Generalized Weyl Algebras and Their Dimensions

Kholood Mobark H Alnefaie

Thesis submitted for the degree of Doctor of Philosophy

School of Mathematics and Statistics

December 2019

Supervisor: Professor Vladimir Bavula

University of Sheffield

To my mother; may her soul rest in peace. She is the person who taught me the meaning of life and gave me all her love, care and support. You left me before the final step of my dream, but I still feel your love and impact surrounding me every day. I anticipated sharing this moment with you. Finally, the day that you were waiting for has come, and I hope you know that I did it.

To my father, who has supported me, given me his faith and been with me since the first day of my journey.

Abstract

The global and Krull dimensions of the generalized Weyl algebra $A = K[H,C](\sigma, a)$ and its localization, where K is an algebraically closed field and σ is an affine automorphism of $K[H,C]$, were studied.

Many classical algebras are examples of the algebra A or its localization (e.g., $U(\mathrm{sl}_2)$, $U_q(\mathrm{sl}_2)$, the Heisenberg algebra and its quantum analogues, the Witten's deformations, the Woronowicz's deformation, $\mathcal{O}_q(\mathrm{SL}_2(K))$ and others). As a consequence, the exact value of the global and Krull dimensions are computed for well-known quantum algebras, their localizations, and their tensor products.

Acknowledgments

First, praise is to Allah, the Almighty, for giving me life, guidance and many generous opportunities to move ahead.

I would like to express my deepest and most sincere appreciation to my supervisor, Prof Vladimir Bavula, for his encouragement, patience and counsel during my study. Without him, none of this work would be possible.

Words are inadequate to express my gratitude to my father for his devotion to me starting the day I arrived in this world and his ongoing support as I travel my life's path. I will not forget when I decided to study abroad, and he said, 'If that is what you want, I will be with you'. He encouraged me and ignored criticism, even though I know he was afraid for me taking this enormous step. However, he believed in and stayed the course with me always. Heaven-sized thanks to the dearest father in the world.

Thanks to my sister Modhi, who gave me succour and guidance from the very beginning of my studies. You let me feel safe when everything around seemed bad. Words cannot describe how grateful I am to have a sister with a mother's heart.

My sister Ohowod, thanks for all you have done and your comforting presence through the good times and the difficult ones. You are always there for me, sharing my laughter and tears. I am so lucky to have you as a sister, making my life wonderful.

Wafa, my little sister, thanks for your love and assurance. I am so glad for all the great memories that we shared together during my studies in the UK. I felt I was home when you were here, and you helped make some amazing days to remember.

I owe my deep appreciation to all of my brothers and sisters for giving me a foundation of love and for championing my goals. Thank you for being beside me whenever I need you.

I would like to thank my best friend, A. Alhabeeb, one who believes in me more than I believe in myself. Thank you for everything you do.

S. Abeid is the best person I have known during my studies. Thank you for being there for me with support in hard times and good ones. You gave me hope when life seemed dark.

Also, I am so grateful to my friends M. Alatawi and T. Alqurashi for their guidance and for cheering me on during my master's and PhD studies.

In addition, I would like to thank The University of Sheffield for its generous support that made my PhD journey more interesting.

Finally, I would like to thank Taibah University for financing my research and giving me the opportunity to work with some outstanding mathematicians.

Contents

Chapter 1

Introduction

1.1 Background to the thesis

The generalized Weyl algebras (GWAs) were introduced by Bavula in the 1990s. Initially, Bavula defined generalized Weyl algebras of degree 1 over the polynomial algebra $K[H]$ and then generalized for any algebra D. Since then, this class of algebras has been studied by various researchers in both pure and applied mathematics. Furthermore, a new class of algebras called weak generalized Weyl algebras was derived from GWAs by replacing the automorphism of D with an endomorphism.

1.1.1 Main objective of the thesis

Definition 1.1. *([44, p. 81].) Let D be a ring, σ be an automorphism of D and a a nonzero element of the centre $Z(D)$ of D. The* **generalized Weyl algebra** *$A := D(\sigma, a) = D[X, Y; \sigma, a]$ of degree 1 is a ring generated by the ring D and the 2 indeterminates X and Y subject to the relations:*

$$X\alpha = \sigma(\alpha)X \quad \text{and} \quad Y\alpha = \sigma^{-1}(\alpha)Y, \text{ for all } \alpha \in D, \quad YX = a \quad \text{and} \quad XY = \sigma(a).$$

The ring D is called the base ring *of the GWA. The automorphism σ and a are called the* defining automorphism *and the* defining element *of the GWA, respectively.*

In our research, we study the generalized Weyl algebra $A = D(\sigma, a)$ of degree 1 where the base ring $D = S^{-1}K[H, C]$ is a localization of the polynomial algebra

$K[H,C]$ in two variables H and C over an algebraically closed field K at a multiplicative set $S \subseteq K[H,C]$. The automorphism σ is an *affine automorphism* of D.

Definition 1.2. *Let X be a multiplicative set in a ring R. Then X satisfies the* right Ore condition *provided that, for each $x \in X$ and $r \in R$, there exist $y \in X$ and $s \in R$ such that $ry = xs$, that is, $rX \cap xR \neq \emptyset$. A multiplicative set satisfying the right Ore condition is called a* right Ore set. *The* left Ore condition *and* left Ore set *are defined symmetrically. An* Ore set *is a multiplicative set which is both a right and a left Ore set.*

Now, let S be a multiplicative subset of $K[H,C]\backslash\{0\}$ such that $\sigma(S) = S$. Then S is an (left and right) Ore subset of the GWA $K[H,C](\sigma,a)$ that consists of regular elements (i.e. non-zero divisors) in $K[H,C](\sigma,a)$. The defining automorphism of the GWA

$$A = S^{-1}(K[H,C](\sigma,a)) = D(\sigma,a)$$

is defined by the rule $\sigma(s^{-1}d) = \sigma(s)^{-1}\sigma(d)$ with the affine automorphism σ of $K[H,C]$; that is,

$$\sigma(H) = \alpha H + \beta C + b \text{ and } \sigma(C) = \gamma H + \delta C + c$$

for some scalars $\alpha, \beta, \ldots, c \in K$ such that $\det \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \neq 0$. As an abstract algebra, the algebra $K[H,C](\sigma,a)$ is generated by the elements H, C, X and Y subject to the defining relations:

$$\begin{aligned} XH &= (\alpha H + \beta C + b)X, \qquad YX = a, \\ YC &= (\gamma H + \delta C + c)Y, \qquad XY = \sigma(a). \end{aligned}$$

Since the field K is an algebraically closed field then up to an affine change of variables in the polynomial algebra $K[H,C]$ ($H' = \alpha' H + \beta' C + b'$, $C' = \gamma' H + \delta' C + c'$) there are only 4 types of affine automorphisms [44, Lemma 4.1] where $\lambda, \mu \in K^*$:

1. $\sigma(H) = H - 1$, $\sigma(C) = \lambda C$,
2. $\sigma(H) = \lambda H$, $\sigma(C) = \mu C$,
3. $\sigma(H) = H - 1$, $\sigma(C) = C + H$,
4. $\sigma(H) = \lambda H + C$, $\sigma(C) = \lambda C$.

