_Y$ which was a weak equivalence by assumption. Suppose that $\lambda_{P_{n-1}}$ is a weak equivalence. Consider the diagram

$$\begin{tikzcd}
F(Y \wedge Q_n(f)/\Sigma_n) \arrow[rr] \arrow[dr] \arrow[dd] & & \widetilde{F}Y \wedge Q_n(\widetilde{F}f)/\Sigma_n \arrow[dr] \arrow[dd] & \\
& FP_{n-1} \arrow[rr, crossing over] & & R_{n-1} \arrow[dd] \\
F(Y \wedge \mathbb{P}^n_{\mathcal{C}}X') \arrow[rr] \arrow[dr] & & \widetilde{F}Y \wedge \mathbb{P}^n_{\mathcal{D}}FX' \arrow[dr] & \\
& FP_n \arrow[rr] \arrow[from=uu, crossing over] & & R_n
\end{tikzcd}$$

in which the leftmost face and the rightmost face are pushouts in $\mathcal{D}$. The horizontal map $FP_{n-1} \to R_{n-1}$ is a weak equivalence by the inductive hypothesis.

The horizontal map $F(Y \wedge Q_n(f)/\Sigma_n) \to \widetilde{F}Y \wedge Q_n(\widetilde{F}f)/\Sigma_n$ factors as the composite

$$F(Y \wedge Q_n(f)/\Sigma_n) \to FY \wedge FQ_n(f)/\Sigma_n \to \widetilde{F}Y \wedge FQ_n(f)/\Sigma_n \to \widetilde{F}Y \wedge Q_n(\widetilde{F}f)/\Sigma_n$$

where the first map is a weak equivalence since $F \dashv G$ is a weak monoidal Quillen pair. The map $FY \to \widetilde{F}Y$ is a weak equivalence between cofibrant objects in $\widetilde{\mathcal{D}}$, and the map $FQ_n(f)/\Sigma_n \xrightarrow{\sim} Q_n(\widetilde{F}f)/\Sigma_n$ is a weak equivalence between cofibrant objects in $\mathcal{D}$. Since cofibrant objects in $\mathcal{D}$ are also cofibrant in $\widetilde{\mathcal{D}}$, both of these maps are weak equivalences between cofibrant objects in $\widetilde{\mathcal{D}}$. By Ken Brown's lemma, tensoring with cofibrant objects preserves weak equivalences between cofibrant objects, and hence the second and third map are weak equivalences.

The horizontal map $F(Y \wedge \mathbb{P}^n_{\mathcal{C}}X') \to \widetilde{F}Y \wedge \mathbb{P}^n_{\mathcal{D}}FX'$ is a weak equivalence since it factors as the composite

$$F(Y \wedge \mathbb{P}^n_{\mathcal{C}}X') \to FY \wedge F\mathbb{P}^n_{\mathcal{C}}X' \to \widetilde{F}Y \wedge \widetilde{F}\mathbb{P}^n_{\mathcal{C}}X' \cong \widetilde{F}Y \wedge \mathbb{P}^n_{\mathcal{D}}FX'.$$

Therefore, the map $FP_n \to R_n$ is a weak equivalence by [19, 5.2.6]. Each filtration map is a cofibration between cofibrant objects and hence by Ken Brown's lemma and [19, 5.1.5], the map $FP \to \widetilde{F}P$ is a weak equivalence.

It remains to show that the property is preserved under the transfinite compositions used to build relative cell complexes which again follows from [19, 5.1.5]. Therefore for any cofibrant commutative monoid object A of $\mathcal{C}$, we have that the map $FA \to \widetilde{F}A$ is a weak equivalence which concludes the proof. $\square$

Remark 6.4. The hypothesis that $\mathcal{C}$ and $\mathcal{D}$ satisfy the commutative monoid axiom and the monoid axiom ensures that the categories of commutative monoids inherit a right lifted model structure [42, 3.2].

Remark 6.5. In some cases such as rational chain complexes, cofibrant commutative algebras are cofibrant as modules, see Example 2.7. In such examples, one can take $\mathcal{C} = \widetilde{\mathcal{C}}$ in the previous theorem.

Theorem 6.6. *There is a zig-zag of Quillen equivalences between the category of commutative $H\mathbb{Q}$-algebras and the category of commutative rational DGAs.*

Proof. Consider the adjunctions

$$\mathrm{Mod}_{H\mathbb{Q}}^{\mathrm{pf}} \underset{U}{\overset{Z}{\rightleftarrows}} \mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})_{\mathrm{pf}} \underset{\varphi^* N}{\overset{L}{\leftrightarrows}} \mathrm{Sp}^\Sigma(\mathrm{Ch}_{\mathbb{Q}}^+)_{\mathrm{pf}} \underset{R}{\overset{D}{\rightleftarrows}} \mathrm{Ch}_{\mathbb{Q}}$$

where pf denotes the positive flat model structure. Recall from Corollary 4.13 that on $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ and $\mathrm{Sp}^\Sigma(\mathrm{Ch}_{\mathbb{Q}}^+)$ the positive stable and positive flat model structures are the same.

Firstly, we must justify that these are Quillen adjunctions. The adjunction $D \dashv R$ is Quillen since it can be viewed as the composite of Quillen adjunctions

$$\mathrm{Sp}^\Sigma(\mathrm{Ch}_{\mathbb{Q}}^+)_{\mathrm{pf}} \underset{1}{\overset{1}{\rightleftarrows}} \mathrm{Sp}^\Sigma(\mathrm{Ch}_{\mathbb{Q}}^+) \underset{R}{\overset{D}{\rightleftarrows}} \mathrm{Ch}_{\mathbb{Q}}$$

where the second adjunction was proved to be Quillen in [39, 4.7].

For the adjunction $Z \dashv U$, by Proposition 2.3 it is sufficient to check that the right adjoint U preserves acyclic positive flat fibrations and positive flat fibrations between positive flat fibrants. Recall that a map is an acyclic positive flat fibration if and only if it is a levelwise acyclic blended fibration for levels $n > 0$, and that a map is a positive flat fibration between positive flat fibrants if and only if it is a levelwise blended fibration for levels $n > 0$ between positively flat fibrant objects. By [28, 3.2.1], an object X of $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is positively flat fibrant if and only if X is levelwise blended fibrant for levels $n > 0$ and $X_n \to \underline{\mathrm{Hom}}(K, X_{n+1})$ is a naive weak equivalence for all $n \geq 0$ where $\underline{\mathrm{Hom}}(K, -)$ is the right adjoint to $K \otimes -$. One notes that all the conditions that must be checked, except for the last condition, are all levelwise. Therefore, applying the arguments given in Lemma 5.1 and to levels $n > 0$ verifies the necessary levelwise conditions. The remaining condition that $X_n \to \underline{\mathrm{Hom}}(K, X_{n+1})$ is a naive weak equivalence is unchanged between the flat model structure and the positive flat model structure. This condition was also verified in Lemma 5.1. For the $L \dashv \varphi^* N$ adjunction one can argue similarly, using [39, 4.4].

We now apply Theorem 6.3. For each of the categories of symmetric spectra, we take $\mathcal{C}$ to be the version equipped with the positive flat model structure, and $\widetilde{\mathcal{C}}$ to be equipped with the flat model structure. For the category of chain complexes, we take $\mathcal{C} = \widetilde{\mathcal{C}}$. In each case, cofibrant commutative algebras forget to flat cofibrant modules by Corollary 4.11. Hypothesis 6.1 holds for the first and last adjunctions since they are strong symmetric monoidal Quillen equivalences and therefore they give Quillen equivalences on the commutative monoids by Theorem 6.3.

For the $L \dashv \varphi^* N$ adjunction, we argue that condition (ii) in Lemma 6.2 holds. We show that for a finite group G, if $f\colon X \to Y$ is an underlying cofibration in $G\text{-}\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})_{\mathrm{pf}}$ then f is a naive cofibration in $G\text{-}\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})_{\mathrm{pf}}$. A G-object X in $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ consists of $G \times \Sigma_n$-objects $X(n)$

in $s\mathbb{Q}$-mod with $G\times\Sigma_n$-equivariant structure maps. Similarly, a map $\varphi\colon X\to Y$ between objects in G-$\mathrm{Sp}^\Sigma(s\mathbb{Q}\text{-mod})$ consists of a collection of $G\times\Sigma_n$-equivariant maps $\varphi(n)\colon X(n)\to Y(n)$ making the evident diagrams commute.

Write U for the forgetful functor G-$\mathrm{Sp}^\Sigma(s\mathbb{Q}\text{-mod})\to\mathrm{Sp}^\Sigma(s\mathbb{Q}\text{-mod})$. Suppose that $f\colon X\to Y$ is an underlying cofibration in G-$\mathrm{Sp}^\Sigma(s\mathbb{Q}\text{-mod})_{\mathrm{pf}}$, i.e., $Uf\colon UX\to UY$ is a positive flat cofibration, and that $p\colon A\to B$ is a naive acyclic fibration in G-$\mathrm{Sp}^\Sigma(s\mathbb{Q}\text{-mod})_{\mathrm{pf}}$, i.e., Up is an acyclic positive flat fibration. Therefore Uf has the left lifting property with respect to Up. It remains to argue that the lift $\theta\colon UY\to UA$ can be made into an equivariant map $\varphi\colon Y\to A$. The lift $\theta\colon UY\to UA$ is a collection $\theta(n)\colon Y(n)\to A(n)$ of Σ_n-equivariant maps. Since the maps are determined levelwise, one can apply the averaging method as in the proof of Proposition 4.6 to construct $G\times\Sigma_n$-equivariant maps $\varphi(n)\colon Y(n)\to A(n)$ and it follows that φ is a map in G-$\mathrm{Sp}^\Sigma(s\mathbb{Q}\text{-mod})$ which is also a lift. Therefore, f is a naive cofibration in G-$\mathrm{Sp}^\Sigma(s\mathbb{Q}\text{-mod})$. Hence by Theorem 6.3, the middle Quillen equivalence also lifts to the commutative monoids. $\square$

7. A symmetric monoidal equivalence for modules

In this section, we give a symmetric monoidal Quillen equivalence between the categories of modules over a commutative $H\mathbb{Q}$-algebra and a commutative DGA. We note that this result has been assumed without proof in the literature; for more details see the introduction. We firstly explain why this result is not an immediate corollary of the zig-zag of Quillen equivalences $\mathrm{Mod}_{H\mathbb{Q}}\simeq_Q \mathrm{Ch}_{\mathbb{Q}}$.

Let $F:\mathcal{C}\rightleftarrows\mathcal{D}:G$ be a strong symmetric monoidal Quillen equivalence and suppose that the unit objects of $\mathcal{C}$ and $\mathcal{D}$ are cofibrant. If S is a *cofibrant* monoid in $\mathcal{C}$, Schwede-Shipley [34, 3.12] show that $F:\mathrm{Mod}_S(\mathcal{C})\rightleftarrows\mathrm{Mod}_{FS}(\mathcal{D}):G$ is a Quillen equivalence. Now suppose that S is a commutative monoid in $\mathcal{C}$, which is not cofibrant as a monoid. Since S is commutative, the category $\mathrm{Mod}_S(\mathcal{C})$ of modules is symmetric monoidal, with tensor product defined by the coequalizer of the two maps

$$M\otimes_S N=\mathrm{coeq}(M\otimes S\otimes N \rightrightarrows M\otimes N)$$

defined by the action of S on M and N.

However, a cofibrant replacement $q\colon cS\xrightarrow{\sim}S$ as a monoid will no longer be commutative, and hence the zig-zag of Quillen equivalences

$$\begin{tikzcd}\mathrm{Mod}_S(\mathcal{C}) \arrow[r, shift right, "q^*"'] & \mathrm{Mod}_{cS}(\mathcal{C}) \arrow[l, shift right, "-\otimes_{cS}S"'] \arrow[r, shift left, "F"] & \mathrm{Mod}_{FcS}(\mathcal{D}) \arrow[l, shift left, "G"]\end{tikzcd}$$

cannot be symmetric monoidal. We explain how to rectify this.

Before we can prove the desired symmetric monoidal Quillen equivalence, we require an abstract lemma about lifting symmetric monoidal Quillen equivalences to the categories of modules. We note that this first statement is a counterpart to [34, 3.12(2)]. The proof is effectively the same.

Lemma 7.1. *Let*

$$\begin{tikzcd}\mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "G"]\end{tikzcd}$$

be a strong symmetric monoidal Quillen equivalence and let S be a commutative monoid in $\mathcal{C}$. Suppose that $\mathcal{C}$ and $\mathcal{D}$ satisfy the monoid axiom. If F preserves all weak equivalences and Quillen invariance holds in $\mathcal{C}$ and $\mathcal{D}$, then

$$\begin{tikzcd}\mathrm{Mod}_S \arrow[r, shift left, "F"] & \mathrm{Mod}_{FS} \arrow[l, shift left, "G"]\end{tikzcd}$$

is a strong symmetric monoidal Quillen equivalence.

Proof. Let $q\colon cS \to S$ be a cofibrant replacement of S as a monoid in $\mathcal{C}$. As F preserves all weak equivalences $Fq\colon FcS \to FS$ is a weak equivalence. Consider the diagram of left Quillen functors

$$\begin{tikzcd} \mathrm{Mod}_S \arrow[d, "F"'] & \mathrm{Mod}_{cS} \arrow[l, "S\otimes_{cS}-"'] \arrow[d, "F"] \\ \mathrm{Mod}_{FS} & \mathrm{Mod}_{FcS} \arrow[l, "FS\otimes_{FcS}-"] \end{tikzcd}$$

which is commutative since F is strong monoidal. By [34, 3.12(1)] the right hand vertical is a Quillen equivalence, and by Quillen invariance the horizontals are Quillen equivalences. Hence by 2-out-of-3 the left vertical is a Quillen equivalence as required. As a functor between the module categories, F is strong symmetric monoidal since the tensor product in the module category Mod_S is defined by a coequalizer which F preserves. Therefore

$$\mathrm{Mod}_S \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{Mod}_{FS}$$

is a strong symmetric monoidal Quillen equivalence. □

We recall from Shipley [39, 1.2] the zig-zag of natural weak equivalences between Zc and $\alpha^*\widetilde{\mathbb{Q}}$ where α is the ring map $H\mathbb{Q} \to \widetilde{\mathbb{Q}}H\mathbb{Q}$ induced by the unit of the monad structure on $\widetilde{\mathbb{Q}}$. Let $\beta\colon \widetilde{\mathbb{Q}}H\mathbb{Q} \to H\mathbb{Q}$ be the ring map induced by the multiplication map of the monad structure.

We have $Zc = \beta_*\widetilde{\mathbb{Q}}c \cong \alpha^*\beta^*\beta_*\widetilde{\mathbb{Q}}c$ since $\beta\alpha = 1$. There is then a natural map $\alpha^*\widetilde{\mathbb{Q}}c \to \alpha^*\beta^*\beta_*\widetilde{\mathbb{Q}}c$ arising from the unit of the $\beta_* \dashv \beta^*$ adjunction. This is a weak equivalence since $\widetilde{\mathbb{Q}}$ preserves cofibrant objects, the $\beta_* \dashv \beta^*$ adjunction is a Quillen equivalence and α^* preserves all weak equivalences. Finally there is a natural map $\alpha^*\widetilde{\mathbb{Q}}c \to \alpha^*\widetilde{\mathbb{Q}}$ which is a weak equivalence as α^* and $\widetilde{\mathbb{Q}}$ preserve all weak equivalences. We can now apply the previous lemma to obtain the desired statement.

Theorem 7.2. *Let A be a commutative $H\mathbb{Q}$-algebra. There are zig-zags of weak symmetric monoidal Quillen equivalences*

$$\mathrm{Mod}_A^{\mathrm{stable}} \simeq_Q \mathrm{Mod}_{\underline{\Theta}A} \quad \textit{and} \quad \mathrm{Mod}_A^{\mathrm{flat}} \simeq_Q \mathrm{Mod}_{\underline{\Theta}A}$$

where $\underline{\Theta}A = D\varphi^*N\alpha^*\widetilde{\mathbb{Q}}A$ is a commutative DGA.

Proof. The proof for each part of the theorem follows the same method. Namely, we apply [34, 3.12(2)] together with Lemma 7.1 to the underlying Quillen equivalences given by Shipley [39] in the stable case, and given by Theorem 1.1 in the flat case. Since the weak equivalences in both the stable model structure and the flat model structures are the same, the following proof applies in both cases.

The first step is the adjunction

$$\mathrm{Mod}_A(\mathrm{Mod}_{H\mathbb{Q}}) \underset{U}{\overset{\widetilde{\mathbb{Q}}}{\rightleftarrows}} \mathrm{Mod}_{\widetilde{\mathbb{Q}}A}\left(\mathrm{Mod}_{\widetilde{\mathbb{Q}}H\mathbb{Q}}\right).$$

Since $\widetilde{\mathbb{Q}}$ preserves all weak equivalences, this is a strong symmetric monoidal Quillen adjunction by Lemma 7.1.

Recall that there is a ring map $\alpha\colon H\mathbb{Q} \to \widetilde{\mathbb{Q}}H\mathbb{Q}$. Since α^* is lax symmetric monoidal it gives rise to a functor

$$\mathrm{Mod}_{\widetilde{\mathbb{Q}}A}\left(\mathrm{Mod}_{\widetilde{\mathbb{Q}}H\mathbb{Q}}\right) \xrightarrow{\alpha^*} \mathrm{Mod}_{\alpha^*\widetilde{\mathbb{Q}}A}(\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})).$$

It follows from [34, §3.3] that the left adjoint to α^* at the level of modules, is given by $\alpha_*^{\widetilde{\mathbb{Q}}A}(M) = \widetilde{\mathbb{Q}}A \otimes_{\alpha^*\alpha_*\widetilde{\mathbb{Q}}A} \alpha_* M$. We claim that $\alpha_*^{\widetilde{\mathbb{Q}}A}$ is strong monoidal. As α_* preserves colimits and is strong monoidal, we have

$$\begin{aligned}
\alpha_*(M \otimes_{\alpha^*\widetilde{\mathbb{Q}}A} N) &= \alpha_* \mathrm{coeq}(M \otimes \alpha^*\widetilde{\mathbb{Q}}A \otimes N \rightrightarrows M \otimes N) \\
&\cong \mathrm{coeq}(\alpha_* M \otimes \alpha_*\alpha^*\widetilde{\mathbb{Q}}A \otimes \alpha_* N \rightrightarrows \alpha_* M \otimes \alpha_* N) \\
&= \alpha_* M \otimes_{\alpha_*\alpha^*\widetilde{\mathbb{Q}}A} \alpha_* N.
\end{aligned}$$

From this, one sees that

$$\alpha_*^{\widetilde{\mathbb{Q}}A}(M \otimes_{\alpha^*\widetilde{\mathbb{Q}}A} N) \cong \alpha_*^{\widetilde{\mathbb{Q}}A}(M) \otimes_{\widetilde{\mathbb{Q}}A} \alpha_*^{\widetilde{\mathbb{Q}}A}(N)$$

and hence $\alpha_*^{\widetilde{\mathbb{Q}}A}$ is strong symmetric monoidal. Since α^* preserves all weak equivalences, it follows from [34, 3.12(2)] that

$$\mathrm{Mod}_{\widetilde{\mathbb{Q}}A}\left(\mathrm{Mod}_{\widetilde{\mathbb{Q}}H\mathbb{Q}}\right) \underset{\alpha^*}{\overset{\alpha_*^{\widetilde{\mathbb{Q}}A}}{\rightleftarrows}} \mathrm{Mod}_{\alpha^*\widetilde{\mathbb{Q}}A}(\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod}))$$

is a strong symmetric monoidal Quillen equivalence.

The next step is the passage along the Dold-Kan type equivalence. Recall that applying the normalization functor levelwise gives a lax monoidal functor

$$\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod}) \to \mathrm{Mod}_{\mathcal{N}}\left((\mathrm{Ch}_{\mathbb{Q}}^+)^\Sigma\right)$$

where $\mathcal{N} = N(\mathrm{Sym}(\widetilde{\mathbb{Q}}S^1)$, and that there is a ring map $\varphi\colon \mathrm{Sym}(\mathbb{Q}[1]) \to \mathcal{N}$. The composite $\varphi^* N\colon \mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod}) \to \mathrm{Sp}^\Sigma(\mathrm{Ch}_{\mathbb{Q}}^+)$ is lax monoidal.

Let S be a commutative monoid in $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$. We now show that the induced functor

$$\varphi^* N\colon \mathrm{Mod}_S(\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})) \to \mathrm{Mod}_{\varphi^* NS}(\mathrm{Sp}^\Sigma(\mathrm{Ch}_{\mathbb{Q}}^+))$$

on the categories of modules is lax symmetric monoidal. Recall that colimits in categories of modules are calculated in the underlying category of symmetric spectra where they are computed levelwise. Therefore, N preserves colimits as it is an equivalence of categories $\mathrm{s}\mathbb{Q}\text{-mod} \to \mathrm{Ch}_{\mathbb{Q}}^+$. The restriction of scalars φ^* also preserves colimits since it is left adjoint to the coextension of scalars functor. Therefore we have the following map

$$\begin{aligned}
\varphi^* NA \otimes_{\varphi^* NS} \varphi^* NB &= \mathrm{coeq}(\varphi^* NA \otimes \varphi^* NS \otimes \varphi^* NB \rightrightarrows \varphi^* NA \otimes \varphi^* NB) \\
&\to \mathrm{coeq}(\varphi^* N(A \otimes S \otimes B) \rightrightarrows \varphi^* N(A \otimes B)) \\
&\cong \varphi^* N(\mathrm{coeq}(A \otimes S \otimes B \rightrightarrows A \otimes B) \\
&= \varphi^* N(A \otimes_S B)
\end{aligned}$$

giving $\varphi^* N$ a lax symmetric monoidal structure as a functor between the categories of modules. We now must show that $L^S \dashv \varphi^* N$ is a weak monoidal Quillen pair, where L^S denotes the left adjoint of $\varphi^* N$. We use the criteria [34, 3.17]. Since the monoidal unit $\varphi^* NS$ is cofibrant in $\mathrm{Mod}_{\varphi^* NS}$ the first condition is that $L^S \varphi^* NS \to S$ is a weak equivalence. Since $\varphi^* N$ preserves all weak equivalences [39, 4.4] and $\varphi^* NS$ is cofibrant, this map is the derived counit of the Quillen equivalence $L^S \dashv \varphi^* N$ and as such is a weak equivalence. The second condition holds since $\varphi^* NS$ is a generator for the homotopy category of $\mathrm{Mod}_{\varphi^* NS}$.

By taking $S = \alpha^*\widetilde{\mathbb{Q}}A$ in the previous discussion, the adjunction

$$\mathrm{Mod}_{\alpha^*\widetilde{\mathbb{Q}}A}(\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})) \underset{\varphi^* N}{\overset{L^{\alpha^*\widetilde{\mathbb{Q}}A}}{\rightleftarrows}} \mathrm{Mod}_{\varphi^* N\alpha^*\widetilde{\mathbb{Q}}A}(\mathrm{Sp}^\Sigma(\mathrm{Ch}_{\mathbb{Q}}^+))$$

is a weak symmetric monoidal Quillen adjunction. Since $\varphi^* N$ preserves all weak equivalences [39, 4.4], it follows from [34, 3.12(2)] that this is moreover a weak symmetric monoidal Quillen equivalence.

