On Representations of Semigroups

Sanaa Ahmed E Bajri

PhD

University of York

Mathematics

February 2020

Abstract

Semigroup representations are one of the oldest areas in semigroup theory. In 1933, Suschkewitch published the first paper on the topic. Since then, the area has been approached largely by Clifford, Munn, Ponizovskii, and then by Hewitt and Zuckerman, Lallement and Petrich, Preston, McAlister, and Rhodes. Following an intense period of development during the 1950's and 1960's, the theory witnessed a dormant era during the 1970's and 1980's. There was a resurgence of interest in the subject in the late 1990's in the work of Putcha, Brown and others. The lack of continuity of research in the theory is intriguing. This thesis addresses the discontinuous development of the theory and the reasons behind it.

The Clifford-Munn-Ponizovskii correspondence states that the irreducible representations of a semigroup are in one-to-one correspondence with the irreducible representations of its maximal subgroups. Since the principal approach to identify representations of semigroups is this correspondence, we start with the observation that the lack of interest in semigroup representation theory could have been because the Clifford-Munn-Ponizovskii theory reduces the whole problem of finding irreducible representations to group representation theory. It turns out that this is a wrong assumption. It is not clear that during the dormant period the correspondence was widely known. By the time Munn and others stopped working on the theory, the correspondence was not stated in a fully-fledged form. The Clifford-Munn-Ponizovskii correspondence was subsequently formulated by others and emerged much later than in the work of Clifford and Munn.

In the thesis we first discuss the subject using modern mathematical language, starting with groups and then with semigroups. With this hindsight, we then turn to the subject as it was developed for groups and semigroups.

Contents

List of Figures

Acknowledgements

Firstly, I offer my deep thanks to God who gave me the ability to complete my thesis. Secondly, it is my pleasure to express my appreciation to the people who collaborated and provided their support and advice, without them this thesis certainly would not be possible.

From the early stage to the final draft, I owe deep gratitude and appreciation to my supervisor Dr Brent Everitt. Dr Brent provided me with enlightening guidance, unwavering support, constant advice, and valuable comments which were crucial to producing this thesis. It has been an honour to have Dr Brent as my supervisor. For all of his kindness and reassurance over the past four years, I would like to thank Dr Brent from the bottom of my heart not just as a supervisor but also as a friend.

The Ministry of Education in Saudi Arabia, Princess Nourah bint Abdulrahman University, the Saudi Embassy and the Saudi Cultural Bureau in London deserve my sincere thanks for their generous financial support and assistance whilst I did my research.

I am thankful to the University of York and mostly to the Department of Mathematics for offering me the opportunity to study in York. My exceptional thanks to the Department Head, Prof. Niall MacKay for securing funding which allowed me to attend the North British Semigroups and Applications Network (NBSAN) on several occasions during my doctoral program. I would also like to thank Prof. Maxim Nazarov and Prof. Victoria Gould, who were on the TAP panel and provided me with useful suggestions and comments.

Special thanks to our postgraduate administrator Nicholas Page and the department administrators Claire Farrar and Linda Elvin for their sustained help. I wish to express my gratitude to Dr Christopher Hollings, Prof. John Fountain, Dr Norman Reilly, Dr Paul Turner, Dr Majed Albaity, Dr Asawer Hamdi, and Dr Rida-E Zenab for their valuable advice and support. Also, thanks to Dr Henning Bostelmann who offered me useful help on LaTeX.

I am thankful to my husband Mohsin and my sons Hazim, Abdulaziz and my daughter Razaan who have encouraged, inspired and helped me all the time. I would like to say sorry to my sons and my daughter for spending too much time on my studies and not with you. Thanks for believing in me, and remaining understanding and patient throughout my PhD journey. I did this for you to have a better life and there is nothing I can give you in return, but I hope that this work at least make you proud.

I feel incredibly grateful to my parents Ahmed and Fawziah Bajri who have been praying intensely for me throughout my life. Their continued support has increased my motivation for completing my PhD program and overcome difficulties. I feel blessed to be your daughter. I would like to express deep appreciation and gratitude to my sisters Nouf and Sarah and my brothers Moqbel and Ibrahim for giving me the positive energy to do my best. Special thanks also to my brother Moqbel who helped me with the technical errors that I came across in my thesis.

And last but not the least, special thanks to all my friends in Saudi Arabia and the UK for being such real friends.

Declaration

I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 1 consists of definitions and results on group representation theory by other authors mostly from [10, 41, 42, 96, 99, 103, 107]. The other works of Chapter 1 are sourced from [9, 11, 12, 31, 32, 43, 44, 48, 49, 63, 103] and are stated and referenced accordingly.

Background material, definitions and results about semigroup theory in Chapter 2 are provided mainly from [37]. Chapter 3 presents a review of the fundamental notions utilised in this thesis. All fundamental assertions of representation of semigroups are from [14, 15, 22, 23, 102]. In particular, examples provided are from [14], which are referenced accordingly.

Chapter 4 is the first chapter which includes new work in this thesis. The fundamental information of this chapter is mostly from [8, 15, 25–27, 37, 79]. Chapters 4 and 5 consist of a survey of the work of Clifford [6, 7] and the work of Munn [65–70]. However, our contribution in these chapters provides a complete picture of the theory. Based on the other chapters, Chapter 6 presents the main results and the gist of the thesis. All other work is my own.

Chapter 1

Groups and Their Representations

The theory of representations of groups is important in group theory and related areas of algebra. Research dates back to the early 19th century. This chapter adopts both a historical perspective and a contemporary one to provide insight into the theory of representations of groups. In the first section, group representations are examined from a modern standpoint. Section 1.2 describes the motivations and development of the theory. Section 1.3 presents a timeline of the development of the theory between approximately 1850 and the present. This is followed by a discussion of applications.

1.1 Group Representation Theory

In this section, we provide several facts and properties concerning group representations and characters. All of these can be found in [10, 41, 42, 96, 99, 103, 107].

The definition of a linear group representation is based on the definition of group action on a set.

Definition 1.1.1. [103] An *action* of a group G on a set X is a function

$$(x, g) \in X \times G \to x \cdot g \in X$$

that takes a pair (x, g) of an element $x \in X$ and a group element $g \in G$ to $x \cdot g \in X$ such that

$$(x \cdot g_1) \cdot g_2 = x \cdot (g_1 g_2) \text{ for all } x \in X, g_1, g_2 \in G$$

and

$$x \cdot e = x \text{ for all } x \in X$$

where e is the identity element of G. We usually write xg instead of $x \cdot g$ and 1, rather than e, for the identity element of G.

Thus, an action of a group G on a set X is in fact equivalent to a homomorphism from G into the group of all bijective transformations of X. To show this, if a group G acts on X, as in the above definition, then define the map

$$\theta : G \to S_X$$

by $g\theta : X \to X$ is the map with $x(g\theta) = x \cdot g$. This is a bijection from X to X and a homomorphism since,

$$\begin{aligned} x((gh)\theta) &= x \cdot (gh) \\ &= (x \cdot g) \cdot h \\ &= (x(g\theta)) \cdot h \\ &= x((g\theta)(h\theta)), \end{aligned}$$

for all $x \in X$ and $g, h \in G$. This leads to:

Definition 1.1.2. Let G be a group and X be a set. Then G acts on X if there is a homomorphism $\theta : G \to S_X$.

According to definition 1.1.1, if we replace the set X with a vector space V and add a few more conditions, we obtain the group representation definition. Note that the action of a group will be on the right and the composition in G is left to right throughout the section.

Let V be an n-dimensional vector space over a field k. Consider the group $GL(V)$ of all invertible linear maps from V to V. We first define a representation as a linear map:

Definition 1.1.3. [99] Let G be a finite group. A (*linear*) *representation* of G on V is a homomorphism φ from G to $GL(V)$. Each g in G acts as a linear transformation. We write vg for the action of g on $v \in V$ in a representation φ, so that $v(g)\varphi = vg$.

The map $\phi : V \times G \to V$ is a representation of G on V if ϕ is a group action and if, for every $g \in G$, the function $\phi : V \to V$ defined by $(v)\phi = (v, g)\phi$ $(v \mapsto vg)$ is a linear function.

From Definition 1.1.3, since φ is a homomorphism, it follows that, for all $g, h \in G$, $(gh)\varphi = (g)\varphi(h)\varphi$. Moreover, if 1 is the identity of G and id is the identity linear map in $GL(V)$, then $(1)\varphi = id$. Hence $(g^{-1})\varphi = (g\varphi)^{-1}$, as

$$id = (1)\varphi = (gg^{-1})\varphi = (g)\varphi(g^{-1})\varphi.$$

Now, for all $v \in V$ and $g, h \in G$ the product $v(g\varphi) \in V$ and the homomorphism defined above show that

$$v((gh)\varphi) = v(g\varphi)(h\varphi).$$

Further, since $(1)\varphi$ is the identity map, it follows that $v(1\varphi) = v$ for all $v \in V$, and the properties of the linear maps in $GL(V)$ give that for all $u, v \in V$, $\lambda \in k$ and $g \in G$, we have

$$(\lambda v)(g\varphi) = \lambda(v(g\varphi) \text{ and } (u+v)(g\varphi) = u(g\varphi) + v(g\varphi)$$

This observation allows us to view a representation φ of G as a G-module, as defined below.

Definition 1.1.4. [42] Let G be a finite group and V be a finite-dimensional vector space over k. Then, V is a *G-module* if there exists a mapping $V \times G \to V$ such that $(v, g) \mapsto vg \in V$, where $v \in V$ and $g \in G$; and it satisfies the following conditions for all $u, v \in V, \lambda \in k$ and $g, h \in G$:

1. $v(gh) = (vg)h$;
2. $v1 = v$, where 1is the identity of a group G;
3. $(\lambda v)g = \lambda(vg)$;
4. $(u + v)g = ug + vg$.

Now, let G be a group and let k be a field. Consider the group $GL_n(k)$ of invertible $n \times n$ matrices with entries in k.

Definition 1.1.5. [42] A *representation* of G over k is a homomorphism φ from G to $GL_n(k)$, for some n. The integer n is the *degree* (or the *dimension*) of the representation.

Since a representation is a homomorphism, it follows that for every representation φ from G to $GL_n(k)$, we have

$$1\varphi = I_n, \text{ and}$$

$$g^{-1}\varphi = (g\varphi)^{-1} \text{ for all } g \in G,$$

where I_n denotes the $n \times n$ identity matrix.

We call this representation a *matrix representation.* Notice that if we consider an n-dimensional vector space V over k and choose a basis for V, then relative to this basis each endomorphism of V is represented by an $n \times n$ matrix over k. This gives rise to an isomorphism between the group $GL(V)$ of all linear maps from V to V and the group $GL_n(k)$. Thus, if we have a matrix representation, then we can think of it as a representation acting on the vector space k^n. Given a matrix representation $\psi : G \to GL_n(k)$, $g \mapsto A_g$, for A an $n \times n$ matrix, we get a linear representation $\varphi : G \to GL(V), g \mapsto \varphi_g$ via $(v)\varphi_g = vA_g$, for $g \in G$ and $v \in V$ with $\dim_k V = n$. Conversely, given a linear representation $\varphi : G \to GL(V)$, if we fix a basis B, we get a matrix representation $\psi : G \to GL_n(k)$ via $g \mapsto [(g)\varphi]_B$, where $[(g)\varphi]_B$ is the matrix of $(g)\varphi$ relative to the basis B . We often refer to V itself as the representation since the important thing is how G acts on V.

Two matrix representations are equivalent if and only if they describe the same representation in different bases:

Definition 1.1.6. [42] Let $\varphi : G \to GL(V)$ be a representation of a group G. Let $A = \{a_1, \ldots, a_n\}$ and $B = \{b_1, \ldots, b_n\}$ be two bases for V. Then the two associated matrix representations

$$\varphi_A : G \to GL_n(k)$$
$$\varphi_B : G \to GL_n(k)$$

are equivalent.

We need the following fundamental notions before we examine Maschke's Theorem below.

If $U_1, \ldots, U_r$ are subspaces of a vector space V, then the sum $U_1 + \ldots + U_r$ is a subspace of V and defined by

$$U_1 + \ldots + U_r = \{u_1 + \ldots + u_r : \ u_i \in U_i \text{ for } 1 \leq i \leq r\}.$$

We say that the sum $U_1 + \ldots + U_r$ is a *direct sum* if every element of the sum can be written in a unique way as $u_1 + \ldots + u_r$ with $u_i \in U_i$ for $1 \leq i \leq r$. If the sum is direct, then it is written as

$$U_1 \oplus \ldots \oplus U_r.$$

Definition 1.1.7. [42] A *subrepresentation* of a representation φ from G to $GL(V)$ is a linear subspace U of V that is invariant under the action of G, i.e. $(U)\varphi_g \subseteq U$ for all g in G.

Maschke's Theorem. [42] *Let G be a finite group and let k be a field of characteristic zero or coprime to $|G|$. Let V be a representation of G. If W is a subrepresentation of V, then there is a subrepresentation U of V such that $V = W \oplus U$.*

Definition 1.1.8. [42] A non-zero G-representation V is said to be *simple* or *irreducible* if the only subrepresentations of V are $\{0\}$ and V itself.

The space V is called *completely reducible* or *semisimple*, if it can be expressed as a direct sum of irreducible subrepresentations. The power of Maschke's Theorem lies in the following consequence.

Complete Reducibility Theorem. [42] *Every finite-dimensional representation of a finite group over a field of characteristic zero is completely reducible.*

Recall that the symmetric group S_n is the set of all permutations of a set of n symbols. Suppose we have a subgroup $G \subset S_n$. Then we can construct an n-dimensional representation of G called a permutation representation. The method of the construction is as follows:

Let V be an n-dimensional vector space over a field k, with a basis $\{v_1, \ldots, v_n\}$ and G be a subgroup of S_n. Every element g in G is a permutation of the set $[n] = \{1, \ldots, n\}$. For each i with $1 \leq i \leq n$ and each permutation g in G, define a linear map

$$(g)\rho : V \to V \text{ by } v_i \mapsto v_{ig}$$

Then $v_i g \in V$ and $v_i 1 = v_i$. Also, for $g, h \in G$,

$$v_i(gh) = v_{i(gh)} = v_{(ig)h} = (v_i g)h.$$

Then ρ is a homomorphism.

Definition 1.1.9. [42] Let G be a subgroup of S_n. The representation V of G with basis $\{v_1, \ldots, v_n\}$ such that $v_i g = v_{ig}$ for all i, and all $g \in G$, is called the *permutation representation* for G over k. We call $\{v_1, \ldots, v_n\}$ the *natural basis* of V.

Matrices representing $(g)\rho$ with respect to the natural basis of V are called *permutation matrices*: entries 0 everywhere except 1 in each row and column, and $((g)\rho)_{ij} = 1$ precisely when $ig = j$, for $1 \leq i, j \leq n$.

Example 1.1.10. Let $G = \{(1), (123), (132)\} \subset S_3$ so that G is a subgroup. Let V be a 3-dimensional permutation representation for G over $\mathbb{C}$. Then, the permutation representation of G is

$$((1))\rho = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, ((123))\rho = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, ((132))\rho = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}.$$

We now move on to defining characters of a finite group which are the essential tools for the study of group representations. In contrast with representations, we intend to write characters as functions on the left. Any two matrices representing a linear map from V to V with respect to different bases have the same trace. Thus we may refer to the *trace* of a linear map. Note that the trace of a linear function is just the trace of any matrix representing it under any basis. That is, given $L : V \to V$ define $\mathrm{tr}(L)$ by trace of a matrix of L under any basis.

Definition 1.1.11. [42] Let $\varphi : G \to GL(V)$ be a representation of a finite group G. The *character* of this representation is a function $\chi : G \to \mathbb{C}$ defined as $\chi(g) = tr((g)\varphi)$, for $g \in G$, where tr denotes the trace of the linear map $(g)\varphi$. Moreover, if V is an irreducible representation, then χ is called an *irreducible character*.

Proposition 1.1.12. [42] *Let V be representations of a finite group G, and suppose that*

$$V = U_1 \oplus \cdots \oplus U_r,$$

a direct sum of irreducible subrepresentations $U_i, 1 \leq i \leq r$ of V. Then the characters of V is equal to the sum of characters of the subrepresentations $U_1, \ldots, U_r$.

If two elements g and h are conjugate in G, then $\chi(g) = \chi(h)$. The dimension of a representation V, which is the character value of the identity matrix on V, is referred to as the *degree* of the character.

Theorem 1.1.13. [42] *Let V and W be representations of a finite group G, with characters χ and ψ, respectively. Then V and W are isomorphic if and only if $\chi = \psi$.*

Definition 1.1.14. [42] A *class function* is a function $\varphi : G \to \mathbb{C}$ such that $(g)\varphi = (hgh^{-1})\varphi$, for all $g, h \in G$.

The set of all class functions into $\mathbb{C}$ for a given group G is denoted by $\mathbb{C}_{class}(G)$. The set $\mathbb{C}_{class}(G)$ is a subspace of the vector space of all functions from G to $\mathbb{C}$. A basis of $\mathbb{C}_{class}(G)$ is given by those functions which take the value 1 on precisely one conjugacy class and zero on all other classes. Thus, if m is the number of conjugacy classes of G, then $\dim \mathbb{C}_{class}(G) = m$.

Since $tr(B^{-1}AB) = tr(A)$, characters of G are invariant under conjugation and hence are class functions on G. For all g and h in G:

$$\begin{aligned}
\chi(hgh^{-1}) &= tr\Big((hgh^{-1})\varphi\Big) \\
&= tr\Big((h)\varphi(g)\varphi(h^{-1})\varphi\Big) \\
&= tr\Big((h^{-1})\varphi(h)\varphi(g)\varphi\Big) \\
&= tr((h^{-1}h)\varphi(g)\varphi) \\
&= tr((g)\varphi) \\
&= \chi(g).
\end{aligned}$$

This shows that character is constant on conjugacy classes of G. Since the dimension of $\mathbb{C}_{class}(G)$ is the number of conjugacy classes in G, which is in fact equal to the number of irreducible characters of G, and the irreducible characters form a linearly independent set in $\mathbb{C}_{class}(G)$, the irreducible characters form a *basis of* $\mathbb{C}_{class}(G)$.

Definition 1.1.15. [42] Let $g \in G$. The *centralizer* of g in G, denoted by $C_G(g)$, is the set of elements of G which commute with g; that is,

$$C_G(g) = \{h \in G : \ gh = hg \ (\text{or } h^{-1}gh = g)\}.$$

Corollary 1.1.16. [42, Corollary 15.4] *Let G be a finite group. The irreducible characters $\chi_1, \dots, \chi_k$ of G form a basis of the vector space of all class functions on G. Indeed, if ψ is a class function, then*

$$\psi = \sum_{i=1}^{k} \lambda_i \chi_i$$

where

$$\lambda_i = \langle \psi, \chi_i \rangle = \sum_{i=1}^{k} \frac{\psi(g_i)\overline{\chi_i(g_i)}}{|C_G(g_i)|}$$

for $1 \leq i \leq k$; g_i are the representatives of the conjugacy classes of G.

As consequence, there is one-to-one correspondence between the irreducible characters of a group G and the conjugacy classes of G.

For the upcoming fact, we define first the tensor product space and then the dual of a representation as a representation. Let V and W be two vector spaces over a field k with dimensions $\dim V = m$ and $\dim W = n$. Fix bases $\{v_1, \dots, v_m\}$ and $\{w_1, \dots, w_n\}$ for V and W respectively. The *tensor product space* $V \otimes W$ of V and W is an mn-dimensional vector space over k with basis $\{v_i \otimes w_j : 1 \leq i \leq m, 1 \leq j \leq n\}$. Thus:

- $V \otimes W = \left\{ \sum\limits_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n}} \lambda_{ij}(v_i \otimes w_j) \ : \ \lambda_{ij} \in k \right\}$ with obvious addition and scalar multiplication;

- if $v = \sum \alpha_i v_i \in V$ and $w = \sum \beta_j w_j \in W$, then $v \otimes w$ is defined as

$$v \otimes w = \sum_{i,j} \alpha_i \beta_j (v_i \otimes w_j).$$

The map $\varphi : V \times W \to V \otimes W$ is a bilinear map which has the property that for any bilinear map $h : V \times W \to Z$, for any vector space Z, there is a unique linear map $\tilde{h} : V \otimes W \to Z$ such that $h = \varphi \tilde{h}$. See [50, Chapter XVI].

Let V be a finite-dimensional representation of a finite group G. The image of $v \in V$ under the action of $g \in G$ is $v \cdot g$, so $v \cdot (gh) = (v \cdot g) \cdot h$. Let

$$V^* = \{\alpha : V \to \mathbb{C} : \alpha \text{ is a linear function}\}.$$

We define the action of G on the vector space V^* by $\alpha \mapsto \alpha \cdot g$ where $(v)(\alpha \cdot g) = (v \cdot g^{-1})\alpha$ and $\alpha \cdot e = \alpha$. Then we want to show that

$$\alpha \cdot (gh) = (\alpha \cdot g) \cdot h.$$

For all $v \in V$,

$$\begin{aligned}(v)(\alpha \cdot (gh)) &= (v \cdot (gh)^{-1})\alpha \\ &= (v \cdot (h^{-1}g^{-1}))\alpha \\ &= ((v \cdot h^{-1}) \cdot g^{-1})\alpha \\ &= (v \cdot h^{-1})(\alpha \cdot g) \\ &= (v)((\alpha \cdot g) \cdot h).\end{aligned}$$

This is the required result and thus V^* is a representation for the group G, called the *dual space* of V. Note that the dimension of V^* is equal to the dimension of V.

Proposition 1.1.17. [99] *Let V and W be representations of a finite group G. Then the following statements hold:*

1. $\chi(V \oplus W) = \chi(V) + \chi(W)$;
2. $\chi(V \otimes W) = \chi(V) \cdot \chi(W)$;
3. $\chi(V^*) = \overline{\chi(V)}$.

Now, suppose that a group G acts on a vector space V with character χ_1, and a group H acts on a vector space W with character χ_2. Then $G \times H$ acts on the vector space $V \otimes W$ with character $\chi(g, h) = \chi_1(g)\chi_2(h)$, for $g \in G$ and $h \in H$. Furthermore, each irreducible representation of $G \times H$ is a tensor product of an irreducible representation of G and with that of H. This is called the *external tensor product* of representations.

Let G be a finite abelian group. If the order of G is n then $g^n = 1$, for every $g \in G$ and hence $\chi(g)^n = 1$ for each $g \in G$ and each character on G. Therefore, a character of the group G maps G to the roots of unity. Two characters can be multiplied pointwise to define a new character. We explain this in the following paragraph.

The product of two characters χ_1, χ_2 of a group G is defined by $(\chi_1\chi_2)(g) = \chi_1(g)\chi_2(g)$ for $g \in G$. With this product, the characters on G, for G an abelian group, form a group called the *character group* of G and denoted by $\widehat{G}$. The identity element of $\widehat{G}$ is the trivial character that maps G to 1. Since any character on G maps G to the roots of unity, the inverse $\chi^{-1} : g \mapsto \chi(g)^{-1}$ of a character χ is equal to its complex conjugate $\overline{\chi} : g \mapsto \overline{\chi(g)}$. A character group of an abelian group G is isomorphic with G and in particular, $|\widehat{G}|= |G|$.

Characters provide the information needed to classify irreducible representations. This information is organized into a square table called the *character table*, in which the left-most column is dedicated to the irreducible characters, and the top row is dedicated to the conjugacy classes of the group. Each box has a number and that number is the value of the character on the respective conjugacy class. The $k \times k$ matrix of this table is denoted by $X = [\chi_i(g_j)]$, where $\chi_1(= 1), \chi_2, \dots, \chi_k$ are the irreducible characters of G, and $\mathcal{C}_1(= \{e_G\}), \mathcal{C}_2, \dots, \mathcal{C}_k$ are the conjugacy classes, with $g_j \in \mathcal{C}_j$. The $(i, j)^{th}$ entry of X is $\chi_i(g_j)$, for $1 \leq i, j \leq k$.

Example 1.1.18. Let G be a symmetric group S_3. Recall that in symmetric groups, conjugacy classes are the same as the cycle-type classes, thus by Corollary 1.1.16 there are 3 irreducible characters of S_3. Hence the character table will be a 3×3 matrix:

S_3	id	(12)	(123)
trivial χ_1	1	1	1
sign χ_2	1	-1	1
geometric χ_3	2	0	-1

To fully understand the next section, the following concepts are required.

Let G be a finite group with elements $g_1, \dots, g_n$, and k be a field. We define a vector space over k with $g_1, \dots, g_n$ as basis, we denote this vector space by $k[G]$. The elements of $k[G]$ are expressions of the form

$$\lambda_1 g_1 + \cdots + \lambda_n g_n, \text{ where all } \lambda_i \in k.$$

The rules for addition and scalar multiplication are defined as: if

$$u = \sum_{i=1}^{n} \lambda_i g_i \text{ and } v = \sum_{i=1}^{n} \mu_i g_i$$

are elements of $k[G]$, and $\lambda_i, \mu_i \in k$, then

$$u + v = \sum_{i=1}^{n} (\lambda_i + \mu_i) g_i \text{ and } \lambda u = \sum_{i=1}^{n} (\lambda\lambda_i)_i g_i.$$

With these rules, $k[G]$ is a a vector space over k of dimension n, with basis $g_1, \ldots, g_n$, called *natural basis* of $k[G]$.

Definition 1.1.19. [42] The vector space $k[G]$, with multiplication defined by

$$\Big(\sum_{g\in G} \lambda_g g\Big)\Big(\sum_{h\in G} \mu_h h\Big) = \sum_{g,h\in G} \lambda_g \mu_h (gh).$$

for $\lambda_g, \mu_h \in k$, is called the *group algebra* of G over k.

Definition 1.1.20. [42] Let G be a finite group and k be a field. The representation $g \mapsto [g]_B$ obtained by taking B to be the natural basis of the group algebra $k[G]$ ($B = G$) is called the *regular representation* of G over k, where $[g]_B$ denotes the matrix of the endomorphism $v \mapsto vg$ of V, relative to the basis B.

If l is a subfield of a field k, then we say k is a *field extension* of l and it is denoted by k/l.

Definition 1.1.21. [96] Let k/l be a field extension. The *Galois group* is

$$Gal(k/l) = \{\text{automorphisms } \sigma \text{ of } k \text{ fixing } l \text{ pointwise}\}$$

under the binary operation of composition. If $f \in l[x]$ has a splitting field k (which means that f is a product of linear factors in $k[x]$), then the Galois group of f is $Gal(k/l)$.

Here $l[x]$ is the polynomial ring over a field l and an automorphism σ of k fixes l pointwise if $c\sigma = c$ for every element c in l.

Let G be a finite group. Then we call a field k a *splitting field* for G if every irreducible representation of G remains irreducible for every extension field of k.

1.2 Motivations and Developments

A character of a finite group is defined as the trace of a matrix representation. However, it was initially introduced via a different approach. The starting point of character theory of finite groups was in 1896 when the algebraist and number theorist Richard Dedekind (1831-1916) posed a problem to the group theorist Ferdinand Georg Frobenius (1849-1917). The problem was about factoring the determinant of a matrix corresponding to a finite arbitrary group, which is called the group determinant. In this section, we discuss briefly the motivation that led Frobenius to his creation of character theory and subsequently the representation theory of finite groups. This section also contains the main points of the study of the abstract problem of factoring group determinant. For further discussions, [9, 11, 12, 31, 32, 43, 44, 48, 49] provide a historical survey of the work. The original works can be found in [17–20,24] [1] (written in German). Some of the mathematical parts of the discussion are presented in their modern formulation.

Before exploring Frobenius's motivations, it is worth discussing his mathematical background, and that of Dedekind, when they started their correspondence.

Dedekind's background [12, 32] was based on Gaussian characters [2] of finite abelian groups, higher reciprocity and the Legendre symbol, Dirichlet's application of analytical methods to number theory, and his contemporary work on hypercomplex systems and on number theory via his editing of Dirichlet's *Vorlesungen*. This explains why the development of character theory has its unexpected origins in number theory. Prior to 1896, Frobenius did not know about group determinants. However, he had earlier worked with a similar concept relating to the factorization of a homogeneous polynomial in theta functions and in linear algebra, and presumably this was the reason that the problem had his direct attention. Frobenius's background [12,32] was the theory of linear differential operators, linear forms with integer coefficients, improved proofs of Sylow's theorem, linear and bilinear forms, and the theory of biquadratic forms.

Before tracing back the development and the progress of representation theory of groups, we look at the letter that is at the heart of the matter. In his March 1896 letter to Frobenius (published in [13]), Dedekind introduced the concept of the *group determinant* of a finite group. He explained how it factors and suggested Frobenius think about the general case. Dedekind defined this fundamental idea using the following statement (the exposition of the following definition is in its modern formulation):

Definition 1.2.1. [9,48] For a finite group G of size n, let $\{x_g : g \in G\}$ be a set of independent variables over the field of complex numbers $\mathbb{C}$. Define the *group matrix* $X_G(= X_{gh})$ as the $n \times n$ matrix with rows and columns indexed by the elements of G such that the (g, h) entry in X_G is x_{gh}. The *group determinant* $\Theta(G)$ of G is then the determinant of X_G, which is therefor a homogeneous polynomial of degree n in the variables x_g.

However, Dedekind preferred using the variable $x_{gh^{-1}}$ instead of x_{gh} in the (g, h) position to interchange the g and g^{-1} columns of the group table which only affects

[1] The original works of Frobenius writings were brought together in Gesammelie Abhandlungen, J. P. Serre, ed., 3 vols. (BerlinHeidelberg-New York, 1968).

[2] The mathematical term *character* was first introduced by Gauss in his 1801 paper *Disquisitiones Arithmeticae* to assign the numerical information to classes of binary quadratic forms, in order to separate classes of forms with the same determinant into different genera [12, Chapter I, page 3-4].

the determinant by a factor of ± 1.

The following example illustrates the idea of a group determinant.

Example 1.2.2. Consider the cyclic group of order 3, $G = C_3 = \{1, a, a^2\}$ with relation $a^3 = 1$. Then the group matrix is

$$X_{gh} = \begin{bmatrix} x_1 & x_a & x_{a^2} \\ x_a & x_{a^2} & x_1 \\ x_{a^2} & x_1 & x_a \end{bmatrix} \text{ or } X_{gh^{-1}} = \begin{bmatrix} x_1 & x_{a^2} & x_a \\ x_a & x_1 & x_{a^2} \\ x_{a^2} & x_a & x_1 \end{bmatrix}.$$

The determinant is $\Theta(G) = x_1^3 + x_a^3 + x_{a^2}^3 - 3x_1x_ax_{a^2}$. This polynomial decomposes into linear factors over $\mathbb{C}$ as:

$$(x_1 + x_a + x_{a^2})(x_1 + \omega x_a + \omega^2 x_{a^2})(x_1 + \omega^2 x_a + \omega x_{a^2}),$$

where ω is a primitive cube root of unity.

It might be asked why Dedekind was interested in a group determinant and how it came about. The motivation came from his study of the discriminant in a normal field (the algebraic number field). How exactly did this happen? The answer to this question takes us to the initial part of the tale. In 1846, Eugène Charles Catalan (1814–1894) introduced for the first time the so-called *circulant* as follows.

Definition 1.2.3. [9] For a positive integer n, let $X_0, \ldots, X_{n-1}$ be indeterminates and consider an $n \times n$ matrix where each row is obtained from the previous one by a cyclic shift one step to the right:

$$\begin{bmatrix} X_0 & X_1 & X_2 & \cdots & X_{n-1} \\ X_{n-1} & X_0 & X_1 & \cdots & X_{n-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ X_1 & X_2 & X_3 & \cdots & X_0 \end{bmatrix}.$$

The determinant of this matrix is called a *circulant* of order n, and it is a homogeneous polynomial of degree n with integer coefficients.

Ten years later, William Spottiswoode (1825-1883) discovered that over the complex numbers, the circulant of order n factors into n homogeneous linear polynomials whose coefficients are nth roots of unity.

Dedekind [3] then considered the extension of the notion of the circulant into group theory. Let k be a finite field of degree n extension of the rational numbers and let

[3]Dedekind, R., 1931. Gesammelte mathematische Werke, 2, Braunschweig.

$G = \{\pi_1, \ldots, \pi_n\}$ be the Galois group of k. Given $\omega_1, \ldots, \omega_n$, any basis for k over $\mathbb{C}$, then the discriminant Δ of $\omega_1, \ldots, \omega_n$ is defined as $\Delta = D^2$, where $D = \det(\omega_i \pi_j)$. Here, $(\omega_i \pi_j)$ is an $n \times n$ matrix whose ith row consists of the entries $(\omega_i \pi_1, \ldots, \omega_i \pi_n)$. By picking a single element ω in k such that the n entries $(\omega\pi_1, \ldots, \omega\pi_n)$, form a basis for k, or if $\omega_i = \omega\pi_i$, then we get $D = \det(\omega\pi_i\pi_j)$. If we let $x_\pi = \omega\pi$, then the determinant will be $D = \det(x_{\pi_i \pi_j})$. Thus, the discriminant of k leads to the group determinant.

In 1879 [4], Dedekind formally defined the character function for a finite abelian group G. The following definition is in its modern formulation:

Definition 1.2.4. [31, 48] A character is a homomorphism χ from G into the multiplicative group of the field $\mathbb{C}$ of nonzero complex numbers; i.e. a map $\chi : G \to \mathbb{C}^*$ that satisfies $\chi(gh) = \chi(g)\chi(h)$ for all g, h in G.

This is in fact the second version of a character's definition and we now need to clarify the relationship between this version and the previous version, Definition 1.1.11, which is in the sense of representations. First, we show that Definition 1.1.11 is an example of Definition 1.2.4:

$$\begin{aligned}\chi(gh) &= tr((gh)\varphi) = tr((g)\varphi(h)\varphi) \\ &= tr((g)\varphi) \cdot tr((h)\varphi) \\ &= \chi(g) \cdot \chi(h).\end{aligned}$$

Secondly, since a character χ is a class function $G \to \mathbb{C}$ that is constant on conjugacy classes of the group G and by using Corollary 1.1.16, we have:

$$\chi = \sum_{i=1}^{k} \lambda_i \chi_i.$$

Note that the character χ here is in the sense of Definition 1.2.4. Now, if the character χ_i comes from a G-representation V_i, then χ is the character of $V_1^{a_1} \bigoplus \ldots \bigoplus V_k^{a_k}$ and this is an example of Definition 1.1.11. We conclude that the two versions of character are equivalent.

Then, around 1880, Dedekind discovered and proved that if a group G is an abelian group, then the associated group determinant factors completely into linear forms with coefficients given by the characters of G.

Theorem 1.2.5. [11, 48] *Let G be a finite abelian group of size n and let $\widehat{G}$ be the*

[4]This was in one of Dedekind's supplements to Dirichlet's lectures in number theory.

character group of G. The factorization of the group determinant is:

$$\Theta(G) = \prod_{\chi \in \widehat{G}} P_\chi, \text{ where } P_\chi = \sum_{g \in G} \chi(g) x_g.$$

Then $\Theta(G)$ is a homogeneous polynomial of degree n and it factors into exactly n homogeneous linear polynomial P_χ over $\mathbb{C}^*$.

For the above example, the cyclic group C_3 has three characters: χ_1, χ_2 and χ_3, determined by the generator element g of C_3 by $\chi_{i+1}(g) = \omega^i$, where ω is a primitive cube root of unity. Thus, the factorization of the group determinant is:

$$\Theta(G) = \prod_{i=1}^{3} \Big(\chi_i(1)x_1 + \chi_i(a)x_a + \chi_i(a^2)x_{a^2} \Big).$$

It is noteworthy that the underlying concept of character here is that of a group determinant, but not yet that of a group representation.

We can now examine the situation in the case of a finite non-abelian group.

Example 1.2.6. [31] Consider the dihedral group $D_3 = \{1, a, a^2, b, ab, a^2b\}$. Then the group determinant $\Theta(D_3)$ is the product of the following homogeneous factors:

$$(x_1 + x_a + x_{a^2} + x_b + x_{ab} + x_{a^2b}), (x_1 + x_a + x_{a^2} - x_b - x_{ab} - x_{a^2b}), \text{ and}$$

$$(x_1^2 + x_a^2 + x_{a^2}^2 - x_1x_a - x_1x_{a^2} - x_ax_{a^2} - x_b^2 - x_{ab}^2 - x_{a^2b}^2 + x_bx_{ab} + x_bx_{a^2b} + x_{ab}x_{a^2b})^2.$$

As above, the linear factors are derived from the homomorphisms $\chi : D_3 \to \mathbb{C}^*$, but the last factor is of degree two. So what does this mean?

In the period 1880-1886, Dedekind worked intermittently on investigating the following questions: how does $\Theta(G)$ – as an element of $\mathbb{C}[x_g]$, the polynomial ring over $\mathbb{C}$ – factor into irreducible components, and what does it tell us about a finite group? He succeeded in finding a solution for the case when G is abelian in Theorem 1.2.5, but he could not resolve the question in general. For example, he studied the behaviour of the group determinants for two non-abelian groups, the symmetric group S_3 of order 6 and the quaternion group Q_8 of order 8, and found that some factors of their group determinants were not linear. Eventually, in March 1896, Dedekind proposed this problem to Frobenius and provided a number of useful results, the examples and conjectures. Without the collaboration between Dedekind and Frobenius theory of finite group characters and representations would not exist. With enthusiasm, Frobenius showed an interest and immediately considered the question.

Frobenius formulated the problem as follows [12, 32, 48]: let G be a finite group. If

$$\Theta(G) = \prod_{\lambda=1}^{l} \Phi_\lambda(x)^{e_\lambda}$$

is the factorization of the determinant into different irreducible factors Φ of degree f_λ, (where l is the number of conjugacy classes of G), then how does the factorization reflect the properties of the group G?

In order to solve the problem of how $\Theta(G)$ factors linearly, Dedekind tried to extend the field $\mathbb{C}$ to be a hypercomplex number system (a linear associative algebra over the complex numbers in modern terminology). By contrast, Frobenius looked at the coefficients of the irreducible factors over $\mathbb{C}$ and investigated the relation between the structure of finite group G and the irreducible factors of the homogeneous polynomials and their number and degree. In about a month, with intensive work, Frobenius succeeded in extending the concept of character to arbitrary finite groups and applied it to provide a full solution to Dedekind's group determinant problem. In the following paragraphs, we review the results of Frobenius in this area.

Dedekind [5] emphasized the following conjecture, which was proven by Frobenius. This was also the first conjecture that Frobenius considered.

Theorem 1.2.7. [11, 31, 48] *The number of linear factors in the factorization of $\Theta(G)$ over $\mathbb{C}$ is equal to the index of the commutator subgroup G' and hence to the order of the abelian group G/G'.*

This theorem relates the property of the group determinant $\Theta(G)$ to the structure of the underlying group. This was later successfully proved by Frobenius. The first main result for Frobenius was on the basic property of irreducible characters, the orthogonality relations [6]:

Theorem 1.2.8. [11, 12] *For two irreducible characters χ and ψ of G, we have*

$$\frac{1}{|G|}\sum_{g\in G}\chi(g)\psi(g^{-1}) = \begin{cases} 1 & \text{if } \chi = \psi; \\ 0 & \text{otherwise.} \end{cases}$$

Using the previous orthogonality relations, he proved the following results [7]:

Theorem 1.2.9. [12, 48] *The number of distinct irreducible factors of $\Theta(G)$ over $\mathbb{C}$ is equal to the number of the conjugacy classes of G.*

[5] Dedekind, Werke, vol.2, page 422.

[6] The original formula of the relations are in *Über Gruppencharaktere* [18].

[7] The original formula of the results are in *Über die Primfactoren der Gruppendeterminante* [19].

Theorem 1.2.10. [12, 32, 48] *The degrees of the irreducible factors of* $\Theta(G)$ *divide the order of* G.

He then obtained a result which he called the *fundamental theorem in the theory of the group determinant*, through which he achieved the desired result [7]:

Theorem 1.2.11. [12,32,48] *The degree* f_λ *of each irreducible factor of* $\Theta(G)$ *and the multiplicity* e_λ *with which it occurs in the factorization of* $\Theta(G)$ *coincide and are both equal to the degree of the corresponding character.*

Frobenius called the positive integer $f_\chi = \chi(1)$ the *degree* [*Grad*] of a character χ, which is the value of χ at the identity element 1 of G.

In terms of matrix representations, this is the theorem that an irreducible representation occurs in a representation as often as its degree. According to Frobenius, this was a difficult theorem to prove and its proof is extremely long and complicated. An important consequence of this fundamental theorem is is that the order of a group is equal the sum of the squares of the degrees of its characters [8].

Corollary 1.2.12. [11, 12] *If* G *is a finite group with order* n*, then*

$$\sum f_\chi^2 = n \ (f_\chi = e_\chi).$$

Frobenius then stated the multiplicative property of the group determinant $\Theta(G)$, given here in the next theorem [8].

Theorem 1.2.13. [9,12] *Let* G *be a finite group. Let* Φ *be a homogeneous irreducible polynomial in the variables* X_g*. Then* $\Phi(xy) = \Phi(x)\Phi(y)$ *if and only if* Φ *is monic in* X_e *and is a factor of* $\Theta(G)$*, where* e *is the identity element of* G *and* x, y *are independent variables (or indeterminates).*

When Frobenius was trying to prove his fundamental Theorem 1.2.11, he did not know that he was studying representations. Let us briefly discuss the observation that led Frobenius to shift from characters to representations. If G is any group of order n, and

$$\Theta(G) = \det(X_G) = \prod_{\lambda=1}^{k} \Phi_\lambda^{f_\lambda}$$

is the factorization of the group determinant into its irreducible factors, then the k characters $\chi^{(\lambda)}$ of G may be used to define a matrix A such that

$$A^{-1}X_GA = \begin{pmatrix} N_1(x) & 0 & \cdots & 0 \\ 0 & N_2(x) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & N_k(x) \end{pmatrix}, \tag{1.1}$$

[8]Originally, it appeared in *Über die Primfactoren der Gruppendeterminante* [19].

where $\det(N_\lambda(x)) = \Phi_\lambda(x)^{f_\lambda}$ [32]. Frobenius proved this result and then defined the matrix A that carried the decomposition of the group matrix further:

Theorem 1.2.14. [32] *If G is any group of order n, then a matrix A exists such that (1.1) holds, where the $f_\lambda^2 \times f_\lambda^2$ matrix $N_\lambda(x)$ has the form*

$$N_\lambda(x) = \begin{pmatrix} x_{ij}^{(\lambda)} & 0 & \cdots & 0 \\ 0 & x_{ij}^{(\lambda)} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & x_{ij}^{(\lambda)} \end{pmatrix},$$

with $x_{ij}^{(\lambda)}$ is an $f_\lambda \times f_\lambda$ matrix whose f_λ^2 entries $x_{ij}^{(\lambda)}$ are linearly independent homogeneous functions of the group variables $x_g, x_h, \ldots$ and $\det(x_{ij}^{(\lambda)}) = \Phi_\lambda(x)$.

From this step, Frobenius changed his perspective from determinants to matrices. For every λ, $x_{ij}^{(\lambda)}$ defines an irreducible representation, which is associated with an irreducible factor Φ_λ of the group determinant $\Theta(G)$ and hence with its characters. This theorem shows that representation of G decomposes into the irreducible representations defined by the matrices $x_{ij}^{(\lambda)}$. In other words, it implies the complete reducibility theorem for the regular representation of G.

One year later in 1897, Frobenius first introduced the idea of a representation of a finite group and explained the equivalence between two representations [12,32]. His main result contains the general version of Dedekind's definition of character and shows that the character is the trace function of the matrix representation.

Each irreducible factor of the group determinant corresponds to an irreducible character. In modern mathematical language, the Maschke-Wedderburn theory of semisimple algebras states that the group algebra $\mathbb{C}[G]$ is isomorphic to a direct product of irreducible matrix algebras over $\mathbb{C}$, and this is in fact equivalent to the reduction of the group matrix X_G to a block diagonal matrix, which in turn is equivalent to the decomposition of the group determinant $\Theta(G)$ into irreducible factors.

Frobenius then introduced a function called a *k-character* using an algorithm to construct the factor corresponding to the character χ.

Definition 1.2.15. [43,44] Let χ be an irreducible character of G. The k-character associated with the character χ is the function $\chi^{(k)} : G^k \to \mathbb{C}$ defined recursively by:

1. $\chi^{(1)} = \chi(g)$ (the ordinary character), and
2. $\chi^k(g_1, g_2, \ldots, g_k) = \chi(g_1)\chi^{(k-1)}(g_2, g_3, \ldots, g_k) - \chi^{(k-1)}(g_1 \cdot g_2, g_3, \ldots, g_k)$
$$- \chi^{(k-1)}(g_2, g_1 \cdot g_3, \ldots, g_k) - \ldots - \chi^{(k-1)}(g_2, g_3, \ldots, g_1 \cdot g_k).$$

We illustrate the algorithm for $k = 2$ and 3:

1. $\chi^{(2)}(g, h) = \chi(g)\chi(h) - \chi(gh)$,
2. $$\begin{aligned}\chi^{(3)}(g, h, k) &= \chi(g)\chi^2(h, k) - \chi^2(gh, k) - \chi^2(h, gk)\\ &= \chi(g)\chi(h)\chi(k) - \chi(g)\chi(hk) - \chi(h)\chi(gk)\\ &\quad - \chi(k)\chi(gh) + \chi(ghk) + \chi(gkh),\end{aligned}$$

here g, h, k are in G. To illustrate analytically, the polynomial P_χ in Theorem 1.2.5 is completely determined by its k-characters and vice versa. Frobenius expressed this fact explicitly in the next theorem [9]:

Theorem 1.2.16. [12, 32, 48] *Let G be a finite group and let $A = \{\chi_1, \ldots, \chi_m\}$ be a complete set of irreducible characters of G. Then, the number of irreducible characters of G equals the number of conjugacy classes of G. Moreover, the complete factorization of the group determinant is*

$$\Theta(G) = \prod_{\chi \in A} P_\chi, \text{ where } P_\chi = \frac{1}{d!} \sum_{\bar{g} \in G^d} \chi^{(d)}(\bar{g}) x_{\bar{g}}$$

and d is the degree of χ with P_χ the corresponding irreducible factor of $\Theta(G)$ and if $\bar{g} = (g_1, \ldots, g_d)$, then $x_{\bar{g}} = x_{g_1} \cdots x_{g_d}$.

It is worth pointing out here that, without representations, Frobenius achieved significant results regarding representations of finite groups. During his development of character theory, he discovered that his generalization of the characters was in fact the trace functions of the irreducible representations of the group.

We can sketch the relation between the determinant, characters and representations of a finite group from Frobenius's perspective. Let G be a finite group of order n and consider the group algebra $\mathbb{C}[G] = \{\sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{C}\}$. Consider a linear transformation T_g that acts on the algebra as right multiplication by $g \in G$:

$$T_g\Big(\sum_{h \in G} \alpha_h h\Big) = \sum_{h \in G} \alpha_h hg.$$

Let $\sigma(g_1)$ be the matrix representation of T_{g_1} with respect to the basis G of $\mathbb{C}[G]$. Thus, $\sigma(g_1)$ is a matrix with 0 and 1 entries. Then $g_1 \mapsto \sigma(g_1)$ is the right regular representation of G and

$$\Theta(G) = \det(X_G), \text{ where } X_G = x_{g_1}\sigma(g_1) + \ldots + x_{g_n}\sigma(g_n)$$

is the $n \times n$ group matrix of G.