According to these four types of affine automorphisms, the algebra $A = D(\sigma,a)$ divides to four classes of algebras. The aim of our research is to find explicit formulas for the global and Krull dimension in each of these classes.

1.1.2 Literature review

• The global dimension of GWA

The global dimension of some examples of GWAs were found at first, such as the primitive factor rings of the universal enveloping algebra $Usl(2)$ of the Lie algebra $sl(2)$.

Let $U = Usl(2)$ be the universal enveloping algebra of the Lie algebra

$$sl(2) = \langle X, Y, H \,|\, [H, X] = X, [H, Y] = -Y, [X, Y] = 2H\rangle$$

over the field K, and let $C = H(H + 1) + YX$ be the Casimir element [42]. So,

$$U \simeq K[H, C](\sigma, a = C - H(H + 1)),$$

where the automorphism σ of the base ring $K[H, C]$ is defined as follows [42]:

$$\sigma(H) = H - 1 \text{ and } \sigma(C) = C.$$

Then, for any $\lambda \in K$, the quotient algebra $U(\lambda) := U/U(C - \lambda)$ is a GWA of degree 1 [42],

$$U(\lambda) \simeq K[H](\sigma, a = \lambda - H(H + 1).$$

$U(\lambda)$ is the infinite dimensional primitive factor ring of $Usl(2)$.

If $K = \mathbb{C}$, then the global dimension of U is 3 [14, Theorem 8.2]. The ideal $U(C - \lambda)$ is maximal if and only if λ is not of the form $n^2 + 2n$ for $n \in \mathbb{N}$ [33]. So, the ring $U(\lambda)$ is simple unless $\lambda = n^2 + 2n$ [21]. In 1973, Ross proved that if λ is transcendental over $\mathbb{Q}$, then gld $(U(\lambda)) = 1$ [20]. In 1981, Smith proved that if $U(\lambda)$ is not simple, then either gld $(U(\lambda)) = 2$ or gld $(U(\lambda)) = \infty$ (see [33, Corollary 6.2]). Later, Smith's result motivated Stafford to study the global dimension of $U(\lambda)$ for $\lambda \in \mathbb{C}$.

In 1982, Stafford [21] wrote $R_n := U/U(C - n^2 - 2n)$ for any $n \in \mathbb{C}$ to denote the factor rings $U(\lambda)$ for $\lambda \in \mathbb{C}$ to make the notation easier. For example, the primitive nonsimple factor rings are precisely $\{R_n \,|\, n \in \mathbb{N}\}$. Furthermore, since $n^2+2n = (-n-2)^2+2(-n-2)$, then $R_n = R_{-n-2}$. Therefore, the nonsimple factor rings can be equally well written as $\{R_n \,|\, n = -2, -3, -4, \cdots\}$. The following theorem presents the global dimension of the infinite dimensional primitive factor rings R_n of U.

Theorem 1.3. *([21, Theorem B].) Let $K = \mathbb{C}$.*

1. *If* R_n *is not simple then* gld $(R_n) = 2$.
2. gld $(R_{-1}) = \infty$.
3. *If* R_n *is simple and* $n \neq -1$ *then* gld $(R_n) = 1$.

After a decade, the global dimension of a class of generalized Weyl algebras had been studied by Bavula. Let $A = D(\sigma, a)$ be a GWA of degree 1, where $D = K[H]$ is a polynomial algebra in one variable, K is an algebraically closed field of characteristic 0 and the automorphism $\sigma \in \mathrm{Aut}(D)$ acts as follows $\sigma(H) = H - 1$. The defining element a is said to have *congruent* roots if it is contained in distinct ideals belonging to the same orbit [42], meaning there is a maximal ideal $\mathfrak{m}$ of D such that $a \in \mathfrak{m}$ and $a \in \sigma^i(\mathfrak{m})$ for some $i \geq 1$ such that $\sigma^i(\mathfrak{m}) \neq \mathfrak{m}$. In 1992, Bavula found the global dimension of $A = K[H](\sigma, a)$ in the following formula [42, Theorem 5],

$$\text{gld } (A) = \begin{cases} \infty & \text{if } a \text{ has at least one multiple root,} \\ 2 & \text{if } a \text{ has no multiple roots but has congruent roots,} \\ 1 & \text{if } a \text{ has neither multiple nor congruent roots.} \end{cases}$$

In 1994, Bavula generalized the previous formula to find the global dimension of GWAs $A = D(\sigma, a)$ of degree 1, where D is a Dedekind domain (see Theorem 4.3).

The method used in the previous studies was reducing the problem of finding the global dimension of algebra A to determining the projective dimensions of simple modules of A. Because the classification of simple modules of noncommutative algebra is not easy, finding the global dimension is a very difficult problem.

In 1996, Bavula found a small class of modules that is responsible for GWAs having an infinite global dimension (see Theorem 4.4). By using Theorem 4.4 and some other results, Bavula also found the global dimension of the GWA $A = K[H, C](\sigma, a)$ of degree 1 over an algebraically closed field K, $\sigma(C) = \lambda C$, $\sigma(H) = H - 1$ when $\lambda = 1$ (see [44, Theorem 5.1]). It was Bavula's result that motivated this research, where we will generalize [44, Theorem 5.1] to any $\lambda \in K^*$. Moreover, we will find the global dimension of $A = K[H, C](\sigma, a)$ for any affine automorphisms σ, in Section 4.4.

• The Krull dimension of GWA

In 1966, the problem of finding the Krull dimension of the universal enveloping algebra $U = Usl(2)$ of the Lie algebra $sl(2)$ over $\mathbb{C}$ was first mentioned by Gabriel and Nouazé [31]. They showed that U has a chain of prime ideals of length 2, and it does not have any chains with a length greater than 2. So they concluded that the Krull dimension of U is 2, although the correct conclusion is only that $\mathcal{K}(U) \geq 2$ [31]. In 1973, Arnal-Pinczon and Ross showed that if the primitive factor ring $U(\lambda)$ over $\mathbb{C}$ is simple, then it has Krull dimension 1 [33]. After that, Smith proved that if $U(\lambda)$ is not simple, then it also has Krull dimension 1 [33, Theorem 3.2]. Finding the Krull dimension of a noncommutative Noetherian ring is usually a very difficult problem. For example, the Krull dimension of the universal enveloping algebra U was believed to be 3 for many years [36]. But in 1981, Smith proved that the correct Krull dimension of U is two [31, Theorem 3.3].

