DERIVED CATEGORIES AND K-GROUPS OF SINGULAR VARIETIES

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Abstract

This thesis consists of three parts and is a collection of papers written by the author of this text during his postgraduate studies, together with an Appendix chapter.

The first chapter is based on [98] and is in collaboration with Evgeny Shinder. It discusses the K-groups K_1, K_0 and K_{-n} of the singularity category of isolated quotient singularities. The second chapter is based on [73] and is joint with Martin Kalck and Evgeny Shinder. It introduces Kawamata type semiorthogonal decompositions for singular varieties and obstructions for such decompositions are studied, mainly for the case of nodal threefolds. Each of these two chapters can be read independently. The third chapter is an Appendix to the first chapter and explains in more detail how the main technical result in chapter one is proven, on which the main theorems rely on.

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Contents

Introduction

The Grothendieck group first appeared in Grothendieck's formulation of the Grothendieck-Riemann-Roch theorem and was since then used in many branches of mathematics and generalized in various ways. Its study lead, for example, to algebraic K-theory and to the notion of derived categories and dg-categories of schemes.

In this thesis, we are mainly interested in K-groups and derived categories of singular varieties. Our main approach is to examine the singularity category and its K-groups.

Singularity K-theory

Let us denote by $\mathrm{G}_0(X)$ and by $\mathrm{K}_0(X)$ the Grothendieck group of coherent sheaves and of vector bundles of a quasi-projective scheme X over a field k. There is a natural homomorphism $\mathrm{K}_0(X) \to \mathrm{G}_0(X)$ induced by the inclusion functor going from vector bundles over X to coherent sheaves of X. By a classical result of Serre, this homomoprhism is an isomorphism in the case when X is regular. If X is singular, this map is in general neither injective nor surjective. In this thesis, we are going to see how the failure of injectivity (resp. surjectivity) is controlled by the K-groups of the so-called singularity category. Let us make this more precise.

Let $\mathcal{D}^b(X)$ denote the bounded derived category of coherent sheaves of X and let $\mathcal{D}^{\mathrm{perf}}(X)$ be the triangulated subcategory of $\mathcal{D}^b(X)$ consisting of perfect complexes. The Buchweitz-Orlov singularity category of X is the Verdier quotient

$$\mathcal{D}^{\mathrm{sg}}(X) = \mathcal{D}^b(X)/\mathcal{D}^{\mathrm{perf}}(X).$$

It was first defined by Buchweitz in [24] and later introduced in a more geometric setting by Orlov in [94]. The singularity K-theory is Schlichting's K-theory of the dg-enhancement of $\mathcal{D}^{\mathrm{sg}}(X)$ and we denote the K-groups by $\mathrm{K}_i^{sg}(X)$. One of the basic properties of singularity K-theory is that there is a long exact sequence

$$\ldots \to \mathrm{G}_1(X) \to \mathrm{K}_1^{sg}(X) \to \mathrm{K}_0(X) \to \mathrm{G}_0(X) \to \mathrm{K}_0^{sg}(X) \to 0,$$

where $\mathrm{K}_i(X) = \mathrm{K}_i(\mathcal{D}^{\mathrm{perf}}(X))$ are the Thomason-Trobaugh K-theory groups of vector bundles and $\mathrm{G}_i(X) = \mathrm{K}_i(\mathcal{D}^b(X))$ are Quillen's K-theory groups of coherent sheaves. There is also an nonconnective version of the singularity K-theory, defined as the K-theory of the dg-enhancement of the idempotent completion $\overline{\mathcal{D}^{\mathrm{sg}}(X)}$ of $\mathcal{D}^{\mathrm{sg}}(X)$. By Schlichting's construction, we have a long exact sequence

$$\ldots \to \mathrm{K}_1^{sg}(X) \to \mathrm{K}_0(X) \to \mathrm{G}_0(X) \to \mathbb{K}_0^{sg}(X) \to \mathrm{K}_{-1}(X) \to 0,$$

where $\mathbb{K}_0^{sg}(X) = \mathrm{K}_0(\overline{\mathcal{D}^{\mathrm{sg}}(X)})$.

In Chapter 1 and partly also in chapter 2, we discuss standard properties for singularity K-theory. In low dimension and for varieties with nice singularities, singularity K-theory can be used to compute the (-1)-th negative K-group of vector bundles in terms of class groups and Picard groups. More concretely, let X be an irreducible variety over an algebraically closed field k. Assume that X is either a normal surface or a threefold with only isolated compound A_n singularities. Then by Proposition 2.3.6 there is a long exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Cl}(X) \to \bigoplus_{x \in \mathrm{Sing}(X)} \mathrm{Cl}(\widehat{\mathcal{O}}_{X,x}) \to \mathrm{K}_{-1}(X) \to 0.$$

This statement was known by Weibel in the case for normal surfaces [124]. Note that there are not many formulas for computing negative K-theory in general. One of the main approaches in the literature is given by the relation between algebraic K-theory and cdh-topology of cdh-differentials. We will discuss the cdh-topological aspect in more detail in chapter 3.

Finally, we study singularity K-theory for quotient varieties X over algebraically closed fields of characteristic zero with only isolated singularities. By computing the local case $\mathrm{K}_i^{sg}(\mathbb{A}^n/G)$ for $i \leq 1$, we obtain:

Theorem (Theorem 1.2.23)**.** *Let X be a n-dimensional quasi-projective variety over an algebraically closed field of characteristic zero with at most isolated quotient singularities. Denote by G_i, $1 \leq i \leq |\mathrm{Sing}(X)|$ the isotropy groups of X. Then*

$$\mathrm{K}_1^{sg}(X) = 0, \quad \mathbb{K}_0^{sg}(X) \simeq \bigoplus_i \mathrm{K}_0^{sg}(\mathbb{A}^n/G_i), \quad \textit{and} \quad \mathrm{K}_{-m}^{sg}(X) = 0, \ \textit{for } m \geq 1.$$

In particular, there is a short exact sequence

$$0 \to \mathrm{K}_0(X) \to \mathrm{G}_0(X) \to \mathrm{K}_0^{sg}(X) \to 0$$

and $\mathrm{K}_0^{sg}(X)$ is finite torsion.

A nontrivial consequence of this result is that the Grothendieck group of vector bundles of weighted projective spaces $\mathbb{P}(a_0, \ldots, a_n)$ with isolated singularities is free abelian of rank $n+1$. To the best of our knowledge, this result wasn't known in the literature before. Another consequence is that the length map $\mathrm{K}_0(X \text{ on } \mathrm{Sing}(X)) \to \mathbb{Z}^{|\mathrm{Sing}(X)|}$ from the supported Grothendieck group of vector bundles is an isomorphism for isolated quotient singularities (see [109, Page 38]). This statement was shown by Levine up to dimension three [82, Theorem 3.3], and that it is an isomorphism up to torsion in general [82, Theorem 2.7]. With our methods, we are able to show the statement in all dimensions.

Bondal-Orlov localization

Let X be a singular variety over a field k of characteristic zero and let $\pi : Y \to X$ be a resolution of singularities. We say that X has rational singularities if $\mathbf{R}\pi_*\mathcal{O}_Y \simeq \mathcal{O}_X$ in $\mathcal{D}^b(X)$. The standard examples of rational singularities are quotient singularities, cones over smooth Fano hypersurfaces and toric varieties. The following conjecture describes the relation between $\mathcal{D}^b(X)$ and $\mathcal{D}^b(Y)$.

Conjecture (Bondal-Orlov localization)**.** *Let X be a variety with at most rational singularities and let $\pi : Y \to X$ be a resolution of singularities. Then there is an equivalence*

$$\mathcal{D}^b(Y)/\ker(\boldsymbol{R}\pi_*) \xrightarrow{\sim} \mathcal{D}^b(X)$$

induced by the functor $\boldsymbol{R}\pi_* : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$.

This conjecture first appeared in Bondal and Orlov's ICM talk in 2002 [19]. It is known to experts that the Bondal-Orlov conjecture is true for surfaces X with rational singularities. In addition, since by a classical result of Artin [4] the exceptional locus of a resolution Y of X is a tree of rational curves E_i, the kernel category $\ker(\mathbf{R}\pi_*)$ is generated by the sheaves $\mathcal{O}_{E_i}(-1)$.

Furthermore, Efimov showed in [45] that the conjecture holds true when X is the cone of a smooth Fano hypersurface. Additionally, in chapter 1 we verify the conjecture for singularities with resolutions with 1-dimensional fibers and for (not necessarily isolated) quotient singularities:

Theorem (Theorem 1.2.30 and Lemma 1.2.32)**.** *Let X be a quasi-projective variety with rational singularities, such that either there is a resolution with at most 1-dimensional fibers, or such that X has at most quotient singularities. Let $\pi : Y \to X$ be a resolution of singularities. Then the Bondal-Orlov conjecture is true, or in other words,*

$$\boldsymbol{R}\pi_* : \mathcal{D}^b(Y)/\ker(\boldsymbol{R}\pi_*) \xrightarrow{\sim} \mathcal{D}^b(X)$$

is an equivalence.

Actually, the case of "at most 1-dimensional fibers" will imply the quotient singularity case by relaxing the definition of a resolution of singularities to Deligne-Mumford stacks. In chapter 1 we explain furthermore that the conjecture is independent of the resolution, i.e. if there is one (stacky) resolution of a rational singularity X such that the conjecture holds, then it holds for all resolutions of X. In the general case however, the conjecture is widely open.

Kawamata type semiorthogonal decompositions

Only recently semiorthogonal decompositions of singular varieties have been studied systematically. The first family of examples, however, was observed by Burban in [25], where chains of smooth rational curves are considered. Karmazyn-Kuznetsov-Shinder discuss the case of toric surfaces completely in [75] and Kawamata gives 2 examples of threefolds in [76, 77]. Namely, the first example is the nodal quadric threefold in $\mathbb{P}^4$ with equation $xy - zw = 0$ and the second example is obtained by blowing up 2 points in $\mathbb{P}^3$ and then contracting the strict transform of the line passing through the 2 points.

All the above definitions are of rather specific form. Namely, they can be summarized by the following definition. We say that a Gorenstein projective variety X admits a *Kawamata type semiorthogonal decomposition* if there is an admissible semiorthogonal decomposition

$$\mathcal{D}^b(X) \simeq \langle \mathcal{A}, \mathcal{D}^b(R_1), \ldots, \mathcal{D}^b(R_m) \rangle,$$

where $\mathcal{A} \subset \mathcal{D}^{\mathrm{perf}}(X)$ and the R_i's are finite-dimensional algebras.

In chapter 2 we mainly focus on necessary conditions for Kawamata type decompositions of threefolds. The main obstruction studied here is $\mathrm{K}_{-1}(X)$. On one hand, we can show that if X admits a Kawamata type decomposition, then $\mathrm{K}_{-1}(X) = 0$ (Corollary 2.4.5). On the other hand, we know how to compute $\mathrm{K}_{-1}(X)$ for curves, surfaces and threefolds using methods from chapter 1. We obtain the following results.

We note first that the same reasoning as in [25] can be used to generalize this example to nodal trees, that is curves with smooth rational components and such that the dual graph has no loops, as we explain in Theorem 2.4.9. On the other hand, by work of Weibel [124] (see also Proposition 2.3.1) one knows that $\mathrm{K}_{-1}(C)$ for a connected projective nodal curve C is a free abelian group of rank λ, where λ is the number of loops (or Betti number) of the dual graph of C. If all the irreducible components of C are isomorphic to $\mathbb{P}^1$, we obtain:

Theorem (Corollary 2.4.11)**.** *Let C be a connected projective nodal curve with only smooth and rational irreducible components, then $\mathcal{D}^b(C)$ admits a Kawamata type decomposition if and only if $\mathrm{K}_{-1}(C) = 0$, or, equivalently, if the dual graph of C has no loops.*

Kawamata type decompositions for toric surfaces have been characterized in a similarly in [75]. There the authors show that a Gorenstein toric surface admits such a decomposition if and only if the Brauer group vanishes. In Proposition 2.3.7 we show that in this case the Brauer group coincides with K_{-1}.

For the 3-dimensional case let us assume that X has only nodal singularities. We can show for the following families of examples that there is no Kawamata type decomposition.

Theorem (Example 2.4.15 and Example 2.4.16)**.** *The following nodal threefolds have no Kawamata type decomposition:*
1) All nodal hypersurfaces in $\mathbb{P}^4$ except for the nodal quadric,
2) All nodal double solids $X \xrightarrow{2:1} \mathbb{P}^3$ except for the nodal quadric,
3) All nodal prime (meaning rank of $\mathrm{Pic}(X)$ is one) Fano threefolds of index 2 of degrees $1 \leq d \leq 4$ and such that the rank of the class group is maximal.

Chapter 1

K-theory and the singularity category of quotient singularities

Abstract

In this paper we study Schlichting's K-theory groups of the Buchweitz-Orlov singularity category $\mathcal{D}^{\mathrm{sg}}(X)$ of a quasi-projective algebraic scheme X/k with applications to Algebraic K-theory.

We prove for isolated quotient singularities over an algebraically closed field of characteristic zero that $\mathrm{K}_0(\mathcal{D}^{\mathrm{sg}}(X))$ is finite torsion, and that $\mathrm{K}_1(\mathcal{D}^{\mathrm{sg}}(X)) = 0$. One of the main applications is that algebraic varieties with isolated quotient singularities satisfy rational Poincaré duality on the level of the Grothendieck group; this allows computing the Grothendieck group of such varieties in terms of their resolution of singularities. Other applications concern the Grothendieck group of perfect complexes supported at a singular point and topological filtration on the Grothendieck groups.

Introduction

In this paper we perform a systematic study of the Schlichting K-theory groups of the dg-enhancement of the Buchweitz-Orlov singularity category $\mathcal{D}^{\mathrm{sg}}(X) = \mathcal{D}^b(X)/\mathcal{D}^{\mathrm{perf}}(X)$; we call the latter K-theory groups the *singularity* K*-theory*.

Let X/k be a quasi-projective scheme. Let $\mathrm{K}_i(X) = \mathrm{K}_i(\mathcal{D}^{\mathrm{perf}}(X))$ denote the Thomason-Trobaugh K-theory of perfect complexes, which in the quasi-projective case coincides with K-theory of vector bundles on X, while $\mathrm{G}_i(X) = \mathrm{K}_i(\mathcal{D}^b(X))$ is Quillen's G-theory, that is K-theory of coherent sheaves. By construction the singularity K-theory groups $\mathrm{K}_i^{sg}(X)$ fit into an exact sequence

$$\cdots \to \mathrm{K}_i(X) \to \mathrm{G}_i(X) \to \mathrm{K}_i^{sg}(X) \to \mathrm{K}_{i-1}(X) \to \dots,$$

for $i \geq 1$, finishing at

$$\cdots \to \mathrm{K}_1^{sg}(X) \to \mathrm{K}_0(X) \to \mathrm{G}_0(X) \to \mathrm{K}_0^{sg}(X) \to 0, \tag{1.0.1}$$

but negative K groups can be taken into account as well, see Lemma 1.1.11.

A classical result going back to Serre is that if X is regular, then $\mathcal{D}^{\mathrm{perf}}(X) = \mathcal{D}^b(X)$, so the canonical maps $\mathrm{K}_i(X) \to \mathrm{G}_i(X)$ are isomorphisms for all i and $\mathrm{K}_i^{sg}(X) = 0$. In general we may think of the singularity K-theory groups $\mathrm{K}_i^{sg}(X)$ as a tool for controlling the difference between K-theory and G-theory. This approach is essentially a homological incarnation of Orlov's definition of the singularity category, and explains the terms "singularity category" and "singularity K-theory".

We develop the theory of singularity K-theory, explaining its functoriality properties and stating relevant exact sequences. Many of these properties follow directly from the work of Orlov [94, 96] once one makes sure that the relevant triangulated functors are induced from dg enhancements. For a similar perspective on studying homological invariants of the singularity category see [114, 115, 54], and for an algebraic approach to K_0 and K_1 of the singularity category via MCM modules see [65, 89, 47].

Let us motivate our study from several viewpoints, relating to earlier work in Algebraic K-theory of singular varieties, and pointing out what singularity K-theory has to offer in each case. As a general rule our most interesting applications concentrate on isolated rational singularities, including quotient and *ADE* singularities.

1. Poincaré duality for quotient singularities.

One of the main questions which motivated this work has been the following one. If X/k is a quasi-projective algebraic variety with quotient singularities, it is a natural guess that canonical maps $\mathrm{K}_i(X) \to \mathrm{G}_i(X)$ are isomorphisms up to torsion; indeed this could be expected as X should be thought of as an analog of a $\mathbb{Q}$-manifold, while $\mathrm{K}_i(X) \to \mathrm{G}_i(X)$ may be thought as the Poincaré duality map; we use the notation $\mathrm{PD} : \mathrm{K}_i(X) \to \mathrm{G}_i(X)$ for this map. In general however it is not true that PD is an isomorphism up to torsion for varieties with quotient singularities.

Indeed, if either X has *nonisolated* quotient singularities or if $i \geq 1$, then examples of Gubeladze [58] (cf Example 1.3.7) and Srinivas [110] (see Remark 1.2.21) respectively show that $\mathrm{K}_i(X) \to \mathrm{G}_i(X)$ is not an isomorphism, even after tensoring with $\mathbb{Q}$. The typical phenomenon is that $\mathrm{G}_i(X)$ are under control while $\mathrm{K}_i(X)$ become counterintuitive. In both examples of Srinivas and Gubeladze $\mathrm{K}_i(X) \to \mathrm{G}_i(X)$ has a "huge" kernel.

One of our main results is that Poincaré duality does hold up to torsion for $i = 0$ in the isolated quotient singularities case:

Theorem 1.0.1 (See Theorem 1.2.23). *Let X be an n-dimensional quasi-projective variety over an algebraically closed field k of characteristic zero. Assume that X has isolated quotient singularities with isotropy groups G_i, $i = 1, \ldots, m$. Then the map*

$$\mathrm{PD} : \mathrm{K}_0(X) \to \mathrm{G}_0(X)$$

is injective, and its cokernel is a finite torsion group annihilated by $\mathrm{lcm}(|G_1|, \ldots, |G_m|)^{n-1}$.

We deduce the following Corollary of the Theorem, which often allows to conclude that $K_0(X)$ is finitely generated:

Corollary 1.0.2 (See Theorem 1.2.28)**.** *Under the same assumptions as in the Theorem, for any resolution of singularities* $\pi : Y \to X$ *the pullback* $\pi^* : K_0(X) \to K_0(Y)$ *is injective.*

This is a strengthening of a result of Levine who proves the result in dimension up to three and shows that π^* has torsion kernel in general [82].

Let us emphasize that there is in principle no easy way of controlling $K_0(X)$ of singular varieties. To illustrate our point, let us note that it is a well-known open question in K-theory of singular varieties, whether every weighted projective space $X = \mathbb{P}(a_0, \ldots, a_n)$ has a finitely generated $K_0(X)$ [58, Acknowledgements], [70, 5.2.3]; note that $G_0(X)$ is finitely generated and has rank $n + 1$.

One important method of computing K-theory of singular varieties has been developed in [33, 34, 35, 36, 37] and consists in relating K-theory to various sheaf cohomology groups. This method has been applied to weighted projective spaces in [86] where $K_0(\mathbb{P}(1, \ldots, 1, a))$ has been computed (it is isomorphic to the Grothendieck group of a projective space of the same dimension, which is the answer one would expect).

Regarding weighted projective spaces, we can prove the following. If $a_0, \ldots, a_n$ are pairwise coprime, so that $X = \mathbb{P}(a_0, \ldots, a_n)$ has isolated quotient singularities, then using the Theorem and the Corollary above we deduce that $K_0(X)$ is a free abelian group of rank $n+1$ (Application 1.3.2).

The Theorem above follows using the exact sequence (1.0.1) once we know that for varieties with isolated quotient singularities over an algebraically closed field of characteristic zero $K_0^{sg}(X)$ is finite torsion (Proposition 1.2.5) and $K_1^{sg}(X) = 0$ (Corollary 1.2.19). In order to study the general case we first study the local case $\mathbb{A}^n/G$, where a finite group G acts on its linear representation. We study this local case in some detail relying on tools such as equivariant K-theory, equivariant Chow groups and cdh topology (Propositions 1.2.8, 1.2.9, 1.2.10, 1.2.18).

In contrast to the isolated quotient singularities case, the group $K_1^{sg}(X)$ in general does not vanish for more general singularities, e.g. for rational isolated singularities (Example 1.3.8) or non-isolated quotient singularities (Example 1.3.7), and the assumption that k is algebraically closed is necessary as well (Example 1.3.9).

2. Cohomology and homology algebraic cycles.

The usual Chow groups have the functoriality property of Borel-Moore homology theory, and it has been asked by Srinivas what is the correct definition of Chow cohomology of singular varieties [111]. Taking insight from the intersection homology, it seems natural that in order to define such a theory one needs to generalize both the algebraic cycles and the rational equivalence relation. For example the Chow group $CH_{\dim(X)-1}(X)$ (which coincides with $Cl(X)$ when X is normal) is the group of "homology divisors", whereas $Pic(X)$ can be thought as the group of "cohomology divisors". Cohomology zero cycles have been introduced and studied in [83].

Let us now take the sheaves rather than cycles perspective, and see what we can say then. This approach is legitimate as one way to define Chow groups (up to torsion) is to take the associated graded groups for the topological filtration on $\mathrm{G}_0(X)$, that is the filtration given by codimension of support. See work of Gillet [52] for a realization of this approach to Chow groups and intersection theory in the regular case and Fulton [49] for the view-point on $\mathrm{K}_0(X) \otimes \mathbb{Q}$ as a variant for Chow-cohomology theory.

In fact the difference between homology and cohomology algebraic cycles in the singular case seems to be of the same nature as between coherent sheaves and perfect complexes. For example, if we ask the question: what makes a homology class cohomological, this can be interpreted as the question about

$$\mathrm{K}_0^{sg}(X) = \mathrm{Coker}(\mathrm{K}_0(X) \to \mathrm{G}_0(X)).$$

We explain that there is an induced "topological" filtration by codimension of support on $\mathrm{K}_0^{sg}(X)$ and study its associated graded groups $\mathrm{gr}^i\mathrm{K}_0^{sg}(X) = F^i\mathrm{K}_0^{sg}(X)/F^{i+1}\mathrm{K}_0^{sg}(X)$; we think of these groups as *obstructions for the codimension i homology algebraic cycles to be cohomological.* This approach can be demonstrated in small dimension and codimension as follows.

Theorem 1.0.3. *(Proposition 1.1.25) Let X/k be a connected reduced quasi-projective scheme of pure dimension over an algebraically closed field, then*
(1) $\mathrm{gr}^0\mathrm{K}_0^{sg}(X) = \mathbb{Z}^{N-1}$*, where N is the number of irreducible components of X.*
(2) If X is irreducible and normal, then $\mathrm{gr}^1\mathrm{K}_0^{sg}(X) = \mathrm{Cl}(X)/\mathrm{Pic}(X)$.
(3) $\mathrm{gr}^{\dim X}\mathrm{K}_0^{sg}(X) = 0$.

In particular, if X is an irreducible normal surface, then $\mathrm{K}_0^{sg}(X)$ is concentrated in a single degree 1 and $\mathrm{K}_0^{sg}(X) \simeq \mathrm{Cl}(X)/\mathrm{Pic}(X)$.

A related and especially amusing phenomenon is that the well-known Knörrer periodicity shifts the topological filtration by one (see Proposition 1.1.31); this puts questions such as factoriality of threefolds and irreducibility of curves on equal footing (Application 1.3.5, Example 1.3.6).

In the case of ordinary double points of arbitrary dimension, the only nontrivial graded group for the topological filtration on $\mathrm{K}_0^{sg}(X)$ is in the middle codimension (Examples 1.1.32, 1.1.33). Returning to the relation between sheaves and algebraic cycles, this predicts that all cycles in codimension up to half the dimension on varieties with ordinary double points are "cohomological". For a very concrete example, note that normal varieties of dimension at least four with ordinary double points are factorial, that is $\mathrm{Pic}(X) = \mathrm{Cl}(X)$.

In the example of an isolated quotient singularity, for instance, in the local case the associated graded groups $\mathrm{gr}^i\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ are typically nonzero in the range $0 < i < n$ and are closely related to the G-equivariant Chow groups of a point [44].

3. Computing $\mathrm{K}_0(X$ on $\mathrm{Sing}(X))$: the Srinivas conjecture.

Let X be a quasi-projective variety with isolated singularities over an algebraically closed field of characteristic zero. In [109] Srinivas introduced and studied the Grothendieck group of the exact category of coherent sheaves supported at the singular locus and having finite projective dimension (i.e. perfect as complexes); for Cohen-Macaulay isolated singularities this group is isomorphic to the Grothendieck group $\mathrm{K}_0(X \text{ on } \mathrm{Sing}(X))$ of the triangulated category of zero-dimensional perfect complexes supported at the singular points [103, Proposition 2].

There is a natural homomorphism

$$l : \mathrm{K}_0(X \text{ on } \mathrm{Sing}(X)) \to \mathbb{Z}^{\mathrm{Sing}(X)}$$

induced by the length of the sheaf.

We call the question whether l is an isomorphism for isolated quotient singularities *the Srinivas conjecture* (see [109, Page 38]). Levine has proved that l is surjective for all isolated Cohen-Macaulay singularities [82, Proposition 2.6]. Furthermore, Levine proved that for isolated quotient singularities of dimension up to three in characterstic zero l is an isomorphism [82, Theorem 3.3], and that it is an isomorphism up to torsion in general [82, Theorem 2.7].

On the other hand, it is known that l is not always injective; for instance for a three-dimensional quadric cone $xy = zw$, $\mathrm{Ker}(l) = \mathbb{Z} \oplus k^*$ [82, Theorem 4.2].

Using singularity K-theory with supports we reprove surjectivity of l and deal with its injectivity. Namely, we prove that l is injective for isolated quotient singularities over an algebraically closed field of characteristic zero (see Proposition 1.3.11); this is a direct consequence of the fact that $\mathrm{K}_1^{sg}(X) = 0$ for such singularities. We also note that in our approach the surjectivity of l follows from the topological filtration considerations in subsection 2 and illustrates the interaction between homology and cohomology cycles in dimension zero: skyscraper sheaf of a singular point (homology cycle) is represented by a class of a perfect complex (cohomology cycle).

In general we show that any example where $\mathrm{K}_0(X) \to \mathrm{G}_0(X)$ has nonzero kernel will automatically have $\mathrm{Ker}(l) \neq 0$ (Remark 1.3.12).

4. Homological Bondal-Orlov localization conjecture.

Given a variety X with rational singularities and $\pi : Y \to X$ a resolution of singularities, it is a natural question whether $\pi_* : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ is essentially surjective and whether there is an equivalence $\mathcal{D}^b(Y)/\mathrm{Ker}(\pi_*) \simeq \mathcal{D}^b(X)$, that is $\mathcal{D}^b(X)$ is a Verdier quotient of $\mathcal{D}^b(Y)$; this question may be called the *Bondal-Orlov localization conjecture* [19]. As a side result, which is morally related, in a certain sense dual to, but not dependent on the singularity category, we prove that Bondal-Orlov localization conjecture holds for quotient singularities (not necessarily isolated) in characteristic zero (Theorem 1.2.30). This implies in particular that $\pi_* : \mathrm{G}_0(Y) \to \mathrm{G}_0(X)$ is surjective, which is the "dual" statement to the injectivity $\pi^* : \mathrm{K}_0(X) \to \mathrm{K}_0(Y)$ for isolated quotient singularities explained in subsection 1 above (but there is no logical link between the two statements).

In the more general setting, to the best of our knowledge it is not known whether the pushforward morphism

$$\pi_* : \mathrm{G}_0(Y) \to \mathrm{G}_0(X)$$

is surjective if X is a variety with rational singularities over an algebraically closed field and $\pi : Y \to X$ is a resolution. We call this question, as well as the long G-theory exact sequence

$$\cdots \to \mathrm{K}_i(\mathrm{Ker}(\pi_*)) \to \mathrm{G}_i(Y) \to \mathrm{G}_i(X) \to \cdots \to \mathrm{K}_0(\mathrm{Ker}(\pi_*)) \to \mathrm{G}_0(Y) \to \mathrm{G}_0(X) \to 0$$

predicted by the Bondal-Orlov conjecture the *Homological Bondal-Orlov conjecture* and we hope return to this question in the future.

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Notation and conventions

Unless specified otherwise, the schemes we consider are quasi-projective over a field k, however most results remain true in the generality of Orlov's (ELF) condition [94]. Furthermore, the base field k is assumed to have characteristic zero; however all general results in Section 1.1 are true without this assumption.

If G is a finite group, we write $\widehat{G}$ for the group of characters $\mathrm{Hom}(G, k^*)$. We write $\mathbb{Z}_n$ for the cyclic group of order n.

All triangulated and dg categories are assumed to be k-linear. All functors such as pullback π^*, pushforward π_* and tensor product $\otimes$ when considered between derived categories are derived functors.

1.1 Singularity K-theory

1.1.1 Triangulated and dg singularity categories

We start by introducing the category whose K-theory we are going to study. Unless stated otherwise, X is a quasi-projective scheme over a field k. We write $\mathcal{D}^b(X)$ for the bounded derived category of coherent sheaves on X and $\mathcal{D}^{\mathrm{perf}}(X)$ for its subcategory of perfect complexes, which in the quasi-projective case coincides with the subcategory of bounded complexes of locally free sheaves.

Definition 1.1.1 (Buchweitz [24], Orlov [94])**.** *The triangulated category of singularities of X is the Verdier quotient*

$$\mathcal{D}^{\mathrm{sg}}(X) := \mathcal{D}^b(X)/\mathcal{D}^{\mathrm{perf}}(X).$$

As we will be interested in K-theory of the singularity category, we need to specify a dg-enhancement for $\mathcal{D}^{\mathrm{sg}}(X)$ to apply Schlichting's machinery of K-theory of dg-categories.

For that we first recall that $\mathcal{D}^b(X)$ has a unique dg-enhancement, up to quasi-equivalence [84, 27]. We denote this dg-enhancement by $\mathcal{D}^b_{dg}(X)$. Considering the full dg-subcategory of perfect objects in $\mathcal{D}^b_{dg}(X)$, we get a dg-enhancement $\mathcal{D}^{\mathrm{perf}}_{dg}(X)$ of $\mathcal{D}^{\mathrm{perf}}(X)$. Finally, applying the Drinfeld quotient construction [41] to the pair $\mathcal{D}^{\mathrm{perf}}_{dg}(X) \subset \mathcal{D}^b_{dg}(X)$ we get a dg-category $\mathcal{D}^{\mathrm{sg}}_{dg}(X)$ which is a dg-enhancement for $\mathcal{D}^{\mathrm{sg}}(X)$.

We note that even though the dg-enhancement of $\mathcal{D}^{\mathrm{sg}}(X)$ may not be unique, our choice is canonical in a way that all enhancements of $\mathcal{D}^{\mathrm{sg}}(X)$ induced by an enhancement of $\mathcal{D}^b(X)$ are quasi-equivalent.

Similarly, we consider the singularity category with supports. For any closed $Z \subset X$ let

$$\mathcal{D}^{\mathrm{sg}}_Z(X) = \mathcal{D}^b_Z(X)/\mathcal{D}^{\mathrm{perf}}_Z(X).$$

Here $\mathcal{D}^b_Z(X)$ consists of complexes in $\mathcal{D}^b(X)$ acyclic away from Z, and $\mathcal{D}^{\mathrm{perf}}_Z(X) = \mathcal{D}^b_Z(X) \cap \mathcal{D}^{\mathrm{perf}}(X)$. A dg-enhancement of $\mathcal{D}^b(X)$ induces one for $\mathcal{D}^b_Z(X)$, and using the Drinfeld quotient construction, $\mathcal{D}^{\mathrm{sg}}_Z(X)$ acquires a dg-enhancement $\mathcal{D}^{\mathrm{sg}}_{Z,dg}(X)$.

Remark 1.1.2. *An alternative definition for the singularity category with supports would be to consider the kernel category of the restriction functor $\mathcal{D}^{\mathrm{sg}}(X) \to \mathcal{D}^{\mathrm{sg}}(X \setminus Z)$. By a result of Chen [30, Theorem 1.3] this kernel category will be the idempotent closure of $\mathcal{D}^{\mathrm{sg}}_Z(X)$ in $\mathcal{D}^{\mathrm{sg}}(X)$ (cf Proposition 1.1.5 below), hence $\mathcal{D}^{\mathrm{sg}}_Z(X)$ carries more information about the singularity.*

We now list some properties of the singularity categories, which are due to Orlov. Even though Orlov formulates these results on the triangulated level, they all lift to the dg-enhancements due to the fact that all well-defined derived pullback and pushforward functors lift to dg-enhancements of $\mathcal{D}^b(X)$ and $\mathcal{D}^{\mathrm{perf}}(X)$ [108].

Proposition 1.1.3 (Orlov [94])**.** *Let $j : U \subset X$ be an open embedding such that $\mathrm{Sing}(X) \subset U$. Then*

$$j^* : \mathcal{D}^{\mathrm{sg}}(X) \xrightarrow{\sim} \mathcal{D}^{\mathrm{sg}}(U)$$

is an equivalence, induced by a functor between dg-enhancements.

Theorem 1.1.4 (Knörrer periodicity, Orlov [94])**.** *Let X/k be a smooth quasi-projective scheme and let $f : X \to \mathbb{A}^1$ be a non-zero morphism. Define $g = f + xy : X \times \mathbb{A}^2 \to \mathbb{A}^1$. Let $Z_f = f^{-1}(\{0\})$ and $Z_g = g^{-1}(\{0\})$, and let $W = Z_f \times \{0\} \times \mathbb{A}^1 \subset X \times \mathbb{A}^2$. Furthermore, denote by $i : W \hookrightarrow Z_g$ the inclusion and $p : W \to Z_f$ the flat projection. Then*

$$i_* p^* : \mathcal{D}^{\mathrm{sg}}(Z_f) \to \mathcal{D}^{\mathrm{sg}}(Z_g)$$

is an equivalence of triangulated categories induced by a functor between dg-enhancements.

Recall that a full triangulated subcategory $\mathcal{T}$ of a triangulated category $\mathcal{D}$ is called dense if any object in $\mathcal{D}$ is a direct summand of an object in $\mathcal{T}$.

Proposition 1.1.5 (Orlov [96])**.** *For any closed subset $Z \subseteq X$, the induced functor*

$$\mathcal{D}^{\mathrm{sg}}_Z(X) \to \mathcal{D}^{\mathrm{sg}}(X)$$

is fully faithful and is induced by a functor between dg-enhancements. Furthermore, if $\mathrm{Sing}(X) \subset Z$*, then this functor has a dense image.*

If $\mathcal{D}$ is a triangulated category, then we write $\overline{\mathcal{D}}$ for the idempotent completion of $\mathcal{D}$; $\overline{\mathcal{D}}$ is a triangulated category by [8]. If $\mathcal{D}$ admits a dg-enhancement, then the fully faithful functor $\mathcal{D} \subset \overline{\mathcal{D}}$ is induced by a dg-functor (see e.g. [10, 1.6.2]).

Theorem 1.1.6 (Orlov [96])**.** *Assume that the formal completions of X and X' along* $\mathrm{Sing}(X)$ *and* $\mathrm{Sing}(X')$ *respectively are isomorphic. Then we have equivalences*

$$\mathcal{D}^{\mathrm{sg}}_{\mathrm{Sing}(X)}(X) \simeq \mathcal{D}^{\mathrm{sg}}_{\mathrm{Sing}(X')}(X')$$

and

$$\overline{\mathcal{D}^{\mathrm{sg}}(X)} \simeq \overline{\mathcal{D}^{\mathrm{sg}}(X')}$$

induced by functors between dg-enhancements.

In light of our interest in idempotent completions we will also need the following celebrated result by Thomason.

Theorem 1.1.7 (Thomason, Theorem 2.1 in [118])**.** *Let $\mathcal{D}$ be an essentially small triangulated category, then there is a one-to-one correspondence*

$$\{\mathcal{T} \subseteq \mathcal{D} \mid \mathcal{T} \text{ dense strictly full triang. subcat.}\} \stackrel{1:1}{\longleftrightarrow} \{H \subseteq \mathrm{K}_0(\mathcal{D}) \mid H \text{ subgroup}\}.$$

The correspondence sends strictly full dense subcategories $\mathcal{T} \subseteq \mathcal{D}$ to the image of $\mathrm{K}_0(\mathcal{T})$ *in* $\mathrm{K}_0(\mathcal{D})$ *and the inverse sends a subgroup H of* $\mathrm{K}_0(\mathcal{D})$ *to the full triangulated subcategory $\mathcal{D}_H$, where* $\mathcal{D}_H := \{A \in \mathcal{D} \mid [A] \in H \subseteq \mathrm{K}_0(\mathcal{D})\}$.

1.1.2 K-Theory of the singularity category

Schlichting's construction of the K-theory spectrum [106, 107] can be applied to produce K-theory groups $\mathbb{K}_i(\mathcal{C})$, $i \in \mathbb{Z}$ for a k-linear pretriangulated dg-category $\mathcal{C}$.

The $\mathbb{K}_i$ groups are covariantly functorial for dg-functors; we summarize their properties as follows. For a pretriangulated dg-category $\mathcal{C}$ we write $H^0(\mathcal{C})$ for its triangulated homotopy category.

(0) $\mathbb{K}_0(\mathcal{C})$ is the Grothendieck group of the idempotent completion of $H^0(\mathcal{C})$.

