Deformations of Asymptotically Conical G_2-Instantons

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To Ciara.

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Abstract

This thesis develops the deformation theory of instantons on asymptotically conical G_2-manifolds, where an asymptotic connection at infinity is fixed. A spinorial approach is adopted to relate the space of deformations to the kernel of a twisted Dirac operator on the G_2-manifold and to the eigenvalues of a twisted Dirac operator on the nearly Kähler link. As an application, we use this framework to study the moduli spaces of known examples of G_2-instantons living on the Bryant-Salamon manifolds and on $\mathbb{R}^7$. We develop two methods for determining eigenvalues of twisted Dirac operators on nearly Kähler 6-manifolds and apply this to calculate the virtual dimension of the moduli spaces that we study. In the case of the instanton of Günaydin-Nicolai, which lives on $\mathbb{R}^7$, we show how knowledge of the virtual dimension of the moduli space can be used to study uniqueness properties of this instanton.

Contents

Chapter 1

Introduction

1.1 Introduction

Instantons are connections whose curvature satisfies a certain algebraic equation. In this thesis we study the deformation theory of instanton connections on manifolds with holonomy group G_2 and which asymptote to a cone. The physical motivation for studying instantons on G_2-manifolds stems from the fact that such connections automatically satisfy the Yang-Mills equation. Furthermore on a compact manifold G_2-instantons are absolute minima of the Yang-Mills energy functional. Interest in these connections has grown significantly in recent years since the suggestion of Donaldson-Thomas [29] and Donaldson-Segal [28] that it may be possible to define invariants from their moduli spaces.

The first example of a non-trivial G_2-instanton was constructed on the principal G_2-bundle over $\mathbb{R}^7$ by Günaydin-Nicolai in 1995 [41] and we shall refer to this example as the standard G_2-instanton. Other examples of G_2-instantons on non-compact manifolds have been found in more recent years. Firstly, Clarke [20] found a family of instanton connections on the manifold $\not{S}(S^3)$ which was shown by Bryant-Salamon to carry a holonomy G_2-metric [15]. The other G_2-manifolds constructed by Bryant-Salamon are $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{C}P^2)$ and examples of G_2-instantons have been constructed on these spaces by Oliveira [77]. Part of the work of Lotay-Oliveira [71] was to study the moduli spaces of instantons on asymptotically conical G_2-manifolds where the connections are required to be invariant under a group action. In particular, they found a limiting connection of Clarke's family of instantons. The important observation here is that all of these examples of G_2-instantons converge to instanton connections on the nearly Kähler 6-manifold at infinity.

The deformation theory that we develop that relies on the analytic framework for elliptic operators on asymptotically conical manifolds that has been developed by Lockhart-McOwen [67] and Marshall [72]. These tools enable one to determine when elliptic operators on non-compact manifolds are Fredholm, and hence a Kuranishi model is applicable to study the moduli space of solutions. In the holonomy G_2 setting, this framework has been used to study the moduli space of associative and coassociative submanifolds by Lotay [68, 69] and the moduli space of G_2-structures by Karigiannis-Lotay [61]. In the gauge theory setting Nakajima [76] has used a similar analysis to study the moduli space of ASD instantons on asymptotically locally Euclidean hyperKähler 4-folds where a flat connection at infinity is fixed.

Since the asymptotic connection in our setting is a nearly Kähler instanton it is interesting to compare deformations of the G_2-instanton with deformations of this asymptotic connection. In particular there is a projection between the moduli spaces of these two instantons and one can try to understand the properties of this map. The deformation theory of nearly Kähler instantons has been developed by Charbonneau-Harland [18] and we use many of the ideas and techniques they develop to analyse the asymptotic connection.

This thesis studies the deformation theory of G_2-instantons on asymptotically conical manifolds by prescribing a fixed rate of decay at infinity. We take a spinorial approach and relate deformations of the G_2-instanton to the kernel of a Dirac operator on a Hilbert space of spinors with fixed decay rate. In contrast to the compact case the index of the Dirac operator controlling the deformation theory is not expected to be zero. The deformation theory that we develop can be applied to the known examples of asymptotically conical G_2-instantons so we then move on to studying examples on each of the Bryant-Salamon manifolds as well as the standard instanton on $\mathbb{R}^7$.

After covering the required background material and providing a literature review we begin in Chapter 4 to study the fundamentals of gauge theory on an asymptotically conical G_2-manifold. Firstly we provide a slice theorem to show that the space of connections modulo gauge is a smooth Hilbert manifold. We also show that the moduli space of asymptotically conical G_2-instantons is locally homeomorphic to the kernel of a twisted Dirac operator. Our analysis shows that the virtual dimension of the space of solutions to the linearised G_2-instanton equation is determined by the spectrum of a twisted Dirac operator on a nearly Kähler 6-manifold and we apply the implicit function theorem to show the moduli space is a smooth manifold when the deformation theory is unobstructed.

In Chapter 5 we develop representation theoretic tools for determining the spectrum of Dirac operators over homogeneous nearly Kähler 6-manifolds which are twisted by the canonical connection on an associate bundle. These results are developed with the known examples of asymptotically conical G_2-instantons in mind, since the asymptotic geometry and connection of each example is of this type. We determine the eigenvalues of an operator which differs from the Levi-Civita Dirac operator by an algebraic term and use this to provide bounds on the eigenvalues coming from group representations under the Frobenius reciprocity theorem.

In Chapter 6 we begin to apply the results we have obtained to specific examples. Firstly we study the examples of Clarke [20] and Lotay-Oliveira [71] and we apply the results of the previous chapter to determine the virtual dimension of the moduli space. In Chapter 7 we consider Oliveira's examples [77] of G_2-instantons on the Bryant-Salamon manifolds $\Lambda^2_-(\mathbb{CP}^2)$ and $\Lambda^2_-(S^4)$. For the example we consider over $\Lambda^2_-(S^4)$ we show that the Lichnerowicz formula is insufficient for determining the relevant eigenvalues and thus much of the work in this chapter is to develop a method for calculating eigenvalues of twisted Dirac operators. For this we generalise a method of Bär [4] to the case of a twisted spinor on a nearly Kähler 6-manifold. As a result we determine the virtual dimension of the given moduli space.

In Chapter 8 we study the standard instanton on $\mathbb{R}^7$, which we view as the metric cone of S^6. Here we must once again determine some eigenvalues of a twisted Dirac operator, this time on S^6, and we do so by providing another method for determining the matrix of the operator on a finite dimensional subspace of the space of sections. We write the operator as a sum of Casimir operators and use representation theoretic techniques to calculate the first few eigenvalues explicitly– to the author's knowledge this method is has not appeared before in the literature. We use this to determine the virtual dimension of the moduli space and as an application we show how, under an assumption of unobstructedness, one can prove a global uniqueness for the standard instanton.

Chapter 2

Geometry of G_2-Manifolds and Nearly Kähler 6-Manifolds

In this chapter we review some basic properties of the group G_2 before introducing G_2-manifolds and nearly Kähler 6-manifolds. Our treatment focuses on the spin geometry of these manifolds. We explain how the two geometries are related via the cone construction and asymptotically conical G_2-manifolds.

2.1 The Octonions and the Group G_2

We begin with a brief introduction to the algebraic structures of normed division algebras and cross products. A more comprehensive treatment can be found in [80]. Let $\mathbb{A} = \mathbb{R}^n$ be given the Euclidean inner product g_0.

Definition 2.1.1. *If $\mathbb{A}$ has the structure of a (possibly non-associative) algebra over $\mathbb{R}$ with multiplicative identity $1 \neq 0$ such that*

$$|ab| = |a||b| \qquad \textit{for all } a, b \in \mathbb{A},$$

then $\mathbb{A}$ is called a normed division algebra.

Normed division algebras are completely classified and the classification consists of precisely 4 possibilities:

$\mathbb{R}$	$\mathbb{C}$	$\mathbb{H}$	$\mathbb{O}$
The Real Numbers	The Complex Numbers	The Quaternions	The Octonions

The last entry in this list, the Octonions, is an 8-dimensional real vector space. The standard basis is given by $\{1, e_1, \ldots, e_7\}$ and the multiplication is described by

$$e_i e_j = -\delta_{ij} 1 + \epsilon_{ijk} e_k$$

for $i, j > 0$ and where ϵ_{ijk} is totally anti-symmetric. The multiplication is not associative but satisfies a weaker condition called power associativity. The Octonions lead to our first definition of the group G_2 :

Definition 2.1.2. *The group G_2 is the automorphism group of the Octonion algebra $\mathbb{O}$.*

If $\mathbb{A}$ is a normed division algebra, we write $\mathrm{Im}\mathbb{A} = \langle 1 \rangle^\perp$ and call this the imaginary part of $\mathbb{A}$.

A cross product on a real inner-product space (V, g) is an alternating map $\times : V \times V \to V$ such that

$$g(u \times v, u) = g(u \times v, v) = 0$$
$$|u \times v|^2 = |u|^2|v|^2 - g(u, v)^2$$

for all $u, v \in V$. If $\mathbb{A}$ is a normed division algebra then one can check that $\mathrm{Im}\mathbb{A}$ carries a natural cross product defined by $a \times b = \frac{1}{2}[a, b]$, where $[\,\cdot\,,\,\cdot\,]$ is the commutator of the algebra multiplication. In fact, this correspondence is a bijection: If V carries a cross product then $\mathbb{R} \oplus V$ can be given the structure of a normed division algebra. It then follows from the classification of normed division algebras that $\mathbb{R}^n$ carries a non-vanishing cross product precisely when $n = 3$ or $n = 7$. Let us package the geometry of the cross product by defining a 3-form

$$\varphi_0(u, v, w) = g_0(u \times v, w).$$

On $\mathbb{R}^3$ one has that $\varphi_0 = \mathrm{Vol}_3$ is the standard volume form. In this sense the geometry of the cross product on $\mathbb{R}^3$ does not define an interesting geometric structure– a choice of cross product is equivalent to an orientation. On $\mathbb{R}^7$ on the other hand, the 3-form φ_0 is non-trivial. Various conventions exists for the 3-form $\varphi_0 \in \Lambda^3(\mathbb{R}^7)^*$, we adopt the convention

$$\varphi_0 = \mathrm{d}x^{127} + \mathrm{d}x^{347} + \mathrm{d}x^{567} + \mathrm{d}x^{145} + \mathrm{d}x^{136} + \mathrm{d}x^{235} - \mathrm{d}x^{246}$$

where $x_1, \cdots, x_7$ are the standard coordinates on $\mathbb{R}^7$ and $\mathrm{d}x^{ijk} = \mathrm{d}x^i \wedge \mathrm{d}x^j \wedge \mathrm{d}x^k$. Using the Euclidean volume form $\mathrm{Vol}_7 = \mathrm{d}x^1 \wedge \ldots \wedge \mathrm{d}x^7$ one can also form a 4-form $\psi_0 = *\varphi_0$ and this has coordinate expression

$$\psi_0 = \mathrm{d}x^{1234} + \mathrm{d}x^{1256} + \mathrm{d}x^{3456} - \mathrm{d}x^{1357} + \mathrm{d}x^{1467} + \mathrm{d}x^{2367} + \mathrm{d}x^{2457}.$$

The relation

$$(u \lrcorner \varphi_0) \wedge (v \lrcorner \varphi_0) \wedge \varphi_0 = 6 g_0(u, v)\mathrm{Vol}_7$$

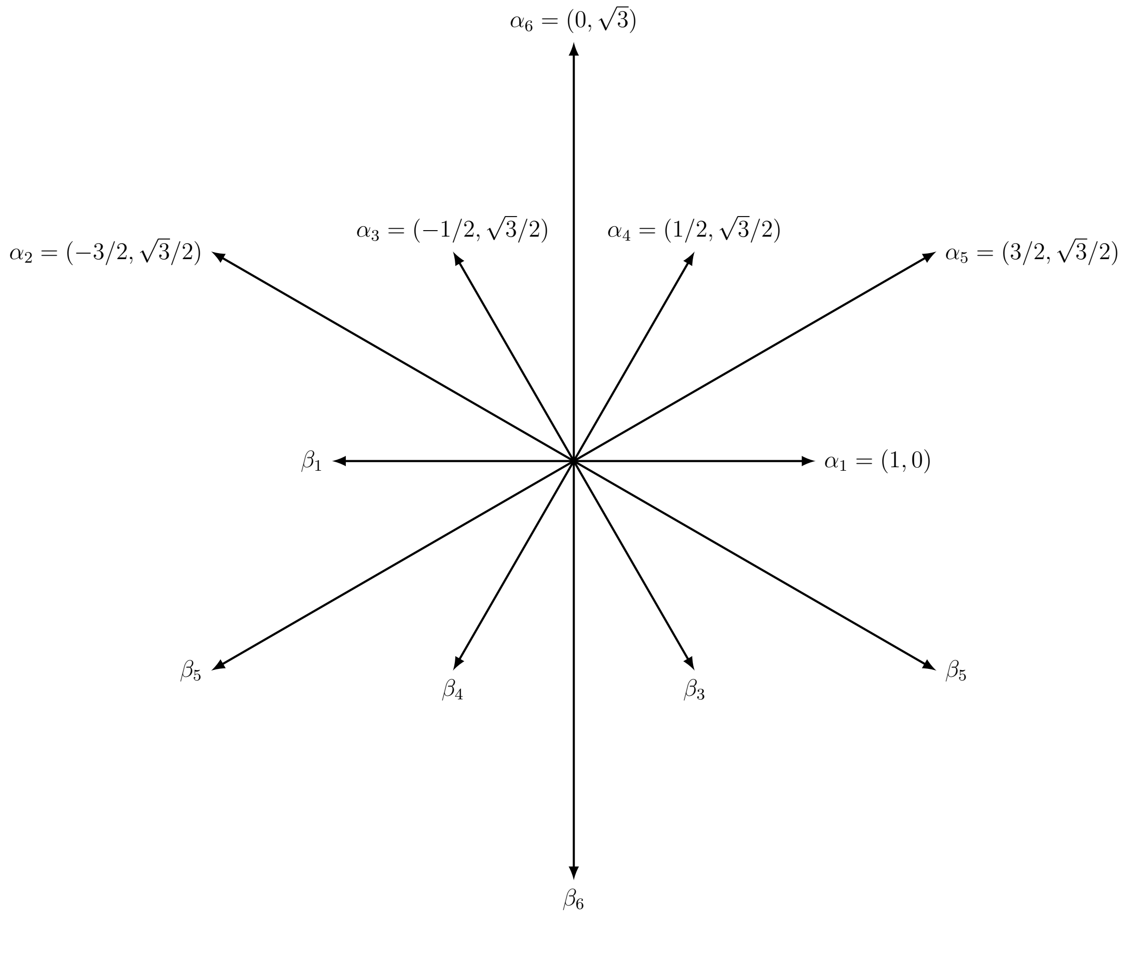
holds for all $u, v \in \mathbb{R}^7$ and this observation hints at a relation between φ_0 and the metric and orientation on $\mathbb{R}^7$. This relation is made precise through the following theorem of Robert Bryant [13]:

Theorem 2.1.3. *Let $G = \{A \in \mathrm{GL}(7, \mathbb{R})\,;\, A^*\varphi_0 = \varphi_0\}$. Then $G = G_2$ and $G \subset \mathrm{SO}(7)$. Thus if $A \in G_2$ then A preserves both the metric g_0 and the orientation* Vol_7.

Thinking of G_2 as the group preserving the 3-form φ_0 is for our purposes the most useful description of the group. As a subgroup of SO(7) we see that G_2 acts on $\mathbb{R}^7$ in the obvious way. This representation is irreducible and called the standard representation of G_2. Furthermore this action is transitive on $S^6 \subset \mathbb{R}^7$ and has isotropy SU(3), thus G_2 fibres over S^6 with fiber SU(3) and this reveals many of the topological properties of G_2. In particular one finds that G_2 is 14-dimensional, compact, path connected and simply connected. Furthermore SU(3) is the maximal subgroup of G_2 and a choice of maximal torus for SU(3) defines a maximal torus of G_2 via the inclusion homomorphism.

The group G_2 is an exceptional Lie group since the Dynkin diagram of the Lie algebra $\mathfrak{g}_2$ does not fall into any of the families A_n, B_n, C_n, D_n. The root diagram of $\mathfrak{g}_2$ is

Figure 2.1: Root Diagram of $\mathfrak{g}_2$



The fundamental roots are α_4 and α_6. Since G_2 is simply connected irreducible representations of the group and Lie algebra are in bijection. For $i, j \geq 0$ we denote by $V_{(i,j)}$ the complex irreducible representation with highest weight $i\alpha_4 + j\alpha_6$. The first few irreducible representations are

- $V_{(0,0)} = \mathbb{C}$, the trivial representation
- $V_{(1,0)} = \mathbb{C}^7$, the standard representation
- $V_{(0,1)} = (\mathfrak{g}_2)_{\mathbb{C}}$, the adjoint representation
- $V_{(2,0)} = \mathrm{Sym}^2_0(\mathbb{C}^7)$, symmetric trace-free 2-tensors.

A maximal subalgebra of $\mathfrak{g}_2$ is given by $\mathfrak{su}(3)$. Let us denote irreducible representations of SU(3) by $W_{(i,j)}$, the first few representations being

- $W_{(0,0)} = \mathbb{C}$
- $W_{(1,0)} = \mathbb{C}^3$
- $W_{(0,1)} = (\mathbb{C}^3)^*$
- $W_{(1,1)} = \mathfrak{su}(3)_{\mathbb{C}} = \mathfrak{sl}(3, \mathbb{C})$.

One can observe from the root diagram that, as an SU(3) module, one has $(\mathfrak{g}_2)_{\mathbb{C}} = W_{(1,1)} \oplus W_{(1,0)} \oplus W_{(0,1)}$. The representation theory of these two Lie algebras is crucial to this thesis and will be revisited in later chapters.

2.2 G_2-Structures On Manifolds

The material covered in this section is based on [54]. Let M be a smooth oriented 7-manifold. For each $p \in M$ define $(\Lambda^3_+(T^*M))_p$ as the set of 3-forms

$$\{\lambda \in \Lambda^3_p(T^*M) \text{ such that there exists an isomorphism } \Phi\colon T_pM \to \mathbb{R}^7 \text{ with } (\Phi^{-1})^*\lambda = \varphi_0\}$$

then $(\Lambda^3_+(T^*M))$ is isomorphic to $GL(7,\mathbb{R})/G_2$ since φ_0 is stabilised by G_2. Let $\Lambda^3_+(T^*M) = \bigcup_{p\in M}(\Lambda^3_+(T^*M))_p$, we call this the bundle of positive 3-forms and a section of this bundle is called a positive 3-form. This is not a vector subbundle of $\Lambda^3(T^*M)$ but rather an open subbundle since both bundles have fibers of dimension 35.

Recall the *frame bundle* $\mathcal{F} \to M$ is the bundle whose fibers is the set of isomorphisms between T_pM and $\mathbb{R}^7$. Let φ be a positive 3-form and let $\mathcal{Q}$ be the subset of $\mathcal{F}$ consisting of the isomorphisms which identify φ_p with φ_0. Then $\mathcal{Q}$ is a principal subbundle of $\mathcal{F}$ whose fibres are isomorphic to G_2. By definition then $\mathcal{Q}$ defines a *G_2-structure* on M.

Conversely a G_2-structure $\mathcal{Q}$ on M allow one to define a 3-form φ by pulling back the G_2-invariant 3-form φ_0. This gives a 1-1 correspondence between positive 3-forms and G_2-structures on M and justifies the following definition (which is technically an abuse of notation but is standard in the literature).

Definition 2.2.1. *A choice of smooth section of $\Lambda^3_+(T^*M)$ is called a G_2-structure on M.*

Remark 2.2.2. *There is a topological obstruction to the existence of a G_2-structure, namely a 7-manifold M carries a G_2-structure if and only if M is spinnable, in other words if and only if the first and second Stiefel-Whitney classes vanish.*

Since $G_2 \subset \mathrm{SO}(7)$ the G_2-structure induces both an orientation and a metric on M. One can in fact use the positive 3-form to explicitly construct a volume form Vol_7 on M (see [60] for details) and the induced metric $g = g_\varphi$ is then defined by the relation

$$(\iota_u\varphi) \wedge (\iota_v\varphi) \wedge \varphi = 6g_\varphi(u, v)\mathrm{Vol}_7.$$

Thus one can also define a 4-form $\psi := *_\varphi\varphi$, but it is important to note that the dependence of g and ψ on φ is non-linear.

A G-structure on a manifold comes with *intrinsic torsion*. In the case at hand the torsion measures the failure of the holonomy group of the Levi-Civita connection (induced by g_φ) to be a subgroup of G_2.

Definition 2.2.3. *A G_2-manifold is a 7-manifold M together with a torsion-free G_2 structure φ. That is, we have $\mathrm{Hol}(\nabla) \subseteq G_2$ where ∇ is the Levi-Civita connection of g_φ.*

A generic G_2-structure will have non-zero torsion, so one can think about torsion-free structures as optimal G_2-structures. Indeed, Hitchin noted in [49] that torsion free G_2-structures are precisely the critical points of a natural functional on $\Gamma(\Lambda^3_+(M))$.

Remark 2.2.4. *One can be more precise with the notion of torsion by defining the torsion tensor $\tau = \nabla\varphi$. By decomposing τ into irreducible components under the action of G_2 one can define several classes of G_2 structure. We shall consider only the torsion free case so we omit further discussion of torsion classes.*

The following result of Fernandez-Gray [32] gives an alternate characterisation of G_2-manifolds:

Theorem 2.2.5. *Let (M, φ) be a G_2-structure manifold. Then the following are equivalent:*

1. $\mathrm{Hol}(\nabla) \subseteq G_2$
2. $\nabla\varphi = 0$
3. $\mathrm{d}\varphi = \mathrm{d}\psi = 0$.

We say that G_2-manifolds are examples of manifolds with *exceptional holonomy*. This is since, from the list of possible Riemannian holonomy groups given by Berger [9] in 1955, the group G_2 is called exceptional since it does not fall into a wider family of holonomy groups. The only other exceptional holonomy group is Spin(7) and manifolds with this holonomy group must necessarily have dimension 8. Having holonomy reduced from

SO(n) determines certain geometric structures, for example a reduction of the holonomy group from SO($2n$) to U(n) determines a Kähler structure. In the case at hand a special holonomy manifold (and in particular a G_2-manifold) carries a parallel spinor and is therefore Ricci flat [48]. A good analogy to keep in mind is that G_2-manifolds are 7-dimensional analogues of Calabi-Yau 3-folds since these two classes of manifold arise from studying different branches of string theory. Furthermore both have parallel spinors, interesting submanifolds (called calibrated submanifolds) which are important from a string theory perspective and also provide natural settings for the study of gauge theory. We list here some of the known examles of G_2-manifolds:

Example 2.2.6.

1. *The first non-trivial examples of G_2-manifolds were given by Bryant-Salamon in [15], where they show that the total space of the vector bundles $\not{S}(S^3)$ (the spinor bundle of S^3), $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ carry metrics with full holonomy G_2, in other words $\mathrm{Hol}(\nabla) = G_2$. Their construction relies on having a large symmetry group acting (more precisely there is an isometric group action of cohomogeneity one) and this allows one to reduce the torsion free condition to an ordinary differential equation which can be solved explicitly.*

2. *Further non-compact examples have been given by Kovalev-Nordström [64] where the geometry is asymptotically cylindrical and Foscolo-Haskins-Nordström in [34] and [35] where the geometries are asymptotically locally conical and asymptotically conical respectively.*

3. *The first compact examples of G_2-manifolds were given by Joyce in [52]. This construction relies on resolving orbifold singularities that occur from quotienting the torus T^7 by a discrete group acting with fixed points. The idea is to resolve the singularites, place a G_2-structure with sufficiently small torsion on the resulting smooth space, and show that one can the perturb to gain a torsion free structure. It is important to note that this construction is non-explicit.*

4. *Further examples of compact G_2-manifolds have been found by Kovalev [63] using the so-called twisted connected sum construction. This was based on a suggestion of Donaldson and has to date proved the most fruitful method of producing compact examples. More recently compact examples have been given (via a third construction method) by Joyce-Karigiannis in [56].*

A rich source of study in G_2-geometry is the study of calibrated submanifolds. These are minimal submanifolds that are absolute minima in their homology classes and were first introduced by Harvey-Lawson in [44]. They are of particular interest since the condition of being a calibrated submanifold is a first order PDE, whereas the minimal submanifold equation is second order, and solutions are automatically stable for the area functional. These remarkable properties are mirrored by G_2-instantons as will be explained in Chapter 3. Furthermore these two objects, calibrated submanifolds and instantons, interact through bubbling phenomena. For that reason we give here a brief overview of calibrated submanifolds in G_2-geometry.

Definition 2.2.7. *Let (M, g) be a Riemannian manifold. A k-form $\kappa \in \Omega^k(M)$ is called a calibration if*

$$\mathrm{d}\kappa = 0 \tag{2.1}$$

$$\kappa(e_1, \dots e_k) \leq 1 \tag{2.2}$$

holds for any set of unit tangent vectors $e_1, \dots e_k$ of T_pM and for each $p \in M$.

It is natural to ask which submanifolds yield equality in (2.2). This motivates the next definition:

Definition 2.2.8. *Let $\kappa \in \Omega^k(M)$ be a calibration. A k-dimensional oriented submanifold N is said to be calibrated by κ if*

$$\kappa|_N = \mathrm{Vol}_N.$$

That is, we have $\kappa(e_1, \dots e_k) = 1$ for every orthonormal basis $e_1, \cdots, e_k$ of T_pN.

The next proposition highlights why calibrated submanifolds are such worthwhile objects of study:

Proposition 2.2.9 ([53, Proposition 3.7.2])**.** *Let N be a compact calibrated submanifold. Then N minimises volume within its homology class.*

The case relevant to this thesis is when $M = (M^7, \varphi)$ is a G_2-manifold. In this case both the 3-form φ and the 4-form ψ are calibrations.

Definition 2.2.10. *Let (M, φ) be a G_2-manifold. We call submanifolds that are calibrated by φ* associative 3-folds*. Similarly submanifolds calibrated by ψ are called* coassociative 4-folds.

Whilst the Poincaré Lemma ensures that $\mathbb{R}^7$ can have no compact calibrated submanifolds, the situation is different for the Bryant-Salamon manifolds.

Example 2.2.11. *Recall that the Bryant-Salamon manifolds are topologically the total spaces of certain vectors bundles. In each case the zero section of the vector bundle defines a compact calibrated submanifold for the G_2-structure.*

1. *On $\$(S^3)$, the spinor space of S^3, the zero section defines an associative 3-fold for the G_2-structure. This is the unique compact associative submanifold for the Bryant-Salamon G_2-structure.*

2. *On the other two Bryant-Salamon manifolds $\Lambda^2_-(\mathbb{C}P^2)$ and $\Lambda^2_-(S^4)$, the zero sections define compact coassociative 4-folds for the G_2-structures. In both cases the zero section is the unique compact coassociative submanifold for the ambient manifold.*

The deformation theory of associative and coassociative submanifolds was initiated by McLean [73], where he showed that the deformations are controlled by Dirac-type operators. When N is a compact associative, the deformation theory is controlled by a self-adjoint twisted Dirac operator- hence (since associatives are odd dimensional) the virtual dimension of the moduli space of deformations is 0. In contrast the moduli space of deformations of a coassociative N is a smooth manifold with dimension $b^2_+(N)$ (the dimension of the space of harmonic self-dual 2-forms on N).

On any manifold with a G_2-structure there is a decomposition of the exterior bundles determined by the irreducible representations of the group G_2. We denote this splitting $\Lambda^k(T^*M) = \bigoplus_d \Lambda^k_d$ where $\Lambda^k_d(T^*M)$ is a rank d vector bundle, fiberwise isomorphic to an irreducible representation of G_2 of dimension d. The splitting is

$$\begin{aligned}\Lambda^1(T^*M) &= \Lambda^1_7\\ \Lambda^2(T^*M) &= \Lambda^2_7 \oplus \Lambda^2_{14}\\ \Lambda^3(T^*M) &= \langle\varphi\rangle_{\mathbb{R}} \oplus \Lambda^3_7 \oplus \Lambda^3_{27}.\end{aligned}$$

Furthermore, the Hodge star operator yields isomorphic splittings

$$\Lambda^{7-k}(T^*M) = \bigoplus_d * \left(\Lambda^k_d(T^*M)\right).$$

We have explicit models for these spaces as follows:

$$\Lambda^2_7 = \{u \lrcorner \varphi \,;\, u \in \Lambda^1(T^*M)\} \tag{2.3}$$

$$\Lambda^2_{14} = \{\alpha \in \Lambda^2(T^*M) \,;\, \alpha \wedge \psi = 0\} = \{\alpha \in \Lambda^2(T^*M) \,;\, *(\alpha \wedge \varphi) = -\alpha\} \tag{2.4}$$

$$\Lambda^3_7 = \{u \lrcorner \psi \,;\, u \in \Lambda^1(T^*M)\} \tag{2.5}$$

$$\Lambda^3_{27} = \{\eta \in \Lambda^3(T^*M) \,;\, \eta \wedge \varphi = \eta \wedge \psi = 0\}. \tag{2.6}$$

Note that the fibers of Λ^2_{14} are isomorphic to $\mathfrak{g}_2$, viewed via the standard association $\mathfrak{g}_2 \subset \mathfrak{so}(7) \cong \Lambda^2(\mathbb{R}^7)^*$. The fibers of Λ^3_{27} are isomorphic to $\mathrm{Sym}^2_0(\mathbb{R}^7)$ and the map between these two spaces is given explicitly in [80].

The spinor bundle of a 7-manifold is constructed from an irreducible representation of Spin(7) arising by restricting a representation of the Clifford algebra $Cl(\mathbb{R}^7) \cong \mathrm{Mat}_{\mathbb{R}}(8) \oplus \mathrm{Mat}_{\mathbb{R}}(8)$. There are two choices of real spinor representation W^+ and W^-– they are both 8 dimensional and distinguished by the fact that the volume form Vol_7 acts as ± 1 on $W^\pm$. The resulting spin bundle is independent of this choice [65]. We make that choice that the volume form acts as $+1$ since this will ensure our formula for Clifford multiplication is the standard one in the literature. We denote by s_7 a fixed unit parallel spinor on M. The stabiliser of s_7 is G_2 and this leads to another description

$$\Lambda^2_{14} = \{\alpha \in \Lambda^2(T^*M)\,;\, \alpha \cdot s_7 = 0\}$$

of the bundle Λ^2_{14} corresponding to the adjoint representation of $\mathfrak{g}_2$. The map $\Lambda^0 \oplus \Lambda^1 \to \$(M), (f+v) \mapsto (f+v)\cdot s_7$ is an isomorphism [2] so that

$$\$(M) \cong \Lambda^0 \oplus \Lambda^1. \tag{2.7}$$

The following result is stated in [2] but no proof is given, so we give a proof here for completeness.

Lemma 2.2.12. *The 3-form φ and 4-form ψ act with the following eigenvalues on the subspaces of $\$(M)$:*

	Λ^0	Λ^1
φ	-7	1
ψ	-7	1

Proof. Since φ is G_2 invariant Schur's lemma says that it preserves this decomposition since Λ^0 and Λ^1 are irreducible representations of G_2. Furthermore it must act as a constant on each space and the action is traceless. We first consider the case Λ^0: We have that $\psi = *\varphi = \varphi \cdot \mathrm{Vol}_7$ and a direct calculation shows that $\varphi \cdot \psi = 7\mathrm{Vol}_7 - 6\varphi$. Let $\varphi \cdot s_7 = \lambda s_7$ then we find

$$\begin{aligned} \varphi \cdot \psi \cdot s_7 &= \varphi^2 \cdot \mathrm{Vol}_7 \cdot s_7 = \lambda^2 s_7 \\ &= (7\mathrm{Vol}_7 - 6\varphi)\cdot s_7 = (7-6\lambda)s_7. \end{aligned}$$

Therefore $\lambda = -7$ or $\lambda = 1$. The eigenvalues of φ acting on Λ^1 satisfy the same equation and since φ is traceless we must have that φ acts as -7 on Λ^0 and as 1 on Λ^1. ☐

Remark 2.2.13. *Had we instead chosen* Vol_7 *to act as -1, the eigenvalues of φ would differ from those above by a minus sign, while the eigenvalues of ψ are independent of this choice.*

An argument similar to those of [69] and [18] yields the following corollary:

Corollary 2.2.14. *Let $\alpha \in \Omega^2(M)$, then*

$$\alpha \cdot s_7 = *(\alpha \wedge \psi)\cdot s_7.$$

Proof. Since the Λ^2_{14} component of α annihilates s_7 we have that

$$\alpha \cdot s_7 = \pi_7(\alpha)\cdot s_7.$$

Now $\pi_7(\alpha) = v \lrcorner \varphi$ for some $v \in \Omega^1(M)$ so Lemma 2.2.12 says

$$\begin{aligned}\alpha \cdot s_7 = (v \lrcorner \varphi) \cdot s_7 &= -\frac{1}{2}\{v, \varphi\} \cdot s_7 \\ &= 3v \cdot s_7.\end{aligned}$$

To find v, note that for $v \in \Lambda^1$ we have $*(*(v \wedge \psi) \wedge \psi) = 3v$ (see [14] for details) and thus we can calculate

$$\begin{aligned}*(\alpha \wedge \psi) = *(\pi_7(\alpha) \wedge \psi) &= *((v \lrcorner \varphi) \wedge \psi) \\ &= 3v\end{aligned}$$

so that $v = \frac{1}{3} * (\alpha \wedge \psi)$ and the result follows. □

Corollary 2.2.15. *Let $f \in \Omega^0(M)$ and $u, v \in \Omega^1(M)$. Then Clifford multiplication of the spinor $(f + v) \cdot s_7$ by u is*

$$\mathrm{cl}(u)(f + v) \cdot s_7 = (-\langle u, v\rangle + fu + *(u \wedge v \wedge \psi)) \cdot s_7.$$

Recall the Dirac operator $D \colon \Gamma(\$(M)) \to \Gamma(\$(M))$ is given in a local orthonormal frame e^i of T^*M by the formula $D(s) = e^i \cdot \nabla_i s$. It is easily verified that

$$D((f + v) \cdot s) = (\mathrm{d}^* v + \mathrm{d}f + *(\mathrm{d}v \wedge \psi)) \cdot s_7$$

so we can write the Dirac operator as the 2×2 matrix

$$D = \begin{pmatrix} 0 & \mathrm{d}^* \\ \mathrm{d} & *(\psi \wedge \mathrm{d}\,\cdot) \end{pmatrix}. \tag{2.8}$$

This agrees with the formula given in [59] for example, although we have found the expression via different methods.

2.3 Nearly Kähler Manifolds

Let (Σ, g) be a Riemannian spin manifold and let $\$(\Sigma)$ denote the real spinor bundle associated to Σ. A spinor $s \in \Gamma(\$(\Sigma))$ is called a *real Killing spinor* if there exists a non-zero real constant λ such that

$$\nabla_X s = \lambda X \cdot s \tag{2.9}$$

for all $X \in \Gamma(T\Sigma)$ and where ∇ is the Levi-Civita connection acting on the spin bundle. From here on we fix Σ to be 6-dimensional.

Definition 2.3.1. *A 6-manifold (Σ, g) together with a real Killing spinor $s_6 \in \Gamma(\$(\Sigma))$ is called a* nearly Kähler 6-manifold.

By scaling the metric, we can also scale the constant λ. In order that the Killing spinor lifts to a parallel spinor on the cone we shall fix

$$\lambda = \frac{1}{2}.$$

Such a manifold is Einstein with $\mathrm{Ric} = 5g$. The group fixing a Killing spinor on a 6-manifold is SU(3) and the spinor defines an SU(3)-structure on Σ as described in [18]. This structure can be described in terms of a non-vanishing holomorphic (3, 0)-form Ω and an almost complex structure J which allows us to define a fundamental 2-form $\omega = g(J\cdot, \cdot)$. In a suitable local orthonormal frame these take the form

$$\begin{aligned}\Omega &= (e^1 + ie^2) \wedge (e^3 + ie^4) \wedge (e^5 + ie^6)\\ \omega &= e^{12} + e^{34} + e^{56}.\end{aligned}$$

On a nearly Kähler manifold ω is not closed but the following equations are satisfied:

$$\mathrm{d}\omega = 3\mathrm{Im}\Omega, \qquad \mathrm{dRe}\Omega = 2\omega^2. \tag{2.10}$$

In fact, a 6-manifold is nearly Kähler if and only if it is an SU(3)-structure manifold such that (2.10) holds [37]. Observe that (2.10) shows that the almost complex structure J is non-integrable, since comparing the type of ω and $\mathrm{d}\omega$ shows that the equation $\mathrm{d} = \partial + \overline{\partial}$ does not hold. This almost complex structure satisfies

$$(\nabla_X J)X = 0 \tag{2.11}$$

and this condition is also equivalent to the existence of a real killing spinor in dimension 6 [40]. Since (2.11) makes sense in any even dimension one can define a nearly Kähler $2n$-manifold to be an almost hermitian manifold such that this equation is satisfied. Nearly Kähler manifolds were first studied by Wolf and Gray [89, 90, 91] and are sometimes referred to as Gray manifolds.

The SU(3)-structure has non-vanishing torsion and therefore the holonomy group of the Levi-Civita connection ∇ need not be an SU(3) subgroup. There is however a distinguished connection on the tangent bundle with skew parallel torsion and holonomy SU(3). This connection ∇^{can} is known as the *canonical connection* and is defined via the formula

$$g(\nabla^{\mathrm{can}}_X Y, Z) = g(\nabla_X Y, Z) + \frac{1}{2}\mathrm{Re}\Omega(X, Y, Z) \tag{2.12}$$

for all $X, Y, Z \in \Gamma(T\Sigma)$. It proves useful to define a one parameter family of connections interpolating between the Levi-Civita connection and the canonical connection by setting

$$g(\nabla^t_X Y, Z) = g(\nabla_X Y, Z) + \frac{t}{2}\mathrm{Re}\Omega(X, Y, Z) \tag{2.13}$$

for $t \in \mathbb{R}$. The torsion tensor T^t of the connection ∇^t is

$$g(X, T^t(Y, Z)) = t\mathrm{Re}\Omega(X, Y, Z).$$

The fact that the holonomy group of the canonical connection $\nabla^{\mathrm{can}} = \nabla^1$ is a subgroup of SU(3) follows from the equation

$$\nabla^t_X s_6 = \frac{1-t}{2} X \cdot s_6.$$

For the purposes of this thesis, the most important examples of nearly Kähler 6-manifolds will be the homogeneous ones. There are precisely four such manifolds

$$S^6 = G_2/\mathrm{SU}(3), \qquad S^3 \times S^3 = \mathrm{SU}(2)^3/\mathrm{SU}(2)$$
$$\mathbb{CP}^3 = \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1), \qquad \mathbb{F}_{1,2,3} = \mathrm{SU}(3)/\mathrm{U}(1)^2.$$

Remark 2.3.2. *Before the work of Foscolo-Haskins [33] in 2015 the only known nearly Kähler 6-manifolds were the homogeneous ones. The authors showed that S^6 and $S^3 \times S^3$ admit cohomogeneity one nearly Kähler structures.*

In each case the homogeneous space G/H is *reductive*, meaning there is a splitting $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ with $\mathfrak{m}$ closed under the adjoint action of H. Here $\mathfrak{h} = \mathrm{Lie}(H)$ is the Lie algebra of $\mathfrak{h}$ and not a Cartan subalgeba of $\mathfrak{g}$. These coset spaces are called *3-symmetric* since in each case the subgroup H is fixed by an automorphism s of G satisfying $s^3 = \mathrm{Id}$. The induced Lie algebra automorphism S also satisfies $S^3 = \mathrm{Id}$. This acts trivially on $\mathfrak{h}$ and non-trivially on $\mathfrak{m}$; one defines a almost complex structure $J \colon \mathfrak{m} \to \mathfrak{m}$ via

$$S|_{\mathfrak{m}} = -\frac{1}{2} + \frac{\sqrt{3}}{2} J. \tag{2.14}$$

The Riemannian metric on each space is determined by the Killing form on $\mathfrak{g}$. In [75] it is shown that $-\frac{1}{12}$ is the normalisation that yields $\lambda = \frac{1}{2}$ in the Killing spinor equation (2.9), so the metric is induced from the bilinear form

$$B(X, Y) = -\frac{1}{12} \mathrm{Tr}_{\mathfrak{g}}(\mathrm{ad}(X)\mathrm{ad}(Y)) \qquad \forall X, Y \in \mathfrak{g}. \tag{2.15}$$

We will refer to (2.15) as the *nearly Kähler metric* on $\mathfrak{g}$. Extending this metric by left translation furnishes G/H with a G-invariant metric. The $(1,1)$-form ω is then defined as $\omega(\cdot, \cdot) = g(J\cdot, \cdot)$ and we require the holomorphic volume form Ω to be a G-invariant $(3,0)$-form with $|\Omega|^2 = 8$ and this determines Ω up to a phase, i.e up to multiplication by a unit complex number.

Following [42] we note that the tensors defining the SU(3)-structure admit local expressions as follows: Let $\{I_A\}$ be a basis for $\mathfrak{g}$ such that I_a for $1 \leq a \leq 6$ forms a basis for $\mathfrak{m}$ and I_i for $7 \leq i \leq \dim(G)$ forms a basis of $\mathfrak{h}$. Assume furthermore that this basis is orthonormal with respect to B. This basis can be represented by left invariant vector fields

$\hat{E}_A$ on G and also by the dual basis $\hat{e}^A$ of left invariant 1-forms. Denote by $\pi\colon G \to G/H$ the natural projection map $g \mapsto gH$ of the canonical bundle. Over an contractible set U the canonical bundle admits a local section, in other words a map $L\colon U \to G$ such that $\pi \circ L = \mathrm{id}_U$. Put $e^A = L^*\hat{e}^A$, then e^a form an orthonormal basis for $T^*(G/H)|_U$ and we can therefore write $e^i = e^i_a e^a$ with locally defined smooth functions e^i_a. Let us denote by E_a the dual frame, then this frame trivialises $T(G/H)$ over U.

Let f^C_{AB} be the structure constants defined by $[I_A, I_B] = f^C_{AB} I_C$ and use the nearly Kähler metric to lower an index $f_{ABC} \coloneqq f^D_{AB}\delta_{DC}$. The Maurer-Cartan equations take the form

$$\mathrm{d}e^a = -f^a_{ib} e^i \wedge e^b - \frac{1}{2} f^a_{bc} e^b \wedge e^c, \tag{2.16}$$

$$\mathrm{d}e^i = -\frac{1}{2} f^i_{bc} e^b \wedge e^c - \frac{1}{2} f^i_{jk} e^j \wedge e^k. \tag{2.17}$$

The components of the complex structure J defined in (2.14) are defined by $J(I_a) = J_{ab} I_b$ and local expressions for the metric, almost complex structure and nearly Kähler form are given by

$$g = \delta_{ab} e^a \otimes e^b \tag{2.18}$$

$$J = J_{ab} e^a \otimes E_b \tag{2.19}$$

$$\omega = \frac{1}{2} J_{ab} e^a \wedge e^b. \tag{2.20}$$

The final local coordinate expression we shall need is for the 3-form $\mathrm{Re}\Omega$ and this is

$$\mathrm{Re}\Omega = -\frac{1}{6} f_{abc} e^a \wedge e^b \wedge e^c. \tag{2.21}$$

Reductive homogeneous spaces come with a distinguished connection, also called the canonical connection, which lives on on the principal H bundle $G \to G/H$. This is the connection whose horizontal distribution is given by left translation of $\mathfrak{m}$. The tangent bundle $T(G/H)$ is associated to $G \to G/H$ via the representation $\mathfrak{m}$ of H. The canonical connection coming from the reductive homogeneous structure therefore defines a connection on $T(G/H)$ and this agrees with the connection (2.12), justifying the nomenclature.

Analogously to the G_2 case the exterior bundles on a nearly Kähler 6-manifold split according to how the fibres split as representations of SU(3). The splitting $\Lambda^k(T^*\Sigma) = \bigoplus_d \Lambda^k_d$, where Λ^k_d has fibre dimension d, is as follows:

$$\Lambda^1(T^*\Sigma) = \Lambda^1_6 \tag{2.22}$$

$$\Lambda^2(T^*\Sigma) = \langle\omega\rangle_{\mathbb{R}} \oplus \Lambda^2_6 \oplus \Lambda^2_8 \tag{2.23}$$

$$\Lambda^3(T^*\Sigma) = \langle\mathrm{Re}\Omega\rangle_{\mathbb{R}} \oplus \langle\mathrm{Im}\Omega\rangle_{\mathbb{R}} \oplus \Lambda^3_6 \oplus \Lambda^3_{12} \tag{2.24}$$

and there are isometric splittings $\Lambda^{6-k} = \bigoplus_d *(\Lambda^k_d)$. These spaces are modelled as follows:

$$ \tag{2.25} $$

$$\Lambda^2_6 = \mathrm{Re}\left(\Lambda^{2,0} \oplus \Lambda^{0,2}\right) = \{v \lrcorner \mathrm{Re}(\Omega)\,;\, v \in \Lambda^1\} \tag{2.26}$$

$$\Lambda^2_8 = \{\alpha \in \Lambda^2\,;\, *(\alpha \wedge \omega) = -\alpha\} \tag{2.27}$$

$$\Lambda^3_6 = \{v \wedge \omega\,;\, v \in \Lambda^1\} \tag{2.28}$$

$$\Lambda^3_{12} = \mathrm{Re}\{\gamma \in \Lambda^{2,1} \oplus \Lambda^{1,2}\,;\, \gamma \wedge \omega = 0\}. \tag{2.29}$$

The bundle Λ^2_8 has fibres isomorphic to $\mathfrak{su}(3)$. Since the action of the group SU(3) fixes the Killing spinor it is clear that the Lie algebra action annihilates it, hence we have $\alpha \cdot s_6 = 0$ for any $\alpha \in \Lambda^2_8$. The Killing spinor s_6 determines a bundle map from $\Lambda^0(T^*\Sigma) \oplus \Lambda^1(T^*\Sigma) \oplus \Lambda^6(T^*M)$ to $\not{S}(\Sigma)$, defined by $\eta \mapsto \eta \cdot s_6$ and Charbonneau-Harland [18] note this is an isomorphism:

$$\not{S}(\Sigma) \cong \Lambda^0(T^*\Sigma) \oplus \Lambda^1(T^*\Sigma) \oplus \Lambda^6(T^*M). \tag{2.30}$$

The almost complex structure can be constructed from this splitting; let Vol_6 be the volume element of the Clifford algebra, then one defines $\mathrm{Vol}_6 \cdot v \cdot s_6 = Jv \cdot s_6$ for any $v \in \Lambda^1$. The forms $\mathrm{Re}\Omega$ and $*\omega$ act as scalar multiples on the summands of this splitting:

Lemma 2.3.3 ([18, Lemma 2])**.** *The subspaces of $\not{S}(\Sigma)$ isomorphic to Λ^0, Λ^1 and Λ^6 are eigenspaces of the operations of Clifford multiplication by $\mathrm{Re}\Omega$ and $*\omega$ with the following eigenvalues*

	Λ^0	Λ^1	Λ^6
$\mathrm{Re}\Omega$	*4*	*0*	*-4*
$*\omega$	*-3*	*1*	*-3.*

We would like to understand Clifford multiplication by 1-forms under this splitting of the spin bundle. We begin with a lemma:

Lemma 2.3.4. *For any $\alpha \in \Omega^2(\Sigma)$ we have that*

$$\alpha \cdot s_6 = (-(\alpha \lrcorner \omega)\mathrm{Vol}_6 - \alpha \lrcorner \mathrm{Re}\Omega) \cdot s_6. \tag{2.31}$$

Proof. Let π_d denote projection from Λ^2 to the subspace Λ^2_d. The description (2.27) shows that $\pi_8(\alpha) \cdot s_6 = 0$, so that

$$\alpha \cdot s_6 = (\pi_1(\alpha) + \pi_6(\alpha)) \cdot s_6.$$

The π_1 component of α is a multiple of ω so we calculate

$$\omega \cdot s_6 = \mathrm{Vol}_6 \cdot *\omega \cdot s_6 = -3\mathrm{Vol}_6 \cdot s_6$$

by Lemma 2.3.3. Since $|\omega|^2 = 3$ we see that $\pi_1(\alpha) = \frac{1}{3}(\alpha \lrcorner \omega)\omega$ and thus

$$\pi_1(\alpha) \cdot s_6 = (\alpha \lrcorner \omega)\mathrm{Vol}_6 \cdot s_6.$$

We have that $\pi_6(\alpha) = v \lrcorner \mathrm{Re}\Omega$ for a unique $v \in \Lambda^1$ and we find

$$v \lrcorner \mathrm{Re}\Omega \cdot s_6 = -\frac{1}{2}\{v, \mathrm{Re}\Omega\} \cdot s_6 = -2v \cdot s_6$$

again by Lemma 2.3.3. An application of Schur's lemma shows that $v = \frac{1}{2}\alpha \lrcorner \mathrm{Re}\Omega$. Therefore

$$\pi_6(\alpha) \cdot s_6 = -(\alpha \lrcorner \mathrm{Re}\Omega) \cdot s_6.$$

Overall we see that

$$\alpha \cdot s_6 = (-(\alpha \lrcorner \omega)\mathrm{Vol}_6 - (\alpha \lrcorner \mathrm{Re}\Omega)) \cdot s_6.$$

□

Corollary 2.3.5. *Clifford multiplication of a spinor $(f + v + h\mathrm{Vol}_6) \cdot s_6$ by a 1-form $u \in \Omega^1(\Sigma)$ is given by*

$$\mathrm{cl}(u)(f+v+h\mathrm{Vol}_6)\cdot s_6 = (-(u \lrcorner v)+fu-hJu-(u\wedge v)\lrcorner \mathrm{Re}\Omega-((u\wedge v)\lrcorner \omega)\mathrm{Vol}_6)\cdot s_6. \quad (2.32)$$

Proof. Since the volume form anti-commutes with 1-forms we have that

$$\mathrm{cl}(u)(f + v + h\mathrm{Vol}) \cdot s_6 = (fu + u \wedge v - (u \lrcorner v) - hJu) \cdot s_6$$

and the term $(u \wedge v) \cdot s_6$ is handled using the previous lemma. □

We can apply these results to understand the Dirac operator on a nearly Kähler manifold. Let $D^t = \mathrm{cl} \circ \nabla^t \colon \Gamma(\$(\Sigma)) \to \Gamma(\$(\Sigma))$ be the Dirac operator constructed from the connection ∇^t acting on the spin bundle. Charbonneau-Harland [18] show that these operators differ by a multiple of the action of $\mathrm{Re}\Omega$ on the spin bundle:

$$D^t = D^0 + \frac{3t}{4}\mathrm{Re}\Omega \quad (2.33)$$

where D^0 is the Levi-Civita Dirac operator. Note that, since $\mathrm{Re}\Omega \cdot v \cdot s_6 = 0$ for any 1-form v, this family of operators have the same action on 1-forms, namely

$$D^t(v \cdot s_6) = (\mathrm{d}v + \mathrm{d}^*v + 2v) \cdot s_6.$$

Corollary 2.3.6. *Under the splitting $\$(\Sigma) \cong \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^6$ the Dirac operator is*

$$D^t = \begin{pmatrix} -3+3t & \mathrm{d}^* & 0 \\ \mathrm{d} & 2-(\mathrm{d}\,\cdot)\lrcorner \mathrm{Re}\Omega & J\,\mathrm{d}* \\ 0 & *\langle \mathrm{d}\,\cdot, \omega\rangle & 3-3t \end{pmatrix}. \quad (2.34)$$

2.4 Asymptotically Conical G_2-Manifolds

The link between Nearly Kähler and G_2 geometry is via the cone construction. It was noted by Hitchin in [48] that manifolds carrying parallel spinors can be defined by their holonomy groups. Following this Bär [5] observed that a manifold admits a real Killing spinor if and only the holonomy of its Riemannian cone fixes a spinor and hence the cone of a nearly Kähler 6-manifold has holonomy group G_2, this being the group that fixes a spinor in dimension 7. One can describe the G_2-structure on the cone of a nearly Kähler 6-manifold explicitly:

Definition 2.4.1. *Let (Σ^6, g_6) be nearly Kähler. The G_2 cone of Σ is $C(\Sigma) = (0,\infty)\times\Sigma$ together with the torsion-free G_2 structure $(C(\Sigma), \varphi_C)$ defined by*

$$\varphi_C = r^2\omega \wedge \mathrm{d}r + r^3 \mathrm{Im}\Omega$$

where r is the coordinate on $(0,\infty)$. The metric determined by φ_C is the cone metric $g_C = \mathrm{d}r^2 + r^2 g_6$. We choose the orientation such that $\mathrm{d}r \wedge r^6 \mathrm{Vol}_6$ is the volume form on C. We call C a G_2 cone and Σ the link of the cone. Finally, we denote the natural projection map $\pi\colon C(\Sigma) \to \Sigma$.

A G_2 cone is of course not complete, but we will consider complete G_2-manifolds whose geometry is *asymptotically* that of a G_2 cone. The next definition makes this notion precise.

Definition 2.4.2. *Let (M, g, φ) be a non-compact G_2-manifold. We call M an asymptotically conical (AC) G_2-manifold with rate $\mu < 0$ if there exists a compact subset $K \subset M$, a compact, connected nearly Kähler 6-manifold Σ, a constant $R > 1$ and a diffeomorphism*

$$h\colon (R,\infty)\times\Sigma \to M\setminus K$$

such that

$$|\nabla_C^j \left(h^*(\varphi|_{M\setminus K}) - \varphi_C\right)|(r,\sigma) = O(r^{\mu-j}) \qquad \text{as } r\to\infty \tag{2.35}$$

for each $\sigma\in\Sigma$, for $j = 0,1,2,\dots,$ where ∇_C is the Levi-Civita connection for the cone metric g_C on $C(\Sigma)$, φ_C is the G_2-structure on the cone and $|\cdot|$ is calculated using g_C. We call $M\setminus K$ the end of M and Σ the asymptotic link of M.

Remark 2.4.3. *We shall often drop the notation showing the dependence of a norm of the form in (2.35) on a point in Σ. Such a norm is always to be understood pointwise in Σ.*

By thinking of $|\nabla_C^j \left(h^*(g|_{M\setminus K}) - g_C\right)|$ as a function of r one can take a Taylor expansion about $r = 0$ to show that $g = g_\varphi$ satisfies the same asymptotic condition [61]

$$|\nabla_C^j \left(h^*(g|_{M\setminus K}) - g_C\right)| = O(r^{\mu-j}) \qquad \text{as } r\to\infty.$$

In the same manner one finds that

$$|\nabla_C^j \left(h^*(\psi|_{M\setminus K}) - \psi_C\right)| = O(r^{\mu-j}) \qquad \text{as } r \to \infty.$$

One must also remark that an AC G_2-manifold M can only have one end. This follows from the Cheeger-Gromoll splitting theorem applied to a complete Ricci flat manifold [61]. If M were to have more than one end it would have reducible holonomy.

Example 2.4.4. 1. $(\mathbb{R}^7, \varphi_0)$ is clearly an AC G_2-manifold with any rate $\mu < 0$ since $C(S^6) = \mathbb{R}^7 \setminus \{0\}$. In our convention $\varphi_C = \varphi_0$ and the induced metric is the Euclidean metric.

