Homological Methods in Algebra

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Abstract

In this thesis, we apply homological methods to the study of groups in two ways: firstly, we generalise the results of [12] to a more general class of categories than posets, including finite groups which satisfy a particular cohomological condition. We then show that the only finite group satisfying this condition is the trivial group, but our results still hold in more generality than the originals, and we suggest a path to further generalisation. Secondly, we study the representation theory of certain groups by passing their actions on certain simplicial complexes to actions on the homologies of those complexes.

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Acknowledgements & Declaration

Firstly, I would like to thank my supervisors, Michael Bate and Brent Everitt for their guidance and support throughout my time at York. I would also like to thank Stephen Donkin for bringing the problem studied in Chapter 7, and a variation of the problem studied in Chapter 6, to my attention.

The support of my family, friends, and Scout group has been invaluable in keeping me (mostly) sane throughout the last few years.

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I declare that this thesis is a presentation of original work and I am the sole author. This work has not previously been presented for an award at this, or any other, University. All sources are acknowledged as References.

Chapter 1

Introduction

The aims of this thesis are twofold. Firstly, we construct an analogue of the cellular homology of CW-complexes in the setting of the cohomology of (a certain class of) categories with coefficients in a presheaf on that category, extending a result of [12]. Our methods in this section closely mirror theirs, with some modifications to deal with new issues that arise in this setting. We had initially hoped to apply this to the cohomology of groups, but we found that there are no non-trivial groups to which our methods apply.

Secondly, we turn our attention to the representations of groups induced by passing their actions on simplicial complexes to actions on the homology groups of those simplicial complexes. We consider two particular examples: firstly, a certain naturally emerging action of the symmetric group, and secondly a naturally emerging action of a finite nilpotent group (or, more generally, a product of groups with nontrivial centres). Both were brought to our attention by Stephen Donkin.

Similar approaches to ours of Chapters 6 and 7 have been used to study the representations of other groups. Notably, there has been considerable study of the representations of braid groups arising by these means, as in [3], [23], and [30], for example, along with the recent unification of [39]. There has also been much interest in homological representations of posets, which has also made use of the concept of shellability that we use in Chapter 7, as in [4] for the shellability and [45] for the representation theory, with the whole being summarised in the survey article of [47]. These methods have also been applied to other areas, such as in the representation theory of reflection groups in [28], Coxeter groups in [20] and [21]. There has also been some recent study of the representations of the symmetric groups by these means, for example in [24].

The cohomology of categories has also been considered in various forms including the cohomology that we define here (beginning with [41]), and its generalisations to the Baues-Wirsching cohomology introduced in [2] and studied further in [37] and the Thomason cohomology studied in [17].

Our work in Chapter 3, and that of [12], builds upon the work of [13]. In this paper, the authors establish a representation of Khovanov homology as the cohomology of a sheaf on a poset. In [12], this process is reversed, in more generality: starting with the cohomologies of sheaves on posets, a cellular way to compute their homologies is constructed. When applied to the sheaf produced in [13], this recovers the original definition of Khovanov homology. Here, we extend this to a more general class of categories.

In Chapter 2 of this thesis, we develop the theory of categories and homological algebra that we shall need throughout, together with some more specialised category-theoretic concepts that we shall need for Chapter 3. In Chapter 3, we shall then use these to prove our first main result, generalising the finite case of [12, Thm. 2].

In Chapters 4 and 5, we then develop the respective topological and group-theoretic material that we shall subsequently require. In Chapter 6 we use these to apply Lemma 5.2.8 above to the action of the symmetric group on a particular naturally-arising complex and analyse the resulting representations. Finally, in Chapter 7, we use the material developed in Chapters 4 and 5 to apply Lemma 5.2.8 above to the action of any finite nilpotent group G (or, more generally, any finite direct product of finite groups which are not centreless) on the complete multipartite simplicial complex $\Delta(G)$ arising from an analogue of decomposition into prime factors in our group, and analyse the resulting representations.

Chapter 2

Homological Background

2.1 Categories

2.1.1 Basic Definitions

We begin by defining some categorical concepts that we shall require. Firstly, a category itself.

Definition 2.1.1. [42, p. 8] A category $\mathcal{C}$ consists of:

1. A class $\mathrm{ob}\,\mathcal{C}$ of **objects**,

2. For each pair of objects $A, B \in \mathrm{ob}\,\mathcal{C}$, a set $\mathrm{Hom}(A, B)$ of **morphisms** from A to B, disjoint from all other such sets, and

3. For each triple of objects $A, B, C \in \mathrm{ob}\,\mathcal{C}$, a binary operation **composition**

$$\circ : \mathrm{Hom}(A, B) \times \mathrm{Hom}(B, C) \to \mathrm{Hom}(A, C),$$

such that for each $A \in \mathrm{ob}\,\mathcal{C}$, there is an **identity** morphism 1_A, such that for any $B \in \mathrm{ob}\,\mathcal{C}$, any $f \in \mathrm{Hom}(A, B)$, and any $g \in \mathrm{Hom}(B, A)$, we have $f \circ 1_A = f$ and $1_A \circ g = g$ which is associative, where it is defined.

There are two size restrictions on categories that will be important. Firstly, a mild condition.

Definition 2.1.2. [35, p. 2] A category $\mathcal{C}$ is **small** if $\mathrm{ob}\,\mathcal{C}$ is a set, rather than a proper class.

And secondly, a more restrictive requirement.

Definition 2.1.3. A category $\mathcal{C}$ is **locally finite** if:

1. For all $A \in \text{ob}\,\mathcal{C}$, there are at most finitely many other objects $B \in \text{ob}\,\mathcal{C}$ such that $\text{Hom}(A, B)$ is non-empty, and

2. For all $A, B \in \text{ob}\,\mathcal{C}$, $\text{Hom}(A, B)$ is a finite set.

The category is **finite** if also $\mathcal{C}$ has finitely many objects.

Secondly, we define some classes of morphisms and objects of a category. Firstly, we shall define the following generalisations of injective, surjective, and bijective functions.

Definition 2.1.4. [42, pp. 22, 303–304] A morphism $f : A \to B$ in a category $\mathcal{C}$ is an **epimorphism** (or is **epic**) if for all $g, h : B \to C$ in $\mathcal{C}$, we have that $g \circ f = h \circ f$ implies $g = h$.

Dually f is a **monomorphism** (or is **monic**) if for all $g, h : C \to A$ in $\mathcal{C}$, we have that $f \circ g = f \circ h$ implies $g = h$.

Finally, f is an **isomorphism** if there is a map $g : B \to A$ such that $f \circ g = id_B$ and $g \circ f = id_A$.

Next, we define the following naturally arising constructions.

Definition 2.1.5. [42, pp. 216-218],[35, p. 14] An object A is **initial** if for every object B, there is a unique morphism $A \to B$, and **terminal** if for every object B there is a unique morphism $B \to A$. If A is both initial and terminal, then it is a **zero object**.

If $\mathcal{C}$ has a zero object 0, then a morphism $f : A \to B$ is a **zero morphism** if the following diagram commutes:

$$\begin{tikzcd} & 0 \arrow[dr] & \\ A \arrow[ur] \arrow[rr, "f"] & & B \end{tikzcd}$$

with the maps to and from 0 the unique such maps.

A category $\mathcal{C}$ **has zero morphisms** if for each pair (A, B) of objects of $\mathcal{C}$, there is a morphism $0_{AB} : A \to B$ such that:

1. For all morphisms $f : C \to A$, we have $0_{AB} \circ f = 0_{CB}$, and

2. For all morphisms $g : B \to C$, we have $f \circ 0_{AB} = 0_{AC}$.

We will generally drop the subscripts.

We note in particular that if $\mathcal{C}$ is locally finite and has an initial object, then $\mathcal{C}$ is in fact finite.

A key property of initial and terminal objects is that they are unique up to isomorphism. The following result is [42, Lemma. 5.3, 5.6].

Lemma 2.1.6. *If both A and B are initial (respectively terminal) in a category C, then there is a unique isomorphism $\varphi : A \to B$.*

Proof. By initiality of A (or terminality of B), there is a unique morphism $\varphi : A \to B$. Similarly, by initiality of B (or terminality of A), there is a unique morphism $\psi : B \to A$. But also, there can be only one morphism $A \to A$ and one morphism $B \to B$, which must be the respective identities, so we have $\varphi \circ \psi = id_B$ and $\psi \circ \varphi = id_A$, so φ is indeed an isomorphism. $\square$

A fundamental concept is that of a (co)limit.

Definition 2.1.7. [48, §A.5] Let $A \in \operatorname{ob}\mathcal{C}$, D be a commuting diagram of objects and morphisms of $\mathcal{C}$, and for each object D_i of D, let $f_i : A \to B_i$ be a morphism such that the diagram formed by adding A and the f_i to D commutes. Then $(A, (f_i))$ is a **limit** of D if, for any other object $B \in \operatorname{ob}\mathcal{C}$ with maps $g_i : B \to D_i$ such that the diagram formed by adding B and the g_i to D commutes, there is a unique morphism $\varphi : B \to A$ such that the combined diagram formed by adding A, B, the f_i, the g_i, and φ to D commutes.

$$\begin{tikzcd}
 & & D_i \arrow[dd] \\
B \arrow[r, dashed, "\varphi"] \arrow[urr, bend left, "g_i"] \arrow[drr, bend right, "g_j"'] & A \arrow[ur, "f_i"] \arrow[dr, "f_j"] & \\
 & & D_j
\end{tikzcd}$$

Dually, if $h_i : D_i \to A$ are morphisms such that the diagram formed by adding A and the h_i to D commutes, then $(A, (h_i))$ is a **colimit** of D if, for any other object $B \in \operatorname{ob}\mathcal{C}$ with maps $k_i : D_i \to B$ such that the diagram formed by adding B and the k_i to D commutes, there is a unique morphism $\psi : A \to B$ such that the combined diagram formed by adding A, B, the h_i, the k_i, and φ to D commutes.

$$\begin{tikzcd}
D_i \arrow[dd] \arrow[dr, "h_i"] \arrow[drr, bend left, "k_i"] & & \\
 & A \arrow[r, dashed, "\psi"] & B \\
D_j \arrow[ur, "h_j"] \arrow[urr, bend right, "k_j"'] & &
\end{tikzcd}$$

This is a useful concept as these (co)limits are unique, up to unique isomorphism, as seen in [42, p. 217].

Lemma 2.1.8. *If both* (A, f) *and* (B, g) *are limits (respectively colimits) of a diagram* D*, then there is a unique isomorphism* $\varphi : A \to B$ *such that* $f_i = g_i \circ \varphi$ *(respectively* $g_i = \varphi \circ f_i$*) for all* i*.*

Proof. We shall prove the limit case: the proof for the colimit case is dual.

Since (B, g) is a limit for D, it in particular has maps to each D_i such that the resulting diagram commutes, so by the definition of A being a limit, there is a unique morphism $\varphi : B \to A$ such that the resulting diagram commutes. But we can also perform this construction with the roles of A and B reversed, so there is a unique morphism ψ from A to B such that the resulting diagrams commute. Composing these two morphisms together, we obtain the following commuting diagram:

$$\begin{tikzcd}
 & & D_i \arrow[dd] \\
A \arrow[r, "\varphi\circ\psi"] \arrow[urr, bend left, "f_i"] \arrow[drr, bend right, "f_j"'] & A \arrow[ur, "f_i"] \arrow[dr, "f_j"] & \\
 & & D_j
\end{tikzcd}$$

Clearly, inserting the identity map on A in the place of $\varphi \circ \psi$ still leaves a commuting diagram, so by the definition of A being a limit, $\varphi \circ \psi$ must be the identity. Similarly, $\psi \circ \varphi$ must be the identity, so φ and ψ are isomorphisms. $\square$

On account of this uniqueness, we will henceforth speak of "the (co)limit of D", rather than "a (co)limit of D".

There are various kinds of (co)limits with names of their own, some of which we now list.

Definition 2.1.9. [42, pp. 214, 217] For any collection of objects $A_i \in \operatorname{ob} \mathcal{C}$, the **product** of the A_i is the limit of the diagram whose objects are the A_i with no morphisms.

Dually, the **coproduct** of the A_i is the colimit of the diagram whose objects are the A_i with no morphisms.

An object which is both the product and the coproduct of the A_i is called their **biproduct**.

We now generalise the concept of a kernel and cokernel to categories.

Definition 2.1.10. [42, pp. 223, 239] For a category $\mathcal{C}$ with zero morphisms and a morphism $f : A \to B$ in $\mathcal{C}$, a **kernel** of f is a limit $(K(f), \ker(f))$ of the diagram

$$\begin{array}{ccc} & & 0 \\ & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

Dually, a **cokernel** of f is a colimit $(C(f), \mathrm{coker} f)$ of the diagram

$$\begin{array}{ccc} A & \longrightarrow & B \\ \downarrow & & \\ 0 & & \end{array}$$

The definition of a category is not quite symmetric: reversing the direction of all morphisms gives the following, different, category.

Definition 2.1.11. [42, p. 23] For a category $\mathcal{C}$, the **opposite category** $\mathcal{C}^{op}$ is the category whose objects are precisely the objects of $\mathcal{C}$, and whose morphism set for each pair of objects A and B in $\mathcal{C}$ is

$$\mathrm{Hom}_{\mathcal{C}^{op}}(A, B) = \mathrm{Hom}_{\mathcal{C}}(B, A).$$

The following definition will be key to everything that we do.

Definition 2.1.12. A sequence of objects and morphisms

$$\cdots \to A \xrightarrow{f} B \xrightarrow{g} C \to \cdots$$

in a category with zero morphisms is **exact** at B if $\ker(g) = \mathrm{coker}(f)$.

2.1.2 Functors

We now move to defining maps between categories, in a way which has a similar asymmetry to the definition of a category, and hence two forms.

Definition 2.1.13. [35, p. 49] Let $\mathcal{C}$ and $\mathcal{D}$ be categories. A **covariant functor** F from $\mathcal{C}$ to $\mathcal{D}$ consists of:

1. A mapping sending each object A of $\mathcal{C}$ to an object $F(A)$ of $\mathcal{D}$,

2. For each pair of objects A and B of $\mathcal{C}$, a map from $\mathrm{Hom}_{\mathcal{C}}(A, B)$ to $\mathrm{Hom}_{\mathcal{D}}(F(A), F(B))$ such that:

   (a) For each $A \in \operatorname{ob} \mathcal{C}$, $F(1_A) = 1_{F(A)}$, and
   (b) If f and g are composable morphisms in $\mathcal{C}$, then $F(f)$ and $F(g)$ are composable in $\mathcal{D}$, and $F(f \circ g) = F(f) \circ F(g)$.

A **contravariant functor** from $\mathcal{C}$ to $\mathcal{D}$ is a covariant functor from the opposite category of $\mathcal{C}$ to $\mathcal{D}$. Equivalently, a contravariant functor $F : \mathcal{C} \to \mathcal{D}$ consists of:

1. A mapping sending each object A of $\mathcal{C}$ to an object $F(A)$ of $\mathcal{D}$.
2. For each pair of objects A and B of $\mathcal{C}$, a map from $\operatorname{Hom}_{\mathcal{C}}(A, B)$ to $\operatorname{Hom}_{\mathcal{D}}(F(B), F(A))$ such that:
   (a) For each $A \in \operatorname{ob} \mathcal{C}$, $F(1_A) = 1_{F(A)}$ still, and
   (b) If f and g are composable morphisms in $\mathcal{C}$, then $F(g)$ and $F(f)$ are composable in $\mathcal{D}$, and $F(f \circ g) = F(g) \circ F(f)$.

The following example will be critical.

Example 2.1.14. Let $\mathcal{C}$ be a category, and let A be an object of $\mathcal{C}$. Then there is a covariant functor $\operatorname{Hom}_{\mathcal{C}}(A, -) : \mathcal{C} \to \mathbf{Set}$ which maps each object B of $\mathcal{C}$ to $\operatorname{Hom}_{\mathcal{C}}(A, B)$ and each morphism $f : B \to C$ of $\mathcal{C}$ to the function $\operatorname{Hom}_{\mathcal{C}}(A, f) : \operatorname{Hom}_{\mathcal{C}}(A, B) \to \operatorname{Hom}_{\mathcal{C}}(A, C)$ given by $g \mapsto f \circ g$.

There is also a contravariant functor $\operatorname{Hom}_{\mathcal{C}}(-, A)$ which maps each object B of $\mathcal{C}$ to $\operatorname{Hom}_{\mathcal{C}}(B, A)$ and each morphism $f : B \to C$ of $\mathcal{C}$ to the function $\operatorname{Hom}_{\mathcal{C}}(f, A) : \operatorname{Hom}_{\mathcal{C}}(C, A) \to \operatorname{Hom}_{\mathcal{C}}(B, A)$ given by $g \mapsto g \circ f$.

We can also define maps between these functors.

Definition 2.1.15. [35, p. 59] Let F and G be functors between categories $\mathcal{C}$ and $\mathcal{D}$. Then a **natural transformation** π from F to G is a collection of morphisms consisting of one morphism from $F(A)$ to $G(A)$ for each $A \in \operatorname{ob} \mathcal{C}$, such that the following diagram commutes for each morphism $f : A \to B$ in $\mathcal{C}$:

$$\begin{tikzcd} F(A) \arrow[r, "F(f)"] \arrow[d, "\pi_A"] & F(B) \arrow[d, "\pi_B"] \\ G(A) \arrow[r, "G(f)"] & G(B) \end{tikzcd}$$

Now, as we have a collection of objects (our functors), and something like morphisms between them, it is natural to assemble these into a category.

Definition 2.1.16. [42, p. 27] If $\mathcal{C}$ and $\mathcal{D}$ are categories, with $\mathcal{D}$ small, the **functor category** $\mathcal{D}^{\mathcal{C}}$ is the category whose objects are the functors $\mathcal{C} \to \mathcal{D}$, and whose morphisms are the natural transformations between those functors.

Above, we require $\mathcal{D}$ to be small only so that the hom-sets of $\mathcal{D}^{\mathcal{C}}$ are sets, rather than proper classes.

We define also the following naturally arising properties of functors.

Definition 2.1.17. [35, pp. 51-52] A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ is **faithful** if, for all objects $X, Y \in \operatorname{ob} \mathcal{C}$, the function

$$F_X^Y : \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is injective, and **full** if F_X^Y is surjective.

A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ between categories with zero morphisms is **exact** if for all exact sequences $X \xrightarrow{f} Y \xrightarrow{g} Z$ in $\mathcal{C}$, the sequence

$$F(X) \xrightarrow{F(f)} F(Y) \xrightarrow{F(g)} F(Z)$$

is exact in $\mathcal{D}$.

A (covariant) functor $F : \mathcal{C} \to \mathcal{D}$ between categories with zero morphisms is **left-exact** if for all exact sequences

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathcal{C}$, the sequence

$$0 \to F(X) \xrightarrow{F(f)} F(y) \xrightarrow{F(g)} F(z)$$

is exact. Dually, F is **right-exact** if for all exact sequences

$$0 \to X \xrightarrow{f} Y \xrightarrow{g} Z \to 0$$

in $\mathcal{C}$, the sequence

$$F(X) \xrightarrow{F(f)} F(y) \xrightarrow{F(g)} F(z) \to 0$$

is exact.

A functor $F : \mathcal{C} \to \mathcal{D}$ is an **embedding** if it is faithful and also the induced function $F : \operatorname{ob} \mathcal{C} \to \operatorname{ob} \mathcal{D}$ is injective.

Another key definition is the following, which allows us to take limits of functors, rather than of diagrams.

Definition 2.1.18. If $F : \mathcal{C} \to \mathcal{D}$ is a contravariant functor, then the **limit** of F is an object $\varprojlim F$ of $\mathcal{D}$ together with morphisms $\varphi_A : \varprojlim F \to F(A)$ for every object A of $\mathcal{C}$ such that for each morphism $\xi : A \to B$ of $\mathcal{C}$, we have $\varphi_A = F\xi\varphi_B$, that is universal with this property. That is, for each object X of $\mathcal{D}$, there is a unique morphism χ making the following diagram commute for all $\xi : A \to B$ of $\mathcal{C}$:

$$\begin{tikzcd}
& & F(A) \\
X \arrow[r, dashed, "\chi"] \arrow[urr, bend left, "\psi_A"] \arrow[drr, bend right, "\psi_B"'] & \varprojlim F \arrow[ur, "\varphi_A"] \arrow[dr, "\varphi_B"] & \\
& & F(B) \arrow[uu, "F(\xi)"']
\end{tikzcd}$$

We can now define the following classes of categories.

Definition 2.1.19. A **subcategory** $\mathcal{D}$ of a category $\mathcal{C}$ is a category whose objects are some subclass of $\operatorname{ob}\mathcal{C}$ and such that, for each pair (A, B) of objects of $\mathcal{D}$, we have

$$\operatorname{Hom}_{\mathcal{D}}(A, B) \subseteq \operatorname{Hom}_{\mathcal{C}}(A, B).$$

In this case, we call the functor $F : \mathcal{D} \to \mathcal{C}$ with $F(A) = A$ for each $A \in \operatorname{ob}\mathcal{C}$ and

$$F(A \xrightarrow{\varphi} B) = A \xrightarrow{\varphi} B$$

for each $\varphi \in \operatorname{Hom}_{\mathcal{D}}(A, B)$ the **inclusion functor**.

Definition 2.1.20. A category $\mathcal{C}$ is **concrete** if it is a subcategory of **Set**: that is, if its objects are sets, and its morphisms are (not necessarily all of the) functions between those sets.

We shall require also the following concept.

Definition 2.1.21. [42, pp. 257, 258] An **adjoint pair** of functors is a pair of functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ such that there are natural isomorphisms

$$\operatorname{Hom}_{\mathcal{C}}(X, GY) \cong \operatorname{Hom}_{\mathcal{D}}(FX, Y)$$

for all $X \in \operatorname{ob}\mathcal{C}$ and all $Y \in \operatorname{ob}\mathcal{D}$. If (F, G) is an adjoint pair, then F is **left-adjoint** to G, and G is **right-adjoint** to F.

2.1.3 Abelian Categories

There is one category that has particularly nice properties.

Definition 2.1.22. [42, Example. 1.4(i)] We denote the category of all abelian groups by **Ab**.

Now, we consider a class of categories that are "close enough" to **Ab** to share its nice properties.

Definition 2.1.23. [42, p. 307] A category $\mathcal{C}$ is **abelian** if:

- There is some **zero object** $0 \in \operatorname{ob} \mathcal{C}$, such that for every $A \in \operatorname{ob} \mathcal{C}$, there is exactly one morphism $0 \to A$, and exactly one morphism $A \to 0$.
- For every $A, B \in \operatorname{ob} \mathcal{C}$, there is some $A \oplus B \in \operatorname{ob} \mathcal{C}$, that is both the product and coproduct of $\{A, B\}$.
- For every $A, B \in \operatorname{ob} \mathcal{C}$, and every $f \in \operatorname{Hom}(A, B)$, there are some $k : K \to A$, and $c : B \to C$ in $\mathcal{C}$ such that k is a kernel for f, and c is a cokernel for f.
- Every monomorphism in $\mathcal{C}$ is the kernel of some morphism in $\mathcal{C}$, and every epimorphism in $\mathcal{C}$ is the cokernel of some morphism in $\mathcal{C}$.

As a first example (other than **Ab** itself), we have the following result, which is [42, Prop. 5.93]:

Lemma 2.1.24. *Let $\mathcal{C}$ be an abelian category, and let $\mathcal{D}$ be a small category. Then the category $\mathcal{C}^{\mathcal{D}}$ of functors (with natural transformations as morphisms) from $\mathcal{D}$ to $\mathcal{C}$ is abelian.*

Proof. We work essentially component-wise: the zero object of $\mathcal{C}^{\mathcal{D}}$ is the functor $\mathbf{0}$ assigning the zero object of $\mathcal{C}$ to every object of $\mathcal{D}$ (with the unique maps between them). For any $F \in \operatorname{ob}(\mathcal{C}^{\mathcal{D}})$, the maps from the zero object of C out to each other object assemble to give a natural transformation from $\mathbf{0}$ to F, and this is unique as any other natural transformation would have to differ at some coordinate, thus contradicting the uniqueness of the maps in $\mathcal{C}$. Similarly, there is a unique natural transformation from F to $\mathbf{0}$.

Our biproduct is then similarly given component-wise: for any two functors F and G, for each $D \in \operatorname{ob} \mathcal{D}$, there is a biproduct of $F(D)$ and $G(D)$ in $\mathcal{C}$, and the functor sending each D to $F(D) \oplus G(D)$ satisfies the universal property for the product in each coordinate, and hence the diagrams assemble to form a diagram in the functor category.

Similarly, given any natural transformation $\pi : F \to G$, we can assemble the (co)kernels of the $\pi_D : F(D) \to G(D)$ into natural transformations, which form a (co)kernel for π.

Finally, for each monomorphism (respectively epimorphism) π in the functor category, each component of π is a monomorphism (respectively epimorphism), so is the kernel (respectively cokernel) for some other morphism, and we can assemble these other morphisms to give a natural transformation of which π is the kernel (respectively cokernel). $\square$

The following results, which are found in [35, Thm. 7.2] and [35, pp. 94, 97] respectively, formalise our concept of abelian categories being "close enough" to **Ab** (or more generally, to the category ${}_R\mathbf{Mod}$ of modules over a ring R), and thereby simplify many proofs, by allowing us to work in ${}_R\mathbf{Mod}$, rather than in some other abelian category that may be less amenable to study.

Theorem 2.1.25. *If $\mathcal{C}$ is a small abelian category, then there is a ring R and a full faithful exact covariant imbedding $F : \mathcal{C} \to {}_R\mathbf{Mod}$.*

See [35, p. 151] for a proof.

Theorem 2.1.26 (Metatheorem)**.** *Let T be a theorem of the form "p implies q", where p is a statement about a finite diagram D that states that some parts of that diagram:*

- *are/are not commutative,*
- *are/are not exact sequences, and/or*
- *are/are not limits/colimits*

and q states that (zero or finitely many) additional morphisms exist between certain objects of D, and that some parts of the diagram resulting from adding those morphisms to D:

- *are/are not commutative,*
- *are/are not exact sequences, and/or*
- *are/are not limits/colimits.*

Then if the theorem is true in the category of R-modules over all rings R, it is true in all abelian categories.

Proof. All statements of this form are preserved by the imbedding of Theorem 2.1.25, so this follows immediately from that result. $\square$

Abelian categories are useful for many reasons, including the following result, which is standard, and is found, for example, as [42, Prop. 2.27].

Lemma 2.1.27 (Five Lemma)**.** *If the following is a commutative diagram in an abelian category with exact rows where β and δ are isomorphisms, α is epic and ε is monic, then there is a map γ making the diagram commute, and γ is an isomorphism.*

$$\begin{tikzcd} A \arrow[r, "\zeta"] \arrow[d, "\alpha"] & B \arrow[r, "\eta"] \arrow[d, "\beta"] & C \arrow[r, "\theta"] \arrow[d, dashed, "\gamma"] & D \arrow[r, "\iota"] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\ F \arrow[r, "\kappa"] & G \arrow[r, "\lambda"] & H \arrow[r, "\mu"] & I \arrow[r, "\nu"] & J \end{tikzcd}$$

Proof. By Theorem 2.1.26, it suffices to show this for modules over a ring R, so we may assume that our categories are concrete, so that "monic" and "epic" are equivalent to "injective" and "surjective" respectively, and we may select elements from our objects.

First, we construct γ

To show that γ is epic, consider any $h \in H$. Then $\delta^{-1}\mu(h) \in D$, and $\varepsilon\iota(\delta^{-1}\mu(h)) = \nu\mu(h) = 0$ since the right-hand square commutes and the bottom row is exact, so $\iota(\delta^{-1}\mu(h)) = 0$ since ε is monic, so there is some $c \in C$ such that $\theta(c) = \delta^{-1}\mu(h)$, so $\mu(h) = \delta\theta(c) = \mu\gamma(c)$, so $\mu(\gamma(c) - h) = 0$, so there is some $g \in G$ such that $\lambda(g) = \gamma(c) - h$, since the bottom row is exact.

