RESULTS ON THE GENERALISED SHIFT GRAPH

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Abstract

In the paper 'On Chromatic Number of Infinite Graphs' (1968), Erdős and Hajnal defined the Shift Graph to be the graph whose vertices are the n-element subsets of some totally ordered set S, regarded as increasing n-tuples, such that $A = (a_1, ..., a_n)$ and $B = (b_1, ..., b_n)$ are neighbours iff $a_1 < b_1 = a_2 < b_2 = a_3 < ... < b_{n-1} = a_n < b_n$ or the other way round. In the paper 'On Generalised Shift Graphs' (2014), Avart, Łuczac and Rödl extend this definition to include all possible arrangements of the a_is and b_is, known as 'types'. In this thesis, we will consider a selection of these types and study the corresponding graphs. All the types we consider will be written as $1^k 3^m 2^k$, where $k + m = n$, which means that the final m entries of $(a_1, ..., a_n)$ are identified with the first m entries of $(b_1, ..., b_n)$. Such a graph with totally ordered set S and type $1^k 3^m 2^k$ is denoted $G(S, 1^k 3^m 2^k)$.

There are two related questions here. One is when the (undirected) graphs $G(S, 1^k 3^m 2^k)$ and $G(S', 1^k 3^m 2^k)$ are distinct (non-isomorphic) for distinct linear orderings S, S'. The other is to what extent we can recognise S inside the graph (called 'reconstruction'). A positive solution to the latter also yields one for the former, since if we can recognise S in its graph, and S' in its graph, and they are distinct, then so must the graphs be. We focus on these main cases: S is finite, S is an ordinal, S is a more general totally ordered set. The tools available for reconstruction depend on whether S is a total ordering, a dense total ordering, or an ordinal. There are additional technical complications in the case where S has endpoints, and similarly for S containing relatively small finite segments.

Since these graphs are undirected, we expect in general only to recover a linear ordering up to order reversal. The natural notion here is of 'linear betweenness', and we spend some time studying linear betweenness relations in their own right, also considering the induced relations on n-tuples. Betweenness relations on n-tuples correspond to shift graphs of the special form $G(S, 1^n 2^n)$ (i.e. in which no identifications are made).

The main contribution of the thesis is to show how it is possible in many instances to reconstruct the underlying linear order (often just up to order-reversal) from the generalized shift graph. A typical example of this is Theorem 4.4. The techniques are to employ graph-theoretical features of the relevant shift graph, such as co-cliques or pairs of co-cliques fulfilling various conditions to 'recognize' points and relations of the underlying linear order. There are many variants depending on the precise circumstances (dense or not, with or without endpoints, well-ordered, only partially ordered).

We show that for ordinals α and β, if $G(\alpha, 1^k 3^m 2^k)$ is isomorphic to $G(\beta, 1^k 3^m 2^k)$ then $\alpha = \beta$. Note that the fact that (in the infinite case) α is not isomorphic to its reversed ordering means that the betweenness relation is enough to give us the ordering. This result does not necessarily extend to all total orderings in full generality, but we obtain many partial results. A suite of techniques is used, which may be adapted suitably depending on circumstances, endpoints or not, density, or finiteness.

In a more open-ended chapter, we generalise as much of the material for total orders to

partial orders, the easiest case being that of trees.

Work by Rubin [15] considers reconstruction in a slightly different sense: that a structure can be reconstructed from its automorphism group. So we have two ‘levels’ of reconstruction: of the graph from its automorphism group, and then if possible of the underlying total order from the graph. With this in mind, we study the automorphism groups of many of the graphs arising, managing in several cases to give quite explicit descriptions, so answering Rubin's reconstruction question - i.e. whether or not a structure can be ‘reconstructed’ from its automorphism group (as in for example [17]) - where possible. For instance, we show that it is possible to determine S from $Aut(G(S, 132))$ if and only if $G(S, 132)$ contains no two points sharing exactly the same neighbour sets.

Finally we return to colouring questions as in the original paper of Erdős and Hajnal, and show that the chromatic number of $G(\kappa, 132)$ is equal to κ for any strong limit cardinal κ.

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1 Introduction

1.1 Background Material and Known Results

Shift graphs were introduced by Erdős and Hajnal in [7] in order to give certain examples in the theory of graph colourings (and compactness type questions about these). Originally these are constructed in the finite case, or for ordinals. The basic example is as follows: a positive integer n, and a finite or infinite ordinal α are given, and the vertices of the graph consist of n-tuples with elements in S, enumerated in increasing order, and $A = (a_1, \ldots, a_n)$ is joined to $B = (b_1, \ldots, b_n)$ if either $b_i = a_{i+1}$ for all $i < n$, or $a_i = b_{i+1}$ for all $i < n$. This pattern of identifications is encapsulated by the 'type' 1333...32, having $n-1$ 3s, the 3s indicating the entries of the two sequences which are identified, and the 1 and 2 telling us that the first and last points are not identified with any element of the other set. Avart, Luczak, and Rödl [3] extended this to more general 'types', which correspond to other arrangements of the a_i and b_i, thus defining the *generalised* shift graph. The formal definition of 'type' is given at the beginning of Chapter 2 (Definition 2.27), and the generalised shift graph on a total linear ordering S with type τ is an undirected graph written $G(S, \tau)$.

A *disjoint type* τ is a type where the two n-tuples are disjoint, i.e. τ contains no 3s.
A *Specker Graph* is a Shift Graph $G(\kappa, \tau)$ where τ is a disjoint type.

Erdős and Hajnal [6] showed:

- The chromatic number $\chi(G(\kappa, \tau))$ of any uncountable Specker graph of cardinality κ is equal to κ for κ an infinite cardinal,
- For every positive integer s there is a disjoint type τ such that no Specker graph $G(\kappa, \tau)$ embeds any odd cycle of length $\leq 2s + 1$, and
- For any infinite cardinal κ, the chromatic number of $G(\kappa, 132)$ is the $\min\{\alpha : \exp(\alpha) \geq \kappa\}$.

In Erdős and Rado [8] showed that for any infinite cardinal κ, the chromatic number of $G(\kappa, 112122)$ is κ.

Komjath has remarked (personal communication) that for any infinite cardinal λ and $n \geq 0$, $\chi(G(\lambda, 13^n 2)) \leq \kappa$ iff $\lambda \leq exp_n(\kappa) = 2^{2^{\cdots 2^\kappa}}$ (n times)

Avart, Łuczak, and Rödl [3] defined a *simple type* as a disjoint type that can be partitioned into subsections of the form $1^n 2^n$ or $2^n 1^n$. For example, 112221111222 is a simple type, and can be partitioned into the subsections 1122 21 111222, but 112122 is not. They then show that if τ is simple with a total number of n 1s (and n 2s), then $\chi(G(k, \tau)) = \lfloor k/n \rfloor$ for finite k. They showed that for any finite z, a disjoint type τ is either simple, or $\chi(G(z, \tau)) \leq 2\log_2(z)$.

The *width* of a type τ is the number of 1s plus the number of 3s. Intuitively, this is the value of n for the graph $G(S,\tau)$ whose vertices are n-tuples.

We can define a *subtype* τ' of a disjoint type τ of width n as follows: Let X be a subset of $\{1,...,n\}$. Then if τ' is obtained by removing all ith occurrences of 1 and ith occurrences of 2 from τ, for all $i \in X$, then τ' is a subtype of τ.

If τ' is a subtype of τ, then for every integer x

$$\chi(G(x,\tau)) \leq \chi(G(x,\tau'))$$

In the final section of [3], they suggest types of the form $1^n 3^m 2^n$. They prove the following results:

For any $n, m \geq 1$ and $k \in \mathbb{N}$ we have

$$\log_{(m/n)} \frac{k}{n} \leq \chi(G(k, 1^n 3^m 2^n)) \leq \log_{(m/n)} k$$

For any $n, m \geq 1$ and infinite κ we have

$$\chi(G(\kappa, 1^n 3^m 2^n)) = \min\{\alpha : exp_{(\lfloor m/n \rfloor)}(\alpha) \geq \kappa\}$$

Finally, they define the type $\delta_{a,b}$ as a 1s followed by b copies of 21, and ending with a 2s (so for example $\delta_{3,4}$ is the type 11121212121222), and show that for any $a \geq 1$ and $b \geq 1$,

$$\log_{(\lceil b/a \rceil)} \frac{k}{2^b} \leq \chi(G(k, \delta_{a,b})) \leq C(\log_{(\lceil b/a \rceil)} k)^{(\frac{\lceil b/a \rceil + 1}{2})}$$

for some constant C depending on a and b.

1.2 Reconstructions

In this thesis we generalise the notion of Shift Graph by allowing the underlying set S to be any totally ordered set, and allowing any type of the form $1^n 3^m 2^n$ as in the final section of [3]. Our main theme is reconstruction. This has been extensively studied by Rubin in a series of papers, for instance [15]. For a general class $\mathcal{C}$ of structures one wishes to know what information about a member X of $\mathcal{C}$ is provided by its automorphism group. Under favourable circumstances, one may be able to prove statements of the form: for any $X_1, X_2 \in \mathcal{C}$, $\text{Aut}(X_1) \cong \text{Aut}(X_2) \Rightarrow X_1 \cong X_2$ (or possibly, just that X_1 and X_2 are elementarily equivalent). Thus one can say in this case that the 'essential' information about a structure is carried by its automorphism group.

In the present context, we have the opportunity to consider reconstruction at *two* levels. First, from the discussion above, it is clear that given a total ordering $(S,<)$, and a type

τ, we can define a corresponding graph $G(S, \tau)$. We have ostensibly 'lost' the information about what S was, as we now just have an (undirected) graph. However, it may be that under certain circumstances, we can show that $G(S_1, \tau) \cong G(S_2, \tau) \Rightarrow (S_1, <) \cong (S_2, <)$, so that the more general linear order can be 'recovered' from the graph. One can also ask whether the type can be recovered. We give results of this kind for both ordinals and linear orders. In addition, in the spirit of Rubin's work ([14], [15], [16], [17], [18]) , one can ask whether the graph (and hence in certain cases also the total order) can be recovered from its automorphism group. This problem is addressed in the final chapter, where some of the automorphism groups are given explicit descriptions. We remark that if any set of two or more elements of the graph share exactly the same neighbour sets, permuting these elements will result in the a non-trivial automorphism, and the graph G can therefore not be recovered from $Aut(G)$.

We also consider the notion of interpretability. Intuitively this says that one structure can be 'recognised' in another, for instance that a total order can be recognised inside a shift graph, or vice versa. The definition we follow comes from Hodges [10], and formalises the above intuition. Two versions are given, first-order and second-order, depending on which formulae arise in the definition. We indicate which of our interpretations are known to be first-order. The rest will be second-order (although in fact these may be first-order, but they appear second-order and we do not know whether they can be reformulated in a first-order way). Thus when we say something is 'second-order interpretable', what we really mean is that it is 'at least second-order interpretable'.

1.3 Structure of Thesis and Summary of Results and Techniques

We start by defining the generalised shift graph in the preliminaries, followed by looking at some very simple cases including $\tau = 3$, $\tau = 3^n$, $\tau = 12$, and $\tau = 123$. We also look at the case where $\tau = 132$, and compare this to the line graph on a set S (Definition 2.26).

In Section 2.4 we introduce some general notation which will be used throughout the thesis within the various reconstruction proofs. Most of the proofs will use this notation in some form or another.

In Chapter 3, we look at total betweenness relations in their own right. We deduce some basic results (many of which have already been shown elsewhere), including Theorem 3.15 and Corollary 3.16, which state that exactly two total orderings arise from a betweenness relation B on a set S, such that if one of them is $<$ the other is $>$. We also consider variations on the betweenness definition, namely how betweenness can function as a relation on more than 3 points. Finally, we show that given these alternative betweenness relations on a set S, we can reconstruct the standard betweenness B on S, in Lemmas 3.22, 3.23, 3.28, and 3.29, and in Theorem 3.32.

We then tackle the main problem in the thesis: whether or not it is possible to recover S in the shift graph $G(S, \tau)$.

We start with the case where S is an infinite linear ordering without endpoints, in Chapter 4. We tackle the problem by considering infinite co-cliques in $G(S, 132)$, and similarly in $G(S, 1^n32^n)$ (the notion of 132 and 1^n32^n is defined in Section 2.2, starting with Definition 2.28). In each case, we start with a lemma describing a pair of co-cliques C and D in G, which will eventually 'represent' a single point in S (such as, for example, Lemma 4.1). We then look at the union of these co-cliques, and we let $A = C \cup D$. We use sets of the form A to represent single points of S, and intersections between varying sets of this form to determine betweenness on points of S.

Due to us essentially talking about sets of sets in these proofs, we see that they are (most likely) not first-order. In the $G(S, 1^n32^n)$ case, we rely on some of the betweeness results from earlier; namely, that we can determine betweenness from betweenness on n-tuples (defined in Section 3, Definition 3.19).

For the $G(S, 13^n2)$ case, we use a different trick (Theorem 4.8). Again, we construct infinite co-cliques C, D with our desired properties, and let $A = C \cup D$; however, we see that for two such sets A, A', their intersection is non-empty if and only if there is an edge between their corresponding vertices in $G(S, 13^{n-1}2)$. Thus we can recursively obtain $G(S, 13^i2)$ for $1 \leq i \leq n$, and hence $G(S, 132)$ and S.

A similar trick can be used in the case $G(S, 1^n3^m2^n)$ to obtain $G(S, 1^n3^k2^n)$, where $n \equiv k \mod m$, again reducing the problem to $G(S, 1^n3^m2^n)$ for $m < n$, as in Lemma 4.9. In this final case, we start by considering dense S, followed by utilising some results on total betweenness relations (namely that we can obtain betweenness from n-betweenness) to obtain S from $G(S, 1^n3^m2^n)$. Finally, we show that S can be obtained from $G(S, 1^n3^m2^n)$ where $m < n$, in Theorem 4.12, utilising some of the results in Chapter 3. We then briefly look at partial orderings in Section 4.2, more specifically trees without endpoints, which behave similarly.

In the ordinal shift graphs chapter, we approach the finite cases one at a time in Theorems 5.1, 5.2, and 5.3, starting with a similar method to Chapter 4 which provides a uniform way of obtaining Z from $G(Z, 13^n2)$.

For infinite ordinals an alternative method is used for proving that we can recover α from $G(\alpha, 132)$, arguing transfinitely. We essentially start at 0 and 'move our way up', obtaining each ordinal recursively and taking limits as needed. Because of this, we will see that once we have reconstructed the graph, we can actually reconstruct each individual vertex within the graph (for example the vertex $(0, 1)$ in $G(\omega, 132)$). This is shown in the discussion following Theorem 5.7.

We initially consider limit ordinals, as endpoints pose some technical problems in the successor case. Once again, we start with lemmas (5.5 and 5.6) about sets of points in $G(\alpha, 13^n2)$ which satisfy some particular properties, and these sets will end up 'representing' ordinals $< \alpha$. Unlike in the general total ordering case, we construct these inductively, starting with 0 and taking unions to obtain limits. We can thus determine α by considering the union of all of these sets, as in Theorem 5.7. We then treat the 1^n32^n case Theorem in

5.10, followed by the $1^n3^m2^n$ case in Theorem 5.12, both approached in a similar manner.

We then consider successor ordinals $\alpha = \alpha_0 + k$ (where α_0 is a limit). The method here is typically to 'remove' the endpoints one at a time, by for instance comparing the set $\{(0, y) : y > 0\}$ to the set $\{(x, \alpha_0 + k - 1) : x < \alpha_0 + k - 1\}$, and noticing that they do not satisfy the same properties. Once we can tell the difference, we can remove the latter set, and repeat k times to obtain the graph $G(\alpha_0, 132)$ 'inside' the graph $G(\alpha, 132)$, and then determining α_0 as in the limit case (Theorems 5.14 and 5.15). This can fairly easily be extended to the 1^n32^n case, except that we will see that $G(\alpha_0 + k_1, 1^n32^n) \cong G(\alpha_0 + k_2, 1^n32^n)$ for all $k_1, k_2 \leq n$, as in Theorem 5.17. Finally, using a sequence of lemmas (5.19 to 5.21) to show that if two points x, z in $G(\alpha, 1^n3^m2^n)$ share exactly one neighbour, and there is no point v sharing exactly one neighbour with both x and z, then x and z must have a certain relationship to one another; namely, they must be neighbours in the graph $G(\alpha, 1^{2n}3^{m-n}2^{2n})$, and so we can recursively reduce the problem to graphs of the form $G(\alpha, 1^n3^m2^n)$ with $m < n$, and showing that α is interpretable from this graph in Theorem 5.22. Again, assuming $\alpha = \alpha_0 + k$, we have some restrictions on k here - so we take $k > n \times z$, where $z \equiv m \bmod n$.

In Chapter 6 we consider whether it is possible to detemine a graph of this form from its automorphism group, and show that $Aut(G(S, 13^n2)) \cong Aut(S, B)$ for any total ordering S and $n \geq 1$ (Theorems 6.6 and 6.8). We also conjecture that if S is any total ordering without endpoints:

$$Aut(G(S, 1^n32^n)) = \prod_{I \subseteq S} (Sym(I)\ Wr\ Aut(S, B))$$

for I a convex subset of S and $n \geq 1$.

In the final chapter we show that $\chi(G(\kappa, 1^n32^n)) = \kappa$ for any strong limit κ (Theorem 7.15). We build up to this result, first by defining measurable cardinals and proving the result for κ measurable using a fairly perspicuous proof (Theorem 7.9), and then by extending the ideas in this proof to the strong limit case.

2 Preliminaries

2.1 Set Theory and Basic Graph Theory

In these thesis we use basic notions in formal logic and set theory as given in [5] (Chapter 2) or [12] (Chapter 1). We also use the elementary theory of ordinals given in both of these. We use the terms 'linear order' and 'total order' interchangeably.

Definition 2.1. A *first-order language* $\mathcal{L}$ is a collection of symbols including:

- Parentheses (,), logical connective symbols $\neg$, $\rightarrow$, $\vee$, $\wedge$, and the equality symbol $=$ (optional)
- Quantifier symbols: $\forall$, $\exists$
- Function symbols: for each positive integer n, some set (possibly empty) of symbols, called n-place function symbols
- Relation symbols: for each positive integer n, some set (possibly empty) of symbols, called n-place predicate symbols
- Variables: $v_1, v_2, ...$ (one for each positive integer n)
- Constant symbols: some set (possibly empty) of symbols

A *second-order language* additionally contains the following symbols:

- Relation variables: for each positive integer n, we have the n-place predicate variables $X_1^n, X_2^n, ...$
- Function variables: for each positive integer n, we have the n-place function variables $F_1^n, F_2^n, ...$

In this thesis the most common language we use is the language of graphs, that is, the language with a single binary relation E representing the edge relation. We also use the language of set theory, consisting of a single binary relation $\in$, and the language of orderings, consisting of a single binary relation $<$.

An *expression* in a first-order language is a finite sequence of symbols from that language. We are interested in expressions that "make sense", and so we define the following:

Definition 2.2. The set of *terms* is defined inductively: variables and constants are terms, and if F is a function symbol, then $F(t_1, ..., t_n)$ is a term, where $t_1, ..., t_n$ are terms.

Definition 2.3. An *atomic formula* is an expression of the form $R(t_1, ..., t_n)$, where R is an n-ary relation symbol and $t_1, ..., t_n$ are terms, or an expression of the form $t_1 = t_2$, where t_1 and t_2 are terms.

Definition 2.4. A *formula* is an expression built up from from atomic formulae by use (zero or more times) of the logical connectives and the quantifier symbols.

Definition 2.5. A *sentence* is a formula where every variable is bound by a quantifier (i.e. it has no *free variables*).

Definition 2.6. A *second-order formula* is a formula in a second-order language where we can also quantify over function and relation variables.

We would now like to interpret these formulae inside a model.

Definition 2.7. A *model* (or *structure*) $\mathfrak{A}$ for a given first-order language $\mathcal{L}$ is a set A (called the *universe* (or *domain*) of $\mathfrak{A}$) along with functions, relations, and constants which interpret the function, relation, and constant symbols in $\mathcal{L}$. More specifically:

- $\mathfrak{A}$ assigns an n-ary operation $f^{\mathfrak{A}} \subseteq A$ to each n-ary function symbol f
$$\text{i.e. } f^{\mathfrak{A}} : A^n \to A$$
- $\mathfrak{A}$ assigns an n-ary relation $R^{\mathfrak{A}} \subseteq A$ to each n-ary relation symbol R, i.e. $R^{\mathfrak{A}}$ is a set of n-tuples of members of the universe
- $\mathfrak{A}$ assigns a member $c^{\mathfrak{A}}$ of the universe A to each constant symbol c

The idea is that $\mathfrak{A}$ assigns meaning to non-logical symbols of the language.

We would now like to determine when a formula φ is true (holds) in a model $\mathfrak{A}$.

Definition 2.8. Let φ be an atomic formula of the form $R(t_1, ...t_n)$ with parameters in $\mathfrak{A}$. Then we say that φ is true in $\mathfrak{A}$, written $\mathfrak{A} \models \varphi$ if $R(t_1, ...t_n)$ holds in A. For detail see [5].

For any given model $\mathfrak{A}$, every sentence will either hold or fail in that model. A formula, does not hold or fail in a model, but for any interpretation of the variables in the formula, it either holds or fails.

Definition 2.9. We now define inductively what it means for a formula φ (with an interpretation of the variables) to hold in $\mathfrak{A}$:

- $\mathfrak{A} \models \neg\varphi$ if φ does not hold in $\mathfrak{A}$
- $\mathfrak{A} \models \varphi \to \psi$ if either φ does not hold in $\mathfrak{A}$, or ψ holds in $\mathfrak{A}$
- $\mathfrak{A} \models \varphi \vee \psi$ if φ and ψ both hold in $\mathfrak{A}$
- $\mathfrak{A} \models \varphi \wedge \psi$ if either φ or ψ hold in $\mathfrak{A}$
- $\mathfrak{A} \models \forall x\varphi(x)$ if either $\varphi(x)$ holds for every x in the universe A of $\mathfrak{A}$
- $\mathfrak{A} \models \exists x\varphi(x)$ if either $\varphi(x)$ holds for some x in the universe A of $\mathfrak{A}$

Definition 2.10. Two models $\mathfrak{A}$ and $\mathfrak{B}$ are *elementarily equivalent*, denoted by $\mathfrak{A} \equiv \mathfrak{B}$, if they satisfy the same first-order sentences, i.e. $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi$ for every sentence φ.

Definition 2.11. Let $\mathfrak{A}$ and $\mathfrak{B}$ be models for the same language $\mathcal{L}$ with respective universes A and B. Then $\mathfrak{A} \subseteq \mathfrak{B}$ if $A \subseteq B$ and for every atomic formula φ with parameters in A, $\mathfrak{A} \models \varphi \Leftrightarrow \mathfrak{B} \models \varphi$.

Definition 2.12. Let S be any set. Then a *total order* or *linear order* $\leq$ on S is a binary relation such that:

- If $a \leq b$ and $b \leq a$ then $a = b$ for all $a, b \in S$ (antisymmetry)
- If $a \leq b$ and $b \leq c$ then $a \leq c$ for all $a, b, c \in S$ (transitivity)
- For all $a, b \in S$, either $a \leq b$ or $b \leq a$ (total condition)

Definition 2.13. Let X be a totally ordered set. Then X is *well-ordered* if every non-empty subset has a minimal element.

Any two well-orderings X and Y are comparable, so that there is an order-isomophism that maps X to an initial segment of Y (in which case $X \leq Y$), or vice versa [12].

For every proper class of isomorphic well-orderings, we would like to choose a representative called an *ordinal*.

Ordinals are defined using transfinite induction, such that each ordinal is the set containing all previous ordinals, for example $3 = \{0, 1, 2\}$ (for details see [12]).

Ordinals generalise the numbers 0,1,2,3,... to include transfinite numbers. The set of all natural numbers (with 0) is called ω, and from there we continue with $\omega+1, \omega+2, \omega+3, ...$.

The proper class of all ordinals is itself well-ordered.

Definition 2.14. An ordinal α is a *cardinal* if there is no bijection from α into any $\beta < \alpha$.

Notice that since the class of ordinals is well-ordered, then so is the class of cardinals.

Definition 2.15. Let S be any set. Then the *cardinality* of S, denoted by $card(S)$, is the unique cardinal κ isomorphic to S.

Definition 2.16. The *power set* $\mathcal{P}(X)$ of a set X is the set of all subsets of X.

In general any ordinal is equal to the set of its predecessors, and this convention is also followed in the finite case, e.g. $5 = \{0, 1, 2, 3, 4\}$. We use ω for $\{0, 1, 2, 3, ...\}$ and $\mathbb{N}$ for $\{1, 2, 3, ...\}$, and we use ω_1 for the least uncountable ordinal.

Definition 2.17. A *graph* G is an ordered pair $(V(G), E(G))$ such that:

- $V(G)$ is a set, known as the *vertices* of G
- $E(G)$ is a symmetric binary relation on $V(G)$, known as the *edge* relation on $V(G)$

Definition 2.18. A *directed graph* G is an ordered pair $(V(G), E(G))$ such that:

- $V(G)$ is a set, known as the *vertices* of G
- $E(G)$ is a (not necessarily symmetric) binary relation on $V(G)$, known as the *edge* relation on $V(G)$, such that if aEb holds then there is an edge *from a to b*

In this thesis, all graphs are simple (there are no edges going from a to itself for all $a \in V(G)$) and undirected unless stated otherwise. We sometimes use G and $V(G)$ interchangably.

Definition 2.19. Let v be some vertex in a graph G. Then the neighbour set N_v of v is defined as the set of neighbours of v in G.

Definition 2.20. Let G be any graph with vertex set $V(G)$. A set of vertices $S \subseteq V(G)$ is a *clique* if there is an edge between every pair of vertices in S.

Definition 2.21. Let G be any graph. A set of vertices $S \subseteq V(G)$ is a *co-clique* if S there is no edge between any pair of vertices in S.

Definition 2.22. Let κ be any cardinal. The complete graph K_κ consists of κ vertices such that there is an edge between every pair of vertices.

Definition 2.23. Let v be a vertex in a graph G. Then the *degree* of v, denoted by $d(v)$, is the number of edges incident to v.

Definition 2.24. Let G be a graph. A *path* in G is a finite or infinite sequence of edges e_i in $E(G)$ such that for distinct vertices v_i, the edge e_i has endpoints v_i and v_{i+1}.

Definition 2.25. Let G be a graph. A *connected component* of G is a subgraph G' of G such that for any two vertices $x, y \in G'$ there is a path from x to y, and such that for every $x \in G'$ there is no path from x to any $y \in G \backslash G'$.

Definition 2.26. Let G be any graph. Then the *line graph* $L(G)$ on G is another graph such that:

- The vertices of $L(G)$ are the edges of G, and
- There is an edge between two vertices of $L(G)$ if and only if their corresponding edges in G share an endpoint in G

2.2 Shift Graphs

Definition 2.27. Let $k, \ell \in \mathbb{N}$ be fixed, $k \leq \ell$. We say that a sequence $\tau = (\tau_i)_{i=1}^{\ell}$ is a *type of width k and length ℓ* if $\tau_i \in \{1,2,3\}$ and $|\{i : \tau_i \in \{1,3\}\}| = |\{i : \tau_i \in \{2,3\}\}| = k$. [3]

This says that a type is a sequence of 1s, 2s, and 3s such that the number of 1s is equal to the number of 2s. This type effectively encodes the edges in the graph by showing what the pattern needs to be between two k-tuples arranged in increasing order for there to be an edge between them.

We thus interpret this type as follows.
Definition 2.28. Let x and y be increasing k-tuples with entries in some totally ordered set $(S,<)$. Note that a k-tuple $x = (x_1, x_2, ..., x_k)$ is said to be *increasing* if $x_1 < x_2 < ... < x_k$, so x and y correspond to k-sets, but enumerated in increasing order.

Let $x \cup y = \{z_1, ..., z_\ell\}$, with $z_1 < z_2 < ... < z_\ell$. Then we say that the pair x, y *has type* τ (denoted by $t(x,y) = \tau$) iff:

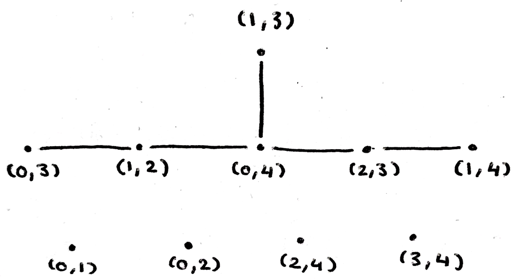
$$\begin{aligned}
\tau_i = 1 &\Rightarrow z_i \in x \setminus y, \\
\tau_i = 2 &\Rightarrow z_i \in y \setminus x, \text{ and} \\
\tau_i = 3 &\Rightarrow z_i \in x \cap y \\
&\text{for } 1 \leq i \leq \ell
\end{aligned}$$

Throughout this thesis, unless otherwise stated we assume that any n-tuple $(x_1, ..., x_n)$ of a totally ordered set S is ordered by the induced ordering. We also adopt the following (non-standard) notation: $S^n = \{(x_1, ..., x_n) : x_i \in S,\ x_1 < x_2 < ... < x_n\}$.

The ordering S doesn't need to be total; this notation is also applicable if S is a partial order. Of course, in this case, the n-tuple is only defined on points that are pairwise comparable, i.e. $x_i < x_j$ or $x_j < x_i$ for all $i, j \in \{1, ..., n\}$ with $i \neq j$.

Definition 2.29. The graph $G(S,\tau)$ is defined to be the graph whose vertices are the k-element subsets of S for some k (determined through τ), and where there is an edge between x and y in $V(G(S,\tau))$ iff $t(x,y) = \tau$. We call such a graph a *shift graph.*

Example 2.30. Consider the graph $G(5, 1221)$ consisting of the 2-tuples with entries in the set $\{0,1,2,3,4\}$. There is an edge between any two vertices $x = (x_1, x_2)$ and $y = (y_1, y_2)$ if $x_1 < y_1 < y_2 < x_2$. The full graph is:



Notice that, intuitively, we would simply see the dots and lines in the example above - we would not know, for example, that the point at the top is the vertex $(1,3)$ (this is part of the information we are trying to recover). Note that several points in this structure can be permuted by an automorphism (for example the isolated points, and the larger connected component can be reversed).

Example 2.31. Consider the graph $G(\omega, 12312)$. Then $\tau = 12312$ has width 3 and length 5. Now let $x = (1,5,6)$ and $y = (3,5,8)$.

$$\begin{array}{cccccccccccc}
(& 1 & < & & & 5 & < & 6 & < & &) \\
 & & & & & \| & & & & & \\
(& & & 3 & < & 5 & < & & & 8 &)
\end{array}$$

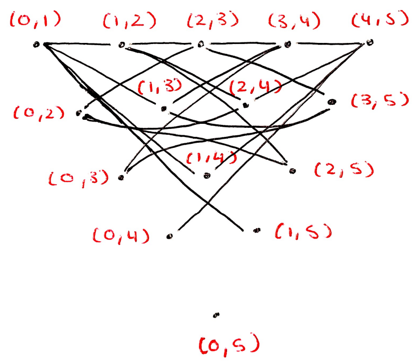
Then $x \cup y = \{1,3,5,6,8\}$, and thus $t(x,y) = \tau$. We see how the pattern above corresponds to the pattern given by 12312. Now, let $x' = (1,3,5)$ and $y' = (2,4,6)$.

$$\begin{array}{cccccccccccccc}
(& 1 & < & & & 3 & < & & & 5 & < & &) \\
(& & & 2 & < & & & 4 & < & & & 6 &)
\end{array}$$

Then $x' \cup y' = \{1,2,3,4,5,6\}$, and so $t(x',y') \neq \tau$. In fact, $t(x',y') = 121212$. We can see how the pattern above does **not** correspond to the the pattern given by 12312, because the middle coordinates aren't equal. Thus in the graph $G(\omega, 12312)$, there is an edge between the vertices x and y but not between x' and y'.

Example 2.32. Consider the graph $G(\mathbb{R}, 13332)$. Then $|\{i : \tau_i \in \{1,3\}\}| = |\{i : \tau_i \in \{2,3\}\}| = 4$, and so the vertices of $G(\mathbb{R}, 13332)$ consist of strictly increasing elements of $\mathbb{R}^4$. There is an edge between two vertices $x = (x_1, x_2, x_3, x_4)$ and $y = (y_1, y_2, y_3, y_4)$ iff $x_1 < y_1 = x_2 < y_2 = x_3 < y_3 = x_4 < y_4$ or vice versa. Any graph like this with type 133...32 is an example of the *shift graph* [7] (as opposed to the *generalised shift graph* where the types can vary more, as defined at the start of Chapter 2).

We now present a finite example of the shift graph, namely $G(6, 132)$:



Again, we have labelled this graph for convenience, but these labels are not part of the graph.

We have so far seen that these graphs contain a certain symmetry, which can be attributed to the symmetry in the underlying finite ordering.

In this thesis, we will mostly focus on the type $1^n3^m2^n$. Note the following cases which provide the 'simplest' case in various scenarios (some of which are trivial), and which we will often consider separately:

$$\tau = 1^n2^n, \ \tau = 3^n, \ \tau = 1^n32^n, \ \tau = 13^n2$$

2.3 Some Very Simple Cases

We will look at for which values of τ and S, S' we have $G(S,\tau) \cong G(S',\tau) \Rightarrow S \cong S'$, starting with some very simple cases.

Perhaps the most natural "simple" case is the graph $G(S, 3)$. It is easy to see that this is simply one giant co-clique of size $card(S)$, and similarly for $G(S, 3^n)$. Thus $G(\alpha, 3^n)$ is a set of $\binom{|\alpha|}{n}$ isolated points, and so $G(\alpha, 3^n) \cong G(\beta, 3^m)$ for all infinite ordinals α, β with $|\alpha| = |\beta|$ and for all finite $n, m \geq 1$. Hence $G(S,\tau) \cong G(S',\tau) \not\Rightarrow S \cong S'$ for the types $3^n, n \in \mathbb{N}$.

Another natural "simple" example to consider is the graph $G(S, 12)$. It is easy to see that this is simply one giant clique of size $card(S)$. Thus, $G(S_1, 12) \cong G(S_2, 12)$ for every pair of ordered sets S_1, S_2 with equal cardinality. Hence $G(S,\tau) \cong G(S',\tau) \not\Rightarrow S \cong S'$ for the type 12.

Adding a '3' onto the end of this type gives us a similar graph, although instead of one complete graph the connected components are all complete graphs (cliques) of possibly different cardinalities. If α is any ordinal, $G(\alpha, 123)$ will contain complete graphs of all cardinalities less than the ordinal α. Once again, we do not have $G(S, 123) \cong G(S', 123) \Rightarrow S \cong S'$, as for example $G(\omega + \omega, 123) \cong G(\omega + \omega + \alpha, 123)$ for all $\alpha < \omega_1$.

We now turn our attention to the graph $G(S, 132)$ for any set S. This is generally the 'simplest' case we consider.

One way to view $G(S, 132)$ is as a modification on a line graph on the complete graph $K_{card(S)}$, although in this case $K_{card(S)}$ a directed graph with an edge going from a to b if $a > b$ for distinct $a, b \in S$. Additionally, there is only an edge between two vertices in $G(S, 132)$ if the corresponding edges in $K_{card(S)}$ are not only incident to one another, but the vertex they share in $K_{card(S)}$ is the head of one edge and the tail of the other. Thus the vertices $(1, 2)$ and $(2, 3)$ are neighbours in $G(\mathbb{R}, 132)$ as the edge going from 2 to 1 and the edge going from 3 to 2 are incident to one another with one being the head of the point 2 and the other being the tail, whereas this is not the case for $(1, 3)$ and $(2, 3)$. Thus $G(\mathbb{R}, 132)$ is a modification on the line graph on the directed graph $K_{2^{\aleph_0}}$ with an edge going from a to b if $a > b$ for distinct $a, b \in R$.

We can similarly view $G(S, 1^n 32^n)$ as the family of increasing paths of $n + 1$ points in the complete graph with two such adjacent if the sink of one is the source of the other, and modifications of this remark also apply in $G(S, 1^n 3^m 2^n)$.

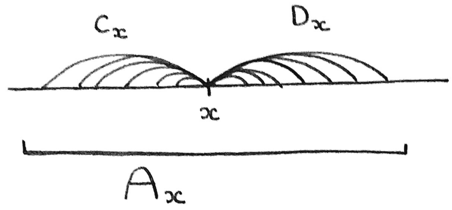
In general, we can determine the size of the underlying set Z of a finite shift graph $G(Z, \tau)$ quite easily by counting the vertices. $G(Z, 132)$ has $\binom{|Z|}{2}$ vertices, and $G(Z, 1^n 3^m 2^n)$ has $\binom{|Z|}{m+n}$ vertices.

2.4 Notation

We will now introduce some general notation which will be used throughout.

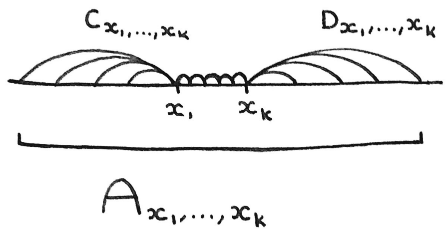
Given a set V of increasing n-tuples $(s_1, ..., s_n)$ with $s_i \in S$ for some linearly ordered set S, we define the following sets for each $x \in S$:

$C_x := \{(s_1, ..., s_{n-1}, x) \in V : s_{n-1} < x\}$,
i.e. C_x is the set of all n-tuples ending in x.
$D_x := \{(x, s_2, ..., s_n) \in V : x < s_2\}$,
i.e. D_x is the set of all n-tuples beginning in x.
$A_x := C_x \cup D_x$,
i.e. A_x is the set of all n-tuples either beginning or ending in x.



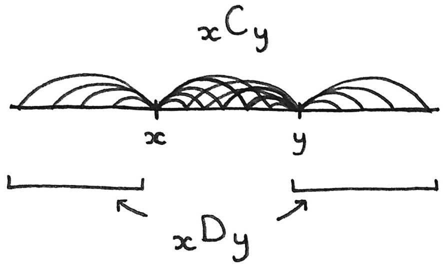
Given a set V of n-tuples $(s_1, ..., s_n)$ with $s_i \in S$ for some set S, for $k < n$ we generalise the above to k-tuples $x_1, ..., x_k \in S$ as follows:

$C_{x_1,...,x_k} := \{(s_1, ..., s_{n-k}, x_1, ..., x_k) \in V : s_{n-k} < x_1\}$,
i.e. C_x is the set of all n-tuples ending in $x_1, ..., x_k$.
$D_{x_1,...,x_k} := \{(x_1, ..., x_k, s_{k+1}, ..., s_n) \in V : x_k < s_{k+1}\}$,
i.e. D_x is the set of all n-tuples beginning in $x_1, ..., x_k$.
$A_{x_1,...,x_k} := C_{x_1,...,x_k} \cup D_{x_1,...,x_k}$
i.e. $A_{x_1,...,x_k}$ is the set of all n-tuples either beginning or ending in $x_1, .., x_k$.



Given a set V of n-tuples $(s_1, ..., s_n)$ with $s_i \in S$ for some set S, we can define the following sets which satisfy similar properties to C_x, D_x:

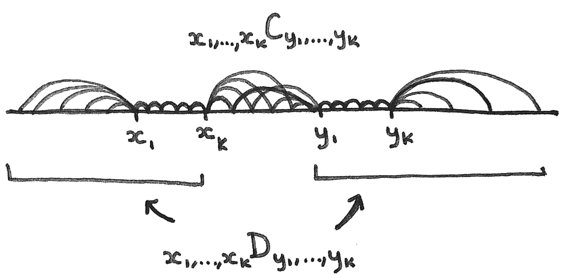
${}_xC_y := \{(x, s_2, ..., s_{n-1}, y) \in V : x < s_1, s_{n-1} < y\} = D_x \cap C_y$
i.e. ${}_xC_y$ is the set of all n-tuples beginning in x and ending in y.
${}_xD_y := \{(s_1, s_2, ..., s_{n-1}, x) \in V : s_{n-1} < x\} \cup (y, s_2, ..., s_n) : y < s_2\} = C_x \cup D_y$,
i.e. ${}_xD_y$ is the set of all n-tuples ending in x or beginning in y.



Finally, given a set V of n-tuples $(s_1, ..., s_n)$ with $s_i \in S$ for some set S, for $2k \leq n$ we can once again generalise the above for k-tuples $x_1, ..., x_k$ and $y_1, ..., y_k$ as follows:

$_{x_1,...,x_k}C_{y_1,...,y_k} := \{(x_1, ..., x_k, s_{k+1}, ..., s_{n-k}, y_1, ..., y_k) : x_k < s_{k+1}, s_{n-k} < y_1\}$,
i.e. $_{x_1,...,x_k}C_{y_1,...,y_k}$ is the set of all n-tuples beginning in $x_1, ..., x_k$ and ending in $y_1, ..., y_k$.