2. Recall the Bryant-Salamon manifolds are the total spaces of certain vector bundles. One can show that the geodesic distance from a point to the zero section of the bundle gives each manifold an AC G_2-structure. In the case of $\not{S}(S^3)$ the asymptotic link $\Sigma = S^3 \times S^3$ and rate -3. The other Bryant-Salamon manifolds $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ have asymptotic links are $\mathbb{CP}^3$ and $\mathbb{F}_{1,2,3}$ respectively and the rate of converge is -4 in both cases.

3. Foscolo, Haskins and Nordström find in [34] infinitely many new diffeomorphism types of AC G_2-manifolds with asymptotic link (a quotient of) $S^3 \times S^3$.

2.5 Analysis on AC Manifolds

For the remainder of this thesis, unless stated otherwise, M will be an AC G_2-manifold with asymptotic link Σ and h the diffeomorphism identifying the end of M with the cone on Σ. Weighted Sobolev spaces provide a natural setting for analysis on AC manifolds so we now give an overview of their properties following [72]. Let us denote by $M_{>t}$ for any $t > R$ the subspace of M given by $h((t, \infty) \times \Sigma)$.

Definition 2.5.1. *A radius function ρ on M is a smooth function on M that satisfies the following conditions. On the compact subset K of M, we define $\rho = 1$. Let x be a point in $M_{>2R}$, then $h^{-1}(x) = (r, y)$ for some $r \in (2R, \infty)$ and we define $\rho(x) = r$ for such a point. Finally, in the region $h((R, 2R) \times \Sigma)$, the function ρ is defined by interpolating smoothly between its definition near infinity and its definition in the compact subset K, in a decreasing fashion.*

We will need to construct spaces of sections suitable for studying elliptic equations on AC manifolds. For this let $E \to M$ be a vector bundle with a bundle metric and metric connection ∇ and denote by the subscript loc the space of sections η such that $f\eta$ lie in the desired space for all smooth compactly supported functions f on M. For example, $C^k_{\text{loc}}(E)$ denotes the space of sections that in this sense lie in $C^k(E)$.

Definition 2.5.2. *Let $p \geq 1, k \in \mathbb{N} \cup \{0\}$ and $\mu \in \mathbb{R}$. Let (M, φ) be an AC G_2-manifold and fix a radius function ρ. Let E be a vector bundle endowed with a bundle metric and a metric connection ∇. Define a norm $\|\cdot\|_{L^p_{k,\mu}}$ on $L^p_{k,loc}$ sections η of E by defining*

$$\|\eta\|_{L^p_{k,\mu}} = \left(\sum_{j=0}^{k} \int_M \left|\rho^{j-\mu} \nabla^j \eta\right|^p \rho^{-7} \, \mathrm{dVol}_g \right)^{\frac{1}{p}}$$

where $|\cdot|$ is calculated using the bundle metric. We let $L^p_{k,\mu}(E)$ denote the completion under this norm.

The weighted Sobolev spaces are Banach spaces and, when $p = 2$, Hilbert spaces. An element $\eta \in L^p_{k,\mu}(E)$ can be thought of as a section that is k times weakly differentiable such that the derivative $\nabla^j \eta$ is growing at most like $r^{\mu-j}$ on the end of M. Indeed if $|\eta| = O(\rho^\mu)$ on the end of M then $\eta \in L^p_{0,\mu+\epsilon}(E)$ for any $\epsilon > 0$.

Denote $L^p_\mu = L^p_{0,\mu}$, then we have a weighted Hölder inequality [7]

$$\|\xi \otimes \eta\|_{L^p_{\mu+\nu}} \leq \|\xi\|_{L^q_\mu} \|\eta\|_{L^{q'}_\nu}$$

where $\frac{1}{p} = \frac{1}{q} + \frac{1}{q'}$. As in the familiar L^p case, this gives a duality pairing provided we use the correct weight $(L^p_\mu(E))^* \cong L^q_{-7-\mu}(E)$ where $\frac{1}{p} + \frac{1}{q} = 1$. We will mostly work with the Hilbert spaces $L^2_{k,\mu}(E)$. For such spaces one has that $\left(L^2_{k,\mu}(E)\right)^* \cong L^2_{-k,-7-\mu}(E)$, but in practice we will only ever be interested in the kernel of an elliptic operator on such a space and we shall see that regularity properties ensure this kernel is in fact independent of k. In this way we can always work with Sobolev spaces with a positive degree of differentiability.

We will also require weighted C^k and C^∞ spaces.

Definition 2.5.3. *Let $\mu \in \mathbb{R}$ and let $k \in \mathbb{N} \cup \{0\}$. The weighted C^k space $C^k_\mu(E)$ is the subspace of $C^k_{loc}(E)$ such that the norm*

$$\|\eta\|_{C^k_\mu} = \sum_{j=0}^{k} \sup_M |\rho^{j-\mu} \nabla^j \eta|$$

is finite. We also define $C^\infty_\mu(E) = \cap_{k \geq 0} C^k_\mu(E)$. The spaces C^k_μ are Banach spaces but C^∞_μ need not be.

The usual embedding theorems for Sobolev spaces can be adapted to the weighted case, we state here only the results needed for our purposes. The theorem is stated using μ-weighted Hölder spaces $C^{l;\alpha}_\mu(E)$, where $\alpha \in (0, 1)$ is the Hölder exponent. These are spaces of sections with l continuous derivatives and controlled growth on the end of M. The definition of a weighted Hölder space is stated here so that we can properly state the relevant embedding theorems.

Definition 2.5.4. *Let $\alpha \in (0,1)$, let $k \in \mathbb{N}$ and let $\mu \in \mathbb{R}$. Let $d(x,y)$ be the geodesic distance between points $x,y \in M$, let $0 < c_1 < 1 < c_2$ be constants and let*

$$H = \{(x,y) \in M \times M\,;\, x \neq y, c_1\rho(x) \leq \rho(y) \leq c_2\rho(x) \text{ and there exists a geodesic in } M \text{ of length } d(x,y) \text{ from } x \text{ to } y\}.$$

A section η of a vector bundle E, endowed with a bundle metric and a metric connection ∇ is called Hölder continuous with exponent α *if*

$$[\eta]^\alpha = \sup_{(x,y)\in H} \frac{|\eta(x) - \eta(y)|_E}{d(x,y)^\alpha} < \infty.$$

Here ∇ is used to identify the fibers E_x and E_y via parallel transport along a geodesic γ connecting x and y (note we can find such a geodesic since $(x,y) \in H$).

The weighted Hölder space $C^{k,\alpha}_\mu(E)$ *of sections of E is the subspace of $C^{k,a}_{loc}(E)$ such that the norm*

$$\|\eta\|_{C^{k,\alpha}_\mu} = \|\eta\|_{C^k_\mu} + [\eta]^{k,\alpha}_\mu$$

is finite, where

$$[\eta]^{k,\alpha}_\mu = [\rho^{k+\alpha-\mu}\nabla^k\eta]^\alpha.$$

The spaces $C^{k,\alpha}_\mu(E)$ is a Banach space and there are embeddings $C^{k,\alpha}_\mu(E) \to C^l_\mu(E)$ whenever $l \leq k$. With this definition in hand we can state the Sobolev embedding theorem we require:

Theorem 2.5.5 (Weighted Sobolev Embedding Theorem). *Let $k,l \geq 0$ and let $\alpha \in (0,1)$.*

1. *If $k - \frac{7}{p} \geq l + \alpha$ then there is a continuous embedding $L^p_{k,\mu}(E) \hookrightarrow C^{l,\alpha}_\mu(E)$.*
2. *If $k \geq l \geq 0$ and $k - \frac{7}{p} \geq l - \frac{7}{q}$, $p \leq q$ and $\mu \leq \nu$ then there is a continuous embedding $L^p_{k,\mu}(E) \hookrightarrow L^q_{l,\nu}(E)$.*

We choose to work with the spaces $L^2_{k,\mu}$ for $k \geq 4$ so that the sections we consider are continuous. We can use the weighted embedding theorem to form a multiplication theorem, adapting the argument of [19] to the weighted setting.

Theorem 2.5.6 (Weighted Sobolev Multiplication Theorem). *Let $\xi \in L^2_{k,\mu}(E)$, $\eta \in L^2_{l,\nu}(F)$ and suppose $l \geq k$. If $k > \frac{7}{2}$ then multiplication $L^2_{k,\mu}(E)\times L^2_{l,\nu}(F) \hookrightarrow L^2_{k,\mu+\nu}(E\otimes F)$ is bounded, i.e there exists a constant $C > 0$ such that*

$$\|\xi \otimes \eta\|_{k,\mu+\nu} \leq C\|\xi\|_{k,\mu}\|\eta\|_{l,\nu}.$$

Proof. It suffices to show that each of the first k derivatives of $\xi \otimes \eta$ are in $L^2_{\mu+\nu}$.

- Claim 1: If $i > \frac{7}{2}$ then there's a continuous pairing $L^2_{i,\mu}(E) \times L^2_{j,\nu}(F) \to L^2_{\mu+\nu}(E \otimes F), (\kappa, \tau) \mapsto \kappa \otimes \tau$ for any $j \geq 0$.
  Proof of claim 1: Let $\xi \in L^2_{i,\mu}(E)$, then since $i > \frac{7}{2}$ one notes that Theorem 2.5.5 ensures $\xi \in C^0_\mu(E)$, so that $\rho^{-\mu}\xi$ is bounded. It follows that $\xi \otimes \eta \in L^2_{\mu+\nu}$ as required.

- Claim 2: If $i \leq \frac{7}{2}, j \leq \frac{7}{2}$ and $i + j > \frac{7}{2}$ then there is a continuous pairing $L^2_{i,\mu}(E) \times L^2_{j,\nu}(F) \to L^2_{\mu+\nu}(E \otimes F)$ given by $(\kappa, \tau) \mapsto \kappa \otimes \tau$.
  Proof of claim 2: Since $i + j > \frac{7}{2}$ there exist q, q' such that $i > \frac{7}{2} - \frac{7}{q},\ \ j > \frac{7}{2} - \frac{7}{q'}$ and $\frac{1}{q} + \frac{1}{q'} = \frac{1}{2}$. Then we are in a position to apply part 2 of Theorem 2.5.5 to see there are embeddings

$$L^2_{i,\mu}(E) \hookrightarrow L^q_{0,\mu}(E) = L^q_\mu(E)$$
$$L^2_{j,\nu}(F) \hookrightarrow L^{q'}_{0,\nu}(F) = L^{q'}_\nu(F)$$

and thus there exists a constant C independent of ξ, η so that

$$\|\kappa\|_{L^q_\mu} \leq C\|\kappa\|_{L^2_{i,\mu}}$$
$$\|\tau\|_{L^{q'}_\nu} \leq C\|\tau\|_{L^2_{j,\nu}}.$$

Now applying the weighted Hölder inequality we get

$$\|\kappa \otimes \tau\|_{L^2_{\mu+\nu}} \leq \|\kappa\|_{L^q_\mu}\|\tau\|_{L^{q'}_\nu} \leq C^2\|\kappa\|_{L^2_{i,\mu}}\|\tau\|_{L^2_{j,\nu}}$$

which proves the claim.

So let $\xi \in L^2_{k,\mu}(E)$ and $\eta \in L^2_{l,\nu}(E)$. To conclude the proof we first note that $\nabla^i(\xi \otimes \eta) = \sum_j C^j_i \nabla^{i-j}\xi \otimes \nabla^j \eta$ and furthermore $\nabla^{i-j}\xi \in L^2_{k-(i-j),\mu}$ and $\nabla^j \eta \in L^2_{l-j,\nu}$ by the hypothesis of the theorem. Now if $k - (i - j) > \frac{7}{2}$ or $l - j > \frac{7}{2}$ then we may appeal to claim 1 to see that $\xi \otimes \eta \in L^2_{\mu+\nu}$. If this does not hold we notice that

$$k - (i - j) + l - j = k + l - i \geq k > \frac{7}{2}$$

so we may apply claim 2 instead. □

At several instances we shall need to apply the implicit function theorem when working with these Banach spaces. We state here two versions that will be used in subsequent chapters.

Theorem 2.5.7 (Implicit Function Theorem, Version 1 [27]). *Let $\mathcal{X}_1, \mathcal{X}_2$ and $\mathcal{Y}$ be Banach spaces and suppose $\mathcal{F}\colon \mathcal{X}_1 \times \mathcal{X}_2 \to \mathcal{Y}$ is a smooth map with partial derivatives $\mathrm{d}_i\mathcal{F}$. If the partial derivative $(\mathrm{d}_2\mathcal{F})$ at a point (ξ_1, ξ_2) is surjective and admits a bounded right inverse then for all η_1 near ξ_1 there is a solution η_2 to the equation*

$$\mathcal{F}(\eta_1, \eta_2) = \mathcal{F}(\xi_1, \xi_2).$$

Theorem 2.5.8 (Implicit Function Theorem, Version 2 [69])**.** *Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces and let $\mathcal{U} \subset \mathcal{X}$ be an open neighbourhood of* 0*. Let $\mathcal{F}\colon \mathcal{U} \to \mathcal{Y}$ be a C^k map, for some $k \geq 1$, with $\mathcal{F}(0) = 0$. Suppose $d\mathcal{F}|_0\colon \mathcal{X} \to \mathcal{Y}$ is surjective with kernel $\mathcal{K}$ such that $\mathcal{X} = \mathcal{K} \oplus \mathcal{Z}$ for some closed subspace $\mathcal{Z}$.*
Then there exists open sets $\mathcal{V} \subset \mathcal{K}$ and $\mathcal{W} \subset \mathcal{Z}$ both containing 0*, with $\mathcal{V} \times \mathcal{W} \subset \mathcal{U}$ and a unique C^k map $\mathcal{G}\colon \mathcal{V} \to \mathcal{W}$ such that*

$$\mathcal{F}^{-1}(0) \cap (\mathcal{V} \times \mathcal{W}) = \{(x, \mathcal{G}(x))\colon x \in \mathcal{V}\}.$$

Chapter 3

Background From Gauge Theory

In this chapter we review the framework for studying gauge theory on G_2-manifolds, nearly Kähler 6-manifolds and homogeneous manifolds. Gauge theory is the study of connections on vector (or principal) bundles together with a gauge invariant curvature condition. Applications of gauge theory in dimension 4 and below have been hugely successful, most notably with applications to topology, with tools such as Donaldson and Seiberg-Witten theory providing novel new invariants. We shall consider gauge theory in dimension 6 and 7, so we first provide an overview of gauge theory in higher dimensions.

The study of gauge theory in dimension greater than four has a long history– the first significant result came from James Simons who announced in 1977 that Yang-Mills connections on spheres of dimension greater than four are unstable and this result was published two years later in [12]. Further results concerning stability of Yang-Mills connections on quotients of S^n and more general homogeneous spaces were obtained by Bourguignon and Lawson in 1981 in [11]. In 1983 Donaldson [24] showed that his theory of ASD instantons could be applied to give strong restrictions on the intersection form of a differentiable four manifold. Around the same time generalisations of the ASD instanton to higher dimensions were first considered by Corrigan, Devchand, Fairlie and Nuyts in [21] and included in their list of examples were the Spin(7) and G_2-instanton equations on $\mathbb{R}^8$ and $\mathbb{R}^7$ respectively. A similar analysis was later provided for a general Riemannian manifold by Reyes Carrion in [17]. The first examples of instantons on higher dimensional Euclidean spaces were given for the Spin(7) case by Fubini-Nicolai [38] and Fairlie-Nuyts [31] and for the G_2 case by Günaydin-Nicolai in [41].

Early suggestions for defining invariants from instanton equations in higher dimensions were given in [29] and [1] and around this time the first thorough study into Spin(7)-instantons was carried out in the PhD thesis of Lewis [66]. The first major success in the program came in 2000 when the Donaldson-Thomas invariant of Calabi-Yau 3-folds was presented in [85]. Some of the technicalities of defining invariants from counting G_2-instantons over compact manifolds were studied in Donaldson-Segal [28]. In particular they observed bubbling phenomena can occur for a 1-parameter family of G_2-structures.

Following this observation there has been much work to overcome the difficulties one encounters from working with these moduli spaces. To define a suitable invariant one must perform the instanton count (when the deformation theory is unobstructed the moduli space is a 0-dimensional manifold) with a weight attached to each instanton- in other words orient the moduli space as is familiar from Morse theory for example. Following important initial work in this direction by Haydys [45] a system for orienting the moduli space, by involving counts to solutions of gauge theoretic equations over associative submanifolds, has been proposed by Haydys-Walpuski in [47] and further work in this area can be found in [46] [23] and [22]. An alternate suggestion for a suitable procedure for orienting the moduli space was given by Joyce in [55] and recent related works [58, 57].

We shall consider G_2-instantons on non-compact manifolds. Early work in this area can be found in [79, 78] and the survey article [70] assesses the current state of play. The manifolds under consideration in [70] fall into two families; the first is when the geometry is asymptotically locally conical, which means it is asymptotic to a circle bundle over a 6-dimensional cone. The second is when the geometry is AC and it is the goal of this thesis to develop the deformation theory for G_2-instantons in this setting. For this we begin by reviewing the basic setup of gauge theory.

Let (X^n, g) be an oriented n dimensional Riemannaian manifold, let $P \to X$ be a principal G-bundle and let A be a connection on P. The bundle P comes with a *gauge group* $\mathcal{G} = \{\text{principal bundle isomorphisms}\colon P \to P \text{ covering the identity}\}$. The gauge group acts on the space of connections $\mathscr{A}$, heuristically via the formula

$$g(A) = gAg^{-1} - \mathrm{d}gg^{-1},$$

and one may form the space of *connections modulo gauge*, $\mathcal{B} = \mathcal{A}/\mathcal{G}$. We denote by $\mathrm{Ad}P$ the vector bundle of Lie algebra valued sections, which is associated to the bundle P via the adjoint representation of G on $\mathfrak{g} = \mathrm{Lie}(G)$. The space of connections is an affine space identified with $\Omega^1(X, \mathrm{Ad}P) = \Gamma(T^*M \otimes \mathrm{Ad}P)$ since any two connections differ by such a section. A choice of connection A on P gives rise to a linear connection acting on sections of $\mathrm{Ad}P$ which we shall denote ∇^A. This is extended in the usual way to $\Omega^*(M, \mathrm{Ad}P)$ and for a section $f \in \Gamma(\mathrm{Ad}P)$ we use the notation $\nabla^A f = \mathrm{d}_A f$ interchangeably.

We assume G to be compact and semi-simple and endow the adjoint bundle with an invariant inner product $\langle\, , \rangle_{\mathfrak{g}}$ coming from the Killing form on $\mathfrak{g}$. Given two Lie algebra valued forms $\alpha \in \Omega^k(M, \mathrm{Ad}P)$ and $\beta \in \Omega^l(M, \mathrm{Ad}P)$ we can form a real valued differential form $\langle \alpha \wedge \beta \rangle_{\mathfrak{g}} \in \Omega^{k+l}(M)$ by taking the inner product of the Lie algebra valued parts of α and β and taking the wedge product of the exterior algebra valued parts as usual. The space of connections comes equipped with a natural functional, called the Yang-Mills functional, defined as

$$\mathcal{YM}(A) = \|F_A\|^2_{L^2} = \int_M \langle F_A \wedge *F_A \rangle_{\mathfrak{g}}$$

where we view F_A as a section of $\Lambda^2(T^*M) \otimes \mathrm{Ad}P$. The Euler-Lagrange equation for this functional is

$$\mathrm{d}_A^* F_A = 0$$

which is called the Yang-Mills equation, and a connection whose curvature satisfies this equation is called a Yang-Mills connection. Note that the Bianchi identity says that $\mathrm{d}_A F_A = 0$, so Yang-Mills connections can be thought of as connections with harmonic curvature. Observe that the Yang-Mills functional is gauge invariant and so descends to $\mathcal{B}$.

To seek out Yang-Mills connections one typically simplifies the problem by choosing an algebraic condition on the curvature that implies that the Yang-Mills equation holds. For example if A is flat, which is to say that F_A vanishes identically, the Yang-Mills equation trivially holds. The condition of flatness is clearly gauge invariant, since the curvature transforms as a tensor, and so the study of flat connections is a typical example of a gauge theory. As previously mentioned one can often define invariants in such a setting, as an example Taubes showed in [84] that one can recover the Casson invariant of homology 3-spheres from the moduli space of flat connections.

Another famous example comes from the study of ASD instanton connections on 4-manifolds. If X is 4-dimensional then the Hodge star operator splits the space of 2-forms on M into its ± 1 eigenspaces, i.e we have a splitting $\Lambda^2(T^*X) = \Lambda^2_+ \oplus \Lambda^2_-$. A connection A is called an ASD (anti-self-dual) instanton if it satisfies the gauge invariant condition $*F_A = -F_A$. It is clear that such a connection is Yang-Mills thanks to the Bianchi identity, but one can use Chern-Weil theory to show that such a connection is in fact an absolute minimum of the energy on a compact manifold. An ASD instanton comes with a notion of topological charge, which is simply the second Chern number $c_2(E)$ of the bundle E on which it lives. On a complex vector bundle an $\mathrm{SU}(r)$-connection has Yang-Mills energy bounded from below by $8\pi^2 c_2(E)$.

Example 3.0.1 (Standard ASD Instanton)**.** The first non-trivial ASD instanton with structure group $\mathrm{SU}(2)$ was constructed in [8], where an ansatz of rotational symmetry was posed. We provide here a very brief introduction to this ASD instanton as it is important for understanding the bubbling behaviour of G_2-instantons as was done in [71] and will be covered in Chapter 6.

Give S^3 the round metric so that $(\mathbb{R}^4 \setminus \{0\})$ is isometric to the Riemannian cone $C(S^3) = (\mathbb{R}^+ \times S^3, \mathrm{d}r^2 + r^2 g_{\mathrm{round}})$, where r is the coordinate on $\mathbb{R}^+$. We may view any connection on $\mathbb{R}^4 \times \mathrm{SU}(2)$ as a path in the space of connections on $S^3 \times \mathrm{SU}(2)$ with bounded curvature at the origin so that the connection extends over the singular point. In [8] a 1-parameter family of solutions to the ASD instanton were given, the parameter describes the concentration of the curvature around the origin. For any scaling parameter we call this the standard ASD instanton and denote it A_{ASD}. This connection extends over 0 and (fixing the scaling parameter) has

$$|F_{A_{\mathrm{ASD}}}| = \frac{1}{(1+r^2)^2}.$$

It follows that A_{ASD} has finite Yang-Mills action and in fact $\mathcal{YM}(A_{\mathrm{ASD}}) = 8\pi^2$, in other words A_{ASD} has topological charge equal to 1. If we view A_{ASD} as a path in the space of connections modulo gauge on $Q \to S^3$ then A_{ASD} is a non-trivial loop starting and ending at the trivial flat connection.

Another way of interpreting the ASD equation is as follows: Under the canonical isomorphism $\Lambda^2(\mathbb{R}^4)^* \cong \mathfrak{so}(4)$ the splitting $\Lambda^2 = \Lambda^2_+ \oplus \Lambda^2_-$ corresponds to the splitting $\mathfrak{so}(4) = \mathfrak{su}(2)_+ \oplus \mathfrak{su}(2)_-$. From this viewpoint an ASD instanton is a connection such that (denoting the bundle Λ^2_- as $\mathfrak{su}(2)_-$)

$$F_A \in \Gamma(\mathfrak{su}(2)_- \otimes \mathrm{Ad}P),$$

in other words 2-form part of the curvature lies in the $\mathfrak{su}(2)_-$ subbundle of Λ^2. One advantage of viewing the ASD equation from this point of view is that the notion of instanton generalises quite readily. Suppose the manifold X carries a K-structure and K is a strict subgroup of $\mathrm{SO}(n)$, then since $\mathfrak{k} \subset \mathfrak{so}(n)$ we get a subbundle of the bundle of 2-forms on M, which we also denote by $\mathfrak{k}$. An instanton connection A on P is then defined to be a connection such that

$$F_A \in \Gamma(\mathfrak{k} \otimes \mathrm{Ad}P).$$

Depending on the K-structure and its torsion classes such a connection may or may not be Yang-Mills. We shall see that the two cases treated in this thesis will have the desirable property of implying the Yang-Mills equation.

3.1 Gauge Theory On G_2-Manifolds

Let (M^7, φ) be a G_2-manifold and recall that M admits a splitting of the bundle of 2-forms

$$\Lambda^2(T^*M) = \Lambda^2_7 \oplus \Lambda^2_{14}$$

and the fibers of Λ^2_{14} are copies of the Lie Algebra $\mathfrak{g}_2$.

Definition 3.1.1. *Let $P \to M$ be a principal bundle and let A be a connection on P. We call A a G_2-instanton if*

$$F_A \in \Gamma(\Lambda^2_{14} \otimes \mathrm{Ad}P).$$

There are many alternative definitions of a G_2-instanton that one could give; it is useful to understand each equivalent condition so we list them here for the convenience of the reader:

- Since the fibers of Λ^2_{14} are copies of $\mathfrak{g}_2 \subset \mathfrak{so}(7)$ and G_2 is the group that stabilises the parallel spinor s_7 we see that a G_2-instanton satisfies

  $$F_A \cdot s_7 = 0$$

  where $\cdot$ denotes Clifford multiplication and only the 2-form part of F_A acts on the spinor bundle.

- A 2-form $\alpha \in \Lambda^2(\mathbb{R}^7)^*$ is in Λ^2_{14} if and only if $\alpha \wedge \psi = 0$ [80] so A is a G_2-instanton if and only if

  $$F_A \wedge \psi = 0.$$

- The space Λ^2_{14} is characterised as the -1 eigenspace of the operator $*(\varphi \wedge \cdot)\colon \Lambda^2 \to \Lambda^2$, so the G_2-instanton equation can also be written

  $$-F_A = *(\varphi \wedge F_A).$$

  This version of the equation can readily be seen as a generalisation of the anti-self-duality equation that defines an ASD instanton; an instanton is a connection whose curvature lies in the -1 eigenspace of an operator on the space of 2-forms.

It is now easy to see that G_2-instantons are Yang-Mills, since $*F_A = -\varphi \wedge F_A$ we can use the Bianchi identity and the fact that φ is closed to see that $\mathrm{d}_A^* F_A = 0$. Similarly to the case of ASD instantons on compact manifolds when the G_2-manifold is compact G_2-instantons are not just critical for the Yang-Mills energy, they are in fact absolute minima. To see this note that, since φ is closed, the characteristic class

$$\kappa(P) = -\int_M \langle F_A \wedge F_A \rangle_{\mathfrak{g}} \wedge \varphi$$

is a topological invariant of the bundle. We assume that the chosen invariant inner product is given by taking the trace (to avoid cumbersome constants) and calculate

$$\begin{aligned}
\kappa(P) &= -\int_M \langle F_A \wedge F_A \rangle_{\mathfrak{g}} \wedge \varphi \\
&= -\int_M \langle F_A \wedge F_A \wedge \varphi \rangle_{\mathfrak{g}} \\
&= -\langle F_A, *2F_A^7 - *F_A^{14}) \rangle_{L^2} \\
&= -2\|F_A^7\|^2 + \|F_A^{14}\|^2
\end{aligned}$$

where F_A^7 and F_A^{14} denote the projections of F_A to the orthogonal subspaces Λ^2_7 and Λ^2_{14} of Λ^2 respectively. Combining this with the Yang-Mills energy one finds that

$$\mathcal{YM}(A) = 3\|F_A^7\|^2 + \kappa(P).$$

One can also write this energy in terms of κ and $\|F_A^{14}\|$ but by assuming $\kappa \geq 0$ we can forget about this possibility. It is now clear that G_2-instantons are stable as claimed. Observe also that the G_2-instanton equation is a first order PDE, whereas the Yang-Mills equation is second order.

Following [79] one can also interpret G_2-instantons from a Chern-Simons viewpoint. Fix a reference connection A, which we assume to be a G_2-instanton, then any other connection B differs from A by an element of $\Omega^1(M, \mathrm{Ad}P)$. Note that $T\mathcal{A} = \mathcal{A} \times \Omega^1(M, \mathrm{Ad}P)$, so thinking of vectors on $\mathcal{A}$ as elements of $\Omega^1(M, \mathrm{Ad}P)$ we define a 1-form ρ by

$$\rho(B, a) = \int_M \mathrm{tr}(F_B \wedge a) \wedge \psi.$$

One can show that ρ is a closed 1-form and since $\mathcal{A}$ is contractible, it must be the exterior derivative of some function ϑ. Moreover ρ vanishes on $\mathrm{d}_B(\Omega^0(M, \mathrm{Ad}P) \cong T_B(\mathcal{G} \cdot \mathcal{A})$, the directions tangent to the gauge orbits. Thus ρ descends to the quotient space $\mathcal{B}$, and so does ϑ, at least locally. If we write $B = A + b$ then ϑ becomes the multi-valued function

$$\vartheta([B]) = \frac{1}{2} \int_M \left(\mathrm{d}_A b \wedge b + \frac{2}{3} b \wedge b \wedge b \right) \wedge \psi$$

and one can show that $\vartheta(g \cdot A)$ differs from $\theta(A)$ by $\int_\sigma \psi$ for some $\sigma \in H^4(M, \mathbb{R})$. Note ψ may not be an integral class, so it may not be possible to think of ϑ as a circle valued function. The gradient $\mathrm{d}\vartheta = \rho$ is none the less well defined on $\mathcal{B}$ [79], so if one is simply interested in the critical points of ϑ, i.e G_2-instantons, this is not too problematic.

We list here some of the known examples of G_2-instantons:

- The Levi-Civita connection of a G_2-manifold is a G_2-instanton. Thinking of the Riemann curvature tensor R as an endomorphism 2-form, the fact that the Levi-Civita connection has holonomy contained in G_2 means that $R(u, v) \in \mathfrak{g}_2$ and the symmetries of the Riemann curvature tensor means that $R \in \Gamma(\mathfrak{g}_2 \otimes \mathfrak{g}_2) \subset \Gamma(\Lambda^2(T^*M) \otimes \mathrm{End}(TM))$.

- The first non-trivial example was given by Gunäydin-Nicolai in 1995, this is a connection on $G_2 \times \mathbb{R}^7 \to \mathbb{R}^7$ and shall be referred to throughout this thesis as the standard G_2-instanton.

- Examples of G_2-instantons on SO(3) bundles over Joyce's compact G_2-manifolds have been given by Walpuski [86]. On twisted connected sum manifolds, examples have been given by Menet et. al in [74] and Walpuski [87]. Similar to the problem of constructing G_2-structures on compact manifolds, the problem of finding G_2-instantons on compact manifold yields non-explicit examples.

- Recall the Bryant-Salamon manifolds are non-compact manifolds admitting cohomegeneity one group actions. It is natural to look for instantons which are also invariant under this action. This is essentially the idea behind the example of Clarke [20] who found a one-parameter family of examples on $\cancel{S}(S^3) = \mathbb{R}^4 \times S^3$. Examples on the other Bryant-Salamon manifolds $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ were found by Oliveria in [77]. In [71] Lotay-Oliveira studied the moduli space of G_2-instantons over $\mathbb{R}^4 \times S^3$ (with both its asymptotically conical and asymptotically locally conical structures) invariant under a given group action. Notably this lead to an instanton on $\cancel{S}(S^3)$ which arises as a limiting connection of Clarke's family.

The important observation from the examples on this list is that those living over an asymptotically conical G_2-manifold have a well defined limit at infinity. In fact, the connection at infinity is an instanton connection for the nearly Kähler link of the AC G_2-manifold.

It is worth noting that the work of Huang [51], which shows that the curvature tensor of a G_2-instanton on $\mathbb{R}^7$ can not lie in L^2. More generally on a non-compact manifold, if the G_2-structure is d-*linear*, which is to say that $\varphi = \mathrm{d}\kappa$ and $|\kappa(x)| \leq 7(1 + \mathrm{dist}(x_0, x))$ for some fixed base point x_0, then the curvature of any G_2-instanton cannot lie in L^2.

3.2 Gauge Theory on Nearly Kähler 6-Manifolds

Throughout this section Σ denotes a compact nearly Kähler 6-manifold and $Q \to \Sigma$ denotes a principal bundle with compact and semi-simple structure group. Since Σ has an SU(3)-structure, the notion of instanton connection is simply that of an instanton for the SU(3)-structure. Recall that there is an orthogonal splitting

$$\Lambda^2(T^*\Sigma) = \Lambda^2_8 \oplus \Lambda^2_6 \oplus \langle\omega\rangle_{\mathbb{R}}$$

and that Λ^2_8 has fibers isomorphic to $\mathfrak{su}(3)$.

Definition 3.2.1. *Let $Q \to \Sigma$ be a principal bundle. A connection A_∞ on Q is called a nearly Kähler instanton (or a pseudo-Hermitian-Yang-Mills connection) if*

$$F_{A_\infty} \in \Gamma(\Lambda^2_8 \otimes \mathrm{Q}).$$

As in the G_2-case there are several equivalent definitions we could have given and we list them below:

- Since the stabiliser of the Killing spinor s_6 is SU(3), nearly Kähler instantons satisfy

$$F_{A_\infty} \cdot s_6 = 0.$$

- Since the decomposition of Λ^2 is orthogonal, and Λ^2_8 consists of real $(1,1)$ forms which are orthogonal to ω, the nearly Kähler instanton equation is equivalent to

$$F^{(2,0)}_{A_\infty} = F^{(0,2)}_{A_\infty} = 0, \qquad \omega \lrcorner F_{A_\infty} = 0.$$

  These equations are called the *Hermitian-Yang-Mills equations*, the name pesudo-Hermitian-Yang-Mills connection is sometimes used since we are studying these equations on a manifold with an SU(3)-structure that has non-vanishing torsion.

- The space Λ^2_8 is the -1 eigenspace for the operator on 2-forms $*(\omega \wedge \cdot)$, thus nearly Kähler instantons satisfy the anti-self-duality equation

$$-F_{A_\infty} = *(\omega \wedge F_{A_\infty}).$$

- Another equivalent formulation is that

$$F_{A_\infty} \wedge \mathrm{Im}\Omega = 0.$$

  That this last viewpoint is equivalent to the other conditions requires a little more work, the reader can find the proof in [92].

Now ω is not closed so it is not immediately obvious that A is Yang-Mills. However Xu [93] observed that $\mathrm{d}_{A_\infty} * F_{A_\infty} = -\,\mathrm{d}\omega \wedge F_{A_\infty}$ and since $\mathrm{d}\omega = 3\mathrm{Im}\Omega$ is a $(3,0)+(0,3)$ form, the wedge product $\mathrm{d}\omega \wedge F_{A_\infty}$ must vanish, so nearly Kähler instantons are Yang-Mills.

Xu [92] also studied a Chern-Simons formulation of nearly Kähler instantons. Define a function on $\mathcal{B} = \mathcal{A}/\mathcal{G}$ by

$$\vartheta_\infty(A_\infty) = \frac{1}{2}\int_\Sigma \mathrm{Tr}(F^2_{A_\infty}) \wedge \omega$$

and observe that, since ω is not closed, this is not a topological invariant. In other words, this is not a constant function. The picture here is somewhat different to the case previously considered. Clearly ϑ_∞ is gauge invariant, so defines a genuine function on $\mathcal{B}$. As noted by Xu [92, Remark 2.2.14] after fixing a reference connection A_∞ one can use $\mathrm{d}\omega = 3\mathrm{Im}\Omega$ to write ϑ_∞ in the more familiar form

$$\vartheta_\infty(A_\infty + a) = \frac{1}{2}\int_\Sigma \left(F_{A_\infty} \wedge a - \frac{2}{3} a \wedge a \wedge a\right) \wedge \mathrm{Im}\Omega.$$

One finds [92, Scetion 2.2] that

$$\mathrm{d}\vartheta_\infty(A_\infty, a) = \int_\Sigma (F_{A_\infty} \wedge \mathrm{d}_{A_\infty} a) \wedge \omega$$

and the Euler-Lagrange equation is $\mathrm{d}_{A_\infty}(F_{A_\infty} \wedge \omega)$ which by the Bianchi identity is equivalent to $F_{A_\infty} \wedge \mathrm{Im}\Omega = 0$. Thus critical points are precisely nearly Kähler instantons.

The most important example of a nearly Kähler instanton is the canonical connection, this is a connection on $T\Sigma$ and when Σ is homogeneous it can also be seen as a connection on the bundle $G \to G/H$ which we will refer to as the canonical bundle (this should not be confused with the canonical bundle from complex geometry). It is shown in [43] that the canonical connection defines a nearly Kähler instanton on $T\Sigma$ and in [42] that it defines an instanton on the canonical bundle. We will revisit the canonical connection in Section 3.3, where it will be placed in the framework of homogeneous bundles.

Remark 3.2.2. *It is easy to verify that a connection A_∞ on $Q \to \Sigma$ is a nearly Kähler instanton if and only if, denoting $\pi\colon C(\Sigma) \to \Sigma$ as the natural projection map, the connection $\pi^* A_\infty$ is a G_2-instanton on $\pi^* Q$. For this reason nearly Kähler instantons are the natural model for G_2-instantons on AC conical G_2-manifolds, with decay conditions imposed at infinity.*

3.3 Homogeneous Bundles and Invariant Connections

This section covers standard material on homogeneous bundles, invariant connections and Wang's theorem. This material is covered in [62] and summaries can also be found in [77] and [71].

Let G be a compact, connected Lie group with closed subgroup H and form the homogeneous space G/H. Furthermore suppose this space is reductive so that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and this splitting is respected by the adjoint action of H. Suppose $E = G \times_{(H,\rho)} V$ is a vector bundle associated via some representation (V, ρ) of H. Examples of such bundles are given by the tensor and spinor bundles, so we have a wealth of examples to work with. We denote by ρ_L (resp. ρ_R) the *left regular representation* (resp. *right regular representation*) acting on $L^2(G, E)$ via

$$\rho_L(g')f(g) = f((g')^{-1}g)$$
$$\rho_R(g')f(g) = f(gg').$$

Then there is a standard association of sections of E and H-equivariant functions $G \to V$ given by

$$L^2(E) \cong L^2(G, V)_H = \{f \in L^2(G, V)\,;\, \rho_R(h)f = \rho_V(h)^{-1}f \text{ for all } h \in H\}$$

(we choose to work with L^2 sections as this is the natural setting in which to apply the Frobenius reciprocity formalism which we use in subsequent chapters). We also denote the induced Lie algebra action by ρ, thus $L^2(G, V)_H$ also has the description

$$L^2(G, V)_H = \{f \in L^2(G, V)\,;\, \rho_R(X)f + \rho_V(X)f = 0 \text{ for all } X \in \mathfrak{h}\}.$$

The isomorphism $L^2(E) \cong L^2(G,V)_H$ works as follows: If $f \in L^2(G,V)_H$ note that $X_f([g]) = [g, f(g)]$ is a well-defined section of $L^2(G \times_{(H,\rho)} V)$. Conversely if $X \in L^2(G \times_{(H,\rho)} V)$ then for each $g \in G$ there is a unique v_g such that $X([g]) = [g, v_g]$. The map $f_X \colon G \to V, g \mapsto v_g$ is H equivariant and hence defines an element of $L^2(G,V)_H$. Furthermore, the two maps $X \mapsto f_X$ and $f \mapsto X_f$ are inverse to each other, so this identification of sections and equivariant maps is indeed an isomorphism.

In this setting it is natural to study G-invariant tensors, which are typically determined by a natural algebraic constraint. For example, it is demonstrated in [62] that there exists a one-to-one correspondence between G-invariant metrics on G/H and $\mathrm{Ad}(H)$-invariant metrics on $\mathfrak{m}$. This fact will be useful for understanding the Bryant-Salamon metrics in Chapters 6 and 7.

Recall on a principal K-bundle $\pi \colon Q \to X$, over any manifold X, the structure group K acts on the right and for any $p \in P$ this action defines a homeomorphism $K \cong Q_{\pi(p)} = \pi^{-1}(\{p\})$.

Definition 3.3.1. *Let $Q \to G/H$ be a principal K-bundle. We say that Q is a G-homogeneous K-bundle if there is a lift of the natural left action of G on G/H to the total space Q which commutes with the right action of K.*

Let Q be a homogeneous K-bundle over G/H. Choose a point $q_0 \in \pi^{-1}(\{eH\})$, then for all $h \in H$, we see that $h \cdot q_0 \in \pi^{-1}(\{eH\})$. Thus, for each $h \in H$ there exists a unique $k \in K$ such that $h \cdot q_0 = q_0 \cdot k$. This defines a map $\lambda \colon H \to K$, we see that this is in fact a homomorphism since $q_0 \cdot \lambda(h_1 h_2) = h_1 h_2 \cdot q_0 = q_0 \cdot \lambda(h_1) \cdot \lambda(h_2)$. We call λ the *isotropy homomorphism*. This allows one to reconstruct the bundle Q: Consider the associated bundle

$$Q \times_{(H,\lambda)} K = (G \times K)/\sim$$

where $\sim$ is the equivalence relation $(gh, k) \sim (g, \lambda(h)k)$ for all $g \in G, h \in H$ and $k \in K$. Then the map

$$G \times_{(H,\lambda)} K \to Q, \qquad [(g,k)] \mapsto g \cdot q_0 \cdot k$$

defines an isomorphism of principal bundles. Thus, G-homogeneous K-bundles over G/H are determined by isotropy homomorphisms $\lambda \colon H \to K$. More precisely, isomorphism classes of homogeneous K-bundles are in bijection with conjugacy classes of homomorphisms $\lambda \colon H \to K$.

Suppose now we have a homogeneous K-bundle $Q \cong G \times_{(H,\lambda)} K$ and a representation (V, ρ) of K. Then the lift of the G-action to Q endows the associated bundle $E = Q \times_{(K,\rho)} V$ with an an action of G. Furthermore there is an isomorphism of homogeneous bundles

$$E \cong G \times_{(H, \rho\circ\lambda)} V.$$

A section $s \in \Gamma(E)$ is then said to be invariant if, once viewed as and H-equivariant map $s \colon G \to V$, it is constant. Thus we can decompose V into irreducible components for the action of H on V and call a section $s \in \gamma(E)$ a G-invariant section if it takes values in a trivial component (if there are any).

We can also understand gauge transformations in the homogeneous setting. Let $Q = G \times_{(H,\lambda)} K$ be a homogeneous K-bundle and consider gauge transformations of Q as sections of $c(Q) = Q \times_{(K,c)} K$ where $c(k_1)k_2 = k_1 k_2 k_1^{-1}$. Using the isomorphism $c(Q) = G \times_{(H, c\circ\lambda)} K$ we can define a G-invariant section of $c(Q)$ from an element k of the centraliser of $\Lambda(H)$ via the map

$$[g] \mapsto [g, k]$$

where $[\,\cdot\,]$ denotes an equivalence class. We say a connection A is G-invariant if its connection 1-form $A \in \Omega^1(Q, \mathfrak{k})$ is left invariant. The most familiar example of an invariant connection is the canonical connection, which we now review.

Let $\Sigma^6 = G/H$ be a reductive nearly Kähler homogeneous space. Recall the canonical 1-form θ on G is the left invariant $\mathfrak{g}$-valued 1-form uniquely determined by

$$\theta(X) = X \qquad \text{for all } X \in \mathfrak{g}.$$

The reductive property of the homogeneous space means that $A_{\text{can}} := \pi_{\mathfrak{h}} \circ \theta$ defines a G-invariant connection on the canonical bundle $G \to G/H$. It was verified in [42] that the canonical connection defines a nearly Kähler instanton on this bundle. If $\lambda \colon H \to K$ is any Lie group homomorphism, there is a canonical H-equivariant mapping

$$i \colon G \to G \times_\lambda K$$

given by

$$g \mapsto [(g, e)]$$

where $e \in K$ is the identity. By first extending A_{can} trivially to $G \times K$ one obtains a connection A^λ_{can} on $G \times_\lambda K$. If λ is an injection we can use i to consider $G \subset G \times_\lambda K$, then by [65, Proposition 4.7] we have

$$A^\lambda_{\text{can}}|_G = \lambda_* A_{\text{can}}$$

and

$$F_{A^\lambda_{\text{can}}}|_G = \lambda_* F_{A_{\text{can}}}$$

where $\lambda_* \colon \mathfrak{g} \to \mathfrak{k}$ is the induced Lie algebra homomorphism. The connection A^λ_{can} defines a nearly Kähler instanton on the associated bundle with $\text{Hol}(A^\lambda_{\text{can}}) = H$.

In the same manner one can also view the canonical connection as living on a vector bundle which has been associated via a linear representation of H. Let (V, λ) be such a representation and form the associated bundle $E = G \times_\lambda V$. Associating vector fields with elements of $C^\infty(G,\mathfrak{m})_H$ and sections with elements of $C^\infty(G,V)_H$, the action of the covariant derivative $\nabla^{A^\lambda_{\mathrm{can}}}$ is [4]

$$\nabla^{A^\lambda_{\mathrm{can}}}_X f(g) = \rho_R(X)f(g) = \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=0} f(ge^{tX(g)}). \tag{3.1}$$

In practice we will work with the canonical connection on a fixed associated bundle and simply write A_{can} to denote the canonical connection living on this bundle.

The broader framework for studying invariant connections is provided by Wang's theorem [88]. This gives an algebraic description of G-invariant connections on homogeneous bundles.

Theorem 3.3.2 (Wang). *Let $Q = G\times_{(H,\lambda)} K$ be a principal homogeneous K-bundle. Then G-invariant connections on P are in one-to-one correspondence with morphisms Φ of H representations*

$$\Phi\colon (\mathfrak{m},\mathrm{Ad}) \to (\mathfrak{k},\mathrm{Ad}\circ\lambda).$$

The connection A_Φ that corresponds to such a morphism Φ satisfies $(A_\Phi - A_{\mathrm{can}})([1]) = \Phi$ where we identify linear maps $\mathfrak{m}\to\mathfrak{k}$ with elements of $(T^*(G/H)\otimes \mathrm{Ad}Q)_{[1]}$.

At various points we shall consider constructing G_2-instantons over an asymptotically conical manifold with a cohomegeneity one group action. The idea will be to apply Wang's theorem and look for a path in the space of invariant connections on the link, which leaves one with an ODE to solve.

More generally let $(G/H,g)$ be a Riemannian homogeneous space, form the Riemannian cone $C(X) = (\mathbb{R}^+\times X, \mathrm{d}r^2 + r^2 g_X)$ (where r is the coordinate on $\mathbb{R}^+$) and let $\pi\colon C(X)\to X$ be the natural projection map. If $Q\to G/H$ is a G-homogeneous K-bundle then π^*Q carries an action of G defined in the obvious way on each radial slice $r =$const. We shall call a connection A on π^*Q invariant if its connection 1-form $A\in\Omega^1(\pi^*Q,\mathfrak{k})$ is left invariant under this action. In looking for solutions to instanton equations in this setting we shall see that imposing invariance allows one to reduce to solving an ODE. Since want a connection defined on a complete manifold we must impose suitable boundary conditions to extend the connection to the entire, complete space.

Chapter 4

Gauge Theory on Asymptotically Conical G_2-Manifolds

In this chapter we begin to study gauge theory on asymptotically conical G_2-manifolds. We begin by introducing the analytic setup under consideration. The study of gauge theory on weighted spaces has been considered by Taubes [83], Nakajima [76] and Donaldson [25] among others so we borrow from these works to set up the basic machinery. The original contributions of this thesis begin in Section 4.2. To specialise to the AC G_2-manifold setting we provide a slice theorem for the action of the gauge group and thus provide local models for the space of connections modulo gauge. We then show that the instanton moduli space is locally homeomorphic to the kernel of an elliptic operator and use the implicit function theorem to show this moduli space is a smooth manifold when the deformation theory is unobstructed. Finally we give a brief overview of what is known about the obstruction spaces in question and adapt the work of [50] and [51] to show that flat connections are unobstructed for certain decay rates.

4.1 Gauge Theory on Weighted Spaces

In this section we adapt the basic setup of gauge theory to the natural boundary conditions that we choose to impose at infinity. Since we require polynomial decay at infinity and study an elliptic equation, a natural choice of function spaces to work with is the weighted Sobolev spaces. We show that, for reasonable decay rates, the machinery familiar from the case of compact manifolds follows through to the present case.

To begin with let us fix some notation. We first adapt the weighted Sobolev spaces from Definition 2.5.2 to sections of twisted bundles. Let $p \geq 1, k \in \mathbb{N} \cup \{0\}$ and $\mu \in \mathbb{R}$. Let (M, φ) be an AC G_2-manifold with radius function ρ and let $P \to M$ be a principal bundle with connection A. In the notation of Definition 2.5.2 the vector bundle E in question is $E = T \otimes \mathrm{Ad}P$ with T being either a tensor or spinor bundle. Thus T inherits a metric g and the Levi-Civita connection and these are combined with the metric $\langle , \rangle_{\mathfrak{g}}$

and connection A on $\mathrm{Ad}P$ to furnish E with a metric and connection. The norm $\|\cdot\|_{L^p_{k,\mu}}$ on $L^p_{k,\mathrm{loc}}$ sections η of $E = T \otimes \mathrm{Ad}P$ is thus

$$\|\eta\|_{L^p_{k,\mu}} = \left(\sum_{j=0}^{k} \int_M \left| \rho^{j-\mu} \nabla^j \eta \right|^p \rho^{-7} \, \mathrm{dVol}_g \right)^{\frac{1}{p}}$$

where $|\cdot|$ is calculated using a combination of g and $\langle , \rangle_{\mathfrak{g}}$ and $\nabla = \nabla^{LC} \otimes \nabla^A$ is the tensor product of the Levi-Civita connection on T and the connection ∇^A on $\mathrm{Ad}P$. As before $L^p_{k,\mu}(T \otimes \mathrm{Ad}P)$ denotes the completion under this norm. For convenience we shall adopt the following notation:

$$\begin{aligned} \Omega^m_\mu(M) &:= C^\infty_\mu(\Lambda^m T^* M), & \Omega^m_{k,\mu}(M) &:= L^2_{k,\mu}(\Lambda^m T^* M) \\ \Omega^m_\mu(M, \mathrm{Ad}P) &:= C^\infty_\mu(T^* M \otimes \mathrm{Ad}P), & \Omega^m_{k,\mu}(M, \mathrm{Ad}P) &:= L^2_{k,\mu}(\Lambda^m T^* M \otimes \mathrm{Ad}P). \end{aligned}$$

Let us now make precise the concept of a G_2-instanton decaying to a nearly Kähler instanton. To do so we first fix a suitable framing of our bundle.

Definition 4.1.1. *Let M be an asymptotically conical G_2-manifold, with asymptotic link Σ, and let $P \to M$ be a principal bundle over M. We call P asymptotically framed if there exists a principal bundle $Q \to \Sigma$ such that*

$$h^* P \cong \pi^* Q$$

where $\pi \colon C(\Sigma) \to \Sigma$ is the natural projection map.

Remark 4.1.2. We loose no generality here since such a framing always exists. The condition here is slightly more general than the setup of Taubes in [83] where it is assumed that Q is trivial. When this is the case Taubes notes that if G is simple and simply connected then P must also be trivial.

From now on we assume that P has a fixed asymptotic framing Q.

Definition 4.1.3. *Let M be an asymptotically conical manifold. A connection A on an asymptotically framed bundle $P \to M$ is called asymptotically conical with rate μ if there exists a connection A_∞ on $Q \to \Sigma$ such that, denoting $A_C = \pi^*(A_\infty)$ we have*

$$|\nabla^j_C (h^*(A|_{M_\infty}) - A_C)|_C = O(r^{\mu-1-j}) \tag{4.1}$$

for all non-negative integers j, for some $\mu < 0$ and where ∇_C is a combination of the Levi-Civita connection on the cone metric and A_C. Here $|\cdot|_C$ is the norm induced by the cone metric and the metric on $\mathfrak{g}$. We say A is asymptotic to A_∞ and call the quantity $\mu_0 := \inf\{\mu\,;\, A$ is asymptotically conical with rate $\mu\}$ is called the fastest rate of converge *of A.*

Remark 4.1.4. Although we choose to measure the rate of convergence with the conical metric and the coordinate r on the cone, we could also have chosen to do so with respect to the AC metric and the radius function, the rate of converge is the same. Note it is natural to require the difference of the connections to be $O(r^{\mu-1})$ since a 1-form a on the link Σ satisfies $|\pi^* a|_C = O(r^{-1})$ where $\pi\colon C(\Sigma) \to \Sigma$ is the projection map. The requirement that $\mu < 0$ is crucial to the analysis that follows; it could be interesting to consider allowing non-negative rates but the methods we develop in this chapter will no longer be applicable.

It follows from [77, Proposition 3] that, on an AC G_2-manifold M, any G_2-instanton with pointwise curvature decay will have as a limit (if it exists) a nearly Kähler instanton on the G_2 on the asymptotic link Σ of M.

The weighted Sobolev spaces provide a suitable framework for studying AC connections. Recall the space of connections is an affine space; a choice of reference connection A identifies the space of connections $\mathcal{A}$ with $\Omega^1(M, \mathrm{Ad}P)$, since any other connection B is $B = A + a$ where a is a uniquely determined Lie algebra valued 1-form. We let

$$\mathcal{A}_{k,\mu-1} = \left\{ A + a\, ;\, a \in \Omega^1_{k,\mu-1}(M, \mathrm{Ad}P) \right\}$$

be the space of $L^2_{k,\mu-1}$ connections and $\mathcal{A}_{\mu-1} = \cap_{k\geq 0}\mathcal{A}_{k,\mu-1}$ the space of $C^\infty_{\mu-1}$ connections.