Since the second square commutes, we have $\gamma(c) - h = \lambda(g) = \gamma\eta\beta^{-1}(g)$, so $h = \gamma(c - \eta\beta^{-1}(g))$, which lies in the image of γ.

To show that γ is monic, consider some $c \in C$ such that $\gamma(c) = 0$. Then $\delta\theta(c) = \mu\gamma(c) = 0$ since the third square commutes, so $\theta(c) = 0$ since δ is monic, so there is some $b \in B$ such that $\eta(b) = c$ by exactness of the top row. Then $\lambda\beta(b) = 0$ since the second square commutes, so by exactness of the bottom row, there is some $f \in F$ such that $\kappa(f) = \beta(b)$. Since α is epic, there is some $a \in A$ such that $\alpha(a) = f$, and since the left-most square commutes, $\zeta(a) = \beta^{-1}\kappa\alpha(a) = b$, so $c = \eta(b) = \eta\zeta(a) = 0$, since the top row commutes. $\square$

2.1.4 Coslice Categories

The following shall be of minor use in our work, but may be of more use in further generalising it.

Definition 2.1.28. [1, p. 17] For $\mathcal{C}$ a category and x_0 an object of $\mathcal{C}$, the **coslice category** $x_0/\mathcal{C}$ is the category whose objects are the $x_0 \to y$ of $\mathcal{C}$

whose source is x_0, and whose morphisms are the commuting triangles (in $\mathcal{C}$) of the form

$$\begin{tikzcd} & x_0 \arrow[dl] \arrow[dr] & \\ y \arrow[rr] & & z \end{tikzcd}$$

The following lemma, which is [32, Lemma. 2.3.5], encodes the property that may make this concept useful for such generalisations of the results of Chapter 3.

Lemma 2.1.29. *For $\mathcal{C}$ a category and x an object of $\mathcal{C}$, the coslice category $x/\mathcal{C}$ has initial object the identity morphism $i : x_0 \to x_0$.*

Proof. For any morphism $f : x_0 \to y$ of $\mathcal{C}$, there is a morphism in $c/\mathcal{C}$ from i to f given by

$$\begin{tikzcd} & x_0 \arrow[dl, "i"'] \arrow[dr, "f"] & \\ x_0 \arrow[rr, "f"] & & y \end{tikzcd}$$

which is clearly unique, so i is initial in $\mathcal{C}$. □

2.1.5 Projectives and Injectives

Projective objects, and projective resolutions, will be key in Chapter 3. We now establish the basic properties of these that we shall require. We shall also briefly require the dual concept of injective objects and resolutions, in order to define our homology.

Definition 2.1.30. [35, p. 69] An object P in a category $\mathcal{C}$ is **projective** if, for every epimorphism $\varphi : A \to B$, the map $\mathrm{Hom}_{\mathcal{C}}(P, A) \to \mathrm{Hom}_{\mathcal{C}}(P, B)$ induced by φ is surjective. Dually, P is **injective** if, for every monomorphism $\varphi : A \to B$, the map $\mathrm{Hom}_{\mathcal{C}}(A, P) \to \mathrm{Hom}_{\mathcal{C}}(B, P)$ induced by φ is surjective.

First, a simple lemma, found as [42, Cor. 3.6].

Lemma 2.1.31. *If $\mathcal{P}$ is a collection of projective objects in an abelian category $\mathcal{C}$, then*

$$\bigoplus_{P \in \mathcal{P}} P$$

is projective.

Proof. Let $\varphi : A \to B$ be an epimorphism. Then

$$\varphi_P := \mathrm{Hom}_{\mathcal{C}}(P, A \xrightarrow{\varphi} B)$$

is an epimorphism (in **Ab**) for all $P \in \mathcal{P}$, so is surjective, since the epimorphisms in **Ab** are the surjectives. Now, the map

$$\Phi : \mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_{P \in \mathcal{P}} P, A\right) \to \mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_{P \in \mathcal{P}} P, B\right)$$

defined by

$$\Phi(f) = \sum_{P \in \mathcal{P}} \varphi \circ f_P$$

is precisely

$$\mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_{P \in \mathcal{P}} P, A \xrightarrow{\varphi} B\right)$$

so it suffices to show that this is surjective.

But for each

$$f \in \mathrm{Hom}_{\mathcal{C}}\left(\bigoplus_{P \in \mathcal{P}} P, B\right),$$

each component $f_P \in \mathrm{Hom}_{\mathcal{C}}(P, B)$ must be given by $f_P = \varphi_P(g_P)$ for some $g_P : P \to A$ (since φ_P is surjective), so with $g := \sum g_P$, we have

$$\Phi(g) = \sum_{P \in \mathcal{P}} \varphi \circ g_P = \sum_{P \in \mathcal{P}} f_P = f,$$

so indeed Φ is surjective, and

$$\bigoplus_{P \in \mathcal{P}} P$$

is projective. $\square$

The following is what we shall need projective and injective objects for.

Definition 2.1.32. [42, p. 325] A **projective resolution** P of an object X in an abelian category $\mathcal{C}$ is an exact sequence

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to X \to 0$$

such that each P_i is a projective object.

Dually, an **injective resolution** Q of an object X in an abelian category $\mathcal{C}$ is an exact sequence

$$0 \to X \to Q_0 \to Q_1 \to \cdots \to Q_{n-1} \to Q_n \to \cdots$$

such that each Q_i is an injective object.

Definition 2.1.33. An abelian category $\mathcal{C}$ has **enough projectives** if for every $X \in \operatorname{ob}\mathcal{C}$, there is a projective $P \in \operatorname{ob}\mathcal{C}$ and an epimorphism $P \to X$.

Dually, $\mathcal{C}$ has **enough injectives** if for every $X \in \operatorname{ob}\mathcal{C}$, there is an injective $Q \in \operatorname{ob}\mathcal{C}$ and a monomorphism $X \to Q$.

To define the cohomology of categories, we will require the following result, which is (the dual form of) [42, Thm. 6.16]

Theorem 2.1.34. *Let $\mathcal{C}$ be an abelian category, and suppose that we have the following diagram in $\mathcal{C}$ with each Q_X^n and Q_Y^n injective and exact columns:*

$$\begin{tikzcd}
0 \arrow[d] & 0 \arrow[d] \\
X \arrow[r, "f"] \arrow[d, "g"] & Y \arrow[d, "h"] \\
Q_X^0 \arrow[d, "d^0"] & Q_Y^0 \arrow[d, "e^0"] \\
\vdots \arrow[d] & \vdots \arrow[d] \\
Q_X^n \arrow[d, "d^n"] & Q_Y^n \arrow[d, "e^n"] \\
\vdots & \vdots
\end{tikzcd}$$

Then there is a chain map $\widehat{f} : Q_X \to Q_Y$ making the completed diagram commute, and all such chain maps are homotopic.

Proof. By Theorem 2.1.26, we work in the case $\mathcal{C} = {}_R\mathbf{Mod}$ for some ring R.

To show the existence of $\widehat{f}$, we proceed by induction on n. If $n = 0$, note that h is a monomorphism, so there is a map $\widehat{f}^0 : Q_X^0 \to Q_Y^0$ with $fg = h\widehat{f}^0$.

Now, for $n > 0$, consider the following diagram:

$$\begin{tikzcd}
Q_X^{n-1} \arrow[r, "\widehat{f}^{n-1}"] \arrow[d, "d^{n-1}"] & Q_Y^{n-1} \arrow[d, "e^{n-1}"] \\
Q_X^n \arrow[r, "\widehat{f}^n"] \arrow[d, "d^n"] & Q_Y^n \arrow[d, "e^n"] \\
Q_X^{n+1} & Q_Y^{n+1}
\end{tikzcd}$$

Define $C = \operatorname{coker}(d^{n-1}) = Q^n_X/\ker(d^n)$, so that the map $\delta : C \to Q^{n+1}_X$ is injective.

And $e^n \widehat{f}^n d^{n-1} = e^n e^{n-1} \widehat{f}^{n-1} = 0$, so $\operatorname{im}(d^{n-1}) \subseteq \ker(e^n \widehat{f}^n)$. Thus, $e^n \widehat{f}^n$ passes to a map $\varepsilon : C \to Q^{n+1}_Y$. But since Q^{n+1}_Y is injective, there is then a map $\widehat{f}^{n+1} : Q^{n+1}_X \to Q^{n+1}_Y$ such that $\widehat{f}^{n+1}\delta = \varepsilon$, and so $\widehat{f}^{n+1} d^n = e^n \widehat{f}^n$, as required.

Now, if $\widetilde{f} : Q_X \to Q_Y$ is another chain map mapping this diagram commute, we construct our homotopy by induction.

First, treat X and 0 as terms -1 and -2 of the left-hand sequence, with $d^{-1} = g$ and $d^{-2} = 0$, and similarly treat Y and 0 as terms -1 and -2 of the right-hand sequence, with $e^{-1} = h$ and $e^{-2} = 0$. Define also

$$\begin{aligned} &\widehat{f}^{-1} = \widetilde{f}^{-1} = f \\ &s^{-1} = 0 s^{-2} = 0. \end{aligned}$$

Then we have $\widetilde{f}^{-1} - \widehat{f}^{-1} = f - f = 0 = e^0 s^{-1} + s^{-2} e^{-1}$, so s^{-1} and s^{-2} can form the first two terms of our homotopy.

Now, if we can show that $(\widetilde{f}^n - \widehat{f}^n - e^{n-1}s^n)(\operatorname{im} d^{n-1}) = 0$, the injectivity of Q^n_Y will give a map $s^{n+1} : Q^{n+1}_X \to Q^n_Y$ such that $s^{n+1}\delta$ is the map $C \to Q^n_Y$ induced by $\widetilde{f}^n - \widehat{f}^n - e^{n-1}s^n$, and so $s^{n+1}d^n + e^{n-1}s^n = \widetilde{f}^n - \widehat{f}^n$, and hence extend our homotopy to all terms.

Now,

$$\begin{aligned} (\widetilde{f}^n - \widehat{f}^n - e^{n-1}s^n)d^{n-1} &= (\widetilde{f}^n - \widehat{f}^n)d^{n-1} - e^{n-1}s^n d^{n-1} \\ &= (\widetilde{f}^n - \widehat{f}^n)d^{n-1} - e^{n-1}(\widetilde{f}^{n-1} - \widehat{f}^{n-1} - e^{n-2}s^{n-1}) \\ &= (\widetilde{f}^n - \widehat{f}^n)d^{n-1} - e^{n-1}(\widetilde{f}^{n-1} - \widehat{f}^{n-1}) \\ &= (\widetilde{f}^n d^{n-1} - e^{n-1}\widetilde{f}^{n-1}) - (\widehat{f}^n d^{n-1} - e^{n-1}\widehat{f}^{n-1}) \\ &= 0 \end{aligned}$$

with the last equality due to $\widehat{f}$ and $\widetilde{f}$ being chain maps. □

Definition 2.1.35. Let $\mathcal{C}$ be an abelian category with enough injectives, let $\mathcal{D}$ be an abelian category, and let $F : \mathcal{C} \to \mathcal{D}$ be a left-exact covariant functor such that for each pair of objects X and Y of $\mathcal{C}$, the map

$$F : \operatorname{Hom}_{\mathcal{C}}(X, Y) \to \operatorname{Hom}_{\mathcal{D}}(F(X), F(Y))$$

is a homomorphism of abelian groups. Then for every $X \in \mathcal{C}$, there is an injective resolution Q_X of X.

Now, for every morphism $f : X \to Y$ of $\mathcal{C}$, for each $n \in \mathbb{N}$, we have the following diagram in $\mathcal{C}$:

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d] & Y \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] \\ Q_X^n & Q_Y^n \end{tikzcd}$$

with the vertical maps given by composing the maps of the injective resolution. By Theorem 2.1.34, there is a (unique up to homotopy) chain map $\widehat{f} : Q_X \to Q_Y$ making the resulting diagram commute. Applying F to this, we obtain the following:

$$\begin{tikzcd} F(X) \arrow[r, "F(f)"] \arrow[d] & F(Y) \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] \\ F(Q_X^n) \arrow[r, "F(\widehat{f})"] & F(Q_Y^n) \end{tikzcd}$$

Now, the vertical sequences of this diagram, with the top row removed, are not in general exact, but are still cochain complexes, and so we can take their cohomology. Further, as $F(\widehat{f})$ is a chain map and is unique up to homotopy, it passes to unique maps on cohomology. That is, we have unique morphisms

$$H^n(Q_X) \mapsto F(\widehat{f})^* H^n(X_Y)$$

for each n. We now define our nth **right derived functor** F^n of F on objects by $F^n(X) = H^n(Q_X)$ and on morphisms by $F^n(f) = F(\widehat{f})^*$.

It remains to show that these are well-defined: that is, that the derived functors as defined above do not depend on the choice of injective resolutions Q_X, which we do in the following result, which is dual to [42, Prop. 6.20].

Lemma 2.1.36. *If $\mathcal{C}$ is an abelian category with enough injectives, $\mathcal{D}$ and abelian category, $F : \mathcal{C} \to \mathcal{D}$ an additive covariant functor, and $\widetilde{Q}_X$ is an injective resolution of each object X of $\mathcal{C}$ (not necessarily agreeing with the Q_X above), then the right derived functors $\widetilde{F}^n$ arising from these new choices are naturally isomorphic to the $\widetilde{D}^n$.*

Proof. First, apply Theorem 2.1.34 to the following diagram to obtain a chain map $i : Q_X \to \widetilde{Q}_X$.

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d] & X \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] \\ Q_X^n & \widetilde{Q}_X^n \end{tikzcd}$$

Let τ_X be the morphism $F^n X \to \widetilde{F}^n X$ induced by $F(i)$ in homology.

But by applying Theorem 2.1.34 to the diagram

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d] & X \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] \\ \widetilde{Q}_X^n & Q_X^n \end{tikzcd}$$

we obtain another chain map $j : \widetilde{Q}_X \to Q_X$. Let ρ_X be the morphism that it induces in homology. But then composing these together, in each order, gives chain maps $ij : Q_X \to Q_X$ and $ji : \widetilde{Q}_X \to \widetilde{Q}_X$ making the following diagrams commute:

$$\begin{tikzcd} X \arrow[r, "1_X"] \arrow[d] & X \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] \\ Q_X^n \arrow[r, "ij"] & Q_X^n \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, "1_X"] \arrow[d] & X \arrow[d] \\ \vdots \arrow[d] & \vdots \arrow[d] \\ \widetilde{Q}_X^n \arrow[r, "ji"] & \widetilde{Q}_X^n \end{tikzcd}$$

But the identity chain maps on Q_X and $\widetilde{Q}_X$ also make these diagrams commute, so by Theorem 2.1.34, ij and ji are homotopic to those respective identity chain maps, hence also $F(ij)$ and $F(ji)$ are homotopic to their respective identities, so $1_{F^n X} = F(ij)^* = \tau_X \rho_X$ and $1_{\widetilde{F}^n X} = F(ji)^* = \rho_X \tau_X$, so τ_X is an isomorphism.

It remains to show that these form a natural isomorphism. That is, we require the following diagram to commute for each morphism $f : X \to Y$ of

$\mathcal{C}$:

$$\begin{array}{ccc} F^n X & \xrightarrow{\tau_X} & \widetilde{F}^n X \\ \Big\downarrow{\scriptstyle F^n f} & & \Big\downarrow{\scriptstyle \widetilde{F}^n f} \\ F^n Y & \xrightarrow{\tau_Y} & \widetilde{F}^n Y \end{array}$$

But applying Theorem 2.1.34 to the outer two columns of each of the following two diagrams (and applying F and taking cohomology) gives the clockwise and anticlockwise compositions around the above diagram:

$$\begin{array}{ccccc} X & \xrightarrow{1_X} & X & \xrightarrow{f} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ Q^n_X & & \widetilde{Q}^n_X & & \widetilde{Q}^n_Y \end{array} \qquad \begin{array}{ccccc} X & \xrightarrow{f} & Y & \xrightarrow{1_Y} & Y \\ \downarrow & & \downarrow & & \downarrow \\ \vdots & & \vdots & & \vdots \\ \downarrow & & \downarrow & & \downarrow \\ Q^n_X & & Q^n_Y & & \widetilde{Q}^n_Y \end{array}$$

And by the uniqueness part of Theorem 2.1.34, these maps are homotopic, so the resulting maps in cohomology agree. □

2.2 Sheaves on Categories

We now proceed to define what will be our primary objects of study in Chapter 3, and prove their basic properties.

Definition 2.2.1. [35, p. 245] For $\mathcal{C}$ a category, a **presheaf** F on $\mathcal{C}$ is a contravariant functor $F : \mathcal{C} \to {}_R\mathbf{Mod}$ for some ring R (usually, we will have $R = \mathbb{Z}$, so ${}_R\mathbf{Mod} = \mathbf{Ab}$). We call the category that they form $\mathbf{PreSh}\,\mathcal{C}$.

The following well-known result, which generalises [42, p. 5.94], allows us to tie in the above results.

Lemma 2.2.2. *If $\mathcal{C}$ is a category, then $\mathbf{PreSh}\,\mathcal{C}$ is an abelian category.*

Proof. This is immediate from 2.1.24, since $\mathbf{PreSh}\,\mathcal{C} = \mathbf{Ab}^{\mathcal{C}^{op}}$, and the concept of an abelian category is self-dual (that is: the opposite category of an abelian category is abelian). □

We shall make use of two particular presheaves:

Definition 2.2.3. [12, Example. 1, 2] For R a ring and M an R-module, and $\mathcal{C}$ a category, the **constant presheaf on $\mathcal{C}$ with value** M is the presheaf $\Delta M : \mathcal{C} \to {}_R\mathbf{Mod}$ given by $\Delta M(X) = M$ for all objects X of $\mathcal{C}$, and $\Delta M(\varphi) = id_M$ for all morphisms φ of $\mathcal{C}$. We define also, for $f : M \to N$ a morphism of ${}_R\mathbf{Mod}$, the natural transformation $\Delta(f) : \Delta M \to \Delta N$ of presheaves with component at each object X given by $\Delta(f)_x = f$, thus making Δ into a functor.

For each object X of $\mathcal{C}$, we define also the **Yoneda presheaf** $\Upsilon_X M : \mathcal{C} \to {}_R\mathbf{Mod}$ to be the presheaf with

$$\Upsilon_X M(Y) = \begin{cases} M & \text{if there is a morphism } X \to Y \text{ in } \mathcal{C} \\ 0 & \text{otherwise} \end{cases}$$

and with

$$\Upsilon_X(\varphi : Y \to Z) = \begin{cases} id_M & \text{if } \Upsilon_X(Y) = \Upsilon_X(Z) = M \\ 0 & \text{otherwise.} \end{cases}$$

We similarly define, for $f : M \to N$ a morphism of ${}_R\mathbf{Mod}$, the natural transformation $\Upsilon_X(f) : \Upsilon_X M \to \Upsilon_X N$ of presheaves with component at each object y given by $\Upsilon_X(f)_Y = f$ if there is a morphism $X \to Y$ in $\mathcal{C}$, and $\Upsilon_X(f)_Y = 0$ otherwise. Again, this makes Υ_X into a functor.

A key property of the latter is the following, which generalises a result found on [12, p. 3]:

Lemma 2.2.4. *Let $ev_X : \mathbf{PreSh}\,\mathcal{C} \to \mathbf{Ab}$ be the evaluation functor sending presheaves F to $F(X)$ and natural transformations κ to κ_X. Then Υ_X is left adjoint to ev_X.*

Proof. Let $A \in \operatorname{ob} \mathbf{Ab}$ and

$$f : \operatorname{Hom}_{\mathbf{PreSh}\,(\mathcal{C})}(\Upsilon_X A, F) \to \operatorname{Hom}_{\mathbb{Z}}(A, F(X))$$

be given by $f(\kappa) = \kappa_X$.

This is injective, since if $f(\kappa) = f(\lambda)$, then $\kappa_X = \lambda_X$, and so $\kappa_Y = \lambda_Y$ for all Y such that there exists a morphism $X \to Y$ in $\mathcal{C}$, and elsewhere $\kappa_Y = \lambda_Y = 0$. It is surjective, since for any $g \in \operatorname{Hom}_{\mathbb{Z}}(A, F(X))$, the morphism $\kappa : \Upsilon_X A$ to F given by

$$\kappa_Y = \begin{cases} g, & \text{if there exists } X \to Y \text{ in } \mathcal{C} \\ 0, & \text{otherwise.} \end{cases}$$

so $f(\kappa) = g$, as required.

Now, for any $g \in \mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Upsilon_X A, F)$, any $\varphi : B \to A$, and any $b \in B$,

$$\begin{aligned} f((\Upsilon_X\varphi)^*(g))(b) &= ((\Upsilon_X\varphi)^*(g))_X(b) \\ &= (g \circ \Upsilon_X(\varphi))_X(b) \\ &= g_X \circ \varphi(X)(b) \\ &= \varphi^*(g_X)(b) \\ &= \varphi^*(f(g))(b), \end{aligned}$$

and for any $\psi : \Upsilon_X A \to \Upsilon_X B$,

$$\begin{aligned} f(\psi^*(g))(b) &= f(g \circ \psi)(b) \\ &= g_X \circ \psi_X(b) \\ &= (\psi_X)^*(g_X)(b) \\ &= ev_X(\psi)^*(f(g))(b). \end{aligned}$$

So f is natural and these functors are an adjoint pair, as required. $\square$

The former has the following analogous key property, which generalises [48, App. 2.6.7].

Lemma 2.2.5. *The functors*

$$\Delta : {}_R\mathbf{Mod} \to \mathbf{PreSh}\,\mathcal{C}$$

and

$$\varprojlim : \mathbf{PreSh}\,\mathcal{C} \to {}_R\mathbf{Mod}$$

form an adjoint pair.

Proof. Define

$$\varphi : \mathrm{Hom}_{{}_R\mathbf{Mod}}(M, \varprojlim F) \to \mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Delta M, F)$$

given by defining $\varphi(f)_X : M \to F(X)$ to be the composition $\psi_X f$, where $\psi_X : \varprojlim F \to F(X)$ is the map of the definition of the limit.

Now, φ is bijective, since if $\kappa : \Delta M \to F$, then by definition of the limit, there is a unique map $M \to \varprojlim F$ that is compatible with the $\kappa_X : M \to F$.

This is natural in M since if $f : M \to N$ then with

$$\begin{aligned} \widehat{f} :& \mathrm{Hom}_{{}_R\mathbf{Mod}}(M, \varprojlim F) \to \mathrm{Hom}_{{}_R\mathbf{Mod}}(N, \varprojlim F) \\ & g \mapsto g \circ f \end{aligned}$$

and

$$\begin{aligned}\bar{f}: &\mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Delta M, F) \to \mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Delta N, F)\\ &G \mapsto G \circ f,\end{aligned}$$

(where $(G \circ f)_X := G_X \circ f$ for each object X) we have

$$\varphi \circ \widehat{f}(g) = \varphi(gf) = \psi_X gf$$

and

$$\bar{f} \circ \varphi(g) = \bar{f}(\psi_X g) = \psi_X gf.$$

Finally, this is natural in F, since if $\kappa : G \to F$ then with

$$\begin{aligned}\widehat{\kappa}: &\mathrm{Hom}_{{}_R\mathbf{Mod}}(M, \varprojlim G) \to \mathrm{Hom}_{{}_R\mathbf{Mod}}(M, \varprojlim F)\\ &f \mapsto (\varprojlim \kappa) \circ f)\end{aligned}$$

and

$$\begin{aligned}\bar{\kappa}: &\mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Delta M, G) \to \mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Delta M, F)\\ &\tau \mapsto \kappa\tau,\end{aligned}$$

we have

$$\varphi \circ \widehat{\kappa}(f)_X = \varphi((\varprojlim \kappa)f) = \psi_X(\varprojlim \kappa)f,$$

and

$$(\bar{\kappa} \circ \varphi(f))_X = \bar{\kappa}_X(\psi_X f) = \kappa_X \psi_X f,$$

and by definition, $\varprojlim \kappa$ is the map $\varprojlim G \to \varprojlim F$ such that

$$\psi_X(\varprojlim \kappa) = \kappa_X \psi_X$$

for all X.

Thus, Δ and $\varprojlim$ form an adjoint pair. □

Further, Υ_x preserves projectivity.

Lemma 2.2.6. *For each $X \in \mathrm{ob}\,\mathcal{C}$ and each $A \in \mathrm{ob}\,\mathbf{Ab}$, the Yoneda presheaf $\Upsilon_X A$ is projective in $\mathbf{PreSh}\,\mathcal{C}$ if and only if A is projective.*

Proof. By Lemma 2.2.4,

$$\mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Upsilon_X A, -) \cong \mathrm{Hom}_{\mathbf{Ab}}(A, ev_X(-)),$$

with the isomorphism being natural.

Now, if A is projective, then for each epimorphism $F \xrightarrow{\kappa} G$ of $\mathbf{PreSh}\,\mathcal{C}$, we have that

$$\mathrm{Hom}_{\mathbf{Ab}}(A, ev_X(F \xrightarrow{\kappa} G))$$

is surjective. Passing this through the above adjoint isomorphism, we have that

$$\mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Upsilon_X A, F \xrightarrow{\kappa} G)$$

is surjective, so $\Upsilon_X A$ is projective.

Conversely, if $\Upsilon_X A$ is projective, then for each epimorphism $B \xrightarrow{f} C$ of $\mathbf{Ab}$, we have that

$$\mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Upsilon_X A, \Delta B \xrightarrow{\Delta f} \Delta C)$$

is surjective. Passing through the above adjoint isomorphism, we have that

$$\mathrm{Hom}_{\mathbf{Ab}}(A, ev_X(\Delta B \xrightarrow{\Delta f} \Delta C)) = \mathrm{Hom}_{\mathbf{Ab}}(A, B \xrightarrow{f} C)$$

is surjective, so A is projective. □

We shall require also the following two key properties of the (contravariant) Hom-functors. The former is [42, Thm. 2.38].

Lemma 2.2.7. *Let $\mathcal{C}$ be an abelian category, and let X be an object of $\mathcal{C}$. Then the functor $\mathrm{Hom}_{\mathcal{C}}(X, -) : \mathcal{C} \to \mathbf{Ab}$ is left-exact.*

Proof. By Theorem 2.1.26, it suffices to show this for $\mathcal{C} = {}_R\mathbf{Mod}$ for all rings R.

Let

$$0 \to A \xrightarrow{f} B \xrightarrow{g} C \to 0$$

be a short exact sequence in $\mathcal{C}$. Then we need to show that

$$0 \to \mathrm{Hom}_{\mathcal{C}}(X, A) \xrightarrow{f\circ -} \mathrm{Hom}_{\mathcal{C}}(X, B) \xrightarrow{g\circ -} \mathrm{Hom}_{\mathcal{C}}(X, C)$$

is exact.

Firstly, if $a \in \mathrm{Hom}_{\mathcal{C}}(X, A)$, and $(f \circ -)(a) = 0$, then $af = 0$, and f is injective, so $a = 0$. Thus, $f \circ -$ is injective.

Secondly, since $gf = 0$, also $(g \circ -)(f \circ -) = 0$. On the other hand, if $(g \circ -)(b) = 0$, then $b(X) \subseteq \ker g = f(A)$, so, since f is an isomorphism onto its image, there is a morphism $f^{-1}b \in \mathrm{Hom}_{\mathcal{C}}(X, A)$, and $(f \circ -)(f^{-1}b) = b$. Thus, we have exactness at b, and our sequence is exact. □

The following is a partial converse to the above.

Lemma 2.2.8. *If $\mathcal{C}$ is an abelian category, and*

$$\mathrm{Hom}_{\mathcal{C}}(X, A \xrightarrow{f} B \xrightarrow{g} C)$$

is an exact sequence in **Ab** *for all $X \in \mathrm{ob}\,\mathcal{C}$, then*

$$A \to B \to C$$

is an exact sequence in $\mathcal{C}$.