The difficulty of calculating the Krull dimension of a noncommutative left Noetherian ring arises from the fact that in order to compute it, we find the largest descending chain of left ideals with simple subfactors. As previously mentioned, dealing with simple modules in noncommutative algebra is usually extremely difficult, whereas with a commutative case, the simple modules can be described easily.

Let $A = D(\sigma, a)$ be a GWA of degree 1 where the base ring D is a commutative ring. For $a = 1$, the GWA $A = D(\sigma, 1)$ is the skew Laurent polynomials ring $D[X, X^{-1}; \sigma]$. In 1982, Hodges computed the Krull dimension of $D[X, X^{-1}; \sigma]$ where D is Noetherian ring that has a finite Krull dimension as follows [35, Theorem 1.1]:

$$\mathcal{K}(D[X, X^{-1}; \sigma]) = \sup \{\mathcal{K}(D), \operatorname{ht}(\mathfrak{q})+1 \mid \mathfrak{q} \text{ is a } \sigma\text{−semistable prime ideal of } D\}. \tag{1.1}$$

Moreover, if a is an invertible element that is not necessarily 1, then the GWA $D[X, Y; \sigma, a]$ is a skew Laurent extension $D[X, X^{-1}; \sigma]$ with $Y = aX^{-1}$. In this case, the Krull dimension of the skew Laurent extension $D[X, X^{-1}; \sigma]$ of D was computed in 1984 by Goodearl and Lenagan (see [23, Corollary 3.3]).

In 1992, Bavula proved that the Krull dimension of GWA $D(\sigma, a)$ of degree 1 with base ring of polynomials $D = K[H]$ in one variable H over an algebraically closed field K of characteristic zero and an arbitrary nonzero defining element a, the defining automorphism σ satisfies $\sigma(H) = H - 1$, is equal to 1 [42, Theorem 2.1].

In 1998, Bavula and Oystaeyen [35] extended (1.1) and found the Krull dimension of the GWA $A = D(\sigma, a)$ of degree 1 (see Theorem 5.2). Moreover, the Krull dimension of $U = Usl(2)$ over any field K was computed as follows [35, Corollary

1.4]:

$$\mathcal{K}(U) = \begin{cases} 2 & \text{if } \operatorname{char}(K) = 0, \\ 3 & \text{if } \operatorname{char}(K) = p > 0. \end{cases}$$

Additionally, as an application of Theorem 5.2, the Krull dimension of some popular algebras such as $U_q sl(2)$, the Woronowicz's deformation, the first and second Witten's deformations, the quantum $\mathcal{O}_{q^2}(\mathrm{so}(K,3))$ and some other algebras was computed in [35].

In 2001, Bavula and Lenagan developed an important formula for the Krull dimension of the skew Laurent extension $A = D[X, X^{-1}; \sigma]$ of the polynomial ring $D = K[x_1, \ldots, x_n]$ over an algebraically closed field K [36, Theorem 3.4].

In general, there is no specific method of finding the Krull dimension of GWAs, and the methods used in the previous studies are difficult to use for GWAs. For example, since we can deal with $A = D(\sigma, a)$ as an induced module $A \otimes_D D$, Goodearl and Lenagan found the Krull dimension of skew Laurent extension by considering the induced A-modules of the form $A \otimes_D M$, where M is a D-module [36]. However, the structure of induced A-modules for GWAs is complicated because of the possible existence of the "stars" and "holes" that are defined and explained in [36].

1.2 The main contribution of the thesis

Let $A = D(\sigma, a)$ be a GWA of degree 1 where $D = S^{-1}K[H, C]$ is a localization of the polynomial algebra $K[H, C]$ over an algebraically closed field K at a multiplicative set S. In general, there are three options for the algebra $D = S^{-1}K[H, C]$: a field, a Dedekind domain or to have Krull dimension 2.

If D is a field, then either $A \simeq D[X, X^{-1}; \sigma]$ or $A = D[X, Y; \sigma, 0]$, so either gld $(A) = 1$ or gld $(A) = \infty$ (Proposition 4.14). If D is a Dedekind domain, then Theorem 4.3 gives a formula for the global dimension of A.

So, the only interesting case is one where D has Krull dimension 2, which is considered in our research.

Research aims and thesis findings.

At the start of this research, we studied the centre of GWA $A = K[H, C](\sigma, a)$

where σ is an affine automorphism of the polynomial algebra $D = K[H, C]$ in the following theorem:

Theorem 1.4. *Let n be the order of the automorphism σ. Then*

1. $Z(A) = \begin{cases} D^\sigma[X^n, Y^n; \mathrm{id}, (-n, n)] & \text{if } n < \infty, \\ D^\sigma & \text{if } n = \infty. \end{cases}$

2. *The algebra $Z(A)$ is finitely generated.*

3. *The algebra $Z(A)$ is a domain if and only if either $n = \infty$ or $a \neq 0$ and $n < \infty$.*

In a description of the centre in Theorem 1.4, the ring of σ-invariants D^σ plays the key role. Thus, we had to find this ring; for example, if $\sigma(H) = H - 1$, and $\sigma(C) = \lambda C$, then

$$D^\sigma = \begin{cases} K[H_p, C^n] & \text{if } p > 0, \, \lambda \in \mathcal{P}_n, \\ K[H_p] & \text{if } p > 0, \, \lambda \notin \mathcal{M}_\infty, \\ K[C^n] & \text{if } p = 0, \, \lambda \in \mathcal{P}_n, \\ K & \text{if } p = 0, \, \lambda \notin \mathcal{M}_\infty, \end{cases}$$

where $H_n := H(H-1)\cdots(H-n+1)$ for $n \geq 1$ and p is the characteristic of K; see Section 3.3 for details.

The ring σ-invariants D^σ and the order of the automorphism σ for each type of affine automorphism are found in Proposition 3.6. These results helped us to achieve our goals.

It is known that the left global dimension of GWA $A = D(\sigma, a)$ is either infinite or equals $\mathrm{lgd}\,(D)$ or $\mathrm{lgd}\,(D) + 1$ [44, Theorem 2.7].

• The first goal of the thesis is to give an explicit criterion (Theorem 1.5) for $\mathrm{gld}\,(A) < \infty$ for some GWAs A.