We also need to introduce gauge transformations with specified decay properties. Recall a gauge transformation g is an automorphism of the principal bundle that covers the identity and that g acts on a connection A via the formula $g \cdot A = gAg^{-1} - \mathrm{d}gg^{-1}$. Following the setup of Nakajima in [76] suppose $P \to M$ is a principal G bundle, pick a faithful representation $G \to GL(V)$ and form the associated bundle $\mathrm{End}(V)$. We define

$$\mathcal{G}_{k+1,\mu} \coloneqq \{ g \in C^0(\mathrm{End}(V))\, ;\, \|\mathrm{Id} - g\|_{k+1,\mu} < \infty,\, g \in G \text{ a.e} \}. \tag{4.2}$$

Furthermore we define $\mathcal{G}_\mu \coloneqq \bigcap_{l\geq 0} \mathcal{G}_{l,\mu}$. The framework of weighted gauge groups has been studied by Taubes in [83], Nakajima [76] and Donaldson [25] among others. As such we are able to borrow a preliminary lemma:

Lemma 4.1.5 ([25, Proposition 4.12])**.** *The pointwise exponential map defines charts making $\mathcal{G}_{k+1,\mu}$ into Hilbert Lie groups with Lie algebra modelled on $\Omega^0_{k+1,\mu}(M, AdP)$ for $k \geq 3$. The group $\mathcal{G}_{k+1,\mu}$ acts smoothly on $\mathcal{A}_{k,\mu-1}$ via gauge transformations when $k \geq 4$.*

Now we can define our main object of study:

Definition 4.1.6. *Let M be an AC G_2-manifold with asymptotic link Σ. Let $P \to M$ be a bundle asymptotically framed by $Q \to \Sigma$ and let A_∞ be a nearly Kähler instanton on Q. The moduli space of G_2-instantons asymptotic to A_∞ with rate μ is*

$$\mathcal{M}(A_\infty, \mu) = \{ G_2\text{-instantons } A \text{ on } P \text{ satisfying (4.1) asymptotic to } A_\infty \}/\mathcal{G}_\mu. \tag{4.3}$$

To begin studying this moduli space we first try to understand it as the zero set of a (non-linear) elliptic operator. Let us pick a reference connection A which we assume to be a G_2-instanton, then we can write any other connection B as $B - A = a$ where $a \in \Omega^1(M, \mathrm{Ad}P)$. The relationship between the curvatures is $F_B - F_A = \mathrm{d}_A a + a \wedge a$ and hence the G_2-instanton equation for B becomes the non-linear equation $\psi \wedge (\mathrm{d}_A a + a \wedge a) = 0$. From an analytic perspective it is advantageous to work instead with the G_2-monopole equation

$$\mathrm{d}_A f + *(\psi \wedge (\mathrm{d}_A a + a \wedge a)) = 0 \tag{4.4}$$

for some $f \in \Omega^0(M, \mathrm{Ad}P)$. This is because adding the (local) gauge fixing condition $\mathrm{d}_A^* a = 0$ to (4.4) yields an elliptic equation. The gauge fixing condition mentioned here will be explored later in the chapter. Ignoring the technicalities of the gauge fixing conditions for the moment, note that the G_2-monopole equation are precisely elements of the zero set of the non-linear operator $\mathcal{D}_A \colon \Gamma((\Lambda^0 \oplus \Lambda^1) \otimes \mathrm{Ad}(P)) \to \Gamma((\Lambda^0 \oplus \Lambda^1) \otimes \mathrm{Ad}(P))$ given by

$$\mathcal{D}_A = \begin{pmatrix} 0 & \mathrm{d}_A^* \\ \mathrm{d}_A & *(\psi \wedge (\mathrm{d}_A + \cdot \wedge \cdot)) \end{pmatrix}. \tag{4.5}$$

To see that the linearisation of $\mathcal{D}_A$ is elliptic we compare the expression for the Dirac operator (2.8) and conclude the linearisation is the *twisted Dirac operator* D_A where

$$D_A = \begin{pmatrix} 0 & \mathrm{d}_A^* \\ \mathrm{d}_A & *(\psi \wedge \mathrm{d}_A) \end{pmatrix}. \tag{4.6}$$

Nothing is lost in moving to this setup for if (f, A) satisfies the G_2-monopole equation (with a decay condition) then $f = 0$. Thus the zero set of $\mathcal{D}_A$ consists of solutions B to the G_2-instanton equation together with the gauge fixing condition $\mathrm{d}_A^*(B - A) = 0$. Similarly if (f, a) satisfies the linearised G_2-monopole equation then $f = 0$. We delay the proof of these facts until later in this section.

Being a twisted Dirac operator, D_A is first order elliptic and formally self adjoint. The kernel of D_A consists of gauge fixed solutions to the linearised G_2-monopole equation. In studying the behaviour of this operator we are lead to consider various other Dirac operators, so we briefly list those operators we will require. If A is an asymptotically conical connection then there is a connection A_∞ on $Q \to \Sigma$ and we define $A_C = \pi^* A_\infty$ as in (4.1). Hence we have operators

Operator	Bundle	Formula
D_A	$\$(M) \otimes \mathrm{Ad}P$	$\mathrm{cl} \circ \nabla^{\mathrm{LC}} \otimes \nabla^A$
D_{A_C}	$\$(C) \otimes \mathrm{Ad}(\pi^* Q)$	$\mathrm{cl} \circ \nabla^{\mathrm{LC}} \otimes \nabla^{A_C}$
$D^t_{A_\infty}$	$\$(\Sigma) \otimes \mathrm{Ad}Q$	$\mathrm{cl} \circ \nabla^t \otimes \nabla^{A_\infty}$

The operator D_A fits into the analytic framework for operators on AC manifolds which has been developed by Lockhart-McOwen in [67] and by Marshall in [72], whose work we now adapt to our setting. Suppose (M, g) is asymptotically conical and ρ is a radius function, then $g_{\mathrm{Acyl}} := \rho^{-2} g$ is asymptotically cylindrical. Let E any vector bundle with fibre metric induced from the Riemannian metric (the cases we shall consider all fall into this category) and let $T_E \colon (E, g) \to (E, g_{\mathrm{Acyl}})$ be the natural isometry from the conformal change of AC to Acyl metrics. Let E and F be two such bundles over M. We will call an operator $K \colon \Gamma(E) \to \Gamma(F)$ *asymptotically conical* with rate ν if

$$\rho^{\nu} (T_F)^{-1} K \, T_E$$

is asymptotic to an operator K_∞ that is invariant under the $\mathbb{R}^+$ action on $M_\infty = h((R, \infty) \times \Sigma)$. Here two smooth order l operators $K, L \colon \Gamma(E) \to \Gamma(F)$ are said to be asymptotic if the following holds: Let U_σ be a finite an open cover of Σ, then over the sets $V_\sigma = h((R, \infty) \times U_\sigma)$ (which form an open cover of the end of M) the operators act as $\mathrm{rank} F \times \mathrm{rank} E$ matrices of operators

$$\begin{aligned} (K_{V_\sigma})_{ij} &= \sum_{0 \leq |\lambda| \leq l} (K)^{\sigma\lambda}_{ij} \partial^\lambda \\ (L_{V_\sigma})_{ij} &= \sum_{0 \leq |\lambda| \leq l} (L)^{\sigma\lambda}_{ij} \partial^\lambda \end{aligned}$$

(here λ is a multi index) and each such matrix satisfies

$$\sup_{t \times U_\sigma} \left| f_\sigma \partial^{\lambda_1} \left((K)^{\nu\lambda_2}_{ij} - (L)^{\nu\lambda_2}_{ij} \right) \right|$$

for all $1 \leq i \leq \mathrm{rank} F, 1 \leq j \leq \mathrm{rank} F, |\lambda_1| \geq 0$ and $0 \leq |\lambda_2| \leq l$ and where $\{f_\sigma\}$ is a translation invariant partition of unity, subordinate to a finite open cover of M which extends the cover $\{V_\sigma\}$ of the end of M. A very detailed treatment of the constructions mentioned here can be found in [72]. An important fact is that an AC rate ν order l operator $K \colon \Gamma(E) \to \Gamma(F)$ admits a bounded extension $K \colon L^2_{k+l,\mu}(E) \to L^2_{k,\mu-\nu}(F)$. Assume the connection A that we work with is AC in the sense of (4.1). Our main cases of interest are:

- $E = \Lambda^0(M) \otimes \mathrm{Ad} P$, $F = \Lambda^1(M) \otimes \mathrm{Ad} P$ and $T_E = \mathrm{id}$, $T_F = \rho$. The operator ∇^A is AC with rate 1, this follows straight from the AC condition on the connection (4.1) in local coordinates. Similarly the operators d_A and d_A^* are AC with rate 1.

- $E = F = \$(M) \otimes \mathrm{Ad}(P) \cong (\Lambda^0 \oplus \Lambda^1) \otimes \mathrm{Ad}(P)$ and $T_E = \begin{pmatrix} 1 & 0 \\ 0 & \rho \end{pmatrix}$. As noted in [61] the Dirac operator D is AC with rate 1. It then follows from the above example that the twisted Dirac operator is AC since if $s \in \$(M)$ and $\eta \in \mathrm{Ad} P$ one has $D_A(s \otimes \eta) = D(s) \otimes \eta + \mathrm{cl} \circ \left(s \otimes \nabla^A \eta \right)$.

- $E = F = \mathrm{Ad}(P)$ and $T_E = \mathrm{Id}$. The coupled Laplace operator $\mathrm{d}_A^* \mathrm{d}_A$ is AC with rate 2. This follows from the fact that it is the composition of the AC rate 1 operators d_A^* and d_A (this is proved in [72, Lemma 4.19]).

Furthermore the above operators are *uniformly elliptic*, which is to say they are elliptic operators that converge to elliptic operators on the cone. The operators in consideration converge to D_{A_C} and $(\mathrm{d}_{A_C})^* \mathrm{d}_{A_C}$ which are built from the connection A_C living on the cone. Such operators come with estimates that ensure desirable regularity properties analogous to the situation on a compact manifold.

Theorem 4.1.7 ([72, Theorem 4.21])**.** *Suppose $K\colon C_c^\infty(E) \to C_c^\infty(F)$ is a smooth uniformly elliptic, asymptotically conical operator of rate γ and order $l \geq 1$. Suppose that $\eta \in L^1_{loc}(F)$ and $\xi \in L^1_{loc}(E)$ is a weak solution of $K\xi = \eta$.*
If $\xi \in L^p_{0,\beta+\gamma}(E)$ and $\eta \in L^p_{k,\beta}(F)$ then $\xi \in L^p_{k+l,\beta+\gamma}(E)$ with

$$\|\xi\|_{L^p_{k+l,\beta+\gamma}(E)} \leq C \left(\|\eta\|_{L^p_{k,\beta}(F)} + \|\xi\|_{L^p_{0,\beta+\gamma}(E)} \right)$$

where the constant $C > 0$ is independent of ξ.

Thus the kernel of an order l AC uniformly elliptic rate ν operator $K\colon L^2_{k+l,\mu} \to L^2_{k,\mu-\nu}$ is independent of k, we therefore denote the kernel simply by $\mathrm{Ker}K_\mu \coloneqq \mathrm{Ker}K\colon L^2_{k+l,\mu} \to L^2_{k,\mu-\nu}$. Using this Sobolev estimate together with the weighted Sobolev embedding theorem we find that the kernel of an AC uniformly elliptic operator has the desirable property of consisting of smooth sections. To study the kernel of our operators we will need to determine the set of *critical weights* which are determined by the asymptotic operators.

Definition 4.1.8. *Let C be a G_2-cone with asymptotic link Σ and $A_C = \pi^*(A_\infty)$. Let K_C be either the twisted Dirac operator D_{A_C} or the coupled Laplace operator $\mathrm{d}_{A_C}^* \mathrm{d}_{A_C}$ acting on sections of $E = \not{S}(C) \otimes \mathrm{Ad}(\pi^*(Q))$ and $E = \mathrm{Ad}(\pi^*(Q))$ respectively. The set $W_{crit}(K_C)$ of critical weights of the operator P_C is*

$$W_{\mathrm{crit}}(K_C) = \{\lambda \in \mathbb{R}\, ; \exists \text{ a non-zero homogeneous order } \lambda \text{ section } \eta \text{ of } E \text{ with } K_C(\eta) = 0\}\,. \tag{4.7}$$

In the case of a twisted spinor, a section η is homogeneous of order λ if

$$\eta = (f + (g\,\mathrm{d}r + v)) \cdot s_C$$

(here s_C denotes the parallel spinor on the cone) with $f = r^\lambda \pi^*(f_\infty), g = r^\lambda \pi^*(g_\infty)$ and $v = r^{\lambda+1}\pi^*(v_\infty)$, where $f_\infty, g_\infty \in \Omega^0(\Sigma, \mathrm{Ad}Q)$ and $v_\infty \in \Omega^1(\Sigma, \mathrm{Ad}Q)$ (equivalently $\eta = r^\mu s_\infty$ where s_∞ is a spinor on Σ lifted to the cone). In general one has to allow for complex critical weights but the formal self-adjoint property of the above operators in question ensures all such weights are real. The set $W_{\mathrm{crit}}(K_C)$ is countable and discrete. Recall an operator between Banach spaces is called Fredholm if it has finite dimensional kernel and cokernel. An operator whose range admits a finite dimensional complementary subspace automatically has closed range, so this is in particular true of Fredholm operators.

Theorem 4.1.9. *Let E and K be as above. Then the extension $K\colon L^p_{k+l,\mu}(E) \to L^p_{k,\mu-\nu}(F)$ is Fredholm whenever $\mu \in \mathbb{R}\backslash W_{\mathrm{crit}}(K_C)$. Furthermore if $[\mu, \mu'] \cap W_{\mathrm{crit}}(P_C) = \emptyset$, then*

$$\mathrm{Ker}K_\mu = \mathrm{Ker}K_{\mu'}.$$

Thus the kernel of K is independent of the weight provided we do not pass through any critical weights. For any $k \geq 4$ and $\mu < 0$ we define:

$$(\mathrm{d}_A^* \, \mathrm{d}_A)_{k,\mu} \coloneqq \mathrm{d}_A^* \, \mathrm{d}_A \colon \Omega^0_{k+2,\mu}(M, \mathrm{Ad}P) \to \Omega^0_{k,\mu-2}(M, \mathrm{Ad}P) \tag{4.8}$$

$$(D_A)_{k,\mu} \coloneqq D_A \colon L^2_{k+1,\mu}(\$(M) \otimes \mathrm{Ad}P) \to L^2_{k,\mu-1}(\$(M) \otimes \mathrm{Ad}P). \tag{4.9}$$

The last result we shall require is a Fredholm alternative for AC manifolds, see [72] and [67] for the proof.

Theorem 4.1.10 ([72, Theorem 4.22])**.** *Let K be an AC uniformly elliptic order l and rate ν operator and suppose that $\mu \notin W_{\mathrm{crit}}(K_C)$ so that the extension*

$$K\colon L^2_{k+l,\mu}(E) \to L^2_{k,\mu-\nu}(F)$$

is Fredholm. Then

1. *There exists a finite dimensional subspace $\mathcal{O}_{\mu-\nu}$ of $L^2_{k,\mu-\nu}$ such that*

$$L^2_{k,\mu-\nu}(F) = K(L^2_{k+l,\mu}(E)) \oplus \mathcal{O}_{\mu-\nu} \tag{4.10}$$

and

$$\mathcal{O}_{\mu-\nu} \cong \mathrm{Ker}K^*_{-7-\mu+\nu}. \tag{4.11}$$

2. *If $\mu > -\frac{7}{2} + \nu$ then we can take*

$$\mathcal{O}_{\mu-\nu} = \mathrm{Ker}K^*_{-7-\mu+\nu}.$$

3. *The image of the extension K is the space*

$$K(L^2_{k+l,\mu}(E)) = \left\{\eta \in L^2_{k,\mu-\nu}(F)\,;\, \langle \eta, \kappa\rangle_{L^2(F)} = 0 \text{ for all } \kappa \in \mathrm{Ker}(K^*)_{-7-\mu+\nu}\right\}. \tag{4.12}$$

4.2 The Space of Connections Modulo Gauge

We now aim to give a description of the space of connections modulo gauge. Given a reference connection A we may view the gauge orbit $\mathcal{G}_{k+1,\mu} \cdot A$ as a subset of $\Omega^1_{k,\mu-1}(M, \mathrm{Ad}P)$. The infinitesimal action of the μ-weighted gauge group is $-\mathrm{d}_A \colon \Omega^0_{k+1,\mu}(M, \mathrm{Ad}P) \to \Omega^1_{k,\mu-1}(M, \mathrm{Ad}P)$ and our strategy is to show this image is closed and hence admits a complement. We aim to find a particular complement for this image, which is called a "slice" of the action and as usual is given by the Coulomb gauge fixing condition. The upshot of this can be seen from the following thereom:

Theorem 4.2.1 ([36, Section 3]). *Suppose G is a group acting on a Banach manifold X. If through each $x \in X$ we can find a slice of the action (i.e an open submanifold Y such that $T_yX = T_y(G \cdot y) \oplus T_yN$ for all $y \in N$ and such that the restriction of the projection $X \to X/G$ to Y is on-to-one) and if the action is free, then X/G is a smooth manifold. If G is a Banach Lie group the same conclusion holds if a slice of the Lie algebra action can be found.*

Applying this theorem to the case of the weighted gauge groups $\mathcal{G}_{k+1,\mu}$ action on the space of connections $\mathcal{A}_{k,\mu-1}$ we see that a splitting of $\Omega^1_{k,\mu-1}(M, \mathrm{Ad}P)$ into an (infinitesimal) orbit space and a complement shows the quotient space is a smooth Hilbert manifold if the action is free. As in the case of a compact manifold the Coulomb gauge fixing condition may not pick out a unique class representative globally, so we also give a sufficient condition for this property to hold. Our strategy for this task is to develop the Fredholm theory of the coupled Laplacian $\mathrm{d}_A^* \mathrm{d}_A$.

To learn when $\mathrm{d}_A^* \mathrm{d}_A$ is Fredholm Theorem 4.1.9 tells us to look for homogeneous order λ elements of the kernel of $\mathrm{d}_{A_C}^* \mathrm{d}_{A_C}$. Such a solution looks like $f = r^\lambda \xi$ for some $\xi \in \Omega^0(\Sigma, \mathrm{Ad}P)$ and we calculate

$$\mathrm{d}_{A_C}^* \mathrm{d}_{A_C} f = r^{\lambda-2} \left(\mathrm{d}_{A_\infty}^* \mathrm{d}_{A_\infty} \xi - \lambda(\lambda+5)\xi \right)$$

so such a solution exists if and only if $\lambda(\lambda+5)$ is an eigenvalue of $\mathrm{d}_{A_\infty}^* \mathrm{d}_{A_\infty}$. Given that the coupled Laplace operator is positive, we find that there are no critical weights in the range $(-5, 0)$. Therefore:

Proposition 4.2.2. *Let A be an AC connection over an AC G_2-manifold. If $\mu \in (-5, 0)$ then the coupled Laplacian $\mathrm{d}_A^* \mathrm{d}_A \colon \Omega^0_{k+2,\mu}(M, \mathrm{Ad}P) \to \Omega^0_{k,\mu-2}(M, \mathrm{Ad}P)$ is Fredholm.*

The next lemma is a gauged version of integration by parts on AC manifolds, the proof goes through identically to [61, Lemma 4.16] but is given here for completeness.

Lemma 4.2.3. *Let $\xi \in \Omega^{m-1}_{k,\mu}(M, \mathrm{Ad}P)$ and $\eta \in \Omega^m_{l,\nu}(M, \mathrm{Ad}P)$. If $k, l \geq 4$ and $\mu + \nu < -6$ then*

$$\langle \mathrm{d}_A \xi, \eta \rangle_{L^2} = \langle \xi, \mathrm{d}_A^* \eta \rangle_{L^2}.$$

Proof. We apply Stoke's theorem to the manifold with boundary $M_{\leq s} = \{x \in M\, ; \rho(x) \leq s\}$ (note this satisfies $\partial(M_{\leq s}) = \{s\} \times \Sigma$). Thus by Stoke's theorem

$$\int_{M_{\leq s}} \langle \mathrm{d}_A \xi, \eta \rangle - \int_{M_{\leq s}} \langle \xi, \mathrm{d}_A^* \eta \rangle = \int_{M_{\leq s}} \mathrm{d} \langle \xi \wedge * \eta \rangle_{\mathfrak{g}} = \int_{\{s\}\times\Sigma} \langle \xi \wedge * \eta \rangle_{\mathfrak{g}}.$$

It remains to show that the integral on the right hand side vanishes at $s \to \infty$. The hypothesis of the theorem ensures $|\xi \wedge \eta| = O(\rho^{-6-\epsilon})$ for some $\epsilon > 0$. We therefore have

$$\left| \int_{\{s\}\times\Sigma} \langle \xi \wedge * \eta \rangle_{\mathfrak{g}} \right| \leq \int_{\{s\}\times\Sigma} |\xi \wedge * \eta| \, \mathrm{dVol}_{\{s\}\times\Sigma} \leq C s^{-\epsilon}$$

which vanishes at $s \to \infty$ as required. □

Corollary 4.2.4. *Let* $f \in \operatorname{Ker}(\mathrm{d}_A^* \mathrm{d}_A)_\mu$. *If* $\mu < 0$ *then* $\mathrm{d}_A f = 0$.

Proof. We have seen that there are no critical weights of $\mathrm{d}_A^* \mathrm{d}_A$ in the region $(-5, 0)$, so $\mathrm{d}_A^* \mathrm{d}_A f = 0$ and $\mu < 0$ then $f \in \Omega^0(M, \mathrm{Ad}P)_{k+2,\mu_0}$ for some $\mu_0 < -\frac{5}{2}$ and for any k. It follows that $\mathrm{d}_A f \in \Omega^1_{k+1,\mu_0-1}(M, \mathrm{Ad}P)$. Integration by parts is valid for such and weight and we see that

$$\|\mathrm{d}_A f\|_{L^2} = \langle \mathrm{d}_A^* \mathrm{d}_A f, f\rangle_{L^2} = 0.$$

□

The following useful lemma which is due to Marshall [72] is a straightforward application of the maximum principle. We denote by $\Delta = \mathrm{d}^* \mathrm{d}$ the usual Laplacian on functions.

Lemma 4.2.5. *Let* (M, g) *be an asymptotically conical manifold. If* $\mu < 0$ *then* $\operatorname{Ker}(\Delta)_\mu = \{0\}$.

As an immediate corollary we find that harmonic sections of the adjoint bundle must vanish:

Corollary 4.2.6. *Let* $f \in \operatorname{Ker}(\mathrm{d}_A^* \mathrm{d}_A)_\mu$. *If* $\mu < 0$ *then* $f = 0$.

Proof. Since $f \in \operatorname{Ker}(\mathrm{d}_A^* \mathrm{d}_A)_\mu$ we know by Corollary 4.2.4 that $\mathrm{d}_A f = 0$ and the connection is compatible with the inner product so that

$$\Delta |f|^2 = \mathrm{d}^* \mathrm{d} |f|^2 = 2\, \mathrm{d}^* \langle \mathrm{d}_A f, f\rangle = 0.$$

So $|f|^2$ is a harmonic function and hence zero. □

Pausing for a moment we can finally justify our switch from the G_2-instanton equation to the G_2-monopole equation (4.4).

Corollary 4.2.7. *If* $\mu < 0$ *and* $(f, A) \in \Omega^0_{k+1,\mu}(M, \mathrm{Ad}P) \oplus \mathcal{A}$ *satisfies the* G_2*-monopole equation*

$$\mathrm{d}_A f + *(\psi \wedge F_A) = 0$$

then $f = 0$.

Proof. We apply d_A^* to the G_2-monopole equation $\mathrm{d}_A f + *(\psi \wedge F_A) = 0$ and use that ψ is closed together with the Bianchi identity to find that $\mathrm{d}_A^* \mathrm{d}_A f = 0$ and hence Corollary 4.2.6 is applicable.

□

Corollary 4.2.8. *Let* A *be an asymptotically conical* G_2*-instanton. If* $\mu < 0$ *and* $(f, a) \in L^2_{k+1,\mu}((\Lambda^0 \oplus \Lambda^1) \otimes \mathrm{Ad}P)$ *satisfies the linearised* G_2*-monopole equation*

$$\mathrm{d}_A f + *(\psi \wedge \mathrm{d}_A a) = 0$$

then $f = 0$.

Proof. Again we apply d_A^* any observe that

$$\mathrm{d}_A^*\, \mathrm{d}_A f + *(\psi \wedge \mathrm{d}_A^2 a) = 0$$

and since A is a G_2-instanton $\psi \wedge \mathrm{d}_A^2 a = \psi \wedge [F_A, a] = 0$ so we may appeal to Corollary 4.2.6. □

We return our attention to splitting the space of 1-forms. It is sufficient for our purpose to work in the regime where $-5 < \mu < 0$, since each known example (except for the twistor lift of ASD instanton as given in [77]) has fastest rate of convergence in this interval. Note from (4.8) that $(\mathrm{d}_A^*\, \mathrm{d}_A)_{k,\mu}$ has trivial cokernel, since the adjoint maps from the space with weight $-5 - \mu < 0$. The bounded inverse theorem then yields:

Lemma 4.2.9. *Let (M, g) be an asymptotically conical G_2-manifold and let A be an asymptotically conical connection on a principal bundle $P \to M$. If $-5 < \mu < 0$ then the coupled Laplacian*

$$\mathrm{d}_A^*\, \mathrm{d}_A \colon \Omega^0_{k+2,\mu}(M, \mathrm{Ad}P) \to \Omega^0_{k,\mu-2}(M, \mathrm{Ad}P)$$

is an isomorphism of topological vector spaces.

This allows us to split the space of 1-forms, heuristically $\Omega^1_{l,\mu-1}(M, \mathrm{Ad}P) = \mathrm{Im}\, \mathrm{d}_A \oplus \mathrm{Ker}\, \mathrm{d}_A^*$. Note however that the splitting may not be orthogonal, since the weights we require need not be in the L^2 integrable regime. Instead we make use of the following basic lemma from Banach space theory:

Lemma 4.2.10. *Suppose $T \colon X \to Y$ is a bounded linear operator between Banach spaces, so that the kernel* Ker$T$ *is closed and admits a closed complement. A closed subspace $X_0 \subset X$ is a complement to* Ker$T$ *if and only if*

1. *$T|_{X_0}$ is injective*
2. *$T(X) = T(X_0)$.*

Proof. In a Banach space X two closed subsets X_0 and K are complementary if and only if they are algebraically complementary, which is to say $X_0 + K = X$ and $X_0 \cap K = \{0\}$. Let K be the kernel of the bounded linear operator T and let X_0 be such that the above conditions hold. Let $x \in X_0 \cap K$, then we must have that $x = 0$. For any $x \in X$ there is a unique $x_0 \in X_0$ such that $T(x) = T(x_0)$, so we can write $x = (x - x_0) + x_0$ and since $x - x_0 \in K$ we are done. □

We would like to apply the above lemma with $X = \Omega^1_{k+1,\mu-1}(M, \mathrm{Ad}P)$, $T = \mathrm{d}_A^*$ and $X_0 = \mathrm{d}_A(\Omega^0_{k+2,\mu}(M, \mathrm{Ad}P))$, thus we must first establish that the image of d_A is closed.

Lemma 4.2.11. *The operator $\mathrm{d}_A \colon \Omega^0_{k+2,\mu}(M, \mathrm{Ad}P) \to \Omega^1_{k+1,\mu-1}(M, \mathrm{Ad})$ has closed image.*

Proof. Let $\{\mathrm{d}_A f_n\}_{n=1}^{\infty}$ be a sequence in $\mathrm{d}_A(\Omega^0_{k+2,\mu}(M, \mathrm{Ad}P))$, and let $a \in \Omega^1_{k+1,\mu-1}(M, \mathrm{Ad}P)$ be such that

$$\|\mathrm{d}_A f_n - a\|_{k+1,\mu-1} \to 0.$$

Applying the bounded operator d_A^* we see that $\mathrm{d}_A^* \,\mathrm{d}_A f_n$ converges to $\mathrm{d}_A^* a$ in $\Omega^0_{k,\mu-1}(M, \mathrm{Ad}P)$. Since $\mathrm{d}_A^* \,\mathrm{d}_A$ admits a bounded inverse we find that f_n converges to $f := (\mathrm{d}_A^* \,\mathrm{d}_A)^{-1} \,\mathrm{d}_A^* a$ in $\Omega^0_{k+2,\mu}(M, \mathrm{Ad}P)$. Finally we apply the bounded operator d_A and see that

$$\|\mathrm{d}_A f_n - \mathrm{d}_A f\|_{k+1,\mu-1} \to 0.$$

So $a = \mathrm{d}_A f$ by uniqueness of limits and hence $\mathrm{Im}\,\mathrm{d}_A$ is closed. □

Theorem 4.2.12 (Slice Theorem)**.** *Let* $-5 < \mu < 0$ *then*

$$\Omega^1_{k+1,\mu-1}(M, \mathrm{Ad}P) = \mathrm{Ker}(\mathrm{d}_A^* \colon \Omega^1_{k+1,\mu-1}(M, \mathrm{Ad}P) \to \Omega^0_{k,\mu-2}(M, \mathrm{Ad}P)) \oplus \mathrm{d}_A(\Omega^0_{k+2,\mu}(M, \mathrm{Ad}P)). \tag{4.13}$$

Proof. We apply Lemma 4.2.10 to the operator $\mathrm{d}_A^* \colon \Omega^1_{k+1,\mu-1}(M, \mathrm{Ad}P) \to \Omega^0_{k,\mu-2}(M, \mathrm{Ad}P)$. Since d_A^* is AC this extension is bounded and hence the kernel is a closed subspace. We claim that $\mathrm{d}_A(\Omega^0_{k+2,\mu}(M, \mathrm{Ad}P))$ satisfies the hypothesis of Lemma 4.2.10. Firstly, as noted above this is a closed subspace.
Claim 1: d_A^* is injective when restricted to $\mathrm{d}_A(\Omega^0_{k+2,\mu}(M, \mathrm{Ad}P))$.
To see this suppose that $\mathrm{d}_A^* \,\mathrm{d}_A f = \mathrm{d}_A^* \,\mathrm{d}_A g$ for $f, g \in \Omega^0_{k+2,\mu}(M, \mathrm{Ad}P)$. Then $f - g$ is harmonic and hence 0, so certainly $\mathrm{d}_A f - \mathrm{d}_A g = 0$.
Claim 2: $\mathrm{d}_A^* \,\mathrm{d}_A(\Omega^0_{k+2,\mu}(M, \mathrm{Ad}P)) = \mathrm{d}_A^*(\Omega^1_{k+1,\mu-1}(M, \mathrm{Ad}P))$.
This follows from Lemma 4.2.9.

□

The importance of this result is that it gives us a local description of the space $\mathcal{B}_{k+1,\mu} = \mathcal{A}_{k+1,\mu-1}/\mathcal{G}_{k+2,\mu}$ of connections modulo gauge. The infinitesimal action of the gauge group $\mathcal{G}_{k+2,\mu}$ is $-\mathrm{d}_A \colon \Omega^0_{k+2,\mu}(M, \mathrm{Ad}P) \to \Omega^1_{k+1,\mu-1}(M, \mathrm{Ad}P)$, so we can interpret Theorem 4.2.12 as a so called "slice" theorem– we have found complements for the action of the gauge group. If the action is free, it will follow from general theory that the quotient space is a smooth manifold.

To see that the action is free set $\Gamma_{A,\mu} = \{g \in \mathcal{G}_{k,\mu}\,;\, g \cdot A = A\}$. Recall we are viewing gauge transformations as sections of $\mathrm{End}(V)$ as defined in (4.2). By a standard argument $\Gamma_{A,\mu}$ is closed Lie subgroup of $\mathrm{End}(V_x)$, for some base point x, whose elements are covariantly constant sections of $\mathrm{End}(V)$ [27, Section 4.2.2] (i.e sections g for which $\mathrm{d}_A g = 0$). Recall G is assumed to be semi-simple and the inner product on the representation V is assumed to be invariant. The connection A has $\mathrm{Hol}(A) \subset G$ so it preserves the inner product on V and thus also preserves the induced inner product on $\mathrm{End}(V) = V \otimes V^*$. Thus, regarding gauge transformations as sections of $\mathrm{End}(V)$ as in (4.2), if $g \in \Gamma_{A,\mu}$ then $|g - \mathrm{Id}| \in \Omega^0_{k+2,\mu}(M, \mathrm{Ad}P)$ and $\Delta|g - \mathrm{Id}|^2 = 2\,\mathrm{d}^*\langle \mathrm{d}_A(g - \mathrm{Id}), g - \mathrm{Id}\rangle = 0$. We have seen that such a function must vanish, so that $\Gamma_{A,\mu} = \mathrm{Id}$. This is in contrast to the case when M is compact since, in the AC case, reducible connections (those with $\mathrm{Hol}(A)$ a proper subgroup of G) do not lead to singularities in the space of connections modulo gauge. As a consequence, if we set

$$T_{A,\mu,\epsilon} = \{a \in \operatorname{Ker} \mathrm{d}_A^* \colon \Omega^1_{k+1,\mu-1}(M, \mathrm{Ad}P) \to \Omega^0_{k,\mu-2}(M, \mathrm{Ad}P)\,;\ \|a\|_{k,\mu-1} < \epsilon\}. \tag{4.14}$$

then $T_{A,\epsilon}$ models a local neighbourhood of A in $\mathcal{B}_{k+1,\mu}$. We summarise this in the following corollary:

Corollary 4.2.13. *Let $P \to M$ be a principal G-bundle with G semisimple. If $-5 < \mu < 0$ then the moduli space $\mathcal{B}_{k+1,\mu} = \mathcal{A}_{k+1,\mu-1}/\mathcal{G}_{k+2,\mu}$ is a smooth manifold and the sets $T_{A,\mu,\epsilon}$ provide charts near $[A]$.*

Working with the local models $T_{A,\mu,\epsilon}$ amounts to solving *Coulomb gauge* condition $\mathrm{d}_A^* a = 0$. This condition picks out a unique gauge equivalence class locally but may not do so sufficiently far away from A. The failure of a global gauge fixing is a reflection of the rich topology of $\mathcal{B}$ [26]. Using the surjectivity of the coupled Laplacian we can prove a weighted version of [27, Proposition 2.3.4] which gives a sufficient condition for solving the Coulomb gauge condition:

Proposition 4.2.14. *Let $P \to M$ be a principal G-bundle with G a compact Lie group and let $A \in \mathcal{A}_{\mu-1}$ for $\mu \in (-5, 0)$. There is a constant $c(A) > 0$ such that if $B \in \mathcal{A}_{\mu-1}$ and $B = A + a$ satisfies*

$$\|a\|_{L^2_{4,\mu-1}} < c(A)$$

then there is a gauge transformation $g \in \mathcal{G}_\mu$ such that $g(B)$ is in Coulomb gauge relative to A.

Proof. We have that

$$g(A + a) = A + gag^{-1} - (\mathrm{d}_A g)g^{-1}.$$

Write $g = \exp(\chi) = e^\chi$ for a section χ of $\mathrm{Ad}P$, then we are looking to solve the nonlinear equation $\mathcal{F}(\chi, a) = 0$ where

$$\mathcal{F}(\chi, a) = \mathrm{d}_A^* \left((\mathrm{d}_A e^\chi) e^{-\chi} - e^\chi a e^{-\chi} \right).$$

The linearisaton of this operator is

$$\mathrm{d}\mathcal{F}_{(0,0)}(\xi, b) = \mathrm{d}_A^* \mathrm{d}_A \xi - \mathrm{d}_A^* b.$$

We are in a position to apply the implicit function theorem. In the notation of Theorem 2.5.7 we pick the Banach spaces $\mathcal{X}_1 = \Omega^1_{5,\mu-1}(M, \mathrm{Ad}P), \mathcal{X}_2 = \Omega^0_{4,\mu}(M, \mathrm{Ad}P)$ and $\mathcal{Y}$ the $L^2_{3,\mu-2}$ closure of $\mathrm{d}_A^*(\Omega^1_{\mu-1}(M, \mathrm{Ad}P))$. By Lemma 4.2.9 $(\mathrm{d}_2\mathcal{F})_{(0,0)}$ is surjective and therefore we can solve for small enough a. More precisely there is a constant $c(A)$ for which we can solve the equation provided $\|a\|_{4,\mu-1} < c(A)$ since we are topologising the space of connections with the $L^2_{4,\mu}$ norm. The solution $g = \exp(\chi)$ is on the outset only in $\mathscr{G}_{5,\mu}$ but we now bootstrap to show in fact $g \in \mathcal{G}_\mu$. For this we let $u \in \mathcal{G}_{k,\mu}$ and note that we can write

$$\mathrm{d}_A^* \mathrm{d}_A u = (\mathrm{d}_A u u^{-1}) \lrcorner \mathrm{d}_A u + u \mathrm{d}_A^* a u^{-1} + \mathrm{d}_A u \lrcorner a + ua \lrcorner u^{-1} \mathrm{d}_A u,$$

so if $k \geq 4$ (so that u is continuous) and a lies in $\Omega^1_{\mu-1}(M, \mathrm{Ad}P)$ (in particular a is smooth) then the right hand side of this equation lies in $L^2_{k,\mu-2}$ and hence by elliptic regularity u lies in $L^2_{k+1,\mu}$. Thus by bootstrapping our solution solution $g \in \mathcal{G}_{5,\mu}$ to the Coloumb gauge equation fixing equation we see that $g \in \mathcal{G}_\mu$ as claimed. □

4.3 A Regularity Result

We observed that elements of the zero set of the operator $\mathcal{D}_A$ from (4.5) consists of (smooth) G_2-instantons B together with the Coulomb gauge condition $\mathrm{d}_A^*(B - A) = 0$ which fixes a gauge near to A. The decay condition (4.1) we impose on connections in the moduli space $\mathcal{M}(A_\infty, \mu)_k$, that a neighbourhood of 0 in the zero set of $\mathcal{D}_A \colon C^\infty_{\mu-1} \to C^\infty_{\mu-2}$ is homeomorphic to a neighbourhood of $[A]$ in $\mathcal{M}(A_\infty, \mu)$. We now show that we may instead study the moduli space using the weighted Sobolev spaces; this is advantageous since these spaces are the natural setting for studying elliptic operators on AC manifolds. Let us denote by $\mathcal{M}(A_\infty, \mu)_k$ the space of $L^2_{k,\mu-1}$ connections by $L^2_{k+1,\mu}$ gauge transformations. Note that the Sobolev multiplication theorem (Theorem 2.5.6) ensures $\mathcal{D}_A \colon L^2_{k+1,\mu-1}(\$(M) \otimes \mathrm{Ad}P) \to L^2_{k,\mu-2}(\$(M) \otimes \mathrm{Ad}P)$ is bounded if $k \geq 3$.

We use the regularity of uniformly elliptic, asymptotically conical operators to obtain a result comparable to Donaldson and Kronheimer [27, Proposition 4.2.16].

Proposition 4.3.1. *Let $k \geq 4$ and $-5 < \mu < 0$. Then the natural inclusion $\mathcal{M}(A_\infty, \mu)_{k+1} \hookrightarrow \mathcal{M}(A_\infty, \mu)_k$ is a homeomorphism.*

Proof. Suppose A is an asymptotically conical G_2-instanton of rate μ and class $L^2_{k,\mu-1}$, we will show that there exists a gauge transform g such that $g(A)$ is in $L^2_{k+1,\mu-1}$. Firstly we know by Proposition 4.2.14 there is an $\varepsilon > 0$ such that any $L^2_{k,\mu-1}$ connection B with $\|A - B\|_{L^2_{k,\mu-1}} < \varepsilon$ can be gauge transformed into Coulomb gauge relative to A. Since $C^\infty_{\mu-1}$ lies densely in $L^2_{k,\mu-1}$ we may pick a smooth connection B satisfying this condition, then we know there exists a g in $L^2_{k+1,\mu}$ with

$$\mathrm{d}_A^*(g^{-1}(B) - A) = 0.$$

The Coulomb gauge condition is symmetric, so that A is also in Coulomb gauge relative to $g^{-1}(B)$, i.e

$$\mathrm{d}^*_{g^{-1}(B)}(A - g^{-1}(B)) = 0. \tag{4.15}$$

Let us denote $g(A) = B + a$, then we can apply g to (4.15) deduce

$$\mathrm{d}_B^* a = 0.$$

Furthermore we have the relation

$$*(\psi \wedge \mathrm{d}_B a) = - * (\psi \wedge F_B) - *(\psi \wedge a \wedge a).$$

Now the weighted multiplication theorem Theorem 2.5.6 ensures that $a \wedge a \in L^2_{k,2(\mu-1)}$ and the curvature of B lies in $C^\infty_{\mu-2}$. Thus $D_B(a) \in L^2_{k,\mu-2}$ and the asymptotically conical uniformly elliptic estimates for the smooth operator D_B allow us deduce that $a \in L^2_{k+1,\mu-1}$. That is, we have bootstraped to gain a degree of differentiability. This establishes the surjectivity of the inclusion.
The map is clearly injective and continuous; to see it is a homeomorphism we show that the two spaces have the same convergent sequences with their induced topologies. Let $\{a_n\}_{n=1}^\infty$ be a sequence in $\mathcal{M}(A_\infty, \mu)$ which is convergent in the $L^2_{k,\mu-1}$ norm, i.e there is some $a_\infty \in \mathcal{M}(A_\infty, \mu)$ with $\|a_n - a_\infty\|_{k\mu-1} \to 0$ as $n \to \infty$. Observe that, since a_n and a_∞ are in the zero set of $\mathcal{D}_B$, we have that

$$|D_B(a_n - a_\infty)| = | * (\psi \wedge (a_n \wedge a_n - a_\infty \wedge a_\infty))| = |\pi_7(a_n \wedge a_n - a_\infty \wedge a_\infty)|$$

where the final equality follows from the facts that the Hodge star operator acts isometrically and since the operation of wedging with ψ is an isomorphism of representations between Λ^2_7 and Λ^6 which preserves the norm. To see that $\{a_n\}_{n=1}^\infty$ also converges in the $L^2_{k+1,\mu}$ norm we observe that

$$\begin{aligned} \|a_n - a_\infty\|_{k+1,\mu-1} &\leq C(\|D_B(a_n - a_\infty)\|_{k,\mu-2} + \|a_n - a_\infty\|_{0,\mu-1}) \\ &\leq C(\|a_n \wedge a_n - a_\infty \wedge a_\infty\|_{k,\mu-2} + \|a_n - a_\infty\|_{0,\mu-1}) \\ &\leq C'(\|a_n\|_{k,-1}\|a_n - a_\infty\|_{k,\mu-1} + \|a_\infty\|_{k,-1}\|a_n - a_\infty\|_{k,\mu-1} + \|a_n - a_\infty\|_{0,\mu-1}) \end{aligned}$$

with the constants $C, C' > 0$ coming from the elliptic estimate for the smooth operator D_B and the weighted Sobolev multiplication Theorem 2.5.6. Now since $\{a_n\}_{n=1}^\infty$ converges in $L^2_{k,\mu-1}$ and there is a continuous embedding $L^2_{k,\mu-1} \hookrightarrow L^2_{k,-1}$, the sequence $\|a_n\|_{k,-1}$ is bounded independent of n and therefore $\|a_n - a_\infty\|_{k+1,\mu-1} \to 0$. Since the spaces $\mathcal{M}(A_\infty, \mu)_k$ and $\mathcal{M}(A_\infty, \mu)_{k+1}$ have the same convergent sequences, the inclusion is a homeomorphism.

□

Note the weighted Sobolev embedding theorem ensures the spaces $\mathcal{M}(A_\infty, \mu)_k$ consists of smooth connections. This yields the following important corollary; the moduli space near $[A]$ is modelled on a neighbourhood of 0 in the zero set of the non-linear elliptic operator $\mathcal{D}_A$ in any Sobolev extension.

Corollary 4.3.2. *Let A be an AC G_2-instanton, asymptotic to A_∞ with rate μ where $-5 < \mu < 0$. The zero set of the operator $\mathcal{D}_A \colon L^2_{k+1,\mu-1}(\$(M) \otimes \mathrm{Ad}P) \to L^2_{k,\mu-2}(\$(M) \otimes \mathrm{Ad}P)$ is independent of $k \geq 3$ and a neighbourhood of A in $\mathcal{M}(A_\infty, \mu)$ is homeomorphic to a neighbourhood of 0 in $(\mathcal{D}_A)^{-1}(0)$.*

4.4 Fredholm and Index Theory of the Twisted Dirac Operator

Recall that the operator $D_A\colon L^2_{k+1,\mu} \to L^2_{k,\mu-1}$ is Fredholm when there are no non-zero solutions to

$$D_{A_C}(r^\mu s_\infty) = 0$$

where $r^\mu s_\infty$ denotes a homogeneous order $\mu - 1$ spinor on the cone. We will find an expression for the Dirac operator on the cone in terms of the operator on the link Σ to determine the solutions to this equation.

We begin by comparing the Dirac operators on the link Σ and the cone C. Let Σ a nearly Kähler 6-manifold and $(C, g_C) = (\Sigma \times \mathbb{R}_{>0}, dr^2 + r^2 g)$ be the cone on Σ. We consider first the case of the ordinary Dirac operators

$$D^0\colon \Gamma(\not{S}(\Sigma)) \to \Gamma(\not{S}(\Sigma))$$
$$D^C\colon \Gamma(\not{S}(C)) \to \Gamma(\not{S}(C))$$

arising from the Levi-Civita connection acting on the spin bundle.

Let e^i be a local orthonormal frame for $T^*\Sigma$, then $E^0 = dr$, $E^i = re^i$ forms a local orthonormal frame for T^*C. We use the convention that an index of μ or ν runs from 0 to 6 whilst an index of i, j or k runs from 1 to 6. Denote by ∂_i differentiation with respect to the vector field dual to e^i using the metric g and denote by D_i differentiation with respect to the vector field dual to E^i with respect to the metric g_C. We write ∇_C for the Levi-Civita connection on the cone and ∇ for the Levi-Civita connection on Σ, similarly we write d_C, d for the exterior derivatives on the cone and Σ respectively.

The connection one form of the Levi-Civita connection on the cone should be metric compatible and torsion free, which here means that $\mathrm{d}_C = \nabla_C\wedge$. Let ω^i_j be the connection one form of the Levi-Civita connection on $T^*\Sigma$, so that $\nabla e^i = -\omega^i_j e^j$ in this frame. Then $\omega^i_j = -\omega^j_i$ and

$$\mathrm{d}e^i + \omega^i_j \wedge e^j = 0 \tag{4.16}$$

(this is equivalent to $\mathrm{d} = \nabla\wedge$).

On the cone we let $\Omega^\mu_\nu = -\Omega^\nu_\mu$ be the Levi-Civita form, then testing that (4.16) holds we first note that $\mathrm{d}E^0 = \mathrm{d}^2 r = 0$ so that $\Omega^0_i \wedge E^i = 0$, and testing when $i \geq 0$ we find

$$\begin{aligned}\mathrm{d}_C E^i &= -\Omega^i_\mu \wedge E^\mu \\ &= -\Omega^i_j \wedge re^j - \Omega^i_0 \wedge \mathrm{d}r\end{aligned}$$

and also

$$\begin{aligned}\mathrm{d}_C E^i &= \mathrm{d}r \wedge e^i + r\,\mathrm{d}e^i \\ &= \mathrm{d}r \wedge e^i - r\omega^i_j \wedge e^j.\end{aligned}$$

Since we know (4.16) holds we conclude that

$$\Omega^i_j = \omega^i_j, \qquad \Omega^i_0 = e^i.$$

The connection acts on one forms as

$$\nabla_C v_\mu e^\mu = \left(\mathrm{d}_C v_\mu - v_\nu \Omega^\nu_\mu\right) \otimes e^\mu.$$

Let $\Gamma^\mu_{\sigma\nu}$ be the Christoffel symbols of the Levi-Civita connection on the cone, so that

$$E^\sigma \Gamma^\mu_{\sigma\nu} = \Omega^\mu_\nu$$

and γ^i_{kj} be the Christoffel symbols on the Levi-Civita connection on Σ so that

$$e^k \gamma^i_{kj} = \omega^i_j.$$

Note that $\Gamma^\mu_{0\nu} = 0$ and $\Gamma^i_{k0} = \frac{1}{r}\delta^{ik}$. We take the natural Clifford algebra embedding of $\mathrm{C}l_n$ into $\mathrm{C}l^{\text{even}}_{n+1}$ given by $e^i \mapsto E^i E^0$. The action of the Dirac operator on Σ is

$$D^0 s = E^i E^0 \left(\partial_i s + \frac{1}{4}\gamma^k_{ij} E^0 E^j E^0 E^k s\right). \tag{4.17}$$

The operator on the cone acts as

$$\begin{aligned}
D^C(s) &= E^0 \nabla_0 s + E^i \left(D_i s + \frac{1}{4}\Gamma^\nu_{i\mu} E^\mu E^\nu s\right) \\
&= E^0 \frac{\partial s}{\partial r} + E^i \left(D_i s + \frac{1}{4}\left(\Gamma^j_{i0} E^0 E^j s + \Gamma^0_{ij} E^j E^0 s + \Gamma^k_{ij} E^j E^k s\right)\right) \\
&= E^0 \frac{\partial s}{\partial r} + E^i \left(\frac{1}{r}\partial_i s + \frac{1}{4r}\left(\delta_{ij} E^0 E^j s - \delta_{ij} E^j E^0 s + \gamma^k_{ij} E^j E^k s\right)\right) \\
&= E^0 \frac{\partial s}{\partial r} - \frac{1}{2r} E^i E^i E^0 s + \frac{1}{r} E^i \left(\partial_i s + \frac{1}{4}\gamma^k_{ij} E^0 E^j E^0 E^k s\right).
\end{aligned}$$

Thus we have that

$$D^C = E^0 \frac{\partial s}{\partial r} + \frac{3}{r} E^0 s + \frac{1}{r} E^i \left(\partial_i s + \frac{1}{4}\gamma^k_{ij} E^0 E^j E^0 E^k s\right)$$

and comparing to (4.17) one finds

$$D^C = E^0 \cdot \left(\frac{\partial}{\partial r} + \frac{1}{r}\left(3 + D^0\right)\right). \tag{4.18}$$

Remark 4.4.1. *The above calculation generalises easily to the case of (X^n, g) an n-dimensional Riemanninan manifold and $(C(X), \mathrm{d}r^2 + r^2 g)$ the $(n+1)$-dimensional cone of X.*

We would like to generalise this to the case of twisted spinors and twisted Dirac operators. Recall the notation $A_C = \pi^* A_\infty$ and consider the operators

$$\begin{aligned} D^0_{A_\infty} &\colon \Gamma(\$(\Sigma) \otimes \mathrm{Ad}Q) \to \Gamma(\$(\Sigma) \otimes \mathrm{Ad}Q) \\ D_{A_C} &\colon \Gamma(\$(C) \otimes \mathrm{Ad}\pi^* Q) \to \Gamma(\$(C) \otimes \mathrm{Ad}\pi^* Q). \end{aligned}$$

These operators factor as follows:

$$\begin{aligned} D^0_{A_\infty} &= \mathrm{cl}_6 \circ (\nabla \otimes 1 + 1 \otimes \nabla^{A_\infty}) = D^0 \otimes 1 + \mathrm{cl}_6 \circ (1 \otimes \nabla^{A_\infty}) \\ D_{A_C} &= \mathrm{cl}_7 \circ (\nabla_C \otimes 1 + 1 \otimes \nabla^{A_C}) = D^C \otimes 1 + \mathrm{cl}_7 \circ (1 \otimes \nabla^{A_C}) \end{aligned}$$

it suffices consider the terms $\mathrm{cl}_6(1 \otimes \nabla^{A_\infty})$ and $\mathrm{cl}_7(1 \otimes \nabla^{A_C})$. We choose a local frame $\{v_a\}$ of $\mathrm{Ad}Q$ and let $\tilde{\omega}^b_a = e^i \tilde{\gamma}^b_{ia}$ the the connection 1-form of ∇^A and let $\tilde{\Omega}^b_a = E^i \tilde{\Gamma}^b_{ia}$ be the connection 1-form of ∇^{A_C}. Then $\pi^* \tilde{\omega} = \tilde{\Omega}$ and we can apply a similar analysis to before. Let s_A be a local frame for the spin bundle. One finds

$$D^0_{A_\infty}(f_{Aa} s_A \otimes v_a) = D^0(f_{Aa} s_A) \otimes v_a + E^i E^0 f_{Aa} s_A \otimes \tilde{\gamma}^b_{ia} v_b$$

whilst

$$\begin{aligned} D_{A_C}(f_{Aa} s_A \otimes v_a) &= E^0 \cdot \left(\frac{\partial}{\partial r} f_{Aa} s_A + \frac{1}{r}(3 + D^0)(f_{Aa} s_A) \right) \otimes v_a + (E^i f_{Aa} s_A) \otimes \tilde{\Gamma}^b_{ia} v_b \\ &= E^0 \cdot \left(\frac{\partial}{\partial r} f_{Aa} s_A + \frac{1}{r}(3 + D^0)(f_{Aa} s_A) \right) \otimes v_a + \frac{1}{r}(E^i f_{Aa} s_A) \otimes \tilde{\gamma}^b_{ia} v_b \\ &= E^0 \cdot \left(\frac{\partial}{\partial r} + \frac{1}{r}\left(3 + D^0_{A_\infty}\right) \right) (f_{Aa} s_A \otimes v_a). \end{aligned}$$

So the twisted Dirac operators satisfy the same relation as in the case of the ordinary spin bundle

$$D_{A_C} = E^0 \cdot \left(\frac{\partial}{\partial r} + \frac{1}{r}\left(3 + D^0_{A_\infty}\right) \right). \tag{4.19}$$

A simple calculation shows that the volume form Vol_6 anti-commutes with the family of Dirac operators $D^t_{A_\infty}$, so the spectrum of this operator is symmetric about 0. Furthermore the spectrum of a Dirac operator is unbounded and discrete [10] and these facts combine with (4.19) to describe the set of critical weights $W_{\mathrm{crit}}(D_{A_C})$,

Proposition 4.4.2. *The set of critical weights for the twisted Dirac operator*

$$D_A \colon L^2_{k+1,\mu}(\$(M) \otimes \mathrm{Ad}P) \to L^2_{k,\mu-1}(\$(M) \otimes \mathrm{Ad}P)$$

is $W_{\mathrm{crit}}(D_{A_C}) = \{\mu \in \mathbb{R} \colon \mu + 3 \in \mathrm{Spec} D^0_{A_\infty}\}$ *where* $D^0_{A_\infty}$ *is the Dirac operator on the link twisted by the asymptotic connection* A_∞. *Furthermore this set is discrete, unbounded and symmetric about* -3. *This operator is therefore Fredholm whenever* $\mu + 3 \in \mathbb{R} \setminus \mathrm{Spec} D^0_{A_\infty}$.

Recall that in studying the deformation theory of $\mathcal{M}(A_\infty, \mu)$ we are lead to work with the operator $D_A \colon L^2_{k+1,\mu-1} \to L^2_{k,\mu-2}$ acting on a $\mu - 1$ weighted Sobolev space, since the kernel of this operator consists solutions to the linearised G_2-instanton equation converging with rate μ. Therefore let us set

$$W := \{\mu \in \mathbb{R}\,;\, \mu + 2 \in \mathrm{Spec}(D^0_{A_\infty})\}, \tag{4.20}$$

so that $D_A \colon L^2_{k+1,\mu-1} \to L^2_{k,\mu-2}$ is Fredholm whenever $\mu \in \mathbb{R} \setminus W$ and W is symmetric about -2.

Remark 4.4.3. In [18] Charbonneau and Harland show that the linearised nearly Kähler instanton equation on the link Σ is solved by one forms a satisfying $D_{A_\infty}(a \cdot s) = 2a \cdot s$. Therefore we may think of rate 0 deformations on cone $C(\Sigma)$ as being deformations of the nearly Kähler instanton A_∞.

Recall the index of a Fredholm operator P is the quantity $\mathrm{ind}P = \dim(\ker P) - \dim(\mathrm{coker}P)$. The Lockhart McOwen theory provides an *index change* formula that describes how the index varies as we vary the weight of the Sobolev space.

Definition 4.4.4. *Let $\mu \in \mathbb{R}$ and define*

$$\mathcal{K}(\mu) = \left\{ \eta_C \in \mathrm{Ker} D_{A_C}\,;\, \eta_C(r,\sigma) = r^{\mu-1} \sum_{j=0}^{m} (\log r)^j \eta_j(\sigma) \text{ each } \eta_j \text{is a section of } \displaystyle{\not}\$(\Sigma) \otimes \mathrm{Ad}Q \right\}.$$

That is, $\mathcal{K}(\mu)$ consists of sections in the kernel of D_{A_C} which are polynomials in $\log r$ whose coefficients are homogeneous order μ spinors.

The importance of these spaces is that they describe the change in index as the weight varies. We let $k(\mu)$ be the dimension of the dimension of the space $\mathcal{K}(\mu)$ in Definition 4.4.4.