Proof. Firstly, we have an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(A, A) \xrightarrow{-\circ f} \mathrm{Hom}_{\mathcal{C}}(A, B) \xrightarrow{-\circ g} \mathrm{Hom}_{\mathcal{C}}(A, C),$$

Now, the identity on A lies in the leftmost set, so in particular we have $0 = (- \circ g)(- \circ f)(id_A) = g \circ f$.

Conversely, if $K \xrightarrow{\ker g} B$ is the kernel, then we have an exact sequence

$$\mathrm{Hom}_{\mathcal{C}}(K, A) \xrightarrow{-\circ f} \mathrm{Hom}_{\mathcal{C}}(K, B) \xrightarrow{-\circ g} \mathrm{Hom}_{\mathcal{C}}(K, C),$$

and $(- \circ g)(\ker g) = 0$, so there is some $h : K \to A$ such that $\ker g = f \circ h$.

Thus, $(\ker g)(K) = (f \circ h)(K) \subseteq f(A)$, so indeed, $A \to B \to C$ is exact. □

We can now prove the following key lemma (found as [35, II, Cor. 12.2]).

Lemma 2.2.9. *If $\mathcal{C}$ is a small category, then $\varprojlim : \mathbf{PreSh}\,\mathcal{C} \to {}_R\mathbf{Mod}$ is a left-exact functor.*

Proof. Clearly, $\varprojlim$ is a functor, when extended to morphisms by defining $\varprojlim(F \xrightarrow{\kappa} G)$ to be the unique morphism from $\varprojlim F$ to $\varprojlim G$ such that

$$\kappa_X \varphi_X = \psi_X \circ \varprojlim \kappa,$$

where φ_X and ψ_X are the morphisms from $\varprojlim F$ and $\varprojlim G$ respectively to $F(X)$ and $G(X)$ respectively given by the definition of $\varprojlim F$.

Now, if

$$0 \to F \to G \to H \to 0$$

is a short exact sequence of presheaves and $C \in \mathrm{ob}\,{}_R\mathbf{Mod}$ then by Lemma 2.2.7, there is an exact sequence

$$0 \to \mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Delta C, F) \to \mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Delta C, G) \to \mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Delta C, H).$$

Now, by Lemma 2.2.5, we also have an exact sequence

$$0 \to \mathrm{Hom}_{{}_R\mathbf{Mod}}(C, \varprojlim F) \to \mathrm{Hom}_{{}_R\mathbf{Mod}}(C, \varprojlim G) \to \mathrm{Hom}_{{}_R\mathbf{Mod}}(C, \varprojlim H).$$

Finally, since this holds for all $C \in \mathrm{ob}\, {}_R\mathbf{Mod}$, we have an exact sequence

$$0 \to \varprojlim F \to \varprojlim G \to \varprojlim H$$

by Lemma 2.2.8, so $\varprojlim$ is left-exact. □

This then allows us to define the cohomology of categories, which will be our primary object of study in Chapter 3.

Definition 2.2.10. [2, p. 188] The i**th cohomology of a category** $\mathcal{C}$ **with coefficients in a presheaf** F, also called the i**th higher limit of** F is the result of applying the ith derived functor $\varprojlim^i$ of $\varprojlim$ to F.

2.3 General Homology

2.3.1 Basic definitions

We shall require also the following basic definitions of homology theory. For this section, we fix a ring R.

Definition 2.3.1. [42, p. 239] A **cochain complex** M is a sequence of R-modules M^i with maps $\varphi^i_M : M^i \to M^{i+1}$ such that $\varphi^i \circ \varphi^{i-1} = 0$ for all i. Dually, a **chain complex** is a sequence of R-modules M_i with maps $\varphi^M_i : M_i \to M_{i-1}$ such that $\varphi_i \circ \varphi_{i+1} = 0$ for all i.

We will later see no fewer than two cochain complexes whose cohomology coincides, with some limits on our categories, with the cohomology of categories defined above.

Definition 2.3.2. [42, p. 343] For M a cochain complex with maps φ^i, the ith **cohomology** module H^iM of M is $\ker\varphi^i/\mathrm{im}\varphi^{i-1}$. Dually, the ith **homology** H_iM of a chain complex M with maps φ_i is $\ker\varphi^i/\mathrm{im}\varphi^{i+1}$.

We can make the collection of chain complexes into a category with the following morphisms.

Definition 2.3.3. [42, p. 318] A **chain map** ρ between cochain complexes M and N is a sequence of maps $\rho^i : M^i \to N^i$ such that, for all i, the following diagram commutes:

$$\begin{tikzcd} M^i \arrow[r, "\rho^i"] \arrow[d, "\varphi^i_M"] & N^i \arrow[d, "\varphi^i_N"] \\ M^{i+1} \arrow[r, "\rho^{i+1}"] & N^{i+1} \end{tikzcd}$$

The key property of chain maps that we shall require is the following, found on [42, p. 330].

Lemma 2.3.4. *Let M and N be cochain complexes, and let $\rho : M \to N$ be a chain map between them. Then there is, for all i, a map $\widehat{\rho}^i : H^iM \to H^iN$ induced by ρ.*

Proof. Define $\widehat{\rho}^i([x]) = [\rho^i(x)]$ for each $x \in \ker \varphi^i_M$. It suffices to show that this is well-defined. For that purpose, suppose that $[x] = [y]$ in H^iM. Then $x - y \in \mathrm{im}\varphi^{i-1}_M$, so there is some $z \in M^{i-1}$ such that $\varphi^{i-1}z = x - y$.

But $\rho^i(x) - \rho^i(y) = \rho^i\varphi^{i-1}_M z = \varphi^{i-1}_N\rho^{i-1}z$, so $\rho^i(x) - \rho^i(y) \in \mathrm{im}\varphi^i_N$, so $[\rho^i(x)] = [\rho^i(y)]$, and $\widehat{\rho}^i$ is well-defined. □

Short exact sequences give us information about homology primarily through the following lemma, which is [42, Thm. 6.10].

Lemma 2.3.5. *If $0 \to A \xrightarrow{\rho} B \xrightarrow{\sigma} C \to 0$ is a short exact sequence of cochain complexes in an abelian category, then there is a* ***long exact sequence of homology***

$$\cdots \to H^iA \xrightarrow{\rho^i} H^iB \xrightarrow{\widehat{\sigma}^i} H^iC \xrightarrow{\delta^i} H^{i+1}A \to \cdots$$

Proof. By Theorem 2.1.26, we work in the category of modules over some ring R.

The maps $\widehat{\rho}^i$ and $\widehat{\sigma}^i$ are precisely those given by Lemma 2.3.4 above. To see that this is exact at H^iB, we note that for each $[a] \in H^iA$, we have $\widehat{\sigma}^i\widehat{\rho}^i[a] = [\sigma^i\rho^i(a)] = [0] = 0$, and for $[b] \in H^iB$, if $\widehat{\sigma}^i[b] = 0$, then $\sigma^i(b) \in \mathrm{im}\varphi^{i-1}_C$, so there is some $c \in C^{i-1}$ such that $\varphi^{i-1}_C(c) = \sigma^i(b)$. Further, σ^{i-1} is surjective, so there is some $b' \in B^{i-1}$ such that $\sigma^{i-1}b' = c$.

Now, $\sigma^i(b - \varphi^{i-1}_B b') = \sigma^i(b) - \varphi^{i-1}_C\sigma^{i-1}b' = 0$, so there is some $a \in A^i$ such that $\rho^i(a) = b - \varphi^{i-1}_B b'$. Also, $\rho^{i+1}\varphi^i_A(a) = \varphi^i_B(b - \varphi^{i-1}_B b') = 0$, and ρ^{i+1} is injective, so $\varphi^i_A(a) = 0$, and $\widehat{\rho}^i[a] = [b - \varphi^{i-1}_B b'] = [b]$.

We now construct δ^i. For this purpose, let $[c] \in H^iC$. Then $\varphi^i_C(c) = 0$, and since σ^i is surjective, there is some $b \in B^i$ such that $\sigma^i(b) = c$.

Now, $\sigma^{i+1}\varphi^i_B(b) = \varphi^i_C(c) = 0$, so there is some $a \in A^{i+1}$ such that $\rho^{i+1}(a) = \varphi^i_B(b)$. We define $\delta^i(c) := a$. We now show that this is well-defined.

Firstly, note that a is uniquely determined by b, since ρ^{i+1} is injective.

Secondly, if $\sigma^i(b') = c$, then as above, there is some $a' \in A^{i+1}$ such that $\rho^{i+1}(a) = \varphi^i_B(b')$.

But further, $\sigma^i(b'-b) = 0$, so there is some $a'' \in A^i$ such that $\rho^i(a'') = b'-b$ by exactness, and $\rho^{i+1}\varphi^i_A(a'') = \varphi^i_B(b'-b) = \rho^{i+1}(a'-a)$. But ρ^{i+1} is injective, so $a' - a = \varphi^i_A(a'')$, so $[a] = [a']$.

Finally, if $[c'] = [c]$, then $\varphi_C^i(c' - c) = 0$, so there is some $c'' \in C^{i-1}$ such that $\varphi_C^{i-1}(c'') = c' - c$. But σ^{i-1} is surjective, so there is some $b'' \in B^{i-1}$ such that $\sigma^{i-1}(b'') = c''$. Then with $b' := \varphi_B^{i-1}b'' + b$, we have

$$\begin{aligned}\sigma^i(b') &= \sigma^i(\varphi_B^{i-1}b'' + b)\\ &= \varphi_C^{i-1}\sigma^{i-1}(b'') + c\\ &= c'.\end{aligned}$$

As before, there is some $a' \in A^{i+1}$ such that $\rho^{i+1}(a') = \varphi_B^i(b')$. But

$$\begin{aligned}\rho^{i+1}(a - a') &= \varphi_B^i(b - b')\\ &= \varphi_B^i(b - \varphi_B^{i-1}b'' - b)\\ &= 0,\end{aligned}$$

and ρ^{i+1} is injective, so in fact $a' = a$. □

2.4 Spectral Sequences

We shall prove our first main result using a spectral sequence, though we shall require only a few simple facts from the theory of such.

Definition 2.4.1. [40, § 4] A (cohomological) **spectral sequence** in an abelian category $\mathcal{C}$ consists of:

- For each non-negative integer r, a **page** E_r of objects $E_r^{p,q}$ (with p and q integers).
- Morphisms $d_r^{p,q} : E_r^{p,q} \to E_r^{p+r,q-r+1}$ such that each $E_r^{p+*,q-*+1}$ is a complex with d as its differential.
- Isomorphisms

  $$E_r^{p,q} \to H^{p,q}(E_r) := \frac{\ker d_r^{p,q}}{\mathrm{im}(d_r^{p-r,q+r-1})}$$

  for each p,q, and r.

A key source of spectral sequences is the following.

Definition 2.4.2. [42, pp. 616, 626] For C a cochain complex, a **filtration** of C is an integer-indexed collection of subcomplexes F^*C of C such that $F^nC \subseteq F^{n-1}C$ for all C.

A filtration F^*C is **bounded** if there exist integers $a < b$ such that $F^aC = 0$ and $F^bC = C$.

We generate spectral sequences from these filtrations of cochain complexes as follows.

Definition 2.4.3. [42, p. 622] If F^*C is a filtration, let

$$G^pC^{p+q} := F^pC^{p+q}/F^{p+1}C^{p+q},$$

let $Z^{p,q}$ be the set of all $[c] \in G^pC^{p+q}$ such that $dc \in F^pC^{p+q+1}$, and let $B^{p,q}$ be the image under d of $F^{p+1}C^{p+q-1}$.

Now, each $B^{p,q}$ is contained in F^pC^{p+q}, since each F^pC is a subcomplex of $F^{p-1}C$.

We can thus define the **spectral sequence associated to** F^*C to be the spectral sequence with E_0 page

$$E_0^{p,q} = \frac{Z^{p,q}}{B^{p,q}} = G^pC^{p+q}.$$

In fact, the E_r page of this spectral sequence is exactly

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}},$$

where $Z_r^{p,q}$ is the set of all $[c] \in G^pC^{p+q}$ such that $dc \in F^{p+r}C^{p+q+1}$, and $B_r^{p,q} = dF^{p+r+1}C^{p+q-1}$, since for $[c] \in Z_r^{p,q}$, we have $[dc] = 0$ if and only if $dc \in F^{p+r+1}C^{p+q+1}$, which holds if and only if $[c] \in Z_{r-1}^{p,q}$. Thus,

$$Z_{r-1}^{p,q} = \ker(d|_{Z_r^{p,q}}),$$

so

$$Z_{r-1}^{p,q}/B_{r-1}^{p,q} = \ker(d_r^{p,q})/\mathrm{im}(d_r^{p-r,q+r-1}) = E_{r-1}^{p,q}$$

for all r.

For convenience, we now restrict our scope: from here onwards, all spectral sequences will be the spectral sequences associated to bounded filtrations.

Definition 2.4.4. [6, p. 163] A spectral sequence E **converges** to a filtered complex G^*D if there is some n such that for all $r \geq n$, we have

$$E_r^{p,q} \cong G^pD^{p+q}/G^{p+1}D^{p+q}.$$

We denote this by $E_r^{p,q} \Rightarrow G^*D$, or $E_r^{p,q} \Rightarrow D$ if the filtration is clear.

We shall make use of spectral sequences through the following result, which is [42, Thm. 10.14].

Theorem 2.4.5. *If E is the spectral sequence associated to a bounded filtration F^*C, then E converges to the homology of C (with the latter inheriting its filtration from F^*C).*

Proof. Since F^*C is bounded, for each $p, q \in \mathbb{Z}$, there is some n such that for all $r \geq n$, $F^r_{p,q}C = C$, so

$$E_r^{p,q} = \frac{Z_r^{p,q}}{B_r^{p,q}} = \frac{\ker(d : C^{p+q} \to C^{p+q+1})}{\mathrm{im}(d : C^{p+q-1} \to C^{p+q})} = H^{p+q}C.$$

□

Chapter 3

Cellular Homology of Categories

In this chapter, we generalise some results of [12] from posets to a larger class of categories.

3.1 Setting

Before we begin, we shall need some definitions, beginning with the classes of categories that we shall consider, which generalise the graded posets of [12, § 2.1].

Definition 3.1.1. For $\mathcal{C}$ a small category, we call $\mathcal{C}$ **graded** if there is a sequence $(\mathcal{C}^n)_{n\in\mathbb{Z}}$ of subcategories such that:

1. $\operatorname{ob}\mathcal{C} = \bigcup_{n\in\mathbb{Z}} \operatorname{ob}\mathcal{C}^n$,
2. $\operatorname{Hom}_{\mathcal{C}}(X,Y) = \bigcup_{n\in\mathbb{Z}} \operatorname{Hom}_{\mathcal{C}^n}(X,Y)$ for any $X, Y \in \operatorname{ob}\mathcal{C}$,
3. If $X \in \operatorname{ob}\mathcal{C}^n$, $y \in \operatorname{ob}\mathcal{C}$, and there exists a morphism $X \to Y$ in $\mathcal{C}$, then $Y \in \operatorname{ob}\mathcal{C}^n$; and
4. If $X, Y \in \operatorname{ob}\mathcal{C}^{n+1} \setminus \operatorname{ob}\mathcal{C}^n$, and there exists a morphism $X \to Y$ in $\mathcal{C}^{n+1}$, then $X = Y$.

The first two points here simply ensure that we do not miss any of $\mathcal{C}$ (analogous to requiring that all elements of a graded poset have a rank), and we extend this analogy by defining, for an object X of a graded category, the **rank** $rk(X)$ of X is the unique integer n such that $X \in \operatorname{ob}\mathcal{C}^n \setminus \operatorname{ob}\mathcal{C}^{n-1}$.

We say that a graded category $\mathcal{C}$ **has corank function** if there are integers N and M such that for all $n < N$, $\mathcal{C}^n = \mathcal{C}^N$, and for all $m > M$, $\mathcal{C}^m = \mathcal{C}^M$ (with N maximal and M minimal with this property). In this case, for each $X \in \mathrm{ob}\,\mathcal{C}$, we say that the **corank** of X is $\mathrm{cr}(X) = M - rk(X)$.

The third and fourth conditions above generalise the requirement of [12] that $x < y$ implies $rk(x) < rk(y)$.

For convenience of notation, we define $\widehat{\mathcal{C}^n} = \mathcal{C}^n \setminus \mathcal{C}^{n-1}$.

We insist also that the non-trivial $\hat{\mathcal{C}^n}$ are together, i.e., such that if we have $\mathcal{C}^n = \mathcal{C}^{n+1}$ for some n, then either $\mathcal{C}^k = \mathcal{C}^n$ for all $k < n$ or $\mathcal{C}^k = \mathcal{C}^n$ for all $k > n$. This generalises the the requirement of [12] that $x \prec y$ implies $rk(y) = rk(x) + 1$.

Definition 3.1.2. Let $\mathcal{C}$ be a category. Generalising the definition of [12, § 1.1], we define the **nerve** $N^*\mathcal{C}$ to be the simplicial set with

1. Simplices $N^n\mathcal{C} = \{\sigma = (\sigma_n \xrightarrow{\sigma^n} \dots \xrightarrow{\sigma^1} \sigma_0)\}$ with each σ_i an object of $\mathcal{C}$, and each σ^i a morphism $\sigma_i \to \sigma_{i-1}$ in $\mathcal{C}$.

2. Face maps $d_i : N^n\mathcal{C} \to N^{n-1}\mathcal{C}$ given by
$$d_i\sigma = (\sigma_n \xrightarrow{\sigma^n} \dots \xrightarrow{\sigma^{i+2}} \sigma_{i+1} \xrightarrow{\sigma^i\sigma^{i+1}} \sigma_{i-1} \xrightarrow{\sigma^{i-1}} \dots \xrightarrow{\sigma^1} \sigma_0).$$

3. Degeneracy maps $s_i : N^n\mathcal{C} \to N^{n+1}\mathcal{C}$ given by
$$s_i\sigma = (\sigma_n \xrightarrow{\sigma^n} \dots \xrightarrow{\sigma^{i+1}} \sigma_i \xrightarrow{id} \sigma_i \xrightarrow{\sigma^i} \dots \xrightarrow{\sigma^1} \sigma_0).$$

We define also the subcomplex $\widehat{N}^n\mathcal{C}$ consisting of all simplices σ such that no σ^i is the identity, and also the subset $N_0^n\mathcal{C} := N^n\mathcal{C} \setminus \widehat{N}^n\mathcal{C}$.

In order to compute the homology of categories, we require the following cochain complex.

Definition 3.1.3. For F a presheaf on a graded category $\mathcal{C}$, let $S^*(\mathcal{C}; F)$ be the cochain complex such that:

1. We have $S^n(\mathcal{C}; F) = \prod_{\sigma \in N^n\mathcal{C}} F(\sigma_n)$.

2. For $s \in S^n(\mathcal{C}; F)$ and $\sigma \in N^n\mathcal{C}$, we denote the component of s at σ by $s \cdot \sigma$.

3. The differential $d : S^{n-1}(\mathcal{C}; F) \to S^n(\mathcal{C}; F)$ is given by

$$ds \cdot \sigma = \sum_{i=0}^{n-1} (-1)^i s \cdot d_i\sigma + (-1)^n F(\sigma^n)(s \cdot d_n\sigma),$$

for $\sigma \in N^n\mathcal{C}$ and $s \in S^{n-1}(\mathcal{C}; F)$.

We define also another complex $T^*(\mathcal{C}; F)$ as the subcomplex of $S^*(\mathcal{C}; F)$ consisting of all elements $s \in S^n(\mathcal{C}; F)$ such that $s \cdot \sigma = 0$ for all degenerate simplices $\sigma \in N_0^n\mathcal{C}$.

This is the natural generalisation of the definitions from [12, pp. 3-4].

The poset case of the following result is mentioned but not proved on [12, p. 3]

Lemma 3.1.4. *$S^*(\mathcal{C}; F)$ and $T^*(\mathcal{C}; F)$ are, indeed, cochain complexes.*

Proof. Firstly, we note that $d_j d_i \sigma = d_{i-1} d_j \sigma$ for all $i > j$, and thus, for $\sigma \in N^{n+1}\mathcal{C}$ and $s \in S^{n-1}(\mathcal{C}; F)$, we have

$$\begin{aligned}
d^2 s \cdot \sigma &= \sum_{i=0}^{n} (-1)^i ds \cdot d_i\sigma + (-1)^{n+1} F(\sigma^n) ds \cdot d_{n+1}\sigma \\
&= \sum_{i=0}^{n} (-1)^i \left(\sum_{j=0}^{n-1} (-1)^j s \cdot d_j d_i \sigma + (-1)^n F((d_i\sigma)^n) s \cdot d_n d_i \sigma \right) \\
&+ (-1)^{n+1} F(\sigma^{n+1}) \left(\sum_{i=0}^{n-1} (-1)^i s \cdot d_i d_{n+1}\sigma + (-1)^n F(\sigma^n) s \cdot d_n d_{n+1}\sigma \right) \\
&= \sum_{0 \le j < i < n} (-1)^{i+j} s \cdot d_j d_i \sigma + \sum_{0 \le i < j < n} (-1)^{i+j+1} d_i d_j \sigma \\
&+ \sum_{0 \le i < n} (-1)^{i+n} F(\sigma^{n+1}) s \cdot d_i d_{n+1}\sigma + \sum_{0 \le i < n} (-1)^{i+n+1} F((d_i\sigma)^n) s \cdot d_i d_{n+1}\sigma \\
&+ (-1)^{2n} F((d_n\sigma)^n) s \cdot d_n d_{n+1}\sigma + (-1)^{2n+1} F(\sigma^{n+1}) F(\sigma^n) s \cdot d_n d_{n+1}\sigma.
\end{aligned}$$

Now, in this final expression, the first two lines are clearly zero, with the latter sum on each line being simply the former multiplied by -1. The final line is also zero, as $(d_n\sigma)^n$ is the nth map of $d_n\sigma$, which by definition of d_n, is $\sigma^{n+1}\sigma^n$, and since F is a functor, we have $F((d_n\sigma)^n) = F(\sigma^{n+1}\sigma^n) = F(\sigma^{n+1})F(\sigma^n)$, so this line cancels in the same way as the previous two. Thus, $S^*(\mathcal{C}; F)$ is a complex.

To show the same for T, it suffices to show that if $t \in T^n(\mathcal{C}; F)$, then $dt \in T^{n+1}(\mathcal{C}; F)$. But this is simple:

Let $\sigma \in N_0^{n+1}\mathcal{C}$. Then

$$dt \cdot \sigma = \sum_{i=0}^{n-1} t \cdot d_i\sigma + (-1)^n F(\sigma^n) t \cdot d_n\sigma.$$

We shall now show that this is zero. First, we note that if $d_i\sigma \in N_0^n\mathcal{C}$, then $t \cdot d_i\sigma = 0$ by definition of $T^n(\mathcal{C}; F)$.

Thus, we need only consider the situation where $d_i\sigma \in \widehat{N}^n\mathcal{C}$. But this can occur in exactly two ways: σ must have exactly one identity morphism, and it must be either σ^i or σ^{i+1}. Firstly, consider the case $i < n-1$. By adjusting i if necessary, suppose that σ^{i+i} is the identity and $i < n-1$. Then $d_i\sigma = d_{i+1}\sigma$, and our sum becomes

$$dt \cdot \sigma = (-1)^i t \cdot d_i\sigma + (-1)^{i+1} t \cdot d_i\sigma = 0.$$

Finally, if $i = n-1$, our sum becomes $(-1)^{n-1} t \cdot d_{n-1}\sigma + (-1)^n F(\sigma^n) t \cdot d_n\sigma$. But now, σ^n is the identity morphism, hence $F(\sigma^n)$ is the identity morphism, so this is again zero, completing the proof. □

The following result is the natural generalisation of the result at the top of [12, p. 4].

Lemma 3.1.5. *The complexes $S^*(\mathcal{C}; F)$ and $T^*(\mathcal{C}; F)$ are homotopy equivalent.*

Proof. We proceed essentially as in the proof of [48, Theorem. 8.3.8]. As noted in that proof, it suffices to show that the complex

$$U^*(\mathcal{C}; F) := S^*(\mathcal{C}; F)/T^*(\mathcal{C}; F)$$

is homotopy equivalent to the zero complex.

So we require a homotopy equivalence between the identity and zero on $U^*(\mathcal{C}; F)$. That is, we require a chain map $h : U^n(\mathcal{C}; F) \to U^{n-1}(\mathcal{C}; F)$ such that $dh - hd = 0$, where d is the differential on $U^*(\mathcal{C}; F)$ inherited from $S^*(\mathcal{C}; F)$.

For $\sigma \in N_0^n\mathcal{C}$, p an integer at most $n-1$, and l a positive integer, we define $P(\sigma, p, l)$ to be 1 if:

- $\sigma^1, \dots, \sigma^p$ are not the identity,
- $\sigma^{p+1}, \dots, \sigma^{p+l}$ are the identity, and
- σ^{p+l+1} is not the identity;

and 0 otherwise.

Now, for $u \in U^n(\mathcal{C}; F)$ and $\sigma \in N_0^{n-1}\mathcal{C}$, we define

$$hu \cdot \sigma = \begin{cases} (-1)^p u \cdot s_p\sigma & \text{if there is some odd } l \text{ such that } P(\sigma, p, l) = 1 \\ 0 & \text{otherwise} \end{cases}$$

Now, we note that, for $\sigma \in N^n\mathcal{C}$, there are unique p and l such that $P(\sigma, p, l) = 1$, and

- For any $i < p$, we have $P(d_i\sigma, p-1, l) = 1$,
- For any $i > p + l$, we have $P(d_i\sigma, p, l) = 1$, and
- If $l \geq 2$, then for any i such that $p \leq i \leq p+l$, we have that $hu \cdot d_i\sigma = 0$ (as the lowest-indexed string of identity maps is now of even length).