1. If $\mathcal{C} \to \mathcal{C}'$ induces a fully faithful embedding $H^0(\mathcal{C}) \to H^0(\mathcal{C}')$ with a dense image, then all $\mathbb{K}_i(\mathcal{C}) \to \mathbb{K}_i(\mathcal{C}')$ are isomorphisms.

2. If $\mathcal{A} \to \mathcal{B} \to \mathcal{C}$ induces a fully faithful embedding $H^0(\mathcal{A}) \to H^0(\mathcal{B})$ such that $H^0(\mathcal{A})$ is the kernel of $H^0(\mathcal{B}) \to H^0(\mathcal{C})$ and a fully faithful functor $H^0(\mathcal{B})/H^0(\mathcal{A}) \to H^0(\mathcal{C})$ with a dense image, then there is a long exact sequence

$$\cdots \to \mathbb{K}_i(\mathcal{A}) \to \mathbb{K}_i(\mathcal{B}) \to \mathbb{K}_i(\mathcal{C}) \to \mathbb{K}_{i-1}(\mathcal{A}) \to \ldots$$

3. $\mathbb{K}_i(\mathcal{D}^b_{dg}(X))$ are isomorphic to $\mathrm{G}_i(X)$, Quillen's G-theory, that is K-theory of coherent sheaves. In particular, $\mathbb{K}_i(\mathcal{D}^b(X)) = 0$ for $i < 0$.

4. $\mathbb{K}_i(\mathcal{D}^{\mathrm{perf}}_{dg}(X))$ are isomorphic to $\mathrm{K}_i(X)$, the Thomason-Trobaugh K-theory [119], which under our assumptions on X (quasi-projective scheme over a field) are isomorphic to Quillen's K-theory of vector bundles.

Remark 1.1.8. *It is well-known that* K*-theory of triangulated categories satisfying the axioms analogous to those listed above cannot be defined [105]; the counterexample is provided by the two singularity categories of schemes which are equivalent as triangulated categories but are forced to have non-isomorphic higher* K*-groups if one assumes the long exact* K*-theory sequence for Verdier quotients of triangulated categories.*

Definition 1.1.9. *We define the singularity* K*-theory groups of X by*

$$\mathrm{K}^{sg}_i(X) = \begin{cases} \mathbb{K}_i(\mathcal{D}^{\mathrm{sg}}_{dg}(X)), & i \neq 0 \\ \mathrm{K}_0(\mathcal{D}^{\mathrm{sg}}_{dg}(X)), & i = 0 \end{cases}$$

and we also define $\mathbb{K}^{sg}_0(X) = \mathbb{K}_0(\mathcal{D}^{\mathrm{sg}}(X))$.

Remark 1.1.10. *We make a special consideration for $i = 0$ since by property (0) above $\mathbb{K}^{sg}_0(X)$ is in fact the Grothendieck group of the idempotent completion of $\overline{\mathcal{D}^{\mathrm{sg}}(X)}$, and not of $\mathcal{D}^{\mathrm{sg}}(X)$ itself. By Theorem 1.1.7 we have $\mathrm{K}^{sg}_0(X) \subset \mathbb{K}^{sg}_0(X)$. On the other hand by property (1) it is true that for $i \neq 0$, $\mathrm{K}^{sg}_i(X) = \mathbb{K}_i(\mathcal{D}^{\mathrm{sg}}_{dg}(X)) \simeq \mathbb{K}_i(\overline{\mathcal{D}^{\mathrm{sg}}_{dg}(X)})$.*

Let us write $\mathrm{PD} : \mathrm{K}_i(X) \to \mathrm{G}_i(X)$ for the canonical "Poincaré duality" morphism induced by $\mathcal{D}^{\mathrm{perf}}_{dg}(X) \subset \mathcal{D}^b_{dg}(X)$. Our main motivation in defining the singularity K-theory is for studying this map.

Lemma 1.1.11 (Singularity K-theory exact sequences)**.** *We have exact sequences*

$$\cdots \to \mathrm{K}_i(X) \overset{\mathrm{PD}}{\to} \mathrm{G}_i(X) \to \mathrm{K}^{sg}_i(X) \to \cdots \to \mathrm{K}_0(X) \overset{\mathrm{PD}}{\to} \mathrm{G}_0(X) \to \mathrm{K}^{sg}_0(X) \to 0, \tag{1.1.1}$$

$$0 \to \mathrm{K}^{sg}_0(X) \to \mathbb{K}^{sg}_0(X) \to \mathrm{K}_{-1}(X) \to 0. \tag{1.1.2}$$

and isomorphisms for $j \geq 1$

$$\mathrm{K}^{sg}_{-j}(X) \simeq \mathrm{K}_{-j-1}(X). \tag{1.1.3}$$

Proof. The statement follows from a single K-theory sequence using the properties of Schlichting K-groups given above and the fact that the image of $\mathrm{G}_0(X)$ in $\mathbb{K}^{sg}_0(X)$ is $\mathrm{K}^{sg}_0(X)$. □

We record the following well-known result:

Lemma 1.1.12. *$\mathcal{D}^{\mathrm{sg}}(X)$ is idempotent complete if and only if $\mathrm{K}_{-1}(X) = 0$.*

Proof. Using (1.1.2) we see that vanishing of $\mathrm{K}_{-1}(X)$ is equivalent to $\mathrm{K}_0(\mathcal{D}^{\mathrm{sg}}(X)) = \mathrm{K}_0(\overline{\mathcal{D}^{\mathrm{sg}}(X)})$ which implies $\mathcal{D}^{\mathrm{sg}}(X) = \overline{\mathcal{D}^{\mathrm{sg}}(X)}$ by the theorem of Thomason (Theorem 1.1.7). □

Similarly to the definition of the singularity K-theory, for every closed $Z \subset X$ we consider the singularity K-theory with supports defined by

$$\mathrm{K}_i^{sg}(X \text{ on } Z) = \begin{cases} \mathbb{K}_i(\mathcal{D}^{\mathrm{sg}}_{Z,dg}(X)), & i \neq 0 \\ \mathrm{K}_0(\mathcal{D}^{\mathrm{sg}}_{Z,dg}(X)), & i = 0 \end{cases}$$

Lemma 1.1.13. *If $\mathrm{Sing}(X) \subset Z$, then we have natural isomorphisms $\mathrm{K}_i^{sg}(X \text{ on } Z) \simeq \mathrm{K}_i^{sg}(X)$ for $i \neq 0$ and $\mathrm{K}_0^{sg}(X \text{ on } Z) \to \mathrm{K}_0^{sg}(X)$ is injective.*

Proof. Follows from Proposition 1.1.5 and property (1) of Schlichting's K-theory. □

Remark 1.1.14. *There are exact sequences for singularity K-theory with supports analogous to (1.1.1), (1.1.2); note that $\mathrm{K}_i(\mathcal{D}^{\mathrm{perf}}_{dg,Z}(X)) = \mathrm{K}_i(X \text{ on } Z)$ are the Thomason-Trobaugh K-theory groups [119] while $\mathrm{K}_i(\mathcal{D}^b_{dg,Z}(X)) \simeq \mathrm{G}_i(Z)$ is Quillen's G-theory [100].*

We now discuss functoriality properties of K_i^{sg}.

Lemma 1.1.15. *K_i^{sg} are contravariantly functorial for morphisms of finite Tor-dimension, and are covariantly functorial for proper morphisms of finite Tor-dimension.*

Proof. This holds because of the triangulated singularity categories have this functoriality [94], and since the pullback and pushforward functors are induced by dg-enhancements [108]. □

Lemma 1.1.16. *Let U be an open subscheme of X containing the singular locus $\mathrm{Sing}(X)$. Then the inclusion $j : U \to X$ induces an isomorphism*

$$j^* : \mathrm{K}_i^{sg}(X) \simeq \mathrm{K}_i^{sg}(U)$$

for all $i \in \mathbb{Z}$.

Proof. By Proposition 1.1.3 j^* is a quasi-equivalence on dg singularity categories, so it must induce an isomorphism on K-theory groups. □

Lemma 1.1.17. *Let X be a quasi-projective variety, and let $p : V \to X$ be a vector bundle over X. Then we have an isomorphism*

$$p^* : \mathrm{K}_0^{sg}(X) \simeq \mathrm{K}_0^{sg}(V).$$

Proof. Let $i : X \to V$ the zero section. Since p is flat and i is a regular embedding, both morphisms p and i are of finite Tor dimension, so by Lemma 1.1.15 we have pullback homomorphisms $p^* : \mathrm{K}_i^{sg}(X) \to \mathrm{K}_i^{sg}(V)$ and $i^* : \mathrm{K}_i^{sg}(V) \to \mathrm{K}_i^{sg}(X)$, and i^* is left-inverse to p^*, in particular p^* is injective.

On the other hand, from the diagram

$$\begin{array}{ccc} \mathrm{G}_0(X) & \twoheadrightarrow & \mathrm{K}_0^{sg}(X) \\ {\scriptstyle p^*}\downarrow{\scriptstyle \simeq} & & \downarrow{\scriptstyle p^*} \\ \mathrm{G}_0(V) & \twoheadrightarrow & \mathrm{K}_0^{sg}(V) \end{array}$$

we see immediately that $p^* : \mathrm{K}_0^{sg}(X) \to \mathrm{K}_0^{sg}(V)$ is surjective as well. □

Remark 1.1.18. *The functors K_i^{sg} are not homotopy invariant for $i \neq 0$ in general. Consider for example the case of K_1^{sg}; if we have $\mathrm{K}_1^{sg}(X \times \mathbb{A}^1) \simeq \mathrm{K}_1^{sg}(X)$, then using the five-lemma applied to the five bottom terms of the sequence* (1.1.1)*, we would deduce that $\mathrm{K}_0(X \times \mathbb{A}^1) \simeq \mathrm{K}_0(X)$ which typically does not hold for singular varieties.*

We will now present a method to compute $\mathrm{K}_j^{sg}(X)$ for a special class of schemes which we call $\mathbb{A}^1$-contractible. This approach generalizes the so-called Swan-Weibel homotopy trick, which is used to show that normal graded domains have vanishing Picard group [88, Lemma 5.1].

Definition 1.1.19. *We say that X is* $\mathbb{A}^1$-contractible*, if there exists a morphism $H : X \times \mathbb{A}^1 \to X$ such that $H|_{X\times 1}$ is the identity map and $H|_{X\times 0}$ is a constant rational point $x_0 \in X$. We also say that H is a* contraction *of X.*

Lemma 1.1.20. *The following affine schemes are $\mathbb{A}^1$-contractible:*

1. *$\mathbb{A}^n/G$, where G acts linearly on $\mathbb{A}^n$*

2. *$V(f) \subset \mathbb{A}^n$, where $f \in k[x_1, \dots, x_n]$ is a weighted homogeneous polynomial*

Proof. (1) $\mathbb{A}^n$ admits a G-equivariant contraction $H_{\mathbb{A}^n} : \mathbb{A}^n \times \mathbb{A}^1 \to \mathbb{A}^n$, $H(v,t) = tv$ (G acts trivially on the $\mathbb{A}^1$ factor), which induces a contraction $H : \mathbb{A}^n/G \times \mathbb{A}^1 \to \mathbb{A}^n/G$.

(2) By assumption the algebra $k[V(f)] = k[x_1, \dots, x_n]/(f)$ is positively graded. Let $w_i > 0$ be the weight of x_i. Then the k-algebra morphism

$$k[X] \to k[X,t], \quad x_i \mapsto t^{w_i} \cdot x_i$$

is well-defined and induces a contraction for $V(f)$. □

Proposition 1.1.21. *Let X be $\mathbb{A}^1$-contractible.*

(1) For every $j \geq 0$ the canonical map $\mathrm{PD} : \mathrm{K}_j(X) \to \mathrm{G}_j(X)$ factors as a composition $\mathrm{K}_j(X) \xrightarrow{x_0^*} \mathrm{K}_j(\mathrm{Spec}(k)) \xrightarrow{p^*} \mathrm{G}_j(X)$, where $p : X \to \mathrm{Spec}(k)$ is the structure morphism.

(2) There is a natural isomorphism

$$\mathrm{K}_0^{sg}(X) \simeq \mathrm{G}_0(X)/(\mathbb{Z}\cdot[\mathcal{O}_X]).$$

(3) If X has a smooth rational point $x_1 \in X$, then

$$\mathrm{G}_0(X) = \mathbb{Z}\cdot[\mathcal{O}_X] \oplus \mathrm{K}_0^{sg}(X)$$

and for every $j \geq 1$ there is a short exact sequence

$$0 \to \mathrm{Coker}(\mathrm{K}_j(k) \xrightarrow{p^*} \mathrm{G}_j(X)) \to \mathrm{K}_j^{sg}(X) \to \mathrm{Ker}(\mathrm{K}_{j-1}(X) \xrightarrow{x_0^*} \mathrm{K}_{j-1}(k)) \to 0.$$

Proof. (1) The proof relies on the fact that the canonical map PD commutes with pullbacks of finite Tor dimension as well as on homotopy invariance of G-theory. Let us write i_0, i_1 for the two embeddings of X into $X \times \mathbb{A}^1$ corresponding to $0, 1 \in \mathbb{A}^1$. These embeddings define Cartier divisors, in particular are regular, hence of finite Tor-dimension.

In the computation below we use the notation $K(f)$ and $G(f)$ for the pullbacks on K and G-theory respectively, and PD_Z for the canonical Poincaré duality map $\mathrm{K}_0(Z) \to \mathrm{G}_0(Z)$ on any Z. We compute:

$$\begin{aligned}
\mathrm{PD}_X &= \mathrm{PD}_X \circ K(i_1) \circ K(H) \\
&= G(i_1) \circ \mathrm{PD}_{X\times\mathbb{A}^1} \circ K(H) \\
&= G(i_0) \circ \mathrm{PD}_{X\times\mathbb{A}^1} \circ K(H) \\
&= \mathrm{PD}_X \circ K(i_0) \circ K(H) \\
&= \mathrm{PD}_X \circ K(p) \circ K(x_0) \\
&= G(p) \circ \mathrm{PD}_{\mathrm{Spec}(k)} \circ K(x_0)
\end{aligned}$$

which is what we had to establish.

Let us now compute Ker(PD), Coker(PD) for $\mathrm{PD} : \mathrm{K}_j(X) \to \mathrm{G}_j(X)$ using (1). The map x_0^* is always surjective (since $x_0^* \circ p^* = \mathrm{id}_{\mathrm{K}_j(k)}$), hence $\mathrm{Coker}(\mathrm{PD}) = \mathrm{Coker}(p^*)$, and applying this to $j = 0$ using the K-theory short exact sequence (1.1.1) we get (2).

If in addition X admits a smooth rational point x_1, then the map p^* is injective (since $x_1^* \circ p^* = \mathrm{id}_{\mathrm{K}_j(k)}$ for the pullback $x_1^* : \mathrm{G}_j(X) \to \mathrm{K}_j(k)$ for the regular embedding of x_1 into X), hence (1) implies $\mathrm{Ker}(\mathrm{PD}) = \mathrm{Ker}(x_0^*)$.

Once we have identified the kernel and cokernel of PD, (3) follows from the K-theory long exact sequence (1.1.1). □

In the two examples below we consider $\mathbb{A}^1$-contractible schemes with no smooth rational points.

Example 1.1.22. *$X = \mathrm{Spec}(k[\epsilon]/\epsilon^n)$ is $\mathbb{A}^1$-contractible by Lemma 1.1.20 (2). In this case the canonical map $\mathrm{K}_0(X) \to \mathrm{G}_0(X)$ is $\mathbb{Z} \xrightarrow{\cdot n} \mathbb{Z}$ and $\mathrm{K}_0^{sg}(X) = \mathbb{Z}_n$.*

Example 1.1.23. *Let X be an affine curve $x^2 + y^2 = 0$ over $k = \mathbb{R}$, the real numbers.*

To compute $\mathrm{G}_0(X)$ *we consider a compactification $X \subset \overline{X}$, where $\overline{X}$ is given by equation $X^2 + Y^2 = 0$ in the real projective plane $\mathbb{P}^2_{\mathbb{R}}$ with coordinates X, Y, Z. The complement $\overline{X} \setminus X$ is the single closed (non-rational) point at infinity $\infty \in \overline{X}$; as a subscheme ∞ is isomorphic to* $\mathrm{Spec}(\mathbb{C})$*.*

It is easy to see that there is an isomorphism

$$\mathrm{G}_0(\overline{X}) \simeq \mathbb{Z} \oplus \mathbb{Z}, \quad [\mathcal{F}] \mapsto (\mathrm{rk}(\mathcal{F}), \deg(\mathcal{F}))$$

(surjectivity is obvious, while injectivity boils down to the fact that every class of a skyscraper sheaf of a closed point $x \in \overline{X}$ is a multiple of the class of the skyscraper sheaf of the rational point $[0:0:1]$*, and this can be checked using the fact that* $\mathrm{CH}_0(\overline{X}) = \mathbb{Z}$*).*

We write the G*-theory localization sequence for $X \subset \overline{X}$:*

$$\mathrm{G}_0(\mathrm{Spec}(\mathbb{C})) \to \mathrm{G}_0(\overline{X}) \to \mathrm{G}_0(X) \to 0.$$

Under the isomorphism $\mathrm{G}_0(\overline{X}) = \mathbb{Z} \oplus \mathbb{Z}$ *the class of structure sheaf of the point at infinity corresponds to* $(0, 2)$*; we deduce that* $\mathrm{G}_0(X) = \mathbb{Z} \oplus \mathbb{Z}_2$ *given by the rank map and degree modulo two.*

The curve X is $\mathbb{A}^1$-contractible by Lemma 1.1.20 (2) so that by Proposition 1.1.21 (2) we obtain

$$\mathrm{K}_0^{sg}(X) = \mathrm{G}_0(X)/\mathbb{Z} \cdot [\mathcal{O}_X] = \mathbb{Z}_2,$$

generated by the class of the structure sheaf of the rational point $(0, 0) \in X$*.*

1.1.3 Topological filtration on K_0^{sg}

We introduce and study the topological filtration on $\mathrm{K}_0^{sg}(X)$, that is the filtration given by the codimension of support. Note that the support of an object in $\mathcal{D}^b(X)$ does not give a well-defined notion of support on $\mathcal{D}^{\mathrm{sg}}(X)$, as any nonzero perfect complex is isomorphic to a zero object in the singularity category. The following example gives a more subtle instance of behaviour of the support.

Example 1.1.24. *Let $X = \{xy = 0\} \subset \mathbb{A}^2$ be the union of two $\mathbb{A}^1$-lines over k. Denote the two affine lines by $L_1 = \mathbb{A}^1 \times 0 \subset X$ and $L_2 = 0 \times \mathbb{A}^1 \subset X$. The structure sheaves of L_1 and L_2 correspond to quotient rings $k[x] = k[x, y]/(y)$ and $k[y] = k[x, y]/(x)$ respectively. We have an exact sequence of $k[x, y]/(xy)$-modules*

$$0 \to k[x] \xrightarrow{x} k[x, y]/(x, y) \to k[y] \to 0$$

which translates into a distinguished triangle in $\mathcal{D}^b(X)$

$$\mathcal{O}_{L_1} \to \mathcal{O}_X \to \mathcal{O}_{L_2} \to \mathcal{O}_{L_1}[1]$$

and yields an isomorphism $\mathcal{O}_{L_2} \simeq \mathcal{O}_{L_1}[1]$ *of objects* $\mathcal{D}^{\mathrm{sg}}(X)$ *(the shift [1] is two-periodic in this example); sheaf-theoretic supports of these two objects are different.*

We can speak about codimension of support of an object of $\mathcal{D}^{\mathrm{sg}}(X)$ without defining the support itself. Let X be a quasi-projective scheme with all irreducible components of the same dimension n.

Recall that $\mathrm{K}_0(X)$, $\mathrm{G}_0(X)$ admit the so-called topological filtration (also called the coniveau or the codimension filtration), which goes back to Grothendieck and is defined as follows. The class $\alpha \in \mathrm{G}_0(X)$ (resp. $\mathrm{K}_0(X)$) belongs to $F^i\mathrm{G}_0(X)$ (resp. $F^i\mathrm{K}_0(X)$) if α can be represented by a bounded complex of coherent sheaves (resp. locally free sheaves) whose support has codimension at least i. It is clear from the definitions that the canonical map $\mathrm{PD} : \mathrm{K}_0(X) \to \mathrm{G}_0(X)$ maps $F^i\mathrm{K}_0(X)$ to $F^i\mathrm{G}_0(X)$.

We consider the natural quotient homomorphism $Q : \mathrm{G}_0(X) \twoheadrightarrow \mathrm{K}_0^{sg}(X)$ and define

$$F^i\mathrm{K}_0^{sg}(X) = Q(F^i\mathrm{G}_0(X)).$$

This gives a filtration

$$0 = F^{n+1}\mathrm{K}_0^{sg}(X) \subseteq F^n\mathrm{K}_0^{sg}(X) \subseteq \ldots \subseteq F^1\mathrm{K}_0^{sg}(X) \subseteq F^0\mathrm{K}_0^{sg}(X) = \mathrm{K}_0^{sg}(X).$$

Explicitly, we say that a class $\alpha \in \mathrm{K}_0^{sg}(X)$ has codimension at least i or that $\alpha \in F^i\mathrm{K}_0^{sg}(X)$, if α can be represented by a complex of coherent sheaves $\mathcal{E}$ on X whose cohomology sheaves are supported in codimension i.

It follows from definitions that we have canonical isomorphisms

$$F^i\mathrm{K}_0^{sg}(X) \simeq \frac{F^i\mathrm{G}_0(X)}{F^i\mathrm{G}_0(X) \cap \mathrm{PD}(\mathrm{K}_0(X))} \tag{1.1.4}$$

We let $\mathrm{gr}^i\mathrm{K}_0^{sg}(X) = F^i\mathrm{K}_0^{sg}(X)/F^{i+1}\mathrm{K}_0^{sg}(X)$, and similarly for $\mathrm{K}_0(X)$, $\mathrm{G}_0(X)$. We have a natural surjection

$$\mathrm{gr}^i\mathrm{G}_0(X) \to \mathrm{gr}^i\mathrm{K}_0^{sg}(X) \tag{1.1.5}$$

and canonical isomorphisms

$$\mathrm{gr}^i\mathrm{K}_0^{sg}(X) \simeq \frac{F^i\mathrm{G}_0(X)}{F^{i+1}\mathrm{G}_0(X) + (F^i\mathrm{G}_0(X) \cap \mathrm{PD}(\mathrm{K}_0(X)))}. \tag{1.1.6}$$

Proposition 1.1.25. *Let X/k be a connected reduced quasi-projective scheme with all irreducible components of the same dimension n.*

(1) Let N be the number of irreducible components of X. Then $\mathrm{gr}^0\mathrm{K}_0^{sg}(X) = \mathbb{Z}^{N-1}$. In particular $\mathrm{gr}^0\mathrm{K}_0^{sg}(X) = 0$ if and only if X is irreducible.

(2) If X is irreducible and normal then $\mathrm{gr}^1\mathrm{K}_0^{sg}(X) \simeq \mathrm{Cl}(X)/\mathrm{Pic}(X)$. In particular in this case $\mathrm{gr}^1\mathrm{K}_0^{sg}(X) = 0$ if and only if X is factorial.

(3) *For any* $i \geq 0$ *there is a surjective homomorphism* $\mathrm{CH}_{n-i}(X) \to \mathrm{gr}^i \mathrm{K}_0^{sg}(X)$ *which is natural with respect to pullback for flat morphisms and pushforward for proper morphisms of finite* Tor*-dimension.*

(4) *If* k *is algebraically closed, then* $\mathrm{gr}^n \mathrm{K}_0^{sg}(X) = 0$.

Proof. (1) Using $i = 0$ case of (1.1.6) we obtain

$$\mathrm{gr}^0 \mathrm{K}_0^{sg}(X) \simeq \frac{\mathrm{G}_0(X)}{F^1\mathrm{G}_0(X) + \mathrm{PD}(\mathrm{K}_0(X))}.$$

We have $\mathrm{gr}^0\mathrm{G}_0(X) = \mathbb{Z}^N$, where the isomorphism is given by generic rank at the irreducible components. Since X is connected, a locally-free sheaf has the same rank at each point, and the image of the composition $\mathrm{K}_0(X) \stackrel{\mathrm{PD}}{\to} \mathrm{G}_0(X) \to \mathrm{gr}^0\mathrm{G}_0(X)$ consists of $\mathbb{Z}$ embedded into $\mathbb{Z}^N$ diagonally, since X is reduced. We conclude that $\mathrm{gr}^0\mathrm{K}_0^{sg}(X) = \mathbb{Z}^{N-1}$.

(2) Since X is irreducible, we have canonical splittings $\mathrm{K}_0(X) = \mathbb{Z} \oplus F^1\mathrm{K}_0(X)$ and $\mathrm{G}_0(X) = \mathbb{Z} \oplus F^1\mathrm{G}_0(X)$, which are respected by PD so that from (1.1.6) we deduce

$$\mathrm{gr}^1\mathrm{K}_0^{sg}(X) \simeq \frac{F^1\mathrm{G}_0(X)}{F^2\mathrm{G}_0(X) + (F^1\mathrm{G}_0(X) \cap \mathrm{PD}(\mathrm{K}_0(X)))} \simeq \frac{F^1\mathrm{G}_0(X)}{F^2\mathrm{G}_0(X) + \mathrm{PD}(F^1\mathrm{K}_0(X))}.$$

By [51, Remark 1 on Page 126], we have a natural isomorphism $\mathrm{gr}^1\mathrm{K}_0(X) = \mathrm{Pic}(X)$. Since X is normal so that its singular locus has codimension at least two we also get the following isomorphisms

$$\mathrm{gr}^1\mathrm{G}_0(X) = \mathrm{gr}^1\mathrm{G}_0(X \setminus \mathrm{Sing}(X)) = \mathrm{Pic}(X \setminus \mathrm{Sing}(X)) = \mathrm{Cl}(X).$$

It follows that the image of $F^1\mathrm{K}_0(X)$ in $\mathrm{gr}^1\mathrm{G}_0(X) = \mathrm{Cl}(X)$ is equal to $\mathrm{Pic}(X)$, and we get $\mathrm{gr}^1\mathrm{K}_0^{sg}(X) = \mathrm{Cl}(X)/\mathrm{Pic}(X)$.

Finally, X is factorial if and only if $\mathrm{Pic}(X) = \mathrm{Cl}(X)$ which is equivalent to $\mathrm{gr}^1\mathrm{K}_0^{sg}(X) = 0$.

(3) There is a surjection $\mathrm{CH}_{n-i}(X) \to \mathrm{gr}^i\mathrm{G}_0(X)$, sending the class of an $(n-i)$-dimensional subvariety to the structure sheaf of this variety (see SGA X [15], [52, Lemma 3.8, Theorem 3.9] or [50, Example 15.1.5]), and composing it with the surjection $\mathrm{gr}^i\mathrm{G}_0(X) \to \mathrm{gr}^i\mathrm{K}_0^{sg}(X)$ gives the desired homomorphism. Naturality of this homomorphism is explained in SGA X [15], and naturality for the pushforward of a proper morphism is also explained in [50, Example 15.1.5].

(4) This is a simple Moving Lemma argument. Assume first that X is irreducible. We fix a closed subvariety $Z \subsetneq X$ containing the singular locus. By De Jong's work (see [40], Theorem 4.1) there is a proper surjective and generically finite morphism $\pi : Y \to X$ where Y is a smooth irreducible and quasi-projective variety. Let $E = \pi^{-1}(Z) \subset Y$.

Let us show that for every closed point $x \in X$ there is a point $x' \in X \setminus Z$ such that $[\mathcal{O}_x] = [\mathcal{O}_{x'}] \in \mathrm{G}_0(X)$. Indeed, since we assume that k is algebraically closed, there is a closed point $y \in Y$ such that $\pi(y) = x$, and using a simple argument (e.g. reducing to the case when Y is a curve, or using the Moving Lemma [31] for Chow groups), we can find $y' \in Y \setminus E$ such that $[\mathcal{O}_y] = [\mathcal{O}_{y'}] \in \mathrm{G}_0(Y)$. Pushing forward this equality to X we get $[\mathcal{O}_x] = [\mathcal{O}_{x'}] \in \mathrm{G}_0(X)$, where $x' = \pi(y')$. Since the structure sheaves of non-singular points are perfect complexes, we get $[\mathcal{O}_x] = [\mathcal{O}_{x'}] = 0 \in \mathrm{K}_0^{sg}(X)$. Finally, every class of a zero-dimensional complex $[\mathcal{F}] \in \mathrm{G}_0(X)$ is a linear combination of structure sheaves of closed points, and this shows that $F^n\mathrm{K}_0^{sg}(X) = 0$ if X is irreducible.

If X is not irreducible, the result is obtained by applying the argument above to each of the irreducible components of $X_i \subset X$ with respect to $Z = X_i \cap \mathrm{Sing}(X)$. □

Corollary 1.1.26. *If k is algebraically closed, $i : x \hookrightarrow X$ is a closed point and $\mathcal{O}_x$ its structure sheaf, then the image of $[i_*\mathcal{O}_x] \in \mathrm{G}_0(X)$ in $\mathrm{K}_0^{sg}(X)$ is zero.*

Proof. This is equivalent to Proposition 1.1.25 (4). □

Remark 1.1.27. *The result of the Corollary does not hold if k is not algebraically closed: see Example 1.1.23, where $X/\mathbb{R}$ is a curve, $\mathrm{K}_0^{sg}(X) = \mathbb{Z}_2$ and the generator is supported in codimension one.*

Corollary 1.1.28. *If k is an algebraically closed field, and X has isolated singularities then $\mathrm{K}_0^{sg}(X$ on $\mathrm{Sing}(X)) = 0$.*

Proof. By definition $\mathrm{K}_0^{sg}(X \text{ on } \mathrm{Sing}(X)) = \mathrm{K}_0(\mathcal{D}^{\mathrm{sg}}_{\mathrm{Sing}(X)}(X))$, and from Proposition 1.1.5 and Theorem 1.1.7 it follows that

$$\mathrm{K}_0^{sg}(X \text{ on } \mathrm{Sing}(X)) \subset \mathrm{K}_0^{sg}(X)$$

is generated by classes of coherent sheaves supported on the singular locus, so the first group has to be zero by Proposition 1.1.25 (4) as the singular locus is zero-dimensional by assumption. □

Corollary 1.1.29. *Let k be an algebraically closed field.*

(1) If X is a connected quasi-projective curve with N irreducible components, then $\mathrm{K}_0^{sg}(X) \simeq \mathbb{Z}^{N-1}$. In particular, $\mathrm{K}_0^{sg}(X) = 0$ if and only if X is irreducible.

(2) If X is a normal irreducible quasi-projective surface, then $\mathrm{K}_0^{sg}(X) \simeq \mathrm{Cl}(X)/\mathrm{Pic}(X)$. In particular, $\mathrm{K}_0^{sg}(X) = 0$ if and only if X is factorial.

Proof. We start by noticing that if $\mathrm{gr}^i\mathrm{K}_0^{sg}(X)$ is the only nontrivial quotient of the topological filtration, then $\mathrm{K}_0^{sg}(X) = \mathrm{gr}^i\mathrm{K}_0^{sg}(X)$. (1) follows as $\mathrm{gr}^i\mathrm{K}_0^{sg}(X)$ are all zero except for $\mathrm{gr}^0\mathrm{K}_0^{sg}(X) = \mathbb{Z}^{N-1}$, and similarly (2) follows using irreducibility of X since $\mathrm{gr}^i\mathrm{K}_0^{sg}(X)$ are all zero except for $\mathrm{gr}^1\mathrm{K}_0^{sg}(X) = \mathrm{Cl}(X)/\mathrm{Pic}(X)$. □

Recall functoriality of the singularity K-groups stated in Lemmas 1.1.15, 1.1.16, 1.1.17. We now explain how the topological filtration is affected by pullback and pushforward.

Lemma 1.1.30. *Let $\phi : X \to Y$ be a morphism of finite* Tor*-dimension.*

1. *If ϕ is flat or a regular closed embedding, then $\phi^*(F^i\mathrm{K}_0^{sg}(Y)) \subset F^i\mathrm{K}_0^{sg}(X)$.*

2. *If ϕ is a vector bundle or an open embedding containing the singular locus of Y, then $\phi^* : F^i\mathrm{K}_0^{sg}(Y) \simeq F^i\mathrm{K}_0^{sg}(X)$.*

3. *If ϕ is proper of codimension $c := \dim(Y) - \dim(X)$, then $\phi_*(F^i\mathrm{K}_0^{sg}(X)) \subset F^{i+c}\mathrm{K}_0^{sg}(Y)$.*

Proof. (1) The result in the case of flat morphisms follows from [52, Lemma 5.28], while in the case of regular embeddings it follows from [52, Theorem 5.27],

(2) The vector bundle case is [52, Lemma 5.29]. Let ϕ be an open embedding, since it is flat by (1) we have $\phi^*(F^i\mathrm{K}_0^{sg}(Y)) \subset F^i\mathrm{K}_0^{sg}(X)$ and we need to show that this is an equality. For that it suffices to show that every coherent sheaf $\mathcal{F}$ on X with support in codimension i can be extended to a coherent sheaf $\mathcal{F}'$ on Y with the same bound on support.

One constructs $\mathcal{F}'$ as a coherent subsheaf of the quasi-coherent sheaf $\phi_*(\mathcal{F})$ (see [62, Ex. Chapter 2, 5.15]. Since $\phi_*(\mathcal{F})$ is supported on the closure of the support of $\mathcal{F}$, we see that $\mathcal{F}'$ is supported in codimension i.

(3) If $\phi : X \to Y$ is a proper morphism of codimension c, then $\mathrm{Supp}(\phi_*\mathcal{E}) \subset \phi(\mathrm{Supp}(\mathcal{E}))$ and thus $\phi_* : F^i\mathrm{G}_0(X) \to F^{i+c}\mathrm{G}_0(Y)$ (see also [51], Ch. VI, Prop. 5.6) which implies the result. □

We now explain how Knörrer periodicity (Theorem 1.1.4) shifts the topological filtration.

Proposition 1.1.31. *The isomorphism $\mathrm{K}_0^{sg}(Z_f) \simeq \mathrm{K}_0^{sg}(Z_g)$ induced by Theorem 1.1.4 shifts the topological filtration by one, that is for all $i \geq 0$ we have natural isomorphisms $F^i\mathrm{K}_0^{sg}(Z_f) \simeq F^{i+1}\mathrm{K}_0^{sg}(Z_g)$ and $\mathrm{gr}^i\mathrm{K}_0^{sg}(Z_f) \simeq \mathrm{gr}^{i+1}\mathrm{K}_0^{sg}(Z_g)$.*

Proof. We know by Lemma 1.1.30 that p^* preserves the topological filtration and that i_* shifts it by one, however this only implies that $i_*p^*(F^i\mathrm{K}_0^{sg}(Z_f)) \subset F^{i+1}\mathrm{K}_0^{sg}(Z_g)$. To show the equality we give a different presentation of the Knörrer periodicity isomorphism.

Let $Y = Bl_{Z_f\times 0}(X \times \mathbb{A}^1)$ be the blow up and let E be the exceptional divisor. Since $Z_f \times \mathbb{A}^1$ is a complete intersection in $X \times \mathbb{A}^1$ the blow up enjoys the same properties which hold in the smooth case. For instance, E is a projective bundle $\pi : E \to Z_f$, and there is a semiorthogonal decomposition [93], [14, Theorem 6.9]

$$\mathcal{D}^b(Y) = \langle \mathcal{D}^b(Z_f), \mathcal{D}^b(X \times \mathbb{A}^1)\rangle.$$

The inclusion of $\mathcal{D}^b(Z_f)$ into $\mathcal{D}^b(Y)$ is given by the fully faithful functor $\Phi : \mathcal{D}^b(Z_f) \to \mathcal{D}^b(Y)$

$$\Phi(-) = i_{E*}(\mathcal{O}_E(-1) \otimes \pi^*(-)),$$

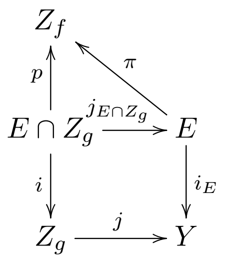
and its left adjoint is

$$\Psi(-) = \pi_*(\mathcal{O}_E(-1) \otimes i_E{}^*(-))[1].$$

Writing the open charts for the blow up one sees that one of the open charts is isomorphic to Z_g while the other one is non-singular. We write $j : Z_g \to Y$ for the open embedding of the first open chart; on the level of singularity categories, j^* is an equivalence by Proposition 1.1.3.

We consider the composition $j^*\Phi : \mathcal{D}^b(Z_f) \to \mathcal{D}^b(Z_g)$ and we now will show that $j^*\Phi = i_* p^*$. We note that restriction of π to $E \cap Z_g$ is a trivial bundle so that $E \cap Z_g = Z_f \times \mathbb{A}^1$ (in fact π itself is a trivial bundle since the normal bundle of $Z_f \times \{0\}$ in $X \times \mathbb{A}^1$ is trivial).

We consider the cartesian diagram



and by flat base change we compute that

$$j^*\Phi(-) = j^* i_{E*}(\mathcal{O}_E(-1) \otimes \pi^*(-)) = i_*(j^*_{E\cap Z_g}\mathcal{O}_E(-1) \otimes j^*_{E\cap Z_g}\pi^*(-)) = i_* p^*(-).$$

where we used that $j^*_{E\cap Z_g}\mathcal{O}_E(-1) \simeq \mathcal{O}_{E\cap Z_g}$.