$_{x_1,...,x_k}D_{y_1,...,y_k} := \{(s_1, ..., s_{n-k}, x_1, ..., x_k) : s_{n-k} < x_1\} \cup (y_1, ...y_k, s_{k+1}, ..., s_{n-k}) : y_k < s_{k+1}\}$,
i.e. $_{x_1,...,x_k}D_{y_1,...,y_k}$ is the set of all n-tuples ending in $x_1, ..., x_k$ or beginning in $y_1, ..., y_k$.



Each of these can be applied to a graph with any type $1^n 3^k 2^n$.

2.5 Definability and Interpretability

Definition 2.33. If $\mathcal{L}$ is a first-order language and $\mathcal{A}$ is an $\mathcal{L}$-structure, then $X \subseteq A^n$ is *first-order definable* (with parameters) if there is a first-order formula φ of $\mathcal{L}$ and finitely many $a_1, ..., a_m \in A$ such that $X = \{(x_1, ..., x_n) : \mathcal{A} \models \varphi(x_1, ..., x_n, a_1, ..., a_m)\}$. If $m = 0$ then it is $\emptyset$-definable, or definable without parameters. A similar definition applies to second-order definable, where φ is a second-order formula.

Example 2.34. The language of graphs has one binary relation symbol $\sim$ (for adjacency). A graph is a model in which $\sim$ is symmetric and irreflexive. The set of vertices of degree 2 is $\emptyset$-definable:

$$x \in X \leftrightarrow (\exists y \exists z)(x \sim y \wedge x \sim z \wedge y \neq z \wedge (\forall t)(x \sim t \rightarrow t = y \vee t = z))$$

Example 2.35. In the language of graphs, 'x, y are in the same connected component $C(x, y)$' is second-order definable, but not first-order definable (for details of how to use Ehrenfeucht-Fraïssé games to prove it's nof first-order definable see [19]).

We start by defining $Connected(X)$:

$$(\forall Y\ \forall Z)((X = Y \cup Z) \wedge (Y \cap Z = \emptyset) \wedge (Y \neq \emptyset) \wedge (Z \neq \emptyset) \rightarrow (\exists y \in Y)(\exists z \in Z)(y \sim z))$$

Note that $X = Y \cup Z$ can be expressed as $(\forall x)(x \in X \leftrightarrow x \in Y \vee x \in Z)$, and similarly with $Y \cap Z$. We now define $C(x, y)$:

$$C(x, y) \leftrightarrow (\exists X)(Connected(X) \wedge x \in X \wedge y \in X)$$

Example 2.36. The set of neighbours of a is first-order definable with a as a parameter.

Example 2.37. Connectedness is not first-order definable. To see this, consider $\mathbb{Z}$ as a graph, with $E(n, n+1)$ for all $n \in \mathbb{Z}$. Then, using Ehrenfeucht-Fraïssé games, we can show that $\mathbb{Z} \equiv \mathbb{Z} + \mathbb{Z}$ [13].

Definition 2.38. Let $\mathfrak{M}$ and $\mathfrak{N}$ be structures with underlying sets M and N respectively. Then $\mathfrak{N}$ is *first-order interpretable* (without parameters) in $\mathfrak{M}$ if for some $n \in \mathbb{N}$, there is:

1. A $\emptyset$-definable subset D of M^n

2. An $\emptyset$-definable equivalence relation E on D

3. A bijection $\gamma : N \to D/E$, such that for every m-ary relation R of the language of $\mathfrak{N}$ there is a formula φ in the language of $\mathfrak{M}$ with mn free variable, such that for every $\overline{a_1}, \overline{a_2}, ..., \overline{a_m}$, where $\overline{a_i}$ have length n and $1 \leq i \leq m$, all entries in M, all $\overline{a_1}$ in D,

$$\mathfrak{N} \models R(\gamma^{-1}(\overline{a_1})_E, \gamma^{-1}(\overline{a_2})_E, ..., \gamma^{-1}(\overline{a_m})_E) \Longleftrightarrow \mathfrak{M} \models \varphi(\overline{a_1}, ..., \overline{a_m})$$

Second-order interpretability is defined similarly:

Definition 2.39. Let $\mathfrak{M}$ and $\mathfrak{N}$ be structures with underlying sets M and N respectively. We say that $\mathfrak{N}$ is *second-order interpretable* in $\mathfrak{M}$ if for some $n \in \mathbb{N}$, there is:

1. A (second-order) definable subset D of M^n

2. A (second-order) definable equivalence relation E on D

3. A bijection $\gamma : N \to D/E$, such that for every $\emptyset$-definable subset R of N^m, the subset of $(M^n)^m$ given by $\hat{R} = \{(\overline{a_1}, ..., \overline{a_m}) \in (M^n)^m : (\gamma^{-1}(\overline{a_1}/E), ..., \gamma^{-1}(\overline{a_m}/E)) \in R\}$ is (second-order) definable in M^n. In other words, N can be identified with a definable subset D of M^n, divided by a suitable $\emptyset$-definable equivalence relation.

We now consider some examples of this.

Example 2.40. $G(\mathbb{Q}, 11322)$ is first-order interpretable inside $(\mathbb{Q}, <)$.

In this case our structure M is $(\mathbb{Q}, <)$ and N is $G(\mathbb{Q}, 11322)$. We take D to be the set $\{(x, y, z) \in \mathbb{Q}^3 : x < y < z\}$, where in this case $n = 3$. E is trivial (equality), and the domain of N is D.

Our m-ary relation is the edge relation, and $m = 2$. Thus $R((x_1, y_1, z_1), (x_2, y_2, z_2)) \Leftrightarrow z_1 = x_2 \vee x_1 = z_2$.

Example 2.41. $(\mathbb{Q}, B)$ is second-order interpretable inside $G(\mathbb{Q}, 132)$, where B is the betweenness relation on $\mathbb{Q}$ (defined in Chapter 3: Definition 3.6).

In this case our structure $\mathfrak{M}$ is $G(\mathbb{Q}, 132)$ and $\mathfrak{N}$ is $(\mathbb{Q}, B)$. We identify $\mathbb{Q}$ with sets of the form A_x, i.e. elements of $\mathcal{P}(G(\mathbb{Q}, 132))$, $n = 1$, and again, E is trivial.

Then our m-ary relation is the betweenness relation, and $m = 3$. Thus, assuming $R(x, y, z)$ means 'x lies between y and z', and $u \sim v$ means 'there is an edge between u and v', in the language of M, we have: $R(x, y, z) \Leftrightarrow \exists u \in A_x \cap A_y \wedge \exists v \in A_x \cap \mathcal{A}_z : u \sim v$

Details of the definability of these notions and their correctness is given in Theorem 4.2.

Definition 2.42. If $\mathfrak{A}, \mathfrak{B}$ are first-order structures such that $\mathfrak{A} \subseteq \mathfrak{B}$, then $\mathfrak{A}$ is *first-order definable* in $\mathfrak{B}$ if the domain A of $\mathfrak{A}$ is definable inside $\mathfrak{B}$, and for every relation R of $\mathfrak{A}$, there is a first-order formula φ of $L(\mathfrak{B})$ such that for all $a_1, ..., a_n$,

$$\mathfrak{A} \models R(a_1, ..., a_n) \Leftrightarrow \mathfrak{B} \models \varphi(a_1, ..., a_n)$$

This is a stronger version of interpretability, where $n = 1$ and E is trivial.

3 Betweenness Relations

Since we are dealing with undirected graphs, the graph $G(S, \tau)$ will be the same as the graph $G(S', \tau)$ where S' is the reversed ordering of S. Consequently, we cannot in general expect to recover the ordering from the graph, but only 'up to order reversal'. The correct context for this is the corresponding 'total betweenness relation', which we now axiomatise as in [1]. We write $xB[y, z]$ to mean 'x is between y and z'.

The material in this chapter is well-known, but in [1] and [11] the details are omitted, and so I am including them here for completeness.

Remark 3.1. We use the standard, reflexive definition of betweenness as in [1], but associated with any such reflexive relation there is automatically a strict one. We will use whichever is most convenient in the context.

Remark 3.2. We sometimes refer to orderings by their ground sets (for example we write $\mathbb{Q}$ when we mean $(\mathbb{Q}, <)$).

Throughout the thesis we will be using the following definitions regarding total orderings:

Definition 3.3. Let S be a total ordering. Then a *maximal finite section* is a convex subset Y of S which is finite and maximal. A *non-trivial* maximal finite section is a maximal finite section of size greater than 1.

This may or may not exist for a given S. For example, $\mathbb{Z}$ contains no non-trivial maximal finite sections, but $\mathbb{Q} + 3 + \mathbb{Q}$ contains one of size 3.

Definition 3.4. Given a, b in some total order S, the *distance* $dist(a, b)$ between a and b with $a < b$ is the cardinality of the set $[a, b)$. In this thesis, distance is always positive.

We can expand the definition of a maximal finite section to include infinite sets as follows:

Definition 3.5. Let $x \in S$, where S is some total ordering. Then the *finite component* x lies in is the set of all points in S of finite distance from x.

If this set is finite, then it is a maximal finite section.

3.1 Total Betweenness and Total Orderings

Definition 3.6. A *total betweenness relation* is a ternary relation B defined on a set S which satisfies:

(i) $\forall x, y, z \in S,\ xB[y, z] \Rightarrow xB[z, y]$ (symmetry)

(ii) $\forall x, y, z \in S,\ xB[y, z] \wedge yB[x, z] \Leftrightarrow x = y$

(iii) $\forall x, y, z \in S,\ xB[y, z] \Rightarrow \forall w,\ xB[y, w] \vee xB[z, w]$

(iv) $\forall x, y, z \in S,\ xB[y, z] \vee yB[x, z] \vee zB[x, y]$ (total condition)

Remark 3.7. $yB[z, y]$ for all $y, z \in S$

This follows from Definition 3.6 (ii), as $x = y \Leftrightarrow xB[y, z] \wedge yB[x, z]$, i.e. $yB[y, z]$, and so $yB[z, y]$ by symmetry.

Lemma 3.8. $xB[y, y] \Rightarrow x = y$

Proof. We know that $yB[x, y]$ by Remark 3.7.

We now have $xB[y, y]$ and $yB[x, y]$, so from the left to right implication in (ii), it follows that $x = y$. □

Lemma 3.9. From the above axioms, we can deduce the following:

$$\forall x, y, z, w \in S,\ yB[x, w] \wedge zB[y, w] \Rightarrow zB[x, w] \quad \text{(transitivity)}$$

Proof. We first deal with all cases in which not all of x, y, z, w are distinct. There are 6 cases.

Assume that $yB[x, w]$ and $zB[y, w]$, and we require $zB[x, w]$. If $x = z$ or $z = w$ then using (i) and Remark 3.7 we see that $zB[x, w]$. If $x = y$ then from $zB[y, w]$ it follows that $zB[x, w]$. If $y = z$ then from $yB[x, w]$ it follows that $zB[x, w]$. If $x = w$ then by applying Lemma 3.8 to $yB[x, w]$ we deduce that $x = y$ which is a case already done. If $y = w$ then applying 3.8 to $zB[y, w]$ we deduce that $y = z$, which has also already been done.

Now let us suppose that all of x, y, z, w are distinct. As usual, we are assuming that $yB[x, w]$ and $zB[y, w]$, and aiming to get $zB[x, w]$. By (iii), from $yB[x, w]$ we get $yB[x, z]$ or $yB[w, z]$. If $yB[w, z]$ then as $zB[y, w]$, by (i) and (ii), $y = z$, contrary to y and z distinct. Therefore $yB[x, z]$. Using (iii) once more, from $zB[y, w]$ we get $zB[y, x]$ or $zB[w, x]$. If $zB[y, x]$ then from $yB[x, z]$ and property (ii), $y = z$, contrary to y and z distinct. Therefore $zB[w, x]$, as required.

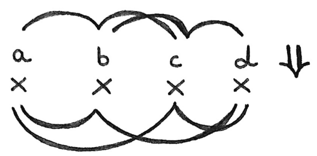
Theorem 3.10. A set S with linear ordering $\leq$ induces a total betweenness relation B on S.

Proof. Define B as follows: $yB[x, z]$ if ($x \leq y$ and $y \leq z$) or ($z \leq y$ and $y \leq x$). □

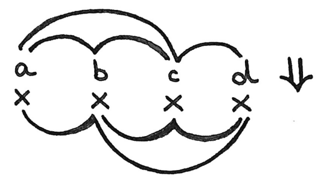
Remark 3.11. S as above with the reverse ordering $\geq$ induces this same total betweenness relation B on S.

Lemma 3.12. : Let S be a set with betweenness relation B, and let $a, b, c, d \in S$, with $b \neq c$. Then

(i) $bB[a, c] \wedge cB[b, d] \Rightarrow bB[a, d] \wedge cB[a, d]$.



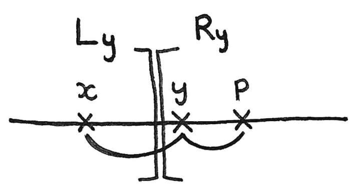
(ii) $bB[a, c] \wedge cB[a, d] \Rightarrow bB[a, d] \wedge cB[b, d]$.



Proof. (i) By Axiom (iii), from $bB[a, c]$ we get $bB[a, d] \vee bB[c, d]$. But as $cB[b, d]$, from $bB[c, d]$ we derive $b = c$ using Axiom (ii), contrary to hypothesis. Hence $bB[a, d]$. The proof that $cB[a, d]$ is similar.

(ii) From $cB[a, d]$ we get $cB[a, b] \vee cB[d, b]$ by Axiom (iii). If $cB[a, b]$ then as $bB[a, c]$ we get $b = c$ again. Consequently, $cB[d, b]$, and to get $bB[a, d]$ we can now use part (i).

Definition 3.13. Let B be a total betweenness relation on a set S and let $x, y \in S$ be distinct. Let R_y be the set of all points p in S such that $yB[x, p]$, and let its complement be L_y.



The idea is that $x < y$, and that R_y should be the set of points greater than or equal to y, and L_y the set of points less than y. Notice that $L_y \cup R_y = S$, and $L_y \cap R_y = \emptyset$.

Lemma 3.14. If $yB[a, b]$ where $a, b \neq y$, then one of a, b lies in R_y and the other lies in L_y.

Conversely, if $a \in L_y$ and $b \in R_y$, then $yB[a, b]$.

Proof. By Axiom (iii), since $yB[a,b]$, we know $yB[a,x] \vee yB[b,x]$ for all x; in particular, for the fixed x in Definition 3.13. It follows that a or b lies in R_y (or both). Suppose for a contradiction that both lie in R_y. Thus both $yB[a,x]$ and $yB[b,x]$ hold. By Axiom (iv), $xB[a,b] \vee aB[x,b] \vee bB[x,a]$. By Axiom (iii), $xB[a,y] \vee xB[b,y] \vee aB[x,y] \vee aB[b,y] \vee bB[x,y] \vee bB[a,y]$. We now use Axiom (ii) to deduce that y is equal to x, a, or b, all of which are ruled out. For instance, if $xB[a,y]$ then since $yB[a,x]$ we get $x = y$, and similarly in the other 5 cases (though in the 4th and 6th cases we appeal directly to $yB[a,b]$).

Now suppose that $a \in L_y$ and $b \in R_y$. Thus $yB[x,b] \wedge \neg yB[x,a]$. By Axiom (iii), $yB[x,a] \vee yB[b,a]$. Since $\neg yB[x,a]$ we deduce that actually, $yB[b,a]$, as required.

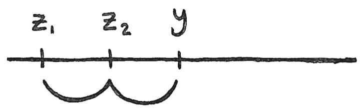
Theorem 3.15. Let B be a total betweenness relation on a set S and let x and y be two fixed, distinct points in S. Then there exists exactly one linear ordering $\leq$ of S with $x \leq y$ and such that B is induced from $\leq$ as in Theorem 3.10.

Proof. We will find a total ordering of S in which $x \leq y$ which induces the given betweenness relation B.

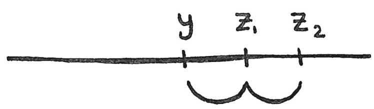
We define L_y, R_y as above. Note that by this definition, $x \leq y$ since $x \in L_y$ and $y \in R_y$. We can now define a relation $\leq$ on S by saying that $z_1 \leq z_2$ iff

$$z_1 \in L_y \text{ and } z_2 \in R_y$$

$$\text{or } z_1, z_2 \in L_y \text{ and } z_2B[z_1, y]$$



$$\text{or } z_1, z_2 \in R_y \text{ and } z_1B[z_2, y]$$



We will start by showing $\leq$ is linear. To show this we have to check:

Reflexivity: $z \leq z$. This holds since $zB[z,y]$ by (ii).

Antisymmetry: Suppose $z_1 \leq z_2$ and $z_2 \leq z_1$. Since L_y and R_y are disjoint, z_1, z_2 lie in the same one of L_y, R_y. Hence $z_1B[z_2,y]$ and $z_2B[z_1,y]$, and from (ii) it follows that $z_1 = z_2$.

Transitivity: Suppose $z_1 \leq z_2 \wedge z_2 \leq z_3$. If $z_1 \in L_y$ and $z_2 \in R_y$ then by definition of $\preceq$, also $z_3 \in R_y$, so $z_1 \leq z_3$ is immediate. Similarly if $z_2 \in L_y$ and $z_3 \in R_y$. The remaining cases that need to be considered are just that all of z_1, z_2, z_3 lie in the same one of L_y, R_y. If $z_1, z_2, z_3 \in L_y$, then $z_2B[z_1, y] \wedge z_3B[z_2, y]$ so by 3.9, $z_3B[z_1, y]$, and $z_1 \leq z_3$. If $z_1, z_2, z_3 \in R_y$, then $z_1B[z_2, y] \wedge z_2B[z_3, y]$ so by 3.9, $z_1B[z_3, y]$, and again, $z_1 \leq z_3$.

Linearity: To see that $\leq$ is total, take any $z_1, z_2 \in S$. If $z_1 \in L_y$ and $z_2 \in R_y$ then $z_1 \leq z_2$, and similarly the other way round. So suppose that they both lie in the same one of L_y, R_y. By (iv) $z_1B[z_2, y] \vee z_2B[z_1, y] \vee yB[z_1, z_2]$. The first two give comparability of z_1, z_2 by definition, and in the third case, by Lemma 3.14 z_1, z_2 lie in different ones of L_y, R_y (which we have ruled out), unless at least one of them equals y, in which case $z_1, z_2 \in R_y$ (since $y \notin L_y$) and if $y = z_1$, then $z_1B[z_2, y]$ by Remark 3.7 (and similarly if $y = z_2$).

Hence $\leq$ is linear.

Thus by Theorem 3.10, $\leq$ induces a betweenness relation $B_{\leq}$. We would now like to show that $B = B_{\leq}$.

To show this, we need to show that $z_2B[z_1, z_3] \Leftrightarrow z_2B_{\leq}[z_1, z_3]$. Now, $B_{\leq}$ is defined as follows: $z_2B_{\leq}[z_1, z_3]$ if $(z_1 \leq z_2 \leq z_3)$ or $(z_3 \leq z_2 \leq z_1)$.

Hence we would like to show that $[(z_1 \leq z_2 \leq z_3) \vee (z_3 \leq z_2 \leq z_1)] \Rightarrow z_2B[z_1, z_3]$.

First assume $[(z_1 \leq z_2 \leq z_3) \vee (z_3 \leq z_2 \leq z_1)]$. If $z_1 = z_2$ or $z_2 = z_3$, then $z_2B[z_1, z_3]$ by Remark 3.7, so we now assume that z_1, z_2, and z_3 are all distinct. We have to show that $z_2B[z_1, z_3]$.

We have two cases.

Case 1: $(z_1 \leq z_2 \leq z_3)$

Since $z_1 \leq z_2$, we know that one of the following holds:

$$\begin{aligned}
&(1)\ z_1 \in L_y \wedge z_2 \in R_y, \text{ or} \\
&(2)\ z_1, z_2 \in L_y \text{ and } z_2B[z_1, y], \text{ or} \\
&(3)\ z_1, z_2 \in R_y \text{ and } z_1B[z_2, y]
\end{aligned}$$

Also since $z_2 \leq z_3$, we know that

$$\begin{aligned} (i)\ & z_2 \in L_y \wedge z_3 \in R_y, \text{ or} \\ (ii)\ & z_2, z_3 \in L_y \text{ and } z_3 B[z_2, y], \text{ or} \\ (iii)\ & z_2, z_3 \in R_y \text{ and } z_2 B[z_3, y] \end{aligned}$$

We tackle these one at a time:

(1) : First suppose $z_1 \in L_y \wedge z_2 \in R_y$. Then we must have $z_2, z_3 \in R_y$ and $z_2 B[z_3, y]$. By Lemma 3.14 we have that $yB[z_1, z_2]$. By Lemma 3.12 (i) applied to $yB[z_1, z_2] \wedge z_2 B[y, z_3]$, we obtain $z_2 B[z_1, z_3]$ as required.

(2) : Second, suppose $z_1, z_2 \in L_y$ and $z_2 B[z_1, y]$. We now have 2 options:

(i) If $z_2 \in L_y \wedge z_3 \in R_y$, then we know $yB[z_2, z_3]$ by 3.14. Again by Lemma 3.12 (i), this time letting $a = z_1, b = z_2, c = y, d = z_3$, we obtain $z_2 B[z_1, z_3]$ as required.

(ii) If $z_2, z_3 \in L_y$ and $z_3 B[z_2, y]$, since $z_2 B[z_1, y]$ and $z_3 B[z_2, y]$ we can use Lemma 3.12 (ii), which states that

$$bB[a, c] \wedge cB[a, d] \Rightarrow bB[a, d] \wedge cB[b, d]$$

Letting $a = y, b = z_3, c = z_2, d = z_1$, we thus obtain $z_2 B[z_3, z_1]$, and so by symmetry $z_2 B[z_1, z_3]$ as required.

(3) : Finally, suppose $z_1, z_2 \in R_y$ and $z_1 B[z_2, y]$. We now only have one option:

(iii) We must have $z_2, z_3 \in R_y$ and $z_2 B[z_3, y]$. Again, we can use Lemma 3.12 (ii), which states that:

$$bB[a, c] \wedge cB[a, d] \Rightarrow bB[a, d] \wedge cB[b, d]$$

By symmetry, $z_2 B[y, z_3]$ and $z_2 B[z_3, y]$. Letting $a = y, b = z_1, c = z_2, d = z_3$, we thus obtain $z_2 B[z_1, z_3]$.

Hence $[(z_1 \leq z_2 \leq z_3) \vee (z_3 \leq z_2 \leq z_1)] \Rightarrow z_2 B[z_1, z_3]$, and so $B = B_{\leq}$.

□

Corollary 3.16. Any total betweenness relation B on a set S arises from exactly two orderings on S, such that if one of them is $\leq$, then the other one is $\geq$.

Proof. Theorem 3.15 states that if B is a total betweenness relation on a set S and distinct $x, y \in S$, then there exists exactly one ordering $\leq$ of S with $x \leq y$ and such that B arises from S. Clearly B also arises from the ordering $\leq^*$ which is the reverse of $\leq$, and in this ordering $y \leq^* x$. Since the ordering which gives rise to B is uniquely determined once we know how x and y are related, it follows that these are the only two possibilities.

3.2 Partial Betweenness Relations

Definition 3.17. A (strict) partial betweenness relation is a ternary relation B defined on a set S which satisfies:

- $\forall x, y, z \in S,\ xB[y,z] \rightarrow (x \neq y \wedge x \neq z \wedge y \neq z)$.
- $\forall x, y, z \in S,\ xB[y,z] \rightarrow xB[z,y]$
- $\forall x, y, z, w \in S,\ (yB[x,w] \wedge zB[y,w]) \rightarrow zB[x,w]$

This is 'partial' betweenness since it is does not contain the total condition, i.e. it is not the case that for all $x, y, z \in S$ either $xB[y,z]$ or $yB[x,z]$ or $zB[y,x]$.

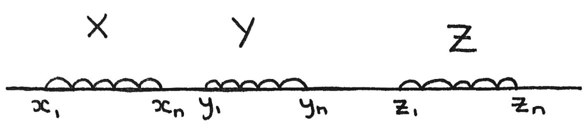
We now apply this to the generalised shift graph.

Definition 3.18. Two n-tuples $x_1, ..., x_n$ and $y_1, ..., y_n$ with $x_1 < ... < x_n$ and $y_1 < ... < y_n$ are *comparable* if either $x_n < y_1$ or $y_n < x_1$.

Definition 3.19. Partial betweenness on n-tuples. Let $(S, <)$ be a totally ordered set. We define betweenness B^n on increasing n-tuples of S as follows:

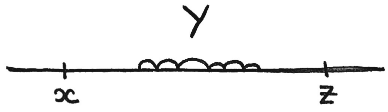
$$(y_1, ..., y_n)B^n[(x_1, ..., x_n), (z_1, ..., z_n)] \text{ if and only if}$$
$$x_1 < ... < x_n < y_1 < ... < y_n < z_1 < .... < z_n$$
$$\text{or } z_1 < ... < z_n < y_1 < ... < y_n < x_1 < .... < x_n$$

Note that the n-tuples must be pairwise comparable for this to hold.



Note that B^n is a partial betweenness relation.

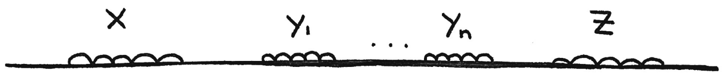
Definition 3.20. We now define a new relation, called n-betweenness B_n on a totally ordered set $(S, <)$. This is a relation on $n + 2$ points (not necessarily increasing) in S, where $\{y_1, ..., y_n\}B_n[x, z]$ holds iff the set of points $\{y_1, ..., y_n\}$ lies between x and z. In this case we call x and z the *endpoints* of the set $\{y_1, ..., y_n, x, z\}$.



Note that $\{y_1, ..., y_n\}B_n[x, z] \to \{y_{\pi(1)}, ..., y_{\pi(n)}\}B_n[x, z]$, where π is any permutation on $\{1, ..., n\}$, and that $\{y_1, ..., y_n\}B_n[x, z] \to \{y_1, ..., y_n\}B_n[z, x]$.

Also note that for 1-betweenness there is a slight difference of notation, in that we write $yB[x, z]$ as opposed to $\{y\}B_1[x, z]$.

Definition 3.21. We can combine the above definitions as follows: B_n^m is defined as partial n-betweenness on increasing m-tuples, i.e. this is a relation on $n + 2$ m-tuples where $\{Y_1, ..., Y_n\}B_n[X, Z]$ holds iff each m-tuple in the set $\{Y_1, ..., Y_n\}$ lies between the m-tuples X and Z, and the Y_is are pairwise comparable.



Similarly, note that $\{Y_1, ..., Y_n\}B_n[X, Z] \to \{Y_{\pi(1)}, ..., Y_{\pi(n)}\}B_n[X, Z]$, where π is any permutation on $\{1, ..., n\}$, and that $\{Y_1, ..., Y_n\}B_n[X, Z] \to \{Y_1, ..., Y_n\}B_n[Z, X]$.

Lemma 3.22. Let $(S, <)$ be a total ordering without endpoints with induced betweenness relation B, and let $n \in \mathbb{N}$. Then (S, B) is 1st-order interpretable from (S^n, B^n).

Proof. We define an equivalence relation $\equiv$ on S^n on n-tuples $X = (x_1, ..., x_n)$, $X' = (x'_1, ..., x'_n)$, $Y = (y_1, ..., y_n)$, $Z = (z_1, ..., z_n)$:

$X \equiv X'$ iff $\forall\ Y, Z \in S^n (XB^n[Y, Z] \leftrightarrow X'B^n[Y, Z])$.

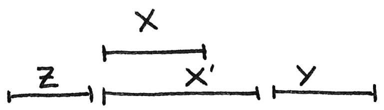
Then it is clear that this holds if and only if $x_1 = x'_1$ and $x_n = x'_n$, so the equivalence classes precisely correspond to intervals of size at least n. This is a first-order formula, as are the ones which follow.

We can also express inclusion and disjointness of intervals (comparability) as follows:

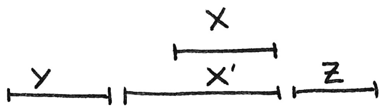
$$X \subseteq X' \;\Leftrightarrow\; \forall\, Y, Z \in S^n \;:\; X'B^n[Y,Z] \rightarrow XB^n[Y,Z]$$
$$comp(X,Y) \;\Leftrightarrow\; \exists\, Z \in S^n \;:\; XB^n[Y,Z]$$

Now we want to represent points x of S by families of intervals of the form $\{(x_1, ..., x_n) : x_1 = x\}$ or $\{(x_1, ..., x_n) : x_n = x\}$. Since this will however give a second-order interpretation, we instead consider pairs of members of these sets at a time, which will sufficiently well represent the whole set. So we use the following formula:

$\rho(X, X') \;\Leftrightarrow\; \exists Y \in S^n \;:\; comp(X,Y) \wedge comp(X',Y) \wedge \forall Z \in S^n\; (XB^n[Y,Z] \leftrightarrow X'B^n[Y,Z])$



(If Y lies to the right of X, X')



(If Y lies to the left of X, X')

We note that if $X = (x_1, ..., x_n)$ and $X' = (x'_1, ..., x'_n)$ share their left endpoint, for instance, so that $x_1 = x'_1$, then this formula holds, since we may take $Y > X, X'$, and then whichever Z we take, if $XB^n[Y,Z]$ then $z_n < x_1$, and hence also $z_n < x'_1$ so $Z < X'$, giving $X'B^n[Y,Z]$, and similarly in the other direction (if $x_n = x'_n$ then we instead take $Y < X, X'$). Conversely, suppose that X and X' do not share either their left or right endpoints. Then we see that the formula is false. Otherwise there is a suitable Y which is disjoint from both X and X'. There are two cases, one in which Y is to the left of both X and X' (similarly to the right of both), and the other in which it is greater than one and less than the other.

If $Y < X, X'$ then as $x_n \neq x'_n$, suppose that $x_n < x'_n$, and then there is Z such that $z_1 = x'_n$. Clearly $XB^n[Y,Z]$ but not $X'B^n[Y,Z]$, violating the formula.

If on the other hand, $X < Y < X'$ we can easily violate the formula by taking Z to the right of X' (or indeed to the left of X).

We observe that for three sets, if $\rho(X, X')$ and $\rho(X, X'')$ and $\rho(X', X'')$, then all of X, X', X'' share the same endpoint, since if for instance $x_1 = x'_1 \neq x''_1$, then as X, X'' share an endpoint, $x_n = x''_n$, and as X', X'' share an endpoint, $x'_n = x''_n$, so all three share their right endpoints (and in fact, $X \equiv X'$).

Two pairs (X, X') and (Y, Y') which both satisfy the formula ρ are said to have the same *parity*, if X and X' share the same left endpoint and so do Y and Y', or else the same statement for right endpoints. We next aim to characterise when two pairs of members of S^n satisfying ρ have the same parity. The following formula gets us part way to this aim.

$$\begin{aligned} \mathit{parity}_1((X,X'),(Y,Y')) \Leftrightarrow \rho(X,X') \ \wedge \ \rho(Y,Y') \ \wedge \forall X''(\rho(X,X'') \ \wedge \ \rho(X',X'') \\ \rightarrow \exists Y''(\rho(Y,Y'') \ \wedge \ \rho(Y',Y'') \ \wedge \ (X'' \subseteq Y'')) \end{aligned}$$

We can see that this formula holds if and only if either (X, X') share the same left endpoint and so do (Y, Y') and $x_1 \geq y_1$, or (X, X') share the same right endpoint and so do (Y, Y') and $x_n \leq y_n$. First suppose that $x_1 = x'_1 \geq y_1 = y'_1$. Then any possible X'' is of the form $(x''_1, ..., x''_n)$ where $x''_1 = x_1$, and we can take $(y''_1, ..., y''_n)$ such that $y''_1 = y_1$ and $y''_n \geq x''_n$. Then the interval given by X'' has endpoints x_1, x''_n, which is contained in that given by Y'', which has endpoints y_1, y''_n. A similar calculation applies in the second case, arguing with respect to right endpoints.

Conversely, suppose that the formula holds, and without loss of generality, suppose that X, X' share left endpoints, but Y, Y' share right endpoints. Then there is X'' which shares its left endpoint with X, but whose right endpoint exceeds the right endpoint of Y, and this cannot be contained in any possible Y''.

Hence Y and Y' also share left endpoints. Clearly $X'' \subseteq Y''$ implies that $y_1 \leq x_1$.

The symmetrised version is as follows:

$$\mathit{sameparity}((X,X'),(Y,Y')) \Leftrightarrow \mathit{parity}_1((X,X'),(Y,Y')) \ \vee \ \mathit{parity}_1((Y,Y'),(X,X'))$$

Then this holds if and only if (X, X') share the same left endpoint and so do (Y, Y'), or (X, X') share the same right endpoint and so do (Y, Y'), so it precisely expresses the fact that (X, X') and (Y, Y') have the same parity.

If we have decided on the parity, we can now represent the points of S by ρ-classes of pairs of members of S^n. Intuitively, this picks out all the members of S^n with a fixed greatest entry, or alternatively, all the members with a fixed least entry. It would be preferable to be able to identify the classes of different parities representing the same point. We can do this as follows:

$$\begin{aligned} \mathit{sameendpoint}((X,X'),(Y,Y')) \Leftrightarrow\ & \rho(X,X') \wedge \rho(Y,Y') \\ & \wedge \neg \mathit{sameparity}((X,X'),(Y,Y')) \wedge \neg \mathit{comp}(X,Y) \\ & \wedge \ \exists Z \ : \ Z \supset X \wedge \rho(X,Z) \wedge \rho(X',Z) \\ & \wedge \forall Z'(Z' \subseteq Z) \ \wedge \ \neg\rho(X,Z') \rightarrow \mathit{comp}(Z',Y) \end{aligned}$$

This is intended to say that X and X' share the same right endpoint x_n and Y, Y' share the same left endpoint y_1, and $y_1 = x_n$ (or the other way round). Assume for instance that the former holds. Since $\neg comp(X, Y), y_1 \leq x_n$, and the hypothesis on Z ensures that it also has x_n as a right endpoint, and then any Z' as given does **not** have x_n as right endpoint. So the formula guarantees that $y_1 = x_n$.

Now, two pairs $((X, X'), (Y, Y'))$ and $((Z, Z'), (W, W'))$ satisfying *sameendpoint* 'encode' the same point if either $\rho(X, Z)$ and $\rho(Y, W)$, or $\rho(X, W)$ and $\rho(Y, Z)$. This forms an equivalence relation E on intervals of size at least n, with equivalence classes corresponding to elements of S. Let $(X_1, X_2), (Y_1, Y_2), (Z_1, Z_2)$ be representatives of equivalence classes encoding points $x, y, z \in S$ respectively. Then $yB[x, z]$ iff there are X, Y, Z such that $\rho(X, X_1) \wedge \rho(X, X_2) \wedge \rho(Y, Y_1) \wedge \rho(Y, Y_2) \wedge \rho(Z, Z_1) \wedge \rho(Z, Z_2)$, and these all have the same parity, and $X \subset Y \subset Z$ or $X \supset Y \supset Z$. □

Lemma 3.23. Let $(S, <)$ be a total ordering without endpoints with induced betweenness relation B, and let $n \in \mathbb{N}$. Then (S, B) is 1st-order interpretable in (S, B_n).

Proof. We start by giving a formal proof using the definition of what it means to be 1st-order interpretable (Definition 2.38), followed by a more intuitive explanation of the proof.

Following Definition 2.38, let $\mathfrak{M} = (S, B_n)$ and $\mathfrak{N} = (S, B)$, and thus $M = N = S$. $D = M$, E is equality, and so the bijection γ going from N to $D \backslash E = M$ is the identity map. Furthermore, $n = 1$ and $m = 3$, and the formula φ containing $mn = 3$ free variables (x, y and z) is as follows:

$$\varphi := \exists\ p_1\ \exists\ p_2\ ...\ \exists\ p_{2n+2}\ :\ \Big(\bigwedge_{i=1}^{n+1} \{p_{i+1}, ..., p_{i+n}\} B_n[p_i, p_{i+n+1}] \Big) \wedge$$
$$\Big[\bigvee_{1 \leq i < j < k \leq 2n+2} ((x = p_i \wedge y = p_j \wedge z = p_k) \vee (x = p_k \wedge y = p_j \wedge z = p_i)) \Big]$$

Thus $\mathfrak{M} \models \varphi$ if and only if $\mathfrak{N} \models yB[z, x]$, where the ternary relation $R(x, y, z)$ is $yB[z, x]$.

Intuitively, given distinct points x, y, and z, we choose distinct points $p_1, ..., p_{2n+2}$ such that x, y, and z appear among these somewhere. This is possible since S has no endpoints. Furthermore we may make this choice so that $p_1 < p_2 < ... < p_{2n+2}$. It is clear that now that φ holds.

Conversely, suppose that φ holds. We note that the hypothesis allows us to find which are the two endpoints given any $n + 2$ distinct points of S (using the version of betweenness assumed).

By assumption, there are (distinct) points p_i validating the formula. Then for each i with $1 \leq i \leq n + 1$, p_i and p_{i+n+1} are the endpoints of $\{p_1, ..., p_{i+n+1}\}$. To ease the argument, let us suppose that $p_1 < p_2, p_3, ..., p_{n+1} < p_{n+2}$. Then p_2 and p_{n+3} are the endpoints of $\{p_2, p_3, ..., p_{n+3}\}$, from which it follows that $p_1 < p_2 < p_3, p_4, ...p_{n+1} < p_{n+2} < p_{n+3}$.

Repeating this argument determines the ordering on all p_i, and since x, y, z appear among them, we can also determine which of these lies between the other two. □

We can extend this result to betweenness on m-tuples:

Lemma 3.24. Let S be a total ordering without endpoints. Then (S, B) is 1st-order interpretable from (S^m, B_n^m).

Proof. Similar to Lemma 3.23.

We finally define a new form of partial betweenness, and show that standard betweenness is definable from this new betweenness. This will come in useful for several future results.

Definition 3.25. Let S be a linearly ordered set under $<$. Then $B_{dist(m)}$ is a partial betweenness relation defined on S as follows: $yB_{dist(m)}[x, z]$ iff $x < y < z$ and $dist(x, y), dist(y, z) \geq m$, or $z < y < x$ and $dist(x, y), dist(y, z) \geq m$.

Definition 3.26. We now define the notion of comparability. We say that for $x, y \in S$, x and y are *comparable* if there exists a $z \in S$ such that $xB_{dist(m)}[y, z] \vee yB_{dist(m)}[x, z] \vee zB_{dist(m)}[x, y]$. In this case we say that x, y are *comparable*.

Lemma 3.27. Let S be any total ordering of size $\geq 4m$. Then $x, y \in S$ are comparable if and only if $dist(x, y) \geq m$.

Proof. First assume x, y are comparable.

Then $xB_{dist(m)}[y, z]$ or $yB_{dist(m)}[x, z]$ or $zB_{dist(m)}[x, y]$ for some z, and so $dist(x, y) \geq m$.

Conversely suppose $dist(x, y) \geq m$. If S has no endpoints in one or both directions, then there will always be a z of distance at least m from x and y, and so we have $xB_{dist(m)}[y, z]$ or $yB_{dist(m)}[x, z]$.

Let S be finite of size at least $4m$. Assume without loss of generality that $x < y$. Then if the distance between x and the 'left endpoint' of S is at least m, choose z to be the left endpoint. If $dist(x, y) \geq 2m$ then choose z to be a point in the middle of x, y, such that $zB_{dist(m)}[x, y]$.

In the remaining case, both the distance between x and the left endpoint of S is less than m, and $dist(x, y) < 2m$. Then, since S has size at least $4m$, there exists a $z > y$ in S with $dist(y, z) \geq m$. Hence x, y are comparable. □

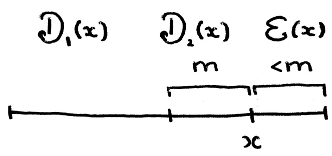
Lemma 3.28. Let $Z = \{0, 1, ..., z-1\}$ be a finite ordinal of size $\geq 4m$. Then (Z, B) is definable in $(Z, B_{dist(m)})$.

Proof. Since $Z \geq 4m$, by Lemma 3.27 any two points in Z are comparable if and only if the distance between them is greater than or equal to m.

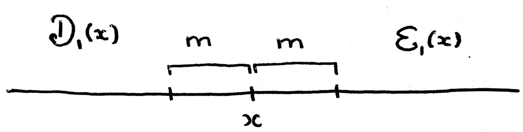
Let $x \in S$. Define the set $\mathcal{C}_1(x)$ as the set of all y comparable to x, and the set $\mathcal{C}_2(x)$ as the set of all y not comparable to x.

We have two cases here; either x is distance $\geq m$ from an endpoint, in which case $\mathcal{C}_1(x)$ has two components, everything lying to the right of x at distance $\geq m$ and everything lying to its left at distance $\geq m$, or x is distance $< m$ from an endpoint.