In the case where l is odd, $l \neq 1$, and $p + l < n$ for any $u \in U^n(\mathcal{C}; F)$, we therefore have

$$\begin{aligned} dhu \cdot \sigma =& \sum_{i=1}^{n-1} (-1)^i hu \cdot d_i\sigma + (-1)^n F(\sigma^n) hu \cdot d_n\sigma \\ =& \sum_{i=1}^{p-1} (-1)^{i+p} u \cdot s_{p-1} d_i\sigma + \sum_{i=p+l+1}^{n-1} (-1)^{i+p} u \cdot s_p d_i\sigma \\ &+ (-1)^{n+p} F(\sigma^n) u \cdot s_p d_i\sigma, \end{aligned}$$

and

$$\begin{aligned} hdu \cdot \sigma =& \sum_{i=1}^{n} (-1)^{i+p} u \cdot d_i s_p\sigma + (-1)^{n+p+1} F((s_p\sigma)^{n+1}) u \cdot d_{n+1} s_p\sigma \\ =& \sum_{i=1}^{n} (-1)^{i+p} u \cdot d_i s_p\sigma + (-1)^{n+p+1} F(\sigma^n) u \cdot d_{n+1} s_p\sigma. \end{aligned}$$

Combining these, we obtain

$$\begin{aligned} (dh - hd)u \cdot \sigma =& \sum_{i=1}^{p-1} (-1)^{i+p} \left(u \cdot s_{p-1} d_i\sigma - u \cdot d_i s_p\sigma \right) \\ &+ \sum_{i=p}^{p+l} (-1)^{i+p} u \cdot \sigma \\ &+ \sum_{i=p+l+1}^{n-1} (-1)^{i+p} (u \cdot s_p d_i\sigma - u \cdot d_i s_p\sigma) \\ &+ (-1)^{n+p} F(\sigma^n) u \cdot s_p d_i\sigma - (-1)^{n+p} u \cdot d_i s_p\sigma \\ &- (-1)^{n+p+1} F(\sigma^n) u \cdot d_n s_p\sigma. \end{aligned}$$

Now, the first sum is zero by the simplicial identities of [48, p. 256], as $u \cdot d_i s_p \sigma = u \cdot s_{p-1} d_i \sigma$ for $i < p$, the second is zero since l is odd (so we have evenly many terms here), and the remainder cancels telescopically, except for the latter half of the first remaining term. That is:

$$\begin{aligned}(dh - hd)u \cdot \sigma &= (-1)^{2p+l+1} u \cdot s_p d_{p+l+1} \sigma \\ &= u \cdot d_{p+l} s_p \sigma \\ &= u \cdot \sigma.\end{aligned}$$

In the case where $l > 1$ is odd and $p + l = n$, we instead have

$$\begin{aligned}dhu \cdot \sigma &= \sum_{i=1}^{n-1} (-1)^i hu \cdot d_i \sigma + (-1)^n F(\sigma^n) hu \cdot d_n \sigma \\ &= \sum_{i=1}^{p-1} (-1)^{i+p} u \cdot s_{p-1} d_i \sigma\end{aligned}$$

and we still have

$$hdu \cdot \sigma = \sum_{i=1}^{n} (-1)^{i+p} u \cdot d_i s_p \sigma + (-1)^{n+p+1} F(\sigma^n) u \cdot d_{n+1} s_p \sigma,$$

and $F(\sigma^n)$ is the identity, so

$$\begin{aligned}(hd - dh)u \cdot \sigma &= \sum_{i=1}^{p-1} (-1)^{i+p} (u \cdot s_{p-1} d_i \sigma - u \cdot d_i s_p \sigma) \\ &+ \sum_{i=p}^{n+1} (-1)^{i+p} u \cdot d_i s_p \sigma.\end{aligned}$$

As before, the first sum is zero. Since $n = p + l$, there are oddly many terms in the latter sum, and all except one cancel, so we have

$$(hd - dh)u \cdot \sigma = (-1)^{2p} u \cdot d_p s_p \sigma = u \cdot \sigma.$$

In the case where l is even, we have $hu \cdot \sigma = 0$ for all u, and $hu \cdot d_i \sigma$ can be non-zero only if $p \leq i \leq p + l$, in which case

$$hu \cdot d_i \sigma = (-1)^p u \cdot s_p d_i \sigma = (-1)^p u \cdot \sigma,$$

so

$$\begin{aligned}
dhu\cdot\sigma &= \sum_{i=1}^{n-1}(-1)^i hu\cdot d_i\sigma + (-1)^n F(\sigma^n)hu\cdot d_i\sigma\\
&= \sum_{i=p}^{p+l}(-1)^{p+i}u\cdot\sigma\\
&= (-1)^{2p}u\cdot\sigma\\
&= u\cdot\sigma.
\end{aligned}$$

(Note that the final term only continues into the second line if $p+l=n$, in which case σ^n is the identity, so $F(\sigma^n)$ is also the identity).

Similarly,

$$hdu\cdot\sigma = 0,$$

so, again, $(dh-hd)u\cdot\sigma = u\cdot\sigma$.

Finally, in the case where $l=1$, we still have all of the above properties except that for i such that $p\le i\le p+1$, we may now have $P(d_i\sigma,q,r)=1$ for some $q>p$. If $p+1=n$, then this cannot happen, so in that case, or if this otherwise does not occur, or if r is even, then the proof for the $l\ge 3$ odd case can be applied without change. If this does occur, then $p+1<n$ and we have

$$\begin{aligned}
dhu\cdot\sigma &= \sum_{i=1}^{n-1}(-1)^i hu\cdot d_i\sigma + (-1)^n F(\sigma^n)hu\cdot d_n\sigma\\
&= \sum_{i=1}^{p-1}(-1)^{i+p}u\cdot s_{p-1}d_i\sigma\\
&\quad +(-1)^{p+q}u\cdot s_q d_p\sigma + (-1)^{p+q+1}u\cdot s_q d_{p+1}\sigma\\
&\quad + \sum_{i=p+2}^{n-1}(-1)^{i+p}u\cdot s_p d_i\sigma + (-1)^{n+p}F(\sigma^n)u\cdot s_p d_i\sigma\\
&= \sum_{i=1}^{p-1}(-1)^{i+p}u\cdot s_{p-1}d_i\sigma + \sum_{i=p+2}^{n-1}(-1)^{i+p}u\cdot s_p d_i\sigma\\
&\quad +(-1)^{n+p}F(\sigma^n)u\cdot s_p d_i\sigma.
\end{aligned}$$

and the rest of the proof goes through as in the first case.

Thus, in all cases, we have $(dh-hd)u\cdot\sigma = u\cdot\sigma$, so h is a homotopy equivalence from the identity map to the zero map on $U^*(\mathcal{C};F)$, so $U^*(\mathcal{C};F)$ is contractible, hence $S^*(\mathcal{C};F)$ and $T^*(\mathcal{C};F)$ are homotopy equivalent. $\square$

Lemma 3.1.6. *For every (finite) graded category $\mathcal{C}$, and every presheaf F on $\mathcal{C}$, the homology $HS^*(\mathcal{C};F)$ coincides with the homology $H^*(\mathcal{C};F)$ of $\mathcal{C}$, as defined in Definition 2.2.10. That is: the homology of $S^*(\mathcal{C};F)$ computes the higher limits of F.*

Proof. We first construct a projective resolution $P^* \to \Delta\mathbb{Z}$ (with $\Delta\mathbb{Z}$ as defined in Definition 2.2.3) such that

$$\mathrm{Hom}(P^*, F) \cong S^*(\mathcal{C};F).$$

For this purpose, define first $P^n := \sum_{\sigma \in N^n\mathcal{C}} \Upsilon_{\sigma_n}\mathbb{Z}$, and define the maps to be those induced by the simplicial structure of $N^*\mathcal{C}$. To see that this is a projective resolution, note that for each fixed object X of $\mathcal{C}$, the abelian group $P^n(X)$ is free on the set of all $n+1$ simplices whose first object is X, so $HP^*(X) \cong H^*(N(X/\mathcal{C});F)$, where $X/\mathcal{C}$ is the coslice category as defined in Definition 2.1.28. But by Lemma 2.1.29, the coslice category $X/\mathcal{C}$ has an initial object, so is contractible, hence $HP^n(X) = 0$ for all n and all X, so P^* is exact. Each $\Upsilon_{\sigma_n}\mathbb{Z}$ is projective since $\mathbb{Z}$ is projective. Thus, P^n is projective by Lemma 2.1.31, so P^* is, indeed, a projective resolution of its colimit, which is $\Delta\mathbb{Z}$.

To see that $\mathrm{Hom}(P^*, F) \cong S^*(\mathcal{C};F)$, note that by Lemma 2.2.4, for each object X of $\mathcal{C}$, we have

$$\mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Upsilon_X\mathbb{Z}, F) \cong \mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(X, F(X)) \cong F(X).$$

Thus,

$$\begin{aligned}
\mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(P^n, F) &= \mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\sum_{\sigma} \Upsilon_{\sigma_n}\mathbb{Z}, F) \\
&\cong \prod_{\sigma} \mathrm{Hom}_{\mathbf{PreSh}\,\mathcal{C}}(\Upsilon_{\sigma_n}\mathbb{Z}, F) \\
&= \prod_{\sigma} F(\sigma_n) \\
&= S^n(\mathcal{C};F).
\end{aligned}$$

Further, the differential on S^* is exactly the image under $\mathrm{Hom}(-, F)$ of the differential of P^*, so indeed, $\mathrm{Hom}(P^*;F) \cong S^n(\mathcal{C};F)$ as complexes.

Finally, we note that, by definition, $H^*(\mathcal{C},F)$ is precisely the homology of $\mathrm{Hom}(P^*, F)$, so we have the result. □

The following generalises [12, Lemma 3].

Lemma 3.1.7. *If $\kappa : F \to G$ is a natural transformation of presheaves over a category $\mathcal{C}$, then the maps $\kappa^* : S^n(\mathcal{C}; F) \to S^n(\mathcal{C}; G)$ given by*

$$\kappa^* s \cdot \sigma := \kappa_{\sigma_n}(s \cdot \sigma)$$

assemble to give a chain map $S^*(\mathcal{C}; F) \to S^*(\mathcal{C}; G)$.

Proof. We require that the following square commutes for all n:

$$\begin{tikzcd}
S^{n-1}(\mathcal{C}; F) \arrow[r, "\kappa^*"] \arrow[d, "d"] & S^{n-1}(\mathcal{C}; G) \arrow[d, "d"] \\
S^n(\mathcal{C}; F) \arrow[r, "\kappa^*"] & S^n(\mathcal{C}; G)
\end{tikzcd}$$

We show this by direct calculation: for $s \in S^{n-1}(\mathcal{C}; F)$ and $\sigma \in N^n\mathcal{C}$, we have

$$\begin{aligned}
\kappa^* ds \cdot \sigma &= \kappa_{\sigma_n}(ds \cdot \sigma) \\
&= \kappa_{\sigma_n}\left(\sum_{i=0}^{n-1}(-1)^i s \cdot d_i\sigma + (-1)^n F(\sigma^n)s \cdot d_n\sigma\right) \\
&= \sum_{i=0}^{n-1}(-1)^i \kappa_{\sigma^n}(s \cdot d_i\sigma) + (-1)^n \kappa_{\sigma_n} F(\sigma^n)s \cdot d_n\sigma \\
&= \sum_{i=0}^{n-1}(-1)^i \kappa_{\sigma^n}(s \cdot d_i\sigma) + (-1)^n G(\sigma^n)\kappa_{\sigma_{n-1}} s \cdot d_n\sigma \\
&= d\kappa^* s \cdot \sigma,
\end{aligned}$$

as required. □

Definition 3.1.8. Let $G : \mathcal{D} \to \mathcal{C}$ be a functor. Generalising [12, § 1.2], we define

$$G^* : \mathbf{PreSh}\,(\mathcal{C}) \to \mathbf{PreSh}\,(\mathcal{D})$$

to be the functor defined on presheaves F by $G^*F := F \circ G$, and on morphisms by taking $G^*\kappa$ (for κ a natural transformation of presheaves) to be the natural transformation with $G^*(\kappa)_x = \kappa(Gx)$ for all $x \in F^*G$.

Finally, we define the **pullback** of G to be the map

$$G^* : S^*(\mathcal{C}; F) \to S^*(\mathcal{D}, G^*F)$$

with

$$G^* s \cdot \sigma = s \cdot G\sigma$$

for all $\sigma \in N^n\mathcal{D}$ and all $s \in S^n(\mathcal{C}, F)$. We similarly define $G^* : T^*(\mathcal{C}; F) \to S^*(\mathcal{D}; G^*F)$ by $G^*t \cdot \sigma = s \cdot g\sigma$ for all $\sigma \in N^n\mathcal{D}$.

In the special case where $\mathcal{D}$ is a subcategory of $\mathcal{C}$ and G is the functor given by $G(D) = D \in \mathrm{ob}\mathcal{C}$ for all $D \in \mathrm{ob}\mathcal{D}$ and for $\varphi \in \mathrm{Hom}_{\mathcal{D}}(D, E)$, by $G(\varphi) = \varphi \in \mathrm{Hom}_{mathcalC}(D, E)$, we define the **relative complex** $S^*(\mathcal{C}, \mathcal{D}; F)$ to be the kernel of $G^* : S^*(\mathcal{C}; F) \to S^*(\mathcal{D}; G^*F)$, and similarly $T^*(\mathcal{C}, \mathcal{D}; F)$ to be the kernel of $G^* : T^*(\mathcal{C}; F) \to T^*(\mathcal{D}; G^*F)$.

These generalise the definitions at the start of [12, § 1.2].

3.2 The Main Result

We now proceed to define our cellular homology, beginning with a generalisation of [12, Lemma 1].

Lemma 3.2.1. *For $F : \mathcal{D} \to \mathcal{C}$ a functor and G a presheaf on $\mathcal{C}$, the pullback $F^* : S^*(\mathcal{C}; G) \to S^*(\mathcal{D}, F^*G)$ is a chain map.*

Proof. Note that, for any $\sigma \in N^n\mathcal{C}$, we have

$$\begin{aligned} F^*ds \cdot \sigma &= ds \cdot F\sigma \\ &= \sum_{i=0}^{n-1}(-1)^i s \cdot d_i F\sigma + (-1)^n G(\sigma^n)(s \cdot d_n F\sigma), \end{aligned}$$

and that

$$\begin{aligned} dF^*s \cdot \sigma &= \sum_{i=0}^{n-1}(-1)^i F^*s \cdot d_i\sigma + (-1)^n G(\sigma^n)(F^*s \cdot d_n\sigma) \\ &= \sum_{i=0}^{n-1}(-1)^i s \cdot Fd_i\sigma + (-1)^n G(\sigma^n)(s \cdot Fd_n\sigma), \end{aligned}$$

and hence that it suffices to show that F commutes with our face maps.

But this is clear - if $\sigma = \sigma_n \xrightarrow{\sigma^n} \dots \xrightarrow{\sigma^1} \sigma_0$, then

$$\begin{aligned} Fd_i\sigma &= F(\sigma_n \xrightarrow{\sigma^n} \dots \xrightarrow{\sigma^{i+2}} \sigma_{i+1} \xrightarrow{\sigma^i\sigma^{i+1}} \sigma_{i-1} \dots \xrightarrow{\sigma^1} \sigma_0) \\ &= F(\sigma_n) \xrightarrow{F(\sigma^n)} \dots \xrightarrow{F(\sigma^{i+2})} F(\sigma_{i+1}) \xrightarrow{F(\sigma^i\sigma^{i+1})} \sigma_{i-1} \to \dots \xrightarrow{F(\sigma^1)} F(\sigma_0) \\ &= F(\sigma_n) \xrightarrow{F(\sigma^n)} \dots \xrightarrow{F(\sigma^{i+2})} F(\sigma_{i+1}) \xrightarrow{F(\sigma^i)F(\sigma^{i+1})} \sigma_{i-1} \to \dots \xrightarrow{F(\sigma^1)} F(\sigma_0) \\ &= d_i(F\sigma_n \xrightarrow{F(\sigma^n)} \dots \xrightarrow{F(\sigma^1)} F(\sigma_0)) \\ &= d_iF\sigma. \end{aligned}$$

□

We now generalise [12, Lemma. 3] to our new setting.

Lemma 3.2.2. *If $\kappa : F \to G$ is a natural transformation of presheaves, and $H : \mathcal{C} \to \mathcal{D}$ a functor , then $\kappa^* H^* = H^* \kappa^*$.*

Proof. We proceed by direct calculation: if $s \in S^n(\mathcal{C}; F)$ and $\sigma \in N^n(\mathcal{C})$, then

$$\begin{aligned} \kappa^* H^* s \cdot \sigma &= \kappa_{\sigma_n}(H^* s \cdot \sigma) \\ &= \kappa_{\sigma_n}(s \cdot H\sigma) \\ &= \kappa^* s \cdot H\sigma \\ &= H^* \kappa^* s \cdot \sigma. \end{aligned}$$

□

The next lemma generalises the result stated at the top of [12, p. 5] .

Lemma 3.2.3. *If $\mathcal{C}$ is a category, $\mathcal{D}$ a subcategory of $\mathcal{C}$, and F a presheaf on $\mathcal{C}$ (and hence on $\mathcal{D}$), then there is a short exact sequence*

$$0 \to S^*(\mathcal{C}, \mathcal{D}; F) \to S^*(\mathcal{C}; F) \to S^*(\mathcal{D}; F) \to 0.$$

Proof. Let $\iota : S^*(\mathcal{C}; F) \to S^*(\mathcal{D}; F)$ be the map induced by the inclusion of $\mathcal{D}$ into $\mathcal{C}$.

Now, for any $s \in S^*(\mathcal{D}; F)$, we define $t \in S^*(\mathcal{C}; F)$ by $t \cdot \sigma = s \cdot \sigma$ if $\sigma \in N^*\mathcal{D}$ and $t \cdot \sigma = 0$. Then $\iota t \cdot \sigma = t \cdot \sigma = s \cdot \sigma$ for any $\sigma \in N^*\mathcal{D}$. Thus, ι is surjective.

This gives exactness of our sequence at $S^*(\mathcal{D}; F)$. Exactness elsewhere follows immediately from the definition of $S^*(\mathcal{C}, \mathcal{D}; F)$ as the kernel of ι. □

As in [12], this gives a long exact sequence.

Lemma 3.2.4. *With $\mathcal{C}$, and $\mathcal{D}$ as above, there is a long exact sequence*

$$\cdots \to HS^n(\mathcal{C}, \mathcal{D}; F) \to HS^n(\mathcal{C}; F) \to HS^n(\mathcal{D}; F) \to \cdots$$

in homology.

Proof. This will be exactly the long exact sequence in cohomology of the short exact sequence of Lemma 3.2.3 above, arising in exactly the standard way, as in Lemma 2.3.5. □

We can extend this a little, to a result (a generalisation of [12, Lemma. 4]) that we note is essentially the third isomorphism theorem in the category of complexes.

Lemma 3.2.5. *If $\mathcal{C}$ is a category, $\mathcal{D}$ is a subcategory of $\mathcal{C}$, $\mathcal{E}$ is a subcategory of $\mathcal{D}$, and F is a presheaf on $\mathcal{C}$, then there is a short exact sequence*

$$0 \to S^*(\mathcal{C}, \mathcal{D}; F) \to S^*(\mathcal{C}, \mathcal{E}; F) \to S^*(\mathcal{D}, \mathcal{E}; F) \to 0.$$

Proof. Let $i : \mathcal{D} \to \mathcal{C}$ and $j : \mathcal{E} \to \mathcal{D}$ be the inclusion functors. These induce pullbacks $i^* : S^n(\mathcal{C}; F) \to S^n(\mathcal{D}; F)$ and $j^* : S^n(\mathcal{D}; F) \to S^n(\mathcal{E}; F)$ as in definition 3.1.8. These assemble to form chain maps, by Lemma 3.2.1.

Now, the identity map on $S^n(\mathcal{C})$ restricts to a map

$$f : S^n(\mathcal{C}, \mathcal{D}; F) \to S^n(\mathcal{C}, \mathcal{E}; F),$$

since

$$S^n(\mathcal{C}, \mathcal{D}; F) = \ker(i^*) \subseteq \ker(j^* i^*) = \ker((ij)^*) = S^n(\mathcal{C}, \mathcal{E}; F)$$

and i^* restricts to a map $S^n(\mathcal{C}, \mathcal{E}; F) \to S^n(\mathcal{D}, \mathcal{E}; F)$ since

$$i^*(S^n(\mathcal{C}, \mathcal{E}; F)) = i^* \ker(j^* i^*) \subseteq \ker(j^*) = S^n(\mathcal{D}, \mathcal{E}; F).$$

Now, for any $s \in S^n(\mathcal{C}, \mathcal{D}; F)$, we have $i^* f(s) = i^*(s) = 0$ since $s \in \ker i^*$.

Further, if $s \in S^n(\mathcal{C}, \mathcal{E}; F) \setminus S^n(\mathcal{C}, \mathcal{D}; F)$, then $s \notin \ker i^*$. Thus, our sequence is exact at $S^n(\mathcal{C}, \mathcal{E}; F)$. Clearly, f is injective, as it is a restriction of the identity map. Finally, for each $s \in S^n(\mathcal{D}, \mathcal{E}; F)$, define t to be the element of $S^n(\mathcal{C}, \mathcal{E}; F)$ with

$$t \cdot \sigma = \begin{cases} s \cdot \sigma & \sigma \in N^n\mathcal{D} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $t \in S^*(\mathcal{D}, \mathcal{E}; F)$ since for $\sigma \in N^n\mathcal{E}$, we have

$$t \cdot \sigma = s \cdot \sigma = 0.$$

Then for $\sigma \in N^n\mathcal{D}$, we have

$$i^*(t) \cdot \sigma = t \cdot i(\sigma) = s \cdot \sigma,$$

so $s = i^*(t)$, and i^* restricts to a surjective map $S^n(\mathcal{C}, \mathcal{E}; F) \to S^n(\mathcal{D}, \mathcal{E}; F)$, and our sequence is indeed exact. □

Corollary 3.2.6. *With $\mathcal{C}$,$\mathcal{D}$, and $\mathcal{E}$ as above, there is a long exact sequence*

$$\cdots \to HS^n(\mathcal{C}, \mathcal{D}; F) \to HS^n(\mathcal{C}, \mathcal{E}; F) \to HS^n(\mathcal{D}, \mathcal{E}; F) \to \cdots$$

Proof. This will be exactly the long exact sequence in cohomology of the short exact sequence of Lemma 3.2.5 above, arising in exactly the standard way, as in Lemma 2.3.5. □

With this long exact sequence in hand, we can now define our cellular homology.

Definition 3.2.7. We define our **cellular cochain complex** $C^*(\mathcal{C}; F)$ for $\mathcal{C}$ a locally finite graded category by $C^n(\mathcal{C}; F) := HS^n(\mathcal{C}^n, \mathcal{C}^{n-1}; F)$, with differential given by the boundary map δ in the long exact sequence of Corollary 3.2.6 applied to the triple of categories $(\mathcal{C}^n, \mathcal{C}^{n-1}, \mathcal{C}^{n-2})$, generalising [12, Def. 2.1].

Lemma 3.2.8. *Indeed, $C^*(\mathcal{C}; F)$ is a cochain complex , generalising the result stated after [12, Def. 2.1] .*

Proof. We proceed by working around the following diagram, which we shall show commutes, and has exact diagonals:

$$\begin{tikzcd}
& \vdots \arrow[d] & \\
& C^{n-1}(\mathcal{C};F) \arrow[dr, "\iota^{n-1}"] \arrow[dd, "\delta^{n-1}"] & \\
& & S^{n-1}(\mathcal{C}^{n-1};F) \arrow[dl, "\beta^{n-1}"] \\
& C^n(\mathcal{C};F) \arrow[dl, "\iota^n"] \arrow[dd, "\delta^n"] & \\
S^n(\mathcal{C}^n;F) \arrow[dr, "\beta^n"] & & \\
& C^{n+1}(\mathcal{C};F) \arrow[d] & \\
& \vdots &
\end{tikzcd}$$

First, let

$$\iota^{n-1} : C^{n-1}(\mathcal{C}; F) \to HS^{n-1}(\mathcal{C}^{n-1}; F)$$

be the map induced by the inclusion

$$i^{n-1} : S^{n-1}(\mathcal{C}^{n-1}, \mathcal{C}^{n-2}; F) \hookrightarrow S^{n-1}(\mathcal{C}^{n-1}; F).$$

Now, let $\beta^{n-1} : HS^{n-1}(\mathcal{C}^{n-1}; F) \to C^n(\mathcal{C}; F)$ be the connecting homomorphism of the long exact sequence given by applying Lemma 3.2.4 to the pair $(\mathcal{C}^n, \mathcal{C}^{n-1})$.

Let $s \in S^{n-1}(\mathcal{C}^{n-1}, \mathcal{C}^{n-2}; F)$. We shall show that $\delta^{n-1}[s] = \beta^{n-1}\iota^{n-1}[s]$. For this purpose, we simply apply the explicit constructions of the boundary maps given in Lemma 2.3.5: let d be the differential on $S^*(\mathcal{C}^{n-1}, \mathcal{C}^{n-2}; F)$, let j be the inclusion of categories $\mathcal{C}^{n-1} \hookrightarrow \mathcal{C}$, and let $\widehat{s}$ be any element of $S^{n-1}(\mathcal{C}^n, \mathcal{C}^{n-2}; F)$ such that $j^*\widehat{s} = s$ (which exists since j^* is surjective).

Then, by that construction, we have $\delta^{n-1}[s] = [d\widehat{s}]$, where the latter homology class is taken in $S^*(\mathcal{C}^n, \mathcal{C}^{n-1}; F)$. Similarly, let $\bar{d}$ be the differential on $S^*(\mathcal{C}^{n-1}; F)$, let $t := i^{n-1}(s)$, and let $\widetilde{t}$ be any element of $S^{n-1}(\mathcal{C}^n, \mathcal{C}^{n-1}; F)$ such that $j^*\widetilde{t} = t$. Then, by the same construction, we have

$$\beta^{n-1}\iota^{n-1}[s] = \beta^{n-1}[i^{n-1}s] = [\bar{d}\widetilde{t}].$$

But by definition, d is the restriction of $\bar{d}$, and $\widetilde{t} = \widehat{s}$, hence

$$\delta^{n-1} = \beta^{n-1}\iota^{n-1}.$$

Now, we have $\delta^n\delta^{n-1} = \beta^n\iota^n\beta^{n-1}\iota^{n-1}$, so we simply need to show that $\iota^n\beta^{n-1} = 0$. But $\iota^n\beta^{n-1}[s] = \iota^n[\bar{d}\widetilde{s}] = [i^n\bar{d}\widetilde{s}] = [di^{n-1}\widetilde{s}] = 0$ since i^* is a chain map, hence the result. $\square$

We now specify the condition that we require for our two homologies to coincide, generalising [12, Def. 3.1]:

Definition 3.2.9. A graded category $\mathcal{C}$ with corank function is **cellular** if

$$HS^i(\mathcal{C}^n, \mathcal{C}^{n-1}; F) = 0$$

for all presheaves F on $\mathcal{C}$, and $i \neq n \in \mathbb{Z}$.

We can now prove our main theorem, which generalises the finite case of [12, Thm. 2].

Theorem 3.2.10. *For a finite cellular category $\mathcal{C}$ and a presheaf F on $\mathcal{C}$, we have*

$$HS^*(\mathcal{C}; F) \cong HC^*(\mathcal{C}; F).$$

Proof. We first filter via $F^pS^* := S^*(\mathcal{C}, \mathcal{C}^p; F)$. Lemma 3.2.5 gives a short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & S^*(\mathcal{C}, \mathcal{C}^{p+1}; F) \arrow[r, hook] \arrow[d, equal] & S^*(\mathcal{C}, \mathcal{C}^p; F) \arrow[r] \arrow[d, equal] & S^*(\mathcal{C}^{p+1}, \mathcal{C}^p; F) \arrow[r] & 0, \\
0 \arrow[r] & F^{p+1}S^* \arrow[r, hook] & F^pS^* & &
\end{tikzcd}$$

and thus, $F^{p+1}S^*$ is a subcomplex of F^pS^* for all p, giving a bounded filtration of $S(\mathcal{C}, \mathcal{C}^p; F)$ of the form $0 \subset F^cS^* \subset \cdots \subset F^0 \subset F^{-1}S^* = S^*$, where c is the maximum corank of any element of $\mathcal{C}$.

The spectral sequence associated to F^* has E_0 page of the form

$$E_0^{p,q} = S^{p+q}(\mathcal{C}^{p+1}, \mathcal{C}^p; F),$$

by the short exact sequence above, with differential that of S^*, and hence E_1 page of the form $E_1^{p,q} = HS^{p+q}(\mathcal{C}^{p+1}, \mathcal{C}^p)$. Note that for $q \neq 1$, the assumption that $\mathcal{C}$ is cellular gives precisely that $E_1^{p,q} = 0$. Now, on the remaining line, the differential of our spectral sequence is precisely the map

$$\delta : E_1^{p,1} = C^p(\mathcal{C}; F) \to E_1^{p+1,1} = C^{p+1}(\mathcal{C}; F)$$

given by the connecting homomorphism of the long exact sequence of homology above, which is precisely the differential of $C^*(\mathcal{C}; F)$.

Thus, our spectral sequence collapses at E_2 with our cellular cohomology on the $q = 1$ line. But also, by Lemma 2.4.5, it must converge to $HS^*(\mathcal{C}; F)$. Thus, we have $HC^*(\mathcal{C}; F) \cong HS^*(\mathcal{C}; F)$. □

3.3 Comments on Generalisations & Applications

In [12], the authors prove these same results for locally finite cellular posets. However, their proof does not generalise, as it relies on the following result about posets that does not cleanly generalise to categories:

Lemma 3.3.1. *If P is a poset with an element $x \in P$ such that $x \leq y$ for every $y \in P$, then $|N^*P|$ is contractible.*

The obvious generalisation of this result to categories, replacing "$x \leq y$" with "there is a map $x \to y$" is not true: a simple counterexample is the category C with two objects x and y, and two non-identity morphisms $x \to y$. In this case, $|N^*C|$ is a circle, so is not contractible.