Theorem 4.4.5. *Let ind_μ denote the index of $D_A \colon L^2_{k+1,\mu-1}(\$(M) \otimes \mathrm{Ad}P) \to L^2_{k,\mu-2}(\$(M) \otimes \mathrm{Ad}P)$. If $\mu, \mu' \in \mathbb{R} \setminus W$ are such that $\mu \le \mu'$, then*

$$\mathrm{ind}_{\mu'} D_A - \mathrm{ind}_\mu D_A = \sum_{\nu \in W \cap (\mu, \mu')} k(\nu).$$

Remark 4.4.6. *Note the operator $D_A \colon L^2_{k+1,-3} \to L^2_{k,-4}$ has index zero, this follows from the fact it is self adjoint since the dual of the target space has the same weight as the domain– in other words the kernel and cokernel are isomorphic. It follows that when $D^0_{A_\infty}$ has non-trivial kernel we have $\mathrm{ind}(D_A)_{-2+\epsilon} = \frac{1}{2}(\dim \mathrm{Ker} D^0_{A\infty})$ for ϵ sufficiently small that $[-2, -2+\epsilon)$ contains no other critical weights. Observe that in this situation the expected dimension is negative for all rates $\mu < -2$. If $-2 \notin W$ then the same observation shows that ind$_{-2+\epsilon} D_A = 0$.*

In analogy with the results of [72] we show that being self adjoint ensures that elements of $\mathcal{K}(\mu)$ have no polynomial terms.

Proposition 4.4.7. *Suppose*

$$D_{A_C}\left(r^{\mu-1}\sum_{j=0}^{m}(\log r)^j v_j(\sigma)\right) = 0.$$

Then $m = 0$.

Proof. Let $v_C \in \mathcal{K}(\mu)$, then we may write

$$v_C(r,\sigma) = r^{\mu-1}\sum_{j=0}^{m}(\log r)^j v_j(\sigma).$$

Suppose for a contradiction that $m > 0$. Thinking of $D_{A_C}v_C$ as a polynomial in $\log r$ we first compare coefficients of $(\log r)^m$ to find

$$D^0_{A_\infty}v_m = -(\mu+2)v_m.$$

Now comparing coefficients of $(\log r)^{m-1}$ we find

$$(\mu+2)v_{m-1} + mv_m + D^0_{A_\infty}v_{m-1} = 0$$

and we us the self-adjointness of $D^0_{A_\infty}v$ to compute

$$\begin{aligned} m\langle v_m, v_m\rangle_{L^2(\Sigma)} &= -\langle D^0_{A_\infty}v_{m-1}, v_m\rangle_{L^2(\Sigma)} - (\mu+2)\langle v_{m-1}, v_m\rangle_{L^2(\Sigma)} \\ &= -\langle v_{m-1}, D^0_{A_\infty}v_m\rangle_{L^2(\Sigma)} - (\mu+2)\langle v_{m-1}, v_m\rangle_{L^2(\Sigma)} \\ &= -\langle v_{m-1}, -(\mu+2)v_m\rangle_{L^2(\Sigma)} - (\mu+2)\langle v_{m-1}, v_m\rangle_{L^2(\Sigma)} = 0. \end{aligned}$$

Thus $v_m = 0$ which yields our contradiction. □

This proposition shows that $\mathcal{K}(\mu)$ is simply the $-(\mu+2)$ eigenspace for the operator $D^0_{A_\infty}$. The dimension of these spaces therefore determines how $\mathrm{ind}_\mu D_A$ varies as we change the weight μ.

4.5 Structure of the Moduli Space

Suppose we work in a small enough neighbourhood of A in the space of $L^2_{k,\mu-1}$ connections so that we may solve the Coulomb gauge condition. Then we have seen that the the zero set of $(\mathcal{D}_A)_{\mu-1}$ consists of smooth sections and that its linearisation is Fredholm whenever $(\mu+2) \notin \mathrm{Spec}D^0_{A_\infty}$.

Definition 4.5.1. *For a given weight $\mu < 0$ we define the rate μ infinitesimal deformation space to be*

$$\mathcal{I}(A,\mu) := \left\{(f,a) \in \Omega^0_{k+1,\mu-1}(M,\mathrm{Ad}P) \oplus \Omega^1_{k+1,\mu-1}(M,\mathrm{Ad}P)\,;\, D_A(f,a) = 0\right\}.$$

By AC uniform elliptic regularity this is independent of k and is finite dimensional.

When $D_A\colon L^2_{k+1,\mu-1} \to L^2_{k,\mu-2}$ is Fredholm it has closed range and finite dimensional kernel, furthermore we can choose a finite dimensional subspace $\mathcal{O}(A,\mu)$ of $\Omega^0_{k,\mu-2}(M,\mathrm{Ad}P)\oplus\Omega^1_{k,\mu-2}(M,\mathrm{Ad}P)$, called the obstruction space, such that

$$\Omega^0_{k,\mu-2}(M,\mathrm{Ad}P)\oplus\Omega^1_{k,\mu-2}(M,\mathrm{Ad}P) = D_A\left(\Omega^0_{k+1,\mu-1}(M,\mathrm{Ad}P)\oplus\Omega^1_{k+1,\mu-1}(M,\mathrm{Ad}P)\right)\oplus\mathcal{O}(A,\mu).$$

Again by elliptic regularity we have that $\mathcal{O}(A,\mu)$ is isomorphic to the kernel of the adjoint map $(D_A)_{l+1,-5-\mu}$ for any $l\in\mathbb{N}$. If $-\frac{5}{2}<\mu<0$ then the kernel of the adjoint is contained in the target space and we may choose $\mathcal{O}(A,\mu)=\ker(D_A)^*_\mu$.

Remark 4.5.2. *One can also describe the framework for study deformations of G_2-instantons in the form of an elliptic complex. In this weighted setting, the complex takes the form*

$$\Omega^0_\mu(M,\mathrm{Ad}P)\xrightarrow{\mathrm{d}_A}\Omega^1_{\mu-1}(M,\mathrm{Ad}P)\xrightarrow{\psi\wedge\mathrm{d}_A}\Omega^6_{\mu-2}(M,\mathrm{Ad}P)\xrightarrow{\mathrm{d}_A}\Omega^7_{\mu-3}(M,\mathrm{Ad}P).$$

Denote the cohomology groups of this complex by $H^k_{A,\mu}$. We have already noted that the zeroth cohomology group is trivial $H^0_{A,\mu}=\{0\}$, whilst $H^1_{A,\mu}\cong\mathcal{I}(A,\mu)$ and $H^2_{A,\mu}\cong\mathcal{O}(A,\mu)$ provided $-5<\mu<0$.

With this in hand we can apply the implicit function theorem to integrate our infinitesimal deformation theory.

Theorem 4.5.3. *Let A_∞ be a nearly Kähler instanton and let A be an AC G_2-instanton converging to A_∞. Suppose that $\mu\in(\mathbb{R}\setminus W)\cap(-5,0)$. There exists a smooth manifold $\hat{\mathcal{M}}(A,\mu)$, which is an open neighbourhood of 0 in $\mathcal{I}(A,\mu)$, and a smooth map $\pi\colon\hat{\mathcal{M}}(A,\mu)\to\mathcal{O}(A,\mu)$, with $\pi(0)=0$, such that an open neighbourhood of 0 in $\pi^{-1}(0)$ is homeomorphic to a neighbourhood of A in $\mathcal{M}(A_\infty,\mu)$. Thus, the virtual dimension of the moduli space is $dim\mathcal{I}(A,\mu)-dim\mathcal{O}(A,\mu)$ and $\mathcal{M}(A_\infty,\mu)$ is smooth if $\mathcal{O}(A,\mu)=\{0\}$.*

Proof. For $k\geq 5$ let

$$\mathcal{X}=\left(\Omega^0_{k+1,\mu-1}(M,\mathrm{Ad}P)\oplus\Omega^1_{k+1,\mu-1}(M,\mathrm{Ad}P)\right)\times\mathcal{O}(A,\mu)$$

and let

$$\mathcal{Y}=\Omega^0_{k,\mu-2}(M,\mathrm{Ad}P)\oplus\Omega^1_{k,\mu-2}(M,\mathrm{Ad}P).$$

Pick a sufficiently small neighbourhood of A so that we may solve the Coulomb gauge condition. This in turn gives an open neighbourhood $\mathcal{U}$ of $(0,0)$ in $\Omega^0_{k,\mu-1}(M,\mathrm{Ad}P)\oplus\Omega^1_{k,\mu-1}(M,\mathrm{Ad}P)$.
We define a map of Banach spaces $\mathcal{F}\colon\mathcal{X}\to\mathcal{Y}$ by

$$\mathcal{F}(v,w)=\mathcal{D}_A(v)+w$$

and note $\mathcal{F}(0,0)=0$. The differential of $\mathcal{F}$ at 0 acts as

$$\begin{aligned} d\mathcal{F}|_{(0,0)}\colon \mathcal{X} &\to \mathcal{Y} \\ (v,w) &\mapsto D_A v + w. \end{aligned}$$

By the definition of the obstruction space $d\mathcal{F}|_{(0,0)}$ is surjective and $d\mathcal{F}|_{(0,0)}(v,w)=0$ if and only if $(D_A v, w)=(0,0)$. Thus $\ker d\mathcal{F}|_{(0,0)} = \mathcal{K} = \mathcal{I}(A,\mu)\times\{0\}$ is finite dimensional and splits $\mathcal{X}$. That is, there exists a closed $\mathcal{Z}\subset\mathcal{X}$ such that $\mathcal{K}\oplus\mathcal{Z}=\mathcal{X}$ and we can moreover write $\mathcal{Z}=\mathcal{Z}_1\times\mathcal{O}(A,\mu)$ for some closed $\mathcal{Z}_1\subset\Omega^0_{k+1,\mu-1}(M,\mathrm{Ad}P)\oplus\Omega^1_{k+1,\mu-1}(M,\mathrm{Ad}P)$. We are now in a position to apply the implicit function theorem Theorem 2.5.8 – we deduce that there are open sets

$$\begin{aligned} \mathcal{V} &\subset \mathcal{I}(A,\mu) \\ \mathcal{W}_1 &\subset \mathcal{Z}_1 \\ \mathcal{W}_2 &\subset \mathcal{O}(A,\mu) \end{aligned}$$

and smooth maps $\mathcal{G}_j\colon\mathcal{V}\to\mathcal{W}_j$ for $j=1,2$ such that

$$\mathcal{F}^{-1}(0)\cap((\mathcal{V}\times\mathcal{W}_1)\times\mathcal{W}_2)=\{((v,\mathcal{G}_1(v)),\mathcal{G}_2(v)) : v\in\mathcal{V}\}$$

in $\mathcal{X}=(\mathcal{I}(A,\mu)\oplus\mathcal{Z}_1)\times\mathcal{O}(A,\mu)$. Therefore the kernel of $\mathcal{F}$ near $(0,0)$ is diffeomorphic to an open subset of $\mathcal{I}(A,\mu)$ containing 0.

Define $\hat{\mathcal{M}}(A,\mu)=\mathcal{V}$ and $\pi\colon\hat{\mathcal{M}}(A,\mu)\to\mathcal{O}(A,\mu)$ by $\pi(v)=\mathcal{G}_2(v)$. Then an open neighbourhood of 0 in $(\mathcal{D}_A)^{-1}(0)$ is homeomorphic to an open neighbourhood of 0 in $\pi^{-1}(0)$. Finally, Theorem 4.5.3 says a neighbourhood of 0 in the zero set of $\mathcal{D}_A$ is homeomorphic to a neighbourhood of A in the moduli space.

□

Remark 4.5.4. *When $\mu\in W$ or when $\mathcal{O}(A,\mu)\neq\{0\}$ the moduli space may not be smooth, or may have larger than expected dimension.*

4.6 A Word On Obstructions

From Theorem 4.5.3 we learn that the moduli space $\mathcal{M}(A_\infty,\mu)$ is a smooth manifold provided the operator $D_A\colon L^2_{k+1,\mu-1}\to L^2_{k,\mu-2}$ is surjective. A typical method for proving surjectivity is to apply Lichnerowicz-Bochner type formulae to prove the vanishing of the kernel of the adjoint operator. In the weighted setting if the weight μ is not too negative, then one has $\mathrm{Ker}((D_A)_{\mu-1})^*\subset\mathrm{Ker}D_A\colon L^2_1\to L^2$ and so one might hope to prove and L^2-vanishing theorem for twisted harmonic spinors. Whilst this is not difficult when the connection A is flat, the presence of a curvature term in the Lichnerowicz formula is problematic in the case of a non-flat G_2-instanton.

The gauged Lichnerowicz formula for Ricci flat manifolds says that

$$D_A^2 s = (\nabla^A)^* \nabla^A s + F_A \cdot s. \tag{4.21}$$

Suppose first that A is flat. In this case one easily obtains a suitable vanishing theorem in the manner of [61, Lemma 4.68]:

Proposition 4.6.1. *Let M be an AC G_2-manifold, let $P \to M$ be an asymptotically framed bundle and let A be an AC flat connection. If $\nu < -\frac{5}{2}$ then the kernel of $D_A\colon L^2_{k+1,\nu}(\not{S} \otimes \mathrm{Ad}P) \to L^2_{k+1,\nu}(\not{S} \otimes \mathrm{Ad}P)$ vanishes.*

Proof. Suppose $\nu < -\frac{5}{2}$ and let

$$(f, v) \in \mathrm{Ker} D_A\colon L^2_{k+1,\nu}\left((\Lambda^0 \oplus \Lambda^1) \otimes \mathrm{Ad}P\right) \to L^2_{k,\nu-1}((\Lambda^0 \oplus \Lambda^1) \otimes \mathrm{Ad}P).$$

Since $\nu < 0$ we have from Corollary 4.2.7 that $f = 0$. Thus by the Lichnerowicz formula we have

$$0 = (\nabla^A)^* \nabla^A v \tag{4.22}$$

and we would like to take the L^2-inner product and integrate by parts to conclude. Note that

$$\begin{aligned} \langle (\nabla^A)^* \nabla^A v, v \rangle &= -g^{ij} \langle (\nabla_i^A \nabla_j^A v_k), v_m \rangle_{\mathfrak{g}} g^{km} \\ &= -g^{ij} \nabla_i (\langle \nabla_j^A v_k, v_m \rangle_{\mathfrak{g}} g^{km}) + g^{ij} g^{km} \langle \nabla_i^A v_k, \nabla_j^A v_m \rangle_{\mathfrak{g}} \\ &= -\, \mathrm{d}^* Y + |\nabla^A v|^2 \end{aligned}$$

where $Y \in \Omega^1(M)$ is the 1-form $Y = \langle \nabla^A v, v \rangle_{\mathfrak{g}}$, which is of order $O(\rho^{2\nu-1})$ as $\rho \to \infty$. Let $M_{\leq s} = \{x \in M\,;\, \rho(x) \leq s\}$ so that for s sufficiently large we have $\partial M_{\leq s} = \{s\} \times \Sigma$. Stokes' theorem gives

$$\left| \int_{M_{\leq s}} (\mathrm{d}^* Y) \mathrm{Vol}_M \right| = \left| \int_{\{s\} \times \Sigma} (Y \lrcorner \mathrm{Vol}_M) \right| \leq C s^{2\nu-1} \left| \int_{\{R\} \times \Sigma} \mathrm{Vol}_{\{s\} \times \Sigma} \right| \leq \tilde{C} s^{2\nu+5} \to 0$$

as $s \to \infty$. Note that $\nu - 1 < -\frac{7}{2}$ and thus $\nabla^A v \in L^2_{k,\nu-1} \subset L^2$ so we can integrate both sides of (4.22) to obtain

$$\|\nabla^A v\|^2_{L^2} = 0.$$

Finally, the Kato inequality gives $|\nabla |v|| \leq |\nabla^A v|$, so $|v|$ is a harmonic function which by Lemma 4.2.5 vanishes. ☐

Corollary 4.6.2. *Let A be an AC flat connection. If $-\frac{5}{2} < \mu < 0$ and $\mu \notin W$ then A is unobstructed and $\mathcal{M}(A_\infty, \mu)$ is a smooth manifold.*

Proof. This follows immediately from Proposition 4.6.1 after recalling that the obstruction space of $D_A\colon L^2_{k+1,\mu-1} \to L^2_{k,\mu-2}$ is isomorphic to the cokernel of this operator which is identified with $\mathrm{Ker} D_A\colon L^2_{l+1,-5-\mu} \to L^2_{l,-6-\mu}$. ☐

If A is not flat, then one must try to deal with the term $\langle F_A \cdot s, s\rangle$. If $f \in \Omega^0(M, \mathrm{Ad}P)$ then $F_A \cdot f \cdot s_7 = 0$ whenever A is a G_2-instanton and for $v \in \Omega^1(M, \mathrm{Ad}P)$ on has $F_A \cdot v \cdot s_7 = -2(F_A \llcorner v) \cdot s_7$, where the differential form parts of v and F_A act by contraction, whilst the Lie algebra parts act adjointly. Arguing along the lines of [50, Lemma 3.5] we show that we can at least control this term when $\mathrm{d}_A v = 0$.

Lemma 4.6.3. *Let $P \to M$ be a principal bundle and let A be a G_2-instanton on P. Let $v \in \Omega^1(M, \mathrm{Ad}P)$ be such that $\mathrm{d}_A v = 0$, then the pointwise identity $\langle F_A \cdot v, v\rangle = 0$ holds.*

Proof. Since $\mathrm{d}_A v = 0$ we certainly have $\mathrm{d}_A^2 v = [F_A, v] = 0$ and we can calculate

$$\begin{aligned} 0 &= \langle *([F_A, v] \wedge \varphi), v\rangle_{\mathfrak{g}} \\ &= -\langle *([F_A \wedge \varphi, v]), v\rangle_{\mathfrak{g}} \\ &= -\langle *([2 * F_A^7 - *F_A^{14}, v]), v\rangle_{\mathfrak{g}} \\ &= -2\langle F_A^7, [v, v]\rangle_{\mathfrak{g}} + \langle F_A^{14}, [v, v]\rangle_{\mathfrak{g}} \end{aligned}$$

where F_A^7 and F_A^{14} are the projections of F_A to $\Omega^2_7(M, \mathrm{Ad}P)$ and $\Omega^2_{14}(M, \mathrm{Ad}P)$ respectively. Since A is a G_2-instanton we have $F_A^7 = 0$ and therefore the above equation says that $\langle F_A^{14}, [v, v]\rangle = -\langle F_A \cdot v, v\rangle = 0$. □

Therefore an L^2-vanishing theorem will follow if one can prove that $v \in \Omega^1(M, \mathrm{Ad}P)$ and $v \cdot s_7 \in \mathrm{Ker} D_A \colon L^2_1 \to L^2$ implies $\mathrm{d}_A v = 0$. Let us set $\mathrm{d}_A^i = \pi_i \circ \mathrm{d}_A$ where $\pi_i \colon \Lambda^2 \to \Lambda^2_i$ is a projection. A priori such a 1-form v satisfies $\mathrm{d}_A^* v = \mathrm{d}_A^7 v = 0$, but from Proposition 4.6.1 we see that when A is flat it follows that $\mathrm{d}_A^{14} v = 0$ also (this is also proved in [51, Lemma 4.9]). In the general case one has to deal with curvature terms and the task is far more complex.

Chapter 5

The Lichnerowicz Formula and Eigenvalue Bounds

The results of the previous chapter inform us that the virtual dimension of the G_2-instanton moduli space is determined by the spectrum of a twisted Dirac operator on a nearly Kähler 6-manifold in an interval determined by the rate of converge of the example being considered. This chapter develops methods for determining the relevant eigenvalues. Since for all known examples the nearly Kähler link in question is a homogeneous space and the limiting connection is the canonical connection we are able to develop a representation theoretic approach.

The problem of computing the spectrum of the Dirac operator on symmetric and homogeneous spaces has received much attention since the advent of spin geometry. To the author's knowledge the first complete computation of the spectrum of a Dirac operator was done in the thesis of Sulanke [82] in 1979, where the Dirac spectrum of spheres was calculated. Shortly after this the spectrum of the Dirac operator on the Grassmannian manifolds $\mathrm{Gr}_2(\mathbb{R}^{2m})$ was calculated in [81]. The idea is to use Frobenius reciprocity to provide a representation theoretic formula for the square of the Dirac operator then provide a so called branching rule for the homogeneous space G/H, which means determining how each irreducible representation of G decomposes when the action is restricted to the subgroup H. Bär [3, 4] gave a formula for the Dirac operator on a homogeneous manifold and used this to calculate the matrix (and hence eigenvalues) of the Dirac operator on finite dimensional subspaces of the space of spinor fields. By calculating the branching rules of various group-subgroup pairs the Dirac spectra of a variety of homogeneous space has been calculated and an excellent reference is [10, Section 15.5] where a list of known examples is provided. The difficulty of determining the spectrum in this way is the calculation of the relevant branching rule; this is a classic problem in representation theory and is only known for a few group-subgroup pairs. It is worth noting that other methods for calculating Dirac spectra exist. For example Bär [6] uses knowledge of the spectrum of the Laplacian to calculate spectrum of Dirac operator on spheres and their quotients and Camporesi-Higuchi [16] use a separation of variables technique to calculate the Dirac spectrum on spheres and hyperbolic spaces.

The methods we shall develop are not designed to calculate the entire twisted Dirac spectrum– since we consider moduli spaces $\mathcal{M}(A_\infty, \mu)$ for μ in some given interval (which is determined by the rates for which the example in question is AC) we need only consider the eigenvalues in this interval (shifted by 2). In most examples it will suffice to calculate eigenvalues in the region $[0, 2)$. For this reason we need not work with general branching rules; knowledge of how the low dimensional irreducible representations branch for each pair (G, H) is sufficient. In later chapters, when the Lichnerowicz formula is insufficient for calculating eigenvalues, we calculate eigenvalues of the Dirac operator explicitly by working on finite dimensional subspaces of the space of sections and calculating the matrix of the induced endomorphism.

In this chapter the link Σ of the AC G_2-manifold M is assumed to be a compact homogeneous nearly Kähler 6-manifold $\Sigma = G/H$. We shall denote by $\widehat{G}$ the set of isomorphism classes of irreducible, complex unitary representations of G and for $\gamma \in \widehat{G}$ we write $(V_\gamma, \rho_{V_\gamma})$ to denote any class representative. Similarly $\widehat{H}$ denotes the set of isomorphism classes of irreducible, complex unitary representations of H and for $\gamma \in \widehat{H}$ a class representative is denoted $(W_\gamma, \rho_{W_\gamma})$.

5.1 Frobenius Reciprocity

To calculate eigenvalues explicitly we utilise results from harmonic analysis. The main tool we shall require is the Frobenius reciprocity theorem which generalises the classical Peter-Weyl theorem to the space of sections of an associated vector bundle.

Let us first briefly review the Peter-Weyl Theorem. For a compact Lie group this works as follows: The left-regular representation ρ_L acts on $L^2(G, \mathbb{C})$ and one can ask how this decomposes as a representation of G. Let (V_γ, ρ_γ) be any irreducible representation of G and note that any non-zero vector $\Phi \in V_\gamma^* = \mathrm{Hom}(V_\gamma, \mathbb{C})$ defines a G equivariant homomorphism $V_\gamma \to L^2(G, \mathbb{C})$, defined by

$$v \mapsto (g \mapsto \Phi(\rho_\gamma(g^{-1})v)).$$

The statement of the Peter-Weyl theorem is that $L^2(G, \mathbb{C})$ decomposes into an orthogonal Hilbert sum of all the irreducible unitary representations, in which the multiplicity of each irreducible representation is equal the dimension of the representation. More precisely

$$L^2(G, \mathbb{C}) \cong \bigoplus_{\gamma \in \widehat{G}} V_\gamma^* \otimes V_\gamma.$$

The Frobenius reciprocity theorem generalises this construction to decompose $(\rho_L, L^2(G, V)_H)$ as a representation of G. The left action of G on this space gives a representation which is said to be *induced* by the representation ρ_V and which is denoted $\mathrm{Ind}_H^G(V)$. Suppose we have an irreducible representation V_γ of G and a non-trivial element Φ of $\mathrm{Hom}(V_\gamma, V)_H$. Here $\mathrm{Hom}(V_\gamma, V)_H$ denotes the space of H-equivariant maps $V_\gamma \to V$

$$\mathrm{Hom}(V_\gamma, V)_H = \{\Phi \in \mathrm{Hom}(V_\gamma, V)\,;\, \Phi \circ \rho_{V_\gamma}(h) = \rho_V(h) \circ \Phi \text{ for all } h \in H\}.$$

Then for any $v \in V_\gamma$ we have a map

$$V_\gamma \to L^2(G, V)_H, \qquad v \mapsto (g \mapsto \Phi(\rho_{V_\gamma}(g^{-1})v)). \tag{5.1}$$

Frobenius reciprocity uses this construction to show that an irreducible representation V_γ is contained in the induced representation if and only if $\mathrm{Hom}(V_\gamma, V)_H \neq \{0\}$ and the multiplicity of V_γ in the induced representation is $\dim(\mathrm{Hom}(V_\gamma, V)_H)$. Thus if we denote by $\mathrm{Res}_H^G(V_\gamma)$ the restriction of (ρ_γ, V_γ) to the subgroup H, we have

$$\mathrm{mult}(V_\gamma, \mathrm{Ind}_H^G(V)) = \mathrm{mult}(\mathrm{Res}_H^G(V_\gamma), V).$$

This construction enables one to decompose the space of sections of $E := G \times_H V$ into an orthogonal Hilbert sum

$$L^2(E) \cong L^2(G, V)_H \cong \bigoplus_{\gamma \in \widehat{G}} \mathrm{Hom}(V_\gamma, V)_H \otimes V_\gamma. \tag{5.2}$$

The element of $L^2(G, V)_H$ that one obtains via (5.1) from an element $\Phi \otimes v \in \mathrm{Hom}(V_\gamma, V)_H \otimes V_\gamma$ will be denoted

$$F^\gamma_{\Phi,v}(g) = \Phi(\rho_{V_\gamma}(g^{-1})v) \tag{5.3}$$

Since G is assumed to be compact any irreducible representation must be finite dimensional. Furthermore, each summand in the above Hilbert space sum in fact lies in $C^\infty(G, E)_H$.

5.2 The Family of Dirac Operators

Let $\Sigma = G/H$ be a compact, homogeneous nearly Kähler 6-manifold. We choose to work with complex spinor bundle $\$_{\mathbb{C}}(\Sigma)$ which is the associated bundle

$$\$_{\mathbb{C}}(\Sigma) = G \times_{(H,\rho)} S.$$

Here S, which we refer to as spinor space, is a complex eight dimensional vector space which as an H-module is

$$S = \mathbb{C} \oplus \mathfrak{m}^*_{\mathbb{C}} \oplus \mathbb{C} \tag{5.4}$$

where $\mathfrak{m}^*_{\mathbb{C}}$ carries the adjoint action of H and ρ is the representation defined by this action. The splitting (5.4) of S as an H-module comes from the splitting $\$_{\mathbb{C}}(\Sigma) = \Lambda^0_{\mathbb{C}} \oplus \Lambda^1_{\mathbb{C}} \oplus \Lambda^6_{\mathbb{C}}$

of the spinor bundle. We shall consider a twisting of the spinor bundle by an associated bundle

$$E = G \times_{(H,\rho_V)} V$$

constructed from some representation V of H. Thus the twisted spinor bundle is given by

$$\not{S}_{\mathbb{C}}(\Sigma) \otimes E = G \times_{(H,\rho_S \otimes \rho_V)} S \otimes V$$

and so the one can hope to apply tools from Harmonic Analysis to study the family of twisted Dirac operators. When the operator is twisted by the canonical connection, that is $A_\infty = A_{\text{can}}$ in the notation of Chapter 4.1, the covariant derivative can be understood using the Frobenius reciprocity theorem. Recall that $D^1_{A_{\text{can}}} = \text{cl} \circ \nabla^{1,A_{\text{can}}}$ is constructed from the canonical connection acting both on the spinor bundle A and the associated bundle E. We can therefore understand this operator using this formalism since (3.1) says that

$$\nabla^{1,A_{\text{can}}}_X s = \rho_R(X)s \tag{5.5}$$

for $s \in L^2(G, S\otimes V)_H$ and where we think of a vector field X as an element of $L^2(G, \mathfrak{m})_H$. Recall $\mathfrak{g}$ is given the nearly Kähler metric (2.15) and that $\{I_A\}$ denotes a basis for $\mathfrak{g}$ such that I_a for $1 \leq a \leq 6$ forms a basis for $\mathfrak{m}$ and I_i for $7 \leq i \leq \dim(G)$ forms a basis of $\mathfrak{h}$. The basis $\{I_a\}$ defines a local frame $\{e^a\}$ of $T^*\Sigma|_U$ as described in Section 2.3 and we can think of these local 1-forms as equivariant maps from $\pi^{-1}U$ to $\mathfrak{m}^*$. By (5.5) the operator $D^1_{A_{\text{can}}}$ takes the form

$$D^1_{A_{\text{can}}} = \text{cl}(e^a)\rho_R(e^a) \tag{5.6}$$

when acting on elements of $L^2(G, S \otimes V)_H$. Furthermore we know from [18] that the family of Dirac operators differ by the action of the 3-form ReΩon the spin bundle

$$D^t_{A_{\text{can}}} = D^1_{A_{\text{can}}} + \frac{3(t-1)}{4}\text{Re}\Omega.$$

By combining these facts with the Frobenius reciprocity theorem we can understand how each of these operators act with respect to the splitting of the space of sections given in (5.3). First we collect some facts about induced actions on the homomorphisms spaces contained in the space of sections.

Since the spinor space is a Clifford module one can act on a spinor with a tangent vector (or equivalently covector) and for an irreducible representation V_γ this in turn endows $\text{Hom}(V_\gamma, S\otimes V)$ with an action $\text{cl}(\,\cdot\,)$ of the Clifford algebra (note the equivariance property may not be preserved). Vector fields (thought of as equivariant maps) act on $\text{Hom}(V_\gamma, S \otimes V)_H$, preserving the equivariance condition, since the spinor bundle carries an action of the Clifford bundle. By identifying $T^*_{[1]}(\Sigma)$ with $\mathfrak{m}^*$ the 3-form $\text{Re}\Omega$ defines an element of $\Lambda^3(\mathfrak{m}^*)$ and this acts on the spinor space and hence on $\text{Hom}(V_\gamma, S \otimes V)_H$ preserving the equivariant condition.

The next result is essentially the same result as [4, Proposition 1] but generalised to the case of a twisted spinor. The proof goes through identically, but is given here since the difference between the Dirac operators defined by the Levi-Civita and canonical connections is in our case more explicit; it is determined by the 3-forms $\mathrm{Re}\Omega$.

Proposition 5.2.1. *Let E be a vector bundle associated via a representation V of H and split the space of sections*

$$L^2(\$_{\mathbb{C}}(\Sigma)\otimes E)\cong L^2(G,S\otimes V)_H\cong\bigoplus_{\gamma\in\widehat{G}}\mathrm{Hom}(V_\gamma,S\otimes V)_H\otimes V_\gamma.$$

For any $\gamma\in\widehat{G}$ and for every $t\in\mathbb{R}$, the operator $D^t_{A_{can}}$ leaves invariant the space $\mathrm{Hom}(V_\gamma,S\otimes V)_H\otimes V_\gamma$ *and*

$$D^t_{A_{can}}|_{\mathrm{Hom}(V_\gamma,S\otimes V)_H\otimes V_\gamma}=(D^t_{A_{can}})_\gamma\otimes\mathrm{Id} \tag{5.7}$$

where $(D^t_{A_{can}})_\gamma\colon\mathrm{Hom}(V_\gamma,S\otimes V)_H\to\mathrm{Hom}(V_\gamma,S\otimes V)_H$ is the operator

$$(D^t_{A_{\mathrm{can}}})_\gamma\Phi=-\mathrm{cl}(I_a)\cdot\big(\Phi\circ\rho_{V_\gamma}(I_a)\big)+\frac{3(t-1)}{4}\mathrm{Re}\Omega\cdot\Phi. \tag{5.8}$$

Proof. This is a consequence of the Frobenius reciprocity theorem, which enables us to determine how the Lie algebra representation ρ_R acts under the decomposition

$$L^2(G,S\otimes V)_H\cong\bigoplus_{\gamma\in\widehat{G}}\mathrm{Hom}(V_\gamma,S\otimes V)_H\otimes V_\gamma.$$

We use Frobenius reciprocity to understand the action of the canonical connection on the map $F^\gamma_{\Phi,v}(g)$ given in (5.3). To understand covariant differentiation by the connection $\nabla^{1,A_{\mathrm{can}}}$ we use (5.5) to identify this with the right regular action. Let $\Phi\otimes v\in\mathrm{Hom}(V_\gamma,S\otimes V)_H\otimes V_\gamma$. Thinking of vector fields as H equivariant maps $G\to\mathfrak{m}$, we let $X\in L^2(G,\mathfrak{m})_H$ be a vector field and calculate

$$\begin{aligned}(\rho_R(X)F^\gamma_{\Phi,v})(g)&=\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}F^\gamma_{\Phi,v}(ge^{tX(g)})\\&=\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\Phi(\rho_{V_\gamma}(e^{-tX(g)}g^{-1})v)\\&=\frac{\mathrm{d}}{\mathrm{dt}}\Big|_{t=0}\Phi(\rho_{V_\gamma}(e^{-tX(g)})\circ\rho_{V_\gamma}(g^{-1})v)\\&=-(\Phi\circ\rho_{V_\gamma}(X(g)))\circ(\rho_{V_\gamma}(g^{-1})v).\end{aligned}$$

Comparing to (5.6) we see that the Dirac operator $D^1_{A_{\mathrm{can}}}$, which yields a globally well defined section $D^1_{A_{\mathrm{can}}}F^\gamma_{\Phi,v}=\mathrm{cl}(e^a)\rho_R(e^a)F^\gamma_{\Phi,v}$, acts on $\mathrm{Hom}(V_\gamma,S\otimes V)_H\otimes V_\gamma$ algebraically as

$$D^1_{A_{\mathrm{can}}}|_{\mathrm{Hom}(V_\gamma,S\otimes V)_H\otimes V_\gamma}(\Phi\otimes v)=-\big(\mathrm{cl}(I_a)\cdot\big(\Phi\circ\rho_{V_\gamma}(I_a)\big)\big)\otimes v.$$

By (2.33) the action of $D^t_{A_{\mathrm{can}}}$ differs from that of $D^1_{A_{\mathrm{can}}}$ by the 3-form $\mathrm{Re}\Omega$. In this way the operator $D^t_{A_{\mathrm{can}}}$ acts on $\mathrm{Hom}(V_\gamma, S\otimes V)_H \otimes V_\gamma$ as

$$D^t_{A_{\mathrm{can}}}|_{\mathrm{Hom}(V_\gamma,S\otimes V)_H\otimes V_\gamma}(\Phi\otimes v) = \left(-\mathrm{cl}(I_a)\cdot\left(\Phi\circ\rho_{V_\gamma}(I_a)\right) + \frac{3(t-1)}{4}\mathrm{Re}\Omega\cdot\Phi\right)\otimes v$$

as required. □

Note that under the isomorphism $\mathrm{Hom}(V_\gamma, S\otimes V)\cong V_\gamma^*\otimes S\otimes V$ an H-equivariant homomorphism corresponds to a vector fixed by the H action and the formula for $(D^t_{A_{\mathrm{can}}})_\gamma$ takes the form

$$(D^t_{A_{\mathrm{can}}})_\gamma = \mathrm{cl}(I_a)\rho_{V_\gamma^*}(I_a) + \frac{3(t-1)}{4}\mathrm{Re}\Omega. \tag{5.9}$$

This point of view will be the one we adopt when calculating eigenvalues explicitly.

5.3 Eigenvalue Bounds

Throughout this chapter A is assumed to be an AC G_2-instanton with fastest rate of convergence $\mu_0 < 0$ and the limiting connection $A_\infty = A_{\mathrm{can}}$ will be assumed to be the canonical connection living on some bundle associated via a representation of H. We shall consider the family of moduli spaces $\mathcal{M}(A_{\mathrm{can}}, \mu)$ with μ ranging from the fastest rate of convergence of the example we are studying, to 0. We have seen the virtual dimension of these moduli spaces jumps as we pass through eigenvalues in this interval shifted by 2. Our method is to develop a representation theoretic Lichnerowicz formula to determine the eigenvalues of a related Dirac operator.

The relevant Lichnerowicz type formula is calculated in [18] and we shall build on this work. The formula gives the square of the Dirac operator as a sum of Casimir operators, so we first remind the reader how these operators are constructed. Any representation (V,ρ) of a Lie algebra $\mathfrak{g}$, equipped with an invariant inner product B, yields a quadratic Casimir operator $\rho(\mathrm{Cas}_\mathfrak{g})$ defined as $\rho(\mathrm{Cas}_\mathfrak{g})v = \rho(I_A)\rho(I_A)v$ for any $v\in V$ and where I_A is an orthonormal basis for $\mathfrak{g}$. In the case at hand the metric on $\mathfrak{g}$ is the nearly Kähler metric $B(X,Y) = -\frac{1}{12}\mathrm{Tr}_\mathfrak{g}(\mathrm{ad}(X)\mathrm{ad}(Y))$ and the metrics on $\mathfrak{m}$ and $\mathfrak{h}$ are the restrictions of B. Since the Casimir operators commute with the group action they act as multiples of the identity on irreducible representations and so we can write

$$\rho_\gamma(\mathrm{Cas}_\mathfrak{g}) = c^\mathfrak{g}_\gamma \mathrm{Id}$$
$$\rho_\gamma(\mathrm{Cas}_\mathfrak{h}) = c^\mathfrak{h}_\gamma \mathrm{Id}.$$

These eigenvalues are calculated, for G and H such that G/H is a nearly Kähler 6-manifold and using the above metric, in [18].

To obtain a suitable Lichnerowicz formula we combine [18, Proposition 8] with the results of [75] to obtain a formula for $(D^{\frac{1}{3}}_{A_{\text{can}}})^2$. The operator we would like to calculate eigenvalues for is $D^0_{A_{\text{can}}}$, however we are unable to calculate these eigenvalues directly using our Lichnerowicz formula, we shall see that we are however able to calculate useful eigenvalue bounds.

Lemma 5.3.1. *Let G/H be a homogeneous nearly Kähler manifold. Let E be the vector bundle obtained from $G \to G/H$ through some representation V of H. Let A_{can} be the canonical connection on E, then $(D^{\frac{1}{3}}_{A_{can}})^2$ preserves the decomposition $\Gamma(\$_{\mathbb{C}}(\Sigma) \otimes E) = \Gamma(\Lambda^0_{\mathbb{C}} \otimes E) \oplus \Gamma(\Lambda^1_{\mathbb{C}} \otimes E) \oplus \Gamma(\Lambda^6_{\mathbb{C}} \otimes E)$ and*

$$(D^{\frac{1}{3}}_{A_{\text{can}}})^2\eta = (-\rho_L(\text{Cas}_{\mathfrak{g}})\eta + \rho_V(\text{Cas}_{\mathfrak{h}})\eta + 4\eta) \tag{5.10}$$

for any $\eta \in \Gamma(\$_{\mathbb{C}}(\Sigma) \otimes E)$.

Proof. The formula is calculated for sections of $\Lambda^1_{\mathbb{C}} \otimes E$ in [18, Proposition 8] so we need only consider the case where $\eta \in (\Lambda^0_{\mathbb{C}} \oplus \Lambda^6_{\mathbb{C}}) \otimes E$. It is also shown in [18] that

$$(D^{\frac{1}{3}}_{A_{\text{can}}})^2\kappa = (\nabla^{1,A_{\text{can}}})^*\nabla^{1,A_{\text{can}}}\kappa + 4\kappa$$

for $\kappa \in \Gamma((\Lambda^0_{\mathbb{C}} \oplus \Lambda^6_{\mathbb{C}}) \otimes E)$ and so we aim to show this admits the required Casimir expression. It is a standard fact, see [75] for example, that the rough Laplacian $(\nabla^{1,A_{\text{can}}})^*\nabla^{1,A_{\text{can}}}$ is identified with the action of a Casimir operator

$$(\nabla^{1,A_{\text{can}}})^*\nabla^{1,A_{\text{can}}} = -\rho_R(\text{Cas}_{\mathfrak{m}})$$

on $L^2(G, (\Lambda^0(\mathbb{R}^6)^* \oplus \Lambda^6(\mathbb{R}^6)^*) \otimes V)_H$. Note if $\kappa \in L^2(G, (\Lambda^0_{\mathbb{C}}(\mathbb{R}^6)^* \oplus \Lambda^6_{\mathbb{C}}(\mathbb{R}^6)^*) \otimes V)_H$ then we have $\rho_R(X)\kappa + \rho_V(X)\kappa = 0$ for any $X \in \mathfrak{h}$ and therefore $\rho_R(\text{Cas}_{\mathfrak{h}}) = \rho_V(\text{Cas}_{\mathfrak{h}})$. Combining this with the fact that $\rho_L(\text{Cas}_{\mathfrak{g}}) = \rho_R(\text{Cas}_{\mathfrak{g}})$ yields the result. □

The operator $(D^{\frac{1}{3}}_{A_{\text{can}}})^2$ preserves the decomposition 5.3, so as in Proposition 5.2.1 we can define an operator $(D^{\frac{1}{3}}_{A_{\text{can}}})^2_\gamma$ on $\text{Hom}(V_\gamma, S \otimes V)_H$ such that

$$(D^{\frac{1}{3}}_{A_{\text{can}}})^2_\gamma \otimes \text{Id} = (D^{\frac{1}{3}}_{A_{\text{can}}})^2|_{\text{Hom}(V_\gamma, S\otimes V)_H \otimes V_\gamma}. \tag{5.11}$$

Since we are considering unitary representations the space $\text{Hom}(V_\gamma, S \otimes V)_H$ carries a natural inner product given by

$$\langle X, Y \rangle = \text{Tr}(X^*Y)$$

where X^* is the hermitian adjoint with respect to the hermitian inner products on V_γ and $S \otimes V$. The self-adjointness of $(D^{\frac{1}{3}}_{A_{\text{can}}})^2$ ensures that the restriction to any of the subspaces $\text{Hom}(V_\gamma, S \otimes V)_H \otimes V_\gamma$ defines a hermitian opertor, hence it is diagonalisable with real eigenvalues. Furthermore the spectrum satisfies

$$\text{Spec}(D^{\frac{1}{3}}_{A_{\text{can}}})^2 = \bigcup_{\gamma \in \widehat{G}} (D^{\frac{1}{3}}_{A_{\text{can}}})^2_\gamma.$$

It follows from Frobenius reciprocity and Lemma 5.3.1 that $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ acts as the endomorphism

$$(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma = -\rho_{V^*_\gamma}(\mathrm{Cas}_{\mathfrak{g}}) + \rho_V(\mathrm{Cas}_{\mathfrak{h}}) + 4 \tag{5.12}$$

and it is not difficult to find a basis of $\mathrm{Hom}(V_\gamma, S\otimes V)_H$ that diagonalises this operator.

For this suppose that V_γ is an irreducible representation of G and let $V_\gamma = \bigoplus_{\sigma\in I} W^\gamma_\sigma$ be the decomposition of V_γ into irreducible representations of H, where I is a finite sequence in $\widehat{H}$ (note I may have repeated entries) and W^γ_σ is any class representative of σ. Similarly let us split $S = \oplus_{\mu\in J} W_\mu$ and $V = \oplus_{\nu\in K} W_\nu$ where J and K are finite sequences in $\widehat{H}$.

Suppose σ, μ, ν are such that one obtains an H-equivariant mapping via the composition

$$q^\sigma_{\mu\nu} \colon V_\gamma \to W^\gamma_\sigma \to W_\mu \otimes W_\nu \to S\otimes V$$

where the first map is a projection, the second is an equivariant embedding and the third is an inclusion. The set of distinct tripples (σ,μ,ν) for which such maps exist lead to a basis $\{q^\sigma_{\mu\nu}\}$ of $\mathrm{Hom}(V_\gamma, S\otimes V)_H$. The maps $q^\sigma_{\mu\nu}$ are readily seen to be eigenvectors of (5.12) and the equation

$$(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma(q^\sigma_{\mu\nu}) = \left(-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\nu + 4\right) q^\sigma_{\mu\nu} \tag{5.13}$$

is satisfied, since the eigenvalues of $\rho_{V_\gamma}(\mathrm{Cas}_{\mathfrak{g}})$ and $\rho_{V^*_\gamma}(\mathrm{Cas}_{\mathfrak{g}})$ are equal. Furthermore, since we are considering only orthogonal decomposition into irreducible H-representations, these map $q^\sigma_{\mu\nu}$ are mutually orthogonal. When calculating eigenvalues of twisted Dirac operators explicitly we shall choose to work with this basis; as a consistency check for the accuracy of our calculations we will show that this basis does indeed diagonalise $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ with the correct eigenvalues. We summarise some of the above discussion as a Corollary to Lemma 5.3.1:

Corollary 5.3.2. *Let V_γ be an irreducible representation of G, and let $V = \bigoplus_{\nu\in K} W_\nu$ be the decomposition of V into irreducible representations of H, where K is a finite sequence in $\widehat{H}$. The eigenvalues and multiplicities of the operator $(D^{\frac{1}{3}}_{A_{can}})^2_\gamma$ are*

Eigenvalue	*Multiplicity*
$-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\nu + 4$	$\dim\mathrm{Hom}(V_\gamma, S\otimes W_\nu)_H$

where $c^{\mathfrak{g}}_\gamma$ is the eigenvalue of the Casimir operator on the irreducible representation V_γ with respect to the inner product B from (2.15) and $c^{\mathfrak{h}}_\nu$ is the eigenvalue of the Casimir operator on the irreducible representation W^V_ν with respect to this metric. The eigenvalues of $(D^{\frac{1}{3}}_{A_{can}})_\gamma$ are $\pm\sqrt{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\nu + 4}$ and the $\sqrt{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\nu + 4}$ and $-\sqrt{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\nu + 4}$ eigenspaces are isomorphic.

Proof. We have a splitting

$$\mathrm{Hom}(V_\gamma, S\otimes V)_H = \bigoplus_{\nu\in K} \mathrm{Hom}(V_\gamma, S\otimes W_\nu)_H$$

which is respected by the operator $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$. Inspection of (5.10) reveals that $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ acts as a constant with the advertised value on each of these summands.

It follows that the eigenvalues of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ are given by $\pm\sqrt{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\nu + 4}$ (verification of this fact is given in [6] for example). Furthermore one can check that $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ anti-commutes with the action of the volume form Vol on the spin bundle and since $\mathrm{Vol}^2 = -1$ one sees that the volume form provides an isomorphism between the $\sqrt{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\nu + 4}$ and $-\sqrt{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\nu + 4}$ eigenspaces. □

Remark 5.3.3. *We see from Corollary 5.3.2 why the task of calculating branching rules is necessary to calculate the entire spectrum of a Dirac operator. One must find out which of the spaces* $\mathrm{Hom}(V_\gamma, S\otimes W_\nu)_H$ *are non-zero and therefore contribute eigenvalues to the spectrum.*

It may be the case that $\gamma \neq \gamma'$ but the operators $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ and $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_{\gamma'}$ have an eigenvalue in common, so one must take this into account in order to calculate the multiplicity of a given eigenvalue of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2$. Suppose for simplicity that a given eigenvalue of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ does not occur for any other $\gamma' \neq \gamma$, then inspection of (5.11) reveals that one must multiply the multiplicity of this eigenvalue by the dimension of the representation V_γ to get the multiplicity of that eigenvalue in the space of sections.

Note that the eigenvalues of $-\rho_{V_\gamma}(\mathrm{Cas}_{\mathfrak{g}})$ change as γ varies, whereas the eigenvalue of $\rho_V(\mathrm{Cas}_{\mathfrak{h}})+4$ are fixed. Furthermore, large dimensional irreducible representations V_γ lead to the Casimir operator $-\rho_{V_\gamma}(\mathrm{Cas}_{\mathfrak{g}})$ having a large positive eigenvalue. Intuitively this should mean that, if V_γ is a large dimensional irreducible representation, the eigenvalues of the operator $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ are large. The following lemma allows us to compare eigenvalues of operators in this way.

Lemma 5.3.4. *Let X and Y be $n\times n$ Hermitian matrices with eigenvalues $\{\lambda^X_1,\dots,\lambda^X_n\}$ and $\{\lambda^Y_1,\dots,\lambda^Y_n\}$ respectively. Let $\{\lambda^{X+Y}_1,\dots\lambda^{X+Y}_n\}$ be the set of eigenvalues of $X+Y$. Then*

$$\min\{|\lambda^{X+Y}_1|,\dots,|\lambda^{X+Y}_n|\} \geq \left|\min\{|\lambda^X_1|,\dots,|\lambda^X_n|\} - \max\{|\lambda^Y_1|,\dots,|\lambda^Y_n|\}\right|.$$

Proof. First recall $\min\{|\lambda^X_i|\}^n_{i=1} = \min_{\|v\|=1}\langle Xv, v\rangle$ and $\max\{|\lambda^Y_i|\}^n_{i=1} = \max_{\|v\|=1}\langle Yv, v\rangle$. Note $\mu_l \geq 0$ is a lower bound on $\{|\lambda^{X+Y}_i|\}^n_{i=1}$ if and only if $|\langle (X+Y)v, v\rangle| \geq \mu_l$ for all $v\in\mathbb{C}^n$ with $\|v\| = 1$.

Now

$$\begin{aligned}|\langle (X+Y)v, v\rangle| &= |\langle Xv, v\rangle + \langle Yv, v\rangle| \\ &\geq ||\langle Xv, v\rangle| - |\langle Yv, v\rangle|| \\ &\geq \left|\min\{|\lambda_i^X|\}_{i=1}^n - \max\{|\lambda_i^Y|\}_{i=1}^n\right|\end{aligned}$$

for all $v \in \mathbb{C}^n$ with $\|v\| = 1$. Hence $\left|\min\{|\lambda_i^X|\}_{i=1}^n - \max\{|\lambda_i^Y|\}_{i=1}^n\right|$ is a lower bound on $\{|\lambda_i^{X+Y}|\}_{i=1}^n$. □

Recall the operators $(D^t_{A_{\mathrm{can}}})_\gamma$ from (5.7), as was the case for $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ these anti-commute with the volume form Vol so have spectra symmetric about 0 and satisfy

$$\mathrm{Spec}(D^t_{A_{\mathrm{can}}}) = \bigcup_{\gamma \in \widehat{G}} \mathrm{Spec}(D^t_{A_{\mathrm{can}}})_\gamma.$$

Recall that our task is to determine the eigenvalues of $D^0_{A_{\mathrm{can}}}$ in the region $(\mu_0 + 2, 2)$ where μ_0 is the fastest rate of convergence of A, therefore we would like to know which representations V_γ have eigenvalues in this region.

Suppose that the operator $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ has eigenvalues $\lambda_1^2, \cdots, \lambda_n^2$, then the eigenvalues of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ are precisely $\pm\lambda_1, \cdots \pm \lambda_n$. Now since

$$(D^0_{A_{\mathrm{can}}})_\gamma = (D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma - \frac{1}{4}\mathrm{Re}\Omega$$

and the eigenvalues of $-\frac{1}{4}\mathrm{Re}\Omega$ are contained in the set $\{\pm 1, 0\}$, we can apply Lemma 5.3.4 to obtain lower bound on the smallest positive eigenvalue of $(D^0_{A_{\mathrm{can}}})_\gamma$.

Theorem 5.3.5. *Let V_γ be an irreducible representation of G. If the quantity*

$$L_\gamma := \sqrt{\min_\nu\{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\nu + 4\}} - 1$$

is positive, then it is a lower bound on the smallest positive eigenvalue of $(D^0_{A_{\mathrm{can}}})_\gamma$.

Proof. Inspection of Corollary 5.3.2 reveals that the smallest eigenvalue of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ is $\sqrt{-c^{\mathfrak{g}}_\gamma + c^{\mathfrak{h}}_\nu + 4}$. Since the eigenvalues of $-\frac{1}{4}\mathrm{Re}\Omega$ are contained in the set $\{\pm 1, 0\}$ we can apply Lemma 5.3.4 to yield the result. □

With this in hand we can outline our strategy for calculating the eigenvalues of $D^0_{A_{\mathrm{can}}}$ in $(\mu_0 + 2, 2)$ is as follows:

1. Use Theorem 5.3.5 to rule out the irreducible representations V_γ such that $(D^0_{A_{\mathrm{can}}})_\gamma$ has no eigenvalues in the interval $(m, 2)$ where $m = \min\{-2, \mu_0 + 2\}$ and μ_0 is the fastest rate of converge of the example we are studying (usually we will just need to consider eigenvalues in the region $[0, 2)$).

2. For the remaining representations, if all maps in $\mathrm{Hom}(V_\gamma, S \otimes V)_H$ factor through $\Lambda^1_{\mathbb{C}} \otimes V \subset S \otimes V$ then $(D^0_{A_{\mathrm{can}}})_\gamma = (D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ on this space (this is thanks to Lemma 2.3.3) and Corollary 5.3.2 informs us of the eigenvalues.

3. If there are maps factoring through $(\Lambda^0_{\mathbb{C}} \oplus \Lambda^6_{\mathbb{C}}) \otimes E \subset S \otimes E$ then $\mathrm{Re}\Omega$ acts non-trivially and one must work harder to calculate the relevant eigenvalues.

In the last case here, when $\mathrm{Re}\Omega$ acts non-trivially, we shall attack the known examples using two different methods. When the nearly Kähler 6-manifold is $\mathbb{CP}^3$ we adapt the work of Bär [4] and develop a representation theoretic framework to calculate directly the matrix (and hence the eigenvalues) of the Dirac operator on one of the homomorphism spaces from via the formula (5.9). When the manifold is S^6 we instead write the Dirac operator as a sum of Casimir operators this again allows us to calculate the matrix and eigenvalues of the Dirac operator on the relevant homomorphism spaces.

Chapter 6

Deformations of Clarke's Instantons

In this chapter we study G_2-instantons on the Bryant-Salamon manifold $\mathbb{R}^4 \times S^3$. We first review the work of Lotay-Oliveira [71] where the moduli space of instantons which are invariant under the action of $\mathrm{SU}(2)^3$ was studied (this being the symmetry group of the Bryant-Salamon metric). In doing so we present the family of G_2-instantons first found by Clarke in [20] and then show that this instanton family admits a limiting connection which can be interpreted as a removeable singularity phenomenon, essentially because bubbling occurs. The original contributions of this thesis are given in Section 6.5. We study the full moduli space of AC G_2-instantons in which these examples live. We show that the Lichnerowicz formula developed in the Chapter 5 is sufficient for determining the virtual dimension of this moduli space.

6.1 Evolution Equations

In this section we show how to view the G_2-instanton equation as an evolution equation, based on the presentation given in [71, 70]. Until now the only $\mathrm{SU}(3)$-structures considered have been nearly Kähler structures, so we first consider $\mathrm{SU}(3)$-structures in full generality. An $\mathrm{SU}(3)$-structure on a manifold Σ is defined as a triple (J, ω, γ_2) where J is an almost complex structure, ω is a $(1,1)$-form with respect to J and γ_2 is a real 3-form such that the relations

$$\omega \wedge \gamma_2 = 0 \tag{6.1}$$

$$\omega^3 = \frac{3}{2}\gamma_1 \wedge \gamma_2 \tag{6.2}$$

are satisfied, where $\gamma_1 = J\gamma_2$. Suppose now that $(J_t, \omega(t), \gamma_2(t))$ is a 1-parameter family of $\mathrm{SU}(3)$ structures where t is the parameter in some interval I, then we may define a G_2

structure on $\Sigma \times I$ with

$$\varphi = \mathrm{d}t \wedge \omega(t) + \gamma_1(t) \tag{6.3}$$

$$\psi = \frac{\omega^2(t)}{2} - \mathrm{d}t \wedge \gamma_2(t). \tag{6.4}$$

Let us denote differentiation of a time dependent form α with respect to t by $\dot{\alpha}$, then the above G_2-structure will be torsion free (i.e $\mathrm{d}\varphi = \mathrm{d}\psi = 0$) precisely when the equations

$$\mathrm{d}\omega = \dot{\gamma_1}, \qquad \omega \wedge \dot{\omega} = -\,\mathrm{d}\gamma_2 \tag{6.5}$$

subject to the constraint $\mathrm{d}\gamma_1 = \mathrm{d}\omega^2 = 0$ for all t, are satisfied. These equations are known as the Hitchin flow (although it is important to note that the system is not parabolic, so does not define a geometric flow) and the constraint $\mathrm{d}\gamma_1 = \mathrm{d}\omega^2 = 0$ is known as the "half flat" condition for the SU(3)-structure, since it implies half of the terms of the torsion tensor must vanish. Furthermore it suffices to impose the half flat condition for initial time since this property is preserved by the flow.