Theorem 1.5. *Let K be an algebraically closed field, let the algebra $D = S^{-1}K[H, C]$ have Krull dimension 2, D^* be its group of units and $A = D[X, Y; \sigma, a]$ be a GWA. Then $\mathrm{gld}\,(A) < \infty$ if and only if either $a \in D^*$ or $a \in D \backslash \{D^*, 0\}$ and $\mathrm{grad}\,(a) := (\frac{\partial a}{\partial H}, \frac{\partial a}{\partial C}) \not\equiv 0 \mod \mathfrak{m}$ (equivalently, $a \notin \mathfrak{m}^2$) for all maximal ideals $\mathfrak{m}$ of D such that $a \in \mathfrak{m}$.*

• The second goal is finding the explicit values for the global dimension of GWA $S^{-1}K[H,C](\sigma,a)$ according to the four types of affine automorphism σ under the assumption that gld $(S^{-1}K[H,C](\sigma,a)) < \infty$.

As mentioned previously, Bavula found the global dimension of the first class of $K[H,C](\sigma,a)$ when $\lambda = 1$, and in the following theorem, we found the global dimension for arbitrary $\lambda \in K^*$.

Theorem 1.6. *Let $A = D(\sigma,a)$ be a GWA such that $D = K[H,C]$, with K an algebraically closed field, $\sigma(H) = H - 1$ and $\sigma(C) = \lambda C$ where $\lambda \in K^*$. Suppose that* gld $(A) < \infty$. *Then* gld $(A) = 2, 3$ *and*

1. gld $(A) = 3$ *if and only if either* char $(K) \neq 0$ *or* char $(K) = 0$ *and there exist elements α, $\beta \in K$ and $i \in \mathbb{N}\backslash\{0\}$ such that $a(\alpha,\beta) = 0$ and $a(\alpha + i, \lambda^{-i}\beta) = 0$.*

2. gld $(A) = 2$ *if and only if* char $(K) = 0$ *and if $a(\alpha,\beta) = 0$ for some α, $\beta \in K$ then $a(\alpha + i, \lambda^{-i}\beta) \neq 0$ for all $i \in \mathbb{N}\backslash\{0\}$.*

3. *Suppose that S is a multiplicative subset of D such that $\sigma(S) = S$ and the algebra $\mathcal{D} = S^{-1}D$ has Krull dimension 2. Let $\mathcal{A} = \mathcal{D}(\sigma,a)$ where $a \in \mathcal{D}$. Suppose that* gld $(\mathcal{A}) < \infty$. *Then* gld $(\mathcal{A}) = 2, 3$. *Furthermore,* gld $(\mathcal{A}) = 3$ *if and only if either* char$(K) \neq 0$ *and λ is a root of unity or* char$(K) \neq 0$, *λ is not a root of unity and $(H - \alpha, C) \neq \mathcal{D}$ for some $\alpha \in K$ or there is a maximal ideal $\mathfrak{m} = (H - \alpha, C - \beta)$ of $\mathcal{D}$ (where $\alpha, \beta \in K$) such that $a \in \mathfrak{m}$ and $a \in \sigma^i(\mathfrak{m}) = (H - \alpha - i, C - \lambda^{-i}\beta)$ for some $i \geq 1$.*

The other three cases are found in Theorem 4.29, Theorem 4.30 and Theorem 4.31, respectively.

There are many classical algebras that belong to the class of GWAs $A = D(\sigma,a)$, where either $D = K[H,C]$ or $D = K[H^{\pm 1},C]$, that we study. For example, the universal enveloping algebra $Usl(2)$ of the Lie algebra $sl(2)$ and its deformation $\Lambda(b)$, $U_q sl(2)$, the universal enveloping algebra $\mathcal{H} = U(\mathcal{N})$ of the 3-dimensional Heisenberg Lie algebra and many other algebras.

As an application of Theorem 1.5 and Theorem 4.2, we compute the global dimension of many classical algebras that are examples of GWAs in Theorem 1.5.

Corollary 1.7. *Let K be an algebraically closed field. Then*

1. gld $(Usl(2)) = 3$.
2. gld $(U_q sl(2)) = 3$.
3. *The global dimension of Woronowicz's deformation V is* 3.
4. *The global dimension of Witten's first deformation E is* 3.
5. *The global dimension of Witten's second deformation W is* 3.
6. *The global dimension of the algebra $\mathcal{O}_{q^2}(\mathrm{so}(K,3))$ is* 3.
7. *The global dimension of the universal enveloping algebra $\mathcal{H} = U(\mathcal{N})$ of the Heisenberg Lie algebra is* 3.
8. *The global dimension of the quantum Heisenberg algebra $\mathcal{H}_q$ is* 3.
9. *Let $\Lambda(b)$ be the deformation of $Usl(2)$ (see [30], [40], [43]) and* char $(K) = 0$. *Then* gld $(\Lambda(0)) = 3$ *and for $b \neq 0$*

$$\text{gld } (\Lambda(b)) = \begin{cases} 3 & \text{if } \alpha(\mu) = \alpha(\mu + i) \text{ for some } \mu \in K \text{ and } i \in \mathbb{N}\backslash\{0\}, \\ 2 & \text{otherwise,} \end{cases}$$

where $\alpha(H)$ is a solution of the equation $\alpha(H) - \alpha(H-1) = b$. In particular, gld $(Usl(2)) = 3$ *since $Usl(2) = \Lambda(2H)$.*

10. gld $(\mathcal{O}_q(\mathrm{SL}_2(K))) = 3$.

The algebras in Corollary 1.7 are examples of the following construction of rings that first appeared in full generality in [44] and are a particular case of the construction of a diskew polynomial ring [41, 45]: Let $\mathcal{D}$ be a ring, $\sigma \in \mathrm{Aut}(\mathcal{D})$. Suppose that elements b and ρ belong to the centre of $\mathcal{D}$; moreover, ρ is a unit of $\mathcal{D}$ and σ-stable (i.e. $\sigma(\rho) = \rho$). Then the ring $E = \mathcal{D}\langle X, Y; \sigma, b, \rho\rangle$ is generated by $\mathcal{D}$, X and Y subject to the defining relations: For all elements $d \in \mathcal{D}$,

$$Xd = \sigma(d)X, \quad Yd = \sigma^{-1}(d)Y \quad \text{and} \quad XY - \rho YX = b. \tag{1.2}$$

The ring E is a very special case of GWAs.