One could restrict to the case where the above result holds, but patching up the proof requires restricting to the case where there is exactly one morphism between each pair of objects: that is, to posets.

This problem could possibly be patched up by adjusting our definitions and using coslice categories as our intervals, rather than subcategories, as coslice categories do have initial objects, so their nerves are contractible.

One might consider applying this result to group cohomology, treating groups as categories with one object. However, this has a fatal flaw:

Lemma 3.3.2. *Let G be a non-trivial finite group, equipped with a grading and corank function (as a one-object category). Let n be minimal such that G^n is non-trivial. Then*

$$HS^k(G^n, G^{n-1}; \mathbb{Z})$$

is non-zero for infinitely many values of k.

Proof. Let $i : 1 \hookrightarrow G^n$ be the inclusion. By 3.1.5, we can use T^* for our calculations, rather than S^*.

Then $i^* : T^k(G^n; \Delta\mathbb{Z}) \to T^k(1; \Delta\mathbb{Z})$ is the zero map, since $T^k(1) = 0$, so $T^k(G^n, 1; \Delta\mathbb{Z}) = T^k(G^n, \Delta\mathbb{Z})$, and

$$\begin{aligned} HS^k(G^n, G^{n-1}; F) &= HS^k(G^n, 1; \Delta\mathbb{Z}) \\ &= HT^k(G^n, 1; \Delta\mathbb{Z}) \\ &= HT^k(G^n, \mathbb{Z}) \\ &= HS^k(G^n, \Delta\mathbb{Z}). \end{aligned}$$

z But this last is simply the group cohomology of G^n with coefficients in $\mathbb{Z}$, and by [11, Thm. 3], this is non-zero for infinitely many values of k, since G is a non-trivial finite group. □

Thus, G cannot be cellular, as our definition of cellular requires that $HS^k(G^n, G^{n-1}; F) = 0$ for all $k \neq n$ and all presheaves F on G.

However, the methods here could still be applied in some cases: our proof actually relies only on G having the above property for the particular sheaf that we are working with, and there are indeed sheaves in which non-trivial finite groups can have cohomologies that are zero in all but one dimension.

Chapter 4

Topological Background

4.1 General Topology

We begin this chapter with a few standard definitions.

Definition 4.1.1. [18, Def. 4.1] An **isometry** between metric spaces (M, d) and (N, e) is a function $f : M \to N$ such that

$$d(x, y) = e(f(x), f(y))$$

for all $x, y \in M$.

Definition 4.1.2. [18, Def. 4.17] A **homeomorphism** between topological spaces X and Y is a bijection $f : X \to Y$ such that both f and f^{-1} are continuous.

An **embedding** of a topological space X into a topological space Y is a map $f : X \to Y$ that is a homeomorphism onto its image.

Definition 4.1.3. [25, p. 108] If X is a topological space, we define a chain complex (see 2.3.1) $C(X)$ by defining $C_n(X)$ to be the free abelian group on the set of all continuous maps σ from the standard n-simplex to X, with boundary map

$$\delta_n : \sigma \mapsto \sum_{i \leq n} (-1)^i d_i \sigma$$

(where $d_i\sigma$ is the restriction of σ to the convex hull of all vertices of the standard n-simplex except for the ith). That this forms a chain complex is standard (and, indeed, the proof is identical to the special case of the proof of Lemma 3.1.4 with F the constant presheaf $\Delta\mathbb{Z}$). See [25, Lem. 2.1] for a proof in this particular case.

We then define the **(singular) homology of** X to be the homology of this chain complex.

4.2 Simplicial Complexes

We first define the basic combinatorial objects with which we shall work.

Definition 4.2.1. [44, § 3.1], [5, § 1] A finite **simplicial complex** C is a finite set $V(C)$ of vertices and a set $C \subseteq 2^V$ such that if $S \in C$ and $T \subseteq S$, then $T \in C$. In this case, we say that T is a **face** of S.

The **dimension** of $S \in C$ is $|S| - 1$.

A **subcomplex** of C is $D \subseteq C$ that is also a simplicial complex.

For $m \leq n$, the m**-skeleton** of C is the subcomplex of C consisting of all simplices of dimension at most m.

A **maximal simplex** of C is a simplex $S \in C$ that is not properly contained in any other simplex of C.

For a simplex $S \in C$, a **maximal face** of S is a face T of S such that if U is a face of S and $T \subsetneq U$, then $U = S$.

A simplicial complex is **pure** (of dimension n) if every maximal simplex has the same dimension n.

An elementary observation is that a simplicial complex C is determined by $V(C)$ and its maximal simplices: the simplices of C are then precisely the subsets of those maximal simplices.

Definition 4.2.2. [14, § 2.2] For C and D simplicial complexes, a **simplicial map** $f : C \to D$ is a function $V(C) \to V(D)$ such that $f(S) \in D$ for each simplex $S \in C$.

We shall also require the topological counterpart to these combinatorial simplices.

Definition 4.2.3. [25, p. 9] For each integer n, the **standard** n**-simplex** Δ^n is the convex hull of the points e_i of the standard orthonormal basis of $\mathbb{R}^{n+1}$.

Definition 4.2.4. [14, p. 4] A **topological** n**-simplex** is a metric space M equipped with an isometry f to the standard n-simplex.

One particular feature of these simplices that we shall make use of is that they can be subdivided into smaller simplices in a natural way.

Definition 4.2.5. [25, p. 103] For a point x in the standard n-simplex, the **barycentric coordinates** of x are the coordinates of x as a point in the ambient $\mathbb{R}^{n+1}$. Note that the barycentric coordinates $(x_0, \dots, x_n)$ of all such points satisfy $\sum x_i = 1$, as this condition determines the affine hyperplane through the e_i, which their convex hull lies in. For a point x

in a topological n-simplex Δ, the **barycentric coordinates** of x are the barycentric coordinates of x under the isometry from M to the standard n-simplex. The **barycentre** of a topological n-simplex Δ is the point whose barycentric coordinates are

$$\left(\frac{1}{n+1}, \ldots, \frac{1}{n+1}\right).$$

Finally, we note the following connection between the combinatorial and topological objects that we consider.

Definition 4.2.6. [34, § 1] The **geometric realisation** of a simplicial complex C is the topological space $|C|$ constructed as follows:

1. For each n-simplex $S \in C$, let $|S|$ be a topological n-simplex with its vertices labelled by the vertices of S.
2. Define $R(C) = \sqcup |S|$, where the disjoint union is over all simplices S of C.
3. For each maximal face T of S, let $\sim_S^T$ be the equivalence relation on $|S| \sqcup |T|$ identifying each point of $|T|$ with barycentric coordinates $(x_1, \ldots, x_n)$ with the unique point in $|S|$ with barycentric coordinates $(x_1, \ldots, x_n, 0)$, with the vertices of S ordered such that those of T come first, in the same order as in the barycentric coordinates of T, and the other vertex of S comes last.
4. Define an equivalence relation $\sim$ by
$$\sim = \bigcup_{(S,T)} \sim_S^T$$
where the union is over all pairs (S, T) of simplices of C such that T is a maximal face of S.
5. Finally, define $|C|$ to be $R(C)/\sim$.

We will also call topological spaces arising from simplicial complexes in this way **simplicial complexes**, where this is not confusing.

4.3 Computational Tools

In establishing our results, we shall require the following tools to compute the homologies of the various simplicial complexes that we shall work with.

4.3.1 Relating homologies of different spaces

We begin by establishing some relations between the homologies of various spaces.

Definition 4.3.1. [25, p. 3] Two continuous maps $f, g : X \to Y$ of topological spaces are **homotopic**, denoted $f \simeq g$, if there is a continuous map

$$H : X \times [0,1] \to Y$$

such that $H(x,0) = f(x)$ and $H(x,1) = g(x)$.

Definition 4.3.2. [25, pp. 3-4] Topological spaces X and Y are **homotopy equivalent** if there are continuous maps $f : X \to Y$ and $g : Y \to X$ such that $fg \simeq id_Y$ and $gf \simeq id_X$.

A topological space that is homotopy equivalent to a point is called **contractible**.

We will make use of homotopy equivalence through the following property, found as [25, Cor. 2.11], though homotopy equivalence is significantly stronger.

Lemma 4.3.3. *If X and Y are homotopy equivalent spaces, then*

$$H_n X \cong H_n Y$$

for all n.

Proof. See, for example, [25, pp. 110-113]. □

Definition 4.3.4. [25, p. 2] A **strong deformation retraction** of a space X to a subspace Y is a continuous map $X \times [0,1] \to X$ such that:

1. $F(x,0) = x$ for all $x \in X$
2. $F(x,1) \in Y$ for all $x \in X$
3. $F(y,t) = y$ for all $y \in Y$ and all $t \in [0,1]$.

In particular, if X strongly deformation retracts to Y, then X and Y are homotopy equivalent, since with $i : Y \hookrightarrow X$ the inclusion, we have

$$F(i(y),1) = i(y) = y$$

for all $y \in Y$ and F is precisely a homotopy between the identity on X and $iF(-,1)$.

The following theorem is [25, Thm. 2.20].

Theorem 4.3.5. *If X is a topological space, Y a subspace of X, and Z a subspace of Y such that the closure in X of Z is contained in the interior of Y, then*

$$H_k(X, Y) \cong H_k(X \setminus Z, Y \setminus Z)$$

for all k.

For a proof, see, for example, [25, p. 124].

Another useful long exact sequence is given by [25, pp. 149-150]:

Theorem 4.3.6 (Mayer-Vietoris)**.** *Suppose that X is a topological space, U and V open subsets of X such that $U \cup V = X$ and $U \cap V \neq \emptyset$. Then we have a long exact sequence of (reduced) homology*

$$\cdots \to \widetilde{H}_k(U \cap V) \to \widetilde{H}_k(U) \oplus \widetilde{H}_k(V) \to \widetilde{H}_k(X) \to \widetilde{H}_{k-1}(U \cap V) \to \cdots$$

Proof. Let $C_n(*)$ be the (singular) chain groups computing $H_n(*)$, and let $C_n(U+V)$ be the subcomplex of $C_n(X)$ consisting of sums of elements $c+c'$ with $c \in C_n(U)$ and $c' \in C_n(V)$, made into a chain complex by inheriting the boundary map from $C_n(X)$.

Then there is a map

$$\varphi : C_n(U \cap V) \to C_n(U) \oplus C_n(V)$$

given by $\varphi(c) = (c, -c)$ (where on the right we think of c as an element of $C_n(U)$ and $C_n(V)$ respectively), and a map

$$\psi : C_n(U) \oplus C_n(V) \to C_n(U+V)$$

given by $\psi(c, c') = c + c'$.

Now, φ is clearly injective, and ψ is surjective by definition of $C_n(U+V)$, and

$$\ker \psi = \{(c, -c) | c \in C_n(U) \cap C_n(V) = C_n(U \cap V)\} = \operatorname{im} \varphi$$

so we have a short exact sequence

$$0 \to C_n(U \cap V) \overset{\varphi}{\hookrightarrow} C_n(U) \oplus C_n(V) \overset{\psi}{\twoheadrightarrow} C_n(U+V) \to 0$$

Lemma 2.3.5 then gives a long exact sequence in homology

$$\cdots \to H_n(U \cap V) \to H_n(C_*(U) \oplus C_*(V)) \to H_n(C_*(U+V)) \to \cdots$$

Now, $H_n(C_*(U) \oplus C_*(V)) \cong H_n(U) \oplus H_n(V)$, and by Theorem 4.3.5, we have $H_n(C_*(U+V)) \cong H_n(X)$, hence the result. $\square$

4.3.2 Combinatorial Tools

We can also compute the homology of a (combinatorial) simplicial complex directly, without the need for passing through the topological realm:

Definition 4.3.7. Let C be a simplicial complex with vertex set $[m]$. Let $M_n(C;R)$ be the set of formal sums of n-simplices of C with coefficients in F, with the usual module structure over some ring R.

Define $d : M_n(C;R) \to M_{n-1}(C;R)$ by

$$d\sigma = \sum_{i=0}^{n}(-1)^i(\sigma \setminus \{\sigma_i\}),$$

where $\sigma = \{\sigma_0, \ldots, \sigma_n\}$.

The following result, which is [25, Lem. 2.1], relates the homologies of our simplicial complexes to the homologies of their geometric realisations.

Lemma 4.3.8. *$(M_n(C;R), d)$ is a complex, and $HM_n(C;R) \cong H_n(|C|;R)$.*

Proof. Firstly, note that

$$\begin{aligned}
d^2\sigma &= \sum_{i=0}^{n}(-1)^i d(\sigma \setminus \{\sigma_i\}) \\
&= \sum_{i=0}^{n}(-1)^i \left(\sum_{j=0}^{i-1}(-1)^j(\sigma \setminus \{\sigma_i, \sigma_j\}) + \sum_{j=i+1}^{n}(-1)^{j-1}(\sigma \setminus \{\sigma_i, \sigma_j\}\right) \\
&= \sum_{i=1}^{n}\sum_{j=0}^{i-1}(-1)^{i+j}(\sigma \setminus \{\sigma_i, \sigma_j\}) + \sum_{j=1}^{n}\sum_{i=0}^{j-1}(-1)^{i+j-1}(\sigma \setminus \{\sigma_i, \sigma_j\}) \\
&= 0,
\end{aligned}$$

so $(M_n(C;R), d)$ is indeed a complex.

Now, define $|\cdot| : M_n(C;R) \to C_n(|C|;R)$ by sending each simplex σ of C to its image $|\sigma|$ in $|C|$. This is clearly injective and forms a chain map, and $d|\sigma| = |d\sigma|$ by definition of the differential on $M_n(C;R)$.

On the combinatorial side, the inclusion $\iota : C^{n-1} \hookrightarrow C^n$ induces a chain map $\iota : M_*(C^{n-1};R) \hookrightarrow M_*(C^n;R)$. We define $M_*(C^n, C^{n-1};R)$ to be the cokernel of this map, so that we have an exact sequence

$$0 \to M_*(C^{n-1};R) \to M_*(C^n;R) \to M_*(C^{n-1}, C^n;R).$$

But the image of $\iota : M_k(C^{n-1};R)$ is all of $M_k(C^n;R)$ for $k < n$, and trivial for $k \geq n$. Thus, $HM_k(C^n, C^{n-1};R)$ is trivial in all degrees except the nth, where it is free on the n-simplices of C.

On the topological side, excision gives an isomorphism

$$HC_k(|C|^n, |C|^{n-1}; R) \to HC_k(|C|^n/|C|^{n-1}; R),$$

where $|C|^m$ is the image under $|\cdot|$ of the m-skeleton C^m. But $|C|^n/|C|^{n-1}$ is a wedge of n-spheres, so $HC_k(|C|^n, |C|^{n-1}; R)$ also is free on the n-simplices of C.

Thus, there is an isomorphism

$$HM_k(C^n, C^{n-1}; R) \to HC_k(|C|^n, |C^{n-1}|; R).$$

Further, C^0 is just the vertex set of C, so has homology that is free on $V(C)$, and similarly $|C|^0$ consists of $|V(C)|$ distinct points, so its homology is also free on $V(C)$.

Inductively, we assume that the result holds for all complexes of dimension at most $n-1$, so that there are isomorphisms

$$HM_k(C^{n-1}; R) \to HC_k(|C|^{n-1}; R)$$

for all k.

Finally, there is a map $f : HM_k(C^n; R) \to HC_k(|C|^n; R)$ given by sending each simplex σ of C^n to the map from the standard simplex to $|C|^n$ sending, for each i, the ith vertex of the standard simplex to the ith vertex of σ.

These maps assemble, with the long exact sequence associated to the above short exact sequence of complexes and the long exact sequence of the pair (C^n, C^{n-1}), to give the following commutative diagram with exact columns:

$$\begin{array}{ccc}
\vdots & & \vdots \\
\downarrow & & \downarrow \\
HM_{k+1}(C^n, C^{n-1}; R) & \xrightarrow{\cong} & HC_{k+1}(|C|^n, |C|^{n-1}; R) \\
\downarrow & & \downarrow \\
HM_k(C^{n-1}; R) & \xrightarrow{\cong} & HC_k(|C|^{n-1}; R) \\
\downarrow & & \downarrow \\
HM_k(C^n; R), & \xrightarrow{f} & HC_k(|C|^n; R) \\
\downarrow & & \downarrow \\
HM_k(C^n, C^{n-1}; R) & \xrightarrow{\cong} & HC_k(|C|^n, |C|^{n-1}; R) \\
\downarrow & & \downarrow \\
HM_{k-1}(C^{n-1}; R) & \xrightarrow{\cong} & HC_k(|C|^{n-1}; R) \\
\downarrow & & \downarrow \\
\vdots & & \vdots
\end{array}$$

Lemma 2.1.27 then gives that

$$HM_k(C^n; R) \to HC_k(|C|^n; R)$$

is an isomorphism, as required. □

4.3.3 Homological Building Blocks

Definition 4.3.9. [25, p. 9] For a simplicial complex X, the **cone** on X is the simplicial complex CX with vertex set $V(X) \sqcup \{*\}$, and whose maximal simplices are the $S \cup \{*\}$ for S a maximal simplex of X.

Similarly, the **cone** on a topological space X is the topological space $CX = (X \times [0,1])/\sim$, where $\sim$ is the equivalence relation generated by $(x, 1) \sim (y, 1)$ for all $x, y \in X$.

These two notions are related as follows:

Lemma 4.3.10. *If X is a simplicial complex, then $|CX| \cong C|X|$.*

Proof. First, we define $\varphi : C|X| \to |CX|$ as follows: for each point $x \in |X|$, let $(x_0, \ldots, x_n)$ be the barycentric coordinates of x in the minimal simplex $|S|$ of $|X|$ containing it (after fixing some ordering on the vertices of S).

Then for $t \in [0,1]$, we define $\varphi(x,t)$ to be the point of $|CX|$ lying in the simplex T whose vertices are $[| * |]$ and the vertices of $|S| \times \{1\}$ whose barycentric coordinates in $|T|$ (with the vertices ordered with those of S in the same order as above, and $[| * |]$ last) are $(y_0, \ldots, y_n, t)$, where

$$y_i = \frac{x_i}{1+t}$$

for all i.

Now, we define $\psi : |CX| \to C|X|$ as follows: for each point x of $|CX|$, let $(x_0, \ldots, x_n)$ be the barycentric coordinates of x in the minimal simplex $S = |T \sqcup \{*\}|$ containing both it and $*$, with $*$ being the final vertex in our ordering.

Then we define $\psi(x)$ to be the point $(y, x_n) \in C|X|$ where y is the point of $|T|$ whose barycentric coordinates (with the vertices ordered as above) are $((1+x_n)x_0, \ldots, (1+x_n)x_{n-1})$.

Both φ and ψ are clearly continuous, and it is equally clear that they are inverse to one another, hence they are homeomorphisms. □

The following standard result is found, for example, on [25, p. 183].

Lemma 4.3.11. *For any topological space X, the cone CX is contractible.*

Proof. We define

$$\begin{aligned} H : CX \times [0,1] &\to CX \\ (x,t) &\mapsto f(t,x,*), \end{aligned}$$

where $f : [0,1] \times (CX)^2 \to CX$ is the map given, in barycentric coordinates on each simplex, by $f(t,x,y) = (1-t)x + ty$.

Now, f is well-defined, since we have

$$\begin{aligned} \sum f(t,x,y)_i &= \sum (1-t)x_i + ty_i \\ &= (1-t)\sum x_i + t\sum y_i \\ &= 1, \end{aligned}$$

so H is also well-defined, and is clearly continuous.

Further, $H(0,x) = x$ for all x, and $H(1,x) = *$ for all x, so H is a homotopy between the identity map and the constant map at $*$, so CX is indeed contractible. □

We now repeat the above with a similar, but more interesting, construction.

Definition 4.3.12. [25, p. 8] For a simplicial complex X, the **suspension** on X is the simplicial complex SX with vertex set $V(X) \sqcup \{A, B\}$ (with A, B arbitrary additional points), whose maximal simplices are the $S \cup \{A\}$ and $S \cup \{B\}$ for S a maximal simplex of X.

Similarly, the **suspension** on a topological space X is the topological space $SX = (X \times [0,1])/\sim$, where $\sim$ is the equivalence relation generated by $(x,1) \sim (y,1)$ and $(x,0) \sim (y,0)$ for all $x, y \in X$.

These two notions are related as follows:

Lemma 4.3.13. *If X is a simplicial complex, then $|SX| \cong S|X|$.*

Proof. We define $\varphi : |SX| \to S|X|$ as follows: for each point $x \in |X|$, let $(x_0, \dots, x_n)$ be the barycentric coordinates of x in the minimal simplex $|S|$ of $|X|$ containing it (after fixing some ordering on the vertices of S).

Then for $t \in [0, \frac{1}{2}]$, we define $\varphi(x,t)$ to be the point of $|SX|$ lying in a simplex T whose vertices are $[|A|]$ and the elements of $\{(|v|, \frac{1}{2}) | v \in Y\}$ with $Y \subseteq X$ whose barycentric coordinates in $|T|$ (with the vertices ordered with those of $|X|$ in the same order as above, and $[|A|]$ last) are $(y_0, \dots, y_n, 2t)$, where

$$y_i = \frac{x_i}{1+2t}$$

for all i.

Similarly, for $t \in [\frac{1}{2}, 1]$, we define $\varphi(x,t)$ to be the point of $|SX|$ lying in the simplex T whose vertices are $[|B|]$ and the elements of $\{(|v|, \frac{1}{2}) | v \in Y\}$ with $Y \subseteq X$ whose barycentric coordinates in $|T|$ (with the vertices ordered with those of $|X|$ in the same order as above, and $[|B|]$ last) are $(y_0, \dots, y_n, 2t-1)$, where

$$y_i = \frac{x_i}{2t}$$

for all i.

Now, we define $\psi : S|X| \to |SX|$ as follows: for each point x of $|SX|$, let $(x_0, \dots, x_n)$ be the barycentric coordinates of x in the minimal simplex $S = S_A = |T \sqcup \{A\}|$ containing both it and $|A|$, if such exists, with A being the final vertex in our ordering. If such does not exist, then we instead take $(x_0, \dots, x_n)$ to be the barycentric coordinates of x in the minimal simplex $S = S_B = |T \sqcup \{B\}|$ containing both it and $|B|$.

Then, if $S = S_A$ we define $\psi(x)$ to be the point $(y, \frac{x_n}{2}) \in S|X|$ where y is the point of $|T|$ whose barycentric coordinates (with the vertices ordered as above) are $((1+2x_n)x_0, \dots, (1+2x_n)x_{n-1})$. Similarly, if $S = S_B$, we define $\psi(x)$ to be the point $(y, \frac{x_n+1}{2}) \in S|X|$ where y is as above.

Both φ and ψ are clearly continuous, and it is equally clear that they are inverse to one another, hence they are homeomorphisms. $\square$

The following result is found as [25, Ex. 2.2.32].

Lemma 4.3.14. *For any topological space X and any $k \geq 0$, we have $\widetilde{H}_k SX \cong \widetilde{H}_{k-1}X$ (where we take the convention that $\widetilde{H}_{-1}X = 0$).*

Proof. Let $a := [(x, 0)]$ for some $x \in X$, and similarly let $b := [(x, 1)]$. Then we apply Theorem 4.3.6 with (in the notation of that theorem) $U = SX \setminus \{a\}$ and $V = SX \setminus \{b\}$. Then both U and V are homeomorphic to CX, so are contractible, and $U \cap V \cong X \times (0, 1)$ strongly deformation retracts to X. Thus, $\widetilde{H}_k U = \widetilde{H}_k V = 0$ and $\widetilde{H}_k(U \cap V) \cong H_k X$ for all $k \geq 0$. Our Mayer-Vietoris sequence thus becomes

$$\cdots \to 0 \to \widetilde{H}_{k+1}SX \to \widetilde{H}_k X \to 0 \to \cdots$$

and since this is exact, the central map above is the desired isomorphism. □

4.3.4 Shellability

In Chapter 7, we shall make use of the concept of shellability of a simplicial complex - we shall make use only of the case where the simplicial complex under consideration is pure, and so, for simplicity, we present this background material in that setting. For a more general treatment, see [5].

Definition 4.3.15. [5, § 2] A **shelling** of a (finite, pure) simplicial complex X is an ordering $S_1, \ldots, S_n$ of the maximal simplices of X such that for each $k \geq 1$, the subcomplex

$$S_{k+1} \cap \left(\bigcup_{i=1}^{k} S_i \right)$$

is pure of dimension $\dim S_{k+1} - 1$.

The **spanning** simplices of a shelling are those S_k such that

$$S_{k+1} \cap \left(\bigcup_{i=1}^{k} S_i \right) = \partial S_{k+1}.$$

If X has a shelling, then it is **shellable**.

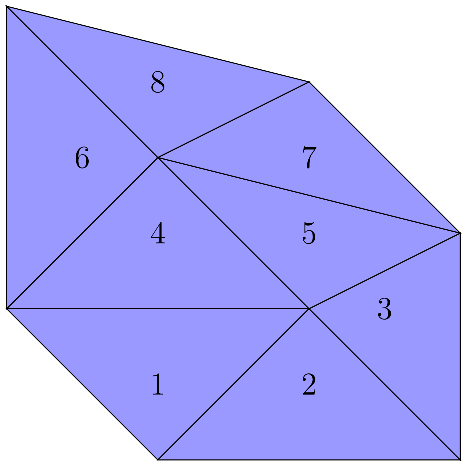
Example 4.3.16. Many common simplicial complexes are shellable:

1. All connected graphs are shellable, with a shelling given by any ordering of the edges E_i such that each $\bigcup_{i=1}^{k} E_i$ is connected.

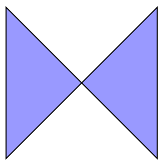
2. Every hollow simplex is shellable, with any ordering of its maximal simplices giving a shelling.

3. All cubes are shellable, with any ordering of its maximal simplices $S_1, \ldots, S_n$ such that S_1 and S_n are opposite giving a shelling.

4. This complex is shellable, with the ordering given in the labels.



However, not all pure connected simplicial complexes are shellable, even in dimension 2. For example, the following complex is not shellable (adding the second simplex gives a 0-dimensional intersection, whereas a shelling would require this to be 1-dimensional):



The following result, which is a special case of [5, Thm. 4.1], is key: it is important to note that not only does it allow us to compute the homology of a shellable complex, but also that it provides explicit generators for said homology, which is the key property that will make it useful to us in Chapter 7.

Theorem 4.3.17. *The geometric realisation of a pure shellable simplicial complex X of dimension d is homotopy equivalent to a wedge of d-spheres with one d-sphere for each spanning simplex in a shelling of X.*

Proof. Firstly, note that we can rearrange our shelling by placing the spanning simplices last, while still having a shelling, as at the stage where we add each spanning simplex S_m to

$$X_{m-1} := \bigcup_{i=1}^{m-1} S_i,$$

the only simplex that is added is S_m itself (since all of its proper faces have already been added), which cannot be a face of any other simplex (as it is maximal). Thus, we suppose, without loss of generality, that $S_1, \ldots, S_k$ are not spanning simplices, and $S_{k+1}, \ldots, S_n$ are spanning simplices.

We now show that X_{m-1} is homotopy equivalent to X_m for all $m \leq k$.

Note that S_m has some proper face F that is not shared with any S_i for $i < m$ (else it would be a spanning simplex). Let F be maximal with this property (such is uniquely defined, as $F := \{x \in S_m | S_m \setminus \{x\} \in X_{m-1}\}$). This must be a maximal face, as X is pure.

We can then strongly deformation retract $|S_m|$ to $\bigcup_T |T|$, where the union is over the faces of S_m other than F, by embedding $|S_m|$ as a standard simplex in $\mathbb{R}^{k+1}$ (where k is the dimension of S_m), and for each point x in the $|S_m|$, sending it to the (unique) point of $\bigcup_T |T|$ lying on the line through x (in the affine plane spanned by the image of $|S_m|$) orthogonal to F.