[9] The original formula is in *Über Gruppencharaktere* [18].

Moreover, let A be a non-singular $n \times n$ matrix over the complex numbers such that for all elements g of G:

$$A\sigma(g)A^{-1} = \begin{pmatrix} \mu(g) & 0 \\ 0 & \nu(g) \end{pmatrix},$$

where $\mu(g)$ and $\nu(g)$ are $r \times r$ and $s \times s$ matrices, respectively, with complex entries. If $\Theta(x_1, \ldots, x_n)$ is a polynomial with integer coefficients in n independent indeterminates $x_1, \ldots, x_n$, where x_i denotes any indeterminate x_g with $g \in G$. Then,

$$\Theta(x_1, \ldots, x_n) = \Phi_1(x_1, \ldots, x_n)\Phi_2(x_1, \ldots, x_n)$$

where $\Phi_1 = \det\Big(\sum x_i\mu(g_i)\Big)$ and $\Phi_2 = \det\Big(\sum x_i\nu(g_i)\Big)$, are polynomials in the x_i with complex coefficients of degree r and s respectively. Thus, the decomposition of a group representation into irreducible representations is in fact equivalent to the decomposition of group determinants into irreducible factors. Additionally, each factor of the group determinant is a homogeneous polynomial with the same degree as its corresponding representation.

The relation between representations and characters of a finite group becomes clear after the following theorem which shows that Frobenius characters are simply the trace functions of the irreducible representations of the group[10].

Theorem 1.2.17. [12, 32, 48] *Let G be a finite group of order n, and let X_G be the group matrix of G. Then there exists an invertible $n \times n$ matrix P such that $P^{-1}X_GP = U$, for a matrix U which is the direct sum of submatrices U_i.Each submatrix U_i is a matrix associated with a representation of G whose determinant is one of the irreducible factors Φ of the group determinant*

$$\Theta(G) = |X_G| = \prod \Phi^f.$$

Each factor of the group determinant is associated with a submatrix U_i, in this way. Moreover, let

$$U_i \leftrightarrow \Phi \leftrightarrow \chi$$

with χ the character of degree f corresponding to Φ. Let $r \mapsto (R)$ be the representation of G such that $U_i = \sum(R)x_r$. Then the coefficient of u^{f-1} in the characteristic polynomial $|U_i - uI|$ of the matrix U_i is $\sum \chi(r)x_r$ and then we have

$$\chi(r) = tr(R) = \sum r_{ii}$$

[10]The original formulation of the theorem can be found in *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen* [20].

for the matrix $(R) = (r_{ij})$ *corresponding to* $r \in G$.

Theorem 1.2.18. [9] *For an irreducible complex representation* σ *of* G*:*

1. *the polynomial* $\det\Big(\sum_g x_g\sigma(g)\Big)$ *is irreducible and*

2. σ *is determined by* $\det\Big(\sum_g x_g\sigma(g)\Big)$.

In the period from 1897 to 1899, Frobenius set up the foundation of representation theory [12, 32]. He published four papers covering representations of finite groups, induced characters, and tensor products of characters. He investigated the relation between the characters of a finite group and characters of its subgroups, which we call today *induced representations* and also, he proved what we now term the *Frobenius Reciprocity Theorem.* His work in this period drove him to the study of characters of symmetric and alternating groups, and application of the theory to the structure of the co-called *Frobenius group*, in 1900 and 1901. Forbenius's inventions mentioned in this paragraph can be found in [12, Chapter II]. According to [12], that Forbenius's original works were brought together in *Gesammelie Abhandlungen* [24].

To sum up the section, Frobenius went from considering Dedekind's question to generalizing the arithmetic idea of a group character. In 1896, he defined the characters of general finite groups, and stated and proved a number of fundamental theorems. He then applied his new theory to solve Dedekind's problem of factoring the determinant of a general group into irreducible factors and published the work in three papers. The analysis of Dedekind's group determinant problem drove Frobenius to introduce formally, in 1897, the modern definition of a matrix representation of a group G. This was followed by the modern definition of the character of a group representation and then by subsequent developments and applications.

The solving of the question of factoring the group determinant of a finite group is considered to be one of many applications of representation theory to the study of groups. In the language of group determinants, most of the basic theorems of the character theory of groups and representations are attributed to the correspondence between Dedekind and Frobenius [12, 32]. According to Hawkins in his book [32], of all the mathematicians who discovered some aspect of the theory of group characters and representations, it was by far Frobenius who developed the theory and its applications most extensively and rigorously. Interestingly, after the invention of representations, the group determinant was abandoned for nearly a century.

1.3 The Timeline of Developments

The foundation of group representation theory had been established by Frobenius in the period (1896-1898) and then by other figures, such as William Burnside

(1852-1927), Heinrich Maschke (1853-1908), Issai Schur (1875-1941), Emmy Noether (1882-1935) and Richard Brauer (1901-1977). Number theory was their inspiration through the course of their research work. Frobenius's discovery of the theory of representations of finite groups had led to a flurry of research in the area and this research enthusiasm does not seem to have dampened. In this section, we discuss briefly the timeline of the development of the theory of representations of finite group involving contributions made by the pioneers mentioned above [11, 12, 32, 48, 49].

Burnside had almost the same period of activity as Frobenius, which was between 1897 and the start of the First World War (1914-1918). However, each of these mathematicians independently put his own stamp on the theory and provided a series of papers and books. It was notable that Burnside and others simplified the proofs of many of Frobenius's results and extended the theory in completely new and different directions.

In 1897, Burnside published the first edition of his book entitled *Theory of groups of finite order*. The second edition of the book appeared in 1911 and it has been described by Curtis [12] as the first book which provides a systematic account of representation theory, and includes many results on abstract groups which were proved using group characters. Additionally, Burnside published around twelve papers on the topic. Here, we mention one of his signature results in group theory:

Burnside's p^aq^b Theorem. [11, 12, 49] *If p and q are prime numbers and a and b are positive integers, then no group of order p^aq^b is simple.*

Burnside proved this theorem for many special choices of the integers a and b, but he only succeeded in proving the theorem in general after he studied Frobenius's new theory of group representations. Burnside's proof of this theorem is considered as the first and outstanding group-theoretic application of representation theory. Furthermore, Burnside's p^aq^b Theorem leads to an informative result about groups of order p^aq^b:

Theorem 1.3.1. [11, 12, 49] *If p and q are prime numbers and a and b are positive integers, then every group of order p^aq^b is solvable.*

By solvable we mean a group G which has subgroups $G_0, G_1, \ldots, G_r$ with

$$1 = G_0 < G_1 < \ldots < G_r = G$$

such that for $1 \leq i \leq r$, $G_{i-1} \lhd G_i$ and the factor group G_i/G_{i-1} is cyclic of prime order.

From the very beginning, both Frobenius and Burnside recognized the importance of the theory of representations of finite group to abstract finite group theory, and they were confident that it would play a vital role. Since that time until today, research activities have confirmed their intuition about the potential of the theory for vast application. It is noteworthy that the range of applications was not confined or limited to just pure mathematics, but extended far beyond the pure boundaries to include physics and chemistry. We will take up this point later in Section 1.4.

In 1899, Maschke discovered his famous theorem about the decomposition of representations of a finite group into irreducible sub-representations. This significant result was the basis of further research by pioneers. It implied the concept of semisimplicity (or complete reducibility) which gave a clear classification of finite group representations. Following this, Frobenius's talented student Schur contributed to the theory and its related subjects between 1904 and 1933. His accomplishments are presented in fourteen papers, plus two joint papers with Frobenius published in 1906.

We summarize Schur's achievement in two points: the first is the introduction of a so-called *projective representation* of a finite group (in 1904 and 1907) – that is, a homomorphism from a finite group into the projective general linear group $PGL_n(\mathbb{C})$. The second is the investigation of the arithmetical properties of representations, linking with algebraic number theory; the main concept here is a splitting field of a finite group. With the mention of Schur's name, we also mention his well-known lemma, which states:

Schur's Lemma. [42] *Let V and W be irreducible $\mathbb{C}[G]$-modules.*

1. *If $\varphi : V \to W$ is a $\mathbb{C}[G]$-homomorphism, then either φ is a $\mathbb{C}[G]$-isomorphism, or $v\varphi = 0$ for all v in V.*
2. *If $\varphi : V \to V$ is a $\mathbb{C}[G]$-isomorphism, then φ is a scalar multiple of the identity endomorphism 1_V.*

The year 1929 marks a real turning point and breakthrough in representation theory, attributed to Noether in her paper, *Hyperkomplexe Grössen and Darstellungstheorie* [72]. The results utilized Wedderburn's Theorems [41] discovered in 1908. It was the start of the study of modules over rings and algebras, which in turn led to new insights into the structure of semisimple rings.

The last pioneer of the foundation of representation theory was Schur's student Brauer. In the period from 1926 to 1933, Brauer studied the transition from representation theory of matrix groups to the theory of simple algebras, and introduced what is today called the *Brauer Group*: an abelian group of equivalence classes of central simple algebras over a field k, denoted $B(k)$.

Brauer and Noether were considered as the leaders in introducing representations of rings and algebras to the area. They collaborated with each other from 1926 to 1933, and have a joint paper with Schur on indecomposable representations, published in 1930. Up to this point the underlying field had characteristic zero or a prime not dividing the order of the group. One of Brauer's initial results over a field of prime characteristic is:

Theorem 1.3.2. [11,12] *The number of equivalence classes of irreducible representations of a finite group G, in a splitting field of characteristic $p > 0$, is equal to the number of conjugacy classes in G containing elements of order prime to p.*

In 1935, Brauer noted that in the case when the characteristic p of a field k divides the order of a group G, then there exist representations of G which are not completely reducible. This means that the group algebra is no longer semisimple (an algebra is semisimple if and only if it is a direct sum of simple algebras). Brauer dealt with this case and started the systematic study of modular representations. During the period 1935-1960, modular representation theory was highly developed by Brauer. He produced many results on the subject and its applications to the theory of finite groups, among them a number of joint papers.

Using MathSciNet, we collected some statistics on the numbers of papers in various periods. Because Mathematical Reviews (MR) covers data from 1940 to the present, our statistics start from 1940. There are approximately 400 papers in the 1940's that mention the phrase "group representation", and the figure for the 1950's is approximately 1100. Figure 1.1 displays the steadily increasing amount of papers from 1970 to 2000. These papers are written in different languages; the main ones are English, German, Russian, and French. In Chapter 6, we will study the situation with semigroups and compare the two cases.

1.4 Applications

The immediate use of the theory to determine information about the structure of a finite group made it distinct, and therein lies its appeal and glamour. There are many applications of group characters and representations within the different branches of mathematics, such as group theory and number theory and also outside the field, such as in physics and in chemistry. As mentioned previously, the primary applications of the theory were purely in the study of the structure of finite groups. Burnside was the first who applied group character theory to pure group problems. However, it is evident that Burnside and Frobenius showed the immediate utility

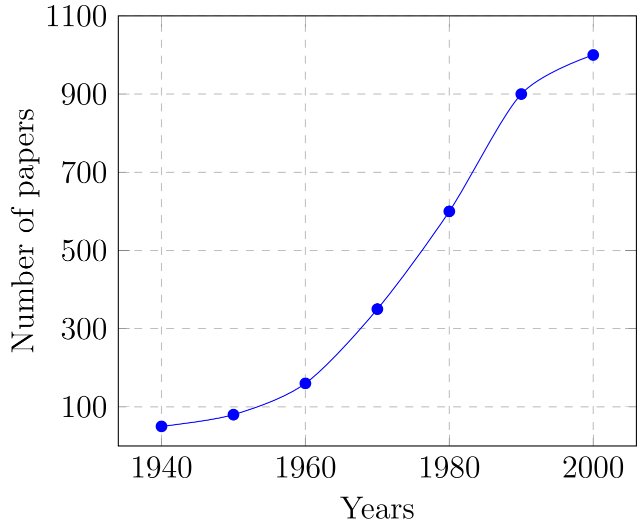

Figure 1.1: The progression of group representation theory

of group representations and characters by establishing some properties of finite groups, simultaneously. For instance, the structure theorem of Frobenius groups (published 1901) and Burnside's $p^a q^b$ Theorem (1.3) (1904) were proved using representation theory.

The original proof of Frobenius's Theorem (found in [12]) using character theory was done by Frobenius himself. Moreover, there is no known proof of Frobenius's Theorem in which character theory is not used. On the contrary, Burnside's $p^a q^b$ Theorem has a proof without the use of representations, due to Helmut Bender (1942-present) in 1972 [12]. But even earlier in 1904, Burnside proved his theorem with the use of the theory. The fact is that Burnside's Theorem shows the importance of representation theory in the classification of finite simple groups. After 1899, Frobenius focused on applying his theory of group characters to the nontrivial problem of computing character tables for particular groups – for example the symmetric and alternating groups.

Burnside used group character results to study groups of odd order. In 1900, he proved that if a group G has odd order, then no irreducible character other than the trivial can be real-valued. Based on this result and some of its consequences, he showed that among subgroups of the symmetric group S_n for $n \leq 100$ there are no simple groups of odd order, and every irreducible group of linear transformations in three, five, or seven variables must be solvable.

According to Curtis [12], Eugene Wigner (1902-1995) was the first who applied representation theory of finite group in physics in two papers on the quantum mechanics of atomic spectra published in 1926 and 1927; it was also used by Hermann

Weyl (1885-1955), in lectures given on related subjects during the 1927-1928 period. Rigorously, Wigner and Weyl formulated the close connection between the theory of group representations and physics. Indeed, their work exposed the importance of group representation theory in physics. Later on, both Weyl and Wigner expanded these works into books in 1928 [108] and in 1931 [109], respectively.

Frobenius and his Ph.D. student Schur had two joint papers on applications of character theory published in 1906 [12, 32]. One year later, Frobenius also wrote a paper on the theme which contains a new link between character theory and the number of solutions of certain equations in a finite group. The last work of Frobenius on applications appeared in 1911 and concerns the classification of crystal classes. It turns out that, since that time, group characters and representations have become increasingly important and have been used extensively in many applied fields, such as spectroscopy, crystallography, molecular orbital theory, ligand field theory, quantum mechanics, and the list continues [12, 32, 103].

The connection with physics was such a large and important area of 20$^{\text{th}}$ century science, it added glamour to the subject and stimulated development. The reason is that representation theory of groups describes the symmetries of the physical field in a natural mathematical language. This leads us to end this section with a brief description of the early application of group representation theory in physics, precisely in quantum theory. Research has shown that representation theory has played a vital role in the development of quantum theory. In fact, it provides an effective structure in which to exploit symmetry –which simplifies problems– in quantum theory and derives implications for the behavior of a quantum mechanical system. To explain this, we require some concepts [63, 103], as given below.

For the rest of this section, we allow our vector spaces V to be infinite dimensional.

Definition 1.4.1. A *Hilbert space* $\mathcal{H}$ is a real or complex vector space with inner product $\langle\cdot,\cdot\rangle$, which is complete as a metric space with respect to the norm $\|\psi\|=\sqrt{\langle\psi|\psi\rangle}$. The space $\mathcal{H}$ is called *separable* if it has a countable orthonormal basis.

The mathematical framework of quantum mechanics is closely related to *unitary group representations*.

Definition 1.4.2. Let G be a group. A representation $\varphi : G \to GL(V)$ on a complex vector space V is called *unitary* if the action of G preserves the inner product on V (viewing V as a Hilbert space). That is

$$\langle vg|wg\rangle = \langle v|w\rangle, \quad \text{for all } \; g \in G \text{ and } v, w \in V,$$

where by vg we mean $v(g\varphi)$.

In a simple way, let us introduce the structure of quantum mechanics and its related terminologies. A quantum mechanical system consists of a separable complex Hilbert space. The *state* of a quantum mechanical system is a nonzero vector in a complex vector space with a Hermitian inner product, the so-called state space. Such a state space will sometimes be a space of functions known as *wave functions.* A *pure state* of the system consists of a one-dimensional subspace.

Let $\mathcal{H}$ be the Hilbert space of the system. We say that G acts as a group of automorphisms if for any two unit vectors ϕ and ψ and $g \in G$, we have

$$|\langle \phi g, \psi g\rangle|^2 = |\langle \phi, \psi\rangle|^2.$$

This is called a *symmetry transformation* and it preserves transition probabilities between the states. The theorem given by Wigner which is considered as a cornerstone of the mathematical formulation of quantum mechanics, states that:

Wigner's theorem. [103] *Any symmetry transformation can be represented on the Hilbert space of physical states by an operator that is either linear and unitary or anti-linear and anti-unitary.*

The theorem was stated and first proved by Wigner himself in his 1931 book [110]. Sternberg in his book [103, Section 3.9, page 149] summarizes the main points of Wigner's work on the subject in what follows: "the logic of physics is quantum mechanics and hence, a symmetry group of the system manifests itself as a unitary representation of the Hilbert space".

Wigner's theorem on quantum mechanical symmetries describes the fundamental relation between quantum mechanics and representation theory, that is, if we have a quantum system which has a Lie group of symmetries, then the state space naturally carries a unitary representation of that symmetry group. Thus, whenever we have a physical quantum system with a group G acting on it, the space of states $\mathcal{H}$ will carry a unitary representation of G. From a physical perspective, this implies that representation theory provides information about quantum mechanical state spaces when G acts on the system. On the other hand, mathematically, this means that physics is considered as a productive source of unitary representations to study since any physical system with a group G acting on it will provide one.

To sum up, the theory of representations of groups has played critical role in different fields for over a century. In this chapter, we discussed the reasons for the continuous development of group representation theory. Later in Chapter 6, we will address the development of semigroup representation theory and then we will compare the two cases.

Chapter 2

Semigroup Theory

2.1 Introduction

Semigroup theory is one of the relatively young areas of study in algebra [35]. Although the initial use of the term *semigroup* can be attributed to de Séguier in 1904, the theory of semigroups really began in 1928 with the work of Suschkewitsch, as explained by Clifford and Preston in the preface of their book [8]. "Suschkewitsch was doing (algebraic) semigroup theory before the rest of the world even knew there was such a thing", says Hollings [35, Section 2].

The attempts to define the term *semigroup* started from 1904 until 1940. From 1940 onwards, a generally accepted definition of the term semigroup emerged, presumably due to the influential works of Rees (1940) [91] and Clifford (1941) [5] (Hollings [36, Chapter 1]). In 1940, the first important structure theorem of semigroup theory, now known as the *Rees Theorem*, was introduced by Rees analogously to the Artin-Wedderburn Theorem for rings. In fact, the Rees Theorem was preceded by the 1928 result of Suschkewitsch in the finite case. This theorem had a significant bearing on the early development of semigroup theory. Nevertheless, the structure theorem provided by Clifford in 1941 is considered as the beginning of the truly independent theory of semigroups, since the prior results and analyses were heavily influenced by both group and ring theories [36, Section 1.3].

The 1950's witnessed the discovery of three highly important concepts in the theory: *Green's relations* and *regular semigroups* by Green in 1951, and *inverse semigroups*, which were introduced independently in 1952 by Wagner – who termed them *generalised groups* – and by Preston in 1954 who named them *inverse semigroups*. By the end of the 1950's, the theory of semigroups had become a self-contained branch of modern algebra.

At the start of the 1960's, a solid foundation for the algebraic theory of semigroup, with unified notations and terminologies, was provided by three highly significant semigroup textbooks: *Semigroups* by Lyapin in 1960 (its English translation appeared in 1963); the first volume of Clifford and Preston's *The algebraic theory of semigroups* [8], in 1961, and the second volume in 1967. Additionally, the establishment of the journal *Semigroup Forum* in 1970 provided the means for a significant body of work to grow on the theory. Since then, a wealth of papers and textbooks have emerged in the literature, such as the texts [37] by Howie and [27] by Grillet.

The purpose of this chapter is to assemble some of the basic ideas from semigroup theory that are particularly relevant to semigroup representation theory. We illustrate some of the concepts mentioned with examples. We start by defining the notion of a semigroup and introducing the standard types of semigroups: groups, and inverse and regular semigroups. The second section is devoted to Green's relations. We present these relations and visualize the intuitive notion of the $\mathcal{D}$-relation in a picture (diagram) called an "Egg-Box". We then state the useful properties and particular facts related to Green's relations that we require before Chapter 3, which outlines the representation theory of semigroups. Also, we focus on idempotents, which play a vital role in semigroup theory, and assert the link between the existence of idempotents and maximal subgroups. Definitions and results are mostly taken from [37]. Throughout the chapter, maps are written on the right and composition is left to right, unless otherwise specified.

2.2 Basic Definitions

Definition 2.2.1. [25] A semigroup is a pair $(S, *)$ where S is a non-empty set and $*$ is an associative binary operation on S (i.e. $*$ is a function $S \times S \to S$ with $(a, b) \mapsto a * b$ and for all a, b, c we have $a * (b * c) = (a * b) * c$).

We write S instead of $(S, *)$. We also write $a * b$ or omit the binary operation and write ab. Any group is a semigroup; the converse is not true. A simple example is the set of natural numbers $\mathbb{N}$ with addition which is a semigroup but not a group.

An element e of a semigroup S is called a *left identity* if $es = s$ for every $s \in S$, and a *right identity* if $se = s$ for every $s \in S$. Moreover, if S has a left identity e and a right identity f, then $e = f$ and e is the unique two-sided identity for S. We normally denote the identity element by 1 when it exists.

Definition 2.2.2. [37] A semigroup S is called a *monoid* if it has an identity $1 \in S$, where $1s = s = s1$ for all $s \in S$.

Observe that an identity element is unique whenever it exists. A monoid G is a group if, in addition, every element $a \in G$ has a unique inverse $a^{-1} \in G$ such that $aa^{-1} = 1 = a^{-1}a$. In any monoid the *units* are the elements that have unique inverses in the sense of group theory, and these form a group called the *group of units*. Note that, groups are monoids and monoids are semigroups. Thus we have:

$$\text{groups} \subset \text{monoids} \subset \text{semigroups}.$$

Definition 2.2.3. [37] For a semigroup S, we define a monoid S^1 by adjoining an identity to S, if S does not have one:

$$S^1 = \begin{cases} S & \text{if } S \text{ is a monoid,} \\ S \cup \{1\} & \text{if } S \text{ is not a monoid.} \end{cases}$$

The multiplication in the semigroup $S \cup \{1\}$ is defined by:

$$a \cdot b = \begin{cases} ab & \text{if } a, b \in S, \\ a & \text{if } b = 1, \\ b & \text{if } a = 1 \end{cases}$$

for all a and b in $S \cup \{1\}$.

A *zero element* of a semigroup S is an element 0 of S such that

$$S \neq \{0\} \text{ and } 0a = a0 = 0 \text{ for all } a \in S.$$

The product of subsets A and B of a semigroup S is defined by:

$$AB = \{ab \in S \mid a \in A, b \in B\}.$$

We write aB for $\{a\}B = \{ab \mid b \in B\}$. As stated in [37, Section 1.1], a non-empty subset T of a semigroup S is called *subsemigroup* if it is closed with respect to multiplication:

$$a, b \in T \Rightarrow ab \in T \ (i.e. : T^2 \subseteq T).$$

Associativity holds throughout T, hence T itself is a semigroup. A subsemigroup of S which is a group with respect to the multiplication of S is called a *subgroup* of S. Alternatively, a non-empty subset T of S is a subgroup of S if and only if

$$\forall\ a \in T :\ aT = T \text{ and } Ta = T. \tag{2.1}$$

To prove this statement, suppose that $T \subseteq S$ is a subgroup of S, then for any $a, b \in T$,

$$b = a(a^{-1}b) \in aT,$$

so that $T \subseteq aT \subseteq T$, hence $T = aT$. Dually, $T = Ta$. Conversely, suppose that for all $a \in T$, $aT = T$ and $Ta = T$. For any $a, b \in T$, the product ab lies in $aT = T$, hence T is a subsemigroup. Let $a \in T$, since $T = aT$ we have $a = ac$ for some $c \in T$. Let $b \in T$, then since $T = Ta$ we have $b = da$ for some $d \in T$. Now,

$$bc = (da)c = d(ac) = da = b,$$

Thus, c is a right identity for T. Dually, we can find a left identity c', such that

$$c = c'c = c'.$$

Thus c is a (two-sided) identity for T. Now let $t \in T$. Since $Tt = T = tT$, we have $c = st = ts'$ for some $s, s' \in T$. Then

$$s = sc = s(ts') = (st)s' = cs' = s',$$

so s is the inverse of t. Therefore T is a group, hence a subgroup of S.

Definition 2.2.4. [37] Let S be a monoid with identity 1. A non-empty subset T of S is said to be a *submonoid* of S if it is a subsemigroup with identity 1.

Two semigroups can be related by a map from one to the other:

Definition 2.2.5. [37] Let S and T be semigroups. A map $\psi : S \to T$ is said to be a *semigroup homomorphism* if, for all $a, b \in S$:

$$(ab)\psi = (a\psi)(b\psi).$$

If, in addition, a semigroup homomorphism ψ is a bijection, we call it an *isomorphism*; the semigroups S and T are then said to be *isomorphic*, denoted $S \cong T$. Moreover, if S, T are monoids, with identity elements 1_S and 1_T, respectively, then ψ is called a *monoid homomorphism* if it is a semigroup homomorphism and $1_S\psi = 1_T$ (note that this doesn't happen automatically, as in the group case). A homomorphism ψ from S into itself is called an *endomorphism*.

There are special elements in semigroup theory.

Definition 2.2.6. [37] An element $e \in S$ is called *idempotent* if $e^2 = e$.

The set of idempotents in S is denoted by $E(S) = \{e \in S \mid e^2 = e\}$. There is a partial order $\leq$ among the elements of $E(S)$ defined by

$$e \leq f \text{ if and only if } ef = fe = e.$$

Definition 2.2.7. [37] An element a of a semigroup S is called *regular* if there exists x in S such that $axa = a$.

Remark 2.2.8.

1. From the previous definition, the elements ax and xa are idempotents. Any idempotent is regular and any unit is regular.

2. Any finite semigroup contains an idempotent [37].

Definition 2.2.9. [37] An *inverse* of an element a in a semigroup S is an element a' of S such that

$$a = aa'a \text{ and } a' = a'aa'.$$

The elements a and a' are called *mutually inverse*. From the definition, it is obvious that an element with an inverse is regular. The opposite is also true. Suppose that a is regular. Then there exists $b \in S$ such that $aba = a$. Let $a' = bab$. Then

$$aa'a = a(bab)a = (aba)ba = aba = a$$

and

$$a'aa' = (bab)a(bab) = b(aba)bab = babab = b(aba)b = bab = a'.$$

Thus a' is an inverse of a.

Remark 2.2.10. To distinguish between inverses in a monoid and inverses in a group, we use a' (or a^*) as an inverse of a in the sense of a monoid, and a^{-1} as an inverse of a in the sense of a group.

Note that inverses need not be unique. Hence, we denote by $V(a)$ the set of all inverses of a. We will illustrate this point below by an example. Regular and inverse elements lead to the particular types of semigroups that are defined in the following paragraph.

A semigroup S is called *regular* if every $a \in S$ is regular. In semigroup theory, regular semigroups are essential because groups are regular semigroups with a unique idempotent. A semigroup S is an *inverse semigroup* if $|V(a)|= 1$ for all $a \in S$, that is, every element has a unique inverse. A fundamental, but not obvious, property of inverse monoids is that their idempotents commute. The following alternative characterization for inverse semigroups provides us with another useful definition of an inverse semigroup.

Theorem 2.2.11. [37, Theorem 5.1.1] *A semigroup S is inverse if and only if S is regular and the idempotents of S commute.*

As mentioned in [37, Section 5.2], if S is an inverse semigroup, then a partial order relation $\leq$ is defined on S by

$$a \leq b \Leftrightarrow a = eb \text{ for some } e \in E(S).$$

Let S be an inverse semigroup S, $a \in S$ and $e \in E(S)$. Let $a \leq e$, we want to show that a is an idempotent.

$$\begin{aligned} a \leq f \Rightarrow a &= ef \qquad \text{[by definition, for some } e \in E(S)] \\ \Rightarrow a^2 &= efef \\ \Rightarrow a^2 &= e^2 f^2 \quad \text{[since idempotents commute in } S] \\ \Rightarrow a^2 &= ef \\ \Rightarrow a^2 &= a. \end{aligned}$$

In fact, inverse semigroups have interesting structural properties first studied by Vagner in 1952 and Preston in 1954.

We have some well-known semigroups on the set $[n] = \{1, ..., n\}$. These are:

1. The *symmetric group* S_n of all permutations of the set $[n]$:

$$S_n = \{\alpha \mid \alpha \text{ is a bijection } [n] \to [n]\}.$$

2. The *full transformation monoid* T_n consisting of all maps from $[n]$ into itself:

$$T_n = \{\alpha \mid \alpha \text{ is a function } [n] \to [n]\}.$$

3. The *symmetric inverse monoid* I_n consisting of all partial bijections:

$$I_n = \{\alpha \mid \alpha \text{ is a partial bijection } X \to Y;\ X, Y \subseteq [n]\}.$$

In all cases, the operation is composition of (partial) maps. According to [37, Section 5.1], we compose elements of I_n by means of the following rule:

1. A partial permutation is defined to be a bijection $\alpha : \operatorname{dom}\alpha \to \operatorname{im}\alpha$, where $\operatorname{dom}\alpha$ and $\operatorname{im}\alpha$ are subsets of $[n]$.

2. Since α is a bijection, it has an inverse $\alpha^{-1} : \operatorname{im}\alpha \to \operatorname{dom}\alpha$ such that the composition $\alpha\alpha^{-1}$ is the identity mapping on $\operatorname{dom}\alpha$ and $\alpha^{-1}\alpha$ is the identity mapping on $\operatorname{im}\alpha$.

3. If $\alpha, \beta \in I_n$, then

$$\operatorname{dom} \alpha\beta = (\operatorname{im} \alpha \cap \operatorname{dom} \beta)\alpha^{-1} \text{ and } \operatorname{im} \alpha\beta = (\operatorname{im} \alpha \cap \operatorname{dom} \beta)\beta.$$

4. The map $\alpha\beta$ is map composition $\operatorname{dom} \alpha\beta \mapsto \operatorname{im} \alpha\beta$.

5. If $\operatorname{im} \alpha \cap \operatorname{dom} \beta = \varnothing$, then $\alpha\beta$ is the *empty transformation*.

6. Idempotents in I_n are the identity functions on $[n]$.

Example 2.2.12. Consider the full transformation monoid T_4 and choose the elements a, b, and c as:

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 3 & 3 \end{pmatrix}, \ b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 4 \end{pmatrix} \text{ and } c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 2 \end{pmatrix}.$$

Then, a simple calculation shows that b and c are both inverses of a.

Cayley's Theorem asserts that a group of order n is isomorphic to some subgroup of S_n. This theorem has an important place in the history of group theory. The analogue of Cayley's theorem in inverse semigroup theory is the Wagner-Preston representation theorem:

Theorem 2.2.13. [37, Theorem 5.1.7] *Let S be an inverse semigroup. Then there exists a symmetric inverse semigroup I_n and a one-to-one homomorphism $\varphi : S \to I_n$.*

2.3 Green's Relations

In this section, we introduce a fundamental tool created by Green [26] for the study of the structure of a semigroup: *Green's relations*. These relations describe how elements of a semigroup interact. Before presenting these relations, we first need the following definitions and facts.

Definition 2.3.1. [37, Section 1.4] Let ψ and φ be binary relations on a set X. Then the composition of relations $\psi \circ \varphi$ is defined by

$$\psi \circ \varphi = \{(a, c) \in X \times X : \exists\, b \in X \text{ such that } a \,\psi\, b \text{ and } b \,\varphi\, c\}.$$

In a special case, if ψ and φ are partial functions, the composition in the above definition is just the ordinary composition of those partial functions. Now let $\psi \vee \varphi$ be the join of equivalence relations ψ and φ:

Proposition 2.3.2. [37, Section 1.5] *Let ψ and φ be equivalences on a set S and $a, c \in S$. Then $(a, c) \in \psi \vee \varphi$ if and only if, for some $n \in \mathbb{N}$, there exist elements $b_1, b_2, \ldots, b_{2n-1} \in S$ such that*

$$a \, \psi \, b_1, \;\; b_1 \, \varphi \, b_2, \;\; b_2 \, \psi \, b_3, \ldots, b_{2n-1} \, \varphi \, c.$$

Moreover:

Corollary 2.3.3. [37, Corollary 1.5.12] *Let ψ and φ be equivalences on a set S such that $\psi \circ \varphi = \varphi \circ \psi$. Then, $\psi \vee \varphi = \psi \circ \varphi$.*

As stated in [37, Section 1.1], a subset $I \subseteq S$ is a *left ideal* if $SI \subseteq I$, a *right ideal* if $IS \subseteq I$ and an *ideal* (or a *two-sided ideal*) if it is both a left and a right ideal. The subset aS^1 of a semigroup S is

$$aS^1 := \{a\}S^1 = \{as : s \in S^1\} = aS \cup \{a\}$$

and is called the *principal right ideal* generated by the element $a \in S$. Dually,

$$S^1a = Sa \cup \{a\}$$

is the *principal left ideal* generated by a. Similarly,

$$S^1aS^1 = SaS \cup Sa \cup aS \cup \{a\}$$

is the *principal two-sided ideal* generated by a. We deal with the monoid S^1 to guarantee that the ideal aS^1 contains its generator a.

Green's relations on a semigroup S are five equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ characterizing the elements of a semigroup in terms of the principal ideals they generate.

Definition 2.3.4. [37, Section 2.1] Let a, b be elements of a semigroup S. Then the relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{H}$, $\mathcal{D}$ and $\mathcal{J}$ are defined as follows.

1. $a \, \mathcal{L} \, b$ if and only if there exists $x, y \in S^1$ such that $xa = b$ and $yb = a$. Equivalently, $S^1a = S^1b$.

2. $a \, \mathcal{R} \, b$ if and only if there exists $u, v \in S^1$ such that $au = b$ and $bv = a$. Equivalently, $aS^1 = bS^1$.

3. $a \, \mathcal{J} \, b$ if and only if there exists $x, y, u, v \in S^1$ such that $xay = b$ and $ubv = a$. Equivalently, $S^1aS^1 = S^1bS^1$.

4. $\mathcal{H} = \mathcal{L} \cap \mathcal{R} := \{(a, b) \in S \times S : \; (a, b) \in \mathcal{L} \text{ and } (a, b) \in \mathcal{R}\}$.

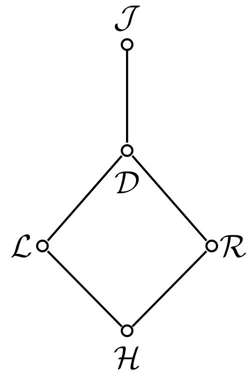

Figure 2.1: Hasse diagram of Green's relations on a semigroup

5. $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$, the join of the $\mathcal{R}$ and $\mathcal{L}$ relations.

Proposition 2.3.5. [37, Proposition 2.1.3] *The equivalence relations $\mathcal{L}$ and $\mathcal{R}$ commute: i.e. for all $a,b \in S$,*

$$a\,\mathcal{L}\,c\,\mathcal{R}\,b \text{ for some } c \in S \Longleftrightarrow a\,\mathcal{R}\,d\,\mathcal{L}\,b \text{ for some } d \in S.$$

By a consideration of the results 2.3.3 and 2.3.5, Green's relation $\mathcal{D}$ can also be described as

$$\mathcal{L} \vee \mathcal{R} = \mathcal{L} \circ \mathcal{R} = \mathcal{R} \circ \mathcal{L}.$$

This allows us to have an alternative definition of the relation $\mathcal{D}$ as follows:

$$\begin{aligned} a\,\mathcal{D}\,b &\text{ if and only if there exists an element } c \in S \text{ such that } a\,\mathcal{R}\,c\,\mathcal{L}\,b \\ &\text{ if and only if there exists an element } d \in S \text{ such that } a\,\mathcal{L}\,d\,\mathcal{R}\,b. \end{aligned}$$

By Definition 2.3.4, it is obvious that $\mathcal{L} \subseteq \mathcal{J}$ and $\mathcal{R} \subseteq \mathcal{J}$. Furthermore, $\mathcal{D} \subseteq \mathcal{J}$ as $\mathcal{D} = \mathcal{L} \vee \mathcal{R}$, and $\mathcal{H} \subseteq \mathcal{R}$ and $\mathcal{H} \subseteq \mathcal{L}$.

Proposition 2.3.6. [37, Proposition 2.1.4] *If S is a finite semigroup, then $\mathcal{D} = \mathcal{J}$.*

We denote by L_a, R_a, H_a, D_a, J_a the $\mathcal{L}$-class, $\mathcal{R}$-class, $\mathcal{H}$-class, $\mathcal{D}$-class and $\mathcal{J}$-class containing the element $a \in S$, respectively. In addition, we denote the principal (two-sided) ideal S^1aS^1 generated by the element a as $J(a)$.

The kernel of $\alpha : X \to Y$; denoted by kerα, is an equivalence relation on X defined by the rule

$$x \ \mathrm{ker}\alpha \ y \Leftrightarrow x\alpha = y\alpha.$$

The *fibers* of a function are the equivalence classes made by the kernel. For example, the fiber of the element y in the set Y under a map $\alpha : X \to Y$ is the inverse image

$y\alpha^{-1} = \{x \in X \mid x\alpha = y\}$. Observe that the intersection of any fiber of a function with its image should be in one point.

According to [14], Green's relations in S_n, I_n and T_n are given by:

1. $$a \,\mathcal{L}\, b \text{ if and only if } \operatorname{im}(a) = \operatorname{im}(b); \tag{2.2}$$

2. $$a \,\mathcal{R}\, b \text{ if and only if } \begin{cases} \text{fibers}(a) = \text{fibers}(b) \text{ when } a, b \in T_n; \text{or} \\ \text{dom}(a) = \text{dom}(b) \text{ when } a, b \in I_n; \end{cases} \tag{2.3}$$

   note that since S_n is a group, $a \,\mathcal{R}\, b \;\forall\; a, b \in S_n$,

3. $a \,\mathcal{J}\, b$ if and only if $|\operatorname{im}(a)| = |\operatorname{im}(b)|$;

4. $a \,\mathcal{H}\, b$ if and only if both equations (2.2) and (2.3) hold.

Example 2.3.7. Consider the full transformation monoid T_3.

1. Choose $e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix} \in E(T_3)$.

2. Since $\operatorname{im}(e) = \{1, 3\}$, the fibers$(e)$ are $\{1, 2\}$ and $\{3\}$.

3. Hence, $a \,\mathcal{H}\, e$ if and only if $\operatorname{im}(a) = \operatorname{im}(e)$ and fibers(a) = fibers(e). Equivalently, $\operatorname{im}(a) = \{1, 3\}$ and fibers(a) are $\{1, 2\}$ and $\{3\}$.

4. Thus the choices for a in step 3 are
$$a = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 1 \end{pmatrix} \text{ or } a = e = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 3 \end{pmatrix}.$$

Observe that the idempotents of T_n are the elements a which are the identity when restricted to $\operatorname{im}(a)$, since $e^2 = e$ if and only if $xe = (xe)e$, for $x \in \operatorname{im}(e)$.

There is a natural partial order $\leq$ on the set $\mathcal{J}$-classes of a finite semigroup S defined by:

$$J_a \leq J_b \text{ if and only if } S^1 a S^1 \subseteq S^1 b S^1, \text{ where } a, b \in S.$$

A $\mathcal{J}$-class is called *minimal* (respectively, *maximal*) if it is a minimal (respectively, maximal) element in this partial order. Observe that every finite semigroup contains minimal and maximal $\mathcal{J}$-classes.

2.3.1 The Structure of $\mathcal{D}$-classes

With the consideration of the definition of $\mathcal{D}$,

$$a\,\mathcal{D}\,b \text{ if and only if there exists an element } c \in S \text{ such that } a\,\mathcal{R}\,c\,\mathcal{L}\,b.$$

Thus, the element $c \in R_a \cap L_b$, which means $R_a \cap L_b \neq \varnothing$, also we have $L_a \cap R_b \neq \varnothing$, since $\mathcal{D}$ is symmetric. Therefore, every $\mathcal{L}$-class and every $\mathcal{R}$-class belonging to the same $\mathcal{D}$-class have nonempty intersection. For $a, b \in S$ we have:

$$a\,\mathcal{D}\,b \text{ if and only if } L_a \cap R_b \neq \varnothing \text{ if and only if } L_b \cap R_a \neq \varnothing.$$

Hence a $\mathcal{D}$-class can be visualized by a rectangular table termed an 'egg-box' by Clifford and Preston (1961) [8]. In the egg-box, the rows correspond to $\mathcal{R}$-classes, the columns correspond to $\mathcal{L}$-classes, and each cell gives the $\mathcal{H}$-class that is the intersection of the $\mathcal{L}$-class and $\mathcal{R}$-class forming the column and row that contain that cell. Since $\mathcal{L} \subseteq \mathcal{D}$ and $\mathcal{R} \subseteq \mathcal{D}$, every $\mathcal{D}$-class must be both a union of $\mathcal{L}$-classes and a union of $\mathcal{R}$-classes. The order of these rows and columns may be placed arbitrarily.

Proposition 2.3.8. [37, Proposition 2.3.1] *Let a be a regular element in a semigroup S. Then every element belonging to the $\mathcal{D}$-class D_a is regular.*

Proof. As a is a regular element, then there exists x in S such that $axa = a$. If $b \in D_a$, then there exists an element $c \in S^1$ such that $a\mathcal{R}c\mathcal{L}b$. Since $a\mathcal{R}c$, there exists $u, v \in S^1$ such that $au = c$ and $cv = a$. Notice that

$$c = au = (axa)u = (ax)au = axc = cvxc = ctc,$$

where $t = vx$. Thus, c is regular. Since $c\mathcal{L}b$, a dual argument shows that b is regular. Hence, the result holds. $\square$

This proposition shows that either all elements of a $\mathcal{D}$-class are regular or no element is regular. We say that a $\mathcal{D}$-class (or an $\mathcal{H}$-class or $\mathcal{R}$-class or $\mathcal{L}$-class) is *regular* if it contains only regular elements. Since idempotents are regular, any $\mathcal{D}$-class containing an idempotent is regular. On the other hand, we have

Proposition 2.3.9. [37, Proposition 2.3.2] *In a regular $\mathcal{D}$-class, each $\mathcal{R}$-class and $\mathcal{L}$-class contains an idempotent.*

Proof. Let $a \in S$ be such that D_a is regular. In particular, a is regular, so $axa = a$ for some $x \in S$. Now we have, $ax\,\mathcal{R}\,a$ and $(ax)^2 = axax = ax$. Hence, ax is an idempotent in R_a. similarly, xa is an idempotent in L_a, thus the result holds. $\square$

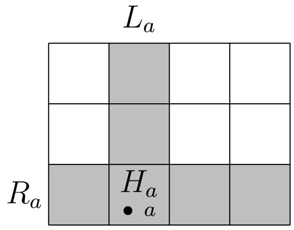

Figure 2.2: The egg-box diagram of a $\mathcal{D}$-class D_a

In particular, any regular $\mathcal{D}$-class contains an idempotent. The above description of $\mathcal{D}$-classes is taken from [37, Sections 2.3].

Figure 2.2 depicts the $\mathcal{R}$-class R_a and the $\mathcal{L}$-class L_a in the egg-box diagram by the row and column that intersect in the box representing the $\mathcal{H}$-class H_a containing the element a.

Example 2.3.10. Let $S = I_3$. Let $a, b, c \in I_3$ be such that $a \in D_3$, $b \in D_2$, and $c \in D_1$, where $D_r = \{\alpha \in I_3 : |\mathrm{im}(\alpha)| = r, 0 \leq r \leq 3\}$. Notice that $\mathcal{D} = \mathcal{J}$ so D_r is a $\mathcal{J}$-class. Then we have:

$$\begin{aligned} J(a) &= D_3 \cup D_2 \cup D_1 \cup D_0, \\ J(b) &= D_2 \cup D_1 \cup D_0, \\ J(c) &= D_1 \cup D_0. \end{aligned}$$

This implies

$$J(c) \leq J(b) \leq J(a),$$

where

$$\begin{aligned} J(a) = S^1 a S^1 &= \{xay \mid x, y \in I_3\} \\ &= D_3 \cup D_2 \cup D_1 \cup D_0. \end{aligned}$$

Similarly, we can get the other principal ideals $S^1 b S^1$ and $S^1 c S^1$. The diagram in Figure 2.3 illustrates the $\mathcal{D}$-classes ($= \mathcal{J}$-classes) of I_3.

We end this section by listing necessary facts associated with Green's relations which are required for Chapter 3. We start with a theorem that shows the importance of the existence of an idempotent in such a semigroup. The theorem below is a crucial result to the representation theory of semigroups which characterises the maximal subgroups within a semigroup S.

Theorem 2.3.11 (Maximal Subgroup Theorem). [14] *Let e be an idempotent in a semigroup S. Then the $\mathcal{H}$-class H_e is the* maximal subgroup *of S with identity*

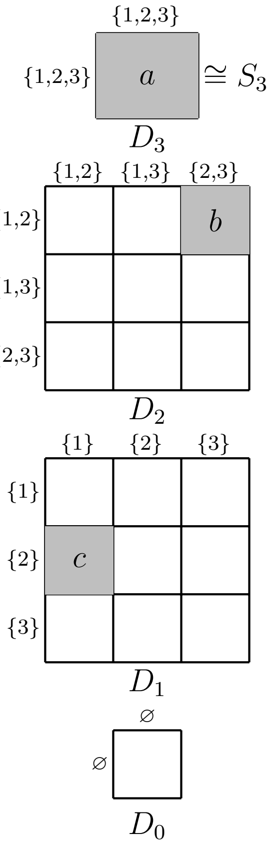

Figure 2.3: The total order in I_3

e. The maximal subgroups of S are precisely the $\mathcal{H}$-classes containing an idempotent e.

Remark 2.3.12. The $\mathcal{H}$-classes with an idempotent are isomorphic to S_r in T_n and I_n, where $1 \leq r \leq n$.

The regular semigroups can be characterized by the Green's relations as given in the following theorem:

Theorem 2.3.13. [37] *Let S be a semigroup. Then the following statements are equivalent:*

1. *S is a regular semigroup;*
2. *every $\mathcal{L}$-class contains at least one idempotent;*
3. *every $\mathcal{R}$-class contains at least one idempotent;*
4. *every $\mathcal{D}$-class contains at least one idempotent.*

Indeed, suppose that a semigroup S is regular, this implies that every $\mathcal{D}$-class is regular. Then by Proposition 2.3.9, every $\mathcal{D}$-class contains an idempotent, hence (1) $\Rightarrow$ (4). Now, suppose that every $\mathcal{D}$-class contains at least one idempotent, then by Proposition 2.3.8 every $\mathcal{D}$-class is regular as idempotents are regular, hence S is regular and this is (4) $\Rightarrow$ (1). Statements (2) and (3) are identical.