The metric on $\Sigma \times I$ determined by this construction is

$$g = \mathrm{d}t^2 + g_t$$

where $g_t(\cdot,\cdot) = \omega_t(\cdot, J_t\cdot)$ is the metric determined by the SU(3)-structure. We have of course already seen a solution to the Hitchin flow; when the SU(3)-structure $(J, \omega, \mathrm{Re}\Omega)$ on Σ is nearly Kähler the G_2-cone $(C(\Sigma), \varphi_C) = (\Sigma \times \mathbb{R}^+, t^2\omega \wedge \mathrm{d}t + t^3\mathrm{Im}\Omega)$ solves the Hitchin flow equations.

It is natural to consider the G_2-instanton equation in the same framework. As in Chapter 4 we shall assume that our principal bundle $P \to \Sigma \times I$ is framed by $Q \to \Sigma$, in other words we have $P = \pi^*Q$ for the natural projection map $\pi\colon \Sigma \times I \to \Sigma$ and also that the connection A on P is in temporal gauge, that is $\mathrm{d}t \lrcorner A = 0$. When this is the case we shall write $A = a(t)$ where $a(t)$ is a path in the space of connections on Q. No generality is lost here– P will always have such a framing and a temporal gauge may always be chosen. One finds

$$F_A = \mathrm{d}t \wedge \dot{a} + F_{a(t)}$$

where $F_{a(t)}$ is the curvature of the connection $a(t)$ on Q. Inserting this into the G_2-instanton equation $F_A \wedge \psi = 0$ yields the evolution equation

$$\dot{a} \wedge \frac{\omega^2}{2} = F_a \wedge \gamma_2, \qquad F_a \wedge \omega^2 = 0. \tag{6.6}$$

Applying the Hodge star operator of the metric g_t to this system yields the following result:

Lemma 6.1.1 ([71, Lemma 1]). *Let Σ be an* SU(3) *structure* 6*-manifold and give $\Sigma \times I$ the G_2 structure* (6.3)*. Then G_2 instantons A on $P \to \Sigma \times I$ are in bijection with 1-parameter families of connections $\{a(t)\}_{t\in\mathbb{R}^+}$ on $Q \to \Sigma$ satisfying the equation*

$$J_t \dot{a} = - *_t (F_a \wedge \gamma_2) \tag{6.7}$$

subject to the constraint $\omega_t \lrcorner F_a = 0$. Furthermore this constraint is compatible with the evolution in the sense that if it holds for some $t_0 \in \mathbb{R}^+$ then it holds for all $t \in I$.

6.2 The Bryant-Salamon $\mathbb{R}^4 \times S^3$

We begin by briefly introducing the construction of this metric, again based on the presentation given in [71]. We consider an SU(3) structure on $S^3 \times S^3 = \mathrm{SU}(2)^2$. Let I_i for $i = 1, 2, 3$ be a basis of $\mathfrak{su}(2)$ with $[I_i, I_j] = 2\epsilon_{ijk} I_k$, then we may split $\mathfrak{su}(2)^2 = \mathfrak{su}(2)^+ \oplus \mathfrak{su}(2)^-$ where $I_i^+ = (I_i, I_i)$ and $I_i^- = (I_i, -I_i)$ provide bases for $\mathfrak{su}(2)^+$ and $\mathfrak{su}(2)^-$ respectively. We define η_i^+ and η_i^- to be dual to I_i^+ and I_i^- respectively. The Maurer-Cartan relations take the form

$$\mathrm{d}\eta_i^+ = -\epsilon_{ijk}(\eta_j^+ \wedge \eta_k^+ + \eta_j^- \wedge \eta_k^+), \tag{6.8}$$
$$\mathrm{d}\eta_i^- = -2\epsilon_{ijk}\eta_j^- \wedge \eta_k^+. \tag{6.9}$$

The group $\mathrm{SU}(2)^3$ acts on $\mathrm{SU}(2)^2$ as follows:

$$(g_1, g_2, g_3) \cdot (\tilde{g}_1, \tilde{g}_2) = (g_1 \tilde{g}_1 g_3^{-1}, g_2 \tilde{g}_2 g_3^{-1})$$

and under this action we can impose $\mathrm{SU}(2)^3$ symmetry on our SU(3) structure; this yields the following expressions:

$$\omega = 4XY \eta_i^- \wedge \eta_i^+ \tag{6.10}$$
$$\gamma_1 = 8Y^3 \eta_1^- \wedge \eta_2^- \wedge \eta_3^- - 4X^2 Y \epsilon_{ijk} \eta_i^+ \wedge \eta_j^+ \wedge \eta_k^- \tag{6.11}$$
$$\gamma_2 = 8X^3 \eta_1^+ \wedge \eta_2^+ \wedge \eta_3^+ - 4Y^2 X \epsilon_{ijk} \eta_i^- \wedge \eta_j^- \wedge \eta_k^+ \tag{6.12}$$

where X and Y are real valued functions of $t \in \mathbb{R}^+$. One finds that the Hitchin flow equation (6.5) reduces to the coupled ODE

$$\dot{X} = \frac{1}{2}\left(1 - \frac{X^2}{Y^2}\right), \qquad \dot{Y} = \frac{X}{Y}. \tag{6.13}$$

Set $Y = s$ and $X = sF(s)$, then (6.13) becomes

$$\frac{\mathrm{d}}{\mathrm{d}s}(sF) = \frac{1 - F^2}{2F}. \tag{6.14}$$

Solutions are given by

$$F(s) = \sqrt{\frac{1 - c^3 s^{-3}}{3}} \tag{6.15}$$

where $c > 0$ and $s \geq c$.This yields

$$X(s) = \frac{s}{\sqrt{3}}\sqrt{1 - c^3 s^{-3}} \qquad \text{and} \qquad Y(s) = s.$$

Let us choose $c = 1$ and define a coordinate $r \in [1, +\infty)$ as follows: recall t is the arclength parameter along the geodesic parameterised by s, so we may set

$$t(r) = \int_1^r \frac{\mathrm{d}s}{\sqrt{1 - s^{-3}}}$$

then our functions take the form

$$X = \frac{r}{3}\sqrt{1 - r^{-3}} \qquad \text{and} \qquad Y = \frac{r}{\sqrt{3}}.$$

This is the form of the solution that appears in Bryant-Salamon's original construction.

This defines an asymptotically conical G_2-manifold where the fastest rate of convergence is $\mu = -3$ [61]. Furthermore the limit as $c \to 0$, where c is the parameter c from (6.15), yields the conical G_2-structure $C(S^3 \times S^3)$. If we view $\mathbb{R}^4 \times S^3$ as $\not{S}(S^3)$, then $(\mathbb{R}^4 \times S^3) \setminus S^3 \cong \mathbb{R}^+ \times \mathrm{SU}(2)^2$ where $S^3 \subset \mathbb{R}^4 \times S^3$ denotes the zero section of the spinor bundle. This is the unique singular orbit for the action of $\mathrm{SU}(2)^3$ on $\mathbb{R}^4 \times S^3$. This singular orbit is modelled as $S^3 = \mathrm{SU}(2)^3/\mathrm{SU}(2)^2$ where $\mathrm{SU}(2)^2 \subset \mathrm{SU}(2)^3$ is the group $\{(g_1, g_1, g_2)\,;\, g_i \in \mathrm{SU}(2)\}$. Furthermore this singular orbit is the unique compact associative submanifold for the G_2-structure.

6.3 Clarke's Instantons from Evolution Equations

Clarke's family of instantons have structure group $\mathrm{SU}(2)$ and live over the Bryant-Salamon manifold $\mathbb{R}^4 \times S^3$. Much like the G_2-structure itself, this family of instantons is $\mathrm{SU}(2)^3$-invariant. In [71] the moduli space of $\mathrm{SU}(2)^3$-invariant solutions was studied and we begin by summarising this part of their work (the reader should note that their work also considers invariant instantons for the asymptotically locally conical G_2-structure on $\mathbb{R}^4 \times S^3$).

The bundle P that we work with is topologically the trivial $\mathrm{SU}(2)$-bundle over $\mathbb{R}^4 \times S^3$. Away from the singular orbit P is given the homogeneous structure $P|_{(\mathbb{R}^4 \times S^3)\setminus S^3} = \pi^* Q$ where Q is the homogeneous bundle $Q = \mathrm{SU}(2)^3 \times_{(\Delta \mathrm{SU}(2), \mathrm{id})} \mathrm{SU}(2)$. We delay for the moment explaining the extension of the group action over the singular orbit. Note topologically both P and Q are trivial $\mathrm{SU}(2)$-bundles.

We first consider the ODEs that arise as the $\mathrm{SU}(2)^3$-invariant G_2-instanton equations for the Bryant-Salamon G_2-structure in this setting. After presenting this result we shall consider the boundary conditions required to extend solutions over the zero section.

Proposition 6.3.1 ([71, Proposition 5]). *Let A be an $\mathrm{SU}(2)^3$-invariant G_2-instanton on $\mathbb{R}^+ \times \mathrm{SU}(2)^3/\mathrm{SU}(2)$, with gauge group $\mathrm{SU}(2)$. Up to an invariant gauge transformation we have*

$$A = xX\left(\sum_{i=1}^{3} I_i \otimes \eta_i^+\right) + yY\left(\sum_{i=1}^{3} I_i \otimes \eta_i^+\right)$$

where $x, y\colon \mathbb{R}^+ \to \mathbb{R}$ satisfy

$$\begin{aligned} \dot{x} &= \tfrac{\dot{X}}{X}x + y^2 - x^2 &&= \tfrac{1}{2X}\left(1 - \tfrac{X^2}{Y^2}\right)x + y^2 - x^2 \\ \dot{y} &= \tfrac{2\dot{X}-3}{X}y + 2xy &&= -\tfrac{1}{X}\left(2 + \tfrac{X^2}{Y^2}\right)y + 2xy. \end{aligned}$$

To determine initial conditions for the above ODEs one first must extend the group action over the singular orbit $S^3 = \mathrm{SU}(2)^3/\mathrm{SU}(2)$. Extensions are determined by isomorphism classes of homogeneous $\mathrm{SU}(2)$-bundles over $\mathrm{SU}(2)^2/\mathrm{SU}(2)$ which are in turn determined by isotropy homomorphisms $\lambda\colon \mathrm{SU}(2) \to \mathrm{SU}(2)$. There are two possibilities, λ is either the trivial homomorphism, which we denote 1, or the identity id. We denote by P_λ the corresponding bundle over $\mathbb{R}^4 \times S^3$ whose extension is determined by λ.

Remark 6.3.2. *A word of caution is advised here. The homomorphism denotes the choice of extension of the group action over the singular orbit, it is not the isotropy homomorphism of the homogeneous bundle at infinity.*

In this section we consider the bundle P_1, where the extension of the group action over the singular orbit is the trivial one. The case of P_{id} will be considered in 6.4.

Lemma 6.3.3 ([71, Lemma 4]). *The connection A in Proposition 6.3.1 extends smoothly over the singular orbit S^3 if $x(t)$ is odd, $y(t)$ is even, and their Taylor expansions around $t = 0$ are*

- $x(t) = x_1 t + x_3 t^3 + \cdots$
- $y(t) = y_2 t^2 + y_4 t^4 + \cdots$.

The solution to these equations was first found by Clarke in [20]. The theorem we state is from [71], as it gives not just the existence of a solution, but also uniqueness in this $\mathrm{SU}(2)^3$-invariant setting.

Theorem 6.3.4 ([71, Theorem 4]). *Let A be an $\mathrm{SU}(2)^3$-invariant G_2-instanton with gauge group $\mathrm{SU}(2)$ on the Bryant-Salamon $\mathbb{R}^4 \times S^3$, extending smoothly over the singular orbit P_1. Then A is one Clarke's examples and in the notation of Proposition 6.3.1 there is an $x_0 \in \mathbb{R}$ such that*

$$x(t) = \frac{2x_0 X(t)}{1 + x_0(Y^2(t) - \frac{1}{3})} \qquad \text{and} \qquad y(t) = 0.$$

The resulting connection, which we denote by A_{x_0}, is globally defined if and only if $x_0 \geq 0$. Furthermore A_0 is the trivial flat connection.

These instantons are, in the notation of Definition 4.1.3, AC with asymptotic connection A_{can} and with fastest rate of convergence $\mu_0 = -2$. This is stated in [71, Proposition 7]. The parameter x_0 describes the concentration of the instanton around the associative S^3.

6.4 The Limiting Connection of Lotay-Oliveira

Consider now the bundle P_{id} which extends the action over the singular orbit via the identity homomorphism $\mathrm{id}\colon \mathrm{SU}(2) \to \mathrm{SU}(2)$. The next theorem gives a local existence theorem for G_2-instantons defined near the singular orbit.

Theorem 6.4.1 ([71, Proposition 6])**.** *Let S^3 be the singular orbit in the Bryant-Salamon manifold $\mathbb{R}^4 \times S^3$. There is a one-parameter family of $\mathrm{SU}(2)^3$-invariant G_2-instantons, defined in a neighbourhood os S^3 and smoothly extending over S^3 on P_{id}. The instantons are parameterised by $y_0 \in \mathbb{R}$ and, using the notation of Proposition 6.3.1, satisfy*

$$x(t) = \frac{2}{t} + \frac{y_0^2 - 1}{4} t + O(t^3), \qquad y(t) = y_0 + \frac{y_0}{2}\left(\frac{y_0^2}{2} - 3\right) t^2 + O(t^4).$$

Lotay-Oliveria note that, except for the case $y_0 = 0$, numerics suggest that the connections do not extend to define G_2-instantons with decaying curvature. The special case of $y_0 = 0$ however does extend and we state this as a proposition.

Proposition 6.4.2 ([71, Theorem 5])**.** *The G_2-instanton A_{lim} arising from the case $y_0 = 0$ has*

$$x(t) = \frac{X(t)}{\frac{1}{2}(Y^2(t) - \frac{1}{3})} \qquad \text{and} \qquad y(t) = 0$$

and defines a G_2-instanton on the Bryant-Salamon G_2-manifold $\mathbb{R}^4 \times S^3$.

One can check that A_{lim} is, in the notation of Definition 4.1.3, AC with asymptotic connection A_{can} and fastest rate of converge $\mu_0 = -3$. Note that this connection converges faster than Clarke's instantons.

This instanton allows one to compactify the invariant moduli space. Recall Clarke's instantons are parameterised by $x_0 \geq 0$, so to define a compactification one needs to glue in a connection at $x_0 = \infty$. This can be understood more precisely by studying the bubbling behaviour of Clarke's instantons. This is the content of the next theorem, to state this we first need to define suitable rescalings: For $p \in S^3$ and $\delta > 0$ define a map s_δ^p from the unit ball $B_1 \subset \mathbb{R}^4$ by

$$s_\delta^p \colon B_1 \to B_\delta \times \{p\} \subset \mathbb{R}^4 \times S^3, \qquad x \mapsto (\delta x, p).$$

Theorem 6.4.3 ([71, Theorem 6])**.** *Let $\{A_{x_i}\}$ be a sequence of Clarke's G_2-instantons with $x_i \to \infty$.*

a *Given any* $\lambda > 0$*, there is a sequence of positive real number* $\delta_i = \delta(x_i, \lambda) \to 0$ *as* $x_i \to \infty$ *such that: for all* $p \in S^3$, $(s^p_{\delta_i})^* A_{x_i}$ *converges uniformly with all derivatives to the standard ASD instanton* A_{ASD} *on* $B_1 \subset \mathbb{R}^4$.

b *The connection* A_{x_i} *converge uniformly with all derivatives to* A_{lim} *given in Proposition 6.4.2 on every compact subset of* $(\mathbb{R}^4 \setminus \{0\}) \times S^3$ *as* $x_i \to \infty$.

In other words, Clarke's instantons converge outside of the associative S^3 to A_{lim} and they "bubble off" an ASD instanton on the normal bundle to this associative. The fact that A_{lim} actually extends over the associative can be interpreted as a removable singularity phenomenon. Moreover, as this bubbling happens curvature concentrates on the associative S^3 and Lotay-Oliveira prove a conservation of energy statement. More precisely [71, Corollary 2] if f is any compactly supported function then, in the sense of currents, we have

$$\lim_{x_i \to \infty} \int_{\mathbb{R}^4 \times S^3} f(|F_{A^{x_1}}|^2 - |F_{A_{\mathrm{lim}}}|^2)\, \mathrm{dVol}_g = 8\pi^2 \int_{\{0\} \times S^3} f\, \mathrm{dVol}_g|_{\{0\} \times S^3}.$$

Remark 6.4.4. *The standard ASD instanton* A_{ASD} *has Yang-Mills energy* $8\pi^2$.

6.5 Calculation of the Virtual Dimension

The goal of this section is to calculate the virtual dimension of the moduli space of instantons asymptotic to A_{can}. Since Clarke's instantons form a 1-parameter family we expect to see this deformation parameter in our calculation. This parameter should appear at rate -2, in other words the kernel of the Dirac operator at infinity should be non-trivial. Note that A_{x_0} and A_{lim} live on (topologically) the same bundle and both converge to the canonical connection on Q but (in the terminology of (4.1)) have fastest rates of convergence -2 and -3 respectively. To determine the virtual dimension of the moduli space $\mathcal{M}(A_{\mathrm{can}}, \mu)$ for $\mu \in (-3, 0)$ we shall determine the eigenvalues of $D^0_{A_{\mathrm{can}}}$ in the interval $(-1, 2)$. Since the eigenvalues of this operator are symmetric about 0, it suffices to determine the eigenvalues in the interval $[0, 2)$.

An asymptotic framing for the bundle P_1 is provided by the homogeneous bundle $Q = \mathrm{SU}(2)^3 \times_{(\Delta\mathrm{SU}(2),\mathrm{id})} \mathrm{SU}(2)$. Since away from the associative S^3, we have $P_1 = \pi^* Q$ where $\pi \colon C(S^3 \times S^3) \to S^3 \times S^3$ is the projection map. For any $x_0 > 0$ the connection A_{x_0} is, in the sense of Definition 4.1.3, asymptotically conical with asymptotic limit $A_\infty = A_{\mathrm{can}}$ and with fastest rate of convergence -2. Thus to calculate the virtual dimension of the moduli space we must calculate the eigenvalues of $D^0_{A_{\mathrm{can}}}$ which lie in the interval $[0, 2)$.

Remark 6.5.1. *The canonical connection on the bundle* Q *is shown in [18, Theorem 3] to be a rigid nearly Kähler instanton.*

We will calculate the relevant eigenvalues using the representation theoretic methods developed in Chapter 5. The bundle $\mathrm{Ad}Q$ is associated to the canonical bundle via the adjoint representation of SU(2) on its Lie algebra. As a complexified representation the vector space is $(\mathfrak{su}(2))_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$. Let us denote by W_i the unique irreducible representation of SU(2) with dimension $(i+1)$. Irreducible representations of $\mathrm{SU}(2)^3$ are then given by $V_{(i,j,k)} := W_i \otimes W_j \otimes W_k$. As a representation of SU(2) we have $\mathfrak{m}^*_{\mathbb{C}} = W_2 \oplus W_2$ so the complexified spinor space is $S = W_0 \oplus W_2 \oplus W_2 \oplus W_0$ and an application of the Clebsch-Gordan rule shows that the twisted spinor space is

$$S \otimes \mathfrak{su}(2)_{\mathbb{C}} = 2W_4 \oplus 4W_2 \oplus 2W_0. \tag{6.16}$$

The Frobenius reciprocity theorem says that the space of sections splits as

$$\begin{aligned} L^2(\$_{\mathbb{C}}(S^3 \times S^3) \otimes \mathrm{Ad}Q) &\cong L^2(\mathrm{SU}(2)^3, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{SU}(2)} \\ &\cong \bigoplus_{\gamma \in \widehat{\mathrm{SU}(2)^3}} \mathrm{Hom}(V_\gamma, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{SU}(2)} \otimes V_\gamma \end{aligned} \tag{6.17}$$

where $\widehat{\mathrm{SU}(2)^3}$ denotes the set of isomorphism classes of irreducible representations of $\mathrm{SU}(2)^3$ and V_γ is a class representative for γ. The action of the Casimir operators on irreducible representations is given by

$$\begin{aligned} \rho_{V_{(i,j,k)}}(\mathrm{Cas}_{\mathfrak{su}(2)^3}) &= c^{\mathfrak{su}(2)^3}_{(i,j,k)}\mathrm{Id} \\ \rho_{W_i}(\mathrm{Cas}_{\mathfrak{su}(2)}) &= c^{\mathfrak{su}(2)}_i \mathrm{Id} \end{aligned}$$

where these eigenvalues of the Casimir operators are with respect to the nearly Kähler metric. The eigenvalues are calculated in [18] to be

$$\begin{aligned} c^{\mathfrak{su}(2)^3}_{(i,j,k)} &= -\frac{3}{2}\left(i(i+2) + j(j+2) + k(k+2)\right) \\ c^{\mathfrak{su}(2)}_i &= -\frac{1}{2} i(i+2). \end{aligned}$$

Since the adjoint representation of SU(2) is the 3-dimensional irreducible representation we see from Lemma 5.3.1 the relevant Lichnerowicz formula for $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2$ is

$$(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2 = -\rho_L(\mathrm{Cas}_{\mathfrak{su}(2)^3}) + \rho_{W_2}(\mathrm{Cas}_{\mathfrak{su}(2)}) + 4 = -\rho_L(\mathrm{Cas}_{\mathfrak{su}(2)^3}). \tag{6.18}$$

Under the splitting (6.17) we have endomorphisms $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ of $\mathrm{Hom}(V_\gamma, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{SU}(2)}$ such that

$$(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma \otimes \mathrm{Id} = (D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2|_{\mathrm{Hom}(V_\gamma, S\otimes\mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{SU}(2)} \otimes V_\gamma}. \tag{6.19}$$

This operator takes the form

$$(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma = -\rho_{V^*_\gamma}(\mathrm{Cas}_{\mathfrak{su}(2)^3}) \tag{6.20}$$

and furthermore $\mathrm{Spec}(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2 = \bigcup_{\gamma \in \widehat{\mathrm{SU}(2)^3}} \mathrm{Spec}(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$. This tells us precisely the eigenvalues of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2$:

Proposition 6.5.2. *The spectrum of* $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2$ *is*

$$\mathrm{Spec}(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2 = \left\{ -c^{\mathfrak{su}(2)^3}_{(i,j,k)} \,;\, \dim \mathrm{Hom}(V_{(i,j,k)}, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{SU}(2)} \neq 0 \right\}.$$

We can use this to rule out representations that do not lead to eigenvalues of the twisted Levi-Civita Dirac operator in the interval $[0,2)$. First let us recall the relation between the various operators that we need:

The operator $D^1_{A_{\mathrm{can}}}$ is constructed from the canonical connection acting on both the spinor space and the adjoint bundle. We can define endomorphisms $(D^1_{A_{\mathrm{can}}})_\gamma$ of $\mathrm{Hom}(V_\gamma, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{SU}(2)}$ such that

$$(D^1_{A_{\mathrm{can}}})_\gamma \otimes \mathrm{Id} = (D^1_{A_{\mathrm{can}}})|_{\mathrm{Hom}(V_\gamma, S\otimes\mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{SU}(2)} \otimes V_\gamma}$$

and thus a family of endomorphisms $(D^t_{A_{\mathrm{can}}})_\gamma$ of $\mathrm{Hom}(V_\gamma, S \otimes \mathfrak{su}(2)_{\mathbb{C}})$ via the formula

$$(D^t_{A_{\mathrm{can}}})_\gamma = (D^1_{A_{\mathrm{can}}})_\gamma + \frac{3(t-1)}{4}\mathrm{Re}\Omega.$$

Then one has

$$\mathrm{Spec}(D^t_{A_{\mathrm{can}}}) = \bigcup_\gamma \mathrm{Spec}(D^t_{A_{\mathrm{can}}})_\gamma$$

and we use this to obtain eigenvalue estimates for the Levi-Civita Dirac operators $(D^0_{A_{\mathrm{can}}})_\gamma$.

Lemma 6.5.3. *If* $V_\gamma \neq \mathbb{C}$ *then* $\mathrm{Spec}(D^0_{A_{\mathrm{can}}})_\gamma \cap [0,2) = \emptyset$.

Proof. Let V_γ be an irreducible representation of $\mathrm{SU}(2)^3$ and suppose that $\dim \mathrm{Hom}(V_\gamma, S \otimes \mathfrak{sl}(2,\mathbb{C}))_{\mathrm{SU}(2)} \neq 0$. Application of Theorem 5.3.5 shows that the smallest non-negative eigenvalue of $(D^0_{A_{\mathrm{can}}})_\gamma$ is bounded below by

$$L_\gamma = \sqrt{-c^{\mathfrak{su}(2)^3}_{(i,j,k)}} - 1$$

where $V_\gamma = V_{(i,j,k)}$. The quantity L_γ does not yield a lower bound when $V_\gamma = \mathbb{C}$ and since $\dim \mathrm{Hom}(V_{(1,0,0)}, S \otimes \mathfrak{sl}(2,\mathbb{C}))_{\mathrm{SU}(2)} = 0$ the representation $V_{(1,0,0)}$ need not be considered. The next representation to consider is $V_{(1,1,0)}$, this yields the bound $L_\gamma = \sqrt{9} - 1 = 2$, so this bound is sufficient for the statement of the theorem. Any other irreducible representation V_γ leads to a large lower bound, which completes the proof. □

To calculate the relevant eigenvalues we therefore only need consider those coming from the trivial representation. Let $V_\gamma = \mathbb{C}$ and let us write $S = W_0^{(1)} \oplus W_2^{(1)} \oplus W_2^{(2)} \oplus W_0^{(2)}$ where $W_i^{(a)}$ are distinct copies of the irreducible representation W_i. A basis of $\mathrm{Hom}(\mathbb{C}, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{SU}(2)} = \mathrm{Hom}(\mathbb{C}, S \otimes W_2)_{\mathrm{SU}(2)}$ is given by the SU(2)-equivariant maps $q_0^{2,2}$ and $\tilde{q}_0^{2,2}$ which factor as maps

$$q_0^{2,2} \colon \mathbb{C} \to W_2^{(1)} \otimes W_2 \to S \otimes W_2$$
$$\tilde{q}_0^{2,2} \colon \mathbb{C} \to W_2^{(2)} \otimes W_2 \to S \otimes W_2.$$

Notice that all maps in this homomorphism space factor through $\Lambda^1 \subset S$. On this space (6.20) says that $(D^{\frac{1}{3}}_{A_{\text{can}}})^2_\gamma \equiv 0$. Furthermore $\text{Ker}(D^{\frac{1}{3}}_{A_{\text{can}}})^2_\gamma = \text{Ker}(D^{\frac{1}{3}}_{A_{\text{can}}})_\gamma$ so we see that $(D^{\frac{1}{3}}_{A_{\text{can}}})_\gamma$ also vanishes on this space. Finally, since all basis vectors factor through $\Lambda^1 \subset S$ Lemma 2.3.3 ensures that $\text{Re}\Omega$ acts trivially on this space. Therefore $(D^t_{A_{\text{can}}})_\gamma = 0$ for all t, when V_γ is the trivial representation. We have proved:

Proposition 6.5.4. *The only eigenvalue of $D^0_{A_{can}}$ in the interval $[0, 2)$ is 0 and has multiplicity 2.*

This result tells, by the observation in Remark 4.4.6 that $\text{ind}_{-2+\epsilon} D_A = \frac{1}{2}\text{dim Ker} D^0_{A_{\text{can}}}$ for ϵ sufficiently small, the virtual dimension of the moduli space:

Theorem 6.5.5. *Let P be the trivial $\mathrm{SU}(2)$-bundle over $\mathbb{R}^4 \times S^3$, framed at infinity by the homogeneous $\mathrm{SU}(2)$-bundle over the nearly Kähler $S^3 \times S^3$ whose isotropy homomorphism is the identity. Let A_{can} be the canonical connection on Q, then the virtual dimension of the moduli space of G_2-instantons asymptotic to A_{can} with rate μ is*

$$_{\text{virt}}\text{dim}\mathcal{M}(A_{\text{can}}, \mu) = \begin{cases} 1 & \text{if } \mu \in (-2, 0) \\ -1 & \text{if } \mu \in (-3, -2). \end{cases}$$

Thus the virtual dimension of our moduli space coincides with that dimension of the $SU(2)^3$-invariant moduli space when $\mu \in (-2, 0)$. Observe that the virtual dimension is negative for the weights $\mu \in (-3, -2)$ and thus the deformation theory is obstructed for these weights.

Chapter 7

Deformations of Oliveira's Instantons

In this chapter we study moduli spaces of known instantons on the Bryant-Salamon manifolds $\Lambda^2_-(\mathbb{CP}^2)$ and $\Lambda^2_-(S^4)$. Firstly we consider Oliveira's G_2-instanton [77] on $\Lambda^2_-(\mathbb{CP}^2)$ with structure group SO(3). We use the methods developed in the previous two chapters to determine the virtual dimension of the moduli space in question. Next we consider Oliveira's G_2-instanton with structure group SU(2) over $\Lambda^2_-(S^4)$. Here we show that the Lichnerowicz formula and eigenvalue bounds are insufficient for determining the virtual dimension and to compensate for this we develop a method for calculating the matrix of a twisted Dirac operator on a finite dimensional subspace of the space of sections. As a consequence we explicitly determine some of the eigenvalues of the operator and hence the virtual dimension. Finally we consider deformations of the spin connection on S^4, which was observed by Oliveira to pull back to a G_2-instanton over $\Lambda^2_-(S^4)$.

7.1 The Bryant-Salamon $\Lambda^2_-(\mathbb{CP}^2)$

We begin this chapter with a brief overview of the Bryant-Salamon manifolds, based on the material covered in [70] and [77]. Let (N, g_N) be an even dimensional Riemannian manifold. The twistor fibration $\pi\colon \mathcal{T}(N, g_N) \to N$ is the fibre bundle whose fibre at a point is the space of orthogonal almost complex structures of T_pN. In other words

$$\mathcal{T}_pN = \{J \in \mathrm{SO}(T_pN)\,;\, J^2 = -\mathrm{Id}_{T_pN}\}.$$

When N has dimension 4 there is an identification of the twistor space with the unit sphere bundle in $\Lambda^2_-(N)$, so the twistor fibration is an S^2-bundle over N. The case of interest to this thesis is when N^4 is a self-dual quaternion Kähler manifold with positive Ricci curvature. In this case it turns out [30] that the total space of the twistor fibration carries a canonical nearly Kähler structure whose metric is

$$g_6 = \frac{1}{2} g_{S^2} + \pi^* g_N$$

where g_{S^2} is the round metric on the fibres of $\mathcal{T}$. There are exactly two possibilities for such a manifold, either $N = S^4$ with the round metric or $N = \mathbb{CP}^2$ with the Fubini-Study metric.

We consider first the case when $N = \mathbb{CP}^2$, then the twistor space is the space of flags in $\mathbb{C}^3$ i.e the space of complex lines contained in planes in $\mathbb{C}^3$ and will be denoted $\mathbb{F}_{1,2,3}$. The standard action of SU(3) on $\mathbb{C}^3$ gives rise to a transitive action of SU(3) on the space of flags with isotropy subgroup T^2. At a suitable point of $\mathbb{F}_{1,2,3}$ the T^2 isotropy subgroup of SU(3) that one obtains is the standard one [77]

$$T^2 = \left\{ \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{-i(\theta+\phi)} & 0 \\ 0 & 0 & e^{i\phi} \end{pmatrix} \right\}$$

and we will work with this fixed subgroup throughout this section.

The Bryant-Salamon metric on the total space of $\Lambda^2_-(\mathbb{CP}^2)$ takes the form

$$g = f^2(s) g_{\mathbb{R}^3} + f^{-2}(s)\pi^* g_{\mathbb{CP}^2}$$

where s is the Euclidean distance to the zero section given by the fiber metric and

$$f(s) = (1+s^2)^{-\frac{1}{4}}. \tag{7.1}$$

The geodesic distance to the zero section then takes the form $t(s) = \int_0^s f(u)\,\mathrm{d}u$ and this allows us to rewrite the metric as

$$g = \mathrm{d}t^2 + s^2(t) f^2(s(t)) g_{S^2} + f^{-2}(s(t))\pi^* g_{\mathbb{CP}^2}$$

where g_{S^2} is the round metric on the unit sphere of a fibre in $\Lambda^2_-(\mathbb{CP}^2)$. One can also describe the G_2-structure explicitly. To gain a suitable local expression we first study homogeneous structure of $\mathrm{SU}(3)/T^2$.

Under the adjoint action the real representation $\mathfrak{m}$ splits into three irreducible subspaces $\mathfrak{m} = \mathfrak{m}_1 \oplus \mathfrak{m}_2 \oplus \mathfrak{m}_3$ and we choose orthonormal bases $\{I_1, I_2\}, \{I_3, I_4\}$ and $\{I_5, I_6\}$ respectively (if the reader is interested this decomposition is covered more explicitly in [77]). As explained in 2.3 this induces a local frame for $T^*\mathbb{F}_{1,2,3}$ which we denote $e^1, \dots e^6$. Away from the zero section of $\Lambda^2_-(\mathbb{CP}^2)$ we can think of the geodesic distance t as a coordinate. Let us define 2-forms

$$\begin{array}{lll} \Omega_1 = e^{12} - e^{34}, & \Omega_2 = e^{13} - e^{42}, & \Omega_3 = e^{14} - e^{23} \\ \overline{\Omega}_1 = e^{12} + e^{34}, & \overline{\Omega}_2 = e^{13} + e^{42}, & \overline{\Omega}_3 = e^{14} + e^{23} \end{array} \tag{7.2}$$

then the 3-form φ of the Bryant-Salamon G_2-structure takes the form

$$\varphi = \mathrm{d}t \wedge (a^2(t)e^{12} - a^2(t)e^{34} + c^2(t)e^{56}) + a^2(t)c(t)(\Omega_2 \wedge e^6 - \Omega_3 \wedge e^5) \tag{7.3}$$

where $a(t) = 2f^{-1}(t)$ and $c(t) = 2s(t)f(t)$. From this viewpoint one sees that the G_2-structure is AC with $|g - g_C|_g = O(t^{-4})$.

7.2 Instantons on $\Lambda^2_-(\mathbb{CP}^2)$

Let us fix the standard basis I_i of $\mathfrak{so}(3)$ with $[I_i, I_j] = 2\epsilon_{ijk}I_k$. We choose the maximal torus in SO(3) to be the one generated by $\frac{1}{2}I_1$. Recall that SU(3)-homogeneous SO(3)-bundles over $\mathbb{F}_{1,2,3}$ are determined by isotropy homomorphisms from T^2 to SO(3). It is shown in [77] that there is a unique isotropy homomorphism λ which yields a bundle $Q = \mathrm{SU}(3) \times_\lambda \mathrm{SO}(3)$ with non-trivial invariant connections and for which the pullback π^*Q extends over the zero section $\mathbb{CP}^2$ of $\Lambda^2_-(\mathbb{CP}^2)$.

To describe this bundle we first note that the singular orbit of $\Lambda^2_-(\mathbb{CP}^2)$ is $\mathrm{SU}(3)/\mathrm{U}(2) = \mathbb{CP}^2$ and so SU(3)-homogeneous SO(3)-bundles over the singular orbit are determined by isotropy homomorphisms $\tilde{\lambda}\colon \mathrm{U}(2) \to \mathrm{SO}(3)$. The adjoint action of U(2) on $\mathfrak{su}(2)$ defines a homomorphism

$$\begin{aligned} \tilde{\lambda}\colon \mathrm{U}(2) &\to \mathrm{SO}(3) \\ g &\mapsto \mathrm{Ad}_g \end{aligned} \tag{7.4}$$

where we view SO(3) as a subgroup of $\mathrm{GL}(\mathfrak{su}(2))$. By viewing T^2 as the subgroup of U(2)

$$T^2 = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{i\phi} \end{pmatrix} \subset \mathrm{U}(2)$$

we obtain a homomorphism $T^2 \to \mathrm{SO}(3)$ by restriction of (7.4), this is

$$\begin{aligned} \lambda\colon T^2 &\to \mathrm{SO}(3) \\ g &\mapsto \mathrm{Ad}_g. \end{aligned} \tag{7.5}$$

From now on λ denotes the homomorphism (7.5) and Q denotes the fixed homogeneous bundle

$$Q = \mathrm{SU}(3) \times_\lambda \mathrm{SO}(3) \tag{7.6}$$

Let us denote by P the extension of π^*Q to $\Lambda^2_-(\mathbb{CP}^2)$, here the group action over the singular orbit is determined by the isotropy homomorphism (7.4). We denote by A_{can} the pullback of the canonical connection on Q, this lives on $P|_{\Lambda^2_-(\mathbb{CP}^2)\setminus\mathbb{CP}^2}$. An invariant connection on P then takes the form

$$A = A_{\mathrm{can}} + h(t)(e^5 \otimes I_2 + e^6 \otimes I_3)$$

and one finds [77, Section 4.3.2] that the G_2-instanton equation for such a connection becomes

$$f^{-4}h^2 = 1, \qquad f^{-4}\frac{\mathrm{d}h}{\mathrm{d}s} + sh = 0 \tag{7.7}$$

with boundary data $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} f^{-2}h = 0, \lim_{s\to\infty} h = 0$. The first equation here, which is algebraic, implies $h = \pm f^2$ and one can check that the second equation is then automatically satisfied. Note that the paired equations have a 0-dimensional space of solutions, whilst the linearisation of the system has a 1-dimensional space of solutions. We summarise this result in the following theorem.

Theorem 7.2.1 ([77] Theorem 8). *The connection A given by*

$$A = A_{\text{can}} \pm f^2(s)(e^5 \otimes I_2 + e^6 \otimes I_3)$$

defines a G_2-instanton on P. Moreover A is AC with limiting connection the canonical connection living on the bundle Q.

The conenction A satisfies $|A - A_{\text{can}}|_g = O(t^{-3})$, so in the notation of (4.1) defines an AC G_2-instanton with fastest rate of convergence -2.

7.3 Eigenvalues of the Twisted Dirac Operator on $\mathbb{F}_{1,2,3}$

Again we denote by Q the bundle defined in (7.6). The basis of $\mathfrak{su}(3)_{\mathbb{C}} = \mathfrak{sl}(3,\mathbb{C})$ that we choose is $\{E_{12}, E_{13}, E_{23}, E_{21}, E_{31}, E_{32}, H_{12}, H_{23}\}$, where E_{ij} is the elementary matrix with a 1 in the $(i,j)^{\text{th}}$ entry and zeros elsewhere, and where

$$H_{12} = \text{diag}(1,-1,0), \qquad H_{23} = \text{diag}(0,1,-1).$$

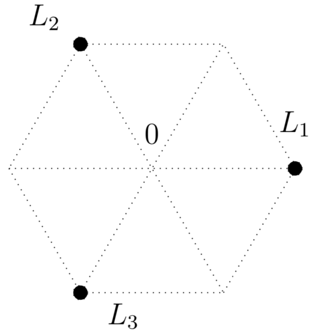
The matrices H_{12} and H_{23} form a basis of a Cartan subalgebra given by the Lie algebra of the group of diagonal matrices in SU(3), we denote this $\mathfrak{t}^2 = \langle H_{12}, H_{23}\rangle$. Irreducible representation are then determined by elements of the lattice

$$(\mathfrak{t}^2)^*_{\mathbb{Z}} = \{\lambda \in (\mathfrak{t}^2)^* \,;\, \lambda(X) \in 2\pi i\mathbb{Z} \text{ for all } X \in \text{Ker}\exp\}.$$

Let us define linear functionals on $\mathfrak{t}$ by

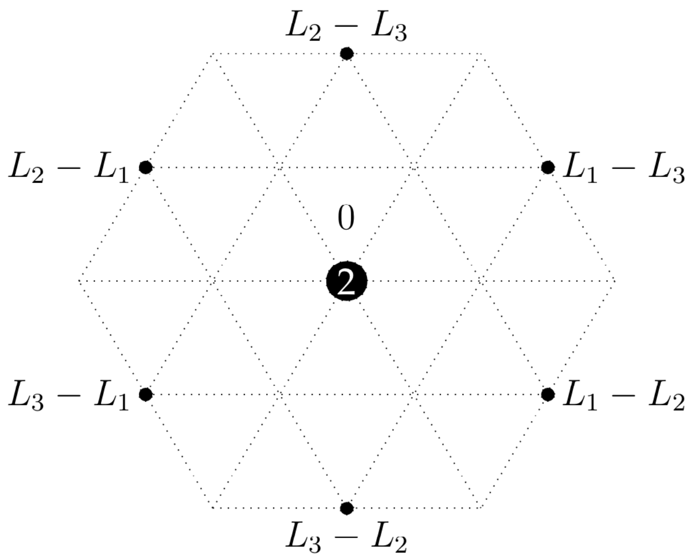
$$L_i \begin{pmatrix} a_1 & 0 & 0 \\ 0 & a_2 & 0 \\ 0 & 0 & a_3 \end{pmatrix} = a_i$$

for $i = 1,2,3$. Then $L_i \in (\mathfrak{t}^2)^*_{\mathbb{Z}}$ and these functionals form a spanning set for $(\mathfrak{t}^2)^*$. Note they are subject to the relation $L_1 + L_2 + L_3 = 0$. We choose to represent the lattice and the functionals L_i in the plane as [1]



[1] All images in this chapter are credited to Jonny Evans https://github.com/jde27/lie-grp/blob/master/notizen.pdf

The fundamental Weyl chamber is the set $\{\alpha L_1 - \beta L_3\,;\, \alpha, \beta \in \mathbb{R}^+\}$ and so irreducible representations of SU(3), which are determined by their highest weight, are in bijection with $(\mathbb{N} \cup \{0\})^2$. This is because a highest weight lies in the intersection of the Weyl chamber with the lattice, and such a weight is precisely $aL_1 - bL_3$ for $(a, b) \in (\mathbb{N} \cup \{0\})^2$. For a pair of natural numbers (a, b) let us denote by $V_{(a,b)}$ the unique complex irreducible representation of SU(3) with highest weight $aL_1 - bL_3$. The branching rule for the T^2-subgroup is easily determined by the weight diagram of given representation. Let us consider as an example the adjoint representation $(\mathfrak{su}(3)_{\mathbb{C}}, \rho_{\mathrm{Ad}})$. Let $H = \mathrm{diag}(a_1, a_2, a_3)$, then one can calculate that $\rho_{\mathrm{Ad}}(H)E_{ij} = (a_i - a_j)E_{ij}$ and it follows that the weight diagram for the adjoint representation is



where 0 has multiplicity 2 (since SU(3) is rank 2) and all other weights have multiplicity one. Let $(W_{(a,b)}, \rho_{(a,b)})$ be the irreducible complex representation of $\mathfrak{t}$ with $\rho_{(a,b)}(H_{12}) = a$ and $\rho_{(a,b)}(H_{23}) = b$, then one uses the relation $L_1 + L_2 + L_3 = 0$ to calculate that

$$\mathfrak{sl}(3, \mathbb{C}) = 2W_{(0,0)} \oplus W_{(2,-1)} \oplus W_{(-1,2)} \oplus W_{(-1,-1)} \oplus W_{(-2,1)} \oplus W_{(1,-2)} \oplus W_{(1,1)}.$$

The representation $\mathfrak{m}_{\mathbb{C}}$ carries the adjoint action of T^2 and, as in [18], one can calculate the weight space decomposition to be

$$\mathfrak{m}_{\mathbb{C}} = W_{(2,-1)} \oplus W_{(-1,2)} \oplus W_{(-1,-1)} \oplus W_{(-2,1)} \oplus W_{(1,-2)} \oplus W_{(1,1)}.$$

Notice that as a representation of T^2 the spinor space is $S = \langle \mathbf{1} \rangle_{\mathbb{C}} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \langle \mathrm{Vol} \rangle_{\mathbb{C}}$, with T^2 acting trivially on the complement of $\mathfrak{m}_{\mathbb{C}}$, and that this decomposes identically to $\mathfrak{sl}(3, \mathbb{C})$.

The eigenvalues of the Casimir operator one obtains from the nearly Kähler metric are calculated in [18]. We may write

$$\rho_{V_{(a,b)}}(\mathrm{Cas}_{\mathfrak{su}(3)}) = c^{\mathfrak{su}(3)}_{(a,b)}\mathrm{Id}$$
$$\rho_{W_{(c,d)}}(\mathrm{Cas}_{\mathfrak{t}^2}) = c^{\mathfrak{t}^2}_{(c,d)}\mathrm{Id}$$

and the eigenvalues are

$$c^{\mathfrak{su}(3)}_{(a,b)} = -\frac{4}{3}\left(a^2+b^2+ab+3a+3b\right) \tag{7.8}$$
$$c^{\mathfrak{t}^2}_{(c,d)} = -\frac{4}{3}\left(c^2+cd+d^2\right). \tag{7.9}$$

Let us now examine the relevant Lichnerowicz formula. In the case at hand this takes the form

$$(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2 = -\rho_L(\mathrm{Cas}_{\mathfrak{su}(3)}) + \rho_{\mathrm{Ad}\circ\lambda}(\mathrm{Cas}_{\mathfrak{t}^2}) + 4$$

where $\rho_{\mathrm{Ad}\circ\lambda}$ acts on $\mathfrak{so}(3)_{\mathbb{C}}$. We split the space of sections using the Frobenius reciprocity theorem

$$L^2(\$_{\mathbb{C}}(\mathbb{F}_{1,2,3})\otimes \mathrm{Ad}Q) \cong \bigoplus_{\gamma\in\widehat{\mathrm{SU}(3)}} \mathrm{Hom}(V_\gamma, S\otimes\mathfrak{so}(3)_{\mathbb{C}})_{T^2}\otimes V_\gamma \tag{7.10}$$

and define an operator $(D^{\frac{1}{2}}_{A_{\mathrm{can}}})^2_\gamma$ by requiring that

$$(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma \otimes \mathrm{Id} = (D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2|_{\mathrm{Hom}(V_\gamma,S\otimes\mathfrak{so}(3))_{\mathbb{C}}\otimes V_\gamma}. \tag{7.11}$$

To calculate the eigenvalues of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2$ we must first determine the decomposition of $(\mathfrak{so}(3)_{\mathbb{C}}, \mathrm{Ad}\circ\lambda)$. One finds that, in a suitable basis of $\mathfrak{so}(3)_{\mathbb{C}}$, we have

$$\mathrm{Ad}\circ\lambda\begin{pmatrix} e^{i\theta} & 0 & 0\\ 0 & e^{-i(\theta+\phi)} & 0\\ 0 & 0 & e^{i\phi}\end{pmatrix} = \begin{pmatrix} e^{i(\phi-\theta)} & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & e^{i(\theta-\phi)}\end{pmatrix}$$

and it follows that

$$(\mathfrak{so}(3)_{\mathbb{C}}, \mathrm{Ad}\circ\lambda) = W_{(-1,-1)}\oplus W_{(0,0)}\oplus W_{(1,1)}.$$

Remark 7.3.1. *The reader is warned that we have chosen different generators of the maximal torus of* SU(3) *to those in [77], so our labelling conventions are not the same.*

With this in hand we can use Corollary 5.3.2 and (7.9) to state the eigenvalues of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2$:

Lemma 7.3.2. *Let $V_\gamma = V_{(a,b)}$ be an irreducible representation of* SU(3) *such that* $\dim \mathrm{Hom}(V_\gamma, S\otimes\mathfrak{so}(3)_{\mathbb{C}}) \neq 0$ *and let $(D^{\frac{1}{3}}_{A_{can}})^2_\gamma$ be the operator defined by (7.11). The eigenvalues and multiplicities of this operator are*

Eigenvalue	*Multiplicity*
$-c^{\mathfrak{su}(3)}_{(a,b)} + 4$	$\dim \operatorname{Hom}(V_\gamma, S)_{T^2}$
$-c^{\mathfrak{su}(3)}_{(a,b)}$	$\dim \operatorname{Hom}(V_\gamma, S \otimes (W_{(-1,-1)} \oplus W_{(1,1)}))_{T^2}$.

Again this allows us to bound the eigenvalues of $(D^0_{A_{\text{can}}})_\gamma$:

Corollary 7.3.3. *Let V_γ be an irreducible representation of* SU(3)*. If V_γ is not one of the representations listed below then* $\operatorname{Spec}(D^0_{A_{can}})_\gamma \cap [0,2] = \emptyset$*:*

- *The trivial representation* $V_{(0,0)} = \mathbb{C}$
- *The standard representation* $V_{(1,0)} = \mathbb{C}^3$
- *The dual of the standard representation* $V_{(0,1)} = (\mathbb{C}^3)^*$

Proof. Let V_γ be an irreducible representation of SU(3), then the lower bound on smallest positive eigenvalue of $(D^0_{A_{\text{can}}})_\gamma$ that is given by Theorem 5.3.5 is

$$L_\gamma = \sqrt{-c^{\mathfrak{su}(3)}_\gamma} - 1.$$

Using (7.8) we see that when $V_\gamma = V_{(1,1)}$ the lower bound that one obtains is $L_\gamma = \sqrt{12} - 1 > 2$. For any other irreducible representation (excluding those stated above) the bound is strictly larger and this completes the proof. □

Let us therefore begin to examine the representations left on our list. The first case to consider is when $V_\gamma = V_{(0,0)}$ is the trivial representation of SU(3). We should expect to see a 2-dimensional kernel for $(D^0_{A_{\text{can}}})_\gamma$ since invariant solutions appear from the trivial representation and we know that the linearisation of the invariant G_2-instanton equations (7.7) has a 1-dimensional space of solutions.

To begin with, note that an application of Schur's lemma shows $\operatorname{Hom}(\mathbb{C}, S \otimes \mathfrak{so}(3)_{\mathbb{C}})_{T^2}$ is four dimensional. Let $m^{(i,j)}$ be a fixed weight vector for the weight $aL_1 - bL_3$ of the representation $\mathfrak{m}_{\mathbb{C}} \subset S$ and let $w^{(i,j)}$ be a fixed weight vector for the weight $aL_1 - bL_3$ of the representation $\mathfrak{so}(3)_{\mathbb{C}}$. We set

$$\begin{aligned} q^{(0,0)(0,0)} &= \mathbf{1} \otimes w^{(0,0)} \\ \tilde{q}^{(0,0)(0,0)} &= \text{Vol} \otimes w^{(0,0)} \\ q^{(1,1)(-1,-1)} &= m^{(1,1)} \otimes w^{(-1,-1)} \\ q^{(-1,-1)(1,1)} &= m^{(-1,-1)} \otimes w^{(1,1)} \end{aligned}$$

then $\{q^{(0,0)(0,0)}, \tilde{q}^{(0,0)(0,0)}, q^{(1,1)(-1,-1)} q^{(-1,-1)(1,1)}\}$ forms a basis of

$$(S \otimes \mathfrak{so}(3)_{\mathbb{C}})_{T^2} \cong \operatorname{Hom}(\mathbb{C}, S \otimes \mathfrak{so}(3)_{\mathbb{C}})_{T^2}$$

where $(S \otimes \mathfrak{so}(3)_{\mathbb{C}})_{T^2}$ is the space of T^2-invariant vectors in $S \otimes \mathfrak{so}(3)_{\mathbb{C}}$. Let $\{I_a\}$ be an orthonormal basis for $\mathfrak{m}$ and recall from (5.9) that

$$(D^0_{A_{\mathrm{can}}})_\gamma = \mathrm{cl}(I_a)\rho_{\mathbb{C}}(I_a) - \frac{3}{4}\mathrm{Re}\Omega$$

on this space. Since the action involved here is the trivial one we see that $(D^0_{A_{\mathrm{can}}})_\gamma = -\frac{3}{4}\mathrm{Re}\Omega$ and the vectors $q^{(0,0)}$ and $q^{(3,3)}$ are, by Lemma 2.3.3, thus eigenvectors with eigenvalues -3 and 3 respectively. In contrast the vectors $q^{(1,1)(-1,-1)}, q^{(-1,-1)(1,1)}$ define equivariant maps that factor through $\Lambda^1 \otimes \mathfrak{so}(3)_{\mathbb{C}} \subset S \otimes \mathfrak{so}(3)_{\mathbb{C}}$ and it follows from Lemma 2.3.3 that $\mathrm{Re}\Omega$ acts trivially on these vectors. These observation yield the next proposition:

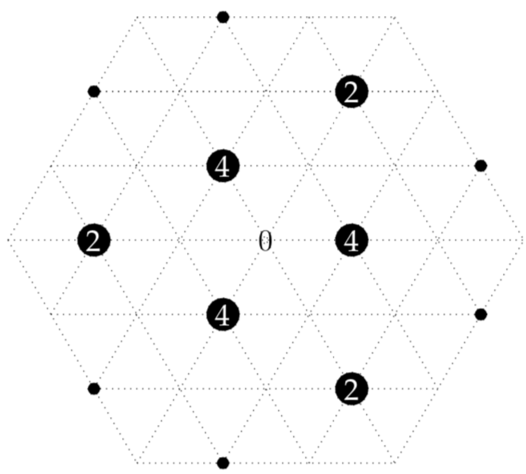
Proposition 7.3.4. *Let $V_\gamma = \mathbb{C}$ and let $(D^0_{A_{can}})_\gamma$ be the operator on* $\mathrm{Hom}(V_\gamma, S \otimes \mathfrak{so}(3)_{\mathbb{C}})_{T^2}$ *given in* (5.7)*. The eigenvalues of this operator are*

Eigenvalue λ	*Multiplicity*
0	2
3	1
-3	1

According to Corollary 7.3.3 the next case to consider is when the representation in question is $V_{(1,0)} = \mathbb{C}^3$, the standard representation of $\mathrm{SU}(3)$. Since the spinor space S is the same as $V_{(1,1)} = \mathfrak{sl}(3,\mathbb{C})$ as a representation of T^2, we find

$$\mathrm{Hom}(V_{(1,0)}, S \otimes \mathfrak{so}(3)_{\mathbb{C}})_{T^2} \cong \mathrm{Hom}(V_{(1,1)} \otimes V_{(1,0)}, W_{(-1,-1)} \oplus W_{(0,0)} \oplus W_{(1,1)})_{T^2}.$$

The weight diagram of $V_{(1,1)} \otimes V_{(1,0)}$ is



and we observe that this does not contain any of the weights of $(\mathfrak{so}(3)_{\mathbb{C}}, \mathrm{Ad} \circ \lambda)$ which are $\{-L_1 + L_3, 0, L_1 - L_3\}$. The only T^2-equivariant map from $V_{(1,0)}$ to $S \otimes \mathfrak{so}(3)_{\mathbb{C}}$ is thus the zero map and we can therefore omit the standard representation from our calculation. An almost identical argument shows that $V_{(0,1)} = (\mathbb{C}^3)^*$ does not contribute any eigenvalues to the spectrum of the twisted Dirac operator.