We can distinguish these two cases by noting that x is distance $< m$ from an endpoint if there do not exist $y, z \in S$ such that $xB_{dist(m)}[y, z]$. If this is the case, we let $\mathcal{D}_1(x) = \mathcal{C}_1(x)$. Now consider all the points in $\mathcal{C}_2(x)$ that are comparable to all of $\mathcal{D}_1(x)$. This is the set of all points between x and the endpoint it is closest to, including x. Let this set be $\mathcal{E}(x)$. Finally, let $\mathcal{D}_2(x)$ be the set $\mathcal{C}_2(x) \backslash \mathcal{E}(x)$, and let $\mathcal{D}(x) = \mathcal{D}_1(x) \cup \mathcal{D}_2(x)$. Thus $\mathcal{D}(x)$ is the set of all points to one side of x, and $\mathcal{E}(x)$ the set of points to its other side (including x).



Now assume x has distance at least m from both endpoints, i.e. there exist $y, z \in S$ such that $xB_{dist(m)}[y, z]$. We define the following equivalence class on $\mathcal{C}_1(x)$: let y and z in $\mathcal{C}_1(x)$ be equivalent if for all t, $xB_{dist(m)}[y, t]$ if and only if $xB_{dist(m)}[z, t]$. Then y and z are equivalent in this sense if and only if they are both comparable with x and they are on the same 'side' of x and of distance at least m from x. Both these equivalence classes will be nonempty as x is distance as least m from each end of S. Call these equivalence classes $\mathcal{D}_1(x)$ and $\mathcal{E}_1(x)$.

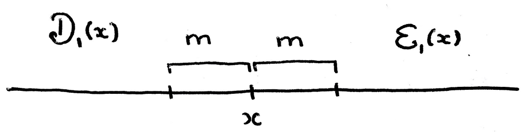


Now let $y \in \mathcal{E}_2(x)$ if and only if y is comparable to *all of* $\mathcal{D}_1(x)$, and $y \in \mathcal{D}_2(x)$ if and only if y is comparable to all of $\mathcal{E}_1(x)$. Let $\mathcal{D}(x) = \mathcal{D}_1(x) \cup \mathcal{D}_2(x)$, and let $\mathcal{E}(x) = \mathcal{E}_1(x) \cup \mathcal{E}_2(x)$. Again, $\mathcal{D}(x)$ is the set of all points to one side of x, and $\mathcal{E}(x)$ the set of points to its other side.

Now let $\mathcal{F}(x) = \{\mathcal{D}(x), \mathcal{E}(x)\}$ for each x, and consider $\mathcal{F}(x), \mathcal{F}(y), \mathcal{F}(z)$ for $x, y, z \in Z$. Then $yB[x, z]$ if and only if there exists $X \in \mathcal{F}(x)$, $Y \in \mathcal{F}(y)$, and $Z \in \mathcal{F}(z)$, such that $X \subseteq Y \subseteq Z$ or $X \supseteq Y \supseteq Z$. □

Lemma 3.29. Let S be a total ordering without endpoints. Then (S, B) is definable in $(S, B_{dist(m)})$.

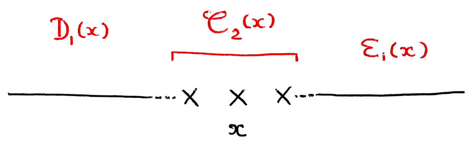
Proof. Let $x \in S$. Again define the set $\mathcal{C}_1(x)$ as the set of all y comparable to x, and $\mathcal{C}_2(x)$ be the set of all y not comparable to x. Let y and z be equivalent if for all t, $xB_{dist(m)}[y, t]$ if and only if $xB_{dist(m)}[z, t]$. Then y and z are equivalent in this sense if and only if they are both comparable with x, and they are on the same 'side' of x and of distance at least m from x. Hence there are two equivalence classes, which we call $\mathcal{D}_1(x)$ and $\mathcal{E}_1(x)$.



We now have two cases:

Case 1: The set $\mathcal{C}_2(x)$ is comparable to all of $\mathcal{D}_1(x)$ and $\mathcal{E}_1(x)$.

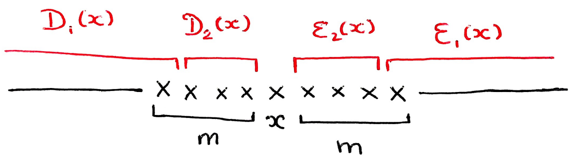
In this case, $\mathcal{C}_2(x)' = \mathcal{C}_2(x) \cup \{x\}$ is a maximal finite section of size $\leq 2m$. We can determine the size of $\mathcal{C}_2(x)'$ as it is simply the number of points in the graph that are comparable with all of $\mathcal{D}_1(x)$ and $\mathcal{E}_1(x)$. Thus we pick a formal labelling of these points with an ordering $<$, which is forced upon us by the existing $B_{dist(m)}$ configuration.



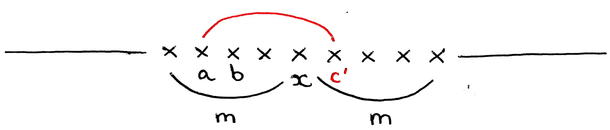
Case 2: The set $\mathcal{C}_2(x)$ is not comparable to all of $\mathcal{D}_1(x)$ or all of $\mathcal{E}_1(x)$.

In this case $\mathcal{C}_2(x) \cup \{x\}$ is a finite section of size $2m+1$, and additionally, there are at least m discrete points to the right of x and m discrete points to its left.

Now, consider the set of all points in $\mathcal{C}_2(x)$ that are comparable to all of $\mathcal{D}_1(x)$. No point lying between x and $\mathcal{D}_1(x)$ will be in this set, so this gives us the set of points between x and $\mathcal{E}_1(x)$. Call this set $\mathcal{E}_2(x)$, and similarly, let $\mathcal{D}_2(x)$ be the set of all points in $\mathcal{C}_2(x)$ that are comparable to all of $\mathcal{E}_1(x)$. Again, let $\mathcal{D}(x) = \mathcal{D}_1(x) \cup \mathcal{D}_2(x)$, and let $\mathcal{E}(x) = \mathcal{E}_1(x) \cup \mathcal{E}_2(x)$, so that $\mathcal{D}(x)$ is the set of all points to one side of x, and $\mathcal{E}(x)$ the set of points to its other side.



Now, given any $a, b, c \in \mathcal{C}_2(x)$, I claim we can determine which lies between the other two. First suppose without loss of generality that two of a, b lie in $\mathcal{D}_2(x)$ and c lies in $\mathcal{E}_2(x)$. Then we know c doesn't lie between a and b. Now consider $c' \in \mathcal{E}_2(x)$ such that c' is comparable to exactly one of a, b. Such a c' always exists because, supposing b lies closer to x than a does, and $dist(a, x) = k < m$, then we can let c' be the point in $\mathcal{E}_2(x)$ of distance $m - k$ from x. It follows that for any two a, b in $\mathcal{D}_2(x)$ or in $\mathcal{E}_2(x)$, we can determine whether $aB[b, x]$ or $bB[a, x]$.



Now suppose a, b, c all lie in $\mathcal{D}_2(x)$. Then we can determine betweenness on any pair in a, b, c and x by the method above. Thus we can determine betweenness on a, b, c as follows: $bB[a, c]$ if and only if $bB[a, x]$ and $cB[b, x]$ (this can be easily justified by the methods in Section 3.1).

So what happens if one of $\mathcal{D}_1(x), \mathcal{E}_1(x)$ is comparable to all of $\mathcal{C}_2(x)$, but not the other? I claim we can ignore those cases, because if the maximal finite section has size $\leq 2m$, then we are in **Case 1** above (and we will notice this when picking a more central point in this maximal finite section), and if the maximal finite section has size $\geq 2m + 1$, then again, when we pick a more central point in the maximal finite section we will obtain betweenness on the entire section as in **Case 2** □

We now expand this Lemma to n-tuples on S:

Definition 3.30. Let $X = (x_1, ..., x_n), Y = (y_1, ..., y_n)$ be n-tuples. Then $dist(X, Y) \geq m$ iff $x_n < y_1$ and $dist(x_n, y_1) \geq m$, or $y_n < x_1$ and $dist(y_n, x_1) \geq m$.

Definition 3.31. Let $B^k_{dist(m)}$ be betweenness on k-tuples of a set S without endpoints, defined for all $X = (x_1, ..., x_k), Y = (y_1, ..., y_k), Z = (z_1, ..., z_k)$ where $dist(X, Y) \geq m, dist(X, Z) \geq m$, and $dist(Y, Z) \geq m$.

Thus $B^k_{dist(1)}$ is the same as B^k.

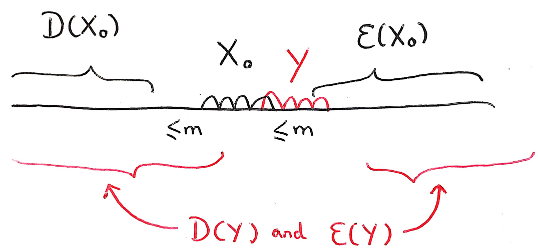
We can also extend the notion of comparability to $B^k_{dist(m)}$ as follows: two tuples X and Y are comparable if $dist(X, Y) \geq m$ (formally: X and Y are comparable if there exists some Z such that $XB^k_{dist(m)}[Y, Z]$ or $YB^k_{dist(m)}[X, Z]$ or $ZB^k_{dist(m)}[X, Y]$).

Theorem 3.32. Let S be a total ordering without endpoints. Then if S has no non-trivial maximal finite section of size less than $m + n$, then $(S, B_{dist(m+n)})$ is interpretable in $(S^n, B^n_{dist(m)})$.

Proof. Given an n-tuple $X = (x_1, ..., x_n)$, let $\mathcal{C}(X)$ be the set of all n-tuples comparable to X.

Now split $\mathcal{C}(X)$ into two halves, $\mathcal{D}(X)$ and $\mathcal{E}(X)$ as in Lemma 3.29 for each $X \in S^n$.

Let $X_0 \in S^n$ be fixed, with corresponding sets $\mathcal{D}(X_0)$ and $\mathcal{E}(X_0)$. It will be true that for every other set $Y \in S^n$, either $\mathcal{D}(Y) \subseteq \mathcal{D}(X_0)$ or $\mathcal{D}(X_0) \subseteq \mathcal{D}(Y)$, or $\mathcal{E}(Y) \subseteq \mathcal{D}(X_0)$ or $\mathcal{D}(X_0) \subseteq \mathcal{E}(Y)$. For every such Y, let $\mathcal{D}(Y)$ be such that either $\mathcal{D}(Y) \subseteq \mathcal{D}(X_0)$ or $\mathcal{D}(X_0) \subseteq \mathcal{D}(Y)$. Thus all the $\mathcal{D}$s in the graph will end up on the same side as one another, and similarly with the $\mathcal{E}$s. Without loss of generality assume $\mathcal{D}(X_0) < X_0 < \mathcal{E}(X_0)$.



We now have two cases for each $X \in S^n$, where $X = (x_1, ..., x_n)$: either $\mathcal{D}(X)$ has a 'maximum' d_1, or not. If $\mathcal{D}(X)$ does have a maximum, then every element of $\mathcal{D}(X)$ ends in something less than or equal to d_1, and $dist(d_1, x_1) = m$. If $\mathcal{D}(X)$ doesn't have a maximum, then $\mathcal{D}(X)$ has no point of distance exactly m from x_1.

In clarification, I remark that if $\mathcal{D}(X)$ has no maximum, there could be any number of points $< m$ (perhaps none) lying between all of $\mathcal{D}(X)$ and x_1. Any n-tuple beginning with one of these points is comparable to all of $\mathcal{D}(X)$, meaning there is no tangible difference between tuples beginning with one of these points and tuples beginning with x_1.

We can distinguish between the two cases as follows: if there exists an n-tuple $Z \in \mathcal{D}(X)$ such that there *aren't* infinitely many n-tuples Y with $YB[X, Z]$, then $\mathcal{D}(X)$ has a maximum, and otherwise it doesn't.

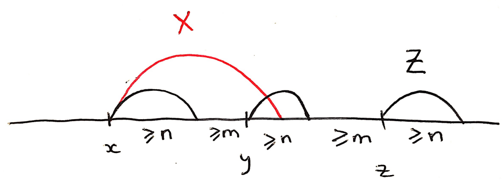
Since every maximal finite section has size at least $m + n$, there is an n-tuple X in every maximal finite section such that $\mathcal{D}(X)$ has a maximum. Similarly, in any finite component with a left endpoint, there is an n-tuple X of distance at least m from the this edge so that $\mathcal{D}(X)$ has a maximum.

We now define an equivalence relation, $\equiv$, on S^n as follows: given $X, Y \in S^n$, $X \equiv Y$ if $\mathcal{D}(X) = \mathcal{D}(Y)$. Thus n-tuples $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ are equivalent if either $x_1 = y_1$ (if $\mathcal{D}(X)$ has a maximum), or x_1, y_1 lie within distance $< m$ of the minimum of a finite component, in which case $\mathcal{D}(X)$ doesn't have a maximum.

The idea is that every time $\mathcal{D}(X)$ doesn't have a maximum, we define the equivalence class for x_1 to be to the minimum of the finite component x_1 lies in.

The $\equiv$-classes are now points $x \in S$, corresponding to the set of all n-tuples beginning with x. These can take any value in S, except for values of distance less than m from the minimum point of a finite component. Let the set of values x can take be S'.

We define comparability on equivalence classes as follows: x and y are comparable if there is some representative in x that is comparable to some representative in y. Any pair of equivalece classes x, y are thus comparable to one another if and only if $dist(x, y) \geq m+n$. Now consider x, y, z, all pairwise comparable. Then $yB[x, z]$ if and only if there exists a representative X of x that is comparable to some representative Z of z but not to *any* representative Y of y, or there exists a representative Z of z that is comparable to some representative X of x but not to any representative Y of y.



We have thus recovered $(S', B_{dist(m+n)})$. We would like to expand this to S, which we can do as follows: formally add $m-1$ points between the minimum point of every finite component, and its successor, and choose some natural ordering on these points $<$ with induced betweenness B_{m+n}. Thus $(S, B_{dist(m+n)})$ is interpretable from $(S^n, B^n_{dist(m)})$.

□

3.3 Betweenness Relations on Trees

We now expand some of the betweenness theorems to partial betweenness on trees.

We start with some basic definitions.

Definition 3.33. A *tree* $(T, <)$ is a partially ordered set such that for each $t \in T$, the set $\{s \in T : s < t\}$ is totally ordered, and any two elements have a common lower bound. A *proper tree* is a tree which is not a totally ordered set i.e. a proper tree has 2 distinct vertices x and y such that $\neg x < y$ and $\neg y < x$.

Definition 3.34. A tree $(T, <)$ is *dense* if for any $s, t \in T$ with $s < t$, the interval $[s, t]$ is dense.

Definition 3.35. Let T be a tree. Then two distinct vertices $x, y \in T$ are *comparable*, denoted by $comp(x, y)$, if either $x < y$ or $y < x$. If x and y are subsets of T whose elements are comparable, then x and y are *comparable* if every element of x is comparable to every element of y.

Note that x is not comparable to itself.

Definition 3.36. Let T be a tree. Then $x \in T$ is a *leaf* if there is no $y \in T$ with $x < y$.

Definition 3.37. Let $(T, <)$ be a tree. Then a point x is a *ramification point* if x is the greatest lower bound of some incomparable $y, z \in T$.

Definition 3.38. We can form a least extension T^+ of T so that any two members of T^+ have a greatest lower bound (so that it is a meet semilattice).

This is a special case of the Dedekind-MacNeille completion [4]. In fact, all 'new' points will be the greatest lower bound of two incomparable members of T.

We call points of $T^+ \backslash T$ *gaps*.

Note that $|T^+| \leq max(\aleph_0, |T|)$.

Definition 3.39. Let $(T, <)$ be a tree. If $x \in T^+$ we define $\sim$ on $\{y : y > x\}$ as follows: $y \sim z$ if there exists a t such that $x < t \leq y, z$.

This is an equivalence relation. Reflexivity and symmetry are clear. To see that it is transitive suppose $y_1 \sim y_2 \sim y_3$. Then $y_1, y_2 \geq t_1 > x$, and $y_2, y_3 \geq t_2 > x$, and so $t_1, t_2 \leq y_2$. Since predecessors of y_2 are linearly ordered, either $t_1 \leq t_2$ or $t_2 \leq t_1$, and so $min\{t_1, t_2\} \leq y_1, y_3$.

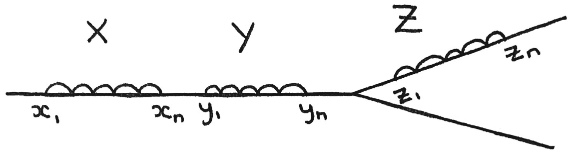
Equivalence classes are called *cones* at x.

Definition 3.40. Let $(T, <)$ be a tree. Then T has a *root* x if x is a minimal point in T.

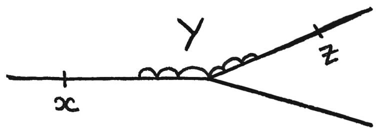
Note that a tree can only have one such minimal point.

Let T^n be the set of all strictly increasing n-tuples in T.

Definition 3.41. Partial betweenness on n-tuples for trees. Let $(T, <)$ be a tree. We define betweenness on n-tuples of T as follows: given $(x_1, ..., x_n), (y_1, ..., y_n), (z_1, ..., z_n)$, $(y_1, ..., y_n)B^n[(x_1, ..., x_n), (z_1, ..., z_n)]$ if and only if $(y_1, ..., y_n)B^n[(x_1, ..., x_n), (z_1, ..., z_n)]$ holds on some linearly ordered subset of T.



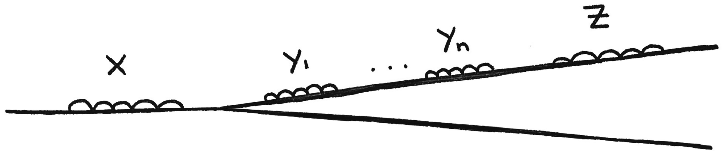
Definition 3.42. We now expand n-betweenness B_n to trees: $\{y_1, ..., y_n\}B_n[x, z]$ holds iff the set of points $\{y_1, ..., y_n\}$ lies between x and z in some linearly ordered subset of T. Again, we call x and z the *endpoints* of the set $\{y_1, ..., y_n, x, z\}$.



Note that $\{y_1, ..., y_n\}B_n[x, z] \to \{y_{\pi(1)}, ..., y_{\pi(n)}\}B_n[x, z]$, where π is any permutation on $\{1, ..., n\}$, and that $\{y_1, ..., y_n\}B_n[x, z] \to \{y_1, ..., y_n\}B_n[z, x]$.

Again note that for 1-betweenness there is a slight difference of notation, in that we write $yB[x, z]$ as opposed to $\{y\}B_1[x, z]$.

Definition 3.43. Again, we can combine the above definitions as follows: B_n^m is defined as partial n-betweenness on m-tuples, i.e. this is a partial relation on $n+2$ m-tuples where $\{Y_1, ..., Y_n\}B_n[X, Z]$ holds iff each m-tuple in the set $\{Y_1, ..., Y_n\}$ lies between the m-tuples X and Z in some linearly ordered subset of T.



Similarly, note that $\{Y_1, ..., Y_n\}B_n[X, Z] \to \{Y_{\pi(1)}, ..., Y_{\pi(n)}\}B_n[X, Z]$, where π is any permutation on $\{1, ..., n\}$, and that $\{Y_1, ..., Y_n\}B_n[X, Z] \to \{Y_1, ..., Y_n\}B_n[Z, X]$.

Lemma 3.44. Let $(T,<)$ be a proper tree without endpoints with induced partial betweenness relation B, and let $n \in \mathbb{N}$. Then $(T,<)$ is 1st-order interpretable in (T^n, B^n).

Proof. We can show that (T,B) is 1st-order interpretable in (T^n, B^n) just as in the proof of Lemma 3.22. Thus it remains to show that $(T,<)$ is interpretable in (T,B).

First note that t_1 and t_2 are not comparable if for every t_3, none of t_1, t_2, t_3 are between the other two, because T has no endpoints.

Since T is a proper tree, there exist x, y, z such that x and y are not comparable to one another, but are both comparable to z. In this case we know that $z < x, y$. Such x, y, z always exists since T is a proper tree.

Now suppose we are given distinct comparable $t_1, t_2 \in T$ and we would like to determine whether $t_1 < t_2$ or $t_2 < t_1$. We have 3 cases:

- t_1, t_2 are both comparable to both x and y. In this case, $t_1 < t_2$ if $t_2 B[t_1, x]$.
- t_1 is comparable to both x and y, and t_2 is only comparable to one of x, y (assume x without loss of generality). Then $t_1 < t_2$ by default.
- t_1, t_2 are both comparable to only one of x, y. Then $t_1 < t_2$ if $t_1 B[z, t_2]$, where z is as above.

Hence for any comparable t_1, t_2, we can determine whether $t_1 < t_2$ or $t_2 < t_1$, and so $(T,<)$ is interpretable from (T^n, B^n). $\square$

Lemma 3.45. Let $(T,<)$ be a tree without leaves with induced betweenness relation B, and let $n \in \mathbb{N}$. Then (T,B) is 1st-order definable from (T, B_n). Moreover, if T is a proper tree then $(T,<)$ is 1st-order definable from (T, B_n).

Proof. The proof that (T,B) is 1st-order definable from (T, B_n) is similar to that of Lemma 3.23, and the proof that $(T,<)$ is 1st-order definable from (T, B_n) if T is a proper tree is similar to that of Lemma 3.44. $\square$

The reason we require T to not have leaves is similar to why we have mainly looked at total orderings without endpoints, namely, that there are quite a few complications as soon as endpoints are considered. T must be a proper tree in order to determine $<$ rather than B, as we must be able to consider 3 elements x, y, z in T as in Lemma 3.44, i.e. z is comparable to x and x, but x and y are not comparable to one another (in which case $z < x$ and $z < y$).

Lemma 3.46. Let T be a tree without endpoints. Then (T,B) is definable from (T^m, B_n^m). Moreover, if T is a proper tree that does not have a minimal element which ramifies, then $(T,<)$ is 1st-order definable from (T^m, B_n^m).

Proof. The proof that (T, B) is 1st-order interpretable from (T^m, B_n^m) is similar to that of Lemma 3.24, and the proof that $(T, <)$ is 1st-order definable from (T^m, B_n^m) if T is a proper tree that doesn't have a minimal element which ramifies is similar to that of Lemma 3.44. □

3.4 The Graph $G(S, 1^n 2^n)$

Proposition 3.47. Let S be any total ordering with no endpoints. Then $(S, <)$ is 1st-order interpretable from the relation $(S^n, <^n)$, where $(x_1, ..., x_n) <_n (y_1, ..., y_n)$ if and only if $x_1 < ... < x_n < y_1 < ... < y_n$.

Proof. Let $a \in S^n$, and define $a^+ := \{y \in S^n : a <^n y\}$, i.e. a^+ is the set of all members of S^n which are greater than a.

We now define the following equivalence relationon S^n: $a \sim b$ iff $a^+ = b^+$ (one can check that this is an equivalence relation). The idea is that we identify two increasing n-tuples if they have the same set of strict upper bounds, which depends only on the final entry.

The equivalence classes of $\sim$ are in natural 1-1 correspondence with members of S. Furthermore, we can also recover the ordering, since the greatest member of a is less than or equal to the greatest member of b if and only if $b^+ \subseteq a^+$.

Note that this is first-order:

$$a^+ = b^+ \leftrightarrow (\forall y)(a <^n y \leftrightarrow b <^n y)$$

And similarly,

$$b^+ \subseteq a^+ \leftrightarrow (\forall y)(b <^n y \rightarrow a <^n y)$$

□

Remark 3.48. Similarly, the total ordering S can be reconstructed from the directed graph $G(S, 1^k 2^k)$ with an edge from $(y_1, ..., y_k)$ to $(x_1, ..., x_k)$ if and only if $x_1 < ... < x_k < y_1 < ... < y_k$.

Theorem 3.49. Let $(S, <)$ be a dense total ordering without endpoints, B_S the associated betweenness relation, and let $n \geq 2$ lie in $\mathbb{N}$. Then (S, B_S) is 1st-order definable inside $G = G(S, 1^n 2^n)$.

Proof. In the graph G, there is an edge between $(a_1, ..., a_n)$ and $(b_1, ..., b_n)$ iff either $a_n < b_1$ or $b_n < a_1$.

Let A be the n-tuple $(a_1, ..., a_n)$, and let N_A be the set of neighbours of A in G. Then N_A is the disjoint union of two sets: the set of n-tuples whose greatest element is less than a_1, and the set of n-tuples whose least element is greater than a_n. From this it follows that

$$N_A = N_B \text{ if and only if } a_1 = b_1 \text{ and } a_n = b_n$$

In this case, we write $A \equiv B$ (we see that this is an equivalence relation). Thus we can identify the $\equiv$-classes with pairs (a_1, a_n) such that $a_1 < a_n$.

Note that this is first-order, as we can write

$$N_A = N_B \Leftrightarrow \ \forall C(A \sim C \Leftrightarrow B \sim C)$$

for elements A, B, C.

Furthermore,

$$N_A \subseteq N_B \text{ if and only if } a_1 \leq b_1 \text{ and } b_n \leq a_n$$

and for this reason, it makes sense to identify each $\equiv$-class with a closed interval $[a_1, a_n]$, rather than the pair of its endpoints, since then the relation $[a_1, a_n] \subseteq [b_1, b_n]$ of inclusion is definable.

Let $\mathcal{I}$ be the family of all these closed intervals:

$$\mathcal{I} := \{[a, b] : a < b,\ a, b \in S \text{ such that there exist } a_1 < ... < a_n \text{ with } a_1 = a \text{ and } a_n = b\}$$

We would now like to define the relations "$A \cup B \in \mathcal{I}$" and "$A \cap B \in \mathcal{I}$" for intervals $A = [a_1, a_n]$ and $B = [b_1, b_n]$.

Let $A \vee B$ be the least upper bound under inclusion of these two intervals. This always exists (for example if $a_1 < b_n$ then this will be the interval $[a_1, b_n]$), and is given by the interval C such that $A \subseteq C$ and $B \subseteq C$, and for every $D \in \mathcal{I}$, if $A \subseteq D$ and $B \subseteq D$, then $C \subseteq D$.

Let $A \wedge B$ be the greatest closed interval C lying in both A and B. This exists if and only if the intersection of A and B is nonempty and not a singleton, and is given by the interval C such that $C \subseteq A$ and $C \subseteq B$, and for every $D \in \mathcal{I}$, if $D \subseteq A$ and $D \subseteq B$, then $D \subseteq C$.

Intersection: We now define $A \cap B$ as follows: $A \cap B \in \mathcal{I}$ if and only if $A \wedge B$ exists, in which case $A \cap B = A \wedge B$, and otherwise $A \cap B$ is undefined.

Consider the statement that $A \wedge B$ does not exist, and for any A' strictly containing A and such that $A \vee B = A' \vee B$, $A' \wedge B$ exists.

This describes two intervals overlapping in a singleton. To see this, note that if they do overlap in a singleton, whenever you try to increase one of the intervals while retaining the same least upper bound (now just the union), you have to increase the 'bit in the middle' where they overlap, so that the greatest lower bound now exists. If however, they are disjoint, then you can increase one of them a little bit (in the middle) so that it is still disjoint from the other, and the greatest lower bound still doesn't exist. *Again, here we utilise the fact that S is dense, as otherwise the statement above would not necessarily describe two intervals overlapping in a singleton.*

Union: This enables us to define $A \cup B$ in cases where this is an interval. Namely, we let $A \cup B = A \vee B$ provided *either* $A \wedge B$ exists (so is an interval), *or* A and B overlap in a singleton.

We can now exploit the definability of intervals overlapping in a point to gain access to points of S inside G. Let us call a pair (A, B) which overlap in a singleton *pointed.* We note that this happens provided that either $a_1 = b_n$ or $b_1 = a_n$, and we say that a pointed pair $([x, y], [y, z])$ *points to* y.

Note that for $n = 1$, this entire argument becomes trivial as (A, B) is pointed iff $A = B$.

We would now like to identify any $y \in S$ with the set of all pairs of intervals $([x, y], [y, z])$ in $\mathcal{I}$ pointing to y.

First define the following equivalence relation $\equiv_\rho$: We say that $(A, B) \equiv_\rho (C, D)$ if (A, C) and (B, D) are pointed and $B \neq C$ or (A, D) and (B, C) are pointed and $A \neq D$.

Since $(A, B) \equiv_\rho (B, A)$ we can switch round the pairs, and assume always that $A = [a, x]$ and $B = [x, b]$ for some a, b, x.

We would like to show that $(A, B) \equiv_\rho (C, D)$ if and only if $A = [a, x], B = [x, b], C = [c, x]$, and $D = [x, d]$ for some a, b, c, d, x. If this is the case, it is immediate that $\equiv_\rho$ is an equivalence relation.

Let's assume then for a contradiction that $(A, B) \equiv_\rho (C, D)$ and $A = [a, x], B = [x, b], C = [c, y]$, and $D = [y, d]$ for some a, b, c, d, x, y with $x \neq y$.

Suppose first that (A, C) and (B, D) are pointed. This implies that $a = y$ or $x = c$, and also that $b = y$ or $x = d$. There appear to be 4 options. However, as $a \neq b$ and $c \neq d$, there are only 2, which are that $a = y$ and $x = d$, or that $x = c$ and $b = y$. But the first implies that $A = D$, which we have ruled out, and the second that $B = C$, also impossible.

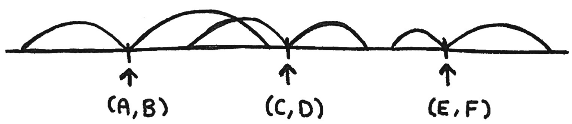
One argues similarly if (A, D) and (B, C) are pointed. Hence $\equiv_\rho$ is an equivalence relation.

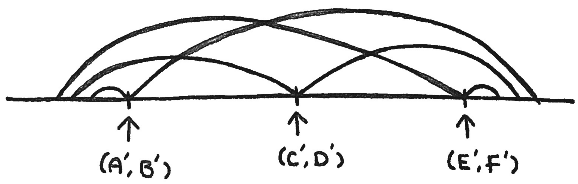
It now remains to recover the betweenness relation, which can be done as follows:

For pointed pairs $(A, B), (C, D), (E, F)$, the element of S pointed to by (C, D) lies between the elements of S pointed to by (A, B) and (E, F) if and only if there exist (A', B'), (C', D'), (E', F') respectively equivalent to $(A, B), (C, D), (E, F)$ such that

$$A' \subseteq C' \subseteq E' \text{ and } F' \subseteq D' \subseteq B'$$
$$\text{or } E' \subseteq C' \subseteq A' \text{ and } B' \subseteq D' \subseteq F'$$

See diagram for intuition:





Thus we can reconstruct (S, B_S) from G. Note that this is first-order. $\square$

We remark that although the dense case without endpoints seems restrictive (in the countable case it is just $\mathbb{Q}$), in the uncountable case there is a vast range of possible orderings to which it could apply.

We would now like to expand this theorem to all total orderings without endpoints. We start with some definitions.

Definition 3.50. Let S be a total ordering. Then the comparability relation C on S^n is the relation which holds for two n-tuples $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ if and only if $x_1, ..., x_n < y_1, ..., y_n$ or $y_1, ..., y_n < x_1, ..., x_n$.

Definition 3.51. Let S be any total ordering, and let $\mathcal{I}(S)$ be the set of all closed intervals on S that are not singletons. Then the comparability relation C on $\mathcal{I}(S)$ is the relation which holds for two intervals $[x_1, x_n]$ and $[y_1, y_n]$ if and only if $x_n < y_1$ or $y_n < x_1$.

Note that comparability on n-tuples of S is the same thing as the edge relation in $G(S, 1^n 2^n)$.

Theorem 3.52. Let $(S, <)$ be any total ordering without endpoints, B_S the associated betweenness relation, and let $n \geq 2$ lie in $\mathbb{N}$. Then (S, B_S) is 1st-order definable inside $G = G(S, 1^n 2^n)$.

Proof. Similarly to the proof of Theorem 3.49, define $\equiv$-classes on closed intervals $[a_1, a_n]$ in S of cardinality at least n, and again let $\mathcal{I}$ be the family of these closed intervals. Note that disjointness and inclusion are easily definable in $\mathcal{I}$.

We would like to represent points of S inside G by means of 'pointed pairs' as in the proof of Theorem 3.49. The problem here arises if a pair of intervals which may appear suitably 'pointed' in fact overlap at a number of points between 2 and $n - 1$.

Let us call a pair $(A, B) \in \mathcal{I}$ *awkward* if:

(i) A and B are not disjoint (i.e. representatives from A and B are not neighbours in G)

(ii) There is no C in $\mathcal{I}$ contained in both A and B

It is clear that awkward pairs are intervals A, B which overlap in a set of size $< n$.

Since clearly $A \cup B \in \mathcal{I}$, we can express that $D = A \cup B$. We simply say that $A \subseteq D$ and $B \subseteq D$, and for any E such that $A \subseteq E$ and $B \subseteq E$, we have $D \subseteq E$. In the dense case, $|A \cap B|$ must be 1, and everything goes through straightforwardly. If S is (in places) non-dense, then we have to use a different trick. We would like to decrease A as far as possible while retaining awkwardness, until the intersection does have size 1. The trouble is that if A is too small, reducing its size may mean that it isn't in $\mathcal{I}$ any more. So we first have to increase it a bit, and after that decrease it again.

Let A' be a proper superset of A such that (A', B) is awkward. Such A' exists because one can extend in the direction away from B, since S has no endpoints. Repeat this n times. Then such A' has size at least $2n$, and overlaps with B at at most $n - 1$ points (so if we 'remove' at most $n - 2$ points from A' so that A' and B overlap at a singleton, we know A' still has size at least n). Now if possible, find $A'' \subset A'$ so that (A'', B) is still awkward, and $A'' \cup B = A' \cup B$ (if this isn't possible, then (A', B) already overlap at a singleton). Since the union is preserved, the endpoints of A' and A'' away from B must agree, so the only way that A' can strictly decrease is in the intersection with B. So this must go down. Repeating n (at most) times, eventually it isn't possible to decrease any more, and this must mean that A'' and B intersect in a singleton.

The rest of the proof is similar to the proof of Theorem 3.49. $\square$

4 Reconstructions of the Shift Graph

We now turn our attention to reconstructing the underlying set S from the shift graph $G(S, 1^n 3^m 2^n)$. We start by considering general linear orderings, and then briefly looking at partial orderings that behave in a similar way.

4.1 Linear Ordering Shift Graphs

We now consider shift graphs in which the underlying set is any linear order. We will focus on infinite linear orderings.

Lemma 4.1. Let $(S, <)$ be any total ordering without endpoints. Then any pair of co-cliques C and D in $G(S, 132)$ with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}\}$$

and where $|C|, |D| \geq 2$ must be of the form

$$C_x = \{(v, x) : v < x\}$$
$$D_x = \{(x, w) : x < w\}$$

where $x \in S$.

Proof. First we see that C_x, D_x satisfy $i)$ and $ii)$. Now, let C and D be a pair of sets satisfying $i)$ and $ii)$. Then we remark that if $(c_1, c_2) \in C$ and $(d_1, d_2) \in D$, then either

$$\begin{array}{ccccc} c_1 & < & c_2 & & \\ & & \| & & \\ & & d_1 & < & d_2 \end{array}$$

or

$$\begin{array}{ccccc} & & c_1 & < & c_2 \\ & & \| & & \\ d_1 & < & d_2 & & \end{array}$$

Note that we cannot have $(c_1, c_2), (c_1', c_2') \in C$ with $c_2 = d_1$ and $c_1' = d_2$ for the same $(d_1, d_2) \in D$, as otherwise (d_1, d_2) is the only point of D adjacent to both (c_1, c_2) and (c_1', c_2), contrary to the assumption that both C and D contain at least 2 elements.

Thus either for all $(c_1, c_2) \in C$ and $(d_1, d_2) \in D$ we have $c_2 = d_1$, or for all $(c_1, c_2) \in C$ and $(d_1, d_2) \in D$ we have $c_1 = d_2$.

Without loss of generality we assume the former. We then note that, by the definition of C and D, any pair ending in c_2 must lie in C, and any pair beginning with c_2 must lie in D. Thus any pair of subsets of G satisfying $i)$ and $ii)$ must be of the form $C_x = \{(v, x) : v < x\}$ and $D_x = \{(x, w) : x < w\}$. $\square$

Theorem 4.2. Let $(S,<)$ be any total ordering without endpoints, and B the associated betweenness relation. Then (S,B) is 2nd-order interpretable inside $G = G(S,132)$.

Proof. We are interested in identifying co-cliques C and D with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

where $|C|, |D| \geq 2$.

By Lemma 4.1, a pair of subsets of $V(G)$ satisfying $i)$ and $ii)$ must be of the form

$$C_x = \{(v,x) : v < x\}$$
$$D_x = \{(x,w) : x < w\}$$

where $x \in S$.

Let $A_x = C_x \cup D_x$ for every such pair $\{C_x, D_x\}$ and $x \in S$. Thus $V(G) = \bigcup\{A_x : x \in S\}$, and for each $x \neq y$, $A_x \cap A_y$ contains exactly one element, namely (x,y) or (y,x).

Now, given A_x, A_y, A_z, we see that $yB[x,z]$ iff the element of $A_x \cap A_y$ is a neighbour of the element of $A_y \cap A_z$, and neither of these elements is a neighbour of the element of $A_x \cap A_z$.

Thus we can reconstruct S with the betweenness relation from G, and so we can reconstruct S up to order reversal by Theorem 3.15. □

Lemma 4.3. Let $(S,<)$ be a dense total ordering without endpoints, and let n be a positive integer. Then any pair of co-cliques C and D in $G(1^n 3 2^n)$ with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

and where C and D are nonempty must either be of the form

$$C_x = \{(v_1, v_2, ..., v_n, x) : v_n < x\}$$
$$D_x = \{(x, w_1, w_2, ..., w_n) : x < w_1\}$$

where $x \in S$, or of the form

$${}_xC_y = \{(x, v_1, v_2, ..., v_{n-1}, y) : x < v_1 < v_2 < ... < v_{n-1} < y\}$$
$${}_xD_y = \{(w_1, w_2, ..., w_n, x) : w_n < x\} \cup \{(y, w_1', w_2', ..., w_n') : y < w_1'\}$$

where $x, y \in S$.

Proof. First we see that C_x, D_x and ${}_xC_y,\ {}_xD_y$ satisfy *i)* and *ii)*. Now, let C and D be a pair of sets satisfying *i)* and *ii)*. Then we remark that if $c = (c_1, ..., c_{n+1}) \in C$ and $d = (d_1, ..., d_{n+1}) \in D$, then either

$$\begin{array}{ccccccccc} c_1 & < & c_2 & \dots & < & c_{n+1} & & & \\ & & & & & \| & & & \\ & & & & & d_1 & < & d_2 \ \dots & < \ d_{n+1} \end{array}$$

in which case we say c is "left adjacent" to d, or

$$\begin{array}{ccccccccc} & & & & & c_1 & < & c_2 \ \dots & < \ c_{n+1} \\ & & & & & \| & & & \\ d_1 & < & d_2 & \dots & < & d_{n+1} & & & \end{array}$$

in which case we say c is "right adjacent" to d.

Note that c, d are adjacent provided c is left or right adjacent to d.

We now have two cases:

Case 1: there exist $c, c' \in C$ and $d \in D$ such that c is left adjacent to d, and d is left adjacent to c'.

$$\begin{array}{ccccccccccc} c_1 & < & c_2 & \dots & < & c_{n+1} & & & & & \\ & & & & & \| & & & & & \\ & & & & & d_1 & < & d_2 & \dots & < & d_{n+1} & & \\ & & & & & & & & & & \| & & \\ & & & & & & & & & & c'_1 & < & c'_2 \ \dots \ < \ c'_{n+1} \end{array}$$

Then any $n+1$-tuple lying in C is adjacent to d, so must end with d_1 or begin with d_{n+1}, and hence any $n+1$-tuple beginning with d_1 and ending with d_{n+1} is adjacent to all members of C, and so lies in D.

Similarly, any $n+1$-tuple in D is adjacent to c and c', so must begin with d_1 and end with d_{n+1}. The set of all $n+1$-tuples which are neighbours of all of these is precisely the set of all $n+1$-tuples beginning with d_{n+1} or ending with d_1, it follows that C is the set of everything adjacent to D.

The argument is similar if $d, d' \in D$ and $c \in C$.

Case 2: neither of the above hold.

Then for every $c \in C$, either every $d \in D$ is left adjacent to c or every $d \in D$ is right adjacent to c.

Similarly, for every $d \in D$, either every $c \in C$ is left adjacent to d or every $c \in C$ is right adjacent to d.