Lemma 1.8. *([44, Lemma 1.2].) The ring $E = \mathcal{D}\langle X, Y; \sigma, b, \rho\rangle$ is a GWA $E = \mathcal{D}[H][X, Y; \sigma, a = H]$ where $\mathcal{D}[H]$ is a polynomial ring in a variable H with coefficients in $\mathcal{D}$ and σ is an extension of the automorphism σ of $\mathcal{D}$ to $\mathcal{D}[H]$ by the rule $\sigma(H) = \rho H + b$.*

The global dimension of the ring E where the ring $\mathcal{D}$ is a localization of a polynomial algebra $K[C]$ in a variable C at a multiplicative subset S of $K[C]$ is computed in the following theorem:

Theorem 1.9. *Let $\mathcal{D} = S^{-1}K[C]$ be a localization of a polynomial algebra $K[C]$ in a variable C at a multiplicative subset S of $K[C]$ and $E = \mathcal{D}\langle X, Y; \sigma, b, \rho\rangle$. Then* gld $(E) = 1, 2$ *or* 3*. Furthermore,*

1. *if $\mathcal{D}$ is a field then* gld $(E) = 1$ *or* 2 *and the exact value of* gld (E) *is given in Theorem 4.3 ($E = \mathcal{D}[H][X, Y; \sigma, a = H]$ is a GWA where $\mathcal{D}[H]$ is a Dedekind domain, by Lemma 1.8).*

2. *If $\mathcal{D}$ is not a field and K is an algebraically closed field then* gld $(E) =$ 2 *or* 3*, and* gld $(E) = 3$ *if and only if there are natural number $n \geq 1$ and an element $\beta \in K$ such that $\mathcal{D}\xi_n + \mathcal{D}(\sigma^n(C) - \beta) \neq \mathcal{D}$ where $\xi_n := \sum_{i=0}^{n-1} \sigma^i(\rho^{n-i-1}b)$.*

• The third goal is to prove that the global dimension of tensor product of GWAs in Theorem 1.5 is equal to the sum of their global dimensions (Theorem 4.34).

• The fourth goal is to compute the Krull dimension of the four classes of algebra $S^{-1}K[H, C](\sigma, a)$ with respect to the four types of affine automorphism σ.

We found the Krull dimension of GWAs $A = K[H, C](\sigma, a)$ (i.e. $S = \{1\}$) with respect to the four types of affine automorphism σ. If σ is of type 1, $\sigma(H) = H - 1$ and $\sigma(C) = \lambda C$, the following theorem will compute the Krull dimension of $A = K[H, C](\sigma, a)$.

Theorem 1.10. *Let $A = D(\sigma, a \neq 0)$ be a GWA such that $D = K[H, C]$, K is an algebraically closed field, $\sigma(H) = H - 1$ and $\sigma(C) = \lambda C$ where $\lambda \in K^*$. Then $\mathcal{K}(A) = 2$ or 3; and $\mathcal{K}(A) = 3$ if and only if either* char $(K) \neq 0$ *or* char $(K) = 0$ *and either λ is a root of unity and $a \in (C - \beta)$ for some $\beta \in K$ or λ is not a root of unity and $a \in (C)$.*

The other three cases are found in Theorem 5.9, Theorem 5.10 and Theorem 5.11, respectively.

If S is a multiplicative subset of $K[H, C]$ that is not necessarily equal to $\{1\}$, we found the Krull dimension of $\mathcal{A} = S^{-1}K[H, C](\sigma, a)$, where σ is of type 1, in the following theorem:

Theorem 1.11. *Let $D = K[H, C]$, K is an algebraically closed field, $\sigma(H) = H - 1$ and $\sigma(C) = \lambda C$ where $\lambda \in K^*$. Suppose that S is a multiplicative subset of $D = K[H, C]$ such that $\sigma(S) = S$, $\mathcal{K}(\mathcal{D}) = 2$ where $\mathcal{D} = S^{-1}D$, and $\mathcal{A} = \mathcal{D}(\sigma, a)$ is a GWA where $0 \neq a \in \mathcal{D}$. Then $\mathcal{K}(\mathcal{A}) = 2$ or 3; and $\mathcal{K}(\mathcal{A}) = 3$ if and only if one of the following conditions holds:*

1. $\mathrm{char}(K) \neq 0$ *and λ is a root of unity,*
2. $\mathrm{char}(K) \neq 0$, *λ is not a root of unity and $(H - \alpha, C) \neq \mathcal{D}$ for some $\alpha \in K$,*
3. $\mathrm{char}(K) \neq 0$, *λ is not a root of unity, $(H - \alpha, C) = \mathcal{D}$ for all $\alpha \in K$, $a \in (H - \alpha)$ and the prime ideal $(H - \alpha)$ is not a maximal ideal of $\mathcal{D}$,*
4. $\mathrm{char}(K) = 0$, *λ is a root of unity, $a \in (C - \beta)$ for some $\beta \in K$ such that the element $C - \beta$ is not a unit of $\mathcal{D}$ and the prime ideal $(C - \beta)$ of $\mathcal{D}$ is not a maximal ideal of $\mathcal{D}$,*
5. $\mathrm{char}(K) = 0$, *λ is not a root of unity, $a \in (C)$, C is not a unit of $\mathcal{D}$ and the prime ideal (C) of $\mathcal{D}$ is not a maximal ideal of $\mathcal{D}$.*

Additionally, we found the Krull dimension of $\mathcal{A}$, where σ is of type 3, in Theorem 5.12.

1.3 Thesis structure

The structure of this thesis is as follows.

In Chapter 2, we give an overview of some of the fundamental terminologies of the homological algebra, Krull dimension and the dynamical Mordell-Lang conjecture. We start Section 2.1 by giving the basic definitions of homological algebra and define the global dimension with some of its properties. Then, in Section 2.2, we give the definition of the Krull dimension of noncommutative algebra with some of its basic properties. The last section of Chapter 2 provides an overview of the dynamical Mordell-Lang conjecture with some of the proven cases of the conjecture that will be used in our research to find the Krull dimension.

Chapter 2 provides background material that will be used later and does not contain any of our results. The main work of the thesis can be found in Section 3.3 and Chapters 4 and 5. The results in Chapter 4 are joint work with Prof Vladimir Bavula and the chapter has been accepted for publication in 'Contemporary Mathematics (AMS)'. Also, Chapter 5 is a part of joint paper with Prof

Bavula.

In Chapter 3, we first define the generalized Weyl algebra of degree n in Section 3.1. In Section 3.2, we state the main properties of GWAs. Finally, in Section 3.3, we describe the ring of σ-invariants D^σ of the ring $D = K[H, C]$ for the four types of affine automorphism σ in Proposition 3.6, and we prove Theorem 1.4.

Chapter 4 contains the main work of finding the global dimension of generalized Weyl algebras $A = S^{-1}K[H, C](\sigma, a)$. We start in Section 4.1 by stating the known results about the global dimension of GWAs. The aim of Section 4.2 is to prove Corollary 1.7. After that, we prove our main result Theorem 1.5, and Theorem 1.9, in Section 4.3. In Section 4.4, we find the explicit values of the global dimension of $A = S^{-1}K[H, C](\sigma, a)$ under the assumption that gld $(A) < \infty$. In Section 4.5, we discuss finding the global dimension of the tensor products of GWAs. In the final section of the chapter, we compute the global dimension of some special cases of GWAs where the field K is not assumed to be algebraically closed.