Observe we have shown 2 is not an eigenvalue of $(D^0_{A_{\mathrm{can}}})_\gamma$ when V_γ is either the trivial representation, the standard representation $\mathbb{C}^3$ or the dual of the standard representation $(\mathbb{C}^3)^*$. Furthermore 2 cannot occur as an eigenvalue coming from any other representation by Corollary 7.3.3. According to [18] perturbations of a nearly Kähler instanton A_∞ on a 6-manifold are given by 1-forms $a \in \Omega^1(\Sigma, \mathrm{Ad}Q)$ such that $D_{A_\infty}(a \cdot s_6) = 2a \cdot s_6$, so this observation yields the following corollary:

Corollary 7.3.5. *The canonical connection living on the homogeneous bundle Q defined in (7.6) is a rigid nearly Kähler instanton.*

We conclude this section by giving the virtual dimension of the moduli space $\mathcal{M}(A_{\mathrm{can}}, \mu)$ of which Oliveira's instanton is an element. This follows from Remark 4.4.6 since 0 is the only eigenvalue of $D^0_{A_{\mathrm{can}}}$ in $[0, 2)$ and has multiplicity 2.

Theorem 7.3.6. *Let A be Oliveria's G_2-instanton with gauge group $\mathrm{SO}(3)$ on $\Lambda^2_-(\mathbb{CP}^2)$ given in Theorem 7.2.1. The virtual dimension of the moduli space is*

$$\mathrm{virtdim}\mathcal{M}(A_{\mathrm{can}}, \mu) = 1$$

for all $\mu \in (-2, 0)$.

Remark 7.3.7. *Observe that inspection of the invariant G_2-instanton equations revealed that the linearised equation has a one-dimensional space of solutions whilst space of solutions to the non-linear equation is zero-dimensional. It follows that the invariant moduli space is obstructed. On the other hand if the AC moduli space is unobstructed then it must be that connections in the moduli space (excluding Oliveira's examples) are not invariant under the given group action. This is because if we assume unobstructedness then the moduli space is a smooth one-dimensional manifold.*

Remark 7.3.8. *Oliveira [77] also constructs a 2-parameter family of G_2-instantons with structure group $SU(3)$ over $\Lambda^2_-(\mathbb{CP}^2)$. These connections also decay to the canonical connection (although this is the canonical connection for a bundle with a different action to that on the bundle over the end of $\Lambda^2_-(\mathbb{CP}^2)$).The calculations involved are likely to be lengthy and these examples are therefore not considered in this thesis. The methods used above are however still applicable to this example. To the author's knowledge, these are the only known non-abelian examples that we do not consider.*

7.4 The Bryant-Salamon $\Lambda^2_-(S^4)$.

Under the isomorphism $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$ we see that $\mathrm{Sp}(2)$ acts transitively on S^4 with isotropy $\mathrm{Spin}(4) \cong \mathrm{Sp}(1)\times\mathrm{Sp}(1)$. We can lift this to an isometric action on the total space of $\Lambda^2_-(S^4)$ by asking that $\mathrm{Sp}(2)$ acts on an ASD 2-form via pull back. One can understand the action of the isotropy group $\mathrm{Sp}(1)\times\mathrm{Sp}(1)$ on a fiber of $\Lambda^2_-(S^4)$ by modelling a fibre as the imaginary quaternions [77], the action is then $(p,q)\cdot x = qx\overline{q}$ for $(p,q) \in \mathrm{Sp}(1)\times\mathrm{Sp}(1)$ and $x \in \mathrm{Im}\mathbb{H}$. The stabiliser of a non-zero imaginary quaternion under this action is $\mathrm{Sp}(1) \times \mathrm{U}(1)$, so away form the zero section the principal orbits are

$$\mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1) \cong \mathbb{CP}^3.$$

Moreover since the action is isometric the principal orbits are diffeomorphic to level sets of the norm function $s = |\cdot|^2$ on the fibers. The metric $|\cdot|$ on the fibers is Euclidean and will be denoted $g_{\mathbb{R}^3}$. The unit sphere bundle of $\Lambda^2_-(S^4)$ is the twistor fibration $\pi\colon \mathbb{CP}^3 \to S^4$ and carries a nearly Kähler structure.

As was the case for $\Lambda^2_-(\mathbb{CP}^2)$ the Bryant-Salamon metric takes the form

$$g = f^2(s)g_{\mathbb{R}^3} + f^{-2}(s)\pi^* g_{S^4}$$

where $f(s) = (1+s^2)^{-\frac{1}{4}}$. The geodesic distance to the zero section then takes the form

$$t(s) = \int_0^s f(u)\,\mathrm{d}u$$

and this allows us to rewrite the metric as

$$g = \mathrm{d}t^2 + s^2(t)f^2(s(t))g_{S^2} + f^{-2}(s(t))\pi^* g_{S^4}$$

where g_{S^2} is the round metric on the unit sphere of a fibre in $\Lambda^2_-(S^4)$. Again we can also describe the G_2-structure explicitly in local coordinates. To gain a suitable local expression we first study the homogeneous structure of $\mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)$.

Recall the Lie algebra $\mathfrak{sp}(2)$ is defined as

$$\mathfrak{sp}(2) = \{X \in \mathrm{Mat}_2(\mathbb{H})\,;\, X + X^\dagger = 0\}$$

where $X^\dagger$ is the quaternionic conjugate transpose of X. Let $\mathfrak{t} = \mathrm{Lie}(\mathrm{U}(1))$ and let $\boldsymbol{i}, \boldsymbol{j}$ and $\boldsymbol{k}$ denote the unit imaginary quaternions. We embed the subalgebra $\mathfrak{sp}(1) \oplus \mathfrak{t}$ as

$$\begin{pmatrix} q & 0 \\ 0 & a\boldsymbol{i} \end{pmatrix}$$

for $a \in \mathbb{R}$. Under the reductive decomposition $\mathfrak{sp}(2) = (\mathfrak{sp}(1) \oplus \mathfrak{t}) \oplus \mathfrak{m}$ a suitable model for $\mathfrak{m}$ is

$$\mathfrak{m} = \begin{pmatrix} 0 & h \\ -h^\dagger & b\boldsymbol{j} + c\boldsymbol{k} \end{pmatrix}$$

where h is a quaternion, $b, c \in \mathbb{R}$ and $h^\dagger$ is the quaternionic conjugate of h. Thus we have a splitting of vector spaces $\mathfrak{m} = \mathbb{H} \oplus \langle \boldsymbol{j}, \boldsymbol{k} \rangle_{\mathbb{R}}$. As described in Section 2.3 a choice of orthonormal basis for $\mathfrak{m}$ determines a local frame for $T^*\mathbb{C}P^3$, we label the 1-forms arising from the standard basis of $\mathbb{H}$ as $e^1, \dots e^4$ and the one-forms corresponding to $\boldsymbol{j}$ and $\boldsymbol{k}$ respectively are labelled e^5 and e^6. Let us define local Sp(2)-invariant 2-forms Ω_i identically to (7.2), then the 3-form φ of the Bryant-Salamon structure takes the form

$$\varphi = \mathrm{d}t \wedge (a^2(t)e^{56} + b^2(t)\Omega_1) + a(t)b^2(t)(e^6 \wedge \Omega_2 - e^5 \wedge \Omega_3) \tag{7.12}$$

where $a(s) = 2sf(s^2)$ and $b(s) = \sqrt{2}f^{-1}(s^2)$. This viewpoint allows us to see that the metric and indeed the G_2-structure are again AC with $|g - g_C|_g = O(t^{-4})$.

7.5 Instantons on $\Lambda^2_-(S^4)$

In [77] examples of Sp(2)-invariant G_2-instantons with gauge group SU(2) were constructed. Here we give an overview of this construction. Again the framework for this is provided by the framework of homogeneous bundles and Wang's theorem.

Homogeneous SU(2)-bundles over $\mathbb{CP}^3 = \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)$ are determined by isotropy homomorphisms $\lambda\colon \mathrm{Sp}(1) \times \mathrm{U}(1) \to \mathrm{SU}(2)$. By [77, Proposition 5] such a homomorphism is either trivial, $\lambda(g, e^{i\theta}) = \mathrm{diag}(e^{il\theta}, e^{-il\theta})$ for $l \in \mathbb{Z}$ or $\lambda(g, e^{i\theta}) = g$ where we use the standard isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$. We consider first the homomorphisms $\lambda_l(g, e^{i\theta}) = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$ and delay consideration of the homomorphism $\lambda(g, e^{i\theta}) = g$ until Section 7.8 By [77, Lemma 1] the only one of the bundles

$$Q_l \coloneqq \mathrm{Sp}(2) \times_{\lambda_l} \mathrm{SU}(2)$$

to admit a non-trivial family of invariant connections is Q_1. Let us therefore fix $Q = Q_1$. In the standard basis I_1, I_2, I_3 of $\mathfrak{su}(2)$ an invariant connection on Q is of the form

$$A = A_{\mathrm{can}} + c(e^5 \otimes I_1 + e^6 \otimes I_2)$$

for $c \in \mathbb{R}$. Let $p\colon (\Lambda^2_-(S^4) \setminus S^4) \to \mathbb{CP}^3$ denote the projection to the unit sphere bundle of $\Lambda^2_-(S^4)$, then the bundle p^*Q admits an extension to the total space of $\Lambda^2_-(S^4)$ which we denote P. Thus an invariant connection on P is of the form

$$A = A_{\mathrm{can}} + h(e^5 \otimes I_2 + e^6 \otimes I_3)$$

with h a function of the geodesic distance coordinate t (or alternatively the Euclidean distance s). The G_2-instanton equation for A then becomes the system [77, Proposition 6]

$$f^{-4}h^2 = 1, \qquad f^{-4}\frac{\mathrm{d}h}{\mathrm{d}s} + sh = 0$$

together with boundary data $\frac{\mathrm{d}}{\mathrm{d}s}\big|_{s=0} f^{-2}h = 0, \lim_{s\to\infty} h = 0$. Note these equations are identical to (7.7)– for more on the origins of this duality see [77, Remark 13]. Again the first equation here, which is algebraic, implies the differential equation is automatically satisfied. Thus:

Theorem 7.5.1 ([77, Theorem 5])**.** *The connection*

$$A = A_{\mathrm{can}} \pm f^2(s)(e^5 \otimes I_2 + e^6 \otimes I_3)$$

is a G_2-instanton on the principal SU(2)*-bundle $P \to \Lambda^2_-(S^4)$. Moreover A is AC with limiting connection the canonical connection living on the bundle $Q \to \mathbb{CP}^3$.*

This connection satisfies $|A - A_{\mathrm{can}}|_g = O(t^{-3})$ so in the notation of (4.1) defines an AC G_2-instanton with fastest rate of convergence -2.

7.6 Eigenvalues of the Twisted Dirac Operator on $\mathbb{CP}^3$

Let us again denote by $Q \to \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1)$ the homogeneous SU(2)-bundle associated via the homomorphism $\lambda_1(g, e^{i\theta}) = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$. Using the framework developed in Chapter 5 we aim here to determine which representations of Sp(2) could lead to eigenvalues of $D^0_{A_{\mathrm{can}}}$ in the interval $[0, 2)$. We look for eigenvalues in this range since the connection A from Theorem 7.5.1 has fastest rate of convergence -2.

To achieve this we must first review the representation theory of the groups Sp(2) and $\mathrm{Sp}(1) \times \mathrm{U}(1)$. Following [18] we choose the Cartan subalgebra of $\mathfrak{sp}(2)_{\mathbb{C}}$ to be the space of 2×2 quaternionic matrices of the form $\mathrm{diag}(z\mathbf{i}, w\mathbf{i})$ with $z, w \in \mathbb{C}$. Let us call a weight positive if it evaluates to a positive real number on the matrix $i\mathrm{diag}(2\mathbf{i}, \mathbf{i})$. The matrices $H_1 = i\mathrm{diag}(0, \mathbf{i})$ and $H_2 = i\mathrm{diag}(\mathbf{i}, -\mathbf{i})$ are dual to the fundamental weights λ_1 and λ_2. The irreducible complex representations of Sp(2) are thus determined by their highest weight $a\lambda_1 + b\lambda_2$ for $a, b \in \mathbb{N}$ and we write $(V_{(a,b)}, \rho_{V_{(a,b)}})$ for such a representation. The first few representations are as follows

- $V_{(0,0)} = \mathbb{C}$ the trivial representation
- $V_{(0,1)} = \mathbb{H}^2 = \mathbb{C}^4$ the standard representation. The group Sp(2) acts on $\mathbb{H}^2$ by matrix multiplication and hence on $\mathbb{C}^4$ via the isomorphism $\mathbb{H}^2 \cong \mathbb{C}^4$ of complex vector spaces.
- $V_{(1,0)}$ is a 5 dimensional representation which under the isomorphism $\mathrm{Sp}(2) \cong \mathrm{Spin}(5)$ corresponds to the vector representation of Spin(5). This means that Spin(5) acts adjointly on $\mathbb{R}^5 \otimes \mathbb{C} \subset \mathrm{Cl}(\mathbb{R}^5) \otimes \mathbb{C}$. As noted in [18] the real 5 dimensional representation
$$V^{\mathbb{R}}_{(1,0)} = \left\{ \begin{pmatrix} x & h \\ h^\dagger & -x \end{pmatrix} ; x \in \mathbb{R} \text{ and } h \in \mathbb{H} \right\}$$
with the action being matrix commutation, satisfies $V^{\mathbb{R}}_{(1,0)} \otimes_{\mathbb{R}} \mathbb{C} = V_{(1,0)}$.

It is worth noting that each irreducible representation of Sp(2) is isomorphic to its dual representation. The final fact we need is that the Lie algebra $\mathfrak{sp}(2)$ is 3-symmetric since the map

$$S\colon \mathfrak{sp}(2) \to \mathfrak{sp}(2) \tag{7.13}$$

$$S(X) = \begin{pmatrix} 1 & 0 \\ 0 & e^{-\frac{2\pi}{3}\mathbf{i}} \end{pmatrix} X \begin{pmatrix} 1 & 0 \\ 0 & e^{\frac{2\pi}{3}\mathbf{i}} \end{pmatrix} \tag{7.14}$$

satisfies $S^3 = \mathrm{Id}$ (here we view an element X of $\mathfrak{sp}(2)$ as a 2×2 skew quaterionic hermitian matrix). This provides a reductive decomposition the Lie algebra $\mathfrak{sp}(2) = (\mathfrak{sp}(1) \oplus \mathfrak{t}) \oplus \mathfrak{m}$.

The representation theory of $\mathrm{Sp}(1)\times \mathrm{U}(1)$ is straightforward since irreducible representations are precisely those of the form $W \otimes W'$ where W is an irreducible representation of Sp(1) and W' is an irreducible representation of U(1). For $a \in \mathbb{N}$ and $b \in \mathbb{Z}$ let us therefore denote by $(W_{(a,b)}, \rho_{W_{(a,b)}})$ the unique $(a+1)$-dimensional representation of $\mathrm{Sp}(1) \times \mathrm{U}(1)$ on which U(1) acts with weight b. We realise the Lie algebra as

$$\mathfrak{sp}(1) \oplus \mathfrak{t} = \begin{pmatrix} p & 0 \\ 0 & x\mathbf{i} \end{pmatrix} \tag{7.15}$$

for $p \in \mathfrak{sp}(1) = \mathrm{Im}\mathbb{H}$ and $x \in \mathbb{R}$ so that $\mathfrak{sp}(1) \oplus \mathfrak{t}$ naturally forms a subalgebra of $\mathfrak{sp}(2)$. The spinor space $S = \mathbb{C} \oplus \mathfrak{m}_{\mathbb{C}} \oplus \mathbb{C}$ is the following representation of $\mathrm{Sp}(1) \times \mathrm{U}(1)$ [18]

$$S = \langle \mathbf{1} \rangle_{\mathbb{C}} \oplus W_{(1,-1)} \oplus W_{(1,1)} \oplus W_{(0,-2)} \oplus W_{(0,2)} \oplus \langle \mathrm{Vol} \rangle_{\mathbb{C}}.$$

Next we consider the Casimir operators on the irreducible representations $V_{(a,b)}$ and $W_{(c,d)}$ of Sp(2) and $\mathrm{Sp}(1) \times \mathrm{U}(1)$ respectively. We know that the Casimir operators act as multiples on the identity on these representations so

$$\rho_{V_{(a,b)}}(\mathrm{Cas}_{\mathfrak{sp}(2)}) = c^{\mathfrak{sp}(2)}_{(a,b)}\mathrm{Id}$$
$$\rho_{W_{(c,d)}}(\mathrm{Cas}_{\mathfrak{sp}(1)\oplus\mathfrak{t}}) = c^{\mathfrak{sp}(1)\oplus\mathfrak{t}}_{(c,d)}\mathrm{Id}$$

and these constants are found in [18] to be

$$c^{\mathfrak{sp}(2)}_{(a,b)} = -\left(2a^2 + 2ab + b^2 + 6a + 4b\right) \tag{7.16}$$

$$c^{\mathfrak{sp}(1)\oplus\mathfrak{t}}_{(c,d)} = -\left(c(c+2) + d^2\right). \tag{7.17}$$

Let us begin to apply these facts to study the spectrum of $D^{\frac{1}{3}}_{A_{\mathrm{can}}}$. By Corollary 5.3.2 we need to decompose $(\mathfrak{su}(2)_{\mathbb{C}}, \mathrm{Ad}\circ\lambda_1)$ as a representation of $\mathrm{Sp}(1) \times \mathrm{U}(1)$. The action is simply the adjoint action of the maximal torus of SU(2) on $\mathfrak{su}(2)_{\mathbb{C}} = \mathfrak{sl}(2,\mathbb{C})$ so the decomposition is

$$(\mathfrak{su}(2)_{\mathbb{C}}, \mathrm{Ad}\circ\lambda_1) = W_{(0,-2)} \oplus W_{(0,0)} \oplus W_{(0,2)}$$

and $\rho_{\mathrm{Ad}\circ\lambda_1}(\mathrm{Cas}_{\mathfrak{sp}(1)\oplus\mathfrak{t}})$ acts as -4 on the $W_{(-2,0)} \oplus W_{(0,2)}$ subspace and acts trivially on $W_{(0,0)}$. As in (5.11) we split the operator $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2$ into endomorphisms $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ of $\mathrm{Hom}(V_\gamma, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ for each irreducible representation V_γ of Sp(2). Let $V_\gamma = V_{(a,b)}$ then using Corollary 5.3.2 we see that the eigenvalues of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ are

Eigenvalue	Multiplicity
$-c^{\mathfrak{sp}(2)}_{(a,b)} + 4$	$\dim\mathrm{Hom}(V_{(a,b)}, S)_{\mathrm{Sp}(1)\times \mathrm{U}(1)}$
$-c^{\mathfrak{sp}(2)}_{(a,b)}$	$\dim\mathrm{Hom}(V_{(a,b)}, S \otimes (W_{(0,-2)} \oplus W_{(0,2)}))_{\mathrm{Sp}(1)\times \mathrm{U}(1)}$

Again this will prove a useful consistency check when we come to calculate the matrices of the twisted Dirac operators $(D^t_{A_{\mathrm{can}}})_\gamma$.

Recall that we are looking for eigenvalues of $D^0_{A_{\mathrm{can}}}$ in the interval $[0, 2)$. We can use Theorem 5.3.5 to eliminate most of the representation $V_{(a,b)}$ under consideration.

Lemma 7.6.1. *Let V_γ be an irreducible representation of* Sp(2)*. If V_γ is not one of the following representations then $(D^0_{A_{\mathrm{can}}})_\gamma$ has no eigenvalues in the interval $[0, 2]$:*

- $V_{(0,0)}$ *the trivial representation*
- $V_{(0,1)}$ *the standard representation*
- $V_{(1,0)}$ *the vector representation of* Spin(5).

Proof. This is a simple application of Theorem 5.3.5. When $V_\gamma = V_{(0,2)}$ we find that $L_\gamma = \sqrt{12} - 1 > 2$ whilst other irreducible representations which are not on the above list yield a greater bound. □

Thus we need only consider the 3 representations stated in Lemma 7.6.1. For notational convenience let us set $[[W_{(a,b)}]] := W_{(a,b)} \oplus W_{(a,-b)}$ and recall the twisted spinor space splits

$$S \otimes \mathfrak{su}(2)_{\mathbb{C}} = S \oplus \left(S \otimes [[W_{(0,2)}]]\right).$$

As representations of $\mathrm{Sp}(1) \times \mathrm{U}(1)$ these spaces are

$$S = 2W_{(0,0)} \oplus W_{(1,1)} \oplus W_{(1,-1)} \oplus W_{(0,2)} \oplus W_{(0,-2)} \tag{7.18}$$

$$S \otimes [[W_{(0,2)}]] = [[W_{(1,3)}]] \oplus [[W_{(1,1)}]] \oplus [[W_{(0,4)}]] \oplus 2\left([[W_{(0,2)}]] \oplus W_{(0,0)}\right). \tag{7.19}$$

The space of sections admits a splitting

$$L^2(S \otimes \mathrm{Ad}Q) \cong \left(\bigoplus_{\gamma \in \widehat{\mathrm{Sp}(2)}} \mathrm{Hom}(V_\gamma, S)_{\mathrm{Sp}(1)\times\mathrm{U}(1)} \otimes V_\gamma \right) \oplus \left(\bigoplus_{\gamma \in \widehat{\mathrm{Sp}(2)}} \mathrm{Hom}(V_\gamma, S \otimes [[W_{(0,2)}]])_{\mathrm{Sp}(1)\times\mathrm{U}(1)} \otimes V_\gamma \right)$$

and this is preserved by the operators $(D^t_{A_{\mathrm{can}}})_\gamma$. For each irreducible representation V_γ of Sp(2) we will therefore consider the operators defined by the restriction of $(D^t_{A_{\mathrm{can}}})_\gamma$ to the spaces $\mathrm{Hom}(V_\gamma, S)_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ and $\mathrm{Hom}(V_\gamma, S \otimes [[W_{(0,2)}]])_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$.

The first case to consider is when $V_\gamma = V_{(0,0)} = \mathbb{C}$ is the trivial representation. The space $\mathrm{Hom}(\mathbb{C}, S)_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ has dimension 2 and a basis is given by the maps $q^{(0,0)}\colon \mathbb{C} \to \Lambda^{(0,0)} \subset S$ and $q^{(3,3)}\colon \mathbb{C} \to \Lambda^{(3,3)} \subset S$ defined in the obvious way. On this space $(D^1_{A_{\mathrm{can}}})_\gamma = \mathrm{cl}(I_a)\rho_{V^*_\gamma}(I_a) \equiv 0$ since the action is trivial. Now

$$(D^0_{A_{\mathrm{can}}})_\gamma = (D^1_{A_{\mathrm{can}}})_\gamma - \frac{3}{4}\mathrm{Re}\Omega$$

and so Lemma 2.3.3 tells us that $q^{(0,0)}$ and $q^{(3,3)}$ are eigenvectors of $(D^0_{A_{\mathrm{can}}})_\gamma$ with eigenvalues -3 and 3 respectively. Note that the sections these maps define correspond to the Killing spinor s_6 and $\mathrm{Vol}\cdot s_6$.

Consider now the space $\mathrm{Hom}(V_\gamma, S \otimes [[W_{(0,2)}]])_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ when V_γ is the trivial representation so that once again $(D^1_{A_{\mathrm{can}}})_\gamma$ acts trivially. This space has dimension 2 since $S\otimes[[W_{(0,2)}]]$ contains two trivial components coming from the subspaces $W_{(0,2)} \otimes W_{(0,-2)}$ and $W_{(0,-2)}\otimes W_{(0,2)}$. Since the spaces in question are actually subspaces of $\mathfrak{m}^*_{\mathbb{C}}\otimes[[W_{(0,2)}]] \subset S\otimes[[W_{(0,2)}]]$ we find from Lemma 2.3.3 that $\mathrm{Re}\Omega$ acts trivially on $\mathrm{Hom}(\mathbb{C}, S\otimes[[W_{(0,2)}]])_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$. Therefore $(D^t_{A_{\mathrm{can}}})_\gamma = 0$ on this space for all t. We conclude that

Proposition 7.6.2. *Let $V_\gamma = \mathbb{C}$ be the trivial representation of* Sp(2)*. The eigenvalues and multiplicities of $(D^0_{A_{can}})_\gamma$ are*

Eigenvalue	*Multiplicity*
0	2
−3	1
3	1

The next case to be considered is when $V_\gamma = V_{(0,1)}$ is the standard representation of Sp(2). Restricting to the action of the subgroup $\mathrm{Sp}(1)\times\mathrm{U}(1)$ one finds that $V_{(0,1)} = W_{(1,0)} \oplus [[W_{(0,1)}]]$ but inspection of (7.18) and (7.19) then reveals that there are no non-trivial $\mathrm{Sp}(1)\times\mathrm{U}(1)$ equivariant maps from $V_{(0,1)} \to S\otimes\mathfrak{su}(2)_{\mathbb{C}}$, so this case need not be considered.

The last case to consider is when the representation in question is $V_\gamma = V_{(1,0)}$ the vector representation of Spin(5). We present this in the next section as the calculations involved here are more complicated.

7.7 Eigenvalues from the Vector Representation of Spin(5)

Throughout this section the representation of Sp(2) under consideration is $V_\gamma = V_{(1,0)}$ the vector representation of Spin(5). Consider first the restriction of $(D^t_{A_{\mathrm{can}}})_\gamma$ to the space $\mathrm{Hom}(V_{(1,0)}, S \otimes [[W_{(0,2)}]])_{\mathrm{Sp}(1)\times \mathrm{U}(1)}$. The branching of $V_{(1,0)}$ as a representation of $\mathrm{Sp}(1)\times\mathrm{U}(1)$ is [18] $V_{(1,0)} = [[W_{(1,1)}]] \oplus W_{(0,0)}$ and one therefore finds that $\mathrm{Hom}(V_{(1,0)}, S \otimes [[W_{(0,2)}]])_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ has dimension four and a basis is given by the $\mathrm{Sp}(1)\times\mathrm{U}(1)$ equivariant maps which factor

$$q^{(i,j)}_{(k,l)(m,n)} \colon V_{(1,0)} \to W_{(i,j)} \to W_{(k,l)} \otimes W_{(m,n)} \hookrightarrow S \otimes [[W_{(0,2)}]]$$

where the first map is a projection and the second map is an embedding of $W_{(i,j)}$ in $W_{(k,l)} \otimes W_{(m,n)} \subset S \otimes [[W_{(0,2)}]]$. The basis one obtains is

$$\left\{ q^{(1,1)}_{(1,-1)(0,2)}, q^{(1,-1)}_{(1,1)(0,-2)}, q^{(0,0)}_{(0,-2)(0,2)}, q^{(0,0)}_{(0,2)(0,-2)} \right\},$$

observe that all of these maps factor through $\Lambda^1 \subset S$ so that $\mathrm{Re}\Omega$ acts trivially on this space. Using Corollary 5.3.2 and (7.16) we find that $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ acts as multiplication by 8 on this space. The eigenvalues of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ are therefore $\pm\sqrt{8}$, each with multiplicity 2. Since $\mathrm{Re}\Omega$ acts trivially on this space the spectrum is identical for each $(D^{t}_{A_{\mathrm{can}}})_\gamma$ for each $t \in \mathbb{R}$.

Let us now consider the action of the operators $(D^{t}_{A_{\mathrm{can}}})_\gamma|_{\mathrm{Hom}(V_{(1,0)},S)_{\mathrm{Sp}(1)\times\mathrm{U}(1)}}$. The space $\mathrm{Hom}(V_{(1,0)}, S)_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ four dimensional. A basis is provided by linearly independent maps which factor

$$q^{(i,j)} \colon V_{(1,0)} \to W_{(i,j)} \hookrightarrow S$$

with the first map being projection and the second being inclusion. This yields the basis

$$\left\{ q^{(1,1)}, q^{(1,-1)} q^{(0,0)}, \tilde{q}^{(0,0)} \right\} \tag{7.20}$$

where $q^{(0,0)}$ maps into $\Lambda^{(0,0)} \subset S$ and $\tilde{q}^{(0,0)}$ maps into $\Lambda^{(3,3)} \subset S$. The situation here is more complicated than those previously considered since an $\mathrm{Sp}(1)\times\mathrm{U}(1)$-equivariant map takes values in the entire spinor space S, not just one of the subspaces $(\Lambda^{(0,0} \oplus \Lambda^{(3,3)})$ or $\Lambda^1_{\mathbb{C}}$. We know from Corollary 5.3.2 that $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma = 12$ on this space and the eigenvalues of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ are thus $\pm\sqrt{12}$. The complication arises because the 3-form $\mathrm{Re}\Omega$ acts non-trivially on this space. To calculate the eigenvalues of $(D^{0}_{A_{\mathrm{can}}})_\gamma$ on this space we work with formula (5.9). We will seek to calculate the action $\rho_{V_{(1,0)}}(I_a) q^{(i,j)}$ where I_a is an orthonormal basis of $\mathfrak{m}$ and $q^{(i,j)}$ is one of the basis vectors from (7.20), as well as understanding the action of Clifford multiplication and the almost complex structure in this basis.

The first job is to find an orthonormal basis with respect to the metric (2.15). We choose to view the Lie algebra $\mathfrak{sp}(2)$ as a subspace of $\mathrm{Mat}_4(\mathbb{C})$ by viewing the quaternionic algebra as an algebra of 2×2 complex matrices in the standard way. From this point of view the Killing form is [39]

$$\mathrm{Tr}(\mathrm{ad}_X \circ \mathrm{ad}_Y) = 6\mathrm{Tr}XY$$

for $X, Y \in \mathfrak{sp}(2) \subset \mathrm{Mat}_4(\mathbb{C})$.

One finds that

$$H_1 = \begin{pmatrix} \mathbf{i} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad H_2 = \begin{pmatrix} \mathbf{j} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$H_3 = \begin{pmatrix} \mathbf{k} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \qquad H_4 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{i} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \end{pmatrix}$$

$$M_1 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{j} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 \\ 0 & \mathbf{k} \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}$$

$$M_3 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \qquad M_4 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathbf{i} \\ \mathbf{i} & 0 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}$$

$$M_5 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathbf{j} \\ \mathbf{j} & 0 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \qquad M_6 = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & \mathbf{k} \\ \mathbf{k} & 0 \end{pmatrix} = \tfrac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \end{pmatrix}.$$

is an orthonormal set of generators with the matrices H_i generating the subalgebra $\mathfrak{sp}(1) \oplus \mathfrak{t}$ in (7.15). The complex structure in defined by restricting the map (7.13) to $\mathfrak{m} = \langle M_1, \ldots, M_6 \rangle_{\mathbb{R}}$. A simple calculation yields

$$\begin{array}{cc} J(M_1) = -M_2 & J(M_2) = M_1 \\ J(M_3) = M_4 & J(M_4) = -M_3 \\ J(M_5) = -M_6 & J(M_6) = M_5 \end{array}$$

and by calculating the structure constants of the algebra and appealing to the formula (2.21) one finds that

$$\mathrm{Re}\Omega = e^{135} + e^{146} + e^{236} + e^{254} \tag{7.21}$$

in the local frame $\{e^a\}$ of $T^*\mathbb{CP}^3$ determined by the basis $\{M_a\}$ of $\mathfrak{m} \cong \mathfrak{m}^*$. With almost complex structure and the 3-form $\mathrm{Re}\Omega$ determined we can determine Clifford multiplication. This follows from the fact that $(u \wedge v) \lrcorner \omega = u \lrcorner Jv$ for any pair of 1-forms u and v, since applying this to the formula for Clifford multiplication (2.32) yields an equivalent formula

$$\mathrm{cl}(u)(f + v + h\mathrm{Vol}) \cdot s_6 = (-u \lrcorner v + fu - hJu - (u \wedge v) \lrcorner \mathrm{Re}\Omega - (u \lrcorner Jv)\mathrm{Vol}) \cdot s_6. \tag{7.22}$$

To calculate the matrix of $(D^1_{A_{\mathrm{can}}})_\gamma$ using (5.9) we must understand the action of $\rho_{V^*_{(1,0)}}(M_a)$ on $\mathrm{Hom}(V_{(1,0)}, S)_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$. For this we note that

$$\mathrm{Hom}(V_{(1,0)}, S)_{\mathrm{Sp}(1)\times\mathrm{U}(1)} \cong \left(S \otimes V^*_{(1,0)}\right)_{\mathrm{Sp}(1)\times\mathrm{U}(1)} \cong \left(S \otimes V_{(1,0)}\right)_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$$

and this last space is the subspace of $S \otimes V_{(1,0)}$ consisting of vectors fixed by the action of the $\mathrm{Sp}(1) \times \mathrm{U}(1)$ subgroup of $\mathrm{Sp}(2)$. For the representation $V_{(1,0)}$ we choose basis vectors

$$I := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad X := \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad Y := \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & i & 0 \\ 0 & -i & 0 & 0 \\ -i & 0 & 0 & 0 \end{pmatrix}$$
$$Z := \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad W := \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & -i \\ -i & 0 & 0 & 0 \\ 0 & i & 0 & 0 \end{pmatrix}$$

and the action is matrix commutation.

To determine what the maps $q^{(i,j)}$ from (7.20) look like in the space $\big(V_{(1,0)} \otimes S\big)_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ we note first that $W_{(i,j)} \otimes W_{(k,l)}$ contains a copy of the trivial representation of $\mathrm{Sp}(1) \times \mathrm{U}(1)$ if and only if $i = k$ and $j = -l$. We will therefore determine explicit decompositions of the representation $V_{(1,0)}$ and the spinor space S into irreducible representations of $\mathrm{Sp}(1) \times \mathrm{U}(1)$.

By calculating the action $\rho_{V_{(1,0)}}(H_i)$ on the basis matrices of $V_{(1,0)}$ we can determine explicitly the decomposition $V_{(1,0)} = [[W_{(1,1)}]] \oplus W_{(0,0)}$. Let us set

$$\begin{aligned} \theta_1 &:= X + iY, & \theta_2 &:= W - iZ \\ \overline{\theta}_1 &:= X - iY, & \overline{\theta}_2 &:= W + iZ \end{aligned}$$

then one finds that $W_{(0,0)} = \langle I \rangle_{\mathbb{C}}$ whilst $W_{(1,1)} = \langle \theta_1, \theta_2 \rangle_{\mathbb{C}}$ and $W_{(1,-1)} = \langle \overline{\theta}_1, \overline{\theta}_2 \rangle_{\mathbb{C}}$. A similar analysis works for the representation $\mathfrak{m}$. If we define

$$\begin{aligned} \phi_1 &:= M_3 + iM_4, & \phi_2 &:= M_6 + iM_5 \\ \overline{\phi}_1 &:= M_3 - iM_4, & \overline{\phi}_2 &:= M_6 - iM_5 \end{aligned}$$

then one finds that $W_{(1,1)} = \langle \phi_1, \phi_2 \rangle_{\mathbb{C}}$ and $W_{(1,-1)} = \langle \overline{\phi}_1, \overline{\phi}_2 \rangle_{\mathbb{C}}$. With this in hand we can look for invariant vectors in $W_{(i,j)} \otimes W_{(i,-j)} \subset S \otimes V_{(1,0)}$. Abusing notation slightly let us use the same notation as in (7.20) after identifying $\mathrm{Hom}(V_{(1,0)}, S)_{\mathrm{Sp}(1)\times\mathrm{U}(1)} \cong (S \otimes V_{(1,0)})_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$, then the basis becomes

$$\begin{aligned} q^{(0,0)} &= \mathbf{1} \otimes I \\ \tilde{q}^{(0,0)} &= \mathrm{Vol} \otimes I \\ q^{(1,1)} &= \phi_1 \otimes \overline{\theta_1} + \phi_2 \otimes \overline{\theta}_2 \\ q^{(1,-1)} &= \overline{\phi}_1 \otimes \theta_1 + \overline{\phi}_2 \otimes \theta_2. \end{aligned}$$

We can now proceed to calculate the matrix of the Dirac operator in this basis using the formula

$$(D^1_{A_{\mathrm{can}}})_\gamma = \mathrm{cl}(M_a)\rho_{V_{(1,0)}}(M_a). \tag{7.23}$$

Inspection of (7.22) reveals that $\mathrm{cl}(M_a)(1 \otimes v) = M_a \otimes v$ for $v \in V_{(1,0)}$ and we find that $\rho_{V_{(1,0)}}(M_1)I = \rho_{V_{(1,0)}}(M_2)I = 0$ whilst

$$\rho_{V_{(1,0)}}(M_3)I = -\sqrt{2}X, \qquad \rho_{V_{(1,0)}}(M_4)I = -\sqrt{2}Y$$
$$\rho_{V_{(1,0)}}(M_5)I = \sqrt{2}Z, \qquad \rho_{V_{(1,0)}}(M_6)I = -\sqrt{2}W.$$

Applying these facts to (7.23) one finds

$$\begin{aligned}(D^1_{A_{\mathrm{can}}})_\gamma q^{(0,0)} &= \sqrt{2}\left(-M_3 \otimes X - M_4 \otimes Y + M_5 \otimes Z - M_6 \otimes W\right)\\ &= -\frac{1}{\sqrt{2}}\left(q^{(1,1)} + q^{(1,-1)}\right).\end{aligned}$$

Again by (7.22) we have that $\mathrm{cl}(M_a)(\mathrm{Vol} \otimes v) = -J(M_a) \otimes v$ for $v \in V_{(1,0)}$ and thus

$$\begin{aligned}(D^1_{A_{\mathrm{can}}})_\gamma \tilde{q}^{(0,0)} &= -\sqrt{2}\left(-M_4 \otimes X + M_3 \otimes Y - M_6 \otimes Z - M_5 \otimes W\right)\\ &= -\frac{i}{\sqrt{2}}\left(q^{(1,1)} - q^{(1,-1)}\right).\end{aligned}$$

To calculate the action of the operator on $q^{(1,1)}$ we note that

$$\rho_{V_{(1,0)}}(M_1)\overline{\theta}_1 = i\theta_2, \qquad \rho_{V_{(1,0)}}(M_2)\overline{\theta}_1 = i\theta_2, \qquad \rho_{V_{(1,0)}}(M_3)\overline{\theta}_1 = \sqrt{2}I$$
$$\rho_{V_{(1,0)}}(M_4)\overline{\theta}_1 = -i\sqrt{2}I, \qquad \rho_{V_{(1,0)}}(M_5)\overline{\theta}_1 = 0, \qquad \rho_{V_{(1,0)}}(M_6)\overline{\theta}_1 = 0$$

and

$$\rho_{V_{(1,0)}}(M_1)\overline{\theta}_2 = -i\theta_1, \qquad \rho_{V_{(1,0)}}(M_2)\overline{\theta}_2 = \theta_1, \qquad \rho_{V_{(1,0)}}(M_3)\overline{\theta}_2 = 0$$
$$\rho_{V_{(1,0)}}(M_4)\overline{\theta}_2 = 0, \qquad \rho_{V_{(1,0)}}(M_5)\overline{\theta}_2 = -i\sqrt{2}I, \rho_{V_{(1,0)}}(M_6)\overline{\theta}_2 = \sqrt{2}I.$$

Since we know the action of the almost complex structure and have a local expression for $\mathrm{Re}\Omega$ we can again use (7.22) to calculate $\mathrm{cl}(M_i)M_j$ and we find

$$\begin{aligned}\mathrm{cl}(M_1)\phi_1 &= -i\overline{\phi}_2, & \mathrm{cl}(M_1)\phi_2 &= i\overline{\phi}_1\\ \mathrm{cl}(M_2)\phi_1 &= -\overline{\phi}_2, & \mathrm{cl}(M_2)\phi_2 &= \overline{\phi}_1\\ \mathrm{cl}(M_3)\phi_1 &= -\mathbf{1} + i\mathrm{Vol}, & \mathrm{cl}(M_5)\phi_2 &= -i\mathbf{1} - \mathrm{Vol}\\ \mathrm{cl}(M_4)\phi_1 &= -i\mathbf{1} - \mathrm{Vol}, & \mathrm{cl}(M_6)\phi_2 &= -\mathbf{1} + i\mathrm{Vol}.\end{aligned}$$

Thus we find

$$\begin{aligned}(D^1_{A_{\mathrm{can}}})_\gamma(\phi_1 \otimes \overline{\theta}_1 + \phi_2 \otimes \overline{\theta}_2) &= \mathrm{cl}(M_a)\rho_{V_{(1,0)}}(M_a)(\phi_1 \otimes \overline{\theta}_1 + \phi_2 \otimes \overline{\theta}_2)\\ &= 2\left(\overline{\phi}_1 \otimes \theta_1 + \overline{\phi}_2 \otimes \theta_2\right) + \sqrt{2}\left(-4\mathbf{1} + 4i\mathrm{Vol}\right) \otimes I\\ &= -4\sqrt{2}q^{(0,0)} + 4\sqrt{2}i\tilde{q}^{(0,0)} + 2q^{(1,1)} + 2q^{(1,-1)}.\end{aligned}$$

In an identical manner we calculate $(D^1_{A_{\mathrm{can}}})_\gamma(\overline{\phi}_1 \otimes \theta_1 + \overline{\phi}_2 \otimes \theta_2)$ and we find

$$(D^1_{A_{\mathrm{can}}})_\gamma(\overline{\phi}_1 \otimes \theta_1 + \overline{\phi}_2 \otimes \theta_2) = -4\sqrt{2}q^{(0,0)} - 4\sqrt{2}i\tilde{q}^{(0,0)} + 2q^{(1,1)} + 2q^{(1,-1)}.$$

Thus we have found the matrix of $(D^1_{A_{\mathrm{can}}})_\gamma$ in the basis $q^{(0,0)}, \tilde{q}^{(0,0)}, q^{(1,1)}, q^{(1,-1)}$ and given that Lemma 2.3.3 informs of the action of $\mathrm{Re}\Omega$ in this basis we can appeal to (5.8) and find the matrix of $(D^t_{A_{\mathrm{can}}})_\gamma$ to be

$$(D^t_{A_{\mathrm{can}}})_\gamma = \begin{pmatrix} 3(t-1) & 0 & -4\sqrt{2} & -4\sqrt{2} \\ 0 & -3(t-1) & 4i\sqrt{2} & -4i\sqrt{2} \\ -\frac{1}{\sqrt{2}} & -\frac{i}{\sqrt{2}} & 0 & 2 \\ -\frac{1}{\sqrt{2}} & \frac{i}{\sqrt{2}} & 2 & 0 \end{pmatrix} \tag{7.24}$$

We perform a consistency check by noting that $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma = \mathrm{diag}(12, 12, 12, 12)$ as predicted by (5.13). By calculating the eigenvalues of $(D^0_{A_{\mathrm{can}}})_\gamma$ we obtain the following proposition:

Proposition 7.7.1. *Let $V_\gamma = V_{(1,0)}$ be the vector representation of $\mathrm{Spin}(5) = \mathrm{Sp}(2)$ and let $(D^0_{A_{can}})_\gamma$ denote the twisted Dirac operator on $\mathrm{Hom}(V_{(1,0)}, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{Sp}(1)\times\mathrm{U}(1)}$ as in (5.9). The eigenvalues are symmetric about zero, the $\pm\lambda$ eigenspaces are isomorphic and the positive eigenvalues and multiplicities are:*

Eigenvalue	*Multiplicity*
$\sqrt{8}$	2
$\frac{1}{2} + \frac{\sqrt{57}}{2}$	1
$-\frac{1}{2} + \frac{\sqrt{57}}{2}$	1

Observe now that 2 is not an eigenvalue of $(D^0_{A_{\mathrm{can}}})_\gamma$ when V_γ is either the trivial representation, the standard representation or the vector representation and Lemma 7.6.1 ensures it is not an eigenvalue for any other representation. Therefore 2 is not in the spectrum of $D^0_{A_{\mathrm{can}}}$. Since deformations of a nearly Kähler instanton correspond to 1-forms in the $+2$ eigenspace of the operator $D^0_{A_{\mathrm{can}}}$ [18] we obtain a corollary:

Corollary 7.7.2. *Let $Q \to \mathbb{CP}^3$ be the homogeneous $\mathrm{SU}(2)$-bundle*

$$Q = \mathrm{Sp}(2) \times_{\lambda_1} \mathrm{SU}(2)$$

where $\lambda_1 \colon \mathrm{Sp}(1) \times \mathrm{U}(1) \to \mathrm{SU}(2)$ is the homomorphism $\lambda_1(g, e^{i\theta}) = \mathrm{diag}(e^{i\theta}, e^{-i\theta})$. Then the canonical connection A_{can} on Q is a rigid nearly Kähler instanton.

We conclude this section by stating the virtual dimension of the moduli space we have been studying. This follows from the observation in Remark 4.4.6:

Theorem 7.7.3. *Let A_{can} be the canonical connection on the homogeneous* $\mathrm{SU}(2)$*-bundle* $Q = \mathrm{Sp}(2) \times_{\lambda_1} \mathrm{SU}(2)$. *Let $p\colon \Lambda^2_-(S^4) \to \mathbb{CP}^3$ denote the natural projection and let P be the extension of the bundle p^*Q as described in [77]. The virtual dimension of the moduli space $\mathcal{M}(A_{\mathrm{can}}, \mu)$ of G_2-instantons on P asymptotic to A_{can} with rate μ is*

$$\mathrm{virtdim}\mathcal{M}(A_{\mathrm{can}}, \mu) = 1$$

for all $\mu \in (-2, 0)$.

7.8 Pullback of the Spin Connection on S^4

In [77] Oliveira showed that the Levi-Civita connection on the spinor bundle of S^4 can be pulled back to $\Lambda^2_-(S^4)$ to define a G_2-instanton for the Bryant-Salamon G_2-structure. More generally, any ASD instanton over S^4 or $\mathbb{CP}^2$ can be pulled back to the Bryant-Salamon manifolds to define a G_2-instanton.

We can interpret the spin connection and its pullback homogeneous bundle language. Using the isomorphism $\mathrm{Sp}(1) \cong \mathrm{SU}(2)$ the isotropy homomorphism $\lambda\colon \mathrm{Sp}(1) \times \mathrm{Sp}(1) \to \mathrm{SU}(2)$ given by $\lambda(g, g') = g$ defines an $\mathrm{Sp}(2)$-homogeneous $\mathrm{SU}(2)$-bundle over $S^4 = \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{Sp}(1)$. This is the frame bundle of the spinor bundle of S^4 and the canonical connection coincides with the Levi-Civita connection as S^4 is a symmetric space. The twistor projection enables one to lift the bundle and connection to an $\mathrm{SU}(2)$ bundle and connection over $\mathbb{CP}^3$. The bundle here is the homogeneous bundle

$$Q = \mathrm{Sp}(2) \times_{\tilde{\lambda}} \mathrm{SU}(2)$$

where $\tilde{\lambda}(g, e^{i\theta}) = g$ and the connection A_{can} is the canonical connection. We pullback Q and A_{can} to $\Lambda^2_-(S^4)$; both the bundle and the connection automatically extend over the zero section and it is verified in [77, Proposition 3] that this yields a G_2-instanton. We label this bundle P and still denote the canonical connection by A_{can}. The situation here is somewhat different to those previously considered since the G_2-instanton is AC with any rate $\mu < 0$ and so to determine the virtual dimension of the moduli space of G_2-instantons asymptotic to A_{can} for all rates one would need to calculate the entire spectrum of D^0_{can}. Doing so would require calculating the branching rule from $\mathrm{Sp}(2)$ to $\mathrm{Sp}(1) \times \mathrm{U}(1)$, however we will not attempt this. We can however quite easily determine the virtual dimension for a small interval of weights using the methods developed in Chapter 5.

As is now familiar, we wish to determine the eigenvalues of $D^0_{A_{\mathrm{can}}}$ in the interval $[0, 2)$ (or possibly a bigger interval containing this one). This operator acts on section of $\displaystyle{\not}\mathcal{S}(\mathbb{CP}^3) \otimes \mathrm{Ad}Q$ where the bundle $\mathrm{Ad}Q$ is associated via the representation $\mathrm{Ad} \circ \tilde{\lambda}$ of $\mathrm{Sp}(1) \times \mathrm{U}(1)$, and it is clear that the isomorphism class of this representation is $W_{(2,0)}$.

Recalling the decomposition of the spinor space as an Sp(1) × U(1) representation from (7.18) we find that the isomorphism class of twisted spinor space to be

$$S \otimes \mathfrak{su}(2)_{\mathbb{C}} = 2W_{(2,0)} \oplus [[W_{(3,1)}]] \oplus [[W_{(1,1)}]] \oplus [[W_{(2,2)}]]. \tag{7.25}$$

Observe that $\mathrm{Hom}(\mathbb{C}, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{Sp}(1)\times \mathrm{U}(1)} = \{0\}$ by Schur's lemma, so this homomorphism space does not contribute any eigenvalues. The next case to consider is $V_{(0,1)}$ which is the standard representation of Sp(2). The branching of this representation is $V_{(0,1)} = W_{(1,0)} \oplus [[W_{(0,1)}]]$ so this representation also contributes no eigenvalues.

The first irreducible representation of Sp(2) that contributes eigenvalues is $V_{(1,0)}$, the vector representation of Spin(5). Recall the branching is $V_{(1,0)} = W_{(0,0)} \oplus [[W_{(1,1)}]]$ so $\mathrm{Hom}(V_{(1,0)}, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{Sp}(1)\times \mathrm{U}(1)}$ has dimension 2 and a basis is given by the maps $q^{(1,1)}_{(1,1)(2,0)}, q^{(1,-1)}_{(1,-1)(2,0)}$, where $q^{(i,j)}_{(k,l)(m,n)}$ which factor

$$V_{(1,0)} \to W_{(i,j)} \to W_{(k,l)} \otimes W_{(m,n)} \hookrightarrow S \otimes \mathfrak{su}(2)_{\mathbb{C}}$$

with the first map being an equivariant projection and the second being an equivariant embedding. Let $V_\gamma = V_{(1,0)}$, from (5.12) and (7.16) we have that

$$(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma = 8\mathrm{Id}.$$

Now all maps in $\mathrm{Hom}(V_{(1,0)}, S \otimes \mathfrak{su}(2)_{\mathbb{C}})_{\mathrm{Sp}(1)\times \mathrm{U}(1)}$ actually map into $\mathrm{m}^*_{\mathbb{C}} \otimes \mathfrak{su}(2)_C$ and it follows that $\mathrm{Re}\Omega$ acts trivially on this space. We conclude that $(D^t_{A_{\mathrm{can}}})_\gamma = (D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ for all t and therefore the eigenvalues of $(D^0_{A_{\mathrm{can}}})_\gamma$ are the same as those of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ which are $\pm\sqrt{8}$.

We do not consider calculating eigenvalues from any more representations but instead provide bounds sufficient for us to state the virtual dimension of the moduli space under consideration. Let $V_\gamma = V_{(0,2)}$, then the lower bound on the positive eigenvalues of $(D^0_{A_{\mathrm{can}}})_\gamma$ that one obtains from Theorem 5.3.5 is $L_\gamma = \sqrt{12} - 1$. For any other irreducible representation of Sp(2) we get a larger lower bound on the smallest positive eigenvalue of the induced operator. Thus we have shown

Proposition 7.8.1. *Let A_{can} be the canonical connection on the bundle $Q = \mathrm{Sp}(2) \times_{\tilde{\lambda}} \mathrm{SU}(2)$. Then the operator $D^0_{A_{\mathrm{can}}}$ has no eigenvalues in the region $(-\sqrt{12}+1, \sqrt{12}-1)$.*

As a result we are able to determine the virtual dimension of the moduli space for certain rates:

Theorem 7.8.2. *Let $\mathcal{M}(A_{\mathrm{can}}, \mu)$ be the moduli space of G_2-instantons on $P \to \Lambda^2_-(S^4)$, asymptotic to A_{can} with rate μ. Then the virtual dimension of the moduli space is*

$$\mathrm{virtdim}\mathcal{M}(A_{\mathrm{can}}, \mu) = 0 \qquad \textit{for all} \quad \mu \in (-\sqrt{12} - 1, 0).$$

Since both the invariant instanton equations and the virtual dimension of the AC moduli space are identical in both cases considered in this chapter, the observations of Remark 7.3.7 are equally applicable to the case considered above.

Chapter 8

The Standard Instanton

In this chapter we consider the moduli space of AC G_2-instantons on $\mathbb{R}^7$ with structure group G_2 and which decay to the canonical connection of S^6 at infinity. We repeat the process of calculating the virtual dimension of the moduli space but we use a different method to that of the previous chapter for determining eigenvalues of the twisted Dirac operator. We write the operator in question as a sum of Casimir operators and find two bases that diagonalise the different terms in this expression. We again find the matrix of the operator in a suitable basis of the relevant homomorphism space and by calculating its eigenvalues we find the virtual dimension of the moduli space. As an application of the deformation theory, we explore uniqueness properties of the G_2-instanton of Günaydin and Nicolai. This culminates with the main result of this chapter, Theorem 8.11.3, that a G_2-instanton on this bundle is either obstructed or is the instanton of Günaydin-Nicolai.

8.1 The Standard Instanton

Throughout this section we work with the AC G_2-manifold $M = \mathbb{R}^7$ whose asymptotic link is the homogeneous nearly Kähler manifold $\Sigma = S^6$. There is a well known example of a G_2-instanton on $\mathbb{R}^7$ constructed by Günaydin and Nicolai [41]. This example was recovered by Harland et al [42] via a different construction method, namely they considered the instanton equation on the cylinder over the nearly Kähler 6-sphere (which is conformally equivalent to the Euclidean $\mathbb{R}^7 \setminus \{0\}$). This viewpoint makes it easier to understand the asymptotics of the connection, in particular one easily observes that it asymptotes to the canonical connection. We give here a very brief overview of the construction before revisiting it in more detail in 8.9. The important fact we shall need here is the rate at which the instanton converges.

Form a bundle Q over S^6 via

$$Q = G_2 \times_{(\mathrm{SU}(3),\iota)} G_2$$

where $\iota\colon \mathrm{SU}(3) \hookrightarrow G_2$ is the inclusion homomorphism. Topologically $Q = G_2 \times S^6 \to S^6$ is the trivial G_2 bundle over S^6 but it helps to keep in mind the homogeneous structure. Since the bundle Q is associated from the canonical bundle $G_2 \to G_2/\mathrm{SU}(3)$ the canonical connection lives on Q. Essentially Q extends the canonical bundle to a G_2 bundle and the canonical connection is the connection whose horizontal spaces are defined by the inclusion of left translates of $\mathfrak{m}$. In particular, the canonical connection on Q has holonomy $\mathrm{SU}(3)$.

Other than the canonical connection A_{can} there is another G_2-invariant connection which is in fact flat. We denote this $A_{\mathrm{flat}} = A_{\mathrm{can}} + a$. Consider $P = \mathbb{R}^7 \times G_2$ so that $P|_{\mathbb{R}^7\setminus\{0\}} = \pi^* Q$, the pullback of Q to the cone. Let $(r, \sigma) \in (0, \infty) \times S^6$ and make the G_2 invariant ansatz

$$A(r, \sigma) = A_{\mathrm{can}}(\sigma) + f(r)a(\sigma)$$

where f is a function on $\mathbb{R}^7$ depending only on the radial coordinate r. It is shown in [42, Section 5.3] that

$$f(r) = \frac{1}{Cr^2 + 1} \tag{8.1}$$

with $C > 0$ a constant yields a G_2 instanton which extends over the origin in $\mathbb{R}^7$.