Without loss of generality assume there is a $d \in D$ such that every $c \in C$ is right adjacent to d.

$$\begin{array}{ccccccccccc} & & & & & & c_1 & < & c_2 & \ldots & < & c_{n+1} \\ & & & & & & \| & & & & & \\ d_1 & < & d_2 & \ldots & < & & d_{n+1} & & & & & \end{array}$$

Then $c_1 = d_{n+1}$, and moreover all members of C must begin with this same c_1. Thus all $n+1$-tuples ending in d_{n+1} lie in D, and similarly all $n+1$-tuples beginning in c_1 lie in C. □

Theorem 4.4. Let $(S, <)$ be a dense total ordering without endpoints, B the associated betweenness relation, and let n be a positive integer. Then (S, B) is 2nd-order interpretable inside $G = G(S, 1^n 32^n)$.

Proof. Consider co-cliques C and D with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

where C and D are nonempty.

By Lemma 4.3, a pair of subsets of $V(G)$ satisfying $i)$ and $ii)$ must either be of the form

$$C_x = \{(v_1, v_2, \ldots, v_n, x) : v_n < x\}$$
$$D_x = \{(x, w_1, w_2, \ldots, w_n) : x < w_1\}$$

where $x \in S$, or of the form

$${}_xC_y = \{(x, v_1, v_2, \ldots, v_{n-1}, y) : x < v_1 < v_2 < \ldots < v_{n-1} < y\}$$
$${}_xD_y = \{(w_1, w_2, \ldots, w_n, x) : w_n < x\} \cup \{(y, w_1', w_2', \ldots, w_n') : y < w_1'\}$$

where $x, y \in S$.

Let $\mathcal{F}$ be the family of all pairs $\{C, D\}$ satisfying $i)$ and $ii)$. Let $\mathcal{F}_1$ be the family of all pairs $\{C_x, D_x\}$, where C_x, D_x are as above, and let $\mathcal{F}_2$ be the family of all pairs $\{{}_xC_y,\ {}_xD_y\}$, where ${}_xC_y,\ {}_xD_y$ are as above. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.

Given some $\{C, D\} \in \mathcal{F}$, we would like to determine whether $\{C, D\} \in \mathcal{F}_1$ or $\{C, D\} \in \mathcal{F}_2$. I claim that $\{C, D\} \in \mathcal{F}_2$ if and only if there exist pairs of sets $\{C', D'\}, \{C'', D''\} \in \mathcal{F}$ with $\{C', D'\}, \{C'', D''\} \neq \{C, D\}$ such that for some $X' \in \{C', D'\}$ and $X'' \in \{C'', D''\}$, we have $X' \cup X'' \in \{C, D\}$.

To see this, suppose $\{C, D\} \in \mathcal{F}_2$. Then $\{C, D\} = \{{}_xC_y,\ {}_xD_y\}$ for some x, y. Now let $\{C', D'\} = \{C_x, D_x\}$ and $\{C'', D''\} = \{C_y, D_y\}$, and let $X' = C_x$ and $X'' = D_y$. Thus we have ${}_xD_y = C_x \cup D_y$, i.e. $C_x \cup D_y \in \{{}_xC_y,\ {}_xD_y\}$.

Conversely, suppose $\{C, D\} \notin \mathcal{F}_2$, i.e. $\{C, D\} = \{C_x, D_x\}$ for some x. We will show that C_x cannot be the union of two sets of the form C_y, D_y, ${}_yC_z$, ${}_yD_z$.

- C_x cannot be the union of any C_y with another set, as $C_x \cap C_y = \emptyset$ for all $x \neq y$.
- C_x cannot be the union of any D_y with another set, as if $x \leq y$ then D_y contains points of the form $(y, z_1, ..., z_n)$ with $x \leq y < z_1$ which do not lie in C_x, and if $x > y$ then D_y contains the points of the form $(y, z_1, ..., z_n)$ where $z_n \neq x$ which do not lie in C_x.
- We see that C_x cannot be the union of any ${}_yD_z$ with another set, as if $x < z$ then ${}_yD_z$ contains points of the form $(z, w_1, ..., w_n)$ where $x < z < w_1$ which do not lie in C_x, and if $x > y$ then ${}_yD_z$ contains points of the form $(v_1, ..., v_n, y)$ where $v_n < y < x$ which don't lie in C_x (note that we have covered all cases here since $y \neq z$).

Thus C_x must be the union of two sets ${}_yC_z$ and ${}_{y'}C_{z'}$, for some y, z, y', z'. Now, by the construction of these sets, we must have $z = z' = x$. But then C_x contains some point $(w_1, ..., w_n, x)$ where $w_1 \neq y, y'$, and so $C_x \neq_y C_z \cup {}_{y'}C_{z'}$, a contradiction.

Hence, given G we can uniquely determine $\mathcal{F}_1$. All pairs $\{C, D\} \in \mathcal{F}_1$ can be indexed by some point in S, such that each pair is of the form $\{C_x, D_x\}$ for some $x \in S$. Thus we have determined the set S (or representatives of the set S), but not the betweenness relation on S. Let $A_x = C_x \cup D_x$ for every such pair. Thus $G = \bigcup\{A_x : x \in S\}$, and for each $x \neq y$, $A_x \cap A_y$ contains the set of vertices $(x, v_2, ..., v_n, y)$ if $x < y$ with $x < v_2 < ... < v_n < y$, or $(y, v_2, ..., v_n, x)$ if $y < x$, with $y < v_2 < ... < v_n < x$. Such $v_2, ..., v_n$ always exists since S is dense.

Now, given A_x, A_y, A_z, we see that $yB[x, z]$ iff every element of $A_x \cap A_y$ is a neighbour of every element of $A_y \cap A_z$, and no element of $A_x \cap A_y$ or $A_y \cap A_z$ has any neighbours in $A_x \cap A_z$. □

Thus we can reconstruct S with the betweenness relation from G (and so we can reconstruct S up to order reversal by Theorem 3.15).

Theorem 4.5. Let $(S, <)$ be any total ordering without endpoints, B the associated betweenness relation, and let n be a positive integer. Then, if S contains no non-trivial maximal finite section under size $2n$, (S, B) is 2nd-order interpretable inside $G = G(S, 1^n 32^n)$.

Proof. We start by introducing sets C_x, D_x, ${}_xC_y$, ${}_xD_y$ as in Theorem 4.4.

We let $\mathcal{F}$ be the family of all pairs $\{C, D\}$ satisfying *i*) and *ii*), and $\mathcal{F}_1$ be the family of all pairs of the form $\{C_x, D_x\}$ and $\mathcal{F}_2$ be the family of all pairs of the form $\{{}_xC_y, {}_xD_y\}$. We can uniquely determine the set $\mathcal{F}_1$ as in Theorem 4.4, and all such pairs can again be indexed by some point in S, such that each pair is of the form $\{C_x, D_x\}$ for some $x \in S$. Let $A_x = C_x \cup D_x$ for every such pair.

We now have a problem that for $x \neq y$, $A_x \cap A_y$ is not necessarily nonempty; if $dist(x, y) \leq n$, then $A_x \cap A_y = \emptyset$. If $dist(x, y) > n$, then $A_x \cap A_y$ is the set of vertices $(x, v_2, ..., v_n, y)$ if $x < y$ with $x < v_2 < ... < v_n < y$ (or $(y, v_2, ..., v_n, x)$ if $y < x$, with $y < v_2 < ... < v_n < x$).

Now, given A_x, A_y, A_z, we have 3 possible cases:

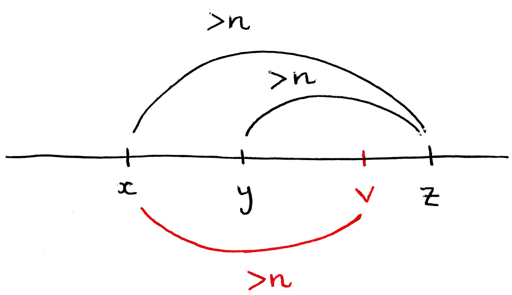
Case 1: $A_x \cap A_y, A_x \cap A_z$, and $A_y \cap A_z$ are all nonempty. Here, similarly to Theorem 4.4, we see that $yB[x, z]$ iff every element of $A_x \cap A_y$ is a neighbour of every element of $A_y \cap A_z$, and neither has any neighbours in $A_x \cap A_z$.

Case 2: Exactly one of $A_x \cap A_y, A_x \cap A_z$, and $A_y \cap A_z$ is nonempty. In this case, whichever of $A_x \cap A_y, A_x \cap A_z$, and $A_y \cap A_z$ is nonempty represents the 'outer two' points, and so $yB[x, z]$ iff only $A_x \cap A_z$ is nonempty.

This is because, without loss of generality, if y is between x and z, then $dist(x, z) \geq dist(x, y)$ and $dist(x, z) \geq dist(y, z)$, and so if one of $A_x \cap A_y, A_y \cap A_z$ is nonempty, then $A_x \cap A_z$ must be nonempty also.

Case 3: Exactly two of $A_x \cap A_y, A_x \cap A_z$, and $A_y \cap A_z$ are nonempty. In this case one of the nonempty intersections represents the 'outer two' points. Without loss of generality, suppose $A_x \cap A_z$ and $A_y \cap A_z$ are both nonempty, but $A_x \cap A_y = \emptyset$. Then, since one of $A_x \cap A_z$ and $A_y \cap A_z$ represents the 'outer two' points, z is not between x and y.

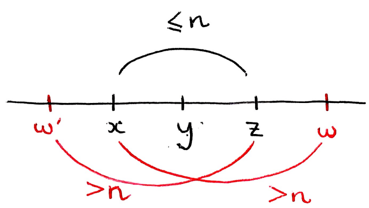
Then y lies between x, z iff there exists a v such that $A_v \cap A_y = \emptyset$ but $A_v \cap A_x \neq \emptyset$, and $\neg xB(v, z)$. Such a v will always exist as, assuming $yB[x, z]$, since $dist(y, z) \geq n + 1$ we must have $dist(x, z) \geq n + 2$. Then we can let v be any point of distance at least $n + 1$ (and less than $dist(x, z)$) from x in the direction of z.



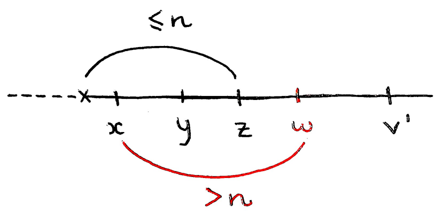
Case 4: All of $A_x \cap A_y, A_x \cap A_z$, and $A_y \cap A_z$ are empty.

In this case, the distance between the two 'outer points' is $\leq n$.

Again, we consider $\{v : A_v \cap A_x = \emptyset\}$. These are all things of distance at most n from x. Since every maximal finite section has size at least $2n$, the size of this set must be at least n. Now, if there exists a $w \notin \{v : A_v \cap A_x = \emptyset\}$ such that $A_w \cap A_y = \emptyset$ and $A_w \cap A_z = \emptyset$, then x is an 'endpoint' of x, y, z, i.e. $\neg xB[y, z]$. It will always be possible to determine at least one 'endpoint' this way as the distance between the two endpoints is at most n, and so if e is an endpoint of e, p_1, p_2, we can simply let w be the point of distance $n + 1$ away from e in the same direction as p_1 and p_2. If there exists such a w for two of x, y, z, then we have determined the two 'endpoints' and thus betweenness on x, y, z.



Suppose such a w only exists for one of x, y, z. Without loss of generality suppose x is an 'endpoint' of x, y, z, i.e. $w \notin \{v : A_v \cap A_x = \emptyset\}$ and $A_w \cap A_y = \emptyset$, $A_w \cap A_z = \emptyset$. Then x lies quite close to the 'edge' of the maximal finite section; close enough that z has distance at most n to the end of the maximal finite section (otherwise we would be able to determine two 'endpoints' in this way). Now, $yB[x, z]$ iff there is a $v' \notin \{v : A_v \cap A_x = \emptyset\}$ such that $A_{v'} \cap A_y \neq \emptyset$ and $A_{v'} \cap A_z = \emptyset$. We know such a v' always exists, as y is distance at most $n - 1$ from the edge, and so v' is distance at most $2n$ from the edge, which is the maximal distance we have allowed for.



□

Remark 4.6. The reason we can't have non-trivial maximal finite sections of size less than $2n$ in Theorem 4.5 is that if there is a maximal finite section smaller than this containing points x, y, z, it is not always possible to determine which lies between the other two.

Suppose Y is a maximal finite section of size $n + k$ for $1 \leq k < n$, and let $x, y, z \in P$. We consider the various cases as in Theorem 4.5:

- If we are in **Case 1**, **Case 2**, or **Case 3** above, then we can determine which lies between the other two as in Theorem 4.5.

- If we are in **Case 4** and x, y, z are all close enough to the 'middle' of the maximal finite section, we can determine the two 'endpoints' of x, y, z as in Theorem 4.5.

If x, y, z are all close to the 'edge' of the maximal finite section and the distance between the middle point of x, y, z and this edge is less than k, then we can also determine which lies between the other two using the same method as in Theorem 4.5. This is because in this case the distance between the middle point and the *other* edge will be $> n$, as the entire maximal finite section has size $n + k$, and so there exists a v' as in the proof.

The problem occurs if we can only determine one endpoint of x, y, z, and the distance between the middle point of x, y, z and the edge x, y, z are all close to is $\geq k$. Assuming $x < y < z$ and x is the point closest to this edge, in this case, we might not be able to find a point v' such that $A_{v'} \cap A_y \neq \emptyset$, but $A_{v'} \cap A_z = \emptyset$. Thus for every single point p in the S, either p lies in the maximal finite section, in which case $A_p \cap A_y = A_p \cap A_z = \emptyset$, or p doesn't, in which case $A_p \cap A_y$ and $A_p \cap A_z$ are both nonempty, and moreover there are no edges between these intersections. We therefore cannot determine betweenness on x, y, z.

If the non-trivial maximal finite section Y has size less than n, we can determine that $yB[x, z]$ for any $y \in Y$ and x, z lying either side of Y, but for any $x, y, z \in Y$, we can not determine which lies between the other two. This is because in this case, we have $A_x \cap A_y = \emptyset$, $A_x \cap A_z = \emptyset$, $A_y \cap A_z = \emptyset$, but for *any* $v \in S$, either $A_v \cap A_x, A_v \cap A_y, A_v \cap A_z$ will *all* be empty or they will *all* be nonempty. Thus there is no way whatsoever of distinguishing the points in Y from one another, and so Y can be permuted however we like in the automorphism group.

This means that any permutation $\pi : Y \to Y$ can be extended to an automorphism $\varphi : G \to G$ fixing all $g \in G \backslash Y$.

Lemma 4.7. Let S be any total ordering without endpoints, and let n be a positive integer. Then any pair of co-cliques C and D in $G(S, 13^n 2)$ satisfying:

$$i)\ C = \bigcap \{N_v : v \in D\}$$
$$ii)\ D = \bigcap \{N_v : v \in C\}$$

and where $|C|, |D| \geq 2$ must be of the form

$$C_{x_1,...,x_n} = \{(v, x_1, x_2, ..., x_n) : v < x_1\}$$
$$D_{x_1,...,x_n} = \{(x_1, x_2, ..., x_n, w) : x_n < w\}$$

where $x_1, ..., x_n \in S$.

Proof. First we see that $C_{x_1,...,x_n}$ and $D_{x_1,...,x_n}$ satisfy $i)$ and $ii)$. Now, let C and D be a pair of sets satisfying $i)$ and $ii)$. Then we remark that if $(c_1, ..., c_{n+1}) \in C$ and $(d_1, ..., d_{n+1}) \in D$, then either

$$\begin{array}{ccccccccccccc}
c_1 & < & c_2 & < & c_3 & < & \ldots & < & c_n & < & c_{n+1} & & \\
 & & \| & & \| & & & & & & \| & & \\
 & & d_1 & < & d_2 & < & d_3 & < & \ldots & < & d_n & < & d_{n+1}
\end{array}$$

or

$$\begin{array}{ccccccccccccc}
 & & c_1 & < & c_2 & < & c_3 & < & \ldots & < & c_n & < & c_{n+1} \\
 & & \| & & \| & & & & & & \| & & \\
d_1 & < & d_2 & < & d_3 & < & \ldots & < & d_n & < & d_{n+1} & &
\end{array}$$

Note that we cannot have both of the above for the same $d_1, ..., d_{n+1}$, because if $c_{i+1} = d_i$ for $1 \leq i \leq n$ and $c'_i = d_{i+1}$ for $1 \leq i \leq n$, where $(c_1, ..., c_{n+1}), (c'_1, ..., c'_{n+1}) \in C$ and $(d_1, ..., d_{n+1}) \in D$, then $(d_1, ..., d_{n+1})$ is the only point of D adjacent to both $(c_1, ..., c_{n+1})$ and $(c'_1, ..., c'_{n+1})$, contrary to the assumption that both C and D contain at least 2 elements.

Thus either for all $(c_1, ..., c_{n+1}) \in C$ and $(d_1, ..., d_{n+1}) \in D$, we have $c_{i+1} = d_i$ for $1 \leq i \leq n$, or for all $(c_1, ..., c_{n+1}) \in C$ and $(d_1, ..., d_{n+1}) \in D$, we have $c_i = d_{i+1}$ for $1 \leq i \leq n$.

Without loss of generality we assume the former. We then note that, by the definition of C and D, any $n+1$-tuple ending in $c_2, ..., c_{n+1}$ must lie in C, and any $n+1$-tuple beginning with $c_2, ..., c_{n+1}$ must lie in D. Thus any pair of subsets of G satisfying $i)$ and $ii)$ must be of the form $C_{x_1,...,x_n} = \{(v, x_1, x_2, ..., x_n) : v < x_1\}$ and $D_{x_1,...,x_n} = \{(x_1, x_2, ..., x_n, w) : x_n < w\}$. □

Theorem 4.8. Let S be any total ordering without endpoints, and let n be a positive integer. Then (S, B) is 2nd-order interpretable inside $G = G(S, 13^n 2)$.

Proof. Consider co-cliques C and D satisfying:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

where $|C|, |D| \geq 2$.

By Lemma 4.7, any pair of subsets of G satisfying $i)$ and $ii)$ must be of the form

$$C_{x_1,...,x_n} = \{(v, x_1, x_2, ..., x_n) : v < x_1\}$$
$$D_{x_1,...,x_n} = \{(x_1, x_2, ..., x_n, w) : x_n < w\}$$

Again, this forms a family $\mathcal{F}$ of pairs $\{C_{x_1,...,x_n}, D_{x_1,...,x_n}\}$ for $x_1, ..., x_n \in S$. Now let $A_{x_1,...,x_n} = C_{x_1,...,x_n} \cup D_{x_1,...,x_n}$ for every such n-tuple, and let $\mathcal{F}'$ be the family of all such $A_{x_1,...,x_n}$.

We start by reconstructing $G(S, 13^{n-1}2)$ from $G(S, 13^n 2)$.

Consider the following isomorphism $\varphi : \mathcal{F}' \to G(S, 13^{n-1}2)$:

$$\varphi : A_{x_1,...,x_n} \mapsto (x_1, ..., x_n)$$

where the relation "the elements $A_{x_1,...,x_n}$ and $A_{x'_1,...,x'_n}$ have a common element in $\mathcal{F}'$" is mapped to the relation "there is an edge between the vertices $(x_1, ..., x_n)$ and $(x'_1, ..., x'_n)$ in $G(S, 13^{n-1}2)$" .

To see that this is an isomorphism, first note that this is a bijection as each $A_{x_1,...,x_n}$ corresponds to exactly the n-tuple $(x_1, ..., x_n)$. Now let $A_{x_1,...,x_n} \cap A_{x'_1,...,x'_n} \neq \emptyset$, i.e. there is some element lying in both, and without loss of generality assume $x_1 < x'_1$. Then $G(S, 13^n 2)$ contains some vertex $(v_1, ..., v_{n+1}) \in A_{x_1,...,x_n}$ and $(v_1, ..., v_{n+1}) \in A_{x'_1,...,x'_n}$. Since $x_1 < x'_1$ the first n coordinates of $(v_1, ..., v_{n+1})$ must be $x_1, ..., x_n$, and the last n coordinates must be $x'_1, ..., x'_n$, thus giving us

$$\begin{array}{ccccccccccccc}
x_1 & < & x_2 & < & x_3 & < & \dots & < & x_{n-1} & < & x_n & & \\
 & & \| & & \| & & & & & & \| & & \\
 & & x'_1 & < & x'_2 & < & x'_3 & < & \dots & < & x'_{n-1} & < & x'_n
\end{array}$$

i.e. if $A_{x_1,...,x_n} \cap A_{x'_1,...,x'_n} \neq \emptyset$ then $(x_1, ..., x_n)$ and $(x'_1, ..., x'_n)$ are neighbours in $G(S, 13^{n-1}2)$. Conversely, suppose $A_{x_1,...,x_n} \cap A_{x'_1,...,x'_n} = \emptyset$, i.e. we do not have

$$\begin{array}{ccccccccccccc}
x_1 & < & x_2 & < & x_3 & < & \dots & < & x_{n-1} & < & x_n & & \\
 & & \| & & \| & & & & & & \| & & \\
 & & x'_1 & < & x'_2 & < & x'_3 & < & \dots & < & x'_{n-1} & < & x'_n
\end{array}$$

Then $(x_1, ..., x_n)$ and $(x'_1, ..., x'_n)$ are not neighbours in $G(S, 13^{n-1}2)$.

Thus we can reconstruct $G(S, 132)$ from $G(S, 13^n 2)$ recursively for any $n \in \mathbb{N}$, and reconstruct S from $G(S, 132)$ by Lemma 4.2. We can also recover n as it takes precisely $n-1$ recursions to "reach" $G(S, 132)$. We can "recognise" when we have reached $G(S, 132)$ because it has the property that every pair of members of $\mathcal{F}'$ has a nonempty intersection, which is false for $G(S, 13^n 2)$ where $n > 1$. □

Lemma 4.9. Let S be a total ordering without endpoints, and let $n > m$ be positive integers. Let $n \equiv k \mod m$ where $m \geq k \geq 1$ (so if m divides n we let $k = m$). Then $G(S, 1^m 3^k 2^m)$ is 2nd-order interpretable inside $G(S, 1^m 3^n 2^m)$.

Proof. We start by reconstructing $G(S, 1^m 3^{n-m} 2^m)$ from $G(S, 1^m 3^n 2^m)$.

Elements of $G(S, 1^m 3^n 2^m)$ consist of $n+m$-tuples, where there is an edge between

$$(x_1, ..., x_m, y_1, ..., y_n) \text{ and } (y_1, ..., y_n, z_1, ..., z_m)$$

with $x_i, y_i, z_i \in S$.

$G(S, 1^m 3^{n-m} 2^m)$ consists of vertices of the form $(x_1, ..., x_{m+(n-m)})$, i.e. vertices of the form $(x_1, ..., x_n)$, with an edge between $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ if and only if

$$\begin{array}{ccccccccccc} x_1 < x_2 < ... < x_m < & x_{m+1} & < & x_{m+2} & < & ... & < & x_n & & \\ & \| & & \| & & & & \| & & \\ & y_1 & < & y_2 & < & ... & < & y_{n-m} & < y_{n-m+1} < ... < y_n \end{array}$$

We can construct the family $\mathcal{F}$ of pairs of maximal co-cliques $\{C_{a_1,...,a_n}, D_{a_1,...,a_n}\}$ as in Theorem 4.8. Let $A_{a_1,...,a_n} = C_{a_1,...,a_n} \cup D_{a_1,...,a_n}$ for every n-tuple in S, and let $\mathcal{F}'$ be the family of all such $A_{a_1,...,a_n}$. Similarly to Theorem 4.8, there is an isomorphism $\varphi : \mathcal{F}' \to G(S^m, 1^m 3^{n-m} 2^m)$:

$$\varphi : A_{x_1,...,x_n} \mapsto (x_1, ..., x_n)$$

where the relation "the elements $A_{x_1,...,x_n}$ and $A_{x'_1,...,x'_n}$ have a common element in $\mathcal{F}'$" is mapped to the relation "there is an edge between the vertices $(x_1, ..., x_n)$ and $(x'_1, ..., x'_n)$ in $G(S, 1^m 3^{n-m} 2^m)$".

To see that this is an isomorphism, first note that this is a bijection as each $A_{x_1,...,x_n}$ corresponds to exactly the n-tuple $(x_1, ..., x_n)$. Now let $A_{x_1,...,x_n} \cap A_{x'_1,...,x'_n} \neq \emptyset$, i.e. there is some element lying in both, and without loss of generality assume $x_1 < x'_1$. Then $G(S, 1^m 3^{n-m} 2^m)$ contains some vertex $(v_1, ..., v_{n+m}) \in A_{x_1,...,x_n}$ and $(v_1, ..., v_{n+m}) \in A_{x'_1,...,x'_n}$. Since $x_1 < x'_1$ the first n coordinates of $(v_1, ..., v_{n+m})$ must be $x_1, ..., x_n$, and the last n coordinates must be $x'_1, ..., x'_n$, thus giving us

$$\begin{array}{ccccccccccc} x_1 < x_2 < ... < x_m < & x_{m+1} & < & x_{m+2} & < & ... & < & x_n & & \\ & \| & & \| & & & & \| & & \\ & x'_1 & < & x'_2 & < & ... & < & x'_{n-m} & < x'_{n-m+1} < ... < x'_n \end{array}$$

i.e. if $A_{x_1,...,x_n} \cap A_{x'_1,...,x'_n} \neq \emptyset$ then $(x_1, ..., x_n)$ and $(x'_1, ..., x'_n)$ are neighbours in $G(S, 1^m 3^{n-m} 2^m)$. Conversely, suppose $A_{x_1,...,x_n} \cap A_{x'_1,...,x'_n} = \emptyset$, i.e. we do not have

$$\begin{array}{ccccccccccc} x_1 < x_2 < ... < x_m < & x_{m+1} & < & x_{m+2} & < & ... & < & x_n & & \\ & \| & & \| & & & & \| & & \\ & x'_1 & < & x'_2 & < & ... & < & x'_{n-m} & < x'_{n-m+1} < ... < x'_n \end{array}$$

Thus $(x_1, ..., x_n)$ and $(x'_1, ..., x'_n)$ are not neighbours in $G(S, 1^m 3^{n-m} 2^m)$. □

We repeat this t times until $n - tm = k \leq m$, leaving us with $G(S, 1^m 3^k 2^m)$.

Thus the general problem of reconstructing S from $G(S, 1^m 3^n 2^m)$ where $n > m$ has been reduced to the general problem of reconstructing S from $G(S, 1^m 3^k 2^m)$ where $k \leq m$.

Lemma 4.10. Let $(S,<)$ be a dense total ordering without endpoints, and let $k \leq m$ be positive integers. Then any pair of co-cliques C and D in $G(1^m 3^k 2^m)$ with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

and where C and D are nonempty must either be of the form

$$C_{x_1,\dots,x_k} = \{(v_1, v_2, \dots, v_m, x_1, \dots, x_k) : v_m < x_1\}$$
$$D_{x_1,\dots,x_k} = \{(x_1, \dots, x_k, w_1, w_2, \dots, w_m) : x_k < w_1\}$$

where $x_1, \dots, x_k \in S$, or of the form

$${}_{x_1,\dots,x_k}C_{y_1,\dots,y_k} = \{(x_1, \dots, x_k, v_1, v_2, \dots, v_{m-k}, y_1, \dots, y_k) : x_k < v_1 < v_2 < \dots < v_{m-k} < y_1\}$$
$${}_{x_1,\dots,x_k}D_{y_1,\dots,y_k} = \{(w_1, w_2, \dots, w_m, x_1, \dots, x_k) : w_m < x_1\} \cup \{(y_1, \dots, y_k, w'_1, w'_2, \dots, w'_m) : y_k < w'_1\}$$

where $x_1, \dots, x_k, y_1, \dots, y_k \in S$.

Proof. First we see that $C_{x_1,\dots,x_k}, D_{x_1,\dots,x_k}$ and ${}_{x_1,\dots,x_k}C_{y_1,\dots,y_k}$ ${}_{x_1,\dots,x_k}D_{y_1,\dots,y_k}$ satisfy $i)$ and $ii)$. Now, let C and D be a pair of sets satisfying $i)$ and $ii)$. Then we remark that if $c = (c_1, \dots, c_{m+k}) \in C$ and $d = (d_1, \dots, d_{m+k}) \in D$, then either

$$\begin{array}{ccccccccccccccc}
c_1 & < & c_2 & \dots & < & c_m & < & c_{m+1} & < & c_{m+2} & \dots & < & c_{m+k} & & \\
 & & & & & & & \| & & \| & & & \| & & \\
 & & & & & & & d_1 & < & d_2 & \dots & < & d_k & < & d_{1+k} < \dots < d_{m+k}
\end{array}$$

in which case we say c is "left adjacent" to d, or

$$\begin{array}{ccccccccccccccc}
 & & & & & & & c_1 & < & c_2 & \dots & < & c_k & < & c_{1+k} \dots < c_{m+k} \\
 & & & & & & & \| & & \| & & & \| & & \\
d_1 & < & d_2 & \dots & < & d_m & < & d_{m+1} & < & d_{m+2} & \dots & < & d_{m+k} & &
\end{array}$$

in which case we say c is "right adjacent" to d.

Note that c, d are adjacent provided c is left or right adjacent to d.

We now have two cases:

Case 1: there exist $c, c' \in C$ and $d \in D$ such that c is left adjacent to d, and d is left adjacent to c'.

$$\begin{array}{ccccccccccccccc}
c_1 < \dots < & c_{m+1} & < \dots & c_{m+k} & & & & & \\
 & \| & & \| & & & & & \\
 & d_1 & < \dots & d_k & < \dots & d_{m+1} & < \dots & d_{m+k} & \\
 & & & & & \| & & \| & \\
 & & & & & c'_1 & < \dots & c'_k & < c'_{1+k} < \dots < c'_{m+k}
\end{array}$$

Then any $m+k$-tuple lying in C is adjacent to d, so must end with $d_1, ..., d_k$ or begin with $d_{m+1}, ..., d_{m+k}$, and hence any $m+k$-tuple beginning with $d_1, ..., d_k$ and ending with $d_{m+1}, ..., d_{m+k}$ is adjacent to all members of C, and so lies in D.

Similarly, any $m+k$-tuple in D is adjacent to c and c', so must begin with $d_1, ..., d_k$ and end with $d_{m+1}, ..., d_{m+k}$. The set of all $m+k$-tuples which are neighbours of all of these is precisely the set of all $m+k$-tuples beginning with $d_{m+1}, ..., d_{m+k}$ or ending with $d_1, ..., d_k$, and it follows that C is the set of everything adjacent to D.

The argument is similar if $d, d' \in D$ and $c \in C$.

Case 2: neither of the above hold.

Then for every $c \in C$, either every $d \in D$ is left adjacent to c or every $d \in D$ is right adjacent to d.

Similarly, for every $d \in D$, either every $c \in C$ is left adjacent to d or every $c \in C$ is right adjacent to d.

Without loss of generality assume there is a $d \in D$ such that every $c \in C$ is right adjacent to c.

$$\begin{array}{ccccccccccccccccc}
 & & & & & & & c_1 & < & c_2 & ... & < & c_k & < & c_{1+k} & ... < & c_{m+k} \\
 & & & & & & & \| & & \| & & & \| & & & & \\
d_1 & < & d_2 & ... & < & d_m & < & d_{m+1} & < & d_{m+2} & ... & < & d_{m+k} & & & &
\end{array}$$

Then $c_1 = d_{m+1}, ... c_k = d_{m+k}$, and moreover all members of C must begin with this same $c_1, ..., c_k$. Thus all $m+k$-tuples ending in $d_{m+1}, ..., d_{m+k}$ lie in D, and similarly all $m+k$-tuples beginning in $c_1, ..., c_k$ lie in C. □

Theorem 4.11. Let S be a dense total ordering without endpoints, and let $k \leq m$ be positive integers. Then S is 2nd-order interpretable inside $G = G(S, 1^m 3^k 2^m)$.

Proof. Consider co-cliques C and D satisfying:

$$i)\ C = \bigcap \{N_v : v \in D\}$$
$$ii)\ D = \bigcap \{N_v : v \in C\}$$

where C and D are nonempty.

By Lemma 4.10, any pair of subsets satisfying $i)$ and $ii)$ must either be of the form

$$C_{x_1,...,x_k} = \{(v_1, v_2, ..., v_m, x_1, ..., x_k) : v_m < x_1\}$$
$$D_{x_1,...,x_k} = \{(x_1, ..., x_k, w_1, w_2, ..., w_m) : x_k < w_1\}$$

where $x_1, ..., x_k \in S$, or of the form

$$
\begin{aligned}
{}_{x_1,...,x_k}C_{y_1,...,y_k} &= \{(x_1, ..., x_k, v_1, v_2, ..., v_{m-k}, y_1, ..., y_k) : x_k < v_1 < v_2 < ... < v_{m-k} < y_1\} \\
{}_{x_1,...,x_k}D_{y_1,...,y_k} &= \{(w_1, w_2, ..., w_n, x_1, ..., x_k) : w_n < x_1\} \cup \\
&\quad \{(y_1, ..., y_k, w'_1, w'_2, ..., w'_n) : x_k < w'_1\}
\end{aligned}
$$

where $x_1, ..., x_k, y_1, ..., y_k \in S$.

Let $\mathcal{F}$ be the family of all pairs $\{C, D\}$ satisfying *i)* and *ii)*. Let $\mathcal{F}_1$ be the family of all pairs $\{C_{x_1,...,x_k}, D_{x_1,...,x_k}\}$, where $C_{x_1,...,x_k}, D_{x_1,...,x_k}$ are as above, and let $\mathcal{F}_2$ be the family of all pairs $\{{}_{x_1,...,x_k}C_{y_1,...,y_k},\ {}_{x_1,...,x_k}D_{y_1,...,y_k}\}$, where ${}_{x_1,...,x_k}C_{y_1,...,y_k},\ {}_{x_1,...,x_k}D_{y_1,...,y_k}$ are as above. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.

Given some $\{C, D\} \in \mathcal{F}$, we would like to determine whether $\{C, D\} \in \mathcal{F}_1$ or $\{C, D\} \in \mathcal{F}_2$. I claim that $\{C, D\} \in \mathcal{F}_2$ if and only if there exist pairs of sets $\{C', D'\}, \{C'', D''\} \in \mathcal{F}$ with $\{C', D'\}, \{C'', D''\} \neq \{C, D\}$ such that for some $X' \in \{C', D'\}$ and $X'' \in \{C'', D''\}$, we have $X' \cup X'' \in \{C, D\}$.

To see this, suppose $\{C, D\} \in \mathcal{F}_2$. Then $\{C, D\} = \{{}_{x_1,...,x_k}C_{y_1,...,y_k},\ {}_{x_1,...,x_k}D_{y_1,...,y_k}\}$ for some x, y. Now let $\{C', D'\} = \{C'_{x_1,...,x_k}, D'_{x_1,...,x_k}\}$ and $\{C'', D''\} = \{C_{y_1,...,y_k}, D_{y_1,...,y_k}\}$, and let $X' = C_{x_1,...,x_k}$ and $X'' = D_{y_1,...,y_k}$. Thus we have ${}_{x_1,...,x_k}D_{y_1,...,y_k} = C_{x_1,...,x_k} \cup D_{y_1,...,y_k}$, i.e. $C_{x_1,...,x_k} \cup D_{y_1,...,y_k} \in \{{}_{x_1,...,x_k}C_{y_1,...,y_k},\ {}_{x_1,...,x_k}D_{y_1,...,y_k}\}$.

Conversely, suppose $\{C, D\} \notin \mathcal{F}_2$, i.e. $\{C, D\} = \{C_{x_1,...,x_k}, D_{x_1,...,x_k}\}$ for some $x_1, ..., x_k$. We will show that $C_{x_1,...,x_k}$ cannot be the union of two sets of the form $C_{y_1,...,y_k}$, $D_{y_1,...,y_k}$, ${}_{y_1,...,y_k}C_{z_1,...,z_k}$, ${}_{y_1,...,y_k}D_{z_1,...,z_k}$.

- $C_{x_1,...,x_k}$ cannot be the union of any $C_{y_1,...,y_k}$ with another set, as $C_{x_1,...,x_k} \cap C_{y_1,...,y_k} = \emptyset$ for all $x_1, ..., x_k, y_1, ..., y_k$ where $x_i \neq y_i$ for some $1 \leq i \leq k$.
- $C_{x_1,...,x_k}$ cannot be the union of any $D_{y_1,...,y_k}$ with another set, as if $x_k \leq y_1$ then $D_{y_1,...,y_k}$ contains points of the form $(y_1, ..., y_k, z_1, ..., z_m)$ which do not lie in $C_{x_1,...,x_k}$, and if $x_k > y_1$ then $D_{y_1,...,y_k}$ contains the points of the form $(y_1, ..., y_k, z_1, ..., z_m)$ where $(z_{m-k+1}, ..., z_m) \neq (x_1, ..., x_k)$, which do not lie in $C_{x_1,...,x_k}$.
- We see that $C_{x_1,...,x_k}$ cannot be the union of any ${}_{y_1,...,y_k}D_{z_1,...,z_k}$ with another set, as if $x_k < z_1$ then ${}_{y_1,...,y_k}D_{z_1,...,z_k}$ contains points of the form $(z_1, ..., z_k, w_1, ..., w_m)$ which do not lie in $C_{x_1,...,x_k}$, and if $x_k > y_k$ then ${}_{y_1,...,y_k}D_{z_1,...,z_k}$ contains points of the form $(v_1, ..., v_m, y_1, ..., y_k)$ which don't lie in $C_{x_1,...,x_k}$.

Thus $C_{x_1,...,x_k}$ must be the union of two sets ${}_{y_1,...,y_k}C_{z_1,...,z_k}$ and ${}_{y'_1,...,y'_k}C_{z'_1,...,z'_k}$, for some $y_1, ..., y_k, z_1, ..., z_k, y'_1, ..., y'_k, z'_1, ..., z'_k$. Now, by the construction of these sets, we must have $z_i = z'_i = x_i$ for all $1 \leq i \leq k$. But then $C_{x_1,...,x_k}$ contains some point $(w_1, ..., w_m, x_1, ..., x_k)$ where $w_1 \neq y_1, y'_1$, and so $C_{x_1,...,x_k} \neq_{y_1,...,y_k} C_{z_1,...,z_k} \cup\ {}_{y'_1,...,y'_k}C_{z'_1,...,z'_k}$, a contradiction.

Thus, given G we can uniquely determine $\mathcal{F}_1$. All pairs $\{C, D\} \in \mathcal{F}_1$ can be indexed by some k-tuple in S, such that each pair is of the form $\{C_{x_1,...,x_k}, D_{x_1,...,x_k}\}$ for some

$x_1, ..., x_k \in S$. Thus we have determined the set S^k (or representatives of the set S^k), but not the betweenness relation on S.

Let $A_{x_1,...,x_k} = C_{x_1,...,x_k} \cup D_{x_1,...,x_k}$ for every such pair. Thus $G = \bigcup\{A_{x_1,...,x_k} : x_1, ..., x_k \in S\}$.

Since the dense case is much simpler than the general case, we will consider this separately (even though the dense case is a subset of the general case).

Then for each $x_1, ..., x_k \neq y_1, ..., y_k$, $A_{x_1,...,x_k} \cap A_{y_1,...,y_k}$ is non-empty iff $x_1, ..., x_k$ and $y_1, ..., y_k$ are non-overlapping, in which case $A_{x_1,...,x_k} \cap A_{y_1,...,y_k}$ contains the set of vertices $(x_1, ..., x_k, v_1, ..., v_{m-k}, y_1, ..., y_k)$ if $x_k < y_1$ with $x_k < v_1 < ... < v_{m-k} < y_1$, or $(y_1, ..., y_k, v_1, ..., v_{m-k}, x_1, ..., x_k)$ if $y_k < x_1$, with $y_k < v_1 < ... < v_{m-k} < x_1$. Such $v_1, ..., v_{m-k}$ always exists since S is dense.

Now, let $X = x_1, ..., x_k, Y = y_1, ..., y_k, Z = z_1, ..., z_k$ be comparable. Given A_X, A_Y, A_Z, we see that $YB[X, Z]$ iff every element of $A_X \cap A_Y$ is a neighbour of every element of $A_Y \cap A_Z$, and neither has any neighbours in $A_X \cap A_Z$. Thus we can get partial betweenness B^k on k-tuples S^k, and so by Lemma 3.22 we can reconstruct (S, B).

Thus we can reconstruct S with the betweenness relation from G (and so we can reconstruct S up to order reversal by Theorem 3.15). □

Theorem 4.12. Let S be a total ordering without endpoints containing no non-trivial maximal finite section of size less than m, and let $k \leq m$ be positive integers. Then (S, B) is 2nd-order interpretable inside $G = G(S, 1^m 3^k 2^m)$.