In Chapter 5, we start by introducing the known results about the Krull dimension of GWA, in Section 5.1. Then, in Section 5.2, we describe maximal ideals of the localization ring $S^{-1}K[H, C]$ of the polynomial algebra $K[H, C]$ over an algebraically closed field K at a multiplicative set $S \subseteq K[H, C]$. In Section 5.3, we present and prove our main results about the Krull dimension of the GWA $S^{-1}K[H, C](\sigma, a)$. After that, we finish the chapter by stating an important result about the Krull dimension of the tensor product of GWAs, in Section 5.4.

Chapter 2

Preliminaries

In this chapter, we provide fundamental definitions and some known results that are useful in our research. The main reference for this chapter is [19].

2.1 Homological algebra

Homological algebra arose as a branch of topology in the late 1800s, but in the 20th century, it became an independent branch of mathematics. Its influence progressively expanded, and presently, it plays an important role in several branches of mathematics, including commutative algebra, algebraic geometry, algebraic number theory and mathematical physics.

Homological algebra concerns the study of homological functors and the algebraic structures that they involve. Chain complex is one of the useful concepts in mathematics, and homological algebra provides the method to extract information from these chain complexes and present it as homological invariants of rings, modules, topological spaces and other mathematical objects. The global dimension (or global homological dimension) is an important homological invariant in the dimension theory of Noetherian rings.

Before we can define the global dimension and the weak global dimension, we need to introduce some basic concepts.

2.1.1 Chain complexes and exact sequences

Let R be a ring. A *chain complex* is a collection of $\{C_i\}_{i\in\mathbb{Z}}$ of R-modules and connecting R-module homomorphisms $\{d_i : C_i \to C_{i-1}\}$ called *differentials* such that $d_{i-1} \circ d_i = 0$. The chain complex is written as follows

$$\cdots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \cdots .$$

The *cochain complex* is the dual notion to a chain complex. It consists of a collection of $\{C^i\}_{i\in\mathbb{Z}}$ of R-modules and connecting R-module homomorphisms $\{d^i : C^i \to C^{i+1}\}$ such that $d^{i+1} \circ d^i = 0$. The cochain complex is written as follows

$$\cdots \leftarrow C^{i+1} \xleftarrow{d^i} C^i \xleftarrow{d^{i-1}} C^{i-1} \leftarrow \cdots .$$

The index i is referred to as the *degree*. The only difference between a chain complex and cochain complex is that, in chain complex the connecting homomorphisms decrease degree, whereas in cochain complex they increase degree.

The complexes may be finite or infinite, if $C_i = 0$ for all sufficiently large (small) values of i then the complex is said to be *bounded above* (*bounded below*). A complex is said to be *bounded* if it is bounded both above and below.

Definition 2.1. *Given a chain complex*

$$\cdots \to C_{i+1} \xrightarrow{d_{i+1}} C_i \xrightarrow{d_i} C_{i-1} \to \cdots ,$$

the ith homology group H_i *is defined to be equal to* $\ker d_i/\operatorname{im} d_{i+1}$. *Similarly, we define the ith* cohomology group *for a cochain complex as follows*

$$H^i = \ker d^i/\operatorname{im} d^{i-1}.$$

An *exact sequence* is a chain complex whose homology groups are all zero. This means $\ker d_i = \operatorname{im} d_{i+1}$ for all i. A *long exact sequence* is an exact sequence with infinitely many R-modules and connecting homomorphisms. A sequence of R-modules of the form

$$0 \to L \xrightarrow{\alpha} M \xrightarrow{\varphi} N \to 0,$$

such that the map α is injective, the map φ is surjective and $\operatorname{Im}\alpha = \ker\varphi$, is called a *short exact sequence*.

2.1.2 Projective, injective and flat dimensions

Before we define the three dimensions we must give the definition of projective, injective and flat modules first and their resolutions.

Definition 2.2. *Let R be a ring.*

- *An R-module P is* projective *if, for each surjective R-module homomorphism $\alpha : M \to N$ and for each $\varphi : P \to N$, there exists $\psi : P \to M$ such that $\alpha\psi = \varphi$.*

Thus a module P is projective if there always exists ψ such that the following diagram commutes.

$$\begin{tikzcd} & P \arrow[dl, dashed, "\exists\psi"'] \arrow[d, "\varphi"] \\ M \arrow[r, two heads, "\alpha"] & N \end{tikzcd}$$

- *An R-module I is* injective *if, for each injective R-module homomorphism $\alpha : N \to M$ and for each $\varphi : N \to I$, there exists $\psi : M \to I$ such that $\psi\alpha = \varphi$.*

Thus a module I is injective if there always exists ψ such that the following diagram commutes.

$$\begin{tikzcd} 0 \arrow[r] & N \arrow[r, "\alpha"] \arrow[d, "\varphi"] & M \arrow[dl, dashed, "\exists\psi"] \\ & I & \end{tikzcd}$$

- *An R-module F is* flat *if, for every injective R-module homomorphism $\alpha : N \to M$, the map $\varphi_F(\alpha) : F \otimes_R N \to F \otimes_R M$ is injective.*

The following lemma is the generalization of Schanuel's Lemma that allows us to compare how far a module from being a projective module.

Lemma 2.3. (The long version of Schanuel's Lemma [19, p. 246].) *Let R be a ring. If*

$$0 \to K \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0,$$
$$0 \to K' \to P'_{n-1} \to \cdots \to P'_1 \to P'_0 \to M' \to 0$$

are exact sequences of R-modules in which each P_i and P_i' is projective and $M \simeq M'$, then

$$K \oplus P_{n-1}' \oplus P_{n-2} \oplus P_{n-3}' \oplus \cdots \simeq K' \oplus P_{n-1} \oplus P_{n-2}' \oplus P_{n-3} \oplus \cdots ,$$

where the last terms are P_0 and P_0', as appropriate. (If $n = 1$, this is Schanuel's Lemma.*)*

Definition 2.4. *Let M be an R-module.*

• *A* projective resolution *of M is an exact sequence of the form*

$$\cdots \to P_n \to P_{n-1} \to \cdots \to P_1 \to P_0 \to M \to 0$$

in which all P_i are projective R-modules.

• *An* injective resolution *of M is an exact sequence of the form*

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to I_{n+1} \to \cdots$$

in which all I_i are injective R-modules.