The final step in the zig-zag is the adjunction

$$\mathrm{Mod}_{\varphi^* N\alpha^*\widetilde{\mathbb{Q}}A}(\mathrm{Sp}^\Sigma(\mathrm{Ch}^+_{\mathbb{Q}})) \underset{R}{\overset{D}{\rightleftarrows}} \mathrm{Mod}_{\underline{\Theta}A}(\mathrm{Ch}_{\mathbb{Q}})$$

which is a strong symmetric monoidal Quillen equivalence by Lemma 7.1, since D preserves all weak equivalences rationally [39, 4.8]. $\square$

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CHAPTER 4

Algebraic models of change of groups functors for (co)free rational equivariant spectra

ALGEBRAIC MODELS OF CHANGE OF GROUPS FUNCTORS IN (CO)FREE RATIONAL EQUIVARIANT SPECTRA

JORDAN WILLIAMSON

Abstract. Greenlees-Shipley [19, 21] and Pol and the author [36] have given an algebraic model for rational (co)free equivariant spectra. We give a model categorical argument showing that the induction-restriction-coinduction functors between categories of (co)free rational equivariant spectra correspond to functors between the algebraic models in the case of connected compact Lie groups.

Contents

1. Introduction

1.1. **Rational equivariant cohomology theories.** Rational equivariant cohomology theories are represented by objects called rational G-spectra. Greenlees [12] conjectured that for each compact Lie group G, there is an abelian category $\mathcal{A}(G)$ and a zig-zag of Quillen equivalences

$$\mathrm{Sp}_G \simeq_Q d\mathcal{A}(G)$$

between rational G-spectra and differential objects in $\mathcal{A}(G)$. The conjecture has been proved in many cases: G finite [4], $G = SO(2)$ [39], $G = O(2)$ [5], $G = SO(3)$ [29] and G a torus of any rank [22].

Rather than focusing on a particular group G, we are interested in certain classes of rational G-spectra: those of free and cofree G-spectra. These are the G-spectra for which the natural map $EG_+ \wedge X \to X$ or the natural map $X \to F(EG_+, X)$ are equivalences respectively. There are several reasons why these objects are of particular interest. Firstly, they represent equivariant cohomology theories on free G-spaces. In addition, these cases provide insight into the general case, where the algebraic models are built from contributions at each closed subgroup, where the model resembles that of the free case. Greenlees-Shipley [19, 21] constructed an algebraic model for free G-spectra and Pol and the author have given an algebraic model for cofree G-spectra [36], where G is any compact Lie group.

1.2. **Change of groups.** The inclusion of a subgroup $i\colon H \to G$ in a compact Lie group gives rise to an adjoint triple. The restriction functor $i^*\colon \mathrm{Sp}_G \to \mathrm{Sp}_H$ has both a left adjoint $i_* = G_+ \wedge_H -$ called induction and a right adjoint $i_! = F_H(G_+, -)$ called coinduction. Moreover the adjoint triple

$$\begin{tikzcd} \mathrm{Sp}_G \arrow[r, "i^*" description] & \mathrm{Sp}_H \arrow[l, shift right=2ex, "i_*"'] \arrow[l, shift left=2ex, "i_!"] \end{tikzcd}$$

is a Quillen adjoint triple. In other words, both adjunctions are Quillen with respect to the *same* model structures. Such a situation is a rare occurrence since it forces the middle functor to be both left and right Quillen. In particular, such a functor must preserve all of the classes of maps in the model structure, including the weak equivalences.

In a setting where we have algebraic models for G-spectra and H-spectra, it is a natural question to ask what functors between the algebraic models correspond to this adjoint triple. Diagramatically, we want to find functors

$$\begin{tikzcd} \mathrm{Sp}_G \arrow[rr, leftrightarrow, "\simeq_Q"] \arrow[d, "i^*" description] & & d\mathcal{A}(G) \arrow[d, "?" description] \\ \mathrm{Sp}_H \arrow[u, shift left=3ex, "i_*"] \arrow[u, shift right=3ex, "i_!"'] \arrow[rr, leftrightarrow, "\simeq_Q"'] & & d\mathcal{A}(H) \arrow[u, shift left=3ex, "?"] \arrow[u, shift right=3ex, "?"'] \end{tikzcd}$$

which are algebraic counterparts of the functors in topology. We emphasise that our approach to this is model categorical; we view the functors as Quillen functors and we want to show that there are natural maps at the point-set level which realise the correspondence of functors. At the homotopy category level, the correspondence of functors has been studied by Greenlees-Shipley [19] and Greenlees [14] in the free case. We also note that in the non-free case, Greenlees [13] has given an account of the correspondence of change of groups functors between $SO(2)$-spectra and H-spectra for H a subgroup of $SO(2)$, again at the derived level.

If G is connected, the algebraic model for free G-spectra is I-power torsion modules over the polynomial ring H^*BG [19, 1.1] where I is the augmentation ideal of H^*BG. The algebraic model for cofree G-spectra is L-complete modules over H^*BG [36, 8.4], where a H^*BG-module is said to be L-complete if the natural map $M \to L_0^I M$ is an isomorphism, where L_0^I is the zeroth left derived functor of the I-adic completion. The inclusion $i\colon H \to G$ gives rise to a ring map $\theta\colon H^*BG \to H^*BH$, and therefore an adjoint triple between the categories of modules. The adjoint triple

$$\begin{tikzcd} \mathrm{Mod}_{H^*BG} \arrow[r, shift left=2ex, "\theta_*"] \arrow[r, shift right=2ex, "\theta_!"'] & \mathrm{Mod}_{H^*BH} \arrow[l, "\theta^*" description] \end{tikzcd}$$

is given by the restriction of scalars θ^*, extension of scalars $\theta_* = H^*BH \otimes_{H^*BG} -$ and coextension of scalars $\theta_! = \mathrm{Hom}_{H^*BG}(H^*BH, -)$.

Despite this, one notices that it is not routine to write down the algebraic models as there is a mismatch in directions. In topology, two of the functors go from H-spectra to G-spectra, but in algebra only one functor goes in this direction. Therefore, one must construct extra functors in algebra to model the adjoint triple in topology.

Before we can state the main theorem of this paper, we must explain what we mean by a correspondence of Quillen functors. We do this by considering the following example. The notation we have chosen is suggestive of the special case we have in mind. In particular, we are not assuming the existence of any group actions on the model categories in this general framework. Suppose that we have a diagram

$$\begin{tikzcd}
\mathcal{C}_G \arrow[r, shift left, "F_G"] \arrow[d, shift right, "L"'] & \mathcal{D}_G \arrow[l, shift left, "U_G"] \arrow[r, shift right, "U'_G"'] & \mathcal{E}_G \arrow[l, shift right, "F'_G"'] \arrow[d, shift right, "L'"'] \\
\mathcal{C}_H \arrow[r, shift left, "F_H"] \arrow[u, shift right, "R"'] & \mathcal{D}_H \arrow[l, shift left, "U_H"] \arrow[r, shift right, "U'_H"'] & \mathcal{E}_H \arrow[l, shift right, "F'_H"'] \arrow[u, shift right, "R'"']
\end{tikzcd}$$

of model categories where each of the horizontal adjunctions is a Quillen equivalence, and (L, R) and (L', R') are Quillen adjunctions. We say that (L', R') corresponds to (L, R) if there exists a Quillen adjunction $L'' : \mathcal{D}_G \rightleftarrows \mathcal{D}_H : R''$ together with natural weak equivalences $F_H L \simeq L'' F_G$ and $L'' F'_G \simeq F'_H L'$ on cofibrant objects. Such a correspondence of Quillen adjunctions gives natural isomorphisms of derived functors

$$\mathbb{R}U'_H \circ \mathbb{L}F_H \circ \mathbb{L}L \cong \mathbb{L}L' \circ \mathbb{R}U'_G \circ \mathbb{L}F_G$$

and

$$\mathbb{R}U'_G \circ \mathbb{L}F_G \circ \mathbb{R}R \cong \mathbb{R}R' \circ \mathbb{R}U'_H \circ \mathbb{L}F_H$$

using the theory of mates. For more details, see Section 4. In a similar way one can define correspondences of Quillen adjunctions along zig-zags of Quillen equivalences of any length. We note that this is a particularly structured form of correspondence since the intermediate steps are required to be Quillen adjunctions too.

We can now state the main theorem of this paper. Recall that the restriction functor i^* is both left and right Quillen. Therefore, there are two functors which correspond to it in algebra; one as a left Quillen functor and one as a right Quillen functor. This can be seen in the diagram below where there are four functors in algebra rather than the three in topology.

Theorem 1.1. *Let $i\colon H \to G$ be the inclusion of a connected subgroup into a connected compact Lie group. We have the following correspondence of Quillen functors*

$$\begin{tikzcd}[column sep=small]
\textit{free } G\textit{-spectra} \arrow[rrr, leftrightarrow, "\simeq_Q"] \arrow[d, "i^*"] & & & \textit{derived torsion } H^*BG\textit{-modules} \arrow[d, "\Sigma^{-a}\theta_!"] \arrow[d, shift left=6, "\theta_*"] \\
\textit{free } H\textit{-spectra} \arrow[rrr, leftrightarrow, "\simeq_Q"'] \arrow[u, shift left=4, "i_*"] \arrow[u, shift right=4, "i_!"'] & & & \textit{derived torsion } QH^*BH\textit{-modules} \arrow[u, shift left=4, "\Sigma^a\theta^*"] \arrow[u, shift right=10, "\theta^*"']
\end{tikzcd}$$

where $a = \dim(G/H)$ and QH^*BH is a cofibrant replacement of H^*BH as a commutative H^*BG-algebra. In other words, (i_*, i^*) corresponds to $(\Sigma^a\theta^*, \Sigma^{-a}\theta_!)$ and $(i^*, i_!)$ corresponds to (θ_*, θ^*). Similarly, when the induction, forgetful functor and coinduction functors are viewed as functors between the categories of cofree spectra, they correspond to the same functors as in the free case, now viewed as functors between the categories of derived complete modules.

Remark 1.2. The functors θ_* and $\Sigma^{-a}\theta_!$ are not isomorphic in general at the model categorical level. However, at the derived level this is true. For more details, see Corollary 3.11.

Remark 1.3. It would be interesting to investigate whether the natural isomorphism $\mathbb{L}\theta_* \cong \mathbb{R}\Sigma^{-a}\theta_!$ in algebra corresponds to the natural isomorphism $\mathbb{L}i^* \cong \mathbb{R}i^*$ in topology. However, since the natural isomorphism $\mathbb{L}\theta_* \cong \mathbb{R}\Sigma^{-a}\theta_!$ is a zig-zag of equivalences due to the relative Gorenstein condition, it is not clear how to approach this at a model categorical level. For more details on these natural isomorphisms see Section 3. We also note that the existing applications do not require this additional correspondence.

If the ranks of G and H are equal there is a stronger statement, see Remark 10.5 and Proposition 10.3. In this case Remark 1.2 does not apply, and the restriction of scalars is both left and right adjoint (without a shift) to the extension of scalars along the ring map $\theta\colon H^*BG \to H^*BH$.

Theorem 1.4. *Let $i\colon H \to G$ be the inclusion of a connected subgroup into a connected compact Lie group and assume that $\mathrm{rk}G = \mathrm{rk}H$. Then we have the correspondence of functors*

$$\begin{tikzcd}
\text{free } G\text{-spectra} \arrow[r, leftrightarrow, "\simeq_Q"] \arrow[d, "i^*"] & I\text{-power torsion } H^*BG\text{-modules} \arrow[d, "\theta_*"] \\
\text{free } H\text{-spectra} \arrow[u, shift left=6, "i_*"] \arrow[u, shift right=6, "i_!"'] \arrow[r, leftrightarrow, "\simeq_Q"'] & J\text{-power torsion } H^*BH\text{-modules} \arrow[u, shift left=6, "\theta^*"] \arrow[u, shift right=6, "\theta^*"']
\end{tikzcd}$$

where I and J are the augmentation ideals of H^*BG and H^*BH respectively. Similarly, when the induction, forgetful functor and coinduction are viewed as functors between the categories of cofree spectra, they correspond to θ^*, θ_* and θ^* respectively, between the categories of L-complete modules.

1.3. **Summary of the method.** The first part of the paper is dedicated to producing the required setup to prove the main theorem. In particular, the results are not restricted to this example, and we expect that they can be applied in other cases, such as the disconnected case or the non-free case. We consider diagrams of model categories and Quillen functors of the form

$$\begin{tikzcd}
\mathcal{C}_G \arrow[r, shift left, "F_G"] \arrow[d, shift right=2, "\theta_{(*)}"'] \arrow[d, shift left=2, "\theta_*"] & \mathcal{D}_G \arrow[l, shift left, "U_G"] \arrow[d, shift right=2, "\varphi_{(*)}"'] \arrow[d, shift left=2, "\varphi_*"] \\
\mathcal{C}_H \arrow[u, shift left=6, "\theta^\dagger"] \arrow[u, shift right=6, "\theta^*"'] \arrow[r, shift left, "F_H"] & \mathcal{D}_H \arrow[l, shift left, "U_H"] \arrow[u, shift left=6, "\varphi^\dagger"] \arrow[u, shift right=6, "\varphi^*"']
\end{tikzcd}$$

where $\mathbb{R}\theta_{(*)} \cong \mathbb{L}\theta_*$ and $\mathbb{R}\varphi_{(*)} \cong \mathbb{L}\varphi_*$, and $F_G \dashv U_G$ and $F_H \dashv U_H$ are Quillen equivalences. As described above, in this general framework the notation is purely suggestive and does not indicate the existence of any group actions.

This encompasses all of the cases that arise in our proof of the correspondence for (co)free spectra and connected groups. There are eight squares of derived functors which one would like to show commute:

(1) $\mathbb{L}\varphi_* \circ \mathbb{L}F_G \cong \mathbb{L}F_H \circ \mathbb{L}\theta_*$,
(2) $\mathbb{R}\theta^* \circ \mathbb{R}U_H \cong \mathbb{R}U_G \circ \mathbb{R}\varphi^*$,
(3) $\mathbb{L}F_G \circ \mathbb{L}\theta^\dagger \cong \mathbb{L}\varphi^\dagger \circ \mathbb{L}F_H$,
(4) $\mathbb{R}\theta_{(*)} \circ \mathbb{R}U_G \cong \mathbb{R}U_H \circ \mathbb{R}\varphi_{(*)}$,
(5) $\mathbb{L}\theta^\dagger \circ \mathbb{R}U_H \cong \mathbb{R}U_G \circ \mathbb{L}\varphi^\dagger$,
(6) $\mathbb{L}F_G \circ \mathbb{R}\theta^* \cong \mathbb{R}\varphi^* \circ \mathbb{L}F_H$,
(7) $\mathbb{R}\varphi_{(*)} \circ \mathbb{L}F_G \cong \mathbb{L}F_H \circ \mathbb{R}\theta_{(*)}$,
(8) $\mathbb{L}\theta_* \circ \mathbb{R}U_G \cong \mathbb{R}U_H \circ \mathbb{L}\varphi_*$.

By virtue of the natural isomorphisms $\mathbb{R}\theta_{(*)} \cong \mathbb{L}\theta_*$ and $\mathbb{R}\varphi_{(*)} \cong \mathbb{L}\varphi_*$, we note that (7) is equivalent to (1) and (8) is equivalent to (4). Since the functors in (1) and (4) have the same handedness, these conditions are the ones which are more natural to check. Therefore, from now on we refer to *six* derived squares, rather than eight.

We show that instead of checking that all six squares commute, it is sufficient to prove that only two do, see Theorem 4.4. Moreover, we check these conditions at a model categorical level; that is, we construct natural weak equivalences which realize the isomorphisms of derived functors.

We then verify that these squares do commute in many general settings. In particular, we treat the cases where the horizontals are strong monoidal Quillen equivalences (Proposition 4.6), weak monoidal Quillen equivalences (Proposition 4.9) and Quillen equivalences arising from Quillen invariance of modules (Proposition 4.7), and the vertical functors are given by change of rings adjunctions. The proof of our main theorem then reduces to showing that the zig-zag of Quillen equivalences between (co)free equivariant spectra and their algebraic models satisfy the relevant hypotheses. This requires the construction of ten different squares of the form shown above which satisfy the relevant hypotheses. See Section 5.2 for a more comprehensive discussion.

1.4. **Conventions.** We follow the convention of writing the left adjoint on the left in an adjoint pair displayed vertically, and on top in an adjoint pair displayed horizontally. In the second part of this paper, everything is rationalized without comment. In particular, $H^*(-)$ denotes the (unreduced) cohomology with rational coefficients. We write $q\colon QX \to X$ for cofibrant replacement. All monoidal model categories are assumed to be *symmetric* monoidal as in [37]. We use the standard convention that 'subgroup' means 'closed subgroup'.

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Part 1. The formal setup

2. Background

In this section we provide the necessary background. In particular, we recap flat model structures and their importance for our method. We also recall key facts about monoidal structures and Quillen pairs from [38].

2.1. **Flat model structures.** We will always use the model structure on (commutative) algebras and modules which is right lifted from the underlying category along the forgetful functor, meaning that the weak equivalences and fibrations are the maps which are weak equivalences and fibrations respectively in the underlying category. We write $\mathrm{CAlg}_S(\mathcal{C})$ and $\mathrm{Mod}_S(\mathcal{C})$ for the categories of commutative S-algebras and S-modules respectively. If the underlying category is clear, we will often omit it from the notation.

Hypothesis 2.1. We shall always assume that the required model structures on (commutative) algebras and modules exist; in particular, we implicitly assume that all our model categories are cofibrantly generated. For general existence theorems see [37] and [44]. See Examples 2.2 for more details on the existence of these model structures in our cases of interest.

We next record the key examples for this paper.

Examples 2.2. There is a right lifted model structure on modules in each of the following cases:

- dg-modules over a DGA [6, 3.3];
- modules over a ring spectrum in the stable model structure [31, III.7.6] and flat model structure, see [40, 2.6] and [35, 3.2.1];
- modules over a ring G-spectrum in the stable model structure [30, III.7.6] and flat model structure [43, 2.3.33].

Similarly, there is a right lifted model structure on commutative algebras in each of the following cases:

- commutative DGAs over a field of characteristic zero [44, §5.1];
- commutative algebra spectra in the positive stable model structure and in the positive flat model structure, see [40, 3.2] and [35, 4.1];
- equivariant commutative algebra spectra in the positive stable model structure and in the positive flat model structure, see [30, III.8.1] and [43, 2.3.40].

Definition 2.3. Let $\mathcal{C}$ be a monoidal model category and suppose that there exists another model structure $\widetilde{\mathcal{C}}$ on the same underlying category as $\mathcal{C}$, which has the same weak equivalences and for which the identity functor $\widetilde{\mathcal{C}} \to \mathcal{C}$ is left Quillen. We say that $(\mathcal{C}, \widetilde{\mathcal{C}})$ is *convenient* if the forgetful functor $\mathrm{CAlg}_S(\widetilde{\mathcal{C}}) \to \mathrm{Mod}_S(\mathcal{C})$ preserves cofibrant objects, for all commutative monoids $S \in \mathcal{C}$.

The following lemma summarizes the key feature of a pair $(\mathcal{C}, \widetilde{\mathcal{C}})$ of convenient model structures.

Lemma 2.4. *Suppose that $(\mathcal{C}, \widetilde{\mathcal{C}})$ is convenient and let $\theta\colon S \to R$ be a map of commutative monoids in $\mathcal{C}$. A cofibrant replacement of R as a commutative S-algebra in the model structure right lifted from $\widetilde{\mathcal{C}}$ is cofibrant as an S-module in the model structure right lifted from $\mathcal{C}$.*

Before turning to the examples, we describe the importance of this property. The restriction of scalars functor along a map of commutative monoids $\theta\colon S \to R$ in a monoidal model category $\mathcal{C}$ is always right Quillen, but it is not left Quillen in general. If the monoidal unit of the underlying category is cofibrant, then restriction of scalars is left Quillen if and only if R is cofibrant as an S-module, see Proposition 3.8. Since a key step in the proof of algebraic models is a formality argument based on the fact that polynomial rings are intrinsically formal as commutative DGAs, one needs to be able to replace R in such a way that it is still a commutative S-algebra, and is cofibrant as an S-module. If $(\mathcal{C}, \widetilde{\mathcal{C}})$ is convenient, then this is possible by replacing R as a commutative S-algebra in the model structure right lifted from $\widetilde{\mathcal{C}}$.

The pair of model structures $(\text{stable}, \text{positive stable})$ on spectra is *not* convenient. To rectify this, Shipley [40] constructs the flat and positive flat model structures on symmetric spectra which are convenient. This has since been generalised by Stolz [43] to equivariant spectra and by Pavlov-Scholbach [35] to symmetric spectra in general model categories. It is important to note that in the flat model structures on (equivariant) spectra, the weak equivalences are the same as in the stable model structure. Therefore for each of these flat model structures, the identity functor is a right Quillen equivalence from it to the stable model structure.

In summary, we have the following crucial result.

Theorem 2.5. *The following pairs of model categories are all convenient in the sense of Definition 2.3:*

- $(\textit{projective}, \textit{projective})$ *on chain complexes over a field of characteristic zero;*
- $(\textit{flat}, \textit{positive flat})$ *on symmetric spectra;*
- $(\textit{flat}, \textit{positive flat})$ *on symmetric spectra in simplicial $\mathbb{Q}$-modules;*
- $(\textit{flat}, \textit{positive flat})$ *on symmetric spectra in non-negatively graded chain complexes of $\mathbb{Q}$-modules;*
- $(\textit{flat}, \textit{positive flat})$ *on equivariant spectra.*

Proof. The proofs can be found in [44, §5.1], [40, 4.1], [35, 4.4] together with [45, 4.11], and [43, 2.3.40] respectively. □

2.2. **Flat cofibrants.** In a monoidal model category, if X is cofibrant, then $X \otimes -$ is left Quillen and hence preserves weak equivalences between cofibrant objects by Ken Brown's lemma. It is often convenient to work with monoidal model categories which satisfy something stronger.

Definition 2.6. We say that *cofibrants are flat* in a monoidal model category, if for every cofibrant object X, the functor $X \otimes -$ preserves all weak equivalences.

This is satisfied in all of the model categories of interest in this paper. In particular it holds in the following examples:

- *Projective model structure on dg-modules*: The cofibrant objects in this model category are the semi-projective modules P, and these have the property that $P \otimes -$ preserves quasiisomorphisms [2, 11.1.6, 11.2.1].
- *Equivariant spectra*: Cofibrants are flat in both the stable and flat model structure on G-spectra and in modules over a ring G-spectrum, see [30, 7.3, 7.7] and [43, 2.3.40].
- *Symmetric spectra*: Cofibrant objects are flat in both the stable and flat model structure on symmetric spectra (in general model categories) and in modules over a ring spectrum, see [27, 5.3.10] and [35, 3.5.1].

2.3. **Lifting Quillen adjunctions to module categories.** Recall that a Quillen adjunction $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ between monoidal model categories is said to be a *weak monoidal Quillen adjunction* if the right adjoint U is lax symmetric monoidal (which gives the left adjoint F an oplax symmetric monoidal structure) and the following conditions hold:

(1) for cofibrant A and B in $\mathcal{C}$, the oplax monoidal structure map $\varphi\colon F(A \otimes B) \to FA \otimes FB$ is a weak equivalence in $\mathcal{D}$;
(2) for a cofibrant replacement $c\mathbb{1}_{\mathcal{C}}$ of the unit in $\mathcal{C}$, the map $F(c\mathbb{1}_{\mathcal{C}}) \to \mathbb{1}_{\mathcal{D}}$ is a weak equivalence in $\mathcal{D}$.