Proof. The first part of the proof is identical to that of Theorem 4.11. Construct sets $A_{x_1,...,x_k}$ as before.

Let $X = x_1, ..., x_k$ and $Y = y_1, ..., y_k$. Then $A_X \cap A_Y$ is non-empty iff X and Y are non-overlapping AND $dist(X, Y) \geq m - k$, in which case $A_{x_1,...,x_k} \cap A_{y_1,...,y_k}$ contains the set of vertices $(x_1, ..., x_k, v_1, ..., v_{m-k}, y_1, ..., y_k)$ if $x_k < y_1$ with $x_k < v_1 < ... < v_{m-k} < y_1$, or $(y_1, ..., y_k, v_1, ..., v_{m-k}, x_1, ..., x_k)$ if $y_k < x_1$, with $y_k < v_1 < ... < v_{m-k} < x_1$.

Thus we obtain betweenness on k-tuples in S of distance at least $m - k$ from one another, i.e. $(S^k, B^k_{dist(m-k)})$. Since S contains no non-trivial finite section of size less than m, by Theorem 3.32, we can interpret $(S, B_{dist(m)})$, and so by Lemma 3.29, (S, B) is interpretable in $G = G(S, 1^m 3^k 2^m)$.

Thus we can reconstruct S with the betweenness relation from G (and so we can reconstruct S up to order reversal by Theorem 3.15). □

Corollary 4.13. Let S be a total ordering without endpoints containing no non-trivial maximal finite sections of size less than m. Then (S, B) is 2nd-order interpretable inside $G(S, 1^m 3^n 2^m)$. If $m = 1$, then S can be any total ordering without endpoints.

Proof. If $n \leq m$, (S, B) is 2nd-order interpretable inside $G(S, 1^m 3^n 2^m)$ by Theorem 4.12.

If $n > m$, then by Lemma 4.9 $G(S, 1^m 3^k 2^m)$ is interpretable inside $G(S, 1^m 3^n 2^m)$, where $n \equiv k \bmod m$. Now, since $k \leq m$, and S contains no non-trivial maximal finite sections of size less than m, by Theorem 4.12 (S, B) is 2nd-order interpretable inside $G(S, 1^m 3^k 2^m)$, and hence from $G(S, 1^m 3^n 2^m)$.

If $m = 1$, then (S, B) is 2nd-order interpretable inside $G(S, 13^n 2)$ by Theorem 4.8.

4.2 Partial Ordering Shift Graphs

We can extend these results to trees, except that with trees we can interpret the original *ordering*, as opposed to just the betweenness relation, although once again we will focus only on trees without endpoints.

We can also determine some additional general properties on T from $G(T, 132)$, namely that T does not have a root which ramifies iff $G(T, 132)$ has one connected component (and similarly for other types of the form $1^n3^m2^n$).

We start by defining the shift graph on trees.

Let T be a tree. We can now define the shift graph $G(T, \tau)$ similarly to the shift graph $G(S, \tau)$ for a total ordering S:

Definition 4.14. Let x and y be comparable k-element subsets (listed in increasing order) of the tree $(T, <)$. Let $x \cup y = \{z_1, ..., z_\ell\}$, with $z_1 < z_2 < ... < z_\ell$. Then we say that the pair x, y *has type* τ (denoted by $t(x, y) = \tau$) iff:

$$\tau_i = 1 \Rightarrow z_i \in x \setminus y,$$
$$\tau_i = 2 \Rightarrow z_i \in y \setminus x, \text{ and}$$
$$\tau_i = 3 \Rightarrow z_i \in x \cap y$$

Definition 4.15. The graph $G(T, \tau)$ is, as before, defined to be the graph whose vertices are the k-element subsets of T *in which all elements are comparable*, and where there is an edge between x and y iff $t(x, y) = \tau$.

We can treat these shift graphs on trees in much the same way as we have been treating the shift graphs on linear orderings, with some minor adjustments.

We also have an easy way of categorising whether or not $G(T, 132)$ has one connected component or not, and similarly for $G(T, 1^n3^m2^n)$.

Lemma 4.16. Let T be a tree without leaves. Then $G(T, 132)$ has one connected component if and only if T does not have a root which ramifies.

Proof. First suppose that r is the root, and it ramifies, so there are at least 2 cones at r. We show at that any path starting at (r, x) where $r < x$ has all co-ordinates of its vertices lying in $C \cup \{r\}$, where C is the cone at r containing x. We do this by induction on the vertices in the path. So suppose that $(y, z) \in P$ and that we already know that $y, z \in C \cup \{r\}$. Then the next point in the path is either (z, t) or (t, y) for some t. In the first case, $t > z \in C \cup \{r\}$, so certainly also $t \in C \cup \{r\}$. In the second case, $t < y$, so $y \neq r$. If $t = r$, then $t \in C \cup \{r\}$. Otherwise, the greatest lower bound of t and y is t, which is greater than r, so by definition of 'cone', t and y lie in the same cone at r, in other words, $t \in C$. Finally, since r ramifies, there are at least two cones at r, C_1 and C_2 say. If a path starts with (r, x) where $x \in C_1$, then all co-ordinates of entries of the path must lie in $C_1 \cup \{r\}$, and hence (r, y) cannot lie in the path if $y \in C_2$. Hence $G(T, 132)$ is disconnected.

Conversely, suppose that T has no leaves, and either has no root, or else has a root which does not ramify. Let (x_1, x_2) and (y_1, y_2) be any members of $G(T, 132)$. Thus $x_1 < x_2$ and $y_1 < y_2$.

As T is a tree there is $z \leq x_1, y_1$. If T has no root, there is $t < z$, and $s < t$ in T. Then we have a path from (x_1, x_2) to (y_1, y_2) as follows:

$$(x_1, x_2) \sim (t, x_1) \sim (s, t) \sim (t, y_1) \sim (y_1, y_2)$$

Now suppose that T has a root r which does not ramify. Since T has no leaves, there are $x_4 > x_3 > x_2$ and $y_4 > y_3 > y_2$. Since r does not ramify and $x_2, y_2 > r$, there is just one cone at r, so there is $t > r$ such that $r < t \leq x_2, y_2$, and this time we get our path as follows:

$$(x_1, x_2) \sim (x_2, x_3) \sim (x_3, x_4) \sim (t, x_3) \sim (r, t) \sim (t, y_3) \sim (y_3, y_4) \sim (y_2, y_3) \sim (y_1, y_2)$$

□

We now define a 'stump':

Definition 4.17. Let T be a tree without leaves, and write S for the set of all vertices comparable with all members of T. This set is called the *stump*.

Lemma 4.18. Let T be a tree without leaves. Then $G = G(T, 13^n2)$ is connected if and only if $|S| \geq n + 1$, or $|S| \leq n$ and there is no minimal ramification point $\geq S$.

Proof. First suppose $|S| \leq n$ and there is a minimal ramification point $r \geq S$. Note that r must be the greatest point of S. Let the other members of S be $s_1 < ... < s_k < r$ where $k < n$ (k might be 0). Then there are at least two cones at r, C_1 and C_2. We show at that any path starting at $(s_1, ...s_k, r, x_1, ..., x_{n-k})$ has all co-ordinates of its vertices lying in $C' = C \cup \{s_1, ..., s_k, r\}$, where C is the cone at r containing $(x_1, ..., x_{n-k})$. We do this by induction on the vertices in the path. Suppose $(y_1, ..., y_{n+1}) \in P$ and that we already know that $y_1, ..., y_{n+1} \in C'$. Then the next point in the path is either $(y_2, ..., y_{n+1}, t)$ or $(t, y_1, ..., y_n)$ for some t. In the first case, $t > y_{n+1}$, so since $k < n$, $y_{n+1} > r$, and so $t \in C$, so certainly $t \in C'$. In the second case, $t < y_1$, so $y_1 \neq s_1$. If $t \in \{s_1, ..., s_k, r\}$, then $t \in C'$. Otherwise, the greatest lower bound of t and y_1 is t, which is greater than r, so by definition of 'cone', $t \in C$. Since there are at least two cones at r, suppose $x_1, ..., x_{n-k} \in C_1$ and $y_1, ..., y_n \in C_2$. If a path starts with $(r_1, ...r_k, r', x_1, ..., x_{n-k})$, then all co-ordinates of entries of the path must lie in C_1', and hence $(s_1, ...s_k, r, y_1, ..., y_{n-k})$ cannot lie in the path. Hence $G(T, 13^n2)$ is disconnected.

Next suppose $|S| \leq n$ and there is no minimal ramification point $\geq S$. Since T has no leaves, there are $z_{n+1} > ... > z_1 > x_{n+1}$ and $w_{n+1} > ... > w_1 > y_{n+1}$. Since T is a tree there is some $p \leq z_1, w_1$. Since T has no minimal ramification point above S, and both $z_1, w_1 \notin S$ due to size restrictions, there are $t_1, ..., t_n < p$. Then the path goes from $(x_1, ..., x_{n+1}) \sim \cdots \sim (z_1, ..., z_{n+1}) \sim \cdots \sim (t_1, ..., t_n, p) \sim \cdots \sim (w_1, ..., w_{n+1}) \sim \cdots \sim (y_1, ..., y_{n+1})$.

Finally suppose $|S| \geq n+1$. Since T has no leaves, there are $z_{n+1} > ... > z_1 > x_{n+1}$ and $w_{n+1} > ... > w_1 > y_{n+1}$. Since T has a stump of length at least $n+1$, there exist $s_0, s_1, ..., s_n$ comparable to all members of T such that $s_0 < s_1 < ... < s_n \leq x_{n+1}, y_{n+1}$. This time we get our path as follows:

$$\begin{gathered}(x_1, ..., x_{n+1}) \sim (x_2, ..., x_{n+1}, z_1) \sim \cdots \\ \sim (x_{n+1}, z_1, ..., z_n) \sim (z_1, ..., z_{n+1}) \sim (s_n, z_1, ..., z_n) \sim (s_{n-1}, s_n, z_1, ..., z_{n-1}) \sim \cdots \\ \sim (s_1, ..., s_n, z_1) \sim (s_0, s_1, ..., s_n) \sim (s_1, ..., s_n, w_1) \sim \cdots \\ \sim (s_{n-1}, s_n, w_1, ..., w_{n-1}) \sim (s_n, w_1, ..., w_n) \sim (w_1, ..., w_{n+1}) \sim (y_{n+1}, w_1, ..., w_n) \cdots \\ \sim (y_2, ..., y_{n+1}, w_1) \sim (y_1, ..., y_{n+1})\end{gathered}$$

□

Lemma 4.19. Let T be a tree without leaves. Then $G(T, 1^n32^n)$ has one connected component if and only if $|S| \geq n+1$, or $|S| \leq n$ and there is no minimal ramification point $\geq S$.

Proof. First suppose $|S| \leq n$ and there is a minimal ramification point $r \geq S$. Note that r must be the greatest point of S. Let the other members of S be $s_1 < ... < s_k < r$ where $k < n$ (k might be 0). Then there are at least two cones at r, C_1 and C_2. We show at that any path starting at $(s_1, ...s_k, r, x_1, ..., x_{n-k})$ has all co-ordinates of its vertices lying in $C' = C \cup \{s_1, ..., s_k, r\}$, where C is the cone at r containing $(x_1, ..., x_{n-k})$. We do this by induction on the vertices in the path. Suppose $(y_1, ..., y_{n+1}) \in P$ and that we already know that $y_1, ..., y_{n+1} \in C'$. Then the next point in the path is either $(y_{n+1}, t_1, ..., t_n)$ or $(t_1, ..., t_n, y_1)$ for some $t_1, ..., t_n$. In the first case, $t_1 > y_{n+1}$, so since $k < n$, $y_{n+1} > r$, and so $t_1 \in C$, so certainly $t \in C'$. In the second case, $t_n < y_1$, so $y_1 \neq s_1$. If $t_n \in \{s_1, ..., s_k, r\}$, then $t \in C'$. Otherwise, the greatest lower bound of t_n and y_1 is t_n, which is greater than r, so by definition of 'cone', $t_n \in C$. Since there are at least two cones at r, suppose $x_1, ..., x_{n-k} \in C_1$ and $y_1, ..., y_n \in C_2$. If a path starts with $(r_1, ...r_k, r', x_1, ..., x_{n-k})$, then all co-ordinates of entries of the path must lie in C_1', and hence $(s_1, ...s_k, r, y_1, ..., y_{n-k})$ cannot lie in the path. Hence $G(T, 1^n32^n)$ is disconnected.

Next suppose suppose $|S| \leq n$ and there is no minimal ramification point $\geq S$. Since T has no leaves, there are $z_{2n} > ... > z_1 > x_{n+1}$ and $w_{2n} > ... > w_1 > y_{n+1}$. Since T is a tree there is some $p \leq z_1, w_1$. Since T has no minimal ramification point above S, and both $z_1, w_1 \notin S$ due to size restrictions, there are $t_1, ..., t_{2n-1} < p$. Then the path goes from $(x_1, ..., x_{n+1}) \sim \cdots \sim (z_1, ..., z_{2n}) \sim \cdots \sim (t_1, ..., t_n) \sim \cdots \sim (w_1, ..., w_{n+1}) \sim \cdots \sim (y_1, ..., y_{n+1})$.

Finally suppose that $|S| \geq n+1$. Since T has no leaves, there are $z_{2n} > ... > z_1 > x_{n+1}$ and $w_{2n} > ... > w_1 > y_{n+1}$. Since T has a stump of length at least $n+1$, there exist $s_0, s_1, ..., s_n$ comparable to all members of T such that $s_0 < s_1 < ... < s_n \leq x_{n+1}, y_{n+1}$. This time we get our path as follows:

$$(x_1, ..., x_{n+1}) \sim (x_{n+1}, z_1, ..., z_n) \sim (z_n, z_{n+1}, ..., z_{2n}) \sim (s_n, z_1, ..., z_n) \sim (s_0, s_1, ..., s_n)$$
$$\sim (s_n, w_1, ..., w_n) \sim (w_n, w_{n+1}, ..., w_{2n}) \sim (y_{n+1}, w_1, ..., w_n) \sim (y_1, ..., y_{n+1})$$

□

Lemma 4.20. Let T be a tree without leaves. Then $G(T, 1^n 3^m 2^n)$ has one connected component if and only if $|S| \geq n+1$, or $|S| \leq n$ and there is no minimal ramification point $\geq S$.

Proof. First suppose $|S| \leq n$ and there is a minimal ramification point $r \geq S$. Note that r must be the greatest point of S. Let the other members of S be $s_1 < ... < s_k < r$ where $k < n$ (k might be 0). Then there are at least two cones at r, C_1 and C_2. We show at that any path starting at $(s_1, ...s_k, r, x_1, ..., x_{n+m-k})$ has all co-ordinates of its vertices lying in $C' = C \cup \{s_1, ..., s_k, r\}$, where C is the cone at r containing $(x_1, ..., x_{n+m-k})$. We do this by induction on the vertices in the path. Suppose $(y_1, ..., y_{n+m}) \in P$ and that we already know that $y_1, ..., y_{n+m} \in C'$. Then the next point in the path is either $(y_{n+1}, ..., y_{n+m}, t_1, ..., t_n)$ or $(t_1, ..., t_n, y_1, ..., y_m)$ for some $t_1, ..., t_n$. In the first case, $t_1 > y_{n+1}$, so since $k < n$, $y_{n+m} > r$, and so $t_1 \in C$, so certainly $t \in C'$. In the second case, $t_n < y_1$, so $y_1 \neq s_1$. If $t_n \in \{s_1, ..., s_k, r\}$, then $t \in C'$. Otherwise, the greatest lower bound of t_n and y_1 is t_n, which is greater than r, so by definition of 'cone', $t_n \in C$. Since there are at least two cones at r, suppose $x_1, ..., x_{n+m-k} \in C_1$ and $y_1, ..., y_{n+m-k} \in C_2$. If a path starts with $(r_1, ...r_k, r', x_1, ..., x_{n+m-k})$, then all co-ordinates of entries of the path must lie in C'_1, and hence $(s_1, ...s_k, r, y_1, ..., y_{n+m-k})$ cannot lie in the path. Hence $G(T, 1^n 3^m 2^n)$ is disconnected.

Next suppose that $|S| \geq n+1$. We split into two cases:

Case 1: $n \geq m$

Since T has no leaves, there are $z_{2n} > ... > z_1 > x_{n+1}$ and $w_{2n} > ... > w_1 > y_{n+1}$. Since r does not ramify and has a stump of length at least $n+1$, there exist $s_0, s_1, ..., s_{n+m}$

comparable to all members of T such that $s_0 < s_1 < ... < s_{n+m} \leq x_{n+m}, y_{n+m}$, and this time we get our path as follows:

$$(x_1, ..., x_{n+m}) \sim (x_{n+1}, ..., x_{n+m}, z_1, ..., z_n) \sim (z_{n-m}, z_{n-m+1}, ..., z_n, ..., z_{2n}) \sim$$
$$(s_n, ..., s_{n+m}, z_1, ..., z_n) \sim (s_0, s_1, ..., s_{n+m}) \sim (s_n, ..., s_{n+m} w_1, ..., w_n) \sim$$
$$(w_{n-m}, w_{n-m+1}, ..., w_n, ..., w_{2n}) \sim (y_{n+1}, ..., y_{n+m}, w_1, ..., w_n) \sim (y_1, ..., y_{n+m})$$

Case 2: $n < m$

Since T has no leaves, there are $z_{(k+2)n} > ... > z_1 > x_{n+m}$ and $w_{(k+2)n} > ... > w_1 > y_{n+m}$. Since T has a stump of length at least $n+m$, there exist $s_1, ..., s_{(k+1)n}, r_1, r_2, ..., r_{k'}$ comparable to everything in T such that $r_1 < r_2 < ... < r_{k'} < s_1 < ... < s_{(k+1)n} \leq x_{n+m}, y_{n+m}$ (note that these indices add up to $m+n$, as $k'+(k+1)n = kn+k'+n = m+n$), and this time we get our path as follows:

$$(x_1, ..., x_{n+m}) \sim (x_{n+1}, ..., x_{n+m}, z_1, ..., z_n) \sim (x_{2n+1}, ..., x_{n+m}, z_1, ..., z_{2n}) \sim \cdots$$
$$\sim (x_{kn+1}, ..., x_{n+m}, z_1, ..., z_{kn}) \sim (x_{(k+1)n+1}, ..., x_{n+m}, z_1, ..., z_{(k+1)n}) \sim (z_{n-k'+1}, ..., z_{(k+2)n}) \sim$$
$$(s_{kn+1}, ..., s_{(k+1)n}, z_{n-k'+1}, ..., z_{(k+1)n}) \sim (s_{(k-1)n+1}, ..., s_{(k+1)n}, z_{n-k'+1}, ..., z_{kn}) \sim \cdots$$
$$\sim (s_{n+1}, ..., s_{(k+1)n}, z_{n-k'+1}, ..., z_{2n}) \sim (s_1, ..., s_{(k+1)n}, z_{n-k'+1}, ..., z_n) \sim$$
$$(r_1, r_2, ..., r_{k'}, s_1, ..., s_{(k+1)n}) \sim$$
$$(s_1, ..., s_{(k+1)n}, w_{n-k'+1}, ..., w_n) \sim (s_{n+1}, ..., s_{(k+1)n}, w_{n-k'+1}, ..., w_{2n}) \sim \cdots$$
$$\sim (s_{(k-1)n+1}, ..., s_{(k+1)n}, w_{n-k'+1}, ..., w_{kn}) \sim (s_{kn+1}, ..., s_{(k+1)n}, w_{n-k'+1}, ..., w_{(k+1)n}) \sim$$
$$(w_{n-k'+1}, ..., w_{(k+2)n}) \sim (y_{(k+1)n+1}, ..., y_{n+m}, w_1, ..., w_{(k+1)n}) \sim (y_{kn+1}, ..., y_{n+m}, w_1, ..., w_{kn}) \sim$$
$$\cdots \sim (y_{2n+1}, ..., y_{n+m}, w_1, ..., w_{2n}) \sim (y_{n+1}, ..., y_{n+m}, w_1, ..., w_n) \sim (y_1, ..., y_{n+m})$$

Finally suppose $|S| \leq n$ and there is no minimal ramification point $\geq S$. The path here is similar to that of Lemmas 4.18 and 4.19, with the modifications as in Case 1 and 2 above. □

Definition 4.21. A (strict) betweenness relation on trees is a ternary relation B_T defined on a tree T which satisfies:

- $\forall x, y, z \in T,\ xB[y, z] \rightarrow (x \neq y \wedge x \neq z \wedge y \neq z)$.
- $\forall x, y, z \in T,\ xB[y, z] \rightarrow xB[z, y]$
- $\forall x, y, z, w \in T,\ (yB[x, w] \wedge zB[y, w]) \rightarrow zB[x, w]$

- $\forall\, T' \subseteq T$ such that $T' = \{x : x < t\}$, some fixed $t \in T$:

$$\forall x, y, z \in T',\ \ xB[y, z] \vee yB[x, z] \vee zB[x, y]$$

Lemma 4.22. Let $(T, <)$ be any tree without endpoints. Then any pair of co-cliques C and D in $G(T, 132)$ with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}\}$$

where $|C|, |D| \geq 2$ must be of the form

$$C_x = \{(v, x) : v < x\}$$
$$D_x = \{(x, w) : x < w\}$$

where $x \in S$.

Proof. Similar to the proof of Lemma 4.1 □

Theorem 4.23. Let $(T, <)$ be any proper tree without endpoints. Then $(T, <)$ is 2nd-order interpretable inside $G = G(S, 132)$.

Proof. We would like to identify co-cliques C and D with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

where $|C|, |D| \geq 2$.

By Lemma 4.22, any pair of subsets of $V(G)$ satisfying $i)$ and $ii)$ must be of the form

$$C_x = \{(v, x) : v < x\}$$
$$D_x = \{(x, w) : x < w\}$$

where $x \in S$.

Now, let $A_x = C_x \cup D_x$ for every such pair $\{C_x, D_x\}$ and $a \in S$. Thus $G = \bigcup\{A_x : x \in T\}$. We now note that for $x \neq y$, x and y are comparable if and only if $A_x \cap A_y$ is nonempty. Furthermore, if this is the case, $A_x \cap A_y$ contains exactly one element, namely (x, y) or (y, x).

Now, given A_x, A_y, A_z with pairwise nonempty intersections, we see that $yB[x, z]$ iff $A_x \cap A_y$ is a neighbour of $A_y \cap A_z$, and neither is a neighbour of $A_x \cap A_z$.

Provided the tree isn't a total ordering (i.e. it splits somewhere), we can do more than this. We can actually work out the ordering on the tree (as opposed to the partial betweenness relation), and we can do so as follows:

Since T splits, there exist 3 points, x, y, and z, such that x is comparable to y, and x is comparable to z, but y is not comparable to z. In this case, $x < y$ and $x < z$. Now, the set $V_{<x}$ of all points v such that $xB[v, y]$ is exactly the set of all things less than x, and thus its complement $V_{>x}$ is the set of all things greater than x. Now let $a, b \in T$. We have three options: $a \in V_{<x}$ and $b \in V_{>x}$, in which case $a < b$ (and similarly for $a \in V_{>x}, b \in V_{<x}$). If a and b are both in $V_{<x}$, then if $bB[a, x]$ then $a < b$ and if $aB[b, x]$ then $b < a$. Finally, if a and b are both in $V_{>x}$ then if $aB[b, x]$ then $a < b$ and if $bB[a, x]$ then $b < a$. This gives us the ordering $<$ on T.

Lemma 4.24. Let $(T, <)$ be any tree without endpoints. Then any pair of co-cliques C and D in $G(T, 1^n32^n)$ with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

where C and D are nonempty must either be of the form

$$C_x = \{(v_1, v_2, ..., v_n, x) : v_n < x\}$$
$$D_x = \{(x, w_1, w_2, ..., w_n) : x < w_1\}$$

where $x \in S$, or of the form

$${}_xC_y = \{(x, v_1, v_2, ..., v_{n-1}, y) : a < x_1 < x_2 < ... < x_{n-1} < b\}$$
$${}_xD_y = \{(w_1, w_2, ..., w_n, x) : w_n < x\} \cup \{(x, w_1', w_2', ..., w_n') : x < w_1'\}$$

where $x, y \in S$.

Proof. Similar to the proof of Lemma 4.3. □

Definition 4.25. Let T be a tree. Then a *maximal finite section* of is a convex subset Y of T such that $y < t$ for every $y \in Y$ and for some fixed $t \in T$ (i.e. elements of Y are pairwise comparable). A *non-trivial maximal finite section* is a maximal finite section of size greater than 1.

Definition 4.26. In a maximal finite section of a tree, the *distance* $dist(a, b)$ between a and b is the size of the set $\{x : a \leq x < b\}$.

Theorem 4.27. Let $(T, <)$ be a tree without endpoints, and let n be a positive integer. Then, if T contains no non-trivial maximal finite section of size under $2n$, $(T, <)$ is 2nd-order interpretable inside $G = G(T, 1^n32^n)$.

Proof. We would like to identify co-cliques C and D with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

where C and D are nonempty.

By Lemma 4.24, such a pair must either be of the form

$$C_x = \{(v_1, v_2, ..., v_n, x) : v_n < x\}$$
$$D_x = \{(x, w_1, w_2, ..., w_n) : x < w_1\}$$

where $x \in S$, or of the form

$${}_xC_y = \{(x, v_1, v_2, ..., v_{n-1}, y) : a < x_1 < x_2 < ... < x_{n-1} < b\}$$
$${}_xD_y = \{(w_1, w_2, ..., w_n, x) : w_n < x\} \cup \{(x, w'_1, w'_2, ..., w'_n) : x < w'_1\}$$

where $x, y \in S$.

Let $\mathcal{F}$ be the family of all pairs $\{C, D\}$ satisfying $i)$ and $ii)$. Let $\mathcal{F}_1$ be the family of all pairs $\{C_x, D_x\}$, where C_x, D_x are as above, and let $\mathcal{F}_2$ be the family of all pairs $\{{}_xC_y,\ {}_xD_y\}$, where ${}_xC_y,\ {}_xD_y$ are as above. Then $\mathcal{F} = \mathcal{F}_1 \cup \mathcal{F}_2$.

Given some $\{C, D\} \in \mathcal{F}$, we would like to determine whether $\{C, D\} \in \mathcal{F}_1$ or $\{C, D\} \in \mathcal{F}_2$. I claim that $\{C, D\} \in \mathcal{F}_2$ if and only if there exist pairs of sets $\{C', D'\}, \{C'', D''\} \in \mathcal{F}$ with $\{C', D'\}, \{C'', D''\} \neq \{C, D\}$ such that for some $X' \in \{C', D'\}$ and $X'' \in \{C'', D''\}$, we have $X' \cup X'' \in \{C, D\}$.

To see this, suppose $\{C, D\} \in \mathcal{F}_2$. Then $\{C, D\} = \{{}_xC_y,\ {}_xD_y\}$ for some x, y. Now let $\{C', D'\} = \{C'_x, D'_x\}$ and $\{C'', D''\} = \{C_y, D_y\}$, and let $X' = C_a$ and $X'' = D_y$. Thus we have ${}_xD_y = C_x \cup D_y$, i.e. $C_x \cup D_y \in \{{}_xC_y,\ {}_xD_y\}$.

Conversely, suppose $\{C, D\} \notin \mathcal{F}_2$, i.e. $\{C, D\} = \{C_x, D_x\}$ for some x. We will show that C_x cannot be the union of two sets of the form $C_y, D_y, {}_yC_z, {}_yD_z$.

- C_x cannot be the union of any C_y with another set, as $C_x \cap C_y = \emptyset$ for all $x \neq y$.
- C_x cannot be the union of any D_y with another set, as if $x \leq y$ then D_y contains points of the form $(y, z_1, ..., z_n)$ with $x < y < z_1$ which do not lie in C_x, and if $y < x$ then D_y contains the points of the form $(y, z_1, ..., z_n)$ where $z_n \neq x$ which do not lie in C_x.

- We see that C_x cannot be the union of any ${}_yD_z$ with another set, as if $x < z$ then ${}_yD_z$ contains points of the form $(z, w_1, ..., w_n)$ where $x < z < w_1$ which do not lie in C_x, and if $x > y$ then ${}_yD_z$ contains points of the form $(v_1, ..., v_n, y)$ where $v_n < y < x$ which don't lie in C_x (note that we have covered all cases here since $y \neq z$).

Thus C_x must be the union of two sets ${}_yC_z$ and ${}_{y'}C_{z'}$, for some y, z, y', z'. Now, by the construction of these sets, we must have $z = z' = x$. But then C_x contains some point $(w_1, ..., w_n, x)$ where $w_1 \neq y, y'$, and so $C_x \neq_y C_z \cup \ {}_{y'}C_{z'}$, a contradiction.

Thus, given G we can uniquely determine $\mathcal{F}_1$. All pairs $\{C, D\} \in \mathcal{F}_1$ can be indexed by some point in T, such that each pair is of the form $\{C_x, D_x\}$ for some $x \in T$. Thus we have determined the set T (or representatives of the set T), but not the betweenness relation on T. Let $A_x = C_x \cup D_x$ for every such pair. Thus $G = \bigcup\{A_x : x \in S\}$, and for each $x \neq y$, $A_x \cap A_y$ is nonempty iff $comp(x, y)$ and $dist(x, y) > n$, and contains the set of vertices $(x, v_2, ..., v_n, y)$ if $x < y$ with $x < v_2 < ... < v_n < y$, or $(y, v_2, ..., v_n, x)$ if $y < x$, with $y < v_2 < ... < v_n < x$.

We can thus determine betweenness on any set of 3 comparable vertices x, y, z with $dist(x, y) > n$, $dist(y, z) > n$, $dist(x, z) > n$ as follows: given A_x, A_y, A_z of pairwise nonempty intersections, we see that $yB[x, z]$ iff every element of $A_x \cap A_y$ is a neighbour of every element of $A_y \cap A_z$, and neither has any neighbours in $A_x \cap A_z$.

Now, similarly to Lemma 4.5, because there is no maximal finite section of size under $2n$, we now obtain n-betweenness on all of T. Thus, by Lemma 3.45, we obtain betweeness on T.

Similarly to Lemma 4.23, provided T splits somewhere we can work out the ordering on T as follows: Since T splits, there exist 3 points, x, y, and z, such that x is comparable to y, and x is comparable to z, but y is not comparable to z. In this case, $x < y$ and $x < z$. Now, since we can work out the betweenness relation on everything else, this will give us the tree $(T, <)$. □

Theorem 4.28. Let T be any tree without endpoints, and let n be a positive integer. Then $(T, <)$ is 2nd-order interpretable inside $G = G(T, 13^n 2)$.

Proof. Consider co-cliques C and D satisfying:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

and let C and D contain at least 2 elements.

By a slight alteration to Lemma 4.7 any pair of subsets of G satisfying $i)$ and $ii)$ must be of the form

$$\begin{aligned} C_{a_1,\dots,a_n} &= \{(x, a_1, a_2, \dots, a_n) : x < a_1\} \\ D_{a_1,\dots,a_n} &= \{(a_1, a_2, \dots, a_n, y) : a_n < y\} \end{aligned}$$

Again, this forms a family $\mathcal{F}$ of pairs $\{C_{a_1,\dots,a_n}, D_{a_1,\dots,a_n}\}$ for $a_1, \dots, a_n \in S$. Now let $A_{a_1,\dots,a_n} = C_{a_1,\dots,a_n} \cup D_{a_1,\dots,a_n}$ for every such n-tuple, and let $\mathcal{F}'$ be the family of all such $A_{a_1,\dots,a_n}$.

We can now reconstruct $G(S, 13^{n-1}2)$ from $G(T, 13^n 2)$.

Consider the following isomorphism $\varphi : \mathcal{F}' \to G(T, 13^{n-1}2)$:

$$\varphi : A_{a_1,\dots,a_n} \mapsto (a_1, \dots, a_n)$$

where the relation "the elements $A_{a_1,\dots,a_n}$ and $A_{a'_1,\dots,a'_n}$ have a common element in $\mathcal{F}'$" is mapped to the relation "there is an edge between the vertices $(a_1, \dots, a_n)$ and $(a'_1, \dots, a'_n)$ in $G(T, 13^{n-1}2)$" .

To see that this is an isomorphism, first note that this is a bijection as each $A_{a_1,\dots,a_n}$ corresponds to exactly the n-tuple $(a_1, \dots, a_n)$. Now let $A_{a_1,\dots,a_n} \cap A_{a'_1,\dots,a'_n} \neq \emptyset$, i.e. there is some element lying in both, and without loss of generality assume $a_1 < a'_1$. Then $G(T, 13^n 2)$ contains some vertex $(x_1, \dots, x_{n+1}) \in A_{a_1,\dots,a_n}$ and $(x_1, \dots, x_{n+1}) \in A_{a'_1,\dots,a'_n}$. Since $a_1 < a'_1$ the first n coordinates of $(x_1, \dots, x_{n+1})$ must be $a_1, \dots, a_n$, and the last n coordinates must be $a'_1, \dots, a'_n$, thus giving us

$$\begin{array}{ccccccccccccc} a_1 & < & a_2 & < & a_3 & < & \dots & < & a_{n-1} & < & a_n & & \\ & & \| & & \| & & & & & & \| & & \\ & & a'_1 & < & a'_2 & < & a'_3 & < & \dots & < & a'_{n-1} & < & a'_n \end{array}$$

i.e. if $A_{a_1,\dots,a_n} \cap A_{a'_1,\dots,a'_n} \neq \emptyset$ then $(a_1, \dots, a_n)$ and $(a'_1, \dots, a'_n)$ are neighbours in $G(S, 13^{n-1}2)$. Conversely, suppose $A_{a_1,\dots,a_n} \cap A_{a'_1,\dots,a'_n} = \emptyset$, i.e. we do not have

$$\begin{array}{ccccccccccccc} a_1 & < & a_2 & < & a_3 & < & \dots & < & a_{n-1} & < & a_n & & \\ & & \| & & \| & & & & & & \| & & \\ & & a'_1 & < & a'_2 & < & a'_3 & < & \dots & < & a'_{n-1} & < & a'_n \end{array}$$

Then $(a_1, \dots, a_n)$ and $(a'_1, \dots, a'_n)$ are not neighbours in $G(S, 13^{n-1}2)$.

Thus we can reconstruct $G(T, 132)$ from $G(T, 13^n 2)$ recursively for any $n \in \mathbb{N}$, and reconstruct T from $G(T, 132)$ by Theorem 4.23. We can also reconstruct n by counting the number of recursions before "reaching" $G(T, 132)$. We can "recognise" $G(T, 132)$ because it has the property that every pair of members of $\mathcal{F}'$ has a nonempty intersection, which is false for $G(T, 13^n 2)$ where $n > 1$. Again, assuming T splits, we can determine the ordering as opposed to the betweenness relation. $\square$

Lemma 4.29. Let T be a tree without endpoints, and let $n > m$ be positive integers. Let $n \equiv k \bmod m$ where $m \geq k \geq 1$ (so if m divides n we let $k = m$). Then $G(S, 1^m 3^k 2^m)$ is 2nd-order interpretable inside $G(S, 1^m 3^n 2^m)$.

Proof. Similar to Lemma 4.9.

This way we can obtain a similar result to Theorem 4.12.

Theorem 4.30. Let T be a tree without endpoints, and let $k \leq m$ be positive integers. Assume T has no non-trivial maximal finite section of size less than m. Then $(T, <)$ is 2nd-order interpretable inside $G = G(T, 1^m 3^k 2^m)$.

Proof. Similar to Theorem 4.12. Again, assuming T is a proper tree we can obtain $(T, <)$, rather than (T, B), as in Theorem 4.23.

Corollary 4.31. Let T be a tree without endpoints containing no non-trivial maximal finite sections of size less than m. Then $(T, <)$ is 2nd-order interpretable inside $G(S, 1^m 3^n 2^m)$. If $m = 1$, then T can be any tree without endpoints.

Proof. If $n \leq m$, (S, B) is 2nd-order interpretable inside $G(S, 1^m 3^n 2^m)$ by Theorem 4.30.

If $n > m$, then by Lemma 4.29 $G(T, 1^m 3^k 2^m)$ is interpretable inside $G(T, 1^m 3^n 2^m)$, where $n \equiv k \bmod m$. Now, since $k \leq m$, and T contains no non-trivial maximal finite sections of size less than m, by Theorem 4.30 $(T, <)$ is 2nd-order interpretable inside $G(T, 1^m 3^k 2^m)$, and hence from $G(T, 1^m 3^n 2^m)$.

If $m = 1$, then $(T, <)$ is 2nd-order interpretable inside $G(T, 1 3^n 2)$ by Theorem 4.28.

Remark 4.32. The case where T has endpoints or 'leaves', or where T is a more general partial order, remains open.

5 Ordinal Shift Graphs

We now consider shift graphs in which the underlying set is an ordinal, and show that in general, it is possible to determine α from $G(\alpha, 1^n 3^m 2^n)$. The case in which this is a limit ordinal works out with the fewest complications. We shall see that the infinite successor case presents more technical problems than the limit case, and that in certain successor cases the underlying ordinal cannot be determined. What we mean by this is that for distinct ordinals α, α', $G(\alpha, 1^n 3^m 2^n) \cong G(\alpha', 1^n 3^m 2^n)$ - thus we 'cannot determine' what the underlying set is from the graph (although we might have some idea, as there may only be finitely many distinct ordinals giving rise to isomorphic graphs, so we know it is one of these).

5.1 Finite Ordinals

First we briefly consider finite ordinals. In some ways this may seem very simple, as here we can distinguish them all merely by means of counting; if z is the finite ordinal $\{0, 1, ..., z-1\}$, then the graph $G(z, 1^n 3^m 2^n)$ has $\binom{z}{n+m}$ verties. This may, however, be regarded as not completely satisfactory, since though for each finite ordinal and type, there will be a formula distinguishing that case, and the formulae will get longer and longer as the finite ordinal increases. So it is far more illuminating to try to do things uniformly, meaning that the formulae will stay consistent no matter what the size of the finite set. It turns out that some of the same techniques as used for other total orders or ordinals can be employed, thereby giving extra information, so this is the approach we adopt. Note that there is an immediate difference here with the case of any infinite ordinal, namely that the finite ordering is isomorphic to its reversed ordering, so we can only ever expect to recover the betweenness relation, as in many of the cases we studied in the previous chapter.

Theorem 5.1. Let $z = \{0, 1, ..., z-1\}$ be a finite set. Then given $G(z, 132)$, there is a uniform procedure for interpreting the betweenness relation on z from the graph.

Thus we can reconstruct $\{A_i : i \in z\}$ as in Theorem 4.2 with the betweenness relation such that A_i corresponds to $i \in z$.

Proof. In all cases for $1 \leq x \leq z-2$, A_x is taken to be $\{(v,x) : v < x\} \cup \{(x,w) : x < w\}$. This is easiest to represent when $2 \leq x \leq z-3$, using ideas from the linear orders chapter. We start by identifying co-cliques C and D with the properties $C = \bigcap\{N_v : v \in D\}$, $D = \bigcap\{N_v : v \in C\}$ of size ≥ 2. As before we can show that these all have the form $C_x = \{(v,x) : v < x\}, D_x = \{(x,w) : x < w\}$ for some $x \in \{2, ..., z-3\}$, and so we can let $A_x = C_x \cup D_x$ in these cases.

We then work out the betweenness relation as in Theorem 4.2: $xB[w,y]$ if and only if $A_w \cap A_y$ is not a neighbour of $A_w \cap A_x$ or $A_x \cap A_y$, but the latter two are neighbours of one another.

Now there are exactly two points, $(0, z-2)$ and $(1, z-1)$, of degree 1, so we can characterize each of A_1 and A_{z-2} as consisting of a vertex which is adjacent to a vertex of degree 1, and all its neighbours.

Although it is possible to determine the set A_0 constisting of all points beginning with 0 and the set A_{z-1} consisting of all points ending with $z-1$, this is both laborious and unnecessary, and it suffices to let A_0 consist of the unique member of A_{z-2} of degree 1, and A_{z-1} consist of the unique member of A_1 of degree 1. We can then extend the definition of betweenness to fit $A_0, A_1, A_{z-2}, A_{z-1}$ so as to interpret the whole of z.

□

Theorem 5.2. Let $Z = \{0, 1, ..., z-1\}$ be a finite ordinal of size at least $4n$. Then given $G(Z, 1^n 32^n)$, there is a uniform procedure for interpreting the betweenness relation on Z from the graph.

Proof. First we adopt the following notation: let 0_n mean "any member of the set $\{0, ..., n-1\}$", and let $z-1_n$ mean any member of the set $\{z-n-1, ..., z-1\}$".

By identifying co-cliques with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

we can isolate pairs of the form

$$C_x = \{(v_1, ...v_n, x) : v_n < x\}$$
$$D_x = \{(x, w_1, ..., w_n) : x < w_1\}$$

where $x \in \{n+1, ..., z-n-2\}$, and again let $A_x = C_x \cup D_x$ for every such x. Note that pairs of the form ${}_xC_y$, ${}_xD_y$ also satisfy $i)$ and $ii)$, but that we can eliminate these as in Lemma 4.3.