Remark 8.1.1. *Harland et al [42] work on the cylinder over S^6, with coordinates (σ, t) for $\sigma \in S^6$ and $t \in \mathbb{R}$, so to change to the conical viewpoint considered here one makes the change of variables $r = e^t$. Even after this change of variables the function given (8.1) differs from the one found in [42] but this is simply a consequence of the normalisations we have chosen. Namely the metric on $\mathfrak{g}_2$ is $-\frac{1}{12}^{th}$ of the Killing form and our* $\mathrm{SU}(3)$ *structure satisfies* $\mathrm{d}\omega = 3\mathrm{Im}\Omega$.

For any such f we define

$$A_{\mathrm{std}} = A_{\mathrm{can}} + fa$$

and call this the standard G_2 instanton. It is clear from (8.1) that A_{std} aymptotes to the canonical connection with fastest rate of convergence -2.

In the notation of Chapter 3 we set $A = A_{\mathrm{std}}$ and therefore $A_\infty = A_{\mathrm{can}}$. Since the fastest rate of convergence is -2 we will consider the family of moduli spaces $\mathcal{M}(A_{\mathrm{can}}, \mu)$ for $\mu \in (-2, 0)$. Recall we denote by W the set of critical weights for the operator $D_A\colon L^2_{k+1,\mu-1} \to L^2_{k,\mu-2}$. In the case at hand we expect that $\{-2, -1\} \subset W$ for the following reason: The deformation defined by the dilation (the $\mathbb{R}_+$ action on the end of M) is $\iota_{\rho\frac{\partial}{\partial\rho}} F_{A_{\mathrm{std}}}$ and this is added with rate -2. The deformations determined by translation in $\mathbb{R}^7$ are $\iota_{\frac{\partial}{\partial x_i}} F_{A_{\mathrm{std}}}$ and these are added with rate -1, therefore we expect the $+1$ eigenspace of $D^0_{A_{\mathrm{can}}}$ has dimension at least 7.

8.2 The Dirac Operator on $G_2/\mathbf{SU}(3)$

This section aims to understand the action of the Dirac operator on spinor fields from a representation theoretic viewpoint. To calculate eigenvalues of the twisted Dirac operator on S^6 we use a different method to the one presented in previous chapters. Instead of working directly with the definition of the Dirac operator we write the operator as a sum of Casimir operators. In doing so we are able to calculate the matrix and hence the eigenvalues of the Dirac operator.

Throughout this section the notation of Fulton and Harris in [39] is used, thus we denote a complex irreducible representation of G_2 by $(V_{(i,j)}, \rho_{V_{(i,j)}})$, so that $V_{(0,0)}$ is the trivial representation, $V_{(1,0)} = \mathbb{C}^7$ is the standard representation and $V_{(0,1)} = (\mathfrak{g}_2)_{\mathbb{C}}$ is the adjoint representation. Similarly we denote complex irreducible representations of SU(3) by $(W_{(i,j)}, \rho_{W_{(i,j)}})$ (note that in Chapter 7 the same representation was denoted $V_{(i,j)}$) so that $W_{(1,0)} = \mathbb{C}^3$ and $W_{(0,1)} = (\mathbb{C}^3)^*$

All representations under consideration are complex representations for simplicity. It follows from Lemma 8.2.1 that the complexified tangent bundle is associated via the representation $(\mathfrak{m}_{\mathbb{C}}, \mathrm{Ad}) = W_{(1,0)} \oplus W_{(0,1)}$ of SU(3) and it is worth noting that $\mathfrak{m}_{\mathbb{C}} \cong \mathfrak{m}^*_{\mathbb{C}}$ as representations. The spinor space $S = \mathbb{C} \oplus \mathfrak{m}^*_{\mathbb{C}} \oplus \mathbb{C}$ is

$$S = W_{(0,0)} \oplus W_{(1,0)} \oplus W_{(0,1)} \oplus W_{(0,0)} \tag{8.2}$$

as an isomorphism class of SU(3) representations.

It will prove useful to have an explicit description of the embedding $\mathfrak{g}_2 \hookrightarrow \mathfrak{so}(7) \cong \Lambda^2(\mathbb{R}^7)^*$ in term of the subalgebra $\mathfrak{su}(3)$. Recall $\mathfrak{so}(7) \cong \Lambda^2(\mathbb{R}^7)^*$ splits as into irreducible representations of G_2 as $\Lambda^2(\mathbb{R}^7)^* = \Lambda^2_{14} \oplus \Lambda^2_7$ where the subscript denotes the dimension of the irreducible component. We pick an orthonormal basis $e^1, \ldots, e^6, \mathrm{d}t$ of $(\mathbb{R}^7)^* \cong (\mathbb{R}^6)^* \oplus \mathbb{R}^*$ so that the summand $\mathbb{R}^*$ is identified with $\langle \mathrm{d}t \rangle$. Under the splitting $\mathbb{R}^7 = \mathbb{R}^6 \oplus \mathbb{R}$ the action of SU(3) is the obvious one. The image of the embedding $\mathfrak{g}_2 \hookrightarrow \Lambda^2(\mathbb{R}^7)^*$ is the space of 2-forms α satisfying $\alpha \wedge \psi_0 = 0$. We decompose the space of 2-forms on $\mathbb{R}^7$ as SU(3) modules:

$$\begin{aligned} \Lambda^2(\mathbb{R}^7)^* &\cong \Lambda^2(\mathbb{R}^6)^* \oplus \left((\mathbb{R}^6)^* \wedge \mathrm{d}t\right) \\ &\cong \Lambda^2_8 \oplus \Lambda^2_6 \oplus \langle \omega \rangle_{\mathbb{R}} \oplus \left((\mathbb{R}^6)^* \wedge \mathrm{d}t\right). \end{aligned}$$

Recall the Lie algebra $\mathfrak{g}_2$ is reductive with respect to the subalgebra $\mathfrak{su}(3)$. This means there is a splitting $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{m}$ and the component $\mathfrak{m}$ is closed under the adjoint action of $\mathfrak{su}(3)$. Furthermore, $\mathfrak{m}$ is the orthogonal complement of $\mathfrak{su}(3)$ with respect to the Killing form on $\mathfrak{g}_2$. We see that

$$\mathfrak{m} \subset \mathrm{Re}\left(\Lambda^{2,0}(\mathbb{R}^6)^*\right) \oplus \langle \omega \rangle \oplus \left((\mathbb{R}^6)^* \wedge \mathrm{d}t\right).$$

The space $\mathfrak{m}$ is a 6-dimensional space such that $\psi_0 \wedge \eta = 0$ for all $\eta \in \mathfrak{m}$. A direct calculation shows

$$\psi_0 \wedge \omega = 3\mathrm{Vol}_6$$
$$\psi_0 \wedge \left(e^i \wedge \mathrm{d}t + \frac{1}{2} e^i \lrcorner \mathrm{Re}\Omega\right) = 0.$$

This yields the following lemma:

Lemma 8.2.1. *Consider the Lie algebra $\mathfrak{g}_2$ as a subspace of $\Lambda^2(\mathbb{R}^7)^*$ via the standard embedding. Then there is a splitting $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{m}$ where $\mathfrak{su}(3)$ is the space of ω anti-self-dual 2-forms on $\mathbb{R}^6$ and*

$$\mathfrak{m} = \mathrm{span}\left\{e^a \wedge \mathrm{d}t + \frac{1}{2} e^a \lrcorner \mathrm{Re}\Omega\,;\, i = 1, \ldots, 6\right\}.$$

The space $\mathfrak{m}$ is closed under the adjoint action of $\mathrm{SU}(3)$ and this splitting is orthogonal with respect to the Killing form on $\mathfrak{g}_2$. Furthermore the map

$$F\colon \Lambda^1(\mathbb{R}^6)^* \to \mathfrak{m} \tag{8.3}$$
$$F(v) = v \wedge \mathrm{d}t + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega \tag{8.4}$$

is an isomorphism of $\mathrm{SU}(3)$ representations and an isometry with respect to the Euclidean inner product on $\mathbb{R}^6$ and $B|_{\mathfrak{m}}$ on $\mathfrak{m}$.

The map F thus allows us to think of $\mathfrak{m}$, as an $\mathrm{SU}(3)$ module, as either $(\mathbb{R}^6)^*$ or as a subspace of $\mathfrak{g}_2$ and it will prove convenient to switch between these two viewpoints. The importance of the above lemma is that it allows one to understand Clifford multiplication on the space of sections of the spinor bundle on $G_2/\mathrm{SU}(3)$ as we now explain.

We seek to extend the spinor space S to a representation of G_2, this will allow elements of $\mathfrak{m} \subset \mathfrak{g}_2$ (which we may view as tangent vectors at a point) to act on the spinor bundle and we can compare this action with that of Clifford multiplication. Extending S to a representation of G_2 means seeking an action that yields the splitting (8.2) when the action is restricted to $\mathrm{SU}(3) \subset G_2$. Note that the standard representation $V_{(1,0)} = \mathbb{C}^7$ of G_2 becomes the representation $V_{(1,0)} = W_{(0,0)} \oplus W_{(1,0)} \oplus W_{(0,1)}$ when one restricts to the subgroup $\mathrm{SU}(3) \subset G_2$ (this is the branching rule for $V_{(1,0)}$). Thus $V_{(1,0)} \oplus V_{(0,0)}$ branches to the correct representation of $\mathrm{SU}(3)$. To view S as the representation $V_{(1,0)} \oplus V_{(0,0)}$ of G_2 we use the isomorphism

$$\mathbb{C} \oplus \mathfrak{m}^*_{\mathbb{C}} \oplus \mathbb{C} = (\mathbb{C}^7)^* \oplus \mathbb{C}$$
$$(z, v, w) \mapsto (z\,\mathrm{d}t + v, w).$$

The action ρ_S of $\mathfrak{g}_2$ on S that yields the above decomposition is

$$\rho_S(\eta)(z\,\mathrm{d}t + v, w) = (\eta \llcorner_7 (z\,\mathrm{d}t + v), 0),$$

where $\eta \in \mathfrak{g}_2 \subset \Lambda^2(\mathbb{R}^7)^*$ and $\llcorner_7$ denotes the contraction from the Euclidean inner product on $(\mathbb{R}^7)^*$ extended complex linearly to $(\mathbb{C}^7)^*$. This action is then extended to sections of the spin bundle $\not{S}_{\mathbb{C}} = \Lambda^0_{\mathbb{C}} \oplus \Lambda^1_{\mathbb{C}} \oplus \Lambda^6_{\mathbb{C}}$.

To compare this action to Clifford multiplication we calculate

$$\rho_S(F(u))(f, v, g\mathrm{Vol}) = \left(-u \lrcorner v, fu - \frac{1}{2}(u \wedge v)\lrcorner \mathrm{Re}\Omega, 0\right). \tag{8.5}$$

Here $\lrcorner$ denotes the contraction map induced by the round metric, extended complex linearly. This does not agree with the formula for Clifford multiplication given by (2.32), so to fix this disparity we consider another representation on the spin bundle. There is another natural representation of $\mathfrak{g}_2$ on given by $\rho_{\tilde{S}}(X) = \mathrm{Vol}^{-1} \cdot \rho_S(X) \cdot \mathrm{Vol}$ and we shall see that the two representations ρ_S and $\rho_{\tilde{S}}$ together recover Clifford multiplication. To see this we first need a simple lemma:

Lemma 8.2.2. *Let $(\Sigma, \Omega, \omega, J)$ be a nearly Kähler 6-manifold. Then the following identities hold*

$$J\left((u \wedge Jv)\lrcorner \mathrm{Re}\Omega\right) = (u \wedge v)\lrcorner \mathrm{Re}\Omega \tag{8.6}$$

$$u \lrcorner Jv = (u \wedge v)\lrcorner \omega \tag{8.7}$$

for all $u, v \in \Lambda^1(T^*\Sigma)$.

Corollary 8.2.3. *After associating spinor fields with elements of $L^2(G_2, S)_{\mathrm{SU}(3)}$ and vector fields with elements of $L^2(G_2, \mathfrak{m})_{\mathrm{SU}(3)}$, Clifford multiplication of a spinor s by a tangent vector u takes the form*

$$\mathrm{cl}(u)s = \left(\rho_S(F(u)) - \rho_{\tilde{S}}(F(u))\right)s \tag{8.8}$$

where ρ_S is the representation (8.5) of $\mathfrak{g}_2$ on S and $\rho_{\tilde{S}} = \mathrm{Vol}^{-1} \cdot \rho \cdot \mathrm{Vol}$.

Proof. Recall that the action of the volume form on the spin bundle is $\mathrm{Vol}(f, v, h\mathrm{Vol}) = (-h, Jv, f\mathrm{Vol})$, one thus calculates that

$$(\rho_S(F(u)) - \rho_{\tilde{S}}(F(u)))(f, v, h\mathrm{Vol}) =$$
$$\left(-u \lrcorner v, fu - \frac{1}{2}(u \wedge v)\lrcorner \mathrm{Re}\Omega - hJu - \frac{1}{2}J((u \wedge Jv)\lrcorner \mathrm{Re}\Omega), -(u \lrcorner Jv)\mathrm{Vol}\right)$$

and an application of Lemma 8.2.2 shows that this exactly matches the formula for Clifford multiplication given by (2.32). □

Having found formulae for both covariant differentiation and Clifford multiplication we can now consider the Dirac operators relevant to this setting. Recall the operators D^t are built from the modified connections ∇^t and that ∇^1 is the canonical connection arising from the reductive homogeneous structure. We fix bases $\{I_a\}_{a=1}^{6}$ for $\mathfrak{m}$ and $\{I_i\}_{i=7}^{14}$ for $\mathfrak{h}$ which are orthonormal with respect to the nearly Kähler metric on $\mathfrak{g}_2$. We know from (5.6) that $D^1 = \mathrm{cl}(I_a)\rho_R(I_a)$ and we can rewrite the form of this operator using Corollary 8.2.3.

Corollary 8.2.4. *After associating spinor fields with elements of $L^2(G_2, S)_{\mathrm{SU}(3)}$ and vector fields with elements of $L^2(G_2, \mathfrak{m})_{\mathrm{SU}(3)}$, the Dirac operator of the canonical connection $D^1 \colon \Gamma(\mathbb{S}_{\mathbb{C}}(\Sigma)) \to \Gamma(\mathbb{S}_{\mathbb{C}}(\Sigma))$ is*

$$D^1 = (\rho_S(I_a) - \tilde{\rho}_S(I_a))\rho_R(I_a)s.$$

Let us therefore define operators

$$D^\rho \colon \Gamma(\mathbb{S}_{\mathbb{C}}(\Sigma)) \to \Gamma(\mathbb{S}_{\mathbb{C}}(\Sigma)) \qquad \tilde{D}^\rho \colon \Gamma(\mathbb{S}_{\mathbb{C}}(\Sigma)) \to \Gamma(\mathbb{S}_{\mathbb{C}}(\Sigma))$$
$$D^\rho = \rho_S(I_a)\rho_R(I_a) \qquad \tilde{D}^\rho = \rho_{\tilde{S}}(I_a)\rho_R(I_a)$$

so that $D^1 = D^\rho - \tilde{D}^\rho$. We now explain how to view these operators as *Casimir operators*:

For $i \neq j$ we set $[[W_{(i,j)}]] = W_{(i,j)} \oplus W_{(j,i)}$. Recall that isomorphism class of the SU(3) representation S is

$$S = W_{(0,0)} \oplus [[W_{(1,0)}]] \oplus W_{(0,0)}.$$

Since the actions ρ_S and ρ_R commute we can rewrite D^ρ as:

$$D^\rho = \frac{1}{2}\left(\rho_{S\otimes R}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_S(\mathrm{Cas}_{\mathfrak{m}}) - \rho_R(\mathrm{Cas}_{\mathfrak{m}})\right)$$

where a representation of G_2 defines a representation of SU(3) by restriction and $\rho(\mathrm{Cas}_{\mathfrak{m}}) := \rho(\mathrm{Cas}_{\mathfrak{g}_2}) - \rho(\mathrm{Cas}_{\mathfrak{su}(3)})$. Similarly the expression for $\tilde{D}^\rho$ is

$$\tilde{D}^\rho = \frac{1}{2}\left(\rho_{\tilde{S}\otimes R}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{\tilde{S}}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_R(\mathrm{Cas}_{\mathfrak{m}})\right)$$

and in fact $\tilde{D}^\rho = \mathrm{Vol}^{-1}D^\rho\mathrm{Vol}$. Combining this with (2.33) yields a representation theoretic formula for the Levi-Civita Dirac operator:

$$D^0 = D^\rho - (\mathrm{Vol}^{-1}D^\rho\mathrm{Vol}) - \frac{3}{4}\mathrm{Re}\Omega. \tag{8.9}$$

The discussions transfers easily to the homomorphism space decomposition of the space of spinor fields provided by Frobenius reciprocity. If V_γ is an irreducible representation of G_2 then $\mathrm{Hom}(V_\gamma, S)_{\mathrm{SU}(3)} \otimes V_\gamma$ embeds into the space of sections and we define operators D^t_γ on $\mathrm{Hom}(V_\gamma, S)_{\mathrm{SU}(3)}$ such that

$$D^t_\gamma \otimes \mathrm{Id} = D^t|_{\mathrm{Hom}(V_\gamma,S)_{\mathrm{SU}(3)}\otimes V_\gamma}.$$

For each such homomorphism space the operator D^ρ also defines an endomorphism D^ρ_γ via the formula

$$D^\rho|_{\mathrm{Hom}(V_\gamma,S)_{\mathrm{SU}(3)}\otimes V_\gamma} = D^\rho_\gamma \otimes \mathrm{Id}$$

and we find $D^\rho_\gamma\colon \mathrm{Hom}(V_\gamma, S)_{\mathrm{SU}(3)} \to \mathrm{Hom}(V_\gamma, S)_{\mathrm{SU}(3)}$ is the operator

$$D^\rho_\gamma = \frac{1}{2}\left(\rho_{S\otimes V^*_\gamma}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_S(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V^*_\gamma}(\mathrm{Cas}_{\mathfrak{m}})\right). \tag{8.10}$$

In the same way $\tilde{D}^\rho$ defines an endomorphism $\tilde{D}^\rho_\gamma$ and one has that

$$\tilde{D}^\rho_\gamma = \mathrm{Vol}^{-1} D^\rho_\gamma \mathrm{Vol}$$

where Vol is the action of the volume form on $\mathfrak{m}$ (induced by the Riemannian volume form on S^6) on spinor space. It then follows from the previous discussion and from (5.9) that

$$D^1_\gamma = D^\rho_\gamma - (\mathrm{Vol}^{-1} D^\rho_\gamma \mathrm{Vol})$$

and thus that

$$D^t_\gamma = D^\rho_\gamma - (\mathrm{Vol}^{-1} D^\rho_\gamma \mathrm{Vol}) + \frac{3(t-1)}{4}\mathrm{Re}\Omega. \tag{8.11}$$

If we can therefore understand the operator D^ρ_γ as well as the action of Vol and ReΩ in a suitable basis then we can calculate the matrices of the family of Dirac operators on the relevant homomorphism space. For this we need the eigenvalues of the $\mathfrak{g}_2$ and $\mathfrak{su}(3)$ Casimir operators with respect to the nearly Kähler metric.

Let $V_{(i,j)}$ be an irreducible representation of $\mathfrak{g}_2$ and let $W_{(i,j)}$ be an irreducible representation of $\mathfrak{su}(3)$, then we have

$$\rho_{(i,j)}(\mathrm{Cas}_{\mathfrak{g}_2}) = c^{\mathfrak{g}_2}_{(i,j)}\mathrm{Id}$$
$$\rho_{(i,j)}(\mathrm{Cas}_{\mathfrak{su}(3)}) = c^{\mathfrak{su}(3)}_{(i,j)}\mathrm{Id}$$

and the eigenvalues are calculated in [18] to be

$$c^{\mathfrak{g}_2}_{(i,j)} = -(i^2 + 3j^2 + 3ij + 5i + 9j) \tag{8.12}$$
$$c^{\mathfrak{su}(3)}_{(i,j)} = -(i^2 + j^2 + ij + 3i + 3j). \tag{8.13}$$

8.3 Some Warm Up Calculations

At this point we pause to outline how to use this formulation of the Dirac operator to calculate some eigenvalues of the Levi-Civita-Dirac operator on S^6. This is a useful exercise, as the method we use is identical to the one we shall for the twisted Dirac operator, but the calculations involved are less cumbersome.

Recall that the nearly Kähler S^6 carries the standard round metric, so our method should recover the well known eigenvalues of the Dirac operator on the round sphere. The spectrum is [6]

$$\mathrm{Spec} D^0 = \{\pm(3+k)\,;\, k \in \mathbb{N}\}$$

so the eigenvalues we calculate should lie in this set. Furthermore, the multiplicity of an eigenvalue $\pm(3+k)$ is

$$\mathrm{mult}(\pm(3+k)) = 8\binom{k+5}{k}.$$

If we can understand how the operator D^ρ_γ from (8.10) and the volume form Vol act on the given homomorphism space then we can calculate the matrix of the operator D^1_γ. Then if we know the action of the 3-form $\mathrm{Re}\Omega$ in this basis, we can calculate the matrix and eigenvalues of the operator D^0_γ via (8.11).

Example 8.3.1. Let $V_\gamma = \mathbb{C}$ be the trivial representation of $\mathfrak{g}_2$. Then the space $\mathrm{Hom}(\mathbb{C}, S)_{\mathrm{SU}(3)}$ is two dimensional with basis $q^{(0,0)}$ and $q^{(3,3)}$, which map into the subspaces $\Lambda^{(0,0)}$ and $\Lambda^{(3,3)}$ of S. We have

$$D^\rho_\gamma = \frac{1}{2}\left(\rho_{S\otimes\mathbb{C}}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{\mathbb{C}}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_S(\mathrm{Cas}_{\mathfrak{m}})\right)$$

and $\rho_{\mathbb{C}}(\mathrm{Cas}_{\mathfrak{m}}) = 0$ and $\rho_{S\otimes\mathbb{C}}(\mathrm{Cas}_{\mathfrak{m}}) = \rho_S(\mathrm{Cas}_{\mathfrak{m}})$ so we see that $D^\rho_\gamma \equiv 0$ and thus $D^1_\gamma \equiv 0$. The basis $q^{(0,0)}$ and $q^{(3,3)}$ correspond to the Killing spinor s_6 and $\mathrm{Vol}\cdot s_6$, which are parallel sections with respect to ∇^1 and hence harmonic spinors for D^1. By Lemma 2.3.3 we see that

$$-\frac{3}{4}\mathrm{Re}\Omega = \mathrm{diag}(-3, 3)$$

in this basis. So the matrix of D^1_γ in this basis is

$$D^0_\gamma = \mathrm{diag}(-3, 3)$$

which corresponds to s_6 and $\mathrm{Vol}\cdot s_6$ being eigenspinors of D^0 with eigenvalues -3 and $+3$ respectively.

Next we carry out the same procedure for the standard representation of G_2 :

Example 8.3.2. Let $V_\gamma = V_{(1,0)}$ be the standard representation of G_2. The space $\mathrm{Hom}(V_\gamma, S)_{\mathrm{SU}(3)}$ has dimension four since $V_\gamma = W_{(1,0)} \oplus W_{(0,1)} \oplus W_{(0,0)}$. It proves helpful to split this space into two components

$$\mathrm{Hom}(V_{(1,0)}, S)_{\mathrm{SU}(3)} = \mathrm{Hom}(V_{(1,0)}, V_{(1,0)})_{\mathrm{SU}(3)} \oplus \mathrm{Hom}(V_{(1,0)}, \mathbb{C})_{\mathrm{SU}(3)}.$$

Let $q^{(i,j)}$ be the projection maps $q^{(i,j)}\colon V_{(1,0)} \to W_{(i,j)}$, then the maps $q^{(1,0)}, q^{(0,1)}, q^{(0,0)}$ form a basis of $\mathrm{Hom}(V_{(1,0)}, V_{(1,0)})_{\mathrm{SU}(3)}$ and we can extend this to a basis of $\mathrm{Hom}(V_{(1,0)}, \mathbb{C})_{\mathrm{SU}(3)}$ by adding the map $\mathrm{Vol}\cdot q^{(0,0)}$. Under this splitting the operator D^ρ_γ splits into a 3×3 and 1×1 block diagonal matrix and arguing identically to the previous calculation we see that D^ρ_γ acts trivially on $\mathrm{Hom}(V_{(1,0)}, \mathbb{C})_{\mathrm{SU}(3)}$. Furthermore D^ρ_γ acts on $\mathrm{Hom}(V_{(1,0)}, V_{(1,0)})_{\mathrm{SU}(3)}$ as

$$\frac{1}{2}\left(\rho_{V^*_{(1,0)}\otimes V_{(1,0)}}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V_{(1,0)}}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V^*_{(1,0)}}(\mathrm{Cas}_{\mathfrak{m}})\right)$$

and the basis maps $q^{(1,0)}, q^{(0,1)}, q^{(0,0)}$ are eigenvectors of $\rho_{V_{(1,0)}}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V^*_{(1,0)}}(\mathrm{Cas}_{\mathfrak{m}})$, the eigenvalue of $q^{(i,j)}$ being $-2(c^{\mathfrak{g}_2}_{(1,0)} - c^{\mathfrak{su}(3)}_{(i,j)})$.

Consider the operator $\rho_{V_{(1,0)}\otimes V_{(1,0)}}(\mathrm{Cas}_{\mathfrak{m}})$, we can use the isomorphism $\mathrm{Hom}(V_{(1,0)}, V_{(1,0)})_{\mathrm{SU}(3)} \cong \mathrm{Hom}(V_{(1,0)} \otimes V_{(1,0)}, \mathbb{C})_{\mathrm{SU}(3)}$ to see that the SU(3) equivariant maps that factor

$$p^{(i,j)} \colon V_{(1,0)} \otimes V_{(1,0)} \to V_{(i,j)} \to \mathbb{C}$$

form a basis of eigenvectors of this operator. Now

$$\begin{aligned} V_{(1,0)} \otimes V_{(1,0)} &\cong \Lambda^2(V_{(1,0)}) \oplus \mathrm{Sym}^2(V_{(1,0)}) \\ &\cong V_{(1,0)} \oplus V_{(0,1)} \oplus V_{(2,0)} \oplus V_{(0,0)} \end{aligned}$$

and since, except for the adjoint representation $V_{(0,1)}$, each summand contains a copy of the trivial representation when restricted to a representation of SU(3), the basis we obtain is $p^{(0,0)}, p^{(1,0)}, p^{(0,1)}$. The eigenvalues are

$$\rho_{V_{(1,0)}\otimes V_{(1,0)}}(\mathrm{Cas}_{\mathfrak{m}})p^{(i,j)} = c^{\mathfrak{g}_2}_{(i,j)}$$

so to calculate the matrix of D^{ρ}_{γ} in either of the bases presented above, it remains to calculate the change of basis matrix.

From the above decomposition of $V_{(1,0)} \otimes V_{(1,0)}$ it is clear that $p^{(0,0)} = \mathrm{Id} = q^{(0,0)} + q^{(1,0)} + q^{(0,1)}$. The trivial SU(3) component contained in $\mathfrak{so}(7)$ lies in $V_{(1,0)}$ which is modelled as the space $\{v \lrcorner_7 \varphi_0 \,:\, v \in (\mathbb{R}^7)^*\}$. If we model $(\mathbb{R}^7)^*$ as $(\mathbb{R}^6)^* \oplus \langle \mathrm{d}t \rangle$, then the only SU(3) invariant unit vector in $(\mathbb{R}^7)^*$ is $\mathrm{d}t$, and $\mathrm{d}t \lrcorner_7 \varphi_0 = \omega$. This is identified with the complex structure J on the space $W_{(1,0)} \oplus W_{(0,1)}$ and thus $p^{(1,0)} = iq^{(1,0)} - iq^{(0,1)}$. Finally write $p^{(2,0)} = aq^{(0,0)} + bq^{(1,0)} + cq^{(0,1)}$. Since $p^{(2,0)}$ defines a traceless map we find $a+3b+3c = 0$ and since $p^{(2,0)}(\overline{v}) = \overline{p^{(2,0)}(v)}, p^{(1,0)}(\overline{v}) = \overline{p^{(1,0)}(v)}$ it must be that $a = \overline{a}, c = \overline{b}$. Self-adjointness of $p^{(2,0)}$ gives $a + 6b = 0$ and so we have calculated the change of basis relations:

$$\begin{aligned} p^{(0,0)} &= q^{(0,0)} + q^{(1,0)} + q^{(0,1)} \\ p^{(1,0)} &= iq^{(1,0)} - iq^{(0,1)} \\ p^{(2,0)} &= -6q^{(0,0)} + q^{(1,0)} + q^{(0,1)}. \end{aligned}$$

Using this together with the eigenvalues given in (8.12) and (8.13) we find that the matrix of D^{ρ}_{γ} in the basis $q^{(i,j)}$ is

$$\begin{pmatrix} 0 & 3 & 3 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It remains to understand the action of the volume form Vol and the 3-form $\mathrm{Re}\Omega$ on $\mathrm{Hom}(V_{(1,0)}, S)_{\mathrm{SU}(3)}$. Recall that the volume form satisfies $\mathrm{Vol}^2 = -\mathrm{Id}$ and that its left action on $\Lambda^1 \subset S$ is precisely the action of the almost complex structure J. It follows that, in the basis $q^{(i,j)}$, the volume form acts as the matrix

$$\begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

One therefore finds that

$$D^1_\gamma = D^\rho_\gamma - \mathrm{Vol}^{-1} D^\rho_\gamma \mathrm{Vol} = \begin{pmatrix} 0 & 3 & 3 & 0 \\ 1 & 0 & 2 & i \\ 1 & 2 & 0 & -i \\ 0 & -3i & -3i & 0 \end{pmatrix}.$$

From Lemma 2.3.3 the action of $\mathrm{Re}\Omega$ in this basis is given by the matrix $\mathrm{diag}(4, 0, 0, -4)$ and so D^0_γ is given by the matrix

$$D^0_\gamma = D^1_\gamma - \frac{3}{4}\mathrm{Re}\Omega = \begin{pmatrix} -3 & 3 & 3 & 0 \\ 1 & 0 & 2 & i \\ 1 & 2 & 0 & -i \\ 0 & -3i & -3i & 3 \end{pmatrix}$$

which has eigenvalues $\{\pm 3, \pm 4\}$.

Note in this second example, when V_γ had dimension 7, the eigenvalues that we calculated (which each had multiplicity 1 for D^0_γ) must have multiplicity at least 7 in the space of sections. Note that ± 3 is also an eigenvalue for the operator D^0_γ when $V_\gamma = \mathbb{C}$. We know from [6] that the multiplicity of the eigenvalues ± 3 in 8, therefore we have found all of the eigenspinors of this eigenspace.

8.4 The Twisted Dirac Operator

This discussion generalises easily to the case of twisted spin bundles. To study the standard G_2 instanton we would like to form a similar representation-theoretic formula for the twisted Dirac operator $D^0_{A_{\mathrm{can}}} : \Gamma(\not{S}_{\mathbb{C}}(\Sigma) \otimes (\mathfrak{g}_2)_{\mathbb{C}}) \to \Gamma(\not{S}_{\mathbb{C}}(\Sigma) \otimes (\mathfrak{g}_2)_{\mathbb{C}})$. Let us fix $(V, \rho_V) = (S \otimes (\mathfrak{g}_2)_{\mathbb{C}}, \rho_S \otimes \mathrm{Ad})$. As a representation of G_2 this twisted spinor space is

$$\begin{aligned} V &= (V_{(1,0)} \oplus V_{(0,0)}) \otimes V_{(0,1)} \\ &= V_{(1,1)} \oplus V_{(2,0)} \oplus V_{(0,1)} \oplus V_{(1,0)}. \end{aligned}$$

The adjoint representation splits $(\mathfrak{g}_2)_{\mathbb{C}} = W_{(1,1)} \oplus [[W_{(1,0)}]]$ as a representation of $\mathrm{SU}(3)$, so as a representation of $\mathrm{SU}(3)$ the twisted spinor space is

$$\begin{aligned} V &= ([[W_{(1,0)}] \oplus 2W_{(0,0)}) \otimes (W_{(1,1)} \oplus [[W_{(1,0)}]]) \\ &= [[W_{(2,1)}]] \oplus 2[[W_{(2,0)}]] \oplus 4W_{(1,1)} \oplus 4[[W_{(1,0)}]] \oplus 2W_{(0,0)}. \end{aligned}$$

In terms of the Lie algebra action the space of sections of the twisted spin bundle is

$$L^2(G_2, V)_{\mathrm{SU}(3)} = \{f \in L^2(G_2, V)\,;\, \rho_R(X)f + \rho_V(X)f = 0 \quad \text{for all } X \in \mathfrak{su}(3)\}. \tag{8.14}$$

Consider the operator $D^1_{A_{\mathrm{can}}}$ acting on sections of E, this takes the form

$$D^1_{A_{\mathrm{can}}} = \mathrm{cl} \circ \nabla^{1,A_{\mathrm{can}}}.$$

Recall the connection $\nabla^{1,A_{\mathrm{can}}}$ is simply the canonical connection acting on sections of $G \times_H E$, associated via the representation E. Thus $\nabla^{1,\mathrm{can}}_X f = \rho_R(X)f$ for $f \in L^2(G_2, E)_{\mathrm{SU}(3)}$ and $X \in C^\infty(G, \mathfrak{m})$. An analysis identical to the (untwisted) case of D^1 yields a representation theoretic formula for the twisted operator $D^1_{A_{\mathrm{can}}}$ via (5.6) and Corollary 8.2.3.

Proposition 8.4.1. *Let $D^\rho_{A_{can}} : L^2(G_2, E)_{\mathrm{SU}(3)} \to L^2(G_2, E)_{\mathrm{SU}(3)}$ be the operator*

$$D^\rho_{A_{\mathrm{can}}} = \frac{1}{2}\left(\rho_{S\otimes R}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_S(\mathrm{Cas}_{\mathfrak{m}}) - \rho_R(\mathrm{Cas}_{\mathfrak{m}})\right).$$

Under the association $\Gamma(\$(\Sigma) \otimes \mathrm{Ad}P) \cong L^2(G_2, E)_{\mathrm{SU}(3)}$ the twisted Levi-Civita Dirac operator $D^t_{A_{can}}$ is

$$D^t_{A_{\mathrm{can}}} = D^\rho_{A_{\mathrm{can}}} - \mathrm{Vol}^{-1} D^\rho_{A_{\mathrm{can}}} \mathrm{Vol} + \frac{3(t-1)}{4}\mathrm{Re}\Omega. \tag{8.15}$$

As in the case of the ordinary spin bundle this can be easily transferred to the homomorphism space decomposition of the space of section. For an irreducible representation V_γ of G_2 we define an endomorphism $(D^\rho_{A_{\mathrm{can}}})_\gamma$ of $\mathrm{Hom}(V_\gamma, S \otimes (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$ via

$$(D^\rho_{A_{\mathrm{can}}})_\gamma = \frac{1}{2}\left(\rho_{S\otimes V^*_\gamma}(\mathrm{Cas}_{\mathfrak{m}}) - \rho_S(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V^*_\gamma}(\mathrm{Cas}_{\mathfrak{m}})\right) \tag{8.16}$$

and it follows that the operators $(D^t_{A_{\mathrm{can}}})_\gamma$ given in (5.9) take the form

$$(D^t_{A_{\mathrm{can}}})_\gamma = (D^\rho_{A_{\mathrm{can}}})_\gamma - \mathrm{Vol}^{-1}(D^\rho_{A_{\mathrm{can}}})_\gamma \mathrm{Vol} + \frac{3(t-1)}{4}\mathrm{Re}\Omega. \tag{8.17}$$

8.5 Application of The Lichnerowicz Formula

Recall that the standard G_2 instanton A_{std} has rate -2 and so we shall consider a family of moduli spaces $\mathcal{M}(A_{\mathrm{can}}, \mu)$ for $\mu \in (-2, 0)$. We have seen that the expected dimension of these moduli spaces varies as we pass through values λ_k which are in $\mathrm{Spec}(D^0_{A_{\mathrm{can}}}) \cap [0, 2)$ so we are lead to study the spectrum of this twisted Dirac operator in this interval. Let us first determine the eigenvalues of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2$ using Corollary 5.3.2, as this will again provide a useful consistency check. We can also use Theorem 5.3.5 to show that most representation give rise to eigenvalues that do not fall into the interval $[0, 2)$ that we are interested in.

Let V_γ be an irreducible representation of G_2, then the eigenvalues and multiplicities of the operator $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ are

Eigenvalue	Multiplicity
$-c_\gamma^{\mathfrak{g}_2}$	$\dim \mathrm{Hom}(V_\gamma, S \otimes (W_{(1,0)} \oplus W_{(0,1)}))_{\mathrm{SU}(3)}$
$-c_\gamma^{\mathfrak{g}_2} - 5$	$\dim \mathrm{Hom}(V_\gamma, S \otimes W_{(1,1)})_{\mathrm{SU}(3)}$

where $c_\gamma^{\mathfrak{g}_2}$ is the eigenvalue of the Casimir operator on the irreducible representation V_γ with respect to the inner product B from (2.15).

To see this we simply apply Corollary 5.3.2 to the case at hand. We have $(\mathfrak{g}_2)_{\mathbb{C}} = W_{(1,0)} \oplus W_{(0,1)} \oplus W_{(1,1)}$ and this yields the splitting

$$\mathrm{Hom}(V_{(i,j)}, S\otimes(\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)} = \mathrm{Hom}(V_{(i,j)}, S\otimes(W_{(1,0)}\oplus W_{(0,1)}))_{\mathrm{SU}(3)}\oplus\mathrm{Hom}(V_{(i,j)}, S\otimes W_{(1,1)})_{\mathrm{SU}(3)}.$$

The operator $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ respects this splitting and inspection of (5.10), together with the fact that $c^{\mathfrak{su}(3)}_{(i,j)} = c^{\mathfrak{su}(3)}_{(j,i)}$, reveals that

$$(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma|_{\mathrm{Hom}(V_\gamma,S\otimes(W_{(1,0)}\oplus W_{(0,1)}))_{\mathrm{SU}(3)}} = -c_\gamma^{\mathfrak{g}_2} + c^{\mathfrak{su}(3)}_{(1,0)} + 4.$$

We know from (8.13) that $c^{\mathfrak{su}(3)}_{(1,0)} = -4$, so the Dirac operator acts as the constant $-c_\gamma^{\mathfrak{g}_2}$ on this space equivariant maps. A similar observation shows that $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ acts as the constant $-c_\gamma^{\mathfrak{g}_2} - 5$ on $\mathrm{Hom}(V_\gamma, S \otimes W_{(1,1)})_{\mathrm{SU}(3)}$.

Lemma 8.5.1. *If V_γ is not one of the following irreducible representations of G_2, then the operator $(D^0_{A_{can}})_\gamma$, acting on $\mathrm{Hom}(V_\gamma, S \otimes (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$, has no eigenvalues in the range $[0, 2)$:*

1. *$V_{(0,0)}$, the trivial representation*
2. *$V_{(1,0)}$, the standard representation*
3. *$V_{(0,1)}$, the adjoint representation.*

Proof. By Corollary 5.3.2 the smallest possible eigenvalue of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$ is $-c_\gamma^{\mathfrak{g}_2} - 5$ and it follows that the smallest possible non-negative eigenvalue of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})_\gamma$ is $(-c_\gamma^{\mathfrak{g}_2} - 5)^{\frac{1}{2}}$. Applying Theorem 5.3.5 to the case when $V_\gamma = V_{(2,0)}$ one finds that the lower one the smallest positive eigenvalue one obtains is $L_\gamma = \sqrt{14-5} - 1 = 2$ so this bound is sufficient for the statement of the lemma. When V_γ is an irreducible representation of higher dimension one obtains a greater lower bound and the only irreducible representations of lower dimension are those given in the statement of the lemma. □

For the three representations in the above list we now compute the matrix of the Dirac operator and the set of eigenvalues explicitly. We briefly outline a few conventions used throughout the calculations:

Firstly $\mathrm{Hom}(V_\gamma, S \otimes (\mathfrak{g}_2)_\mathbb{C})_{\mathrm{SU}(3)} \cong \mathrm{Hom}(S \otimes V_\gamma, (\mathfrak{g}_2)_\mathbb{C})_{\mathrm{SU}(3)}$ and it will prove more convenient to view the homomorphism space from the latter viewpoint when constructing basis vectors. Furthermore since $S = V_{(1,0)} \oplus V_{(0,0)}$ as a representation of G_2 it will prove convenient to decompose

$$\mathrm{Hom}(S \otimes V_\gamma, (\mathfrak{g}_2)_\mathbb{C})_{\mathrm{SU}(3)} \cong \mathrm{Hom}(V_{(1,0)} \otimes V_\gamma, (\mathfrak{g}_2)_\mathbb{C})_{\mathrm{SU}(3)} \oplus \mathrm{Hom}(V_{(0,0)} \otimes V_\gamma, (\mathfrak{g}_2)_\mathbb{C})_{\mathrm{SU}(3)}$$

as G_2 modules. Recall we have the following models for irreducible representations of G_2 :

- $V_{(1,0)} = \mathbb{C}^7$
- $V_{(0,1)} = (\mathfrak{g}_2)_\mathbb{C} = \{\alpha \in \mathbb{C} \otimes \Lambda^2(\mathbb{R}^7)^* \,;\, \alpha \wedge \psi_0 = 0\}$
- $V_{(2,0)} = \{\eta \in \mathbb{C} \otimes \Lambda^3(\mathbb{R}^7)^* \,;\, \eta \wedge \varphi_0 = \eta \wedge \psi_0 = 0\}$.

For irreducible representations of SU(3) we model

- $W_{(1,0)} = \Lambda^{(1,0)}(\mathbb{R}^6)^*$
- $W_{(0,1)} = \Lambda^{(0,1)}(\mathbb{R}^6)^*$
- $W_{(1,1)} = \mathfrak{su}(3) = \{\alpha \in \Lambda^2(\mathbb{C}^6)^* \,;\, *(\alpha \wedge \omega) = 0\}$

where ω is the standard Kähler form on $\mathbb{R}^6$. Note that $W_{(1,1)}$ embeds into $V_{(0,1)}$ by inclusion and the embedding of $W_{(1,0)}$ into $V_{(0,1)}$ is given by the restriction of the embedding F of $(\mathbb{R}^6)^* \otimes \mathbb{C}$ into $(\mathfrak{g}_2)_\mathbb{C}$

$$F(v) = v \wedge \mathrm{d}t + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega.$$

8.6 Eigenvalues from the Trivial Representation

Let $V_\gamma = V_{(0,0)}$ be the trivial representation of G_2. Then Schur's lemma tells us that $\mathrm{Hom}(\mathbb{C}, S \otimes (\mathfrak{g}_2)_\mathbb{C})_{\mathrm{SU}(3)} \cong \mathrm{Hom}(S, (\mathfrak{g}_2)_\mathbb{C})_{\mathrm{SU}(3)}$ is two-dimensional. A basis for this space is given by the maps that factor through projections

$$q^{(1,0)}_{(1,0)} \colon S \to W_{(1,0)} \to (\mathfrak{g}_2)_\mathbb{C}$$
$$q^{(0,1)}_{(0,1)} \colon S \to W_{(0,1)} \to (\mathfrak{g}_2)_\mathbb{C}.$$

When V_γ is the trivial representation the operators $\big(\rho_{S \otimes V_\gamma^*}(\mathrm{Cas}_\mathfrak{m}) - \rho_S(\mathrm{Cas}_\mathfrak{m})\big)$ and $\rho_{V_\gamma^*}(\mathrm{Cas}_\mathfrak{m})$ both vanish. We know that $(D^\rho_{A_{\mathrm{can}}})_\gamma$ is built from precisely these operators and hence vanishes. Note also that Lemma 2.3.3 ensures that $\mathrm{Re}\Omega$ acts as 0 on this space since $q^{(1,0)}_{(1,0)}$ and $q^{(0,1)}_{(0,1)}$ factor through $\Lambda^1 \subset S$. Overall then $(D^0_{A_{\mathrm{can}}})_\gamma$ vanishes identically on this space.

Proposition 8.6.1. *Let $V_\gamma = V_{(0,0)}$, then the unique eigenvalue of $(D^0_{A_{can}})_\gamma$ is 0 and has multiplicity 2.*

8.7 Eigenvalues from the Standard Representation

Now let $V_\gamma = V_{(1,0)}$ be the standard representation of G_2. The space $\mathrm{Hom}(S\otimes V_{(1,0)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$ is ten dimensional. It is convenient to split this space as

$$\mathrm{Hom}(S \otimes V_{(1,0)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)} \cong \mathrm{Hom}(V_{(1,0)} \otimes V_{(1,0)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)} \oplus \mathrm{Hom}(\mathbb{C} \otimes V_{(1,0)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$$

where $V_{(1,0)}$ is modelled as $\Lambda^0 \oplus \Lambda^1 \subset S$. The matrix of $(D_{A_{\mathrm{can}}})_\gamma$ will be block diagonal with respect to this splitting, and the part acting on $\mathrm{Hom}(\mathbb{C} \otimes V_{(1,0)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$ is 0 by the previous calculation.

It remains to calculate $(D_{A_{\mathrm{can}}})_\gamma$ on $\mathrm{Hom}(V_{(1,0)} \otimes V_{(1,0)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$. Notice that

$$\rho_{S\otimes V^*_{(1,0)}}(\mathrm{Cas}_{\mathfrak{su}(3)}) = \rho_{(\mathfrak{g}_2)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{su}(3)}) \qquad \text{on } \mathrm{Hom}(S \otimes V_{(1,0)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$$

so we can write the operator $(D^\rho_{A_{\mathrm{can}}})_\gamma$ as

$$(D^\rho_{A_{\mathrm{can}}})_\gamma = \frac{1}{2}(\rho_{S\otimes V^*_{(1,0)}}(\mathrm{Cas}_{\mathfrak{g}_2}) - \rho_{(\mathfrak{g}_2)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{su}(3)}) - \rho_S(\mathrm{Cas}_{\mathfrak{m}}) - \rho_{V^*_{(1,0)}}(\mathrm{Cas}_{\mathfrak{m}})). \tag{8.18}$$

Our strategy for calculating the eigenvalues of this operator is:

- Find a basis diagonalising $\rho_{(\mathfrak{g}_2)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{su}(3)}) + \rho_S(\mathrm{Cas}_{\mathfrak{m}}) + \rho_{V^*_{(1,0)}}(\mathrm{Cas}_{\mathfrak{m}})$
- Find a basis diagonalising $\rho_{S\otimes V^*_{(1,0)}}$
- Calculate the change of basis matrix and hence find the matrix of $(D^0_{A_{\mathrm{can}}})_\gamma$.
- Calculate the eigenvalues of $(D^0_{A_{\mathrm{can}}})_\gamma$.

To begin this task we construct a basis of $\mathrm{Hom}(V_{(1,0)} \otimes V_{(1,0)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$ by finding non-zero SU(3)-equivariant maps

$$q^{(i,j)(k,l)}_{(m,n)} : V_{(1,0)} \otimes V_{(1,0)} \to W_{(i,j)} \otimes W_{(k,l)} \to W_{(m,n)} \to (\mathfrak{g}_2)_{\mathbb{C}}$$

and observe that on this space:

- $q^{(i,j)(k,l)}_{(m,n)}$ are eigenvectors of $\rho_S(\mathrm{Cas}_{\mathfrak{m}})$ with eigenvalue $c^{\mathfrak{g}_2}_{(1,0)} - c^{\mathfrak{su}(3)}_{(i,j)}$
- $q^{(i,j)(k,l)}_{(m,n)}$ are eigenvectors of $\rho_{V^*_{(1,0)}}(\mathrm{Cas}_{\mathfrak{m}})$ with eigenvalue $c^{\mathfrak{g}_2}_{(1,0)} - c^{\mathfrak{su}(3)}_{(k,l)}$
- $q^{(i,j)(k,l)}_{(m,n)}$ are eigenvectors of $\rho_{(\mathfrak{g}_2)_{\mathbb{C}}}(\mathrm{Cas}_{\mathfrak{su}(3)})$ with eigenvalue $c^{\mathfrak{su}(3)}_{(m,n)}$.

These maps are constructed from the following projection maps:

- $V_{(1,0)} \to W_{(0,0)}, (u + adt) \mapsto a\,\mathrm{d}t$
- $V_{(1,0)} \to W_{(1,0)}, (u + adt) \mapsto \frac{1}{2}(1 + iJ)u$
- $V_{(1,0)} \to W_{(0,1)}, (u + adt) \mapsto \frac{1}{2}(1 - iJ)u$

- $\Lambda^{1,1}(\mathbb{R}^6) \to W_{(1,1)}, \alpha \mapsto \alpha - \frac{1}{3}\langle \alpha, \omega \rangle \omega.$

The basis of $\mathrm{Hom}(V_{(1,0)} \otimes V_{(1,0)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$ that we get is

Map	Factorisation and Formula
$q_1 = q^{(0,0)(1,0)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(1,0)} \to W_{(0,0)} \otimes W_{(1,0)} \to W_{(1,0)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto adt \otimes \frac{1}{2}(1 + iJ)v \mapsto \frac{1}{2}a(1 + iJ)v \mapsto F(\frac{1}{2}a(1 + iJ)v)$
$q_2 = q^{(0,0)(0,1)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(1,0)} \to W_{(0,0)} \otimes W_{(0,1)} \to W_{(0,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto adt \otimes \frac{1}{2}(1 - iJ)v \mapsto \frac{1}{2}a(1 - iJ)v \mapsto F(\frac{1}{2}a(1 - iJ)v)$
$q_3 = q^{(1,0)(0,0)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(1,0)} \to W_{(1,0)} \otimes W_{(0,0)} \to W_{(1,0)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto \frac{1}{2}(1 + iJ)u \otimes bdt \mapsto \frac{1}{2}b(1 + iJ)u \mapsto F(\frac{1}{2}b(1 + iJ)u)$
$q_4 = q^{(0,1)(0,0)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(1,0)} \to W_{(0,1)} \otimes W_{(0,0)} \to W_{(0,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto \frac{1}{2}(1 - iJ)u \otimes bdt \mapsto \frac{1}{2}b(1 - iJ)u \mapsto F(\frac{1}{2}b(1 - iJ)u)$
$q_5 = q^{(1,0)(1,0)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(1,0)} \to W_{(1,0)} \otimes W_{(1,0)} \to W_{(0,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u+adt)\otimes(v+bdt) \mapsto \frac{1}{2}(1+iJ)u\otimes\frac{1}{2}(1+iJ)v \mapsto \frac{1}{4}[((1+iJ)u)\wedge((1+iJ)v)]\lrcorner\overline{\Omega} \mapsto$ $F(\frac{1}{4}[((1 + iJ)u) \wedge ((1 + iJ)v)]\lrcorner\overline{\Omega})$
$q_6 = q^{(0,1)(0,1)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(1,0)} \to W_{(0,1)} \otimes W_{(0,1)} \to W_{(1,0)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u+adt)\otimes(v+bdt) \mapsto \frac{1}{2}(1-iJ)u\otimes\frac{1}{2}(1-iJ)v \mapsto \frac{1}{4}[((1-iJ)u)\wedge((1-iJ)v)]\lrcorner\Omega \mapsto$ $F(\frac{1}{4}[((1 - iJ)u) \wedge ((1 - iJ)v)]\lrcorner\Omega)$
$q_7 = q^{(1,0)(0,1)}_{(1,1)}$	$V_{(1,0)} \otimes V_{(1,0)} \to W_{(1,0)} \otimes W_{(0,1)} \to W_{(1,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto \frac{1}{2}(1 + iJ)u \otimes \frac{1}{2}(1 - iJ)v \to \frac{1}{4}[((1 + iJ)u) \wedge ((1 - iJ)v) -$ $\frac{1}{3}\langle((1 + iJ)u) \wedge ((1 - iJ)v), \omega\rangle\omega]$
$q_8 = q^{(0,1)(1,0)}_{(1,1)}$	$V_{(1,0)} \otimes V_{(1,0)} \to W_{(0,1)} \otimes W_{(1,0)} \to W_{(1,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto \frac{1}{2}(1 - iJ)u \otimes \frac{1}{2}(1 + iJ)v \to \frac{1}{4}[((1 - iJ)u) \wedge ((1 + iJ)v) -$ $\frac{1}{3}\langle((1 - iJ)u) \wedge ((1 + iJ)v), \omega\rangle\omega]$

Furthermore we can extend this to a basis of $\mathrm{Hom}(S \otimes V_{(1,0)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$ by adding the maps $q_9 = \mathrm{Vol} \cdot q^{(0,0)(1,0)}_{(1,0)}$ and $q_{10} = \mathrm{Vol} \cdot q^{(0,0)(0,1)}_{(0,1)}$.

The only term of the decomposition (8.18) of $(D^{\rho}_{A_{\mathrm{can}}})_{\gamma}$ that this basis does not diagonalise is $\rho_{S\otimes V^*_{(1,0)}}(\mathrm{Cas}_{\mathfrak{g}_2})$. We choose a basis diagonalising this operator by considering projections through the splitting of $V_{(1,0)} \otimes V_{(1,0)}$ into irreducible representations of G_2:

$$p^{(i,j)}_{(k,l)} \colon V_{(1,0)} \otimes V_{(1,0)} \to V_{(i,j)} \to W_{(k,l)} \to \mathfrak{g}_2,$$

these maps are eigenvectors of $\rho_{S\otimes V^*_{(1,0)}}(\mathrm{Cas}_{\mathfrak{g}_2})$ with eigenvalue $c^{\mathfrak{g}_2}_{(i,j)}$. To relate the two bases it is necessary to understand each of the projection maps involved in the above construction, then by composition we will be able to understand how they are on an element of $V_{(1,0)} \otimes V_{(1,0)}$.