For an inverse semigroup we have:

Theorem 2.3.14. [37, Theorem 5.1.1] *Let S be a semigroup. Then the following statements are equivalent:*

1. *S is an inverse semigroup;*
2. *S is a regular semigroup, and its idempotents commute;*
3. *every $\mathcal{L}$-class and every $\mathcal{R}$-class contains exactly one idempotent;*
4. *every element of S has a unique inverse.*

2.3.2 Green's Lemmas and their Consequences

Green's Lemmas provide an explicit description of the structure of a $\mathcal{D}$-class in a semigroup and determine the relationships between Green's relations.

Definition 2.3.15. [37] Let S be a semigroup and $s, t \in S^1$. The map $\rho_s : S \to S$ defined by $a\rho_s = as$ for all $a \in S$ is called a *right translation.* Dually, the map $\lambda_t : S \to S$ defined by $a\lambda_t = ta$ for all $a \in S$ is called a *left translation.*

The following shows that multiplication by suitable elements induces bijections between certain $\mathcal{R}$-classes, $\mathcal{L}$-classes, and $\mathcal{H}$-classes.

Lemma 2.3.16 (Green's Lemma). [37, Lemma 2.2.1] *Let S be a semigroup and $a, b \in S$ be such that $a\mathcal{R}b$. Let $s, s' \in S^1$ where $as = b$ and $bs' = a$. Then:*

1. *the map $\rho_s : L_a \to L_b$, defined by $a\rho_s = as$, and the map $\rho_{s'} : L_b \to L_a$, defined by $b\rho_{s'} = bs'$, are mutually inverse, hence bijections;*
2. *both maps ρ_s and $\rho_{s'}$ preserve $\mathcal{R}$-classes; that is, for all $c \in L_a$ and $d \in L_b$, we have $c\,\mathcal{R}\,c\rho_s$ and $d\,\mathcal{R}\,d\rho_{s'}$.*

In other words,

$$\rho_s|_{L_a}\rho'_s|_{L_b} = \mathrm{id}_{L_a} \text{ and } \rho'_s|_{L_b}\rho_s|_{L_a} = \mathrm{id}_{L_b},$$

that is the right translation maps ρ_s and $\rho_{s'}$ restrict to mutually inverse bijections between the $\mathcal{L}$-classes L_a and L_b, and both of these restricted maps preserve $\mathcal{R}$-classes.

We illustrate Green's Lemma in T_n:

Example 2.3.17.

1. Let $S = T_4$ and choose $a, b \in T_4$ to be

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 4 & 4 \end{pmatrix} \text{ and } b = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 3 & 3 \end{pmatrix}.$$

2. Then a has image $= \{2, 4\}$ and fibers $\{1\}, \{2, 3, 4\}$ and b has image $= \{2, 3\}$ and fibers $\{1\}, \{2, 3, 4\}$.

3. Therefore, $a \mathcal{R} b$ since fibers(a) = fibers(b).

4. Let $s, s' \in T_4$ be

$$s = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \text{ and } s' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 4 & 3 \end{pmatrix},$$

so that $as = b$ and $bs' = a$.

5. If $c = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 2 \end{pmatrix} \in L_a$, then $c \mathcal{R} c\rho_s$ since

$$c\rho_s = cs = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 3 & 2 \end{pmatrix}$$

has fibers $\{1, 2, 4\}$ and $\{3\}$ that are equal to the fibers of c.

6. Moreover, if $d = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & 3 & 2 \end{pmatrix} \in L_b$, then $d \mathcal{R} d\rho_{s'}$ since

$$d\rho_{s'} = ds' = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 2 \end{pmatrix}$$

has fibers $\{1, 2, 3\}$ and $\{4\}$ and they are equal to the fibers of d.

7. We can see that $cs \mathcal{L} b$, since $\text{im}(cs) = \text{im}(b) = \{2, 3\}$. Also, $ds' \mathcal{L} a$, as $\text{im}(ds') = \text{im}(a) = \{2, 4\}$.

The next result is the dual version of Green's Lemma:

Lemma 2.3.18. [37, Lemma 2.2.2] *Let S be a semigroup and $a, b \in S$ be such that $a \mathcal{L} b$. Let $t, t' \in S^1$ where $ta = b$ and $t'b = a$. Then:*

1. *The map $\lambda_t : R_a \to R_b$, defined by $a\lambda_t = ta$, and the map $\lambda_{t'} : R_b \to R_a$, defined by $b\lambda_{t'} = t'b$, are mutually inverse bijections. That is, $\lambda_t \lambda_{t'}$ is the identity map on R_a, and dually $\lambda_{t'} \lambda_t$ is the identity map on R_b.*

2. *Both maps λ_t and $\lambda_{t'}$ preserve $\mathcal{L}$-classes; that is, for all $c \in R_a$ and $d \in R_b$, we have $c \mathcal{L} c\lambda_t$ and $d \mathcal{L} d\lambda_{t'}$.*

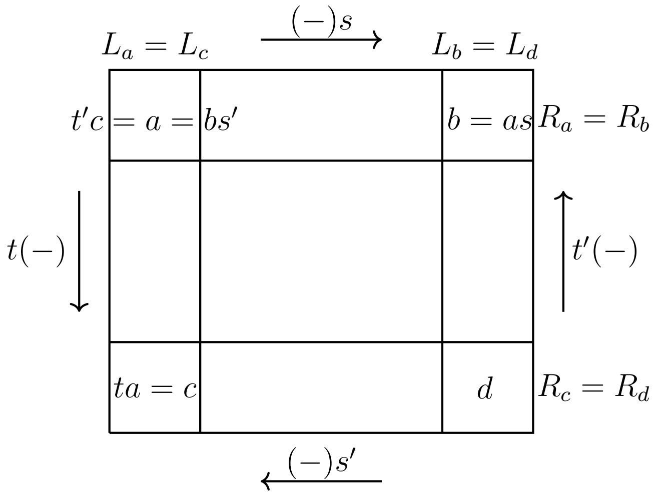

Figure 2.4: Combination of Green's Lemmas

See Figure 2.4 for the illustration of Green's Lemmas. The combined effect of the two Green's Lemmas 2.3.16 and 2.3.18 gives bijections between all $\mathcal{H}$-classes in the same $\mathcal{D}$-class:

Lemma 2.3.19. [37, Lemma 2.2.3] *If a, b are $\mathcal{D}$-equivalent elements in a semigroup S, then $|H_a|= |H_b|$.*

Proof. Assume $a\,\mathcal{D}\,b$. Then there exists an element $c \in S$ such that $a\,\mathcal{R}\,c$ and $c\,\mathcal{L}\,b$. Let $s, s', t, t' \in S^1$ be such that $as = c, cs' = a, tc = b$, and $t'b = c$. Then by Green's Lemmas 2.3.16 and 2.3.18 we have:

$$\rho_s|_{H_a} : H_a \to H_c \text{ is a bijection, and}$$
$$\lambda_t|_{H_c} : H_c \to H_b \text{ is a bijection.}$$

Then

$$\rho_s|_{H_a}\lambda_t|_{H_c}\colon H_a \to H_b \text{ is a bijection.}$$

Hence $|H_a|= |H_b|$. □

Green's Lemmas 2.3.16 and 2.3.18 also provide a nice consequence focusing on the multiplicative properties of an $\mathcal{H}$-class.

Lemma 2.3.20. [37, Lemma 2.2.4] *Let a,b be elements of a semigroup S.*

1. *If $ab \in H_a$, then $\rho_b|_{H_a}$ is a bijection of H_a onto itself.*

2. *If $ab \in H_b$, then $\lambda_a|_{H_b}$ is a bijection of H_b onto itself.*

The following results allow us to apply group theory to semigroups:

Theorem 2.3.21 (Green's Theorem). [37, Theorem 2.2.5] *If H is an $\mathcal{H}$-class in a semigroup S, then either $H^2 \cap H = \varnothing$ or $H^2 = H$ and H is a subgroup of S.*

Proof. If $H^2 \cap H = \varnothing$, then there is nothing further to prove. Suppose that $H^2 \cap H \neq \varnothing$. Then there exist $a, b \in H$ such that $ab \in H$. Then $a \,\mathcal{H}\, ab$, in particular $a \,\mathcal{R}\, ab$. By Green's Lemma, $\rho_b|_H$ is a bijection from H onto itself. Similarly, since $b \,\mathcal{L}\, ab$, $\lambda_a|_H$ is a bijection from H onto itself, by Lemma 2.3.18. Now let $c \in H$. Then $ac = c\lambda_a|_H$ and $cb = c\rho_b|_H$ are both in H. By Green's Lemmas, $\rho_c|_H$ and $\lambda_c|_H$ are bijections from H onto itself. Since $c \in H$ is arbitrary, we have $cH = Hc = H$ for all $c \in H$. Hence $H^2 = H$, which certainly implies $H^2 \cap H = H$ and H is a subsemigroup. By statement (2.1), H is a subgroup of S. $\square$

Let e be an idempotent in a semigroup S. For any $x \in R_e$, there exists $s \in S^1$ such that $x = es$. Then

$$x = es \Rightarrow ex = e(es) = e^2 s = es = x.$$

Thus, e is a left identity for the $\mathcal{R}$-class R_e. We can apply a dual argument to prove that $ye = y$ for $y \in L_e$ and so:

Proposition 2.3.22. [37] *Every idempotent e in a semigroup S is a left identity for the $\mathcal{R}$-class R_e and a right identity for the $\mathcal{L}$-class L_e.*

The following theorem determines the location of the inverses of a regular element a in a semigroup S by determining the location of the idempotents in the $\mathcal{D}$-class of a.

Theorem 2.3.23. [37, Theorem 2.3.4] *Let a be an element of a regular $\mathcal{D}$-class D in a semigroup S.*

1. *If $a' \in V(a)$, then $a' \in D$ and the two $\mathcal{H}$-classes $R_a \cap L_{a'} and L_a \cap R_{a'}$ contain, respectively, the idempotents aa' and $a'a$.*

2. *If $b \in D$ is such that $R_a \cap L_b$ and $L_a \cap R_b$ contain idempotents e and f, respectively, then H_b contains an inverse a^* of a such that $aa^* = e$ and $a^*a = f$.*

3. *No $\mathcal{H}$-class contains more than one inverse of a.*

Figure 2.5 shows the location of idempotents and inverse elements in a $\mathcal{D}$-class. By Theorems 2.3.14 and 2.3.23 and since the order of rows and columns is arbitrary, we can deduce that the picture of the $\mathcal{D}$-class of an inverse semigroup is square, with all $\mathcal{H}$-classes containing idempotents appear in the diagonal of the $\mathcal{D}$. With respect to the diagonal, the $\mathcal{H}$-classes of mutually inverse elements are located symmetrically. As an immediate consequence of Theorem 2.3.23, we have:

Proposition 2.3.24. [37, Proposition 2.3.5] *Let e and f be idempotents in a semigroup S. Then e and f belong to the same $\mathcal{D}$-class if and only if there exist an element a in S and an inverse a' of a such that $aa' = e$ and $a'a = f$.*

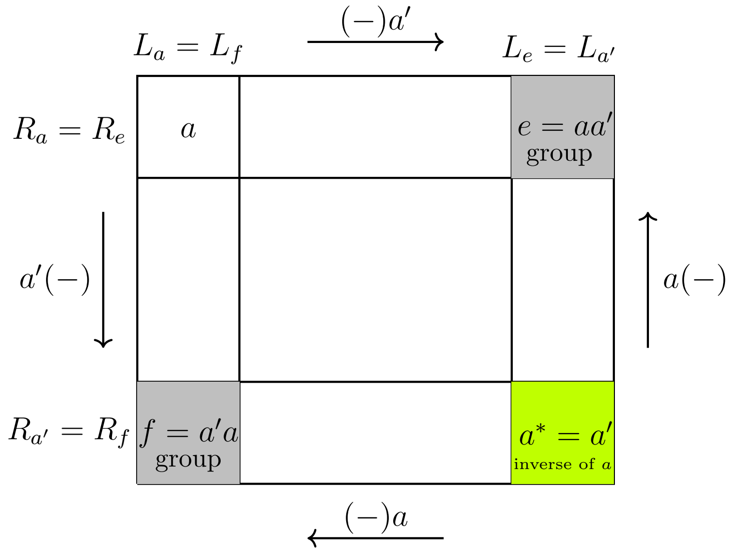

Figure 2.5: Locating semigroup inverses

Proof. Suppose that $e \mathcal{D} f$. Since idempotents are regular, the $\mathcal{D}$-classes D_e and D_f are regular. Choose $a \in R_e \cap L_f$ and $b \in R_f \cap L_e$. Then by Theorem 2.3.23, H_b contains some $a' \in V(a)$ such that $aa' = e$ and $a'a = f$. Conversely, suppose that a in S and the inverse a' of a are such that $aa' = e$ and $a'a = f$. Since $e = aa'$ and $ea = aa'a = a$, we have $e \mathcal{R} a$. Similarly, since $f = a'a$ and $af = aa'a = a$, it implies that $a \mathcal{L} f$. Hence $e \mathcal{R} a \mathcal{L} f$ and therefore $e \mathcal{D} f$. ☐

Proposition 2.3.25. [37, Proposition 2.3.6] *If H and K are two group $\mathcal{H}$-classes in the same (regular) $\mathcal{D}$-class, then H and K are isomorphic.*

In view of the above results, we observe the following:

Remark 2.3.26. [14]

1. Every row and column in an egg-box containing an idempotent contains a group.

2. Products are located by idempotents: $ab \in R_a \cap L_b$ if and only if $R_b \cap L_a$ contains $e \in E(S)$.

3. Any two $\mathcal{L}$-classes contained in the same $\mathcal{D}$-class have the same cardinality. Similarly, for any two $\mathcal{R}$-classes, and for any two $\mathcal{H}$-classes.

4. Two $\mathcal{H}$-classes containing idempotents that are in the same $\mathcal{D}$-class are isomorphic subgroups (maximal subgroups).

5. Let e be an idempotent in S. If $e \mathcal{R} a$, then $ea = a$ and $a = ex$ for some $x \in S^1$.

6. If H_e and H_a are two $\mathcal{H}$-classes in the same $\mathcal{R}$-class, then every element of the $\mathcal{H}$-class H_a has a unique expression as ga where $g \in H_e$. Dually, if H_e and H_b are two $\mathcal{H}$-classes in the same $\mathcal{L}$-class, then every element of the $\mathcal{H}$-class H_b has a unique expression as bg for some $g \in H_e$.

Having examined several basic definitions and results in semigroup theory, we are now ready to introduce semigroup representation theory in the next chapter.

Chapter 3

Semigroup Representation Theory

This chapter is devoted to the basic aspects of representations of semigroups and some of the related terminologies. We also provide different examples across the chapter to illustrate the various ideas put forward. For further detail in this area, see [14, 15, 22, 23, 102]. Throughout the chapter, the action of a semigroup will be on the right, unless otherwise indicated.

3.1 Basic Definitions

Definition 3.1.1. Let S be a (finite) semigroup and V be a finite-dimensional vector space over a field k. A *representation* of S or S-*representation* is a homomorphism φ from S to $\mathrm{End}(V)$, the semigroup of all linear transformations of V.

We identify $a \in S$ and $a\varphi \in \mathrm{End}(V)$; for $v \in V$ we write $v \cdot a$ or va instead for the effect of $a\varphi$ on the vector v. If S is a monoid, an additional requirement is that φ map the identity element from S to the identity transformation on V. We call $\dim V$ the *degree* of the representation φ. A representation φ is called *faithful* if besides being a homomorphism, φ is also a monomorphism.

Remark 3.1.2.

- We exclude the *null* representation which maps every element to zero. Thus $\mathrm{im}\,\varphi \neq \{0\}$. Indeed, when S is a monoid, this cannot happen since φ must send the identity element 1 of a monoid S to the identity linear map $\mathrm{id} : V \to V$.
- The elements of $\mathrm{End}(V)$, like those of S, need not have inverses. If S is a group, then $\mathrm{im}\,\varphi \subseteq GL(V)$, the group of all invertible maps from V to V.

We call a representation in the sense of the previous definition a *representation as linear maps*.

If we choose a basis for V, then relative to this basis each endomorphism of V is represented by an $n \times n$ matrix over k and this gives an isomorphism between

End(V) and $M_n(k)$. There is another version of representation which is called a *matrix representation* as follows:

Definition 3.1.3. A *matrix representation* of S over k is a homomorphism ρ from S to $M_n(k)$, the multiplicative monoid of all $n \times n$ matrices over k for some $n \geq 1$.

An equivalent notion of representation is that of a *module*. If V is a vector space, then V is an S-*module* (or *module over* S) if a multiplication $V \times S \to V$ given by $(v, a) \mapsto v \cdot a$ is defined, satisfying the following conditions for all $v, w \in V$, $\lambda \in k$ and $a, b \in S$:

1. $(\lambda v) \cdot a = \lambda(v \cdot a)$;

2. $(v \cdot a) \cdot b = v \cdot (a \cdot b)$;

3. $v \cdot \mathbf{1} = v$, where $\mathbf{1}$ is the identity element of S;

4. $(v + w) \cdot a = v \cdot a + w \cdot a$.

Note that the above conditions (2) and (4) ensure that for $v \in V$ the function

$$v \mapsto v \cdot a$$

is an endomorphism of V, for all $a \in S$.

In fact, the concepts of linear representations, matrix representations and representation modules are equivalent. Thus, we can phrase their related terminologies in terms of endomorphisms, matrices or modules. Throughout this chapter, we provide the notions in terms of endomorphisms, however we sometimes use the equivalent concepts interchangeably.

Let V and W be two S-modules. A linear mapping $\mu : V \longrightarrow W$ is called a *homomorphism* or an S-*homomorphism* if it commutes with the action of all elements from S. Thus, for each element $s \in S$ the following diagram commutes:

$$\begin{tikzcd} V \arrow[r, "(-)s"] \arrow[d, "\mu"'] & V \arrow[d, "\mu"] \\ W \arrow[r, "(-)s"] & W \end{tikzcd}$$

where $(-)s$ denotes multiplication by s. Moreover, μ is an *isomorphism* if and only if it is bijective. The set of all S-homomorphisms from V to W is denoted by $\mathrm{Hom}_S(V, W)$.

Two representations $\varphi : S \to \mathrm{End}(V)$ and $\psi : S \to \mathrm{End}(W)$ are said to be *equivalent*, and we write $\varphi \sim \psi$, if there exists a vector space isomorphism $T : V \to W$

such that $(s)T\psi = (s)\varphi T$ for all elements s in S (which means $\mathrm{Hom}_S(V, W)$ contains an isomorphism).

Let S be a finite semigroup and k be a field. Define a vector space $k[S]$ where the basis vectors are elements of S. That is, if $S = \{a_1, \dots, a_n\}$, then let

$$k[S] = \Big\{ \sum_{i=1}^{n} \lambda_i a_i \ : \ \lambda_i \in k \Big\}$$

where

$$\sum_{i=1}^{n} \lambda_i a_i + \sum_{i=1}^{n} \mu_i a_i = \sum_{i=1}^{n} (\lambda_i + \mu_i) a_i \text{ and } \lambda\Big(\sum_{i=1}^{n} \lambda_i a_i \Big) = \sum_{i=1}^{n} (\lambda \lambda_i) a_i,$$

and $\dim k[S] = n$. Define a multiplication on $k[S]$ as follows:

$$\Big(\sum_{a \in S} \lambda_a a \Big) \Big(\sum_{b \in S} \lambda_b b \Big) = \Big(\sum_{a,b \in S} \lambda_a \lambda_b (ab) \Big), \tag{3.1}$$

where $\lambda_a, \lambda_b \in k$. Then $k[S]$ is called the *semigroup algebra* of S over k.

Each representation of the semigroup algebra $k[S]$ induces or restricts to a representation on the underlying semigroup S. Conversely, each representation of S over k uniquely extends by linearity to a representation of the semigroup algebra $k[S]$. This produces an equivalence between representations of S and those of $k[S]$. It then follows that a representation of S on a vector space V over k is the same thing as a $k[S]$-module structure on V and that two representations are equivalent if and only if the corresponding $k[S]$-modules are isomorphic. Explicitly, if φ is an S-representation, then the $k[S]$-module structure on V over k is given by

$$v \cdot \left(\sum_{a \in S} \alpha_a a \right) = \sum_{a \in S} \alpha_a (v \cdot a), \text{ for } v \in V. \tag{3.2}$$

We continue presenting the definitions related to representations of semigroups.

Definition 3.1.4. If V is an S-representation and U a subspace of V with the property that $US \subseteq U$, then U is an S*-subrepresentation* of V.

Obviously, the whole of V and the zero vector space $\{0\}$ are subrepresentations of any representation.

If V is a vector space and $V_1, \dots, V_n$ are subspaces of V, then

$$V_1 + \cdots + V_n = \{v_1 + \cdots + v_n \ : \ v_i \in V_i \text{ where } 1 \leq i \leq n\}.$$

This vector space is a subspace of V.

Definition 3.1.5. A vector space V is the direct sum of a family $\{V_i \,:\, 1 \leq i \leq n\}$ of subspaces of V if and only if

1. $V = \sum_i V_i$, and
2. for each i, where $1 \leq i \leq n$,

$$V_i \cap \Big(\sum_{j \neq i} V_j\Big) = 0.$$

Then V is denoted by $\bigoplus_i V_i$, with $1 \leq i \leq n$.

The direct sum of representations is the simplest way of constructing new representations from given ones.

Definition 3.1.6. Let V be an S-representation, and suppose that as vector spaces

$$V = \bigoplus_i V_i, \text{ with } 1 \leq i \leq n,$$

where V_i are S-subrepresentations of V. Then, V is called a *direct sum* of subrepresentations.

Now, we are ready to introduce the irreducibility of a representation.

Definition 3.1.7. An S-representation V is an *irreducible* S-representation if it is not null and the only subrepresentations are $\{0\}$ and V itself. Otherwise V is *reducible.*

Example 3.1.8. Let S be any semigroup. Then mapping all elements of S to the identity transformation $k \xrightarrow{id} k$ defines on k the structure of an S-representation. This representation is called the *trivial* S-representation and it is irreducible because it has only two subrepresentations, namely, k and $\{0\}$. A one-dimensional representation is automatically irreducible.

In terms the language of modules, an S-module V is called *simple* when it is not null and the only submodules of V are $\{0\}$ and V, thus the corresponding representation is irreducible. Moreover, we have the following essential definition:

Definition 3.1.9. A representation V is called *completely reducible* if it is equivalent to a direct sum of irreducible subrepresentations. In other words, V can be written (decomposed) as $V = V_1 \oplus \cdots \oplus V_n$ for some irreducible subrepresentations $V_1, \ldots, V_n$.

Again, in terms of module language, modules corresponding to completely reducible S-representations are called *semisimple* S-modules. The preceding definitions apply to representations of algebras, and an S-representation is irreducible (completely reducible) if and only if the corresponding algebra representation is reducible (completely reducible).

The *character* of a representation $\varphi : S \to \mathrm{End}(V)$ of a semigroup S is the function $\chi : S \to k$ defined by $\chi(s) = \text{trace } s\varphi$ for all $s \in S$. The trace of an endomorphism is equal to the trace of the matrix representing it. If χ is the character of an irreducible representation, then χ is said to be an *irreducible character*. The aim of the following example is to illustrate the concept of the reducibility of a semigroup representation.

Example 3.1.10. Mapping representations [14]
Let S be S_n, I_n or T_n. Consider the vector space V over k with basis $\{v_1, v_2, \ldots, v_n\}$. For every $a \in S$ define a linear map on V by prescribing its action on the basis elements of V in the following way: $v_i \cdot a = v_{ia}$, when $a \in S_n$ or T_n, or

$$v_i \cdot a = \begin{cases} v_{ia} & \text{if } i \in \mathrm{dom}(a); \\ 0, & \text{otherwise,} \end{cases}$$

when $a \in I_n$.

Consider the following two subspaces of V.

- The subspace
$$U = k\text{-span of } v_1 + v_2 + \ldots + v_n. \tag{3.3}$$
- The hyperplane W with equation $x_1 + x_2 + \ldots + x_n = 0$; that is,
$$W = \Big\{ w = \sum_{i=1}^{n} \lambda_i v_i \in V : \sum_{i=1}^{n} \lambda_i = 0, \lambda_i \in k \Big\}. \tag{3.4}$$

Let us verify whether both subspaces U and W can be S-subrepresentation when S is S_n, I_n and T_n, respectively.

When $S = S_n$ (permutation representation) [14]:

Since $\dim U = 1$, the only subspaces of U are $\{0\}$ and U itself. Hence, U is an irreducible subrepresentation of V as $US_n \subseteq U$. Then V is a reducible S_n-representation.

First, we claim that W is a subrepresentation of V. Let $w = \lambda_1 v_1 + \ldots + \lambda_n v_n \in W$ with $\lambda_1 + \ldots + \lambda_n = 0$ and $\sigma \in S_n$. Then $w \cdot \sigma = \lambda_1 v_{1\sigma} + \ldots + \lambda_n v_{n\sigma}$ with $\lambda_1 + \ldots + \lambda_n = 0$. This implies that $w \cdot \sigma \in W$, which means $WS_n \subseteq W$. Second, we claim that W is an irreducible S_n-subrepresentation. To prove this, we need to show that the only S_n-invariant subspaces of W is are the trivial one and the whole space W.

Assume that the characteristic char(k) of the field k does not divide n, the dimension of V. Suppose that $v \in W$ with $v \neq 0$ and $v = \sum_{i=1}^{n} \lambda v_i$ (the coordinates are the same). Then, as $v \in W$ and $\sum_{i=1}^{n} \lambda = 0$, we have $n\lambda = 0$. By the restriction on char(k) we must have $\lambda = 0$ and thus $v = 0$ which is a contradiction to our assumption. This means that any $v \in W$ must have at least two coordinates that are different. That is, we write the vector v as $\sum_{i=1}^{n} \lambda_i v_i$ with $\lambda_j \neq \lambda_k$ for some $1 \leq j < k \leq n$. For any i there is a $\sigma_i \in S_n$ $(1 \leq i \leq n-1)$ such that $j\sigma_i = i$ and $k\sigma_i = i+1$. Hence, if we apply σ_i to v we get that $v \cdot \sigma_i$ has i-th and $(i+1)$-st coordinates that are different. Any subrepresentation of W, say $Z \neq 0$, that contains $v \neq 0$ also contains $v \cdot \sigma_i$ for $1 \leq i \leq n$.

Consider $v \cdot \sigma_i\ (i, i+1) - v \cdot \sigma_i$. Then if $v \cdot \sigma_i$ has coordinates the row vector

$$(\mu_1, \mu_2, \ldots, \mu_i, \mu_{i+1}, \ldots, \mu_n),$$

we get

$$\begin{aligned} v \cdot \sigma_i(i, i+1) - v \cdot \sigma_i &= (\mu_1, \mu_2, \ldots, \mu_{i+1}, \mu_i, \ldots, \mu_n) \\ &\quad - (\mu_1, \mu_2, \ldots, \mu_i, \mu_{i+1}, \ldots, \mu_n) \\ &= (0, 0, \ldots, \mu_{i+1} - \mu_i, \mu_i - \mu_{i+1}, \ldots, 0) \\ &= \kappa(0, 0, \ldots, 1, -1, \ldots, 0), \end{aligned}$$

for $\kappa = \mu_{i+1} - \mu_i \neq 0$, as $\mu_{i+1} \neq \mu_i$. Thus, $v \cdot \sigma_i(i, i+1) - v \cdot \sigma_i = \kappa(v_i - v_{i+1})$. In other words, a nonzero multiple of $v_i - v_{i+1}$ is contained in the subspace Z. This implies that $v_i - v_{i+1} \in Z \subseteq W$, for all i with $1 \leq i \leq n-1$. We claim that the set $\{v_i - v_{i+1}, 1 \leq i \leq n-1\}$ forms a basis for W, to show this:

1. the vectors $v_i - v_{i+1}$ are independent as $\{v_i, \ldots, v_n\}$ is the basis of V. It follows that all the coefficients of $v_i - v_{i+1}$ are zeros.

2. the vectors $v_i - v_{i+1}$ span $(n-1)$-dimensional subspace of W:

$$\mathrm{Span}_k\{v_i - v_{i+1}, 1 \leq i \leq n-1\} \leq W.$$

As W is a hyperplane so $\dim W = n-1$. Hence,

$$\mathrm{Span}_k\{v_i - v_{i+1}, 1 \leq i \leq n-1\} = W$$

This means that $Z = W$, indicating that W is an irreducible S_n-subrepresentation.

We conclude that $V = U \oplus W$ is a direct sum of irreducible subrepresentations, which means that V is completely reducible, for $S = S_n, n \geq 2$.

When $S = I_n$, $n>1$ (partial permutation representation) [14]:

Consider the previous subspaces W (3.4) and U (3.3). Let $u \in U$ and $a \in I_n$ such that

$$a = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & - & - & \cdots & - \end{pmatrix} \in I_n. \tag{3.5}$$

Let us see the action of I_n on U and W, respectively. The element $u \cdot a = (v_1 + v_2 \ldots + v_n) \cdot a = v_1$ does not belong to the subspace U since it has the coordinates $(1, 0, 0, \ldots, 0)$. The element $v_1 - v_3$ is in the subspace W but its image under the action of a is $(v_1 - v_3) \cdot a = v_1$, and is not in W since the sum of its coordinates is not zero. Thus, U and W are not I_n-subrepresentations since $UI_n \nsubseteq U$ and $WI_n \nsubseteq W$.

In fact, we will prove that V is an irreducible partial permutation representation for I_n. Let $V' \subset V$ be an I_n-subrepresentation with $V' \neq \{0\}$. We need to show that V' is indeed the whole space V. Take $v' \in V'$ with $v' \neq 0$; then $v' = \sum_{i=1}^{n} \lambda_i v_i$ with $\lambda_j \neq 0$ for some j. Let

$$a_i = \begin{pmatrix} 1 & 2 & \cdots & j-1 & j & j+1 & \cdots & n \\ - & - & \cdots & - & i & - & \cdots & - \end{pmatrix} \in I_n, 1 \leqslant i \leqslant n. \tag{3.6}$$

Then, $v'a_i = \lambda_j v_i \in V'$ and hence each $v_i \in V', 1 \leq i \leq n$. But the v_i are the basis of V and therefore $V = V'$. Thus, the only two subrepresentations, that V has are $\{0\}$ and V. Hence, we get the desired result.

When $S = T_n$, $n>1$ (mapping representation) [14]:

First, we need the following preliminary results:

1. If T is a submonoid of S and V is an S-representation, then restricting the S-action to T gives V the structure of a T-representation.

2. Let V be an S-representation and $V = \bigoplus_i V_i$ with the V_i irreducible subrepresentations. If $W \subset V$ is an irreducible subrepresentation, then $W \cong V_j$ for some j [107].

Consider the subspaces W (3.4) and U (3.3). We claim that U is not a T_n-subrepresentation of V. To see this, let $a \in T_n$ such that

$$a = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ 1 & 1 & 1 & \cdots & 1 \end{pmatrix} \text{and } u = v_1 + \ldots + v_n \in U. \tag{3.7}$$

We have $u \cdot a= v_1 + \ldots + v_1 = nv_1 \notin U$ and this implies that $UT_n \nsubseteq U$. On the other hand, W is an irreducible T_n-subrepresentation of V. To prove this: let $w \in W$ such that $w = \sum_{i=1}^{n} \lambda_i v_i$ with $\sum_{i=1}^{n} \lambda_i = 0$ and $a \in T_n$ with $\mathrm{im}(a) = \{i_1, i_2, \ldots, i_k\} \subset [n]$. The set $[n] = \bigcup i_j a^{-1} = \mathrm{dom}(a)$ is a disjoint union where $i_j a^{-1} = \{l \in [n] : la = i_j, 1 \leq j \leq k\}$ are the fibers of the element a.

Aside:

Example 3.1.11. Let $a \in T_5$ be such that

$$a = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 3 & 1 & 5 \end{pmatrix}.$$

Then $\mathrm{im}(a) = \{1, 3, 5\}$ and the fibers of a are $\{1, 4\}, \{2, 3\},$ and $\{5\}$.

Now, we compute

$$w \cdot a = \Big(\sum_{m \in i_1 a^{-1}} \lambda_m \Big) v_{i_1} + \ \ldots \ + \Big(\sum_{m \in i_k a^{-1}} \lambda_m \Big) v_{i_k} \tag{3.8}$$

where $\sum_{n=1}^{k} \Big(\sum_{m \in \ i_n a^{-1}} \lambda_m \Big) = \sum \lambda_i = 0$. That is, when we add the coordinates of $w \cdot a$ we have $\sum_{i=1}^{n} \lambda_i = 0$ and hence $w \cdot a \in W$. This implies that $WT_n \subseteq W$, which means that W is a T_n-subrepresentation of V, hence a T_n-representation V is reducible. Moreover, W is irreducible since if $W' \subset W$ is a T_n-subrepresentation, then, given that $S_n \leq T_n$ we have that W' is an S_n-subrepresentation of W, which is an irreducible S_n-representation. Thus, $W' = \{0\}$ or W.

Let X be a 1-dimensional subrepresentation of V. Let $X = \mathbb{C}[v]$ with $v = \sum \lambda_i v_i$ where $\sum \lambda_i = 0$; thus $v \in W$, which implies that X is a subrepresentation of W. But this is a contradiction since W is irreducible, hence $\sum \lambda_i \neq 0$. Now, let $a \in T_n$ be such that $(i)a = 1$ for all i, $1 \leqslant i \leqslant n$. Then $v \cdot a = (\sum \lambda_i) v_1 \in X$ and since $\sum \lambda_i \neq 0$ we get $v_1 \in X$. If we repeat the previous step with $(i)a = 2$ for all i, we get $v_2 \in X$. This is a contradiction since X has dimension one.

Thus V has no 1-dimensional T_n-subrepresentations if $n > 2$, and exactly one T_n-subrepresentation namely, W, if $n = 2$ [14]. Furthermore, if we can write $V = \bigoplus_i V_i$ with the V_i irreducible subrepresentations, then W is isomorphic to one of the V_i, which is therefore $(n-1)$-dimensional. Now, we have $V = V_1 \bigoplus V_2$ with V_2, say, a 1-dimensional subrepresentation and $W \cong V_1$. But, no such V_2 exists and this means that V cannot be decomposed, i.e, V is not completely reducible. This example also shows that a non-completely reducible representation of a semigroup need not be irreducible.

3.2 Semisimplicity of Semigroup Algebras

In the theory of representations of a finite group, **Maschke's Theorem** plays a fundamental role.

Maschke's Theorem. [8] *Let G be a finite group and k be a field. Then the algebra $k[G]$ is semisimple if and only if the characteristic of k does not divide the order of G.*

Consequently, representations of G over k are completely reducible. If we replace the group G by a finite monoid, how can we describe semisimplicity for the monoid algebra $k[S]$? Steinberg has addressed this topic [102] and we describe here his main results. Throughout this section, we write mapping symbols on the left and the action of a semigroup is on the left.

Let S be a finite monoid. We need to introduce the notion of an S-set. An *S-set* is a set X together with a mapping $S \times X \to X$, written $(s, x) \mapsto sx$, such that $1x = x$ and $s_1(s_2x) = (s_1s_2)x$ for all $x \in X$ and $s_1, s_2 \in S$. A mapping $\theta : X \to Y$ of *S-set* is said to be *S-equivariant* if $\theta(sx) = s(\theta x)$ for all $x \in X$ and $s \in S$. We denote by $\mathrm{Hom}_S(X, Y)$ the set of all S-equivariant mappings from X to Y. Note that a special case of this is the earlier S-homomorphism.

Let V be a simple S-module. An idempotent $e \in E(S)$ is called an *apex* for V if $eV \neq 0$ and $I_eV = 0$ where $I_e = eSe \setminus G_e$. We sometime refer to an apex e by its $\mathcal{J}$-class J_e. The following proposition provides the characterization of the apex of an S-module V.

Proposition 3.2.1. [23] *An S-module V has apex e if and only if J_e is the unique minimal $\mathcal{J}$-class, with respect to the order of $\mathcal{J}$-classes, that does not annihilate V.*

The theorem below shows the existence of apexes in a finite semigroup.

Theorem 3.2.2. [23] *Let S be a finite semigroup. Then every simple S-module V has an apex.*

It turns out that there exists an apex; so, we can fix an apex e in $E(S)$ and its related maximal subgroup G_e. Then we define some functors associated to e as follows [102, Chapter 5],

$$\mathrm{Ind}_{G_e} : k[G_e]\text{-mod} \longrightarrow k[S]\text{-mod}; \tag{3.9}$$

$$\mathrm{Coind}_{G_e} : k[G_e]\text{-mod} \longrightarrow k[S]\text{-mod}, \tag{3.10}$$

by putting

$$\mathrm{Ind}_{G_e}(V) = k[L_e] \otimes_{k[G_e]} V;$$

$$\mathrm{Coind}_{G_e}(V) = \mathrm{Hom}_{G_e}(R_e, V).$$

Here, V is a $k[G_e]$-module and $\mathrm{Hom}_{G_e}(R_e, V)$ is the vector space of G_e-equivariant mappings $\theta : R_e \to V$. Also, $k[L_e]$ and $k[R_e]$ are the left and right *Schützenberger representations* associated to the $\mathcal{J}$-class J_e of S. Note that $k[L_e]$ is the set of all formal sums $\sum_i \lambda_i s_i$ where $\lambda_i \in k$ and $s_i \in L_e$, similarly for $k[R_e]$. The function Ind_{G_e} is called the *induction functor* and Coind_{G_e} is called the *coinduction functor*.

Let W be a $k[G_e]$-module, $e \in E(S)$. Then, if we put $A_e = k[S]/k[I(e)]$, where $I(e) = \{a \in S \mid e \notin SaS\}$ (it is an ideal of S if it is not empty), we get a homomorphism of A_e-modules

$$\varphi_W : \mathrm{Ind}_{G_e}(W) \longrightarrow \mathrm{Coind}_{G_e}(W).$$

For $l \in L_e$, $w \in W$, and $r \in R_e$, define

$$\varphi_W(l \otimes w)(r) = (r \diamond l)w,$$

where

$$r \diamond l = \begin{cases} rl, & \text{if } \ rl \in G_e; \\ 0, & \text{else.} \end{cases}$$

Note that $rl \in eSe$. An arbitrary element of $\mathrm{Ind}_{G_e}(W)$ is a k-linear combination of terms $(l \otimes w)$.

All preparations for stating the main results of semisimplicity are now completed.

Definition 3.2.3. [15] Let S is a finite regular monoid and k is a field. Then the semigroup algebra $k[S]$ is *semisimple* when every S-module V over k is completely reducible.

This means there exists essentially a unique way to write V as a direct sum of irreducible submodules as $V \cong V_1 \oplus \cdots \oplus V_n$.

Theorem 3.2.4. [102, Chapter 5] *Let S be a finite monoid and k be a field. Then $k[S]$ is semisimple if and only if all the following hold:*

1. *S is regular;*
2. *the characteristic of k does not divide the order of G_e, for any $e \in E(S)$;*
3. *the homomorphism $\varphi_{k[G_e]} : \mathrm{Ind}_{G_e}(k[G_e]) \longrightarrow \mathrm{Coind}_{G_e}(k[G_e])$ is an isomorphism for all $e \in E(S)$.*

The following theorem is the analogous semigroup theorem to **Maschke's Theorem** of finite group representations.

Corollary 3.2.5. [8, Oganesyan] *Let S be a finite inverse monoid and k be a field. $k[S]$ is semisimple if and only if the characteristic of k does not divide the order of any of the maximal subgroups G_{e_i}.*

Example 3.2.6. Let $S = I_n$ and let V be an I_n-representation. Let k be a field . All maximal subgroups of I_n are isomorphic to S_r, $1 \leq r \leq n$. Thus, V is semisimple if and only if $\text{char}(k) \nmid r!$, $1 \leq r \leq n$. Consequently, V is semisimple if and only if $\text{char}(k) \nmid n!$. In particular, when $k = \mathbb{C}$, every I_n-representation is semisimple.

Example 3.2.7. Since $S = T_n$ is not inverse and the mapping representation is not completely reducible for $n > 2$, (T_n, k) is not semisimple for any field k (even for the complex vector space with characteristic 0).

In terms of matrices, the following theorem shows that the monoid algebra $k[S]$ is isomorphic to a product of matrix algebras $M_{n_i}(k[G_{e_i}])$ over the group algebras of its maximal subgroups.

Theorem 3.2.8. [102, Chapter 5] *Let S be a finite monoid and k be a field such that $k[S]$ is semisimple. Let $\{e_1, \ldots, e_s\}$ be a complete set of representatives of the $\mathcal{J}$-classes of idempotents of S and suppose that J_{e_i} contains n_i $\mathcal{L}$-classes. Then there is an isomorphism of k-algebras*

$$k[S] \cong \prod_{i=1}^{s} M_{n_i}\Big(k[G_{e_i}]\Big).$$

In the case of finite inverse monoids we have:

Corollary 3.2.9. [102, Chapter 9] *Let S be a finite inverse monoid and k be a field. Let $\{e_1, \ldots, e_s\}$ be the set of idempotent representatives of the $\mathcal{J}$-classes of S and let $n_i = |E(J_{e_i})|$. Then there is an isomorphism*

$$k[S] \cong \prod_{i=1}^{s} M_{n_i}\Big(k[G_{e_i}]\Big).$$

3.3 Reduction and Induction of Representations

In this section, we introduce the notion of *reduction* (some authors call it *restriction*) and describe the induction of (3.9) in a more elementary way. We also illustrate each notion with an example in I_n [14].

Reduction of a representation is a technical procedure for constructing a representation of a maximal subgroup from a representation of a semigroup. In the reverse direction, induction is a technical procedure for constructing a representation of a semigroup from a representation of a maximal subgroup. We start with the reduction procedure.

3.3.1 Reduction Process

Let S be a finite monoid and V be an irreducible S-representation. Consider a $\mathcal{J}$-class J of S and fix an idempotent e belonging in J. Consider the maximal subgroup G_e with $e \in G_e$. We need two steps to construct a representation of G_e, as follows [1]:

- consider the subspace $Ve = \{v \cdot e \mid v \in V\}$ of V and
- define the action of G_e on Ve.

These steps are required in the reduction process. Note that we do not know whether the resulting G_e-representation is irreducible or reducible, we will discuss this point later in this chapter.

The following is illustration of the process in the symmetric inverse monoid I_n.

Example 3.3.1. [14] Let $S = I_n$ and consider the previous vector space V of Example 3.1.10, the I_n- irreducible partial permutation representation with dimension n. Fix $e \in E(S)$ so that G_e is a maximal subgroup of S. Let $X = \{1, \dots, \ell\}$, we have $e = id_X : X \to X$, $X \subseteq [n]$, the identity of G_e, where $|X|= \ell$ and $G_e = \{\text{all the bijections from } X \text{ onto } X\}$, hence is isomorphic to S_ℓ, $0 \leq \ell \leq n$. Let $Ve = \{v \cdot e \mid v \in V\}$ be the k-space with the basis $\{v_1 e, \dots, v_\ell e\} = \{v_1, \dots, v_\ell\}$, so that $\dim(Ve) = \ell$. For $g \in G_e$ and $v \cdot e \in Ve$, we define the action of G_e on Ve (group action)

$$(v \cdot e) \cdot g = v \cdot (eg). \tag{3.11}$$

Note that $v \cdot (eg) = v \cdot (ge) = (v \cdot g) \cdot e \in Ve$. It follows that Ve is a G_e-representation that is isomorphic to the permutation representation of S_ℓ. Note that, S_ℓ-representation is reducible when $\ell \geq 2$ and irreducible when $\ell = 1$, and if $e \in J_0$ is the zero map then $Ve = 0$ (see Example 3.1.10).

We conclude that the procedure above turns the irreducible partial permutation representation for I_n into a reducible permutation representation for S_ℓ where $2 \leq \ell \leq n$.

Let f be another idempotent in the same $\mathcal{J}$-class as e, f is the identity of G_f. Let a^* is the inverse of an element $a : Y \to X$ such that $aa^* = f$ and $a^*a = e$. Define an action of G_f on $Vf = \{v \cdot f \mid v \in V\}$ gives a G_f-representation, as follows. For $h \in G_f$ and $v \cdot f \in Vf$,

$$(v \cdot f) \cdot h = v \cdot (fh) = v \cdot (hf) = (v \cdot h) \cdot f \in Vf. \tag{3.12}$$

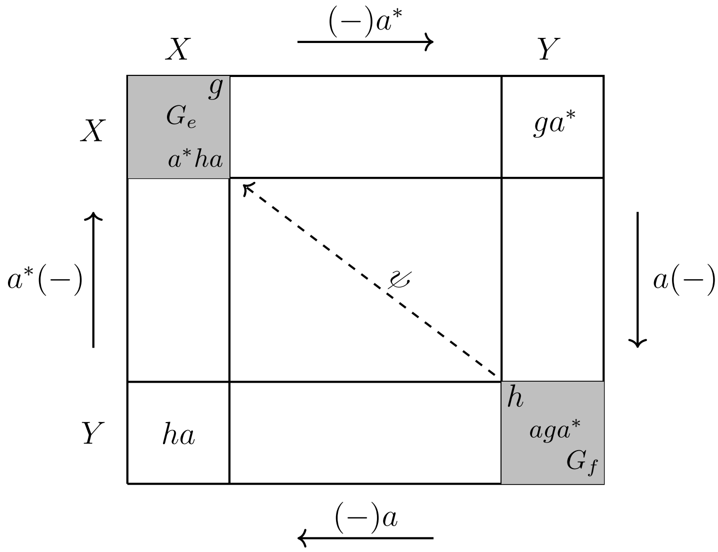

Figure 3.1: Applying Green's Lemmas in our case

We have G_f and G_e are isomorphic via $\psi : G_f \to G_e$, given by $h \mapsto a^*ha$, where its inverse $\psi^{-1} : G_e \to G_f$ is given by $g \mapsto aga^*$, for $g \in G_e$. So that G_f, S_Y, S_X and G_e are all isomorphic. Now, by applying Green's Lemmas in our case we obtain the diagram in Figure 3.1.

Also, Vf is isomorphic to Ve via: $v \cdot f \mapsto v \cdot (fa) = v \cdot (ae) = (v \cdot a)e \in Ve$. Moreover, the following commutative diagram shows that Ve does not depend on the choice of e inside the $\mathcal{J}$-class we choose:

$$\begin{tikzcd} Vf \arrow[r, "(-)h"] \arrow[d, "\cong"'] & Vf \arrow[d, "\cong"] \\ Ve \arrow[r, "(-)a^*ha"] & Ve \end{tikzcd}$$

where $(-)h$ and $(-)a^*ha$ denote the multiplications by h and a^*ha, respectively. This implies that the G_f-representation Vf is isomorphic to the G_e-representation Ve (for more detail see [14]).

In the remaining parts of this section, we present the induction procedure that can be more complicated when there is a possibility that the constructed representation is reducible. Then we apply these processes to I_n.

3.3.2 Induction Process

Let S be a finite regular monoid and k be a field. Fix $e \in J_e$ and consider R_e, the $\mathcal{R}$-class of e and the maximal subgroup G_e. Let V be an irreducible G_e-representation. Our aim is to extend the G_e representation to an S-representation. The steps of induction are as described below [1]:

- let $A = \{a_j : j \in J\}$ be a set of representatives from each of the $\mathcal{H}$-classes inside R_e, with e itself as the representative in G_e. Note that each $\mathcal{H}$-class has only one representative.