Now we check the effect of $j^*\Phi$ on the toplogical filtration. We rely on Lemma 1.1.30. Since j^* strictly preserves the filtration it is sufficient to check that Φ strictly shifts the filtration by one: this holds true since π^* preserves the filtration while i_{E*} shifts it by one, and the left adjoint Ψ of Φ which will become its inverse on the level of singularity categories, shifts the filtration by negative one: this holds since i_E^* preserves the filtration while π_* shifts it by negative one. □

The next two examples consider the singularity Grothendieck group of split nodal affine quadrics, that is ordinary double points (cf [94, 3.3]).

Example 1.1.32 (Even-dimensional ordinary double points)**.** *Let $n = 2m$ and consider the split quadratic form*

$$q_n = \sum x_i y_i + z^2 \in k[x_1, y_1, \dots, x_m, y_m, z]$$

and let $Q_n \subset \mathbb{A}^{n+1}$ be the n-dimensional nodal quadric defined by $q_n = 0$.

From Knörrer periodicity we get

$$\mathrm{K}_0^{sg}(Q_n) \simeq \mathrm{K}_0^{sg}(k[z]/(z^2)) = \mathbb{Z}_2,$$

see Example 1.1.22. Furthermore, since by dimension reasons the nonzero element of $\mathrm{K}_0^{sg}(k[z]/(z^2))$ has support in codimension zero, by the shift of the topological filtration of Proposition 1.1.31 we get

$$\mathrm{K}_0^{sg}(Q_n) = \mathrm{gr}^{n/2}\mathrm{K}_0^{sg}(Q_n) = \mathbb{Z}_2.$$

Explicitly $\mathrm{K}_0^{sg}(Q_n)$ can be seen to be generated by the structure sheaf of $n/2$-codimensional subvariety $V(y_1, \dots, y_m, z) \subset Q_n$.

Example 1.1.33 (Odd-dimensional ordinary double points)**.** *Let $n = 2m - 1$ and consider*

$$q_n = \sum x_i y_i \in k[x_1, y_1, \dots, x_m, y_m]$$

and let $Q_n \subset \mathbb{A}^{n+1}$ be the n-dimensional nodal quadric defined by $q_n = 0$.

By Knörrer periodicity we get

$$\mathrm{K}_0^{sg}(Q_n) \simeq \mathrm{K}_0^{sg}(k[x, y]/(xy)),$$

and the latter Grothendieck group is isomorphic to $\mathbb{Z}$ by Corollary 1.1.29 (1), generated by the structure sheaves of one of the two irreducible components (cf Example 1.1.24). Using Proposition 1.1.31 we obtain

$$\mathrm{K}_0^{sg}(Q_n) = \mathrm{gr}^{(n-1)/2}\mathrm{K}_0^{sg}(Q_n) = \mathbb{Z},$$

generated by the structure sheaf of a codimension $(n-1)/2$ linear space $V(y_1, \dots, y_m) \subset Q_n$.

1.2 Singularity K-theory of quotient singularities

1.2.1 The local case: non-positive K-groups

For a finite group $G \subset \mathrm{GL}_n(k)$ we consider the quotient variety $\mathbb{A}^n/G = \mathrm{Spec}(k[x_1, \dots, x_n]^G)$. In this subsection we study $\mathrm{G}_0(\mathbb{A}^n/G)$ as well as K-groups $\mathrm{K}_i(\mathbb{A}^n/G)$ and $\mathrm{K}_i^{sg}(\mathbb{A}^n/G)$ for $i \leq 0$. Recall that we assume the ground field k to have characteristic zero.

Proposition 1.2.1. *Assume that $\mathbb{A}^n/G$ has an isolated singularity at the origin. Then*

$$\mathrm{K}_0(\mathbb{A}^n/G) \simeq \mathbb{Z} \quad \textit{and} \quad \mathrm{K}_{-j}(\mathbb{A}^n/G) = \mathrm{K}_{-j}^{sg}(\mathbb{A}^n/G) = 0 \textit{ for } j > 0.$$

Proof. Since $\mathbb{A}^n/G$ has an isolated singularity, $k[x_1, \dots, x_n]^G$ is a positively graded k-algebra and since the base field k has characteristic zero, we can use [36, Theorem 1.2] to express non-positive K-theory groups as follows:

$$\mathrm{K}_0(\mathbb{A}^n/G) = \mathbb{Z} \oplus \mathrm{Pic}(\mathbb{A}^n/G) \oplus \bigoplus_{i=1}^{n-1} H^i_{\mathrm{cdh}}(\mathbb{A}^n/G, \Omega^i_{/\mathbb{Q}})/dH^i_{\mathrm{cdh}}(\mathbb{A}^n/G, \Omega^{i-1}_{/\mathbb{Q}})$$

and for $j > 0$

$$\mathrm{K}_{-j}(\mathbb{A}^n/G) = H^j_{\mathrm{cdh}}(\mathbb{A}^n/G, \mathcal{O}) \oplus \bigoplus_{i=1}^{n-j-1} H^{i+j}_{\mathrm{cdh}}(\mathbb{A}^n/G, \Omega^i_{/\mathbb{Q}})/dH^{i+j}_{\mathrm{cdh}}(\mathbb{A}^n/G, \Omega^{i-1}_{/\mathbb{Q}}).$$

Here H^*_{cdh} denotes cohomology of $\mathbb{A}^n/G$ defined via the cdh-topology on Sch/k [113] and the group $dH^j_{\mathrm{cdh}}(\mathbb{A}^n/G, \Omega^{i-1}_{/\mathbb{Q}})$ is the image of the map $d : H^j_{\mathrm{cdh}}(\mathbb{A}^n/G, \Omega^{i-1}_{/\mathbb{Q}}) \to H^j_{\mathrm{cdh}}(\mathbb{A}^n/G, \Omega^i_{/\mathbb{Q}})$ induced by the Kähler differential.

Let us show that cohomology groups $H^q_{\mathrm{cdh}}(\mathbb{A}^n/G, \Omega^p_{/\mathbb{Q}})$ are zero for all $q > 0$, $p \geq 0$. Note first that one can identify $H^*_{\mathrm{cdh}}(X, \Omega^p_{/k_0}) \simeq H^*_{\mathrm{eh}}(X, \Omega^p_{/k_0})$ for all $X \in \mathrm{Sch}/k$, where the right hand side denotes the cohomology of X via the eh-topology (see [66]). To compute the cohomology of a smooth variety M endowed with a finite group action G we use the simplicial scheme $\mathrm{Ner}(G, M)$ [39, 43]; with rational coefficients we have $\mathbb{Q}_{\mathrm{eh}}(M/G) \simeq \mathbb{Q}_{\mathrm{eh}}(\mathrm{Ner}(G, M))$. Let us supress $\mathbb{Q}$ in the notation of differentials. Since we work in characteristic zero, for any smooth quasiprojective M with a G-action we have a chain of isomorphisms

$$H^*_{\mathrm{eh}}(M/G, \Omega^p) \simeq H^*_{\mathrm{eh}}(\mathrm{Ner}(G, M), \Omega^p) \simeq H^*(\mathrm{Ner}(G, M), \Omega^p_{\mathrm{Ner}(G,M)}) \simeq H^*(M, \Omega^p_M)^G.$$

Note that we used [37, Corrolary 2.5] for the second isomorphism. In particular, for $M = \mathbb{A}^n$, the latter cohomology groups vanish for all $p \geq 0$ in positive degrees.

Finally, using the formulas for K-theory groups presented at the beginning of the proof we get $\mathrm{K}_{-j}(\mathbb{A}^n/G) = 0$ and $\mathrm{K}_0(\mathbb{A}^n/G) = \mathbb{Z} \oplus \mathrm{Pic}(\mathbb{A}^n/G) = \mathbb{Z}$, since the Picard group of a normal graded k-algebra is zero. □

Remark 1.2.2. *In the previous version of this paper we claimed that every vector bundle on $\mathbb{A}^n/G$ is trivial. We do not know if this statement is true. We thank Sasha Kuznetsov for pointing out an error in our argument.*

Corollary 1.2.3. *If $\mathbb{A}^n/G$ is an isolated singularity, then the singularity category $\mathcal{D}^{\mathrm{sg}}(\mathbb{A}^n/G)$ is idempotent complete.*

Proof. As $\mathrm{K}_{-1}(\mathbb{A}^n/G) = 0$ by Proposition 1.2.1, the result follows from Lemma 1.1.12. □

Remark 1.2.4. *It is not true that every affine quasi-homogeneous or $\mathbb{A}^1$-contractible singularity has an idempotent complete singularity category: the simplest example is provided by the so-called Bloch-Murthy surface singularity X given by $x^2 + y^3 + z^7 = 0$ which has non-vanishing $K_{-1}(X)$ [124, Example 6.1].*

Proposition 1.2.5. *Let G be a finite group acting linearly on the affine space $\mathbb{A}^n$ over a field k. Then we have*

$$\mathrm{G}_0(\mathbb{A}^n/G) = \mathbb{Z} \oplus \mathrm{K}_0^{sg}(\mathbb{A}^n/G),$$

and $\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ is a finite torsion group.

Proof. By Proposition 1.1.21 (3) there is a split short exact sequence

$$0 \to \mathbb{Z} \to \mathrm{G}_0(\mathbb{A}^n/G) \to \mathrm{K}_0^{sg}(\mathbb{A}^n/G) \to 0$$

where the first map is split by the rank map. Let us show that $\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ is finite torsion. The fact that $\mathrm{G}_0(\mathbb{A}^n/G)$ is finitely-generated is well-known [7] and follows e.g. from the fact that the pushforward functor from the equivariant category to the category of coherent sheaves on the quotient variety $\pi_* : \mathcal{D}^b_G(\mathbb{A}^n) \to \mathcal{D}^b(\mathbb{A}^n/G)$ is essentially surjective by Theorem 1.2.30 (1). Thus it suffices to show that $G_0(\mathbb{A}^n/G)$ is of rank one. For that we can work rationally and compare G_0 to the Chow groups. We have a chain of isomorphisms

$$G_0(\mathbb{A}^n/G) \otimes \mathbb{Q} \simeq \mathrm{CH}_*(\mathbb{A}^n/G) \otimes \mathbb{Q} \simeq \mathrm{CH}_*(\mathbb{A}^n)^G \otimes \mathbb{Q} \simeq \mathbb{Q},$$

where the first isomorphism is the Grothendieck-Riemann-Roch Theorem for singular varieties [9, Chapter III] and the second isomorphism is [50, Example 1.7.6]. We conclude that $G_0(\mathbb{A}^n/G)$ is a finitely generated abelian group of rank one and that $K_0^{sg}(\mathbb{A}^n/G)$ is finite torsion. □

We call an element $g \in \mathrm{GL}_n(k)$ a reflection if g has finite order and acts trivially on a hyperplane. We need the following well-known Lemmas.

Lemma 1.2.6 ([11, Theorem 3.9.2])**.** *Let G be a finite subgroup of $\mathrm{GL}_n(k)$ and let N be the subgroup of G generated by reflections. There is a natural isomorphism $\mathrm{Cl}(\mathbb{A}^n/G) \simeq \widehat{G/N}$.*

Proof. Our proof relies on equivariant Chow groups [44]. Let us first assume that G does not contain reflections. In this case there is a G-invariant subvariety $Z \subset \mathbb{A}^n$ of codimension at least two, such that G acts freely on $\mathbb{A}^n \setminus Z$. Let $\pi : \mathbb{A}^n \to \mathbb{A}^n/G$ be the quotient map. Since removing locus of codimension two does not change $(n-1)$-st Chow groups, we have a chain of isomorphisms

$$\mathrm{Cl}(\mathbb{A}^n/G) = \mathrm{CH}_{n-1}(\mathbb{A}^n/G) \simeq \mathrm{CH}_{n-1}(\mathbb{A}^n/G \setminus \pi(Z)) = \mathrm{CH}_{n-1}((\mathbb{A}^n \setminus Z)/G) \simeq \mathrm{CH}^G_{n-1}(\mathbb{A}^n).$$

Since $\mathbb{A}^n$ is smooth, we have $\mathrm{CH}^G_{n-1}(\mathbb{A}^n) = \mathrm{Pic}^G(\mathbb{A}^n)$, and the latter group of G-equivariant line bundles on $\mathbb{A}^n$ is isomorphic to the group $\widehat{G}$ of characters of G.

In the general case the fixed locus of the action of G/N on $\mathbb{A}^n/N \simeq \mathbb{A}^n$ does not contain divisors and the same argument applies to show that $\mathrm{Cl}(\mathbb{A}^n/G) \simeq \widehat{G/N}$. □

Lemma 1.2.7. *For every $0 \leq i \leq n-1$, $\mathrm{CH}_i(\mathbb{A}^n/G)$ is annihilated by $|G|$.*

Proof. Let $V \subset \mathbb{A}^n/G$ be a subvariety, let $\pi^{-1}(V)$ be the scheme theoretic preimage of V under the quotient morphism $\pi : \mathbb{A}^n \to \mathbb{A}^n/G$, and let W be a reduced irreducible component of $\pi^{-1}(V)$. Let us show that the degree of the field extension $[k(W) : k(V)]$ divides $|G|$.

Since we assume k to be of characteristic zero, G is a linearly reductive k-group scheme. Thus according to [48, Proof of Theorem 1.1, p. 28], the G-invariant ring of $A := k[x_1, \ldots, x_n] \otimes_{k[\mathbb{A}^n/G]} k(V)$ is just $k(V)$. If A_{red} is the quotient of A by the nilradical, then

$$A_{red} = \prod_{i=1}^{r} k(W_i)$$

where $W_1, \ldots, W_r$ are all components of $\pi^{-1}(V)$. The action of G on A induces an action on A_{red}, and $A^G_{red} = A^G = k(V)$. Since G acts on the components $W_1, \ldots, W_r$ transitively, the degrees $[k(W_i) : k(V)]$ are equal to each other, and it is easy to see that they divide $|G|$.

Since $\mathrm{CH}_i(\mathbb{A}^n) = 0$ in the considered range, by definition of pushforward on Chow groups we get

$$0 = \pi_*([W]) = [k(W) : k(V)] \cdot [V]$$

so that $[V] \in \mathrm{CH}_i(\mathbb{A}^n/G)$ is $|G|$-torsion. $\square$

Proposition 1.2.8. *There is a well-defined surjective first Chern class homomorphism*

$$c_1 : \mathrm{K}_0^{sg}(\mathbb{A}^n/G) \to \widehat{G/N}.$$

If $n = 2$ and k is algebraically closed, then c_1 is an isomorphism.

Proof. We claim that there is a Grothendieck-Riemann-Roch without denominators style surjection

$$(\mathrm{rk}, c_1) : \mathrm{G}_0(\mathbb{A}^n/G) \twoheadrightarrow \mathbb{Z} \oplus \mathrm{CH}_{n-1}(\mathbb{A}^n/G).$$

Indeed, since $\mathbb{A}^n/G$ is normal, to construct c_1 one may simply remove the singular locus of $\mathbb{A}^n/G$ and thus reduce to the smooth case, and the surjectivity follows easily.

Splitting off the direct summand $\mathbb{Z}$ corresponding to the trivial bundles and using Proposition 1.2.5 and Lemma 1.2.6 we get the desired surjection.

By construction of the topological filtration on $\mathrm{K}_0^{sg}(X)$ we have $\mathrm{Ker}(c_1) = F^2\mathrm{K}_0^{sg}(X)$ (cf. proof of Proposition 1.1.25 (2)), in particular if $n = 2$, then $\mathrm{Ker}(c_1) = F^2\mathrm{K}_0^{sg}(X) = 0$ by Proposition 1.1.25 (4). $\square$

Proposition 1.2.9. *Let k be an algebraically closed field of characteristic zero. Then every element of $\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ is annihilated by $|G|^{n-1}$.*

Proof. We consider the topological filtration on $\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ and its associated graded pieces $\mathrm{gr}^i\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$. By Proposition 1.1.25 we have $\mathrm{gr}^0\mathrm{K}_0^{sg}(\mathbb{A}^n/G) = \mathrm{gr}^n\mathrm{K}_0^{sg}(\mathbb{A}^n/G) = 0$ so that the filtration has the form

$$0 = F^n\mathrm{K}_0^{sg}(\mathbb{A}^n/G) \subset F^{n-1}\mathrm{K}_0^{sg}(\mathbb{A}^n/G) \subset \ldots \subset F^1\mathrm{K}_0^{sg}(\mathbb{A}^n/G) = \mathrm{K}_0^{sg}(\mathbb{A}^n/G).$$

By Proposition 1.1.25, each subquotient $\mathrm{gr}^i\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$, $1 \leq i \leq n-1$, admits a surjection $\mathrm{CH}_{n-i}(\mathbb{A}^n/G) \to \mathrm{gr}^i\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ and by Lemma 1.2.7, each of these groups is annihilated by $|G|$. This means that multiplication by $|G|$ shifts the filtration: $|G| \cdot F^i\mathrm{G}_0(\mathbb{A}^n/G) \subset F^{i+1}\mathrm{G}_0(\mathbb{A}^n/G)$, in particular multiplication by $|G|^{n-1}$ acts as the zero map. $\square$

The next proposition gives the formula for $\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ in the isolated singularity case. For other approaches to how to compute this group see [7], [85], [63].

Proposition 1.2.10 ([53])**.** *Let G be a finite group acting linearly on $\mathbb{A}^n$ such that the G-action on $\mathbb{A}^n \setminus \{0\}$ is free. Let ρ be the corresponding representation of G. Then we have*

$$\mathrm{G}_0(\mathbb{A}^n/G) \simeq R(G)/rR(G),$$

where $R(G)$ is the representation ring of G and $r \in R(G)$ is the Koszul class

$$r = \sum_{i=0}^{n} (-1)^i [\Lambda^i(\rho^\vee)].$$

Proof. The proof uses equivariant algebraic K-theory [116]. Since G acts freely away from 0, there is an isomorphism

$$\mathrm{K}_0^G(\mathbb{A}^n \setminus \{0\}) \simeq \mathrm{G}_0\left(\frac{\mathbb{A}^n \setminus \{0\}}{G}\right).$$

Let $i : \{0\} \to \mathbb{A}^n$, $\bar{i} : \{0\} \to \mathbb{A}^n/G$ be the closed embeddings. Consider the localization exact sequences of G_0 and K_0^G:

$$\begin{array}{ccccccc} \mathrm{K}_0^G(0) & \xrightarrow{i_*^G} & \mathrm{K}_0^G(\mathbb{A}^n) & \longrightarrow & \mathrm{K}_0^G(\mathbb{A}^n \setminus \{0\}) & \longrightarrow & 0 \\ \downarrow & & \downarrow & & \downarrow \simeq & & \\ \mathrm{G}_0(0) & \xrightarrow{\bar{i}_*} & \mathrm{G}_0(\mathbb{A}^n/G) & \longrightarrow & \mathrm{G}_0(\mathbb{A}^n/G - 0) & \longrightarrow & 0 \end{array} \tag{1.2.1}$$

related by pushforward maps followed by taking G-invariants. The pushforward $\bar{i}_*$ is a zero map since it factor through $i_* = 0$. By equivariant homotopy invariance [116, 4.1] we have an isomorphism $\mathrm{K}_0^G(\mathbb{A}^n/G) \simeq \mathrm{K}_0^G(0) \simeq R(G)$ and under these identifications the pushforward $i_*^G : \mathrm{K}_0^G(0) \to \mathrm{K}_0^G(\mathbb{A}^n)$ corresponds to the multiplication by the class $[\mathcal{O}_0] = r$.

Putting everything together we obtain

$$\mathrm{G}_0(\mathbb{A}^n/G) \simeq \mathrm{G}_0((\mathbb{A}^n \setminus \{0\})/G) \simeq R(G)/rR(G).$$

□

Remark 1.2.11. *If G has no reflections, then the condition that G acts freely on $\mathbb{A}^n \setminus \{0\}$ is equivalent to the quotient $\mathbb{A}^n/G$ to have an isolated singularity at the origin.*

Remark 1.2.12. *Since $\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ is a finite group by Proposition 1.2.5 we see that under the assumptions of Proposition 1.2.10 the linear map $r : R(G) \to R(G)$ has cokernel of rank one, and so it has a one-dimensional kernel.*

Example 1.2.13 ([53]). *Computing $R(G)/rR(G)$ for a two-dimensional ADE singularity $\mathbb{A}^2/G$, one can compute $\mathrm{K}_0^{sg}(\mathbb{A}^2/G)$ using Proposition 1.2.10 as follows:*

Type	$\mathrm{K}_0^{sg}(\mathbb{A}^2/G)$
A_n	$\mathbb{Z}/(n+1)\mathbb{Z}$
D_n, *n even*	$\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$
D_n, *n odd*	$\mathbb{Z}/4\mathbb{Z}$
E_6	$\mathbb{Z}/3\mathbb{Z}$
E_7	$\mathbb{Z}/2\mathbb{Z}$
E_8	0

(note a typo in [53] in the E_7 case on page 415). The same result can be obtained using Proposition 1.2.8, and another way is given by Yoshino using Auslander-Reiten sequences [127, Chapter 13].

Corollary 1.2.14. *Let k be an algebraically closed field of characteristic zero and let X be the local $\frac{1}{m}(\overbrace{1,\dots,1}^{n})$ singularity, that is $X = \mathbb{A}^n/\mathbb{Z}_m$ with the diagonal action by the primitive root of unity. Then $\mathrm{K}_0^{sg}(X)$ is a finite abelian group of order m^{n-1}.*

Proof. Let us fix a primitive character ρ of $\mathbb{Z}_m$. Then the representation ring is $R(\mathbb{Z}_m) = \mathbb{Z}[x]/(x^m - 1)$ where we choose x to be the class $[\rho^\vee]$. Then

$$r = \sum_{i=0}^{n} (-1)^i [\Lambda^i(\rho^\vee)] = \sum_{i=0}^{n} \binom{n}{i} (-1)^i x^i = (1-x)^n \in R(G)$$

and after making a substitution $y = 1 - x$, we obtain

$$\mathrm{G}_0(X) = \mathbb{Z}[y]/(y^n, my - \binom{m}{2} y^2 + \dots) = \mathbb{Z}\cdot 1 \oplus \mathrm{K}_0^{sg}(X)$$

so that $\mathrm{K}_0^{sg}(X)$ is a quotient of a free $\mathbb{Z}$-module with the basis $y, y^2, \dots, y^{n-1}$ by the upper-triangular relations $my^i - \binom{m}{2} y^{i+1} + \dots$ for $i \geq 1$. This means that $\mathrm{K}_0^{sg}(X)$ is a finite abelian group of order m^{n-1}. ☐

We abuse the notation slightly by writing $X = \frac{1}{m}(1,\dots,1)$ for the corresponding local singularity. The precise structure of $\mathrm{K}_0^{sg}(\frac{1}{m}(\overbrace{1,\dots,1}^{n}))$ will vary depending on m and n.

Example 1.2.15. *For $n = 2$ we have $\mathrm{K}_0^{sg}(\frac{1}{m}(1,1)) \simeq \mathbb{Z}_m$, in accordance with Proposition 1.2.9.*

Example 1.2.16. *For $n = 3$ one can see that*

$$\mathrm{K}_0^{sg}(\tfrac{1}{m}(1,1,1)) = \begin{cases} (\mathbb{Z}_m)^2, & m \text{ odd} \\ \mathbb{Z}_{m/2} \times \mathbb{Z}_{2m}, & m \text{ even} \end{cases}$$

Example 1.2.17. *If $m = 2$ and n is arbitrary, one can see that $\mathrm{K}_0^{sg}(\mathbb{A}^n/\mathbb{Z}_2) \simeq \mathbb{Z}_{2^{n-1}}$ (here the action of $\mathbb{Z}_2$ on $\mathbb{A}^n$ is $v \mapsto -v$).*

1.2.2 The local case: positive K-groups

Proposition 1.2.18. *Let k be an algebraically closed field of characteristic zero, and let $G \subset \mathrm{GL}_n(k)$ be a finite group such that the G-action on $\mathbb{A}^n \setminus \{0\}$ is free. For every $j \geq 0$ consider the group $T_j = \mathrm{Tor}(\mathrm{K}_0^{sg}(\mathbb{A}^n/G), \mathrm{K}_j(k))$.*

(1) T_j is a finite torsion group annihilated by $|G|^{n-1}$, and $T_j = 0$ for all even j.

(2) For every $j \geq 1$ there is a short exact sequence

$$0 \to \mathrm{K}_j(k) \to \mathrm{G}_j(\mathbb{A}^n/G) \to T_{j-1} \to 0, \tag{1.2.2}$$

where the first map is the pullback from $\mathrm{Spec}(k)$*. In particular for all* $j \geq 1$ *we have* $\mathrm{G}_j(\mathbb{A}^n/G) \otimes \mathbb{Z}[1/|G|] \simeq \mathrm{K}_j(k) \otimes \mathbb{Z}[1/|G|]$ *and for all odd* $j \geq 1$ *we have* $\mathrm{G}_j(\mathbb{A}^n/G) \simeq \mathrm{K}_j(k)$*.*

(3) For every $j \geq 1$*, there is an exact sequence*

$$0 \to T_{j-1} \to \mathrm{K}_j^{sg}(\mathbb{A}^n/G) \to \mathrm{K}_{j-1}(\mathbb{A}^n/G) \to \mathrm{K}_{j-1}(k) \to 0, \tag{1.2.3}$$

where the last morphism in the sequence is induced by restriction to the rational point $0 \in \mathbb{A}^n/G$*.*

We prove the Proposition at the end of this subsection.

Corollary 1.2.19. *If* $\mathbb{A}^n/G$ *is an isolated singularity over an algebraically closed field of characteristic zero then* $\mathrm{K}_1^{sg}(\mathbb{A}^n/G) = 0$*.*

Proof. We may assume that G acts freely on $\mathbb{A}^n \setminus \{0\}$ (see Remark 1.2.11 and proof of Lemma 1.2.6). The result follows from (1.2.3) using the fact that $T_0 = 0$ and Proposition 1.2.1 which says that $\mathrm{K}_0(\mathbb{A}^n/G) = \mathrm{K}_0(k) = \mathbb{Z}$. □

Remark 1.2.20. *For non-algebraically closed field, see Example 1.3.9. We do not know if* $\mathrm{K}_1^{sg}(\mathbb{A}^n/G) = 0$ *in the non-isolated singularity case.*

Remark 1.2.21. *The structure of the groups* $\mathrm{K}_j(\mathbb{A}^n/G)$ *for* $j \geq 1$ *is in general not known. Since the work of Srinivas it is known that* $\mathrm{Ker}(\mathrm{K}_1(\mathbb{A}^n/G) \to \mathrm{K}_1(k))$ *is "huge", that is as large as the base field k, even in the simplest* $\frac{1}{2}(1,1)$ *case [110], and the same follows for* $\mathrm{K}_2^{sg}(\mathbb{A}^n/G)$ *from Proposition 1.2.18.*

In order to prove Proposition 1.2.18, we use the language of equivariant K-theory [116] which for finite groups can also be interpreted as K-theory of Deligne-Mumford stacks [70].

Lemma 1.2.22. *In the assumptions of Proposition 1.2.18, let* $i : \mathrm{Spec}(k) \to \mathbb{A}^n$ *be the closed embedding of the origin* 0*. Then the following is true.*

(1) We have natural $R(G)$*-module isomorphisms* $K_j^G(\mathbb{A}^n) \overset{i_G^*}{\simeq} K_j^G(k) \simeq R(G) \otimes K_j(k)$ *and a commutative diagram*

$$\begin{array}{ccc} \mathrm{K}_j^G(k) & \xrightarrow{\ i_*^G\ } & \mathrm{K}_j^G(\mathbb{A}^n) \\ \simeq\uparrow & & \simeq\uparrow \\ R(G) \otimes \mathrm{K}_j(k) & \xrightarrow{\ r_j\ } & R(G) \otimes \mathrm{K}_j(k) \end{array}$$

where r_j *is multiplication by the Koszul class* $r \in R(G)$ *defined in Proposition 1.2.10.*

(2) Let π_0 *be the projection from the Deligne-Mumford stack* $[\mathrm{Spec}(k)/G]$ *to its coarse moduli space* $\mathrm{Spec}(k)$ *and let* α_j *be the restriction of* $\pi_{0,*} : \mathrm{K}_j^G(k) \to \mathrm{K}_j(k)$ *to* $\mathrm{Ker}(i_*^G) = \mathrm{Ker}(r_j)$*. Then for every* $j \geq 0$ *there is a commutative diagram*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{K}_j(k) & \longrightarrow & \mathrm{Ker}(r_j) & \longrightarrow & T_j & \longrightarrow & 0 \\ & & & \overset{\simeq}{\searrow} & \downarrow{\scriptstyle \alpha_j} & & & & \\ & & & & \mathrm{K}_j(k) & & & & \end{array} \tag{1.2.4}$$

where T_j is defined as in Proposition 1.2.18 and the top row is exact.

(3) Let π be the projection from the Deligne-Mumford stack $[\mathbb{A}^n/G]$ to its coarse moduli space $\mathbb{A}^n/G$. For every $j \geq 0$ consider the subgroup $\mathrm{K}_j(k) \simeq 1 \otimes \mathrm{K}_j(k) \subset R(G) \otimes \mathrm{K}_j(k) = \mathrm{K}_j^G(\mathbb{A}^n)$, and let β_j be the restriction of $\pi_* : \mathrm{K}_j^G(\mathbb{A}^n) \to \mathrm{G}_j(\mathbb{A}^n/G)$ to this subgroup. Then β_j is isomorphic to pullback morphism $p^* : \mathrm{K}_j(k) \to \mathrm{G}_j(\mathbb{A}^n/G)$ induced by the structure morphism $p : \mathbb{A}^n/G \to$ $\mathrm{Spec}(k)$.

Furthermore, for $j \geq 1$ the embedding $1 \otimes \mathrm{K}_j(k) \subset R(G) \otimes K_j(k)$ induces an isomorphism $\mathrm{K}_j(k) \simeq \mathrm{Coker}(r_j)$.

Proof. (1) i_G^* being an isomorphism is the standard homotopy invariance of K-theory in the regular case [116, 4.1], $\mathrm{K}_j^G(k) \simeq R(G) \otimes \mathrm{K}_j(k)$ holds e.g. by [121, Proposition 1.6]. The commutative diagram follows from [121, Lemma 1.7].

(2) Since the map r_j is isomorphic to $r \otimes \mathrm{id}$, $\mathrm{Ker}(r_j)$ can be computed via the Universal Coefficient Theorem applied to the complex $[r : R(G) \to R(G)]$ as follows. We have

$$0 \to \mathrm{Ker}(r) \otimes K_j(k) \to \mathrm{Ker}(r_j) \to \mathrm{Tor}(\mathrm{Coker}(r), K_j(k)) \to 0.$$

By Remark 1.2.12, $\mathrm{Ker}(r) = \mathbb{Z} \cdot t$, for some element $t \in R(G)$, and by Proposition 1.2.10, $\mathrm{Coker}(r) \simeq \mathbb{Z} \oplus \mathrm{K}_0^{sg}(\mathbb{A}^n/G)$. We see that $\mathrm{Ker}(r) \otimes \mathrm{K}_j(k) = \mathrm{K}_j(k)$ and

$$\mathrm{Tor}(\mathrm{Coker}(r), K_j(k)) = \mathrm{Tor}(\mathrm{K}_0^{sg}(\mathbb{A}^n/G)), \mathrm{K}_j(k)) = T_j$$

so that the top row of (1.2.4) is exact. We compute α_j as follows

$$\alpha_j|_{t \otimes \mathrm{K}_j(k)} = \pi_{0,*}|_{t \otimes \mathrm{K}_j(k)} = \pi_{0,*}(t) \cdot \mathrm{id}_{\mathrm{K}_j(k)},$$

and for commutativity of (1.2.4) it remains to show that $\pi_{0,*}(t) = \pm 1$. This follows easily by extending the commutative diagram (1.2.1) on term to the left [116, Theorem 2.7] (cf $j = 1$ case in (1.2.5) in the Proof of Proposition 1.2.18).

(3) The fact that β_j is equal to p^* follows from the projection formula. Since tensor product is right exact we have

$$\mathrm{Coker}(r_j) = \mathrm{Coker}(r) \otimes \mathrm{K}_j(k) \simeq (\mathbb{Z} \oplus \mathrm{K}_0^{sg}(\mathbb{A}^n/G)) \otimes \mathrm{K}_j(k).$$

Since k is algebraically closed, by a result of Suslin [112], for every $j \geq 1$, $\mathrm{K}_j(k)$ is a divisible group, so that since $\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ is torsion, $\mathrm{K}_0^{sg}(\mathbb{A}^n/G)) \otimes \mathrm{K}_j(k) = 0$, and we have

$$\mathrm{Coker}(r_j) \simeq \mathrm{K}_j(k),$$

induced by tensoring $\mathrm{rk} : R(G) \to \mathbb{Z}$ by $\mathrm{K}_j(k)$. Since $\mathrm{rk}(1) = 1$, the composition

$$1 \otimes \mathrm{K}_j(k) \subset R(G) \otimes \mathrm{K}_j(k) \to \mathrm{Coker}(r_j)$$

provides a splitting, and hence the inverse to this morphism. □

Proof of Proposition 1.2.18. (1) By Proposition 1.2.5, $\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ is a torsion group annihilated by $|G|^{n-1}$, hence the same is true for T_j.

For even j, $\mathrm{K}_j(k)$ of an algebraically closed field is torsion-free by a result of Suslin [112], hence $T_j = 0$ for even j.

For odd j, and every $n \geq 1$, the n-torsion subgroup $\mathrm{K}_j(k)$ is finite [112], and since $\mathrm{K}_0^{sg}(\mathbb{A}^n/G)$ is a finite abelian group, T_j is finite as well.

(2) The key in proving (1.2.2) is to compare the localization sequence for G-theory of $\mathbb{A}^n/G$ to the G-equivariant localization sequence for K-theory of $\mathbb{A}^n$. The two sequences are related by pushforward morphisms:

$$\begin{array}{ccccccccc} \mathrm{K}_j^G(k) & \xrightarrow{i_*^G} & \mathrm{K}_j^G(\mathbb{A}^n) & \longrightarrow & \mathrm{K}_j^G(\mathbb{A}^n \setminus \{0\}) & \longrightarrow & \mathrm{K}_{j-1}^G(k) & \xrightarrow{i_*^G} & \mathrm{K}_{j-1}^G(\mathbb{A}^n) \\ {\scriptstyle \pi_{0,*}}\downarrow & & {\scriptstyle \pi_*}\downarrow & & \downarrow & & {\scriptstyle \pi_{0,*}}\downarrow & & {\scriptstyle \pi_*}\downarrow \\ \mathrm{K}_j(k) & \xrightarrow{i_*} & \mathrm{G}_j(\mathbb{A}^n/G) & \longrightarrow & \mathrm{G}_j((\mathbb{A}^n \setminus \{0\})/G) & \longrightarrow & \mathrm{K}_{j-1}(k) & \xrightarrow{i_*} & \mathrm{G}_{j-1}(\mathbb{A}^n/G) \end{array} \tag{1.2.5}$$

Here $i : \mathrm{Spec}(k) \to \mathbb{A}^n/G$ is the origin (unique fixed point of the action), and π (resp. π_0) is the canonical morphism from the quotient Deligne-Mumford stack $[\mathbb{A}^n/G]$ (resp. $[\mathrm{Spec}(k)/G]$) to its coarse moduli space, as in Lemma 1.2.22.

The morphisms $i_* : \mathrm{G}_j(k) \to \mathrm{G}_j(\mathbb{A}^n/G)$ are zero as they factor through the pushforward $\mathrm{G}_j(k) \to \mathrm{G}_j(\mathbb{A}^n)$ which are zero maps by the Bass formula in G-theory [100, chapter 6 Theorem 8. ii]. Thus the localization sequence for G-theory of $\mathbb{A}^n/G$ splits into short exact sequences.

Using isomorphisms given by Lemma 1.2.22, from the commutative ladder (1.2.5) for every $j \geq 1$ we obtain the diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \mathrm{Coker}(r_j) & \longrightarrow & \mathrm{K}_j^G(\mathbb{A}^n \setminus \{0\}) & \longrightarrow & \mathrm{Ker}(r_{j-1}) & \longrightarrow & 0 \\ & & {\scriptstyle \beta_j}\downarrow & & \downarrow & & {\scriptstyle \alpha_{j-1}}\downarrow & & \\ 0 & \longrightarrow & \mathrm{G}_j(\mathbb{A}^n/G) & \longrightarrow & \mathrm{G}_j((\mathbb{A}^n \setminus \{0\})/G) & \longrightarrow & \mathrm{K}_{j-1}(k) & \longrightarrow & 0 \end{array}$$

Since 0 is the only fixed point of the action, the action of G on $\mathbb{A}^n \setminus \{0\}$ is free, so that the middle vertical map is an isomorphism and using the Snake Lemma we deduce an isomorphism

$$\mathrm{Ker}(\alpha_{j-1}) \simeq \mathrm{Coker}(\beta_j).$$

From Lemma 1.2.22 (2) we get $\mathrm{Ker}(\alpha_j) \simeq T_j$ and from Lemma 1.2.22 (3) we get $\mathrm{Coker}(\beta_j) \simeq \mathrm{Coker}(p^* : \mathrm{K}_j(k) \to \mathrm{G}_j(\mathbb{A}^n/G))$. Since p^* is injective (it is split by any smooth point $x_1 \in \mathbb{A}^n/G$), we obtain the exact sequence (1.2.2).