If moreover the oplax monoidal structure maps are isomorphisms, we say that it is a *strong monoidal Quillen adjunction*.

Throughout the paper we will make use of categories of modules and how Quillen pairs are lifted to the categories of modules, for more detail see [38, §3.3]. If $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ is a weak monoidal Quillen adjunction then U preserves commutative monoids. Let S be a commutative monoid in $\mathcal{D}$. The adjoint lifting theorem [7, 4.5.6] shows that the Quillen adjunction lifts to a weak monoidal Quillen adjunction

$$\mathrm{Mod}_{US}(\mathcal{C}) \underset{U}{\overset{F^S}{\rightleftarrows}} \mathrm{Mod}_S(\mathcal{D}).$$

If the original pair (F, U) is a strong monoidal Quillen pair, then we have the formula $F^S M = S \otimes_{FUS} FM$ and the lifted Quillen pair (F^S, U) is strong monoidal. For example, this arises in the case of the fixed points-inflation adjunction, see Section 7.

In addition, when (F, U) is a strong monoidal Quillen pair, F preserves commutative monoids. It follows that for S a commutative monoid in $\mathcal{C}$, the adjunction lifts to a strong monoidal Quillen adjunction

$$\mathrm{Mod}_S(\mathcal{C}) \underset{U}{\overset{F}{\rightleftarrows}} \mathrm{Mod}_{FS}(\mathcal{D}).$$

3. Functors between categories of modules

3.1. **The derived story.** Let R and S be commutative ring spectra (for concreteness, commutative monoids in the category of orthogonal spectra) with a ring map $\theta\colon S \to R$. This ring map induces a restriction of scalars functor $\theta^*\colon \mathrm{Mod}_R \to \mathrm{Mod}_S$ and this has left and right adjoints. If R is (derived) small as an S-module, i.e., the natural map

$$\oplus [R, M_i]^S \to [R, \oplus M_i]^S$$

is an isomorphism for any set of S-modules $\{M_i\}$, then there are further adjoints (at least in the derived categories).

We write $\mathbb{D}R$ for the relative dualizing complex $\mathbb{R}\mathrm{Hom}_S(R,S)$, which is an (S,R)-bimodule. We begin with a discussion in the derived categories. Ultimately, we want to lift these derived functors to Quillen functors. In this subsection, all functors are implicitly derived.

There are functors

$$\mathcal{D}(R) \begin{array}{c} \xrightarrow{\quad \theta^\dagger \quad} \\ \xleftarrow{\quad \theta_{(*)} \quad} \\ \xleftarrow{\quad \theta_* \quad} \\ \xrightarrow{\quad \theta^* \quad} \\ \xleftarrow{\quad \theta_! \quad} \end{array} \mathcal{D}(S)$$

defined as follows, where $\mathcal{D}(S)$ denotes the derived category of S-modules.

(1) *Twisted extension of scalars*: $\theta^\dagger(M) = \mathbb{D}R \otimes_R M$
(2) *Twisted coextension of scalars*: $\theta_{(*)}(N) = \mathrm{Hom}_S(\mathbb{D}R, N)$
(3) *Extension of scalars*: $\theta_*(N) = R \otimes_S N$
(4) *Restriction of scalars*: $\theta^*(M)$ is the S-module with underlying object M and action defined by the composite $S \otimes M \to R \otimes M \to M$
(5) *Coextension of scalars*: $\theta_!(N) = \mathrm{Hom}_S(R, N)$

Since $\mathbb{D}R$ is an (S,R)-bimodule, we have an adjoint pair $\theta^\dagger \dashv \theta_{(*)}$. In addition, we have the adjoint triple $\theta_* \dashv \theta^* \dashv \theta_!$.

We always require R to be small as an S-module. This can often be checked using the following result.

Proposition 3.1 ([15, 10.2])**.** *Let S be a ring spectrum such that π_*S is regular. Then an S-module M is small if and only if π_*M is a finitely generated π_*S-module.*

By Venkov's theorem, H^*BH is a finitely generated H^*BG-module. Since H^*BG is polynomial, it is regular. Therefore, any map $\theta\colon S \to R$ of ring spectra with $\pi_*S = H^*BG$ and $\pi_*R = H^*BH$ has the property that R is a small S-module. The majority of ring maps that we use in this paper fall into this category.

3.2. **Relatively Gorenstein maps.** Throughout this paper we will rely upon a helpful relationship between the dual of an S-algebra and itself. In particular, this will allow us to show that the derived functors $\mathbb{L}\theta_*$ and $\mathbb{R}\theta_!$ are isomorphic up to a shift.

Definition 3.2. Let $\mathcal{C}$ be a stable, monoidal model category. A map $\theta\colon S \to R$ of commutative monoids in $\mathcal{C}$ is *relatively Gorenstein of shift a* if $\mathbb{D}R \simeq \Sigma^a R$ as R-modules, where $\mathbb{D} = \mathbb{R}\mathrm{Hom}_S(-,S)$ is the derived hom of S-modules.

All the relatively Gorenstein maps that we use will arise from one particular example. Whilst we do not state the theorem in its full generality, we note that the fact that we work with connected compact Lie groups is vital. We write $DBG_+ = F(BG_+, S^0)$ and note that this is a special case of the cochain spectrum $C^*(BG,k) = F(BG_+, Hk)$ for $k = \mathbb{Q}$, since the rational sphere spectrum is equivalent to $H\mathbb{Q}$.

Theorem 3.3 ([16, 6.1])**.** *Let H be a subgroup of G where both H and G are connected compact Lie groups. The ring map $\theta\colon DBG_+ \to DBH_+$ is relatively Gorenstein of shift d, where d is the codimension of H in G.*

We now prove that relatively Gorenstein maps are preserved by monoidal Quillen equivalences. Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a weak monoidal Quillen equivalence. Under certain hypotheses, given a monoid $S' \in \mathcal{D}$, there is an induced Quillen equivalence $\mathrm{Mod}_{S'}(\mathcal{D}) \simeq_Q \mathrm{Mod}_{US'}(\mathcal{C})$; see [38, 3.12]. There is an analogous result where one begins with a monoid in $\mathcal{C}$. Recall that whilst U passes to a functor of the module categories (as it is lax monoidal), the left adjoint of U will not be F anymore in general.

Remark 3.4. As the Gorenstein condition is a *derived* condition, we may assume that objects are suitably (co)fibrant in order to satisfy the hypotheses of [38, 3.12].

Proposition 3.5.

(1) Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a strong monoidal Quillen equivalence between stable, monoidal model categories and let $\theta\colon S \to R$ be a relatively Gorenstein map of shift a of commutative monoids in $\mathcal{C}$. Then $F\theta\colon FS \to FR$ is relatively Gorenstein of shift a.

(2) Let $F : \mathcal{C} \rightleftarrows \mathcal{D} : U$ be a weak monoidal Quillen equivalence between stable, monoidal model categories and let $\varphi\colon S' \to R'$ be a relatively Gorenstein map of shift a of commutative monoids in $\mathcal{D}$. Then $U\varphi\colon US' \to UR'$ is relatively Gorenstein of shift a.

Proof. Throughout this proof all functors are derived, however we abuse notation and fail to record this in the notation. For the first statement we have

$$\mathbb{D}FR \simeq FU\mathbb{D}FR = FU\mathrm{Hom}_{FS}(FR, FS) \simeq F\mathrm{Hom}_S(R, UFS) \simeq F\mathrm{Hom}_S(R, S) \simeq \Sigma^a FR$$

since the Quillen equivalence provides equivalences of derived categories $\mathcal{D}(S) \simeq \mathcal{D}(FS)$ and $\mathcal{D}(R) \simeq \mathcal{D}(FR)$, see Remark 3.4. This shows that $F\theta$ is relatively Gorenstein of shift a.

Similarly we have

$$\mathbb{D}UR' = \mathrm{Hom}_{US'}(UR', US') \simeq U\mathrm{Hom}_{S'}(F^{S'}UR', S') \simeq U\mathrm{Hom}_{S'}(R', S') \simeq \Sigma^a UR'$$

where $F^{S'} : \mathcal{D}(US') \to \mathcal{D}(S')$ is the left adjoint to U. This shows that $U\theta$ is relatively Gorenstein of shift a. □

The previous proposition shows that all of the ring maps we consider in the second part of this paper are relatively Gorenstein of the same shift, since they all arise via monoidal Quillen equivalences from the ring map $DBG_+ \to DBH_+$.

3.3. **Quillen functors.** In this section we show how to give Quillen functors whose derived functors present the functors of interest. One can define the extension, restriction and coextension of scalars functors at the model categorical level. However, defining Quillen functors whose derived functors are the twisted extension and twisted coextension of scalars is more complex, due to the fact that the (underived) dual has poor homotopical properties.

Let $\theta\colon S \to R$ be a map of monoids in a monoidal model category. Recall that unless otherwise stated, we use the projective model structure on modules in which the weak equivalences and fibrations are created by the forgetful functor to the underlying category.

Proposition 3.6. *The adjoint pair $\theta_* \dashv \theta^*$ is Quillen.*

Proof. It is immediate that θ^* preserves weak equivalences and fibrations since they are tested in the underlying model category. □

In order to discuss the $\theta^* \dashv \theta_!$ adjunction we first need a lemma about enriched model categories. We refer the reader to [23, §4.3] for definitions and basic properties of enriched model categories. Recall that we implicitly assume that our model categories are cofibrantly generated, see Hypothesis 2.1.

Lemma 3.7. *Let $\mathcal{C}$ be a monoidal model category and let S be a monoid in $\mathcal{C}$. Then Mod_S is a $\mathcal{C}$-enriched model category.*

Proof. By cofibrant generation, it suffices to check the required condition on generating (acyclic) cofibrations. The generating cofibrations (resp. acyclic cofibrations) of Mod_S are of the form $S \wedge f$ where f is a generating cofibration (resp. acyclic cofibration) for $\mathcal{C}$. Therefore, it suffices to show that for a cofibration $i\colon X \to Y$ in $\mathcal{C}$ and a generating cofibration $S \wedge f\colon S \wedge M \to S \wedge N$ of Mod_S, that $i\square(S \wedge f)$ is a cofibration in Mod_S which is acyclic if either i or f is.

We have that $i\square(S \wedge f) = S \wedge (i\square f)$. As $\mathcal{C}$ is a monoidal model category, $i\square f$ is a cofibration in $\mathcal{C}$ which is acyclic if either i or f is. Since $S \wedge -$ is left Quillen, $i\square(S \wedge f)$ is a cofibration in Mod_S which is acyclic if either i or f is. □

Proposition 3.8. *Let $\mathcal{C}$ be a monoidal model category with cofibrant unit. The adjoint pair $\theta^* \dashv \theta_!$ is Quillen if and only if $\theta^* R$ is cofibrant as an S-module.*

Proof. As Mod_S is an $\mathcal{C}$-enriched model category by Lemma 3.7, it follows by lifting properties that for $i\colon M \to N$ a cofibration in Mod_S and $p\colon E \to B$ a fibration in Mod_S that

$$\mathrm{Hom}_S(N, E) \to \mathrm{Hom}_S(M, E) \times_{\mathrm{Hom}_S(M,B)} \mathrm{Hom}_S(N, B)$$

is a fibration which is acyclic if either i or p is. Taking i to be the map $* \to \theta^* R$ shows that $\mathrm{Hom}_S(R, -)$ is right Quillen if $\theta^* R$ is cofibrant as an S-module, as fibrations and weak equivalences are determined in $\mathcal{C}$.

Conversely, as $\mathbb{1}$ is cofibrant in $\mathcal{C}$ we have that R is cofibrant as an R-module. Therefore if θ^* is left Quillen, then $\theta^* R$ is a cofibrant S-module. □

The (underived) dual $\mathrm{Hom}_S(R, S)$ behaves badly in general at the model category level. For example, consider the ring map $\theta\colon \mathbb{Q}[c] \to \mathbb{Q}$ where $|c| = -2$. This example is pertinent since it is the map $H^*BSO(2) \to H^*B1$. Then $\mathrm{Hom}_{\mathbb{Q}[c]}(\mathbb{Q}, \mathbb{Q}[c]) = 0$ but $\mathbb{R}\mathrm{Hom}_{\mathbb{Q}[c]}(\mathbb{Q}, \mathbb{Q}[c]) \simeq \Sigma\mathbb{Q}$. We now explain how to resolve these issues and produce Quillen functors which model $\theta^\dagger$ and $\theta_{(*)}$.

Write $\mathbb{D}$ for the functor $\mathbb{R}\mathrm{Hom}_S(-, S)$. There is a natural morphism $\psi_{M,N}\colon \mathbb{D}M \otimes^{\mathbb{L}}_S N \to \mathbb{R}\mathrm{Hom}_S(M, N)$ which is defined to be the transpose of the natural map

$$\mathbb{D}M \otimes^{\mathbb{L}}_S N \otimes^{\mathbb{L}}_S M \cong \mathbb{D}M \otimes^{\mathbb{L}}_S M \otimes^{\mathbb{L}}_S N \xrightarrow{\mathrm{ev}\otimes 1} S \otimes^{\mathbb{L}}_S N \cong N.$$

There is also a natural morphism $\varphi_M\colon M \to \mathbb{D}^2 M$ defined to be the transpose of the evaluation map

$$\mathbb{D}M \otimes^{\mathbb{L}}_S M \xrightarrow{\mathrm{ev}} S.$$

The following result is well known. Its proof can be found in [26, 2.1.3] for instance. Note that we implicitly assume that our categories are generated by small objects (i.e., are algebraic stable homotopy categories in the sense of [26]).

Lemma 3.9.

(1) *The natural map $\psi_{M,N}\colon \mathbb{D}M \otimes^{\mathbb{L}}_S N \to \mathbb{R}\mathrm{Hom}_S(M, N)$ is an equivalence if M is a small S-module.*
(2) *The natural map $\varphi_M\colon M \to \mathbb{D}^2 M$ is an equivalence if M is a small S-module.*

Proposition 3.10. *Let $\theta\colon S \to R$ be a map of commutative monoids such that R is a small S-module. There is a natural isomorphism of derived functors $\mathbb{L}\theta_* \xrightarrow{\sim} \mathbb{R}\theta_{(*)}$.*

Proof. Note that since R is a small S-module by assumption, we have that $\mathbb{D}R$ is also a small S-module. We define $\alpha\colon \mathbb{L}\theta_* \Rightarrow \mathbb{R}\theta_{(*)}$ to be the composite

$$\mathbb{L}\theta_*(N) = R \otimes^{\mathbb{L}}_S N \xrightarrow{\varphi_R \otimes 1} \mathbb{D}^2 R \otimes^{\mathbb{L}}_S N \xrightarrow{\psi_{\mathbb{D}R,N}} \mathbb{R}\mathrm{Hom}_S(\mathbb{D}R, N) = \mathbb{R}\theta_{(*)}(N).$$

By Lemma 3.9, α is an equivalence. □

The following result summarises the key points of this section.

Corollary 3.11. *Let $\theta\colon S \to R$ be a relatively Gorenstein map of shift a, such that R is a small S-module and R is cofibrant as an S-module. Then $\theta_* \dashv \theta^*$ and $\theta^* \dashv \theta_!$ are Quillen pairs and there are natural isomorphisms of derived functors $\mathbb{L}\theta_* \cong \Sigma^{-a}\mathbb{R}\theta_!$ and $\mathbb{L}\theta^* \cong \mathbb{R}\theta^* \cong \Sigma^{-a}\mathbb{L}\theta^\dagger$.*

Proof. The fact that $\theta_* \dashv \theta^*$ and $\theta^* \dashv \theta_!$ are Quillen pairs is Propositions 3.6 and 3.8. By Proposition 3.10 we have a natural isomorphism $\mathbb{L}\theta_* \cong \mathbb{R}\theta_{(*)}$. Then

$$\mathbb{R}\theta_{(*)}M = \mathbb{R}\mathrm{Hom}_S(\mathbb{D}R, M) \cong \mathbb{R}\mathrm{Hom}_S(\Sigma^a R, M) \cong \Sigma^{-a}\mathbb{R}\theta_! M.$$

Since θ^* is both left and right Quillen, $\mathbb{L}\theta^* \cong \mathbb{R}\theta^*$. Therefore the other natural isomorphism follows by adjunction. □

4. Comparing Quillen functors

In this section, we set up the general techniques for showing when Quillen functors correspond.

4.1. **The calculus of mates.**

We firstly recap the calculus of mates, see [28, §2] for a comprehensive account.

Consider the diagram

$$\begin{tikzcd} \mathcal{C}_G \arrow[r, shift left, "F_G"] \arrow[d, "L"'] & \mathcal{D}_G \arrow[l, shift left, "U_G"] \arrow[d, "L'"] \\ \mathcal{C}_H \arrow[r, shift left, "F_H"] & \mathcal{D}_H \arrow[l, shift left, "U_H"] \end{tikzcd}$$

in which $F_G \dashv U_G$ and $F_H \dashv U_H$ are adjunctions and L and L' are functors (not necessarily left adjoints). As stated in the introduction, in this general framework the notation is only suggestive and does not indicate the existence of any group actions.

Given a natural transformation $\alpha : F_H L \Rightarrow L' F_G$, one can define its *mate* $\overline{\alpha} : LU_G \Rightarrow U_H L'$ to be the natural transformation

$$LU_G \xRightarrow{\eta LU_G} U_H F_H LU_G \xRightarrow{U_H \alpha U_G} U_H L' F_G U_G \xRightarrow{U_H L' \varepsilon} U_H L'.$$

Conversely, given $\beta : LU_G \Rightarrow U_H L'$ one defines its mate $\overline{\beta} : F_H L \Rightarrow L' F_G$ by the composite

$$F_H L \xRightarrow{F_H L \eta} F_H LU_G F_G \xRightarrow{F_H \beta F_G} F_H U_H L' F_G \xRightarrow{\varepsilon L' F_G} L' F_G.$$

These two operations are inverse to one another by the triangle identities and therefore give a bijection between the two kinds of natural transformation.

In general, the mate of a natural isomorphism need not be a natural isomorphism. However, this does hold in certain cases.

Proposition 4.1. *If $F_G \dashv U_G$ and $F_H \dashv U_H$ are adjoint equivalences, then a natural transformation $F_H L \Rightarrow L' F_G$ is a natural isomorphism if and only if its mate is.*

Proof. Since $F_G \dashv U_G$ and $F_H \dashv U_H$ are adjoint equivalences, the units and counits are natural isomorphisms. The result then follows by the 2-out-of-3 property of isomorphisms. □

Unfortunately, there is no analogous result at the level of model categories, saying that a map is a natural weak equivalence if and only if its mate is. However, the mates correspondence does give us a way to attack questions about the commutativity of derived functors.

Note that natural weak equivalences of Quillen functors of the same handedness pass to natural isomorphisms of the derived functors by virtue of the natural isomorphism $\mathbb{L}(F \circ F') \cong \mathbb{L}F \circ \mathbb{L}F'$. However, a natural weak equivalence between composites of left and right Quillen functors does not imply that the composites of derived functors are naturally isomorphic. An explicit counterexample can be found in [33, 0.0.1].

We give a short overview of how we will use the machinery of mates. Suppose we are given a square

$$\begin{tikzcd} \mathcal{C}_G \arrow[r, shift left, "F_G"] \arrow[d, "L"'] & \mathcal{D}_G \arrow[l, shift left, "U_G"] \arrow[d, "L'"] \\ \mathcal{C}_H \arrow[r, shift left, "F_H"] & \mathcal{D}_H \arrow[l, shift left, "U_H"] \end{tikzcd}$$

in which L and L' are left Quillen functors and $F_G \dashv U_G$ and $F_H \dashv U_H$ are Quillen equivalences. Suppose we want to know that $\mathbb{L}L \circ \mathbb{R}U_G \cong \mathbb{R}U_H \circ \mathbb{L}L'$. Instead we can check that there is a natural weak equivalence $L'F_G \simeq F_H L$ on cofibrant objects, and then since they have the same handedness, we have a natural isomorphism $\mathbb{L}L' \circ \mathbb{L}F_G \cong \mathbb{L}F_H \circ \mathbb{L}L$. Since the Quillen equivalences will descend to adjoint equivalences at the level of derived categories, it follows from Proposition 4.1 that taking mates gives the desired natural isomorphism $\mathbb{L}L \circ \mathbb{R}U_G \cong \mathbb{R}U_H \circ \mathbb{L}L'$.

Remark 4.2. Shulman [42] develops a method for comparing composites of left and right derived functors. Since we will be interested in the case where the horizontal adjunctions are Quillen equivalences, we can always compare composites of left and right derived functors by instead comparing their mates, as described above. This allows us to only ever have to consider Quillen functors of the same handedness. Therefore, Shulman's method is unnecessary for our purposes.

4.2. **Proving commutation of derived functors.** Consider the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge] \mathcal{C}_G \arrow[r, shift left, "F_G"] \arrow[d, shift left=1, "\theta_{(*)}"] \arrow[d, shift left=3, "\theta_*"] & \mathcal{D}_G \arrow[l, shift left, "U_G"] \arrow[d, shift left=1, "\varphi_{(*)}"] \arrow[d, shift left=3, "\varphi_*"] \\ \mathcal{C}_H \arrow[r, shift left, "F_H"] \arrow[u, shift left=1, "\theta^\dagger"] \arrow[u, shift right=5, "\theta^*"'] & \mathcal{D}_H \arrow[l, shift left, "U_H"] \arrow[u, shift left=1, "\varphi^\dagger"] \arrow[u, shift right=5, "\varphi^*"'] \end{tikzcd}$$

of model categories and Quillen functors, where $\mathbb{R}\theta_{(*)} \cong \mathbb{L}\theta_*$ and $\mathbb{R}\varphi_{(*)} \cong \mathbb{L}\varphi_*$, and $F_G \dashv U_G$ and $F_H \dashv U_H$ are Quillen equivalences. We note that we have chosen names in keeping with the example that we have in mind, where the top horizontal is a Quillen equivalence arising in the construction of an algebraic model for G-spectra. We are in interested in using a model categorical approach to show that all of the six squares of derived functors listed in Section 1.3 commute. In this section we prove that in order to check that all six of the derived squares commute, it is sufficient to check that only two squares commute.

Lemma 4.3. *Let*

$$\begin{tikzcd}[column sep=huge, row sep=huge] \mathcal{C}_G \arrow[r, shift left, "F_G"] \arrow[d, shift right, "\theta_*"'] & \mathcal{D}_G \arrow[l, shift left, "U_G"] \arrow[d, shift right, "\varphi_*"'] \\ \mathcal{C}_H \arrow[r, shift left, "F_H"] \arrow[u, shift right, "\theta^*"'] & \mathcal{D}_H \arrow[l, shift left, "U_H"] \arrow[u, shift right, "\varphi^*"'] \end{tikzcd}$$

be a diagram of model categories and Quillen functors.

(1) There is a natural weak equivalence $\varphi_* F_G \overset{\sim}{\Longrightarrow} F_H \theta_*$ on cofibrant objects if and only if there is a natural weak equivalence $\theta^* U_H \overset{\sim}{\Longrightarrow} U_G \varphi^*$ on fibrant objects.