Now given A_x, A_y, A_w, all of distance at least n apart, we can determine which lies between the other two as follows: $yB[x, w]$ if and only if $A_x \cap A_w$ is not a neighbour of $A_x \cap A_y$ or $A_y \cap A_w$, but the latter two are neighbours of one another.

This gives us $B_{dist(n)}$ on $Z \setminus \{0_n, z-1_n\}$. By Lemma 3.28, we can determine B on $Z \setminus \{0_n, z-1_n\}$ from this. We then 'add' n points on to either end, with labels $0, ..., n-1$ and $z-n-1, ..., z-1$ and the usual ordering.

Theorem 5.3. Let $Z = \{0, 1, ..., z-1\}$ be a finite set, and let $Z > 2n$. Then given $G(Z, 13^n 2)$, there is a uniform procedure for interpreting the betweenness relation on Z from the graph.

Proof. By identifying co-cliques C and D with the following properties:

$$i)\ C = \bigcap\{N_v : v \in D\}$$
$$ii)\ D = \bigcap\{N_v : v \in C\}$$

where $|C|, |D| \geq 1$, we can isolate pairs of the form

$$\begin{aligned} C_{x_1,...,x_n} &= \{(y, x_1, ..., x_n) : y < x_1\} \\ D_{x_1,...,x_n} &= \{(x_1, ..., x_n, y') : x_n < y'\} \end{aligned}$$

where $x_1, ..., x_n \in \{1, ..., z-2\}$.

Note that pairs of the form ${}_{x_1,...,x_n}C_{y_1,...,y_n}$, ${}_{x_1,...,x_n}D_{y_1,...,y_n}$ also satisfy $i)$ and $ii)$, but that we again can eliminate this case in a similar way to Lemma 4.3.

Now let $A_{x_1,...,x_n} = C_{x_1,...,x_n} \cup D_{x_1,...,x_n}$, such that $A_{x_1,...,x_n}$ is the set of all vertices beginning or ending with $x_1, ..., x_n$.

We start by noting that $A_{x_1,...,x_n}$ and $A_{y_1,...,y_n}$ have a non-empty intersection if and only if either

$$\begin{array}{ccccccccccc} x_1 & < & x_2 & < & ... & < & x_{n-1} & < & x_n & & \\ & & \| & & & & & & \| & & \\ & & y_1 & < & y_2 & < & ... & < & y_{n-1} & < & y_n \end{array}$$

in which case their intersection is the point $(x_1, ..., x_n, y_n)$, or

$$\begin{array}{ccccccccccc} & & x_1 & < & x_2 & < & ... & < & x_{n-1} & < & x_n \\ & & \| & & & & & & \| & & \\ y_1 & < & y_2 & < & ... & < & y_{n-1} & < & y_n & & \end{array}$$

in which case their intersection is the point $(y_1, ..., y_n, x_n)$.

Let $\mathcal{F}$ be the family of $A_{x_1,...,x_n}$ for $1 \leq x_1, ..., x_n \leq z-2$

We can thus construct the following isomorphism $\varphi : \mathcal{F} \to G(z \setminus \{0, z-1\}, 13^{n-1}2)$

$$\varphi : A_{x_1,...,x_n} \mapsto (x_1, ..., x_n)$$

where the relation "the elements $A_{x_1,...,x_n}$ and $A_{y_1,...,y_n}$ have a non-empty intersection in $\mathcal{F}$ is mapped to the relation "there is an edge between the vertices $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ in $G(Z \setminus \{0, z-1\}, 13^{n-1}2)$. After repeating this process $n-1$ times we obtain all the A_is in $G(Z \setminus \{0_n, z-1_n\}, 132)$, and so by Theorem 5.1 we can work out the betweenness relation on these. Note that $Z \setminus \{0_n, z-1_n\}$ is nonempty since $Z > 2n$. Finally, we 'add' n points to either end of $Z \setminus \{0_n, z-1_n\}$ to obtain Z. □

Remark 5.4. The more general $1^n3^m2^n$ case remains open, but it looks like z can in general be obtained from $G(z, 1^n3^m2^n)$, using similar methods to those in Chapter 4. Additionally, the finite case shares some similarities with the 'endpoints' case, which also remains open.

5.2 Limit Ordinals

In this section, we will start by considering all graphs of the form $G(\alpha, 13^n2)$, where $n \geq 1$. We will start by defining a set having some desired properties - namely, the set of all vertices beginning with 0. We can also repeat this inductively for any ordinal less than α. These are presented as Lemmas, after which the reconstruction theorem follows. We then repeat with $G(\alpha, 1^n32^n)$ and $G(\alpha, 1^n3^m2^n)$, using a similar structure.

Lemma 5.5. Let α be a limit ordinal. Then there is a unique sequence of pairwise disjoint subsets D_γ of $G(\alpha, 132)$ for $\gamma < \alpha$ such that:

(i) The induced subgraph on D_γ is a maximal co-clique of $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta$

(ii) Any two vertices of D_γ have disjoint neighbour sets in $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta$.

Proof. First note that $\{(0, \beta) : \beta > 0\}$ satisfies (i) and (ii). Now let D_0 be any set satisfying (i) and (ii). Suppose $D_0 \not\subseteq \{(0, \beta) : \beta > 0\}$. Then D_0 must contain some vertex (δ, β) with $\delta > 0$. Now, since D_0 is a maximal co-clique, it must contain $(0, \beta)$; but then (δ, β) and $(0, \beta)$ have a common neighbour, (β, β') for some $\beta' > \beta$ (which always exists as α is a limit), a contradiction of the second condition above. If $D_0 \subsetneq \{(0, \beta) : \beta > 0\}$ then D_0 would not be maximal, and so $D_0 = \{(0, \beta) : \beta > 0\}$.

Now assume inductively that we have chosen D_δ for $\delta < \gamma$ and $D_\delta = \{(\delta, \beta) : \beta > \delta\}$. Then $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta = \{(\beta_0, \beta_1) : \beta_0 \geq \gamma\}$. Then G restricted to this vertex set is isomorphic to $G' = G(\alpha', 132)$, where $\gamma + \alpha' = \alpha$, and α' is still a limit ordinal. By the same argument as for D_0, there is a unique $D_\gamma \subseteq V(G') \subseteq V(G)$ such that the induced subgraph on D_γ is a maximal co-clique and any two vertices in D_γ have disjoint neighbour sets.

Hence there is a unique sequence of pairwise disjoint subsets D_γ of $G(\alpha, 132)$ for $\gamma < \alpha$ satisfying (i) and (ii), and this sequence is $D_\gamma = \{(\gamma, \beta) : \beta > \gamma\}$. □

Lemma 5.6. Let α be a limit ordinal. Then there is a unique sequence of pairwise disjoint subsets D_γ of $G(\alpha, 13^n2)$ for $\gamma < \alpha$ such that:

(i) The induced subgraph on D_γ is a maximal co-clique of $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta$

(ii) Any two vertices of D_γ have disjoint neighbour sets in $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta$.

Proof. First note that $\{(0, \beta_1, ..., \beta_n) : \beta_1 > 0\}$ satisfies (i) and (ii). Now let D_0 be any set satisfying (i) and (ii), and suppose $D_0 \not\subseteq \{(0, \beta_1, ..., \beta_n) : \beta_1 > 0\}$. Then D_0 must contain some vertex $(\delta, \beta_1, ..., \beta_n)$ with $\delta > 0$. Now, since D_0 is a maximal co-clique, it must contain $(0, \beta_1, ..., \beta_n)$; but then $(\delta, \beta_1, ..., \beta_n)$ and $(0, \beta_1, ..., \beta_n)$ have a common neighbour, $(\beta_1, ..., \beta_n, \beta)$ for some $\beta > \beta_n$ (which always exists as α is a limit ordinal), a contradiction of the second condition above. If $D_0 \subsetneq \{(0, \beta_1, ..., \beta_n) : \beta_1 > 0\}$ then D_0 would not be maximal, and so $D_0 = \{(0, \beta_1, ..., \beta_n) : \beta_1 > 0\}$.

Now assume inductively that we have chosen D_δ for $\delta < \gamma$ and $D_\delta = \{(\delta, \beta_1, ..., \beta_n) : \beta_1 > \delta\}$. Then $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta = \{(\beta_0, \beta_1, ..., \beta_n) : \beta_0 \geq \gamma\}$. Then G restricted to this vertex set is isomorphic to $G' = G(\alpha', 132)$, where $\gamma + \alpha' = \alpha$, and α' is still a limit ordinal. By the same argument as for D_0, there is a unique $D_\gamma \subseteq V(G') \subseteq V(G)$ such that the induced subgraph on D_γ is a maximal co-clique and any two vertices in D_γ have disjoint neighbour sets.

Hence there is a unique sequence of pairwise disjoint subsets D_γ of $G(\alpha, 132)$ for $\gamma < \alpha$ satisfying (i) and (ii), and this sequence is $D_\gamma = \{(\gamma, \beta_1, ..., \beta_n) : \beta_1 > \gamma\}$. □

Theorem 5.7. Any limit ordinal α is 2nd-order interpretable inside $G = G(\alpha, 13^n2)$ for n a positive integer.

Proof. We may define $D_\gamma \subseteq V(G)$ as in Theorem 5.6, and then α is the least ordinal γ such that $D_\gamma = \emptyset$, and α is the order-type of the family $(D_\gamma : \gamma < \alpha)$. □

Remark 5.8. Given $v \in G$ as in Theorem 5.7, we can determine the unique $(n+1)$-tuple "corresponding to" v as follows: first, find the unique D_{γ_1} containing v. Then, consider the set V_2 of N_v in $V(G) \setminus \bigcup_{\xi<\gamma_1} D_\xi$. This set V_2 lies entirely inside one D_{γ_2} for some $\gamma_2 > \gamma_1$. Now, pick any element v_2 of V_2, and consider N_{v_2} in $V(G) \setminus \bigcup_{\xi<\gamma_2} D_\xi$. This set, once again, lies entirely within one D_{γ_3} for some $\gamma_3 > \gamma_2$. Repeat this process n times, until we consider N_{v_n} in $V(G) \setminus \bigcup_{\xi<\gamma_n} D_\xi$. Finally, this set lies entirely within some $D_{\gamma_{n+1}}$. We now have a sequence, $\gamma_1, \gamma_2, ..., \gamma_{n+1}$. Thus we see that v was "generated" by the $n+1$-tuple $(\gamma_1, \gamma_2, ..., \gamma_{n+1})$, which is represented by $(D_{\gamma_1}, D_{\gamma_2}, ..., D_{\gamma_{n+1}})$.

We can illustrate the above with a more concrete example. Suppose our graph G is $G(\omega, 132)$, and that v is the pair $(5, 10)$. Now, the smallest D_γ containing v is D_5. If we consider the set of neighbours of v which lie in $V(G) \setminus \bigcup_{\xi<5} D_\xi$, we see that this is precisely the set D_{10}. We thus have that the original pair was $(5, 10)$ as required, which is represented by (D_5, D_{10}).

Let us take a more complicated example. Suppose our graph G is $G(\omega, 133332)$, and that v is is the 5-tuple $(1, 3, 5, 7, 9)$. Now, the smallest D_{γ_1} containing v is D_1. If we consider the set of neighbours of $(1, 3, 5, 7, 9)$ which lie in $V(G) \setminus \bigcup_{\xi<1} D_\xi$, i.e. which lie in $V(G) \setminus D_0$, we see that this is the set $V_2 := \{(3, 5, 7, 9, y) : 9 < y\}$. This set lies entirely in D_3 (but is not equal to D_3, as the second, third, and fourth "coordinates" must be $5, 7, 9$).

Pick any element v_2 of V_2, for example, $(3, 5, 7, 9, 10)$. Once again, if we consider N_{v_2} inside $V(G) \setminus \bigcup_{\xi<3} D_\xi$, we see that it is the set $V_3 := \{(5, 7, 9, 10, y) : 10 < y\}$, and that this set lies entirely within D_5.

Again, choose an element v_3 of V_3, say $(5, 7, 9, 10, 11)$. Once again, if we consider N_{v_3} inside $V(G) \setminus \bigcup_{\xi<5} D_\xi$, we see that it is the set $V_4 := \{(7, 9, 10, 11, y) : 11 < y\}$, and that this set lies entirely within D_7.

Choose an element v_4 of V_4, say $(7, 9, 10, 11, 12)$. Once again, if we consider N_{v_4} inside $V(G) \setminus \bigcup_{\xi<7} D_\xi$, we see that it is the set $V_5 := \{(9, 10, 11, 12, y) : 12 < y\}$. Finally, we see that this set lies entirely within D_9.

We thus have $1, 3, 5, 7, 9$ as our sequence $\gamma_1, \gamma_2, ..., \gamma_{n+1}$, and therefore that the original vertex v is the $n+1$-tuple $(1, 3, 5, 7, 9)$ as required, which is represented by $(D_1, D_3, D_5, D_7, D_9)$.

This cannot be done in general for $1^n3^m2^n$ with $n > 1$.

Lemma 5.9. Let α be a limit ordinal. Then there is a unique sequence of pairwise disjoint subsets D_γ of $G(\alpha, 1^n32^n)$ for $\gamma < \alpha$ such that:

(i) The induced subgraph on D_γ is a maximal co-clique of $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta$

(ii) Any two vertices of D_γ have identical or disjoint neighbour sets in $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta$. Furthermore, D_γ is equal to the set of all $(\beta_0, \beta_1, ..., \beta_n) \in V(G)$ for which $n\gamma \leq \beta_0 < n\gamma+n$.

Proof. First note that $\{(\beta_0, \beta_1, ..., \beta_n) : \beta_0 < n\}$ satisfies (i) and (ii). Now let D_0 be any set satisfying (i) and (ii), and suppose $D_0 \not\subseteq \{(\beta_0, \beta_1, ..., \beta_n) : \beta_0 < n\}$. Then D_0 must contain some vertex $(\delta, \beta_1, ..., \beta_n)$ with $\delta \neq 0, 1, ..., n-1$. Now, since D_0 is a maximal co-clique, it must contain $(0, \beta_1, ..., \beta_n)$; but since $\delta \geq n$, $N_{(\delta,\beta_1,...,\beta_n)}$ and $N_{(0,\beta_1,...,\beta_n)}$ are not equal as $(0, 1, ..., n-1, \delta)$ lies in $N_{(\delta,\beta_1,...,\beta_n)}$ and not in $N_{(0,\beta_1,...,\beta_n)}$, yet have a non-empty intersection as $(\beta_n, \beta_n+1, ..., \beta_n+n)$ lies in both, contradicting the second condition above. Note that since α is a limit, $\beta_n < \alpha$ and so $\beta_n + 1, ..., \beta_n + n \in \alpha$. If $D_0 \subset \{(\beta_0, \beta_1, ..., \beta_n) : \beta_0 < n\}$ then D_0 would not be maximal, and so $D_0 = \{(\beta_0, \beta_1, ..., \beta_n) : \beta_0 < n\}$.

Now assume inductively that we have chosen D_δ for $\delta < \gamma$ and $D_\delta = \{(\beta_0, ..., \beta_n) : n\delta \leq \beta_0 < n\delta + n\}$. Then $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta = \{(\beta_0, ..., \beta_n) : \beta_0 \geq n\gamma\}$. Then G restricted to this vertex set is isomorphic to $G' = G(\alpha', 1^n32^n)$, where $n\gamma + \alpha' = \alpha$, and α' is still a limit ordinal. By the same argument as for D_0, there is a unique $D_\gamma \subseteq V(G') \subseteq V(G)$ such that the induced subgraph on D_γ is a maximal co-clique and for any two vertices $v_1, v_2 \in D_\gamma$, either $N_{v_1} = N_{v_2}$ or $N_{v_1} \cap N_{v_2} = \emptyset$, and this equals $\{(\beta_0, ..., \beta_n) : n\gamma \leq \beta_0 < n\gamma + n\}$.

Hence there is a unique sequence of pairwise disjoint subsets D_γ of $G(\alpha, 1^n32^n)$ for $\gamma < \alpha$ satisfying (i) and (ii), and this sequence is $\{(\beta_0, ..., \beta_n) : n\gamma \leq \beta_0 < n\gamma + n\}$. □

Theorem 5.10. Any limit ordinal α is 2nd-order interpretable inside $G = G(\alpha, 1^n32^n)$ for n a positive integer.

Proof. We may define $D_\gamma \subseteq V(G)$ as in Theorem 5.9, and then α is the least ordinal γ such that $D_\gamma = \emptyset$, and α is the order-type of the family $(D_\gamma : \gamma < \alpha)$. □

Lemma 5.11. Let α be a limit ordinal. Then there is a unique sequence $(D_\gamma : \gamma < \alpha)$ of pairwise disjoint sets of vertices in $G(\alpha, 1^n 3^m 2^n)$ with the following properties:

(i) The induced subgraph on D_γ is a maximal co-clique of $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta$

(ii) Any two vertices of D_γ have identical or disjoint neighbour sets in $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta$. Furthermore, D_γ is equal to the set of all $(\beta_0, \beta_1, ..., \beta_n) \in V(G)$ for which $n\gamma \leq \beta_0 < n\gamma+n$.

Proof. First note that $\{(\beta_0, \beta_1, ..., \beta_{n+m-1}) : \beta_0 < n\}$ satisfies (i) and (ii). Now let D_0 be any set satisfying (i) and (ii), and suppose $D_0 \not\subseteq \{(\beta_0, \beta_1, ..., \beta_{n+m-1}) : \beta_0 < n\}$. Then D_0 must contain some vertex $(\delta, \beta_1, ..., \beta_{n+m-1})$ with $\delta \neq 0, 1, ..., n-1$. Now, since D_0 is a maximal co-clique, it must contain $(0, \beta_1, ..., \beta_{n+m-1})$; but since $\delta \geq n$, $N_{(\delta,\beta_1,...,\beta_{n+m-1})}$ and $N_{(0,\beta_1,...,\beta_{n+m-1})}$ are not equal as $(0, 1, ..., n-1, \delta, \beta_1, ..., \beta_{m-1})$ lies in $N_{(\delta,\beta_1,...,\beta_{n+m-1})}$ and not in $N_{(0,\beta_1,...,\beta_{n+m-1})}$, yet have a non-empty intersection as $(\beta_n, \beta_{n+1}, ..., \beta_{n+m-1}, \beta_{n+m-1} + 1, ..., \beta_{m+n-1} + n)$ lies in both, contradicting the second condition above. Note that since α is a limit, $\beta_{n+m-1} < \alpha$ and so $\beta_{n+m-1}+1, ..., \beta_{m+n-1}+n \in \alpha$. If $D_0 \subsetneq \{(\beta_0, \beta_1, ..., \beta_{n+m-1}) : \beta_0 < n\}$ then D_0 would not be maximal, and so $D_0 = \{(\beta_0, \beta_1, ..., \beta_{n+m-1}) : \beta_0 < n\}$.

Now assume inductively that we have chosen D_δ for $\delta < \gamma$ and $D_\delta = \{(\beta_0, ..., \beta_{n+m-1}) : n\delta \leq \beta_0 < n\delta+n\}$. Then $V(G) \setminus \bigcup_{\delta<\gamma} D_\delta = \{(\beta_0, ..., \beta_{n+m-1}) : \beta_0 \geq n\gamma\}$. Then G restricted to this vertex set is isomorphic to $G' = G(\alpha', 1^n 3 2^n)$, where $n\gamma+\alpha' = \alpha$, and α' is still a limit ordinal. By the same argument as for D_0, there is a unique $D_\gamma \subseteq V(G') \subseteq V(G)$ such that the induced subgraph on D_γ is a maximal co-clique and for any two vertices $v_1, v_2 \in D_\gamma$, either $N_{v_1} = N_{v_2}$ or $N_{v_1} \cap N_{v_2} = \emptyset$, and this equals $\{(\beta_0, ..., \beta_n) : n\gamma \leq \beta_0 < n\gamma + n\}$.

Hence there is a unique sequence of pairwise disjoint subsets D_γ of $G(\alpha, 1^n 3^m 2^n)$ for $\gamma < \alpha$ satisfying (i) and (ii), and this sequence is $\{(\beta_0, ..., \beta_{n+m-1}) : n\gamma \leq \beta_0 < n\gamma + n\}$. □

Theorem 5.12. Any limit ordinal α is 2nd-order interpretable inside $G = G(\alpha, 1^n 3^m 2^n)$ for positive integers m, n.

Proof. By the Lemma there is a unique sequence $(D_\gamma : \gamma < \alpha)$ having the stated properties, and we can identify α as the least ordinal such that $\bigcup_{\gamma<\alpha} D_\gamma = G$. □

The theorems above apply only to limit ordinals. We would now like to generalise the result for the type $\tau = 132$ to all ordinals.

5.3 Successor Ordinals

In this section, we will start by taking an example, namely, the graph $G(\alpha_0 + 5, 132)$, where α_0 is a limit ordinal. We can work out that there is a +5 at the end by looking at the degrees of vertices. Similarly, we can apply the same logic to $G(\alpha_0 + k, 132)$, and so generalise this to all successor ordinals. A similar method is applied in the case of $13^n 2$.

We then take $G(\alpha_0 + k, 11322)$ as an example, and show that $\alpha_0 + k$ can be reconstructed iff $k \geq 3$. Similarly, we can once again generalise this case to $G(\alpha_0 + k, 1^n 3 2^n)$, which can be reconstructed iff $k \geq n + 1$.

Finally, we generalise to $G(\alpha_0 + k, 1^n 3^m 2^n)$. We start by showing that for $m < n < k$, α is interpretable from $G(\alpha_0 + k, 1^n 3^m 2^n)$.

Next we consider the case where $m \geq n$. Our goal here is to 'reconstruct' $G(\alpha_0 + k, 1^{2n} 3^{m-n} 2^{2n})$ inside $G(\alpha_0 + k, 1^n 3^m 2^n)$. We can do this by isolating $n + m$-tuples overlapping in this way:

$$\begin{array}{cccccccccccccc}
(x_1, & \ldots & x_n, & y_1, & \ldots & y_n, & y_{n+1}, & y_{n+2}, & \ldots & y_m) & & & & \\
 & & & & & & \| & \| & & \| & & & & \\
 & & & & & & (y'_1, & y'_2, & \ldots & y'_{m-n}, & y'_{m-n+1}, & \ldots & y'_m, & z_1, \ \ldots \ z_n)
\end{array}$$

Two vertices of this form will share a neighbour $(y_1, ..., y_m, y'_{m-n+1}, ..., y'_m)$. However, this is not the only arrangement of vertices sharing a neighbour - so we impose the following conditions one at a time, thus isolating the case above:

i) the two vertices x and z must share exactly one neighbour

ii) there must be no other vertex v sharing a neighbour with both x and z

We see in this proof that we must have $n \geq 2$ in $G(\alpha_0 + k, 1^n 3^m 2^n)$.

We finish off with Corollary 5.23, grouping all these cases together and thus the general result for all successor ordinals.

Example 5.13. Before we consider the successor case, let us first take the following example: consider the graph $G(\alpha_0 + 5, 132)$, where $\alpha_0 \neq 0$ is some limit ordinal.

First note that for the degree of a vertex (x, y) to be finite in general, we must both have $dist(0, x) < \omega$, and $dist(y, \alpha_0 + 4) < \omega$.

We now count the number of vertices of each finite degree. Starting with all pairs of the form $(n, \alpha_0 + 4)$, we get:

$$\begin{array}{c}
1 \text{ vertex of degree } 0, \text{ namely the vertex } (0, \alpha_0 + 4) \\
1 \text{ vertex of degree } 1, \text{ namely the vertex } (1, \alpha_0 + 4) \\
1 \text{ vertex of degree } 2, \text{ namely the vertex } (2, \alpha_0 + 4) \\
\vdots
\end{array}$$

Now considering pairs of the form $(n, \alpha_0 + 3)$, we get:

$$\begin{gathered}
1 \text{ vertex of degree } 1, \text{ namely the vertex } (0, \alpha_0 + 3) \\
1 \text{ vertex of degree } 2, \text{ namely the vertex } (1, \alpha_0 + 3) \\
1 \text{ vertex of degree } 3, \text{ namely the vertex } (2, \alpha_0 + 3) \\
\vdots
\end{gathered}$$

Again, considering pairs of the form $(n, \alpha_0 + 2)$, we get:

$$\begin{gathered}
1 \text{ vertex of degree } 2, \text{ namely the vertex } (0, \alpha_0 + 2) \\
1 \text{ vertex of degree } 3, \text{ namely the vertex } (1, \alpha_0 + 2) \\
1 \text{ vertex of degree } 4, \text{ namely the vertex } (2, \alpha_0 + 2) \\
\vdots
\end{gathered}$$

Considering pairs of the form $(n, \alpha_0 + 1)$, we get:

$$\begin{gathered}
1 \text{ vertex of degree } 3, \text{ namely the vertex } (0, \alpha_0 + 1) \\
1 \text{ vertex of degree } 4, \text{ namely the vertex } (1, \alpha_0 + 1) \\
1 \text{ vertex of degree } 5, \text{ namely the vertex } (2, \alpha_0 + 1) \\
\vdots
\end{gathered}$$

Finally, considering pairs of the form (n, α_0), we get:

$$\begin{gathered}
1 \text{ vertex of degree } 4, \text{ namely the vertex } (0, \alpha_0) \\
1 \text{ vertex of degree } 5, \text{ namely the vertex } (1, \alpha_0) \\
1 \text{ vertex of degree } 6, \text{ namely the vertex } (2, \alpha_0) \\
\vdots
\end{gathered}$$

Now, adding all these up, we see that we have a total of:

$$\begin{gathered}
1 \text{ vertex of degree } 0 \\
2 \text{ vertices of degree } 1 \\
3 \text{ vertices of degree } 2 \\
4 \text{ vertices of degree } 3 \\
5 \text{ vertices of degree } 4 \\
5 \text{ vertices of degree } 5 \\
5 \text{ vertices of degree } 6 \\
\vdots
\end{gathered}$$

Thus there are

$$\begin{aligned} j+1 &\text{ vertices of degree } j \text{ for all } j < 4 \\ 5 &\text{ vertices of degree } j \text{ for all } j \geq 4 \end{aligned}$$

And so supposing we hadn't known what the original ordinal was, we would have been able to determine that it had form $\alpha + 5$, where α is some limit.

We see that this holds in general for any $G(\alpha_0 + k, 132)$, where k is finite.

Theorem 5.14. Any infinite ordinal α is 2nd-order interpretable inside $G = G(\alpha, 132)$.

Proof. Let $\alpha = \alpha_0 + k$ where $\alpha_0 \neq 0$ is a limit ordinal and $k < \omega$.

We start by determining k; this can be done by looking at the degrees of the vertices of G.

Claim 1: k is the largest finite number such that for some finite j there are exactly k vertices of degree j. If no such number exists, then $k = 0$.

Proof of Claim 1. For finite x, the vertex $(x, \alpha_0 + k - 1)$ has degree x. If x were infinite, the degree would also be infinite. In general, the vertex $(x, \alpha_0 + k - v - 1)$ has degree $x + v$. Counting the total number of vertices of each degree, we see that there are

$$\begin{aligned} j+1 &\text{ vertices of degree } j \text{ for all } j < k-1 \\ k &\text{ vertices of degree } j \text{ for all } j \geq k-1 \end{aligned}$$

This proves **Claim 1**.

We would now like to work out α_0. We do this by "recognising" $G(\alpha_0, 132)$ inside $G(\alpha_0 + k, 132)$ as an induced subgraph.

We will start by "recognising" $G(\alpha_0 + k - 1, 132)$ inside $G(\alpha_0 + k, 132)$, and repeating the process k times. We can then determine α_0 as in the proof of Theorem 5.7.

Claim 2: In $G(\alpha_0 + k, 132)$, excluding isolated points, any maximal co-clique whose neighbour sets are pairwise disjoint is of the form $\{(0, \gamma) : 0 < \gamma < \alpha_0 + k - 1\}$ or $\{(\delta, \alpha_0 + k - 1) : 0 < \delta < \alpha_0 + k - 1\}$.

Proof of Claim 2: First of all, each of these is a co-clique, because no two vertices of the form $(0, \gamma)$ are joined, not are any two of the form $(\delta, \alpha_0 + k - 1)$. Each of them is a maximal co-clique, as we cannot add any vertex (x, y) with $x > 0$ to the first and retain its being a co-clique, and similarly for the second. The neighbour sets are pairwise disjoint, as any neighbour of $(0, \gamma)$ must begin with γ; similarly in the second case.

We will now show that any maximal co-clique S with pairwise disjoint neighbour sets is equal to one of these two sets. Let $(a, b) \in S$ with $a > 0$. Then $(0, b)$ is not joined to any member of S, since if it was, then it would have to be (b, c), but this is joined to (a, b). By maximality of S, $(0, b) \in S$. If $b < \alpha_0 + k - 1$, then $(0, b)$ and (a, b) have overlapping neighbour sets, a contradiction. We deduce that $b = \alpha_0 + k - 1$. To sum up, any $(a, b) \in S$ has $a = 0$ or $b = \alpha_0 + k - 1$ (but not both, as we have excluded isolated points).

Hence S is the union of two subsets, those pairs beginning with 0, and those ending in $\alpha_0 + k - 1$. In fact, for every $\beta < \alpha$, either $(0, \beta)$ or $(\beta, \alpha_0 + k - 1)$ lies in S. To see this, suppose $(0, \beta)$ does not lie in S. By maximality, it must have a neighbour in S, and this can only be $(\beta, \alpha_0 + k - 1)$.

It follows that α can be written as the disjoint union of two subsets A and B such that $S = \{0\} \times A \cup B \times \alpha_0 + k - 1$. Since the neighbour sets of S are pairwise disjoint, if $(0, a)$ and $(b, \alpha_0 + k - 1)$ lie in S, then $b \leq a$, as otherwise (a, b) would be a common neighbour. Thus A is closed upwards, and B is closed downwards. Since $\alpha_0 + k - 1 \in A$, A is non-empty. Let β be its least member. If $1 < \beta < \alpha_0 + k - 1$, consider $(1, \beta + 1)$. Since S is a maximal co-clique, this is joined to a member of S. If it is joined to $(0, a)$ then $a = 1 \in A$, contrary to $\beta > 1$ the least member of A. If it is joined to $(b, \alpha_0 + k - 1)$, then $b = \beta + 1 \in B$, contrary to members of B being $\leq$ members of A.

This proves **Claim 2**.

We can distinguish between these sets, $\{(0, \gamma) : 0 < \gamma < \alpha_0 + k - 1\}$ and $\{(\delta, \alpha_0 + k - 1) : 0 < \delta < \alpha_0 + k - 1\}$, since $\{(\delta, \alpha_0 + k - 1) : 0 < \delta < \alpha_0 + k - 1\}$ has vertices of all finite degrees, whereas $\{(0, \gamma) : 0 < \gamma < \alpha_0 + k - 1\}$ only has vertices of finite degrees up to k.

Now we remove $\{(\delta, \alpha_0 + k - 1) : 0 < \delta < \alpha_0 + k - 1\}$ from $G(\alpha_0 + k, 132)$, along with all isolated points, leaving us with the graph $G(\alpha_0 + k - 1, 132)$. Repeat this process k times, so that we are left with the graph $G(\alpha_0, 132)$. We can now determine α_0 as in the proof of Theorem 5.10. Thus we can determine the ordinal α from the graph $G(\alpha, 132)$. □

We can extend this theorem to all types of the form 13^n2 fairly easily:

Theorem 5.15. Any infinite ordinal α is 2nd-order interpretable inside $G = G(\alpha, 13^n2)$.

Proof. Let $\alpha = \alpha_0 + k$, where α_0 is a limit ordinal and k is finite.

A path in G is said to be 'balanced' if all its vertices have the same (finite) degree. A maximal balanced path is one which is as long as possible. If $k \leq n$, then this length is k, and if $k > n$, it is n. So the problem is determining which of these two cases applies, and determining k in each.

Now, a maximal balanced path must be of the form:

$$\begin{aligned}(r, r+1, ..., r+k-1, \beta_k, ..., \beta_{n-1}, \alpha_0) &\sim (r+1, r+2, ..., r+k-1, \beta_k, ..., \beta_{n-1}, \alpha_0, \alpha_0+1) \\ &\sim ... \\ &\sim (r+k-1, \beta_k, ..., \beta_{n-1}, \alpha_0, ..., \alpha_0+k-1)\end{aligned}$$

if $k \leq n$, or of form:

$$\begin{aligned}(r_0, r_1, ..., r_{n-1}, \alpha_0+s_0) &\sim (r_1, r_2, ..., r_{n-1}, \alpha_0+s_0, \alpha_0+s_1) \\ &\sim ... \\ &\sim (r_{n-1}, \alpha_0+s_0, \alpha_0+s_1, ..., \alpha_0+s_{n-1})\end{aligned}$$

if $k > n$, where $r_i - s_i$ is constant.

We remark that interior vertices in a maximal balanced path have neighbours which have finite degree. However, it is possible for the endpoints of these paths to have neighbours (not in the path, of course) which have infinite degree.

This is possible since we know explicitly what the options are for maximal balanced paths.

Consider the following condition on a vertex v. It must be the endpoint of a maximal balanced path, and have EXACTLY ONE neighbour of infinite degree. Call this property $Unique_\infty(v)$. This happens for $k \leq n$ if $r = 1$, since $v = (1, 2, ..., k, \beta_k, ..., \beta_{n-1}, \alpha_0)$ has the vertex $(0, 1, 2, ..., k, \beta_k, ..., \beta_{n-1})$ as its unique neighbour of infinite degree, and for $k > n$, if $r_0 = 1$ then $(1, r_1, ..., r_{n-1}, \alpha_0 + s_0)$ has just one neighbour of infinite degree, namely $(0, 1, r_1, ..., r_{n-1})$ and if $s_{n-1} = k-2$, $(r_{n-1}, \alpha_0+s_0, ..., \alpha_0+s_{n-2}, \alpha_0+k-2)$ has just one neighbour of infinite degree, namely $(\alpha_0+s_0, ..., \alpha_0+s_{n-2}, \alpha_0+k-2, \alpha_0+k-1)$.

For a vertex v fulfilling $Unique_\infty(v)$, let v_∞ stand for its unique neighbour of infinite degree.

We can tell the difference between the two cases $k \leq n$ and $k > n$ as follows:

For any maximal balanced path P, there is at most one endpoint v fulfilling $Unique_\infty(v)$. This holds if and only if $k \leq n$, and furthermore, k can then be recovered as the length of such a path.

Now suppose that $k > n$. Then a path can have two endpoints fulfilling $Unique_\infty(v)$, and we can tell which is which as follows. The first case 'small end' is characterised by saying that there are only finitely many neighbours of v_∞ which have *finite* degree. To see this, let $v = (1, r_1, ..., r_{n-1}, \alpha_0 + s_0)$ be our vertex at the 'small end' such that $v_\infty = (0, 1, r_1, ..., r_{n-1})$. The neighbours of v_∞ having finite degree are precisely vertices of the form $(1, r_1, ..., r_{n-1}, \alpha_0 + s)$ for $0 \leq s \leq k-1$. But for a $Unique_\infty$ vertex at the large end, $v = (r_{n-1}, \alpha_0+s_0, ..., \alpha_0+s_{n-2}, \alpha_0+k-2)$, and $v_\infty = (\alpha_0+s_0, ..., \alpha_0+s_{n-2}, \alpha_0+k-2, \alpha_0+k-1)$, which has infinitely many neighbours of finite degree, namely all of the form $(r, \alpha_0+s_0, ..., \alpha_0+s_{n-2}, \alpha_0+k-2)$ for $r < \omega$. So we can distinguish these two types of endpoint, and we can recover k as the number of possible neighbours of v_∞ of finite degree in the first case, since s can take k possible values.

We now turn our attention to α_0. We consider separately the cases where $k = 1$ and where $k \geq 2$.

Case 1: $k \geq 2$

We will show that $x \in G(\alpha_0, 13^n 2) \Leftrightarrow$ there's a path x, y, z, u, v in G such that:

- $z \neq x, v$
- x, v have infinite degree
- u has degree 2
- v has exactly k neighbours of finite degree which have degrees $1, 2, ..., k$

First suppose that $x \in G(\alpha_0, 13^n 2)$. Then $x = (\beta_0, ..., \beta_n)$, where $\beta_n < \alpha_0$. Let

$$\begin{aligned} y &= (\beta_1, ..., \beta_n, \alpha_0 + k - 2) \\ z &= (\beta_2, ..., \beta_n, \alpha_0 + k - 2, \alpha_0 + k - 1) \\ u &= (1, \beta_2, ...\beta_n, \alpha_0 + k - 2) \\ v &= (0, 1, \beta_2, ..., \beta_n) \end{aligned}$$

We see that all finite degree neighbours of v are of the form $(1, \beta_2, ..., \beta_n, \alpha_0 + i)$ where $i = 0, ..., k-1$, meaning these neighbours have degree $1, 2, ..., k$.

Now suppose that x, y, z, u, v are as stated. v must be of the form $(\beta_0, ..., \beta_n)$ for some $\beta_0, ..., \beta_n$. Neighbours of v are either of the form $(\beta, \beta_0, ..., \beta_{n-1})$, or of the form $(\beta_1, ..., \beta_n, \gamma)$. Thus if these have finite degree then $\beta < \omega$ and $\beta_{n-1} \geq \alpha_0$, OR $\beta_1 < \omega$ and $\gamma \geq \alpha_0$.

As v has infinite degree, we must either have $\beta_0 \geq \omega$ or $\beta_n < \alpha_0$.

If $\beta_0 \geq \omega$, then neighbours of v of finite degree are of the form $(\beta, \beta_0, ..., \beta_{n-1})$. But β could have infinitely many values, and we know v only has finitely many neighbours of finite degree.

Thus $\beta_0 < \omega$, and so $\beta_n < \alpha_0$.

The neighbours of v of finite degree are $(\beta_1, ..., \beta_n, \alpha_0 + i)$, and we know $\beta_1 < \omega$. Thus to fulfil the stipulations, we must have $\beta_1 = 1$ (as otherwise there will be a neighbour of v of finite degree greater than k).

Thus $v = (0, 1, \beta_2, ..., \beta_n)$, where $\beta_n < \alpha_0$.

Thus $u = (1, \beta_2, ..., \beta_n, \alpha_0 + k - 2)$, which is where we use $k \geq 2$.

Since $v \neq z$, we must have $z = (\beta_2, ..., \beta_n, \alpha_0 + k - 2, \alpha_0 + k - 1)$.

Then $y = (\gamma_1, \beta_2, ..., \beta_n, \alpha_0 + k - 2)$, for some $\gamma_1 < \beta_2$.

And so, since $x \neq z$, we must have $x = (\gamma_0, \gamma_1, \beta_2, ..., \beta_n)$, where $\beta_n < \alpha_0$. Thus $x \in G(\alpha_0, 13^n 2)$, as desired.

<u>**Case 2:** $k = 1$</u>

We consider a maximal co-clique C in G such that its members have pairwise disjoint neighbour sets and the only vertices of C having finite degree are isolated. The aim is to recover the set of all vertices beginning with 0, which clearly satisfies these properties.

Now the isolated vertices in this case have the form $(0, \beta_1, ..., \beta_{n-1}, \alpha_0)$. These must all clearly lie in C, so we concentrate on the other members of C.

First we show that if $v = (\beta_0, ..., \beta_n) \in C$ is not isolated, then $\beta_0 = 0$ or $\beta_n = \alpha_0$. If not, then $0 < \beta_0$, and $(0, \beta_1, ..., \beta_n)$ shares a neighbour with v, namely $(\beta_1, ..., \beta_n, \beta_n + 1)$ (here using $\beta_n < \alpha_0$). It follows that $(0, \beta_1, ..., \beta_n)$ does not lie in C. Since C is a maximal co-clique, $(0, \beta_1, ..., \beta_n)$ must be joined to some member of C, which must have the form $(\beta_1, ..., \beta_n, \beta)$. However, this is joined to v, so this gives a contradiction.

Let $C_1 = \{(0, \beta_1, ..., \beta_n) \in C : \beta_n < \alpha_0\}$ and $C_2 = \{(\beta_1, ..., \beta_n, \alpha_0) \in C : 0 < \beta_1\}$. We have therefore shown that C is the (disjoint) union of C_1, C_2, together with the set of isolated vertices.

Given $\beta_1, ..., \beta_n$ such that $\beta_1 > 0$ and $\beta_n < \alpha_0$ we see that exactly one of $(0, \beta_1, ..., \beta_n)$ and $(\beta_1, ..., \beta_n, \alpha_0)$ lies in C. For if the first does not lie in C, then as C is a maximal co-clique, it must be joined to a member of C, but this can only be $(\beta_1, ..., \beta_n, \alpha_0)$. The two points cannot both lie in C since they are joined.