(3) Since $\mathbb{A}^n/G$ is contractible by Lemma 1.1.20, one gets (1.2.3) by plugging in (1.2.2) into the exact sequence of Proposition 1.1.21 (3). □

1.2.3 The global case

Theorem 1.2.23. *Let k be an algebraically closed field of characteristic zero and let X be an n-dimensional quasi-projective variety. Assume that X has only isolated quotient singularities $x_1, \ldots, x_m$ with isotropy groups $G_1, \ldots, G_m$, i.e. the completions $\widehat{\mathcal{O}}_{X,x_i}$ are isomorphic to $\widehat{\mathcal{O}}_{\mathbb{A}^n/G_i,0}$ where each $G_i \subset \mathrm{GL}_n(k)$ is a finite group acting freely away from the origin. Then*

(1) $\mathrm{K}_0^{sg}(X) \subset \mathbb{K}_0^{sg}(X)$ *are finite abelian groups, annihilated by* $\mathrm{lcm}(|G_1|, \ldots, |G_m|)^{n-1}$.

(2) $\mathrm{K}_1^{sg}(X) = 0$.

(3) For all $j \geq 1$, we have $\mathrm{K}_{-j}^{sg}(X) = 0$.

In addition, if $\dim(X) = 2$, *then* $\mathbb{K}_0^{sg}(X) \simeq \widehat{G_1} \times \ldots \times \widehat{G_m}$.

Proof. By Orlov's Completion Theorem 1.1.6 and Corollary 1.2.3 we obtain equivalences

$$\overline{\mathcal{D}^{\mathrm{sg}}(X)} \simeq \bigoplus_{i=1}^{m} \overline{\mathcal{D}^{\mathrm{sg}}(\mathbb{A}^n/G_i)} \simeq \bigoplus_{i=1}^{m} \mathcal{D}^{\mathrm{sg}}(\mathbb{A}^n/G_i),$$

induced by functors between dg-enhancements.

Thus by definition of the singularity K-theory groups and Remark 1.1.10 we have

$$\mathrm{K}_0^{sg}(X) \subset \mathbb{K}_0^{sg}(X) \simeq \bigoplus_{i=1}^{m} \mathrm{K}_0^{sg}(\mathbb{A}^n/G_i)$$

and for $j \neq 0$

$$\mathrm{K}_j^{sg}(X) \simeq \bigoplus_{i=1}^{m} \mathrm{K}_j^{sg}(\mathbb{A}^n/G_i).$$

Now (1) follows from Propositions 1.2.5, 1.2.9, (2) follows from Proposition 1.2.19 and (3) follows from Proposition 1.2.1.

Finally if $\dim(X) = 2$, we have isomorphisms $\mathrm{K}_0^{sg}(\mathbb{A}^n/G_i) = \widehat{G_i}$ by Proposition 1.2.8 (G_i acts freely on $\mathbb{A}^n \backslash \{0\}$ and in particular has no reflections) so that in this case $\mathbb{K}_0^{sg}(X) \simeq \widehat{G_1} \times \cdots \times \widehat{G_m}$. □

Corollary 1.2.24. *Under the assumptions of Theorem 1.2.23 the following is true.*

(1) We have an exact sequence $0 \to \mathrm{K}_0(X) \to \mathrm{G}_0(X) \to \mathrm{K}_0^{sg}(X) \to 0$.

(2) $\mathrm{K}_{-1}(X)$ *is a finite torsion abelian group satisfying the same condition on orders as* $\mathrm{K}_0^{sg}(X)$ *(see Theorem 1.2.23 (1)).*

(3) For all $j \geq 2$, we have $\mathrm{K}_{-j}(X) = 0$.

Proof. This follows from Theorem 1.2.23 and Lemma 1.1.11. □

Remark 1.2.25. *The injectiviy of the canonical map* $\mathrm{K}_0(X) \to \mathrm{G}_0(X)$ *will generally fail if either:*

(a) X has quotient singularities which are not isolated, see Example 1.3.7

(b) X has an isolated rational singularity which is not a quotient singularity, see Example 1.3.8

Remark 1.2.26. *We do not know if $\mathrm{K}_0^{sg}(X) = \mathrm{Coker}(\mathrm{K}_0(X) \to \mathrm{G}_0(X))$ is torsion for any variety X with quotient singularities, not necessarily isolated ones. The result is known to be true for simplicial toric varieties [23].*

Example 1.2.27. *One of the simplest examples of a projective surface X with quotient singularities and non-vanishing $\mathrm{K}_{-1}(X)$ is the following one. Consider $G = \mathbb{Z}_2$ acting on $\mathbb{P}^1$ via $[x : y] \mapsto [x : -y]$ and let $X = (\mathbb{P}^1 \times \mathbb{P}^1)/\mathbb{Z}_2$ where the action is diagonal. Thus X has four ordinary double points as singularities. Using [124] one can compute that $\mathrm{K}_{-1}(X) = \mathbb{Z}_2$.*

1.2.4 Relation to the resolution of singularities

Let $\pi : Y \to X$ be a resolution of singularities. Here X is a variety and Y is a variety or more generally a Deligne-Mumford stack. If we assume that singularities of X are rational which by definition means that $\pi_*\mathcal{O}_Y = \mathcal{O}_X$, using the projection formula we get a commutative diagram

$$\begin{array}{ccccc} & & \mathcal{D}^b(Y) & & \\ & \overset{\pi^*}{\nearrow} & & \overset{\pi_*}{\searrow} & \\ \mathcal{D}^{\mathrm{perf}}(X) & & \longrightarrow & & \mathcal{D}^b(X) \end{array}$$

where the functors $\mathcal{D}^{\mathrm{perf}}(X) \to \mathcal{D}^b(X)$ and π^* are both fully faithful.

We also get an induced diagram on the Grothendieck groups

$$\begin{array}{ccccc} & & \mathrm{K}_0(Y) & & \\ & \overset{\pi^*}{\nearrow} & & \overset{\pi_*}{\searrow} & \\ \mathrm{K}_0(X) & & \xrightarrow{\mathrm{PD}} & & \mathrm{G}_0(X) \end{array} \qquad (1.2.6)$$

Theorem 1.2.28. *If X is a quasi-projective variety over an algebraically closed field of characteristic zero and with only isolated quotient singularities, then $\pi^* : \mathrm{K}_0(X) \to \mathrm{K}_0(Y)$ is injective.*

Proof. By Corollary 1.2.24, $\mathrm{K}_0(X) \overset{\mathrm{PD}}{\to} \mathrm{G}_0(X)$ is injective. Injectivity of π^* follows from the diagram (1.2.6). □

Remark 1.2.29. *The injectivity of $\pi^* : \mathrm{K}_0(X) \to \mathrm{K}_0(Y)$ does not follow from the fact that $\pi^* : \mathcal{D}^{\mathrm{perf}}(X) \to \mathcal{D}^b(Y)$ is fully faithful and will generally fail for rational singularities. Indeed in Examples 1.3.7, 1.3.8 varieties with rational singularities have huge $\mathrm{K}_0(X)$, but admit resolutions with finitely generated $\mathrm{K}_0(Y)$.*

In dimension up to three, Theorem 1.2.28 has been known since the work of Levine [82, Corollary 3.4] and for normal surfaces with rational singularities an analogous result follows from the work of Krishna and Srinivas [80, Corollary 1.5].

There is an apparent duality between π_* and π^* in the diagram (1.2.6). Instead of injectivity of π^* we can ask about surjectivity of π_*, which indeed sometimes holds.

Theorem 1.2.30. *Let X be a variety over a field k characteristic zero with quotient singularities (not necessarily isolated) and let $\pi : Y \to X$ be a resolution of singularities, where Y is a variety or more generally a Deligne-Mumford stack. Then:*

(1) The pushforward $\pi_* : \mathcal{D}^b(Y) \to \mathcal{D}^b(X)$ is essentially surjective.

(2) The pushforward induces an exact equivalence

$$\mathcal{D}^b(Y)/\ker(\pi_*) \xrightarrow{\simeq} \mathcal{D}^b(X).$$

In particular $\pi_* : \mathrm{K}_0(Y) \to \mathrm{G}_0(X)$ is surjective.

Lemma 1.2.31. *If k is a field of characteristic zero and the statement (1) (resp. (2)) of Theorem 1.2.30 holds for a single resolution $\pi : Y \to X$, then (1) (resp. (2)) holds for all resolutions of X.*

Proof. The proof is a standard application of the Weak Factorization Theorem [126, 1], extended to Deligne-Mumford stacks in [12]. Given a birational isomorphism between Deligne-Mumford orbifolds, that is Deligne-Mumford stacks with trivial generic stabilizers, it can be decomposed into a sequence of stacky blows ups and blow downs with smooth centers. This means that given two resolutions $\pi : Y \to X$, $\pi' : Y' \to X$ we may assume that Y' is obtained from Y by a single smooth stacky blow up $\gamma : Y' \to Y$. Recall that by definition a stacky blow up is either a blow up of a substack, or a root stack along a smooth divisor, and in each case we have $\gamma_* \mathcal{O}_{Y'} \simeq \mathcal{O}_Y$ (see e.g. [13, Example 4.6]).

We get a commutative diagram

$$\begin{array}{ccc} \mathcal{D}^b(Y') & \xrightarrow{\gamma_*} & \mathcal{D}^b(Y) \\ & {\scriptstyle \pi'_*}\searrow \quad \swarrow{\scriptstyle \pi_*} & \\ & \mathcal{D}^b(X) & \end{array}$$

Furthermore the adjoint pair γ^*, γ_* satisfies $\gamma_*\gamma^* = \mathrm{id}$ so that γ^* is fully-faithful and there is a semi-orthogonal decomposition

$$\mathcal{D}^b(Y') = \langle \mathrm{Ker}(\gamma_*), \gamma^* \mathcal{D}^b(Y) \rangle.$$

In particular γ_* is essentially surjective and condition (1) for Y is equivalent to condition (1) for Y'. Furthermore we have a semi-orthogonal decomposition

$$\mathrm{Ker}(\pi'_*) = \langle \mathrm{Ker}(\gamma_*), \gamma^* \mathrm{Ker}(\pi_*) \rangle$$

which induces an equivalence of Verdier localizations

$$\begin{array}{ccc} \mathcal{D}^b(Y')/\mathrm{Ker}(\pi'_*) & \xrightarrow{\simeq} & \mathcal{D}^b(Y)/\mathrm{Ker}(\pi_*) \\ & {\scriptstyle \pi'_*}\searrow \quad \swarrow{\scriptstyle \pi_*} & \\ & \mathcal{D}^b(X) & \end{array}$$

so that conditions (2) for Y and Y' are equivalent as well. □

Lemma 1.2.32. *If $\pi : Y \to X$ is a resolution of rational singularities and $\mathcal{D}^-(Y)$ admits a t-structure which induces a bounded t-structure on $\mathcal{D}^b(Y)$ and for which $\pi_* : \mathcal{D}^-(Y) \to \mathcal{D}^-(X)$ is t-exact, then (1) and (2) of Theorem 1.2.30 are true.*

Proof. We temporarily use the notation $\mathrm{Ker}^b(\pi_*) := \mathrm{Ker}(\pi_*) \cap \mathcal{D}^b(Y)$. We will show that the functor $\overline{\pi_*} : \mathcal{D}^b(Y)/\mathrm{Ker}^b(\pi_*) \to \mathcal{D}^b(X)$ is essentially surjective and fully faithful.

Essential surjectivity is proved in the same way as in [79, Corollary 2.5]. For every $\mathcal{E} \in \mathcal{D}^b(X)$ and $N \geq 1$ we consider the distinguished triangle

$$\pi^*\mathcal{E} \to \tau_{\mathcal{A}}^{\geq -N}\pi^*\mathcal{E} \to C,$$

where $\tau_{\mathcal{A}}^{\geq -N}$ denotes the truncation with respect to the corresponding t-structure $\mathcal{A}$ on $\mathcal{D}^-(Y)$. We apply π_* to this triangle. Since $\pi_*\pi^* = \mathrm{id}$ and π_* is t-exact, in particular π_* commutes with truncation functors, the pushforward of the triangle above has the form:

$$\mathcal{E} \to \tau^{\geq -N}\mathcal{E} \to \pi_*C,$$

where $\tau^{\geq -N}$ is the truncation with respect to the standard t-structure on $\mathcal{D}^-(X)$. Since $\mathcal{E}$ is bounded, for sufficiently large N we have $\pi_*C = 0$ so that $C \in \mathrm{Ker}(\pi_*)$. In particular, $\overline{\pi_*} : \mathcal{D}^b(Y)/\mathrm{Ker}^b(\pi_*) \to \mathcal{D}^b(X)$ is essentially surjective.

On the other hand, one observes by the diagram

$$\begin{array}{ccc} \mathcal{D}^-(Y)/\mathrm{Ker}(\pi_*) & \xrightarrow[\simeq]{\overline{\pi_*}} & \mathcal{D}^-(X) \\ \uparrow & & \uparrow \\ \mathcal{D}^b(Y)/\mathrm{Ker}^b(\pi_*) & \xrightarrow{\overline{\pi_*}} & \mathcal{D}^b(X) \end{array}$$

that $\overline{\pi_*}$ is fully faithful if and only if the natural functor

$$\mathcal{D}^b(Y)/\mathrm{Ker}^b(\pi_*) \to \mathcal{D}^-(Y)/\mathrm{Ker}(\pi_*)$$

is fully faithful. To show this, we use Verdier's criterion [120, Theorem 2.4.2]. We see that if $C \to B$ is a morphism in $\mathcal{D}^-(Y)$ with $C \in \mathrm{Ker}(\pi_*)$ and $B \in \mathcal{D}^b(Y)$, then for a big enough N (depending on B) this morphism factors through $\tau_{\mathcal{A}}^{\geq -N}C$. Since π_* commutes with $\tau_{\mathcal{A}}^{\geq -N}$, one easily sees that $\tau_{\mathcal{A}}^{\geq -N}C \in \mathrm{Ker}^b(\pi_*)$. □

Proof of Theorem 1.2.30. By Lemma 1.2.31 it suffices to check the statement for a single resolution. We consider the canonical stack $\pi : \mathcal{X}_{can} \to X$ over X [46, Remark 4.9]. Since we assume k has characteristic zero, the pushforward $\pi_* : \mathcal{X}_{can} \to X$ is exact. The proof is finished using Lemma 1.2.32. □

Remark 1.2.33. *Statements (1) and (2) of Theorem 1.2.30 for a resolution of arbitrary rational singularities $\pi : Y \to X$ is an old open question going back to Bondal and Orlov [19]. In addition to quotient singularities the answer is positive in the case of cones over smooth Fano varieties [45], and for rational singularities such that fibers of a resolution $Y \to X$ have dimension at most one [79] (in [79, Corollary 2.5] property (1) is proved, while property (2) follows from Lemma 1.2.32).*

1.3 Examples and Applications

In this section k is an algebraically closed field of characteristic zero.

1.3.1 Torsion-free $\mathrm{K}_0(X)$

Application 1.3.1 ([81, 82])**.** *Let X be a projective rational surface with isolated quotient singularities. Then $\mathrm{K}_0(X)$ is a free abelian group of the same rank as $\mathrm{G}_0(X)$.*

Indeed if $\pi : Y \to X$ is a resolution, then by Theorem 1.2.28 we have an injection $\pi^* : \mathrm{K}_0(X) \to \mathrm{K}_0(Y)$. Since Y is a smooth projective rational surface, $\mathrm{K}_0(Y)$ is a free abelian group of finite rank, and the same is true for $\mathrm{K}_0(X)$. Finally by Corollary 1.2.24, we have an inclusion $\mathrm{K}_0(X) \subset \mathrm{G}_0(X)$ and the ranks of the two groups are equal.

Note that $\mathrm{G}_0(X)$ will typically have non-zero torsion.

Application 1.3.2 (Weighted projective spaces with coprime weights)**.** *Let $X = \mathbb{P}(a_0, \ldots, a_n)$ be a weighted projective space. Let us assume that the weights $a_0, \ldots, a_n$ are pairwise coprime. In this case singularities of X are isolated, and using our results we show that $\mathrm{K}_0(X)$ is a free abelian group of rank $n + 1$.*

Indeed if we let $Y = [\mathbb{P}(a_0, \ldots, a_n)]$ to be the weighted projective stack, the natural morphism $\pi : Y \to X$ is a resolution of singularities, and by Theorem 1.2.28, π^* is injective. Since $\mathrm{K}_0(Y)$ is a free abelian of finite rank [70, Theorem 5.6], the same is true for $\mathrm{K}_0(X)$. To compute the rank of $\mathrm{K}_0(X)$ we can use the following argument: by Corollary 1.2.24 we have an isomorphism $\mathrm{K}_0(X) \otimes \mathbb{Q} \simeq \mathrm{G}_0(X) \otimes \mathbb{Q}$ and the latter space is $(n + 1)$-dimensional, which can be seen by comparing $\mathrm{G}_0(X)$ to Chow groups [2]. Thus we get $\mathrm{K}_0(X) \simeq \mathbb{Z}^{n+1}$.

1.3.2 ADE curves and threefolds

We consider one-dimensional ADE singularities over an algebraically closed field of characteristic zero. For each such curve C we compute $\mathrm{Pic}(C)$ as well as $\mathrm{K}_0^{sg}(C)$ and $\mathrm{K}_1^{sg}(C)$. If N is the number of irreducible components of the curve, then $\mathrm{K}_0^{sg}(C)$ is computed using Corollary 1.1.29. We record the results in the table:

C	Equation	N	$\mathrm{K}_0^{sg}(C)$	$\mathrm{Pic}(C)$	$\mathrm{K}_1^{sg}(C)$
$A_{2l},\ l \geq 1$	$y^2 + z^{2l+1}$	1	0	k^l	k^l
$A_{2l-1},\ l \geq 1$	$y^2 + z^{2l}$	2	$\mathbb{Z}$	0	$k^* \oplus \mathbb{Z}$
$D_{2l},\ l \geq 2$	$y^2 z + z^{2l-1}$	3	$\mathbb{Z}^2$	0	$(k^* \oplus \mathbb{Z})^2$
$D_{2l-1},\ l \geq 3$	$y^2 z + z^{2l-2}$	2	$\mathbb{Z}$	k^{l-2}	$[k^* \oplus \mathbb{Z}; k^{l-2}]$
E_6	$y^3 + z^4$	1	0	k^3	k^3
E_7	$y^3 + yz^3$	2	$\mathbb{Z}$	k	$[k^* \oplus \mathbb{Z}; k]$
E_8	$y^3 + z^5$	1	0	k^4	k^4

(1.3.1)

We use the notation $[A; B]$ to denote an abelian group which has a subgroup A with quotient B. The first singularity K-theory groups $\mathrm{K}_1^{sg}(C)$ are computed using the following Proposition.

Proposition 1.3.3. *For every ADE singularity we have a natural exact sequence*

$$0 \to (k^* \oplus \mathbb{Z})^{N-1} \to \mathrm{K}_1^{sg}(C) \to \mathrm{Pic}(C) \to 0.$$

Proof. By Proposition 1.1.21 (3) and using the fact that $\mathrm{K}_0(C) = \mathbb{Z} \oplus \mathrm{Pic}(C)$ (see [51, Remark 1 on page 126]) we get a short exact sequence

$$0 \to \mathrm{G}_1(C)/k^* \to \mathrm{K}_1^{sg}(C) \to \mathrm{Pic}(C) \to 0.$$

To finish the proof we show that $\mathrm{G}_1(C) = (k^*)^N \oplus \mathbb{Z}^{N-1}$, and the morphism $\mathrm{K}_1(k) \to \mathrm{G}_1(C)$ maps k^* into $(k^*)^N$ diagonally. This is done using the localization sequence for the closed embedding $i : \{0\} \to C$. Since for every $j \geq 0$, $i_* : \mathrm{G}_j(k) \to \mathrm{G}_j(C)$ factors through any component $\mathbb{A}^1$ of the normalization of C, it is a zero map, and we get a short exact sequence

$$0 \to \mathrm{G}_1(C) \to \mathrm{G}_1(\mathbb{A}^1 \setminus \{0\})^N \to \mathrm{G}_0(k) \to 0$$

which finishes the proof as $\mathrm{G}_1(\mathbb{A}^1 \setminus \{0\}) = k^* \oplus \mathbb{Z}$. □

Lemma 1.3.4. *If C is a curve with equation $x^a - y^b = 0$ where $\gcd(a, b) = 1$, then we have an isomorphism*

$$\mathrm{Pic}(C) \simeq k^{\frac{1}{2}(a-1)(b-1)}.$$

Proof. Consider $\pi : \mathbb{A}^1 \to C$ given by $\pi(t) = (t^b, t^a)$. Under the condition $\gcd(a, b) = 1$, π is surjective which implies irreducibility of C, and since π is finite of degree one, π is the normalization morphism. By [92, Corollary 3.3] we get an isomorphism of abelian groups

$$\mathrm{Pic}(C) \simeq k[t]/k[t^a, t^b].$$

Thus $\mathrm{Pic}(C)$ obtains a k-vector space structure with a k-basis corresponding of t^i, for every $i \geq 0$ which can not be represented as a non-negative integer combination of a and b.

By a classical theorem of Sylvester, the number of positive integers not representable by non-negative integer combinations of a and b is equal to $\frac{1}{2}(a-1)(b-1)$ (see [91] for a modern treatment) so that we have an isomorphism of abelian groups

$$k[t]/k[t^a, t^b] \simeq k^{\frac{1}{2}(a-1)(b-1)}.$$

□

Every ADE curve C is a union of components isomorphic to $\mathbb{A}^1$ and at most one component C_0 with equation $x^a - y^b = 0$. Trivializing line bundles on each affine line component, we deduce that $\mathrm{Pic}(C) = \mathrm{Pic}(C_0)$. Proposition 1.3.3 and Lemma 1.3.4 allow us to fill in the table (1.3.1).

We demonstrate what the singularity K-theory has to do with the question of computing class groups. The applications below can be obtained by other methods too, however, we demonstrate the approach which relies on Knörrer periodicity shifting the topological filtration (Proposition 1.1.31).

Application 1.3.5. *Let $C \subset \mathbb{A}^2$ with coordinates z, w be given by $g(z,w) = 0$ and let $X \subset \mathbb{A}^4$ with coordinates x, y, z, w be given by $xy + g(z,w) = 0$.*

Let us assume that C is reduced. Since we have $\mathrm{Sing}(X) = \{(0,0)\} \times \mathrm{Sing}(C)$, *the latter condition is equivalent to X having isolated singularities, and since X is a hypersurface, it is irreducible and normal.*

Let N be the number of irreducible components of C. By Proposition 1.1.31 and Proposition 1.1.25 we have an isomorphism

$$\mathrm{Cl}(X)/\mathrm{Pic}(X) = \mathrm{gr}^1 \mathrm{K}_0^{sg}(X) \simeq \mathrm{gr}^0 \mathrm{K}_0^{sg}(C) = \mathbb{Z}^{N-1},$$

in particular X is factorial if and only if C is irreducible.

Example 1.3.6. *We can compute the class group of the standard forms of three-dimensional ADE singularities. Since X is given by a weighted homogeneous equation,* $\mathrm{Pic}(X) = 0$ *[88, Lemma 5.1] we have* $\mathrm{Cl}(X) \simeq \mathrm{K}_0^{sg}(X) = \mathbb{Z}^{N-1}$. *We put the results in the table (cf. table 1.3.1):*

X	*Equation*	$\mathrm{Cl}(X)$
A_{2k} $(k \geq 1)$	$xy + z^2 + w^{2k+1}$	0
A_{2k-1} $(k \geq 1)$	$xy + z^2 + w^{2k}$	$\mathbb{Z}$
D_{2k} $(k \geq 2)$	$xy + z^2w + w^{2k-1}$	$\mathbb{Z}^2$
D_{2k-1} $(k \geq 3)$	$xy + z^2w + w^{2k-2}$	$\mathbb{Z}$
E_6	$xy + z^3 + w^4$	0
E_7	$xy + z^3 + zw^3$	$\mathbb{Z}$
E_8	$xy + z^3 + w^5$	0

(1.3.2)

1.3.3 Non-vanishing $\mathrm{K}_1^{sg}(X)$

In this section we collect some examples where $\mathrm{K}_1^{sg}(X)$ is nonzero. From the singularity K-theory exact sequence (1.1.1) it follows that $\mathrm{K}_1^{sg}(X)$ surjects onto $\mathrm{Ker}(\mathrm{K}_0(X) \overset{\mathrm{PD}}{\to} \mathrm{G}_0(X))$.

Example 1.3.7 (Non-isolated quotient singularity with huge kernel $\mathrm{Ker}(\mathrm{K}_0(X) \to \mathrm{G}_0(X))$**.** *The first such example has been constructed by Gubeladze [58]. We present an example given by Cortiñas, Haesemeyer, Walker and Weibel [34, Example 5.10].*

Let $E = \mathcal{O} \oplus \mathcal{O}(2)$ be the rank two bundle over $\mathbb{P}^1$. Let $\mathbb{Z}_2$ act on E fiberwise via $v \mapsto -v$. Then $X = E/\mathbb{Z}_2$, has quotient singularities and its singular locus isomorphic to $\mathbb{P}^1$.

The canonical map $\mathrm{K}_0(X) \to \mathrm{G}_0(X)$ is not injective, and furthermore, the kernel $\mathrm{Ker}(\mathrm{K}_0(X) \to \mathrm{G}_0(X))$ is huge, that is contains the base field k as a subgroup.

Example 1.3.8 (Isolated rational singularity with huge kernel $\mathrm{Ker}(\mathrm{K}_0(X) \to \mathrm{G}_0(X))$**.** *Consider a smooth cubic hypersurface $S \subset \mathbb{P}^3$, and let $X \subset \mathbb{A}^4$ be the affine cone over S. Then X has an isolated rational singularity.*

The Grothendieck group of a cone over a smooth variety has been computed in [36]. In particular since $\chi(\mathcal{T}_S) = -4$ by Riemann-Roch so that $H^1(S, \Omega^1_{S/\mathbb{Q}}(1)) = H^1(S, \Omega^1_{S/k}(1)) = H^1(S, \mathcal{T}_S) \neq 0$, where the first equation follows from the short exact sequence

$$0 \to \Omega^1_{k/\mathbb{Q}}(1) \to \Omega^1_{S/\mathbb{Q}}(1) \to \Omega^1_{S/k}(1) \to 0,$$

see [56, Proposition 20.6.2]. The main result of [36] implies that $\mathrm{K}_0(X)$ is huge, that is it contains a nonzero k-vector space.

Finally by Proposition 1.1.21 (1) the canonical map $\mathrm{K}_0(X) \to \mathrm{G}_0(X)$ factors through $\mathbb{Z}$, so that $\mathrm{Ker}(\mathrm{K}_0(X) \to \mathrm{G}_0(X))$ is huge as well.

Example 1.3.9 (Non-vanishing $\mathrm{K}_1^{sg}(X)$ for isolated quotient singularities over non-algebraically closed fields)**.** *Let $X = \mathbb{A}^2/\mathbb{Z}_2$ be the quotient by the action $v \mapsto -v$. We claim that $\mathrm{K}_1^{sg}(X) \simeq k^*/(k^*)^2$.*

Indeed, X is isomorphic to the affine surface $xy + z^2 = 0$, and using the Knörrer periodicity Theorem 1.1.4 we have

$$\mathrm{K}_1^{sg}(X) \simeq \mathrm{K}_1^{sg}(R),$$

where $R = k[\epsilon]/(\epsilon^2)$. We compute the singularity K*-theory via the* K*-theory sequence* (1.1.1)*, plugging in $\mathrm{G}_i(R) = \mathrm{K}_i(k)$, as* G*-theory is independent of the non-reduced scheme structure:*

$$\mathrm{K}_1(R) \to k^* \to \mathrm{K}_1^{sg}(R) \to \mathrm{K}_0(R) \to \mathbb{Z} \to \mathrm{K}_0^{sg}(R) \to 0.$$

Now $\mathrm{K}_0(R) = \mathbb{Z}$ and the map $\mathbb{Z} = \mathrm{K}_0(R) \to \mathbb{Z}$ is multiplication by two (cf Example 1.1.22), and similarly $\mathrm{K}_1(R) = R^*$, and the map $\mathrm{K}_1(R) \to k^*$ is $a + b\epsilon \mapsto a^2$ [65, Example 10.2]. We get

$$\mathrm{K}_1^{sg}(X) \simeq \mathrm{K}_1^{sg}(R) \simeq k^*/(k^*)^2,$$

which is in general a non-finitely generated 2*-torsion group.*

1.3.4 Proof of a conjecture of Srinivas for quotient singularities

In the 1980s Srinivas considered the question whether for an isolated quotient singularity $x_0 \in X$ the length homomorphism $l : \mathrm{K}_0(X \text{ on } x_0) \to \mathbb{Z}$ is an isomorphism [109, Page 38]. Here $\mathrm{K}_0(X \text{ on } x_0)$ stands for the Grothendieck group of perfect complexes supported at x_0 (originally Srinivas has considered the Grothendieck group of coherent sheaves which are supported at the singular points and which are perfect as complexes, but by [103, Proposition 2] these two groups are isomorphic).

Levine has proved that l is an isomorphism if X is two-dimensional with isolated quotient singularities [81, Theorem 3.2], that l is always surjective for isolated Cohen-Macaulay singularities, and that it has torsion kernel [82, Proposition 2.6, Theorem 2.7] in the case of isolated quotient-singularities.

The language of the singularity K-theory is well-adapted to deal with this kind of questions.

Lemma 1.3.10. *Let k be an algebraically closed field. Let X be a quasi-projective variety with isolated singularities. There is an exact sequence*

$$\mathrm{K}_1^{sg}(X) \to \mathrm{K}_0(X \text{ on } \mathrm{Sing}(X)) \xrightarrow{l} \mathbb{Z}^{\mathrm{Sing}(X)} \to 0 \tag{1.3.3}$$

and a natural surjective homomorphism $\mathrm{Ker}(l) \to \mathrm{Ker}(\mathrm{PD} : \mathrm{K}_0(X) \to \mathrm{G}_0(X))$.

Proof. We consider the diagram of pretriangulated dg-categories

$$\begin{array}{ccccc}
\mathcal{D}^{\mathrm{perf}}_{dg}(X \text{ on } \mathrm{Sing}(X)) & \longrightarrow & \mathcal{D}^{b}_{dg}(X \text{ on } \mathrm{Sing}(X)) & \longrightarrow & \mathcal{D}^{\mathrm{sg}}_{dg}(X \text{ on } \mathrm{Sing}(X)) \\
\downarrow & & \downarrow & & \downarrow \\
\mathcal{D}^{\mathrm{perf}}_{dg}(X) & \longrightarrow & \mathcal{D}^{b}_{dg}(X) & \longrightarrow & \mathcal{D}^{\mathrm{sg}}_{dg}(X)
\end{array}$$

and the associated long exact sequences of Schlichting's K-groups:

$$\begin{array}{ccccccc}
\mathrm{K}_1^{sg}(X \text{ on } \mathrm{Sing}(X)) & \longrightarrow & \mathrm{K}_0(X \text{ on } \mathrm{Sing}(X)) & \longrightarrow & \mathrm{G}_0(X \text{ on } \mathrm{Sing}(X)) & \longrightarrow & \mathrm{K}_0^{sg}(X \text{ on } \mathrm{Sing}(X)) \\
\simeq\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{K}_1^{sg}(X) & \longrightarrow & \mathrm{K}_0(X) & \xrightarrow{\mathrm{PD}} & \mathrm{G}_0(X) & \longrightarrow & \mathrm{K}_0^{sg}(X)
\end{array}$$

where the left vertical arrow is an isomorphism and the right vertical arrow is injective by Lemma 1.1.13.

We have $\mathrm{K}_0^{sg}(X \text{ on } \mathrm{Sing}(X)) = 0$ (Corollary 1.1.28), and we have a natural isomorphism

$$\mathrm{G}_0(X \text{ on } \mathrm{Sing}(X)) = \mathrm{G}_0(\mathrm{Sing}(X)) = \mathbb{Z}^{\mathrm{Sing}(X)}$$

given by the length (dimension) of zero-dimensional coherent sheaves, so that exact sequence

(1.3.3) is the first row in the diagram above.

Finally, the diagram above also induces the surjection $\mathrm{Ker}(l) \to \mathrm{Ker}(\mathrm{PD})$. □

The next result deals with the injectivity part of the Srinivas conjecture for quotient singularities, and thus gives a stronger version of [81, Theorem 3.2].

Proposition 1.3.11. *If X is a quasi-projective variety with isolated quotient singularities then the length map*

$$l : \mathrm{K}_0(X \text{ on } \mathrm{Sing}(X)) \to \mathbb{Z}^{\mathrm{Sing}(X)}$$

is an isomorphism.

Proof. By Theorem 1.2.23 (2), $\mathrm{K}_1^{sg}(X) = 0$. Lemma 1.3.10 implies the result. □

Remark 1.3.12. *By Lemma 1.3.10 non-vanishing of* $\mathrm{Ker}(\mathrm{PD} : \mathrm{K}_0(X) \to \mathrm{G}_0(X))$ *implies non-vanishing of* $\mathrm{Ker}(l)$*. This applies for instance in the case of a cone over a smooth cubic surface, see Example 1.3.8.*

Example 1.3.13. *Let k be an arbitrary field with* $\mathrm{char}(k) \neq 2$*. Let Q_n be the n-dimensional affine split quadric cone as in Examples 1.1.32, 1.1.33. It is a result of Levine [82, Theorem 4.2] that*

$$\mathrm{K}_0(Q_n \text{ on } 0) \simeq \begin{cases} \mathbb{Z} \oplus k^*/(k^*)^2, & n \text{ even} \\ \mathbb{Z}^2 \oplus k^*, & n \text{ odd} \end{cases}$$

This result can be reproved using exact sequence (1.3.3) *and the fact that*

$$\mathrm{K}_1^{sg}(Q_n) \simeq \begin{cases} k^*/(k^*)^2, & n \text{ even (cf Example 1.3.9)} \\ \mathbb{Z} \oplus k^*, & n \text{ odd (cf } A_1 \text{ case in (1.3.1))} \end{cases}$$

Similarly, one can compute $K_0(X \text{ on } 0)$ for other ADE singularities of arbitrary dimension. We omit the details of this computation.

Chapter 2

Obstructions to semiorthogonal decompositions for singular threefolds I: $\mathbb{K}$-theory

Abstract

We investigate necessary conditions for Gorenstein projective varieties to admit semiorthogonal decompositions introduced by Kawamata, with main emphasis on threefolds with isolated compound A_n singularities. We introduce obstructions coming from Algebraic K-theory and translate them into the concept of maximal nonfactoriality.

Using these obstructions we show that many classes of nodal threefolds do not admit Kawamata type semiorthogonal decompositions. These include nodal hypersurfaces and double solids, with the exception of a nodal quadric, and del Pezzo threefolds of degrees $1 \leq d \leq 4$ with maximal class group rank.

We also investigate when does a blow up of a smooth threefold in a singular curve admit a Kawamata type semiorthogonal decomposition and we give a complete answer to this question when the curve is nodal and has only rational components.

2.1 Introduction

Semiorthogonal decompositions for derived categories of singular projective algebraic varieties have recently began to be extensively studied. One important type of such semiorthogonal decomposition is

$$\mathcal{D}^b(X) = \langle \mathcal{D}^b(R_1), \dots, \mathcal{D}^b(R_m) \rangle \tag{2.1.1}$$

where X/k is a projective variety and all R_i's are finite-dimensional k-algebras. One can think of (2.1.1) as a generalization of a full exceptional collection which is the case when all $R_i = k$.

A typical construction of (2.1.1) proceeds through constructing a full exceptional collection on a resolution of singularities $\pi : \widetilde{X} \to X$ and pushing it forward to X. Burban has constructed decompositions (2.1.1) for nodal chains of rational curves [25], while Kawamata [76], Kuznetsov [79] and Karmazyn-Kuznetsov-Shinder [75] studied rational surfaces with isolated rational singularities; the exhaustive answer for toric surfaces is given in [75]. Finally Kawamata [76, 77] has also studied two examples of Fano threefolds with a single ordinary double point which admit decomposition (2.1.1). These examples are the nodal quadric threefold and a blow up of $\mathbb{P}^3$ in two points followed by contraction of the proper preimage of a line passing through the two points (this variety can be also described as a nodal linear section of a Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \subset \mathbb{P}^8$), see Example 2.4.13.

In this paper we investigate necessary conditions for (2.1.1) to hold on a Gorenstein projective variety X. In fact we allow more general decompositions

$$\mathcal{D}^b(X) = \langle \mathcal{A}, \mathcal{D}^b(R_1), \ldots, \mathcal{D}^b(R_m) \rangle \tag{2.1.2}$$

where $\mathcal{A} \subset \mathcal{D}^{\mathrm{perf}}(X)$ and which we call *Kawamata type semiorthogonal decompositions* (because it is similar to what Kawamata has studied in [77]). Here again the R_i's are finite-dimensional k-algebras. We assume that semiorthogonal decompositions we consider are admissible; if $m = 1$ the latter condition is automatic, see Proposition 2.4.7. We think of (2.1.2) as a splitting of the derived category into its "nonsingular part" $\mathcal{A}$ and the algebras R_i which carry information about the singular points of X.

We concentrate on obstructions coming from Algebraic K-theory, namely on the negative $\mathrm{K}_{-1}(X)$ group. The latter group is a part of the package of the Thomason-Trobaugh K-theory machinery, and the negative K-groups including $\mathrm{K}_{-1}(X)$ have been extensively studied, in particular in the seminal work of Weibel [124].