- for $a_j \in A$ let $V_j = \{v_j = v \otimes a_j : v \in V\} \cong V$ be a copy of V, a k-space with

$$v_j + v'_j = (v \otimes a_j) + (v' \otimes a_j) = (v + v') \otimes a_j$$

  and

$$\lambda v_j = \lambda(v \otimes a_j) = (\lambda v) \otimes a_j,$$

  where $v, v' \in V$ and $\lambda \in k$. Note that $\dim V_j = \dim V$.

- consider all the copies V_j and define the vector space

$$U = \bigoplus_{a_j \in A} V_j.$$

  Then any $u \in U$ can be written uniquely in the form $u = \sum_{j \in J} v_j$, where $v_j \in V_j$. Note that $\dim \bigoplus V_j = \sum_{j \in J} \dim V_j = |J| \cdot \dim V$.

- recall that by Green's Lemma [37] every $a \in R_e$ can be written uniquely as ga_j, for some $a_j \in A$ and a unique element $g \in G_e$ (see Chapter 2). Now, we define an S-action on U in the following way: for $b \in S$ define

$$(v \otimes a_j) \cdot b = \begin{cases} v \cdot g \otimes a_k \text{ if } a_j b \in R_e \text{ and } a_j b = g a_k, & \text{for some } a_k \in A \text{ and } g \in G_e; \\ 0, & a_j b \notin R_e. \end{cases} \tag{3.13}$$

This gives an S-representation U. Up to this point, we do not know whether the resulting S-representation is irreducible or reducible.

Example 3.3.2. [14] Let $S = I_n$ and J_1 be the $\mathcal{J}$-class consisting of all partial bijections between subsets of $[n]$ with size 1. Fix an idempotent $e : \{k\} \to \{k\} \in J_1$, where $k \in [n]$ so that G_e is a maximal subgroup of S and consider its $\mathcal{R}$-class R_e. Let V be the trivial representation of $G_e \cong S_1$. Notice that V is a one-dimensional irreducible representation with basis $\{v\}$ and $v \cdot e = v$. The set of representatives a_j as $A = \{a_j : \{k\} \to \{j\}; k, j \in [n]\}$. Observe that there are only one element in each $\mathcal{H}$-class in R_e and the domain $\{k\}$ for these representatives is fixed and the images are vary.

To do the induction process, we need a copy V_j of V that has basis

$$\{v_j = v \otimes a_j : v \in V, a_j \in A\}$$

and $V_j \cong V$. If we consider all the copies V_j, we have

$$\dim \bigoplus_{a_j \in A} V_j = n$$

For any $b \in I_n$, we have $a_j b \in R_e$ if and only if $\mathrm{dom}(a_j b) = \mathrm{dom}(e) = \{k\}$ if and only if $j \in \mathrm{dom}(b)$. Thus the partial map $a_j b$ can be described as

$$a_j b = \begin{cases} a_j b : \{k\} \to \{jb\} & \text{if } j \in \mathrm{dom}(b); \\ 0, & \text{otherwise.} \end{cases} \tag{3.14}$$

In this case, there is a unique element $e \in G_e$ such that $a_j b = a_{jb} = e a_{jb}$, with $a_j \in A$. Hence, the action of I_n on V_j can be written as

$$v_j \cdot b = (v \otimes a_j) \cdot b \begin{cases} (v \cdot e) \otimes a_{jb} = v \otimes a_{jb} & \text{if } j \in \mathrm{dom}(b); \\ 0, & \text{else.} \end{cases} \tag{3.15}$$

The vector space $\bigoplus_{a_j \in A} V_j$ carries the partial permutation representation for I_n and, in particular, it is irreducible by Example 3.1.10.

Now, if the induction procedure produces a reducible representation, how can we make it irreducible? The following paragraphs will explain and illustrate this situation.

In general, when we do the induction process, the previous steps is insufficient, and we need an extra step to ensure that we end up with an irreducible representation. Before starting this step, we list some required definitions and facts.

- If V is an S-representation and $U \subset V$ is an S-subrepresentation, then the *quotient space* V/U is an S-representation via the action: $(v+U) \cdot a = v \cdot a + U$.
- $U \subset V$ is a *maximal subrepresentation* of V if and only if when W is another subrepresentation of V such that $U \subset W \subset V$, we have that $W = U$ or $W = V$.
- The relation between maximality and quotient S-representations is: U is a maximal subrepresentation of V if and only if the quotient representation V/U is irreducible.
- Let S be a semigroup and V be an S-representation. Then the set $\mathrm{Ann}_S(V) = \{s \in S : Vs = 0\}$ is called the *annihilator* of V.

Now, back to induction process steps. If we define $\mathrm{Ann}(L_e)$ as

$$\mathrm{Ann}(L_e) = \{v \in \bigoplus_A V_j : v \cdot b = 0 \text{ for all } b \in L_e\},$$

then $\mathrm{Ann}(L_e)$ is the unique maximal subrepresentation of $\bigoplus_A V_j$. Hence, by using the above we end up with the irreducible S-representation:

$$V \uparrow S \stackrel{\text{def}}{:=} \bigoplus_A V_j/\mathrm{Ann}(L_e).$$

Observe that we commence the induction process with an irreducible G_e-representation V and by inducing it up we obtain the irreducible S-representation

$$V \uparrow S = \bigoplus_A V_j/\mathrm{Ann}(L_e).$$

The steps required for the induction process are completed.

Example 3.3.3. [14] Let S be the inverse monoid I_n and V be the trivial irreducible representation of G_e, where $e : X \xrightarrow{id} X$. We claim that $\mathrm{Ann}(L_e) = \{0\}$. Let $v \in \mathrm{Ann}(L_e)$, so that $v \in \bigoplus_{a_i \in A} V_i$ with $v \cdot b = 0$ for all $b \in L_e$ and we have $v = \sum_i v_i \otimes a_i$, for some $i \in I$. For any i and fixed j, if a_j^* is the inverse of $a_j \in R_e$ (with $a_j : X \longrightarrow Y$), then $a_i a_j^* \in R_e$ if and only if $\mathrm{dom}(a_i a_j^*) = X$ if and only if $\mathrm{im}(a_i) = \mathrm{dom}(a_j^*) = Y$. This happens if and only if $i = j$ which means that when $i \neq j$, we have $a_i a_j^* \notin R_e$. Now, we have $v \in \mathrm{Ann}(L_e)$ and $a_j^* \in L_e$, hence, $v \cdot a_j^* = 0$. So,

$$\begin{aligned}
0 &= v \cdot a_j^* \\
&= \Big(\sum_i v_i \otimes a_i\Big) \cdot a_j^* \\
&= \sum_i (v_i \otimes a_i) \cdot a_j^* \\
&= (v_j \otimes a_j) \cdot a_j^* \quad [\text{since } a_i = a_j \text{ and by formula (3.13)}] \\
&= (v_j \cdot e) \otimes e \qquad [\text{since } a_j a_j^* = e = ee \text{ and by formula (3.13)}] \\
&= v_j \otimes e. \qquad\qquad (3.16)
\end{aligned}$$

This means that the vector $v_j \otimes e$ is the zero vector, and we obtain this when $v_j = 0$ in the trivial G_e-representation V. It follows that the copy $v_j \otimes a_j$ in V_j is also zero. When we let j vary, we have $v = 0$. This completes the proof that $\mathrm{Ann}(L_e) = \{0\}$.

We deduce that when S is finite inverse monoid, then $\mathrm{Ann}(L_e) = \{0\}$.

Theorem 3.3.4. [1, Chapter 4] *Let S be finite inverse monoid and V be an irreducible G_e-representation. Then, $(V \uparrow S)$ is irreducible.*

The following section explains the concept of Clifford-Munn-Ponizovskii correspondence theory with respect to the above ideas.

3.4 Clifford-Munn-Ponizovskii Theory

The Clifford-Munn-Ponizovskii correspondence gives a bijection between the irreducible representations of a finite monoid and the irreducible representations of its maximal subgroups. In this section, we provide two expositions of this theory. The first one is given in an abstract approach by Steinberg in his book mentioned previously [102]. The second one is provided by Everitt in his paper [14] where he deals pragmatically with this theory by applying it to the symmetric inverse monoid I_n.

Prior to examining the main theory, we state the requirements for understanding it. Note that in the first exposition, the action of a semigroup will be on the left.

Definition 3.4.1. Let M be a module. A descending chain

$$M = M_1 \supset M_2 \supset \ldots \supset M_k \supset M_{k+1} = \varnothing$$

of submodules is called a *composition series* of M if all factor modules M_i/M_{i+1} are simple. These quotient modules M_i/M_{i+1} are called *composition factors* of M, for $i = 1, \ldots, k$.

Let S be a finite monoid and k be a field. In addition to the previous functors in Section (3.2) which are associated to $k[S]/k[I_{(e)}]$ and $k[G_e]$, we introduce new ones [102]. Fix an idempotent $e \in E(S)$ and define:

$$\mathrm{Ind}_{G_e} : k[G_e]\text{-mod} \longrightarrow k[S]\text{-mod} \tag{3.17}$$

$$\mathrm{Coind}_{G_e} : k[G_e]\text{-mod} \longrightarrow k[S]\text{-mod} \tag{3.18}$$

$$\mathrm{Res}_{G_e} : k[S]\text{-mod} \longrightarrow k[G_e]\text{-mod} \tag{3.19}$$

$$T_e : k[S]\text{-mod} \longrightarrow k[S]\text{-mod} \tag{3.20}$$

$$N_e : k[S]\text{-mod} \longrightarrow k[S]\text{-mod} \tag{3.21}$$

by putting

$$\begin{aligned}
\mathrm{Ind}_{G_e}(V) &= k[L_e] \otimes_{k[G_e]} V \\
\mathrm{Coind}_{G_e}(V) &= \mathrm{Hom}_{G_e}(R_e, V) \\
\mathrm{Res}_{G_e}(V) &= eV \\
T_e(V) &= SeV \\
N_e(V) &= \{v \in V \mid eSv = 0\}.
\end{aligned}$$

Observe that

- $\mathrm{Ind}_{G_e}(V) = V \uparrow k[S]$ (induction function) and
- $\mathrm{Res}_{G_e}(V) = eV = V \downarrow k[G_e]$, is called *restriction*(reduction) of V.

The Clifford-Munn-Ponizovskii Theory is the following:

Theorem 3.4.2. [102, Chapter 5] *Let S be a finite monoid and k be a field.*

1. *There is a bijection between isomorphism classes of simple $k[S]$-modules with apex $e \in E(S)$ and isomorphism classes of simple $k[G_e]$-modules given by*

$$W \mapsto \mathrm{Res}_{G_e}(W) = eW, \tag{3.22}$$
$$V \mapsto V^\sharp = \mathrm{Ind}_{G_e}(V)/N_e(\mathrm{Ind}_{G_e}(V)) \cong T_e(\mathrm{Coind}_{G_e}(V)), \tag{3.23}$$

for W a simple $k[S]$-module with apex e and V a simple $k[G_e]$-module.

2. *Every simple $k[S]$-module has an apex (unique up to $\mathcal{J}$-equivalence).*
3. *If V is a simple $k[G_e]$-module, then every composition factor of $\mathrm{Ind}_{G_e}(V)$ and $\mathrm{Coind}_{G_e}(V)$ has apex f with $SeS \subseteq SfS$. Moreover, $V^\sharp$ is the unique composition factor of the two modules W and V with apex e.*

The isomorphism in (3.23) is given by Corollary 4.12 of [102]. Consequently, there are two corollaries to this theorem. The first one states that there is a bijection between irreducible representations of S and irreducible representations of the maximal subgroups of S. The other gives an example of obtaining the irreducible representations of an $\mathcal{R}$-trivial monoid. Before proceeding further with the corollaries, we need to introduce this notion.

Let M be a monoid. If $m \in M$, then Mm, mM and MmM are the principal left, right and two-sided ideals, respectively, generated by m. A monoid M is called *$\mathcal{R}$-trivial* if $mM = nM$ implies $m = n$, that is, the $\mathcal{R}$-relation is equality. The notion of *$\mathcal{L}$-trivial* is defined dually. Moreover, a monoid is *$\mathcal{J}$-trivial* if $MmM = MnM$ implies that $m = n$.

Let $\mathrm{Irr}_k(S)$ be the set of isomorphism classes of simple $k[S]$-modules and $\mathrm{Irr}_k(G_{e_i})$ be the set of isomorphism classes of simple $k[G_e]$-modules.

Corollary 3.4.3. [102, Chapter 5] *Let S be a finite monoid and k be a field. Let $\{e_1, \ldots, e_s\}$ be a complete set of idempotent representatives of the regular $\mathcal{J}$-classes of idempotents of S. Then there is a bijection between $\mathrm{Irr}_k(S)$ and the disjoint union $\bigcup_{i=1}^{s} \mathrm{Irr}_k(G_{e_i})$.*

Corollary 3.4.4. [102, Chapter 5] *Let S be an $\mathcal{R}$-trivial monoid and k be a field. The simple $k[S]$-modules are in bijection with regular $\mathcal{J}$-classes of S. More precisely, for each regular $\mathcal{J}$-class J_e with $e \in E(S)$, there is a one-dimensional simple $k[S]$-module W_{J_e} with corresponding representation $\chi_{J_e} : S \longrightarrow k$ given by*

$$\chi_{J_e}(a) = \begin{cases} 1, & \text{if } J_e \subseteq SaS; \\ 0, & a \in I(e). \end{cases} \tag{3.24}$$

This example is a simple application of the Clifford-Munn-Ponizovskii correspondence to a particular class, the $\mathcal{R}$-trivial monoid.

The following subsection provides an implementation of the Clifford-Munn-Ponizovskii correspondence, which yields an alternative approach to the subject in a simple way [14]. We start with an explanation of the method using slightly different notations than the above and then apply it to the symmetric inverse monoid.

3.4.1 The proof of the Clifford-Munn-Ponizovskii correspondence in the case of the symmetric inverse monoid

Before we start applying the correspondence, we need to clarify the idea of the apex defined in Section (3.2). Let S be a finite semigroup. Then any irreducible S-representation V has a unique apex J_e, where e is idempotent. The reason for the need for this apex resides in the fact that when we reduce an irreducible S-representation V to be G_e-representation Ve, we have many G_e-representations associated with different $\mathcal{J}$-classes. Which one should we therefore use in the reduction to end with irreducible G_e-representation? This is the point of the apex; it navigates us to where exactly to send the V and which maximal subgroup to associate with it. Thus the transition from an irreducible S-representation to an irreducible G_e-representation is via reduction process with an apex J_e. Then the correspondence will run perfectly.

The apex J_e of V is a unique minimal $\mathcal{J}$-class that satisfies the following conditions [1]:

- its idempotent representative e ($:= e \in J_e$) determines a nonzero G_e-representation Ve;
- for all other $\mathcal{J}$-classes that are greater than the $\mathcal{J}$-class J_e (with respect to partial ordering $\leq_{\mathcal{J}}$), their idempotent representatives yield nonzero G-representations: In other words,

$$\text{for each } f \in J_f \text{ where } J_f > J_e; \text{ we have } Vf \neq 0;$$

- for all remaining $\mathcal{J}$-classes that are not greater than the $\mathcal{J}$-class J_e, their idempotent representatives annihilate V: In other words,

$$\text{for each } f \in J_f \text{ where } J_f \not> J_e; \text{ we have } Vf = 0.$$

Hence, an idempotent representative e belonging to the $\mathcal{J}$-class J_e can also identify the apex of V, and is unique up to $\mathcal{J}$-equivalence. The following theorem shows the important of the apexes when S is finite inverse monoid.

Theorem 3.4.5. *Let S be a finite inverse monoid. If V is a irreducible S-representation with apex J_e, then the G_e-representation Ve is irreducible.*

Let S be a finite regular monoid and e be an idempotent in the set $E(S)$. Let G_e be a maximal subgroup of S with idempotent e and T be the set of idempotent representatives for $\mathcal{J}$-classes of S. Let V be an irreducible S-representation and U be an irreducible G_e-representation. We need to recall some facts:

- for any idempotent $f \in T$, let Vf be a G_f-representation. In fact, if $f' \in T$ and f' is $\mathcal{J}$-related to f, then Vf is isomorphic to Vf'.
- define an apex of a representation in a different way: an apex is a unique minimal $\mathcal{J}$-class J_e of S, where e is idempotent, such that for any idempotent $f \in J_f$ the following rule holds:

$$Vf \neq 0 \text{ if and only if } \; J_e \leqslant J_f.$$

We now fix the field k. We have three main sets: the set of all irreducible representations of S over k is denoted by $\mathrm{Irr}(S)$; the set $\mathrm{Irr}_e(S) = \{V \in \mathrm{Irr}(S)$ such that V has an apex J_e with $e \in J_e\}$, and for each $e \in T$ we have the set $\mathrm{Irr}(G_e)$ of all irreducible G_e-representations over k.

In this correspondence, we have two mappings, namely

$$\alpha_1 : V \xmapsto{\text{reduction}} V \downarrow G_e \text{ and } \alpha_2 : U \xmapsto{\text{induction}} U \uparrow S.$$

In the first map, we take V to be an irreducible representation of S with apex J_e and reduce it to be an irreducible G_e-representation $Ve = V \downarrow G_e$. In the second map, we start with U, an irreducible representation of G_e, and induce it up to an irreducible representation of S. Thus, the result $U \uparrow S = \bigoplus_A U_j/\mathrm{Ann}(L_e)$ ends up in the set $\mathrm{Irr}_e(S)$ of irreducible S-representations with apex J_e.

The next aim is to show that, for a fixed e in T, both maps are inverses of each other, hence bijectives. We do this in two steps as follows:

1. $(U \uparrow S) \downarrow G_e \cong U$.
   Here, we start with U an irreducible representation of G_e and induce it to an irreducible S-representation and then reduce that to end up with an irreducible G_e-representation that is isomorphic to U. Therefore, this step gives the identity map on the set $\mathrm{Irr}(G_e)$.

2. $(V \downarrow G_e) \uparrow S \cong V$.
   In this step, we start with the irreducible S-representation V, having apex J_e, and reduce to an irreducible G_e-representation we then induce that to end up with an irreducible S-representation that is isomorphic to V. This map is the identity map on the set $\mathrm{Irr}_e(S)$.

Thus, the Clifford-Munn-Ponizovskii correspondence equips us with the following machine, for a fixed e:

$$\begin{tikzcd} \boxed{\mathrm{Irr}_e(S)} \arrow[rr, bend left, "\text{Reduction}"] & & \boxed{\mathrm{Irr}(G_e)} \arrow[ll, bend left, "\text{Induction}"] \end{tikzcd}$$

Figure 3.2: Clifford-Munn-Ponizovskii Correspondence Machine

This implies that

$$\mathrm{Irr}(S) = \bigcup_{e \in T} \mathrm{Irr}_e(S), \text{ a disjoint union.}$$

Then,

$$|\mathrm{Irr}(S)| = \sum_{e \in T} |\mathrm{Irr}_e(S)| = \sum_{e \in T} |\mathrm{Irr}(G_e)|,$$

where $T = \{e : e \text{ is an idempotent representative for each } \mathcal{J}\text{-class}\}$.

Now it is time to translate the method mathematically and prove it for $S = I_n$.

Example 3.4.6. Let $S = I_n$ and V be an irreducible S-representation Let U be an irreducible G_e-representation which is isomorphic to a representation of the symmetric group S_k, $0 \leq k \leq n$. Fix the $\mathcal{J}$-class J_k, let $X = \{1, \ldots, k\}$ and e be the identity map on X. Note that if a is an element of I_n, then $0 \leq |\mathrm{im}(a)| = |\mathrm{dom}(a)| \leq n$. Consider $Y = \{i_1, i_2, \ldots, i_k\}$ contains distinct elements of $[k]$ such that $i_1 < i_2 < \ldots < i_k$. Pick a very particular element $a_Y : (j)a_Y \mapsto i_j$, where $1 \leq j \leq k$:

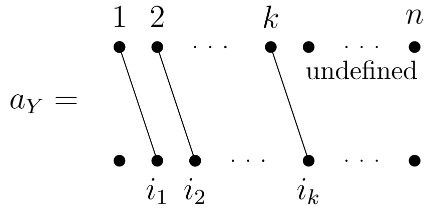

Figure 3.3: Induction Process: $a_Y \in I_n$

and let $A = \{a_Y : |Y| = k\}$.

1. Claim: $V \downarrow G_e = Ve$ is an irreducible G_e-representation when J_e is the apex of V. Fix $V \in \mathrm{Irr}_e(S)$ an irreducible S-representation with apex J_e and reduce it to Ve, an irreducible G_e-representation. The maximal subgroup G_e acts on Ve via:

$$\begin{aligned}(v \cdot e) \cdot a &= v \cdot \underbrace{(ea)}_{\in\, G_e} \\ &= v \cdot (ae) \\ &= \underbrace{\underbrace{(v \cdot a)}_{\in\, V} \cdot e}_{\in\, Ve}\end{aligned}$$

   for $(v \cdot e) \in Ve$ and $a \in G_e$. Since V is irreducible, Ve is an irreducible G_e-representation by [14, Exercise 3]. Thus Ve belongs to the set $\mathrm{Irr}(G_e)$ and this implies that $\alpha_1 : V \mapsto V \downarrow G_e$ is a map from the set $\mathrm{Irr}_e(S)$ to the set $\mathrm{Irr}(G_e)$.

2. Claim: $U \in \mathrm{Irr}(G_e)$ implies that $U \uparrow S$ is irreducible and has an apex J_e which means $U \uparrow S \in \mathrm{Irr}_e(S)$. The irreducibility follows from [14, Example 16]. To do the induction process, we have $U \uparrow S = \bigoplus_{a_Y \in A} U_Y$ where U_Y has basis $\{u \otimes a_Y : u \in U\}$ (recall that $\mathrm{Ann}(L_e) = 0$). Let $f : \{1, \ldots, \ell\} \xrightarrow{id} \{1, \ldots, \ell\}$ be another idempotent; we prove that $U \uparrow S$ has apex J_e in three steps, as follows:

   - $(U \uparrow S) \cdot f = 0$ when $\ell < k$, which means $J_f < J_e$. By considering $a_Y \in S$, compute $(u \otimes a_Y) \cdot f$: we have (see Figure 3.3 and 3.4)

$$\mathrm{dom}(a_Y f) \subseteq \mathrm{dom}(a_Y) = \{1, \ldots, k\}, \text{but } k \notin \mathrm{dom}(a_Y f).$$

     This means

$$\begin{aligned}&\mathrm{dom}(a_Y f) \subsetneq X = \{1, \ldots, k\} \\ &\Longrightarrow a_Y f \notin R_e\end{aligned}$$

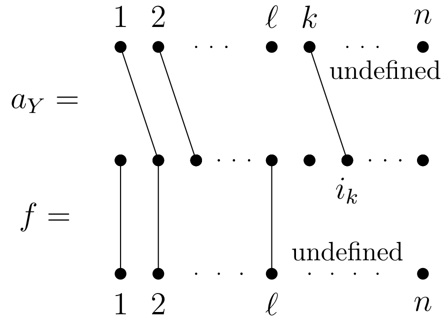

Figure 3.4: Computing $a_Y f$

$$
\begin{aligned}
&\Longrightarrow (u \otimes a_Y) \cdot f = 0 \\
&\Longrightarrow U_Y f = 0 \text{ for all } Y.
\end{aligned}
$$

Hence $(U \uparrow S) \cdot f = 0$ when $\ell < k$.

- $(U \uparrow S) \cdot e \neq 0$, which implies that J_e is the apex of $U \uparrow S$. Take an element $u \otimes a_Y$ from U_Y when $u \neq 0$. Then we compute the following:

$$
\begin{aligned}
(u \otimes a_Y) \cdot e &= u \cdot e \otimes a_Y \cdot e \ (a_Y e \in R_e \Rightarrow \operatorname{dom}(a_Y e) = X) \\
&= u \cdot e \otimes e \\
&= u \otimes e \neq 0 \\
&\Longrightarrow (U \uparrow S) \cdot e \neq 0.
\end{aligned}
$$

- $(U \uparrow S) \cdot f \neq 0$ when $k < \ell \leq n$. By contrapositive, we will show that if $(U \uparrow S) \cdot e \neq 0$, then $(U \uparrow S) \cdot f \neq 0$ when $e < f$. Suppose that $(U \uparrow S) \cdot f = 0$. We have $e < f$ if and only if $fe = e$. Then $(U \uparrow S) \cdot f = 0$ implies that $((U \uparrow S) \cdot f) \cdot e = 0$, hence $(U \uparrow S) \cdot (fe) = 0$. Thus $(U \uparrow S) \cdot e = 0$, and the result holds.

We conclude that $U \uparrow S$ has an apex J_e which implies that $U \uparrow S$ belongs to the set $\operatorname{Irr}_e(S)$ and hence α_2 is a map from the set $\operatorname{Irr}(G_e)$ to the set $\operatorname{Irr}_e(S)$.

3. For a fixed e, the two maps $\alpha_1 : V \mapsto V \downarrow G_e$ and $\alpha_2 : U \mapsto U \uparrow S$ are bijections. In this part, we need to prove the following:

   - $(U \uparrow S) \downarrow G_e \cong U$.
     For an element $b \in S$ and $g \in G_e$, the action of S on $U \uparrow S$ is

$$
(u \otimes a_Y) \cdot b = \begin{cases} u \cdot g \otimes a_k, & \text{where } a_Y b = g a_k \in R_e; \\ 0, & a_Y b \notin R_e. \end{cases} \tag{3.25}
$$

Since in the vector space $(U \uparrow S) \cdot e$ we have $(u \otimes a_Y) \cdot e = u \otimes e$ $(a_Y e \notin R_e$

if $Y \neq X$), then we define an isomorphism $(U \uparrow S) \cdot e \xrightarrow{\cong} U$ by $u \otimes e \mapsto u$ and the following diagram commutes:

$$\begin{tikzcd}
(U \uparrow S) \cdot e \ni u \otimes e \arrow[r, mapsto, "(-)g"] \arrow[d, mapsto] & (u \otimes e) \cdot g = u \cdot g \otimes e \arrow[d, mapsto] \\
U \ni u \arrow[r, mapsto, "(-)g"] & u \cdot g
\end{tikzcd}$$

We have that $(U \uparrow S) \downarrow G_e \cong U$ is an irreducible G_e-representation and the map

$$U \mapsto (U \uparrow S) \mapsto (U \uparrow S) \downarrow G_e \cong U$$

is the identity map on the set $\mathrm{Irr}(G_e)$.

- $(V \downarrow G_e) \uparrow S \cong V$.
  For V an irreducible S-representation with apex J_e, we need to reconstruct $(V \downarrow G_e) \uparrow S$ inside the vector space V. Consider the subspace $V \cdot (ea_Y)$ of the vector space V for the element $a_Y \in A$ and let a_Y^* be the inverse map of a_Y, where A is the set of representatives from each of the $\mathcal{H}$-class inside R_e. Then:
  (a) The two maps

$$V \cdot e \xrightarrow{a_Y} V \cdot (ea_Y) \text{ and } V \cdot (ea_Y) \xrightarrow{a_Y^*} V \cdot e$$

  are vector space maps with

$$v \cdot e \mapsto v \cdot (ea_Y) \mapsto v \cdot (ea_Y a_Y^*) = v \cdot e^2 = v \cdot e.$$

  These maps are mutually inverse and hence are isomorphisms. Each $V \cdot (ea_Y)$ is thus an isomorphic copy of $V \cdot e$.
  (b) Consider the sum $\sum_Y V \cdot (ea_Y)$, we prove the following claim: for $Z \neq Y$, where $|Z| = k = |Y|$, we have

$$V \cdot (ea_Y) \cap \sum_{Z \neq Y} V \cdot (ea_Z) = 0.$$

  To show this we need the following fact:
  Fact 3.4.7. Let S be any finite semigroup and V be an S-representation. Let f be an idempotent of S with $Vf = 0$. If $a \in S$ is a $\mathcal{J}$-related to f, then $Va = 0$.
  To prove this fact: since $a\mathcal{J}f$ if and only if there exists s, s' and t, t' in S^1 such that $a = sft$ and $f = s'at'$. Suppose that $Vf = 0$, then

$$Va = V(sft) = Vs(ft) \subseteq V(ft) = (Vf)t = 0t = 0.$$

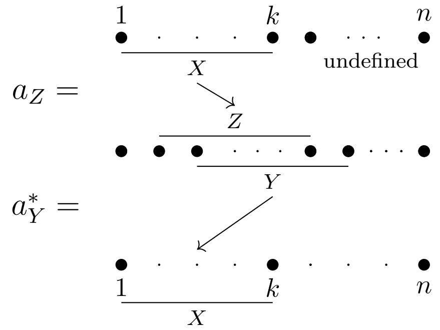

Figure 3.5: Domain $a_Z a_Y^*$

Hence, $Va = 0$. Now, consider the map:

$$V \cdot (ea_Y) \cap \sum_{Z \neq Y} V \cdot (ea_Z) \xrightarrow[\cong]{(-)a_Y^*} \Big(V \cdot (ea_Y) \cap \sum_{Z \neq Y} V \cdot (ea_Z)\Big) a_Y^*.$$

The vector space $\Big(V \cdot (ea_Y) \cap \sum_{Z \neq Y} V \cdot (ea_Z)\Big)$ is a subspace of $V \cdot (ea_Y)$. But

$$\begin{aligned}\Big(V \cdot (ea_Y) \cap \sum_{Z \neq Y} V \cdot (ea_Z)\Big) a_Y^* &\subseteq V \cdot (ea_Y a_Y^*) \cap \sum_{Z \neq Y} V \cdot (ea_Z a_Y^*)\\ &= V \cdot e \cap \sum_{Z \neq Y} V \cdot (ea_Z a_Y^*).\end{aligned}$$

For $Z \neq Y$ we have $\mathrm{dom}(a_Z a_Y^*) \subsetneq \mathrm{dom}(e) = \mathrm{dom}(a_Z) = X$ with $|X| = k$ and $\mathrm{im}(a_Z a_Y^*) \subsetneq \mathrm{im}(a_Y^*)$. Thus $ea_Z a_Y^*$ is located in a lower $\mathcal{J}$-class than e, (which is in the $\mathcal{J}$-class with size k). Next, let $f^2 = f$ be an idempotent in the $\mathcal{J}$-class of $ea_Z a_Y^*$. We have the following:

$$\begin{aligned}&\xRightarrow[\text{of } V]{J_e \text{ is an apex}} V \cdot f = 0\\ &\Longrightarrow V \cdot (ea_Z a_Y^*) = 0 \text{ for all } Z\\ &\Longrightarrow V \cdot e \cap \sum_{Z \neq Y} V \cdot (ea_Z a_Y^*) = 0\\ &\Longrightarrow V \cdot (ea_Y a_Y^*) \cap \sum_{Z \neq Y} V \cdot (ea_Z a_Y^*) = 0\\ &= \Big(V \cdot (ea_Y) \cap \sum_{Z \neq Y} V \cdot (ea_Z)\Big) \cdot a_Y^* = 0.\end{aligned}$$

Since the partial map a_Y^* is injective, we have

$$V \cdot (ea_Y) \cap \sum_{Z \neq Y} V \cdot (ea_Z) = 0.$$

This shows that

$$\sum_Z V \cdot (ea_Z) = \bigoplus_Z V \cdot (ea_Z) \subseteq V.$$

(c) Consider the vector space $\bigoplus_{a_Y \in A} V \cdot (ea_Y) \subset V$ and note that $V \cdot e \neq 0$ is one of them when $a_Y = e$. For $b \in S$, the S-action on $\bigoplus_{a_Y \in A} V \cdot (ea_Y)$ is defined as follows: first, the partial map $a_Y b$ is described as

$$a_Y b = \begin{cases} \in R_e \Rightarrow a_Y b = g a_Z, \text{ where } g \in G_e \text{ and } a_Z \in A; \\ \notin R_e \Rightarrow \mathrm{dom}(a_Y b) \subsetneq \mathrm{dom}(a_Y) = X \Rightarrow a_Y b \in J < J_e. \end{cases} \tag{3.26}$$

Second, for $v \in V$ the action of $b \in S$ on the vector $V \cdot (ea_Y)$ is

$$v \cdot (ea_Y) \cdot b = \begin{cases} v \cdot (ea_Y b) = v \cdot (ega_Z) \\ \qquad\qquad\quad = v \cdot (gea_Z) \\ \qquad\qquad\quad = (v \cdot g) \cdot (ea_Z) \in V \cdot (ea_Z) \text{ if } ea_Y b \in R_e; \\ 0, \text{ as } J_e \text{ is the apex of } V \text{ and if } ea_Y b \notin R_e. \end{cases} \tag{3.27}$$

After this, we deduce the following points:

i. The vector space $\bigoplus_{a_Y \in A} V \cdot (ea_Y) \cdot S$ is a subspace of $\bigoplus_{a_Y \in A} V \cdot (ea_Y)$ since it is left invariant under the action of S. This implies that $\bigoplus_{a_Y \in A} V \cdot (ea_Y)$ is subrepresentation of V.

ii. Since J_e is the apex of the vector space V, we have that $Ve \neq 0$ is an irreducible G_e-representation. Hence,

$$0 \neq Ve \subset \bigoplus_{a_Y \in A} V \cdot (ea_Y) \subset V \text{ (which means it is not equal to zero)}$$
$$\xRightarrow[\text{irreducible}]{V \text{ is}} V = \bigoplus_{a_Y \in A} V \cdot (ea_Y).$$

This shows that

$$V \cong \bigoplus_Z V \cdot (ea_Z),$$

as vector spaces.

iii. Finally, the following diagram is commutative

$$\begin{array}{ccc} v \cdot (ea_Y) & \xmapsto{(-)b} & (v \cdot g) \cdot (ea_Z) \text{ if } ea_Y b \in R_e \text{ or } 0, \text{else} \\ \cong \big\downarrow & & \big\downarrow \cong \\ (v \cdot e) \otimes a_Y & \xmapsto{(-)b} & v \cdot (eg) \otimes a_Z \text{ if } a_Y b = ga_Z \in R_e \text{ or } 0, \text{else} \end{array}$$

This shows that

$$V \cong \bigoplus_{Z} V \cdot (ea_Z),$$

as S-representations. Here, $Ve = V \downarrow G_e$ induces up to S by taking copies. It turns out that the vector space V is isomorphic to $\bigoplus_{a_Y \in A} V \cdot (ea_Y)$ and thus is isomorphic to $(V \downarrow G_e) \uparrow S$ as an irreducible S-representation. This implies that the map

$$V \mapsto (V \downarrow G_e) \mapsto (V \downarrow G_e) \uparrow S \cong V$$

is the identity map on the set $\mathrm{Irr}_e(S)$.

This completes the proof of the result for $S = I_n$.

To conclude [1], the irreducible representations of a finite regular monoid S are determined by the irreducible representations of its maximal subgroups. Hence, to construct all irreducible representations of S, it is sufficient to consider all irreducible representations of the representative maximal subgroups of each $\mathcal{J}$-class and apply the induction process to each individual one.

In the next chapters, we will discuss the Clifford-Munn-Ponizovskii correspondence from a historical perspective.

Chapter 4

Simple Semigroups and Clifford's Contributions

In order to answer the research questions posed in this thesis, we track back the development of semigroup representation theory. To gain insight into the topic, we investigate who was contributing to this area, what they were doing mathematically, and what was the nature of semigroup representation theory. We focus mainly on the work that pertains to the Clifford-Munn-Poniszovskii correspondence. We intend to see how this correspondence evolved over the timeline of the development of the theory.

The present chapter consists of four sections. The first two sections give an account of the necessary background on semigroups, concerning Rees matrix semigroups and their related terminologies; the reader is referred to [8, 15, 25–27, 37, 79] for more detail. Section 4.3 briefly outlines Clifford's biography [62, 86] and his work on representation theory of semigroups [6–8]. Clifford's own symbols, and semigroup actions, will be used throughout Section 4.3. The chapter ends with a discussion about Clifford's contribution in representation theory of semigroups.

4.1 Simple Semigroups

Let S be a semigroup. We say that a semigroup S without zero is *simple* if it has no proper ideals, or equivalently its only (two-sided) ideal is itself, that is, if $SaS = S$ for every a of S. A semigroup S with zero is called 0-*simple* if S is not null (i.e. $S^2 = \{st \mid s, t \in S\} \neq 0$) and S has only two ideals, namely: $\{0\}$ and S. Note that S^2 is always an ideal, so the condition $S^2 \neq 0$ is only required to eliminate the 2-element null semigroup, where every product equals zero. An example of a simple semigroup is a group since a group does not contain any proper ideals. Let G be a group and I be a right ideal, so $IG \subseteq I$. Let $g \in G, a \in I$ then we have

$$g = a(a^{-1}g) \in I$$

and so $G = I$. Thus G is simple.

In terms of the Green's relation $\mathcal{J}$ on S, we have S is simple if and only if $\mathcal{J} = S \times S$ (or if any two elements are $\mathcal{J}$-related). Similarly, if S has a zero, then $\{0\}$ and $S \setminus \{0\}$ are the only $\mathcal{J}$-classes of S (or, any two non-zero elements are $\mathcal{J}$-related). An ideal of a semigroup S is called *minimal* if it contains no other ideal of S. Clearly, every finite semigroup S has a minimal ideal. The minimal ideal of a finite semigroup S is a simple semigroup and is a regular $\mathcal{J}$-class. In the case where S has a zero, then an ideal is called a 0-*minimal ideal* if the only ideal of S contained in it is $\{0\}$ (and it is not equal to $\{0\}$). Equivalently with 0-simple, $S^2 \neq 0$ and S is a 0-minimal ideal of itself. According to Clifford and Preston [8], any 0-minimal ideal I of a finite semigroup S is either null (meaning $I^2 = 0$), or it is a 0-simple semigroup (so then $I \setminus \{0\}$ is a regular $\mathcal{J}$-class). We note that if a semigroup S contains a minimal ideal, then it is unique and called the *kernel* of S and if S has a zero, the kernel is equal to $\{0\}$.

With the above as background, we now introduce the term *principal factor* of a finite semigroup S. Let a be an element of a semigroup S. Suppose that the $\mathcal{J}$-class J_a of a is minimal among the $\mathcal{J}$-classes of S. Then $J(a) = J_a$ is the least ideal of S (the kernel of S). On the other hand, if J_a is not minimal in $S/\mathcal{J}$, then the set

$$I(a) = \{b \in J(a) : J_b < J_a\}$$

is an ideal of S such that $J(a) = I(a) \cup J_a$, and this union is disjoint. If B is a proper ideal of $J(a)$ and $I(a) \subseteq B$, then $I(a) = B$. This implies that $J(a)/I(a)$ is a 0-minimal ideal of $S/I(a)$, i.e. $J(a)/I(a)$ is either a null semigroup or it is a 0-simple semigroup. Each semigroup $J(a)/I(a)$ is called the *principal factor* of S.

We can think of the principal factor $J(a)/I(a)$ as consisting of the $\mathcal{J}$-class $J_a = J(a) \setminus I(a)$ with zero adjoined (if $I(a) \neq \varnothing$). The principal factor $J(a)/I(a)$ is null if and only if the product of any two elements of J_a always falls into a lower $\mathcal{J}$-class. In particular, if J_a is a subsemigroup of S, then the principal factor $J(a)/I(a)$ is not null. We say that the $\mathcal{J}$-class J_a is regular if and only if its principal factor is a 0-simple semigroup. Thus, J_a is non-regular if and only if its principal factor is a null semigroup. Note that a semigroup cannot be both 0-simple and null.

The following results summarize the above:

Theorem 4.1.1. [37, Theorem 3.1.6] *If a is an element of a semigroup S, then either:*

1. *the $\mathcal{J}$-class J_a is the kernel of S; or*
2. *the set $I(a) = \{b \in J(a) : J_b < J_a\}$ is non-empty and is an ideal of $J(a) = S^1aS^1$ such that $J(a)/I(a)$ is either 0-simple or null.*

Lemma 4.1.2. [8, Section 2.6] *Each principal factor of any semigroup S is* 0-*simple, simple, or null. If S has a kernel, then the only simple principal factor is the kernel.*

With respect to the natural partial order among the elements of $E(S)$, an idempotent $e \neq 0$ of a semigroup S with zero is *primitive* if it is minimal within the set of non-zero idempotents of S. Thus a primitive idempotent e has the property that for all non-zero idempotents f of S,

$$ef = fe = f \neq 0 \Rightarrow e = f.$$

Definition 4.1.3. [37, Section 3.2] A semigroup S is called *completely* (0)*-simple* if it is (0)-simple and contains a primitive idempotent.

Rees [91] showed that every finite 0-simple semigroup is completely 0-simple. A semigroup is called a *completely semisimple* if each of its principal factors is either (completely) 0-simple or (completely) simple. Thus completely 0-simple semigroups are the building blocks of finite semigroups.

4.2 Rees Matrix Semigroups

A *Rees matrix semigroup* is constructed as follows: let G be a group, let I, Λ be non-empty sets and let P be a $\Lambda \times I$ matrix over $G^0 = G \cup \{0\}$ such that no row or column of P consists entirely of zeros (in fact, P is a function $P : \Lambda \times I \to G^0$ with $P : (\lambda, i) \mapsto (p_{\lambda i})$ which we consider as a matrix $P = (p_{\lambda i})$), in which case, P is said to be *regular*. We define a multiplication on the set $(I \times G \times \Lambda) \cup \{0\}$ by letting 0 act as a zero

$$(i, g, \lambda)0 = 0(i, g, \lambda) = 00 = 0$$

and

$$(i, g, \lambda)(k, h, \mu) = \begin{cases} 0 & \text{if } \ p_{\lambda k} = 0, \\ (i, gp_{\lambda k}h, \mu) & \text{if } \ p_{\lambda k} \neq 0 \end{cases} \tag{4.1}$$

where $g, h \in G$; $i, k \in I$; $\lambda, \mu \in \Lambda$ (see Figure 4.1). This gives a semigroup called a *Rees matrix semigroup* over G^0 and denoted by $\mathcal{M}^0(G; I, \Lambda; P)$. The group G is

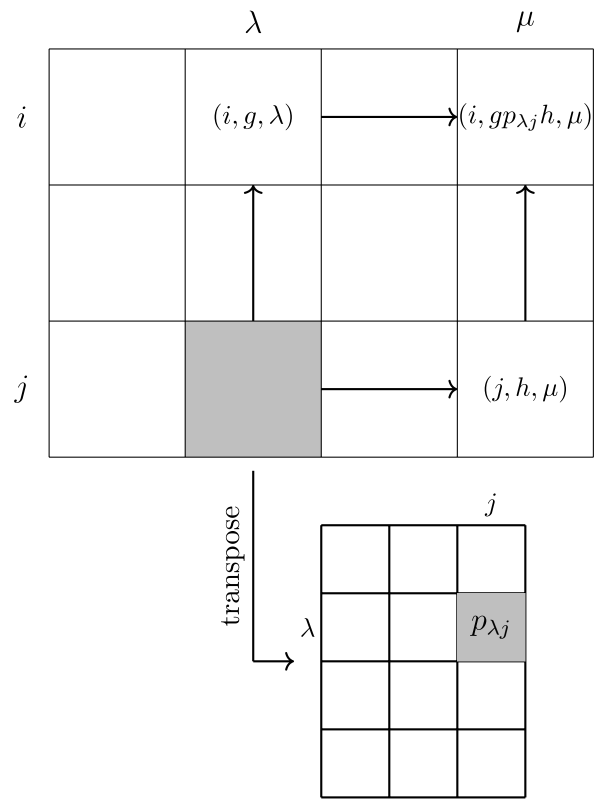

Figure 4.1: The multiplication in a Rees matrix semigroup

called the *structure group* of the semigroup, and the matrix P is called the *sandwich matrix*. The elements of $\mathcal{M}^0(G; I, \Lambda; P)$ are all triples (i, g, λ) with $g \neq 0$. A Rees matrix semigroup is *regular* if and only if the sandwich matrix P is regular.

Example 4.2.1. The product $(1, g, 2)(3, h, 4)$ is either $(1, gp_{23}h, 4)$ or 0, depending on the value of p_{23}. The shape of the transpose of P is the same as the shape of the grid, and the cells containing the multiplicands, the cell corresponding to p_{23}, and the cell containing the product (if it is not zero) form the corners of a rectangle (see Figure 4.2).

A Rees matrix semigroup has the following properties [37, Section 3.2]:

1. (i, g, λ) is idempotent $\Leftrightarrow p_{\lambda i} \neq 0$ and $g = p_{\lambda i}^{-1}$;
2. $\mathcal{M}^0$ is regular;
3. $(i, g, \lambda)\mathcal{R}(j, h, \mu) \Leftrightarrow i = j$;
4. $(i, g, \lambda)\mathcal{L}(j, h, \mu) \Leftrightarrow \lambda = \mu$;
5. $(i, g, \lambda)\mathcal{H}(j, h, \mu) \Leftrightarrow i = j$ and $\lambda = \mu$;
6. the $\mathcal{D} = \mathcal{J}$-classes are $\{0\}$ and $\mathcal{M}^0 \setminus \{0\}$ (so $\{0\}$ and $\mathcal{M}^0$ are the only ideals);
7. $\mathcal{M}^0$ is 0-simple.

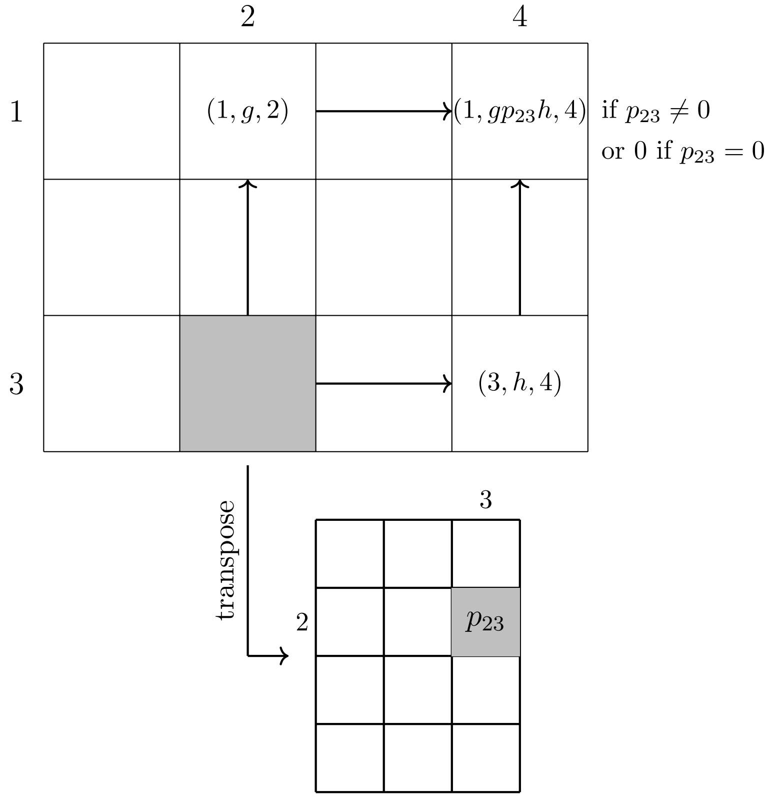

Figure 4.2: The product $(1, g, 2)(3, h, 4)$ in a Rees matrix semigroup

The following is the Rees Theorem, which plays an important part in the representation theory of semigroups.