Thus, X_{m-1} is homotopy equivalent to X_m. But also, $X_1 = S_1$ is a solid simplex, so is contractible, and so X_k is contractible, and so X is homotopy equivalent to X/X_k.

But each S_m for $m > k$ has its entire boundary glued to X_{m-1}. Thus, in the quotient, S_{k+1} (and, inductively, S_m) has its entire boundary glued to the single vertex that X_k contracts to.

Thus, X/X_k is homotopic to a wedge of $n - k$ spheres of dimension d, as required. □

This result is useful primarily through the following immediate corollary (found as [5, Thm. 4.3] in more generality).

Corollary 4.3.18. *The homology of a pure shellable simplicial complex X of dimension d is given by*

$$H_d(X) = \mathbb{Z}^n$$

with n the number of spanning simplices in any shelling of X, and

$$\widetilde{H}_k(X) = 0$$

for $k \neq d$. Further, the generators of $H_d(X)$ correspond to those spanning simplices.

Proof. As the homology of a wedge of spheres is precisely this, with n the number of d-spheres, this follows immediately from Theorem 4.3.17 and Lemma 4.3.3. □

Chapter 5

Representation Theory Background

In this chapter, we will provide an overview of the concepts and definitions from the theory of groups and their representations that we shall need later. We will first present the purely group-theoretic results.

5.1 Group Theory

5.1.1 Elementary Results

To begin with, we shall require some standard results of group theory, beginning with the following definitions.

Definition 5.1.1. For G a group and p a prime, a **p-element** of G is $g \in G$ such that the order of g is a power of p.

A **p'-element** of G is $g \in G$ such that the order of g is coprime to p.

The following result will be key to the construction of the simplicial complex studied in Chapter 7.

Lemma 5.1.2. *Let G be a finite group, and let p be a prime dividing the order of G. Then for every $g \in G$, there are unique elements $u(g)$ and $s(g)$ of G such that:*

- $g = u(g)s(g) = s(g)u(g)$,
- *$u(g)$ is a p-element of G,*
- *$s(g)$ is a p'-element of G, and*

- *$u(g)$ and $s(g)$ are both powers of g.*

This result, and the proof below, was communicated to us by S. Donkin.

Proof. By the uniqueness of prime factorisation in the integers, there are unique integers n and q, with q coprime to p, such that $|g| = p^n q$. Now, p^n and q are coprime, so, by Bézout's identity, there are integers a and b, with a unique modulo q and b unique modulo p^n, such that $ap^n + bq = 1$. Let $s(g) := g^{ap^n}$, and let $u(g) := g^{bq}$.

Then $u(g)$ and $s(g)$ commute, since both are powers of g, and

$$u(g)s(g) = g^{ap^n+bq} = g.$$

Further, $u(g)^{p^n} = g^{bp^n q} = 1^b = 1$, and $s(g)^q = g^{ap^n q} = 1^a = 1$, so indeed $u(g)$ is a p-element, and $s(g)$ is a p'-element of G.

Further, for any elements $u'(g)$ and $s'(g)$ with these properties, there are integers c and d such that $s'(g) = g^c$, and $u'(g) = g^d$. But since

$$g = u'(g)s'(g) = g^{c+d}$$

we must have $c + d \cong 1 \,(\mathrm{mod}\, |g|)$ for some integer h.

Also, since $u'(g)$ is a p-element, we have $q|d$, and since $s'(g)$ is a p'-element, we have $p^n|c$, say $c = ep^n$ and $d = fq$.

Combining these, we have $ep^n + fq \cong 1 \,(\mathrm{mod}\, |g|)$, so there is some integer h such that $(f - hp^n)q + ep^n = 1$. Thus, by the uniqueness of a and b, we have $f \cong a \,(\mathrm{mod} p^n)$, and $e \cong b \,(\mathrm{mod} q)$, and hence $u'(g) = g^d = g^{fq} = g^{ap^n} = u(g)$, and $s'(g) = g^c = g^{ep^n} = g^{bq} = s(g)$, so $u(g)$ and $s(g)$ are, indeed, unique. ☐

Definition 5.1.3. For G a group and $g \in G$, the **centraliser** of g in G is $Cg := \{h \in G | h^{-1}gh = g\}$.

Let G be a group and H a subgroup. Then the **normaliser** of H in G is $NH := \{g \in G : g^{-1}Hg = H\}$.

The following lemma is standard, and may be found, for example, as [22, Prop. 3.8].

Lemma 5.1.4. *If G is a finite group, then*

$$|G| = |ZG| + \sum |G : Cg_i|$$

where the sum is over the non-trivial conjugacy classes of G and the g_i are representatives of those conjugacy classes.

5.1.2 Nilpotent Groups

Definition 5.1.5. [10, p. 190] Let G be a group, and define $G_0 = G$. Then, for all $n > 0$, define G_n to be the commutator subgroup $[G_{n-1}, G]$. If there is some finite n such that G_n is the trivial group, then G is called **nilpotent**.

The following result is found on [7, p. 1].

Lemma 5.1.6. *Let G be a nilpotent group. Then every Sylow-p-subgroup P of G is normal in G.*

Proof. Let $g \in NNP$. Then $g^{-1}Pg$ is a Sylow-p-subgroup of G, and $g^{-1}Pg \subseteq g^{-1}NPg = NP$. Thus, $g^{-1}Pg$ is a Sylow-p-subgroup of NP, so $g^{-1}Pg = P$, so $NP = NNP$.

Now, since P is normal in NP, it suffices to show that $NP = G$. Suppose not. Then there is some minimal $k > 0$ such that $G_k \leq NP$, so there is some $g \in G_{k-1} \setminus NP$. But $[g, NP] \leq [g, G_{k-1}] \leq G_k \leq NP$, so for all $h \in NP$, we have $g^{-1}hgh^{-1} \in NP$, so $g^{-1}hg \in NPh = NP$, so $g \in NNP = NP$, a contradiction. Thus, $NP = G$, so P is normal in G. □

The following Corollary is what will allow us to construct our complex in Chapter 7 with an easy-to-study action.

Corollary 5.1.7. *A finite nilpotent group G is the direct product of its Sylow subgroups.*

Proof. Let H be the product of the Sylow subgroups of G. As all Sylow subgroups of G are normal and intersect trivially, this product is direct. But the order of H is then the product of the orders of the Sylow subgroups, which is precisely the order of G. □

5.2 Representation Theory

We now move to the concepts and results of the representation theory of groups that we shall require. We begin with some standard definitions. Proofs of standard results will be omitted: see the citations for detailed proofs.

Definition 5.2.1. [10, p. 41] An **action** of a group G on a set X is a homomorphism $\varphi : G \to \mathrm{Sym}(X)$. If G acts on X, we denote $g \cdot x = \varphi(g)(x)$.

Definition 5.2.2. [10, p. 840] A **representation** of a group G with coefficients in a ring R is an R-module M, together with an action $\cdot$ of G on M

such that for all $g \in G$ and all $x, y \in M$, we have $g \cdot (x + y) = g \cdot x + g \cdot y$ and $g \cdot (ab) = (g \cdot a)(g \cdot b)$.

A **subrepresentation** of a representation $(M, \cdot)$ is an R-submodule N of M such that $G \cdot N = N$, with action given by the restriction of $\cdot$ to N.

Definition 5.2.3. [10, p. 847] A representation $(M, \cdot)$ of a group G is **irreducible** if it has no subrepresentations other than itself and the trivial representation.

A representation $(M, \cdot)$ of a group G is **indecomposable** if there do not exist subrepresentations S and T of M such that $N = S \oplus T$ as R-modules.

The following standard result was first proven in [33].

Lemma 5.2.4. *Let G be a finite group and K a field whose characteristic does not divide $|G|$. Let M be a finite-dimensional (as a K-vector space) representation of G. Then M decomposes as a direct sum of irreducible modules.*

In particular, over such fields as described in the lemma, the indecomposable representations of G are precisely the irreducible representations of G.

The following standard result of representation theory may be found, for example, as [16, Prop. 2.30].

Lemma 5.2.5. *Let G be a finite group. Then the number of irreducible representations of G over $\mathbb{C}$ is equal to the number of conjugacy classes of G.*

See [16, § 2.2, 2.4] for a proof.

A large part of our work in the subsequent chapters will be to find free generators of our homology modules such that our representations belong to the following class.

Example 5.2.6. [10, p. 43] If G is a subgroup of the symmetric group S_n, a **permutation representation** of G is the representation on the free R-module on n generators with action given by permuting those generators.

We pass this terminology to groups isomorphic to G.

The following mild condition enables all of our following work.

Definition 5.2.7. [25, p. 96] A **simplicial action** of a group G on a simplicial complex X is an action of G on X such that for each $g \in G$ and each $\sigma = \{\sigma_0, \ldots, \sigma_n\} \in X$, we have $g \cdot \sigma = \{g \cdot \sigma_0, \ldots, g \cdot \sigma_n\}$.

Equivalently, this is an action of G on the vertex set V of X such that for all $\sigma = \{\sigma_0, \ldots, \sigma_n\} \in X$, we have $\{g\sigma_0, \ldots, g\sigma_n\} \in X$.

The following is key to our approach in Chapters 6 and 7, and may be found on [25, p. 96][9, p. 179].

Lemma 5.2.8. *If a group G acts simplicially on a simplicial complex X, then it also acts on each homology module $H_n(X; R)$, with action induced by the action on X.*

Proof. For the purposes of this proof, for σ a k-simplex of X, we denote by $[\sigma]$ the generator of the kth simplicial chain group corresponding to σ.

Let the vertices of X be $v_1, \ldots, v_n$. Then the action of G on X is fully determined by its action on $\{v_1, \ldots, v_n\}$, as the action is simplicial. Given a list of distinct vertices $(v_{i_1}, \ldots, v_{i_k})$, define the orientation of $(v_{i_1}, \ldots, v_{i_k})$ to be $(-1)^{t(v_{i_1}, \ldots, v_{i_k})}$, where $t(v_{i_1}, \ldots, v_{i_k})$ is the number of transpositions needed to put $(i_1, \ldots, i_k)$ into ascending order.

Now, for $\sigma = \{v_{j_0}, \ldots, v_{j_k}\}$ with $j_0 < \ldots < j_k$, the action of G on X passes to an action of G on the (simplicial) k-chains of X, as a k-chain is simply an R-weighted sum of simplices of X. , so we can define $g \sum_\sigma r\sigma = \sum_\sigma (-1)^{t(gv_{j_0}, \ldots gv_{j_k})} rg\sigma$ for all $g \in G$ and all k-chains $\sum_\sigma r\sigma$.

It now suffices to show that this action sends cycles to cycles and boundaries to boundaries. But note that, for any $\sigma = \{v_{j_0}, \ldots, v_{j_k}\} \in X$ with $j_0 < \ldots < j_k$ and any $g \in G$ and $u_i = v_{j_i}$ for all i, we have (with $m(i)$ the difference between i and the position of gu_i in the ordered list of vertices $g\sigma$ in the order as defined above):

$$\begin{aligned}
gd[\sigma] &= g\left(\sum_{i=0}^{k} (-1)^i [\sigma \setminus \{u_i\}]\right) \\
&= \sum_{i=0}^{k} (-1)^i (-1)^{t(gu_0, \ldots, gu_{i-1}, gu_{i+1}, \ldots, gu_k)} [g(\sigma \setminus \{u_i\})] \\
&= \sum_{i=0}^{k} (-1)^i (-1)^{t(gu_0, \ldots, gu_{i-1}, gu_{i+1}, \ldots, gu_k)} [(g\sigma) \setminus \{gu_i\}] \\
&= (-1)^{t(gu_0, \ldots, gu_k)} \sum_{i=0}^{k} (-1)^i (-1)^{m(i)} [(g\sigma) \setminus \{gu_i\}] \\
&= d((-1)^{t(gu_0, \ldots, gu_k)} [g\sigma]) \\
&= dg[\sigma].
\end{aligned}$$

Thus, in particular, if $d\sigma = \tau$, then $dg\sigma = g\tau$, so τ is a boundary if and only if σ is a boundary, and $d\sigma = 0$ if and only if $dg\sigma = g0 = 0$, so indeed, our action preserves cycles and boundaries, so passes to our homology by $g[\sum_\sigma r\sigma] = [\sum_\sigma rg\sigma]$.

□

5.2.1 Young Tableaux

We shall now require some concepts and results from the theory of Young Tableaux, which we shall use in Chapter 6 to analyse our representations.

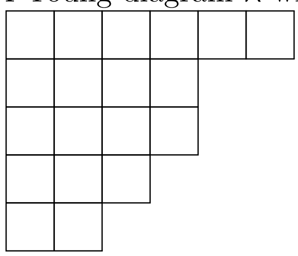
Definition 5.2.9. [19, p. 244] A **partition** of a set X is a collection U of non-empty subsets, called **blocks**, of X such that for every $A \neq B \in U$, the intersection $A \cap B$ is empty, and such that

$$\bigcup_{A \in U} A = X.$$

We define also the **Stirling Number (of the second kind)** $S(n, k)$ to be the number of partitions U of a set X with $|X| = n$ such that $|U| = k$.

Definition 5.2.10. [15, pp. 1-2, 25, 85] A **Young diagram** is a collection of boxes arranged in left-aligned rows, with a weakly decreasing number of boxes in each row. For λ such a diagram, denote by λ_i the number of boxes in the ith row of λ.

Figure 5.1: A Young diagram λ with 20 boxes.



A **filling** F of a Young diagram is the result of writing one natural number in each box of that diagram. The Young diagram so filled is called the **shape** of the filling. We denote it by $\lambda(F)$. A **numbering** of a Young diagram is a filling whose entries do not repeat.

Figure 5.2: A filling (left) and numbering (right) of shape λ

6	3	6	2	7	4
1	7	5	3		
2	7	7	4		
1	4	6			
3	4				
3					

6	3	8	2	7	4
1	9	5	10		
15	20	17	21		
23	26	30			
25	24				
16					

A **Young tableau** is a filling of a Young diagram such that the entries in each row weakly increase and the entries in each column strictly increase.

Figure 5.3: A tableau of shape λ

6	6	7	8	9	40
7	7	8	9		
8	8	9	10		
9	9	10			
10	10				
11					

A numbering of a Young diagram with n boxes is **standard** if its entries are precisely the integers $1, \ldots, n$, each appearing exactly once. A **tabloid** is

Figure 5.4: A standard numbering (left) and tableau (right) of shape λ

2	4	10	3	9	12
14	8	1	13		
16	6	15	11		
7	17	18			
19	5				
20					

1	4	5	6	7	20
2	8	9	10		
3	11	12	13		
14	15	16			
17	19				
18					

an equivalence class of numberings of a Young diagram where two diagrams are equivalent if they put the same numbers in each row, which we indicate by missing the vertical lines between boxes. The tabloid of a tableau T is denoted $\{T\}$. The **type** of a tabloid $\{T\}$ is a sequence $\mu = (\mu_1, \ldots, \mu_n)$

Figure 5.5: A tabloid $\{T\}$ of shape λ (left), and a numbering equivalent to T (right)

1	4	5	6	7	20
2	8	9	10		
3	11	12	13		
14	15	16			
17	19				
18					

1	20	5	4	7	6
10	8	2	9		
3	13	12	11		
15	14	16			
17	19				
18					

such that μ_i of the boxes of T contain the number i for each i.

We note that Young diagrams with n boxes correspond to partitions of n, with a diagram λ of k rows corresponding to the partition $(\lambda_1, \dots, \lambda_k)$. We shall move without further note between the two.

We define orderings on the sets of all Young diagrams and numberings as follows:

Definition 5.2.11. [15, pp. 26, 84–85] We define two orderings on the set of all Young diagrams:

- The **lexicographic order**: $\lambda \leq \mu$ if there is some i such that $\lambda_j = \mu_j$ for all $j < i$, and $\lambda_i < \mu_i$, or if $\lambda = \mu$.
- The **dominance order**: $\lambda \trianglelefteq \mu$ if for all i,

$$\sum_{j \leq i} \lambda_i \leq \sum_{j \leq i} \mu_i.$$

We define also an ordering on the set of all numberings by saying that $N \leq M$ if either

1. $\lambda(N) < \lambda(M)$ or
2. $\lambda(N) = \lambda(M)$ and the largest entry that is different between N and M appears earlier in M than in N.

The following numbers are key to the results of the subsequent section.

Definition 5.2.12. [29] For λ and μ Young diagrams, we define the **Kostka number** $K(\mu, \lambda)$ to be the number of Young tableaux of shape μ and type λ.

5.2.2 Representations of the Symmetric group

To analyse the representations of the symmetric group, we first require the following definitions, which will enable us to make use of the machinery of Young Tableaux for this purpose.

Definition 5.2.13. [15, § 7.1] For λ any Young diagram with n boxes, the symmetric group S_n acts on the set of all standard fillings of λ with $\sigma \cdot T$ being the filling which puts $\sigma(i)$ in the box where T puts i for each i.

This action passes to an action on the set of all tabloids via

$$\sigma \cdot \{T\} = \{\sigma \cdot T\}.$$

Definition 5.2.14. [15, p. 84] For each standard numbering T of a diagram with n boxes, define the **column group** of T to be the subgroup $C(T)$ of S_n consisting of all those elements which send all entries of each column of T to entries of that same column, and dually define the **row group** of T to be the subgroup $R(T)$ of S_n consisting of the elements which sends all entries of each row of T to entries of that same row.

Definition 5.2.15. Let T be a standard numbering of a Young diagram with n boxes. Then define b_T to be the element of the group ring $\mathbb{C}[S_n]$ given by the formula

$$b_T = \sum_{\sigma \in C(T)} \mathrm{sgn}(\sigma)\sigma,$$

where $\mathrm{sgn}(\sigma)$ is the sign of σ.

Definition 5.2.16. [15, § 7.2] For λ a Young diagram, let M^λ be the complex vector space with basis the set of all tabloids of shape λ, with the action of S_n induced by the action on the tabloids.

The **Specht Module** S^λ corresponding to a Young diagram λ is the subspace of M^λ spanned by the

$$v_T := b_T \cdot \{T\} = \sum_{\sigma \in C(T)} \mathrm{sgn}(\sigma)\{\sigma \cdot T\}$$

for all numberings T of shape λ.

The following is the standard classification of irreducible representations of the symmetric group, and may be found as [43, Thm. 2.4.6].

Theorem 5.2.17. *The non-isomorphic irreducible representations of S_n are precisely the Specht modules S^λ corresponding to diagrams λ of n boxes.*

The following theorem is [43, Thm. 2.10.1].

Theorem 5.2.18. *There is a basis of* $\mathrm{Hom}_{\mathbb{C}}(S^\lambda, M^\mu)$ *in bijection with the set of all semisimple λ-tableaux of type μ.*

For a proof, see [43, Ch. 2.10].

The following result will be key to our analysis of the representations arising in Chapter 6, and is found on [15, p. 92].

Theorem 5.2.19. *The module M^λ decomposes into irreducible representations as*

$$M^\lambda = S^\lambda \oplus \bigoplus_{\mu \trianglerighteq \lambda} (S^\mu)^{K_{\mu,\lambda}},$$

with $K_{\mu,\lambda}$ the Kostka number.

Chapter 6

Homological Representations of the Symmetric Group

In this chapter, and the chapter that follows, we shall compute group representations by passing group actions on simplicial complexes to actions on the homologies of those complexes. Similar methods have been used elsewhere many times. Notably, in [38], these techniques are applied to compute the characters of (co)homological representations of various groups given by their actions on hyperplane arrangements. In [31], this is similarly used to compute representations of the symmetric group given by its action on a particular hyperplane arrangement.

In [45, § 4], a similar analysis to ours of this chapter is performed with the action of the symmetric group on a different complex, and similar results are obtained, with the usual irreducible representations arising. [27, § 7.5] summarises the application of these methods to groups with BN-pairs acting on their Tits complexes, and in particular notes that (in characteristic zero) the homology representation arising in the top dimension is the Steinberg representation, as proven in [8, § 66C].

The subject has also received recent attention, as in the preprint [36], which applies these methods to the study of groups acting on a matroid via the action on the simplicial complex formed by its family of independent sets.

6.1 Preliminaries

We consider the following simplicial complex:

Definition 6.1.1. Let $[n]$ denote the set $\{1, \ldots, n\}$.

Then we define $T(n)$ to be the simplicial complex whose k-simplices are the $\sigma = a_0, \ldots, a_k$, with $\emptyset \neq a_i \subsetneq [n]$ and $a_i \cap a_j = \emptyset$ for all $i \neq j$.

We define the **total size** of such a simplex $\sigma = a_0, \ldots, a_k$ to be

$$|\sigma| = \sum_{i=0}^{k} |a_i|,$$

and the **type** of a simplex

$$a = \{a_0^1, \ldots, a_0^{k_0}\}, \ldots \{a_j^1, \ldots, a_j^{k_j}\}$$

with $k_i \geq k_m$ for all $m > i$, to be $(k_0, \ldots, k_j)$.

To facilitate later discussions, we also fix a global ordering on the vertices of $T(n)$, and somewhat arbitrarily choose this to be in increasing order of size first, then increasing order of smallest element, then increasing order of second-smallest element, and so on to increasing order of largest element. For ease of later notation, we adopt the following convention: for a point x contained in some simplex $|\sigma|$ of $|T(n)|$, when we say "the barycentric coordinates of x in $|\sigma|$" (or in images thereof), we mean the vector $(x_{\{1\}}, \ldots, x_{[n-1]})$, where $x_v = 0$ if v is not a vertex of σ, and the x_v for $v \in \sigma$ are such that

$$x = \sum_{v \in \sigma} x_v |v|, \qquad \sum_{v \in \sigma} x_v = 1,$$

and $x_v \geq 0$ for all v.

Note that the requirement that a_i is a proper subset of $[n]$ removes only a single disconnected point from $T(n)$, and is done for reasons of convenience, as it slightly simplifies the formula for our homology in the 0th degree, and removes a trivial factor from the resulting representation.

We define also an action of S_n on this complex, by

$$\sigma \cdot \{\{a_0^1, \ldots, a_0^{k_0}\}, \ldots, \{a_j^1, \ldots, a_j^{k_j}\}\} = \{\{\sigma a_0^1, \ldots, \sigma a_1^{k_0}\}, \ldots, \{\sigma a_j^1, \ldots, \sigma a_j^{k_j}\}\}.$$

By Lemma 5.2.8, this action passes to an action of S_n on the homology of $T(n)$, and hence representations of S_n, which we wish to investigate.

6.2 Homology Calculation

6.2.1 A Homotopy Equivalence

In order to compute the homology of $T(n)$, we will first construct another complex $\widetilde{T}(n)$, then show that this is homotopy equivalent to $T(n)$.

We define $\widetilde{T}(n)$ to be the simplicial complex whose 0-simplices are the $a_0 \subsetneq [n]$ with $|a_0| > 1$, together with an extra 0-simplex $*$, and whose k-simplices, for $k > 0$, are:

1. The $\sigma = a_0, \dots, a_k$ in $T(n)$ such that $|a_i| > 1$ for all i, and
2. the $a_0, \dots, a_{k-1}, *$, where $\sigma = a_0, \dots, a_{k-1}$ is a $k-1$-simplex of $T(n)$ such that $|\sigma| < n$ (that is: the simplices required to make $*$ the vertex of a cone on the simplices of $\widetilde{T}(n)$ corresponding to simplices of $T(n)$ of total size at most $n-1$ with no singleton vertices).

We define a global ordering on the vertices of $\widetilde{T}(n)$, similar to the ordering on $T(n)$: we order $*$ first, and then the other vertices according to the order on the vertices of $T(n)$. We apply the same convention for barycentric coordinates here as in $T(n)$.

Once the homotopy equivalence of $T(n)$ and $\widetilde{T}(n)$ is established, it is simple to compute our homology, and the resulting representations: our equivalence will map the simplices with no singleton vertices to the corresponding simplices in $\widetilde{T}(n)$, and send all other simplices into the cone formed by the simplices of $\widetilde{T}(n)$ with a vertex at $\{*\}$. Thus, any set of generators for the homology of $\widetilde{T}(n)$ correspond directly to generators for the homology of $T(n)$.

In order to establish this homotopy equivalence, we take the following steps:

1. We split $T(n)$ into three subcomplexes C_n, B_n and σ_0.
2. We define a continuous map φ from each of $|C_n|$, $|B_n|$ and $|\sigma_0|$ to $|\widetilde{T}(n)|$.
3. We verify that these three definitions of φ agree on the intersections of $|C_n|$, $|B_n|$, and $|\sigma_0|$, and hence that φ is a continuous map
$$|T(n)| \to |\widetilde{T}(n)|.$$
4. We similarly split $\widetilde{T}(n)$ into two subcomplexes $\widetilde{C}_n$ and $\widetilde{B}_n$, and define a continuous map ψ from $|\widetilde{B}_n|$ to $|T(n)|$.
5. We embed $|\sigma_0 \cup C_n|$ into $\mathbb{R}^{2^n}$.
6. We similarly embed $|\widetilde{C}_n|$ into $\mathbb{R}^{2^n}$ in such a way that the image is contained in our embedded image of $|\sigma_0 \cup C_n|$.
7. We use these embeddings to extend ψ continuously to all of $|\widetilde{T}(n)|$.
8. We give explicit homotopies from $\psi \circ \varphi$ and $\varphi \circ \psi$ to the identity maps on $|T(n)|$ and $|\widetilde{T}(n)|$ respectively.

To split $T(n)$, we define

- σ_0 to be the simplex $\{\{1\}, \ldots \{n\}\}$,
- B_n to be the sub-simplicial complex of $T(n)$ consisting of all simplices $\sigma = a_0, \ldots, a_k$ such that $|a_i| > 1$ for all i, and
- C_n to be the sub-simplicial complex of $T(n)$ consisting of all simplices $\sigma = a_0, \ldots, a_k$ with some $|a_i| = 1$ for some i, together with all sub-simplices thereof, apart from σ_0.

We now define φ such that it collapses $|\sigma_0|$ to the vertex $|*|$, leaves $|B_n|$ essentially unchanged, and adjusts $|C_n|$ appropriately to connect the two.

First, as the simplices of B_n are all included in $\widetilde{T}(n)$, we can simply define $\iota : |B_n| \to |\widetilde{T}(n)|$ to be the map induced by that inclusion.

We now define a map p from $|C_n|$ to $|\widetilde{T}(n)|$ as follows: for each point $x \in |C_n|$, choose any simplex $\sigma = a_0, \ldots, a_k$ (with $|a_0|, \ldots, |a_j| > 1$, and $|a_i| = 1$ for all $i > j$) such that x is in $|\sigma|$. Writing

$$x = (x_{\{1\}}, \ldots, x_{[n-1]})$$

in barycentric coordinates in $|\sigma|$, we define $p_\sigma(x)$ to be the point in

$$|\widetilde{\sigma}| = |a_0, \ldots, a_j, *|$$

whose barycentric coordinates $(y_*, \ldots, y_{[n-1]})$ with respect to $|\widetilde{\sigma}|$ are given by

$$y_* = \sum_{i=1}^{n} x_v$$

and $y_v = x_v$ for all other vertices v. After verifying that it is well-defined, we shall take $p(x)$ to be $p_\sigma(x)$ for any choice of σ.

This then allows us to define our map by

$$\varphi(x) = \begin{cases} * & x \in |\sigma_0| \\ p(x) & x \in |C_n| \\ \iota(x) & x \in |B_n|. \end{cases}$$

This map is clearly continuous on σ_0, on $|B_n|$, and on each simplex of $|C_n|$.

To verify that φ is well-defined on the intersections of these, we consider each case in turn:

If x lies in two simplices σ and τ of $|C_n|$, let μ be their intersection. We will show that the images given by taking x as a point of σ and μ agree: symmetrically, those of τ and μ agree, hence those of σ and τ.