After recalling some preliminary results on semiorthogonal decompositions and saturatedness in the singular setting, Orlov's singularity category and various K-theory groups in Section 2.2, in Section 2.3 we translate vanishing of K_{-1} into geometric properties of X. This has already been done by Weibel for curves and surfaces [124], and our study concentrates on isolated threefold singularities, while reproving some of Weibel's results for curves and surfaces along the way. This relies on previous joint work of the second and third authors [98], where K-theory of Orlov's singularity category is studied. We recall geometric description of K_{-1} for curves in Proposition 2.3.1 and Corollary 2.3.3 and K_{-1} for surfaces can be computed using Proposition 2.3.6.

In general we show that vanishing of $\mathrm{K}_{-1}(X)$ implies that X is what we call *maximally nonfactorial*, see Definition 2.3.4 and Proposition 2.3.5, and that for certain types of singularities, including three-dimensional compound A_n singularities vanishing of $\mathrm{K}_{-1}(X)$ is equivalent to X being maximally nonfactorial (Corollary 2.3.8).

Informally, maximally nonfactorial varieties have as many Weil non-Cartier divisors as the local class groups allow. In particular, in the nodal threefold case each local class group is isomorphic to $\mathbb{Z}$, and maximally nonfactorial nodal threefolds are characterized by having sufficiently many Weil divisors to separate singularities, that is for every ordinary double point

$p \in X$ there exists a Weil divisor which generates the local class group at p and is Cartier at all other nodes. This is stronger than just being non-factorial which only requires existence of a Weil divisor which is non-Cartier. On the other hand, for singularities for which the local class groups vanish, the variety is automatically factorial, while maximal nonfactoriality is a vacuous condition holding trivially.

More generally we relate $\mathrm{K}_{-1}(X)$ to the so-called *defect* of X, that is the codimension of $\mathrm{Pic}(X)$ in $\mathrm{Cl}(X)$, see Definition 2.3.9 and Corollary 2.3.8. It follows that in the language of defect, maximal nonfactoriality for nodal threefolds implies that defect is equal to the number of singular points, which is the maximal value the defect can take.

In Section 2.4 we show that existence of a decomposition (2.1.2) implies that $\mathrm{K}_{-1}(X) = 0$, see Corollary 2.4.5. This is obtained by passing to Orlov's singularity category in (2.1.2), and using idempotent completeness of the singularity category of a finite-dimensional algebra.

Combining the results explained so far we can state our main result as follows:

Theorem 2.1.1 (Proposition 2.3.5 and Corollary 2.4.5)**.** *If a normal Gorenstein projective variety X has a Kawamata type decomposition* (2.1.2)*, then $\mathrm{K}_{-1}(X) = 0$. If in addition X has isolated singularities, then X is maximally nonfactorial.*

This explains why the two nodal threefolds with a Kawamata type decomposition studied by Kawamata [77] are nonfactorial. In both cases the threefold X has a single ordinary double with defect of X being equal to one (in the nodal quadric threefold case $\mathrm{Pic}(X) = \mathbb{Z}$, $\mathrm{Cl}(X) = \mathbb{Z}^2$, while in the other example $\mathrm{Pic}(X) = \mathbb{Z}^2$, $\mathrm{Cl}(X) = \mathbb{Z}^3$), which illustrates the maximal nonfactoriality of X. Furthermore using the theorem above we show that many types of threefolds do not admit decompositions (2.1.2).

Application 2.1.2 (Example 2.4.15, 2.4.16, 2.5.5)**.** *The following types of nodal threefolds do not admit a Kawamata type semiorthogonal decomposition:*

1. *All nodal threefold hypersurfaces $X \subset \mathbb{P}^4$, except for the nodal quadric.*
2. *All nodal threefold double solids $X \xrightarrow{2:1} \mathbb{P}^3$, except for the nodal quadric.*
3. *Del Pezzo threefolds V_d of degrees $1 \leq d \leq 4$ with maximal class group rank [99].*
4. *Threefolds obtained by blowing up a nodal irreducible curve in a smooth threefold.*

Del Pezzo threefolds in (3) can also be described as follows [99, Theorem 7.1]: V_d is a blow up of $8 - d$ general points on $\mathbb{P}^3$ followed by contraction of proper preimages of lines passing through pairs of points and twisted cubics through six-tuples of points (for $d = 1, 2$). Thus we negatively answer a question of Kawamata [77, Remark 7.5], in all cases except for $d = 5$ which is a 3-nodal V_5. In fact we expect that only a few types of nodal Fano threefolds admit Kawamata type semiorthogonal decompositions. Looking at the potential cases of Fano threefolds with maximal defect, I. Cheltsov has suggested the following.

Conjecture 2.1.3. *The only nodal Fano threefolds of Picard rank one with Kawamata type decompositions are the quadric, V_5 and V_{22}.*

However, in spite of the sparsity of the Fano examples, we can construct lots of nodal threefolds with a Kawamata decomposition using the blow up construction with a locally complete intersection center as soon as the base variety and the center of the blow up both admit Kawamata type decompositions (see Theorem 2.5.1 and Corollary 2.5.3). In particular, blowing up a smooth threefold in a disjoint union of nodal trees of smooth rational curves produces nodal threefolds with an arbitrary large number of ordinary double points and admitting a Kawamata type decomposition:

Theorem 2.1.4 (Corollary 2.5.4). *Let X be a smooth projective threefold and C is a disjoint union of nodal curves in X such that all irreducible components of C are rational curves. Then the blow up $\widetilde{X}$ of X along C admits a Kawamata type semiorthogonal decomposition if and only if C is a disjoint union of nodal trees with smooth rational components.*

Relation to other work

The link between idempotent completeness of the Orlov singularity category and nonfactoriality is already present in the work of Iyama and Wemyss [67]. It follows from [67, Theorem 1.2] that nodal threefolds with idempotent complete singularity categories are nonfactorial. However from the perspective of our applications our results are sharper in a sense that we show maximal nonfactoriality, which is strictly stronger than nonfactoriality for varieties with several ordinary double points.

The Grothendieck group of the singularity category has been used by the first author of this paper and Karmazyn [72, Corollary 5.3] to show that some types of surface quotient singularities most notably D_n, $n \geq 4$ and E_n, $n = 6, 7, 8$ do not allow a decomposition (2.1.1) with *local* possibly noncommutative algebras R_i's. Even though all existing Kawamata type decompositions for Gorenstein surfaces only admit A_n singularities [75], we do not currently know how to rule out D_n and E_n singularities without assuming that the algebras R_i are local.

A similar sort of obstruction to K_{-1} has been used by Karmazyn, Kuznetsov and the third author of the present paper [75], where it is shown that a necessary condition for existence of a decomposition (2.1.1) on a projective normal rational surface X with rational singularities is vanishing of the Brauer group $\mathrm{Br}(X)$. We explain in Proposition 2.3.7 that for such surfaces $\mathrm{Br}(X) \simeq \mathrm{K}_{-1}(X)$, so in this paper we generalize the obstruction from [75] from surfaces to higher-dimensional varieties.

In the sequel to this paper [74] we study restrictions on types of singularities that are forced by Kawamata type decompositions, using representation theory of finite-dimensional algebras.

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2.2 Preliminaries

2.2.1 Notation

We work over an algebraically closed field k of characteristic zero. By an (algebraic) variety we mean a reduced, but not necessarily irreducible, scheme of finite type over k.

All triangulated categories are assumed to be k-linear. The opposite category of a category $\mathcal{T}$ will be denoted $\mathcal{T}^\circ$. We denote by $\mathcal{D}(\mathrm{Qcoh}(X))$ the unbounded derived category of quasi-coherent sheaves, by $\mathcal{D}^b(X)$ the bounded derived category of coherent sheaves of a variety X and by $\mathcal{D}^{\mathrm{perf}}(X)$ its full subcategory consisting of perfect complexes. Similarly, for a k-algebra R we denote by $\mathcal{D}^b(R) = \mathcal{D}^b(\text{mod-}R)$ the bounded derived category of finitely generated right modules over R and $\mathcal{D}^{\mathrm{perf}}(R)$ is again the full subcategory of perfect complexes in $\mathcal{D}^b(R)$.

All functors such as pull-back π^*, pushforward π_* and tensor product $\otimes$ when considered between derived categories are derived functors.

2.2.2 Semiorthogonal decompositions and saturatedness

Following [16, 17, 78], we recall standard definitions and properties of semiorthogonal decompositions of triangulated categories, of saturated categories and relations between these two notions.

Let $\mathcal{T}$ be a triangulated category. We call $\mathcal{T}$ Hom-*finite* if $\dim_k \mathrm{Hom}(A,B) < \infty$ for all $A, B \in \mathcal{T}$; we call $\mathcal{T}$ *of finite type* if $\bigoplus_i \dim_k \mathrm{Hom}(A, B[i]) < \infty$ for all $A, B \in \mathcal{T}$. For example, if X is projective, then $\mathcal{D}^b(X)$ is Hom-finite, and $\mathcal{D}^{\mathrm{perf}}(X)$ is of finite type. From now we assume all triangulated categories to be Hom-finite, but not necessarily of finite type.

A triangulated category $\mathcal{T}$ is called *idempotent complete* (or Karoubian) if every idempotent $e \in \mathrm{Hom}(A, A)$ gives rise to a direct sum decomposition of A. It is well-known that for every triangulated category $\mathcal{T}$ has a triangulated idempotent completion $\mathcal{T} \subset \overline{\mathcal{T}}$ [8].

Let $\mathcal{A} \subset \mathcal{T}$ be a full triangulated subcategory. The left and right orthogonals to $\mathcal{A}$ are defined as

$$
\begin{aligned}
{}^{\perp}\mathcal{A} &= \{T \in \mathcal{T} \mid \forall A \in \mathcal{A},\ \operatorname{Hom}(T, A) = 0\}, \\
\mathcal{A}^{\perp} &= \{T \in \mathcal{T} \mid \forall A \in \mathcal{A},\ \operatorname{Hom}(A, T) = 0\}.
\end{aligned}
$$

Definition 2.2.1 ([18])**.** *A collection $\mathcal{A}_1, \ldots, \mathcal{A}_m$ of full triangulated subcategories of $\mathcal{T}$ is called a semiorthogonal decomposition if for all $1 \leq i < j \leq m$*

$$\mathcal{A}_i \subset \mathcal{A}_j^{\perp}$$

and if the smallest triangulated subcategory of $\mathcal{T}$ containing $\mathcal{A}_1, \ldots, \mathcal{A}_m$ coincides with $\mathcal{T}$. We use the notation

$$\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$$

for a semiorthogonal decomposition of $\mathcal{T}$ with components $\mathcal{A}_1, \ldots, \mathcal{A}_m$.

The next Lemma is well-known and follows immediately from the definitions:

Lemma 2.2.2. *If $\mathcal{T}$ admits a semiorthogonal decomposition into components $\mathcal{A}_1, \ldots, \mathcal{A}_m$ then $\mathcal{T}$ is idempotent complete if and only if all $\mathcal{A}_i$'s are idempotent complete.*

Definition 2.2.3 ([16, 17])**.** *A full triangulated subcategory $\mathcal{A}$ of $\mathcal{T}$ is called left (resp. right) admissible, if the inclusion functor $\mathcal{A} \subset \mathcal{T}$ has a left (resp. right) adjoint. If $\mathcal{A}$ is both left and right admissible, then we call $\mathcal{A}$ admissible in $\mathcal{T}$.*

Lemma 2.2.4 ([17, Proposition 1.5])**.** *Let $\mathcal{A}$ be a full triangulated subcategory of $\mathcal{T}$, then $\mathcal{A}$ is left (resp. right) admissible in $\mathcal{T}$ if and only if there is a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}, {}^{\perp}\mathcal{A} \rangle$ (resp. $\mathcal{T} = \langle \mathcal{A}^{\perp}, \mathcal{A} \rangle$).*

Definition 2.2.5. *We call a semiorthogonal decomposition $\mathcal{T} = \langle \mathcal{A}_1, \ldots, \mathcal{A}_m \rangle$ admissible if every $\mathcal{A}_i$ is admissible in $\mathcal{T}$.*

Admissible decompositions are called strong in [78]. Let us recall in what follows the relation between (left/right) admissible subcategories and representability of (co)homological functors of finite type. Note that in the following definition we do not assume that our triangulated category $\mathcal{T}$ is of finite type (which is assumed in [17]).

Definition 2.2.6 ([17])**.** *$\mathcal{T}$ is called left (resp. right) saturated if any exact functor $\mathcal{T} \to \mathcal{D}^b(k)$ (resp. $\mathcal{T}^{\circ} \to \mathcal{D}^b(k)$) is representable. If $\mathcal{T}$ is both left and right saturated, then we call $\mathcal{T}$ saturated.*

Theorem 2.2.7 (Rouquier)**.** *If X is a projective variety, then $\mathcal{D}^b(X)$ is saturated.*

Proof. Left staturatedness is shown in [90, Theorem 7.1]. Let us explain in the following how left staturatedness of $\mathcal{D}^b(X)$ implies right saturatedness.

Let $F : \mathcal{D}^b(X)^\circ \to \mathcal{D}^b(k)$ be an exact functor. Let us denote by $(\text{-})^\circ : \mathcal{D}^b(k) \to \mathcal{D}^b(k)^\circ$ the dualizing functor $(\text{-})^\circ = R\mathrm{Hom}_{\mathcal{D}^b(k)}(\text{-}, k)$. Denote further by $\omega_X^\bullet \in \mathcal{D}^b(X)$ the dualizing complex of X and write $\mathcal{P}^\vee = \mathcal{H}om(\mathcal{P}, \mathcal{O}_X) \in \mathcal{D}^b(X)$ for a perfect complex $\mathcal{P} \in \mathcal{D}^b(X)$.

We observe that F° is a covariant exact functor on $\mathcal{D}^b(X)$ and thus by [90, Theorem 7.1] F° is represented by a perfect complex $\mathcal{P} \in \mathcal{D}^b(X)$. Furthermore, $(\text{-})^\circ$ is inverse to itself and we get thus

$$\begin{aligned} F \simeq (F^\circ)^\circ &\simeq R\mathrm{Hom}_{\mathcal{D}^b(k)}(R\mathrm{Hom}_{\mathcal{D}^b(X)}(\mathcal{P}, \text{-}), k) \\ &\simeq R\mathrm{Hom}_{\mathcal{D}^b(k)}(Rp_*((\text{-}) \otimes \mathcal{P}^\vee), k) \\ &\simeq R\mathrm{Hom}_{\mathcal{D}^b(X)}((\text{-}) \otimes \mathcal{P}^\vee, \omega_X^\bullet) \\ &\simeq R\mathrm{Hom}_{\mathcal{D}^b(X)}(\text{-}, \mathcal{P} \otimes \omega_X^\bullet), \end{aligned}$$

where we used the fact that $\mathcal{P}$ is a perfect complex in the second and fourth equality and Grothendieck-Verdier duality with respect to the projection $p : X \to \mathrm{Spec}(k)$ in the third equality. Hence F is represented by $\mathcal{P} \otimes \omega_X^\bullet \in \mathcal{D}^b(X)$. □

Lemma 2.2.8 ([17])**.** *Let $\mathcal{T}$ be saturated and let $\mathcal{A}$ be a left (resp. right) admissible full triangulated subcategory of $\mathcal{T}$. Then $\mathcal{A}$ is saturated.*

Proof. See e.g. [78, Lemma 2.10]. □

Corollary 2.2.9. *Let X be a projective variety. Then any left (resp. right) admissible subcategory of $\mathcal{D}^b(X)$ is saturated.*

Finite type saturated categories are universally admissible in the following sense.

Proposition 2.2.10 ([17, Proposition 2.6])**.** *Let $\mathcal{A}$ be a full triangulated subcategory of $\mathcal{T}$, where $\mathcal{T}$ is of finite type and let moreover $\mathcal{A}$ be left (resp. right) saturated. Then $\mathcal{A}$ is left (resp. right) admissible in $\mathcal{T}$.*

Definition 2.2.11 ([17])**.** *Let $\mathcal{T}$ be a triangulated category. Then an autoequivalence $S : \mathcal{T} \to \mathcal{T}$ is called a Serre functor if there is a functorial equivalence*

$$\mathrm{Hom}(A, B) \simeq \mathrm{Hom}(B, S(A))^\star$$

for $A, B \in \mathcal{T}$.

Lemma 2.2.12 ([17, Proposition 3.7])**.** *If $\mathcal{T}$ has a Serre functor then for every admissible decomposition $\mathcal{T} = \langle \mathcal{A}_1, \dots, \mathcal{A}_m \rangle$ each component $\mathcal{A}_i$ has a Serre functor.*

2.2.3 Gorenstein varieties and algebras

Definition 2.2.13. *A two-sided noetherian ring R satisfying $\mathsf{inj.dim}_R R < \infty$ and $\mathsf{inj.dim}\, R_R < \infty$ is called Gorenstein. A variety X is called Gorenstein if all its local rings are Gorenstein.*

Gorenstein property is preserved under regular embeddings, projective bundles and blow ups with locally complete intersection centers.

Let $\omega_X^\bullet$ denote the dualizing complex $p^!(k)$ of X, where $p : X \to \mathrm{Spec}(k)$ (for the definition of $p^!$, see [61]). It is well-known that X is Gorenstein if and only if $\omega_X^\bullet$ is a shift of a line bundle [61, Proposition V.9.3]. Let $S_X(-) = (-) \otimes \omega_X[\dim(X)]$ the Serre functor on $\mathcal{D}^{\mathrm{perf}}(X)$. By abuse of notation, we also write S_X for the autoequivalence on $\mathcal{D}^b(X)$ defined by the same formula. S_X is not a Serre functor on $\mathcal{D}^b(X)$, however by the Grothendieck-Verdier duality there is a weaker statement: for all E in $\mathcal{D}^{\mathrm{perf}}(X)$ and all F in $\mathcal{D}^b(X)$ (resp. for all E in $\mathcal{D}^b(X)$ and all F in $\mathcal{D}^{\mathrm{perf}}(X)$), we have

$$\mathrm{Hom}_{\mathcal{D}^b(X)}(E, F) \simeq \mathrm{Hom}_{\mathcal{D}^b(X)}(F, S_X(E))^*. \tag{2.2.1}$$

This isomorphism typically fails when neither E nor F are perfect, for instance it always fails for structure sheaves of singular points.

The homological meaning of the Gorenstein condition is the following result:

Lemma 2.2.14. *A projective variety X (resp. finite-dimensional algebra R) is Gorenstein if and only if $\mathcal{D}^{\mathrm{perf}}(X)$ (resp. $\mathcal{D}^{\mathrm{perf}}(R)$) has a Serre functor.*

Proof. For finite-dimensional algebras this is a result of Chen [30, Corollary 3.9], which goes back to Happel [60, Section 3.6].

For varieties the "only if" direction is clear. For the "if" direction, let us denote by S the Serre functor on $\mathcal{D}^{\mathrm{perf}}(X)$ and let $\omega_X^\bullet$ be as above. By the definition of the Serre functor S and by Grothendieck-Verdier duality, we have a functorial isomorphism

$$\mathrm{Hom}(E, \omega_X^\bullet) \simeq \mathrm{Hom}(E, S(\mathcal{O}_X)) \tag{2.2.2}$$

for $E \in \mathcal{D}^{\mathrm{perf}}(X)$. In particular we obtain a canonical map $f : S(\mathcal{O}_X) \to \omega_X^\bullet$ corresponding to the identity morphism of $\mathcal{O}_X$. Let C be the cone of f. By (2.2.2) we see that $\mathrm{Hom}(E, C) = 0$ for all $E \in \mathcal{D}^{\mathrm{perf}}(X)$. By [20, Corollary 3.1.2] we have that $C = 0$ and, in other words, $S(\mathcal{O}_X) \simeq \omega_X^\bullet$. In particular the dualizing complex of X is perfect. Since $\omega_X^{\bullet\vee} \otimes \omega_X^\bullet \simeq R\mathcal{H}om(\omega_X^\bullet, \omega_X^\bullet) \simeq \mathcal{O}_X$ by the definition of a dualizing complex, it is easy to deduce that $\omega_X^\bullet$ is a shift of a line bundle. Equivalently, X is Gorenstein. □

In the Gorenstein case one can mutate semiorthogonal decompositions as follows:

Lemma 2.2.15. *Let X be a Gorenstein projective variety. If $\mathcal{D}^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$, and either $\mathcal{A}$ or $\mathcal{B}$ is contained in $\mathcal{D}^{\mathrm{perf}}(X)$, then both $\mathcal{A}$ and $\mathcal{B}$ are admissible and there is a semiorthogonal decomposition $\mathcal{D}^b(X) = \langle \mathcal{B} \otimes \omega_X, \mathcal{A} \rangle$.*

Proof. Let $\mathcal{A} \subset \mathcal{D}^{\mathrm{perf}}(X)$. Applying Corollary 2.2.9, Proposition 2.2.10 and Lemma 2.2.12 we obtain that $\mathcal{A}$ is saturated, admissible in $\mathcal{D}^{\mathrm{perf}}(X)$ and has a Serre functor $S_{\mathcal{A}}$ (alternatively, instead of relying on Lemma 2.2.12 we can deduce existence of the Serre functor on $\mathcal{A}$ using [17, Corollary 3.5] immediately from $\mathcal{A}$ being saturated and of finite type).

We can define the right adjoint of $I : \mathcal{A} \to \mathcal{D}^b(X)$ using the following standard construction of [17]. Let $L : \mathcal{D}^b(X) \to \mathcal{A}$ be the left adjoint of I and define the functor $R : \mathcal{D}^b(X) \to \mathcal{A}$ by the formula $R = S_{\mathcal{A}} \circ L \circ S_X^{-1}$. It follows from definitions and (2.2.1) that R is right adjoint to I, and that there is a semiorthogonal decomposition

$$\mathcal{D}^b(X) = \langle \mathcal{B} \otimes \omega_X, \mathcal{A} \rangle.$$

Since ω_X is a line bundle, we obtain also $\langle \mathcal{B} \otimes \omega_X, \mathcal{A} \rangle \simeq \langle \mathcal{B}, \mathcal{A} \otimes \omega_X^\vee \rangle$ and hence $\mathcal{B} \hookrightarrow \mathcal{D}^b(X)$ is admissible as well.

The case $\mathcal{B} \subset \mathcal{D}^{\mathrm{perf}}(X)$ can be proven similarly. □

2.2.4 Singularity categories

We recall standard facts about singularity categories. The basic references for these results are [24, 94]. Let X be k-scheme satisfying Orlov's ELF condition [94]; in particular we can take X to be a quasi-projective variety, or the Spec of a completion of a local ring for a point in a variety. For every closed $Z \subset X$ the triangulated category of singularities of X supported at Z is the Verdier quotient

$$\mathcal{D}_Z^{\mathrm{sg}}(X) = \mathcal{D}_Z^b(X)/\mathcal{D}_Z^{\mathrm{perf}}(X).$$

We write $\mathcal{D}^{\mathrm{sg}}(X)$ for $\mathcal{D}_X^{\mathrm{sg}}(X)$. If R is a ring, then we define its singularity category by the same formula $\mathcal{D}^{\mathrm{sg}}(R) = \mathcal{D}^b(R)/\mathcal{D}^{\mathrm{perf}}(R)$.

Let us denote by $\overline{\mathcal{D}^{\mathrm{sg}}(X)}$ the idempotent completion of $\mathcal{D}^{\mathrm{sg}}(X)$. As we will see in the next section, idempotent completeness of $\mathcal{D}^{\mathrm{sg}}(X)$ is controlled by the first negative K-theory group of X.

The following is an important property of the singularity category, called Knörrer periodicity.

Theorem 2.2.16 ([94, Theorem 2.1])**.** *Let X be regular and let $f : X \to \mathbb{A}^1$ be a non-zero morphism. Define $g = f + xy : X \times \mathbb{A}^2 \to \mathbb{A}^1$. Let $Z_f = f^{-1}(\{0\})$ and $Z_g = g^{-1}(\{0\})$. Then we have a canonical equivalence*

$$\mathcal{D}^{\mathrm{sg}}(Z_f) \to \mathcal{D}^{\mathrm{sg}}(Z_g).$$

The following result goes back to Auslander.

Proposition 2.2.17. *If X is n-dimensional Gorenstein with only isolated singularities, then $\mathcal{D}^{\mathrm{sg}}(X)$ is a Calabi-Yau-$(n-1)$ category, that is $[n-1]$ is its Serre functor, or in other words, for every two objects $E, F \in \mathcal{D}^{\mathrm{sg}}(X)$ we have a functorial isomorphism*

$$\mathrm{Hom}(E, F) \simeq \mathrm{Hom}(F, E[n-1])^*.$$

Proof. By [94, Proposition 1.14] we can reduce to the affine case. The statement is then a result of Iyama and Wemyss [68, Theorem 1.4], which follows essentially from a theorem of Auslander [6, Theorem 3.1] combined with Buchweitz' famous result [24, Theorem 4.4.1 (2)]. □

Example 2.2.18. *If Q is an affine nodal n-dimensional quadric (that is an ordinary double point of dimension n), then by Knörrer periodicity Theorem 2.2.16, $\mathcal{D}^{\mathrm{sg}}(Q)$ only depends on the parity of n. In particular, let us assume $n \equiv 1 \pmod 2$, so that*

$$\mathcal{D}^{\mathrm{sg}}(Q) \simeq \mathcal{D}^{\mathrm{sg}}(A),$$

where $A = k[x,y]/(xy)$. By Proposition 2.2.17, $\mathcal{D}^{\mathrm{sg}}(Q)$ is a Calabi-Yau-0 category, that is $\mathrm{Hom}(E,F) \simeq \mathrm{Hom}(F,E)^*$. *In fact $\mathcal{D}^{\mathrm{sg}}(Q)$ is equivalent to the category of $\mathbb{Z}/2$-graded finite-dimensional vector spaces, with the shift functor* $[1]$ *exchanging the graded pieces.*

Lemma 2.2.19 (Chen [29, Corollary 2.4])**.** *For any finite dimensional k-algebra R, $\mathcal{D}^{\mathrm{sg}}(R)$ is idempotent complete.*

In the Gorenstein case, the Lemma above also follows from the famous result of Buchweitz [24, Theorem 4.4.1] that $\mathcal{D}^{\mathrm{sg}}(R)$ of a Gorenstein ring R is equivalent to the stable category of stable maximal Cohen-Macaulay R-modules (also called Gorenstein projectives) $\underline{\mathrm{MCM}}(R)$, and the latter category is well-known to be idempotent complete for finite-dimensional algebras (see e.g. [71, Lemma 2.68]).

2.2.5 Grothendieck groups and the topological filtration

We assume that X is an ELF k-scheme. Let $Z \subset X$ be a closed subscheme.

We define the following Grothendieck groups of X with supports on Z; the first two are classical, and the last two are defined and studied in [98]. We define

$$\begin{aligned}
\mathrm{K}_0(X \text{ on } Z) &= \mathrm{K}_0(\mathcal{D}^{\mathrm{perf}}_Z(X)) \\
\mathrm{G}_0(X \text{ on } Z) &= \mathrm{K}_0(\mathcal{D}^b_Z(X)) \simeq \mathrm{G}_0(Z) \\
\mathrm{K}^{sg}_0(X \text{ on } Z) &= \mathrm{K}_0(\mathcal{D}^{\mathrm{sg}}_Z(X)) \\
\mathbb{K}^{sg}_0(X \text{ on } Z) &= \mathrm{K}_0(\overline{\mathcal{D}^{\mathrm{sg}}_Z(X)})
\end{aligned}$$

where the the isomorphism in the second line is Quillen's devissage. The last two groups are called *singularity Grothendieck groups.* We write $\mathrm{K}^{sg}_0(X)$ (resp. $\mathbb{K}^{sg}_0(X)$) for these groups when $Z = X$. Essentially from definitions (see [98, Remark 1.13]) we get a canonical exact sequence

$$\mathrm{K}_0(X \text{ on } Z) \to \mathrm{G}_0(Z) \to \mathrm{K}^{sg}_0(X \text{ on } Z) \to 0. \tag{2.2.3}$$

Let $\mathrm{K}_{-1}(X)$ (resp. $\mathrm{K}_{-1}(X \text{ on } Z)$) be the (-1)-st K-group of X (resp. of X with supports in Z) [119]. We have the following well-known relation between the two singularity Grothendieck groups defined above, going back to Thomason [118], Schlichting [106] and Orlov [96].

Lemma 2.2.20 ([98, Lemma 1.10, 1.11 and Remark 1.13])**.** *There is a canonical short exact sequence*

$$0 \to \mathrm{K}^{sg}_0(X \text{ on } Z) \to \mathbb{K}^{sg}_0(X \text{ on } Z) \to \mathrm{K}_{-1}(X \text{ on } Z) \to 0. \tag{2.2.4}$$

Moreover, $\mathcal{D}^{\mathrm{sg}}_Z(X)$ *is idempotent complete if and only if* $\mathrm{K}_{-1}(X \text{ on } Z) = 0$.

We note that all categories and Grothendieck groups with supports on Z used above only depend on the set of points of Z rather than its scheme structure.

For a noetherian commutative k-algebra A of finite Krull dimension, we write $\mathrm{K}_0^{sg}(A)$ for $\mathrm{K}_0^{sg}(\mathrm{Spec}(A))$. For complete local rings we have the following result.

Proposition 2.2.21. *Let* $\widehat{A}$ *be the completion of a commutative noetherian local* k*-algebra* A *of Krull dimension* n.
1) There is an isomorphism

$$\mathrm{K}_0^{sg}(\widehat{A}) \simeq \mathbb{K}_0^{sg}(\widehat{A})$$

2) If $\widehat{A}$ *is reduced, then*

$$F^n\mathrm{K}_0^{sg}(\widehat{A}) = 0.$$

Proof. 1) This is a consequence of [42, Theorem 3.7] and the short exact sequence (2.2.4).

2) It is well-known by Nakayama's Lemma that any finitely generated projective module over a local ring B is free and thus $\mathrm{K}_0(B) \simeq \mathbb{Z}$. Using (2.2.3) we obtain $F^i\mathrm{G}_0(B) \simeq F^i\mathrm{K}_0^{sg}(B)$ for all $i \geq 1$. In particular $F^n\mathrm{G}_0(A) \simeq F^n\mathrm{K}_0^{sg}(A)$ and $F^n\mathrm{G}_0(\widehat{A}) \simeq F^n\mathrm{K}_0^{sg}(\widehat{A})$. Moreover, by definition $F^n\mathrm{G}_0(\widehat{A})$ and $F^n\mathrm{G}_0(A)$ are generated by $[k]$ and the flat pullback of the canonical morphism $\mathrm{Spec}(\widehat{A}) \to \mathrm{Spec}(A)$ induces a surjective map $F^n\mathrm{G}_0(A) \twoheadrightarrow F^n\mathrm{G}_0(\widehat{A})$. By [98, Proposition 1.24 (4)] we know however that $F^n\mathrm{G}_0(A) = 0$ and thus $F^n\mathrm{G}_0(\widehat{A}) = 0$. We conclude that $F^n\mathrm{K}_0^{sg}(\widehat{A}) = 0$. Note that the statement becomes false without the assumption that k is algebraically closed, see [98, Example 1.22]. □

Assume that all irreducible components of X have the same dimension. There is a so-called *topological filtration* $F^i\mathrm{K}_0^{sg}(X)$ on $\mathrm{K}_0^{sg}(X)$ induced by the topological filtration on $\mathrm{G}_0(X) = \mathrm{K}_0(\mathcal{D}^b(X))$ (see [98, Subchapter 1.3]). Recall that $F^i\mathrm{K}_0^{sg}(X)$ is generated by elements $[\mathcal{O}_T]$, where $T \subset X$ is a closed subscheme of codimension at least i. Let us denote the associated graded groups by $\mathrm{gr}^i\mathrm{K}_0^{sg}(X)$. A topological filtration on $\mathrm{K}_0^{sg}(X \text{ on } Z)$ can be defined in the same way. We have the following useful properties of the associated graded groups of $\mathrm{K}_0^{sg}(X)$:

Proposition 2.2.22 ([98])**.** *1) Assume* X *has only isolated singularities and let* $Z \subset X$ *be a closed subscheme. Then there is an isomorphism*

$$\mathbb{K}_0^{sg}(X \text{ on } Z) \simeq \bigoplus_{p \in \mathrm{Sing}(X) \cap Z} \mathrm{K}_0^{sg}(\widehat{\mathcal{O}}_{X,p}). \tag{2.2.5}$$

Furthermore for all $i \geq 0$ *we have*

$$F^i\mathrm{K}_0^{sg}(X \text{ on } Z) \subset \bigoplus_{p \in \mathrm{Sing}(X) \cap Z} F^i\mathrm{K}_0^{sg}(\widehat{\mathcal{O}}_{X,p}), \tag{2.2.6}$$

and in particular, $F^n\mathrm{K}_0^{sg}(X \text{ on } Z) = 0$*, where* $n = \dim(X)$.

2) Let C be reduced and connected one-dimensional ELF k-scheme with N irreducible components and let $Z \subset C$ be a reduced subscheme of C of dimension 1. Denote by N_Z the number of irreducible components of Z. Then

$$\mathrm{K}_0^{sg}(C \text{ on } Z) = \mathrm{gr}^0\mathrm{K}_0^{sg}(C \text{ on } Z) = \begin{cases} \mathbb{Z}^{N_Z} & Z \subsetneq C \\ \mathbb{Z}^{N_C - 1} & Z = C \end{cases},$$

generated by the structure sheaves of the irreducible components of Z.
3) If X is normal irreducible then $\mathrm{gr}^1\mathrm{K}_0^{sg}(X) \simeq \mathrm{Cl}(X)/\mathrm{Pic}(X)$, functorially with respect to flat pullbacks.
4) The isomorphism $\mathrm{K}_0^{sg}(Z_f) \simeq \mathrm{K}_0^{sg}(Z_g)$ induced by Theorem 2.2.16 shifts the topological filtration by one, that is for all $i \geq 0$ we have natural isomorphisms $F^i\mathrm{K}_0^{sg}(Z_f) \simeq F^{i+1}\mathrm{K}_0^{sg}(Z_g)$ and $\mathrm{gr}^i\mathrm{K}_0^{sg}(Z_f) \simeq \mathrm{gr}^{i+1}\mathrm{K}_0^{sg}(Z_g)$.

Proof. 1) Let us denote by S the singular locus of X. We have a well-defined, fully-faithful functor $\mathcal{D}^{\mathrm{sg}}_{Z\cap S}(X) \to \mathcal{D}^{\mathrm{sg}}_Z(X)$ [96, Lemma 2.6] and its image is dense in $\mathcal{D}^{\mathrm{sg}}_Z(X)$ (see [96, Proposition 2.7]). In particular, these two categories have the same idempotent completion. Moreover, since $Z \cap S$ is a finite set of closed points, we have

$$\overline{\mathcal{D}^{\mathrm{sg}}_Z(X)} \simeq \overline{\mathcal{D}^{\mathrm{sg}}_{Z\cap S}(X)} \simeq \bigoplus_{p \in Z\cap S} \mathcal{D}^{\mathrm{sg}}_{\{p\}}(\widehat{\mathcal{O}}_{X,p}),$$

where we used [96, Theorem 2.10] and Proposition 2.2.21 1) in the second equivalence. Passing to the Grothendieck group yields (2.2.5).

By (2.2.4), we get (2.2.6) for $i = 0$, and then since flat pullbacks of morphisms preserve the topological filtration [98, Lemma 1.29 (1)], (2.2.6) follows for all $i \geq 0$. By Proposition 2.2.21 2) we get thus that $F^n\mathrm{K}_0^{sg}(X \text{ on } Z) = 0$.

2) By 1) we see that $F^1\mathrm{K}_0^{sg}(C \text{ on } Z) = 0$, or, equivalently, that $\mathrm{K}_0^{sg}(C \text{ on } Z) \simeq \mathrm{gr}^0\mathrm{K}_0^{sg}(C \text{ on } Z)$. The result now follows using (2.2.3), since classes of perfect complexes on X with one-dimensional support and supported on Z generate $\mathbb{Z}\cdot[\mathcal{O}_C]$ (resp. have trivial image) in $\mathrm{gr}^0\mathrm{G}_0(C) = \mathbb{Z}^{N_C}$ for $Z = C$ (resp. $Z \subsetneq C$).

3) For the isomorphism see [98, Proposition 1.24 (2)]. Functoriality follows easily by construction.

4) See [98, Proposition 1.30]. ☐

2.2.6 Local geometry of compound A_n singularities

Recall that a threefold X has a *compound A_n* (abbreviated as *cA_n*) *singularity* at $p \in X$ if the complete local ring $\widehat{\mathcal{O}}_{X,p}$ is isomorphic to a hypersurface singularity given by the equation

$$f = xy + z^{n+1} + wh(x, y, z, w),$$

where h is an arbitrary power series (see [102, Definition 2.1]). It is well-known in the isolated singularity case that the equation f can, after a change of coordinates, be expressed as $f = xy + g(z,w)$ for some $g \in (z,w)^2 \subset k[[z,w]]$ (Morse Lemma [3, Section 11.1]). Conversely, any isolated hypersurface given by the equation $f = xy + g(z,w)$ is a cA_n singularity, where $n = \mathrm{ord}(g) - 1$ and $\mathrm{ord}(g)$ is the lowest term of the power series of $g \in (z,w)^2$ [26, Proposition 6.1 (e)]. Of particular interest are nodal singularities (also called ordinary double points) given complete locally by $xy + zw = 0$ and more generally ADE singularities, see table (2.2.8). Since cA_n singularities are given by one equation they are automatically Gorenstein.