• *A* flat resolution *of M is an exact sequence of the form*

$$\cdots \to F_n \to F_{n-1} \to \cdots \to F_1 \to F_0 \to M \to 0$$

in which all F_i are flat R-modules.

Definition 2.5. *Let ${}_RM$ be a left R-module.*

• The projective dimension *of ${}_RM$, written as $\mathrm{pd}\,{}_RM$, is the shortest length n (if it exists) of a projective resolution*

$$0 \to P_n \to P_{n-1} \to \cdots \to P_0 \to M \to 0.$$

• The injective dimension *of ${}_RM$, written as $\mathrm{id}\,{}_RM$, is the shortest length n (if it exists) of an injective resolution*

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to 0.$$

• The flat dimension *of ${}_RM$, written as $\mathrm{fd}\,{}_RM$, is the shortest length n (if it exists) of a flat resolution*

$$0 \to F_n \to F_{n-1} \to \cdots \to F_0 \to M \to 0.$$

If no finite resolution exists, we put $\mathrm{pd}\,{}_RM$, $\mathrm{id}\,{}_RM$ and $\mathrm{fd}\,{}_RM$ equal to ∞.

A partially ordered set (poset, for short) P is said to satisfy the *ascending chain condition* (ACC) if for every ascending chain

$$a_1 \leq a_2 \leq a_3 \leq \cdots ,$$

of elements of P there exists $n \in \mathbb{N}$ such that $a_n = a_{n+1} = a_{n+2} = \cdots$. Similarly, P is said to satisfy the *descending chain condition* (DCC) if there is no infinite descending chain of elements of P, that is, every descending chain

$$a_1 \geq a_2 \geq a_3 \geq \cdots$$

stabilizes. The set of ideals of a ring R is a poset under inclusion and R is said to be a left (respectively, right) *Noetherian ring* if R satisfies the ascending chain condition on left (respectively, right) ideals. R is said to be *Noetherian ring* if it is both left and right Noetherian. A left (respectively, right) *Artinian ring* is a ring satisfies the descending chain condition on left (respectively, right) ideals. R is an *Artinian ring* if it is both left and right Artinian.

Remarks 2.6. *([19, p. 247].) 1. Since a projective module is flat, then clearly,*

$$\mathrm{fd}\,_R M \leq \mathrm{pd}\,_R M.$$

Furthermore, if $_RM$ is finitely generated and R is left Noetherian, then $\mathrm{fd}\,_R M = \mathrm{pd}\,_R M$.

2. In a short exact sequence of left R-modules

$$0 \to A \to B \to C \to 0,$$

if two modules have finite projective, injective or flat dimensions, so does the third. In fact, $\mathrm{pd}\, B = \sup\,\{\mathrm{pd}\, A, \mathrm{pd}\, C\}$ *unless* $\mathrm{pd}\, B < \mathrm{pd}\, C = 1 + \mathrm{pd}\, A$. *The same relationship holds for flat dimension and for injective dimension, except that the equality occurs unless* $\mathrm{id}\, B < \mathrm{id}\, A = 1 + \mathrm{id}\, C$. *Note that if*

$$0 \to A_n \to A_{n-1} \to \cdots \to A_0 \to M \to 0$$

and $\mathrm{pd}\, A_i \leq k$ *for all i, then* $\mathrm{pd}\, M \leq n + k$. *The same applies for* $\mathrm{fd}\, M$ *and, similarly (in reverse), for* $\mathrm{id}\, M$.

3. Given a family of left R-modules $\{M_i \,|\, i \in I\}$, then

- $\mathrm{pd}\,(\oplus M_i) = \sup\,\{\mathrm{pd}\, M_i\}$,

- $\mathrm{fd}\,(\oplus M_i) = \sup\,\{\mathrm{fd}\,M_i\}$, *and*

- $\mathrm{id}\,(\prod M_i) = \sup\,\{\mathrm{id}\,M_i\}$.

In the special case of a left Noetherian ring R, it is also true that $\mathrm{fd}\,(\prod M_i) = \sup\,\{\mathrm{fd}\,M_i\}$, *and for a right Noetherian ring,* $\mathrm{id}\,(\oplus M_i) = \sup\,\{\mathrm{id}\,M_i\}$. *Indeed, the property that a direct sum of injective modules is injective characterizes right Noetherian rings.*

2.1.3 The global and the weak global dimensions

The global dimension of a ring R is defined by the projective or injective dimensions of all R-modules. If R is a noncommutative ring, then two versions of the definition will appear. The right global dimension that originates from the right R-modules, and the left global dimension from the left R-modules. We will define the left global dimension, and the right global dimension is similarly defined, but by the right R-modules instead of the left R-modules.

Definition 2.7. *[19] Let R be a ring. The following numbers are all equal:*

1. $\sup\,\{\mathrm{pd}\,M \mid M$ *any left R-module*$\}$,

2. $\sup\,\{\mathrm{pd}\,M \mid M$ *any cyclic left R-module*$\}$,

3. $\sup\,\{\mathrm{id}\,M \mid M$ *any left R-module*$\}$, *and*

4. $\sup\,\{\mathrm{id}\,M \mid M$ *any cyclic left R-module*$\}$.

The common number (possibly ∞) is called the left global dimension *of R, written as* $\mathrm{lgd}\,(R)$. *The* right global dimension *of R that is defined by right R-modules is denoted by* $\mathrm{rgd}(R)$.

A left R-module M is called a *semisimple* module if every submodule of M is a direct summand, that is for every submodule N of M, there is a complement P such that $M = N \oplus P$. A ring R is left *semisimple* if it is semisimple as a left module over itself. The left global dimension of any left semisimple ring is 0 [3]. However, $\mathrm{lgd}\ (R) = \sup\,\{\mathrm{pd}\,I \mid {}_R I \subseteq R\} + 1$ unless R is semisimple [19].

Definition 2.8. *[19] Let R be a ring. The following numbers are all equal:*

1. $\sup\{\mathrm{fd}\, M \mid M$ *any left* R*-module*$\}$, *and*

2. $\sup\{\mathrm{fd}\, M \mid M$ *any cyclic left* R*-module*$\}$.

The common number (possibly ∞*) is called* the weak global dimension *of* R*, written as* $\mathrm{wd}\,(R)$.

$\mathrm{wd}\,(R) \leq \mathrm{lgd}\;(R)$ with equality if R is left Noetherian [3]–[19]. The weak global dimension is symmetric, so there is no distinction between "left" and "right" dimension [26]. Therefore, also, $\mathrm{wd}\,(R) \leq \mathrm{rgd}\,(R)$ with equality if R is right Noetherian.