(2) There is a natural weak equivalence $F_H \theta_* \overset{\sim}{\Longrightarrow} \varphi_* F_G$ on cofibrant objects if and only if there is a natural weak equivalence $U_G \varphi^* \overset{\sim}{\Longrightarrow} \theta^* U_H$ on fibrant objects.

Proof. Given a natural weak equivalence $\alpha : \varphi_* F_G \xRightarrow{\sim} F_H \theta_*$, we have a natural map $\widetilde{\alpha} : \theta^* U_H \Rightarrow U_G \varphi^*$ defined by

$$\theta^* U_H \xRightarrow{\eta\theta^* U_H} U_G\varphi^*\varphi_* F_G \theta^* U_H \xRightarrow{U_G\varphi^*\alpha\theta^* U_H} U_G\varphi^* F_H\theta_*\theta^* U_H \xRightarrow{U_G\varphi^*\varepsilon'} U_G\varphi^*$$

where η is the unit of the $\varphi_* F_G \dashv U_G\varphi^*$ adjunction and ε' is the counit of the $F_H\theta_* \dashv \theta^* U_H$ adjunction. To check that this is a natural weak equivalence on fibrant objects, it suffices to check that

$$\widetilde{\alpha}_* \colon \mathrm{map}(X, \theta^* U_H Y) \xrightarrow{\sim} \mathrm{map}(X, U_G\varphi^* Y)$$

is a weak equivalence of homotopy function complexes for all cofibrant X and fibrant Y [25, 17.7.7]. Left Quillen functors preserve cosimplicial resolutions of cofibrant objects, so the adjunction passes to the level of homotopy function complexes [25, 17.4.16] to give the following diagram

$$\begin{tikzcd}
\mathrm{map}(F_H\theta_* X, Y) \arrow[r, "\sim"] \arrow[d, "\cong"'] & \mathrm{map}(\varphi_* F_G X, Y) \arrow[d, "\cong"] \\
\mathrm{map}(X, \theta^* U_H Y) \arrow[r] & \mathrm{map}(X, U_G\varphi^* Y).
\end{tikzcd}$$

Therefore, the natural map $\theta^* U_H \xRightarrow{\sim} U_G\varphi^*$ is a weak equivalence on fibrant objects by 2-out-of-3. The converse to (1) and the proof of (2) follow similarly. $\square$

We now state the theorem which allows us to reduce the problem of checking that all six squares commute, to only checking that two squares commute. Note that in the following theorem, saying 'there is a natural weak equivalence $F \simeq G$' means that there is either a natural weak equivalence $F \Rightarrow G$ or $G \Rightarrow F$. We do not permit zig-zags of weak equivalences.

Theorem 4.4. *Consider the square*

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathcal{C}_G \arrow[r, shift left, "F_G"] \arrow[d, shift right=1, "\theta_{(*)}"'] \arrow[d, shift left=1, "\theta_*"] & \mathcal{D}_G \arrow[l, shift left, "U_G"] \arrow[d, shift right=1, "\varphi_{(*)}"'] \arrow[d, shift left=1, "\varphi_*"] \\
\mathcal{C}_H \arrow[r, shift left, "F_H"] \arrow[u, shift left=3, "\theta^\dagger"] \arrow[u, shift right=3, "\theta^*"'] & \mathcal{D}_H \arrow[l, shift left, "U_H"] \arrow[u, shift left=3, "\varphi^\dagger"] \arrow[u, shift right=3, "\varphi^*"']
\end{tikzcd}$$

in which all the adjoint pairs are Quillen, $F_G \dashv U_G$ and $F_H \dashv U_H$ are Quillen equivalences, and suppose that there are natural isomorphisms $\mathbb{L}\theta_* \cong \mathbb{R}\theta_{(*)}$ and $\mathbb{L}\varphi_* \cong \mathbb{R}\varphi_{(*)}$. Consider the statements:

a) There is a natural weak equivalence $\varphi_* F_G \simeq F_H\theta_*$ on cofibrant objects.
b) There is a natural weak equivalence $\theta^* U_H \simeq U_G\varphi^*$ on fibrant objects.
c) There is a natural weak equivalence $F_G\theta^\dagger \simeq \varphi^\dagger F_H$ on cofibrant objects.
d) There is a natural weak equivalence $\theta_{(*)} U_G \simeq U_H\varphi_{(*)}$ on fibrant objects.

If either (a) or (b) is true and either (c) or (d) is true, then each of the six derived squares listed in Section 1.3 commutes.

Proof. Statement (a) is equivalent to statement (b) by taking adjoints (see Lemma 4.3), and similarly (c) is equivalent to (d). Since all of the functors in each statement have the same handedness, the natural weak equivalences descend to isomorphisms of derived functors.

It just remains to show that these statements are sufficient to conclude that we have natural isomorphisms $\mathbb{L}\theta^\dagger \circ \mathbb{R}U_H \cong \mathbb{R}U_G \circ \mathbb{L}\varphi^\dagger$ and $\mathbb{L}F_G \circ \mathbb{R}\theta^* \cong \mathbb{R}\varphi^* \circ \mathbb{L}F_H$. By statement (c) we have a natural isomorphism $\beta \colon \mathbb{L}F_G \circ \mathbb{L}\theta^\dagger \cong \mathbb{L}\varphi^\dagger \circ \mathbb{L}F_H$. Since $F_G \dashv U_G$ and $F_H \dashv U_H$ are Quillen equivalences and hence their derived adjunctions are adjoint equivalences, by Proposition 4.1 we find that the mate of β is also a natural isomorphism $\overline{\beta}\colon \mathbb{L}\theta^\dagger \circ \mathbb{R}U_H \cong \mathbb{R}U_G \circ \mathbb{L}\varphi^\dagger$. Similarly, applying Proposition 4.1 to the natural isomorphism $\mathbb{R}U_G \circ \mathbb{R}\varphi^* \cong \mathbb{R}\theta^* \circ \mathbb{R}U_H$ from statement (b) we obtain a natural isomorphism $\mathbb{L}F_G \circ \mathbb{R}\theta^* \cong \mathbb{R}\varphi^* \circ \mathbb{L}F_H$. □

4.3. **Checking commutativity.** In this section we give some lemmas which verify when the hypotheses of Theorem 4.4 hold. These fall into three types: strong monoidal Quillen pairs, weak monoidal Quillen pairs and Quillen equivalences which arise from Quillen invariance of modules.

If (F, U) is a strong monoidal adjunction, then $U\underline{\mathrm{Hom}}(FX, Y) \cong \underline{\mathrm{Hom}}(X, UY)$ where $\underline{\mathrm{Hom}}$ denotes the internal hom. However, if the adjunction is a weak monoidal Quillen pair, there is a natural weak equivalence relating the two.

Lemma 4.5. *Let (F, U) be a weak monoidal Quillen pair. For X cofibrant, there is a natural weak equivalence*

$$U\underline{\mathrm{Hom}}(FX, Y) \xrightarrow{\sim} \underline{\mathrm{Hom}}(X, UY).$$

Proof. There is a natural map $U\underline{\mathrm{Hom}}(FX, Y) \to \underline{\mathrm{Hom}}(X, UY)$ which is adjunct to the natural map

$$F(X \otimes U\underline{\mathrm{Hom}}(FX, Y)) \to FX \otimes FU\underline{\mathrm{Hom}}(FX, Y) \xrightarrow{1\otimes\varepsilon} FX \otimes \underline{\mathrm{Hom}}(FX, Y) \xrightarrow{\mathrm{ev}} Y$$

constructed using the oplax monoidal structure map of F. We use the Yoneda lemma for homotopy function complexes [25, 17.7.7] to show that this is a weak equivalence. Let A be cofibrant. We then have the following string of equivalences:

$$\begin{aligned}
\mathrm{map}(A, U\underline{\mathrm{Hom}}(FX, Y)) &\cong \mathrm{map}(FA, \underline{\mathrm{Hom}}(FX, Y)) \\
&\cong \mathrm{map}(FA \otimes FX, Y) \\
&\xrightarrow{\sim} \mathrm{map}(F(A \otimes X), Y) \\
&\cong \mathrm{map}(A \otimes X, UY) \\
&\cong \mathrm{map}(A, \underline{\mathrm{Hom}}(X, UY))
\end{aligned}$$

which completes the proof. □

Firstly we deal with the case of strong monoidal Quillen pairs.

Proposition 4.6. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left, "F"] & \mathcal{D} \arrow[l, shift left, "U"] \end{tikzcd}$$

be a strong monoidal Quillen pair and let $\theta\colon S \to R$ be a map of commutative monoids in $\mathcal{C}$. Write $F\theta = \varphi\colon FS \to FR$. There are natural isomorphisms $\theta^* U \cong U\varphi^*$ and $F\theta^* \cong \varphi^* F$, i.e., the diagrams

$$\begin{tikzcd} \mathrm{Mod}_S(\mathcal{C}) & \mathrm{Mod}_{FS}(\mathcal{D}) \arrow[l, "U"'] \\ \mathrm{Mod}_R(\mathcal{C}) \arrow[u, "\theta^*"] & \mathrm{Mod}_{FR}(\mathcal{D}) \arrow[l, "U"] \arrow[u, "\varphi^*"'] \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Mod}_S(\mathcal{C}) \arrow[r, "F"] & \mathrm{Mod}_{FS}(\mathcal{D}) \\ \mathrm{Mod}_R(\mathcal{C}) \arrow[u, "\theta^*"] \arrow[r, "F"'] & \mathrm{Mod}_{FR}(\mathcal{D}) \arrow[u, "\varphi^*"'] \end{tikzcd}$$

commute up to natural isomorphism.

Proof. Take $N \in \mathrm{Mod}_{FR}$. The underlying objects of $\theta^* UN$ and $U\varphi^* N$ are the same so it remains to check that the module structures agree. We write Φ for the lax monoidal structure map on U, and $a\colon FR \wedge N \to N$ for the action map.

$$\begin{tikzcd}
S \wedge U\varphi^* N \arrow[r, "\eta\wedge 1"] \arrow[d, "1"'] \arrow[dr, "\theta\wedge 1"] & UFS \wedge U\varphi^* N \arrow[r, "\Phi"] \arrow[dr, "UF\theta\wedge 1"] & U(FS \wedge \varphi^* N) \arrow[r, "U(\varphi\wedge 1)"] \arrow[dr, "U(F\theta\wedge 1)"] & U(FR \wedge N) \arrow[r, "Ua"] \arrow[d, "1"] & UN \arrow[d, "1"] \\
S \wedge \theta^* UN \arrow[r, "\theta\wedge 1"'] & R \wedge UN \arrow[r, "\eta\wedge 1"'] & UFR \wedge UN \arrow[r, "\Phi"'] & U(FR \wedge N) \arrow[r, "Ua"'] & UN
\end{tikzcd}$$

The first triangle commutes by definition, the following square by naturality of η, the following square by naturality of Φ, and the remaining triangle and square by definition.

The second natural isomorphism can be proved similarly. ☐

We now turn to the case of Quillen equivalences which arise from Quillen invariance of modules.

Proposition 4.7. *Let $\mathcal{C}$ be a monoidal model category. Suppose that we have a commutative square of commutative monoids*

$$\begin{tikzcd}
S \arrow[d, "\theta"'] & S' \arrow[l, "f", "\sim"'] \arrow[d, "\psi"] \\
R & R' \arrow[l, "\sim", "g"']
\end{tikzcd}$$

in $\mathcal{C}$ whose horizontal arrows are weak equivalences.

(1) There is a natural isomorphism $f^*\theta^* \cong \psi^* g^*$, i.e., the diagram

$$\begin{tikzcd}
\mathrm{Mod}_S \arrow[r, "f^*"] & \mathrm{Mod}_{S'} \\
\mathrm{Mod}_R \arrow[u, "\theta^*"] \arrow[r, "g^*"'] & \mathrm{Mod}_{R'} \arrow[u, "\psi^*"']
\end{tikzcd}$$

commutes up to natural isomorphism.

(2) If cofibrants are flat in $\mathrm{Mod}_{R'}$ (see Definition 2.6) and there is a natural weak equivalence $S \otimes_{S'} R' \xrightarrow{\sim} R$, then there is a natural weak equivalence $f_*\psi^* M \xrightarrow{\sim} \theta^* g_* M$ for cofibrant R'-modules M, i.e., the diagram

$$\begin{tikzcd}
\mathrm{Mod}_S & \mathrm{Mod}_{S'} \arrow[l, "f_*"'] \\
\mathrm{Mod}_R \arrow[u, "\theta^*"] & \mathrm{Mod}_{R'} \arrow[u, "\psi^*"'] \arrow[l, "g_*"]
\end{tikzcd}$$

commutes up to natural weak equivalence on cofibrant objects.

Proof. The isomorphism $f^*\theta^* \cong \psi^* g^*$ follows from the fact that $f^*\theta^* \cong (\theta f)^*$. The natural weak equivalence follows from applying $- \otimes_{R'} M$ to the natural weak equivalence $S \otimes_{S'} R' \xrightarrow{\sim} R$. We note that $- \otimes_{R'} M$ preserves weak equivalences since M is cofibrant and cofibrants are flat. ☐

Remark 4.8. We note that in the category of commutative monoids, the pushout of a span $S \leftarrow S' \rightarrow R'$ is given by the tensor product $S \otimes_{S'} R'$. Therefore, the condition that there is a natural weak equivalence $S \otimes_{S'} R' \xrightarrow{\sim} R$ is satisfied if f is an acyclic cofibration of commutative S'-algebras as pushouts preserve acyclic cofibrations, or if the model category of commutative S'-algebras is left proper and ψ is a cofibration of commutative S'-algebras. Alternatively, the condition clearly holds if the square is a pushout of commutative S'-algebras.

Finally, we treat the case of weak monoidal Quillen pairs. In this case we have to argue with the right adjoint since the left adjoint at the level of modules is different to the underlying left adjoint.

Proposition 4.9. *Let*

$$\mathcal{C} \underset{U}{\overset{F}{\rightleftarrows}} \mathcal{D}$$

be a weak monoidal Quillen pair and let $\theta\colon S \to R$ be a map of commutative monoids in $\mathcal{C}$. Write $\varphi = U\theta\colon US \to UR$. Suppose that there exists another model structure $\widetilde{\mathcal{C}}$ on the same underlying category as $\mathcal{C}$ so that $(\mathcal{C}, \widetilde{\mathcal{C}})$ is convenient (see Definition 2.3). Let $q\colon QUR \to UR$ be a cofibrant replacement of UR in $\mathrm{CAlg}_{US}(\widetilde{\mathcal{C}})$ and write $\psi\colon US \to QUR$ for the unit map of the US-algebra structure on QUR.

(1) There is a natural isomorphism $\psi^* q^* U \cong U\theta^*$, i.e., the diagram

$$\begin{tikzcd}
\mathrm{Mod}_S(\mathcal{C}) \arrow[rr, "U"] & & \mathrm{Mod}_{US}(\mathcal{D}) \\
\mathrm{Mod}_R(\mathcal{C}) \arrow[u, "\theta^*"] \arrow[r, "U"'] & \mathrm{Mod}_{UR}(\mathcal{D}) \arrow[r, "q^*"'] & \mathrm{Mod}_{QUR}(\mathcal{D}) \arrow[u, "\psi^*"']
\end{tikzcd}$$

commutes up to natural isomorphism.

(2) Suppose that R is cofibrant as an S-module and that

$$\mathrm{Mod}_S(\mathcal{C}) \underset{U}{\overset{F^S}{\rightleftarrows}} \mathrm{Mod}_{US}(\mathcal{D})$$

is a Quillen equivalence. Assume that either R is fibrant, or that U preserves all weak equivalences. Then there is a natural weak equivalence $q^* U\theta_! M \xrightarrow{\sim} \psi_! UM$ for M fibrant in $\mathrm{Mod}_S(\mathcal{C})$, i.e., the diagram

$$\begin{tikzcd}
\mathrm{Mod}_S(\mathcal{C}) \arrow[rr, "U"] \arrow[d, "\theta_!"'] & & \mathrm{Mod}_{US}(\mathcal{D}) \arrow[d, "\psi_!"] \\
\mathrm{Mod}_R(\mathcal{C}) \arrow[r, "U"'] & \mathrm{Mod}_{UR}(\mathcal{D}) \arrow[r, "q^*"'] & \mathrm{Mod}_{QUR}(\mathcal{D})
\end{tikzcd}$$

commutes up to natural weak equivalence on fibrant objects.

Proof. For an R-module N the underlying objects of $\psi^* q^* UN$ and $U\theta^* N$ are equal so it suffices to check that the module structures agree:

$$\begin{tikzcd}
US \wedge U\theta^* N \arrow[r, "\Phi"] \arrow[d, "1"'] \arrow[dr, "\varphi\wedge 1"] & U(S \wedge \theta^* N) \arrow[r, "U(\theta\wedge 1)"] \arrow[dr, "U(\theta\wedge 1)"] & U(R \wedge N) \arrow[r, "Ua"] \arrow[d, "1"] & UN \arrow[d, "1"] \\
US \wedge \psi^* q^* UN \arrow[r, "q\psi\wedge 1"'] & UR \wedge UN \arrow[r, "\Phi"'] & U(R \wedge N) \arrow[r, "Ua"'] & UN
\end{tikzcd}$$

This diagram commutes by using the naturality of the lax monoidal structure map Φ on U, which completes the proof of (1).

For (2), note that since $(\mathcal{C}, \widetilde{\mathcal{C}})$ is convenient, QUR is a cofibrant US-module. Since $F^S \dashv U$ is a Quillen equivalence, the derived counit $F^S QUR \to R$ is a weak equivalence (using either that R is fibrant or that U preserves all weak equivalences). Since M is fibrant, Ken Brown's lemma shows that $\mathrm{Hom}_S(-, M)$ sends weak equivalences between cofibrant objects to weak equivalences between fibrant objects, where $\mathrm{Hom}_S(-,-)$ denotes the internal hom of S-modules. Therefore, $\mathrm{Hom}_S(R, M) \to \mathrm{Hom}_S(F^S QUR, M)$ is a weak equivalence between fibrant objects. Another application of Ken Brown's lemma gives

$$q^* U\theta_! M = q^* U\mathrm{Hom}_S(R, M) \xrightarrow{\sim} q^* U\mathrm{Hom}_S(F^S QUR, M) \xrightarrow{\sim} \mathrm{Hom}_{US}(QUR, M) = \psi_! UM$$

where the last equivalence follows from Lemma 4.5. ∎

4.4. **Quillen pairs post localization.** In this section we will give conditions under which the Quillen pairs $\theta_* \dashv \theta^*$ and $\theta^* \dashv \theta_!$ descend to Quillen pairs between Bousfield localizations.

We first must recall the projection formula.

Definition 4.10. Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories with adjunctions

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2ex, "i_*" description] \arrow[r, shift right=2ex, "i_!" description] & \mathcal{D} \arrow[l, "i^*" description] \end{tikzcd}$$

where i^* is strong monoidal. We say that the *projection formula for i_* holds* if the natural map $p\colon i_*(i^*(X) \wedge Y) \to X \wedge i_*(Y)$ defined by

$$i_*(i^*(X) \wedge Y) \xrightarrow{i_*(1\wedge\eta)} i_*(i^*(X) \wedge i^*i_*(Y)) \xrightarrow[\cong]{i_*\Phi} i_*i^*(X \wedge i_*(Y)) \xrightarrow{\varepsilon} X \wedge i_*(Y)$$

is an isomorphism for all $X \in \mathcal{D}$ and $Y \in \mathcal{C}$, where Φ denotes the monoidal structure map of i^*.

Remark 4.11. There is also a natural map $p'\colon X \wedge i_!(Y) \to i_!(i^*(X) \wedge Y)$ for $X \in \mathcal{D}$ and $Y \in \mathcal{C}$, defined by

$$X \wedge i_!(Y) \xrightarrow{\eta} i_!i^*(X \wedge i_!(Y)) \xrightarrow[\cong]{i_!\Phi^{-1}} i_!(i^*(X) \wedge i^*i_!(Y)) \xrightarrow{i_!(1\wedge\varepsilon)} i_!(i^*(X) \wedge Y)$$

so it also makes sense to ask when $i_!$ satisfies the projection formula. However, this is not relevant for our purposes.

The projection formula clearly holds for the extension of scalars functor along a map of commutative monoids in a symmetric monoidal category. It also holds for the induction functor from H-spectra to G-spectra, see [30, V.2.3].

We can now deal with the case of left Bousfield localizations. We write $\langle E \rangle$ for the Bousfield class of E, that is, for the class of objects X for which $E \wedge X \simeq 0$.

Theorem 4.12. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2ex, "\theta_*" description] \arrow[r, shift right=2ex, "\theta_!" description] & \mathcal{D} \arrow[l, "\theta^*" description] \end{tikzcd}$$

be a Quillen adjoint triple between stable monoidal model categories, and suppose that θ_* satisfies the projection formula. Let $E \in \mathcal{C}$ and $E' \in \mathcal{D}$ be cofibrant. Then

$$\begin{tikzcd} L_E\mathcal{C} \arrow[r, shift left=2ex, "\theta_*" description] \arrow[r, shift right=2ex, "\theta_!" description] & L_{E'}\mathcal{D} \arrow[l, "\theta^*" description] \end{tikzcd}$$

is a Quillen adjoint triple if $\langle E' \rangle = \langle \theta_* E \rangle$.

Proof. As $\langle E' \rangle = \langle \theta_* E \rangle$, the left Bousfield localizations $L_{E'}\mathcal{D}$ and $L_{\theta_*E}\mathcal{D}$ are *equal* as model categories. Therefore it suffices to prove the result for the case when $E' = \theta_*E$.

By [36, 3.1] the adjunction $\theta_* \dashv \theta^*$ is Quillen between the localizations as the objects correspond. For the $\theta^* \dashv \theta_!$ adjunction, by Hirschhorn [25, 3.3.18], it suffices to check that θ^* sends θ_*E-equivalences between cofibrant objects to E-equivalences. Let $f\colon A \to B$ be an θ_*E-equivalence between cofibrant objects. As $\theta^*(\theta_*E \otimes_R f) \cong E \otimes_S \theta^* f$ by the projection formula, the result follows. $\square$

We now give a means of checking the hypotheses of the previous theorem in a certain case. We note that if $E' \simeq \theta_*E$ (up to suspension) then they have the same Bousfield class. The following proposition shows that the condition that $E' \simeq \theta_*E$ is preserved by Quillen equivalences.

Proposition 4.13.

(1) *Let $F \dashv U$ be a strong monoidal Quillen pair. Let $\theta\colon S \to R$ and write $\varphi = F\theta\colon FS \to FR$. Let $E \in \mathrm{Mod}_S$ and $E' \in \mathrm{Mod}_R$, with both cofibrant. If $E' \simeq \theta_*E$, then $FE' \simeq \varphi_*FE$.*
(2) *Let $F \dashv U$ be a weak monoidal Quillen equivalence. Let $\theta\colon S \to R$ and write $\varphi = U\theta\colon US \to UR$. Let $E \in \mathrm{Mod}_S$ and $E' \in \mathrm{Mod}_R$, with both cofibrant. If $E' \simeq \theta_*E$, then $QU\widehat{f}E' \simeq \varphi_*QU\widehat{f}E$ where $\widehat{f}$ denotes fibrant replacement.*
(3) *Suppose that we have a commutative square of commutative monoids*

$$\begin{tikzcd} S \arrow[d, "\theta"'] & S' \arrow[l, "f"', "\sim"] \arrow[d, "\psi"] \\ R & R' \arrow[l, "\sim"', "g"] \end{tikzcd}$$

in which the horizontal maps are weak equivalences and Quillen invariance of modules holds.
  (a) *Let $E \in \mathrm{Mod}_{S'}$ and $E' \in \mathrm{Mod}_{R'}$ with both cofibrant, such that $E' \simeq \psi_*E$. Then $g_*E' \simeq \theta_*f_*E$.*
  (b) *Let $E \in \mathrm{Mod}_S$ and $E' \in \mathrm{Mod}_R$ with both cofibrant, such that $E' \simeq \theta_*E$. Then $\psi_*Qf^*E \simeq g^*E'$.*

Proof. Statement (1) is a consequence of the fact that F is strong monoidal, together with Ken Brown's lemma.