Let A be a complete local ring isomorphic to $k[[x,y,z,w]]/(f)$, where $f \in A$ is of the form $xy + g(z,w)$, for some $g \in k[[z,w]]$. One sees that A has an isolated singularity at the origin if and only if the ring $k[[z,w]]/(g)$, which we denote by A', has an isolated singularity at the origin. The latter condition is equivalent to g being a nonconstant power series with no multiple factors.

Let $\mathrm{br}_0(A')$ be the number of irreducible components of A'. Here 0 stands for the closed point $0 \in \mathrm{Spec}(A')$.

Lemma 2.2.23. *We have a chain of equivalences*

$$\mathbb{Z}^{\mathrm{br}_0(A')-1} \simeq \mathrm{K}_0^{sg}(A') \simeq \mathrm{K}_0^{sg}(A) \simeq \mathrm{Cl}(A). \tag{2.2.7}$$

Proof. Follows from Proposition 2.2.22 2), 3) and 4). □

We call $\mathrm{br}_0(A) := \mathrm{br}_0(A')$, that is the number of irreducible components of A', also *branch number* of A (resp. A'). More generally, if X is a normal threefold with isolated cA_n singularities, we denote by $\mathrm{br}_p(X)$ the branch number of $\widehat{\mathcal{O}}_{X,p}$ and we call it the *branch number of* X *at* p. The (total) *branch number of* X, denoted $\mathrm{br}(X)$, is the sum of the $\mathrm{br}_p(X)$ running over $p \in \mathrm{Sing}(X)$.

It is well-known and easy to see that isolated cA_1 singularities are precisely A_n threefold singularities (Morse Lemma [3, Section 11.1]). More generally, the following table lists the local class groups of the ADE threefold singularities.

Type	Equation	$\mathrm{Cl}(A)$	$\mathrm{br}_0(A)$
A_{2k} $(k \geq 1)$	$x^2 + y^2 + z^2 + w^{2k+1}$	0	1
A_{2k-1} $(k \geq 1)$	$x^2 + y^2 + z^2 + w^{2k}$	$\mathbb{Z}$	2
D_{2k} $(k \geq 2)$	$x^2 + y^2 + z^2 w + w^{2k-1}$	$\mathbb{Z}^2$	3
D_{2k-1} $(k \geq 3)$	$x^2 + y^2 + z^2 w + w^{2k-2}$	$\mathbb{Z}$	2
E_6	$x^2 + y^2 + z^3 + w^4$	0	1
E_7	$x^2 + y^2 + z^3 + zw^3$	$\mathbb{Z}$	2
E_8	$x^2 + y^2 + z^3 + w^5$	0	1

(2.2.8)

The global geometry of cA_n singularities in relation to their class groups, the so-called defect δ and K_{-1} is considered at the end of the next section.

2.3 Class groups and K_{-1}

Throughout this section we assume that our schemes satisfy Orlov's ELF condition [94]. Furthermore, the words curve, surface, threefold are reserved for reduced quasi-projective schemes of dimensions one, two and three respectively. Our goal in this section is to study K_{-1} for curves, surfaces and threefolds. The results for threefolds with cA_n singularities are new, whereas results for curves and surfaces mostly go back to Weibel [124].

For a curve C we denote by $\mathrm{br}_p(C)$ the branch number of $\widehat{\mathcal{O}}_{C,p}$ and call it *branch number of C at p* and by $\mathrm{br}(C) = \sum \mathrm{br}_p(C)$ the (total) *branch number of C*. Let us now consider K_{-1} of a curve.

Proposition 2.3.1. *Let C be a connected curve. Then $\mathrm{K}_{-1}(C)$ is a free abelian group of rank*

$$\mathrm{br}(C) - |\mathrm{Sing}(C)| - N + 1, \tag{2.3.1}$$

where N is the number of irreducible components of C. In particular, if C has at most nodal singularities, then $\mathrm{K}_{-1}(C)$ is free abelian of rank $|\mathrm{Sing}(C)| - N + 1$.

Example 2.3.2. *Let $C \subset \mathbb{P}^2$ be a union of N projective lines intersecting in one point. Then $\mathrm{br}(C) = N$, $|\mathrm{Sing}(C)| = 1$, hence $\mathrm{K}_{-1}(C) = 0$.*

Proof. The statement will follow from a result of Weibel [124, Lemma 2.3 (2)] by comparing the number of loops of the graph constructed in [124] with the number given in the statement. We will, however, give a different proof here using Grothendieck groups of the singularity category.

By Proposition 2.2.22 1) and 2) we see that $\mathbb{K}_0^{sg}(C) \simeq \bigoplus_p \mathrm{K}_0^{sg}(\widehat{\mathcal{O}}_{C,p})$ with each component being a free abelian group of rank $\mathrm{br}_p(C) - 1$ and that $\mathrm{K}_0^{sg}(C)$ is a free abelian group of rank $N - 1$. Using the short exact sequence (2.2.4) it is clear that the rank of $\mathrm{K}_{-1}(C)$ is just $\sum(\mathrm{br}_p(C) - 1) - N - 1$, which is equal to (2.3.1). Furthermore, if C is irreducible, the above argument also shows that $\mathrm{K}_{-1}(C)$ is torsion-free.

To show that $\mathrm{K}_{-1}(C)$ is torsion-free in general, we proceed by induction on the number of irreducible components N of C. Assume that $N \geq 2$ and let $C_0 \subset C$ be an irreducible component of C. We may choose C_0 in such a way that $C - C_0$ is still connected (for that we can take C_0 to be a component with minimal number of intersections with other components). Consider the commutative diagram

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{K}_0^{sg}(C \text{ on } C_0) & \longrightarrow & \mathbb{K}_0^{sg}(C \text{ on } C_0) & \longrightarrow & \mathrm{K}_{-1}(C \text{ on } C_0) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{K}_0^{sg}(C) & \longrightarrow & \mathbb{K}_0^{sg}(C) & \longrightarrow & \mathrm{K}_{-1}(C) & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{K}_0^{sg}(C - C_0) & \longrightarrow & \mathbb{K}_0^{sg}(C - C_0) & \longrightarrow & \mathrm{K}_{-1}(C - C_0) & \longrightarrow & 0. \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
 & & 0 & & 0 & & 0 & &
\end{array} \tag{2.3.2}$$

The rows are exact by (2.2.4). Let us now consider exactness of the columns. By Proposition 2.2.22 2) $\mathrm{K}_0^{sg}(C) \simeq \mathbb{Z}^{N-1}$, $\mathrm{K}_0^{sg}(C - C_0) \simeq \mathbb{Z}^{N-2}$ and $\mathrm{K}_0^{sg}(C \text{ on } C_0) \simeq \mathbb{Z}$ generated by the structure sheaves of the components of C, $C - C_0$ and C_0 respectively, and the maps between the groups are the obvious ones, so that the left column is split exact. By Proposition 2.2.22 1) the middle column is also split exact.

Applying the Snake Lemma to the first two columns we get exactness of the right column. Finally, by the induction hypothesis $\mathrm{K}_{-1}(C - C_0)$ is torsion-free and since $\mathrm{K}_{-1}(C \text{ on } C_0)$ is also torsion-free because the top row is split exact, we obtain that $\mathrm{K}_{-1}(C)$ is torsion-free. □

Recall that the *dual graph* Γ of a nodal curve C is defined to be the following (undirected) graph. Vertices of Γ correspond to irreducible components of C. Edges between distinct vertices correspond to intersections of components. Finally, for every self-intersection point on a component the corresponding vertex has a loop. Usually Γ is decorated by indicating the genus of each component, but we do not need this for our purposes.

The following corollary implies for example that K_{-1} of a nodal cubic (Γ has one vertex with a loop), or any cycle of smooth curves (Γ is a cycle) is $\mathbb{Z}$ while K_{-1} of any tree (that is Γ is a tree) of smooth curves is zero.

Corollary 2.3.3. *Let C be a curve with at most nodal singularities and let $\Gamma = \Gamma(C)$ be the dual graph of C, then*

$$\mathrm{K}_{-1}(C) \simeq \mathbb{Z}^{\lambda},$$

where $\lambda = \lambda(\Gamma)$ is the first Betti number of Γ.

Proof. Both sides are additive for finite disjoint unions so we may assume that C is connected.

By definition of Γ, N is its number of vertices and $|\mathrm{Sing}(C)| = \sum(\mathrm{br}_p(C) - 1)$ is its number of edges. The result follows by Proposition 2.3.1 since

$$1 - \lambda = N - |\mathrm{Sing}(C)|$$

so that $\lambda = |\mathrm{Sing}(C)| - N + 1$. $\square$

Let us consider the higher dimensional case.

Definition 2.3.4. *Let X be normal with at most isolated singularities. We say that X is maximally nonfactorial if the natural map $\mathrm{Cl}(X) \to \bigoplus_p \mathrm{Cl}(\widehat{\mathcal{O}}_{X,p})$ is surjective, where the direct sum runs over all $p \in \mathrm{Sing}(X)$.*

Proposition 2.3.5. *Let X be normal irreducible with at most isolated singularities. Assume that $\mathrm{K}_{-1}(X) = 0$. Then X is maximally nonfactorial.*

Proof. Let $X' = \mathrm{Spec}\left(\prod_{p\in\mathrm{Sing}(X)} \widehat{\mathcal{O}}_{X,p}\right)$. By Proposition 2.2.22 1), $\mathbb{K}_0^{sg}(X) \simeq \mathrm{K}_0^{sg}(X')$.

Since X is irreducible, we have $\mathrm{gr}^0\mathrm{K}_0^{sg}(X) = 0$, so that $\mathrm{K}_0^{sg}(X) = F^1\mathrm{K}_0^{sg}(X)$. Similarly, since X' is a disjoint union of irreducible components, $\mathrm{gr}^0\mathrm{K}_0^{sg}(X') = 0$ and $\mathrm{K}_0^{sg}(X) = F^1\mathrm{K}_0^{sg}(X')$. In particular as $\mathrm{K}_0^{sg}(X) \to \mathbb{K}_0^{sg}(X)$ is surjective by (2.2.4), we also get that $\mathrm{gr}^1\mathrm{K}_0^{sg}(X) \to \mathrm{gr}^1\mathrm{K}_0^{sg}(X')$ is surjective.

By Proposition 2.2.22 3) we have a commutative diagram

$$\begin{array}{ccc} \mathrm{gr}^1\mathrm{K}_0^{sg}(X) & \longrightarrow & \mathrm{gr}^1\mathrm{K}_0^{sg}(X') \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Cl}(X)/\mathrm{Pic}(X) & \longrightarrow & \mathrm{Cl}(X') \end{array} \tag{2.3.3}$$

where we used that $\mathrm{Pic}(X') = 0$. Since we know that the top horizontal arrow is surjective, the bottom horizontal arrow is surjective as well, which is equivalent to X being maximally nonfactorial. $\square$

If the local singularity Grothendieck groups are generated by codimension one cycles, then $\mathrm{K}_{-1}(X)$ is controlled by codimension one cycles as well:

Proposition 2.3.6. *Let X be normal irreducible with at most isolated singularities and such that $F^2\mathrm{K}_0^{sg}(\widehat{\mathcal{O}}_{X,p}) = 0$ for all p in $\mathrm{Sing}(X)$. Then there is an isomorphism $\mathrm{K}_0^{sg}(\widehat{\mathcal{O}}_{X,p}) \simeq \mathrm{Cl}(\widehat{\mathcal{O}}_{X,p})$ for all p in $\mathrm{Sing}(X)$ and an exact sequence*

$$0 \to \mathrm{Pic}(X) \to \mathrm{Cl}(X) \to \bigoplus_{p\in\mathrm{Sing}(X)} \mathrm{Cl}(\widehat{\mathcal{O}}_{X,p}) \to \mathrm{K}_{-1}(X) \to 0. \tag{2.3.4}$$

In particular, X is maximally nonfactorial if and only if $\mathrm{K}_{-1}(X) = 0$.

Proof. We keep the notation of the previous proof. By (2.2.6) we have injective maps

$$F^i\mathrm{K}_0^{sg}(X) \to F^i\mathrm{K}_0^{sg}(X').$$

Thus using $F^2\mathrm{K}_0^{sg}(\widehat{\mathcal{O}}_{X,p}) = 0$ for all p in $\mathrm{Sing}(X)$, so that $F^2\mathrm{K}_0^{sg}(X') = 0$ we see that $F^2\mathrm{K}_0^{sg}(X) =$

0. Therefore diagram (2.3.3) is isomorphic to

$$\begin{array}{ccc} \mathrm{K}_0^{sg}(X) & \longrightarrow & \mathrm{K}_0^{sg}(X') \\ \downarrow \simeq & & \downarrow \simeq \\ \mathrm{Cl}(X)/\mathrm{Pic}(X) & \longrightarrow & \mathrm{Cl}(X') \end{array} \tag{2.3.5}$$

The upper horizontal map is injective with cokernel $\mathrm{K}_{-1}(X)$ by (2.2.4), hence we get a short exact sequence

$$0 \to \mathrm{Cl}(X)/\mathrm{Pic}(X) \to \mathrm{Cl}(X') \to \mathrm{K}_{-1}(X) \to 0,$$

which implies (2.3.4). □

If S is a normal surface, then the exact sequence (2.3.4) holds for S by Proposition 2.2.21 2), which recovers a result of Weibel [124, Corollary 5.4]. Furthermore we have the following result.

Proposition 2.3.7. *If X is normal rational projective surface with rational singularities, then we have an isomorphism* $\mathrm{K}_{-1}(X) \simeq \mathrm{Br}(X)$.

Proof. The proof is a combination of a result of Weibel computing $\mathrm{K}_{-1}(X)$ with a result of Bright computing $\mathrm{Br}(X)$.

Let $\pi : \widetilde{X} \to X$ be a resolution of singularities of X, such that the exceptional divisor $E = \pi^{-1}(\mathrm{Sing}(X))$ is a normal crossing divisor. By Artin [4], E is a tree of smooth rational curves. Let N be the number of irreducible components of E.

By [124, Example 2.13 and Proposition 5.1] there is an exact sequence

$$0 \to \mathrm{Pic}(X) \to \mathrm{Pic}(\widetilde{X}) \to \mathrm{Pic}(E) \to \mathrm{K}_{-1}(X) \to 0. \tag{2.3.6}$$

It is well known that $\mathrm{Pic}(E) \simeq \mathbb{Z}^N$ spanned by the tautological bundles of the components of E. This group is also canonically isomorphic to the free abelian group $\mathbf{E}^*$ generated by the components of the exceptional divisor defined in [22], and comparing (2.3.6) to [22, Proposition 1] (the setup in [22] includes minimality of the resolution, but it is not required in the proof), where we use that $\mathrm{Br}(\widetilde{X}) = 0$ since $\widetilde{X}$ is a smooth projective rational surface yields $\mathrm{K}_{-1}(X) \simeq \mathrm{Br}(X)$.
□

The following result allows to compute K_{-1} of threefolds with isolated compound A_n singularities (in particular for nodal threefolds), in terms of their Picard group, Class group and the branch number defined in subsection 2.2.6.

Corollary 2.3.8. *Let X be normal threefold with at most isolated cA_n singularities. Then we have an exact sequence*

$$0 \to \mathrm{Pic}(X) \to \mathrm{Cl}(X) \to \mathbb{Z}^L \to \mathrm{K}_{-1}(X) \to 0,$$

where $L = \mathrm{br}(X) - |\mathrm{Sing}(X)|$ *is the difference between the branch number and the number of the singular points of* X*. In particular, if* X *has at most nodal singularities, then* $L = |\mathrm{Sing}(X)|$*.*

Proof. Using (2.2.7) we obtain that $\mathrm{Cl}(\widehat{\mathcal{O}}_{X,p}) \simeq \mathbb{Z}^{\mathrm{br}_p(X)-1}$. Since $L = \sum_{p\in \mathrm{Sing}(X)}(\mathrm{br}_p(X) - 1)$, the result follows from Proposition 2.3.6. □

Definition 2.3.9. *Let* X *be a normal threefold with at most isolated* cA_n *singularities. We define the defect* δ *of* X *by*

$$\delta := \mathrm{rk}\,\mathrm{Cl}(X)/\mathrm{Pic}(X).$$

Remark 2.3.10. *Note that the defect is well-defined by Corollary* 2.3.8*. It was first defined by Clemens in [32] for double solids, then by Werner [125] for nodal 3-dimensional hypersurfaces and later it was extended by Rams to 3-dimensional hypersurfaces with ADE singularities [101]. By [32, Corollary 2.32] and [101, Theorem 4.1] one sees that the classical definition of the defect agrees with Definition* 2.3.9*.*

Remark 2.3.11. *Let* X *be as in Definition* 2.3.9*. We can rewrite Corollary* 2.3.8 *as a short exact sequence*

$$0 \to \mathbb{Z}^{\delta} \to \mathbb{Z}^{L} \to \mathrm{K}_{-1}(X) \to 0 \tag{2.3.7}$$

where $L = \mathrm{br}(X) - |\mathrm{Sing}(X)|$*. Explicitly, the first group* $\mathbb{Z}^{\delta} \simeq \mathrm{Cl}(X)/\mathrm{Pic}(X)$ *is generated by the classes of Weil divisors which are not Cartier, the second group* $\mathbb{Z}^{L}$ *is the sum of local class groups of the singular points, and the map between them is given by restricting Weil divisors to the local class groups.*

By definition, X *is factorial if and only if* $\delta = 0$*. On the other hand, if* X *is maximally nonfactorial, then* $\delta = L$*. Conversely, if* $\delta = L$*, then* X *is maximally nonfactorial up to torsion.*

It is worth noticing that if $L = 0$*, then* X *is factorial and maximally nonfactorial at the same time. Indeed, from* (2.3.7) *we see that* $\delta = 0$*, as well as* $\mathrm{K}_{-1}(X) = 0$*. For isolated* cA_n *singularities* $L = 0$ *if and only if all branch numbers of the singular points are equal to one, that is singularities are of type* $xy + g(z, w) = 0$*, where* $g(z, w)$ *is irreducible. For example, this is the case for* A_{2k} *singularities, see* (2.2.8)*.*

We collect examples for known defect of nodal 3-folds.

Example 2.3.12. *If* X *is a nodal quadric threefold in* $\mathbb{P}^4$*, then* $\mathrm{Pic}(X) = \mathbb{Z}$ *generated by the class of the hyperplane section* H*;* $\mathrm{Cl}(X) = \mathbb{Z}^2$*, generated by the two planes* D_1*,* D_2 *passing through the singular point, so that* $H = D_1 + D_2$*. Therefore,* $\delta = 1$*,* $L = 1$*, the first map in* (2.3.7) *is an isomorphism,* $\mathrm{K}_{-1}(X) = 0$ *and* X *is maximally nonfactorial.*

Example 2.3.13. *Let* X *be a nodal hypersurface in* $\mathbb{P}^4$ *or a nodal double cover of* $\mathbb{P}^3$*, which is not the nodal quadric hypersurface in* $\mathbb{P}^4$*. Let* r *be the number of nodes of* X*. The defect* δ *in these cases has been studied in detail and it is known that* $\delta < r$ *(see [38, Definition 1 and Theorem 9] for the hypersurface case and [32, Corollary 2.32] for double solids). Thus by Corollary* 2.3.8 *we get that* $\mathrm{rk}\,\mathrm{K}_{-1}(X) = r - \delta > 0$*, i.e. 3-dimensional nodal hypersurfaces and* *nodal double covers of $\mathbb{P}^3$ are never maximally nonfactorial, except for the 3 dimensional nodal quadric hypersurface.*

Lemma 2.3.14. *Let $\pi : \widetilde{X} \to X$ be a small resolution of a nodal projective threefold with r nodes. Assume that $\widetilde{X}$ is obtained as a blow up of a smooth projective threefold Y in μ points. Let ρ_X, ρ_Y are Picard ranks of X and Y respectively. Then we have*

$$\operatorname{rk} \mathrm{K}_{-1}(X) = r - \delta = r - \mu + \rho_X - \rho_Y,$$

where $\delta = \mu + \rho_Y - \rho_X$ is the defect of X.

Proof. Since $\pi : \widetilde{X} \to X$ is a small resolution, we have that $\mathrm{Cl}(\widetilde{X}) \simeq \mathrm{Cl}(X)$. Moreover, since $\widetilde{X}$ is a smooth blow up of Y at μ points, we see further that $\mathrm{Cl}(\widetilde{X}) \simeq \mathrm{Cl}(Y) \oplus \mathbb{Z}^\mu$. The result is then a direct consequence of Corollary 2.3.8. □

Example 2.3.15. *According to Prokhorov [99, Theorem 7.1], del Pezzo threefolds of degree $1 \le d \le 5$, that is Fano threefolds of Picard rank one and index two, with maximal class group rank are obtained by blowing up $8 - d$ general points P_i on $\mathbb{P}^3$ followed by blowing down proper preimages of lines and twisted cubics passing through the points P_i (the latter contraction is realized as an algebraic variety by taking half-anticanonical model of the blow up). The number of nodes of X is 28 for $d = 1$, 16 for $d = 2$ and $\binom{8-d}{2}$ for $3 \le d \le 5$ [99, Theorem 7.1 (iii)].*

Using Lemma 2.3.14 we see that the rank of K_{-1} is 21 for $d = 1$, 10 for $d = 2$ and $\frac{(8-d)(5-d)}{2}$ for $3 \le d \le 5$. In particular, the cases $1 \le d \le 4$ are not maximally nonfactorial (see also table (2.4.3)).

2.4 Kawamata type semiorthogonal decompositions

In this section X is a Gorenstein projective variety. The following definition is motivated by [76] and [77].

Definition 2.4.1. *We say that X has a Kawamata type semiorthogonal decomposition if*

$$\mathcal{D}^b(X) = \langle \mathcal{A}, \mathcal{B}_1, \ldots, \mathcal{B}_m \rangle$$

is an admissible semiorthogonal decomposition, such that $\mathcal{A} \subset \mathcal{D}^{\mathrm{perf}}(X)$ and the $\mathcal{B}_j$'s are equivalent to $\mathcal{D}^b(R_j)$, where the R_j's are (possibly noncommutative) finite-dimensional k-algebras.

Remark 2.4.2. *Note that a smooth projective variety X trivially admits a Kawamata type decomposition. Indeed, in this case $\mathcal{D}^{\mathrm{perf}}(X) = \mathcal{D}^b(X)$ and we can set $m = 0$, $\mathcal{A} = \mathcal{D}^b(X)$.*

Remark 2.4.3. *Any admissible decomposition of $\mathcal{D}^b(X)$ into components which are subcategories of $\mathcal{D}^{\mathrm{perf}}(X)$ and components equivalent to $\mathcal{D}^b(R)$ can be rearranged to make a Kawamata decomposition. This follows from a result of Bondal and Kapranov [17, Lemma 1.9], which implies more generally that any admissible semiorthogonal decomposition can be mutated.*

Theorem 2.4.4. *If X admits a Kawamata type semiorthogonal decomposition*

$$\mathcal{D}^b(X) = \langle \mathcal{A}, \mathcal{B}_1 \ldots, \mathcal{B}_m \rangle,$$

then the following holds.
1) There is an admissible semiorthogonal decomposition

$$\mathcal{D}^{\mathrm{perf}}(X) = \langle \mathcal{A}, \mathcal{B}_1 \cap \mathcal{D}^{\mathrm{perf}}(X), \ldots, \mathcal{B}_m \cap \mathcal{D}^{\mathrm{perf}}(X) \rangle,$$

and $\mathcal{B}_j \cap \mathcal{D}^{\mathrm{perf}}(X)$ is equivalent to $\mathcal{D}^{\mathrm{perf}}(R_j)$ for all $1 \leq j \leq m$. The Serre functor on $\mathcal{D}^{\mathrm{perf}}(X)$ induces Serre functors on $\mathcal{A}$ and on all $\mathcal{B}_j \cap \mathcal{D}^{\mathrm{perf}}(X)$.
2) The finite-dimensional k-algebras R_j are Gorenstein.
3) There is an equivalence of singularity categories $\mathcal{D}^{\mathrm{sg}}(X) \simeq \langle \mathcal{D}^{\mathrm{sg}}(R_1), \ldots, \mathcal{D}^{\mathrm{sg}}(R_m) \rangle$. Furthermore, if X has only isolated singularities, then the decomposition above is completely orthogonal, that is

$$\mathcal{D}^{\mathrm{sg}}(X) \simeq \mathcal{D}^{\mathrm{sg}}(R_1) \oplus \ldots \oplus \mathcal{D}^{\mathrm{sg}}(R_m) \simeq \mathcal{D}^{\mathrm{sg}}(R_1 \times \ldots \times R_m). \tag{2.4.1}$$

Proof. 1) The decomposition and its admissibility follows immediately from Orlov's characterization of perfect complexes as homologically finite objects [95, Proposition 1.10 and 1.11]. Moreover, by the analogous characterization of $\mathcal{D}^{\mathrm{perf}}(R_j)$ in $\mathcal{D}^b(R_j)$ [67, Proposition 2.18] and by admissibility of $\mathcal{B}_j$ it is easy to see that $\mathcal{D}^{\mathrm{perf}}(R_j) = \mathcal{B}_j \cap \mathcal{D}^{\mathrm{perf}}(X)$. By Lemma 2.2.12 it follows that the components $\mathcal{A}$ and $\mathcal{B}_j \cap \mathcal{D}^{\mathrm{perf}}(X)$ have Serre functors.
2) By 1) we see that $\mathcal{D}^{\mathrm{perf}}(R_j)$ has a Serre functor, and by Lemma 2.2.14 this is equivalent to R_j being Gorenstein.
3) The decomposition of $\mathcal{D}^{\mathrm{sg}}(X)$ follows by [95, Proposition 1.10]. Let us assume now that X has isolated singularities. By Proposition 2.2.17, $\mathcal{D}^{\mathrm{sg}}(X)$ is a Calabi-Yau category. By a standard argument going back to Bridgeland [21] it is easy to see that in this case decomposition is completely orthogonal. The second equivalence in (2.4.1) is clear. ☐

Corollary 2.4.5. *If X admits a Kawamata type semiorthogonal decomposition, then $\mathcal{D}^{\mathrm{sg}}(X)$ is idempotent complete, or, equivalently,* $\mathrm{K}_{-1}(X) = 0$.

Proof. This follows from Theorem 2.4.4 using Lemma 2.2.2 and Lemma 2.2.19. The final equivalence is Lemma 2.2.20. ☐

Example 2.4.6. *If X has trivial canonical bundle, then it admits a Kawamata decomposition if and only if X is smooth. Indeed if we assume that X has a Kawamata decomposition, it follows by Theorem* 2.4.4 *1) that there is an induced semiorthogonal decomposition of $\mathcal{D}^{\mathrm{perf}}(X)$. While the Serre functor on $\mathcal{D}^{\mathrm{perf}}(X)$ is just the shift by $n = \dim(X)$, we see that for a finite-dimensional algebra as in Definition* 2.4.1*, we have that*

$$\mathrm{Hom}_{R_j}(R_j, R_j) = \mathrm{Ext}^n_{R_j}(R_j, R_j)^*,$$

which is only possible if $\dim(X) = 0$ *or* $m = 0$*. In both cases* X *is smooth.*

The next proposition shows that admissibility is automatic in the case when we have only one algebra.

Proposition 2.4.7. *Assume that* X *has a semiorthogonal decomposition*

$$\mathcal{D}^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle,$$

where $\mathcal{A} \subset \mathcal{D}^{\mathrm{perf}}(X)$*,* $\mathcal{B} \simeq \mathcal{D}^b(R)$ *(resp.* $\mathcal{B} \subset \mathcal{D}^{\mathrm{perf}}(X)$*,* $\mathcal{A} \simeq \mathcal{D}^b(R)$*), and* R *is a finite-dimensional algebra. Then* $\mathcal{A}$ *and* $\mathcal{B}$ *are admissible subcategories in* $\mathcal{D}^b(X)$ *so that the semiorthogonal decomposition* $\mathcal{D}^b(X) = \langle \mathcal{A}, \mathcal{B} \rangle$ *(resp.* $\mathcal{D}^b(X) = \langle \mathcal{B} \otimes \omega_X, \mathcal{A} \rangle$*) is of Kawamata type.*

Proof. This follows from Lemma 2.2.15. □

Remark 2.4.8. *Kawamata type decompositions generalize tilting objects in the following sense. Recall that a classical tilting object* $\mathcal{E}$ *of* $\mathcal{D}(\mathrm{Qcoh}(X))$ *is a perfect complex of* $\mathcal{D}(\mathrm{Qcoh}(X))$*, such that it generates* $\mathcal{D}(\mathrm{Qcoh}(X))$ *(i.e. if* $\mathrm{Hom}(\mathcal{E}, \mathcal{F}) = 0$*, then* $\mathcal{F} \simeq 0$*) and such that* $\mathrm{Hom}(\mathcal{E}, \mathcal{E}[i]) = 0$ *for all* $i \neq 0$*. It is well known that, if* $\mathcal{D}(\mathrm{Qcoh}(X))$ *possesses a classical tilting object* $\mathcal{E}$*, then there is an equivalence* $\mathcal{D}(\mathrm{Qcoh}(X)) \simeq \mathcal{D}(\mathrm{Mod}\text{-}R)$ *which restricts to an equivalence* $\mathcal{D}^b(X) \simeq \mathcal{D}^b(R)$*, where* R *is the finite-dimensional algebra* $\mathrm{End}(\mathcal{E})$ *(see e.g. [64, Theorem 7.6 (2)]). This means that* X *has a Kawamata semiorthogonal decomposition with trivial* $\mathcal{A} \subset \mathcal{D}^{\mathrm{perf}}(X)$ *part as soon as* $\mathcal{D}(\mathrm{Qcoh}(X))$ *has a classical tilting object.*

We collect the known examples of Gorenstein projective varieties with Kawamata type semiorthogonal decompositions. We start in dimension one.

Theorem 2.4.9 (Burban [25])**.** *Let* X *be a nodal tree of projective lines, that is a connected nodal curve with all irreducible components isomorphic to* $\mathbb{P}^1$ *and with the dual graph* Γ *of* X *forming a tree. Then* $\mathcal{D}^b(X)$ *has a tilting object, and furthermore admits a Kawamata type semiorthogonal decomposition*

$$\mathcal{D}^b(X) = \langle \mathcal{O}_X, \mathcal{D}^b(R_\Gamma) \rangle.$$

The algebra R_Γ *is the path algebra of the quiver* Q *with relations, obtained by the following construction from* Γ*:* Q *has the same vertices as* Γ *and for each two vertices* p*,* q *in* Γ *connected by an edge there is an arrow* a *from* p *to* q *and an arrow* a^* *from* q *to* p *in* Q*. The relations are that all compositions* aa^* *and* a^*a *are set equal zero.*

Proof. The proof is the same as that of Theorem 2.1 in [25], where only chains of projective lines are considered. □

Example 2.4.10. *Let* $X = X_1 \cup X_2$ *be the* A_2 *tree of projective lines, that is a union of 2 copies of* $\mathbb{P}^1$ *intersecting transversely. Then the algebra* R_Γ *in Theorem 2.4.9 has the form*

$$R: \qquad 1 \underset{a}{\overset{a^*}{\leftrightarrows}} 2 \qquad a^*a = 0, \quad aa^* = 0. \tag{2.4.2}$$

By Theorem 2.4.4 we have $\mathcal{D}^{\mathrm{sg}}(X) \simeq \mathcal{D}^{\mathrm{sg}}(R) \simeq \underline{\mathrm{MCM}}(R)$ *(we used Buchweitz' equivalence between the singularity category and the stable category of MCM modules in the Gorenstein case for the second equivalence). On the other hand, by [94, Proposition 1.14], we have an equivalence*

$$\mathcal{D}^{\mathrm{sg}}(X) \simeq \mathcal{D}^{\mathrm{sg}}(A)$$

where $A = k[x, y]/(xy)$ *is the ring considered in Example 2.2.18.*

Explicitly the generators of the singularity category considered in Example 2.2.18 correspond to the two MCM R-modules which are the two simple modules given by the vertices of the quiver.

Corollary 2.4.11. *Let* C *be a connected nodal projective curve such that all its irreducible components are rational curves. Then the following are equivalent:*
1) C *is a nodal tree of projective lines.*
2) $\mathcal{D}^b(C)$ *admits a Kawamata type semiorthogonal decomposition.*
3) $\mathrm{K}_{-1}(C) = 0$.

Proof. This is a direct consequence of Theorem 2.4.9, Corollary 2.4.5 and Corollary 2.3.3. □

The following result gives a source of examples of Kawamata type semiorthogonal decompositions in dimension two.

Theorem 2.4.12 (Karmazyn-Kuznetsov-Shinder [75])**.** *Let* X *be a projective Gorenstein toric surface. Let* $n_1, \ldots, n_m$ *be the orders of the cyclic quotient singularities of* X*. Then* X *has a Kawamata type semiorthogonal decomposition if and only if* $\mathrm{K}_{-1}(X) = 0$ *and in this case the decomposition is of the form*

$$\mathcal{D}^b(X) \simeq \langle \mathcal{A}, \mathcal{D}^b(R_1), \ldots \mathcal{D}^b(R_m) \rangle,$$

where the category $\mathcal{A} \subset \mathcal{D}^{\mathrm{perf}}(X)$ *is a collection of exceptional objects and such that* $R_i = k[z]/(z^{n_i})$.

Proof. By Proposition 2.3.7 we have $\mathrm{Br}(X) = \mathrm{K}_{-1}(X)$. If $\mathrm{K}_{-1}(X) = 0$, the semiorthogonal decomposition in Theorem 2.4.12 is [75, Corollary 5.10] and admissibility of the components is [75, Theorem 2.12].

Conversely, existence of a Kawamata type decomposition implies $\mathrm{K}_{-1}(X) = 0$ by Corollary 2.4.5. □

For threefolds, we have the following two Fano examples due to Kawamata.

Example 2.4.13 (Kawamata)**.** *(1) Let* X *be the nodal quadric threefold in* $\mathbb{P}^4$ *with the equation* $xy - zw = 0$*. In [76, Example 5.6] (see also [77, Example 7.1]) it has been shown that there is an admissible semiorthogonal decomposition*

$$\mathcal{D}^b(X) \simeq \langle \mathcal{O}_X(-2H), \mathcal{O}_X(-H), \mathcal{D}^b(R), \mathcal{O}_X \rangle,$$

where $\mathcal{O}_X(H)$ is a hyperplane section bundle and R is the same algebra as (2.4.2) *in Example* 2.4.10*. By Remark 2.4.3, X has a Kawamata type decomposition.*
(2) Let $\widetilde{X}$ be the blow up of two points in $\mathbb{P}^3$ and let $L \subset \widetilde{X}$ be the strict transform of a line passing through the two points. Let X be the contraction of L to a node given by the half-anticanonical embedding in $\mathbb{P}^7$. By [77, Example 7.2] X has a Kawamata type decomposition

$$\mathcal{D}^b(X) \simeq \langle \mathcal{A}, \mathcal{D}^b(R) \rangle \simeq \langle \mathcal{O}_X(C_1), \ldots, \mathcal{O}_X(C_5), \mathcal{D}^b(R) \rangle,$$

where the $\mathcal{O}_X(C_i)$'s are line bundles on X which are push-forwards of line bundles from $\widetilde{X}$, and R is again the algebra (2.4.2)*.*

Remark 2.4.14. *Let us contemplate here on the fact that the algebra occurring in Examples* 2.4.13 *(1) and (2) coincides with the algebra which shows up in the union C of 2 rational curves intersecting at a node (Example* 2.4.10*). This observation is related to Knörrer periodicity. More concretely, the singularity category of C will agree with the singularity category of Kawamata's examples via Knörrer periodicity (Theorem* 2.2.16*). On the other hand, Knörrer periodicity can be realized via a blow up construction (see [98, Proof of Proposition 1.30] and Remark 2.5.2) which provides an explicit link between Kawamata type decompositions of ordinary double points with the same parity.*

More generally, this viewpoint of blowing up projective Gorenstein schemes at locally complete intersection subschemes will provide further examples of Kawamata type semiorthogonal decomposition, as shown in Section 2.5.

In the next two examples we consider typical singular threefolds: hypersurfaces, double covers and contractions of blow ups.

Example 2.4.15. *Nodal hypersurfaces in $\mathbb{P}^4$ of degree $d \geq 3$ and nodal double covers of $\mathbb{P}^3$ branched in a surface of degree at least four have no Kawamata type decomposition by Example* 2.3.13 *and Corollary* 2.4.5*.*

This generalizes an example of Kawamata [77, Example 7.8], constructed as follows. One considers a cubic threefold with two nodes $p, q \in X_0$; it is well-known that such cubics are always factorial. Let X be the blow up of q. In the commutative square from (2.3.4)

$$\begin{array}{ccc} \mathrm{Cl}(X) & \longrightarrow & \mathrm{Cl}(\widehat{\mathcal{O}}_{X,p}) \\ \downarrow & & \downarrow \\ \mathrm{Cl}(X_0) & \longrightarrow & \mathrm{Cl}(\widehat{\mathcal{O}}_{X_0,p}) \oplus \mathrm{Cl}(\widehat{\mathcal{O}}_{X_0,q}) \end{array}$$

the bottom horizontal map is zero, the right vertical map is an embedding of a direct summand, hence the top horizontal map is also zero, so that X is factorial as well. From (2.3.4) *we deduce that $\mathrm{K}_{-1}(X) = \mathbb{Z}$, so that X has no Kawamata type decomposition by Corollary 2.4.5.*

Example 2.4.16. *Del Pezzo threefolds as in Example* 2.3.15 *of degree $1 \leq d \leq 4$ have no Kawamata type decomposition. This follows from Corollary* 2.4.5*.*

This relates to a question of Kawamata about derived categories of blow ups of $\mathbb{P}^3$ *in more than* 2 *points [77, Remark 7.5]. We have shown that the half-anticanonical contraction of a blow up of* $\mathbb{P}^3$ *in* 4 *or more points has no Kawamata type decomposition. The remaining case, that is the nodal del Pezzo threefold of rank* 5*, seems to be the most interesting one, as we cannot detect obstructions with our methods. The following table gives a summary:*

$d(X)$	$\lvert\mathrm{Sing}(X)\rvert$	$\mathrm{rk}\,\mathrm{Pic}(X)$	$\mathrm{rk}\,\mathrm{Cl}(X)$	$\mathrm{rk}\,\mathrm{K}_{-1}(X)$	*Kawamata decomp.*
1	*28*	*1*	*8*	*21*	*No*
2	*16*	*1*	*7*	*10*	*No*
3	*10*	*1*	*6*	*5*	*No*
4	*6*	*1*	*5*	*2*	*No*
5	*3*	*1*	*4*	*0*	*?*
6	*1*	*2*	*3*	*0*	*Yes*

(2.4.3)

Here the $d = 6$ *case refers to Example* 2.4.13 *2).*

2.5 Kawamata decompositions, K_{-1} and blow ups

We start with the following well-known result.