Theorem 4.2.2. [37, Theorem 3.2.3] *A semigroup S is completely 0-simple if and only if S is isomorphic to a regular Rees matrix semigroup over a group with zero adjoined.*

We now describe the equivalent version of a Rees matrix semigroup. Let I, Λ and G, P be as above. The set $\mathcal{M}^0[G; I, \Lambda; P]$ denotes the set of all $I \times \Lambda$-matrices over G^0 with at most one nonzero entry. The symbol $(g)_{i\lambda}$ denotes the $I \times \Lambda$-matrix with $g \in G$ in the (i, λ)-position and zero elsewhere. Moreover, $(0_{i\lambda})$ denotes the $I \times \Lambda$ zero matrix; it is also denoted by $\mathbf{0}$. Let $P = (p_{\lambda i})$ be a fixed $\Lambda \times I$ matrix over G^0. This matrix is called a *defining* matrix since it is used to define a multiplication in the set of $I \times \Lambda$ matrices over G^0 as follows:

$$A \circ B = APB,$$

where A and B are $I \times \Lambda$ matrices over G^0. If $A = (a)_{i\lambda}$ and $B = (b)_{j\mu}$, then the multiplication is defined by

$$(a)_{i\lambda} \circ (b)_{j\mu} = (a_{i\lambda})P(b_{j\mu}) = (ap_{\lambda j}b)_{i\mu} \ (a, b \in G; i, j \in I; \lambda, \mu \in \Lambda).$$

Let $P = (p_{\lambda i})$ and $a, b \in G$; $i, j \in I$; $\lambda, \mu \in \Lambda$. Then

$$(a)_{i\lambda} \circ (b)_{j\mu} \circ (a)_{i\lambda} = (ap_{\lambda j}bp_{\mu i}a)_{i\lambda}$$

is equal to $(a)_{i\lambda}$ if and only if $p_{\lambda j}bp_{\mu i} = a^{-1}$. With the given element $(a)_{i\lambda}$, there exists such an element $(b)_{j\mu}$ in $\mathcal{M}^0[G; I, \Lambda; P]$ if and only if $p_{\lambda j} \neq 0$ and $p_{\mu i} \neq 0$, that is, if and only if the λ-th row and i-th column of P each contain a non-zero element in G^0. Hence, every matrix P in $\mathcal{M}^0[G; I, \Lambda; P]$ is regular if and only if each row and column of P contains a non-zero entry.

Lemma 4.2.3. [79, Lemma 3.3] *The mapping*

$$(i, a, \lambda) \to (a)_{i\lambda}, \ 0 \to 0$$

is an isomorphism of $\mathcal{M}^0(G; I, \Lambda; P)$ *onto* $\mathcal{M}^0[G; I, \Lambda; P]$.

Note that we will not distinguish between the two versions of Rees matrix semigroups later in the thesis.

The semigroup $\mathcal{M}(G; I, \Lambda; P)$ is called a *Rees matrix semigroup without zero* over the group G. This semigroup consists of all the nonzero elements of the semigroup $\mathcal{M}^0(G; I, \Lambda; P)$ constructed from the same data by deleting all the zeros in the above definition. Note that the matrix P has to be regular since there are no zeros. Note that a Rees matrix semigroup is finite if the group G is finite and the index sets I and Λ are finite.

For the purpose of the next chapter, we introduce the following notion based on Rees matrix semigroups. Let S be a Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ over a group G^0 with $\Lambda \times I$ sandwich matrix $P = (p_{\lambda i})$, where $\lambda \in \Lambda$, $i \in I$. Then the nonzero elements of S are the matrices $(g)_{i\lambda}$ having g in the (i, λ)- position and zero elsewhere with multiplication defined by

$$(g)_{i\lambda} \circ (h)_{j\mu} = (g)_{i\lambda}P(h)_{j\mu} = (gp_{\lambda j}h)_{i\mu},$$

where $g, h \in G$. Now let U be any algebra over a field k and P be a fixed $\Lambda \times I$ matrix over U. Let $\mathcal{M}[U; I, \Lambda; P]$ be a vector space of all $I \times \Lambda$ matrices over U. The matrices in $\mathcal{M}[U; I, \Lambda; P]$ have at most one nonzero entry. The product in $\mathcal{M}$ is defined by $A \circ B = APB$, where A and B are in $\mathcal{M}$. Therefore, $\mathcal{M}[U; I, \Lambda; P]$ is an algebra over k and it is called the *Munn* $I \times \Lambda$ *matrix algebra* over U with sandwich matrix P. The construction of a Munn matrix algebra is similar to the construction of a Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$ where a group with zero adjoined stands instead of an algebra U, and the condition on the sandwich matrix

P is that it contain a nonzero entry in each row and each column. The elements in both are matrices with at most one nonzero entry and their multiplication is the same; we multiply matrices of arbitrary size by ignoring the sums of an arbitrary number of zeros [8, Chapter 5, page 162].

4.3 Representations of Completely Simple Semigroups

4.3.1 Clifford's Biography

Two distinguished well-known algebraists share the same surname of "Clifford". The first one is William Kingdon Clifford (1845-1879) who was a group algebraist and the second is Alfred Hoblitzelle Clifford (1908-1992) who was mainly a semigroup algebraist. This section focuses on the latter.

Miller wrote of Clifford [62, Page 4]: "With a very few others, notably A. K. Suschkewitsch and D. Rees, he may be counted as a founder of [semigroup] theory". Clifford was also one of the founding editors of the journal *Semigroup Forum*, during the years 1972 to 1976. In addition, he was a former editor of the *Transactions of the American Mathematical Society.* He dedicated about forty-five years of his life to the semigroup research community as a scholar, teacher and colleague.

In 1933, he was awarded a doctorate from the California Institution of Technology. His thesis, conducted under the supervision of Eric Temple Bell, was entitled *Arithmetic of ova. Arithmetic and ideal theory of abstract multiplication* is the title that Clifford gave to a summary of the thesis that he published the Bulletin of the *American Mathematical Society.* His publication phase ran from 1933 to 1979. The majority of Clifford's published work was on algebra, some of it was in collaboration with other mathematicians. According to MathSciNet, forty-five of his fifty-three papers are on group theory and generalizations thereof.

According to different sources [58, 60, 65], matrix representations of completely simple semigroups are linked to Clifford's input to the theory. Nevertheless, it is important to point out that the prior work of Suschkewitsch in 1933 and 1935 played a catalytic role in that achievement. Clifford presented his contribution to representation theory of semigroups precisely in two papers and a joint book with Preston: *Matrix representations of completely simple semigroups* [6] in 1942, *Basic representations of completely simple semigroups* [7] in 1960, and the first volume of *The algebraic theory of semigroups* [8] in 1961. This book, considered as an advanced textbook, has had a strong influence on the development of semigroup theory. Prior to Clifford and Preston's 1961 book, there were only two books on the subject, and both of them were in Russian. These were Suschkewitsch's *Theory of generalized groups* in 1937 and Lyapin's *Semigroups* in 1960 [62].

4.3.2 Clifford's Representation Theory of Semigroups

In the introduction to his 1960 paper [7, Page 431], Clifford wrote of Suschkewitsch's 1933 paper: "the underlying ideas and methods of [6] should be attributed to Suschkewitsch['s 1933 paper]". This is confirmed by Preston who described Clifford's 1942 work as [86, Page 38]: "an extension and elaboration of that of Suschkewitsch". Thus we start this subsection with a brief outline of Suschkewitsch's 1933 work.

This 1933 paper entitled *Über die Matrizendarstellung der verallgemeinerte Gruppen*, contained Suschkewitsch's first work in the representation of generalized groups where he realised the importance of the matrices in representing semigroups in a concrete form. Suschkewitsch began the paper by providing some preliminary results on matrices which followed by studying the matrix representations of ordinary groups, and then utilizing these to obtain similar representations of a special type of semigroups called *Kerngruppen* ((finite) simple semigroup) [36, Section 3.3]. Suschkewitsch considered matrices with rank strictly less than their order to determine all representations of Kerngruppen by these matrices. We will discuss the work of Suschkewitsch briefly in more detail later in Chapter 6.

The following provides "a simpler characterisation of these representations due to Clifford appears in [6, 7]", says Hollings [36, Section 11.1, page 282].

As indicated by the two papers' titles, Clifford addressed representations of completely (0-)simple semigroups. He managed to provide a method for finding all irreducible representations of a 0-simple semigroup from those of its structure group. The following is a synthesis of Clifford's key results on semigroup representation theory.

Let S be a Rees matrix semigroup $\mathcal{M}^0[G; I, \Lambda; P]$. According to Clifford and Preston [8], it is possible for two Rees $(I \times \lambda)$ matrix semigroups over the same group G^0 to be isomorphic without the sandwich matrices being the same. This isomorphism allowed Clifford to make a useful normalization of the sandwich matrix P. Thus, P can be normalized in the sense that all the elements in a given row and in a given column are either 0 or $e(= 1)$, the identity element of the structure group G, and so that $p_{11} = e$ and the elements $p_{1i},\ p_{\lambda 1}$ are either 0 or 1. Consider the set

$G_{11} = \{(x)_{11}; x \in G\}$ which forms a maximal subgroup of S with identity $(e)_{11}$ and is isomorphic to G.

Let Γ^* be a matrix representation of the completely 0-simple semigroup S of degree m over a field Φ, as $\Gamma^* : S \longrightarrow M_m(\Phi),\ s \mapsto \Gamma^*(s)$, for $s \in S$, where $M_m(\Phi)$ denotes the multiplicative semigroup of $m \times m$ matrices with entries from Φ. By restricting the representation Γ^* to the group G_{11} we have:

$$\Gamma^* : (e)_{11} \longrightarrow \begin{pmatrix} I_n & 0 \\ 0 & 0 \end{pmatrix},$$

since the matrix $\Gamma^*(e)_{11}$ is an idempotent matrix and hence is diagonalizable with eigenvalues 0 or 1. As

$$(e)_{11} \cdot (x)_{11} = (x)_{11} = (x)_{11} \cdot (e)_{11},$$

we therefore get:

$$\Gamma^* : (x)_{11} \longrightarrow \begin{pmatrix} \Gamma(x) & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence, Γ^* induces a representation Γ of G_{11} given by $\Gamma : x \mapsto \Gamma(x)$ of degree n, say, as for $x, y \in G_{11}$ we have

$$\Gamma(x)\Gamma(y) = \Gamma(xy) \text{ and } \Gamma(e) = I_n,$$

for all $x, y \in G_{11}$. The representation Γ^* is called an *extension representation* to S of the representation Γ of G .

After a number of calculations using a series of results that hold in S, Clifford obtained an $n \times t$ matrix Q_λ and a $t \times n$ matrix R_i; where $t = m - n$, and put forward the following theorem:

Theorem 4.3.1. [6, Theorem 3.1] *Let Γ^* be given by*

$$\Gamma^* : (x)_{i\lambda} \longmapsto \begin{pmatrix} \Gamma(p_{1i}xp_{\lambda 1}) & \Gamma(p_{1i}x)Q_\lambda \\ R_i\Gamma(xp_{\lambda 1}) & R_i\Gamma(x)Q_\lambda \end{pmatrix} \text{where } Q_1 = 0 \text{ and } R_1 = 0. \tag{4.2}$$

Then formula (4.2) defines a representation of S if and only if the matrices Q_λ and R_i satisfy the following equations for all $i \neq 1$ and $\lambda \neq 1$:

$$Q_\lambda R_i = \Gamma(p_{\lambda i}) - \Gamma(p_{\lambda 1}p_{1i}). \tag{4.3}$$

Conversely, every representation of S is equivalent to one of this form.

For $\lambda = 1$ and $i = 1$, we have

$$\Gamma^*(e)_{i1} = \begin{pmatrix} \Gamma(p_{1i}) & 0 \\ R_i & 0 \end{pmatrix} \text{ and } \Gamma^*(e)_{1\lambda} = \begin{pmatrix} \Gamma(p_{\lambda 1}) & Q_\lambda \\ 0 & 0 \end{pmatrix}.$$

This fact shows the relation between the matrices Q_λ and R_i, on one side, and the sandwich matrix P on the other. Clifford credits the creation of his theory, in particular the first part of Theorem 4.3.1, to Suschkewitsch's 1933 paper [8, Section 5.4].

From Theorem 4.3.1, we observe the following.

- The representation Γ^* is determined by $\Gamma, R_1, \ldots, R_i$ and $Q_1, \ldots, Q_\lambda$. The matrices $R's$ and $Q's$ must satisfy equation (4.3).
- Conversely, given a representation Γ of G and matrices $R_1, \ldots, R_i, Q_1, \ldots, Q_\lambda$ satisfying equation (4.3), then Γ^* given by formula (4.2) is a representation of S.
- Moreover, every S-representation Γ^* has this form up to equivalence.

Thus Clifford's terminology is firstly the representation Γ^* restricts to Γ; this is always possible. Secondly, the representation Γ extends to Γ^*; here we need to find the matrices $R_1, \ldots, R_i, \ldots, Q_1, \ldots, Q_\lambda, \ldots$ satisfying (4.3), then we get Γ^* by formula (4.2).

Now let:

$$H_{\lambda i} = \Gamma(p_{\lambda i}) - \Gamma(p_{\lambda 1} p_{1i}), \tag{4.4}$$

and define H to be the matrix of matrices

$$H = \begin{pmatrix} & \vdots & \\ \ldots & H_{\lambda i} & \ldots \\ & \vdots & \end{pmatrix}, \tag{4.5}$$

for $\lambda \neq 1, i \neq 1$. Then we need matrices Q and R, with t columns and t rows, respectively, such that

$$H = QR \tag{4.6}$$

where

$$\begin{pmatrix} \vdots \\ Q_\lambda \\ \vdots \end{pmatrix} \text{ and } \begin{pmatrix} \ldots & R_i & \ldots \end{pmatrix},$$

for $\lambda \neq 1, i \neq 1$ if and only if rank $H \leq t$. Using Theorem 1.1 of [6], we have finally:

Theorem 4.3.2. [6, Theorem 3.2] *A given representation* Γ *of* G *of degree* n *has an extension* Γ^* *to* S *of finite degree if and only if the rank* h *of the matrix* H *defined previously by (4.4) and (4.5) is finite. The defining matrices* Q_λ *and* R_i *of any extension* Γ^* *of* Γ *of degree* $n+t$ *are obtained from a factorization (4.6) of* H*. This is possible if and only if* $h \leq t$*, and so every extension of* Γ *has degree at least* $n+h$*.*

The above is an analysis of Clifford's work which appeared in [6, Section 3].

Furthermore, Clifford discussed the equivalence and reducibility of extension representations.

Theorem 4.3.3. [6, Theorem 4.1] *Let* S *be a completely* 0*-simple semigroup. Let* Γ^* *and* $\Gamma^{'*}$ *be two extension representations to* S *of representations* Γ *and* Γ' *of* G*, respectively. Let* Γ^* *be defined by (4.2), and* $\Gamma^{'*}$ *is defined analogously:*

$$\Gamma^{'*} : (x)_{i\lambda} \longmapsto \begin{pmatrix} \Gamma'(p_{1i}xp_{\lambda 1}) & \Gamma'(p_{1i}x)Q'_\lambda \\ R'_i\Gamma'(xp_{\lambda 1}) & R'_i\Gamma'(x)Q'_\lambda \end{pmatrix}.$$

Then, Γ^*_1 *and* $\Gamma^{'*}$ *are equivalent if and only if there exist invertible matrices* C_1 *and* C_2 *such that the following hold:*

1. $\Gamma'(x) = C_1\Gamma(x)C_1^{-1}$ *for all* $x \in G$*;*
2. $Q'_\lambda = C_1Q_\lambda C_2^{-1}$ *for all* $\lambda \neq 1$*;*
3. $R'_i = C_2R_iC_1^{-1}$ *for all* $i \neq 1$*.*

Theorem 4.3.4. [6, Theorem 7.1] *If a representation of* G *is irreducible, then its extension representation to* S *is also irreducible.*

On the other hand, if the representation Γ of G decomposes into Γ_1 and Γ_2, say, then the extension representation Γ^* to S decomposes into the extensions Γ_1^* and Γ_2^* of Γ_1 and Γ_2, respectively. Clifford then applied his theory to Brandt groupoids, named here for the first time (in the 1942 paper), and found all their representations (for a discussion about Brandt groupoids, see [8, Section 3.3]).

The above results derive from Clifford's 1942 paper. In the 1960 paper, Clifford completed the picture of his representation theory. He introduced and discussed the theory of a *basic representation* of a completely 0-simple semigroup. Let Γ be a representation of G; this representation can be extended, and among its extensions, there is one with least possible degree over a field Φ and it is uniquely determined by Γ to within equivalence. This extension is called the *basic extension of* Γ, denoted by Γ_0^*. Additionally, any representation of S which is the basic extension to S of some representation of G is called a *basic representation* of S. It turns out that any other extension of a representation Γ of G decomposes into the basic representation Γ_0^* and null representations.

Clifford proved the converse of Theorem 4.3.4 via the next theorem.

Theorem 4.3.5. [8, Theorem 5.51] *Let S be a completely simple semigroup. Let Γ be a representation of G and Γ^* be any extension representation of Γ to S. If Γ^* is an irreducible representation of S, then Γ is an irreducible representation of G. Moreover, all irreducible representations of S over a field Φ are obtained as the basic extensions to S of the irreducible representations of the basic group G.*

This theorem shows the relationship between the basic representations of a completely (0-)simple semigroup S and the representations of its basic group G. Furthermore, the following corollary states the correspondence between the basic representations of S and representations of G.

Corollary 4.3.6. [8, Corollary 5.47] *Let Γ and Γ' be representations of G. Let Γ_0^* and $\Gamma_0'^*$ be their respective basic extensions to S. Then Γ_0^* and $\Gamma_0'^*$ are equivalent if and only if Γ and Γ' are equivalent.*

We end this section with two results about complete reducibility and semisimplicity, from Clifford's perspective.

Theorem 4.3.7. [8, Theorem 5.52] *Complete reducibility holds for representations of S over a field Φ if and only if:*

1. *complete reducibility holds for representations of its basic group G over Φ, and*

2. *the only extension representation to S of a representation of G is the basic extension.*

Corollary 4.3.8. [8, Corollary 5.53] *Let S be a finite semigroup, and assume that the characteristic of Φ does not divide the order of G. Then the algebra of S is semisimple if and only if the only representation of S extending any given representation of G is its basic extension.*

4.4 Discussion

One might ask why Clifford was only interested in representations of completely 0-simple semigroups. The possible and simple answer is that completely 0-simple semigroups are built from groups, hence these were semigroups that could be approached via groups. Thus Clifford was able to construct representations of completely 0-simple semigroup in terms of those of its associated structure group. Also, at the time when Suschkewitsch started using matrices in the semigroup context

Clifford was working with Weyl on group representations [36, Chapter 11, page 277].

The explanations so far indicate that representations of completely 0-simple semigroups were the starting point for the representation theory of semigroups. The previous section shows that Suschkewitsch and Clifford provided a good picture of the very earliest work in this direction. In his first work on the subject in 1942 [6], Clifford built on Suschkewitsch's 1933 work. He was able to find a method of constructing all representations of a Rees matrix semigroup via representations of its maximal subgroup. Then, he established a one-to-one correspondence between representations of a Rees matrix semigroup and representations of its structure group.

The first class of semigroups whose representations were understood were the completely 0-simple ones. They were first considered by Suschkewitsch in 1933 but their representation theory was completely worked out by Clifford in 1942 and then 1960. Although Clifford wrote two papers on the subject, separated by two decades, he settled for just the completely 0-simple case without further progress towards a complete description for an arbitrary semigroup. What we now understand to be the Clifford-Munn-Poniszovskii correspondence is therefore not found in Clifford's work. Clifford's work was picked up later by Munn and Poniszovskii and this thesis will consider their contributions to the topic. Also, we will revisit the work of Suschkewitsch and Clifford mentioned here later in the discussion of Chapter 6.

Since most of the contributions on the subject were made by Munn, the main goal of the next chapter is to fit Munn's work in this area into a broader picture. Our focus will be on what Munn envisioned and wanted to achieve in semigroup representation theory. Our strategic overview of the "flow" of ideas through his work is to see where his mathematics was going and how it led him to the Clifford-Munn-Poniszovskii correspondence. As mentioned previously, we want to see how this correspondence evolved over the course of Munn's work.

To achieve our objectives, we need to contextualize Munn's work. For instance, we need to assess what Munn knew at the start of his work. We have already discussed Clifford's work. But we have to understand the studies of representations of semigroups (if any) that were being carried out during Munn's time and determine whether he was aware of such studies.

Chapter 5

Munn's Contributions to Semigroup Representation Theory

In this chapter, our basic consideration will be devoted to the work of Munn in the representation theory of semigroups. We recall that the Clifford-Munn-Ponizovskii correspondence states that there is a one-to-one correspondence between the equivalence classes of the irreducible representations of a finite regular monoid and the equivalence classes of the irreducible representations of its maximal subgroups. As an introduction to Munn's contribution to this significant result, we commence with the paper: *On Semigroup Algebras* [65]. Then we move chronologically to the following papers, *Matrix Representations of Semigroups* [66], *Characters of the Symmetric Inverse Semigroup* [67], *Irreducible Matrix Representations of Semigroups* [68], *A Class of Irreducible Matrix Representations of an Arbitrary Inverse Semigroup* [69], and finally, *Matrix Representations of Inverse Semigroups* [70]. In our own words, with comments and observations, we provide a summary account of Munn's six papers, each in an individual section. However, we will use Munn's own notation, symbols, and semigroup action, unless otherwise indicated. The chapter begins with Munn's biography and finishes with a discussion and synopsis.

5.1 Munn's Biography

Howie's introduction of Munn [38, Page 2] is a nice start to the biography: "Douglas Munn is arguably the most influential semigroup theorist of his generation". Walter Douglas Munn was born on April 25, 1929 and passed away on October 26, 2008. In 1951, he graduated from the University of Glasgow with an undergraduate degree in Mathematics and Natural Philosophy. He then moved to Cambridge for postgraduate study and his Ph.D. journey started. After reading Clifford's work on algebra, Munn found his calling and made a decision on his area. Together with Clifford, Munn was inspired by Rees and Green [36]. In 1955, he was awarded a doctorate by the University of Cambridge for his thesis *Semigroups and their algebras*, under the supervision of Derek Taunt.

Munn's research in the 1950's until the mid 1960's was on semigroup representation theory. Since then until the mid 1970's, he was interested in the theory of inverse and regular semigroups. From the 1980's onwards, Munn worked on semigroup rings and returned to his initial interest in semigroup algebras. Munn's publication period was from 1955 to 2008, comprising around eighty-one papers, which made contributions in different directions. If we could classify his papers, it would be into three main topics: representations of semigroups, inverse semigroups, and semigroup algebras [15]. It is noticeable from his papers that he had numerous collaborations with Michael Crabb (we will discuss the significance of this point in Section 5.9). Munn had four Ph.D. students: Norman Reilly, John Hickey, P. Mclean, Pedro Silva [92].

Munn held several academic positions at the University of Glasgow and the University of Stirling. Specifically, in 1966, he established a department of mathematics at the new University of Stirling. During Munn's tenure there, John Howie and Gordon Preston were around and that made the department full of activities in semigroup theory. Munn's work at Government Communications Headquarters in Cheltenham, 90 miles west of London, showed his computing skills (details on this part of his life are few) [36]. Apart from Munn's scientific life, he was talented in music and the mathematical community enjoyed his piano performance during musical evenings at conferences [92].

5.2 On Semigroup Algebras

The paper [65] was published in 1955 which was the same year that Munn submitted his Ph.D. thesis [64]. It has two main objectives which are, firstly, finding necessary and sufficient conditions for the algebra of a semigroup S to be semisimple, and as a result, the matrix representations of S and its algebra are completely reducible and secondly, obtaining the complete set of irreducible representations of a semigroup S from the irreducible representations of groups associated with S.

The following paragraphs will provide the key points of this paper. Munn worked over a field F, with suitable characteristic, and dealt with finite semigroups. Most of the concepts are based on the existence of a descending series:

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \varnothing. \tag{5.1}$$

This is a chain of subsemigroups $S_i, i = 1, \ldots, n$, of a semigroup S and such that S_{i+1} is an ideal of S_i. This series is defined to be a *composition series* or a *prin-*

cipal series depending on the Rees factor semigroups S_i/S_{i+1}; $i = 1, \ldots n$, which are called the *factors* of the series. If each factor S_i/S_{i+1} is simple semigroup (or irreducible, in other words), then the series is a composition series. In addition to that condition, if each S_i is an ideal of S, then the series is principal, and in this case the semigroup is said to be *semisimple*. The factors of a principal series are called *principal factors* of a semigroup S. As an observation, a principal factor is either simple or 0-simple. Note that the principal factor of a semigroup S described in Section 4.1 and the principal factor of the principal series 5.1 are isomorphic in some order, see [8, Section 2.6]

Munn described the notion of a semigroup algebra in a manner entirely analogous to that of a group algebra. Let S be a finite semigroup and F be a field; a vector space with basis S together with an associative multiplication over F forms the semigroup algebra $U(S)$. The elements are formal sums $\sum\limits_i \lambda_i s_i$, where s_i are elements of S and λ_i are coefficients in F. The multiplication is defined by the rule (as seen in Section 3.1):

$$\Big(\sum_i \lambda_i s_i\Big)\Big(\sum_j \mu_j s_j\Big) = \sum_{i,j} \lambda_i \mu_j s_i s_j.$$

When S has a zero z, Munn preferred to work with the *contracted algebra*, which is the quotient algebra $U(S)/U(z)$, where $U(z) = \{\lambda z : \lambda \in F\} \cong F$. In this case, $U(S)$ is semisimple if and only if the quotient algebra $U(S)/U(z)$ is semisimple. In addition, if I is an ideal of S, $U(S)/U(I)$ is isomorphic to the contracted algebra of the quotient S/I.

One of the basic results about semisimplicity that he proved is:

Theorem 5.2.1. [65, Lemma 3.3] *If $U(S)$ is the algebra of a semigroup S, then $U(S)$ is semisimple if and only if the algebra of each of the principal factors of S is semisimple.*

Now, we turn to a new matrix algebra denoted by $M_{mn}[U, P]$, where U is an algebra over F with an identity element e; we call it a *Munn matrix algebra.* Munn defined this algebra as follows: it is the algebra consisting of $m \times n$ matrices with entries from U; P is any fixed $n \times m$ matrix over U and the multiplication is defined by the rule, for A and $B \in M_{mn}$: we define $A \circ B = APB$. Munn denoted the $n \times n$ unit matrix over U by U_n and wrote $M_n(U)$ for the algebra $M_{nn}[U, U_n]$ of $n \times n$ matrices over U. The matrix P is termed *non-singular* if there exists an $m \times n$ matrix Q over U such that either $PQ = I_n$ or $QP = I_m$.

Recall from Chapter 4 the equivalent notion of Rees regular matrix semigroup $S_{mn}[G, P]$. This is a semigroup that consists of $m \times n$ matrices $(x)_{ij}$ with just one nonzero entry $x \in G^0 = G \cup \{0\}$ in the ij-th position and zeros elsewhere. The Rees matrix semigroup $S_{mn}[G, P]$ is called *regular* because P has at least one element in each row and column, which is not zero. In other words, the matrix P is regular in the case that for each $i \in I$ there exists $\lambda \in \Lambda$ such that $p_{\lambda i} \neq 0$, and for each $\lambda \in \Lambda$ there exists $i \in I$ such that $p_{\lambda i} \neq 0$. See Chapter 4.

We point out here the result of Rees that every finite completely 0-simple semigroup is isomorphic to the semigroup $S_{mn}[G, P]$ over a group G with zero [91]. Note that this result is due to Rees in the precise form given here, but the result was proved by Suschkewitsch in 1928 in slightly different terms. Munn then indicated that the contracted algebra of $S_{mn}[G, P]$ is isomorphic to $M_{mn}[U, P]$, where $U = U(G)$ is the algebra of the structure group G of the Rees matrix semigroup.

We sketch the proof as follows. Any element in $S = S_{mn}[G, P]$ is written as an $m \times n$ matrix $(x)_{ij}$. Let α be an arbitrary element of the algebra $U(S)$; it can be written as an F-linear sum of elements of S: $\alpha = \sum\limits_{k} \lambda_k \, (x_k)_{i_k j_k}$. Thus, if we compute this linear combination of matrices, we get an $m \times n$ matrix, say A_α, with entries from the algebra $U(G)$. In other words, the result of the sum of scaler multiple of matrices is one matrix with entries are linear combinations of elements of G, which is one matrix with entries from $U(G)$. Now we can define a map: $U(S) \longrightarrow M_{mn}[U(G), P]$ by $\alpha \mapsto A_\alpha$. This map will give us the desired isomorphism $U(S) \cong M_{mn}[U(G), P]$.

Furthermore, Munn discussed the effect of the non-singularity of the fixed matrix P on the semisimplicity of the algebra $M_{mn}[U, P]$. For instance, he demonstrated the following:

Theorem 5.2.2. [65, Theorem 4.7] *The algebra $M_{mn}[U, P]$ is semisimple if and only if U is semisimple and P is non-singular.*

By the previous theorem, Munn reduced the problem of determining the semisimplicity of the algebra of an arbitrary semigroup to the problem of determining the semisimplicity of the contracted algebra of a simple semigroup $S_{mn}[G, P]$, which is isomorphic to $M_{mn}[U, P]$. Different tests for the non-singularity of matrices over any algebra U were provided, such as the following result:

Lemma 5.2.3. [65, Lemma 5.3] *Let $A = \begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix} \in M_n(U)$, where $A_{11} \in M_r(U), 1 \leqslant r < n$. Then A is non-singular if and only if both A_{11} and A_{22} are non-singular.*

Moreover, via a number of theorems, Munn illustrated the importance of semisimplicity and non-singularity in the representation theory of algebras. For example:

Theorem 5.2.4. [65, Theorem 6.1] *Let $S = S_{mn}[G, P]$, and let F be a field of characteristic zero or a prime not dividing the order of G. Let G^* be any subgroup of G containing all the non-zero entries of P, and let $U^* = U(G^*)$ be the algebra of G^* over F. Let $\{\Gamma_i^*; i = 1, \ldots, k^*\}$ be a complete set of inequivalent irreducible representations of G^* over F. Then the algebra of S over F is semisimple if and only if each of the matrices $\Gamma_i^*(P)$ is non-singular.*

Note that if $\Gamma_i^*(P)$ is non-singular then P must be square. As a corollary of the previous theorem, Munn proved that if the algebra $U(S)$ of a simple semigroup without zero is semisimple, then the semigroup S is in fact a group. The term non-singular had been extended as follows: a simple semigroup S will be termed *c-non-singular* if it is isomorphic to a Rees matrix semigroup of the form $S_{nn}[G, P]$, where P is a non-singular square matrix over the group algebra $U(G)$ over any field of characteristic c. Munn used this notion in Theorem 6.4 of [65]:

Theorem 5.2.5. [65, Theorem 6.4] *Let U(S) be the algebra of a semigroup S over a field of characteristic c. Then U(S) is semisimple if and only if*

1. *each principal factor of S is a c-non-singular simple semigroup, and*

2. *the characteristic is zero or does not divide the orders of any of the basic groups of the principal factors.*

As a corollary to this result, he proved that if the algebra $U(S)$ is semisimple, then the kernel of S is a group (the kernel being the minimal ideal of a semigroup).

Clifford demonstrated that if a semigroup S is completely 0-simple then every representation of S can be obtained from representations of its maximal subgroups [6]. Accordingly, Munn gave an overview of how Clifford constructed all representations of a completely 0-simple semigroup S as extensions of those of its maximal subgroup.

We summarize Clifford's construction procedure in a few steps as follows:

1. Let $S = S_{mn}[G, P]$ and $p_{11} = e$ be the identity of G. Consider the set $G_{11} = \{(x)_{11}; x \in G\}$ which forms a maximal subgroup of S with identity $(e)_{11}$ and is isomorphic to G^0.

2. Let $\Gamma' : S \longrightarrow M_k(F)$ be a matrix representation of the completely 0-simple semigroup S. By restricting the representation Γ' to the group G_{11} we have:

$$\Gamma' : (x)_{11} \mapsto \begin{pmatrix} \Gamma(x) & 0 \\ 0 & 0 \end{pmatrix}.$$

   Hence, Γ' induces a representation Γ of G_{11} given by $\Gamma : x \mapsto \Gamma(x)$.

3. After a number of calculations, Clifford obtained matrices Q_λ ($1 \le \lambda \le n$) and R_i ($1 \le i \le m$) where $Q_1 = 0$ and $R_1 = 0$ such that

$$\Gamma' : (x)_{i\lambda} \mapsto \begin{pmatrix} \Gamma(p_{1i}xp_{\lambda 1}) & \Gamma(p_{1i}x)Q_\lambda \\ R_i\Gamma(xp_{\lambda 1}) & R_i\Gamma(x)Q_\lambda \end{pmatrix} \tag{5.2}$$

defines a representation of S if and only if the matrices Q_λ and R_i satisfy the following:

$$Q_\lambda R_i = \Gamma(p_{\lambda i}) - \Gamma(p_{\lambda 1}p_{1i}).$$

Conversely, every representation of S is equivalent to one of this form. For more details about Clifford's theory, the reader refer to Subsection 4.3.2. Employing that, Munn extended the results to a representation of the algebra $M_{mn}[U, P]$ and imposed its semisimplicity in order to find the complete set of irreducible representations of a semigroup $S = S_{mn}[G, P]$. Theorem 8.7 of [65] states:

Theorem 5.2.6. [65, Theorem 8.7] *Let $S = S_{mn}[G, P]$ and $\{\Gamma_i : i = 1, \dots, k\}$ be a complete set of inequivalent irreducible representations of G over F whose characteristic is zero or a prime not dividing the order of G. Let the contracted algebra $U(S)$ be semisimple. Then $\{\Gamma_i' : i = 1, \dots, k\}$ is a complete set of inequivalent irreducible representations of S over F, where Γ_i' is the basic extension of Γ_i.*

At the end of this paper, Munn discussed the notion of a semigroup that admits relative inverses and then he applied some of his results on this type of semigroup. A semigroup S (not necessarily finite) is said to *admit relative inverses* if for any a in S there exist elements e and a' in S such that $ea = a = ae$ and $a'a = e = aa'$. Today, such semigoups are called *completely regular* (to a wider work on completely regular semigroup, see [5]). Among the applications is the following theorem:

Theorem 5.2.7. [65, Theorem 9.5] *Let S be a finite semigroup which admits relative inverses. Let F have characteristic zero or a prime not dividing the orders of any of the basic groups of the principal factors of S. Then $U(S)$ is semisimple if and only if all the idempotents of S commute.*

As a consequence, if F is a field with characteristic zero or a prime not dividing the orders of any of related groups of a semigroup S, then the representations of S over F are completely reducible. In the previous theorem, as a semigroup S is regular and the idempotents commute, therefore S is inverse semigroup.

On Semigroup Algebras was the first work of Munn on the development of representation theory of semigroups. Whenever a semigroup algebra is semisimple, there is an ideal series such that the quotients (principal factors) are matrix algebras over the group algebra. Munn related his work on semisimplicity to Clifford's representation theory for a completely (0-)simple semigroup and showed that semigroups with semisimple algebras form a special class. Further elaboration on this topic will be provided in the next section.

5.3 Matrix Representations of Semigroups

In this section, we intend to highlight the main points of Munn's paper entitled: *Matrix Representations of Semigroups* [66]. Munn continued his work on semigroup representation theory making use of his results in the previous paper [65]. The goal of the current paper is to construct all the inequivalent irreducible representations of a certain semigroup S, whose algebras are semisimple, in terms of those of the basic groups of the principal factors of S. The semigroups in question include finite Rees matrix semigroups $S_{mn}[G, P]$ with square non-singular matrix P, and also inverse semigroups. All the approaches in this area build upon Rees's result of characterizing simple or 0-simple semigroups up to isomorphism [91].

Munn commenced with preliminaries and recalled some basic notions. He assumed that all the semigroups throughout the paper [66] are finite and any algebra U is over a field F with a specific characteristic. Let $(x)_{ij}$ denote the $m \times n$ matrix over a group G^0 with entry $x \in G$ in the ij-th position and zero elsewhere, and let P be any fixed $n \times m$ matrix over G^0. Define a multiplication by the rule:

$$(x)_{ij} \circ (y)_{kl} = (x)_{ij} P (y)_{kl} = (x p_{jk} y)_{il}, \text{where } x, y \in G.$$

Then the elements $(x)_{ij}$ form the well-known Rees matrix semigroup $S_{mn}[G, P]$. Any simple or 0-simple finite semigroup is isomorphic to some $S_{mn}[G, P]$. By theorem 4.2.2, a semigroup S is completely 0-simple if and only if S is isomorphic to a regular Rees matrix semigroup over a group with zero adjoined. Although, here we are using Munn's notation not Howie's notation mentioned from Section 4.2.

Let U be any algebra over a field F, and let A and B be $m \times n$ matrices with entries from U; P is any fixed $n \times m$ matrix over U. Then the multiplication $A \circ B = APB$ gives the Munn matrix algebra $M_{mn}[U, P]$ over U. Recall our Theorem 5.2.2, originally from the previous paper [65], that the Munn matrix algebra $M_{mn}[U, P]$ is semisimple if and only if U is semisimple and the defining matrix P is non-singular.

The link between Rees matrix semigroups $S_{mn}[G, P]$ and Munn matrix algebras $M_{mn}[U, P]$ is that the contracted algebra of $S_{mn}[G, P]$ over a field F is isomorphic to $M_{mn}[U, P]$, where $U = U(G)$ [65]. As usual, Munn was concerned with the contracted algebra and pointed out that there is a one-to-one correspondence between the representations of the algebra of a semigroup S and those of its contracted algebra. According to Maschke's theorem, $U(G)$ is semisimple if and only if the characteristic of F does not divide the order of G. If the matrix P is non-singular over the algebra $U(G)$, then the Rees matrix semigroup $S_{mn}[G,P]$ is called a non-singular simple semigroup; in fact the non-singularity of P depends only on the characteristic of F.

Before Munn discussed the method of obtaining the complete set of irreducible representations of a non-singular simple semigroup $S_{mn}[G,P]$, he defined the notation whereby if A is a matrix over the algebra U, and if θ is a matrix representation of U, then $\theta(A)$ is the block matrix whose (i,j)th entry is $\theta(a_{ij})$. He then stated and proved the following theorem:

Theorem 5.3.1. [66, Theorem 2.1] *Let $S = S_{nn}[G,P]$. Let $\{\gamma_r : r = 1,\dots,k\}$ be a complete set of inequivalent irreducible representations of the semisimple algebra U over F, and let P be a non-singular $n\times n$ matrix over U. Define the mapping γ_r' on $M_{nn}[U,P]$ by the rule $\gamma_r'(X) = \gamma_r(XP)$. Then $\{\gamma_r' : r = 1,\dots,k\}$ is a complete set of inequivalent irreducible representations of $M_{nn}[U,P]$ over F.*

Here, since P is non-singular, the mapping $X \to XP$ is an isomorphism from $M_{nn}[U,P]$ to $M_n(U)$, the algebra of $n\times n$ matrices over U. In the case where the semigroup $S_{mn}[G,P]$ is non-singular, we can use the following corollary to obtain the complete set of inequivalent irreducible representations of the contracted algebra of $S_{mn}[G,P]$:

Corollary 5.3.2. [66, Corollary 2.2] *Let $S = S_{nn}[G,P]$ and let $\{\gamma_r : r = 1,\dots,k\}$ be a complete set of inequivalent irreducible representations of G over a field F whose characteristic is zero or a prime not dividing the order of G. Let P be a non-singular $n\times n$ matrix over $U(G)$. Define the mapping γ_r' on $S_{nn}[G,P]$ by the rule*

$$\gamma_r'\{(x)_{ij}\} = \gamma_r\{(x)_{ij}P\} = \sum_{s=1}^{n} \gamma_r\{(xp_{js})_{is}\}$$

for $(x)_{ij} \in S$. Then $\{\gamma_r' : r = 1,\dots,k\}$ is a complete set of inequivalent irreducible representations of S over F.

To make the representation in the above corollary clear, we point out that the matrix representation $\gamma_r\{(x)_{ij}P\}$ is a block matrix with one non-zero row of blocks (rather than a row of entries) and the other rows are blocks of zero matrices. Each block in this row (precisely the ith row) is dedicated to the matrix representations $\gamma_r(xp_{js})_{is}$, where $s = 1,\dots,n$. The matrix $(x)_{ij}P$ is:

$$\begin{array}{cc} s & \begin{array}{ccccc} 1 & 2 & 3 & \cdots\cdots & n \end{array} \\ \begin{array}{c} \\ \\ i \\ \\ \\ \end{array} & \begin{pmatrix} 0 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ xp_{j1} & xp_{j2} & xp_{j3} & \cdots\cdots & xp_{jn} \\ 0 & \cdots & \cdots & \cdots & 0 \\ 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \end{array},$$

where $xp_{js} \in G$. Hence, the matrix representation $\gamma_r\{(x)_{ij}P\}$ is:

$$\begin{array}{cc} s & \begin{array}{cccccc} 1 & 2 & 3 & \cdots\cdots & n & \end{array} \\ \begin{array}{c} \\ i \\ \\ \end{array} & \begin{pmatrix} & & 0 & & & \\ \gamma_r(xp_{j1}) & \gamma_r(xp_{j1}) & \cdots\cdots & \gamma_r(xp_{js}) & \cdots\cdots & \\ & & 0 & & & \end{pmatrix} \end{array}.$$

Each $\gamma_r\{(xp_{js})_{is}\}$ is a block matrix whose (i, s)-th block is $\gamma_r(xp_{js})$ and looks like:

$$\begin{array}{cc} \begin{array}{ccc} & s & \end{array} & \\ \begin{pmatrix} & \vdots & \\ & \vdots & \\ \cdots\cdots\cdots & \gamma_r(xp_{js}) & \cdots \\ & \vdots & \\ & \vdots & \end{pmatrix} & \begin{array}{c} \\ \\ i. \\ \\ \\ \end{array} \end{array}$$

Munn then extended the problem of obtaining the representations of the non-singular semigroup to finding the representations of an arbitrary semigroup S whose algebra $U(S)$ is semisimple over a field F of characteristic zero or a prime not dividing the order of any of the basic groups of the principal factors of S. This implies in particular that S is also a semisimple semigroup. He started with a complete set of inequivalent irreducible representations of the principal factors of the algebra $U(S)$ and ended up with a complete set of inequivalent irreducible representations of S over F. The method is provided in the following theorem.

Theorem 5.3.3. [66, Theorem 3.1] *Let S be a semisimple semigroup whose algebra $U(S)$, over a field F of characteristic zero or a prime not dividing the order of any of the basic groups of the principal factors of S, is semisimple. Let*

$$U(S) = U(S_1) \supset U(S_2) \supset \cdots \supset U(S_n) \supset U(S_{n+1}) = U(\varnothing)$$

be the series of ideals of $U(S)$ corresponding to the principal series

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \varnothing$$

of S, and let e_i be the identity element of the algebra $U(S_i)/U(S_{i+1})$ $(i = 1, \ldots, n)$. Let $\{\gamma'_{ir} : r = 1, \ldots, k_i\}$ be a complete set of inequivalent irreducible representations of $U(S_i)/U(S_{i+1})$ over F. Define the mapping γ^*_{ir} on S by the rule

$$\gamma^*_{ir}(x) = \gamma'_{ir}(x^\theta e_i),$$

where θ is the natural homomorphism of S onto S/S_{i+1}. Then

$$\{\gamma^*_{ir}; i = 1, \ldots, n; r = 1, \ldots, k_i\}$$

is a complete set of inequivalent irreducible representations of S over F.

Now we turn our attention to an important type of semigroup: inverse semigroups. Recall that an inverse semigroup is a semigroup in which each element has precisely one inverse in the sense that for each element $a \in S$ there is a unique element $a' \in S$ such that $a = aa'a$ and $a' = a'aa'$. It has the equivalent property that it is regular and its idempotents commute. Before Munn proceeded with his discussion about the representations of inverse semigroups, he stated and proved a number of results. For example:

Lemma 5.3.4. [66, Lemma 4.2] *A (finite) simple inverse semigroup is isomorphic to a Rees matrix semigroup $S_{nn}[G, U_n]$, where U_n is the $n \times n$ unit matrix over G.*

Munn introduced the notion of *Brandt semigroup* as a completely 0-simple inverse semigroup with zero. By Lemma 5.3.4, these are precisely the completely 0-simple inverse semigroups. Thus a Brandt semigroup is a semigroup isomorphic to a Rees matrix semigroup $B = \mathcal{M}^0(G; I, I; \Delta)$ over a group G, where Δ is the $I \times I$ identity matrix over G. The group G may be referred to as the structure group of $\mathcal{M}^0$. Thus, B is a completely 0-simple inverse semigroup. The *rank* of a Brandt semigroup is defined to be the cardinal of its set of non-zero idempotents $E \setminus \{0\}$. Thus, the rank of B is $|I|$. In particular, B is a group with zero ($\cong G^0$) if and only if $\text{rank}(B) = 1$. Moreover, when I is finite, with $|I| = k$ say, Munn replaced $\mathcal{M}^0(G; I, I; \Delta)$ by $\mathcal{M}^0(G; k, k; \Delta_k)$, denoting by Δ_k the $k \times k$ identity matrix over G^0. The elements of $\mathcal{M}^0(G; k, k; \Delta_k)$ are denoted by $(a; i, j)$, where $a \in G^0, 1 \leqslant i, j \leqslant k$, and the non-zero idempotents are of the form $(u; \ell, \ell)$, where u is the identity element of G and $1 \leqslant \ell \leqslant k$.

Each principal factor of an inverse semigroup is also an inverse semigroup, and in particular is simple, and this implies that S is a semisimple semigroup. As a consequence we have:

Theorem 5.3.5. [66, Theorem 4.4] *Let S be an inverse semigroup and F be a field of characteristic zero or a prime not dividing the order of any of the basic groups of the principal factors of S. Then U(S) is semisimple.*

The representations of inverse semigroups were derived directly from the results of Theorem 5.3.1 and Corollary 5.3.2 mentioned above. The following theorem illustrates an application to the inverse semigroup $S_{nn}[G, U_n]$.

Theorem 5.3.6. [66, Theorem 4.5] *Let $S = S_{nn}[G, U_n]$, and let $\{\gamma_r : r = 1, \ldots, k\}$ be a complete set of inequivalent irreducible representations of G over F whose characteristic is zero or a prime not dividing the order of G. Define the mapping γ'_r on S by the rule*

$$\gamma'_r\{(x)_{ij}\} = \gamma_r\{(x)_{ij}\}$$

for $(x)_{ij} \in S$. Then $\{\gamma'_r : r = 1, \ldots, k\}$ is a complete set of inequivalent irreducible representations of S over F.

Notice that $\gamma_r\{(x)_{ij}\}$, where $x \in G$, is a block matrix whose $(i, j)th$ block is the representation matrix $\gamma_r(x)$, and the remaining blocks are zero matrices.