We now show that $C_2 = \emptyset$. If not, it has a member $(\beta_1, ..., \beta_n, \alpha_0)$ say. Since C has no non-isolated vertices of finite degree, $\beta_1 \geq \omega$. Now $(1, \beta_1, ..., \beta_n)$ is a neighbour of $(\beta_1, ..., \beta_n, \alpha_0)$, so is not a neighbour of any member of C_1. Hence $(0, 1, \beta_1, ..., \beta_{n-1})$ doesn't lie in C_1. Therefore by the above calculation, $(1, \beta_1, ..., \beta_n, \alpha_0)$ lies in C_2. But this has finite degree, and is non-isolated, so this contradicts the stipulation on C.

Hence $C_2 = \emptyset$. We deduce that $C = \{(0, \beta_1, ..., \beta_n) : \beta_n < \alpha_0\}$, and this fulfils all the rules for our set.

We can now recover $G(\alpha_0, 13^n2)$ inside $G(\alpha_0 + k, 13^n2)$ as follows: starting this time with a maximal co-clique whose neighbour sets are pairwise disjoint and which is disjoint from C_1, the above calculations show that this must be equal to the set of all vertices ending in α_0, and the complement of this set in $G(\alpha_0 + k, 13^n2)$ gives us all vertices NOT ending in α_0, which is precisely $G(\alpha_0, 13^n2)$.

We can thus isolate $G(\alpha_0, 13^n2)$ in $G(\alpha_0 + k, 13^n2)$, and so determine α using Theorem 5.7.

□

Theorem 5.16. Let $\alpha = \alpha_0 + k$ where $\alpha_0 \neq 0$ is a limit ordinal and $k < \omega$. Then α is 2nd-order interpretable inside $G = G(\alpha, 11322)$ if $k \geq 3$. In the case where $k = 1$ or 2, we have $G(\alpha_0 + 1, 11322) \cong G(\alpha_0 + 2, 11322)$.

Proof. We will start with the case $k \geq 3$.

Consider the induced subgraph $G_{fin} \subseteq G$ consisting of all vertices V_{fin} of finite degree, i.e. vertices of the form $a = (a_1, a_2, a_3)$ where $a_1 < \omega$ and $a_3 \geq \alpha_0$. Now, $N(a)$ is the union of two subsets:

$$N_\ell(a) = \{(b_1, b_2, b_3) : b_3 = a_1\}$$
$$N_r(a) = \{(b'_1, b'_2, b'_3) : a_3 = b'_1\}$$

Note that $N_\ell(a)$ is empty for $a_1 = 0, 1$, and $N_r(a)$ is empty for $a_3 = \alpha_0 + k - 2, \alpha_0 + k - 1$.

Identify two members of G_{fin} if they have the same neighbour sets by the relation $\sim$. $\sim$-classes now correspond to pairs (a_1, a_3) such that $2 \leq a_1 < \omega$ or $a_1 = 0_1$, and $\alpha_0 \leq a_3 \leq \alpha_0 + k - 3$ or $a_3 = \alpha_0 + k - 2_{\alpha_0+k-1}$, where for (a_1, a_3) and (a'_1, a'_3), corresponding neighbour sets have a trivial (empty) intersection unless $a_1 = a'_1$ or $a_3 = a'_3$. Note that this holds since $a_1, a'_1 < \omega$ and $a_3, a'_3 \geq \alpha_0$ - otherwise we could have had $a_1 < a_3 < a'_1 < a'_3$ for example. Here we use 0_1 to mean "0 or 1" and $\alpha_0 + k - 2_{\alpha_0+k-1}$ to mean "$\alpha_0 + k - 2$ or $\alpha_0 + k - 1$".

Turn $G_{fin}(\sim)$ into a graph with relation E and say that $(a_1, a_3)E(a'_1, a'_3)$ if their neighbour sets are unequal, but intersect non-trivially.

Then $(a_1, a_3)E(a'_1, a'_3)$ iff $a_1 = a'_1$ or $a_3 = a'_3$ or $a_1, a'_1 \in \{0, 1\}$ or $a_1, a'_1 \in \{\alpha_0 + k - 2, \alpha_0 + k - 1\}$. Thus a clique under E is a set of vertices (a_1, b) for fixed a_1, or a set of vertices (b, a_3) for fixed a_3. If the clique is maximal, b takes all possible values.

Now we can verify that for a fixed $x < \omega$, $X = \{(x, a_3) : a_3 \geq \alpha_0\}$, and for a fixed $y \geq \alpha_0$, $Y = \{(a_1, y) : a_1 < \omega\}$ are maximal cliques under the relation E, and X is finite of size k. Thus we can recover k as the unique size of such a maximal clique which is finite (and non-trivial).

Once we have recovered k, we would like to 'recognise' $G(\alpha_0, 11322)$ inside $G(\alpha_0+k, 11322)$.

Now expand the relation $\sim$ to all of G in the natural way, and similarly expand the relation E to all of G. Consider an infinite maximal clique under relation E in G_{fin}, i.e. of the form $Y = \{(a_1, y) : a_1 < \omega, \alpha_0 < y \leq \alpha_0 + k - 3\}$ for fixed y, and expand this to a maximal clique Y' under relation E in G. Then Y' will be of the form $\{(a_1, y) : a_1 < \omega\}$, where y is fixed and $\alpha_0 \leq y \leq \alpha_0 + k - 3$. Remove every such clique Y' from G. We are now left with $G(\alpha_0 + 2, 11322)$.

Now we turn our attention to the case where $k = 1$ or 2.

Claim: $G(\alpha_0 + 1, 11322) \cong G(\alpha_0 + 2, 11322)$.

Proof of Claim. Let $G_1 = G(\alpha_0 + 1, 11322)$ and $G_2 = G(\alpha_0 + 2, 11322)$. We would like to find an isomorphism θ from G_1 to G_2. Let θ fix all points of $G(11322, \alpha_0)$. The remaining points in G_1 are of the form (a, b, α_0) and in G_2 are of the form (a, b, α_0) or

$(a, b, \alpha_0 + 1)$. Now all of these points for fixed a have the same neighbour sets, namely (a_1, a_2, a) for $a_1 < a_2 < a$ (which will be empty if $a = 0$ or 1). Since b can take infinitely many values, we find ANY bijection θ taking $\{(a, b, \alpha_0) : a < b < \alpha_0\}$ to $\{(a, b, \alpha_0) : a < b < \alpha_0\} \cup \{(a, b, \alpha_0 + 1) : a < b < \alpha_0 + 1\}$ for each a, and this will be an isomorphism.

Finally, we determine α_0 from $G(\alpha_0 + 1, 11322)$.

In the following, we use the notation 0_1 to indicate "0 or 1" - this is because 0 and 1 effectively act as a single point here. Thus for example, $\{(0_1, \gamma_1, \gamma_2) : \beta \leq \gamma_2 \leq \alpha_0\}$ really means $\{(0, \gamma_1, \gamma_2) : \beta \leq \gamma_2 \leq \alpha_0\} \cup \{(1, \gamma_1, \gamma_2) : \beta \leq \gamma_2 \leq \alpha_0\}$.

First we show that any maximal co-clique whose neighbour sets are pairwise either equal or entirely disjoint is of the form $\{(0_1, \gamma_1, \gamma_2) : 2 \leq \gamma_2 \leq \alpha_0\}$ or $\{(\delta_1, \delta_2, \alpha_0) : \delta_1 < \alpha_0\}$. We can see that this is a co-clique, because no two vertices of the form $(0_1, \gamma_1, \gamma_2)$ are joined, nor are any two of the form $(\delta_1, \delta_2, \alpha_0)$. Each of them is maximal, as we cannot add any vertex (x_1, x_2, x_3) to the first and retain its being a co-clique, and similarly for the second. The neighbour sets are either equal or pairwise disjoint, as any neighbour of $(0_1, \gamma_1, \gamma_2)$ must begin with γ_2, and so two vertices of the form $(0_1, \gamma_1, \gamma_2), (0_1, \gamma_1', \gamma_2)$ have identical neighbour sets, whereas two vertices of the form $(0_1, \gamma_1, \gamma_2), (0_1, \gamma_1, \gamma_2')$ have disjoint neighbour sets, and similarly in the second case.

We will now show that any maximal co-clique S with neighbour sets either equal or pairwise disjoint is equal to one of these two sets S. Let $(x_1, x_2, x_3) \in S$ with $x_1 \geq 2$. Then $(0_1, x_2, x_3)$ is not joined to any member of S, since if it was, it would have to be (x_3, y_1, y_2), but this is joined to (x_1, x_2, x_3). Then $(0_1, x_2, x_3) \in S$ by maximality. If $x_3 < \alpha_0$, then the neighbour sets of $(0_1, x_2, x_3)$ and (x_1, x_2, x_3) are not disjoint, and so they must be equal; but this is not possible since $x_1 \geq 2$ and so $(0, 1, x_1)$ is a neighbour of (x_1, x_2, x_3) but not $(0_1, x_2, x_3)$. We deduce that $x_3 = \alpha_0$. To sum up, any $(x_1, x_2, x_3) \in S$ has $x_1 = 0_1$ or $x_3 = \alpha_0$ (or both).

Hence S is the union of two subsets, those tuples beginning with 0 or 1, and those ending in α_0. In fact, for every $\beta < \alpha$, either $(0_1, b, \beta)$ or (β, b', α_0) lies in S, for some b, b'. To see this, suppose $(0, b, \beta)$ does not lie in S. Then by maximality, it must have a neighbour in S, and this can only be of the form (β, b', α_0).

Now, α can be written as the (not necessarily disjoint) union of two subsets A and B such that $S = \{(0, x, a) : a \in A\} \cup \{(b, y, \alpha_0) : b \in B\}$. Since the neighbour sets of S are pairwise disjoint, if $(0, x, a)$ and (b, y, α_0) both lie in S then $b \leq a + 1$, as otherwise $(a, a+1, b)$ would be a common neighbour. Thus A is closed upwards, and B is closed downwards. Since $\alpha_0 \in A$, A is nonempty. Let β be its least member. If $2 < \beta < \alpha_0$, consider $(2, \beta+1, \beta+2)$. Since S is a maximal co-clique, this is joined to a member of S. If it is joined to $(0, x, a)$, then $a = 2 \in A$, contrary to β being the least member of A. If it is joined to (b, y, α_0), then $b = \beta + 2$, contrary to the assertion that $b \leq a + 1$ (since β is the least member of A).

This proves the **Claim**.

We can distinguish between these sets, $\{(0_1, \gamma_1, \gamma_2) : 1 \leq \gamma_2 \leq \alpha_0\}$ and $\{(\delta_1, \delta_2, \alpha_0) : \delta_1 < \alpha_0\}$, since$\{(\delta_1, \delta_2, \alpha_0) : \delta_1 < \alpha_0\}$, contains vertices of finite degree which are not isolated points, and $\{(0_1, \gamma_1, \gamma_2) : 1 \leq \gamma_2 \leq \alpha_0\}$ does not.

Now we remove $\{(\delta_1, \delta_2, \alpha_0) : \delta_1 < \alpha_0\}$ from $G(\alpha_0 + 1, 11322)$, leaving us with the graph $G(\alpha_0, 11322)$, and so we can determine α_0 by Theorem 5.10. □

One can extend this theorem fairly easily:

Theorem 5.17. Let $\alpha = \alpha_0 + k$ where $\alpha_0 \neq 0$ is a limit ordinal and $k < \omega$. Then α is 2nd-order interpretable inside $G = G(\alpha, 1^n 32^n)$ if $k \geq n + 1$. For all $k, k' \leq n$, we have $G(\alpha_0 + k, 1^n 32^n) \cong G(\alpha_0 + k', 1^n 32^n)$.

Proof. We will start with the case $k \geq n + 1$.

Consider the induced subgraph $G_{fin} \subseteq G$ consisting of all vertices V_{fin} of finite degree, i.e. vertices of the form $a = (a_1, ..., a_{n+1})$ where $a_1 < \omega$ and $a_{n+1} \geq \alpha_0$. Now, $N(a)$ is the union of two subsets:

$$N_\ell(a) = \{(b_1, ..., b_{n+1}) : b_{n+1} = a_1\}$$
$$N_r(a) = \{(b'_1, ..., b'_{n+1}) : a_{n+1} = b'_1\}$$

Note that $N_\ell(a)$ is empty for $a_1 = 0, ..., n - 1$, and $N_r(a)$ is empty for $a_{n+1} = \alpha_0 + k - n, ..., \alpha_0 + k - 1$.

We would like to identify the set of all vertices $a = (a_1, ..., a_{n+1}) \in G_{fin}$ such that exactly one of $N_\ell(a)$ and $N_r(a)$ is nonempty. Let us call the sets of all such vertices G^ℓ_{fin} and G^r_{fin} respectively.

First note that if both $N_\ell(a) = N_r(a) = \emptyset$, then a is an isolated point.

Excluding isolated points, identify two members of G_{fin} if they have the same neighbour sets by the relation $\sim$.

We will show that

$$a \in G^\ell_{fin} \cup G^r_{fin} \Leftrightarrow \text{every pair of neighbours } n_1, n_2 \text{ of } a$$
$$\text{have a common neighbour } v \text{ such that } v \not\sim a$$

$\Rightarrow$: Since either $N_\ell(a)$ or $N_r(a)$ is empty, any two neighbours of a must either begin with the same point a_{n+1} or end with the same point a_1. Without loss of generality assume the former. Then take v to be any vertex ending in a_{n+1} and not beginning in a_1, and such that $n \leq v_1 < \omega$. Note that in the latter case the argument relies on $n \geq 2$.

$\Leftarrow$: Assume $a \notin G^\ell_{fin} \cup G^r_{fin}$, i.e. both $N_\ell(a)$ and $N_r(a)$ are nonempty, and take $n_1 \in N_\ell(a)$ and $n_2 \in N_r(a)$. Then every common neighbour v of n_1 and n_2 must begin with a_1 and end with a_{n+1}, i.e. $v \sim a$. Thus not every pair of neighbours of a have a common neighbour v such that $v \not\sim a$.

Now, there is a natural restriction of $\sim$ to $G^{\ell}_{fin} \cup G^{r}_{fin}$, with $\sim$-classes corresponding to pairs (a_1, a_{n+1}) with either $n \leq a_1 < \omega$ or $\alpha_0 \leq a_{n+1} < \alpha_0 + k - n$.

For (a_1, a_{n+1}) and (a'_1, a'_{n+1}) respectively, corresponding neighbour sets have a trivial (empty) intersection unless $a_1 = a'_1$ or $a_{n+1} = a'_{n+1}$. Note that this holds since $a_1, a'_1 < \omega$ and $a_{n+1}, a'_{n+1} \geq \alpha_0$ - otherwise we could have had $a_1 < a_{n+1} < a'_1 < a'_{n+1}$ for example.

Turn $G_{fin}(\sim)$ into a graph with relation E and say that $(a_1, a_{n+1})E(a'_1, a'_{n+1})$ if their neighbour sets are unequal, but intersect non-trivially.

Then $(a_1, a_{n+1})E(a'_1, a'_{n+1})$ iff $a_1 = a'_1$ or $a_{n+1} = a'_{n+1}$, but not both. Thus a clique under E is a set of vertices (a_1, b) for fixed a_1, or a set of vertices (b, a_{n+1}) for fixed a_{n+1}. If the clique is maximal, b takes all possible values.

Now let $a \in G^{\ell}_{fin} \cup G^{r}_{fin}$, and let b be a neighbour of a. Then the set of all neighbours v of b lying in G_{fin} forms a maximal clique under the relation E. We can see this as if $a \in G^{\ell}_{fin}$, then $n \leq a_1 < \omega$ and $a_{n+1} \geq \alpha_0 + k - n$, and so $b_{n+1} = a_1$. Every v either has $v_{n+1} = b_1$ or $v_1 = b_{n+1}$, and so every such $v \in G_{fin}$ must start with $b_{n+1} = a_1$. This is obviously a clique, and since we are considering the set of all neighbours of b, this clique is maximal. The case where $a \in G^{r}_{fin}$ is similar.

It is easy to see that if $a \in G^{\ell}_{fin}$, then the clique is finite of size k, whereas if a is in G^{r}_{fin}, then the clique is infinite. Thus k can be recovered as the unique size of such a maximal clique which is finite.

Once we have recovered k, we would like to 'recognise' $G(\alpha_0, 1^n 32^n)$ inside $G(\alpha_0 + k, 1^n 32^n)$.

Now expand the relation $\sim$ to all of G in the natural way, and similarly expand the relation E to all of G. Consider an infinite maximal clique under relation E in G^{r}_{fin}, i.e. of the form $Y = \{(a_1, y) : a_1 < \omega\}$ where y is fixed such that $\alpha_0 \leq y \leq \alpha_0 + k - n - 1$, and expand this to a maximal clique Y' under relation E in G. Then Y' will be of the form $\{(a_1, y) : a_1 < y\}$, where $\alpha_0 \leq y \leq \alpha_0 + k - n - 1$ is fixed. Remove every such clique Y' from G. We are now left with $G(\alpha_0 + n, 1^n 32^n)$.

Repeat this m times, where $k \equiv k' \bmod n$ (and $mn + k' = k$), until we are left with $G(\alpha_0 + k', 1^n 32^n)$.

We now turn our attention to the case where $k \leq n$.

Claim: $G(\alpha_0 + 1, 1^n 32^n) \cong G(\alpha_0 + 2, 1^n 32^n) \cong ... \cong G(\alpha_0 + n, 1^n 32^n)$.

Proof of Claim. Let $G_1 = G(\alpha_0 + 1, 1^n 32^n)$ and $G_2 = G(\alpha_0 + k, 1^n 32^n)$ for some $1 < k \leq n$. We would like to find an isomorphism θ from G_1 to G_2. Let θ fix all the points of $G(\alpha_0, 1^n 32^n)$. The remaining points in G_1 are of the form $(a, b_2, ..., b_n, \alpha_0)$, and in G_2 are of the form $(a, b_2, ..., b_n, \alpha_0 + j)$, where $0 \leq j < k$. Now all of these points for fixed

a have the same neighbour sets, namely $(a_1, ..., a_n, a)$ for $a_n < a$. Since the $b_i s$ can take infinitely many values, we can find ANY bijection θ taking $\{(a, b_2, ..., b_n, \alpha_0) : b_n < \alpha_0\}$ to $\{(a, b_2, ..., b_n, \alpha_0 + j) : b_n < \alpha_0, j \leq k\}$ for each a, and this will be an isomorphism.

Finally, we determine α_0 from $G(\alpha_0 + 1, 1^n 32^n)$.

We start by identifying the set $\{(x, y_2, ..., y_n, \alpha_0) : y_n < \alpha_0\}$.

In the following, we use the notation 0_{n-1} to indicate "0 or 1" - this is because the set $\{0, 1, ..., n-1\}$ effectively acts as a single point here. Thus for example, $\{(0_n, \gamma_1, \gamma_2) : \beta \leq \gamma_2 \leq \alpha_0\}$ really means $\{(0, \gamma_1, \gamma_2) : \beta \leq \gamma_2 \leq \alpha_0\} \cup \{(1, \gamma_1, \gamma_2) : \beta \leq \gamma_2 \leq \alpha_0\} \cup ... \cup \{(n-1, \gamma_1, \gamma_2) : \beta \leq \gamma_2 \leq \alpha_0\}$.

First we show that any maximal co-clique whose neighbour sets are pairwise either completely equal or entirely disjoint is of the form $\{(0_{n-1}, \gamma_1, ..., \gamma_n) : n \leq \gamma_n \leq \alpha_0\}$ or $\{(\delta_1, ..., \delta_n, \alpha_0) : \delta_1 < \alpha_0\}$. We can see that this is a co-clique, because no two vertices of the form $(0_{n-1}, \gamma_1, ..., \gamma_n)$ are joined, nor are any two of the form $(\delta_1, ..., \delta_n, \alpha_0)$. Each of them is maximal, as we cannot add any vertex $(x_1, x_2, ..., x_{n+1})$ to the first and retains its being a co-clique, and similarly for the second. The neighbour sets are either equal or pairwise disjoint, as any neighbour of $(0_{n-1}, \gamma_1, ..., \gamma_n)$ must begin with γ_n, and so two vertices of the form $(0_{n-1}, \gamma_1, ..., \gamma_n), (0_{n-1}, \gamma_1', ..., \gamma_n)$ have identical neighbour sets, whereas two vertices of the form $(0_{n-1}, \gamma_1, ..., \gamma_n), (0_{n-1}, \gamma_1, ..., \gamma_n')$ have disjoint neighbour sets, and similarly in the second case.

We will now show that any maximal co-clique S with neighbour sets either equal or pairwise disjoint is equal to one of these two sets S. Let $(x_1, ..., x_{n+1}) \in S$ with $x_1 \geq n$. Then $(0_{n-1}, x_2, ..., x_{n+1})$ is not joined to any member of S, since if it was, it would have to be $(x_{n+1}, y_1, ..., y_n)$, but this is joined to $(x_1, ..., x_{n+1})$. Then $(0_1, x_2, ..., x_{n+1}) \in S$ by maximality. If $x_{n+1} < \alpha_0$, then the neighbour sets of $(0_{n-1}, x_2, ..., x_{n+1})$ and $(x_1, x_2, ..., x_{n+1})$ are not disjoint, and so they must be equal; but this is not possible since $a \geq n$ and so $(0, 1, ..., n-1, x_1)$ is a neighbour of $(x_1, x_2, ..., x_{n+1})$ but not $(0_{n-1}, x_2, ..., x_{n+1})$. We deduce that $x_{n+1} = \alpha_0$. To sum up, any $(x_1, ..., x_{n+1}) \in S$ has $x_1 = 0$ or $x_{n+1} = \alpha_0$ (or both).

Hence S is the union of two subsets, those tuples beginning with 0 or 1, and those ending in α_0. In fact, for every $\beta < \alpha$, either $(0_{n-1}, x_1, ..., x_{n-1}, \beta)$ or $(\beta, y_1, ..., y_{n-1}, \alpha_0)$ lies in S, for some $x_1, ..., x_{n-1}, y_1, ..., y_{n-1}$. To see this, suppose $(0, x_1, ..., x_{n-1}, \beta)$ does not lie in S. Then by maximality, it must have a neighbour in S, and this can only be of the form $(\beta, y_1, ..., y_{n-1}, \alpha_0)$.

Now, α can be written as the (not necessarily disjoint) union of two subsets A and B such that $S = \{(0_{n-1}, x_1, ..., x_{n-1}, a) : a \in A\} \cup \{(b, y_1, ..., y_{n-1}, \alpha_0) : b \in B\}$. Since the neighbour sets of S are pairwise disjoint, if $(0_{n-1}, x_1, ..., x_{n-1}, a)$ and $(b, y_1, ..., y_{n-1}, \alpha_0)$ both lie in S then $b \leq a + n$, as otherwise $(a, a+1, ..., a_n, b)$ would be a common neighbour. Thus A is closed upwards, and B is closed downwards. Since $\alpha_0 \in A$, A is nonempty. Let β be its least member. If $n < \beta < \alpha_0$, consider $(n, \beta+1, \beta+2, ..., \beta+n)$. Since S is a maximal co-clique, this is joined to a member of S. If it is joined to $(0_{n-1}, x_1, ..., x_{n-1}, a)$, then $a = n \in A$, contrary to β being the least member of A. If it is joined to $(b, y_1, ..., y_{n-1}, \alpha_0)$, then $b = \beta + n$, contrary to the assertion that $b \leq a + 1$ (since β is the least member of A). This proves the **Claim**.

We can distinguish between these sets, $\{(0_{n-1}, \gamma_1, ..., \gamma_n) : n \leq \gamma_n \leq \alpha_0\}$ or $\{(\delta_1, ..., \delta_n, \alpha_0) : \delta_1 < \alpha_0\}$, since $\{(\delta_1, ..., \delta_n, \alpha_0) : \delta_1 < \alpha_0\}$, contains vertices of finite degree which are not isolated points, and $\{(0_{n-1}, \gamma_1, ..., \gamma_n) : n \leq \gamma_n \leq \alpha_0\}$ does not.

Now we remove $\{(\delta_1, ..., \delta_n, \alpha_0) : \delta_1 < \alpha_0\}$ from $G(\alpha_0 + 1, 1^n 32^n)$, leaving us with the graph $G(\alpha_0, 1^n 32^n)$, and so we can determine α_0 by Theorem 5.10. □

Theorem 5.18. Let $\alpha = \alpha_0 + k$ be a successor ordinal with α_0 a limit and finite $k > n$. Then α is interpretable in $G = G(\alpha, 1^n 3^m 2^n)$ for all $m \leq n$.

Proof. Consider the induced subgraph $G_{fin} \subseteq G$ consisting of all vertices V_{fin} of finite degree, i.e. vertices of the form $a = (x_1, ..., x_m, y_1, ..., y_{n-m}, z_1, ..., z_m)$ where $x_1, ..., x_m < \omega$ and $z_1, ..., z_m \geq \alpha_0$. Now, $N(a)$ is the union of two subsets:

$$N_\ell(a) = \{(v_1, ..., v_n, x_1, ..., x_m) : v_1, ..., v_n < x_1\}$$
$$N_r(a) = \{(z_1, ..., z_m, w_1, ..., w_n) : w_1, ..., w_n > z_m\}$$

Note that $N_\ell(a)$ is empty for $x_1 = 0, ..., n-1$, and $N_r(a)$ is empty for $z_m = \alpha_0 + k - n, ..., \alpha_0 + k - 1$.

Identify two members of G_{fin} if they have the same neighbour sets by the relation $\sim$. $\sim$-classes now correspond to pairs $(x_1, ..., x_m, z_1, ..., z_m)$ such that $n - 1 \leq x_1, ..., x_m < \omega$ and $\alpha_0 \leq z_1, ..., z_m \leq \alpha_0 + k - n$. For $(x_1, ..., x_m, z_1, ..., z_m)$ and $(x'_1, ..., x'_m, z'_1, ..., z'_m)$, corresponding neighbour sets have a trivial (empty) intersection unless $x_1 = x'_1, x_2 = x'_2, ..., x_m = x'_m$ or $z_1 = z'_1, z_2 = z'_2, ..., z_m = z'_m$. Note that this holds since $x_1, ..., x_m, x'_1, ..., x'_m < \omega$ and $z_1, ..., z_m, z'_1, ..., z'_m \geq \alpha_0$ - otherwise we could have had $x_1, ..., x_m < z_1, ..., z_m < x'_1, ..., x'_m < z'_1, ..., z'_m$ for example.

Turn $G_{fin}(\sim)$ into a graph with relation E and say that

$$(x_1, ..., x_m, z_1, ..., z_m) E (x'_1, ..., x'_m, z'_1, ..., z'_m)$$

if their neighbour sets are unequal, but intersect non-trivially.

Now we can verify that for fixed $v_1, ..., v_m < \omega$, $V = \{(v_1, ..., v_m, z_1, ..., z_m) : z_1 \geq \alpha_0\}$, and for fixed $w_1, ..., w_m \geq \alpha_0$, $W = \{(x_1, ..., x_m, w_1, ..., w_m) : x_m < \omega$ are maximal cliques under the relation E, and V is finite of size $\binom{k}{m}$. Thus we can recover k from the unique size of such a maximal clique which is finite (and non-trivial). □

We now turn our attention to the case where $k \le n$.

Claim: $G(\alpha_0+1, 1^n3^m2^n) \cong G(\alpha_0+2, 1^n3^m2^n) \cong ... \cong G(\alpha_0+k, 1^n3^m2^n)$.

Proof of Claim. Let $G_x = G(\alpha_0+x, 1^n3^m2^n)$. We would like to find an isomorphism θ from G_x to G_{x+1}. Let θ fix all points of $G(\alpha_0+x-1, 1^n3^m2^n)$. The remaining points in G_x are of the form $(a_1, ..., a_m, b_1, ..., b_{n-1}, \alpha_0+x)$ and in G_{x+1} are of the form $(a_1, ..., a_m, b_1, ..., b_{n-1}, \alpha_0+x)$ or $(a_1, ..., a_m, b_1, ..., b_{n-1}, \alpha_0+x+1)$. Now all of these points for fixed $a_1, ..., a_m$ have the same neighbour sets, namely $(y_1, y_2, ..., y_n, a_1, ..., a_m)$ for $y_1 < ... < y_n < a_1, ..., a_m$ (which will be empty if $a_1 \le n$). Since the b_is can take infinitely many values (or none!), we again find ANY map θ taking $\{(a_1, ..., a_m, b_1, ..., b_{n-1}, \alpha_0) : a_m < b_1 < ... < b_n < \alpha_0\}$ to $\{(a_1, ..., a_m, b_1, ..., b_{n-1}, \alpha_0) : a_m < b_1 < ... < b_{n-1} < \alpha_0\} \cup \{(a_1, ..., a_m, b_1, ..., b_{n-1}, \alpha_0+x) : a_m < b_1 < ... < b_{n-1} < \alpha_0+x+1\}$ for each $a_1, ..., a_m$, and this will be an isomorphism.

Lemma 5.19. Let $\alpha = \alpha_0 + k$ be a successor ordinal with α_0 a limit and $k < \omega$. Then for $n < m$, any two points in $G = G(\alpha, 1^n3^m2^n)$ which share a neighbour must be either of the form

$$\begin{array}{cccccccccccccc}
(x_1, & ... & x_n, & y_1, & ... & y_n, & y_{n+1}, & y_{n+2}, & ... & y_m) & & & & \\
 & & & & & & \| & \| & & \| & & & & \\
 & & & & & & (y'_1, & y'_2, & ... & y'_{m-n}, & y'_{m-n+1}, & ... & y'_m, & z_1, \; ... \; z_n)
\end{array}$$

or of the form

$$\begin{array}{ccccc}
(y_1, & ... & y_m, & x_1, & ... \; x_n) \\
\| & & \| & & \\
(y_1, & ... & y_m, & z_1, & ... \; z_n)
\end{array}$$

for $y_1 \ge n$, or of the form

$$\begin{array}{ccccc}
(x_1, \; ... & x_n, & y_1, & ... & y_m) \\
 & & \| & & \| \\
(z_1, \; ... & z_n, & y_1, & ... & y_m)
\end{array}$$

for $y_m \le \alpha_0 + k - n - 1$.

Furthermore, the neighbour in the first case is the point $(y_1, ..., y_n, y'_1, ..., y'_m)$, the neighbours in the second case are the points $(w_1, ..., w_n, y_1, ..., y_m)$ for $w_1, ..., w_n < y_1$, and the neighbours in the third case are the points $(y_1, ..., y_m, v_1, ..., v_n)$ for $v_1, ..., v_n > y_m$.

Proof. Let $x = (x_1, ..., x_n, y_1, ..., y_m)$ be a vertex in G. Then any neighbour of x must either be of the form

$$\begin{array}{cccccc}
(x_1, \; ... & x_n, & y_1, & ... & y_m) & \\
 & & \| & & \| & \\
 & & (y_1, & ... & y_m, & v_1, \; ... \; v_n)
\end{array}$$

or of the form

$$\begin{array}{ccccccccccc}
 & & & (x_1, & \dots & x_m, & \dots & x_n, & y_1, & \dots & y_m) \\
 & & & \| & & \| & & & & & \\
(v_1, & \dots & v_n, & x_1, & \dots & x_m) & & & & &
\end{array}$$

Without loss of generality assume the former, and let $n(x) = (y_1, ...y_m, v_1, ...v_n)$. Now, any neighbour of $n(x)$ must again be of one of the forms above.

Case 1:

$$\begin{array}{cccccccccccc}
(y_1, & \dots & y_n, & y_{n+1}, & \dots & y_m, & v_1, & \dots & v_n) & & & \\
 & & & \| & & \| & \| & & \| & & & \\
 & & & (y_{n+1}, & \dots & y_m, & v_1, & \dots & v_n, & z_1, & \dots & z_n)
\end{array}$$

Relabelling, we get:

$$\begin{array}{cccccccccccc}
(y_1, & \dots & y_n, & y_{n+1}, & \dots & y_m, & v_1, & \dots & v_n) & & & \\
 & & & \| & & \| & \| & & \| & & & \\
 & & & (y'_1, & \dots & y'_{m-n}, & y'_{m-n+1}, & \dots & y'_m, & z_1, & \dots & z_n)
\end{array}$$

and thus the two points sharing a neighbour are of the form

$$\begin{array}{ccccccccccccccc}
(x_1, & \dots & x_n, & y_1, & \dots & y_n, & y_{n+1}, & y_{n+2}, & \dots & y_m) & & & & & \\
 & & & & & & \| & \| & & \| & & & & & \\
 & & & & & & (y'_1, & y'_2, & \dots & y'_{m-n}, & y'_{m-n+1}, & \dots & y'_m, & z_1, \dots & z_n)
\end{array}$$

with neighbour $(y_1, ...y_m, z_1, ...z_n)$.

Case 2:

$$\begin{array}{ccccccccc}
 & & & (y_1, & \dots & y_m, & v_1, & \dots & v_n) \\
 & & & \| & & \| & & & \\
(z_1, & \dots & z_n, & y_1, & \dots & y_m) & & &
\end{array}$$

Then the two points sharing a neighbour are of the form

$$\begin{array}{cccccc}
(x_1, & \dots & x_n, & y_1, & \dots & y_m) \\
 & & & \| & & \| \\
(z_1, & \dots & z_n, & y_1, & \dots & y_m)
\end{array}$$

with common neighbour $(y_1, ...y_m, z_1, ...z_n)$. Note that we must have $y_m \leq \alpha_0 + k - n - 1$, as otherwise we could not have distinct $z_1, ..., z_n > y_m$.

The remaining case is similar to case 2 except we start by assuming

$$\begin{array}{ccccccccccc}
 & & & (x_1, & \dots & x_m, & \dots & x_n, & y_1, & \dots & y_m) \\
 & & & \| & & \| & & & & & \\
(v_1, & \dots & v_n, & x_1, & \dots & x_m) & & & & &
\end{array}$$

and therefore end up with

$$\begin{array}{cccccc}
(y_1, & \dots & y_m, & x_1, & \dots & x_n) \\
\| & & \| & & & \\
(y_1, & \dots & y_m, & z_1, & \dots & z_n)
\end{array}$$

for $y_1 \geq n$, $\square$

Lemma 5.20. Let $\alpha = \alpha_0 + k$ be a successor ordinal with α_0 a limit and $k < \omega$. Then any two points in $G = G(\alpha, 1^n 3^m 2^n)$ which share *exactly one* neighbour must be either of the form

$$\begin{array}{cccccccccccccccc}
(x_1, & \dots & x_n, & y_1, & \dots & y_n, & y_{n+1}, & y_{n+2}, & \dots & y_m) & & & & & & \\
 & & & & & & \| & \| & & \| & & & & & & \\
 & & & & & & (y'_1, & y'_2, & \dots & y'_{m-n}, & y'_{m-n+1}, & \dots & y'_m, & z_1, & \dots & z_n)
\end{array}$$

or of the form

$$\begin{array}{cccccc}
(y_1, & \dots & y_m, & x_1, & \dots & x_n) \\
\| & & \| & & & \\
(y_1, & \dots & y_m, & z_1, & \dots & z_n)
\end{array}$$

for $y_1 = n$, or of the form

$$\begin{array}{cccccc}
(x_1, & \dots & x_n, & y_1, & \dots & y_m) \\
 & & & \| & & \| \\
(z_1, & \dots & z_n, & y_1, & \dots & y_m)
\end{array}$$

for $y_m = \alpha_0 + k - n - 1$.

Furthermore, the neighbour in the first case is the point $(y_1, ..., y_n, y'_1, ..., y'_m)$, the neighbour in the second case is the point $(0, ..., n-1, y_1, ..., y_m)$, and the neighbour in the third case is the point $(y_1, ..., y_m, \alpha_0 + k - n, ..., \alpha_0 + k - 1)$.

Proof. By Lemma 5.19, any two points in G sharing a neighbour must be of the form

$$\begin{array}{cccccccccccccccc}
(x_1, & \dots & x_n, & y_1, & \dots & y_n, & y_{n+1}, & y_{n+2}, & \dots & y_m) & & & & & & \\
 & & & & & & \| & \| & & \| & & & & & & \\
 & & & & & & (y'_1, & y'_2, & \dots & y'_{m-n}, & y'_{m-n+1}, & \dots & y'_m, & z_1, & \dots & z_n)
\end{array}$$

or of the form

$$\begin{array}{cccccc} (y_1, & \ldots & y_m, & x_1, & \ldots & x_n) \\ \| & & \| & & & \\ (y_1, & \ldots & y_m, & z_1, & \ldots & z_n) \end{array}$$

for $y_1 \geq n$, or of the form

$$\begin{array}{cccccc} (x_1, & \ldots & x_n, & y_1, & \ldots & y_m) \\ & & & \| & & \| \\ (z_1, & \ldots & z_n, & y_1, & \ldots & y_m) \end{array}$$

for $y_m \leq \alpha_0 + k - n - 1$ (and $y_m < \alpha_0$ if $k \leq n$).

In the first case, they have exactly one common neighbour, namely the point

$$(y_1, ..., y_n, y'_1, ..., y'_m)$$

In the second case, neighbours are of the form $(w_1, ..., w_n, y_1, ..., y_m)$ for $w_1, ..., w_n < y_1$. Thus there is exactly one common neighbour if and only if $y_1 = n$, in which case the neighbour is the point $(0, ..., n-1, y_1, ..., y_m)$

Similarly, in the third case neighbours are of the form $(y_1, ..., y_m, v_1, ..., v_n)$ for $v_1, ..., v_n > y_m$. Thus there is exactly one common neighbour if and only if $y_m = \alpha_0 + k - n - 1$, in which case the neighbour is the point $(y_1, ..., y_m, \alpha_0 + k - n, ..., \alpha_0 + k - 1)$. □

Lemma 5.21. Let $\alpha = \alpha_0 + k$ be a successor ordinal with α_0 a limit and $k < \omega$. Let $m > n \geq 2$. Then any two points x, z in $G = G(\alpha, 1^n 3^m 2^n)$ which share exactly one neighbour, and such that there is no point $v \in G$ such that v shares exactly one neighbour with both x and z, must be of the form:

$$\begin{array}{cccccccccccccccc} (x_1, & \ldots & x_n, & y_1, & \ldots & y_n, & y_{n+1}, & y_{n+2}, & \ldots & y_m) & & & & & & \\ & & & & & & \| & \| & & \| & & & & & & \\ & & & & & & (y'_1, & y'_2, & \ldots & y'_{m-n}, & y'_{m-n+1}, & \ldots & y'_m, & z_1, & \ldots & z_n) \end{array}$$

with common neighbour $(y_1, ..., y_n, y'_1, ..., y'_{m-n})$.

Proof. By Lemma 5.20, any two points sharing exactly one neighbour must be of the form

$$\begin{array}{cccccccccccccccc} (x_1, & \ldots & x_n, & y_1, & \ldots & y_n, & y_{n+1}, & y_{n+2}, & \ldots & y_m) & & & & & & \\ & & & & & & \| & \| & & \| & & & & & & \\ & & & & & & (y'_1, & y'_2, & \ldots & y'_{m-n}, & y'_{m-n+1}, & \ldots & y'_m, & z_1, & \ldots & z_n) \end{array}$$

with neighbour $(y_1, ..., y_n, y'_1, ..., y'_{m-n})$, or of the form

$$\begin{array}{cccccc} (y_1, & \ldots & y_m, & x_1, & \ldots & x_n) \\ \| & & \| & & & \\ (y_1, & \ldots & y_m, & z_1, & \ldots & z_n) \end{array}$$

for $y_1 = n$, with neighbour $(0, ..., n-1, y_1, ..., y_m)$, or of the form

$$\begin{array}{cccccc} (x_1, & ... & x_n, & y_1, & ... & y_m) \\ & & & \| & & \| \\ (z_1, & ... & z_n, & y_1, & ... & y_m) \end{array}$$

for $y_m = \alpha_0 + k - n - 1$. with neighbour $(y_1, ..., y_m, \alpha_0 + k - n, ..., \alpha_0 + k - 1)$.

In the first case, there cannot be a point $v \in G$ such that v shares exactly one neighbour with both x and z.

In the second case, take $v = (y_1, ..., y_m, v_1, ..., v_n)$ with $v_1, ..., v_n \neq x_1, ..., x_n$ and $v_1, ..., v_n \neq z_1, ..., z_n$.

Similarly, in the third case take $v = (v_1, ..., v_n, y_1, ..., y_m)$ with $v_1, ..., v_n \neq x_1, ..., x_n$ and $v_1, ..., v_n \neq z_1, ..., z_n$.

□

Note that we must have $n \geq 2$ in this case. If $n = 1$, we could have the following:

Suppose we are in $G(\alpha_0 + k, 13^m 2)$.