Now, as the intersection of the k-simplex σ with another simplex, μ is necessarily a j-face of σ, for some $j \leq k$. If the barycentric coordinates (x_v) of x in σ have $x_v > 0$ for some $v \notin \mu$, since the barycentric coordinates on σ are an extension of those on μ, we must have

$$\sum_{w \in \sigma} x_w \geq x_v + \sum_{w \in \mu} x_w > \sum_{w \in \mu} x_w = 1,$$

contradicting the definition of our barycentric coordinates on $|\sigma|$.

Thus, the barycentric coordinates of x in σ are identical to its barycentric coordinates in μ, and so $p_\sigma(x) = p_\mu(x)$, so our map is well-defined on all of $|C_n|$.

Any x in the intersection of $|\sigma_0|$ and $|C_n|$ is necessarily contained in some simplex σ of $|C_n|$ whose vertices all correspond to singletons, and by the above, we can use p_σ to compute $p(x)$, but $p_\sigma : \sigma \to | * |$ is necessarily the constant map, as it maps into the one-point set, so indeed, $p(x) = *$, agreeing with our definition of φ on σ_0,

Finally, any x in the intersection of $|C_n|$ and $|B_n|$ must lie in some simplex $|\sigma|$ of $|C_n|$ containing no singleton vertices, and by the above, we can use p_σ to compute $p(x)$. But p_σ is exactly the map induced by the inclusion of σ into $\widetilde{T}(n)$, agreeing with our definition of φ on $|B_n|$.

Thus, φ is, indeed, well-defined.

To split $\widetilde{T}(n)$ into two subcomplexes, we define:

- A subcomplex $\widetilde{C}_n$ consisting of all simplices with a vertex at $*$, and all subsimplices thereof (that is: $|\widetilde{C}_n| = \varphi(|C_n|)$), and

- A subcomplex $\widetilde{B}_n$ consisting of all simplices without a vertex at $*$ (that is: $|\widetilde{B}_n| = \varphi(|B_n|)$).

To define ψ separately on each of these, we first note that ι restricts to a homeomorphism from $|B_n|$ to $|\widetilde{B}_n|$, and we therefore can simply define $\psi(x) = \iota^{-1}(x)$ for all $x \in |\widetilde{B}_n|$, and this too will be continuous on $|\widetilde{B}_n|$. Defining ψ on $\widetilde{C}_n$ will take rather more work.

To embed $|C_n \cup \sigma_0|$ into $\mathbb{R}^{2^n}$ (with coordinates indexed by the subsets of $[n]$) by a map ξ_n, we proceed as follows:

We send $|\sigma_0|$ to form a standard $n-1$-simplex, with vertices the standard basis elements $e_{\{1\}}, \dots, e_{\{n\}}$, with the vertex $\{k\}$ sent to e_k, and interpolate linearly between these points.

Let b_0 be the image under this map of the barycentre of $|\sigma_0|$, and let π be the composition of the natural projection π_1 of $\mathbb{R}^{2^n}$ onto $\mathbb{R}^n$ (given by $\pi_1(e_{\{i\}}) = e_i$, and $\pi_1(e_v) = 0$ for non-singleton v) with the orthogonal projection π_2 from $\mathbb{R}^n$ onto the affine plane through the e_v for singletons v. Note, in particular, that π_2 fixes $\pi_1\xi_n(|\sigma_0|)$, as this is already contained in that affine plane.

For each vertex v in the remainder of C_n, there is a unique maximal simplex whose vertices are v and some subset $A(v)$ of the vertices of σ_0 ($A(v)$ is the set of singletons of elements of $[n] \setminus v$). Take $b(v)$ to be the barycentre of the images under our embedding of $A(v)$, and map v to the point $2b(v) + e_{\{v\}} - b_0$.

We then map the simplices of C_n to the convex hull of the images of their vertices.

If τ has only one non-singleton vertex v, and ρ likewise has a single non-singleton vertex $w \neq v$, then all points x of $\xi_n|\tau|$ have $x \cdot e_w = 0$, and all points y of ρ have $y \cdot e_v = 0$, so these can intersect only at points z of τ with $z \cdot e_v = 0$, which occurs, by construction of ξ_n, only in the face that τ shares with σ_0. Thus, ξ_n embeds $|B_n^1 \cup \sigma_0|$, where B_n^1 is the collection of all simplices of B_n with at most one non-singleton vertex.

We now proceed by induction. In the $n = 1$ and $n = 2$ cases, C_n is empty, so the result is vacuously true. In the $n = 3$ case, C_3 consists of three maximal 1-simplices: $(\{1\}, \{2, 3\})$, $(\{2\}, \{1, 3\})$, and $(\{3\}, \{1, 2\})$. By the case above, since each of these has only a single non-singleton vertex, ξ_n embeds C_3.

In general, suppose that ξ_n embeds C_m for all $m < n$.

If τ and ρ are simplices of C_n which share a common vertex v, then we can remove v from both to give two simplices τ' and ρ', which lie in an isometric copy D of $C_{n-|v|}$ inside C_n, given by fixing some set N_v of $n - |v|$ elements of $[n]$ that includes the union of all of the vertices of τ and ρ, with the isometry given by relabelling the elements of $[n] - v$ to match those of $[n - |v|]$. Since this map is an isometry, and $\xi_{n-|v|}$ is an embedding, so is their composition, $\xi_n|_D$.

Now, ξ_n, by definition, sends τ and ρ to the convex hull of

$$\xi_n|\tau'| \cup \xi_n|v|$$

and

$$\xi_n|\rho'| \cup \xi_n|v|$$

respectively. Since $\xi_n|_D$ is an embedding, we have

$$\xi_n(|\tau'|) \cap \xi_n(|\rho'|) = \xi_n(|\tau' \cap \rho'|).$$

Thus,

$$\xi_n(|\tau|) \cap \xi_n(|\rho|)$$

is the convex hull of

$$(\xi_n(|\tau'|) \cap \xi_n(|\rho'|)) \cup \{\xi_n(|v|)\} = \xi_n(|\tau' \cap \rho'|) \cup |v|,$$

which is precisely

$$\xi_n(|\tau \cap \rho|).$$

Finally, if τ and ρ are simplices of C_n sharing no common vertices, then since $\xi_n(|\tau|)$ lies in the linear span of the e_v for $v \in \tau$, and $\xi_n(|\rho|)$ lies in the linear span of the e_w for $w \in \tau$, and these two spans intersect only at the origin, $\xi_n(|\tau|)$ and $\xi_n(|\rho|)$ can only intersect if both meet the origin.

But ξ_n sends all vertices of $|C_n|$ to positively-signed basis vectors, and all other simplices to the convex hull of their vertices. Thus, for any point $x = (x_1, \ldots, x_{2^n})$ in $\xi_n(C_n)$, for any simplex τ containing x, we may write x in barycentric coordinates for τ as (a_v) such that $\sum_{v \in \tau} a_v = 1$. In particular there is some $a_j > 0$, so x cannot be the origin.

Thus, $\xi_n(|\tau|)$ and $\xi_n(|\rho|)$ do not intersect, and hence ξ_n is, indeed, an embedding.

To embed $|\widetilde{C}_n|$ into $C'_n = \xi_n(|C_n \cup \sigma_0|)$, we proceed as follows.

Firstly, consider any vertex $v \neq *$ of $\widetilde{C}_n$. This is also a vertex of C_n. We map the line segment L_v from $|v|$ to $|*|$ into C_n by first subdividing it into two segments, L_v^1, consisting of the half closest to $|*|$ and L_v^2, consisting of the half closest to v. With $m(v)$ the barycentre of the singleton face $A(v)$ of the highest-dimensional simplex $\rho(v)$ of C'_n containing $\xi_n(v)$, as before, we map L_v^1 to the line segment from $*$ to $m(v)$, and L_v^2 to the line segment from $m(v)$ to $\xi_n(|v|)$. All points of this lie in C'_n since $*$ and $m(v)$ both lie in $\xi_n|\sigma_0|$, which is convex, and since $m(v)$ and $\xi_n(|v|)$ both lie in $\rho(v)$, which is also convex.

We shall now construct our map on the rest of $\widetilde{C}'_n$ by interpolating between these lines. Specifically, for any $x \in |\widetilde{C}_n \cap \widetilde{B}_n|)$, let $|\tau|$ be any simplex of $|\widetilde{C}_n|$ containing x. We define a map $\omega_\tau : L_x \to C'_n$ by sending L_x^1 to the line segment from b_0 to $m(x) := \sum_{w\in\tau} \alpha_w(x)m(w)$, and L_x^2 to the line segment from $m(x)$ to $v(x) := \sum_{w\in\tau} \alpha_w(x)\xi_n(|w|)$, with $\alpha_w(x)$ the barycentric coordinate of x at the vertex w. Note that, in particular, both m and v are continuous on all of $|C_n|$, once we have established that they are well-defined.

As the map ω formed by assembling the ω_τ is clearly continuous on each simplex of $\widetilde{C}'_n$, being piece-wise linear, it suffices to show that it is well-defined to establish its continuity everywhere.

For that purpose, consider any $x \in |\widetilde{C}_n \cap \widetilde{B}_n|$ that is contained in two simplices $|\tau|$ and $|\theta|$. Then it is also contained in their intersection, which is itself some simplex $|\zeta|$. Thus, to show that $\omega_\tau(x) = \omega_\theta(x)$, it suffices to show that both are equal to $\omega_\zeta(x)$. We shall show this for $\omega_\tau(x)$: the argument for $\omega_\theta(x)$ is the same.

As before, we note that, since the barycentric coordinates on τ extend those on θ, the barycentric coordinates of x on τ and θ must be identical, so $m(x)$ and $v(x)$ are well-defined, and hence $\omega_\tau(x) = \omega_\theta(x)$.

We therefore define $\omega : |\widetilde{C}_n| \hookrightarrow C'_n$ by $\omega(x) = \omega_\tau(x)$ for any simplex τ of $\widetilde{C}_n$ such that $x \in |\tau|$. This is continuous as it is piece-wise linear on each simplex of $|\widetilde{C}'_n|$.

We now define our map ψ by

$$\begin{aligned} \psi :& |\widetilde{T}(n)| \to |T(n)| \\ & x \mapsto \begin{cases} \xi_n^{-1}(\omega(x)) & x \in |\widetilde{C}_n| \\ \iota'(x) & x \in |\widetilde{B}_n|. \end{cases} \end{aligned}$$

with ι' the map induced by the inclusion of $\widetilde{B}_n$ into $T(n)$.

As we defined ω to send each point x of $\widetilde{B}_n \cap \widetilde{C}_n$ to $\xi_n\iota'(x)$, ψ is well-defined, and it is continuous since both $\xi_n^{-1} \circ \omega$ and ι' are continuous. To show that $T(n)$ and $\widetilde{T}(n)$ are homotopy equivalent, then, we need only show that $\varphi\psi$ and $\psi\varphi$ are each homotopic to the identity.

We define our first homotopy

$$H : T(n) \times [0,1] \to T(n)$$

as follows:

First, note that we can inherit a scalar multiplication and an addition on $|T(n)|$ from $\mathbb{R}^{2^n}$ via our embedding, and that since each simplex of $|T(n)|$ is convex, all operations of the form $f(t,x,y) := tx + (1-t)y$, with x and y lying in the same simplex, and $t \in [0,1]$ give another point of $|T(n)|$.

For $x \in |B_n|$, we define $H_B(x,t) = x$ for all t. Note that since $\psi\varphi(x) = x$, we have $H_B(x,1) = \psi\varphi(x)$.

For $x \in |\sigma_0|$, we define

$$H_\sigma(x,t) = \begin{cases} x & t < \frac{1}{2} \\ f(2-2t, x, b|\sigma_0|) & t \geq \frac{1}{2}, \end{cases}$$

with $b\rho$ denoting the barycentre of ρ. Note that H_σ is continuous, since $H_\sigma(x, \frac{1}{2}) = x$.

For $x \in |C_n|$, there is some unique $v(x) \in |\widetilde{B}_n \cap \widetilde{C}_n|$ such that $\varphi(x) \in L_{v(x)}$. Let m_x be the point where $L_{v(x)}$ meets the boundary of $|\sigma_0|$. If $\varphi(x) \in L^1_{v(x)}$, then we define

$$H^1_C(x,t) = \begin{cases} f(1-2t, x, \psi(m_{v(x)})) & t < \frac{1}{2} \\ f(2-2t, \psi(m_{v(x)}), \psi\varphi(x)) & t \geq \frac{1}{2}. \end{cases}$$

If, on the other hand, $\psi(x) \in L^2_{v(x)}$, then we define

$$H^2_C(x,t) = \begin{cases} f(1-2t, x, \psi\varphi(x)) & t < \frac{1}{2} \\ \psi\varphi(x) & t \geq \frac{1}{2} \end{cases}$$

Again, note that both of these are continuous.

Finally, if $x \in |B_n|$, then we define $H_B(x,t) = x$ for all t.

We shall then assemble H from H_B, H^1_C, H^2_C, and H_σ, after checking that it is well-defined.

For that purpose, first note that for $x \in |\sigma_0 \cap C_n|$, we have $\psi(m_y) = x$, so for $t < \frac{1}{2}$, we have

$$H^1_C(x,t) = f(1-2t, x, x) = x = H_\sigma(x,t),$$

and for $t \geq \frac{1}{2}$, we have

$$H^1_C(x,t) = f(2-2t, x, \psi\varphi(x)) = H_\sigma(x,t).$$

Now, the intersection of the domains of H^1_C and H^2_C is precisely the set of all x such that $\varphi(x) = m_y$ for some $y \in |\widetilde{B}_n \cap \widetilde{C}_n|$. For such x, and $t < \frac{1}{2}$, we have

$$H^1_C(x,t) = f(1-2t, x, \psi(m_y)) = f(1-2t, x, \psi\varphi(x)) = H^2_C(x,t).$$

For $t \geq \frac{1}{2}$, we have

$$H^1_C(x,t) = f(2-2t, \psi(m_y), \psi\varphi(x)) = \psi\varphi(x) = H^2_C(x,t).$$

Finally, for $x \in |B_n \cap C_n|$, we have $\psi\varphi(x) = x$, so for $t < \frac{1}{2}$, we have

$$H_C^2(x,t) = f(1-2t, x, \psi\varphi(x)) = f(1-2t, x, x) = x = H_B(x,t),$$

and for $t \geq \frac{1}{2}$, we have

$$H_C^2(x,t) = \psi\varphi(x) = x = H_B(x,t).$$

Thus, H is well-defined.

Further, H is clearly continuous on $|\sigma_0|$ and $|B_n|$, so we need only confirm that it is continuous on $|C_n|$. It is clear that H is continuous on

$$\{(x,t)|t \in [0,1]\}$$

for each $x \in |C_n|$. As shown previously, the maps v and m are continuous on $|C_n|$. Thus, H must also be continuous on all of $|C_n|$, and hence we have $\psi\varphi \simeq id_{T(n)}$.

Finally, we define our other homotopy

$$G : |\widetilde{T}(n)| \times [0,1] \to |\widetilde{T}(n)|$$

to be the straight-line homotopy $G(x,t) = f(1-t, x, \varphi\psi(x))$, with all arithmetic done in the barycentric coordinates on $|\widetilde{T}(n)|$.

This is well-defined on $|\widetilde{B}_n| \times [0,1]$, since $\varphi\psi(x) = x$ here, so $G(x,t) = x$. To show that it is well-defined on $|\widetilde{C}_n \setminus \widetilde{B}_n|$, we shall show that $\varphi\psi(x)$ lies in the minimal simplex τ of $|\widetilde{C}_n|$ containing x, which is convex.

For this purpose, note that for $x \in L_y^1$, we have $\psi(x) \in |\sigma_0|$, and hence $\varphi\psi(x) = |*|$. If, on the other hand, $x \in L_y^2$, then $\psi(x) = \xi_n^{-1}\omega(x)$, which lies in $|C_n|$, so $\varphi\psi(x) = p\xi_n^{-1}\omega(x)$. Now, p maps each simplex $|\rho|$ of $|C_n|$ to the simplex of $|\widetilde{C}_n|$ given by the geometric realisation of the simplex given by removing all singletons from ρ, and replacing them by $*$.

Thus, we need only show that the non-singleton vertices of $\xi_n^{-1}\omega(\tau)$ are those corresponding (under ι) to the non-$*$ vertices v of τ. But for such v, we have $\omega(v) = \xi\iota(v)$, so this is, indeed, the case.

Thus, we have $\varphi\psi \simeq id_{\widetilde{T}_n}$, so indeed, $T(n)$ and $\widetilde{T}_n$ are homotopic.

6.2.2 The Homology of $\widetilde{T}(n)$

To compute generators for the homology of $\widetilde{T}(n)$, we note that, by construction, the vertex $*$ is the vertex of a cone whose base is all of those simplices of total size at most $n-1$ (as each such simplex can be made into a simplex of $T(n)$ with a singleton vertex, and hence a simplex of $\widetilde{T}(n)$ with a vertex at $*$, by adding into it the singleton containing one of the elements of $[n]$ not already contained in any of its vertices). There is, therefore, a homotopy equivalence $x \mapsto [x]$ from $|\widetilde{T}(n)|$ to its quotient (as a cellular complex) $\widehat{T}(n)$ by the equivalence relation identifying the entire cone to $*$.

In $\widehat{T}(n)$, we have only one remaining vertex $*$ (as there is no singleton of total size n in $T(n)$), and for $k > 0$, the k-simplices of $T(n)$ of total size n containing no singletons (the only simplices of $\widetilde{T}(n)$ not identified to $*$ in $\widehat{T}(n)$) have now been mapped to k-cells in $\widehat{T}(n)$ with their entire boundary glued to $*$ (as the boundary in $\widehat{T}(n)$ is covered by the images of the faces in $\widetilde{T}(n)$, and each such face is a $k-1$ simplex of total size at most $n-1$, so is mapped to $*$ by our quotient).

Thus, $\widehat{T}(n)$ is a wedge of spheres of varying dimensions, and to compute its homology, we need only count the number of spheres that appear in each dimension. That is, for each $k < n$, we need to count the number $\beta(n,k)$ of partitions of $\{1,\ldots,n\}$ into k non-singleton subsets.

For this purpose, let $S(n,k)$ be the Stirling number of the second kind, which counts the number of partitions of $\{1,\ldots,n\}$ into k subsets (possibly including singletons). Then we can use an inclusion-exclusion counting argument to compute our $\beta(n,k)$. First, note that for each choice of i elements from $\{1,\ldots,n\}$, there are $S(n-i,k-i)$ partitions of $\{1,\ldots,n\}$ into k subsets including each of our i elements as a singleton.

We therefore have

$$\beta(n,k) = \sum_{i=0}^{k} (-1)^i \begin{pmatrix} n \\ i \end{pmatrix} S(n-i,k-i).$$

Thus, we have the following theorem (with the zeroth homology following from the connectedness of $T(n)$:

Theorem 6.2.1. *The homology of $T(n)$ with coefficients in a module M over a commutative ring R is given by*

$$\widetilde{H}_k(T(n)) = M^{\beta(n,k)},$$

with, for $k > 0$, generators for the kth homology labelled by the partitions of $\{1,\ldots,n\}$ into subsets of size at least 2.

6.3 The Associated Representations

To compute the representations of S_n arising from its action on $T(n)$, we first pass a set of generators for the homologies of $\widehat{T}(n)$ back through our sequence of homotopy equivalences to a set of generators for the homologies of $T(n)$. For this purpose, note that the generators given above are precisely the images under our sequence of homotopies of the k-simplices of $T(n)$ of total size n with no singleton vertices, and thus, we have a set of generators $A^k(n)$ of the kth homology of $T(n)$ indexed by the collection of all such k-simplices. We therefore write elements of the homology as formal sums

$$\sum_{\sigma \in A(n)} \alpha_\sigma[\sigma],$$

with $\alpha_\sigma \in M$.

Thus, the corresponding representation ρ_n^k of S_n is given, with respect to these generators, by

$$\rho_n^k(\tau)\left(\sum_{\sigma \in A^k(n)} \alpha_\sigma \sigma\right) = \sum_{\sigma \in A^k(n)} \alpha_\sigma (-1)^{t(\tau\sigma)} \tau(\sigma).$$

Since our action on $T(n)$, and hence ρ_n^k, permutes simplices of each type separately, ρ decomposes as the direct sum of the representations ρ_λ, for each type λ of k-simplices corresponding to generators of $H^k(T(n), M)$. Let N^λ be the $\mathbb{C}[S_n]$ module corresponding to ρ_λ.

The representations of S_n arising from the standard action of S_n on such families of Young tableaux are well-understood. See, for example, the treatment in [15, § 7].

Let Y^λ be the set of standard tabloids of shape λ, and let M^λ be the $\mathbb{C}[S_n]$ module with basis the Y^λ and the action induced by the standard action of S_n on Y^λ.

We then define an equivalence relation on M^λ generated by

$$\{T\} \sim \operatorname{sgn}(\sigma)(\sigma\{T\})$$

for $\{T\} \in M^\lambda$ and σ a permutation of the rows of $\{T\}$ preserving its shape, extended linearly. We note in particular that if λ has no repeated values, there are no non-trivial such σ, so our equivalence relation is the trivial one in this case.

We note that this is equivariant: if $\tau \in S_n$ and T and σ are as above, then

$$\begin{aligned}
\tau \cdot (\operatorname{sgn}(\sigma)(\sigma T)) &= \operatorname{sgn}(\sigma)(\tau \cdot \sigma T) \\
&= \operatorname{sgn}(\sigma)(\sigma(\tau \cdot T)) \\
&\sim \tau \cdot T.
\end{aligned}$$

Thus, the quotient $L^\lambda := Y^\lambda / \sim$ is also a $\mathbb{C}[S_n]$ module, with basis the equivalence classes of the tabloids. Again, we note that if λ has no repeated values, then $L^\lambda = Y^\lambda$.

Proposition 6.3.1. *There is an isomorphism of* $\mathbb{C}[S_n]$ *modules*

$$f : N^\lambda \to L^\lambda.$$

Proof. For σ one of our homological generators of shape λ, define $f(\sigma)$ to be the equivalence class in L^λ represented by the tabloid given by ordering the vertices of σ in decreasing order of length then in increasing order of minimal element, then entering the elements of the k-th such vertex as the values in the k-th row. This defines a map between the bases of N^λ and L^λ, which extends to a $\mathbb{C}$-isomorphism $N^\lambda \to L^\lambda$.

It remains to show that this map is compatible with the S_n actions. But f sends a generator which has each entry i in a vertex of size m_i to a tabloid with the entry i in a row of length m_i, so in particular, $f(g\cdot\sigma)$ has each entry $g(i)$ in a row of length $|\sigma_i|$, as does $g \cdot f(\sigma)$. Additionally, if i and j share a vertex in σ, then $g(i)$ and $g(j)$ share a row in both $g\cdot f(\sigma)$ and $f(g\cdot\sigma)$. Thus, $g \cdot f(\sigma)$ and $f(g \cdot \sigma)$ differ only in permuting rows of the same length. But generators differing in this way are exactly the generators that are identified by $\sim$, so $f(g \cdot \sigma) = g \cdot f(\sigma)$, as required. □

Let H_λ be the group of permutations of the rows of λ that preserve its shape. Then the equivalence relation $\sim$ can equivalently be defined by $A \sim B$ if and only if $A - B$ lies in the linear span K^λ of

$$J^\lambda := \{\{T\} - \mathrm{sgn}(\sigma)(\sigma\{T\}) | \{T\} \in Y^\lambda, \sigma \in H_\lambda\},$$

where H_λ acts in the natural way. Since $\sim$ is equivariant, K^λ is a $\mathbb{C}[S_n]$-submodule of M^λ.

Now, H_λ is a subquotient of S_n as follows: let A_λ be the Young subgroup associated to the partition λ (if $\lambda = (\lambda_1^{r_1}, \dots, \lambda_k^{r_k})$, then we have $A_\lambda = (S_{\lambda_1})^{r_1} \times \cdots \times (S_{\lambda_k})^{r_k}$ [15, p. 84]. Let Z_λ be the normalizer of A_λ. Then $H_\lambda = Z_\lambda / A_\lambda$, since Z_λ is precisely the group of all elements that preserve our partition of $[n]$, and A_λ is the subgroup of those that preserve the ordering of the subsets.

Further, H_λ decomposes as the product of the permutation groups of rows of each length, so $H_\lambda = S_{r_1} \times \ldots \times S_{r_k}$. Each factor S_{r_i} then acts via its regular permutation representation on each subset of our generators that fixes all entries outside of rows of length λ_i, and H_λ acts faithfully as the tensor product of those representations. The orbits under this action are the subsets of our generators in which, for each $i \in [n]$, we fix the length of the row in which i lies and the entries that it shares that row with. On each orbit, H_λ acts by its regular representation, and so M^λ decomposes (as a H_λ module) as the sum of some number of copies of the regular representation R_λ of H_λ, which we label by $R_1, \ldots, R_k$.

However, in N^λ, this group of row permutations acts also with sgn (in each S_{r_i} factor). Thus, we shall instead consider the module $Q^\lambda = M^\lambda \otimes \operatorname{sgn} = Q_1 \otimes \cdots \otimes Q_k$, with $Q_i := R_i \otimes \operatorname{sgn}$, which M^λ similarly decomposes into a product of (as a H_λ module). We note also that each Q_i is generated by some orbit O_i of the standard permutation action of H_λ.

Lemma 6.3.2. *K^λ is a H_λ-submodule of Q^λ, and hence its intersections $K_i := Q_i \cap K^\lambda$ with the Q_i are H_λ-submodules of the Q_i.*

Proof. For each tabloid $\{T\}$ and each $g, h \in H_\lambda$, with the Q^λ action, we have

$$\begin{aligned} g \cdot (\{T\} - \operatorname{sgn}(h)\{T\}) &= \operatorname{sgn}(g) g\{T\} - \operatorname{sgn}(g)\operatorname{sgn}(h) gh\{T\} \\ &= \operatorname{sgn}(g)(\{gT\} - \operatorname{sgn}(ghh^{-1}) ghg^{-1}\{gT\}) \end{aligned}$$

which lies in K^λ. Thus, this action of H_λ sends each generator of K^λ into K^λ, this is a H_λ-submodule of M^λ, and so each K_i is a submodule of Q_i. $\square$

We now fix some generator $\{T_i\}$ from each orbit O_i.
Define $G_i := \{\{T_i\} - \operatorname{sgn}(\sigma)\sigma\{T_i\} | \sigma \in H_\lambda \setminus \{1\}\}$.

Lemma 6.3.3. *G_i is a vector space basis for K_i.*

Proof. Firstly, any element $\{S\} - \operatorname{sgn}(\sigma)(\sigma\{S\})$ of J^λ can be written (with $\sigma(S,T)$ the element of H_λ such that $\sigma(S,T)\{S\} = \{T\}$) as

$$\begin{aligned} &\{S\} - \operatorname{sgn}(\sigma)(\sigma\{S\}) \\ &\quad = \{S\} - \operatorname{sgn}(\sigma(S,T_i))\{T_i\} + \operatorname{sgn}(\sigma(S,T_i))\{T_i\} - \operatorname{sgn}(\sigma)(\sigma\{S\}) \\ &\quad = -\operatorname{sgn}(\sigma(S,T_i))(\{T_i\} - \operatorname{sgn}(\sigma(T_i,S))\{S\}) \\ &\qquad + \operatorname{sgn}(\sigma(S,T_i))(\{T_i\} - \operatorname{sgn}(\sigma(T_i,S)\sigma)(\sigma\{S\})) \\ &\quad = -\operatorname{sgn}(\sigma(S,T_i))(\{T_i\} - \operatorname{sgn}\sigma(T_i,S)\{S\}) \\ &\qquad + \operatorname{sgn}(\sigma(S,T_i))(\{T_i\} - \operatorname{sgn}\sigma(T_i,\sigma S))(\sigma\{S\}) \end{aligned}$$

and each we note that if $\sigma(T_i, S)$ or $\sigma(T_i, \sigma S)$ is the identity, then the respective term in the final line is zero, so the union of the G_i generate K^λ, as a vector space. As each G_i is contained in Q_i, and the Q_i intersect trivially, each G_i then generates K_i, as a vector space.