Once Munn was able to construct all the irreducible representations of a simple inverse semigroup $S_{nn}[G, U_n]$, he was able to deal with the case of a general arbitrary inverse semigroup. Munn stated the method in the following theorem and provided its proof:

Theorem 5.3.7. [66, Theorem 4.7] *Let S be an inverse semigroup. Let*

$$S = S_1 \supset S_2 \supset \cdots \supset S_n \supset S_{n+1} = \varnothing$$

be any principal series for S, and let $\{e_{ij} : j = 1, \ldots, m_i\}$ be the set of non-zero idempotents of S_i/S_{i+1} $(i = 1, \ldots, n)$. (We take $m_n = 1$ and e_{n1} to be the identity of the group S_n.) Let F be a field of characteristic zero or a prime not dividing the order of any of the basic groups of any of the principal factors S_i/S_{i+1}, and let $\{\gamma'_{ir} : r = 1, \ldots, k_i\}$ be a complete set of inequivalent irreducible representations of S_i/S_{i+1} over F. Define the mapping γ^*_{ir} on S by the rule

$$\gamma^*_{ir}(x) = \sum_{j=1}^{m_i} \gamma'_{ir}(x^\theta e_{ij}),$$

where θ is the natural homomorphism of S onto S/S_{i+1}. Then

$$\{\gamma^*_{ir}; i = 1, \ldots, n; r = 1, \ldots, k_i\}$$

is a complete set of inequivalent irreducible representations of S over F.

Theorem 5.3.7 starts with the irreducible representations of the principal factors S_i/S_{i+1} of an inverse semigroup S. Each such representation then gives a representation of S of same dimension of that we start with. This is in contrast to induction in the Clifford-Munn-Ponizovskii correspondence from Chapter 3, where we start instead with irreducible representations of maximal subgroups and the induced representations of S then have dimensions some multiple of the dimensions of that we start with.

In the final part of the paper [66], Munn gave some results on the identity element of the algebra of an inverse semigroup S over our restricted field F. By Theorem 5.3.5, the algebra of S is semisimple and hence it contains an identity. Let E denote the semigroup of idempotents of S. Since S is an inverse semigroup, E is a semilattice of groups of order one and again by Theorem 5.3.5, $U(E)$ is semisimple (for more discussion about semilattice, see [8, Section 1.8]). Let e and f be two elements in E, we say that e *covers* f when $e > f$ and there exists no $g \in E$ such that $e > g > f$. Now let u be the identity of $U(S)$. Then the result is that we can express u as a linear combination of the elements of E as follows:

$$u = \sum_{e \in E} (1 - n_e)e,$$

where n_e is the number of elements of E covering e [66, Section 4, page 11].

The moral of the second paper is that Munn determined semigroup representations of completely 0-simple semigroups by the rules of Theorem 5.3.1 and Corollary 5.3.2. Although these rules apparently just link and connect representations of a finite semigroup with representations of its related groups and because we know previously the Clifford-Munn-Ponizovskii correspondence from Chapter 3, they might be the starting point of the idea of the correspondence which is the core of the theory of semigroup representations. We will discuss this point in Section 5.9.

5.4 Characters of the Symmetric Inverse Semigroup

In April 1956, Munn completed his paper: *Characters of the Symmetric Inverse Semigroup* [67]. The title clearly states the purpose of the paper. Munn provided a concrete method for constructing all the characters of the irreducible representations of the symmetric inverse semigroup in terms of the characters of the irreducible representations of the symmetric group over a field F with characteristic zero. An explicit application was given to the case $n = 4$. This section aims at summarizing the underlying results related to this special inverse semigroup. In addition, we reformulate Munn's work from our perspective, in a separate subsection.

Throughout this section, we work with a finite set $X = \{1, \ldots, n\}$ with cardinality n. Munn denoted the symmetric inverse semigroup (or symmetric inverse monoid) by $A^{(n)}$, but the standard notation is I_n. The order of $A^{(n)}$ can be computed by the formula $\sum_{r=0}^{n} \binom{n}{r}^2 r!$. In terms of principal series for $A^{(n)}$, each principal factor $A_r^{(n)} - A_{r-1}^{(n)}$ is a 0-simple inverse semigroup, so that the principal factors are Brandt semigroups, and hence are isomorphic to Rees matrix semigroups over groups. In fact, they are of the form $S_{\binom{n}{r}\binom{n}{r}}[G_r, U_{\binom{n}{r}}](r = 1, \ldots, n)$, where G_r is the symmetric group of size r and $U_{\binom{n}{r}}$ is the $\binom{n}{r} \times \binom{n}{r}$ unit matrix over G_r. The idempotents of $A^{(n)}$ are the identity mappings of subsets of X of cardinal r onto themselves. For any two subsets A and B of X of the same cardinality, the element x of $A^{(n)}$ is written as

$$x = \begin{pmatrix} a_1, \ldots, a_r \\ b_1, \ldots, b_r \end{pmatrix}, \text{where } a_i \in A \text{ and } b_i \in B;\ (i = 1, \ldots, r).$$

The inverse of x is the mapping:

$$x^{-1} = \begin{pmatrix} b_1, \ldots, b_r \\ a_1, \ldots, a_r \end{pmatrix}.$$

The cardinal of the subsets A and B is called the *rank* of x.

The product of two partial transformations x and x' is defined as follows:

1. Suppose x maps A onto B and x' maps A' onto B'; all the sets are subsets of X.
2. Then $x \cdot x'$ maps $(A' \cap B)x^{-1} \subseteq A$ onto $(A' \cap B)x' \subseteq B'$.
3. If $A' \cap B$ is the empty set, then the product is equal to the zero map.
4. The rank of this product is

$$\operatorname{rank}(x \cdot x') \leqslant \min(\operatorname{rank} x, \operatorname{rank} x').$$

Now, how might we decompose a partial transformation x on X? We have two cases with respect to subsets A and B of X:

1. When $A = B$, then x is a permutation.

2. When $A \neq B$, pick any element a_1 in A but not in $A \cap B$ ($a_1 \in A - (A \cap B)$) and let $x : a_1 \mapsto b_1$. We have now two cases:

   (a) if $b_1 \in B - (A \cap B)$, then we cannot proceed any further;

   (b) if $b_1 \in A \cap B$, then $b_1 = a_2$, say, and we continue in a similar way with $b_2, \ldots, b_s$ until we have a sequence of elements of X: $a_1, \ldots, b_s$, starting with an element of $a_1 = A - (A \cap B)$ and ending with an element of $a_s = B - (A \cap B)$.

By finiteness, this procedure must terminate. The sequence is called a *link* and is written in square brackets $[a_1, \ldots, b_s]$, to be distinct from the round brackets of a cyclic permutation. The number of elements of $A \cap B$ permuted by x is called the *subrank* of x. Analogously to group permutations, these elements can be decomposed into disjoint cycles. The partition of the disjoint cycles is called the *cycle pattern* of x. In the decomposition of a partial transformation we cannot omit cycles of length one.

If C is a subset of X and x is a partial transformation on X, so the restriction of x to C maps C to itself. Then we say that x *induces* a permutation of C.

To make the above notions clear, if

$$x = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 8 & 9 & 4 & 5 & 3 & 6 & 7 & - & - \end{pmatrix},$$

then x can be written as $x = [18][29](345)(6)(7)$. The rank of x is 7, the subrank is 5, and the cycle pattern is (31^2). If $C = \{3, 4, 5, 6, 7\}$, then x induces the permutation $(345)(6)(7)$ of C with cycle pattern (31^2). Similarly, if $C = \{3, 4, 5, 6\}$, then x induces the permutation $(345)(6)$ of C with cycle pattern (31). It turns out that any partial transformation of a finite set can be expressed as a conjunction of links and disjoint cycles (note that the order is not important here).

The rest of this section is dedicated to the characters of the symmetric inverse semigroup $A^{(n)}$.

Generally, the character of a matrix representation of any algebraic structure over a field F is the trace of the representation matrix; this is the sum of the diagonal elements of that matrix. Munn utilized the results of paper [66], to discuss the matrix representations and the characters of $A^{(n)}$. He started with the semisimplicity of $A^{(n)}$ and its ideals and gave the following theorem:

Theorem 5.4.1. [67, Theorem 3.1] *The algebra of $A^{(n)}$ over a field of characteristic* *zero or a prime greater than n is semisimple.*

Moreover, the contracted algebra of any principal factor of $A^{(n)}$ is semisimple.

To proceed, we need to set up some notations before we reach the main results of the paper [67]. Munn obtained the irreducible representations of $A^{(n)}$ in terms of those of its principal factors. Let $A_r^{(n)}/A_{r-1}^{(n)}$ be the principal factors of $A^{(n)}$, where $A_r^{(n)} = \{\sigma \in A^{(n)} : \text{ rank } \sigma \leq r\}$, for $r = 1, \ldots, n$. Let $\{\gamma'_{rs} : s = 1, \ldots, h_r\}$ be a complete set of irreducible representations of the contracted algebra of $A_r^{(n)}/A_{r-1}^{(n)}$ over a field of characteristic zero. He defined the mapping γ^*_{rs} on $A^{(n)}$ by the rule that for all $x \in A^{(n)}$:

$$\gamma^*_{rs}(x) = \sum_{i=1}^{\binom{n}{r}} \gamma'_{rs}\{(x.e_{ri})^{\theta}\}, \tag{5.3}$$

where θ is the natural homomorphism of $A^{(n)}$ onto $A_r^{(n)}/A_{r-1}^{(n)}$ and $\{e_{ri} : i = 1, \ldots, \binom{n}{r}\}$ is the set of all idempotents of $A^{(n)}$ of rank r. Then by our Theorem 5.3.7, originally from [66], we deduce that $\{\gamma^*_{rs} : s = 1, \ldots, h_r; r = 0, \ldots, n\}$ is a complete set of inequivalent irreducible representations of $A^{(n)}$ over F.

Next, Munn showed the natural one-one correspondence between the representations of the contracted algebra of $A_r^{(n)}/A_{r-1}^{(n)}$ and those of the group G_r. To make this correspondence clear, we have the following steps:

1. Consider the J-class with size r $(0 \leqslant r \leqslant n)$ with corresponding principal factor $A_r^{(n)}/A_{r-1}^{(n)}$.

2. Let $M_1, \ldots, M_{\binom{n}{r}}$ be the subsets of $X = \{1, \ldots, n\}$ of size r $(r = 0, \ldots, n)$.

3. Let u_i be a partial bijection mapping M_1 onto M_i and u_j be another partial bijection mapping M_1 onto M_j, where $1 \leqslant i, j \leqslant \binom{n}{r}$.

4. If $\mathbf{y}$ is a partial bijection mapping M_i onto M_j, then there is a unique element $\mathbf{x} : M_1 \to M_1 \in G_r(M_1)$ such that $\mathbf{y} = u_i^{-1}\mathbf{x}u_j$, where $G_r(M_1)$ is a symmetric group G_r on the subset M_1.

5. We write $[\mathbf{x}]_{ij}$ instead of $u_i^{-1}\mathbf{x}u_j$, so that

   $$[\mathbf{x}]_{ij}[\mathbf{x}']_{jk} = u_i^{-1}\mathbf{x}u_j \cdot u_j^{-1}\mathbf{x}'u_k = u_i^{-1}\mathbf{x}\mathbf{x}'u_k = [\mathbf{x}\mathbf{x}']_{ik}.$$

   Otherwise, if $j \neq l$, then $[\mathbf{x}]_{ij}[\mathbf{x}']_{lk}$ has domain a proper subset of M_i, or has rank $< r$.

Apparently at this step, Munn started his induction process, without being aware of it.

If χ_{rs} is a character of an irreducible representation of G_r, then $\chi_{rs}^{(\lambda)} \in F$ will denote its value on the conjugacy class of elements of G_r having a cycle pattern defined by the partition (λ) of r. The next result gives the value of the characters of irreducible representations of the principal factor of $A^{(n)}$.

Lemma 5.4.2. [67, Lemma 3.4] *Let χ_{rs} and χ'_{rs} be the characters of the corresponding irreducible representations of G_r and $A_r^{(n)}/A_{r-1}^{(n)}$ over F, respectively. If $x \in A_r^{(n)}/A_{r-1}^{(n)}$, then*

$$\chi'_{rs}(x) = \begin{cases} \chi_{rs}^{(\lambda)} & \text{if } x \text{ has subrank } r \text{ and cycle pattern } (\lambda), \\ 0 & \text{if } x \text{ has subrank less than } r. \end{cases}$$

Now, the preparation is finished and we are ready to state the theorem that describes the process of obtaining the character values of the irreducible representations of $A^{(n)}$.

Theorem 5.4.3. [67, Theorem 3.5] *Let γ_{rs}^* be the representation of $A^{(n)}$ of rank r derived from γ'_{rs} as defined in Formula (5.3) and let χ_{rs}^* be its character. Then, for any $x \in A^{(n)}$,*

$$\chi_{rs}^*(x) = \sum \chi_{rs}^{(\lambda)}, \tag{5.4}$$

where the summation is over the partitions (with repetitions) corresponding to the cycle patterns of all permutations of rank r induced by x.

In the following paragraphs, we explain how we can obtain the irreducible representations of the symmetric inverse semigroup $A^{(n)}$ in terms of irreducible representations of its maximal subgroups, the symmetric groups G_r, for $1 \leqslant r \leqslant n$. The idea is that we start with irreducible representations of the maximal subgroup G_r and these will give us irreducible representations of the principal factors $A_r^{(n)}/A_{r-1}^{(n)}$ of $A^{(n)}$ and these give irreducible representations of the symmetric semigroup.

To illustrate the above process, we have the following example. Let x be the element

$$x = [01](234)(56)(7)(8)(9) \in A^{(10)}.$$

Let χ_{3s}^* be the character of a matrix representation of $A^{(10)}$ of rank (= subrank) 3, and let χ_{3s} be the corresponding character of the matrix representation of G_3. As there are three partitions of 3, namely, $1+1+1, 2+1$, and 3 , there are three conjugacy classes. We denote the partition $1+1+1$ as (1^3), $2+1$ as (21), and 3 as (3). This implies that there are three irreducible representations of $A^{(10)}$ with subrank = 3. We calculate the following:

$$\chi'(xe_{\{2,3,4\}})^\theta = \chi'(234) = \chi(123);$$

$$\chi'(xe_{\{5,6,7\}})^\theta = \chi'\{(56)(7)\} = \chi\{(12)(3)\};$$
$$\chi'(xe_{\{7,8,9\}})^\theta = \chi'\{(7)(8)(9)\} = \chi e_{\{1,2,3\}}.$$

By Formula (5.4), the character of $A^{(10)}$ with rank 3 is

$$\chi^*_{3s}(x) = \chi^{(3)}_{3s} + 3\chi^{(21)}_{3s} + \chi^{(1^3)}_{3s}.$$

(The summand $\chi^{(21)}_{3s}$ occurs with multiplicity 3 because in Formula (5.4) we are taking the sum with repetitions over partitions).

Now, we need the character table of the symmetric group G_3 [42]:

G_3	id	(12)	(123)
trivial χ_{31}	1	1	1
sign χ_{32}	1	-1	1
geometric χ_{33}	2	0	-1

Thus,

$$\chi^*_{31}(x) = 5, \ \chi^*_{32}(x) = -1, \text{ and } \chi^*_{33}(x) = 1.$$

In the following subsection, we will take up this example in detail. Since the characters of matrix representations of the symmetric inverse semigroup are expressible as sums of characters of the matrix representations of the symmetric group, we have:

Corollary 5.4.4. [67, Corollary 3.6] *The values of the characters of the matrix representations of $A^{(n)}$ are integers.*

Corollary 5.4.5. [67, Corollary 3.7] *The elements of $A^{(n)}$ with the same subrank and the same cycle pattern have the same character value in every representation of $A^{(n)}$.*

This means that the symmetric inverse semigroup $A^{(n)}$ can be partitioned into classes according to subrank and cycle pattern. The number of these classes is $\sum_{s=0}^{n} p_s$, where p_s is the number of partitions of s with $p_0 = 1$. In fact, the number of classes of $A^{(n)}$ is also the number of inequivalent irreducible representations of $A^{(n)}$. Precisely, there are p_r representations of rank r ($r = 0, \ldots n$). Observe that the links play no role. At the end of the paper [67], Munn applied the above character theory and provided the complete character table for the semigroup $A^{(4)}$ of order 209.

We observe that in the present work Munn demonstrated that there is a one-to-one correspondence between the irreducible representations of $A^{(n)}$ and the irreducible representations of G_r, $0 \leqslant r \leqslant n$. This may be the point where he first realised what would become the Clifford-Munn-Ponizovskii correspondence. More comments about this point will be given in the last section.

5.4.1 The Characters of the Symmetric Inverse Monoid, from our Perspective

The goal of this subsection is to elaborate on the technique described by Munn in his 1957 paper [67] to induce the characters of the symmetric group S_r to characters of the symmetric inverse semigroup I_n over $\mathbb{C}$, $0 \leq r \leq n$. We explicitly give the method by example on I_{10} using modern notation. To do this, we start with some preliminary concepts of group characters and then provide some results about characters of the symmetric groups and Young Tableaux which are the motivation behind the article [67].

In Chapter 1, we saw that characters are considered as a significant part of the representation theory of finite groups. Specifically, characters are essential tools in many applications of group theory to different problems, in mathematics, but also physics and chemistry. The question is: why are characters so useful in applications? What does their importance stem from? The simple answer is that since a character is obtained from a representation by taking the trace of a matrix representation, it is more convenient and easier to deal with a character (a number) than a representation (a matrix). The utility of characters comes mainly from the following reasons.

First, equivalent representations have the same characters; also, any character is constant on conjugacy classes. This feature shows that characters of a group representation are intimately linked with the conjugacy classes of the group. Second, using characters is the better way to classify representations, whether the representation is reducible or irreducible. Third, any reducible characters of a group can be written uniquely as a sum of irreducible characters of that group. Finally, irreducible characters of a group encapsulate information about the structure of a group itself. For example, once the character table of a group is known, it can be used to determine whether a group is abelian, simple or solvable.

To give the article of Munn a flavour, we provide a strategic picture of the symmetric inverse monoid I_n. The diagram in Figure 5.1 illustrates the $\mathcal{J}$-classes of the symmetric inverse monoid I_n.

According to Munn [67], the elements of a symmetric inverse semigroup can be written as a conjunction of links in square brackets and disjoint cycles in round brackets. For example, we write

$$\alpha = \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ 1 & - & 3 & 4 & 2 & 6 & 5 & 7 & 8 & 9 \end{pmatrix}$$

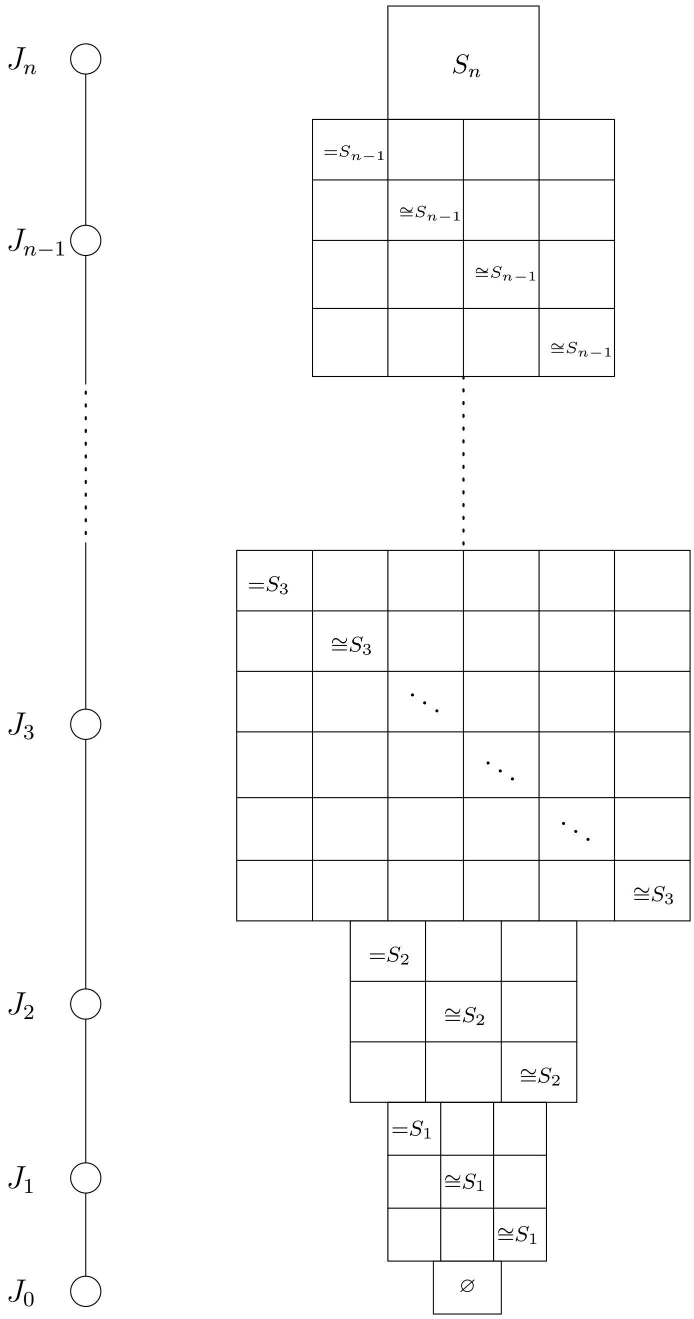

Figure 5.1: The $\mathcal{J}$-classes of the symmetric inverse monoid I_n

as $\alpha = [01](234)(56)(7)(8)(9)$ and, as usual, (234) is a cycle and $[01]$ is a link. Before we present our example, we need the following notions.

Let $\lambda = \{\lambda_1, \ldots, \lambda_k\}$ be a partition of a positive integer n such that $\sum_{j=1}^{k} \lambda_j = n$ and $\lambda_1 \geq \ldots \geq \lambda_k > 0$. Any cyclic decomposition of a permutation in S_n is associated with a partition of n, where $\{\lambda_j\}$ are the lengths of the individual cycles. This partition can be expressed using a *Young diagram*, which is a set of empty boxes arranged in rows such that there are λ_1 boxes in row 1, etc. Thus, the Young diagram associated with the partition λ has k rows and λ_j boxes in the jth row. Clearly, there is a one-to-one correspondence between partitions and Young diagrams. We call λ the *shape* of the permutation and each shape specifies a conjugacy class.

Since the number of conjugacy classes is equal to the number of irreducible representations, each partition λ corresponds to an irreducible character of S_n. For S_3, there are three conjugacy classes, corresponding to the Young diagrams $\ydiagram{1,1,1}$, $\ydiagram{2,1}$, and $\ydiagram{3}$ of sizes $1, 3$, and 2, respectively. For S_4, there are five conjugacy classes, corresponding to the Young diagrams $\ydiagram{1,1,1,1}$, $\ydiagram{2,1,1}$, $\ydiagram{3,1}$, $\ydiagram{4}$ and $\ydiagram{2,2}$ of sizes $1, 6, 8, 6$ and 3, respectively. These diagrams will be used later on in our example.

We use Munn's formula (5.4), with some modification in symbols, to calculate the values of characters χ_* on any element α of the symmetric inverse semigroup I_n in terms of the corresponding characters of the symmetric group S_r, $0 \leq r \leq n$:

$$\chi_*^{\lambda}(\alpha) = \sum_{\text{(cycle type)}} \chi^{\lambda}.$$

The sum runs over the cycle types counted with multiplicity of the permutations of degree r induced by α. Note that χ^{λ} is the value of the irreducible character of S_r on the conjugacy class of shape λ and χ_*^{λ} is the corresponding character of I_n. Our aim now is to induce a character χ of a symmetric group S_r to a character χ_*.

Let α be the element of I_{10} such that

$$\alpha = [01](234)(56)(7)(8)(9).$$

We have one link $[01]$ and five cycles $(234)(56)(7)(8)(9)$. For $r = 3$, the partial permutation α has five permutations of three elements: (234), $(56)(7)$, $(56)(8)$, $(56)(9)$, and $(7)(8)(9)$. Hence, we have, respectively, one permutation with cycle type (3), three permutations with cycle type $(2, 1)$, and one with cycle type $(1, 1, 1)$.

The following table displays the above information about these permutations:

r	cycle type		
	$(1,1,1)$	$(7)(8)(9)$	1
3	$(2,1)$	$(56)(7), (56)(8), (56)(9)$	3
	(3)	(234)	1

In the table above, the third column shows the permutations of rank $r = 3$ induced by α of the given cycle type. The forth column contains the number of such permutations.

The character table for the symmetric group S_3 is as follows [42]:

S_3	$(1)(2)(3)$	$(12)(3)$	(123)
cycle type	$(1,1,1)$	$(2,1)$	(3)
trivial $\chi^{\ydiagram{3}}$	1	1	1
sign $\chi^{\ydiagram{1,1,1}}$	1	-1	1
geometric $\chi^{\ydiagram{2,1}}$	2	0	-1

This is the same character table mentioned before Corollary 5.4.4 except with different notation.

In Figure 5.2, we have an equilateral triangle whose vertices have been labeled by $1, 2$, and 3. If σ is an element of S_3, then it acts on the equilateral triangle by permuting the vertices as σ permutes the numbers $1, 2$, and 3. According to that permutation, this is a linear map of $\mathbb{R}^2$ to itself sending the equilateral triangle to itself by choosing basis v_1 and v_2. Then the matrices of the representation of the symmetric group S_3 corresponding to $\ydiagram{2,1}$ with respect to the basis v_1 and v_2 are:

$$(1)(2)(3) \mapsto \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \text{the trace is } 2,$$

$$(12)(3) \mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \Rightarrow \text{the trace is } 0,$$

and

$$(123) \mapsto \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix} \Rightarrow \text{the trace is } -1.$$

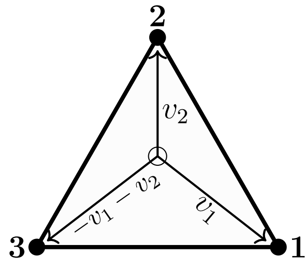

Figure 5.2: Geometric representation of S_3

All actions are on the right when computing the matrices above.

Thus,

$$\chi_*^{\ydiagram{2,1}}(\alpha) = \chi^{\ydiagram{2,1}}(1)(2)(3) + 3\,\chi^{\ydiagram{2,1}}(12)(3) + \chi^{\ydiagram{2,1}}(123) = 1.$$

Similarly, we compute the character of the trivial representation $\chi_*^{\ydiagram{3}}$ and the character of the sign representation $\chi_*^{\ydiagram{1,1,1}}$:

$$\begin{aligned} \chi_*^{\ydiagram{3}}(\alpha) &= 1 + 3 \cdot (1) + 1 = 5, \\ \chi_*^{\ydiagram{1,1,1}}(\alpha) &= 1 + 3 \cdot (-1) + 1 = -1. \end{aligned}$$

We remind the reader that this is a completely analogous calculation to that in the previous section but using modern notation.

For $r = 4$, the partial permutation α has six permutations of four elements: (234)(7), (234)(8), (234)(9), (56)(7)(8), (56)(7)(9), and (56)(8)(9). So, we have three permutations with cycle type $(3, 1)$ and three permutations with cycle type $(2, 1, 1)$. The character table for the symmetric group S_4 is as follows [42]:

S_4	(1)(2)(3)(4)	(12)	(123)	(1234)	(12)(34)
cycle type	$(1,1,1,1)$	$(2,1,1)$	$(3,1)$	(4)	$(2,2)$
$\chi^{\ydiagram{4}}$	1	1	1	1	1
$\chi^{\ydiagram{1,1,1,1}}$	1	-1	1	-1	1
$\chi^{\ydiagram{3,1}}$	3	1	0	-1	-1
$\chi^{\ydiagram{2,1,1}}$	3	-1	0	1	-1
$\chi^{\ydiagram{2,2}}$	2	0	-1	0	2

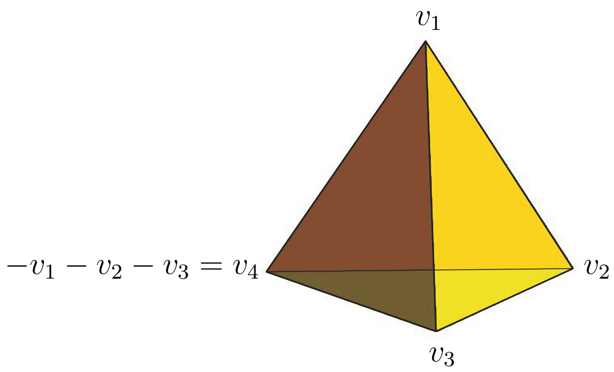

Figure 5.3: Geometric representation of S_4

Here we compute $\chi^{\yng(3,1)} \uparrow I_{10} = \chi_*^{\yng(3,1)}$ and the others can be obtained in the same way. In Figure 5.3, we have a tetrahedron whose vertices have been labeled by v_1, v_2, v_3, and v_4. If σ is an element of S_4, then it acts on the tetrahedron by permuting the vertices as σ permutes the numbers $1, 2, 3$, and 4. According to that permutation, this is a linear map of $\mathbb{R}^3$ to itself sending the tetrahedron to itself. Then the matrices of the representation of the symmetric group S_4 corresponding to $\yng(3,1)$ with respect to the basis v_1, v_2, v_3 and $v_4 = -v_1 - v_2 - v_3$ are:

$$(1)(2)(3)(4) \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{the trace is } 3,$$

$$(12)(3)(4) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \Rightarrow \text{the trace is } 1,$$

$$(123)(4) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix} \Rightarrow \text{the trace is } 0,$$

$$(1234) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix} \Rightarrow \text{the trace is } -1,$$

and

$$(12)(34) \mapsto \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ -1 & -1 & -1 \end{pmatrix} \Rightarrow \text{the trace is } -1.$$

From the above representation of the element (1234), we have:

$$v_1 \mapsto v_2, \; v_2 \mapsto v_3, \; \text{and } v_3 \mapsto v_4 = -v_1 - v_2 - v_3.$$

Because we are acting on the right that means that each row is the image of the corresponding basis vector and this is the reason that we get the matrix:

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & -1 & -1 \end{pmatrix}$$

Thus,

$$\begin{aligned} \chi_*^{\yng(3,1)}(\alpha) &= 3\,\chi^{\yng(3,1)}(123) + 3\,\chi^{\yng(3,1)}(12) \\ &= 3. \end{aligned}$$

5.5 Irreducible Matrix Representations of Semigroups

In this section, we discuss the highlights of Munn's paper: *Irreducible Matrix Representations of Semigroups* [68].

In his previous papers [65] and [66], Munn considered the problem of constructing the complete set of irreducible representations of a finite semigroup, whose algebra over a certain field is semisimple, via the irreducible representations of its maximal subgroups. In this next paper, he demonstrated how to do this for an arbitrary semigroup (maybe infinite), even when its algebra is not semisimple.

The main result of this paper is that for a semigroup S satisfying the minimal condition on its principal one-sided ideals, there is a natural bijective correspondence between the irreducible principal representations of S and the irreducible representations vanishing at zero of the (0-)simple principal factors of S.

We begin with some notational conventions and definitions. Let S be a semigroup and J be a fixed $\mathcal{J}$-class of S. Then:

1. $P(J)$ denotes the principal ideal in S generated by the elements of J.
2. $N(J)$ denotes the set of non-generators of $P(J)$ (it is the ideal $P(J) - J$).
3. $Q(J)$ denotes the Rees quotient semigroup $P(J)/N(J)$. It is in fact the principal factor of S corresponding to J.
4. $x \mapsto \bar{x}$ denotes the natural homomorphism from $P(J)$ to $Q(J)$ and is defined by
$$\bar{x} = \begin{cases} x & \text{for all } x \text{ in } J, \\ z & \text{otherwise,} \end{cases}$$
where $z = N(J)$ is the zero element of $Q(J)$.

Let S and T be semigroups and k be a field. If a map $\theta : S \to T$ is a semigroup homomorphism, then θ extends uniquely to $\bar{\theta} : k[S] \to k[T]$. Thus by linearity, we can extend the homomorphism in (4) uniquely to a homomorphism from the algebra of $P(J)$ to the algebra of $Q(J)$. Moreover, if the Rees factor $Q(J)$ is simple, then J is a simple $\mathcal{J}$-class of S.

According to Green [26], a semigroup S is said to have the property M_f if every set of principal two-sided ideals of S has a minimal member. This is called the *minimal condition on the principal two-sided ideals of* S. The minimal conditions M_r and M_l on principal right and left ideals are defined in a similar manner. These conditions M_r, M_l, and M_f, are respectively, equivalent to the descending chain conditions on principal right, left, and two-sided ideals of S. In other words, a semigroup S has M_l if and only if there are no infinite chains

$$S^1a_1 \supset S^1a_2 \supset S^1a_3 \supset \cdots$$

of principal left ideals. M_l is the *descending chain condition* (d.c.c.) on principal left ideals. The condition M_r is the dual of this.

Munn denoted the algebra of all $n \times n$ matrices over a field F by $(F)_n$. Then he defined a representation Γ of a semigroup S of degree n over F to be a homomorphism of S into the multiplicative semigroup of $(F)_n$. It is clear that every representation of a semigroup S over F can be extended uniquely to a representation of the semigroup algebra of S. Let T be a subset of S; then $\Gamma(T)$ is defined to be the set of all matrices $\Gamma(t)$, where $t \in T$ and $[\Gamma(T)]$ is the subspace of $(F)_n$ spanned by the elements of $\Gamma(T)$. Moreover, we say that two representations Γ and Γ' of S are equivalent if and only if there exists a non-singular matrix A over F such that $\Gamma'(x) = A^{-1}\Gamma(x)A$ for all x in S. Note that the space $F[T]$ is the linear subspace of the algebra of S spanned by T (i.e. the set of all finite linear combinations of elements of T with coefficients in F).

Before we state the results of the paper currently under consideration, we point out that they make use of the existence of elements e_i in an ideal of S and $\alpha_i \in F$ such that $\sum \alpha_i \Gamma(e_i) = I_n$, where I_n denotes the $n \times n$ identity matrix of $(F)_n$.

Lemma 5.5.1. [68, Lemma 1] *Let Γ be a representation of S of degree n over F and let T be a subset of S such that $[\Gamma(T)]$ is an irreducible subalgebra of $(F)_n$. Then there exists an element e of $F[T]$ such that $\Gamma(e) = I_n$.*

We now describe the idea of an irreducible principal representation of a semigroup. Let Γ be an S-representation. We let $V(\Gamma)$ denote the ideal of S consisting of all elements x in S such that $\Gamma(x) = 0$; we call this ideal the *vanishing ideal* of Γ. The representation Γ of S is called *principal* if $S - V(\Gamma)$ contains a unique minimal $\mathcal{J}$-class of S. If such a $\mathcal{J}$-class J of S exists, we call it the *apex* of Γ. This representation Γ is described by the rule: $\Gamma(x) \neq 0$ if and only if $J \leqslant J_x$. In particular, if S has a kernel K and $V(\Gamma) = \varnothing$, then Γ is a principal representation with apex K. (Recall that the kernel K is the unique minimal $\mathcal{J}$-class of S.)

Now, let $\Gamma(x) = 0$ for all x in the ideal $N(J)$. Then we restrict the representation Γ to a representation Γ^* of $Q(J)$ by the following relation:

$$\Gamma^*(\bar{x}) = \Gamma(x), \text{where } x \in P(J).$$

Γ^* is said to be the representation of $Q(J)$ induced by Γ and we say that Γ is an *extension* of Γ^* to S. It is clear that $V(\Gamma^*) = \{z\}$, thus Γ^* is a 0-restricted representation in the sense that only the zero of $Q(J)$ is mapped onto the zero matrix. Note that we need $\Gamma(x) = 0$ for all x in $N(J)$, so that the representation Γ^* of $Q(J)$ extends uniquely to the representation Γ of $P(J)$. Here, the representation Γ is a unique well-defined extension of Γ^*. In particular, if Γ is a principal representation of S with apex J, then we call it a principal extension of Γ^*.

The following theorem is the main result of the paper [68]; it establishes the one-to-one correspondence mentioned earlier:

Theorem 5.5.2. [68, Theorem 1] *Let S be a semigroup and F be a field.*

1. *Let Γ be an irreducible principal representation of S of degree n over F, and let J be the apex of Γ. Then J is simple, and the representation Γ^* of $Q(J)$ induced by Γ is irreducible. Also, there exists an element e of the algebra of J such that $\Gamma^*(e) = I_n$, and for any such element e we have*

$$\Gamma(x) = \Gamma^*(\bar{x}e), \text{where } x \in S. \tag{5.5}$$

2. *Let J be a simple $\mathcal{J}$-class of S and let Γ^* be an irreducible representation of $Q(J)$ of degree n over F such that $\Gamma^*(z) = 0$, where z is the zero of $Q(J)$. Then there exists an element e of the algebra of J such that $\Gamma^*(e) = I_n$, and for any such element e equation (5.5) serves to define an irreducible principal extension Γ of Γ^*.*

3. *Two irreducible principal representations of S are equivalent if and only if they have the same apex J and induce equivalent representations of $Q(J)$.*

When a semigroup S obeys the minimal condition M_f on principal two-sided ideals, then the representations of such a semigroup lie in a special class, as in the following result:

Theorem 5.5.3. [68, Theorem 2] *Let S be a semigroup satisfying the condition M_f and let F be a field. Then every irreducible representation of S over F is principal.*

Munn then observed that our Theorems 5.5.2 and 5.5.3, together with the work attributed to Suschkewitsch, 1933 paper, and Clifford [8] on the theory of representations of completely simple semigroups, allow us to obtain the complete set of irreducible representations of a semigroup S satisfying the conditions M_r and M_l in terms of those of its maximal subgroups. In addition, since conditions M_r and M_l together imply M_f [26, Theorem 4], and by Theorem 5.5.3 above, we deduce that every irreducible representation of S is principal. In a special case, if a semigroup S is finite and its algebra is semisimple, the element e mentioned previously is unique, and it is in fact the identity element of the algebra of $Q(J)$. This case had been studied before by Munn [65, 66] and Ponizovskii [68].

The characterization of complete reducibility was discussed in the last theorem of this paper. Munn first stated a required lemma and then the key theorem which contains a sufficient condition for the complete reducibility of representations of a semisimple semigroup S.

Lemma 5.5.4. [68, Lemma 2] *Let J be a simple $\mathcal{J}$-class of a semigroup S. Let*

$$\Gamma : x \to \Gamma(x) = (\gamma_{ij}(x))$$

be a representation of S of degree m over a field F. If $\gamma_{ij}(x) = 0$ for all i, j such that $1 \leqslant i \leqslant j \leqslant m$ and for all x in J, then $\Gamma(x) = 0$ for all x in J.

Theorem 5.5.5. [68, Theorem 3] *Let S be a semisimple semigroup which satisfies condition M_f and let F be a field. Further, let every representation of every principal factor of S over F be completely reducible. Then every representation of S over F is completely reducible.*

We have a special case of this theorem: if the representation of a semigroup S is over the real or the complex field, then the representation will become a bounded representation (a representation is bounded if there exists a positive real number k such that $|\gamma_{ij}(x)| < k$ for all x in S and all i, j). Hence, every bounded representation of S is completely reducible and even the induced representation is also bounded and so completely reducible.

The paper ends with an application of the theoretical techniques discussed above. As in his previous papers [66, 67], Munn illustrated the results on an inverse semigroup obeying the minimal condition M_l and he provided the method of constructing all irreducible representations of this special type of semigroup over a field F via those of its related groups. Moreover, he also studied the case when the field is the real or the complex field and showed that all bounded representations of the inverse semigroup are completely reducible. In fact, these results are an extended version of Munn's works in [66].

We recall that the paper under discussion is the fourth in Munn's series of works on the development of semigroup representation theory. Our principal observations on this paper are that it has the first mention of Ponizovskii's work on semisimplicity, which shows Munn's awareness of Ponizovskii's work in the area. It also contains the first mention of the term 'apex'. The determination of the type of representations discussed here is based on the existence of the apex of a representation, which is a certain $\mathcal{J}$-class. A question arises here: what if a semigroup does not satisfy the minimal condition on principal ideals and as a result its irreducible representations may not be principal? This case encouraged Munn to introduce a new type of representation without minimal conditions, as explained in the following section.

5.6 A Class of Irreducible Matrix Representations of an Arbitrary Inverse Semigroup

From his series of papers on the representation theory of semigroups, it can be noted that Munn dealt with irreducible representations of different classes of semigroups. However, the study of the irreducible representations of inverse semigroups was the major theme in Munn's works on this area; the next paper to consider is one of them [69]. In contrast to the earlier [68], this paper studies the representations of inverse semigroups without minimal conditions.

We start the discussion with preliminaries. Recall that an inverse semigroup S is defined by the following equivalent properties:

1. S is regular (i.e. $a \in aSa$ for every $a \in S$) and its idempotents commute;

2. every element of S has a unique inverse;

3. each principal left ideal and each principal right ideal of S is generated by one and only one idempotent.

The first theorem discusses the idea of the maximal group homomorphic image of an inverse semigroup:

Theorem 5.6.1. [69, Theorem 1] *Let S be an inverse semigroup and let a relation σ be defined on S by the rule that:*

$$x \, \sigma \, y \textit{ if and only if there exists an idempotent } e \in S \textit{ such that } ex = ey.$$

Then we have:

1. *the relation σ is a congruence relation and S/σ is a group;*

2. *if τ is any congruence on S with the property that S/τ is a group, then $\sigma \subseteq \tau$, and so S/τ is isomorphic to a quotient group of S/σ. The quotient S/σ is called the* maximal group homomorphic image *of S and is denoted by G_S;*

3. *if M is an ideal of S, then M is an inverse semigroup and G_M is isomorphic to G_S.*

For more detail about the maximal group homomorphic image of an inverse semigroup S, see [37, Sections 1.4 and 1.5].

The map $x \to \bar{x}$ denotes the natural homomorphism of an inverse semigroup S onto its maximal group homomorphic image. In a special case, if the inverse semigroup S has kernel K, then K is a group, then G_S is isomorphic to K. By showing that for any two elements x and $y \in S$: $\bar{x} = \bar{y}$ if and only if $ex = ey$, where e is the identity of K, the mapping $\bar{x} \to ex$ gives the desired isomorphism, where $\bar{x}$ is the equivalence class of x.

Munn was then motivated to introduce the concept of a prime irreducible representation of an inverse semigroup S. At the beginning of this part, he drew attention to the case where S has an identity, in which case it is required that it is mapped to the identity matrix. Also, he excluded the null representation of degree one (which maps every element to the zero matrix) from the concept of irreducibility. Munn made use of the notations of paper [68].

In representation theory, we can think of Γ as a matrix (as Munn did) or as a linear transformation. This means that each element s of S corresponds to an $n \times n$ matrix or a linear transformation. Let W be a representation space for Γ and U be an invariant subspace of W under Γ. If we choose a basis $\{w_1, \ldots, w_n\}$ of W such that $\{w_1, \ldots, w_r\}$ with $1 \leq r < n$ is a basis of U, then the representation matrix $\Gamma(s)$ takes the partitioned form

$$\Gamma(s) = \begin{pmatrix} \Gamma_1(s) & 0 \\ \Gamma_{12}(s) & \Gamma_2(s) \end{pmatrix},$$

where $\Gamma_1(s)$ is an $r \times r$ matrix, and $\Gamma_2(s)$ is an $(n-r) \times (n-r)$ matrix $(1 \leq r < n)$. A matrix representation Γ is said to be *reducible* if it has the previous block form. Otherwise, Γ is *irreducible.*

The vanishing set $V(\Gamma)$ is called a *prime* ideal of S if it is not equal to S and the complement set $S \setminus V(\Gamma)$ is a subsemigroup of S. If the vanishing set $V(\Gamma)$ is empty or a prime ideal, then the representation Γ is called a *prime representation* of S. Munn utilized the following lemma for the proofs of Theorems 2 and 3 of his paper.

Lemma 5.6.2. [8, Theorem 5.7] *An irreducible subalgebra of $(F)_n$ is a simple algebra over F.*

The next theorem provides a means of obtaining all the prime irreducible representations of an arbitrary inverse semigroup via certain unique irreducible representations of the maximal group homomorphic image of $S \setminus V$.

Theorem 5.6.3. [69, Theorem 2] *Let S be an inverse semigroup and F be a field.*

1. *Let V be the empty set or a prime ideal of S. Then $S \setminus V$ is an inverse semigroup.*

2. *Let Γ be a prime irreducible representation of S over F and let V=$V(\Gamma)$. Then $S \setminus V$ is an inverse semigroup and*

$$\Gamma(x) = \begin{cases} \Gamma^*(\bar{x}) & \text{if } x \in S \setminus V, \\ 0 & \text{if } x \in V, \end{cases}$$

where $x \to \bar{x}$ is the natural homomorphism of $S \setminus V$ onto $G_{S\setminus V}$, and Γ^* is an irreducible representation of $G_{S\setminus V}$.

3. *Also, if Γ^* is any irreducible representation of $G_{S\setminus V}$, then the mapping Γ defined previously is a prime irreducible representation of S.*

Generally, every semigroup S has a trivial prime irreducible representation which is the representation Γ of degree one defined by $\Gamma(x) = 1$ for all x in S. Munn ended this part with an example of an inverse semigroup that possesses an irreducible representation that is not prime.

Munn closed this paper with a discussion of a significant class of inverse semigroups for which every irreducible representation is prime, and he applied the earlier results to this class. According to Clifford and Preston [8], a semigroup S is *intraregular* if and only if $a \in Sa^2S$ for every $a \in S$. Using Clifford's terminology [5], such semigroups are semilattices of simple semigroups (a semilattice means a commutative semigroup of idempotents, see [8, Section 1.8]). In particular, an intraregular inverse semigroup is a semilattice of simple inverse semigroups, and conversely.

Since a group is a special type of a simple inverse semigroup, a semilattice of groups is an intraregular inverse semigroup. Such a semigroup S is a union of disjoint groups S_α, say, where α belongs to a semilattice Y. To each pair of elements α and β in Y such that $\alpha \geqslant \beta$, there exists a homomorphism $\phi_{\alpha\beta}$ of S_α into S_β. The transitivity relation holds for these homomorphisms: for $\alpha \geqslant \beta \geqslant \gamma$, we illustrate the transitivity by the following commutative diagram:

$$\begin{tikzcd} S_\alpha \arrow[rr, "\phi_{\alpha\beta}"] \arrow[dr, "\phi_{\alpha\gamma}"'] & & S_\beta \arrow[dl, "\phi_{\beta\gamma}"] \\ & S_\gamma & \end{tikzcd}$$

In addition, I_{S_α} is the identity of S_α, for all $\alpha \in Y$. Then the multiplication in S is defined by the rule:

$$x_\alpha y_\beta = (x_\alpha \phi_{\alpha\gamma})(y_\beta \phi_{\beta\gamma}),$$

where x_α and y_β are elements of S_α and S_β, respectively, and $\gamma = \alpha\beta$. Consequently, $S_\alpha S_\beta \subseteq S_{\alpha\beta}$. The groups S_α are exactly the $\mathcal{J}$-classes of S. A semilattice Y is a special example of a poset and all posets can be given a structure of a category in a standard way. In fact, what we have is a contravariant functor from Y, as a category, to the category of groups. The structure of a semilattice of groups is taken from Theorem 3 of Clifford's paper [5].