For (2), by Proposition 4.9 we have that $U\theta^* \cong \varphi^*U$. Taking mates, we have an equivalence $\varphi_*QU\widehat{f}E \simeq QU\widehat{f}\theta_*E \simeq QU\widehat{f}E'$ as required.

For (3), we have an isomorphism of functors $f^*\theta^* \cong \psi^*g^*$ by Proposition 4.7. Taking left adjoints yields an equivalence $\theta_*f_*E \simeq g_*\psi_*E \simeq g_*E'$ as required for (a). For (b), taking mates of $f^*\theta^* \cong \psi^*g^*$ gives an equivalence $\psi_*Qf^*E \simeq g^*\theta_*E \simeq g^*E'$. ∎

Now we turn to the case of cellularizations. Let $\mathcal{D}$ be a stable model category and X and Y be objects in $\mathcal{D}$. We say that X *builds* Y if Y is in the localizing subcategory generated by X, that is, if Y is in the smallest replete, triangulated full subcategory of $h\mathcal{D}$ which is closed under arbitrary coproducts and contains X.

Theorem 4.14. *Let*

$$\begin{tikzcd} \mathcal{C} \arrow[r, shift left=2, "\theta_*"] \arrow[r, shift right=2, "\theta_!"'] & \mathcal{D} \arrow[l, "\theta^*" description] \end{tikzcd}$$

be a Quillen adjoint triple between stable monoidal model categories. Let $K \in \mathcal{D}$ be cofibrant. If K builds $\theta_*\theta^*K$, then

$$\begin{tikzcd} \mathrm{Cell}_{\theta^*K}\mathcal{C} \arrow[r, shift left=2, "\theta_*"] \arrow[r, shift right=2, "\theta_!"'] & \mathrm{Cell}_K\mathcal{D} \arrow[l, "\theta^*" description] \end{tikzcd}$$

is a Quillen adjoint triple.

Proof. By [20, 2.7], the adjunction $\theta^* \dashv \theta_!$ is Quillen between the cellularizations as the objects correspond. To check that the adjunction $\theta_* \dashv \theta^*$ is Quillen between the cellularizations, we must check that θ^* sends K-cellular equivalences between fibrant objects to θ^*K-cellular equivalences by Hirschhorn [25, 3.3.18].

Suppose that $M \to N$ is a K-cellular equivalence between fibrant objects. By adjunction, $\theta^*M \to \theta^*N$ is a θ^*K-cellular equivalence if and only if $M \to N$ is a $\theta_*\theta^*K$-cellular equivalence. As K builds $\theta_*\theta^*K$, $M \to N$ is an $\theta_*\theta^*K$-cellular equivalence as required. ∎

Remark 4.15. If $\theta\colon S \to R$ is a map of commutative ring spectra such that R is small over S, then for any R-module K, K finitely builds $\theta_*\theta^*K$.

Part 2. The correspondence of functors

5. The general strategy

5.1. **Recap of the algebraic model.** In this section we recap the construction of the algebraic model for (co)free G-spectra for G connected. The free case is due to Greenlees-Shipley [19, 21], and the cofree case is work of Pol and the author [36]. We note that in the free case, we follow the Eilenberg-Moore approach taken in [21], rather than the approach via Koszul duality taken in [19]. It is important to note that the correspondence of functors we obtain by using the zig-zag of Quillen equivalences in [21] is *different* from the correspondence obtained in [19] by using the Koszul duality approach. In the cofree case we follow the direct approach given in [36], rather than the approach given by first passing to free G-spectra and then using the equivalence between derived torsion and derived complete modules.

Free G-spectra are modelled by the cellularization $\mathrm{Cell}_{G_+}\mathrm{Sp}_G$, and cofree G-spectra are modelled by the homological localization $L_{EG_+}\mathrm{Sp}_G$. We now recall the Quillen equivalences used in the construction of the algebraic model. Note that not all of the Quillen adjunctions below are Quillen equivalences, but they all become so after appropriate localization/cellularization.

(1) *Change of rings:* Beginning in G-spectra, the first step is to change rings along the map $\kappa\colon S^0 \to DEG_+$ where $DEG_+ = F(EG_+, S^0)$. Therefore the first stage is the extension and restriction of scalars adjunction

$$\mathrm{Sp}_G \underset{\kappa^*}{\overset{DEG_+\wedge -}{\rightleftarrows}} \mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G).$$

(2) *Fixed points-inflation adjunction:* The next step is to use categorical fixed points to remove equivariance. More precisely, the next stage is the adjunction

$$\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \underset{(-)^G}{\overset{DEG_+\otimes_{DBG_+}-}{\rightleftarrows}} \mathrm{Mod}_{DBG_+}(\mathrm{Sp})$$

where we have suppressed notation for the inflation functors in the left adjoint.

(3) *Shipley's algebraicization theorem:* The next stage is to use Shipley's algebraicization theorem [41] to pass from modules over the commutative ring spectrum DBG_+ to modules over a commutative DGA which we denote ΘDBG_+. This is a zig-zag of Quillen equivalences. See Section 8 for more details on the zig-zag of Quillen equivalences.

(4) *Formality:* One can next use the fact that polynomial algebras are strongly intrinsically formal as *commutative* DGAs to construct a quasiisomorphism $z\colon H^*BG \to \Theta DBG_+$. This gives a Quillen equivalence

$$\mathrm{Mod}_{\Theta DBG_+} \underset{z^*}{\overset{\Theta DBG_+\otimes_{H^*BG}-}{\rightleftarrows}} \mathrm{Mod}_{H^*BG}.$$

via extension and restriction of scalars.

(5) *Torsion and completion:* The final step is to identify the resulting localization or cellularization with an abelian model. In the localization case, this is the category of L_0^I-complete modules, and in the cellularization case this is the category of torsion modules.

5.2. **Proving the correspondence of functors.** In this section we explain the general process for proving that the functors correspond. We give the details here rather than in the following sections since the principle remains the same throughout the zig-zag of Quillen equivalences.

The general process is that we will have a square

$$\begin{tikzcd}
\mathcal{C}_G \arrow[rr, shift left, "F_G"] \arrow[d, shift right=3, "\theta_{(*)}"'] \arrow[d, shift left=3, "\theta_*"] & & \mathcal{D}_G \arrow[ll, shift left, "U_G"] \arrow[d, shift right=3, "\varphi_{(*)}"'] \arrow[d, shift left=3, "\varphi_*"] \\
\mathcal{C}_H \arrow[u, shift left=9, "\theta^*"] \arrow[u, shift right=9, "\theta^\dagger"'] \arrow[rr, shift left, "F_H"] & & \mathcal{D}_H \arrow[ll, shift left, "U_H"] \arrow[u, shift left=9, "\varphi^*"] \arrow[u, shift right=9, "\varphi^\dagger"']
\end{tikzcd}$$

of model categories and Quillen functors, where $\mathbb{R}\theta_{(*)} \cong \mathbb{L}\theta_*$ and $\mathbb{R}\varphi_{(*)} \cong \mathbb{L}\varphi_*$, and $F_G \dashv U_G$ and $F_H \dashv U_H$ are Quillen equivalences. Depending on the type of square we can then apply Theorem 4.4 in conjunction with Propositions 4.6, 4.7 and 4.9 to conclude that all six squares of derived functors commute.

In general the vertical functors will be the extension-restriction-coextension of scalars functors along a map of commutative rings $\theta\colon S \to R$. In general, R need not be cofibrant as an S-module, so that $\theta^* \dashv \theta_!$ will not be a Quillen adjunction by Proposition 3.8. To rectify this, we must cofibrantly replace R as a commutative S-algebra to obtain $S \to QR$. In a convenient model structure, this implies that QR is also cofibrant as an S-module, see Section 2.1. Quillen invariance of modules also shows that extension and restriction of scalars gives a Quillen equivalence $\mathrm{Mod}_R \simeq_Q \mathrm{Mod}_{QR}$.

In summary, each step will consist of:

(1) Construct a square of Quillen functors. This may involve taking cofibrant replacements of commutative algebras in a convenient model structure.
(2) Check that under the relevant localizations/cellularizations which make the horizontals Quillen equivalences, the vertical functors are still Quillen. In general, this can be achieved by using Theorem 4.12, Proposition 4.13 and Theorem 4.14.
(3) Use Propositions 4.6, 4.7 and 4.9 to verify that the hypotheses of Theorem 4.4 are satisfied and hence deduce that the Quillen functors correspond.

6. Change of rings

6.1. **The setup.** The first step in the series of Quillen equivalences is a change of rings along the map $S^0 \to DEG_+$. The construction of adjoints between $\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G)$ and $\mathrm{Mod}_{DEH_+}(\mathrm{Sp}_H)$ requires some explanation, because we not only have to change the underlying category between Sp_G and Sp_H, but also the rings that we take modules over.

Lemma 6.1. *Let $\mathcal{C}$ and $\mathcal{D}$ be monoidal categories and R be a monoid in $\mathcal{C}$. Suppose that we have adjunctions*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, shift left=2, "i_*"] \arrow[r, shift right=2, "i_!"'] & \mathcal{D} \arrow[l, "i^*" description]
\end{tikzcd}$$

where i^* is strong monoidal. If the projection formula for i_* is satisfied (see Definition 4.10) then we have adjunctions

$$\begin{tikzcd}
\mathrm{Mod}_{i^*R}(\mathcal{C}) \arrow[r, shift left=2, "i_*"] \arrow[r, shift right=2, "i_!"'] & \mathrm{Mod}_R(\mathcal{D}) \arrow[l, "i^*" description]
\end{tikzcd}$$

between the categories of modules.

Proof. Since i^* is strong monoidal it sends monoids in $\mathcal{C}$ to monoids in $\mathcal{D}$. Moreover, it follows that i_* is oplax monoidal and $i_!$ is lax monoidal. Let M be an i^*R-module.

To give i_*M a R-module structure define the action map by

$$R \wedge i_*M \xrightarrow[\cong]{p^{-1}} i_*(i^*R \wedge M) \xrightarrow{i_*(a)} i_*M$$

where p is the projection formula map and $a\colon i^*R \wedge M \to M$ is the module structure map for M. Similarly, we define a R-module structure on $i_!M$ by

$$R \wedge i_!M \xrightarrow{p'} i_!(i^*R \wedge M) \xrightarrow{i_!(a)} i_!M$$

where p' is the natural map in the projection formula for $i_!$, see Remark 4.11.

It remains to check that the action maps defined above are associative and unital. So as to avoid interrupting the flow, we defer the remainder of the proof to Appendix A. $\square$

Proposition 6.2. *There is a Quillen adjoint triple of functors*

$$\begin{tikzcd} \mathrm{Mod}_{DEH_+}(\mathrm{Sp}_H) \arrow[r, shift left=2, "i_*"] \arrow[r, shift right=2, "i_!"'] & \mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \arrow[l, "i^*"'] \end{tikzcd}$$

where i^* is the restriction from G-spectra to H-spectra, $i_* = G_+ \wedge_H -$ and $i_! = F_H(G_+, -)$.

Proof. Recall that the projection formula holds for $i_*\colon \mathrm{Sp}_H \to \mathrm{Sp}_G$, see [30, V.2.3]. By Lemma 6.1, it follows that we have adjunctions as described, so it only remains to check that they are Quillen. Since the weak equivalences and fibrations in the categories of modules are created by the forgetful functors to the underlying equivariant spectra, i^* and $i_!$ are right Quillen, since they are right Quillen when viewed as functors between G-spectra and H-spectra. $\square$

6.2. **Quillen functors.** In this section we show that all the functors of interest are Quillen after the relevant localizations and cellularizations.

Proposition 6.3. *There are Quillen adjoint triples of functors*

$$\begin{tikzcd} L_{EH_+}\mathrm{Sp}_H \arrow[r, shift left=2, "i_*"] \arrow[r, shift right=2, "i_!"'] & L_{EG_+}\mathrm{Sp}_G \arrow[l, "i^*"'] \end{tikzcd}$$

and

$$\begin{tikzcd} L_{EH_+}\mathrm{Mod}_{DEH_+}(\mathrm{Sp}_H) \arrow[r, shift left=2, "i_*"] \arrow[r, shift right=2, "i_!"'] & L_{EG_+}\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \arrow[l, "i^*"'] \end{tikzcd}$$

Proof. The Bousfield classes $\langle EG_+ \rangle$ and $\langle G_+ \wedge_H EH_+ \rangle$ are equal since for a free G-spectrum F, $X \wedge F \simeq 0$ if and only if X is non-equivariantly contractible. Therefore the result follows from Theorem 4.12. $\square$

Proposition 6.4. *There are Quillen adjoint triples of functors*

$$\begin{tikzcd} \mathrm{Cell}_{H_+}\mathrm{Sp}_H \arrow[r, shift left=2, "i_*"] \arrow[r, shift right=2, "i_!"'] & \mathrm{Cell}_{G_+}\mathrm{Sp}_G \arrow[l, "i^*"'] \end{tikzcd}$$

and

$$\begin{tikzcd} \mathrm{Cell}_{H_+}\mathrm{Mod}_{DEH_+}(\mathrm{Sp}_H) \arrow[r, shift left=2, "i_*"] \arrow[r, shift right=2, "i_!"'] & \mathrm{Cell}_{G_+}\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \arrow[l, "i^*"'] \end{tikzcd}$$

Proof. Since $i_*H_+ \simeq G_+$, we have that the $i_* \dashv i^*$ adjunction passes to the cellularizations. For the $i^* \dashv i_!$ adjunction, by Hirschhorn [25, 3.3.18] it suffices to check that $i_!$ sends H_+-cellular equivalences between fibrant objects to G_+-cellular equivalences. We have that i^*G_+ is built from H_+ as $EH_+ \wedge i^*G_+ \simeq i^*G_+$. Therefore $i_!$ sends H_+-cellular equivalences between fibrant objects to G_+-cellular equivalences. $\square$

6.3. **The corresponding adjoints.** We consider the square

$$\begin{tikzcd}
\mathrm{Mod}_{S^0}(\mathrm{Sp}_G) \arrow[r, shift left, "\kappa_*^G"] \arrow[d, "i^*"] & \mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \arrow[l, shift left, "\kappa_G^*"] \arrow[d, "i^*"] \\
\mathrm{Mod}_{S^0}(\mathrm{Sp}_H) \arrow[r, shift left, "\kappa_*^H"] \arrow[u, shift left=2, "i_*"] \arrow[u, shift right=2, "i_!"'] & \mathrm{Mod}_{DEH_+}(\mathrm{Sp}_H) \arrow[l, shift left, "\kappa_H^*"] \arrow[u, shift left=2, "i_*"] \arrow[u, shift right=2, "i_!"']
\end{tikzcd}$$

where the horizontals are given by extension and restriction of scalars along the ring maps $\kappa_G\colon S^0 \cong F(S^0,S^0) \to F(EG_+,S^0) = DEG_+$ and $\kappa_H\colon S^0 \to DEH_+$ and the verticals are the change of groups adjunctions of equivariant spectra.

Firstly, the forgetful functor i^* commutes with the restrictions κ^* on underlying spectra since none of the functors change the underlying spectra. It remains to show that the module structures agree. Let M be an object of $\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G)$ with module action map $a\colon DEG_+ \wedge M \to M$. We write Φ for the monoidal structure map of i^*.

$$\begin{tikzcd}
S^0 \wedge \kappa_H^* i^* M \arrow[r, "\kappa_H \wedge 1"] \arrow[d, "1"'] & DEH_+ \wedge i^* M \arrow[r, "1"] & i^*DEG_+ \wedge i^*M \arrow[r, "\Phi", "\cong"'] & i^*(DEG_+ \wedge M) \arrow[r, "i^*(a)"] \arrow[d, "1"] & i^*M \arrow[d, "1"] \\
S^0 \wedge i^*\kappa_G^* M \arrow[r, "1"'] & i^*S^0 \wedge i^*\kappa_G^*M \arrow[r, "\cong", "\Phi"'] \arrow[u, "\kappa_H \wedge 1"] & i^*(S^0 \wedge \kappa_G^* M) \arrow[r, "i^*(\kappa_G \wedge 1)"'] \arrow[ur, "i^*(\kappa_G\wedge 1)"] & i^*(DEG_+ \wedge M) \arrow[r, "i^*(a)"'] & i^*M
\end{tikzcd}$$

The top row is the module structure defined by $\kappa_H^* i^* M$ and the bottom row is the module structure defined by $i^*\kappa_G^*$. This diagram commutes by naturality of Φ using that $i^*\kappa_G = \kappa_H$, and hence the module structures agree (i.e., the identity is a module map). Therefore we have a natural isomorphism $\kappa_H^* i^* \cong i^* \kappa_G^*$.

We now check that the identity gives a natural isomorphism $i_!\kappa_H^* \cong \kappa_G^* i_!$. Let M be a DEH_+-module with action map $a\colon DEH_+ \wedge M \to M$. Then the underlying objects of $i_!\kappa_H^* M$ and $\kappa_G^* i_! M$ are the same so it remains to check the module structures agree.

$$\begin{tikzcd}
S^0 \wedge i_!\kappa_H^* M \arrow[r, "p^{-1}"] \arrow[d, "1"'] \arrow[dr, "\kappa_G \wedge 1"] & i_!(S^0 \wedge \kappa_H^* M) \arrow[r, "i_!(\kappa_H \wedge 1)"] \arrow[dr, "i_!(\kappa_H\wedge 1)"] & i_!(DEH_+ \wedge M) \arrow[r, "i_!(a)"] & i_!M \arrow[d, "1"] \\
S^0 \wedge \kappa_G^* i_! M \arrow[r, "\kappa_G \wedge 1"'] & DEG_+ \wedge i_!M \arrow[r, "p^{-1}"'] & i_!(DEH_+ \wedge M) \arrow[r, "i_!(a)"'] & i_!M
\end{tikzcd}$$

The top row is the module structure on $i_!\kappa_H^* M$ and the bottom row is the module structure on $\kappa_G^* i_! M$. The diagram commutes by the naturality of p, and hence we have a natural isomorphism $i_!\kappa_H^* \cong \kappa_G^* i_!$. It follows from the special case of Theorem 4.4 in which the two downward pointing arrows are the same, that all the squares commute at the derived level.

7. The fixed points-inflation adjunction

In this section we describe how the passage from equivariant module spectra to non-equivariant module spectra interacts with change of groups functors.

7.1. **Quillen pairs.** Stolz [43] constructs a flat model structure on orthogonal G-spectra such that the identity functor is a left Quillen equivalence from the stable model structure to the flat model structure. The flat model structure has all the necessary properties to apply our general results from Sections 3 and 4. In particular, the flat model structure on orthogonal G-spectra is a monoidal model structure which satisfies the monoid axiom (so that the categories of modules inherit a right lifted model structure), is convenient, and has flat cofibrant objects, see Section 2 for more details. Before we can show a correspondence of functors we need to prove that certain functors are Quillen in the flat model structure on equivariant spectra.

The flat model structure is built from the blended model structure on G-spaces. In the blended model structure, the weak equivalences are the naive weak equivalences and the fibrations are the maps which are genuine fibrations and for which

$$\begin{tikzcd} X \arrow[r] \arrow[d] & F(EG_+, \widehat{f}X) \arrow[d] \\ Y \arrow[r] & F(EG_+, \widehat{f}Y) \end{tikzcd}$$

is a homotopy pullback (in the genuine model structure), where $\widehat{f}$ denotes a genuine fibrant replacement functor.

Lemma 7.1. *There is a Quillen adjoint triple*

$$\mathrm{Sp}_G^{\mathrm{flat}} \underset{\longleftarrow i_! \longrightarrow}{\overset{\longleftarrow i_* \longrightarrow}{\xrightarrow{\quad i^* \quad}}} \mathrm{Sp}_H^{\mathrm{flat}}$$

where both categories are equipped with the flat model structure.

Proof. Firstly we claim that $i_!$ is right Quillen between the level flat model structures. To prove this it is sufficient to check that $i_!$ is right Quillen as a functor $\mathrm{Top}_H^{\mathrm{blended}} \to \mathrm{Top}_G^{\mathrm{blended}}$. Let $g\colon X \to Y$ be a blended H-fibration between blended H-fibrant objects, i.e., g is a genuine H-fibration and

$$\begin{tikzcd} X \arrow[r] \arrow[d] & F(EH_+, \widehat{f}X) \arrow[d] \\ Y \arrow[r] & F(EH_+, \widehat{f}Y) \end{tikzcd}$$

is a homotopy pullback (in the genuine model structure). We know that $i_!$ sends genuine H-fibrations to genuine G-fibrations. Since the objects in the homotopy pullback square are fibrant, $i_!$ sends this homotopy pullback square to a homotopy pullback square. In other words,

$$\begin{tikzcd} i_!X \arrow[r] \arrow[d] & i_!F(EH_+, \widehat{f}X) \arrow[d] \\ i_!Y \arrow[r] & i_!F(EH_+, \widehat{f}Y) \end{tikzcd}$$

is a homotopy pullback square. A simple adjunction argument shows that $i_!F(EH_+, \widehat{f}X) \cong F(EG_+, i_!\widehat{f}X)$ and therefore $i_!g$ is a blended G-fibration. In a similar fashion, one sees that $i_!$ sends blended H-fibrant objects to blended G-fibrant objects. Therefore, $i_!$ sends blended H-fibrations between blended H-fibrants to blended G-fibrations between blended G-fibrants. As the blended model structure is a left Bousfield localization of the genuine model structure, the acyclic fibrations in both model structures coincide and hence $i_!$ preserves the acyclic fibrations in the blended model structure. Therefore by [9, A.2], $i_!\colon \mathrm{Top}_H^{\mathrm{blended}} \to \mathrm{Top}_G^{\mathrm{blended}}$ is right Quillen. It follows from [36, 3.1] that as $i_!$ is right Quillen between the level flat model structures, it is right Quillen between the flat model structures.

By [8, 2.6.11] the adjunction $i_* \dashv i^*$ is Quillen with respect to the level flat model structures. Let S_G denote the set of morphisms with the property that $L_{S_G}\mathrm{Sp}_G^{\text{level flat}} = \mathrm{Sp}_G^{\mathrm{flat}}$. By Hirschhorn [25, 3.3.18] it is sufficient to check that i_* sends S_H-local equivalences between flat cofibrant objects to S_G-local equivalences. Since $L_{S_G}\mathrm{Sp}_G^{\text{level flat}} = \mathrm{Sp}_G^{\mathrm{flat}}$, the S_G-local equivalences are precisely the $\underline{\pi}_*$-isomorphisms (the usual weak equivalences of G-spectra). It follows that i_* sends S_H-local equivalences between flat cofibrant objects to S_G-local equivalences as induction i_* preserves all $\underline{\pi}_*$-isomorphisms. □

Proposition 7.2. *There is a Quillen adjunction*

$$\mathrm{Sp}_G^{\mathrm{flat}} \underset{(-)^G}{\overset{\mathrm{inf}}{\leftrightarrows}} \mathrm{Sp}^{\mathrm{flat}}$$

where the categories are equipped with the flat model structure.