Secondly, we note that G_i is linearly independent: if

$$\sum_{\{S\}\in O_i\setminus\{\{T\}\}} a_{\{S\}}(\{T_i\} - \operatorname{sgn}(s(T_i, S))\{S\}) = 0,$$

then looking at the coefficients of $\{S\}$, we see that $\operatorname{sgn}(s(T_i, S))a_{\{S\}} = 0$, so $a_{\{S\}} = 0$. □

Let C_i denote the linear span of the element

$$c_i := \sum_{\{S\}\in O_i} \operatorname{sgn}(s(T_i, S))\{S\}.$$

Lemma 6.3.4. *C_i is the $\mathbb{C}[H_\lambda]$-complement of K_i in Q_i.*

Proof. Firstly, we note that, with the standard inner product, we have

$$\begin{aligned}\langle c_i, \{T_i\} - \operatorname{sgn}(s(T_i, S))\{S\}\rangle &= \langle c_i, \{T_i\}\rangle - \operatorname{sgn}(s(T_i, S))\langle c_i, \{S\}\rangle \\ &= 1 - \operatorname{sgn}(s(T_i, S))^2 \\ &= 0.\end{aligned}$$

So c_i is orthogonal to each of the $\{T_i\} - \operatorname{sgn}(s(T_i, S))\{S\}$, so in particular is linearly independent of them, so C_i and K_i intersect trivially.

Further, the dimension of Q_i is $|O_i|$, and the dimension of K_i is $|G_i| = |O_i| - 1$, so $G_i \cup \{c_i\}$ must generate Q_i, hence C_i is the orthogonal complement of K_i in Q_i.

But also, for any $\sigma \in H_\lambda$, we have

$$\begin{aligned}\sigma c_i &= \sum_{\{S\}\in O_i} \operatorname{sgn}(s(T_i, S))\sigma\{S\} \\ &= \operatorname{sgn}(\sigma) \sum_{\{S\}\in O_i} \operatorname{sgn}(s(T_i, S)) \operatorname{sgn}(\sigma)\sigma\{S\} \\ &= \operatorname{sgn}(\sigma) \sum_{\{S\}\in O_i} \operatorname{sgn}(s(T_i, S)\sigma)\sigma\{S\} \\ &= \operatorname{sgn}(\sigma) \sum_{\{S\}\in O_i} \operatorname{sgn}(s(T_i, \sigma S))\{\sigma S\} \\ &= \operatorname{sgn}(\sigma)c_i.\end{aligned}$$

So C_i is also a $\mathbb{C}[H_\lambda]$-subspace of Q_i, hence the $\mathbb{C}[H_\lambda]$-complement of K_i. □

Thus, the representation on C_i is exactly the sign representation of H_λ. Meanwhile, the representation on Q_i is the tensor product of the regular representation with the sign representation, so K_i is the complement of the sign representation in that representation (which is exactly the tensor product with sign of the complement of the trivial representation in the regular representation).

Finally, we note that K^λ is the sum of the K_i, so is a sum of $z(\lambda)$ copies of the complement of that sign representation, where $z(\lambda)$ is the number of orbits of the standard action of H_λ on our generators. Thus, the quotient Q^λ/K^λ (as a H_λ-module) is the sum of $z(\lambda) = [S_n : H_\lambda]$ copies of that sign representation.

Further, as an S_n-module, Q^λ/K^λ is precisely the tensor product (over $\mathbb{C}[H_\lambda]$) of Q_i/K^λ by $\mathbb{C}[S_n]$. That is: it is precisely $Ind_{Z_\lambda}^{S_n}(Q_i/K^\lambda)$, where we obtain our Z_λ action by having $g \in Z_\lambda$ act as $[g] \in H_\lambda$ does.

Therefore, our homological representation is precisely the result of inducing the one-dimensional representation ρ_λ of Z_λ (given by the sign representation of $H_\lambda = Z_\lambda/A_\lambda$) up to S_n (so is $z(\lambda)$-dimensional).

We can also compute $z(\lambda)$ explicitly: it is the quotient of the dimensions of M^λ and R_λ. The dimension of M^λ is $\dfrac{n!}{\prod_{i=1}^{k}(\lambda_i!)^{r_i}}$ (it is precisely the number of tabloids of shape λ, which is the number of standard numberings of shape λ, divided by the order of $A_\lambda \cong S_{\lambda_1}^{r_1} \times \ldots \times S_{\lambda_k}^{r_k}$ [15, p. 84]), and R_λ is the tensor product of the R_i, so has dimension

$$\prod_{i=1}^{k} r_i!$$

Thus, we have

$$z(\lambda) = \frac{n!}{\prod_{i=1}^{k}(\lambda_i!)^{r_i} r_i!}.$$

We have thus proven the following.

Theorem 6.3.5. *Let Z_λ be the normaliser of the Young subgroup of S_n associated to the partition λ, and let ρ_λ be the one-dimensional representation of Z_λ given by the sign representation of Z_λ/A_λ. Then the homological representation given by the action of S_n on $T(n)$ in degree k is given by*

$$\bigoplus_{\lambda} \left(\mathrm{Ind}_{Z_\lambda}^{S_n}\rho_\lambda\right),$$

where the direct sum is over all partitions λ of n into k non-singleton components, and

$$\dim \operatorname{Ind}_{Z_\lambda}^{S_n} \rho_\lambda = \frac{n!}{\prod_{i=1}^{k} (\lambda_i!)^{r_i} r_i!}.$$

Chapter 7

Homological Representations of Nilpotent Groups

7.1 Definitions

We begin the construction of the simplicial complex that we shall study in this chapter by defining the following set.

Definition 7.1.1. For G a finite group, we define $D(G)$ to be the collection of all $(g_1, \ldots, g_n)$, where each g_i is a p_i element of G.

We now recall the following simple lemma (previously seen as Lemma 5.1.2):

Lemma 7.1.2. *Let G be a finite group, and let p be a prime dividing the order of G. Then for every $g \in G$, there are unique elements $u(g)$ and $s(g)$ of G such that:*

- $g = u(g)s(g) = s(g)u(g)$,
- *$u(g)$ is a p-element of G,*
- *$s(g)$ is a p'-element of G, and*
- *$u(g)$ and $s(g)$ are both powers of g.*

In the case where G is a finite group whose order has distinct prime divisors $p_1, \ldots, p_n$ (whose order we fix once and for all, though we shall see at the end that this is unnecessary), we can apply the above to each $g \in G$ for each p_i in turn, defining $u_1(g)$ and $s_1(g)$ via the prime p_1, then defining each $u_i(g)$ and $s_i(g)$ to be the elements of G given by applying the above lemma to $s_{i-1}(g)$, using the prime p_i. Note that in the last case, $s_{n-1}(g)$ is necessarily a p_n-element, as it cannot be divided by any of $p_1, \ldots, p_{n-1}$, so this produces a unique sequence of elements $u_1(g), \ldots, u_n(g)$ such that each $u_i(g)$ is a power of g, each $u_i(g)$ is a p_i element, and $g = u_1(g) \ldots u_n(g)$.

Conversely, if $g_1, \ldots, g_n$ are pairwise commuting elements of G such that each g_i is a p_i element, with $|g_i| = p_i^{e_i}$, let $g := g_1 \ldots g_n$. Then for each i, we have $g^{\alpha_i} = g_i^{\alpha_i}$, where

$$\alpha_i = \prod_{j \neq i} p_j^{e_j}.$$

As α_i is coprime to $p_i^{e_i}$, there is some integer a such that $a\alpha_i \cong 1 \pmod{p_i^{e_i}}$, so $g^{a\alpha_i} = g_i$. Thus, by the lemma above, we have the following result:

Lemma 7.1.3. *There is a bijection between G and $D(G)$.*

Denote this bijection by $u : G \to D(G)$.

We can now define our simplicial complex.

Definition 7.1.4. For any finite group G, we define $\Delta(G)$ to be the simplicial complex whose k-simplices are the $(g_1, \ldots, g_n) \in D(G)$ where exactly $k+1$ of the g_i are not the identity. In particular, all elements of $D(G)$ except for the identity are simplices of $\Delta(G)$ of some dimension.

The ith face map of $\Delta(G)$ is given by replacing the $(i-1)$st non-identity element of a k-simplex (with $k \geq i$) $(g_1, \ldots, g_n)$ by the identity.

We define also an action of G on $\Delta(G)$ by conjugation:

$$g \cdot (g_1, \ldots, g_n) = (gg_1g^{-1}, \ldots, gg_ng^{-1}).$$

7.2 Topological Constructions

In order to explore the structure of this complex, we begin with the simplest example: if G has prime-power order, then $u(g) = (g)$ for each $g \in G$, so $\Delta(G)$ consists of $|G| - 1$ discrete points. More generally, if G is a finite nilpotent group, then it is the direct sum of its Sylow-p-subgroups. Let G_i be the Sylow-p_i-subgroup of G for each $p_i \mid |G|$.

Define also

$$G^k = \bigoplus_{i \leq k} G_i$$

for each k. We shall proceed to construct $\Delta(G^k)$ out of $\Delta(G^{k-1})$, and therefore inductively construct $\Delta(G)$.

First, note that the subcomplex $\Delta_1(G^k)$ of $\Delta(G^k)$ consisting of all simplices of the form $(g_1, \ldots, g_{k-1}, 1)$ is isomorphic to $\Delta(G^{k-1})$ via the map

$(g_1, \ldots, g_{k-1}) \mapsto (g_1, \ldots, g_{k-1}, 1)$. For each non-identity element g of G_k, we have a subcomplex $\Delta_g(G^k)$ of $\Delta(G^k)$ consisting of all simplices whose kth entry is either g or the identity. The vertices of $\Delta_g(G^k)$ are precisely those of $\Delta_1(G^k)$, together with the single additional vertex $(1, 1, \ldots, g)$. Every simplex $(g_1, \ldots, g_{k-1}, 1)$ of $\Delta_1(G^k)$ is a maximal face of the simplex $(g_1, \ldots, g_{k-1}, g)$ of $\Delta_g(G^k)$, and all simplices of $\Delta_g(G^k)$ are either simplices of $\Delta_1(G^k)$ or are produced from such a simplex in this manner. Thus, $\Delta_g(G^k)$ is precisely a cone on $\Delta_1(G^k)$.

Further, as

$$\Delta(G^k) = \bigcup_{g \in G_k \setminus \{1\}} \Delta_g(G^k),$$

and for any $g \neq h \in G_k$, we have

$$\Delta_g(G^k) \cap \Delta_h(G^k) = \Delta_1(G^k),$$

our complex $\Delta(G^k)$ consists of $|G_k| - 1$ cones sharing a common base isomorphic to $\Delta(G^{k-1})$.

7.3 Homology

We shall compute the homology of $\Delta(G)$ by considering a more general class of complexes. We shall, throughout this section, identify each simplicial complex with its geometric realisation in our notation. We shall compute our homology with coefficients in $\mathbb{Q}$, for convenience, but an identical argument holds with coefficients in $\mathbb{Z}$, with a little care taken to ensure the splitting of an exact sequence.

Definition 7.3.1. For positive integers $a_1, \ldots, a_n$, we define the **n-partite simplicial complex** $\Gamma(a_1, \ldots, a_n)$ to be the simplicial complex with vertices $v_1^1, \ldots, v_1^{a_1}, \ldots, v_n^1 \ldots, v_n^{a_n}$, and maximal simplices the $\{v_1^{i_1}, \ldots, v_n^{i_n}\}$, where $i_j \in [a_j]$ for all j.

Lemma 7.3.2. *The homology of* $\Gamma(a_1, \ldots, a_n)$ *is given by*

$$\widetilde{H}_k \Gamma(a_1, \ldots, a_n) = 0$$

for all $k \neq n-1$, *and*

$$\widetilde{H}_{n-1} \Gamma(a_1, \ldots, a_n) = \mathbb{Q}^{\alpha},$$

where $\alpha = \prod(a_i - 1)$.

Proof. First, we note that

$$H_k\Gamma(a_1) = \begin{cases} \mathbb{Q}^{a_1} & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

as $\Gamma(a_1)$ consists of a_1 discrete points. In particular, this has non-zero homology only in degree 0.

Secondly, we note that if $a_n = 1$, then v_n^1 is the vertex of a cone on $\Gamma(a_1, \ldots, a_{n-1})$, and so $\Gamma(a_1, \ldots, a_{n-1}, 1)$ is contractible, and if $a_n = 2$, then v_n^1 and v_n^2 are the endpoints of a suspension of $\Gamma(a_1, \ldots, a_{n-1})$, so $H_k\Gamma(a_1, \ldots, a_{n-1}, 2) = H_{k-1}\Gamma(a_1, \ldots, a_{n-1})$. In particular, if $\Gamma(a_1, \ldots, a_{n-1})$ has non-zero homology only in degree $n-2$, then $\Gamma(a_1, \ldots, a_{n-1}, 2)$ has non-zero homology only in degree $n-1$.

We now proceed by a double induction, supposing that our result holds for both $\Gamma(a_1, \ldots, a_{n-1})$ and for $\Gamma(a_1, \ldots, a_{n-1}, a_n - 1)$.

For $n > 2$, we define two subsets of $\Gamma(a_1, \ldots, a_n)$:

We define

$$U = \Gamma(a_1, \ldots, a_n) \setminus \{v_n^{a_n}\},$$

and

$$V = \Gamma(a_1, \ldots, a_n) \setminus \Gamma(a_1, \ldots, a_{n-1}, a_n - 1),$$

where the latter is interpreted to be the subset of the $\Gamma(a_1, \ldots, a_n)$ corresponding to the subcomplex of $\Gamma(a_1, \ldots, a_n)$ given by removing $v_n^{a_n}$. Now, we see that both U and V are open in $\Gamma(a_1, \ldots, a_n)$.

Secondly, note that V is contractible, as it is homeomorphic to the cone space

$$(\Gamma(a_1, \ldots, a_{n-1}) \times [0, 1))/((x, 0) \sim (y, 0)).$$

Also, it is clear that the union of U and V is $\Gamma(a_1, \ldots, a_n)$, since $v_n^{a_n}$ does not lie in $\Gamma(a_1, \ldots, a_{n-1}, a_n - 1)$. Further, the intersection of U and V is homeomorphic to

$$\Gamma(a_1, \ldots, a_{n-1}) \times (0, 1),$$

so strongly deformation retracts to $\Gamma(a_1, \ldots, a_{n-1})$. Finally, note that U strongly deformation retracts to

$$\Gamma(a_1, \ldots, a_{n-1}, a_n - 1),$$

via the map $F : U \times [0, 1] \to U$ given by $F(x, t) = (1 - t)x + tb(x)$, where:

- If x lies in $\Gamma(a_1, \ldots, a_{n-1}, a_n - 1)$, then $b(x) = x$.
- Otherwise, $b(x)$ is the unique point in $\Gamma(a_1, \ldots, a_{n-1})$ such that x lies on the line from $b(x)$ to $v_n^{a_n}$, which exists since $\Gamma(a_1, \ldots, a_n) \setminus \Gamma(a_1, \ldots, a_{n-1}, a_n - 1)$ is the cone on $\Gamma(a_1, \ldots, a_{n-1}])$.

So the homology of U is equal to the homology of $\Gamma(a_1, \dots, a_{n-1}, a_n - 1)$, the homology of $U \cap V$ is equal to the homology of $\Gamma(a_1, \dots, a_{n-1})$, and the reduced homology of V is trivial. Thus, the Mayer-Vietoris reduced homology exact sequence for U and V reads

$$\cdots \to \widetilde{H}_{k+1}U \xrightarrow{\alpha} \widetilde{H}_{k+1}\Gamma(a_1, \dots, a_n) \xrightarrow{\beta} \widetilde{H}_k U \cap V \xrightarrow{\gamma} \widetilde{H}_k U \to \cdots$$

Now, since $\Gamma(a_1, \dots, a_{n-1})$ and $\Gamma(a_1, \dots, a_{n-1}, a_n - 1)$ (hence $U \cap V$ and U) have non-zero homologies only in degrees $n-2$ and $n-1$ respectively, for all $k \neq n-1$, our sequence is of the form

$$\cdots \to 0 \to \widetilde{H}_k\Gamma(a_1, \dots, a_n) \to 0 \to \cdots$$

so $\Gamma(a_1, \dots, a_n)$ has non-zero homology only in degree $n-1$. Thus, the only non-zero portion of our exact sequence is

$$0 \to \widetilde{H}_{n-1}U \xrightarrow{\alpha} \widetilde{H}_{n-1}\Gamma(a_1, \dots, a_n) \xrightarrow{\beta} \widetilde{H}_{n-2}(U \cap V) \to 0.$$

As a short exact sequence of abelian groups, this splits, so

$$\begin{aligned}\widetilde{H}_{n-1}\Gamma(a_1, \dots, a_n) &\cong \widetilde{H}_{n-1}U \oplus \widetilde{H}_{n-2}(U \cap V)\\ &\cong \widetilde{H}_{n-1}\Gamma(a_1, \dots, a_{n-1}, a_n - 1) \oplus \widetilde{H}_{n-2}\Gamma(a_1, \dots, a_{n-1}).\end{aligned}$$

Finally, we can inductively compute our homology:

$$\begin{aligned}\widetilde{H}_{n-1}\Gamma(a_1, \dots, a_n) &= \widetilde{H}_{n-2}\Gamma(a_1, \dots, a_{n-1}) \oplus \widetilde{H}_{n-1}\Gamma(a_1, \dots, a_{n-1}, a_n - 1)\\ &= \bigoplus_{a_n - 2} \widetilde{H}_{n-2}\Gamma(a_1, \dots, a_{n-1}) \oplus \widetilde{H}_{n-1}\Gamma(a_1, \dots{}_{n-1}, 2)\\ &= \bigoplus_{a_n - 2} \widetilde{H}_{n-2}\Gamma(a_1, \dots, a_{n-1}) \oplus \widetilde{H}_{n-2}S\Gamma(a_1, \dots, a_{n-1})\\ &= \bigoplus_{a_n - 1} \widetilde{H}_{n-2}\Gamma(a_1, \dots, a_{n-1})\\ &= \bigoplus_{i=2}^{n} \bigoplus_{a_i - 1} \widetilde{H}_0\Gamma(a_1)\\ &= \bigoplus_{i=2}^{n} \bigoplus_{a_i - 1} \mathbb{Z}^{a_1 - 1}\\ &= \mathbb{Q}^{\alpha}\end{aligned}$$

as required. $\square$

Finally, we note that $\Delta(G) = \Gamma(p_1^{e_1} - 1, \ldots, p_k^{e_k} - 1)$, where $|G| = \prod p_i^{e_i}$, and the p_i are distinct primes, and so we have

Theorem 7.3.3.

$$\widetilde{H}_k \Delta(G) = \begin{cases} \mathbb{Q}^\alpha & k = n-1 \\ 0 & \textit{otherwise} \end{cases}$$

where

$$\alpha = \prod_{i=1}^{n} (p_i^{e_i} - 2).$$

7.4 Shellability

To identify the action of G on $\Delta(G)$, we require a set of free generators for our homology. For this purpose, we shall provide a shelling of $\Gamma(a_1, \ldots, a_n)$.

First, we define a valuation function ν on the set of maximal simplices of $\Gamma(a_1, \ldots, a_n)$.

- We define $\nu(\sigma_0) = 0$, where $\sigma_0 = \{v_1^1, \ldots, v_n^1\}$.
- Where σ_i^j is the simplex given by replacing v_i^1 by v_i^j in σ_0, we define $\nu(\sigma_i^j) = i + n(j-1)$.
- Inductively, if ρ is a simplex which shares $k < n-1$ vertices with σ_0, then for each vertex v_i^j of ρ with $j > 1$, define ρ_i to be the simplex given by replacing v_i^j by v_i^1 in ρ. Define

$$\nu(\rho) = \max_i(\nu(\rho_i)) + \frac{1}{2^{n-k}}.$$

We now order our maximal simplices according to this valuation, breaking ties arbitrarily. Label the ith simplex of this ordering σ_i (beginning at 0, so that σ_0 keeps its name). We shall now show that this order gives a shelling of $\Gamma(a_1, \ldots, a_n)$.

Now, for $i > 0$, let k_i be the number of vertices that σ_i shares with σ_0. Let τ be any face of σ_i. Then we have two possibilities:

1. Every vertex of σ_i is contained in $\tau \cup \sigma_0$. Then every maximal simplex σ_j containing τ with $i \neq j$ can differ from σ_i only in replacing $\sigma_i \cap \sigma_0 = \{v_{i_1}^1, \ldots, v_{i_m}^1\}$ by some $\{v_{i_1}^{j_1}, \ldots, v_{i_m}^{j_m}\}$ with $j_t > 1$ for some t.

   Define $\rho_0 = \sigma_i$ and inductively define each ρ_k to be the result of replacing $v_{i_k}^1$ by $v_{i_k}^{j_k}$ in ρ_{k-1}, so that $\rho_m = \sigma_j$. By the third part of the definition of ν, we have $\rho_k < \rho_{k+1}$ for all k, and hence $\nu(\sigma_i) < \nu(\sigma_j)$.

   Thus, the face τ is not contained in the intersection of σ_i with $\bigcup_{j<i} \sigma_j$.

2. There is some vertex v_s^t contained in σ_i but not contained in either τ or σ_0. Let $\rho = \sigma_i \setminus \{v_s^t\}$. Then $\tau \subset \rho$, and $\rho' = \rho \cup \{v_s^1\}$ is a maximal simplex which contains τ, has $|\rho' \cap \sigma_0| = k+1$, and has $|\rho' \cap \sigma_i| = n-1$. Thus, by the third part of our definition of ν, we have $\nu(\rho') < \nu(\sigma_i)$.

Thus,

$$\sigma_i \cap \bigcup_{j<i} \sigma_j$$

is, for each $i > 0$, exactly the union of the ρ in case 2 above, each of which is $n-2$-dimensional.

Thus, our ordering of our maximal simplices is, indeed, a shelling, and our spanning simplices are precisely those simplices σ_i for which no proper face of σ_i contains all vertices of $\sigma_i \setminus \sigma_0$, which are precisely those with $\sigma_i \cap \sigma_0 = \emptyset$. By 4.3.18, this set of spanning simplices corresponds to a set of free generators of $H_k\Gamma(a_1, \ldots, a_n)$.

7.5 Representations

In the $\Delta(G)$ case, we choose σ_0 to be any maximal simplex whose vertices are all central in G (this is possible because each G_i is a p_i-group, so is not centreless). Thus, our maximal simplices are the $\{h_1, \ldots, h_n\}$ of G with $h_i \in G_i \setminus \{1, g_i\}$ for each i.

For any $g \in G$, then, we have

$$g \cdot (h_1, \ldots, h_n) = (u_1(g)h_1u_1(g)^{-1}, \ldots, u_n(g)h_nu_n(g)^{-1}),$$

and since each g_i is central, we cannot have $u_i(g)h_iu_i(g)^{-1} = g_i$ for any i, so this is again one of our generators, so G acts on $H_n(\Delta(G))$ by permuting our generators.

To further analyse this representation, it is helpful to consider the action of G on each $\Delta(G_i) \subset \Delta(G)$ separately. On $\Delta(G_i)$, each G_j with $j \neq i$ acts trivially (since G is the direct product of the G_i), so we need only consider the action of G_i on $H_0\Delta(G_i)$. This is simply the conjugation action of G_i on $G_i \setminus \{1, g_i\}$, so the associated representation $\widehat{\rho_i} : G_i \to GL_{|G_i|-2}(\mathbb{Z})$ is the quotient of the conjugation permutation representation of G_i by a two-dimensional trivial representation. This extends to a representation

$$\rho_i : G \to GL_{|G_i - 2|}(\mathbb{Z})$$

by defining $\rho_i(g) = I_{|G_i|-2}$ for $g \notin G_i$.

Summarising the above, we have the following result.

Theorem 7.5.1. *Let $p_1, \ldots, p_n$ be distinct primes, and for each i, let P_i be a group of order $p_i^{e_i}$, and define*

$$\alpha = \prod_{i=1}^{n} (p_i^{e_i} - 2).$$

Let G be the direct product of the P_i. Further, let ρ_i be the conjugation permutation representation of P_i, and let τ_i be the quotient of ρ_i by a two-dimensional trivial factor.

Then $\Delta(G) = \Gamma(p_1^{e_i} - 1, \ldots, p_n^{e_n} - 1)$, so

$$\widetilde{H}_k \Delta(G) = \begin{cases} \mathbb{Z}^\alpha & k = n-1 \\ 0 & \textit{otherwise}, \end{cases}$$

and the representation of G induced by its action on $\Delta(G)$ in degree $n-1$ is the tensor product of of the τ_i.

Proof. Let M be the module associated to our representation, and let $N = \bigotimes_i N_i$ be the module associated to the tensor product of the τ_i. Given the above, we need only construct a $\mathbb{C}[G]$-isomorphism between the two.

This is simple: M has basis B_ρ given by the simplices $(h_1, \ldots, h_n)$ with no h_i equal to g_i or 1. Define a map $\varphi : M \to N$ by $\varphi(h_1, \ldots, h_n) = h_1 \otimes \ldots \otimes h_n$, extended linearly. Then the image of our basis under φ is precisely the standard basis $B_\otimes := \{(h_1, \ldots, h_n) | h_i \in G_i \setminus \{1, g_i\}\}$ of $N = \bigotimes_i N_i$, since $B_i := G_i \setminus \{1, g_i\}$ is a basis for $N - i$. As φ is clearly injective, it is therefore a vector space isomorphism.

Finally, we note that

$$\begin{aligned}
(\otimes_i \tau_i)(g) \circ \varphi \left(\sum_{h=(h_1,\ldots,h_n) \in B_\rho} a_h h \right) &= \sum_{h \in B_\otimes} a_h (\otimes_i \tau_i)(g) \circ \varphi(h) \\
&= \sum_{h \in B_\otimes} a_h (\otimes_i \tau_i)(g)(h_1 \otimes \ldots \otimes h_n) \\
&= \sum_{h \in B_\otimes} a_h (\tau_1(g)(h_1) \otimes \ldots \otimes \tau_n(h_n)) \\
&= \sum_{h \in B_\otimes} a_h \varphi(\tau_1(g)(h_1), \ldots, \tau_n(g)(h_n)) \\
&= \sum_{h \in B_\otimes} a_h \varphi \circ (\rho(g))(h) \\
&= \varphi \circ (\rho(g))(h) \left(\sum_{h \in B_\otimes} a_h h \right)
\end{aligned}$$

so in fact φ is a $\mathbb{C}[G]$-isomorphism, as required. $\square$

We note, in particular, that this is independent of the ordering on the prime factors, so our initial fixing of the orders is unnecessary.

There is also a simple generalisation of this: if we have a finite group G which decomposes into a direct sum of groups

$$G = \bigoplus_{i=1}^{n} G_i$$

such that no G_i is centreless, then we can similarly construct a complex

$$\Delta(G_1, \ldots, G_n) = \Gamma(|G_1| - 1, \ldots, |G_n| - 1),$$

(which is no longer uniquely determined by G, but also by the choice of our G_i) on which G acts as in the above case, and hence obtain the same result in the representation theory of G, by a proof identical to the above:

Theorem 7.5.2. *Let*

$$G = \prod_{i=1}^{n} G_i$$

be any finite direct product of finite groups with $Z(G_i)$ non-trivial for all i, and

$$\alpha := \prod_{i=1}^{n} (|G_i| - 1).$$

Then

$$\widetilde{H}_k \Delta(G_1, \ldots, G_n) = \begin{cases} \mathbb{Z}^\alpha & k = n - 1 \\ 0 & \text{otherwise} \end{cases}$$

and if ρ_i is the conjugation permutation representation of G_i and τ_i is the quotient of ρ_i by a two-dimensional trivial factor. Then the representation of G induced by its action on $\Delta(G_1, \ldots, G_n)$ is the tensor product of the τ_i.

The only remaining obvious potential generalisation is to the case where some or all of the G_i are centreless. To extend our result to this case, it suffices to compute the representation induced in reduced homology by the conjugation action of a centreless group G on its non-identity elements.

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