We point out here that the semigroup in the next theorem need not be an inverse semigroup.

Theorem 5.6.4. [69, Theorem 3] *Every irreducible representation of an intraregular semigroup is a prime representation.*

By combining Theorems 5.6.3 and 5.6.4, we are able to obtain all the irreducible representations of an intraregular inverse semigroup S. Munn deduced that if Γ_1 and Γ_2 are equivalent irreducible representations of S, then their vanishing sets are equal ($V(\Gamma_1) = V(\Gamma_2) = V$) and the corresponding irreducible representations Γ_1^* and Γ_2^* of $G_{S\setminus V}$, defined in Theorem 5.6.3, are also equivalent. Conversely, if Γ_1^* and Γ_2^* are equivalent irreducible representations of $G_{S\setminus V}$, where V is empty or a prime ideal of S, then the Γ_1 and Γ_2, defined in Theorem 5.6.3, are equivalent irreducible representations of S.

The last part of this section is devoted to the construction of the principal irreducible representations of an intraregular inverse semigroup, where a representation Γ is called principal if $S \setminus V(\Gamma)$ contains a unique minimal $\mathcal{J}$-class, which is the apex of Γ. Let Γ be a principal irreducible representation of S over a field F. By Theorem 5.6.4, this representation is prime. Furthermore, since Γ is principal, there exists an element $\omega \in Y$ such that S_ω is the kernel of $S \setminus V$. By a former result, the mapping $\bar{x}_\alpha \to e_\omega x_\alpha = x_\alpha \phi_{\alpha\omega}$ ($x_\alpha \in S_\alpha \subseteq S \setminus V$) is an isomorphism of $G_{S\setminus V}$ onto S_ω, where e_ω is the identity of S_ω.

Therefore, by part (1) of Theorem 5.6.3, there is an irreducible representation Γ^* of S_ω over F such that

$$\Gamma(x_\alpha) = \begin{cases} \Gamma^*(x_\alpha \phi_{\alpha\omega}) & \text{if } x_\alpha \in S_\alpha, \alpha \geqq \omega, \\ 0 & \text{if } x_\alpha \in S_\alpha, \alpha \ngeqq \omega. \end{cases}$$

Conversely, if ω is an element of a semilattice Y and if Γ^* is an irreducible representation of S_ω over F, then the previous representation defines a principal irreducible representation of S over F with apex S_ω. In a special case, if a semigroup S obeys the minimal condition on principal ideals, then every irreducible representation is principal and so is defined as above. A trivial example is provided of non-principal irreducible representations.

The study of representations of inverse semigroups remained a productive research area in the mathematical work of Munn. The next section reviews the last paper of Munn on this theme, *Matrix Representations of Inverse Semigroups* [70].

5.7 Matrix Representations of Inverse Semigroups

The objectives of this paper are, for an arbitrary field F, to construct all representations of a 0-simple inverse semigroup and all the irreducible representations of an arbitrary inverse semigroup from those of its associated Brandt semigroup. To provide a method for finding the desired representations, Munn used the results and concepts of [69, 71].

For a semigroup S, we write $S = S^0$ when S is a semigroup which has a zero and at least one other element. Munn showed that the following conditions are important in the theory of representations of a semigroup $S = S^0$.

- C1. If a, b, c are elements of S such that $abc = 0$, then either $ab = 0$ or $bc = 0$.
- C2. If M and N are non-zero ideals of S, then so is $M \cap N$.

The first condition is called a *categorical at* 0. In the case where $\{0\}$ is a prime ideal of S (which means that the complement $S \setminus \{0\}$ is a subsemigroup) both conditions are trivially satisfied.

Let Γ be a representation of a semigroup $S = S^0$ of degree n over a field F. By convention, we require $\Gamma(0)$ to be the $n \times n$ zero matrix 0. If $S \neq S^0$, then we may extend any representation Γ of S to a representation of S^0 by defining $\Gamma(0) = 0$. Throughout this paper, Munn dealt with the case $S = S^0$.

Recall the notation of the vanishing ideal of Γ, $V(\Gamma) = \{x \in S : \Gamma(x) = 0\}$. Now, we define some notions on S which play a prominent part in the discussion.

1. $r(\Gamma)$ = least integer s greater than zero such that, for some $x \in S \setminus V(\Gamma)$, the matrix representation $\Gamma(x)$ has rank s.
2. $M(\Gamma) = \{x : x \in S,\ \operatorname{rank} \Gamma(x) = r(\Gamma)\} \cup V(\Gamma)$. If $r(\Gamma) = n$, then $M(\Gamma) = S$ and $\Gamma(x)$ is non-singular for all x in $S \setminus V(\Gamma)$.

In preparation for the main purpose of the paper, we state some required results related to these subsets.

Lemma 5.7.1. [70, Lemma 1.5] *Let Γ be a non-null representation of a semigroup $S = S^0$. Then*

1. *$V(\Gamma)$ and $M(\Gamma)$ are ideals of S and $M(\Gamma)/V(\Gamma) = \{M(\Gamma)/V(\Gamma)\}^0$;*
2. *$M(\Gamma)/V(\Gamma)$ satisfies condition C1,*
3. *if a, b, c, d are elements of $M(\Gamma)$ such that $ab = ac \notin V(\Gamma)$ and $bd = cd \notin V(\Gamma)$; then $\Gamma(b) = \Gamma(c)$.*

Lemma 5.7.2. [70, Lemma 1.7] *Let Γ be an irreducible representation of a semigroup $S = S^0$. Then $S/V(\Gamma)$ satisfies condition C2.*

Now we start our discussion about representations of inverse semigroups. For the case of inverse semigroups, Munn reformulated part 3 of Lemma 5.7.1 as follows:

Lemma 5.7.3. [70, Lemma 2.1] *Let Γ be a non-null representation of an inverse semigroup $S = S^0$ and let e, x, y be elements of $M(\Gamma)$ such that $e^2 = e$ and $ex = ey \notin V(\Gamma)$. Then $\Gamma(x) = \Gamma(y)$.*

Munn provided a summary of Clifford's construction for non-null representations of a completely 0-simple inverse semigroup [6]:

Theorem 5.7.4. [70, Lemmas 2.2 and 2.3] *Let $B = \mathcal{M}^0(G; k, k; \Delta_k)$, a Brandt semigroup and let F be a field.*

1. *A Brandt semigroup B admits a non-null representation if and only if the rank of B is finite.*

2. *Let $\Gamma^\sharp$ be an irreducible representation of G^0 of degree l over F and let Γ^* be a mapping of B into $(F)_{kl}$, where $(F)_{kl}$ is the set of all $kl \times kl$ matrices over F, defined by the rule $\Gamma^*\{(a; i, j)\}$ is the $k \times k$ matrix of $l \times l$ blocks which has $\Gamma^\sharp(a)$ as its (i,j)-th block and zeros elsewhere. Then Γ^* is an irreducible representation of B, called the* basic extension *of $\Gamma^\sharp$.*

3. *Every irreducible representation Γ^* of B is, to within equivalence, of the form described in* (1)*.*

4. *The correspondence $\Gamma^* \longleftrightarrow \Gamma^\sharp$ established in* (1) *and* (2) *preserves reducibility.*

Recall from Section 4.3 that Clifford considered Rees matrix semigroup $S = \mathcal{M}^0(G; m, n; P)$ over a group G^0. Also, he defined the basic extension of a representation Γ of a group G^0 to a semigroup S as the unique extension of least possible degree over F. Every other extension of Γ reduces to the basic extension and null representations. Moreover, if the representation Γ of G^0 decomposes into two representations Γ' and Γ'' say, then the basic extension of Γ to S decomposes into the basic extensions of Γ' and Γ''. Based on Theorem 5.7.4, Munn inferred two further results.

Corollary 5.7.5. [70, Corollary 2.4] *Let B be a Brandt semigroup of finite rank k and let F be a field. Let Γ^* be a representation of B of degree n over F. Then k divides n and $\Gamma^*(B)$ is a homogeneous subset of $(F)_n$ such that for $x \in B \setminus \{0\}$*

$$\text{rank } \Gamma^*(x) = \frac{n}{k}.$$

Corollary 5.7.6. [70, Corollary 2.5] *Let B be a Brandt semigroup of finite rank k and let $f_1, \ldots, f_k$ be the distinct non-zero idempotents of B. Let Γ^* be an irreducible representation of B of degree n over F. Then*

$$\sum_{i=1}^{k} \Gamma^*(f_i) = I_n.$$

The notions of the following paragraph are attributed to Munn [71]. A congruence ρ on a semigroup $S = S^0$ is *proper* if $\{0\}$ is a ρ-class of S; that is, if $0\rho_\sharp\rho_\sharp^{-1} = 0$, or in other words, if $a\rho_\sharp = 0\rho_\sharp$ implies $a = 0$, where $\rho_\sharp$ denotes the natural homomorphism of S onto S/ρ. If S/ρ is a Brandt semigroup, then a congruence ρ is called a *Brandt congruence*. Conditions C1 and C2 are necessary conditions for the existence of a proper Brandt congruence on an inverse semigroup $S = S^0$. Let σ be a congruence satisfying: $\{0\}$ is a σ-class and if $x, y \in S \setminus \{0\}$ then $x\sigma y$ if and only if there exists an idempotent $e \in S$ such that $ex = ey \neq 0$. Then σ is a proper Brandt congruence on S and is the unique finest such congruence. That is, if τ is any proper Brandt congruence on S, then $\sigma \subseteq \tau$. The corresponding largest homomorphic image S/σ is denoted by B_S.

For any 0-simple inverse semigroup S of finite rank, not equal to zero, the following theorem shows that there is a natural relation between the non-null representations of S and those of B_S.

Theorem 5.7.7. [70, Theorem 2.6] *Let S be a* 0*-simple inverse semigroup and F be a field.*

1. *The semigroup S admits a non-null representation if and only if S has a finite rank, not equal to zero.*

2. *Let S have finite non-zero rank and let Γ^* be a non-null representation of B_S of degree n over F. Let $x \to \bar{x}$ denote the natural homomorphism of S onto B_S. Then a representation Γ, defined on S by:*

$$\Gamma(x) = \Gamma^*(\bar{x})$$

   for all x in S, is a non-null representation of S of degree n over F.

3. *Every non-null representation Γ of S is obtained from a non-null representation Γ^* of B_S in the above way.*

So, the representations of S depend only on those of B_S. Thus we have a one-to-one correspondence between representations of S and representations of B_S.

Suppose that S is a simple inverse semigroup and $S \neq S^0$. In this case $\{0\}$ is a prime ideal of S^0 and hence S^0 satisfies conditions C1 and C2. Thus, by deleting the zero elements from S^0 and B_{S^0}, we get that $B_{S^0} \setminus 0$ ($\cong G_S$) is a group. Thus, we can use the above result to find the representations of S in terms of the representations of G_S, the maximal group homomorphic image of S.

In the last part of this paper, Munn worked with an arbitrary inverse semigroup $S = S^0$. We need to set up some notation before stating the main result, which gives a method of obtaining the irreducible representations of an inverse semigroup $S = S^0$. Let $V \neq S$ be an ideal of S such that S/V satisfies condition C2. If S/V has non-zero rank, then there is an ideal M of S, strictly containing V (so that $M/V = (M/V)^0$) and having the property that M/V is the unique maximal ideal of S/V satisfying condition C1. Furthermore, M/V is an inverse semigroup satisfying condition C2 and thus the Brandt semigroup $B_{M/V}$ exists.

Theorem 5.7.8. [70, Theorem 3.1] *Let $S = S^0$ be an inverse semigroup and F be a field.*

1. *Let $V(\neq S)$ be an ideal of S such that S/V satisfies condition C2 and has finite non-zero rank k. Let M be an ideal of S containing V such that M/V is the maximal ideal of S/V satisfying condition C1.*

2. *Let $a \to \bar{a}$ denote the natural homomorphism of M onto $B_{M/V}$. Let $e_1, \ldots, e_k$ be idempotents of M such that $\bar{e_1}, \ldots, \bar{e_k}$ are the distinct non-zero idempotents of $B_{M/V}$. Let Γ^* be an irreducible representation of $B_{M/V}$ of degree n over F. Then Γ, defined on S by the rule that*

$$\Gamma(x) = \sum_{i=1}^{k} \Gamma^*(\overline{e_i x})$$

   for all x in S, is an irreducible representation of S of degree n over F. Furthermore, $V(\Gamma) = V$, $M(\Gamma) = M$, $r(\Gamma) = n \div k$, and for any x in S, rank $\Gamma(x)$ is a multiple of $r(\Gamma)$.

3. *Conversely, every irreducible representation of S over F is of the type described above.*

Theorem 5.7.8 shows that there is a natural bijective correspondence between the set of all irreducible representations of an inverse semigroup S and those of the associated Brandt semigroups $B_{M/V}$ of finite rank. In part 1 of this theorem, if instead of the ideal M we put an ideal T strictly containing V and such that T/V satisfies condition C1, then $T/V \subseteq M/V$ and so $B_{T/V} \cong B_{M/V}$ and the formula mentioned above holds. Then, Munn concluded the paper by utilizing Theorem

5.7.8 to find the principal irreducible representations of S and the prime irreducible representations of S.

We have now finished our review of the last paper in a course of six written by Munn between 1955 and 1964 about the theory of semigroup representations. Next we need to discuss the overall context of Munn's work and try to analyze his vision and line of thought. This is the content of the last section which also includes a synopsis, but before that we briefly discuss the work of Ponizovskii.

5.8 Ponizovskii's Contribution

Paying tribute to the achievement of Munn and Ponizovskii on semigroup algebras, Hollings says in [36, Chapter 11, page 279]: "the two main authors of [the] new representation theory were Munn and Ponizovskii.".

One might wonder why we have not so far devoted a complete section (or even a subsection) to Ponizovskii. The answer is twofold. First, we had poor access to Ponizovskii's work. In fact, we know little about his life: we did not find a piece of biographical information about Ponizovskii in the literature except for a small paragraph in [36], where Hollings also points out this absence of materials on the researcher's life. Using the limited available resources, we outline his early contribution to the theory of semigroup representations.

Ponizovskii's first paper was in 1956 and was entitled *On matrix representations of associative systems*. The second paper was *On irreducible matrix representations of finite semigroups*, in 1958. Both papers were written in Russian. What we found is that from the beginning of their work, Munn and Ponizovskii were mindful of the importance of the semisimplicity of semigroup algebras in studying semigroup representations. Regarding semisimplicity, in a direct parallel to Munn's work [65], Ponizovskii's 1956 work covered very similar ground to Munn. Additionally, with respect to the construction of all irreducible representations of Rees matrix semigroups, Ponizovskii's 1958 result corresponded to Munn's 1957 [36].

Therefore, since our main concern is in Munn's work and given the above reason, we limit ourselves to tackle briefly Ponizovskii's work on representations of semigroups later in Section 6.1.

5.9 Discussion and Synopsis

Each of Munn's papers has been treated in isolation, so we need to retrace the flow of ideas across the papers as a group. Chapter 3 addresses the image that we have in our minds concerning the Clifford-Munn-Ponizovskii correspondence. Let us see how Munn's papers led up to the correspondence.

At the start [65], Munn was motivated to develop the concepts of Maschke's Theorem, and its consequences, which enable us to study representations of a finite group via its irreducible representations. Thus, he determined the representations of a finite semigroup, for which the corresponding semigroup algebra is semisimple, in terms of representations of its principal factors. As every principal factor of S is a completely 0-simple semigroup or null, all the irreducible representations of S can be expressed in terms of irreducible representations of maximal subgroups by utilizing Clifford's results [6]. This was a combined work of Clifford's results and Munn's results. We recall that Clifford in his work [6] restricted a representation of a completely simple semigroup to a representation of its maximal subgroups, hence it was a restriction process and not the reduction process described in Chapter 3. Also, Clifford extended representations of the maximal subgroups of a completely simple semigroup to obtain representations of the whole semigroup. This gives a one-to-one correspondence between representations of a completely simple semigroup and the representations of its maximal subgroups. Thus Clifford's terminology in his theory of semigroup representations was the restriction and extension of a representation and not the reduction and induction presented in Chapter 3.

In [66], Munn gave a theoretical technique to connect, without recourse to reduction or induction presented in Chapter 3, the irreducible representations of a finite semigroup (whose algebra is semisimple) to the representations of its associated groups. He provided a one-to-one correspondence between the irreducible representations of a finite semigroup whose algebras are semisimple and the irreducible representations of its principal factors. The formulas in his theorems tell us that representations of a finite semigroup exist, but they do not explicitly describe what the representations look like. We can say that the idea here is similar to the Clifford-Munn-Ponizovskii correspondence, which relates representations of a semigroup to representations of its maximal subgroups but with different formulation. Wherein lies the difference? Munn's version of what we call the Clifford-Munn-Ponizovskii correspondence is different from the version in Chapter 3 which is in a modern form. The fundamental difference is that the one given in Chapter 3 is directly between a semigroup and its maximal subgroups using induction and reduction processes; whereas Munn's version which appeared in the results of [66, Section 2 and 3] is between a semigroup and its principal factors using the results of [66] and Clifford's result [6]. Thus until this stage, there was no explicit detailed explanation about reduction and induction process of representations, as given in Chapter 3. This came later in the paper of Rhodes and Zalcstein [95].

As indicated previously, Munn's paper, *The characters of the symmetric inverse semigroup* [67] is distinguished. It seems to be the first time in the three first papers that there is an explicit connection made between the irreducible representations of the symmetric inverse monoid I_n and the irreducible representations of the maximal subgroups, the symmetric groups. Munn provided a formula for irreducible characters of the symmetric inverse monoid in terms of the irreducible characters of the symmetric groups. This was a special case of his general theory developed in [66, Section 4] of representations of finite semigroups. We can say that Munn provided the induction process in a different form from the one given in Chapter 3.

In the previous two papers [65,66], we have a completely simple semigroup (Rees matrix semigroup) and one maximal subgroup. Munn is showing that there is a one-to-one correspondence between representations of the basic group and representations of the Rees matrix semigroup itself. In [67] he is presenting again the same idea, in a slightly more complicated way. Furthermore, the paper does not contain any hint or reference to reduction – the opposite process of induction – and we would anticipate that Munn would provide in his next work a reduction method or at least would generalize his induction process, but he seemingly did not.

For an arbitrary semigroup S, Munn [68] introduced the concept of an apex of a representation, and he called its associated representation a principal representation. Furthermore, he showed that when a semigroup S satisfies a minimal condition on its principal ideals, every irreducible representation of S is principal, thus has an apex. As described in Chapter 3, the apex of a representation is required in the reduction process. Munn however used the apex to construct irreducible representations of S in terms of irreducible representations of its principal factors, and he established a one-to-one correspondence between these representations. In this paper [68, Section 2], Munn proved that every irreducible representation has an apex. Independently, the proof was provided by Ponizovskii in his 1956 paper. The previous result is in fact Theorem 3.2.2 mentioned in Section 3.2. Munn then [69, 70] studied further the representations of inverse semigroups. The last paper [70] concluded with an open-ended statement: "the question of determining the reducible representations remains open". Contrary to our expectations, Munn quit the subject and left the question to his successors.

The theoretical treatment of the Clifford-Munn-Ponizovskii correspondence theory presented in Chapter 3 connects the irreducible representations of finite monoids with the irreducible representations of their maximal subgroups. After the above survey on Munn's contribution to the representation theory of semigroups, the ultimate question is to what extent Munn had this version of the correspondence. The answer is that the full form of the correspondence is not in Munn's papers examined above. Thus because we already know the correspondence, and this may not have been obvious to Munn, it is easy for us to recognize a partial version of Munn and others and determine the basis for attributing any aspect of this correspondence to Munn. The basis and partial versions of the commonly referred to as the Clifford-Munn-Ponizovskii correspondence theory that we have seen in Munn's work are in [66, Sections 2 and 3], [67, Sections 3] and [68, Section 2].

It is time now to identify the people other than Rees [91], Green [26], and Clifford [5, 6] who were involved in Munn's development of semigroup representation theory.

Clifford's representations of completely simple semigroups by matrices over a suitable field had been referred to by Munn in his first published paper [65], and afterwards. It is obvious that, from the very beginning and throughout these papers, Clifford's contribution was the prime and essential motivator for Munn's development of semigroup representation theory. Clifford in [6] obtained representations of Rees matrix semigroups as extension of representations of the basic groups. Munn then in [65], adapted Clifford's result and found irreducible representations of semisimple algebra of Rees matrix semigroups (Munn algebras). Munn's 1955 results about semisimplicity were found independently by Ponizovskii in his 1956 paper. The following result is due to Munn [65] and Ponizovskii's 1956 paper [58, Page 223]:

Theorem 5.9.1. *Let $S = S_{mn}[G, P]$ be a finite 0-simple semigroup and let F be a field. Then the full reducibility holds for representations of S if and only if P is invertible and the characteristic of F does not divide the order of G.*

As we mentioned previously that until this point Munn was not aware of Ponizovskii's work.

In the introduction of his 1960 paper [68], Munn wrote that "a different approach in constructing all representations of a finite semigroup whose algebra is semisimple was done independently by Ponizovskii in his 1956 paper entitled: *On matrix representations of associative systems*". In the same introduction, Munn indicated that the proofs of Theorem 1 5.5.2 and 2 5.5.3 of [68] include a generalization of the technique used by Hewitt and Zuckerman in their 1957 paper [33], where they obtained the irreducible representations of the full transformation monoid T_n. Hewitt and Zuckerman observed that if Γ is an irreducible representation of T_n with apex J, then $\Gamma(F[J^0])$, is a simple algebra over a field F and therefore has an identity, where $J^0 = J \cup \{0\}$. Thus there exist $\alpha_i \in F$ and $e_i \in J$ such that $\sum \alpha_i \Gamma(e_i)$ is the appropriate identity matrix. Then

$$\Gamma(a) = \sum \alpha_i \Gamma(ae_i),$$

for every $a \in T_n$ and Γ is determined by the action of a on J^0. This result was used by Ponizovskii (1958) and also by Munn in [68]. In [33], Hewitt and Zuckerman utilized their 1955 joint paper *Finite dimensional convolution algebras*. At the end of paper [68, Page 309] Munn mentioned that "a method of obtaining all the irreducible representations of a finite semigroup over an arbitrary field has recently been given by Ponizovskii, *On irreducible matrix representations of finite semigroups*, 1958".

During his Ph.D. project, Munn met Preston for the first time and Preston told him about the new invention inverse semigroups which became preferable examples for Munn to apply his results. Also during that time, Munn was aware of the Wagner-Preston representation theorem 2.2.13 and Preston's 1954 work [81] on representations of inverse semigroups by partial transformations. In his last paper [70], Munn showed his knowledge of Warne's 1963 work [106] on matrix representations of d-simple semigroups. Note that the modern term for such semigroups is bisimple semigroups. Warne obtained irreducible representations of a bisimple inverse semigroup with an identity. Munn started researching bisimple semigroups just two years after the last paper on semigroup representation theory. Munn wrote seven papers investigating bisimple semigroups, one of them was a joint paper with his first Ph.D. student Norman Reilly. Munn also utilized Warne's work on bisimple inverse semigroups with identity.

Norman Reilly, in his paper [92] pointed out that Munn in his last few years returned to his first interest in semigroup algebras. He focused on linking semigroup properties to ring-theoretic properties of their algebras and enjoyed working jointly with Michael Crabb between 1995-2007 [38, 92] with a total of nine papers. In fact, Crabb was essentially a functional analyst, but his work with Munn was mainly on semigroup algebras. Since semigroup algebras are the key tool in semigroup representation theory, it seems that Munn's successful collaboration with Michael Crabb made him return indirectly to representations of semigroups.

We conclude that the Clifford-Munn-Ponizovskii correspondence was not stated in a fully-fledged form in any of Munn's papers. Looking back at the work of Clifford in Chapter 4, we can see only glimmers of the Clifford-Munn-Ponizovskii correspondence in their works. Now, the following questions arise: if Clifford and Munn did not have the full correspondence, so where did it first appear? And what was the first modern formulation of the correspondence? The answers will be in Section 6.1.

In the next chapter, we will revisit some of the works mentioned here.

Chapter 6

Some Conclusions

This chapter examines the main arguments of the thesis and is organized as follows. We begin with a discussion of the development of semigroup representation theory including, in particular, the Clifford-Munn-Ponizovskii correspondence theory. The results of Chapters 4 and 5 are then summarized to connect the flow of ideas through the timeline of the theoretical development. The following section addresses the reasons why semigroup representation theory lay dormant during the seventies and the eighties. Finally, we show that the theory was revived and redeveloped in the late nineties. We especially refer to [8, 36, 102] for the discussion of the main points in this chapter.

6.1 Development of the Representation Theory of Semigroups

The theory of semigroup representations developed significantly during the fifties and sixties, due largely to the efforts of Clifford, Munn, Ponizovskii, Hewitt and Zuckerman, and then to Lallement and Petrich, Preston, McAlister, and Rhodes and Zalcstein. Although this thesis concentrates mainly on the work of Munn on the representation theory of semigroups, other parallel works that strongly depend on Munn's ideas are also mentioned. In this section, we try to assess every contribution that impacts significantly on the theory. In order to tie up the story and to provide a complete and detailed picture of the development, we recall and outline some of the work discussed previously in Chapters 4 and 5.

We start with an overview of the work of Munn. Let us now recall from Chapter 1 the main representation theorem of semisimple algebras: Maschke's Theorem is one of the fundamental results in group representation theory. It states that if G is a finite group and k is a field, then the group algebra of G is semisimple if and only if the characteristic of k does not divide the order of G. Thereafter, Van der Waerden

(1903-1996) reformulated this important result and stated the main representation theorem of semisimple algebras:

Theorem 6.1.1. [8, Section 5.2] *Every representation of a finite group G over a field k is completely reducible if and only if the characteristic of k does not divide the order of G.*

In fact, this result shows the significant role of semisimplicity in the representation theory of algebras, whether in semigroup or group theory. In terms of groups, this result and Maschke's Theorem enable us to study representations of finite groups over a certain field via its irreducible representations. This was later developed for semigroup algebras by Munn [64, 65], in parallel with Ponizovskii (1956). Theorem 6.1.1 provided the starting point for Munn. In the introduction of his PhD thesis in 1955, he stated that the central problem is extending Theorem 6.1.1 to the case of a finite semigroup. Munn studied this case and found necessary and sufficient conditions for a finite semigroup and its semigroup algebra to be semisimple.

Munn then studied the representations of finite semigroups and semigroup algebras in the semisimple case. In order to illustrate the concept, he applied his results to inverse semigroups. Basically, Munn's results draw on the early works of Clifford on semigroup theory, especially the work on matrix representations of completely simple semigroups [6]. As we saw in Chapter 4 and in comparison with the early work of Suschkewitsch, Clifford's results constitute the core paradigm shift in representations of semigroups. However, Suschkewitsch's 1933 work laid the ground for the theory of semigroup representations, as seen in Section 4.3.

Next, Munn addressed the problem of turning the known representations of the maximal subgroups into representations of the given semigroup. Basically, he obtained the irreducible representations of a finite semigroup S from the irreducible representations of the completely (0-)simple principal factors of S, and these factors are connected, via Clifford's results [6], with the irreducible representations of the maximal subgroups of S. Munn started with representations of an associated group and ended up with all representations of of a finite semigroup. The idea was looking for representations for a Rees matrix semigroup via the representations of its basic group. Here, Munn was demonstrating that there is a one-to-one correspondence between the representations of a Rees matrix semigroup and representations of its basic group. From our point of view, this was the start of the well-known Clifford-Munn-Ponizovskii correspondence.

Munn obtained representations of different types of simple semigroups: non-singular semigroups and arbitrary semigroups whose algebras are semisimple [65,66].

As usual, inverse semigroups were the ideal semigroups to which to apply the results. Then, he studied as a particular example the symmetric inverse semigroup. Using the fact that every finite inverse semigroup is completely reducible (we saw this result in Chapter 3), Munn succeeded in describing all irreducible characters of the symmetric inverse semigroup from irreducible characters of its maximal subgroups, which are isomorphic to symmetric groups [67]. As we have indicated in Chapter 5, this is the point where Munn might have formulated what we now understand as the Clifford-Munn-Ponizovskii correspondence, but he did not.

The procedure where we take a representation of a maximal subgroup and turn it into a representation of the whole semigroup is the induction process, as seen in Chapter 3. The induction process for the symmetric inverse semigroup I_n in Munn's sense is slightly different from the modern one which is provided in [22, Section 11.2]. First, he relates the representations of the symmetric group to the representations of principal factors of the symmetric inverse semigroup. Second, he defines the representations of the symmetric inverse semigroup in terms of those of its principal factors. This could be the first time where there was an explicit connection between the irreducible representations of a semigroup and the irreducible representations of the maximal subgroups. During that time, Munn was using again the same idea of the Clifford-Munn-Ponizovskii correspondence, but in a slightly more complicated example, as seen in Section 5.4.

In 1960, Munn was aware of the work of Ponizovskii (1956) on semigroup representations. For the first time in his paper [68], Munn pointed out that both of them worked independently on the same problem of finding all representations of a finite semigroup whose algebra is semisimple over a specific field. He also stated that Ponizovskii addressed independently the same problem of obtaining all representations of a finite semigroup whose algebra is semisimple [68, page 295]. Moreover, Munn took advantage of the technique of Hewitt and Zuckerman in constructing the irreducible representations of the full transformation monoid T_n over the complex field [33] (we will discuss this work later in this chapter). After defining the notion of a principal representation, Munn obtained the irreducible principal representations for an arbitrary semigroup and showed that there is a one-to-one correspondence between these and the irreducible 0-restricted representations of the simple principal factors of the semigroup. Most of Munn's work continued to include applications of the results to inverse semigroups.

Apparently, the study of representations of inverse semigroups was the most popular theme in the mathematical works of Munn on the theory of semigroup representations. Also, in 1964 he introduced a new type of representation – the prime representations [70]. For an arbitrary inverse semigroup, he defined the prime irreducible representations in terms of those of a certain group – the maximal group homomorphic image. Next, he gave a definition of a class of inverse semigroups for which every irreducible representation is prime. These are called intraregular inverse semigroups. Munn then applied the previous result on prime representations to them. We can thus see that the years 1956 to 1960 were productive years for Munn.

The year 1964 was the last one in which Munn studied matrix representations of semigroups. Here [70] we have more about representations of 0-simple inverse semigroups. First, he showed that there is a natural one-to-one correspondence between representations of 0-simple inverse semigroups and those of the associated quotient semigroup (a Brandt semigroup). Second, for a simple inverse semigroup, Munn proved that there is a natural one-to-one correspondence between representations of 0-simple inverse semigroup and those of the maximal group homomorphic image. Finally, Munn demonstrated the correspondence between the irreducible representations of an inverse semigroup and the irreducible representations of the associated Brandt semigroups of finite rank.

After our survey of the development of semigroup representations, and taking into consideration that it was developed principally by Munn, we can divide the stages of the development of the theory as follows:

- The Pre-Munn era; the initial phase (1933-1954): This contains the early work of Suschkewitsch and the work of Rees and Green on the structure of semigroups. This stage includes the work of Clifford in 1942, described by Hollings [36, Section 11.2, page 285]: "Clifford's work might therefore be viewed as bridging the initial work of Suschkewitsch and that of Munn and Ponizovskii".

- The Munn era (1955-1964): In addition to Munn's work, this stage includes the work of Clifford, Hewitt and Zuckerman, Ponizovskii and Preston.

- The Post-Munn era (1965-1975): This period contains the work of McAlister, Lallement and Petrich, Preston, and Rhodes and Zalcstein.

- The dormant era (1975-1995).

- The revival era (1995-current): This follows a two-decade period of inactivity in the development of the theory of semigroup representations. Here, we will have the work of Putcha, Rhodes, Brown, Steinberg and others. This is discussed in the third section of the present chapter.

We will review each stage individually.

The Pre-Munn era (1933-1954):

The first paper on semigroup theory was published in 1928 and the first on matrix representations of semigroups was in 1933. Both papers were by Suschkewitsch. According to Hollings (in an unpublished paper), Munn as a student was aware of the first paper but he did not know about the other papers of Suschkewitsch listed under his name in the bibliography of [8]. The most influential works on semigroup theory in this period were those of Rees [91], Clifford [5], and Green [26], [36, Section 11.4] . Because of the absence of abstract algebra in the syllabus of the University of Glasgow, where Munn graduated as an undergraduate student, Munn started his PhD journey in Cambridge with little knowledge of algebra. The contributions of Rees [91], Clifford [5], and Green [26] helped him to build a strong foundation in semigroup theory. The paper of Rees in 1940 attracted Munn to become a semigroup theorist, and Clifford's papers [5, 6] provided him with the starting point of his PhD project.

We now briefly discuss Suschkewitsch's 1933 paper *Über die Matrizendarstellung der verallgemeinerten Gruppen*. In this paper, Suschkewitsch showed the importance of matrices in the study of semigroup theory, which becomes in a concrete form after involving matrices. His goal was to characterize all representations of two certain types of finite semigroups which were called *left groups* (in a modern terminology, a left group is a semigroup which is both left simple and right cancellative) and *Kerngroups* (the union of all the minimal left ideals of a semigroup S; this is in fact a finite simple semigroup without a zero element) [36, Section 11.1].

Firstly, Suschkewitsch stated the following theorem for an arbitrary group:

Theorem 6.1.2. [36, Theorem 11.3] *All representations of a (finite) group by means of $m \times m$ matrices of rank $n < m$ may be obtained from the representations of the same group by $n \times n$ matrices of rank n.*

Secondly, he extended this result to the case of left groups. According to Suschkewitsch, a Kerngruppe may be written as a union of left groups, so he used the result on left groups also to obtain representations of Kerngruppen by means of $m \times m$ matrices of rank $n < m$. According to [36, Section 11.1, page 282], Suschkewitsch's construction of representations of Kerngruppen is very long and Clifford's 1942 paper provides a simpler description of these representations, as appeared in Section 4.3. Suschkewitsch ended his paper by applications first to Klein 4-group in the form of 2×2 matrices of rank 2, then to the Kerngruppe formed from four isomorphic copies of Klein-4-group in terms of 3×3 matrices of rank 2 [36, Section 11.1].

Clifford, in his 1942 paper [6], provided a clear definition of a representation close to the modern one, as follows:

Definition 6.1.3. [6, Page 327] Let S be a semigroup, a be an element in S and F be a field. Let $T(a)$ be an $n \times n$ matrix corresponds to a. If, for all $a, b \in S$,

$$T(ab) = T(a)T(b),$$

then the correspondence $\Gamma : a \to T(a)$ is called a *matrix representation* of S over F of degree n.

This is similar to Definition 3.1.3 provided in Section 3.1. Clifford was influenced by Rees [91] and Suschkewitsch (1933) and by his work on group representations. He showed his awareness of these works by developing their results. He used his own notation for the Rees matrix semigroup: it was denoted by $(a)_{ij}$, instead of the triple form (i, a, j), where $a \in G^0 = G \cup \{0\}$. Clifford's aim was to construct all finite-dimensional irreducible representations of a Rees matrix semigroup over a group with zero element as extensions of irreducible representations of its basic group (its maximal subgroup). He also obtained irreducible representations of the maximal subgroup as restrictions of the irreducible representations of the Rees matrix semigroup, as seen in Section 4.3.

Clifford discussed also the irreducibility and equivalence of these representations. He then ended the paper with illustrations involving Brandt groupoids. This work is regarded as the core of the development of the theory because it was an inspiration for many and led to a great productive phase by Munn and others.

The Munn era (1955-1964):

The outstanding contributions in this period were made by Munn and he left a prominent mark on the theory. In this part, we discuss the works of other mathematicians in the field.

Any collection of one-one partial mappings, with or without 0, whose inclusion is not always necessary, was termed simply a *semigroup of 1-1 mappings* if it was closed under composition. If, in addition, it was closed under inversion, then the semigroup was called to be *complete* [36, Section 10.6, page 274].

In 1954, Preston published his first paper on the representation theory of semigroups [81], where he investigated representations of a new discovery at that time –

the inverse semigroups introduced by Wagner in 1952 (Russian) and independently by Preston in 1954. Preston proved the following result:

Theorem 6.1.4. [81, Theorem 1] *A semigroup admits a faithful representation as a complete semigroup of 1-1 mappings if and only if it is an inverse semigroup.*

Independently, this result also obtained by Wagner in his 1952 paper *Generalised groups.* In fact, the representation in the previous theorem is indeed the analogue of Cayley's theorem in inverse semigroup theory, the Wagner-Preston representation theorem 2.2.13 mentioned in Section 2.1.

Due to a lack of communication, Ponizovskii (1956) achieved very similar results on the semisimplicity of semigroups to those of Munn in his paper [65]. Both described the semisimplicity of semigroup algebras in terms of the invertibility of the sandwich matrices of Rees matrix semigroups over appropriate group algebras. Although Ponizovskii published his first paper, *On matrix representations of associative systems* (Russian) in 1956, he underlined that he obtained these results in 1952 and 1953 – that is, before the publication of Munn's 1955 work on the topic. For a given field P, Ponizovskii termed a semigroup whose semigroup algebra over P is semisimple a *P-system.* He investigated the semisimplicity of the symmetric inverse semigroup and of Rees matrix semigroups [36, Section 11.5]. Interestingly, even though Munn and Ponizovskii were aware of each other's work by the end of the 1950's, "both continued to study representation theoretic problems, seemingly without worrying that they might be duplicating the work of their counterpart" [36, Section 11.5, page 301]. This is because there was no real prospect of being able to communicate.

Among the results of Ponizovskii's 1956 paper is the following theorem:

Theorem 6.1.5. [36, Section 11.5] *An associative system S with a principal series is a P-system if and only if all its principal factors are P-systems.*

In modern terminology, an associative system S means a semigroup. Ponizovskii studied the P-system in the cases of a field with characteristic zero. A general method was given to find all irreducible matrix representations of a finite semigroup whose algebra over an algebraically closed field is semisimple.

Utilizing only the results of Rees [91] and the results of Clifford [6], Ponizovskii published another paper in 1958 where he studied irreducible matrix representations of finite semigroups equipped with a zero. The title of this paper is: *On irreducible matrix representations of finite semigroups.* We use an (unpublished) translation of this paper by Hollings.

Over a fixed field, Ponizovskii proved that every representation of every Rees quotient of a finite semigroup S with zero generates, in one way only, an irreducible representation of the semigroup S. As mentioned, this study was based on Clifford's fundamental construction of irreducible matrix representations of completely 0-simple semigroups [6]. Thus, all of the irreducible matrix representations of S can be obtained. He also stated that the total number of inequivalent irreducible representations of a semigroup S (which has a principal series) is equal to the sum of the number of inequivalent irreducible representations of each of its principal factors.

The study of representation theory of the full transformation monoid T_n has a long history, beginning with the joint work of Hewitt and Zuckerman [33]. In 1957, Hewitt and Zuckerman published the paper, *The irreducible representations of a semigroup related to the symmetric group* [33]. In it, they described the representations of the full transformation monoid T_n on a finite set $[n]$ and determined all representations of this semigroup. As stated previously, in the same year, Munn constructed all characters of irreducible representations of the symmetric inverse monoid I_n [67]. Regarding the term *symmetric inverse semigroup*, Hollings [36] indicates that the first appearance of this term can be traced back to Munn's 1957 paper [67]. Also, Munn in [68] generalized Hewitt and Zuckerman main theorem to an arbitrary finite monoids. In this paper [68, Page 296], Munn pointed out that the algebra of the full transformation monoid T_n cannot be semisimple, hence its representations are not completely reducible. This was examined in Sections 3.1 and 3.2. Around 40 years later, Putcha in [87] computed the character table of the full transformation monoid T_n.

Schützenberger [97, 98], Preston [83], and Tully [104, 105] dealt with a different approach to matrix representations of semigroups than that of Munn, called a *monomial representation*. According to Schützenberger in his 1957 paper [97]: *$\mathcal{D}$-représentation des demigroupes*, for each $\mathcal{D}$-class of a semigroup S with identity, there is a homomorphism of S into a semigroup of matrices over a group with zero. Based on this result, and without Schützenberger's restriction that $\mathcal{D}$-classes should be finite, Preston constructed the direct sum of the Schützenberger representations determined by the $\mathcal{D}$-classes of S and discussed the duals of these representations. These results are presented in Preston's paper [83].

Corresponding to each $\mathcal{D}$-class in S, Preston constructed two representations M and M'. Let $\Gamma = \oplus M$ and $\Gamma' = \oplus M'$ be the direct sums over the $\mathcal{D}$-classes of S. Furthermore, $\Gamma \oplus \Gamma'$ denotes the direct sum of Γ and Γ'. Preston stated a number of necessary and sufficient conditions for each of the representations Γ, Γ', and $\Gamma \oplus \Gamma'$ to be faithful representations.

Preston then investigated the faithfulness of these representations when a semigroup S is regular and inverse. If Γ and Γ' are the representations defined in the previous paragraph then:

Theorem 6.1.6. [83, Theorem 2] *Let S be a regular semigroup. A representation $\Gamma \oplus \Gamma'$ is a faithful representation of S.*

Generally, when S is a regular semigroup, both representations Γ and Γ' are not faithful. In the case that S is an inverse semigroup, however, they are faithful (this was an observation of Clifford).

In 1960, Clifford published his second paper [7] on representations of semigroups which is supplemental to his earlier results in [6]. He proved that all irreducible representations of a completely simple semigroup S are obtained as the basic extensions to S of the irreducible representations of the basic group G.

We end this subsection with a brief outline of two works. The first is the book of Clifford and Preston: *The Algebraic Theory of Semigroup*, volume 1 [8]. We have observed that since this book was published in 1961, it has become an important reference in semigroup theory, particularly in the representation theory of semigroups. It can be said that it was considered as the Semigroup Bible at that time. According to Munn, the notations of semigroup theory became largely standardized after the publication of this book. The first volume of the book contains five chapters on the structure of semigroups, whereas the fifth chapters is about semigroup representations by matrices.

Miller wrote on the occasion of Clifford's sixty-fifth birthday [62, Page 9]: "the Clifford [and] Preston work, which goes beyond its predecessors in breadth and depth, is both a treatise and a textbook". Hollings says "it is arguably the most influential semigroup textbook to date" [35, Section 2, page 501]. That is why we consult Clifford and Preston's book [8] as our main reference for the study of the development of semigroup representation theory during Munn's time and even before that.

The second work of this part is a paper written by Warne in 1963 [106], where he considered a type of semigroup called a d-simple semigroup. Let S be any semigroup with identity element. Two elements a and b in S are said to be d-equivalent if there exists an element d in S such that $Sa = Sd$ and $bS = dS$. Then a semigroup S is d-simple if it consists of a single class of d-equivalent elements. In other words, a semigroup S is d-simple if any two elements in S are $\mathcal{D}$-related. Warne determined matrix representations of d-simple semigroups. We remind the reader that the modern term for such semigroups is bisimple semigroups. Regarding this paper, Warne wrote [106, Page 434]: "our paper is the first paper in matrix representations of semigroups that makes use of the technique of embedding a semigroup in more special semigroup. We feel this technique may have further applications in the representation theory".

The Post-Munn era (1965-1975):

In this subsection, we provide a general outline of the work done after Munn ceased to work on representation theory. In 1967, McAlister produced his first paper [55], where he made use of Munn's paper [70]. McAlister generalized the method that was given by Munn, to obtain the irreducible representations of an inverse semigroup, to the case of an arbitrary semigroup. Throughout McAlister's paper, all semigroups are with zero. As a preparation for the main objective, McAlister studied first the nature of a semigroup $S = S^0$ that obeys a number of conditions. The conditions are:

C1. For any $a, b, x \in S$, if $axb = 0$, then $ax = 0$ or $xb = 0$. This condition is due to Munn [70].

C2. If $a \in S$ and $aSa = \{0\}$, then $a = 0$.

C3. If $a, b, x, y \in S$, then the relations $ax = bx \neq 0$ and $ya = yb \neq 0$ together imply that $a = b$.

A semigroup $S = S^0$ satisfies $C1$ is called *categorical at zero.* If S obeys $C2$, it is called *indecomposable at zero.* Moreover, if S obeys both $C1$ and $C2$, it is called *0-primary.* If S obeys $C3$, it is called *weakly* 0-*cancellative* or *weakly reductive.*

A representation θ of a semigroup $S = S^0$ is said to be 0-restricted if it is a 0-restricted homomorphism: if $a\theta = 0\theta$ then $a = 0$. McAlister investigated the existence of this type of representation and then characterized all 0-restricted irreducible representations of an arbitrary semigroup $S = S^0$ via the 0-restricted irreducible representations of certain associated semigroups. Also, he discussed the non-null representations in the case of a 0-simple semigroup.

Two years later, Preston [84] studied matrix representations of inverse semigroups. He started by studying the nature of a type of inverse semigroup called a *primitive inverse semigroup of matrices*, and he dealt with inverse semigroups with zero. Preston pointed out that he developed the methods used previously by Munn [70] to determine the matrix representations of inverse semigroups. According to Preston, Munn's results on 0-simple inverse semigroups are a special case of the results of [84].

In 1969, Lallement and Petrich published [47] which concerns irreducible matrix representations of finite semigroups. They provided a number of results on the irreducible representations of finite 0-simple semigroups. Factorizations of a matrix H of finite rank are described as solutions to the matric equation $XY = H$, where X has a finite number of columns, and Y has a finite number of rows. Clifford used this factorization problem to determined all representations of a completely 0-simple semigroup [6, 7]. On the other hand, Lallement and Petrich managed to give expressions for these representations without the factorization conditions, instead employing Schützenberger representations by monomial matrices.

Let S be a finite semigroup and k be a field. Fix a $\mathcal{J}$-class J of S and let M_J be the k-vector space with basis the $\mathcal{J}$-class J. An element $s \in S$ acts on a basis vector x of M_J by

$$x \cdot s = \begin{cases} xs \text{ if } xs \in J, \\ 0 \text{ otherwise.} \end{cases}$$

The M_J are called *Schützenberger* representations of S.

Lallement and Petrich defined a new representation called a *standard representation* as follows:

Definition 6.1.7. [47, Definition 1.1] Let J be a regular $\mathcal{J}$-class of a finite semigroup S and G be its Schützenberger group (G is isomorphic to the maximal subgroups of S contained in J). Let M_J be the Schützenberger representation of S defined by J. If γ is a representation of G^0 by matrices over a field k, define $\Gamma(x)$ for x in S to be $\Gamma(x) = \gamma[M_J(x)]$, the matrix obtained by replacing each entry of $M_J(x)$ by its image under γ. Then Γ is a representation of S by matrices over k, and is called the *standard representation* defined by J and γ.