Proof. By mimicking the proof that $i_!\colon \mathrm{Sp}_H^{\text{level flat}} \to \mathrm{Sp}_G^{\text{level flat}}$ is right Quillen, one can show that $(-)^G\colon \mathrm{Sp}_G^{\text{level flat}} \to \mathrm{Sp}^{\text{level flat}}$ is right Quillen. To complete the proof, by Hirschhorn [25, 3.3.18] it is sufficient to check that inflation sends π_*-isomorphisms between flat cofibrant spectra to $\underline{\pi}_*$-isomorphisms. This is clear since inflation preserves weak equivalences. □

7.2. **A coinduction Quillen equivalence.** The coinduction functor $i_!\colon \mathrm{Sp}_H \to \mathrm{Sp}_G$ is lax symmetric monoidal since the forgetful functor $i^*\colon \mathrm{Sp}_G \to \mathrm{Sp}_H$ is strong symmetric monoidal. Therefore the coinduction functor $i_!$ takes (commutative) ring H-spectra to (commutative) ring G-spectra.

Proposition 7.3. *Let $H \leq G$ and R be a ring H-spectrum. If R generates $\mathrm{Mod}_R(\mathrm{Sp}_H)$, then $i_!R$ generates $\mathrm{Mod}_{i_!R}(\mathrm{Sp}_G)$.*

Proof. Any $i_!R$-module M is a retract of $i_!R \wedge M$. By the Wirthmüller isomorphism and the projection formula for i_*, we have $i_!R \wedge M \simeq i_!(R \wedge i^*M)$ and the result follows. □

As H is connected (and we work rationally), the commutative ring H-spectrum DEH_+ generates its category of modules [16, 3.1]. Therefore we obtain the following corollary.

Corollary 7.4. *The commutative ring G-spectrum $F_H(G_+, DEH_+)$ generates its category of modules.*

Proposition 7.5. *The adjunction*

$$\mathrm{Mod}_{DEH_+}(\mathrm{Sp}_H) \underset{i_!}{\overset{DEH_+\otimes_{i^*F_H(G_+,DEH_+)} i^*(-)}{\leftrightarrows}} \mathrm{Mod}_{F_H(G_+,DEH_+)}(\mathrm{Sp}_G)$$

is a Quillen equivalence.

Proof. The existence of the Quillen adjunction follows from [38, §3], also see Section 2.3. The derived counit is an equivalence on DEH_+ and therefore as DEH_+ and $F_H(G_+, DEH_+)$ generate their categories of modules, by the Cellularization Principle [20, 2.7] the adjunction is a Quillen equivalence. □

Remark 7.6. Since the adjunction

$$\mathrm{Mod}_{DEH_+}(\mathrm{Sp}_H) \underset{i_!}{\overset{DEH_+\otimes_{i^*F_H(G_+,DEH_+)} i^*(-)}{\leftrightarrows}} \mathrm{Mod}_{F_H(G_+,DEH_+)}(\mathrm{Sp}_G)$$

is Quillen in both the stable and flat model structures, the previous proposition holds in both the stable and flat model structures.

Remark 7.7. Proposition 7.5 could be viewed as an analogue of the results of Balmer-Dell'Ambrogio-Sanders [3, 1.1] on étale extensions, also see [32, 5.32].

7.3. **Commutativity.** Let $\omega\colon DEG_+ \to F_H(G_+, DEH_+)$ be the natural map which is adjoint to the identity map on DEH_+ (i.e., the unit map $DEG_+ \to i_! i^* DEG_+$). By taking a cofibrant replacement of $F_H(G_+, DEH_+)$ as a commutative DEG_+-algebra, we obtain a commutative diagram

$$\begin{tikzcd}
DEG_+ \arrow[d, "\omega"'] \arrow[dr, hook, "\psi"] & \\
F_H(G_+, DEH_+) & QF_H(G_+, DEH_+). \arrow[l, two heads, "\sim", "q"']
\end{tikzcd}$$

Taking categorical G-fixed points yields a map

$$DBG_+ \xrightarrow{\psi^G} (QF_H(G_+, DEH_+))^G \xrightarrow{\sim} F_H(G_+, DEH_+)^G \xrightarrow{\cong} DBH_+.$$

The isomorphism follows by the fact that there is a natural isomorphism $(i_! M)^G \cong M^H$ which can be proved by an adjointness argument using the Yoneda lemma.

By factoring this map into a cofibration followed by an acyclic fibration of commutative DBG_+-algebras, we obtain a commutative diagram

$$\begin{tikzcd}
DBG_+ \arrow[d] \arrow[dr, hook, "\varphi"] & \\
DBH_+ & QDBH_+. \arrow[l, two heads, "\sim"']
\end{tikzcd}$$

Since $DBG_+ \to QDBH_+$ is a cofibration and $(QF_H(G_+, DEH_+))^G \to F_H(G_+, DEH_+)^G$ is an acyclic fibration, we may use the lifting properties to obtain a natural map $\chi\colon QDBH_+ \to (QF_H(G_+, DEH_+))^G$.

Notation 7.8. We set the notation $a = \dim(G/H)$. Note that this is the shift that arises in the Wirthmüller isomorphism and in the Gorenstein condition for the map $DBG_+ \to DBH_+$, see Theorem 3.3.

To complete the necessary setup, we must check that $F_H(G_+, DEH_+)$ is a small DEG_+-module. We note that this is not covered by Proposition 3.1 as we are in the land of *equivariant* spectra. Nonetheless, one sees that $F_H(G_+, DEH_+)$ is a small DEG_+-module since

$$[F_H(G_+, DEH_+), -]^{DEG_+} \cong [\Sigma^{-a} G/H_+ \wedge DEG_+, -]^{DEG_+} \cong [\Sigma^{-a} G/H_+, -]$$

by the Wirthmüller isomorphism. Since G/H_+ is a small G-spectrum (as induction is left adjoint to the restriction functor i^* which preserves sums), the result follows.

Now that we have set up the appropriate groundwork, we can move on to showing that we have a correspondence of functors. The first stage is the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \arrow[r, shift right, "1"'] \arrow[d, shift left=2, "i^*"] & \mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \arrow[l, shift right, "1"'] \arrow[d, shift left=1, "\Sigma^{-a}\psi_!"] \arrow[d, shift left=4, "\psi_*"] \\
\mathrm{Mod}_{DEH_+}(\mathrm{Sp}_H) \arrow[u, shift left=5, "i_*"] \arrow[u, shift right=1, "i_!"'] \arrow[r, shift right, "q^* i_!"'] & \mathrm{Mod}_{QF_H(G_+, DEH_+)}(\mathrm{Sp}_G) \arrow[l, shift right, "DEH_+ \otimes_{i^* F_H(G_+, DEH_+)} i^* q_*(-)"'] \arrow[u, shift left=4, "\Sigma^a \psi^*"] \arrow[u, shift right=6, "\psi^*"']
\end{tikzcd}$$

We must firstly show that the functors pass to the localizations and cellularizations.

Proposition 7.9. *There is a Quillen adjoint triple of functors*

$$\begin{tikzcd} L_{q^*F_H(G_+,EH_+)}\mathrm{Mod}_{QF_H(G_+,DEH_+)}(\mathrm{Sp}_G) \arrow[r, "\psi^*" description] & L_{EG_+}\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \arrow[l, bend right, "\psi_*"'] \arrow[l, bend left, "\psi_!"] \end{tikzcd}$$

Proof. Since the diagram

$$\begin{tikzcd} DEG_+ \arrow[d, "\omega"'] \arrow[dr, hook, "\psi"] & \\ F_H(G_+, DEH_+) & QF_H(G_+, DEH_+). \arrow[l, two heads, "\sim", "q"'] \end{tikzcd}$$

commutes, we have an isomorphism of functors $\omega_* \cong q_*\psi_*$. Taking mates, we have an equivalence $\psi_* \simeq q^*\omega_*$ on cofibrant objects. Note that $\omega_* EG_+ \simeq F_H(G_+, EH_+)$ since

$$F_H(G_+, DEH_+) \otimes_{DEG_+} EG_+ \simeq F(G/H_+, S^0) \wedge EG_+ \simeq F_H(G_+, EH_+)$$

by smallness of G/H_+. Therefore $\psi_* EG_+ \simeq q^*F_H(G_+, EH_+)$ and the result follows from Theorem 4.12. $\square$

Since $i_! H_+ \simeq G_+$ (up to shift) by the Wirthmüller isomorphism, Theorem 4.14 yields the following.

Proposition 7.10. *There is a Quillen adjoint triple of functors*

$$\begin{tikzcd} \mathrm{Cell}_{q^*i_!H_+}\mathrm{Mod}_{QF_H(G_+,DEH_+)}(\mathrm{Sp}_G) \arrow[r, "\psi^*" description] & \mathrm{Cell}_{G_+}\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \arrow[l, bend right, "\psi_*"'] \arrow[l, bend left, "\psi_!"] \end{tikzcd}$$

We can now show that the functors correspond. We note that for a DEH_+-module M, the coinduction $i_!M$ is both a DEG_+-module and a $F_H(G_+, DEH_+)$-module: the DEG_+-module structure arises from the projection formula map and the $F_H(G_+, DEH_+)$-module structure by using the lax monoidal structure on $i_!$.

Proposition 7.11. *Let $M \in \mathrm{Mod}_{DEH_+}$. Then $i_!M \cong \psi^*q^*i_!M$ as DEG_+-modules.*

Proof. The underlying objects are the same so it is sufficient to check that the module structures agree. Note that $\psi^*q^* \cong (q\psi)^* = \omega^*$. We write $R = DEG_+$ to ease the notation. We must check that

$$\begin{tikzcd} R \wedge i_!M \arrow[r, "p'"] \arrow[d, "1"'] \arrow[dr, "\eta\wedge 1"] & i_!(i^*R \wedge M) \arrow[r, "i_!(a)"] \arrow[dr, "1"] & i_!M \arrow[dr, "1"] & \\ R \wedge \omega^* i_!M \arrow[r, "\omega\wedge 1"'] & i_!i^*R \wedge i_!M \arrow[r, "l"'] & i_!(i^*R \wedge M) \arrow[r, "i_!(a)"'] & i_!M \end{tikzcd}$$

commutes, where p' is the projection formula map and l is the lax monoidal structure on $i_!$. By definition ω is the adjoint of the identity map on i^*R, or in other words it is the unit $\eta\colon R \to i_!i^*R$ and hence the first triangle commutes. From this one sees that it is enough to just check that the middle square commutes. Spelling out the definitions of p' and l, this amounts to checking that

$$\begin{tikzcd} R \wedge i_!M \arrow[r, "\eta"] \arrow[d, "\eta\wedge 1"'] & i_!i^*(R \wedge i_!M) \arrow[rr, "i_!\Phi^{-1}"] \arrow[d, "i_!i^*(\eta\wedge 1)"] & & i_!(i^*R \wedge i^*i_!M) \arrow[dl, "i_!(\eta\wedge 1)"'] \arrow[d, "i_!(1\wedge\varepsilon)"] \\ i_!i^*R \wedge i_!M \arrow[r, "\eta"'] & i_!i^*(i_!i^*R \wedge i_!M) \arrow[r, "i_!\Phi^{-1}"'] & i_!(i^*i_!i^*R \wedge i^*i_!M) \arrow[r, "i_!(\varepsilon\wedge\varepsilon)"'] & i_!(i^*R \wedge M) \end{tikzcd}$$

commutes, where going right and down is the map p' and going along the bottom row is the lax monoidal structure map l. This diagram commutes using (from left to right) naturality of η, naturality of Φ^{-1} and the triangle identities. $\square$

Proposition 7.12. *For any fibrant DEG_+-module M, there is a natural weak equivalence $\Sigma^{-a}\psi_! M \xrightarrow{\sim} q^* i_! i^* M$.*

Proof. The Wirthmüller isomorphism gives a natural weak equivalence $G_+ \wedge_H DEH_+ \to \Sigma^a F_H(G_+, DEH_+)$. Since $\Sigma^a QF_H(G_+, DEH_+) \to \Sigma^a F_H(G_+, DEH_+)$ is an acyclic fibration and $G_+ \wedge_H DEH_+$ is cofibrant, by liting properties we have a natural weak equivalence $G_+ \wedge_H DEH_+ \to \Sigma^a QF_H(G_+, DEH_+)$ between cofibrant objects. We then have the string of natural weak equivalences

$$\begin{aligned}
\Sigma^{-a}\psi_! M &= \Sigma^{-a}\mathrm{Hom}_{DEG_+}(QF_H(G_+, DEH_+), M) \\
&\xrightarrow{\sim} q^*\mathrm{Hom}_{DEG_+}((G_+ \wedge_H DEH_+), M) \\
&\cong q^*\mathrm{Hom}_{DEG_+}(G/H_+ \wedge DEG_+, M) \\
&\cong q^* F(G/H_+, M) \\
&\cong q^* i_! i^* M
\end{aligned}$$

where the first equivalence follows from Ken Brown's lemma. $\square$

Therefore, by Theorem 4.4 all six squares of derived functors commute.

The next square in this step is the diagram

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Mod}_{DEG_+}(\mathrm{Sp}_G) \arrow[r, shift left, "(-)^G"] \arrow[d, shift left=3, "\Sigma^{-a}\psi_!"] \arrow[d, shift right=1, "\psi_*"'] & \mathrm{Mod}_{DBG_+}(\mathrm{Sp}) \arrow[l, shift left, "DEG_+ \otimes_{DBG_+} -"] \arrow[d, shift left=3, "\Sigma^{-a}\varphi_!"] \arrow[d, shift right=1, "\varphi_*"'] \\
\mathrm{Mod}_{QF_H(G_+,DEH_+)}(\mathrm{Sp}_G) \arrow[r, shift right, "\chi^*((-)^G)"'] \arrow[u, shift left=7, "\Sigma^a\psi^*"] \arrow[u, shift right=5, "\psi^*"'] & \mathrm{Mod}_{QDBH_+}(\mathrm{Sp}) \arrow[l, shift right, "QF_H(G_+,DEH_+) \otimes_{QDBH_+} (-)"'] \arrow[u, shift left=7, "\Sigma^a\varphi^*"] \arrow[u, shift right=5, "\varphi^*"']
\end{tikzcd}$$

where $\chi\colon QDBH_+ \to (QF_H(G_+, DEH_+))^G$ is the natural map constructed by lifting properties of the cofibration $DBG_+ \to QDBH_+$ against the acyclic fibration $(QF_H(G_+, DEH_+))^G \to F_H(G_+, DEH_+)^G$.

Firstly we check that the vertical functors are Quillen after localization. In order to do this we require a lemma. We write Γ_k for the k-cellularization functor, i.e., the right adjoint to the inclusion of the localizing subcategory generated by k.

Lemma 7.13. *Let $\theta\colon S \to R$ be a map of ring spectra such that R is small over S, and let k be an R-algebra. Then for M an S-module, we have $R \otimes_S \Gamma_k M \simeq \Gamma_k(R \otimes_S M)$.*

Proof. We must check that $R \otimes_S \Gamma_k M$ satisfies the universal properties of the cellularization. Firstly, there is a natural map $R \otimes_S \Gamma_k M \to R \otimes_S M$ as there is a natural map $\Gamma_k M \to M$. Since S finitely builds R, k finitely builds $R \otimes_S k$, so $R \otimes_S \Gamma_k M$ is k-cellular. It remains to check that $R \otimes_S \Gamma_k M \to R \otimes_S M$ is a k-cellular equivalence.

Note that there are natural equivalences

$$\mathrm{Hom}_R(k, R \otimes_S -) \simeq \mathrm{Hom}_R(k, \mathrm{Hom}_S(DR, -)) \simeq \mathrm{Hom}_S(DR, \mathrm{Hom}_S(k, -))$$

since R is a small S-module. It follows that $R \otimes_S \Gamma_k M \to R \otimes_S M$ is a k-cellular equivalence as $\Gamma_k M \to M$ is a k-cellular equivalence. $\square$

Corollary 7.14. *There is an equivalence $DBH_+ \otimes_{DBG_+} \Sigma^{\dim(G)} BG_+ \simeq \Sigma^{\dim(H)} BH_+$.*

Proof. This follows from Lemma 7.13 and the fact that $\Gamma_{S^0} DBG_+ \simeq \Sigma^{\dim(G)} BG_+$ by Gorenstein duality [11]. □

Proposition 7.15. *There is a Quillen adjoint triple of functors*

$$L_{BH_+}\mathrm{Mod}_{QDBH_+}(\mathrm{Sp}) \begin{array}{c} \xleftarrow{\quad\varphi_*\quad} \\ \xrightarrow{\quad\varphi^*\quad} \\ \xleftarrow{\quad\varphi_!\quad} \end{array} L_{BG_+}\mathrm{Mod}_{DBG_+}(\mathrm{Sp})$$

Proof. By Corollary 7.14, we have that $\varphi_* BG_+ \simeq BH_+$ up to suspension. The result then follows from Theorem 4.12. □

Applying Theorem 4.14 we have the following.

Proposition 7.16. *There is a Quillen adjoint triple of functors*

$$\mathrm{Cell}_{S^0}\mathrm{Mod}_{QDBH_+}(\mathrm{Sp}_H) \begin{array}{c} \xleftarrow{\quad\varphi_*\quad} \\ \xrightarrow{\quad\varphi^*\quad} \\ \xleftarrow{\quad\varphi_!\quad} \end{array} \mathrm{Cell}_{S^0}\mathrm{Mod}_{DBG_+}(\mathrm{Sp}_G)$$

It follows from Proposition 4.9 and Theorem 4.4 that all six derived squares commute.

8. Shipley's algebraicization theorem

In this section we use the results of [45] to pass between modules over the commutative ring spectra DBG_+ and $QDBH_+$ to modules over commutative DGAs, whilst keeping track of the change of rings adjunctions. This consists of three steps which we will deal with separately. The first step involves passage from symmetric spectra in simplicial sets to symmetric spectra in simplicial $\mathbb{Q}$-modules, the second step is a stabilized version of the Dold-Kan equivalence and the final step is the passage to algebra. Shipley [41] proved that these Quillen equivalences hold in the stable model structure. For our purposes, the flat model structure is crucial, and the Quillen equivalences are proven to hold in the flat model structures in [45].

8.1. Recap of the equivalences.

We provide a brief recap of the Quillen equivalences used in Shipley's algebraicization theorem and use this as an opportunity to set notation. For more details see [41] and [45].

Let $\mathcal{C}$ be a bicomplete, closed symmetric monoidal category. Let Σ be the category whose objects are the finite sets $\underline{n} = \{1, \dots, n\}$ for $n \geq 0$ where $\underline{0} = \emptyset$, and whose morphisms are the bijections of $\underline{n}$. The category of *symmetric sequences* in $\mathcal{C}$ is the functor category $\mathcal{C}^\Sigma$. For an object $K \in \mathcal{C}$, the category of *symmetric spectra* $\mathrm{Sp}^\Sigma(\mathcal{C}, K)$ is the category of modules over $\mathrm{Sym}(K)$ in $\mathcal{C}^\Sigma$, where $\mathrm{Sym}(K) = (\mathbb{1}, K, K^{\otimes 2}, \cdots)$ is the free commutative monoid on K.

We write $\widetilde{\mathbb{Q}} \colon \mathrm{sSet}^\Sigma \to \mathrm{s}\mathbb{Q}\text{-mod}^\Sigma$ for the functor which takes the free simplicial $\mathbb{Q}$-module levelwise. Recall that we define $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$ to be the category of modules over $\mathrm{Sym}(\widetilde{\mathbb{Q}}S^1)$ in $\mathrm{s}\mathbb{Q}\text{-mod}^\Sigma$. The object $\mathrm{Sym}(\widetilde{\mathbb{Q}}S^1)$ is equivalent to $H\mathbb{Q}$. There is a ring map $\alpha \colon H\mathbb{Q} \to \widetilde{\mathbb{Q}}H\mathbb{Q}$, and the composite $\alpha^*\widetilde{\mathbb{Q}}$ gives a zig-zag of strong monoidal Quillen equivalences between $\mathrm{Mod}_{H\mathbb{Q}}$ and $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod})$.

We write $\mathrm{Sp}^\Sigma(\mathrm{Ch}^+_\mathbb{Q})$ for the category of modules over $\mathrm{Sym}(\mathbb{Q}[1])$ in $(\mathrm{Ch}^+_\mathbb{Q})^\Sigma$ where $\mathbb{Q}[1]$ is the chain complex which contains a single copy of $\mathbb{Q}$ in degree 1. Applying the normalized chains functor $N \colon \mathrm{s}\mathbb{Q}\text{-mod} \to \mathrm{Ch}^+_\mathbb{Q}$ levelwise gives a functor $\mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod}) \to \mathrm{Mod}_{\mathcal{N}}((\mathrm{Ch}^+_\mathbb{Q})^\Sigma)$ where $\mathcal{N} = N\mathrm{Sym}(\widetilde{\mathbb{Q}}S^1)$. There is a ring map $\varphi \colon \mathrm{Sym}(\mathbb{Q}[1]) \to \mathcal{N}$, and the composite

$$\varphi^* N \colon \mathrm{Sp}^\Sigma(\mathrm{s}\mathbb{Q}\text{-mod}) \to \mathrm{Sp}^\Sigma(\mathrm{Ch}^+_\mathbb{Q})$$

is a right Quillen equivalence. This forms a weak monoidal Quillen equivalence.

Finally, there is a functor $R\colon \mathrm{Ch}_{\mathbb{Q}} \to \mathrm{Sp}^{\Sigma}(\mathrm{Ch}_{\mathbb{Q}}^{+})$ which is defined by $RY_n = C_0(Y \otimes \mathbb{Q}[n])$ where C_0 denotes the connective cover, and this functor is a right Quillen equivalence. Together with its left adjoint D, this forms a strong monoidal Quillen equivalence.

Since all of these functors are appropriately monoidal, they lift to give Quillen equivalences between the categories of modules over monoids as described in Section 2.3.