Take x to be the point $(1, y_1, ..., y_{m-2}, \alpha_0 + k - 3, \alpha_0 + k - 2)$, and z to be the point $(1, y_1, ..., y_{m-2}, \alpha_0 + k - 3, \alpha_0 + k - 1)$, where the y_is can be anything but are the same for x and z. Then x and z share exactly one neighbour, namely the point $(0, 1, y_1, ..., y_{m-2}, \alpha_0 + k - 3)$. Additionally, there is no point v such that v shares exactly one neighbour with both x and z, as we have 'run out' of points on the right hand side. Normally the point v would look like $(1, y_1, ..., y_{m-2}, \alpha_0 + k - 3, w)$, where $w > \alpha_0 + k - 3$.

So why isn't this a problem when $n \geq 2$?

When considering the example above, the reason for there being no such point v is that we can 'fix' $\alpha_0 + k - 3$ so that it is exactly 'two away from the end', and since $n = 1$, this final point in the $m + 1$-tuple can take either of these values. If $n \geq 2$, we would have a minimum of two points to fill on the right hand side. So x and z will look something like:

$$x = (n + 1, y_2, y_3, ...y_m, x_1, ..., x_n)$$
$$z = (n + 1, y_2, y_3, ...y_m, z_1, ..., z_n)$$

where y_m is 'fixed' at being something like $\alpha_0 + k - n - 2$ (or similar)

Now, the common neighbour of x and z will be the point $(0, 1, ..., n, n+1, y_2, ..., y_m)$. Since we must have two points x and z having this common neighbour, the point y_m is 'fixed' and must be strictly less than $\alpha_0 + k - n - 1$, as otherwise the points $x_1, ..., x_n$ and $z_1, ..., z_n$ would not be able to differ. Now, since there must be at least $n+1$ points to 'choose' from, and n values to insert, we end up with $\binom{n+1}{n}$. For $n \geq 2$, $\binom{n+1}{n}$ will always be at least 3 (this is easy to see, as $\binom{n+1}{n} = n+1$ and $n+1$ is at least 3). Thus, we will always be able to find a point v as described above, as v needs to be of the form:

$$v = (n+1, y_2, y_3, ...y_m, v_1, ..., v_n)$$

with $v_1, ..., v_n$ differing from $x_1, ..., x_n$ and $z_1, ..., z_n$.

Theorem 5.22. Let $\alpha = \alpha_0 + k$ be a successor ordinal with α_0 a limit, and k finite. Let $m > n \geq 2$, and k be such that $k > n \cdot z$, where $z \equiv m \bmod n$. Then $G(\alpha, 1^{2n}3^{m-n}2^{2n})$ is interpretable from $G = G(\alpha, 1^n 3^m 2^n)$ for all $m \geq n$.

Proof. Say that xE_1z for two points x, z in G if their neighbour sets intersect at exactly one point, and such that there is no point $v \in G$ such that v shares exactly one neighbour with both x and z.

By Lemma 5.21, these are of the form

$$\begin{array}{cccccccccccccccc}
(x_1, & \ldots & x_n, & y_1, & \ldots & y_n, & y_{n+1}, & y_{n+2}, & \ldots & y_m) & & & & & & \\
 & & & & & & \| & \| & & \| & & & & & & \\
 & & & & & & (y'_1, & y'_2, & \ldots & y'_{m-n}, & y'_{m-n+1}, & \ldots & y'_m, & z_1, & \ldots & z_n)
\end{array}$$

with mutual neighbour $(y_1, ..., y_n, y'_1, y'_2, ..., y'_m)$.

Now, there is an isomorphism between the graph $G(\alpha, 1^n 3^m 2^n)$ with relation E_1, and the graph $G(\alpha, 1^{2n}3^{m-n}2^{2n})$ with the usual edge relation. This isomophism should be fairly obvious, as vertices in the graph $G(\alpha, 1^{2n}3^{m-n}2^{2n})$ also consist of $(m+n)$-tuples with edges of the form

$$\begin{array}{cccccccccc}
(x_1, & \ldots & x_{2n}, & y_1, & y_2, & \ldots & y_{m-n}) & & & \\
 & & & \| & \| & & \| & & & \\
 & & & (y_1, & y_2, & \ldots & y_{m-n}, & z_1, & \ldots & z_{2n})
\end{array}$$

and there is an edge between two vertices in $G(\alpha, 1^{2n}3^{m-n}2^{2n})$ if and only if the relation E_1 holds for those same two vertices in $G(\alpha, 1^n 3^m 2^n)$. $\square$

Note that we must have $k > n \cdot z$, where $z \equiv m \bmod n$, as we repeat this process z times to get $G(\alpha, 1^{zn}3^{m-zn}2^{zn})$ so that we get a graph of the form $G(\alpha, 1^{n'}3^{m'}2^{n'})$, with $m' < n'$. We can thus apply Theorem 5.18 to determine α. Note that by Theorem 5.18, we must have $k > n'$, i.e. $k > n \cdot z$.

Now we repeat this until we have a graph $G(\alpha, 1^n 3^m 2^n)$ with $m < n$.

$\square$

Corollary 5.23. Let $\alpha = \alpha_0 + k$ be a successor ordinal with α_0 a limit and finite k. Then, for some restrictions on k, α is interpretable in $G = G(\alpha, 1^n 3^m 2^n)$ for all finite m, n.

If $n = 1$, then α is interpretable by Theorem 5.15. If $m = 1$ and $k > n$, α is interpretable by Theorem 5.17 (if $k \leq n$ this can't be done). If $m \leq n$ and $k > n$, then α is interpretable by Theorem 5.18 (if $k \leq n$ this can't be done). If $m \geq n$ and $k > n \cdot z$, where $z \equiv m$ mod n, then $G(\alpha, 1^{zn} 3^{m-zn} 2^{zn})$ is interpretable by Theorem 5.22, and thus α is interpretable by Theorem 5.18. If $k \leq n \cdot z$ this can't be done, by the claim following Theorem 5.18. □

Thus we have shown that any ordinal α is interpretable inside the shift graph with type $1^n 3^m 2^n$. The cases where the type is any string of 1s, 2s, and 3s not of the form $1^n 3^m 2^n$, and where the number of 1s equals the number of 2s, remain open (for example 1232112).

6 Automorphism Groups

We now approach the question of whether it is possible to work out a generalised shift graph from its automorphism group. The simple answer is 'no', but it is interesting to look at these different groups and see how the automorphism groups vary depending on the graph and underlying set.

The simplest automorphism groups are those coming from ordinals in which case as we have shown, the ordinal can often be recovered from the shift graph, for instance those of type 13^n2 as in Theorem 5.7.

Things get a little more interesting in the ordinal graph $G = G(\alpha, 1^n32^n)$ with $n \geq 2$. This is very similar to $G(\alpha, 13^n2)$, in that we can say *something* about each vertex in G - more specifically, for each vertex in G, there is a unique pair (x, y) where $x, y \in \alpha$ such that the $n+1$-tuple generating this vertex must begin with x and end with y. The middle coordinates can however be permuted as we wish to give an automorphism of the graph. We will show more formally below that in general $Aut(G(S, 1^n32^n)) \not\cong Aut(G(S))$, but intuitively, we can think of this as follows: any 'twins' (vertices that share exactly the same neighbour sets) can be completely freely permuted. Thus, for fixed x and z, the vertices of the form $(x, y_1, ..., y_{n-1}, z)$, whose neighbour sets are of the form $\{(v_1, ..., v_n, x) : v_n < x\} \cup \{(z, w_1, ..., w_n) : w_1 > z\}$, all share the same neighbour sets for any $y_1, ..., y_{n-1}$, and can thus be permuted freely. For example, in $G(\mathbb{Q}, 1113222)$, the vertices $(1, 2, 3, 5)$, $(1, 3, 4, 5)$, and $(1, 2, 4, 5)$ can all be permuted, meaning that the automorphism group contains the copy of S_3 which permutes these vertices. Thus we see that for any ordinal α, $Aut(G(\alpha, 1^n32^n)) \not\cong Aut(\alpha)$, for $n > 1$, as $Aut(\alpha) = \{id\}$ whereas $Aut(G(\alpha, 1^n32^n))$ clearly contains all the permutations as described above.

Similarly, in the ordinal case with $G(\alpha, 1^n3^m2^n)$, we can determine the first n and last n coordinates of each $n+m$-tuple, and so whether $Aut(G(S, 1^n3^m2^n)) \cong Aut(S)$ or not will depend on whether $m < n$.

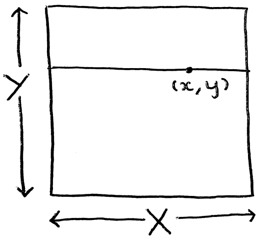
In order to approach these automorphism groups, we must first recall the notion of *wreath product*, denoted by Wr.

Definition 6.1. Let X and Y be sets, and let $G \leq Sym(X)$ and $H \leq Sym(Y)$, where $Sym(X)$ is the group of permutations on X.

We define $G\ Wr\ H$ to be the set of all permutations of $X \times Y$ which have the form (f, h), where f is a mapping from Y to G, and $h \in H$, and which acts as follows:

$$(f, h)(x, y) = (f(y)(x), h(y))$$

It can be straightforwardly verified that (f, h) defined like this is a permutation of $X \times Y$, as follows on noting that $f(y) \in G$ which is a permutation of X and h is a permutation of Y, and then verifying bijectivity.



Intuitively, we can think of the wreath product as follows: Let the diagram above represent X and Y, where a permutation on X corresponds to 'shuffling' points around on a horizontal line, and vice versa. Then h shuffles the horizontal lines up and down, as h acts on Y (and H is a bunch of permutations of Y), and f shuffles the points inside individual horizontal lines, i.e. individual copies of X, but it does this for all Y copies of X simultaneously.

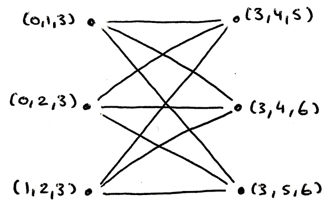
We take a couple of examples to illustrate the difference in automorphism groups for different types:

Example 6.2. The automorphism group of $G(7, 11322)$ is

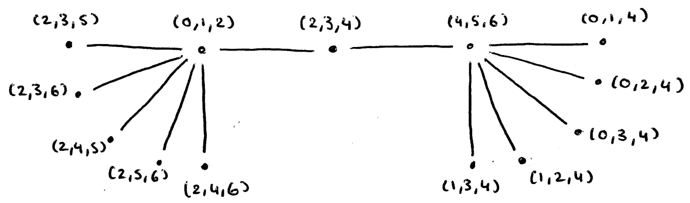
$$(S_5 \; Wr \; \mathbb{Z}_2) \times (S_3 \; Wr \; \mathbb{Z}_2) \times S_{16}.$$

There are three kinds of connected components.

- There are 16 isolated points
- One connected component whose automorphism group is $S_3 \; Wr \; \mathbb{Z}_2$



- An additional component whose automorphism group is $S_5 \; Wr \; \mathbb{Z}_2$.



However, the automorphism group of $G(Z, 132)$ for finite Z is simply $\mathbb{Z}_2$ as the entire graph can be reversed but there are no 'interchangeable' vertices. The automorphism group of $G(Z, 13^n2)$ on the other hand is $\mathbb{Z}_2 \times S_{\binom{z-2}{n-1}}$ for all finite values of n, as the entire graph can be reversed and additionally the singletons, of which there are $\binom{z-2}{n-1}$, are interchangeable.

Example 6.3. $Aut(G(\mathbb{Q}, 11322)) = Sym(\mathbb{Q})\ Wr\ Aut(\mathbb{Q}, B)$

This is because we can interpret $\mathbb{Q}$ inside G, and all vertices of the form (a, b_1, c) and (a, b_2, c) are interchangeable and can be permuted ad lib inside $Aut(G(\mathbb{Q}, 11322))$. Therefore $Aut(G(\mathbb{Q}, 11322))$ is built up from $Aut(\mathbb{Q}, B)$ and $Sym(\mathbb{Q})$, and so the automorphism group is $Sym(\mathbb{Q})\ Wr\ Aut(\mathbb{Q}, B)$.

A similar result holds for the generalisation to all types of the form 1^n32^n: $Aut(G(\mathbb{Q}, 1^n32^n)) = Sym(\mathbb{Q})\ Wr\ Aut(\mathbb{Q}, B)$

Example 6.4. Let's consider any automorphism g of $G(\mathbb{Z}, 11322)$. We know that g must respect the interpretation of $(\mathbb{Z}, B)$, so corresponds to some $h \in Aut(\mathbb{Z}, B)$. To see what else can happen, let's consider $h = id$. Given $a < c$, g can permute $\{(a, b, c) : a < b < c\}$. This will give a copy of S_k, $k = c - a - 1$ (if $c - a = 1$, this doesn't even arise). Thus

$$Aut(G(\mathbb{Z}, 11322)) = \prod_{k=1}^{\infty}(S_k\ Wr\ Aut(\mathbb{Z}, B))$$

We can extend the example above to all types as follows: consider any automorphism of $G(\mathbb{Z}, 1^n32^n)$. Then vertices of the form $(x, y_1, ..., y_{n-1}, z)$ and $(x, y'_1, ..., y'_{n-1}, z)$ can be permuted ad lib. However, there are some restrictions on the number of options we have for the y_is, namely, $\binom{z-x}{n-1}$. Thus

$$Aut(G(\mathbb{Z}, 1^n32^n)) = \prod_{k=1}^{\infty}(S_{\binom{k}{n-1}}\ Wr\ Aut(\mathbb{Z}, B))$$

Remark 6.5. We note that if the ordering S is 'asymmetrical', then $Aut(S, B) = Aut(S, <)$. For example, $Aut(\mathbb{Q}+\mathbb{R}, B) = Aut(\mathbb{Q}+\mathbb{R}, <)$, whereas $Aut(\mathbb{Q}, B) \neq Aut(\mathbb{Q}, <)$.

We take the following example: consider the right long line $\mathbb{L}^+ = \overline{\omega_1 \times \mathbb{R}}$ consisting of the order completion of ω_1 copies of $\mathbb{R}$. $\mathbb{L}^+$ is order-complete, dense without endpoints, and every initial segment is isomorphic to $\mathbb{R}$. Then $Aut(\mathbb{L}^+, B) = Aut(\mathbb{L}^+, <)$.

$$Aut(G(\mathbb{L}^+, 1^n32^n)) = Sym(\mathbb{R})\ Wr\ Aut(\mathbb{L}^+, <)$$

Let $G(S)$ be the shift graph on S for some type τ, such that S is interpretable in $G(S, \tau) = G(S)$. Consider $Aut(S)$ and $Aut(G(S))$.

First note that each automorphism on S induces an automorphism on $G(S)$, since $G(S)$ is interpretable in S. Let us call this natural homomorphism θ, and note that $\theta : Aut(S) \to Aut(G(S))$ is injective but isn't necessarily surjective.

Fix a graph $G(S)$ and an interpretation of S in that graph. This induces a homomorphism ψ going from $Aut(G(S))$ to $Aut(S)$.

Let $\psi : Aut(G(S)) \to Aut(S)$ be a homomorphism such that for each $\varphi_S \in Aut(S), \psi(\theta(\varphi_S)) = \varphi_S$. Such a ψ always exists as S is interpretable in $G(S)$. Note that ψ is surjective, but not necessarily injective. It is clear that $Aut(G(S)) \cong Aut(S)$ if and only if this homomorphism ψ, provided by the interpretation, is injective. Thus, we can determine for various values of τ, whether or not $Aut(G(S)) \cong Aut(S)$.

Theorem 6.6. Let S be a total ordering without endpoints, B the associated betweenness relation, and let $G = G(S, 132)$. Then $Aut(G) \cong Aut(S, B)$.

Proof. Let $\psi : Aut(G) \to Aut(S)$ be a homomorphism provided by the interpretation. We would like to show that ψ is injective, which we will do by showing that $Ker(\psi) = id_G$.

First note that if $\varphi_G : A_x \mapsto A_y$, then $\psi : \varphi_G \mapsto \varphi_S$ where $\varphi_S(x) = y$.

We will now show that for $(x_1, x_2) \neq (y_1, y_2)$, if $\varphi_G((x_1, x_2)) = (y_1, y_2)$, then $\varphi_G : A_i \to A_j$ for some distinct i, j. This is obvious as if $\varphi_G((x_1, x_2)) = (y_1, y_2)$, we must have $\varphi_G : A_{x_1} \to A_{y_1}$ and $\varphi_G : A_{x_2} \to A_{y_2}$, and since $(x_1, x_2) \neq (y_1, y_2)$, we will either have $x_1 \neq y_1$ or $x_2 \neq y_2$, hence $\varphi_G : A_i \to A_j$ for distinct i, j.

Thus if $\varphi_G \in Ker(\psi)$ and $\varphi_G \neq id_G$, there must be some distinct $(x_1, x_2), (y_1, y_2) \in G$ such that $\varphi_G((x_1, x_2)) = (y_1, y_2)$, but such that $\psi : \varphi_G \to id_S$. However, as we have shown above, this cannot happen, hence $\varphi_G = id_G$. □

Theorem 6.7. Let S be a total ordering without endpoints, and let $G = G(S, 1^n 32^n)$, $n \geq 2$. Then $Aut(G) \ncong Aut(S, B)$.

Proof. We show that $Aut(S, B)$ has no element of order 3, but $Aut(G(S, 1^n 32^n))$ does.

First suppose $Aut(S, B)$ has an element g of order 3. Then g is either order-preserving or order-reversing. Assume g is order-preserving. As $g \neq id$, it moves something. Let $g(x) \neq x$, and assume without loss of generality that $g(x) > x$. Then $g^2(x) > g(x)$ and $g^3(x) > g^2(x)$, and so $g^3(x) > x$, a contradiction.

Now suppose g is order-reversing. Then g^2 is order-preserving. Thus if g has order 3, then g^2 must also have order 3, which can't happen by the above argument.

However, $Aut(G(S, 1^n 32^n))$ does have an element of order 3. Choose $x < y < z < s < t$. Then $(x, y, t), (x, z, t), (x, s, t)$ are vertices in G, and the permutation $((x, y, t)(x, z, t)(x, s, t))$ has order 3. □

We now extend this to a more general case of $G(S, 13^n 2)$:

Theorem 6.8. Let S be a total ordering without endpoints, and let $G = G(S, 13^n 2)$. Then $Aut(G) \cong Aut(S, B)$.

Proof. Let $\psi : Aut(G) \to Aut(S, B)$ be a homomorphism provided by the interpretation. We would like to show that ψ is injective, which we will do by showing that $Ker(\psi) = id_G$.

First note that if $\varphi_G : A_{x_1,...,x_n} \mapsto A_{y_1,...,y_n}$, then $\psi(\varphi_G) = \varphi_S$ where $\varphi_S(x_1) = y_1$,..., $\varphi_S(x_n) = y_n$. However, there are also a bunch of other restrictions this gives - namely, that $\varphi_G : A_{x_1,z_2,...,2_n} \mapsto A_{y_1,z'_1,...,z'_n}$ for all $z_2, ..., z_n$ (and some z'_j corresponding to each z_i), and so on. Essentially, the restrictions on tuples $A_{x_1,...,x_n}$ are similar to the restrictions on tuples $(x_1, ..., x_n)$ in $G(S, 13^{n-1}2)$. As in Lemma 4.7, $G(S, 13^{n-1}2)$ is interpretable inside $G(S, 13^n 2)$, and so we can construct a series of homomorphisms (and thus an induced homomorphism ψ as required) as follows:

Let $G(S, 13^n 2) = G_n$, and similarly $G(S, 13^k 2) = G_k$ for all $k < n$. Elements of G_k are in general denoted by φ_{G_k}.

Suppose we have $\varphi_{G_n}((x_1, ..., x_{n+1})) = (y_1, ..., y_{n+1})$ in $Aut(G_n)$. Then we must have $\varphi_{G_n} : A_{x_1,...,x_n} \to A_{y_1,...,y_n}$ and $\varphi_{G_n} : A_{x_2,...,x_{n+1}} \to A_{y_2,...,y_{n+1}}$. To see this, assume that $\varphi_{G_n}((x_1, ..., x_n, x_{n+1})) = (y_1, ..., y_n, y_{n+1})$ but $\varphi_{G_n}((x_1, ..., x_n, x'_{n+1})) \neq (y_1, ..., y_n, y'_{n+1})$, some y'_{n+1}. Then $(z, x_1, ..., x_n)$ is a neighbour of both $(x_1, ..., x_n, x_{n+1})$ and $(x_1, ..., x_n, x'_{n+1})$, but $\varphi_{G_n}((z, x_1, ..., x_n))$ is not a neighbour of $\varphi_{G_n}((x_1, ..., x_n, x_{n+1}))$ and $\varphi_{G_n}((x_1, ..., x_n, x'_{n+1}))$. But homomorphisms must preserve neighbour relations, a contradiction.

Now, this means for distinct $(x_1, ..., x_{n+1}), (y_1, ..., y_{n+1})$, if $\varphi_{G_n}((x_1, ..., x_{n+1})) = (y_1, ..., y_{n+1})$ then $\varphi_{G_n} : A_{(i_1,...,i_n)} \to A_{(j_1,...,j_n)}$, for some distinct $(i_1, ..., i_n), (j_1, ..., j_n)$. Thus $Ker(\psi_n) = id_{G_{n-1}}$, and so we can consider all of $Aut(G_n)$ in terms of automorphisms of entire sets of the form $A_{x_1,...,x_n}$.

We now use the standard homomorphism from G_n to G_{n-1} where $A_{x_1,...,x_n} \to (x_1, ..., x_n)$, and $A_{x_1,...,x_n} \cap A_{y_1,...,y_n} \neq \emptyset$ if and only if there is an edge between $(x_1, ..., x_n)$ and $(y_1, ..., y_n)$ in G_{n-1}. Thus if $\varphi_{G_n} : A_{x_1,...,x_n} \to A_{y_1,...,y_n}$ in G_n, we have $\varphi_{G_{n-1}}((x_1, ..., x_n)) = (y_1, ..., y_n)$ in G_{n-1}. Now, as before, we either have $\varphi_{G_{n-1}} : A_{x_1,...,x_{n-1}} \to A_{y_1,...,y_{n-1}}$, or $\varphi_{G_{n-1}} : A_{x_2,...,x_n} \to A_{y_2,...,y_n}$. Again, we get that $Ker(\psi_{n-1}) = id_{G_{n-2}}$, and so we can consider all of $Aut(G_{n-1})$ in terms of automorphisms of entire sets of the form $A_{x_1,...,x_{n-1}}$.

Repeating this process n times we obtain a chain as follows:

$$Aut(G_n) \xrightarrow{\psi_n} Aut(G_{n-1}) \xrightarrow{\psi_{n-1}} ... \xrightarrow{\psi_3} Aut(G_2) \xrightarrow{\psi_2} Aut(G_1) \xrightarrow{\psi_1} Aut(S)$$

Using the proof of Lemma 6.6, we can show the final step in this chain, i.e. ψ_1 is injective. Thus if we let $\psi = \psi_n \circ \psi_{n-1} \circ ... \circ \psi_1$, we get an induced homomorphism $\psi : Aut(G_n) \to Aut(S)$, whose kernel is id_S by composition, and so $Aut(G) = Aut(G_n) = Aut(S, B)$.

Conjecture 6.9. *Our examples suggest that in the case where S is any total ordering without endpoints,*

$$Aut(G(S, 1^n 3 2^n)) = \prod_{I \subseteq S} (Sym(I)\ Wr\ Aut(S, B))$$

where I is a convex subset of S and $n \geq 1$. However, we leave this open.

The more general $1^n 3^m 2^n$ case also remains open.

7 Chromatic Number of the Cardinal Shift Graph

We will now consider the chromatic number of $G(\alpha, 1^n 3^m 2^n)$, where α is an ordinal. It might be an interesting question to look at the chromatic number of the Finite, Linear Ordering, and Partial Ordering Shift Graphs. In [3] it is stated that Erdős and Hajnal showed that for any infinite cardinal κ,

$$\chi(G(\kappa, 132)) = \min\{\alpha : exp(\alpha) \geq \kappa\}$$

and that in general $\chi(G(\kappa, 13^m 2)) = \min\{\alpha : exp_{(m)}(\alpha) \geq \kappa\}$. This of course covers just types of the form $13^m 2$. Here we will give a fairly perspicuous proof that $\chi(G(\kappa, 1^n 3 2^n)) = \kappa$ for κ measurable, and a modification for κ a strong limit. We will start with the measurable case, which is more direct.

Definition 7.1. Let G be a graph. Then the *chromatic number* of G, denoted by $\chi(G)$, is the minimal number of colours required to colour the vertices of G so that no two neighbours share the same colour.

Remark 7.2. Any clique of size κ must be coloured with κ many colours, and any co-clique of size κ can be coloured with 1 colour.

Definition 7.3. Let κ be a cardinal. Then $\mathcal{U}$ is an *ultrafilter* on κ if:

1. $\mathcal{U} \subseteq \mathcal{P}(\kappa)$
2. $\emptyset \notin \mathcal{U}$
3. $x \cap y \in \mathcal{U}$ for $x, y \in \mathcal{U}$ (i.e. $\mathcal{U}$ is ω-complete)
4. For each $x \in \mathcal{P}(\kappa)$ either $x \in \mathcal{U}$ or $\kappa \backslash x \in \mathcal{U}$

Definition 7.4. $\mathcal{U}$ is *principal* if there is some $\alpha < \kappa$ such that $\{\alpha\} \in \mathcal{U}$.

This is a way for an ultrafilter to be trivial, as the sets in $\mathcal{U}$ are precisely the sets containing α.

Definition 7.5. $\mathcal{U}$ is κ-complete if $\bigcap_{\alpha<\lambda} X_\alpha \in \mathcal{U}$ for $\lambda < \kappa$ and $X_\alpha \in \mathcal{U}$.

This brings us to the following definition:

Definition 7.6. An uncountable cardinal κ is *measurable* iff it has a κ-complete, non-principal ultrafilter.

Lemma 7.7. Let κ be a measurable cardinal, and let $\mathcal{U}$ be a κ-complete, non-principal ultrafilter on κ. Then every set in $\mathcal{U}$ has size κ.

Proof. Suppose $X \in \mathcal{U}$ with $|X| = \lambda < \kappa$. For each $y \in X$, let $X_y = X \backslash \{y\}$.

Then $\bigcap_{y \in X} X_y$ is empty, a contradiction on $\mathcal{U}$ being κ-complete. □

Lemma 7.8. Let κ be a measurable cardinal, and let $\mathcal{U}$ be a κ-complete, non-principal ultrafilter on κ. Then if $X \in \mathcal{U}$, any partition on X into $< \kappa$ pieces contains exactly one set in $\mathcal{U}$.

Proof. Let $\{X_\alpha : \alpha < \lambda\}$ be a partition of $X \in \mathcal{U}$ into $\lambda < \kappa$ pieces.

Suppose for a contradiction that no such X_α lies in $\mathcal{U}$. For each $\alpha < \lambda$, let $Y_\alpha = X \backslash X_\alpha$. By condition 4. in the definition of $\mathcal{U}$, each Y_α must lie in $\mathcal{U}$. But $\bigcap_{\alpha < \lambda} Y_\alpha = \emptyset$, a contradiction on $\mathcal{U}$ being κ-complete. □

Theorem 7.9. $\chi(G(\kappa, 132)) = \kappa$ for every measurable cardinal κ.

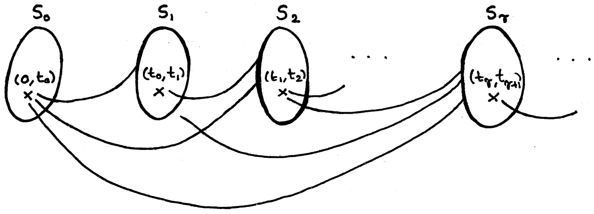
Proof. Let $G = G(\kappa, 132)$, and assume for a contradiction that $\chi(G) = \lambda < \kappa$. Let $\mathcal{U}$ be a κ-complete, non-principal ultrafilter on κ. We choose by transfinite induction a decreasing sequence $(T_\alpha : \alpha < \kappa)$ of members of $\mathcal{U}$. We shall write t_α for the least element of T_α.

Let $T_0 = \kappa$, and so $t_0 = 0$.

Now suppose that T_α (and hence also t_α) has been chosen, and consider the set $\{(t_\alpha, t) : t > t_\alpha\}$ which is a set of κ vertices of G, which is coloured by (at most) λ colours. By Lemma 7.7, the set $\{t : t > t_\alpha\}$ lies in $\mathcal{U}$ (since its complement has size $< \kappa$). We can partition the set $\{t : t > t_\alpha\}$ such that for each set T in the partition, $\{(t_\alpha, t) : t > t_\alpha, t \in T\}$ is monochromatic. Since there are at most λ many sets in this partition, by Lemma 7.8 exactly one of these lies in $\mathcal{U}$. Let $T \in \mathcal{U}$ be such that $\{(t_\alpha, t) : t > t_\alpha, t \in T\}$ is monochromatic.

Let $T_{\alpha+1} = T_\alpha \cap T$ for some such T, which also lies in $\mathcal{U}$. Then $t_\alpha \notin T_{\alpha+1}$, so that $T_{\alpha+1}$ is a proper subset of T_α, and also $t_\alpha < t_{\alpha+1}$. If γ is a limit ordinal $< \kappa$, we let $T_\gamma = \bigcap_{\alpha < \gamma} T_\alpha$. This is a proper subset of each T_α for $\alpha < \gamma$, and hence again $t_\gamma > t_\alpha$. The fact that $T_\gamma \in \mathcal{U}$ follows by κ-completeness of $\mathcal{U}$.

Now let $S_\alpha = \{(t_\alpha, t) : t \in T_{\alpha+1}\}$. By construction, S_α is monochromatic. Furthermore, if $\alpha < \beta$, then $t_\beta \in T_{\alpha+1}$, so that $(t_\alpha, t_\beta) \in S_\alpha$. Thus (t_α, t_β) is joined to some (all actually) members of S_β. Hence the colours assigned to S_α and S_β are different. Therefore κ colours in all must have been used, which gives a contradiction. □



Theorem 7.10. $\chi(G(\kappa, 1^n32^n)) = \kappa$ for every measurable cardinal κ and for every $n \geq 1$.

Proof. Assume for a contradiction that $\chi(G) = \lambda < \kappa$, and choose a decreasing sequence of sets T_α for $\alpha < \kappa$, all lying in $\mathcal{U}$, again writing t_α for the least member of T_α. Let $T_0 = \kappa$. Assuming that T_α has been chosen, find $T \in \mathcal{U}$ such that $\{(t_\alpha, t_\alpha{+}1, ..., t_\alpha{+}n{-}1, t) : t \in T\}$ is monochromatic, and let $T_{\alpha+1} = T_\alpha \cap T$. At limits we take intersections as before. Note that in this case, $t_{\alpha+1} \geq t_\alpha + n$. To conclude the proof, we let $S_\alpha = \{(t_\alpha, t_\alpha + 1, ..., t_\alpha + n - 1, t) : t \in T_{\alpha+1}\}$, and for the same reason as in the proof of Theorem 7.9, these sets must all be coloured by different colours, which gives a contradiction. $\square$

Recall the following definition:

Definition 7.11. A cardinal κ is a *strong limit* if for all $\lambda < \kappa$, $2^\lambda < \kappa$.

Lemma 7.12. If κ is a strong limit cardinal, then $\lambda^\gamma < \kappa$ for all $\lambda, \gamma < \kappa$.

Proof. Since κ is a strong limit cardinal, $2^\lambda < \kappa$ for all $\lambda < \kappa$. But

$$\lambda^\gamma \leq 2^{(\lambda^\gamma)} = (2^\lambda)^\gamma = 2^{\lambda \cdot \gamma}.$$

But $\lambda \cdot \gamma < \kappa$, so $2^{\lambda \cdot \gamma} < \kappa$ by the definition of strong limit. Hence $\lambda^\gamma < \kappa$. $\square$

We can improve Theorem 7.10 by showing that it holds not just for measurable cardinals, but for all strong limit cardinals as well. We will require the following definition:

Definition 7.13. Let λ, κ be cardinals. Then ${}^\kappa[\lambda]$ is the set of all functions from κ to λ.

First, we start with the 132 case.

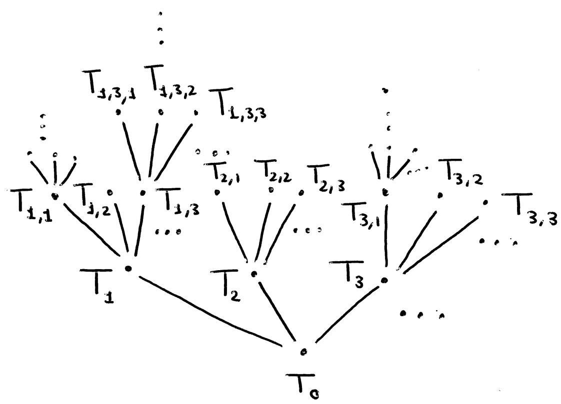
Theorem 7.14. $\chi(G(\kappa, 132)) = \kappa$ for every strong limit cardinal κ.

Proof. In this proof we adapt the method of Theorem 7.9. There we made a choice at each stage of a 'large' set (in the ultrafilter). Here, we shall not in general know which set to take, so instead, we take *all* of them, and arrange them as the points on the next level of a tree. If the resulting tree has a long enough branch, then we can derive a contradiction as before. In fact the vertices will be subsets of κ, the tree order will be reverse inclusion, and distinct vertices on the same level will be disjoint.

Assume for a contradiction that $\chi(G(\kappa, 132)) = \lambda < \kappa$. We will choose a series of subsets $T_{(s)}$ of κ by transfinite induction, each representing a node of the tree, where (s) is a sequence of points in λ. Again, let $t_{(s)}$ be the least element of $T_{(s)}$ for each (s). Each branch of the tree is a decreasing sequence of subsets of κ.

Let the root of the tree be $T_0 = \kappa$, with minimal point $t_0 = 0$.

Now consider a partition of $\kappa \backslash \{0\}$ into monochromatic sets of the form $\{(0, x) : x > 0\}$. Given our assumption that $\chi(G(\kappa, 132)) = \lambda$, there are at most λ many sets of this form. Let us label them $T_1, T_2, T_3, ...$ with minimal points $t_1, t_2, t_3, ...$ respectively. These are all on level 1 of the tree. Now, for each T_α of this form, $\alpha \leq \lambda$, again consider monochromatic sets of the form $\{(t_\alpha, x) : x > t_\alpha\}$. Again, there are at most λ many of these. Call these sets $T_{\alpha,1}, T_{\alpha,2}, T_{\alpha,3}, ...$ with respective minimal points $t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, ...$ These are all the nodes on level 2 of the tree which are joined to T_α. Continue like this recursively for all levels of the tree. Thus at level γ, each verticex is of the form $T_{(s)}$ where (s) is a sequence of points in λ of length γ, and where restricting (s) to its first $\alpha < \gamma$ entries gives us a vertex $T_{(s')}$ on level α of the tree such that $T_{(s')} < T_{(s)}$. Note that since each vertex of the tree splits into at most λ many branches, at level γ of the tree there are at most λ^γ many nodes.



Since κ is a strong limit, $\lambda^\gamma < \kappa$ by Lemma 7.12. Hence the tree must have height κ, and therefore it must have a branch of length λ^+. But by the construction of the tree, this gives us λ^+ many monochromatic sets, all joined by an edge, a contradiction on the chromatic number of $G(\kappa, 132)$ being λ. □

Theorem 7.15. $\chi(G(\kappa, 1^n 32^n)) = \kappa$ for every strong limit cardinal κ and every positive integer n.

Proof. This proof is similar to that of Theorem 7.14.

Again, we assume for a contradiction that $\chi(G(\kappa, 1^n 32^n)) = \lambda < \kappa$. We will choose a series of subsets $T_{(s)}$ of κ by transfinite induction as before.

Let $T_0 = \kappa$ with minimal element t_0, and consider a partition of $\kappa \backslash \{0\}$ into monochrnomatic sets of the form $\{(0, x_1, ..., x_n) : x_1 > 0\}$. Given our assumption that $\chi(G(\kappa, 132)) = \lambda$, there are at most λ many sets of this form, so again, label them $T_1, T_2, T_3, ...$ with minimal points $t_1, t_2, t_3, ...$ respectively. These are all on level 1 of the tree. Again, for each T_α of this form, $\alpha \leq \lambda$, again consider monochromatic sets of the form $\{(t_\alpha, x_1, ..., x_n) : x_1 > t_\alpha\}$ (there are at most λ many of these). Call these sets $T_{\alpha,1}, T_{\alpha,2}, T_{\alpha,3}, ...$ with respective minimal points $t_{\alpha,1}, t_{\alpha,2}, t_{\alpha,3}, ...$ These are all the nodes on level 2 of the tree which are joined to T_α. Continue like this recursively for all levels of the tree as in Theorem 7.14.

Thus at level γ, each verticex is of the form $T_{(s)}$ where (s) is a sequence of points in λ of length γ, and where restricting (s) to its first $\alpha < \gamma$ entries gives us a vertex $T_{(s')}$ on level α of the tree such that $T_{(s')} < T_{(s)}$. Note that since each vertex of the tree splits into at most λ many branches, at level γ of the tree there are at most λ^γ many nodes.

Again, since κ is a strong limit, $\lambda^\gamma < \kappa$ by Lemma 7.12. Hence the tree must have height κ, and therefore it must have a branch of length λ^+. But by the construction of the tree, this gives us λ^+ many monochromatic sets, all joined by an edge, a contradiction on the chromatic number of $G(\kappa, 1^n 32^n)$ being λ. □

Conclusions and Future Work

Our main results concern interpretability of linear orders (sometimes just up to order-reversal) within shift graphs. For ordinals, we have proved that any ordinal α is interpretable inside $G(\alpha, 1^n 3^m 2^n)$, which is covered in a succession of cases in Theorems 5.7, 5.10, 5.12, 5.14, 5.15, 5.16, 5.17, 5.22, and Corollary 5.23. The techniques and precise results obtained vary according to the type, $132, 13^n2, 1^n23^n$, or $1^m3^n2^n$, as well as whether the ordinal is a limit ordinal or a successor. The finite case is treated separately in Theorems 5.1, 5.2, and 5.3. Rather different techniques are used for the case of general linear orders, and once again there are several different cases which need to be distinguished, in Theorems 4.2, 4.4, 4.5, 4.8, 4.11, 4.12, and Corollary 4.13. These are extended where possible to trees in Theorems 4.23, 4.27, 4.28, 4.30, and Corollary 4.31.

It is conjectured that these results can be extended to all graphs of the form $G(z, 1^n 3^m 2^n)$. The obstacles to doing this are illustrated by the arguments that we were compelled to adopt in some of the above (and the reason why so many cases had to be considered, with specific restrictions concerning density, lack of endpoints, absence of certain sub-intervals, and so on). One first needs to tackle to possible presence of endpoints. This has already arisen in the ordinals case (for left and sometimes right endpoints), particularly the finite case. The techniques required are likely to need to be adapted from these cases, as well as from those where there are no 'short' discrete subintervals.

We have shown that $Aut(G(S, 13^n2)) \cong Aut(S, B)$ for any total ordering S and $n \geq 1$, and conjectured that graphs of the form $G(S, 1^n32^n)$ can be reconstructed from their automorphism groups up to order reversal, where if S is any total ordering without endpoints:

$$Aut(G(S, 1^n32^n)) = \prod_{I \subseteq S} (Sym(I) \; Wr \; Aut(S, B))$$

for I a convex subset of S and $n \geq 1$. The more general $1^n3^m2^n$ case is left open.

It has already been shown that for any infinite cardinal κ, the chromatic number of $G(\kappa, 132)$ is the $\min\{\alpha : \exp(\alpha) \geq \kappa\}$, and for any infinite cardinal λ and $n \geq 0$, $\chi(G(\lambda, 13^n2)) \leq \kappa$ iff $\lambda \leq exp_n(\kappa) = 2^{2^{\cdots 2^{\kappa}}}$ (n times). We have extended these results to show that the chromatic number of $G(\kappa, 1^n32^n)$ is κ for any strong limit cardinal κ and $n \in \mathbb{N}$.

The most obvious next step leading on from this work is to consider total orderings with endpoints. We may be able to use techniques similar to the ones used in the successor ordinals chapter, although these will need to be adjusted. There may also be cases with endpoints in which we cannot determine the underlying ordering.

It should also be possible to extend the results on finite ordinals to all types of the form $1^n3^m2^n$, with some additional restrictions on z, using similar tools.

Other future work might consist of considering $G(P, 1^n3^m2^n)$ for all partial orderings P. It would be interesting to look at the chromatic number of the ordinal, finite, linear, and partial ordering graphs. Going further, one might also consider the more general shift graph, that is, *not* with type $1^n3^m2^n$.

It might also be interesting to consider directed graphs - on the surface they seem easier to deal with, as we are given additional information about each pair of vertices - for example, in $\overrightarrow{G}(\alpha, 132)$ we can very easily recognise the set $\{(0, x) : x > 0\}$ as the set of all vertices that don't point at anything - but they might present other problems of their own. It is, however, likely that we will be able to recover the underlying set with the original total ordering, as opposed to the betweenness relation, in this case.

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