Theorem 2.5.1 (Thomason, Orlov). *Let* X *be a Gorenstein projective variety and* $Z \subset X$ *a locally complete intersection closed subvariety of pure codimension* c*. Let* $\pi : \widetilde{X} \to X$ *be the blow up of* X *with center* Z*. Then there is an admissible semiorthogonal decomposition*

$$\mathcal{D}^b(\widetilde{X}) = \langle \underbrace{\mathcal{D}^b(Z), \ldots, \mathcal{D}^b(Z)}_{c-1}, \mathcal{D}^b(X) \rangle$$

and for all $j \in \mathbb{Z}$ *we have*

$$\mathrm{K}_j(\widetilde{X}) \simeq \mathrm{K}_j(X) \oplus \mathrm{K}_j(Z)^{\oplus(c-1)}.$$

Proof. The semiorthogonal decomposition is proved in the same way as [93] (see also [14, Theorem 6.9] or [69, Corollay 3.4]). Let us give just a few words on the well-definedness of the functors involved.

Indeed, note that all the morphisms $Z \subset X$, $E \subset \widetilde{X}$, where E is the exceptional locus of π, $p : E \to Z$, and $\pi : \widetilde{X} \to X$ are proper of finite Tor dimension. This is because the first two morphisms are regular embeddings and the morphism $p : E \to Z$ is a projective bundle. For the blow up $\pi : \widetilde{X} \to X$, we can write it locally as a composition $\widetilde{U} \to \mathbb{P}(\mathcal{E}) \to U$ of a regular embedding and a projective bundle, where $\mathcal{E}$ is a vector bundle on $U \subset X$ such that the zero locus of global section $0 \neq s \in H^0(\mathcal{E}^\vee)$ coincides with Z. Thus the pushforward and pull-back functors of these morphisms on $\mathcal{D}^b$ are well-defined.

Finally the decomposition for K-theory is proved by Thomason [117, Theorem 2.1], and it also can be deduced from the semiorthogonal decomposition lifted to dg-enhancements of the relevant categories, and applying Schlichting's machinery [107, 106]. $\square$

Remark 2.5.2. *In this paper we mostly deal with isolated singularities, however the blow up of a smooth variety in a center with isolated singularties does not necessarily have isolated singularities. For example, if we blow up $\mathbb{A}^3$ at the thick point given by the ideal (x, y, z^2), then one can see that the singular locus of the blow up is 1-dimensional.*

A local computation shows however that, if $Z \subset X$ is a locally complete intersection of codimension 2 in a smooth variety X and such that Z has at most isolated hypersurface singularities given complete locally by an ideal $\big(f(x_1, \ldots, x_{n-1}), x_n\big) \subset k[[x_1, \ldots, x_n]]$, then the blow up $\widetilde{X} \to X$ along Z has at most isolated hypersurface singularities given by the ideal $\big(x_n \cdot x_{n+1} + f(x_1, \ldots, x_{n-1})\big) \subset k[[x_1, \ldots, x_{n+1}]]$.

Corollary 2.5.3. *Under the conditions of Theorem 2.5.1 if both X and Z admit Kawamata decompositions, so does $\widetilde{X}$.*

Proof. While all the components are admissible in $\mathcal{D}^b(\widetilde{X})$, we can use Remark 2.4.3 to rearrange them to obtain the form as in Definition 2.4.1. □

Corollary 2.5.4. *Let X be a smooth projective threefold and let $C \subset X$ be a disjoint union of nodal curves such that each irreducible components of C is a rational curve. Then $\widetilde{X}$ admits a Kawamata type semiorthogonal decomposition if and only if C is a disjoint union of nodal trees of smooth rational curves.*

Proof. If C is a disjoint union of nodal trees of smooth rational curves, then the blow up has a Kawamata type decomposition by Corollary 2.5.3 and Theorem 2.4.9.

Conversely, if the blow up admits a Kawamata type decomposition, then $\mathrm{K}_{-1}(\widetilde{X}) = 0$ by Corollary 2.4.5 hence $\mathrm{K}_{-1}(C) = 0$ by Theorem 2.5.1 and finally C is a nodal tree by Corollary 2.4.11. □

Example 2.5.5. *If X is a smooth projective threefold, and C is a disjoint union of nodal trees of projective lines, then the blow up $\widetilde{X} = \mathrm{Bl}_C(X)$ is a threefold with ordinary double points (see Remark* 2.5.2*) and by Corollary 2.5.4 it admits a Kawamata type semiorthogonal decomposition.*

On the other hand, if C is nodal and irreducible (of arbitrary genus), then the blow up of X in C does not have a Kawamata type decomposition by Corollary 2.4.5 since $\mathrm{K}_{-1}(\widetilde{X}) = \mathrm{K}_{-1}(C) \neq 0$ *where we used Theorem 2.5.1 and Corollary 2.3.3.*

Chapter 3

Appendix: cdh-topology and K-theory

In a series of papers [33, 34, 35, 36, 37] the interplay between K-theory and cdh-cohomology is studied. This relation has lead to positive and negative answers of various conjectures and questions in K-theory, at least in the characteristic 0 case. Examples involve a proof of Weibel's conjecture on the bound of negative K-groups [33], a proof of a conjecture of Vorst [37] and a counterexample to a question of Bass [35].

An important part of this collection of papers are formulas for K_0 and negative K-groups in terms of cdh-cohomology groups in "nice" cases (e.g. Theorem 3.5.6). On the other hand, cdh-cohomology has properties such as Mayer-Vietoris long exact sequences for abstract blow-ups and natural isomorphisms to Zariski cohomology for smooth schemes evaluated at Kähler differentials (Corollary 3.4.4 and Prop. 3.5.1). These two properties make such cohomology groups computable in many examples. This is one of the main approaches for the computation of algebraic K-Theory of singular varieties.

Notation and conventions

Throughout this chapter we assume that k is a field of characteristic 0. We denote by Sch/k the category of separated schemes of finite type over k. We say that $X \in \mathrm{Sch}/k$ is a variety, if it is reduced, but not necessarily irreducible. We denote further by $(\mathrm{Sch}/k)_{\mathrm{Zar}}$ the big Zariski site, that is the category Sch/k endowed with the Zariski topology and by $(X)_{\mathrm{Zar}}$ the small Zariski site over a fixed scheme $X \in \mathrm{Sch}/k$, i.e. the category of open immersions of X endowed with the Zariski topology. Denote further by Ab the category of abelian groups and by $\mathrm{Vect}(k)$ the category of k-vector spaces. Finally, denote by $\mathrm{Sh}(X)$ the category of abelian sheaves over $(X)_{\mathrm{Zar}}$ and by $\Gamma : \mathrm{Sh}(X) \to \mathrm{Ab}$ the global section functor.

3.1 Semi-simplicial schemes and hyperresolutions

Semi-simplicial schemes

Let us remind the reader about semi-simplicial schemes and related notions. We follow [39, 59, 97].

A semi-simplicial scheme $X_\bullet$ is a family $(X_n)_{n\in\mathbb{Z}_{\geq 0}}$ of schemes $X_n \in \mathrm{Sch}/k$ together with morphisms $\delta_i : X_{n+1} \to X_n$, called projection maps, where $0 \leq i \leq n$ and such that $\delta_i \circ \delta_j = \delta_{j-1} \circ \delta_i$, for all $0 \leq i < j \leq n+1$. One can visualize $X_\bullet$ as

$$\ldots\ X_2 \overset{\delta_2}{\underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}}} X_1 \underset{\delta_0}{\overset{\delta_1}{\rightrightarrows}} X_0,$$

where the δ_i statisfy the commutative relation described above. Note that every simplicial scheme (see [39] for the definition) can be viewed as semi-simplicial scheme by forgetting the section maps (also called degeneration maps).

A morphism of semi-simplicial schemes $f_\bullet : X_\bullet \to Y_\bullet$ is a family of morphisms $\{f_n : X_n \to Y_n\}_{n\in\mathbb{Z}_{\geq 0}}$, where all the f_n's commute with respect to the projection maps of $X_\bullet$ and $Y_\bullet$. For any scheme $S \in \mathrm{Sch}/k$, one can define the constant semi-simplicial scheme $S_\bullet$ by setting $S_n = S$ for all $n \in \mathbb{Z}_{\geq 0}$ and by setting all projection maps to be the identity map. An augmentation $X_\bullet \to S$, where $S \in \mathrm{Sch}/k$ and $X_\bullet$ is a semi-simplicial scheme, is a morphism of semi-simplicial schemes $X_\bullet \to S_\bullet$. Note that for all semi-simplicial schemes $X_\bullet$ there is an augmentation $X_\bullet \to \mathrm{Spec}(k)$.

For a scheme $S \in \mathrm{Sch}/k$, one can also just consider the semi-simplicial scheme with S in degree 0 and empty set in all non-negative degrees. This describes a fully faithful embedding of Sch/k into the category of semi-simplicial schemes.

A semi-simplicial scheme $X_\bullet$ is called smooth, affine, quasi-projective, etc., if all its components X_n of $X_\bullet$ are smooth, affine, quasi-projective, etc. and similarly a morphism $f_\bullet : X_\bullet \to Y_\bullet$ of semi-simplicial schemes is called affine, finite, proper etc., if all its components $f_n : X_n \to Y_n$ are affine, finite, proper etc.

Let us give two important examples of semi-simplicial schemes:

Example 3.1.1. *(i) Let X be a scheme (or more generally a topological space) and let $\{U_i\}_{i\in I}$ be an open cover of X with finite ordered index set I. Define the semi-simplicial scheme $\mathcal{U}_\bullet$ componentwise as*

$$\mathcal{U}_i := \coprod_{i_0<\ldots<i_n} U_{i_0\ldots i_n} := \coprod_{i_0<\ldots<i_n} U_{i_0} \times_X \ldots \times_X U_{i_n},$$

with projection maps given by the inclusion maps $U_{i_0\ldots i_n} \to U_{i_0\ldots i_{k-1}i_{k+1}\ldots i_n}$. It is clear that the inclusion maps $\mathcal{U}_i \to X$ define an augmentation $\epsilon : \mathcal{U}_\bullet \to X$. We call a semi-simplicial scheme of this form an open hypercover, or Čech simplicial scheme, of X.

(ii) Let G be a finite group acting linearly on a smooth quasi-projective variety M over k and

let M/G be the quotient variety. We define the semi-simplicial scheme $\mathrm{Ner}(G, M)$ *as*

$$\mathrm{Ner}(G, M)_n = G^n \times M,$$

with projection maps $\delta_i : G^{n+1} \times M \to G^n \times M$ *defined as follows:*

$$\begin{aligned} \delta_0 &: (g_1, \ldots, g_{n+1}, m) \mapsto (g_2, \ldots, g_{n+1}, m) \\ \delta_{n+1} &: (g_1, \ldots, g_{n+1}, m) \mapsto (g_1, \ldots, g_n, g_{n+1}m) \\ \delta_i &: (g_1, \ldots, g_{n+1}, m) \mapsto (g_1, \ldots, g_i g_{i+1}, \ldots, g_{n+1}, m) \end{aligned}$$

for $1 \leq i \leq n$. *The quotient map* $M \to M/G$ *defines an augmentation* $\epsilon : \mathrm{Ner}(G, M) \to M/G$. *Moreover* $\mathrm{Ner}(G, M)$ *is smooth and* $\epsilon : \mathrm{Ner}(G, M) \to M/G$ *is finite. Note further that the* G*-action on* M *lifts to a* G*-action on* $\mathrm{Ner}(G, M)$*, which is defined componentwise by* $g(g_1, \ldots, g_n, m) = (gg_1g^{-1}, \ldots, gg_ng^{-1}, gm)$.

Sheaves on semi-simplicial schemes and cohomology

A sheaf $\mathcal{F}^\bullet$ on a semi-simplicial scheme $X_\bullet$ is a family of sheaves $(\mathcal{F}^n)_{n\in\mathbb{Z}_{\geq 0}}$, where $\mathcal{F}^n$ is a sheaf on X_n, together with morphisms of sheaves $u_i : \mathcal{F}^n \to \delta_{i*}\mathcal{F}^{n+1}$. A morphism of sheaves $F : \mathcal{F}^\bullet \to \mathcal{G}^\bullet$ on $X_\bullet$ is a family of morphisms $\{f^n : \mathcal{F}^n \to \mathcal{G}^n\}_{n\in\mathbb{Z}_{\geq 0}}$ satisfying the commutative relation $u_i(\mathcal{G}) \circ f^n = \delta_{i*}(f^{n+1}) \circ u_i(\mathcal{F})$. Note that the pushforward (resp. pull-back) of a sheaf $\mathcal{F}_\bullet$ on $Y_\bullet$ (resp. $X_\bullet$) with respect to a morphism of semi-simplicial schemes $f : Y_\bullet \to X_\bullet$ is naturally defined by $f_*\mathcal{F}^\bullet = (f_{n*}\mathcal{F}^n)_{n\in\mathbb{Z}_{\geq 0}}$ (resp. $f^*\mathcal{F}^\bullet = (f_n^*\mathcal{F}^n)$) with morphisms $f_{n*}(u_i)$ (resp. ${f_n}^*(u_i)$).

Remark 3.1.2. *Let* $\mathcal{F}$ *be a sheaf on a scheme* $X \in \mathrm{Sch}/k$ *and let* $\mathcal{U}_\bullet$ *be an open hypercover of* X*. One can induce a sheaf on* $\mathcal{U}$ *by* $\mathcal{F}$ *as follows. Define by* $\mathcal{F}|_{\mathcal{U}_\bullet}$ *the sheaf on* $\mathcal{U}_\bullet$ *given by the family* $(\mathcal{F}|_{\mathcal{U}_n})_{n\in\mathbb{Z}_{\geq 0}}$ *and by morphisms* $\mathcal{F}|_{\mathcal{U}_n} \to \delta_{i*}\mathcal{F}|_{\mathcal{U}_{n+1}}$ *induced by the morphisms* $\delta_i : \mathcal{U}_{n+1} \to \mathcal{U}_n$.

By the same construction, one can restrict a sheaf $\mathcal{F}$ *on the big Zariski site* $(\mathrm{Sch}/k)_{\mathrm{Zar}}$ *to a sheaf* $\mathcal{F}|_{X_\bullet}$ *on a semi-simplicial scheme* $X_\bullet$ *(see section 3.2).*

It is easy to see that the category of sheaves on $X_\bullet$, denoted $\mathrm{Sh}(X_\bullet)$, is abelian and has enough injectives. We can thus speak about the derived category $\mathcal{D}^*(X_\bullet) = \mathcal{D}^*(\mathrm{Sh}(X_\bullet))$ of $X_\bullet$, where $* = -, +, b$. For a moprhism of semi-simplicial schemes $f : Y_\bullet \to X_\bullet$, we denote by $Rf_* : \mathcal{D}^+(Y_\bullet) \to \mathcal{D}^+(X_\bullet)$ the induced push-forward functor.

Given an augmentation $a : X_\bullet \to S$ and a sheaf (or more generally a bounded below complex of sheaves) $\mathcal{F}$ on S, we define the pullback functor of a by $a^{-1}\mathcal{F} = (a_n^{-1}\mathcal{F})_{n\in\mathbb{Z}_{\geq 0}}$ with obvious maps $a_n^{-1}\mathcal{F} \to \delta_{i*}a_{n+1}^{-1}\mathcal{F}$ (compare to Remark 3.1.2). Moreover, for a sheaf (or more generally bounded below complex of sheaves) $\mathcal{F}^\bullet$ one can define the push-forward functor of a by

$$Ra_*\mathcal{F} = \mathrm{Tot}(Ra_{0*}\mathcal{F}^0 \xrightarrow{d_0} Ra_{1*}\mathcal{F}^1 \xrightarrow{d_1} \ldots),$$

where the differentials d_n are defined by taking the alternating sum of the maps induced by the maps $\mathcal{F}^n \to \delta_{i*}\mathcal{F}^{n+1}$ and Tot denotes the construction of the total complex of a double complex. The cohomology of $\mathcal{F} \in \mathcal{D}^+(X_\bullet)$ is defined by

$$H^i(X_\bullet, \mathcal{F}) := R^i\Gamma(Ra_*\mathcal{F}) = H^i(\mathrm{Tot}(R\Gamma(X_0, \mathcal{F}^0) \to R\Gamma(X_1, \mathcal{F}^1) \to \ldots)).$$

for all $i \geq 0$.

Example 3.1.3. *Let $a : \mathcal{U}_\bullet \to X$ be an affine open hypercover of $X \in \mathrm{Sch}/k$ and let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-module. The pullback $a^{-1}\mathcal{F}$ is given by $\mathcal{F}|_{\mathcal{U}_n}$ in each component and the corresponding maps are given by restriction. We get thus*

$$\begin{aligned} H^i(\mathcal{U}_\bullet, a^{-1}\mathcal{F}) &= H^i(\mathrm{Tot}(R\Gamma(\mathcal{U}_0, \mathcal{F}) \to R\Gamma(\mathcal{U}_1, \mathcal{F}) \to \ldots)) \\ &\simeq H^i(\Gamma(\mathcal{U}_0, \mathcal{F}) \to \Gamma(\mathcal{U}_1, \mathcal{F}) \to \ldots) \\ &\simeq H^i(X, \mathcal{F}) \end{aligned}$$

for all $i \geq 0$, where we used that $\mathcal{U}_\bullet$ is affine in the second equality and the third equality follows as the third complex is just computing Čech cohomology.

Let M be a smooth quasi-projective variety over k and let G be a finite group acting on M and denote by $\sigma : G \times M \to M$ the action and by $p : G \times M \to M$ the projection. Recall that a G-equivariant sheaf on M is a sheaf $\mathcal{F}$ on M together with an isomoprhism $\varphi : p^*\mathcal{F} \xrightarrow{\sim} \sigma^*\mathcal{F}$ satisfying the cocycle condition on. We can view $\mathcal{F}$ as a sheaf $\mathcal{F}^\bullet$ on $\mathrm{Ner}(G, M)$ (Example 3.1.1 (ii)) as follows. Set $\mathcal{F}^0 = \mathcal{F}$, where we forget the G-equivariant structure of $\mathcal{F}$ here. For $n > 0$ we can define inductively $\mathcal{F}^n = \delta_0^*\mathcal{F}^{n-1}$. Note that, by definition of G-equivariant sheaves, $\mathcal{F}^n \simeq \delta_i^*\mathcal{F}^{n-1}$ for all i. The morphisms $u_i : \mathcal{F}^{n-1} \to \delta_{i*}\mathcal{F}^n$ are then defined in the obvious way. We denote by abuse of notation the sheaf $\mathcal{F}^\bullet$ on $\mathrm{Ner}(G, M)$ by $\mathcal{F}$.

The following result is well-known.

Proposition 3.1.4. *Let M, G and $\mathrm{Ner}(G, M)$ be as in Example 3.1.1 (ii) and let $\mathcal{F}$ be a G-equivariant quasi-coherent sheaf on M. Then there is a canonical isomoprhism*

$$H^i(\mathrm{Ner}(G, M), \mathcal{F}) \simeq H^i(M, \mathcal{F})^G$$

for all $i \geq 0$.

Proof. Let us write $\mathrm{Ner}(G, k)$ instead of $\mathrm{Ner}(G, \mathrm{Spec}(k))$. Consider the composition $\mathrm{Ner}(G, M) \xrightarrow{p} \mathrm{Ner}(G, k) \to \mathrm{Spec}(k)$ induced by the projection $M \to \mathrm{Spec}(k)$. One obtains by flat base-change that $Rp_*(\mathcal{F})_n \simeq R\Gamma(M, \mathcal{F}) \otimes k[G^n]$ for all $n \in \mathbb{Z}_{\geq 0}$, where $k[G^n]$ denotes the group algebra of G^n.

We get thus

$$\begin{aligned} R\Gamma(\mathrm{Ner}(G,M),\mathcal{F}) &\simeq R\Gamma(\mathrm{Ner}(G,k),Rp_*\mathcal{F}) \\ &\simeq \mathrm{Tot}(\Gamma(k,R\Gamma(M,\mathcal{F})) \to \Gamma(k,R\Gamma(M,\mathcal{F})\otimes k[G]) \to \ldots) \\ &\simeq R\mathrm{Hom}(G,R\Gamma(M,\mathcal{F})) \\ &\simeq \Gamma(M,\mathcal{F})^G, \end{aligned}$$

where we used in the third equation that k, viewed as G-equivariant vector space, has a standard G-equivaraint resolution

$$\ldots \to k[G^3] \to k[G^2] \to k[G] \to k \to 0.$$

In the fourth equation we use that k is of characteristic 0 and thus taking G-invariants is exact.
□

3.2 cdh-topology and cdh-sheaves

Recall that a Nisnevich covering of a scheme $X \in \mathrm{Sch}/k$ is a family of étale morphisms $\{\varphi_i : U_i \to X\}$ and such that for every (not necessarily closed) point $x \in X$ there is an i and $y \in U_i$, such that $\varphi_i(y) = x$ and such that the induced map $k(x) \to k(y)$ is an isomorphism. One can check that these covers will define a pretopology on Sch/k, which generates the so-called Nisnevich topology.

Definition 3.2.1. *The* cdh*-topology on* Sch/k*, denoted* $(\mathrm{Sch}/k)_{\mathrm{cdh}}$*, is the weakest Grothendieck topology on* Sch/k *generated by the following coverings:*

(1) Nisnevich coverings.

(2) Abstract blow up squares: For every Cartesian square

$$\begin{array}{ccc} E & \hookrightarrow & Y \\ \downarrow & & \downarrow \pi \\ Z & \overset{i}{\hookrightarrow} & X \end{array} ,$$

such that $\pi : Y \to X$ *is a proper morphism,* $i : Z \hookrightarrow X$ *is a closed subscheme and* π *induces an isomorphism* $Y - E \xrightarrow{\sim} X - Z$*, the morphism* $Y \sqcup Z \xrightarrow{(\pi,i)} X$ *is a* cdh *covering.*

Remark 3.2.2. *(i) Let* X *be a non-reduced scheme in* Sch/k*. Setting* $Z = X_{\mathrm{red}}$ *and* $Y = \emptyset$*, one can see that every* $X \in \mathrm{Sch}/k$ *is* cdh*-locally reduced.*

(ii) Let $X = X_1 \cup X_2$ *be a reduced and reducible scheme in* Sch/k *with irreducible components* X_1 *and* X_2*. Setting* $Z = X_1$ *and* $Y = X_2$*, one can see that every* $X \in \mathrm{Sch}/k$ *is* cdh*-locally irreducible.*

(iii) Consider an integral scheme X in Sch/k and set $Y \to X$ to be a resolution of singularities (which we can, as k has characterisitic 0) and set $Z = \mathrm{Sing}(X)$ to be the singular locus of X. By induction on the dimension of X and using the previous 2 remarks, one sees that every $X \in \mathrm{Sch}/k$ is cdh*-locally smooth over k.*

Remark 3.2.3. *Replacing Nisnevich covers with étale covers in the defintion of* cdh*-topology, we obtain the* eh*-topology. Note that all results in this chapter concerning the* cdh*-topology are also valid for the* eh*-topology. The difference between* cdh *and* eh *will be apparent in Lemma 3.3.3.*

Note that every Zariski covering of an $X \in \mathrm{Sch}/k$ is a Nisnevich covering, and thus, by definition a cdh covering. In other words, the identity functor on Sch/k defines a morphism of topologies $\mathrm{id} : (\mathrm{Sch}/k)_{\mathrm{Zar}} \to (\mathrm{Sch}/k)_{\mathrm{cdh}}$. This corresponds to a morphism of sites $a : (\mathrm{Sch}/k)_{\mathrm{cdh}} \to (\mathrm{Sch}/k)_{\mathrm{Zar}}$ [5] (note that the arrow is drawn in the opposite direction). We denote by $(\mathrm{Sch}/k)^{\sim}_{\star}$ the category of abelian sheaves on $(\mathrm{Sch}/k)_{\star}$, where $\star$ stands for Zar or cdh.

Remark 3.2.4. *The functor a defines a functor $a_* : (\mathrm{Sch}/k)^{\sim}_{\mathrm{cdh}} \to (\mathrm{Sch}/k)^{\sim}_{\mathrm{Zar}}$ given by viewing a* cdh *sheaf F as a sheaf on $(\mathrm{Sch}/k)_{\mathrm{Zar}}$. It is well-known that a_* has a left adjoint $a^* : (\mathrm{Sch}/k)^{\sim}_{\mathrm{Zar}} \to (\mathrm{Sch}/k)^{\sim}_{\mathrm{cdh}}$ which coincides with the sheafification functor and is thus an exact functor [5, Théorème 3.4 and Théorème 4.1 1)].*

Moreover, we denote by $(\mathrm{Sch}/k)^{\sim}_{\mathrm{Zar}} \xrightarrow{|_X} (X)^{\sim}_{\mathrm{Zar}}$ the restriction functor given by the morphism of sites $(\mathrm{Sch}/k)_{\mathrm{Zar}} \to (X)_{\mathrm{Zar}}$ defined by the inclusion functor $i : (X)_{\mathrm{Zar}} \to (\mathrm{Sch})_{\mathrm{Zar}}$ (see [5]).

Let us now introduce the sheaf $\mathbb{Z}_{\star}(X)$ on a Grothendieck topology $(\mathrm{Sch}/k)_{\star}$, where $X \in \mathrm{Sch}/k$ (as usual, we think here of Zar or cdh as $\star$). We denote by $\mathbb{Z}(X)$ the presheaf on Sch/k defined by the mapping $U \mapsto \mathbb{Z}[\mathrm{Hom}(U, X)]$. Here, $\mathbb{Z}[\mathrm{Hom}(U, X)]$ is the free abelian group generated by the set $\mathrm{Hom}(U, X)$, or, in symbols,

$$\mathbb{Z}[\mathrm{Hom}(U, X)] = \bigoplus_{f : U \to X} \mathbb{Z}f.$$

We further denote by $\mathbb{Z}_{\star}(X)$ the sheafification of $\mathbb{Z}(X)$ with respect to a Grothendieck topology $\star$. We have the following results for $\mathbb{Z}_{\mathrm{cdh}}(X)$:

Lemma 3.2.5 (Suslin-Voevodsky [113, Lemma 12.1])**.** *Let*

$$\begin{array}{ccc} E & \overset{j}{\hookrightarrow} & Y \\ \downarrow{\scriptstyle p} & & \downarrow{\scriptstyle \pi} \\ Z & \overset{i}{\hookrightarrow} & X \end{array},$$

be an abstract blow up square (Definition 3.2.1 (2)). Then there is a short exact sequence of abelian sheaves

$$0 \to \mathbb{Z}_{\mathrm{cdh}}(E) \xrightarrow{(j(-), p(-))} \mathbb{Z}_{\mathrm{cdh}}(Y) \oplus \mathbb{Z}_{\mathrm{cdh}}(Z) \xrightarrow{\pi(-) - i(-)} \mathbb{Z}_{\mathrm{cdh}}(X) \to 0.$$

Proof. It is straightforward to show that the sequence of presheaves

$$0 \to \mathbb{Z}(E) \to \mathbb{Z}(Y) \oplus \mathbb{Z}(Z) \to \mathbb{Z}(X)$$

is exact, using the fact that E is the fiber product of Z and Y over X. As sheafification is exact, we get that the sequence

$$0 \to \mathbb{Z}_{\text{cdh}}(E) \to \mathbb{Z}_{\text{cdh}}(Y) \oplus \mathbb{Z}_{\text{cdh}}(Z) \to \mathbb{Z}_{\text{cdh}}(X)$$

is exact. Exactness on the right follows from the fact that $Y \sqcup Z \to X$ is a cdh-covering (see [5] for the definition of surjectivity of sheaves). □

Corollary 3.2.6. *Let $E \in \text{Sch}/k$. Then $\mathbb{Z}_{\text{cdh}}(E_{\text{red}}) \simeq \mathbb{Z}_{\text{cdh}}(E)$, and if $E = E_1 \cup \ldots \cup E_n$, where E_i are closed subschemes of E, then there is a long exact sequence*

$$0 \to \mathbb{Z}_{\text{cdh}}(E_1 \times_E \ldots \times_E E_n) \to \ldots \bigoplus_{i<j} \mathbb{Z}_{\text{cdh}}(E_i \times_E E_j) \to \bigoplus_i \mathbb{Z}_{\text{cdh}}(E_i) \to \mathbb{Z}_{\text{cdh}}(E) \to 0.$$

Proof. The first statement, that is $\mathbb{Z}_{\text{cdh}}(E_{\text{red}}) \simeq \mathbb{Z}_{\text{cdh}}(E)$, follows directly from Lemma 3.2.5 (see also Remark 3.2.2 (i)). The long exact sequence follows by induction on n. That is, we can apply Lemma 3.2.5 to the abstract blow up square

$$\begin{array}{ccc} E_1 \times_E E_2 \cup \ldots \cup E_1 \times_E E_n & \hookrightarrow & E_1 \\ \downarrow & & \downarrow \\ E_2 \cup \ldots \cup E_n & \hookrightarrow & E \end{array}$$

(compare to Remark 3.2.2 (ii)), and as there are long exact sequence for long exact sequence for $\mathbb{Z}_{\text{cdh}}(E_2 \cup \ldots \cup E_n)$ and $\mathbb{Z}_{\text{cdh}}(E_1 \times_E E_2 \cup \ldots \cup E_1 \times_E E_n)$ (induction hypothesis), the result follows. □

Remark 3.2.7. *By the same algorithm as in Remark 3.2.2, for any scheme $X \in \text{Sch}/k$, we can resolve $\mathbb{Z}_{\text{cdh}}(X)$ by a finite resolution with components consisting of (direct sums of) $\mathbb{Z}_{\text{cdh}}(Y_i)$'s, where all the Y_i's are smooth. More explicitly, let $X \in \text{Sch}/k$ be integral with singular locus $Z = \text{Sing}(X)$ and let $\pi : Y \to X$ be a resolution of singularities, such that $E = \pi^{-1}(Z)_{\text{red}}$ is a strict normal crossing divisor with irreducible components $E_1, \ldots, E_n$. Then by Lemma 3.2.5 and Corollary 3.2.6 there is a long exact sequence*

$$0 \to \mathbb{Z}_{\text{cdh}}(E_1 \times_Y \ldots \times_Y E_n) \to \ldots \to \bigoplus_i \mathbb{Z}_{\text{cdh}}(E_i) \to \mathbb{Z}_{\text{cdh}}(Y) \oplus \mathbb{Z}_{\text{cdh}}(Z) \to \mathbb{Z}_{\text{cdh}}(X) \to 0.$$

Note that all components of this long exact sequence, except for maybe $\mathbb{Z}_{\text{cdh}}(Z)$ and $\mathbb{Z}_{\text{cdh}}(X)$, are given by smooth schemes. By induction on the dimension of X and using Corollary 3.2.6, one can resolve $\mathbb{Z}_{\text{cdh}}(Z)$ by a finitely many $\mathbb{Z}_{\text{cdh}}(Z_i)$'s, where the Z_i's are smooth.

3.3 cdh-hyperresolutions

Definition and examples

Note first that for a semi-simplicial scheme $Y_\bullet$ one can define $\mathbb{Z}_\star(Y_\bullet)$ as the complex

$$\cdots \to \mathbb{Z}_\star(Y_2) \to \mathbb{Z}_\star(Y_1) \to \mathbb{Z}_\star(Y_0) \to 0,$$

where the differentials of this complex are given by the alternating sum of the morphisms induced by the projection maps $\delta_i : Y_{n+1} \to Y_n$.

Definition 3.3.1. *Let $X \in \mathrm{Sch}/k$ be a variety and let $\pi : Y_\bullet \to X$ be a an augmentation, such that $Y_\bullet$ smooth over k and π is proper. We say that π is a* cdh*-hyperresolution, if the induced map*

$$\mathbb{Z}_{\mathrm{cdh}}(Y_\bullet) \to \mathbb{Z}_{\mathrm{cdh}}(X)$$

is a quasi-isomorphism.

Let us discuss how to construct rather nice cdh-hyperresolutions of irreducible varieties with smooth singular locus from a resolution of singularities with exceptional divisor being strict normal crossing.

That is, assume that $X \in \mathrm{Sch}/k$ is integral and such that its singular locus $Z = \mathrm{Sing}(X)$ is smooth over k. By Hironaka's theorem there is a resolution of singularities $\pi : Y \to X$, such that the exceptional locus $E = \pi^{-1}(Z)_{\mathrm{red}}$ is a simple normal crossing divisor and denote by $p : E \to Z$ the induced morphism. That is, the irreducible components E_i, $1 \le i \le n$, of E are smooth codimension 1 subvarieties and all possible combinations of intersections $E_{i_1} \times_Y \ldots \times_Y E_{i_k}$ are smooth. We can construct the following semi-simplicial scheme $Y_\bullet$:

$$\ldots \coprod_{i_1 < \ldots < i_k} E_{i_1} \times_Y \ldots \times_Y E_{i_k} \sqcup Z \ldots \underset{\delta_0}{\overset{\delta_2}{\underset{\delta_1}{\longrightarrow}}} \coprod_i E_i \sqcup Z \underset{\delta_0}{\overset{\delta_1}{\longrightarrow}} Y \sqcup Z \sqcup Z, \qquad (3.3.1)$$

where the n-th term is $E_1 \times_Y \ldots \times_Y E_n$ if n is even and $E_1 \times_Y \ldots \times_Y E_n \sqcup Z$ else. For $k > 1$, the projection map $\delta_l : Y_{k+1} \to Y_k$ for $0 \le l \le k$ is given componentwise by the inclusions

$$E_{i_1} \times_Y \ldots \times_Y E_{i_{k+1}} \to E_{i_1} \times_Y \ldots \times_Y E_{i_{l-1}} \times_Y E_{i_{l+1}} \times_Y \ldots \times_Y E_{i_{k+1}}$$

and the Z component will be mapped identically onto Z. Furthermore, $\delta_{k+1} : Y_{k+1} \to Y_k$ is given by projecting $E_{i_1} \times_Y \ldots \times_Y E_{k+1}$ via p to Z. For $k = 1$, δ_0 is given by the inclusion $E_1 \sqcup \ldots \sqcup E_n \to Y$ and maps Z isomorphically to the first component of Z and δ_1 is given by the projection $E_1 \sqcup \ldots \sqcup E_n \to Z$ to the first component of Z and by mapping Z isomorphically to the second component of Z. We have:

Lemma 3.3.2. *Let $X \in \mathrm{Sch}/k$ be integral and assume that the singular locus of X is smooth over k. Let $Y_\bullet \to X$ be as in* (3.3.1)*. Then $Y_\bullet \to X$ is a* cdh*-hyperresolution.*

Proof. This is a consequence of Remark 3.2.7. □

The definition of cdh-hyperresolution can in the same way be defined for the eh-topology. Working with sheaves of $\mathbb{Q}$-vector spaces rather than abelian sheaves, one can see that the quotient semi-simplicial variety $\mathrm{Ner}(G, M) \to M/G$ (Example 3.1.1 (ii)) is a eh-hyperresolution. More precisely:

Lemma 3.3.3. *Let M, G and $\mathrm{Ner}(G, M)$ be as in Example 3.1.1 (ii). Then there is a quasi-isomorphism*

$$\mathbb{Q}_\star(\mathrm{Ner}(G, M)) \xrightarrow{\text{qis}} \mathbb{Q}_\star(M/G)$$

given by the projection $\pi : M \to M/G$, where $\star = \mathrm{et}$ or eh, with et denoting the étale topology.

Proof. Let us first show that on the level of presheaves there is a quasi-isomorphism $\mathbb{Q}(\mathrm{Ner}(G, M)) \to \mathbb{Q}(M)^G$.

Let $U \in \mathrm{Sch}/k$. As G is finite, we have an identification $\mathbb{Q}(M \times G^n)(U) \simeq \bigoplus_{g \in G^n} \mathbb{Q}(M)(U)$ with an induced G-action. In other words, we have

$$\mathbb{Q}(M \times G^n)(U) \simeq \mathbb{Q}[G^n] \otimes \mathbb{Q}(M)(U).$$

We thus can identify the complex $\mathbb{Q}(\mathrm{Ner}(G, M))(U)$ with

$$\ldots \to \mathbb{Q}[G^2] \otimes \mathbb{Q}(M)(U) \to \mathbb{Q}[G] \otimes \mathbb{Q}(M)(U) \to \mathbb{Q}(M)(U) \to 0. \tag{3.3.2}$$

As this complex is just the tensor product of $\mathbb{Q}(M)(U)$ with the standard G-equivariant resolution of $\mathbb{Q}$, we have that the cohomology of (3.3.2) is just the group homology of the G-vector space $Q(M)(U)$. As we work over $\mathbb{Q}$ we have that this complex of G-modules is exact in degrees strictly bigger than 0, and the cohomology in degree 0 is $\mathbb{Q}(M)^G(U)$.