8.2. **To simplicial $\mathbb{Q}$-modules.** To ease the notation, we will work in the general setting of a map of commutative ring spectra $\varphi\colon S \to R$. Suppose that R is a cofibrant S-module and write $\kappa = \widetilde{\mathbb{Q}}\varphi\colon \widetilde{\mathbb{Q}}S \to \widetilde{\mathbb{Q}}R$. Since $\widetilde{\mathbb{Q}}$ is left Quillen, $\widetilde{\mathbb{Q}}R$ is cofibrant as a $\widetilde{\mathbb{Q}}S$-module. Cofibrantly replacing $\alpha^*\widetilde{\mathbb{Q}}R$ as a commutative $\alpha^*\widetilde{\mathbb{Q}}S$-algebra gives a commutative diagram

$$\begin{tikzcd}
\alpha^*\widetilde{\mathbb{Q}}S \arrow[d, "\alpha^*\kappa"'] \arrow[dr, hook, "\delta"] & \\
\alpha^*\widetilde{\mathbb{Q}}R & Q\alpha^*\widetilde{\mathbb{Q}}R \arrow[l, two heads, "\sim", "q"']
\end{tikzcd}$$

The passage to simplicial $\mathbb{Q}$-modules can be seen as the two squares

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Mod}_S \arrow[r, shift left, "\widetilde{\mathbb{Q}}"] \arrow[d, shift right=6, leftarrow, "\Sigma^a\psi^*"'] \arrow[d, shift right=2, "\Sigma^{-a}\psi_!"'] \arrow[d, shift left=2, "\psi_*"] \arrow[d, shift left=6, leftarrow, "\psi^*"] & \mathrm{Mod}_{\widetilde{\mathbb{Q}}S} \arrow[l, shift left, "U"] \arrow[r, shift right, "\alpha^*"'] \arrow[d, shift right=6, leftarrow, "\Sigma^a\kappa^*"'] \arrow[d, shift right=2, "\Sigma^{-a}\kappa_!"'] \arrow[d, shift left=2, "\kappa_*"] \arrow[d, shift left=6, leftarrow, "\kappa^*"] & \mathrm{Mod}_{\alpha^*\widetilde{\mathbb{Q}}S} \arrow[l, shift right, "\alpha_*"'] \arrow[d, shift right=6, leftarrow, "\Sigma^a\delta^*"'] \arrow[d, shift right=2, "\Sigma^{-a}\delta_!"'] \arrow[d, shift left=2, "\delta_*"] \arrow[d, shift left=6, leftarrow, "\delta^*"] \\
\mathrm{Mod}_R \arrow[r, shift left, "\widetilde{\mathbb{Q}}"] & \mathrm{Mod}_{\widetilde{\mathbb{Q}}R} \arrow[l, shift left, "U"] \arrow[r, shift right, "q^*\alpha^*"'] & \mathrm{Mod}_{Q\alpha^*\widetilde{\mathbb{Q}}S} \arrow[l, shift right, "\alpha_*q_*"']
\end{tikzcd}$$

The vertical functors are still Quillen after the appropriate localization and cellularization by Theorems 4.12 and 4.14 and Proposition 4.13. The first square satisfies the hypotheses of Proposition 4.6 and the second square satisfies the hypotheses of Proposition 4.9, so it follows from Theorem 4.4 that each of the six squares of derived functors commutes.

8.3. **The Dold-Kan type equivalence.** Since φ^*N is lax symmetric monoidal, we obtain a map of commutative monoids $\varphi^*N\delta$, and by cofibrant replacement in commutative algebras, we obtain a commutative diagram

$$\begin{tikzcd}
\varphi^*N\alpha^*\widetilde{\mathbb{Q}}S \arrow[d, "\varphi^*N\delta"'] \arrow[dr, hook, "\gamma"] & \\
\varphi^*NQ\alpha^*\widetilde{\mathbb{Q}}R & Q\varphi^*NQ\alpha^*\widetilde{\mathbb{Q}}R \arrow[l, two heads, "\sim", "q"']
\end{tikzcd}$$

From these we get the square which gives the passage from symmetric spectra in simplicial $\mathbb{Q}$-modules to symmetric spectra in non-negatively graded chain complexes of $\mathbb{Q}$-modules, as shown in the following diagram. Note that since (L, φ^*N) is only a weak monoidal Quillen pair as an adjunction between $\mathrm{Sp}^{\Sigma}(\mathrm{s}\mathbb{Q}\text{-mod})$ and $\mathrm{Sp}^{\Sigma}(\mathrm{Ch}_{\mathbb{Q}}^{+})$, the left adjoint to φ^*N at the level of modules is not L.

$$\begin{tikzcd}[column sep=large, row sep=huge]
\mathrm{Mod}_{\alpha^*\widetilde{\mathbb{Q}}S} \arrow[rr, shift right, "\varphi^*N"'] \arrow[d, shift right=6, leftarrow, "\Sigma^a\delta^*"'] \arrow[d, shift right=2, "\Sigma^{-a}\delta_!"'] \arrow[d, shift left=2, "\delta_*"] \arrow[d, shift left=6, leftarrow, "\delta^*"] & & \mathrm{Mod}_{\varphi^*N\alpha^*\widetilde{\mathbb{Q}}S} \arrow[ll, shift right, "L^{\alpha^*\widetilde{\mathbb{Q}}S}"'] \arrow[d, shift right=6, leftarrow, "\Sigma^a\gamma^*"'] \arrow[d, shift right=2, "\Sigma^{-a}\gamma_!"'] \arrow[d, shift left=2, "\gamma_*"] \arrow[d, shift left=6, leftarrow, "\gamma^*"] \\
\mathrm{Mod}_{Q\alpha^*\widetilde{\mathbb{Q}}R} \arrow[r, shift right, "\varphi^*N"'] & \mathrm{Mod}_{\varphi^*NQ\alpha^*\widetilde{\mathbb{Q}}R} \arrow[l, shift right, "L^{Q\alpha^*\widetilde{\mathbb{Q}}S}"'] \arrow[r, shift right, "q^*"'] & \mathrm{Mod}_{Q\varphi^*NQ\alpha^*\widetilde{\mathbb{Q}}R} \arrow[l, shift right, "q_*"']
\end{tikzcd}$$

The vertical functors are Quillen after the appropriate localization and cellularization by Theorems 4.12 and 4.14 and Proposition 4.13. Using Theorem 4.4 along with Proposition 4.9, we obtain that all six possible squares of derived functors commute.

8.4. **To algebra.** The final stage in the process is the passage from symmetric spectra in non-negatively graded chain complexes to chain complexes. This can be realized by a single square

$$\begin{tikzcd}[column sep=huge, row sep=huge]
\mathrm{Mod}_{\varphi^* N\alpha^*\widetilde{\mathbb{Q}}S} \arrow[r, shift left, "D"] \arrow[d, shift left=1ex, "\Sigma^{-a}\gamma_!"] \arrow[d, shift left=3ex, "\gamma_*"] & \mathrm{Mod}_{D\varphi^* N\alpha^*\widetilde{\mathbb{Q}}S} \arrow[l, shift left, "R"] \arrow[d, shift left=1ex, "\Sigma^{-a}\lambda_!"] \arrow[d, shift left=3ex, "\lambda_*"] \\
\mathrm{Mod}_{Q\varphi^* NQ\alpha^*\widetilde{\mathbb{Q}}R} \arrow[r, shift left, "D"] \arrow[u, shift left=1ex, "\Sigma^{a}\gamma^*"] \arrow[u, shift right=3ex, "\gamma^*"'] & \mathrm{Mod}_{DQ\varphi^* NQ\alpha^*\widetilde{\mathbb{Q}}R} \arrow[l, shift left, "R"] \arrow[u, shift left=1ex, "\Sigma^{a}\lambda^*"] \arrow[u, shift right=3ex, "\lambda^*"']
\end{tikzcd}$$

in which $\lambda = D\gamma$. Since D is left Quillen, $DQ\varphi^* NQ\alpha^*\widetilde{\mathbb{Q}}R$ is a cofibrant $D\varphi^* N\alpha^*\widetilde{\mathbb{Q}}S$-module.

The vertical functors are still Quillen after the appropriate localization and cellularization by Theorems 4.12 and 4.14 and Proposition 4.13. By Theorem 4.4 and Proposition 4.6 we have that all of the derived squares commute.

Taking $S = DBG_+$ and $R = QDBH_+$ shows that the functors correspond through Shipley's algebraicization theorem in our setting. We note that despite starting in the flat model structure on spectra, we are now in the usual projective model structure on chain complexes.

9. The formality square

To ease notation we write S_a and R_a for the images of DBG_+ and $QDBH_+$ respectively under the Quillen equivalences in Shipley's algebraicization theorem described in the previous section. They are commutative models for the cochains on BG and BH respectively but they are not good models for us since it is not clear how build a commutative square

$$\begin{tikzcd}
S_a \arrow[d] & H^*BG \arrow[l] \arrow[d] \\
R_a & H^*BH. \arrow[l]
\end{tikzcd}$$

We now describe how to alter these models so that a commutative square can be built with the desired properties.

Firstly factor the map $\lambda\colon S_a \to R_a$ as an acyclic cofibration followed by a fibration

$$S_a \xrightarrow{j_G} S'_a \xrightarrow{\nu} R_a.$$

Since $\nu\colon S'_a \to R_a$ is a fibration (i.e., a surjection) we can build a commutative square

$$\begin{tikzcd}
S_a \arrow[r, hook, "j_G"] \arrow[d, "\lambda"'] & S'_a \arrow[dl, two heads, "\nu"] & H^*BG \arrow[l, "w"', "\sim"] \arrow[d, "\theta"] \\
R_a & & H^*BH. \arrow[ll, "\sim", "w'"']
\end{tikzcd}$$

To do this first define $w'\colon H^*BH \to R_a$ by choosing cocycle representatives $\tilde{y}_i$ of the polynomial generators y_i of H^*BH. Write $H^*BG = \mathbb{Q}[x_1, ..., x_r]$. Choosing cocycle representatives $\tilde{x}'_i$ for the polynomial generators will not yield a commutative square in general. However, the cohomology classes $\nu(\tilde{x}'_i)$ and $w'\theta(x_i)$ are cohomologous since the map $DBG_+ \to DBH_+$ which gives rise to $\lambda\colon S_a \to R_a$ via Shipley's algebraicization theorem represents the map θ in homotopy. Therefore the differences $\nu(\tilde{x}'_i) - w'\theta(x_i)$ are coboundaries $d(b_i)$. As $\nu\colon S'_a \to R_a$ is a surjection, we can lift the coboundary $d(b_i)$ to give a coboundary a_i in S'_a such that $\nu a_i = d(b_i)$. Define $w(x_i) = \tilde{x}'_i - a_i$. Then

$$\nu w(x_i) = \nu\tilde{x}'_i - \nu a_i = \nu\tilde{x}'_i - d(b_i) = w'\theta(x_i)$$

so the square commutes.

Next factor the map $\theta\colon H^*BG \to H^*BH$ into a cofibration followed by an acyclic fibration

$$H^*BG \xrightarrow{\varphi} QH^*BG \xrightarrow{q} H^*BH.$$

Taking the pushout of the span $S'_a \xleftarrow{w} H^*BG \xrightarrow{\varphi} QH^*BH$ gives the commutative diagram

$$\begin{tikzcd}
 & S'_a \arrow[dl, bend right, "\nu"'] \arrow[d, hook, "\delta"'] & H^*BG \arrow[l, "w", "\sim"'] \arrow[d, hook, "\varphi"] \\
 & R'_a \arrow[dl, dashed, "\alpha"] & QH^*BH \arrow[l, "\sim", "z"'] \arrow[dll, bend left, "w'q"] \\
R_a & &
\end{tikzcd}$$

where z is a weak equivalence by left properness.

From this it follows that we have the commutative squares

$$\begin{tikzcd}
S_a \arrow[d, hook, "\lambda"'] & S_a \arrow[l, "1", "\sim"'] \arrow[r, hook, "j_G", "\sim"'] \arrow[d, hook, "\delta j_G"'] & S'_a \arrow[d, hook, "\delta"] & H^*BG \arrow[l, "w", "\sim"'] \arrow[d, hook, "\varphi"] \\
R_a & R'_a \arrow[l, "\sim", "\alpha"'] \arrow[r, "\sim", "1"'] & R'_a & QH^*BH \arrow[l, "\sim", "z"']
\end{tikzcd}$$

and we can see the formality step as the following three squares. We write $\beta = \delta j_G$ to ease notation.

$$\begin{tikzcd}[column sep=large, row sep=large]
\mathrm{Mod}_{S_a} \arrow[r, shift left, "1"] \arrow[d, shift right=3, "\lambda_{(*)}"'] \arrow[d, shift left=1, "\lambda_*"] & \mathrm{Mod}_{S_a} \arrow[l, shift left, "1"] \arrow[r, shift left, "j^G_*"] \arrow[d, shift right=3, "\beta_{(*)}"'] \arrow[d, shift left=1, "\beta_*"] & \mathrm{Mod}_{S'_a} \arrow[l, shift left, "j^*_G"] \arrow[r, shift left, "w^*"] \arrow[d, shift right=3, "\delta_{(*)}"'] \arrow[d, shift left=1, "\delta_*"] & \mathrm{Mod}_{H^*BG} \arrow[l, shift left, "w_*"] \arrow[d, shift right=3, "\varphi_{(*)}"'] \arrow[d, shift left=1, "\varphi_*"] \\
\mathrm{Mod}_{R_a} \arrow[u, shift left=5, "\lambda^\dagger"] \arrow[u, shift right=5, "\lambda^*"'] \arrow[r, shift right, "\alpha^*"'] & \mathrm{Mod}_{R'_a} \arrow[u, shift left=5, "\beta^\dagger"] \arrow[u, shift right=5, "\beta^*"'] \arrow[l, shift right, "\alpha_*"'] \arrow[r, shift left, "1"] & \mathrm{Mod}_{R'_a} \arrow[u, shift left=5, "\delta^\dagger"] \arrow[u, shift right=5, "\delta^*"'] \arrow[l, shift left, "1"] \arrow[r, shift right, "z^*"'] & \mathrm{Mod}_{QH^*BH} \arrow[u, shift left=5, "\varphi^\dagger"] \arrow[u, shift right=5, "\varphi^*"'] \arrow[l, shift right, "z_*"']
\end{tikzcd}$$

The vertical functors are still Quillen after the appropriate localization and cellularization by Theorems 4.12 and 4.14 and Proposition 4.13. All three squares satisfy the hypotheses of Proposition 4.7, so by Theorem 4.4 all of the derived squares commute.

Remark 9.1. The splitting of this formality step into three pieces is necessary to ensure that the vertical functors are Quillen. In order to guarantee this, the vertical maps of commutative DGAs need to be cofibrations, see Proposition 3.8.

Combining all of the results of the previous sections, this completes the proof of Theorem 1.1.

10. Torsion and completion

There are two options for the final step: either to pass to torsion H^*BG-modules, or L-complete H^*BG-modules. We show that unless the ranks of G and H are equal, there are obstructions to forming the correct setup in this step. In this section we will complete the proof of Theorem 1.4.

10.1. **Koszul complexes and local (co)homology.** In this section we recall some key definitions which we require for the rest of the section. For more detail see [10, §6].

For $I = (x_1, \cdots, x_n)$ a finitely generated homogeneous ideal in a graded commutative ring S, the *stable Koszul complex* denoted $K_\infty(I)$ is defined by

$$K_\infty(I) = K_\infty(x_1) \otimes_S \cdots \otimes_S K_\infty(x_n)$$

where $K_\infty(x_i)$ is the complex $S \to S[1/x_i]$ in degrees 0 and -1.

The *local homology* of an S-module N is defined by

$$H_n^I N = H_n \mathrm{Hom}_S(PK_\infty(I), N)$$

where $PK_\infty(I)$ is a projective replacement of $K_\infty(I)$, see [18, §1] for instance. We write $\Lambda_I = \mathbb{R}\mathrm{Hom}_S(K_\infty(I), -)$ for the derived completion functor. The *local cohomology* is defined by

$$H_I^n N = H_{-n}(K_\infty(I) \otimes_S N).$$

We note that when S is Noetherian the 0th local homology is the L-completion functor

$$L_0^I = L_0((-)_I^\wedge))$$

which is the 0th left derived functor of I-adic completion [17, 2.5]. We say that an S-module M is L_0^I-complete if the natural map $M \to L_0^I M$ is an isomorphism, and write $\mathrm{Mod}_S^\wedge$ for the full subcategory of L_0^I-complete S-modules.

The 0th local cohomology is the I-power torsion functor

$$\Gamma_I M = \{m \in M \mid I^n m = 0 \text{ for some } n\}$$

by a result of Grothendieck [24]. We say that an S-module M is (I-power) torsion if the natural map $\Gamma_I M \to M$ is an isomorphism, and write $\mathrm{Mod}_S^{\mathrm{torsion}}$ for the full subcategory of torsion S-modules.

We now turn to the interaction of local homology with change of rings. Given a map of rings $\theta \colon S \to R$ and an ideal $I = (x_1, \ldots, x_n)$ of S, we write IR for the ideal $(\theta(x_1), \ldots, \theta(x_n))$ of R.

Lemma 10.1. *Let $\theta \colon S \to R$ be a map of rings and I be an ideal in S. There is an isomorphism $H_*^I(\theta^* M) \cong H_*^{IR}(M)$ for any R-module M. Furthermore, if R is a projective S-module, there is an isomorphism $\mathrm{Hom}_S(R, H_*^I(N)) \cong H_*^{IR}(\mathrm{Hom}_S(R, N))$ for any S-module N.*

Proof. Throughout this proof we neglect to indicate that homs are derived. Recall that there is an isomorphism $R \otimes_S K_\infty(I) \cong K_\infty(IR)$. Therefore

$$\Lambda_I(\theta^* M) = \mathrm{Hom}_S(K_\infty(I), \theta^* M) \cong \mathrm{Hom}_R(R \otimes_S K_\infty(I), M) \cong \mathrm{Hom}_R(K_\infty(IR), M) = \Lambda_{IR} M$$

which gives the first desired isomorphism.

For the other isomorphism, note that

$$\begin{aligned}
\theta_! \Lambda_I N &= \mathrm{Hom}_S(R, \mathrm{Hom}_S(K_\infty(I), N)) \\
&\cong \mathrm{Hom}_S(R \otimes_S K_\infty(I), N) \\
&\cong \mathrm{Hom}_S(K_\infty(I), \mathrm{Hom}_S(R, N)) \\
&\cong \mathrm{Hom}_R(R \otimes_S K_\infty(I), \mathrm{Hom}_S(R, N)) \\
&= \Lambda_{IR}(\theta_! N).
\end{aligned}$$

The result then follows from the fact that $\theta_!$ is exact since R is projective as an S-module. □

We now recall the change of base theorem for local cohomology. We omit the proof as it is similar to Lemma 10.1.

Lemma 10.2. *Let $\theta \colon S \to R$ be a map of rings and I be an ideal in S. There is an isomorphism $H_I^*(\theta^* M) \cong H_{IR}^*(M)$ for any R-module M. Furthermore, if R is a flat S-module, there is an isomorphism $R \otimes_S H_I^*(N) \cong H_{IR}^*(R \otimes_S N)$ for any S-module N.*

10.2. **Cofibrancy.** We have a natural map $\theta\colon H^*BG \to H^*BH$ induced by the inclusion of H into G. Throughout the remainder of this section, we write I for the augmentation ideal of H^*BG and J for the augmentation ideal of H^*BH. We note that $\sqrt{I \cdot H^*BH} = J$ by Venkov's theorem, using the general fact that if $(S, \mathfrak{n}) \to (R, \mathfrak{m})$ is a map of local rings with R a finitely generated S-module, then $\sqrt{\mathfrak{n}R} = \mathfrak{m}$. Up to quasi-isomorphism, the stable Koszul complex only depends on the radical of the ideal, so we can apply the base change results from the previous section in this example.

Recall from Proposition 3.8 that the restriction-coextenstion of scalars adjunction along θ is Quillen (in the projective model structure) if and only if H^*BH is cofibrant as a H^*BG-module. A dg-module M over a dga S is called *semi-projective* if $\mathrm{Hom}_S(M, -)$ preserves surjective quasiisomorphisms. The semi-projective dg-S-modules are precisely the cofibrant objects in the projective model structure on Mod_S [1, 3.15].

Proposition 10.3. *Let $H \to G$ be the inclusion of a connected compact Lie group into a connected compact Lie group. The following are equivalent:*

(1) $\mathrm{rk}G = \mathrm{rk}H$
(2) H^*BH is a cofibrant H^*BG-module (in the projective model structure)
(3) H^*BH is a free H^*BG-module

Proof. (1) $\Rightarrow$ (3): If $\mathrm{rk}G = \mathrm{rk}H$ then $H^*BH \cong H^*BG \otimes H^*(G/H)$ as a H^*BG-module (see [34, 8.3]) so H^*BH is a free H^*BG-module.

(3) $\Rightarrow$ (2): Both H^*BH and H^*BG have trivial differential so by [2, 9.8.1], since H^*BH is underlying projective as a H^*BG-module (as it is free) it is semi-projective and hence cofibrant.

(2) $\Rightarrow$ (3): Recall that semi-projective implies underlying projective [2, 9.6.1, 9.4.1]. By Venkov's theorem H^*BH is a finitely generated H^*BG-module, so H^*BH is free over H^*BG as finitely generated projective modules over local rings are free.

(3) $\Rightarrow$ (1): Firstly, note that the rank of G is the Krull dimension of H^*BG. As H^*BH is a free H^*BG-module, we have that $\dim_{\mathbb{Q}} H^sBH \geq \dim_{\mathbb{Q}} H^sBG$ for all s. It follows from the characterization of Krull dimension in terms of Hilbert series that $\mathrm{rk}H \geq \mathrm{rk}G$. As H is a subgroup of G, we also have $\mathrm{rk}H \leq \mathrm{rk}G$. □

Corollary 10.4. *If H^*BH is a cofibrant H^*BG-module, then $\theta_*M \cong \theta_!M$.*

Proof. If H^*BH is a cofibrant H^*BG-module then H^*BH is finitely generated and free over H^*BG by Proposition 10.3. Therefore H^*BH is a finite sum of shifted copies of H^*BG and the result follows. □

Remark 10.5. One may expect that if H^*BH is not cofibrant as a H^*BG-module that we may cofibrantly replace it in the category of commutative H^*BG-algebras. Note that H^*BG and H^*BH have zero differential, but the cofibrant replacement QH^*BH will not have zero differential. This means that it is not possible to talk about L-complete or torsion modules over QH^*BH. This is why Theorem 1.1 has a stronger statement in the case that the ranks of G and H are equal.

10.3. **Completion.** Firstly we must show that the functors pass to the category of L-complete modules.

Lemma 10.6. *Suppose that* $\mathrm{rk}G = \mathrm{rk}H$*. There are Quillen adjunctions*

$$\begin{tikzcd} \mathrm{Mod}^{\wedge}_{H^*BG} \arrow[r, shift left=2, "\theta_*"] \arrow[r, shift right=2, "\theta_!"'] & \mathrm{Mod}^{\wedge}_{H^*BH}. \arrow[l, "\theta^*"'] \end{tikzcd}$$

Proof. By Lemma 10.1 the restriction functor θ^* and the coextension functor $\theta_!$ send L-complete modules to L-complete modules. Combining this with Corollary 10.4 shows that θ_* also sends

L-complete modules to L-complete modules. Since the weak equivalences and fibrations of L-complete modules are created by the inclusion to all modules, it is immediate that θ^* and $\theta_!$ are still both right Quillen. □

It remains to consider the square

$$\begin{tikzcd}
\mathrm{Mod}_{H^*BG} \arrow[rr, shift left, "L_0^I"] \arrow[d, shift right=2, "\theta_*"] & & \mathrm{Mod}^{\wedge}_{H^*BG} \arrow[ll, shift left, "i"] \arrow[d, shift right=2, "\theta_*"] \\
\mathrm{Mod}_{H^*BH} \arrow[rr, shift left, "L_0^J"] \arrow[u, shift left=2, "\theta^*"] \arrow[u, shift right=4, "\theta^*"'] & & \mathrm{Mod}^{\wedge}_{H^*BH} \arrow[ll, shift left, "j"] \arrow[u, shift left=2, "\theta^*"] \arrow[u, shift right=4, "\theta^*"']
\end{tikzcd}$$

where i and j denote the inclusions and $\mathrm{rk} G = \mathrm{rk} H$.