Recall $V_{(1,0)}$ is the $+2$ eigenspace and $V_{(0,1)}$ the -1 eigenspace of the operator $*_7(\varphi_0 \wedge \cdot)$ on $\Lambda^2(\mathbb{R}^7)^*$. We have equivariant maps:

- $V_{(1,0)} \otimes V_{(1,0)} \to V_{(1,0)},\ u \otimes v \mapsto (u \wedge v)\lrcorner\phi$
- $V_{(1,0)} \otimes V_{(1,0)} \to V_{(0,1)},\ u \otimes v \mapsto \frac{2}{3}(u \wedge v) - \frac{1}{3} *_7 (\phi \wedge u \wedge v)$

- $V_{(1,0)} \otimes V_{(1,0)} \to V_{(2,0)},\ u \otimes v \mapsto *_7((u \lrcorner \phi) \wedge (v \lrcorner \phi)) - \frac{6}{7}\langle u, v\rangle \phi.$

The last map arises as follows: The space $\Lambda^3(V)^*$ decomposes as $\Lambda^3(V^*) \cong \Lambda^3_{27} \oplus \Lambda^3_7 \oplus \langle\varphi\rangle$, where $\Lambda^3_7 = \{v \lrcorner \psi \colon v \in \Lambda^1(M)$ and $\Lambda^3_{27} = V_{(2,0)}$. It is well known that $\Lambda^3_{27} \cong \mathrm{Sym}^2_0(V)$, see [80] for example. Let $u, v \in V$, the map $u \otimes v \mapsto *_7((u \lrcorner \varphi) \wedge (v \lrcorner \varphi))$ is symmetric in u and v, so its image lies in $\Lambda^3_{27} \oplus \langle\varphi\rangle$. For the correct value μ therefore, $*_7((u \lrcorner \varphi) \wedge (v \lrcorner \varphi)) - \mu\langle u, v\rangle\varphi \in \Lambda^3_{27}$. The condition to be checked is $[*_7((u \lrcorner \varphi) \wedge (v \lrcorner \varphi)) - \mu\langle u, v\rangle\varphi] \wedge \psi = 0$. Using that $\|\varphi\|^2 = 7$ and $*_7((u \lrcorner \varphi) \wedge (v \lrcorner \varphi)) \wedge \psi = (u \lrcorner \varphi) \wedge (v \lrcorner \varphi) \wedge \varphi = 6\langle u, v\rangle \mathrm{Vol}_7$ one finds that $\mu = \frac{6}{7}$. We also use the following projections

- $\Lambda^2(\mathbb{R}^7)^* \to W_{(1,0)},\ v \wedge dt + \alpha \mapsto \frac{1}{2}(1 + iJ)v$
- $\Lambda^2(\mathbb{R}^7)^* \to W_{(0,1)},\ v \wedge dt + \alpha \mapsto \frac{1}{2}(1 - iJ)v$
- $V_{(0,1)} \to W_{(1,1)},\ v \wedge dt + \alpha \mapsto \frac{1}{2}\alpha - \frac{1}{2} *_6 (\omega \wedge \alpha)$
- $V_{(2,0)} \to W_{(1,0)},\ \alpha \wedge dt + \beta \mapsto \frac{1}{2}(1 + iJ)\,[(*_6\beta) \llcorner \omega]$
- $V_{(2,0)} \to W_{(0,1)},\ \alpha \wedge dt + \beta \mapsto \frac{1}{2}(1 - iJ)\,[(*_6\beta) \llcorner \omega]$
- $V_{(2,0)} \to W_{(1,1)},\ \alpha \wedge dt + \beta \mapsto \frac{1}{3}\alpha - \frac{1}{2} *_6 (\omega \wedge \alpha) + \frac{1}{6} *_6 (\omega \wedge *_6(\omega \wedge \alpha)).$

The overall maps are then

Map	Factorisation and Formula
$p_1 = p^{(1,0)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(1,0)} \to V_{(1,0)} \to W_{(1,0)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto F\left(\frac{1}{2}(1 + iJ)\,[(u \wedge v) \lrcorner \mathrm{Im}\Omega + bJu - aJv]\right)$
$p_2 = p^{(1,0)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(1,0)} \to V_{(1,0)} \to W_{(0,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto F\left(\frac{1}{2}(1 - iJ)\,[(u \wedge v) \lrcorner \mathrm{Im}\Omega + bJu - aJv]\right)$
$p_3 = p^{(0,1)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(1,0)} \to V_{(0,1)} \to W_{(1,0)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto F\left(\frac{1}{2}(1 + iJ)\left[\frac{2}{3}(bu - av) - \frac{1}{3}\,[(u \wedge v) \lrcorner \mathrm{Re}\Omega]\right]\right)$
$p_4 = p^{(0,1)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(1,0)} \to V_{(0,1)} \to W_{(0,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto F\left(\frac{1}{2}(1 - iJ)\left[\frac{2}{3}(bu - av) - \frac{1}{3}\,[(u \wedge v) \lrcorner \mathrm{Re}\Omega]\right]\right)$
$p_5 = p^{(0,1)}_{(1,1)}$	$V_{(1,0)} \otimes V_{(1,0)} \to V_{(0,1)} \to W_{(1,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto \frac{1}{3}u \wedge v - \frac{1}{2} *_6 (\omega \wedge u \wedge v) + \frac{1}{6} *_6 (\omega \wedge *_6(\omega \wedge u \wedge v))$
$p_6 = p^{(2,0)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(1,0)} \to V_{(2,0)} \to W_{(1,0)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto F\left(\frac{1}{2}(1 + iJ)\,[Ju \wedge b\omega + a\omega \wedge Jv] \llcorner \omega\right)$
$p_7 = p^{(2,0)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(1,0)} \to V_{(2,0)} \to W_{(0,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto F\left(\frac{1}{2}(1 - iJ)\,[Ju \wedge b\omega + a\omega \wedge Jv] \llcorner \omega\right)$
$p_8 = p^{(2,0)}_{(1,1)}$	$V_{(1,0)} \otimes V_{(1,0)} \to V_{(2,0)} \to W_{(1,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (v + bdt) \mapsto *_6((u \lrcorner \mathrm{Im}\Omega) \wedge (v \lrcorner \mathrm{Im}\Omega)) - \frac{1}{3}\langle *_6(u \lrcorner \mathrm{Im}\Omega) \wedge (v \lrcorner \mathrm{Im}\Omega), \omega\rangle\omega$

Each map $p^{(i,j)}_{(k,l)}$ is an eigenvector of $\rho_{S \otimes V^*_{(1,0)}}(\mathrm{Cas}_{\mathfrak{g}_2})$ with eigenvalue $c^{\mathfrak{g}_2}_{(i,j)}$.

Lemma 8.7.1. *Let $u, v \in \Lambda^1(\mathbb{R}^6)^*$. The following identities hold:*

$$i(1+iJ)\left[(u\wedge v)\lrcorner\mathrm{Im}\Omega\right] = \frac{1}{4}\left[((1-iJ)u)\wedge((1-iJ)v)\right]\lrcorner\Omega \tag{8.19}$$

$$-i(1-iJ)\left[(u\wedge v)\lrcorner\mathrm{Im}\Omega\right] = \frac{1}{4}\left[((1+iJ)u)\wedge((1+iJ)v)\right]\lrcorner\overline{\Omega} \tag{8.20}$$

$$(1+iJ)\left[(u\wedge v)\lrcorner\mathrm{Re}\Omega\right] = \frac{1}{4}\left[((1-iJ)u)\wedge((1-iJ)v)\right]\lrcorner\Omega \tag{8.21}$$

$$(1-iJ)\left[(u\wedge v)\lrcorner\ \mathrm{Re}\Omega\right] = \frac{1}{4}\left[((1+iJ)u)\wedge((1+iJ)v)\right]\lrcorner\overline{\Omega} \tag{8.22}$$

$$u\wedge Jv - Ju\wedge v = *_6((u\lrcorner\mathrm{Im}\Omega)\wedge(v\lrcorner\mathrm{Im}\Omega)) \tag{8.23}$$

$$(u\wedge\omega)\llcorner\omega = 2u. \tag{8.24}$$

Combining all of these facts one finds that the bases $p^{(i,j)}_{(k,l)}$ and $q^{(i,j)(k,l)}_{(m,n)}$ are related as follows:

$$\begin{aligned}
p_1 &= iq_1 - iq_3 - \frac{i}{2}q_6\\
p_2 &= -iq_2 + iq_4 + \frac{i}{2}q_5\\
p_3 &= -\frac{2}{3}q_1 + \frac{2}{3}q_3 - \frac{1}{6}q_6\\
p_4 &= -\frac{2}{3}q_2 + \frac{2}{3}q_4 - \frac{1}{6}q_5\\
p_5 &= \frac{1}{2}q_7 + \frac{1}{2}q_8\\
p_6 &= 2iq_1 + 2iq_3\\
p_7 &= -2iq_2 - 2iq_4\\
p_8 &= \frac{1}{2}iq_7 - \frac{1}{2}iq_8.
\end{aligned}$$

Recall the maps $p^{(i,j)}_{(k,l)}$ are eigenvectors of $\rho_{S\otimes V^*_{(0,1)}}(\mathrm{Cas}_{\mathfrak{g}_2})$ with eigenvalue $c^{\mathfrak{g}_2}_{(i,j)}$ so that in the basis $p_1,\cdots,p_8$ we have

$$\rho_{S\otimes V^*_{(0,1)}}(\mathrm{Cas}_{\mathfrak{g}_2}) = \mathrm{diag}(-6,-6,-12,-12,-12,-14,-14,-14).$$

We have seen that the maps $q^{(i,j)(k,l)}_{(m,n)}$ are eigenvectors of $-\rho_{(\mathfrak{g}_2)_\mathbb{C}}(\mathrm{Cas}_{\mathfrak{su}(3)}) - \rho_S(\mathrm{Cas}_\mathfrak{m}) - \rho_{V^*_\gamma}(\mathrm{Cas}_\mathfrak{m})$ with eigenvalue $c^{\mathfrak{su}(3)}_{(i,j)} + c^{\mathfrak{su}(3)}_{(k,l)} - c^{\mathfrak{su}(3)}_{(m,n)} - 2c^{\mathfrak{g}_2}_{(1,0)}$. In the basis $q_1,\dots q_8$ we find

$$-\rho_{\mathfrak{g}_2}(\mathrm{Cas}_{\mathfrak{su}(3)}) - \rho_S(\mathrm{Cas}_\mathfrak{m}) - \rho_{V^*_\gamma}(\mathrm{Cas}_\mathfrak{m}) = \mathrm{diag}(12,12,12,12,8,8,13,13)$$

and (8.18) says that $(D^\rho_{A_{\mathrm{can}}})_\gamma$ is the sum of these two operators.

Furthermore $\mathrm{Re}\Omega$ acts in the basis $q_1,\dots,q_{10}$ as the matrix $\mathrm{diag}(4,4,0,0,0,0,0,0,-4,-4)$. Since we know the action of each term in (8.18) in the different bases and the change of basis matrix between the bases we can calculate the matrix of the twisted Levi-Civita

Dirac operator in the basis $q, \ldots, q_{10}$ to be

$$(D^t_{A_{\text{can}}})_\gamma = \begin{pmatrix} 3(t-1) & 0 & -1 & 0 & 0 & -4 & 0 & 0 & 0 & 0 \\ 0 & 3(t-1) & 0 & -1 & -4 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 8 & 0 & 0 & -i & 0 \\ 0 & -1 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & i \\ 0 & -\frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & -\frac{i}{4} \\ -\frac{1}{4} & 0 & \frac{1}{2} & 0 & 0 & 0 & 0 & 0 & \frac{i}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 & -4i & 0 & 0 & -3(t-1) & 0 \\ 0 & 0 & 0 & -i & 4i & 0 & 0 & 0 & 0 & -3(t-1) \end{pmatrix}. \tag{8.25}$$

A consistency check is obtained by observing that Corollary 5.3.2 ensures the basis $q_1, \cdots q_{10}$ diagonalises $(D^{\frac{1}{3}}_{A_{\text{can}}})^2_\gamma$. We find

$$(D^{\frac{1}{3}}_{A_{\text{can}}})^2_\gamma = \text{diag}(6,6,6,6,6,6,1,1,6,6)$$

as expected. By calculating the eigenvalues of (8.25) we obtain the following proposition:

Proposition 8.7.2. *Let $V_\gamma = V_{(1,0)}$, the eigenvalues of $(D^0_{A_{can}})_\gamma$ are symmetric about 0, the $\pm\lambda$ eigenspaces are isomorphic and the non-negative eigenvalues and multiplicities are:*

Eigenvalue λ	*Multiplicity*
1	1
$-\frac{1}{2} + \frac{\sqrt{33}}{2}$	2
$\frac{1}{2} + \frac{\sqrt{33}}{2}$	2

8.8 Eigenvalues from the Adjoint Representation

We now consider the case when $V_\gamma = V_{(0,1)}$ is the adjoint representation. We work on the space $\text{Hom}(S \otimes V_{(0,1)}, (\mathfrak{g}_2)_\mathbb{C})_{\text{SU}(3)}$. We have $V_{(1,0)} \otimes V_{(0,1)} = V_{(1,1)} \oplus V_{(2,0)} \oplus V_{(1,0)}$ and the summands split as reps of SU(3) as follows:

$$\begin{aligned} V_{(1,1)} &= [[W_{(2,1)}]] \oplus [[W_{(2,0)}]] \oplus 2W_{(1,1)} \oplus [[W_{(1,0)}]] \\ V_{(2,0)} &= [[W_{(2,0)}]] \oplus W_{(1,1)} \oplus [[W_{(1,0)}]] \oplus W_{(0,0)} \\ V_{(1,0)} &= [[W_{(1,0)}]] \oplus W_{(0,0)}. \end{aligned}$$

We therefore see that $\text{Hom}(S \otimes V_{(0,1)}, (\mathfrak{g}_2)_\mathbb{C})_{\text{SU}(3)}$ has dimension 12. As before we split $\text{Hom}(S \otimes V_{(0,1)}, (\mathfrak{g}_2)_\mathbb{C})_{\text{SU}(3)} \cong \text{Hom}(V_{(1,0)} \otimes V_{(0,1)}, (\mathfrak{g}_2)_\mathbb{C})_{\text{SU}(3)} \oplus \text{Hom}(\mathbb{C} \otimes V_{(0,1)}, \mathfrak{g}_2)_{\text{SU}(3)}$ so that the operator $(D^\rho_{A_{\text{can}}})_\gamma$ is block diagonal and its action on $\mathrm{Hom}(\mathbb{C} \otimes V_{(0,1)}, (\mathfrak{g}_2)_\mathbb{C})_{\mathrm{SU}(3)}$ is 0.

To calculate the action of the operator on $\mathrm{Hom}(V_{(1,0)} \otimes V_{(0,1)}, (\mathfrak{g}_2)_\mathbb{C})_{\mathrm{SU}(3)}$ we again pick two bases for this space which diagonalise the various Casimir operators from which (8.18) tells us $(D^\rho_{A_{\text{can}}})_\gamma$ is constructed. As before we first pick a basis consisting of maps that factor as follows:

$$q^{(i,j)(k,l)}_{(m,n)} \colon V_{(1,0)} \otimes V_{(0,1)} \to W_{(i,j)} \otimes W_{(k,l)} \to W_{(m,n)} \to (\mathfrak{g}_2)_\mathbb{C}.$$

This time these maps are eigenvectors of the operator $-\rho_{\mathfrak{g}_2}(\mathrm{Cas}_{\mathfrak{su}(3)}) - \rho_S(\mathrm{Cas}_\mathfrak{m}) - \rho_{V^*_\gamma}(\mathrm{Cas}_\mathfrak{m})$ with eigenvalue

$$c^{\mathfrak{su}(3)}_{(i,j)} + c^{\mathfrak{su}(3)}_{(k,l)} - c^{\mathfrak{su}(3)}_{(m,n)} - c^{\mathfrak{g}_2}_{(1,0)} - c^{\mathfrak{g}_2}_{(0,1)}.$$

These maps are constructed from the projection maps from V to its subspaces as before and, writing an element of $V_{(0,1)}$ as $\alpha + F(v)$ for $\alpha \in \Lambda^2_8$ and $v \in \Lambda^1$, the maps

- $V_{(0,1)} \to W_{(1,0)}, \alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega \mapsto \frac{1}{2}(1 + iJ)v$
- $V_{(0,1)} \to W_{(0,1)}, \alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega \mapsto \frac{1}{2}(1 - iJ)v$
- $V_{(0,1)} \to W_{(1,1)}, \alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega \mapsto \alpha.$

Map	Factorisation and Formula
$q_1 = q^{(0,0)(1,0)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(0,1)} \to W_{(0,0)} \otimes W_{(1,0)} \to W_{(1,0)} \to (\mathfrak{g}_2)_\mathbb{C}$ $(u + a dt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto F\left(a\frac{1}{2}(1 + iJ)v\right)$
$q_2 = q^{(0,0)(0,1)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to W_{(0,0)} \otimes W_{(0,1)} \to W_{(0,1)} \to (\mathfrak{g}_2)_\mathbb{C}$ $(u + a dt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto F\left(a\frac{1}{2}(1 - iJ)v\right)$
$q_3 = q^{(0,0)(1,1)}_{(1,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to W_{(0,0)} \otimes W_{(1,1)} \to W_{(1,1)} \to (\mathfrak{g}_2)_\mathbb{C}$ $(u + a dt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto a\alpha$
$q_4 = q^{(1,0)(1,1)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(0,1)} \to W_{(1,0)} \otimes W_{(1,1)} \to W_{(1,0)} \to (\mathfrak{g}_2)_\mathbb{C}$ $(u + a dt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto F\left((\frac{1}{2}(1 + iJ)u) \lrcorner \alpha\right)$
$q_5 = q^{(0,1)(1,1)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to W_{(0,1)} \otimes W_{(1,1)} \to W_{(0,1)} \to (\mathfrak{g}_2)_\mathbb{C}$ $(u + a dt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto F\left((\frac{1}{2}(1 - iJ)u) \lrcorner \alpha\right)$
$q_6 = q^{(1,0)(0,1)}_{(1,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to W_{(1,0)} \otimes W_{(0,1)} \to W_{(1,1)} \to (\mathfrak{g}_2)_\mathbb{C}$ $(u + a dt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto \frac{1}{4}[((1 + iJ)u) \wedge ((1 - iJ)v) - \frac{1}{3}\langle((1 + iJ)u) \wedge ((1 - iJ)v), \omega\rangle\omega]$
$q_7 = q^{(0,1)(0,1)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(0,1)} \to W_{(0,1)} \otimes W_{(0,1)} \to W_{(1,0)} \to (\mathfrak{g}_2)_\mathbb{C}$ $(u + a dt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto F\left(\frac{1}{4}[((1 - iJ)u) \wedge ((1 - iJ)v)] \lrcorner \Omega\right)$
$q_8 = q^{(1,0)(1,0)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to W_{(1,0)} \otimes W_{(1,0)} \to W_{(0,1)} \to (\mathfrak{g}_2)_\mathbb{C}$ $(u + a dt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto F\left(\frac{1}{4}[((1 + iJ)u) \wedge ((1 + iJ)v)] \lrcorner \overline{\Omega}\right)$
$q_9 = q^{(0,1)(1,0)}_{(1,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to W_{(0,1)} \otimes W_{(1,0)} \to W_{(1,1)} \to (\mathfrak{g}_2)_\mathbb{C}$ $(u + a dt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto \frac{1}{4}[((1 - iJ)u) \wedge ((1 + iJ)v) - \frac{1}{3}\langle((1 - iJ)u) \wedge ((1 + iJ)v), \omega\rangle\omega].$

We extend this set to a basis of $\mathrm{Hom}(S \otimes V_{(0,1)}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$ by adding the three maps

$$\begin{aligned} q_{10} &= \mathrm{Vol} \cdot q^{(0,0)(1,0)}_{(1,0)} \\ q_{11} &= \mathrm{Vol} \cdot q^{(0,0)(0,1)}_{(0,1)} \\ q_{12} &= \mathrm{Vol} \cdot q^{(0,0)(1,1)}_{(1,1)}. \end{aligned}$$

To diagonalise the operator $\rho_{S\otimes V^*_\gamma}(\mathrm{Cas}_{\mathfrak{g}_2})$ we choose maps that factor as follows:

$$p^{(i,j)}_{(k,l)} : V_{(1,0)} \otimes V_{(0,1)} \to V_{(i,j)} \to W_{(k,l)} \to (\mathfrak{g}_2)_{\mathbb{C}}.$$

These maps are eigenvectors of $\rho_{S\otimes V^*_\gamma}(\mathrm{Cas}_{\mathfrak{g}_2})$ with eigenvalue $c^{\mathfrak{g}_2}_{(i,j)}$.

Let $w \in V$ and $\beta \in V_{(0,1)}$, applying Schur's lemma where necessary gives the required projection maps to be:

- $V_{(1,0)} \otimes V_{(0,1)} \to V_{(2,0)}$, $w \otimes \alpha \mapsto \wedge\beta - \frac{1}{4}(w \lrcorner \beta) \lrcorner \psi$
- $V_{(1,0)} \otimes V_{(0,1)} \to V_{(1,0)}$, $w \otimes \beta \mapsto w \lrcorner \beta$
- $V_{(2,0)} \to W_{(0,1)}$, $dt \wedge \kappa + \eta \mapsto \frac{1}{2}(1 + iJ)(\kappa \lrcorner \mathrm{Re}\Omega)$
- $V_{(2,0)} \to W_{(1,1)}$, $dt \wedge \kappa + \eta \mapsto \frac{1}{3}\kappa - \frac{1}{2} *_6 (\omega \wedge \kappa) + \frac{1}{6} *_6 (\omega \wedge *_6(\omega \wedge \kappa))$
- $V_{(1,0)} \to W_{(1,0)}$, $u + adt \mapsto \frac{1}{2}(1 + iJ)u$

The difficulty in working with this basis is that the space $V_{(1,1)}$ cannot be modelled as a subspace of $\Lambda^*(\mathbb{R}^7)^*$. Instead we work with maps that factor through $V_{(1,1)}$ by noticing that they must be orthogonal with respect to the natural inner product on the space given by $\langle X, Y\rangle = \mathrm{Tr}(X^*Y)$, since they factor through orthogonal subspaces. We can therefore find expressions for the maps $p^{(1,1)}_{(i,j)}$ by ensuring the basis vectors are mutually orthogonal. The basis vectors can be described as follows:

Map	Factorisation and Formula
$p_1 = p^{(1,0)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(0,1)} \to V_{(1,0)} \to W_{(1,0)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto \frac{1}{2}(1 + iJ)[u \lrcorner \alpha - \frac{1}{2}(u \wedge v) \lrcorner \mathrm{Re}\Omega - av]$
$p_2 = p^{(2,0)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(0,1)} \to V_{(2,0)} \to W_{(1,0)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto \frac{1}{2}(1 + iJ)[u \wedge v + a\alpha - \frac{1}{4}(-(u \lrcorner \alpha) \lrcorner \mathrm{Re}\Omega +$ $av \lrcorner \mathrm{Re}\Omega)]$
$p_3 = p^{(1,1)}_{(1,0)}$	$V_{(1,0)} \otimes V_{(0,1)} \to V_{(1,1)} \to W_{(1,0)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ orthogonal to p_1 and p_2
$p_4 = p^{(1,0)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to V_{(1,0)} \to W_{(0,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto \frac{1}{2}(1 - iJ)[u \lrcorner \alpha - \frac{1}{2}(u \wedge v) \lrcorner \mathrm{Re}\Omega - av]$
$p_5 = p^{(2,0)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to V_{(2,0)} \to W_{(0,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto \frac{1}{2}(1 - iJ)[u \wedge v + a\alpha - \frac{1}{4}(-(u \lrcorner \alpha) \lrcorner \mathrm{Re}\Omega +$ $av \lrcorner \mathrm{Re}\Omega)]$
$p_6 = p^{(1,1)}_{(0,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to V_{(1,1)} \to W_{(0,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ orthogonal to p_4 and p_5
$p_7 = p^{(2,0)}_{(1,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to V_{(2,0)} \to W_{(1,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ $(u + adt) \otimes (\alpha + v \wedge dt + \frac{1}{2} v \lrcorner \mathrm{Re}\Omega) \mapsto a\alpha + \frac{1}{3}(u \wedge v) - \frac{1}{2} *_6 (\omega \wedge u \wedge v) + \frac{1}{6} *_6$ $(\omega \wedge *_6(\omega \wedge u \wedge v))$
$p_8 = p^{(1,1)}_{(1,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to V_{(1,1)} \to W_{(1,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ orthogonal to p_7 and p_9
$p_9 = \tilde{p}^{(1,1)}_{(1,1)}$	$V_{(1,0)} \otimes V_{(0,1)} \to V_{(1,1)} \to W_{(1,1)} \to (\mathfrak{g}_2)_{\mathbb{C}}$ orthogonal to p_7 and p_8

Here $p^{(1,1)}_{(1,1)}$ and $\tilde{p}^{(1,1)}_{(1,1)}$ factor through two different copies of $W_{(1,1)}$ contained in $V_{(1,1)}$. For the maps $p^{(i,j)}_{(k,l)}$ with $(i,j) \neq (1,1)$ we can determine their expression in terms of q_i as before. The result is the following:

- $p_1 = -q_1 + q_4 - \frac{1}{4} q_7$
- $p_2 = \frac{1}{2} q_1 + \frac{1}{2} q_4 + \frac{3}{} q_7$
- $p_4 = -q_2 + q_5 - \frac{1}{4} q_8$
- $p_5 = \frac{1}{2} q_2 + \frac{1}{2} q_5 + \frac{3}{8} q_8$
- $p_7 = q_3 + q_6 + q_9$.

By explicitly computing the matrices of the maps $q^{(i,j)(k,l)}_{(m,n)}$ we are able to calculate the norm of each map. They are as follows:

Lemma 8.8.1. *Let* $\|q\|^2 := \mathit{Trace}(q^\dagger q)$*, then the basis* $q^{(ij)(kl)}_{(mn)}$ *is orthogonal and*

- $\|q^{(0,0)(1,0)}_{(1,0)}\|^2 = 3$
- $\|q^{(0,1)(0,1)}_{(1,0)}\|^2 = \|q^{(1,0)(1,0)}_{(1,0)}\|^2 = 48$

- $\|q_{(1,0)}^{(1,0)(1,1)}\|^2 = \|q_{(0,1)}^{(0,1)(1,1)}\|^2 = 12$
- $\|q_{(1,1)}^{(0,0)(1,1)}\|^2 = 8$
- $\|q_{(1,1)}^{(1,0)(0,1)}\|^2 = \|q_{(1,1)}^{(0,1)(1,0)}\| = \frac{16}{3}$.

By ensuring the maps $p_{(k,l)}^{(1,1)}$ are orthogonal to the other basis maps, we obtain the following :

- $p_3 = -8q_1 - q_4 + q_7$
- $p_6 = -8q_2 - q_5 + q_8$
- $p_8 = q_6 - q_9$
- $p_9 = -\frac{4}{3}q_3 + q_6 + q_9$.

We calculate the matrix of $(D^1_{A_{\mathrm{can}}})_\gamma$ in the basis $q_1, \cdots, q_{12}$ in the same way as we have done for the standard representation and find

$$(D^1_{A_{\mathrm{can}}})_\gamma = \begin{pmatrix} 0 & 0 & 0 & -4 & 0 & 0 & 8 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4 & 0 & 0 & 8 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & -i & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & i & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & \frac{3i}{2} \\ \frac{1}{2} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -\frac{i}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & \frac{i}{2} & 0 \\ 0 & 0 & \frac{3}{2} & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & -\frac{3i}{2} \\ 0 & 0 & 0 & 4i & 0 & 0 & 8i & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -4i & 0 & 0 & -8i & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -i & 0 & 0 & i & 0 & 0 & 0 \end{pmatrix}.$$

The action of $\mathrm{Re}\Omega$ in this basis is $\mathrm{Re}\Omega = \mathrm{diag}(4,4,4,0,0,0,0,0,0,-4,-4,-4)$ and this allows us to calculate the matrix of $(D^t_{A_{\mathrm{can}}})_\gamma$ via (2.33). Again it is a useful consistency check to calculate the matrix of $(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma$, we find that

$$(D^{\frac{1}{3}}_{A_{\mathrm{can}}})^2_\gamma = \mathrm{diag}(12,12,7,12,12,7,12,12,7,12,12,7)$$

as expected.

Proposition 8.8.2. *Let $V_\gamma = V_{(0,1)}$, the eigenvalues of $(D^0_{A_{can}})_\gamma$ are symmetric about 0, the $\pm\lambda$ eigenspaces are isomorphic and the non-negative eigenvalues and multiplicities are:*

Eigenvalue λ	*Multiplicity*
$\frac{1}{2} + \frac{\sqrt{57}}{2}$	*2*
$-\frac{1}{2} + \frac{\sqrt{57}}{2}$	*2*
$\frac{1}{2} + \frac{\sqrt{37}}{2}$	*1*
$-\frac{1}{2} + \frac{\sqrt{37}}{2}$	*1*

Note we have shown that 1 occurs only as an eigenvalue of $(D^0_{A_{\mathrm{can}}})_\gamma$ when $V_\gamma = V_{(1,0)}$ is the standard representation. Since this representation has dimension 7, we deduce that the eigenvalue 1 of $D^0_{A_{\mathrm{can}}}$ has multiplicity 7 in the space of sections. Similarly the only representation for which $(D^0_{A_{\mathrm{can}}})_\gamma$ has non-trivial kernel is when $V_\gamma = \mathbb{C}$ is the trivial representation and the nullity is 2, so the dimension of the kernel of $D^0_{A_{\mathrm{can}}}$ in the space of sections is 2. Furthermore there are no other eigenvalues of $D^0_{A_{\mathrm{can}}}$ in the interval $[0, 2)$. With this in hand our next theorem follows immediately:

Theorem 8.8.3. *The virtual dimension of the moduli space $\mathcal{M}(A_{\mathrm{can}}, \mu)$ of AC G_2-instantons on P, decaying to A_{can} with rate $\mu \in (-2, 0) \setminus \{1\}$ is*

$$\mathrm{virtdim}\mathcal{M}(A_{\mathrm{can}}, \mu) = \begin{cases} 1 & \text{if } \mu \in (-2, -1) \\ 8 & \text{if } \mu \in (-1, 0). \end{cases}$$

Establishing the virtual dimension of the moduli space is an important step towards proving a uniqueness theorem for the standard instanton. If we assume this instanton to be unobstructed then the above result provides a local uniqueness theorem for this instanton. In other words, there are no other genuinely different instantons nearby in the moduli space since the only deformations are those defined by the obvious scaling and translation maps. Proving the unobstructedness of connections in the moduli space is a difficult task since curvature terms complicate the usual method of applying Lichnerowicz type formulae to L^2 twisted harmonic spinors. We are however still able to apply the deformation theory to study the class of unobstructed instantons, in the next section we aim to build on Theorem 8.8.3 to attain a uniqueness result in this setting.

8.9 Applications of the Deformation Theory

This section proves that, under the assumption of unobstructedness, the standard instanton is the unique G_2 instanton on $P = G_2 \times \mathbb{R}^7 \to \mathbb{R}^7$ which is asymptotic to A_{can}.

8.10 Invariance of AC Instantons

Let A be an AC G_2 instanton on $P = G_2 \times \mathbb{R}^7 \to \mathbb{R}^7$, converging to A_{can}. Throughout this section we shall assume that A is unobstructed. Recall we may study the deformations of A in terms an elliptic complex. The cohomology group

$$H^1_{A,\mu} = \frac{\operatorname{Ker}\left(\psi \wedge \mathrm{d}_A \colon \Omega^1_{\mu-1}(M, \operatorname{Ad}P) \to \Omega^6_{\mu-2}(M, \operatorname{Ad}P)\right)}{\mathrm{d}_A(\Omega^0_\mu(M, \operatorname{Ad}P))}$$

satisfies $H^1_{A,\mu} \cong I(A,\mu)$. Since we assume A to be unobstructed we know from Theorem 8.8.3 that the dimension of these vector spaces are

$$\dim H^1_{A,\mu} = \begin{cases} 1 & \text{if } \mu \in (-2,-1) \\ 8 & \text{if } \mu \in (-1,0). \end{cases}$$

Furthermore we know by [18, Proposition 9] that these deformations are given by the cohomology classes

$$\begin{aligned} \left[\iota_{\frac{\partial}{\partial x_i}} F_A\right] &= \left[\mathcal{L}_{\frac{\partial}{\partial x_i}} A\right] \\ \left[\iota_{r\frac{\partial}{\partial r}} F_A\right] &= \left[\mathcal{L}_{r\frac{\partial}{\partial r}} A\right] \end{aligned}$$

where $\frac{\partial}{\partial x_i}$, for $i = 1, \ldots, 7$, are coordinate vector fields and $r^2 = x_i x_i$. In fact any Killing field determines a deformation of A and one can ask which Killing fields actually preserve the connection. Here we think of Killing fields X as elements of $\operatorname{Lie}(G_2 \ltimes \mathbb{R}^7)$ and define a map

$$\begin{aligned} L \colon \operatorname{Lie}(G_2 \ltimes \mathbb{R}^7) &\to H^1_{A,-\frac{1}{2}} \\ L(X) &= [\mathcal{L}_X A]\,. \end{aligned}$$

Before investigating the properties of this map we pause to collect some facts about the Lie group $G_2 \ltimes \mathbb{R}^7$, the group generated by translations and rotation by a G_2 matrix. More precisely an element of $G_2 \ltimes \mathbb{R}^7$ consists of a pair (g, v) where g is an element of G_2 and $v \in \mathbb{R}^7$. Denote by R the standard representation of G_2, then the action of (g, v) on a point $p \in \mathbb{R}^7$ is

$$(g, v) \cdot p = R(g)p + v.$$

Acting with two elements gives the composition formula

$$(g', v') \cdot (g, v) = (g'g, R(g')v + v').$$

Denote by $(G_2)_p$ the elements of $G_2 \ltimes \mathbb{R}^7$ that fix a point $p \in \mathbb{R}^7$. Then $(G_2)_p = \{(g, p - R(g)p)\,;\, g \in G_2\}$ is a subgroup of $G_2 \ltimes \mathbb{R}^7$ isomorphic to G_2. Other connected subgroups are of the form $H \ltimes U$ for H a proper subgroup of G_2 and $U \subset \mathbb{R}^7$ a vector subspace, or H_p the isotropy subgroup of H fixing p. If a subgroup $H \ltimes U \subset G_2 \ltimes \mathbb{R}^7$ does not fix a point in $\mathbb{R}^7$ then U is a vector space of positive dimension and it follows that the subgroups of $G_2 \ltimes \mathbb{R}^7$ that are isomorphic to G_2 are precisely the groups $(G_2)_p$ for some $p \in \mathbb{R}^7$.

The set of connected proper Lie subgroups of G_2 is

$$\{\mathrm{SU}(3), \mathrm{SO}(4), \mathrm{U}(2), \mathrm{SU}(2) \times \mathrm{U}(1), \mathrm{SU}(2), \mathrm{SO}(3), \mathrm{U}(1)^2, \mathrm{U}(1)\} \tag{8.26}$$

and we can use this to understand the kernel of the map L.

Proposition 8.10.1. *The kernel of the map L is a Lie subalgebra of* $\mathrm{Lie}(G_2 \ltimes \mathbb{R}^7)$ *isomorphic to* $\mathfrak{g}_2$.

Proof. First we show that Ker$L$ is a Lie subalgebra of $\mathrm{Lie}(G_2 \ltimes \mathbb{R}^7)$. To avoid notational clutter we shall write elements of $H^1_{A,-\frac{1}{2}}$ without square brackets indicating that they are equivalence classes. Suppose that $X, Y \in \mathrm{Ker}L$, then there are $f_X, f_Y \in \Omega^0(M, \mathrm{Ad}P)$ with $L(X) = \mathrm{d}_A f_X$ and $L(Y) = \mathrm{d}_A f_Y$. Observe that

$$\begin{aligned} \mathcal{L}_X(\mathrm{d}_A f_Y) &= \{\iota_X, \mathrm{d}\}(\mathrm{d}f_Y + [A, f_Y]) \\ &= [\{\iota_X, \mathrm{d}\}A, f_Y] + \mathrm{d}\{\iota_X, \mathrm{d}\}f_Y + [A, \{\iota_X, \mathrm{d}\}f_Y] \\ &= [\mathcal{L}_X A, f_Y] + \mathrm{d}_A(\mathcal{L}_X f_Y). \end{aligned}$$

Therefore we find

$$\begin{aligned} L([X,Y]) &= \mathcal{L}_X\mathcal{L}_Y A - \mathcal{L}_Y\mathcal{L}_X A \\ &= \mathrm{d}_A(\mathcal{L}_X f_Y - \mathcal{L}_Y f_X) + [\mathcal{L}_X A, f_Y] - [\mathcal{L}_Y A, f_X] \\ &= \mathrm{d}_A((\mathcal{L}_X f_Y - \mathcal{L}_Y f_X + [f_X, f_Y]) \end{aligned}$$

so $L([X,Y])$ is in the trivial cohomology class. Suppose now that A is unobstructed, then L maps from a 21 dimensional vector space to an 8 dimensional vector space Ker$L$ must have dimension at least 13. Looking through the list (8.26) of subgroups of G_2 we observe that the only possibilities are

1. $\mathrm{Ker}L = \mathrm{Lie}(\mathrm{SU}(3) \ltimes \mathbb{R}^6)$ where SU(3) acts in the obvious way
2. $\mathrm{Ker}L = \mathrm{Lie}(\mathrm{SU}(3) \ltimes (\mathbb{R}^6 \oplus \mathbb{R}))$ where SU(3) acts on $\mathbb{R}^6$ in the obvious way and trivially on $\mathbb{R}$
3. $\mathrm{Ker}L = \mathrm{Lie}(G_2 \ltimes \mathbb{R}^n)$ for $1 \leq n \leq 7$ where G_2 acts trivially
4. $\mathrm{Ker}L = \mathrm{Lie}(\mathrm{SO}(4) \ltimes \mathbb{R}^7)$ where SO(4) acts on $\mathbb{R}^7$ either trivially or by restriction of the standard G_2 action
5. $\mathrm{Ker}L = \mathrm{Lie}(G_2)_p$ for some $p \in \mathbb{R}^7$.

If any of the cases $1-4$ were to hold then A would have translational symmetries, so we must rule this out. Suppose then for a contradiction that A has translational symmetries, then A is a (globally defined) 1-form such that $\mathcal{L}_X A = 0$, where X is the vector field generating the translations under which A is invariant, thus $X = c_i \frac{\partial}{\partial x_i}$ where c_i are constants and $\frac{\partial}{\partial x_i}$ are the coordinate vector fields on $\mathbb{R}^7$. Note that $\mathcal{L}_X A = 0$ implies $\mathcal{L}_X F = 0$ and one can check that if $F = F_{ij}\, \mathrm{d}x^i \wedge \mathrm{d}x^j$ is any 2-form on $\mathbb{R}^n$ then $\mathcal{L}_X F = 0$ if and only if $X(F_{ij}) = 0$ for all i, j. It follows that

$$X(|F|^2) = X(F_{ij}F_{ij}) = 2F_{ij}X(F_{ij}) = 0$$

so $|F|$ is constant in the direction X. Pick $p \in S^6$ such that $\{tp\,;\, t \in \mathbb{R}\}$ is the line through the origin generated by X, then

$$|F_A|(tp) = c(A)$$

where $c(A) \geq 0$ is a constant depending only on A.

Since A is asymptotically conical (up to gauge) there is a gauge transformation g such that $g \cdot A = gAg^{-1} - \mathrm{d}g g^{-1}$ satisfies

$$|g \cdot A - A_C| = |a| = O(r^{\mu-1})$$

where $A_C = \pi^* A_{\mathrm{can}}$ and a is defined as the difference of $g \cdot A$ and A_C. Observe that

$$|F_{g\cdot A} - F_{A_C}| = |\,\mathrm{d}_A a + a \wedge a| = O(r^{\mu-2}),$$

in other words $r^{2-\mu}|F_{g\cdot A} - F_{A_C}|$ is a bounded function of r for some $R > 0$ and where $r \in [R, \infty)$. Recall that $F_{A_C} = \pi^*(F_{A_{\mathrm{can}}})$ and therefore $r^2|F_{A_C}|(rp) = c(A_C)$ where $c(A_C) = |F_{A_{\mathrm{can}}}|_{g_{\mathrm{round}}} > 0$ is a constant independent of both r and p. We calculate

$$\begin{aligned} r^{2-\mu}|F_{g\cdot A} - F_{A_C}|(rp) &\geq r^{2-\mu}||F_{g\cdot A}| - |F_{A_C}||(rp) \\ &= r^{-\mu}|r^2|F_A| - r^2|F_{A_C}||)(rp) \\ &= r^{-\mu}|r^2 c(A) - c(A_C)|. \end{aligned}$$

However $-\mu > 0$ and $c(A_C) > 0$ ensures $r^{-\mu}|r^2 c(A) - c(A_C)|$ is an unbounded function of r, which yields our contradiction. □

This proves the G_2-invariance of unobstructed connections in the moduli space:

Proposition 8.10.2. *Let A be an unobstructed AC G_2-instanton on P, converging to A_{can}. Then A is invariant under the action of $(G_2)_p$ for some $p \in \mathbb{R}^7$.*

Remark 8.10.3. *Instead of the above symmetry argument one might hope to prove invariance of connections in the moduli space by showing that sufficiently fast decay ensures this property holds. The author has not investigated this suggestion but it could prove interesting and useful to do so.*

8.11 A Uniqueness Theorem

The constructions covered in this section come from [42]. Here we review their work using the framework of Wang's theorem, as has been done to study the moduli space of invariant monopoles on the Bryant-Salamon manifolds $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$ in [77].

If $Q \to G_2/\mathrm{SU}(3)$ is a homogeneous bundle we shall use Wang's theorem to study

$$\mathcal{M}_{\text{inv}}(G_2/\mathrm{SU}(3), Q)$$

the space of G_2-invariant nearly Kähler instantons modulo invariant gauge transformations. Principal G_2-homogeneous G_2-bundles are determined by homomorphisms $\lambda \colon \mathrm{SU}(3) \to G_2$. There are exactly two conjugacy classes of such a homomorphism; the class of the trivial homomorphism and the class of the inclusion homomorphism. Hence there are exactly two equivalence classes of principal homogeneous G_2-bundles over S^6.

In the first case, when $\lambda(h) = 1$ for all $h \in \mathrm{SU}(3)$, Wang's theorem says to look for morphisms of $\mathrm{SU}(3)$ representations

$$\Phi \colon (\mathfrak{m}, \mathrm{Ad}) \to \mathbb{R}^{14}$$

where $\mathbb{R}^{14}$ denotes 14 copies on the trivial representation. Since there are no such non-zero maps the only invariant connection corresponds to $\Phi = 0$ and this yields the trivial flat connection.

The other case to be considered is when λ is the inclusion map $\iota \colon \mathrm{SU}(3) \to G_2$ and we denote the associated bundle

$$Q = G_2 \times_{(\mathrm{SU}(3),\iota)} G_2. \tag{8.27}$$

In this case, Wang's theorem instructs us to look for morphism of $\mathrm{SU}(3)$ representations

$$\Phi \colon (\mathfrak{m}, \mathrm{Ad}) \to (\mathfrak{g}_2, \mathrm{Ad}).$$

Working with complexified representations we see that a basis for $\mathrm{Hom}(\mathfrak{m}_{\mathbb{C}}, (\mathfrak{g}_2)_{\mathbb{C}})_{\mathrm{SU}(3)}$ is given by the set $\{\mathrm{Id}, J\}$, the identity map and the complex structure. If we identify such a map $a = x\mathrm{Id} + yJ$, where $x, y \in \mathbb{R}$, with the complex number $z = x + iy$, then the G_2 invariant connection with

$$A([1]) = A_{\text{can}}([1]) + a \tag{8.28}$$

is a nearly Kähler instanton precisely when $\overline{z}^2 - z = 0$ [42]. Other than the canonical connection $z = 0$ the solutions to this equation are precisely the cube roots of unity. Let us write A_0, A_1, A_2 for the connections obtained from the solutions $\exp(\frac{2n\pi i}{3})$ for $n = 0, 1, 2$. These connections are in fact flat and and since S^6 is simply connected these connections must be gauge equivalent. To see that the connections are in fact equivalent through G_2-invariant gauge transformations we construct them explicitly. Let P_0 denote the canonical bundle $P_0 = G_2 \to G_2/\mathrm{SU}(3)$ so that SU(3) acts on the right of P_0. Let $\omega_n = \exp(\frac{2n\pi i}{3})\mathrm{Id}_3 \in \mathrm{SU}(3)$, then $\{\omega_n\}_{n=0,1,2}$ is the centre of SU(3) so the group of invariant gauge transformations on Q is isomorphic to $\mathbb{Z}_3$. To understand the action of this group on the space of invariant connections, note that the adjoint bundle of Q, as defined in (8.27), is associated to P_0 via the adjoint action of SU(3) on $\mathfrak{g}_2$ and therefore $\mathrm{Aut}(P_0)$ acts on this bundle. This induces a linear action of $\mathrm{Aut}(P_0)$ on $\Omega^1(S^6, \mathrm{Ad}Q)$ and so forms a subgroup of the gauge group of Q. Write A_n for the G_2-invariant connection with $A_n([1]) = A_{\mathrm{can}}([1]) + a_n$, where the a_n are linear maps $a_n \colon \mathfrak{m} \to \mathfrak{g}_2$. Then $\mathbb{Z}_3$ acts on the $\mathfrak{g}_2$ part of $\mathfrak{m}^* \otimes \mathfrak{g}_2$. Since this action is a restriction of the adjoint action of SU(3) on $\mathfrak{g}_2 = \mathfrak{su}(3) \oplus \mathfrak{m}$ we observe that the $\mathbb{Z}_3$ subgroup acts trivially on $\mathfrak{su}(3) \subset \mathfrak{g}_2$ and non-trivially on $\mathfrak{m} \subset \mathfrak{g}_2$. Indeed, this action is precisely the action inducing the 3-symmetry map (2.14) since for the coset space in question $S \colon \mathfrak{m} \to \mathfrak{m}$ is the map

$$S(X) = \omega_1 X \omega_1^{-1} = -\frac{1}{2} + \frac{\sqrt{3}}{2} J.$$

The action of S^n maps a_0 which is the inclusion $\mathfrak{m} \to \mathfrak{g}_2$, to the maps a_n for $n = 1, 2$ so the connections A_0, A_1, A_2 are gauge equivalent (through invariant gauge transformations).

Now let $\pi \colon C(\Sigma) \to \Sigma$ denote projection from the cone to Σ. The action of G_2 on $Q := G_2 \times_{(H,\iota)} G_2$ lifts to an action on $P = \mathbb{R}^7 \times G_2 \to \mathbb{R}^7$ since the usual action of G_2 on $\mathbb{R}^7$ preserves length. Recall A is said to be invariant if its connection 1-form $A \in \Omega^1(P, \mathfrak{g}_2)$ is left invariant under this action. It suffices to consider only connections A on P which are in radial gauge, i.e $\mathrm{d}r \lrcorner A = 0$, since such a gauge may always be chosen. Then an invariant connection is determined as before in (8.28) but now we identify $f_1(r)\mathrm{Id} + f_2(r)J$ with the complex valued function $f(r) = f_1(r) + i f_2(r)$. As demonstrated in [42] such a connection is a G_2 instanton if and only if f satisfies the differential equation

$$r f'(r) = 2\left(\bar{f}^2(r) - f(r)\right). \tag{8.29}$$

Again we must remark that the coefficient of 2 is different from that found in [42, Section 5.3] due to our normalisation of the metric and the constraint $\mathrm{d}\omega = 3\mathrm{Im}\Omega$. In [42, Section 5.3] it is shown that this is in fact the gradient flow equation for a real superpotential "superpotential" $W \colon \mathbb{C} \to \mathbb{R}$ where $W(z) = \frac{1}{3}(z^3 + \bar{z}^3) - |z|^2$. Clearly (8.29) is equivalent to

$$\frac{r}{2}\frac{\mathrm{d}}{\mathrm{d}r} f = \frac{\partial W}{\partial \bar{f}}.$$

If we view W as a function of two real variables then a quick calculation shows its has exactly four critical points which are (viewed as complex numbers) 0 and the three cube roots of unity. The unique local maximum is 0 whilst the other critical points are all saddle points. To see the advantage of interpreting (8.29) as a gradient flow equation we must first determine the relevant boundary data.

Note that this construction only yields a connection A on $\pi^*(Q) = (\mathbb{R}^7 \setminus \{0\}) \times G_2$. To get a connection on P we require A to extend over the origin. For this to happen it is necessary and sufficient that the curvature be bounded at $r = 0$. The curvature of the connection satisfies

$$|F_A|^2(\sigma, r) = \frac{c_1}{r^2}(|\bar{f}^2 - f|^2 + |\bar{f}f - 1|^2) + \frac{c_2}{r}|f'|$$

with constants $c_1, c_2 > 0$. Thus f is required to satisfy

- $|\bar{f}^2 - f| = O(r)$ as $r \to 0$
- $|\bar{f}f - 1| = O(r)$ as $r \to 0$
- $|f'| = O(r)$ as $r \to 0$.

As in the previous section we ask that the connection we obtain decays to the canonical connection, in other words we also impose the boundary condition

$$\lim_{r\to\infty} f(r) = 0.$$

Thus our boundary data requires that $f(r)$ tends to a cube root of unity as $r \to 0$ and that $f(r) \to 0$ as $r \to \infty$. The space of invariant connections can therefore be identified with the space of solutions to (8.29) satisfying the above boundary conditions. The solution to this system

$$f_0(r) = \frac{1}{Cr^2 + 1}$$

is the unique one (up to the scaling parameter $C \in \mathbb{R}_+$) with $f(r) \to 1$ as $r \to 0$ whilst other solutions are obtain by applying the 3-symmetry: $f_1(r) = \exp(\frac{2\pi i}{3})f_0(r)$ and $f_2(r) = \exp(\frac{4\pi i}{3})f_0(r)$. Uniqueness follows from observing that the solutions are subject to gradient flow away from a critical point and towards the unique local maximum of W. Note that f_0 is precisely the solution we have seen in (8.1) and that this solution yields the standard instanton

$$A_{\text{std}}(r[1]) = A_{\text{can}}([1]) + f_0(r)\text{Id}.$$

Let the G_2 instantons defined by the functions f_i be denoted $\tilde{A}_i$, then the invariant gauge transformations that related the connections A_i lift to the cone to relate the connections $\tilde{A}_i$. Thus we have uniqueness of solutions for the given boundary data together with the fact that the 3 families of solutions are gauge equivalent. Therefore, the invariant moduli space is determined precisely by the paramater C:

Proposition 8.11.1. *Let A be a G_2-instanton on the homogeneous principal bundle P which decays to the canonical connection of the nearly-Kähler S^6 at infinity. If A is invariant under the action of G_2 on P, then $A = A_{std}$ is the standard G_2-instanton.*

Remark 8.11.2. *The constant C can be interpreted as the "size" of the instanton, in other words how concentrated it is around the origin, and is related to the conformal invariance of the G_2-instanton equation [18]. It is worth noting that this parameter also arises from the dilation map, so the invariant moduli space coincides with the moduli space* (4.3) *for rates in the range $-2 < \mu < -1$.*

The importance of this result is that it can be combined with the results of the previous section to prove a global uniqueness result for unobstructed instantons. Namely, Proposition 8.10.2 says that any unobstructed instanton on P, AC to A_{can}, must be invariant under the action of G_2 on $\mathbb{R}^7$ which fixes some point, so uniqueness follows from the above result.

Theorem 8.11.3. *Let A be an AC G_2-instanton on P, converging to A_{can}. Then either A is obstructed or A is the standard G_2-instanton.*

Chapter 9

Appendices

9.1 A Quaternionic Model of Spinor Space

We present here a model for the spinor space of $\mathbb{R}^6$ as $\mathbb{H} \oplus \mathbb{H}$ which, to the author's knowledge, has not appeared in the literature before. The basic idea is to model the space as an $\mathrm{SU}(2)^3$-module.

Let us collect some facts about the group $\mathrm{SU}(2)^3$ and give a description of the spinor space compatible with the action of this group. Recall that $\mathrm{SU}(2) \cong \mathrm{Sp}(1)$ where $\mathrm{Sp}(1)$ is the group of unit quaternions which has Lie algebra $\mathfrak{sp}(1) = \mathrm{Im}\mathbb{H} = \langle \boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k} \rangle$. A natural action of $\mathrm{Sp}(1)^3$ on $\mathbb{H}^2$ is

$$\rho_S(g_1, g_2, g_3)(h_1, h_2) = (g_1 h_1 g_3^{-1}, g_2 h_2 g_3^{-1}).$$

We label irreducible representations of $\mathrm{Sp}(1) = \mathrm{SU}(2)$ by V_i for $i \in \mathbb{N} \cup \{0\}$. Here V_i is the unique $i+1$ dimensional irreducible representation which is isomorphic to $\mathrm{Sym}^i(\mathbb{C}^2)$. Irreducible representations of $\mathrm{Sp}(1)^3$ are then given by the representations $V_{i,j,k} := V_i^{(1)} \otimes V_j^{(2)} \otimes V_k^{(3)}$ and the branching rule for the diagonal subgroup $\Delta\mathrm{Sp}(1)$ is easily calculated using the standard Clebsch-Gordan rule for tensor products.

As a representation of $\mathrm{Sp}(1)^3$ we have that $(\mathbb{H}^2, \rho_S) = V_{(1,0,1)} \oplus V_{(0,1,1)}$ and the branching rule says that, as a representation of $\Delta\mathrm{Sp}(1)$, we have $(\mathbb{H}^2|_{\Delta\mathrm{Sp}(1)}, \rho_S) = V_0 \oplus V_2 \oplus V_2 \oplus V_0$. This corresponds to the splitting $\mathbb{H}^2 = \mathrm{Re}\mathbb{H} \oplus \mathrm{Im}\mathbb{H} \oplus \mathrm{Im}\mathbb{H} \oplus \mathrm{Re}\mathbb{H}$ and also to the splitting $\not{S}(\Sigma) = \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^6$. Moreover, this is precisely the (isomorphism class of) representation of $\mathrm{Sp}(1)$ that the spinor space should be. If $e^1, \ldots, e^6$ denotes the standard orthonormal basis of $\mathbb{R}^6$ then, under this splitting, the two copies of the adjoint representation V_2 are $V_2 = \mathrm{Im}\mathbb{H}_{\mathrm{odd}} :=< e^1, e^3, e^5 >$ and $V_2 = \mathrm{Im}\mathbb{H}_{\mathrm{even}} :=< e^2, e^4, e^6 >$. To relate S to $\mathbb{H}^2$ we make the associations $e^1 = \boldsymbol{i}, e^3 = \boldsymbol{j}, e^5 = \boldsymbol{k}$ and $e^2 = \boldsymbol{i}, e^4 = \boldsymbol{j}, e^6 = \boldsymbol{k}$ as well as $\mathrm{Re}\mathbb{H} = \Lambda^0$ and $\mathrm{Re}\mathbb{H} = \Lambda^6$. Given that we know from Corollary 2.3.5 how to multiply a spinor by a 1-form, we can understand Clifford multiplication in this setting. A spinor $s \in \Lambda^0 \oplus \Lambda^1 \oplus \Lambda^6$ can be written (h_1, h_2) where $h_1 = \pi_0(s) + \pi_{\mathrm{odd}}(s)$, $h_2 = \pi_6(s) + \pi_{\mathrm{even}}(s)$

where π_0 and π_6 are projections to Λ^0 and Λ^6 respectively and π_{odd} and π_{even} are projections to $\text{Im}\mathbb{H}_{\text{odd}}$ and $\text{Im}\mathbb{H}_{\text{even}}$ respectively. Using this association one finds that

$$\text{cl}(e^1)\begin{pmatrix}h_1\\h_2\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\boldsymbol{i}h_1\\\boldsymbol{i}h_2\end{pmatrix} \tag{9.1}$$

$$\text{cl}(e^2)\begin{pmatrix}h_1\\h_2\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}h_1\boldsymbol{i}\\h_2\boldsymbol{i}\end{pmatrix} \tag{9.2}$$

$$\text{cl}(e^3)\begin{pmatrix}h_1\\h_2\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\boldsymbol{j}h_1\\\boldsymbol{j}h_2\end{pmatrix} \tag{9.3}$$

$$\text{cl}(e^4)\begin{pmatrix}h_1\\h_2\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}h_1\boldsymbol{j}\\h_2\boldsymbol{j}\end{pmatrix} \tag{9.4}$$

$$\text{cl}(e^5)\begin{pmatrix}h_1\\h_2\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}\boldsymbol{k}h_1\\\boldsymbol{k}h_2\end{pmatrix} \tag{9.5}$$

$$\text{cl}(e^6)\begin{pmatrix}h_1\\h_2\end{pmatrix} = \begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0&-1\\1&0\end{pmatrix}\begin{pmatrix}h_1\boldsymbol{k}\\h_2\boldsymbol{k}\end{pmatrix}. \tag{9.6}$$

Note the matrix $\begin{pmatrix}0&-1\\1&0\end{pmatrix}$ corresponds to the action of the volume form on the spin bundle. Clearly $\text{cl}(e^i)^2 = -1$ and a quick calculation shows that the Clifford relations $\{\text{cl}(e^i), \text{cl}(e^j)\} = -2\delta_{ij}$ are satisfied.

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