For any nonempty subset W of a vector space V, $[W]$ is the subspace of V spanned by W. The following theorem provides a description of all irreducible representations of a finite semigroup:

Theorem 6.1.8. [47, Theorem 1.7] *Let S be a finite semigroup. Let Γ be the standard representation defined as in Definition 6.1.7 by a regular $\mathcal{J}$-class J and an irreducible representation γ over k of the Schützenberger group G of J. Then Γ has a unique non-null irreducible constituent Γ^* such that $[\Gamma^*(S)]$ coincides with $[\Gamma^*(J)]$, where $[\Gamma^*(T)]$ $(T \subseteq S)$ denotes the linear closure of $\Gamma^*(T)$. Conversely, every irreducible representation of S is equivalent to the constituent Γ^* of a standard representation Γ defined above.*

The proof of this theorem ends with a general formula for an irreducible representation Γ^* of S defined by its apex J and an irreducible representation γ of the group of J. For every x in S, the formula is

$$\Gamma^*(x) = I_{t,nr} A\gamma[M_J(x)]A^{-1} I_{nr,t},$$

where r is the degree of γ, nr is the degree of Γ, $I_{nr,t}$ is the $nr \times t$ matrix whose entries are $(I_{nr,t})_{ij} = 1$ when $i = j$ and 0 elsewhere, and $t = \operatorname{rank} \gamma(P)$. Here, P is a matrix of the principal factor $Q(J)$ relative to J, and since J is a regular $\mathcal{J}$-class of S, we have $Q(J) \cong$ Rees matrix semigroup $\mathcal{M}^0(G; I, \Lambda; P)$. We may always choose a basis so that

$$\gamma(P) = \begin{pmatrix} I_r & 0 \\ 0 & 0 \end{pmatrix},$$

and then the matrix A is the appropriate change-of-basis matrix. In Theorem 6.1.8, if S is a finite 0-simple semigroup, then the standard representation Γ has only one non-null constituent Γ^*. Generally, if S is not 0-simple semigroup, then Γ has non-null constituents distinct from Γ^*.

In 1969, McAlister published his second paper on the theme [56]. He explained the concept of basic representations from his own perspective. Moreover, he proved that when S is a completely 0-simple semigroup, then his definition of this type of representation is equivalent to Clifford's concept of basic extension of matrix representations of a group G to matrix representations of $\mathcal{M}^0(G; I, \Lambda; P)$ [6, 7].

Definition 6.1.9. [56, Section 2] A representation Γ of a semigroup $S = S^0$ by linear transformations of the finite-dimensional vector space V over a field k is called *basic* if for each ideal N of S such that $\Gamma(N) \neq 0$, we have:

1. $U\Gamma(N) \neq 0$ whenever U is a nonzero subspace of V;

2. $[V\Gamma(N)] = V$.

For a completely 0-simple semigroup, the next theorem shows that a basic representation is equivalent to the definition provided by Clifford in [6, 7]:

Theorem 6.1.10. [56, Theorem 3.12] *Let $S = \mathcal{M}^0(G; I, \Lambda; P)$ be a completely 0-simple semigroup and Γ be a representation of S of degree n over a field k. Then the following conditions on Γ are equivalent:*

1. *Γ is basic;*

2. *Γ is basic in the sense of Clifford (it is a basic extension of matrix representation of G).*

Recall that a representation Γ is called *proper* if

1. $\Gamma(z) = 0$ when S has a zero element z;

2. Γ is not decomposable into a direct sum of two representations, one of which is the null representation.

For finite 0-simple semigroups, we have the following characterization:

Theorem 6.1.11. [56, Theorem 2.3] *Let $S = \mathcal{M}^0(G; m, n; P)$ be a finite 0-simple semigroup and let k be a field. Then the following are equivalent:*

1. *the basic radical $B_k(S) = \{x \in k_0[S] : SxS = 0\}$ is equal to zero, where $k_0[S]$ is the contracted algebra of S over k;*

2. *P is invertible over the algebra $k[G]$;*

3. *$k_0[S]$ has an identity;*

4. *$m = n$ and $k_0[S]$ is isomorphic to the algebra of all $n \times n$ matrices over $k[G]$;*

5. *S is quasi-simple over k; that is, all proper representations of S over k decompose into basic representations;*

6. *each proper representation of S over k is basic.*

We refer the reader to Section 5.2 for the concept of contracted algebra of a finite semigroup.

Additionally, we have the next result for finite semigroups:

Theorem 6.1.12. [56, Theorem 2.4] *Let $S = S^0$ be a finite semigroup and let k be a field. Then each proper representation of S over k is basic if and only if S is 0-simple and satisfies the six conditions of Theorem 6.1.11.*

McAlister constructed all basic representations of an arbitrary semigroup in terms of basic representations of completely 0-simple semigroups. This construction is a generalization of Clifford's 1942 results [6]. A number of results on homogeneous semigroups of linear transformations and on their representations were given. He then defined a *fully basically reducible representation* as follows:

Definition 6.1.13. [56, Section 7] Let $S = S^0$ be a semigroup and let Γ be a representation of S of degree n over a field k. Then Γ is fully basically reducible if each non-null indecomposable representation of Γ is basic. Any fully reducible representation of S is fully basically reducible.

Among the results in [56] regarding fully basically reducible representations, we select the following:

Theorem 6.1.14. [56, Theorem 7.1] *Any basic representation of a semigroup $S = S^0$ of degree n over a field k is fully basically reducible.*

Corollary 6.1.15. [56, Corollary 2] *A representation of S of degree n over a field k is fully basically reducible if and only if it is the direct sum of basic representations.*

In the last section of [56], McAlister applied his results to inverse semigroups.

McAlister published his third paper [57] in 1970 also regarding basic representations of finite semigroups. The objective of the paper was the study of basic representations of finite semigroups. He started with the basic representations of completely 0-simple semigroups. The reason for this is that the 0-simple principal factors of S determine the basic representations of S. Furthermore, the paper contains a discussion about the co-called *quasisimplicity* of semigroups and their algebras. McAlister investigated first the quasisimplicity of completely 0-simple semigroups and then of arbitrary finite semigroups.

Let $S = S^0$ be a finite semigroup and let k be a field. If all representations of S over k decompose into basic representations, then S is said to be *quasisimple* over k. Explicitly, McAlister indicated that the description of the quasisimplicity of a finite semigroup is similar to that of the semisimplicity of a finite semigroup as described by Munn in [65]. The main result is:

Theorem 6.1.16. [57, Theorem 2.4] *Let $S = \mathcal{M}^0(G; m, n; P)$ be a completely 0-simple semigroup and let k be a field. Then the following are equivalent:*

1. *S is quasisimple over k;*
2. *the regular representation γ of G over k extends properly only to a basic representation of S;*
3. *$m = n$ and $\gamma(P)$ is invertible over k;*
4. *$m = n$ and P is invertible over the algebra $k[G]$;*
5. *the (contracted) semigroup algebra $k[S]$ of S over k has an identity;*
6. *$m = n$ and $k[S] \cong (k[G])_n$, where $(k[G])_n$ is the algebra of all $n \times n$ matrices over $k[G]$.*

The following theorem shows necessary and sufficient conditions for an arbitrary finite semigroup to be a quasisimple semigroup:

Theorem 6.1.17. [57, Theorem 4.2] *Let $S = S^0$ be a finite semigroup and let k be a field. Then S is quasisimple over k if and only if each principal factor of S is quasisimple over k.*

Corollary 6.1.18. [57, Corollary 4.2.1] *Let $S = S^0$ be a finite semigroup and let k be a field. If S is quasisimple over k, then each ideal of S is quasisimple over k.*

McAlister then provided a description of the algebra of a finite quasisimple semigroup over a field k:

Theorem 6.1.19. [57, Theorem 5.1] *Let $S = S^0$ be a finite semigroup and let k be a field. Then S is quasisimple over k if and only if the algebra $k[I]$ has an identity for each ideal I of S.*

The reason for the similarity of the results of this paper to those of Munn in [65] is given by the following:

Corollary 6.1.20. [57, Corollary 5.2.1] *Let $S = S^0$ be a finite semigroup and let k be a field. Then the algebra $k[S]$ is semisimple if and only if S is quasisimple over k and the characteristic of k does not divide the order of any maximal subgroup of S.*

We continue with McAlister's work. He published three papers in 1971. The first [58] is in fact the first part of a survey article which deals with representations of completely 0-simple semigroups and basic representations of arbitrary semigroups. The second [59] contains a discussion of representations of inverse semigroups and other special cases, and representations of finite semigroups. The subject of the third 1971 paper [60] is that of constructing the representations of the algebra of a completely 0-simple semigroup (Munn ring) in a different and a simpler way from Clifford's theory in [6, 7]. Unlike Clifford, representations in McAlister's paper [60] do not need to be finite dimensional or over a field. McAlister studied the morphisms between representations of the Munn ring, over a ring with identity. He also defined the notion of a basic representation in a slightly different way than Definition 6.1.9 above.

Definition 6.1.21. [60, Definition 1.3] Let U be a ring and R be a ring with identity. Then a representation Γ of U over R is a morphism of the ring U into the ring $\mathrm{Hom}(V, V)$ of endomorphisms of a vector space V.

Definition 6.1.22. [60, Definition 1.4] For each representation $\Gamma : U \to \mathrm{Hom}(V, V)$ of a ring U over a ring R with identity we define

$$N = N(\Gamma) = \{v \in V : v\Gamma(x) = 0 \text{ for each } x \in U\},$$

$$I = I(\Gamma) = \text{submodule generated by } \{v\Gamma(x) : v \in V, x \in U\}.$$

A representation Γ is a null representation if $N(\Gamma) = V$ or, equivalently, if $I(\Gamma) = 0$. Γ is basic if $N(\Gamma) = 0$ and $I(\Gamma) = V$.

Basic representations are a more general type than irreducible representations, and it turns out that every irreducible representation is basic. McAlister then showed that the category of basic representations of a completely 0-simple semigroup S is equivalent to the category of proper representations of any of the maximal subgroups of S.

Based on the work of Munn [66, 67] and others, McAlister's 1972 paper [61] is on characters of finite semigroups. He developed further the character theory of monoids over the complex field. Independently, this theory was also developed by Rhodes and Zalcstein in [95]. The set of characters of a semigroup S forms an additive commutative cancellative semigroup under the operation of pointwise addition. The pointwise multiplication of two characters also gives a character. Adjoining the negatives of these functions gives a ring $\mathrm{ch}(S)$ called the *character ring of S*. A character of a representation of S is irreducible if and only if the representation is irreducible. Thus, the irreducible characters of S generate the character ring $\mathrm{ch}(S)$.

In [64, Theorem 8.10, page 101], Munn proved the following result:

Theorem 6.1.23. *Let S be a simple semigroup with zero and F be a splitting field of characteristic zero for the basic group of S. Then if the contracted algebra of S over F is semisimple, every matrix representation of S over F is determined to within equivalence by its character.*

McAlister then extended Munn's result and obtained the main result of [61] which describes the character ring of S via the character rings of its maximal subgroups:

Theorem 6.1.24. [61, Theorem 3.4] *Let S be a finite semigroup, let $J_1, \ldots, J_r$ be the regular $\mathcal{J}$-classes of S, and let $H_1, \ldots, H_r$ be maximal subgroups of $J_1, \ldots, J_r$, respectively. Then*

$$\mathrm{ch}(S) \cong \mathrm{ch}(H_1) \times \ldots \times \mathrm{ch}(H_r).$$

Consequently, we have the following corollary:

Corollary 6.1.25. [61, Corollary 3.5] *Let S be a finite semigroup, let $J_1, \ldots, J_r$ be the regular $\mathcal{J}$-classes of S, and let $H_1, \ldots, H_r$ be maximal subgroups of $J_1, \ldots, J_r$, respectively. Then two representations Γ and Δ of S over $\mathbb{C}$ have the same irreducible constituents if and only if $\Gamma|_{H_i}$ is equivalent to $\Delta|_{H_i}$, $1 \leqslant i \leqslant r$.*

Moreover, the character theory was developed further by Rhodes and Zalcstein in [95].

Thus far, we have not yet completed the whole picture of the early development of the theory during the fifties and sixties. We now turn to the Clifford-Munn-Ponizovskii correspondence.

6.1.1 Development of the Clifford-Munn-Ponizovskii Correspondence

As stated in Chapter 3, the Clifford-Munn-Ponizovskii correspondence is as follows. Let M be a finite monoid, and let U denote the poset of regular $\mathcal{J}$-classes of M. Fix a $\mathcal{J}$-class J of M and let e be an idempotent in $E(J)$. Then the set $\mathrm{Irr}(M)$ of irreducible representations of M is in one-to-one correspondence with the set of irreducible representations of the various H_J, the $\mathcal{H}$-class of e, hence:

$$\mathrm{Irr}(M) = \bigcup_{J \in U} \mathrm{Irr}(H_J).$$

We recall from the third chapter the key idea to understanding the Clifford-Munn-Ponizovskii correspondence. Let S be a finite semigroup. There is a procedure called induction which turns a representation of a maximal subgroup into a representation of the semigroup S. Moreover, there is another procedure, called reduction, which turns a representation of S into a representation of any of the maximal subgroups of S. If we take a maximal subgroup from each regular $\mathcal{D}$-class of S and apply induction to the irreducible representations of the given maximal subgroups, then we obtain all irreducible representations of S. So, the way to understand the representation of S is to understand the representation of its maximal subgroups and the induction process.

From Chapters 4 and 5, we summarize the fact that the study of this theory started with Clifford's construction of irreducible representations of a completely 0-simple semigroup [6]. Munn then extended this work to finite semigroups and showed that in order to obtain their representations, it is sufficient to construct the representations of 0-simple semigroups [65, 66]. Munn preferred principal factors to maximal subgroups. We recall here that Clifford and then Munn both used restriction of representations. In the following, we explain the fundamental difference between reduction and restriction processes. In restriction, we just restrict a representation of a semigroup S to a part of the semigroup which is principal factor (in Munn case) and the outcome matrix has the same size as the matrix representation of S which we start with. While in reduction as in Chapter 3, we restrict a representation of a semigroup S to two things at once, a part of the semigroup; its maximal subgroups and also to part of the vector space; subspace, that we are acting on and hence the dimension of the matrix representation of S becomes small.

Munn had a mechanism that produced the irreducible representations of a finite semigroup S, but in three steps: starting with a principal series, going next to the principal factors and then using the work of Clifford on completely 0-simple semigroups. Hence, it is going to be a one-to-one correspondence between the principal factors of S and their maximal subgroups. Munn has a different formulation of the induction process described in Chapter 3. His method starts with the irreducible representations of the principal factors of a semigroup finite S. Each such representation then gives a representation of S of same dimension of that we start with [66]. On the contrary, in the modern method of induction in the Clifford-Munn-Ponizovskii correspondence, we start instead with irreducible representations of maximal subgroups and the induction process gives representations of S with dimensions some multiple of the dimensions of the representations which we start with.

Before his interest in the field waned, Munn wrote a paper on the symmetric inverse monoid [67] where he utilized [66, 81] and Wagner's 1952 paper to completely describe all characters of the symmetric inverse monoid in terms of symmetric groups characters. Munn used the induction process which resembles closely the modern standpoint of the induction of the Clifford-Munn-Ponizovskii correspondence, as described in Chapter 3, but with a different formulation and it was limited to one particular semigroup, the symmetric inverse monoid. One might have expected Munn to demonstrate that the method works for any semigroup, but he did not. Munn's paper [67] is totally different from his other papers and is also a special case using a self-contained technique which Munn did not generalize. That is why we consider [67] as an oddity and as a cryptic paper.

The work of Clifford [6, 7], Munn [65, 67] and Ponizovskii (1958) parameterized the irreducible representations of finite monoids via group representation theory and determined which finite monoids have semisimple algebras. All of these approaches make use of Rees's theorem from Chapter 4 which characterizes 0-simple semigroups up to isomorphism and Wedderburn theory from Chapter 1. The definitive summary of their approach appears in Chapter 5 of Clifford and Preston's book [8].

We know that the underlying idea of the Clifford-Munn-Ponizovskii correspondence is that the irreducible representations of a finite monoid are in one-to-one correspondence with the irreducible representations of the maximal subgroups. Now, which parts of the Clifford-Munn-Ponizovskii correspondence existed at the time Munn stopped working on the subject? The answer is as follows. The Clifford-Munn-Ponizovskii correspondence theory developed in Clifford [6], Munn [65, 66, 68] and Ponizovskii (1956). From the context of their work, we credit all these authors with obtaining the description of the concept of the commonly referred to as the Clifford-Munn-Ponizovskii correspondence but without writing it down and specifying it explicitly. Then the correspondence developed further via monomial representations in the works of Lallement and Petrich [47] and Rhodes and Zalcstein [95].

The first time the Clifford-Munn-Ponizovskii correspondence is found in a recognizable modern form is in the paper [95, Section 2] of Rhodes and Zalcstein. We call this paper the "mysterious" paper because although it was written in the late sixties, it was not published until the early nineties. In the following paragraphs, we will examine this paper. *Monoids and semigroups with applications* [94] is a 1991 survey book edited by John Rhodes. It contains several papers and conference talks and includes relations to other mathematical branches and applications to different fields. Among these is the paper entitled *Elementary representation and character theory of finite semigroups and its application* [95]. It is a joint work of Rhodes and Zalcstein. According to the book editor (Rhodes), this paper was written in the very late 1960's but surprisingly it was not directly published until 1991. Also, the paper [95] was based on lectures given by Rhodes in seminar at the University of California, Berkeley in the spring of 1968 [95, Introduction, page 335].

The paper [95] is different from the others mentioned as it is the first to contain a modern formulation of the Clifford-Munn-Ponizovskii correspondence. In contrast, the non-modern formulations are the partial correspondences of Munn and others. Rhodes and Zalcstein stated the result as follows:

Theorem 6.1.26. [95, Corollary 2.13] *Let S be a finite semigroup and $G_1, \ldots, G_n$ be a choice of exactly one maximal subgroup from each regular $\mathcal{J}$-class of S. Then there is a one-to-one correspondence between the irreducible representations of G_i and the irreducible representations of S having apex J such that G_i is a maximal subgroup of J. In particular, if k_i is the number of conjugacy classes of G_i, then the number of irreducible representations of S is $\sum_{i=1}^{n} k_i$.*

In this paper [95], and based on an unpublished argument of Munn, Rhodes and Zalcstein managed to provide an independent construction of representations of 0-simple semigroups. They also gave a new method for obtaining the irreducible representations of finite semigroups. In addition, [95] includes a development of the character theory of semigroups and applications to a so-called *group complexity* of finite semigroups.

At the end of our tracing of the development of the theory and in assessing the overall theory, we have two key points. We emphasize again that the highly influential pioneering works on the early development of semigroup representation theory were done, in the following order with respect to high impact, by Clifford, Munn, Ponizovskii, Hewitt and Zuckerman, Preston, McAlister, Lallement and Petrich, and Rhodes and Zalcstein. Therefore, "this area may justifiably be referred to as [Clifford-]Munn-Ponizovskii Theory", as Hollings writes [36, Section 11.5, page 302].

6.2 The Dormant Era (1975-1995)

Why did Munn stop working on semigroup representations after the late 1960's? In Subsection 6.2.1 below, we show that in the 1950's and 1960's there was a certain amount of work done on the subject, but subsequently it went relatively quiet. In fact, it was not completely silent during the 1970's and 1980's, but the work done in the area was sporadic. Then all of a sudden, in the middle of the 1990's, interest in the area re-blossomed. The dormant period lasted approximately twenty-five years, during which no significant work was done in the field; several papers were simply repetitions or slight reformulations of the work that had already been done by Munn. Thus, due to the general perceived time frame of the subject, the question above became: why was the quiet period not just limited to Munn but also included other mathematicians working in the area? Therefore, the question naturally needs to be broadened.

6.2.1 The Evidence for the Existence of the Dormant Era

Using MathSciNet, we collected statistics on the number of papers in various periods. Because *Mathematical Reviews* (MR) covers data from 1940 to the present, our statistics start from 1940. There are approximately two papers in the 1940's that mention the phrase "semigroup representation", and approximately 15 papers in the 1950's. Figure 6.1 displays the number of papers published on the subject from 1950 to 2000. They are written in different languages; the main ones are English and Russian. From Chapter 1, we can clearly observe the difference between the interest in group representations and that in semigroup representations from the start. In group case, Figure 1.1 shows that from the beginning of the development of group representations, there were hundreds of publications and then the number of papers increased consistently and there is no indication that there was an inactive period. On the other hand, Figure 6.1 of the development of semigroup representations shows that there were tens of publications at the start of the development followed by an inactive period during the 1970's and the 1980's, then from the mid of 1990's semigroup representations was revived.

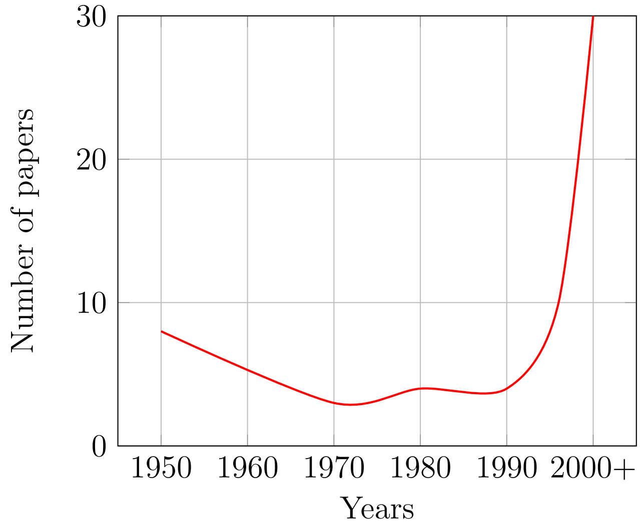

Figure 6.1: The progress of semigroup representation theory

6.2.2 Comparing Group and Semigroup Representation Theory

In order to understand why Munn and others stopped working on the subject, and also to give evidence to our conclusions, we compare the situation of semigroup theory with that of group theory. Let us recall the development of group representation theory from Chapter 1. The collaboration between Dedekind and Frobenius on group character theory started seriously in 1896. Dedekind took material from Galois Theory and tried to understand it – dealing with certain polynomials and their factorizations when a group is abelian. He noted something and asked Frobenius if he could develop the ideas for an arbitrary group. Very quickly, the idea of representations of a group grew out of this.

The mechanism of a group determinant is an important part on the development of group representations. One starts with a finite group and a matrix whose rows and columns are indexed by the elements of the group, and then inserts the variable $x_{gh^{-1}}$ into the (g, h)-position. Then we take the determinant of this matrix, which is a polynomial in these variables. The question that arises here is: can the polynomial be factorized? This was the problem that Dedekind and Frobenius were considering. The polynomial can indeed be factorized, with an irreducible factor of degree d for every irreducible representation of degree d of the group. That was what motivated group representation theory.

Dedekind showed this in the case that the group is abelian. In responding to Dedekind's question about the generalizability of the idea, Frobenius invented group character theory. He directly realized the importance of generalized group characters and that it is in fact part of a deeper theory, which was later to become group representation theory. Ponizovskii wrote of comparing the situation of semigroup theory with that of group theory: “The theory of matrix groups is a rather vast area of the group theory. This is not the case with matrix semigroups. There is a small number of papers only devoted to this subject”, [80, Introduction, page 117].

Again, the motivation for group representation theory was from outside group theory. It came from Galois Theory, thus making it more interesting. Dedekind and others reformulated the notion of a character in number theory to the context of finite abelian groups. This indicates that the theory had a promising and rigorous start. Once Frobenius – and subsequently Burnside – started publishing papers, there was an uninterrupted and increasing amount of material, showing continuous development. Thus, there was no dormant period and there were also immediate applications to group theory (such as Burnside $p^a q^b$ Theorem 1.3) and then to physics and chemistry. This seems quite different from the situation of semigroups. As mentioned in Section 1.4, this was the main reason that group representation theory had genuinely a special appeal and glamour from the very beginning of its development.

6.2.3 The Reasons for the Dormant Era

One might think that the reason for the dormant period was that the Clifford-Munn-Ponizovskii correspondence tells us everything we need to know about the representations of a sensible semigroup via the irreducible representations of the groups contained in it. In other words, the question is reduced back to group theory. Eventually, we decided that this could not be the right reason as the correspondence was not common enough knowledge at the time of the dormant period. Only a few selected people were working on the subject and it was not written in the literature and not accessible to students for them to read and understand. So, what are the real underlying causes of the dormancy during the 1970’s and 1980’s? We have identified three broad reasons regarding this situation. These will be discussed in the following three subsections.

6.2.4 Motivations

We speculate that one reason for the lack of applications after Munn’s era is the genesis of the theory of semigroup representations itself. It seems that both Clifford and Munn were trying to generalize from groups to semigroups; this is a slightly less appealing motivation than is the case for group representation theory. As we mentioned above, the motivation for group representations came from outside group theory. This was a strong and interesting start and also the main reason of its success. On the other hand, semigroup representation theory was merely motivated by the desire to generalize that of group representations.

We found some evidence which supports our claim here. First, the central problem in Munn's thesis [64] is extending Maschke's theorem for a finite group to the case of a finite semigroup. Second, Okniński wrote in [73, Chapter V, page 257] that "the strength of [group representation] theory and its broad applications motivated several authors to develop the theory of representations of semigroups". Thus, for semigroup representations, we identify that the motivation and the applications, as we will explain below, were less vigorous and its development was less active than in the group case.

Generally, a good motivation for the development of a theory is for there to be a question, external to the area, that needs answering. This is what happened with groups. Questions in Galois theory lead to the development of group representations. With semigroups there did not seem to be such questions that needed answering, external to semigroup theory. Instead, the theory was generalising an existing one, and this, while worthwhile, is never as good a reason.

6.2.5 Applications

During the early renaissance of the theory, we found only three applications to the study of finite semigroups. The first application was indicated by McAlister in his 1971 survey [58] and presented in Rhodes's 1969 paper entitled: *Characters and complexity of finite semigroups* [93]. In this paper, Rhodes applied semigroup representation theory to finite semigroups. He also provided a formula for a congruence induced on a finite semigroup S by the direct sum of all irreducible representations of S over the field of complex numbers. This type of congruence is today called the *Rhodes radical* of the semigroup. Rhodes then computed the Krohn-Rhodes complexity (defined in [46]) of completely regular monoid via character theory.

Later in 1971, the second application was produced by Zalcstein [111] where he applied semigroup representation theory to finite semigroup. Surprisingly, the abstract of [111] mentions that the paper itself is a continuation of the results of the mysterious paper of Rhodes and Zalcstein [95] which was listed in the references with an indication that it would be published. As mentioned previously, the paper [95] was written in the late sixties but it was not made public until 1991. Given the time when the paper was written, it is considered the third attempt to apply representations of finite semigroups. Rhodes and Zalcstein developed the character theory of finite semigroups and then applied their results to the study of group complexity of finite semigroups.

There is some evidence regarding the paucity of applications. In a paper [34] entitled: *The relationship of Al Clifford's work to the current theory of semigroups*, commemorating the work of Clifford and regarding representation theory, Rhodes stated that [34, Page 48]:

> It is a beautiful theory, but unlike the situation in finite groups, it has not had any serious applications to date. This is a mystery.

Nevertheless, Rhodes was positive and believed that representation theory of semigroups via the work of Clifford, Munn, McAlister and Putcha would become important over the next 25 years, and indeed he was right. In addition, Okniński wrote with astonishment in [73, Chapter V, page 257] that, unlike in the case of groups, the results obtained in the early development of the theory by Munn and others had not been used as tools to prove important facts on semigroups and their algebras. In [58, Preface, page 191], McAlister wrote:

> Although the representation theory of semigroups has given rise to many important concepts in semigroup theory, it has not yet proved nearly as useful as has group representation theory.

As a result of this lack of direct and serious applications, the progression of the theory subsided for several years until 1996 when it was revived again by Putcha [87]. This point will be taken up in the last section.

Arguably, the modest attempts to apply semigroup representation theory reflect perhaps the lack of awareness of the potential of the theory at that time.

6.2.6 Accessibility

From an early stage, there were interesting texts on group representation theory, making it easier for students and those from different disciplines to engage with the topic. On the other hand, similar books for semigroups have only just started to appear. This might be considered as another reason for the dormancy period. According to [23, Introduction, page 3585], since the early contributions on the theory of semigroup representations utilized and owed a great debt to Rees's theorem for 0-simple semigroups, the results were somewhat inaccessible to the non-specialist in semigroup theory. This evidence boosts our claim for the unpopularity of semigroup representation theory as compared with groups.

When people started reducing their use of the heavy semigroup-theoretic language in their own work, semigroup representation theory reemerged and was restored to its former glory. For example, the new approach to semigroup representation theory avoids the use of principal factors as Munn did. Also, semigroup representation theory has become attractive to people outside semigroup theory because of the use of Green's relations in the formulation of results as seen in Chapter 2. Accordingly, since the early 2000's, the number of research papers on the application of semigroup representation theory increased to an average annual production of about thirty papers and most of them are written by non-specialist semigroup-theorists, according to MathSciNet. As a consequence, general mathematicians became able to interact and easily deal with the recent results in an accessible way that was previously hard to approach because the early results on semigroup representation theory in the literature were technically complicated and required more advanced knowledge about algebras.

The last section of this chapter shows that the theory is gaining ground again and widespread application. Once we have understood the revival of semigroup representation theory, there will be no need to wonder about the cause of its dormancy anymore.

6.3 The Revival of Semigroup Representation Theory

Our investigation of the representation theory of semigroups is divided chronologically into three periods: an early active time, an inactive time and then a renewed period which includes recent times. The early time (1940-1970) is associated with Clifford, Munn, Ponizovskii, and then Hewitt and Zuckerman, Lallement and Petrich, Preston, McAlister, Rhodes and Zalcstein. In this section, we intend to show that, after many inactive years, the theory redeveloped in the late 1990's and the early 2000's. The revival was initiated by Putcha and then by Brown, Steinberg, Solomon and others.

Semigroup representation theory was successfully revived by Putcha in a series of four papers [87–90] between 1996 to 2001. Putcha's work on the topic raises the following question: why and how did he become interested in the theory or why did he break the silence over the theory? The predictable answer would be that he simply got interested in the theory itself. In general, the reason for the rapid growth of semigroup representation theory in the 1990's is that interesting examples started to appear in semigroup theory and outside it, in for example combinatorics, automata theory and probability theory [102, Introduction, page xxi] . Thus we can speculate that this is why the theory has taken off again then.

Putcha addressed complex representations of arbitrary finite monoids in [87] and he then applied his results, specifically, to the full transformation monoids T_n to obtain all their irreducible characters. In fact, Putcha here re-derived by different means the prior work of Hewitt and Zuckerman in [33]. Additionally, his paper [89] has an application of semigroup representation theory to finding the weights of a finite group of Lie type. In 2001, he dealt with irreducible character theory of finite semigroups [90]. One year later and for the purposes of algebraic combinatorics and representations of the symmetric group, Solomon produced a paper [101] about the representation theory of the symmetric inverse monoid. The early 2000's saw the publication of many papers on the topic, for instance [2, 14, 23]. Also, a wealth of general texts has emerged, such as [22, 102].

In [23], the authors indicated that as a consequence of the heavy semigroup-theoretic language used in the early works by Clifford, Munn and others, it seems that when researchers from other areas needed to use semigroup representation theory, they were forced to reinvent parts of the theory for themselves. A perfect example is provided by the group theorist Brown, who was a leader in adopting this trend in his 2000's papers [3,4] where he studied and analyzed *random walks* on finite semigroups via semigroup representation theory. According to Steinberg [102, Introduction, page xix], "Brown was forced to redevelop from scratch a very special case of monoid representation theory in order to analyze [the so-called] Markov chains". It seems that the papers by Brown from the early 2000's have broadly reinvigorated people's interest. In the same manner, Steinberg emphasizes that his recent book [102] is designed for "a fairly broad audience [and] a lot of the technical jargon of semigroup theory is deliberately avoided to make the text accessible to as wide readership as possible" [102, Preface, page vii]. This simplification of the theory might be considered as a leading cause of its redevelopment. For further detail on this topic, the reader is referred to a survey in the introduction of [102].

To conclude: motivations, applications, and accessibility (or, the lack of them) have proven to be the reasons for the continuous rapid development of representation theory in the case of groups and the sporadic slow development of representation theory in the case of semigroups. Arguably, semigroup representation theory has several advantages and a promising future for further developments in various areas of mathematics and even beyond. We hope by the end of this last chapter that we have provided a clear and comprehensive overview of the progress in semigroup representation theory, starting with the modern standpoint and ending with a discussion of its origins and unusual or unpredictable development.

Bibliography

[1] Albaity M. *Representations of reflection monoids* (Doctoral dissertation, University of York), 2018.

[2] Almeida J, Margolis S, Steinberg B, Volkov M. *Representation theory of finite semigroups, semigroup radicals and formal language theory.* Transactions of the American Mathematical Society. 2009; 361(3): 1429-1461.

[3] Brown KS. *Semigroups, rings, and Markov chains. Journal of Theoretical Probability.* 2000; 13(3): 871-938.

[4] Brown KS. *Semigroup and ring theoretical methods in probability.* Representations of finite dimensional algebras and related topics in Lie theory and geometry. 2004; 40: 3-26.

[5] Clifford AH. *Semigroups admitting relative inverses.* Annals of Mathematics. 1941: 1037-1049.

[6] Clifford AH. *Matrix representations of completely simple semigroups.* American Journal of Mathematics. 1942; 64(1): 327-342.

[7] Clifford AH. *Basic representations of completely simple semigroups.* American Journal of Mathematics. 1960; 82(3): 430-434.

[8] Clifford AH, Preston GB. *The algebraic theory of semigroups*, Volume I. American Mathematical Soc.; 1961.

[9] Conrad K. *On the origin of representation theory.* L' Enseignement Mathematique. 1998;44: 361-392.

[10] Curtis CW, Reiner I. *Representation theory of finite groups and associative algebras.* American Mathematical Soc.; 1966.

[11] Curtis CW. *Representation theory of finite groups: from Frobenius to Brauer.* The Mathematical Intelligencer. 1992; 14(4):48-57.

[12] Curtis CW. *Pioneers of Representation Theory: Frobenius, Burnside, Schur, and Brauer.* American Mathematical Soc.; 1999.

[13] Dedekind, R. *Gesammelte Mathematische Werke*, Friedr. Viewe g and Sohn, Braunschweig; 1931.

[14] Everitt B. *The sympathetic sceptics guide to semigroup representations.* arXiv: 1909. 13279. 2019.

[15] Fountain J. *The work of Douglas Munn and its legacy.* Semigroup Forum 2010; 81(1): 2-25.

[16] Fountain J. *Sandy Green: his impact on semigroup theory.* Semigroup Forum 2017; 94(1): 1-5.

[17] Frobenius, G. *Über vertauschbare Matrizen.* S'ber. Akad. Wiss. Berlin (1896), 601-614; Ges. Abh. II, 705-718 .

[18] Frobenius, G. *Über Gruppencharaktere.* S'ber. Akad. Wiss. Berlin (1896), 985-1021; Ges. Abh. Ill , 1-37.

[19] Frobenius, G. *Über die Primfactoren der Gruppendeterminante.* S'ber. Akad. Wiss. Berli n (1896), 1343-1382; Ges. Abh. Ill, 38-77.

[20] Frobenius, G. *Über die Darstellung der endlichen Gruppen durch lineare Substitutionen*, S'ber. Akad. Wiss. Berlin (1897), 944-1015; Ges. Abh. Ill, 82-103.

[21] Fulton MW, Fulton W. *Young tableaux: with applications to representation theory and geometry.* Cambridge University Press; 1997.

[22] Ganyushkin O, Mazorchuk V. *Classical finite transformation semigroups: an introduction.* Springer Science & Business Media; 2008.

[23] Ganyushkin O, Mazorchuk V, Steinberg B. *On the irreducible representations of a finite semigroup.* Proceedings of the American Mathematical Society. 2009; 137(11): 3585-3592.

[24] *Gesammelie Abhandlungen*, JP. Serre, ed. BerlinHeidelberg-New York, 1968.

[25] Gould V. *Semigroup Theory.* University of York; 2017.

[26] Green JA. *On the structure of semigroups.* Annals of Mathematics. 1951: 163-72.

[27] Grillet PA. *Semigroups: an introduction to the structure theory.* Routledge; 2017.

[28] Hammerlindl A. *Group Determinants and k-Characters*; 2005.

[29] Hawkins T. *The origins of the theory of group characters.* Archive for History of Exact Sciences. 1971; 7(2): 142-170.

[30] Hawkins T. *New light on Frobenius' creation of the theory of group characters.* Archive for history of exact sciences. 1974; 12(3): 217-243.

[31] Hawkins T. *The creation of the theory of group characters.* Rice Institute Pamphlet-Rice University Studies. 1978; 64: 57-71.

[32] Hawkins T. *The Mathematics of Frobenius in Context.* Berlin: Springer; 2015.

[33] Hewitt E, Zuckerman HS. *The irreducible representations of a semigroup related to the symmetric group.* Illinois Journal of Mathematics. 1957; 1(2): 188-213.

[34] Hofmann KH, Mislove MW, editors. *Semigroup theory and its applications: proceedings of the 1994 conference commemorating the work of Alfred H. Clifford.* Cambridge University Press; 1996.

[35] Hollings C. *The early development of the algebraic theory of semigroups.* Archive for history of exact sciences. 2009; 63(5): 497-536.

[36] Hollings C. *Mathematics across the Iron Curtain: A history of the algebraic theory of semigroups.* American Mathematical Society; 2014.

[37] Howie JM. *Fundamentals of semigroup theory.* Oxford University Press; 1995.

[38] Howie JM. *Walter Douglas Munn.* Semigroup Forum 1999; 59(1): 1-8.

[39] Howie JM., Reilly NR., Silva PV. ,and Volkov MV. *A special issue dedicated to the memory of Walter Douglas Munn: a note from the editors.* Semigroup Forum 2010; 81(1).

[40] Hum M. *The Representation Theory of Finite Semigroups* (Doctoral dissertation, McGill University); 2000.

[41] Isaacs IM. *Character theory of finite groups.* Academic Press [Harcourt Brace Jovanovich, Publishers], New York-London, 1976.

[42] James G, Liebeck MW, Liebeck M. *Representations and characters of groups.* Cambridge University Press; 2001.

[43] Johnson KW. *On the group determinant.* Mathematical Proceedings of the Cambridge Philosophical Society 1991; 109(2): 299-311.

[44] Johnson KW, Sehgal SK. *The 2-characters of a group and the group determinant.* European Journal of Combinatorics. 1995; 16(6): 623-631.

[45] Krohn K, Rhodes JL, Arbib MA. *Algebraic theory of machines, languages, and semigroups.* Academic Press; 1968.

[46] Krohn K, Rhodes J. *Complexity of finite semigroups*. Annals of Mathematics. 1968: 128-60.

[47] Lallement G, Petrich M. *Irreducible matrix representations of finite semigroups*. Transactions of the American Mathematical Society. 1969;139: 393-412.

[48] Lam TY. *Representations of finite groups: A hundred years*, Part I. Notices of the AMS. 1998; 45(3): 361-372.

[49] Lam TY. *Representations of finite groups: a hundred years*, Part II. Notices of the AMS. 1998; 45(4): 465-474.

[50] Lang S. *Graduate Texts in Mathematics: Algebra*. Springer; 2002.

[51] Lawson M. *Inverse semigroups: the theory of partial symmetries*. World Scientific; 1998.

[52] Ljapin [Liapin] ES. *Representations of Semigroups by Partial Mappings*. AMS. Transl. 1963; 27(2).

[53] Ljapin [Liapin] ES. *On representations of semigroups by partial transformations*. AMS Transl. 1963; 27(2): 289-296.

[54] Li Z, Li Z. *Bijection between Conjugacy Classes and Irreducible Representations of Finite Inverse Semigroups*. arXiv:1208. 5735. 2012.

[55] McAlister DB. *Matrix representations of semigroups*. Glasgow Mathematical Journal. 1967; 8(1): 1-13.

[56] McAlister DB. *Basic representations of arbitrary semigroups*. AMS Transl. 1969; 135: 311-342.

[57] McAlister DB. *Basic representations of finite semigroups*. In Proceedings of the Royal Irish Academy. Section A: Mathematical and Physical Sciences 1970: 97-111.

[58] McAlister DB. *Representations of semigroups by linear transformations: Part I*. Semigroup Forum 1971; 2(1): 189-263.

[59] McAlister DB. *Representations of semigroups by linear transformations: Part II*. Semigroup Forum 1971; 2(1): 283-320.

[60] McAlister DB. *The category of representations of a completely 0-simple semigroup*. Journal of the Australian Mathematical Society. 1971; 12(2): 193-210.

[61] McAlister DB. *Characters of finite semigroups*. Journal of Algebra. 1972; 22(1): 183-200.

[62] Miller D. D. *A. H. Clifford: The first sixty-five years*. Semigroup Forum 1974; 7(1): 4-9.

[63] Mukhopadhyay AK. *Wigner's Symmetry Representation Theorem*. Resonance. 2014; 19(10): 900-916.

[64] Munn WD. *Semigroups and their algebras*. PhD diss., University of Cambridge, 1955.

[65] Munn WD. *On semigroup algebras*. Mathematical Proceedings of the Cambridge Philosophical Society. 1955; 51,(1): 1-15.

[66] Munn WD. *Matrix representations of semigroups*. Mathematical Proceedings of the Cambridge Philosophical Society. 1957; 53(1): 5-12.

[67] Munn WD. *The characters of the symmetric inverse semigroup*. Mathematical Proceedings of the Cambridge Philosophical Society. 1957; 53(1): 13-18.

[68] Munn WD. *Irreducible matrix representations of semigroups*. The Quarterly Journal of Mathematics. 1960; 11(1): 295-309.

[69] Munn WD. *A class of irreducible matrix representations of an arbitrary inverse semigroup*. Glasgow Mathematical Journal. 1961; 5(1): 41-48.

[70] Munn WD. *Matrix representations of inverse semigroups*. Proceedings of the London Mathematical Society. 1964; 3(1): 165-181.

[71] Munn WD. *Brandt congruences on inverse semigroups*. Proceedings of the London Mathematical Society. 1964; 3(1): 154-164.

[72] Noether E. *Hyperkomplexe Grössen und Darstellungstheorie*. Mathematische Zeitschrift. 1929; 30(1): 641-92.

[73] Okniński J. *Linear representations of semigroups*. Monoids and semigroups with applications. 1991: 257-277.

[74] Okniński J, Putcha MS. *Complex representations of matrix semigroups*. AMS Transl. 1991; 323(2): 563-581.

[75] Okniński J, Ponizovskii JS. *A new matrix representation theorem for semigroups*. Semigroup Forum 1996; 52(1): 293-305.

[76] Pastijn F. *A representation of a semigroup by a semigroup of matrices over a group with zero*. Semigroup Forum 1975; 10(1): 238-249.

[77] Petrich M. *Representations of semigroups and the translational hull of a regular Rees matrix semigroup.* AMS Transl. 1969; 143: 303-318.

[78] Petrich M, Reilly NR. *A representation of E-unitary inverse semigroups.* The Quarterly Journal of Mathematics. 1979; 30(3): 339-350.

[79] Petrich M. *Some constructions related to Rees matrix semigroups.* Semigroup Forum 1999; 59(2): 244-263.

[80] Ponizovskii IS. *On irreducible matrix semigroups.* Semigroup Forum 1982; 24(1): 117-148.

[81] Preston GB. *Representations of Inverse Semigroups.* Journal of the London Mathematical Society. 1954; 1(4): 411-419.

[82] Preston GB. *A note on representations of inverse semigroups.* Proceedings of the American Mathematical Society. 1957; 8(6): 1144-1147.

[83] Preston GB. *Matrix representations of semigroups.* Quart. J. Math. Oxford Ser. 1958; 9(2): 169-176.

[84] Preston GB. *Matrix representations of inverse semigroups.* Journal of the Australian Mathematical Society. 1969; 9(1-2): 29-61.

[85] Preston GB. *Representations of inverse semigroups by one-to-one partial transformations of a set.* Semigroup Forum. 1973; 6(1): 240-245.

[86] Preston GB. *A.H. Clifford: an appreciation of his work on the occasion of his sixty-fifth birthday.* Semigroup Forum 1974; 7(1-4): 32-57.

[87] Putcha MS. *Complex representations of finite monoids.* Proceedings of the London Mathematical Society. 1996; 3(3): 623-641.

[88] Putcha MS. *Complex representations of finite monoids II. Highest weight categories and quivers.* Journal of Algebra. 1998; 205(1): 53-76.

[89] Putcha M. *Semigroups and weights for group representations.* Proceedings of the American Mathematical Society. 2000;128(10): 2835-2842.

[90] Putcha MS. *Reciprocity in character theory of finite semigroups.* Journal of Pure and Applied Algebra. 2001; 163(3): 339-51.

[91] Rees D. *On semi-groups.* Mathematical Proceedings of the Cambridge Philosophical Society 1940; 36(4): 387-400

[92] Reilly NR. *Walter Douglas Munn.* Semigroup Forum 2009; 78(1): 1-6.

[93] Rhodes J. *Characters and complexity of finite semigroups.* Journal of Combinatorial Theory. 1969; 6(1): 67-85.

[94] Rhodes J, editor. *Monoids and Semigroups with Applications*: Proceedings of the Berkeley Workshop in Monoids, Berkeley, Calif., 1989. World Scientific; 1991.

[95] Rhodes J, Zalcstein Y. *Elementary representation and character theory of finite semigroups and its application.* Monoids and semigroups with applications (Berkeley, CA, 1989). 1991: 334-367.

[96] Rotman J. *Galois theory.* Springer Science & Business Media; 1998.

[97] Schützenberger MP. *$\mathcal{D}$-relation representation des demi-groupes.* Comptes rendus hebdomadaires des séances de l'Académie des sciences. 1957; 244(15): 1994-6.

[98] Schützenberger MP.*Sur la représentation monomiale des demi-groupes.* Comptes rendus hebdomadaires des séances de l'Académie des sciences. 1958; 246(6): 865-7.

[99] Serre JP. *Linear representations of finite groups.* New York: Springer; 1977.

[100] Sizer WS. *Representations of semigroups of idempotents.* Czechoslovak Mathematical Journal. 1980; 30(3): 369-375.

[101] Solomon L. *Representations of the rook monoid.* Journal of Algebra. 2002; 256(2): 309-342.

[102] Steinberg B. *Representation theory of finite monoids.* New York: Springer; 2016.

[103] Sternberg S. *Group theory and physics.* Cambridge University Press; 1995.

[104] Tully EJ. *Representation of a semigroup by transformations acting transitively on a set.* American Journal of Mathematics. 1961; 83(3): 533-541.

[105] Tully EJ. *Representation of a Semigroup by Row-Monomial Matrices over a Group.* Proceedings of the Japan Academy. 1964; 40(3): 157-160.

[106] Warne RJ. *Matrix representations of d-simple semigroups.* AMS Transl. 1963; 106(3): 427-435.

[107] Weintraub SH. *Representation Theory of Finite Groups: Algebra and Arithmetic.* AMS Transl; 2003.

[108] Weyl H. *Gruppentheorie und Quantenmechanik*, Zurich, 1928; English translation by H . Robertson, Theory of Groups and Quantum Mechanics, E. P. Dutton, New York, 1931.

[109] Wigner E. *Gruppentheorie und ihre Anwendung auf der Quantenmechanik der Atomspektren*, Braunschweig, 1931; English translation by J. Griffin, Group theory and its applications to the quantum mechanics of atomic spectra, Academic Press, New York, 1959 .

[110] Wigner E. *Group theory: and its application to the quantum mechanics of atomic spectra*. Elsevier; 2012.

[111] Zalcstein Y. *Studies in the representation theory of finite semigroups*. AMS Transl. 1971;161: 71-87.