It is clear that the inclusions commute with all vertical functors, and therefore applying Theorem 4.4 shows that all the derived squares commute.

Remark 10.7. We note that each square here commutes up to natural isomorphism rather than just natural weak equivalence.

10.4. **Torsion.** Another possible model instead of L-complete modules is that of torsion modules. The category of torsion modules does not have enough projectives, but it has does have enough injectives, so it supports an injective model structure [19, 8.6]. Since the injective model structure is not right lifted from the underlying category of chain complexes, the results of Section 3.3 do not apply. Therefore, we must next recall when the extension-restriction-coextension functors are Quillen in the injective model structure. A dg-S-module M is said to be *semi-flat* if $M \otimes_S -$ preserves injective quasiisomorphisms.

Proposition 10.8. *Let $\theta\colon S \to R$ be a map of DGAs. The adjunction $\theta^* \dashv \theta_!$ is Quillen in the injective model structure. The adjunction $\theta_* \dashv \theta^*$ is Quillen in the injective model structure if and only if R is semi-flat as an S-module.*

Proof. Since the weak equivalences and cofibrations are underlying, the restriction of scalars functor θ^* preserves them. Therefore, $\theta^* \dashv \theta_!$ is Quillen.

If R is a semi-flat S-module, $R \otimes_S -$ preserves injective quasiisomorphisms. It follows from [2, 11.2.1] that R is linearly flat, meaning that $R \otimes_S -$ preserves injections (cofibrations). Therefore $R \otimes_S -$ preserves cofibrations and acyclic cofibrations so is a left Quillen functor.

Conversely, if the adjunction is Quillen, then $R \otimes_S -$ preserves injective quasiisomorphisms, so R is a semi-flat S-module. □

Note that this proposition shows that if the ranks of G and H are the same, then the extension-restriction and restriction-coextension adjunctions along the ring map $H^*BG \to H^*BH$ are Quillen in the injective model structure.

Lemma 10.9. *Suppose that* $\mathrm{rk} G = \mathrm{rk} H$. *There are Quillen adjunctions*

$$\begin{tikzcd}
\mathrm{Mod}^{\mathrm{torsion}}_{H^*BG} \arrow[r, shift left=3, "\theta_*"] \arrow[r, shift right=3, "\theta_!"'] & \mathrm{Mod}^{\mathrm{torsion}}_{H^*BH}. \arrow[l, "\theta^*"']
\end{tikzcd}$$

Proof. By Lemma 10.2 and Corollary 10.4 we have that extension, restriction and coextension preserve torsion modules. Since the weak equivalences and cofibrations of torsion modules are created by the inclusion to all modules, it is immediate that θ_* and θ^* are still both left Quillen. □

It remains to consider the square

$$\begin{tikzcd}
\mathrm{Mod}_{H^*BG} \arrow[d, shift right=4ex, "\theta^*"'] \arrow[d, "\theta_*"] \arrow[rr, shift right, "\Gamma_I"'] & & \mathrm{Mod}^{\mathrm{torsion}}_{H^*BG} \arrow[ll, shift right, "i"'] \arrow[d, "\theta_*"] \\
\mathrm{Mod}_{H^*BH} \arrow[u, shift right=4ex, "\theta^*"'] \arrow[u, shift left=4ex, "\theta^*"] \arrow[rr, shift right, "\Gamma_J"'] & & \mathrm{Mod}^{\mathrm{torsion}}_{H^*BH} \arrow[ll, shift right, "j"'] \arrow[u, shift right=4ex, "\theta^*"'] \arrow[u, shift left=4ex, "\theta^*"]
\end{tikzcd}$$

where i and j denote the inclusions and $\mathrm{rk}G = \mathrm{rk}H$.

It is clear that the inclusions commute with all vertical functors, and therefore applying Theorem 4.4 shows that all the derived squares commute.

Remark 10.10. We note that each square here commutes up to natural isomorphism rather than just natural weak equivalence.

Combining the results of this section with Theorem 1.1 completes the proof of Theorem 1.4.

Appendix A. Proof of Lemma 6.1

We conclude the proof of Lemma 6.1.

Proof. We must check that the module structures defined in the proof of Lemma 6.1 satisfy the associativity and unit axiom. We prove this for i_*. The proof for $i_!$ is similar and therefore we omit it. Associativity amounts to the outer square commuting in the following diagram.

$$\begin{tikzcd}
R \wedge R \wedge i_*M \arrow[r, "1\wedge p^{-1}"] \arrow[dd, "\mu\wedge 1"'] \arrow[dr, "p^{-1}"'] & R \wedge i_*(i^*R \wedge M) \arrow[r, "1\wedge i_*(a)"] \arrow[d, "p^{-1}"] & R \wedge i_*M \arrow[d, "p^{-1}"] \\
 & i_*(i^*R \wedge i^*R \wedge M) \arrow[r, "i_*(1\wedge a)"] \arrow[d, "i_*(\mu\wedge 1)"] & i_*(i^*R \wedge M) \arrow[d, "i_*(a)"] \\
R \wedge i_*M \arrow[r, "p^{-1}"'] & i_*(i^*R \wedge M) \arrow[r, "i_*(a)"'] & i_*M
\end{tikzcd}$$

The top right square and the bottom left square commute by naturality of p, and the bottom right square commutes by the associativity axiom for M. The triangle in the top left requires us to use the construction of p, by considering the following diagram.

$$\begin{tikzcd}
i_*(i^*R \wedge i^*R \wedge M) \arrow[rr, "1"] \arrow[d, "i_*(1\wedge\eta)"'] & & i_*(i^*R \wedge i^*R \wedge M) \arrow[d, "i_*(1\wedge 1\wedge\eta)"] \\
i_*(i^*R \wedge i^*i_*(i^*R \wedge M)) \arrow[r, "i_*(1\wedge i^*p)"] \arrow[d, "i_*\Phi"'] & i_*(i^*R \wedge i^*(R \wedge i_*M)) \arrow[r, "i_*(1\wedge\Phi)"] \arrow[dr, "i_*\Phi"] & i_*(i^*R \wedge i^*R \wedge i^*i_*M) \arrow[d, "i_*\Phi"] \\
i_*i^*(R \wedge i_*(i^*R \wedge M)) \arrow[rr, "i_*i^*(1\wedge p)"] \arrow[d, "\varepsilon"'] & & i_*i^*(R \wedge R \wedge i_*M) \arrow[d, "\varepsilon"] \\
R \wedge i_*(i^*R \wedge M) \arrow[rr, "1\wedge p"'] & & R \wedge R \wedge i_*M
\end{tikzcd}$$

The bottom square commutes by naturality of ε, the middle square commutes by naturality of Φ and the triangle on the right commutes since i^* is strong monoidal (with structure map Φ). It remains to check that the top square commutes, for which is it sufficient to check that the following diagram commutes.

$$
\begin{tikzcd}
i^*R \wedge M \arrow[r, "1\wedge\eta"] \arrow[dd, "\eta"'] & i^*R \wedge i^*i_*M \arrow[rr, "\Phi"] \arrow[dd, "\eta"] & & i^*(R \wedge i_*M) \arrow[dl, "1"] \\
& & i^*(R \wedge i_*M) \arrow[dr, "\eta"] & \\
i^*i_*(i^*R \wedge M) \arrow[r, "i^*i_*(1\wedge\eta)"'] & i^*i_*(i^*R \wedge i^*i_*M) \arrow[rr, "i^*i_*\Phi"'] & & i^*i_*i^*(R \wedge i_*M) \arrow[uu, "i^*\varepsilon"']
\end{tikzcd}
$$

The left hand square commutes by naturality of η, the right hand square commutes by naturality of η, and the triangle commutes by the triangle identities. This completes the proof of the associativity axiom.

For the unit axiom, we require that the outer square in the following diagram commutes, where $\mathbb{S}$ denotes the monoidal unit of $\mathcal{D}$.

$$
\begin{tikzcd}
\mathbb{S} \wedge i_*M \arrow[rr, "\eta\wedge 1"] \arrow[dr, "p^{-1}"] \arrow[dd, "\lambda_{\mathcal{C}}"'] & & R \wedge i_*M \arrow[dd, "p^{-1}"] \\
& i_*(i^*\mathbb{S} \wedge M) \arrow[dl, "i_*\lambda_{\mathcal{D}}"] \arrow[dr, "i_*(i^*\eta\wedge 1)"] & \\
i_*M & & i_*(i^*R \wedge M) \arrow[ll, "i_*(a)"]
\end{tikzcd}
$$

The top right square commutes by naturality of p and the bottom triangle commutes by the unit axiom for M. For the left triangle we show that it commutes with p instead since we have an explicit construction for this using the unit and counit of the adjunction. Consider the diagram

$$
\begin{tikzcd}
i_*(i^*\mathbb{S} \wedge M) \arrow[rrr, "i_*(1\wedge\eta)"] \arrow[dr, "i_*(1\wedge\eta)"] \arrow[ddd, "i_*(\lambda_{\mathcal{D}})"'] & & & i_*(i^*\mathbb{S} \wedge i^*i_*M) \arrow[dll, "1"'] \arrow[d, "\Phi"] \\
& i_*(i^*\mathbb{S} \wedge i^*i_*M) \arrow[dr, "i_*\lambda_{\mathcal{D}}"] & & i_*i^*(\mathbb{S} \wedge i_*M) \arrow[dl, "i_*i^*\lambda_{\mathcal{C}}"'] \arrow[d, "\varepsilon"] \\
& & i_*i^*i_*M \arrow[dr, "\varepsilon i_*"] & \mathbb{S} \wedge i_*M \arrow[d, "\lambda_{\mathcal{C}}"] \\
i_*M \arrow[urr, "i_*\eta"] \arrow[rrr, "1"'] & & & i_*M
\end{tikzcd}
$$

in which the outer square gives the left triangle in the previous diagram with p instead of p^{-1}. We see that the top triangle commutes immediately, the left hand square commutes by naturality of η, the top right square commutes by definition of i^* being strong monoidal, the bottom right square commutes by naturality of ε, and the bottom triangle commutes by the triangle identities.

We must also check that i_* sends module maps to module maps. Let $f\colon M \to N$ be a map of i^*R-modules. Then the diagram

$$\begin{tikzcd}
R \wedge i_*M \arrow[r, "1\wedge i_* f"] \arrow[d, "p^{-1}"'] & R \wedge i_*N \arrow[d, "p^{-1}"] \\
i_*(i^*R \wedge M) \arrow[r, "i_*(1\wedge f)"] \arrow[d, "i_*(a)"'] & i_*(i^*R \wedge N) \arrow[d, "i_*(a)"] \\
i_*M \arrow[r, "i_* f"'] & i_*N
\end{tikzcd}$$

commutes, by the naturality of p and since f is an i^*R module map.

It remains to show that these functors form an adjoint triple. It is enough to check that the units and counits are module maps.

For the unit of the $i_* \dashv i^*$ adjunction we consider the diagram

$$\begin{tikzcd}
i^*R \wedge N \arrow[r, "1\wedge\eta"] \arrow[dddd, "a"'] \arrow[ddr, "\eta"] & i^*R \wedge i^*i_*N \arrow[d, "\Phi"] \\
& i^*(R \wedge i_*N) \arrow[d, "i^*p^{-1}"] \\
& i^*i_*(i^*R \wedge N) \arrow[d, "i^*i_*(a)"] \\
& \\
N \arrow[r, "\eta"'] & i^*i_*N
\end{tikzcd}$$

in which the square commutes by naturality of η. For the triangle, it suffices to check that the following diagram commutes.

$$\begin{tikzcd}
i^*R \wedge N \arrow[r, "1\wedge\eta"] \arrow[dd, "\eta"'] & i^*R \wedge i^*i_*N \arrow[r, "\Phi"] \arrow[ddr, "\eta"] & i^*(R \wedge i_*N) \\
& & i^*i_*i^*(R \wedge i_*N) \arrow[u, "i^*\varepsilon"'] \\
i^*i_*(i^*R \wedge N) \arrow[rr, "i^*i_*(1\wedge\eta)"'] & & i^*i_*(i^*R \wedge i^*i_*N) \arrow[u, "i^*i_*\Phi^{-1}"']
\end{tikzcd}$$

The left hand square commutes by naturality of η and checking that the right hand square commutes can be done by considering the diagram

$$\begin{tikzcd}
i^*R \wedge i^*i_*N \arrow[rr, "\Phi^{-1}"] \arrow[dd, "\eta"'] & & i^*(R \wedge i_*N) \arrow[dl, "1"'] \\
& i^*(R \wedge i_*N) \arrow[dr, "\eta"] & \\
i^*i_*(i^*R \wedge i^*i_*N) \arrow[rr, "i^*i_*\Phi^{-1}"'] & & i^*i_*i^*(R \wedge i_*N) \arrow[uu, "i^*\varepsilon"']
\end{tikzcd}$$

in which the left hand square commutes by naturality of η and the triangle commutes by the triangle identities. The argument for the counit of the $i_* \dashv i^*$ adjunction follows similarly, as does the other adjunction. □

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CHAPTER 5

Future directions

In this chapter we discuss possible extensions of the work presented in this thesis. There are two potential extensions to the theorem identifying the change of groups functors in the algebraic model; removing the hypothesis of connectedness or extending beyond the (co)free case. We then discuss a potential application beyond rational equivariant stable homotopy theory.

1. Disconnected groups

When finding the counterparts to the change of groups functors, we assumed that both G and H are connected. This assumption has two main consequences. Firstly, there is no action of the Weyl group. When constructing an algebraic model for (co)free G-spectra for disconnected G, one must consider the action of the Weyl group on all modules and rings considered. This leads to an extra layer of complexity. Secondly, the ring map $DBG_+ \to DBH_+$ need not be relatively Gorenstein unless G and H are connected.

The author has some preliminary work on extending the algebraic models of change of groups functors between (co)free spectra to the disconnected case. The main complication now arises from the fact that the action of the group of components of G and H must be taken into account. Nonetheless, since we are working in the free case, we expect that the action of the group of components can be 'removed' and treated in a purely categorical manner as follows.

Firstly, we note that the algebraic models for free and cofree G-spectra still exist, and the general strategy of proof is the same. We write G_e for the identity component of G and $G_d = G/G_e$ for the component group. Since the cofree case is proved in Chapter 2 we work in the cofree rather than free case. The free case is analogous.

The first step in constructing the algebraic model is still a change of rings along $S^0 \to DEG_+$ but the main difference arises in removing equivariance. Taking G-categorical fixed points is too brutal to record the action of the component group, and so we instead must take G_e-fixed points. This provides a Quillen equivalence to $L_{G_{d+}}\mathrm{Mod}_{DBG_{e+}}$ where $DBG_{e+} = (DEG_+)^{G_e}$ is a commutative ring G_d-spectrum. The next step is to use the following lemma to allow us to work with G_d-objects in non-equivariant spectra.

Lemma ([10, 5.3])**.** *There is a Quillen equivalence*

$$L_{EG_{d+}}\mathrm{Sp}_{G_d} \simeq_Q \mathrm{Sp}[G_d].$$

Applying this lemma, we obtain a Quillen equivalence

$$L_{G_{d+}}\mathrm{Mod}_{DBG_{e+}}(\mathrm{Sp}_{G_d}) \simeq_Q L_{G_{d+}}\mathrm{Mod}_{DBG_{e+}}(\mathrm{Sp})[\widetilde{G_d}].$$

By pulling out this action, we have reduced to the connected case and we now expect that proving the correspondence of functors in the disconnected case follows from this formally. In order to check this, one must carefully verify that each step is appropriately equivariant.

A layer of complexity which we have so far hidden in this discussion is that when dealing with a subgroup inclusion $i\colon H \to G$, it does not follow that $i_d\colon H_d \to G_d$ is a monomorphism. For example, if we take $H = SO(2)$ and $G = O(2)$, we have $G_d = C_2$ but H_d is the trivial group. We expect that carefully reconciling the actions of the component groups will be the key step in extending the correspondence to the disconnected case.

Further evidence of the fact that the action can be pulled out and treated formally is that the algebraic models for (co)free equivariant spectra do not detect whether the sequence

$$1 \to G_e \to G \to G_d \to 1$$

is split exact. For example, we have that free $O(2)$-spectra and free $\mathrm{Pin}(2)$-spectra are equivalent.

2. Beyond the (co)free case

The next generalization of the work in this thesis is to find algebraic models for change of groups functors without the (co)free assumption. In this case the algebraic models are more complex and are diagram categories. The slogan is that the algebraic model for G-spectra is a sheaf on the space of (connected) subgroups of G.

Let us focus on the case where G is a torus. An algebraic model for torus-equivariant spectra was given by Greenlees [4, 5, 6] and a zig-zag of Quillen equivalences was constructed by Greenlees-Shipley [8]. We give a brief description of the objects in the algebraic model for rational G-spectra. We define a ring

$$\mathcal{O}_{\mathcal{F}} = \prod_{F \in \mathcal{F}} H^*(BG/F)$$

where $\mathcal{F}$ is the family of finite subgroups of G. One may define a multiplicative set of Euler classes for each connected subgroup K of G, and using this define a diagram of rings indexed on $\mathrm{ConnSub}(G)$ by

$$\widetilde{\mathcal{O}}_{\mathcal{F}}(K) = \mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}.$$

A module over the diagram of rings $\widetilde{\mathcal{O}}_{\mathcal{F}}$ is a collection of modules $M(K)$ indexed by $\mathrm{ConnSub}(G)$, where $M(K)$ is a $\mathcal{E}_K^{-1}\mathcal{O}_{\mathcal{F}}$-module, together with structure maps $M(L) \to M(K)$ for each $L \subseteq K$. The algebraic model $\mathcal{A}(G)$ is then given by a subcategory of $\widetilde{O}_{\mathcal{F}}$-modules. Namely, they are the quasi-coherent and extended modules. These two conditions say that the value at any connected subgroup is determined by the value at the trivial group, and relates the value at any connected subgroup to the value at a quotient. These conditions mirror the Borel-Hsiang-Quillen localization theorem for Borel cohomology which says that if X is a finite complex, then $H^*_G(X) \to H^*_G(X^G) = H^*(BG) \otimes H^*(X^G)$ is an isomorphism after inverting $\mathcal{E}_G^{-1}$. Indeed, one may define $M(K) = H^*_{G/K}(X^K)$ for a finite G-space X and one then sees that this is quasi-coherent and extended by the Borel-Hsiang-Quillen localization theorem.

We write $\mathbb{T}^r$ for the torus of rank r. Take $r \leq s$ and consider the inclusion $i\colon \mathbb{T}^r \to \mathbb{T}^s$. We expect that constructing functors between the algebraic models will require some complex homological algebra, but much of this probably already exists in [5, §6]. It is unclear whether the extra complication which arises from working with modules over diagrams of rings makes the correspondence of functors significantly harder to prove. We expect that once the correct setup has been established, the correspondence should follow in a reasonably formal manner from the results of Chapter 4.

3. Singularity categories and the BGG correspondence

The singularity category of a commutative Noetherian ring R is the Verdier quotient

$$\mathcal{D}_{\mathrm{sg}}(R) = \frac{\mathcal{D}^b(R)}{\mathcal{D}^c(R)}$$

of the bounded derived category (the complexes with bounded cohomology) by the perfect complexes (the bounded complexes of finitely generated projectives) [3]. The singularity category measures how far the ring R is from being regular. Indeed, R is regular if and only if $\mathcal{D}_{\mathrm{sg}}(R) = 0$. Greenlees-Stevenson [9] have recently extended the definition of the singularity category to DGAs and ring spectra by a Noether normalization procedure.

We follow [9] and work in the general setting of a map of ring spectra $R \to k$. We note that one must impose hypotheses on the Noether normalization, but we ignore these technicalities here. One can define the cosingularity category

$$\mathcal{D}_{\mathrm{cosg}}(R) = \frac{\mathcal{D}^b(R)}{\mathrm{thick}(k)}$$

and the BGG correspondence relates the singularity and cosingularity categories via Morita theory. More precisely, given a ring R define its Morita pair to be $\mathcal{E} = \mathrm{Hom}_R(k, k)$. There are equivalences

$$\mathcal{D}_{\mathrm{sg}}(R) \simeq \mathcal{D}_{\mathrm{cosg}}(\mathcal{E}) \quad \text{and} \quad \mathcal{D}_{\mathrm{sg}}(\mathcal{E}) \simeq \mathcal{D}_{\mathrm{cosg}}(R).$$

We note that this is a vast generalization of the original BGG correspondence as stated in [2].

We now give two examples of interest.

Example 3.1. Let $R = \Lambda(t_1, \cdots, t_n)$ be an exterior algebra. Its Koszul dual is $\mathcal{E} = k[x_1, \cdots, x_n]$ where $|x_i| = -1$. The BGG correspondence recovers the usual Koszul duality statement

$$\mathcal{D}_{\mathrm{sg}}(\Lambda(t_1, \cdots, t_n)) \simeq \mathcal{D}^b(\mathbb{P}_k^{n-1}).$$

Example 3.2. Let k be a field of characteristic p and G be a p-group. Let $R = C^*(BG; k) = F(BG_+, Hk)$. Its Morita pair is $\mathcal{E} = kG$. Note that

$$\mathcal{D}_{\mathrm{sg}}(\mathcal{E}) = \frac{\mathcal{D}^b(kG)}{\mathcal{D}^c(kG)} \simeq \mathrm{stmod}(kG)$$

so the BGG correspondence gives an equivalence $\mathcal{D}_{\mathrm{cosg}}(C^*(BG; k)) \simeq \mathrm{stmod}(kG)$. Greenlees-Stevenson [9, 10.6] also treat the case where G is a finite group which is not necessarily a p-group. In this case though, $\mathcal{E} = C_*(\Omega BG_p^\wedge) \not\simeq kG$ since BG is not p-complete in general.

We hope to apply the general machinery developed in this thesis to approach the question of correspondence of functors along the BGG correspondence. For example, given a map of rings $S \to R$, this induces a map on Morita pairs $\mathcal{E}_R \to \mathcal{E}_S$ in the *opposite* direction. Immediately, this is reminiscent of the situation for free G-spectra where the inclusion $H \to G$ induced a ring map $H^*BG \to H^*BH$. We now discuss some of the challenges and obstacles we foresee.

Firstly, all discussions in this section have been in *bounded* derived categories. In order to approach this via the model categorical techniques we developed, we must find model categories which represent the objects we are interested in. Instead of working in the bounded case, we expect to work in the so-called *big singularity category* $\mathrm{K_{ac}Inj}(R)$ as defined by Krause [11]. Becker [1] has discussed this at a model categorical level for DGAs, but we would like to approach it more generally, so as to encompass the examples given by Greenlees-Stevenson. Greenlees-Shipley [7, 5.4] prove a general Koszul duality style result for model categories, and we expect this to be a valuable template in constructing an appropriate version of the BGG correspondence at the model categorical level.

The next issue is the commutativity of rings which played a crucial role in the general theory developed in Chapter 4. Given a commutative ring R, its Morita pair $\mathcal{E}$ is not commutative in general. However, this is true in some interesting cases via a Cartan commutativity argument [7, 5.2]. For example, both R and $\mathcal{E}$ are commutative in Example 3.1. Similarly in Example 3.2, whilst kG is not commutative in general, the stable module category $\mathrm{stmod}(kG)$ is a closed symmetric monoidal category.

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