It remains to show that $\mathbb{Q}_{\mathrm{eh}}(M)^G \simeq \mathbb{Q}_{\mathrm{eh}}(M/G)$. To show this, it is enough to show that the sequence of morphisms

$$\mathbb{Q}_{\mathrm{et}}(M \times G) \xrightarrow{p_1 - \sigma} \mathbb{Q}_{\mathrm{et}}(M) \xrightarrow{\pi} \mathbb{Q}(M/G)_{\mathrm{et}} \to 0$$

is exact, where $p_1 : G \times M \to M$ is the projection and $\sigma : G \times M \to M$ the multiplication map. Exactness in the middle is straightforward and can be shown on the level of presheaves.

For surjectivity of $\pi : \mathbb{Q}(M)_{\mathrm{et}} \to \mathbb{Q}(M/G)_{\mathrm{et}}$, it is enough to show the statement for $U = \mathrm{Spec}(A)$, A a strictly Henselian local ring, as we work over the étale topology. We denote by T the fiber porduct $U \times_{M/G} M$. As U is Henselian and as $T \to U$ is finite, we have that T is a product of local rings. Furthermore, as the residue field of A is algebraically closed (A is strictly Henselian), the covering $T \to U$ splits. In other words, the map $\mathrm{Hom}(U, M) \to \mathrm{Hom}(U, M/G)$ is surjective, and thus $\mathbb{Q}_{\mathrm{et}}(M) \to \mathbb{Q}_{\mathrm{et}}(M/G)$ is surjective. The statement for the eh-topology follows by sheafification from étale to eh-topology. □

Existence of cdh-hyperresolutions

Let us contemplate now about the existence of cdh-hyperresolutions, which is less straightforward as in the case where $\mathrm{Sing}(X)$ is smooth, however it is still explicit. To achieve this, it is helpful to use the language of cubical varieties.

Let n be a positive integer and denote by $[n]$ the set $\{0, \dots, n-1\}$. An *n-cubical scheme* $X_\bullet$ is a family $\{X_I\}_{I \subset [n-1]}$ of schemes in Sch/k together with morphisms $\delta_{IJ} : X_I \to X_J$, $J \subset I$, with the usual commutativity property of the δ_{IJ}'s, that is for all $I, J, K \subset [n-1]$ such that $J \subset K \subset I$ we have that $\delta_{KJ} \circ \delta_{IK} = \delta_{IJ}$.

A morphism of n-cubical schemes $f_\bullet : Y_\bullet \to X_\bullet$ is a family of morphisms $f_I : X_I \to Y_I$, $I \subset [n-1]$, commuting with the δ_{IJ}'s. Analogous to semi-simplicial schemes, an n-cubical scheme is called smooth over k, proper, etc. if all its components are smooth over k, proper, etc. and morphisms of n-cubical schemes are called of finite type, proper, etc., if they are componentwise of finite type, proper, etc.

Remark 3.3.4. *(i) One can see that an n-cubical scheme $X_\bullet$ is just a morphism $Y_\bullet \to Z_\bullet$ of $(n-1)$-cubical schemes. Indeed, one defines $Z_\bullet := \{X_I\}_{I \subset [n-2]}$ and $Y_\bullet := \{X_{I \cup \{n-1\}}\}_{I \subset [n-2]}$ and the corresponding morphism between them is given by $X_{I \cup \{n-1\}} \xrightarrow{\delta} X_I$. This means that a 1-cubical scheme is nothing but a morphism of schemes, a 2-cubical scheme is just a commutative diagram of schemes, a 3-cubical scheme is a commutative cube of schemes, etc.*

(ii) An n-cubical scheme $X_\bullet$ naturally defines a semi-simplicial scheme $\hat{X}_\bullet$ with components

$$\hat{X}_n := \coprod_{|I|=n+1} X_I$$

and with projection morphisms given by $X_I \to X_{I - \{i_{k+1}\}}$, where $I = \{i_1, \dots, i_{|I|}\}$. Moreover, we have an obvious augmentation $\hat{X}_\bullet \to X_\emptyset$.

If $X_\bullet$ is an n-cubical scheme, then we denote $\mathbb{Z}_\star(X_\bullet) := \mathbb{Z}_\star(\hat{X}_\bullet)$, where $\hat{X}_\bullet$ is the associated semi-simplicial scheme of $X_\bullet$ (Remark 3.3.4 (ii)). Denote further by $C(X_\bullet)$ the cone of the induced morphism of complexes $\mathbb{Z}_\star(X_\bullet) \to \mathbb{Z}_\star(X_\emptyset)$.

Lemma 3.3.5. *(1) Let $X_\bullet$ be a $(n+1)$-cubical variety and let us view $X_\bullet$ as a morphism of n-cubical varieties $X^2_\bullet \to X^1_\bullet$ as in Remark 3.3.4 (i). Then there is a distinguished triangle*

$$C(X^2_\bullet) \to C(X^1_\bullet) \to C(X_\bullet) \in \mathcal{D}^-((\mathrm{Sch}/k)^\sim_\star).$$

(2) Let $X_\bullet$ be a $(n+2)$-cubical scheme considered as commutative diagram

$$\begin{array}{ccc} E_\bullet & \longrightarrow & Y_\bullet \; . \\ \downarrow & & \downarrow \\ Z_\bullet & \longrightarrow & S_\bullet \end{array}$$

of n-cubical schemes. Then there is a distinguished triangle

$$C(E_\bullet)[1] \to Cone(C(Y_\bullet) \oplus C(Z_\bullet) \to C(S_\bullet)) \to C(X_\bullet) \in \mathcal{D}^-((\mathrm{Sch}/k)^\sim_\star).$$

Proof. (1): It is straightforward to see that $C(X_\bullet)$, that is the cone of $\mathbb{Z}_\star(X_\bullet) \to \mathbb{Z}_\star(X_\emptyset)$, is the complex

$$0 \to \mathbb{Z}_\star(X_{\{0,\dots,n\}}) \to \dots \to \bigoplus_{0\le i\le n} \mathbb{Z}_\star(X_i) \to \mathbb{Z}_\star(X_\emptyset) \to 0$$

with differentials given by the alternating sums of the projection morphisms. $C(X^1_\bullet)$ and $C(X^2_\bullet)$ have a similar form, but with appropriate indices. It is an easy exercise to show then that the cone of the induced map $C(X^2_\bullet) \to C(X^1_\bullet)$ is just $C(X_\bullet)$.

(2): Consider the diagram

$$\begin{array}{ccc} C(E_\bullet) & \xrightarrow{f} & C(Y_\bullet) \\ \downarrow & & \downarrow \\ C(Z_\bullet) & \xrightarrow{g} & C(S_\bullet) \end{array}$$

and note that, by (1), there is a distinguished triangle $Cone(f) \to Cone(g) \to C(X_\bullet)$. By the octahedral axiom there are distinguished triangles

$$Cone(f) \to C(E_\bullet)[1] \to C(Y_\bullet)[1]$$

and

$$Cone(g) \to Cone(C(Y_\bullet) \oplus C(Z_\bullet) \to C(S_\bullet)) \to C(Y_\bullet[1]).$$

Comparing these two triangles and applying octahedral axiom again one obtains the result. □

Let us now define and discuss resolutions of singularities of n-cubical varieties. The discriminant of a proper morphism of n-cubical varieties $\pi : Y_\bullet \to X_\bullet$ is the smallest closed n-cubical subscheme $Z_\bullet$ of $X_\bullet$, such that for all $I \subset [n-1]$ there is an isomorphism $Y_I - \pi^{-1}(Z_I) \xrightarrow{\sim} X_I - Z_I$ induced via π. We call the fiber diagram given by π and $Z_\bullet \subset X_\bullet$ the abstract cubical blow up square given by $\pi : Y_\bullet \to X_\bullet$. Furthermore, we call $\pi : Y_\bullet \to X_\bullet$ an n-cubical resolution of singularities, if in addition Y_I is smooth and such that $\dim(\pi^{-1}(Z_I)) < \dim(X_I)$ for all I.

As we work over a ground field of characteristic 0, there is always a desingularization of cubical varieties. More precisely:

Theorem 3.3.6 ([59, Exposé I, Théorème 2.6]). *Let $X_\bullet$ be an n-cubical variety. Then there exists a resolution of singularities $Y_\bullet \to X_\bullet$.*

Remark 3.3.7. *Note that the resolution of singularities in Theorem 3.3.6 is not just resolving the components of $X_\bullet$. Actually, this idea will not work in general.*

Indeed, consider the 1-cubical variety $\mathbb{A}^2 \xrightarrow{q} \mathbb{A}^2/\mathbb{Z}_2$, where $A^2/\mathbb{Z}_2$ is the A_1 singularity and q is the quotient morphism. Let $Y \to \mathbb{A}^2/\mathbb{Z}_2$ any resolution of singularities. Note that there is *no morphism $\mathbb{A}^2 \to Y$ which is compatible with the quotient map and thus we have no 1-cubical resolution of singularities of $\mathbb{A}^2 \to \mathbb{A}^2/\mathbb{Z}_2$ with components Y and $\mathbb{A}^2$.*

With the notation of n-cubical varieties, we have a more general version of Lemma 3.2.5:

Lemma 3.3.8. *Let $X_\bullet$ be a n-cubical variety, let $\pi : Y_\bullet \to X_\bullet$ be proper morphism with discriminant $i : Z_\bullet \hookrightarrow X_\bullet$ and let $E_\bullet$ be the fiber product of π and i. Then there is a distinguished triangle*

$$\mathbb{Z}_{\mathrm{cdh}}(E_\bullet) \to \mathbb{Z}_{\mathrm{cdh}}(Y_\bullet) \oplus \mathbb{Z}_{\mathrm{cdh}}(Z_\bullet) \to \mathbb{Z}_{\mathrm{cdh}}(X_\bullet)$$

in $\mathcal{D}^b((\mathrm{Sch}/k)^{\sim}_{\mathrm{cdh}})$.

Proof. We proceed by induction over n. The case $n = 1$ is just Lemma 3.2.5.

For $n > 1$, we view the fiber diagram given by π and i as morphism of fiber diagrams of $(n-1)$-cubical diagrams

$$\begin{array}{ccccccc} E^2_\bullet & \longrightarrow & Y^2_\bullet & & E^1_\bullet & \longrightarrow & Y^1_\bullet \\ \downarrow & & \downarrow & \longrightarrow & \downarrow & & \downarrow \\ Z^2_\bullet & \longrightarrow & X^2_\bullet & & Z^1_\bullet & \longrightarrow & X^1_\bullet \end{array}$$

where $Y^i_\bullet \to X^i_\bullet$ is a resolution of singularities with discriminant $Z^i_\bullet$, where $i = 1, 2$. By induction hypothesis we have distinguished triangles

$$\mathbb{Z}_{\mathrm{cdh}}(E^i_\bullet) \to \mathbb{Z}_{\mathrm{cdh}}(Y^i_\bullet) \oplus \mathbb{Z}_{\mathrm{cdh}}(Z^i_\bullet) \to \mathbb{Z}_{\mathrm{cdh}}(X^i_\bullet)$$

for $i = 1, 2$. Comparing these two triangles, we get by the octahedral axiom that

$$C(E_\bullet) \to C(Y_\bullet) \oplus C(Z_\bullet) \to C(X_\bullet)$$

is a distinguished triangle. This is however equivalent to the statement we want to show. □

Let us now state and show the main result of this section, which is a consequence of Theorem 3.3.6.

Theorem 3.3.9. *Let $X \in \mathrm{Sch}/k$ be a variety. There exists a* cdh*-hyperresolution $\pi : Y_\bullet \to X$.*

Proof. We construct inductively an m-cubical variety $Y^{(m)}_\bullet$ for $m > 1$, such that, viewed as a morphism of $(m-1)$-cubical varieties $U^{(m-1)}_\bullet \to V^{(m-1)}_\bullet$, $V^{(m-1)}_I$ is smooth for all $I \neq \emptyset$ and $V^{(m-1)}_\emptyset = X$ and $\dim U^{(m-1)}_J < \dim V^{(m-1)}_J$ for all J, and such that there is a quasi-isomorphism $\mathbb{Z}_{\mathrm{cdh}}(Y^{(m)}_\bullet) \to \mathbb{Z}_{\mathrm{cdh}}(X)$. By the condition on the dimension, one sees that this process stops after $m = \dim(X)$ steps and we set $Y_\bullet = Y^{(\dim(X))}_\bullet$.

The case $m = 2$ follows by Hironaka's result; let $\pi : Y \to X$ be a resolution of singularities, such that the exceptional locus $E = \pi^{-1}(Z)_{\mathrm{red}}$ is a simple normal crossing divisor. We define by $Y_\bullet^{(2)}$ the 2-cubical variety

$$\begin{array}{ccc} E & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}.$$

As in Remark 3.3.4 (i), we can view $Y_\bullet^{(2)}$ as a morphism between the 1-cubical varieties $U_\bullet^{(1)} = E \to Z$ and $V_\bullet^{(1)} = Y \to X$. Note that the components of $U_\bullet^{(1)}$ have strictly smaller dimension of the components of $V_\bullet^{(1)}$. The quasi-isomorphism $\mathbb{Z}_{\mathrm{cdh}}(Y_\bullet^{(2)}) \to \mathbb{Z}_{\mathrm{cdh}}(X)$ is just Lemma 3.2.5.

Let us assume now that we have constructed inductively an m-cubical variety $Y_\bullet^{(m)}$, such that, viewed as a morphism of $(m-1)$-cubical varieties $U_\bullet^{(m-1)} \to V_\bullet^{(m-1)}$, $V_I^{(m-1)}$ is smooth for all $I \neq \emptyset$ and $V_\emptyset^{(m-1)} = X$ and $\dim U_J^{(m-1)} < \dim V_J^{(m-1)}$ for all J. By Theorem 3.3.6 there is a resolution $p : \widetilde{U}_\bullet^{(m-1)} \to U_\bullet^{(m-1)}$ with discriminant $i : D_\bullet^{(m-1)} \hookrightarrow U_\bullet^{(m-1)}$. Denote by $E_\bullet^{(m-1)}$ the reduced fiber product of p and i. We then define the $(m+1)$-cubical variety $Y_\bullet^{(m+1)}$ as the following diagram:

$$\begin{array}{ccc} E_\bullet^{(m-1)} & \hookrightarrow & \widetilde{U}_\bullet^{(m-1)} \\ \downarrow & & \downarrow \\ D_\bullet^{(m-1)} & \hookrightarrow & V_\bullet^{(m-1)} \end{array}. \tag{3.3.3}$$

It is not hard to check the inductive properties on the components of $Y_\bullet^{(m+1)}$. Finally, let us show that the morphism $\mathbb{Z}_{\mathrm{cdh}}(Y_\bullet^{(m+1)}) \to \mathbb{Z}_{\mathrm{cdh}}(X)$ is a quasi-isomorphism. First note that applying Lemma 3.3.8 on the abstract cubical blow up square given by $\widetilde{U}_\bullet^{(m-1)} \to U_\bullet^{(m-1)}$ gives a distinguished triangle

$$C(E_\bullet^{(m-1)}) \to C(D_\bullet^{(m-1)}) \oplus C(\widetilde{U}_\bullet^{(m-1)}) \to C(U_\bullet^{(m-1)}). \tag{3.3.4}$$

Note further that $\mathbb{Z}_{\mathrm{cdh}}(Y_\bullet^{(m)}) \to \mathbb{Z}_{\mathrm{cdh}}(X)$ is a quasi-isomorphism by induction hypothesis, or in other words $C(Y_\bullet^{(m)}) \simeq 0$ in $\mathcal{D}^b((\mathrm{Sch})^\sim_{\mathrm{cdh}})$. By Lemma 3.3.5 (1), we get thus that there is a quasi-isomorphism $C(U_\bullet^{(m-1)}) \xrightarrow{\mathrm{qis}} C(V_\bullet^{(m-1)})$. Combining this quasi-isomorphism with (3.3.4) we get

$$Cone(C(D_\bullet^{(m-1)}) \oplus C(\widetilde{U}_\bullet^{(m-1)}) \to C(V_\bullet^{(m-1)})) \simeq C(E_\bullet^{(m-1)})[1] \in \mathcal{D}^b((\mathrm{Sch})^\sim_{\mathrm{cdh}}).$$

By Lemma 3.3.5 (2) this is equivalent to say that $C(Y_\bullet^{(m+1)}) \simeq 0$ in $\mathcal{D}^b((\mathrm{Sch}/k)^\sim_{\mathrm{cdh}})$, or in other words that $\mathbb{Z}_{\mathrm{cdh}}(Y_\bullet^{(m+1)}) \to \mathbb{Z}_{\mathrm{cdh}}(X)$ is a quasi-isomorphism. □

The proof of Theorem 3.3.9 can be generalized formally to the cubical case. That is, for an n-cubical variety $X_\bullet$, there is a cubical cdh-hyperresolution $Y_\bullet \to X_\bullet$ (compare to cubical hyperresolutions of a cubical varieties [59, Proof of Theorem I.2.5]). This yields:

Corollary 3.3.10. *Let $X \to S$ be a morphism of varieties. Then there are* cdh*-hyperresolutions*

$Y_\bullet \to X$ and $Y'_\bullet \to S$, and a morphism $Y_\bullet \to Y'_\bullet$, such that the diagram

$$\begin{array}{ccc} Y_\bullet & \longrightarrow & Y'_\bullet \\ \downarrow & & \downarrow \\ X & \longrightarrow & S \end{array}$$

commutes.

Remark 3.3.11. *The algorithm of Theorem 3.3.9 of constructing a* cdh*-hyperresolution of a variety X is not the most efficient one in terms of redundance appearing in the components of the* cdh*-hypperesolution. For example, if we consider the nodal union of two lines $X = \{xy = 0\}$ in $\mathbb{A}^2$, we see that the* cdh*-hyperresolution from Theorem 3.3.9, denoted $Y_\bullet^{(1)}$, is given by*

$$\{0\} \sqcup \{0\} \rightrightarrows \mathbb{A}^1 \sqcup \mathbb{A}^1 \sqcup \{0\}.$$

On the other hand, there is a semi-simplicial hyperresolution $Y_\bullet$ of X given by

$$\{0\} \rightrightarrows \mathbb{A}^1 \sqcup \mathbb{A}^1$$

with obvious projection maps. It is not hard to see that $\mathrm{Sh}(Y_\bullet^{(1)}) \simeq \mathrm{Sh}(Y_\bullet)$.

3.4 cdh-cohomology

Let us recall now the definition of sheaf cohomology of a Grothendieck topology $\star$ on Sch/k. Let us fix an $X \in \mathrm{Sch}/k$. As usual, we can define the global section functor $\Gamma_\star(X, -) : (\mathrm{Sch}/k)^\sim_\star \to \mathrm{Ab}$ by $\Gamma_\star(X, \mathcal{F}) = \mathcal{F}(X)$. Note that this functor is left exact. Moreover, the category $(\mathrm{Sch}/k)^\sim_\star$ is abelian and has enough injectives, so for any bounded below complex $C^\bullet$ with components in $(\mathrm{Sch}/k)^\sim_\star$ we can define the hypercohomology of $C^\bullet$ as

$$H^i_\star(X, C^\bullet) := R\Gamma^i_\star(X, C^\bullet) = H^i(\mathrm{Tot}(\Gamma_\star(X, \mathcal{I}^{\bullet,\bullet}))),$$

where $C^\bullet \to \mathcal{I}^{\bullet,\bullet}$ is the Cartan-Eilenberg resolution of $C^\bullet$.

Remark 3.4.1. *In the case when $\star = \mathrm{Zar}$ it follows from definitions that for $\mathcal{F} \in (\mathrm{Sch}/k)^\sim_{\mathrm{Zar}}$ one has $\Gamma_{\mathrm{Zar}}(X, \mathcal{F}) = \Gamma(X, \mathcal{F}|_X)$. Furthermore, injectives in $(\mathrm{Sch}/k)^\sim_{Zar}$ will be restricted to injectives in $\mathrm{Sh}(X)$ via $(\mathrm{Sch}/k)^\sim_{\mathrm{Zar}} \xrightarrow{|_X} \mathrm{Sh}(X)$. Therefore, and since $|_X$ is exact, we see that for all bounded below complexes with components in $(\mathrm{Sch}/k)^\sim_{\mathrm{Zar}}$ one has*

$$H^i_{\mathrm{Zar}}(X, C^\bullet) \simeq H^i(X, C^\bullet|_X)$$

for all $i \in \mathbb{Z}$, where the right hand side is the standard (Zariski) hypercohomology of $C^\bullet|_X \in \mathcal{D}^+(\mathrm{Sh}(X))$.

Proposition 3.4.2 (Voevodsky [122]). *Let $C^\bullet$ be a bounded below complex with components in $(\mathrm{Sch}/k)^\sim_\star$, let $X \in \mathrm{Sch}/k$ and let $\mathbb{Z}_\star(X)$ be as in the previous section. Then there is a canonical isomorphism*

$$H^i_\star(X, C^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}^+((\mathrm{Sch}/k)^\sim_\star)}(\mathbb{Z}_\star(X), C^\bullet[i]),$$

for all $i \in \mathbb{Z}$.

Proof. By [122, Proposition 2.1.3] the statement is true for sheaves $\mathcal{F} \in (\mathrm{Sch}/k)^\sim_\star$ (an application of Yoneda's Lemma). The statement for bounded below complexes $C^\bullet$ follows then from the Leray spectral sequence. □

As an immediate consequence of this proposition, we get a relation between cdh-cohomology and Zariski hypercohomology.

Corollary 3.4.3. *Denote by $a : (\mathrm{Sch}/k)_{\mathrm{cdh}} \to (\mathrm{Sch}/k)_{\mathrm{Zar}}$ the morphism of sites defined by the identity functor and let $X \in (\mathrm{Sch}/k)_{\mathrm{cdh}}$ and $\mathcal{F} \in (\mathrm{Sch}/k)^\sim_{\mathrm{cdh}}$. Then*

$$H^i_{\mathrm{cdh}}(X, \mathcal{F}) \simeq H^i(X, Ra_*\mathcal{F}|_X)$$

for all $i \geq 0$.

Proof. The statement follows by Prop. 3.4.2, the isomorphism $\mathbb{Z}_{\mathrm{cdh}}(X) \simeq a^*\mathbb{Z}_{\mathrm{Zar}}(X)$, left adjointness of a^* with respect to Ra_*, and Remark 3.4.1. □

Moreover, we get the following propoerty of cdh-cohomology for abstract blow up squares:

Corollary 3.4.4 (Suslin-Voevodsky [113, Lemma 12.1]). *Consider the abstract blow up square*

$$\begin{array}{ccc} E & \hookrightarrow & Y \\ \downarrow & & \downarrow \\ Z & \hookrightarrow & X \end{array}$$

in Sch/k. For any $\mathcal{F} \in (\mathrm{Sch}/k)^\sim_{\mathrm{cdh}}$ (or more generally a bounded below complex with components in $(\mathrm{Sch}/k)^\sim_{\mathrm{cdh}}$) there is a long exact Mayer-Vietoris sequence

$$\ldots \to H^i_{\mathrm{cdh}}(X, \mathcal{F}) \to H^i_{\mathrm{cdh}}(Y, \mathcal{F}) \oplus H^i_{\mathrm{cdh}}(Z, \mathcal{F}) \to H^i_{\mathrm{cdh}}(E, \mathcal{F}) \to \ldots$$

Proof. This follows from Lemma 3.2.5 and Prop. 3.4.2. □

Let $\mathcal{F}$ be a sheaf (or more generally a complex) on $(\mathrm{Sch}/k)_{\mathrm{Zar}}$ and let $X_\bullet$ be a semi-simplicial scheme over k. Let us denote the cohomology of $\mathcal{F}|_{X_\bullet}$ (Remark 3.1.2) by $H^i(X_\bullet, \mathcal{F})$ instead of $H^i(X_\bullet, \mathcal{F}|_{X_\bullet})$. We have:

Corollary 3.4.5. *Let $C^\bullet$ be a bounded below complex with components in $(\mathrm{Sch}/k)^\sim_{\mathrm{Zar}}$, let $X_\bullet$ be a semi-simplicial scheme and assume that $\mathbb{Z}_{\mathrm{Zar}}(X_\bullet)$ has bounded cohomology. Then there is a* *canonical isomorphism*

$$H^i(X_\bullet, C^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}^+((\mathrm{Sch}/k)^\sim_{\mathrm{Zar}})}(\mathbb{Z}_{\mathrm{Zar}}(X_\bullet), C^\bullet[i]),$$

for all $i \in \mathbb{Z}$*.*

Proof. By Prop. 3.4.2 there is a canonical isomorphism

$$H^q(X_p, C^\bullet) \simeq \mathrm{Hom}_{\mathcal{D}^+((\mathrm{Sch}/k)^\sim_{\mathrm{Zar}})}(\mathbb{Z}_{\mathrm{Zar}}(X_p), C^\bullet[q])$$

for all p and q. Note that the left hand side spans the $E_1^{p,q}$ page converging to $H^{p+q}(X_\bullet, C^\bullet)$ and similarly the right hand side converges to $\mathrm{Hom}_{\mathcal{D}^+((\mathrm{Sch}/k)^\sim_{\mathrm{Zar}})}(\mathbb{Z}_{\mathrm{Zar}}(X_\bullet), C^\bullet[p+q])$. The result follows. □

Corollary 3.4.6. *Let* $Y_\bullet \to X$ *be a* cdh*-hyperresolution and let* $\mathcal{F} \in (\mathrm{Sch})^\sim_{\mathrm{cdh}}$*. Then there is a canonical isomorphism*

$$H^i_{\mathrm{cdh}}(X, \mathcal{F}) \simeq H^i(Y_\bullet, Ra_*\mathcal{F})$$

for all $i \geq 0$*.*

Proof. By Prop. 3.4.2 we have canonical isomorphisms

$$\begin{aligned} H^i_{\mathrm{cdh}}(X, \mathcal{F}) &\simeq \mathrm{Hom}_{\mathcal{D}^+((\mathrm{Sch}/k)^\sim_{\mathrm{cdh}})}(\mathbb{Z}_{\mathrm{cdh}}(X), \mathcal{F}[i]) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^+((\mathrm{Sch}/k)^\sim_{\mathrm{cdh}})}(\mathbb{Z}_{\mathrm{cdh}}(Y_\bullet), \mathcal{F}[i]) \\ &\simeq \mathrm{Hom}_{\mathcal{D}^+((\mathrm{Sch}/k)^\sim_{\mathrm{Zar}})}(\mathbb{Z}_{\mathrm{Zar}}(Y_\bullet), Ra_*\mathcal{F}[i]) \\ &\simeq H^i(Y_\bullet, Ra_*\mathcal{F}), \end{aligned}$$

where we used right adjointness of Ra_* with respect to a^* in the third and Corollary 3.4.5 the fourth equality. □

3.5 cdh-Kähler differentials and K-theory

In this subsection, when we say sheaf we mean a sheaf with entries in $\mathbb{Q}$-vector spaces instead of abelian groups.

Let $X \in \mathrm{Sch}/k$ and let us as usual denote by Ω^p_{X/k_0} the p-th exterior power of the sheaf of Kähler differentials on X over k_0, where $k_0 \subset k$ is a subfield. We define by $\Omega^p_{/k_0} : \mathrm{Sch}/k \to \mathrm{Vect}(k)$ the presheaf which sends $X \mapsto \Omega^p_{k_0}(X)$. We will write $\Omega^p_{/k_0}$ instead of $a^*\Omega_{/k_0} \in (\mathrm{Sch}/k)^\sim_{\mathrm{cdh}}$ when it is clear from the context.

The following result allows us to compute cdh-cohomology groups of $\Omega^p_{/k}$ evaluated at X in terms of Zariski cohomology groups of $\Omega^p_{X/k}$.

Proposition 3.5.1 (Cortinas-Haesemeyer-Weibel [37, Corollary 2.5])**.** *Let* $X \in \mathrm{Sch}/k$ *be a smooth scheme over* k*, let* $k_0 \subset k$ *be a subfield and let* $a : (\mathrm{Sch}/k)_{\mathrm{cdh}} \to (\mathrm{Sch}/k)_{\mathrm{Zar}}$ *be the* *morphism of sites induced by the identity functor. Then there is a canonical isomorphism*

$$H^i(X, \Omega^p_{X/k_0}) \simeq H^i_{\mathrm{cdh}}(X, \Omega^p_{/k_0}).$$

for all $p, i \geq 0$.

Corollary 3.5.2. *Let* $Y_\bullet$ *be a smooth semi-simplicial scheme over* k *and let* $k_0 \subset k$ *be a subfield. Denote by* $a : (\mathrm{Sch}/k)_{\mathrm{cdh}} \to (\mathrm{Sch}/k)_{\mathrm{Zar}}$ *the morphism of sites induced by the identity functor and let* $\Omega^p_{/k_0} \in (\mathrm{Sch}/k)^{\sim}_{\mathrm{cdh}}$. *Then there is a canonical isomorphism*

$$H^i(Y_\bullet, \Omega^p_{Y_\bullet/k_0}) \simeq H^i(Y_\bullet, Ra_*\Omega^p_{/k_0}).$$

for all $p, i \geq 0$. *If* $Y_\bullet \to X$ *is in addition a* cdh*-hyperresolution, then there is a canonical isomorphism*

$$H^i_{\mathrm{cdh}}(X, \Omega^p_{/k_0}) \simeq H^i(Y_\bullet, \Omega^p_{Y_\bullet/k_0})$$

for all $p, i \geq 0$.

Proof. The first statement follows directly from Prop. 3.5.1. The second statement follows from the first one and from Corollary 3.4.6. □

Let us discuss the Künneth formula for Kähler differentials. Let $k_0 \subset k$ be a subfield and let $X_0 \in \mathrm{Sch}/k_0$ be smooth over k_0. Denote by $X \in \mathrm{Sch}/k$ the fiber product $X_0 \times_{k_0} k$. Consider the fiber diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X_0 \\ \downarrow{\scriptstyle p} & & \downarrow \\ \mathrm{Spec}(k) & \longrightarrow & \mathrm{Spec}(k_0) \end{array} \quad .$$

Note that there is a fundamental exact sequence for Kähler differentials

$$0 \to p^*\Omega^1_{k/k_0} \to \Omega^1_{X/k_0} \to \Omega^1_{X_k/k} \to 0 \tag{3.5.1}$$

induced by the morphisms $X \to \mathrm{Spec}(k) \to \mathrm{Spec}(k_0)$ (see [56, Proposition 20.6.2]). Moreover, the base-change property for Kähler differentials [56, Proposition 20.5.5] yields $\Omega^1_{X/k} \simeq f^*\Omega^1_{X_0/k_0}$ and gives a splitting of the short exact sequence (3.5.1). After applying the p-th exterior power, we get a Künneth decomposition:

$$\Omega^p_{X/k_0} \simeq \bigoplus_{p=i+j} \Omega^i_{X/k} \otimes p^*\Omega^j_{k/k_0}. \tag{3.5.2}$$

It is clear that, by passing to (Zariski) cohomology, there is a Künneth decomposition for (Zariski) cohomology groups. A similar statement is true for the cdh-cohomology groups:

Proposition 3.5.3. *Let $k_0 \subset k$ be a subfield, let $X_0 \in \mathrm{Sch}/k_0$ and let $X \in \mathrm{Sch}/k$ be the fiber product $X = X_0 \times_{k_0} k$. Then there is an isomorphism*

$$H^q_{\mathrm{cdh}}(X, \Omega^p_{/k_0}) \simeq \bigoplus_{p=i+j} H^q_{\mathrm{cdh}}(X, \Omega^i_{/k}) \otimes \Omega^j_{k/k_0}$$

for all $p, q \geq 0$, which is functorial with respect to morphisms (of finite type, separated and) definable over k_0.

Proof. Let $n = \dim(X_0)$. Let $Y_{0\bullet}$ be a cdh-hyperresolution of X_0 constructed as in Theorem 3.3.9. Observe that $Y_\bullet = Y_{0\bullet} \times_{k_0} k$ is a cdh-hyperresolution of X. We have a sequence of isomorphisms

$$\begin{aligned} H^q_{\mathrm{cdh}}(X, \Omega^p_{/k_0}) &\simeq H^q(Y_\bullet, \Omega^p_{Y_\bullet/k_0}) \\ &\simeq \bigoplus_{p=i+j} H^q(Y_\bullet, \Omega^i_{Y_\bullet/k}) \otimes \Omega^j_{k/k_0} \\ &\simeq \bigoplus_{p=i+j} H^q_{\mathrm{cdh}}(X, \Omega^i_{/k}) \otimes \Omega^j_{k/k_0}, \end{aligned}$$

where we used the third equation of Corollary 3.5.2 for the first and the third equality, and (3.5.2) is used in the second equality. Functoriality follows from Corollary 3.3.10. □

For quotient varieties, we have a particularly nice description of their cdh-cohomology with values in $\Omega^p_{/k_0}$ in terms of Zariski cohomology groups:

Proposition 3.5.4. *Let $k_0 \subset k$ be a subfield, let $M \in \mathrm{Sch}/k$ be a smooth quasi-projective scheme over k and let G be a finite group acting k-linearly on M. Denote by M/G the corresponding quotient variety. Then*

$$H^q_{\mathrm{cdh}}(M/G, \Omega^p_{k_0}) \simeq H^q(M, \Omega^p_{M/k_0})^G$$

for all $p, q \geq 0$.

Proof. First note that there is an isomorphism $H^q_{\mathrm{cdh}}(X, \Omega^p_{/k_0}) \simeq H^q_{\mathrm{eh}}(X, \Omega^p_{/k_0})$ for all $X \in \mathrm{Sch}/k$ (see e.g. [66, Corollary 2.8]), which basically follows from the fact that the $\Omega^p_{/k_0}$'s satisfy descent for étale covers.

By the analogue statement of Corollary 3.5.2 for the eh-topology and by Lemma 3.3.3 one has

$$H^q_{\mathrm{eh}}(M/G, \Omega^p_{/k_0}) \simeq H^q(\mathrm{Ner}(G, M), \Omega^p_{\mathrm{Ner}(G,M)/k_0}).$$

The result follows from Prop. 3.1.4. □

Corollary 3.5.5. *Let G be a finite group acting linearly on $\mathbb{A}^n = \mathbb{A}^n_k$ and denote by $\mathbb{A}^n/G$ the quotient variety. Then*

$$H^q_{\mathrm{cdh}}(\mathbb{A}^n/G, \Omega^p_{/\mathbb{Q}}) = 0$$

for all $p \geq 0$ and $q > 0$.

Let us finally describe the relation between cdh-cohomology groups and negative K-theory.

Theorem 3.5.6 (Cortinas-Haesemeyer-Walker-Weibel [36, Theorem 1.2])**.** *Let R be a finitely generated positively graded algebra over k and let R_0, the 0-th component of R, be a local artinian ring with residue field k. Then there is a decomposition*

$$\mathrm{K}_0(R) \simeq \mathbb{Z} \oplus \mathrm{Pic}(R) \oplus \bigoplus_{i=1}^{\dim(R)-1} H^i_{\mathrm{cdh}}(R, \Omega^i_{/\mathbb{Q}})/dH^i_{\mathrm{cdh}}(R, \Omega^{i-1}_{/\mathbb{Q}}),$$

and for $m > 0$, there is a decomposition

$$\mathrm{K}_{-m}(R) \simeq H^m_{\mathrm{cdh}}(R, \mathcal{O}) \oplus \bigoplus_{i=1}^{\dim(R)-m-1} H^{m+i}_{\mathrm{cdh}}(R, \Omega^i_{/\mathbb{Q}})/dH^{m+i}_{\mathrm{cdh}}(R, \Omega^{i-1}_{/\mathbb{Q}}),$$

where $dH^j_{\mathrm{cdh}}(R, \Omega^{i-1}_{/\mathbb{Q}})$ is the image of the map $d : H^j_{\mathrm{cdh}}(R, \Omega^{i-1}_{/\mathbb{Q}}) \to H^j_{\mathrm{cdh}}(R, \Omega^i_{/\mathbb{Q}})$ induced by the Kähler differential.

Corollary 3.5.7. *Let G be a finite group acting linearly on $\mathbb{A}^n = \mathbb{A}^n_k$ and let $\mathbb{A}^n/G$ be the quotient variety. Assume that $\mathbb{A}^n/G$ has an isolated singularity at 0. Then*

$$\mathrm{K}_0(\mathbb{A}^n/G) \simeq \mathbb{Z} \quad \textit{and} \quad \mathrm{K}_{-m}(\mathbb{A}^n/G) = 0,$$

for all $m > 0$.

Proof. By Theorem 3.5.6 and Corollary 3.5.5 it follows that

$$\mathrm{K}_0(\mathbb{A}^n/G) \simeq \mathbb{Z} \oplus \mathrm{Pic}(\mathbb{A}^n/G) \quad \text{and} \quad \mathrm{K}_{-m}(\mathbb{A}^n/G) = 0$$

for all $m > 0$. Moreover, the Picard group of a normal positively graded k-algebra is zero and the result follows. □

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