2.1.4 Ext and Tor groups

Ext and Tor groups are some of the fundamental objects in homological algebra. They give equivalent conditions for $\mathrm{lgd}\;(R) < n$ and $\mathrm{wd}\,(R) < n$, that is $\mathrm{lgd}\;(R) < n$ if and only if $\mathrm{Ext}_R^n = 0$ and $\mathrm{wd}\,(R) < n$ if and only if $\mathrm{Tor}_n^R = 0$ [26].

Definition 2.9. *Let* M *and* N *be* R*-modules and*

$$\cdots \to P_{n+1} \to P_n \to \cdots \to P_1 \to P_0 \to M \to 0$$

be a projective resolution of M*.*

- *Define the sequence*

$$0 \to \mathrm{Hom}_R(P_0, N) \xrightarrow{\alpha_1} \mathrm{Hom}_R(P_1, N) \xrightarrow{\alpha_2} \cdots \xrightarrow{\alpha_n} \mathrm{Hom}_R(P_n, N) \xrightarrow{\alpha_{n+1}} \mathrm{Hom}_R(P_{n+1}, N) \xrightarrow{\alpha_{n+2}} \cdots,$$

then the group $\mathrm{Ext}_R^i(M, N)$ *is defined to be equal to* $\ker \alpha_i / \mathrm{Im}\, \alpha_{i-1}$*, i.e. equal to the* i*th cohomology group* H^i*.*

- *Define the sequence*

$$\cdots \xrightarrow{\alpha_{n+2}} P_{n+1} \otimes N \xrightarrow{\alpha_{n+1}} P_n \otimes N \xrightarrow{\alpha_n} \cdots \xrightarrow{\alpha_2} P_1 \otimes N \xrightarrow{\alpha_1} P_0 \otimes N \to 0,$$

then the group $\mathrm{Tor}_i^R(M, N)$ *is defined to be equal to* $\ker \alpha_i / \mathrm{Im}\, \alpha_{i+1}$*, i.e. equal to the* i*th homology group* H_i*.*

Some facts about Tor **groups**: ([19, p. 248].)

1. If N is a right R-module, then $\mathrm{fd}\, N_R \leq n$ if and only if $\mathrm{Tor}_{n+1}^R(N, M) = 0$ for all left R-modules M.

2. If $0 \to A \to B \to C \to 0$ is an exact sequence of left R-modules and N is a right R-module, then there is an exact sequence of abelian groups

$$\begin{aligned}\cdots \to \mathrm{Tor}_2^R(N, C) \to \mathrm{Tor}_1^R(N, A) \to \mathrm{Tor}_1^R(N, B) \to \mathrm{Tor}_1^R(N, C)& \\ \to N \otimes A \to N \otimes B \to N \otimes C \to 0.&\end{aligned}$$

In particular, if ${}_RB$ is projective, then $\mathrm{Tor}_i^R(N, A) \simeq \mathrm{Tor}_{i+1}^R(N, C)$ for all $i > 0$.

3. If $A_R, {}_RB_S, {}_SC$ are given with ${}_RB$ and B_S flat, then $\mathrm{Tor}_n^S(A \otimes_R B, C) \simeq \mathrm{Tor}_n^R(A, B \otimes_S C)$.

Some facts about Ext **groups**: ([19, p. 248].)

1. $\mathrm{pd}\, {}_RM \leq n$ if and only if $\mathrm{Ext}_R^{n+1}(M, N) = 0$ for all ${}_RN$.

2. $\mathrm{Ext}_R^1(M, N) = 0$ if and only if every short exact sequence of R-modules

$$0 \to N \to X \to M \to 0$$

is split; that is the R-module X is isomorphic to a direct sum $N \oplus M$ of the modules N and M.

3. If $0 \to A \to B \to C \to 0$ is exact, then the sequence

$$\begin{aligned}0 \to \mathrm{Hom}_R(C, N) \to \mathrm{Hom}_R(B, N) \to \mathrm{Hom}_R(A, N) \to \mathrm{Ext}_R^1(C, N) \to \mathrm{Ext}_R^1(B, N)& \\ \to \mathrm{Ext}_R^1(A, N) \to \cdots \to \mathrm{Ext}_R^n(C, N) \to \mathrm{Ext}_R^n(B, N) \to \mathrm{Ext}_R^n(A, N) \to \cdots&\end{aligned}$$

is exact for any ${}_RN$.

2.1.5 The global dimension of Noetherian rings

Generally, if R is a noncommutative ring, $\mathrm{lgd}\ (R)$ is not equal to $\mathrm{rgd}\,(R)$; however, if R is both left and right Noetherian, then $\mathrm{lgd}\ (R) = \mathrm{rgd}\,(R) = \mathrm{gld}\ (R)$ for short [19]. Also, $\mathrm{gld}\ (R) = \mathrm{wd}\,(R)$ [26].

Definition 2.10. *Let R be a commutative ring and $P_0 \subset P_1 \subset \cdots \subset P_n$ be a strictly ascending chain of prime ideals of R. The number n is called the* length *of the chain. The* (classical) Krull dimension *of the ring R, denoted by* $\mathrm{Kdim}\,(R)$, *is the maximum of lengths of ascending chains of prime ideals of R. The* height *of a proper prime ideal P of R, denoted by* $\mathrm{ht}\,(P)$, *is the maximum of lengths of ascending chains of prime ideals contained in P.*

Example 2.11. *Let $P_n = K[x_1, \cdots, x_n]$ be a polynomial algebra in n variables over a field K. Then* $\mathrm{Kdim}\,(P_n) = n$.

Definition 2.12. *A ring R is called a* local *ring if there is only one maximal ideal in R. Let R be a commutative ring, it is called a* regular local *ring if it is a Noetherian local ring such that* $\mathrm{Kdim}\,(R)$ *is equal to the minimal number of generators of its maximal ideal.*

Within the class of commutative Noetherian rings, Serre's Theorem shows that the global dimension can be used as a tool to distinguish the local rings.

Theorem 2.13. *(Serre) ([15, Theorem 19.2].) Let R be a commutative Noetherian local ring. Then*

$$R \text{ is regular} \iff \mathrm{gld}\ (R) = \mathrm{Kdim}\,(R) \iff \mathrm{gld}\ (R) < \infty.$$

Definition 2.14. *Let R be a ring and M be an R-module. Let N and L be submodules of M such that $N \subseteq L$. The factor module L/N is called a* subfactor *of M.*

Proposition 2.15. *([19, Proposition 7.1.13].) Let R be a Noetherian ring of global dimension $n < \infty$, and let $A_R, {}_RB$ be such that* $\mathrm{Tor}_n^R(A, B) \neq 0$. *Then* $\mathrm{Tor}_n^R(C, B) \neq 0$ *for some simple subfactor module C of A.*

The following corollary gives a characterisation of the global dimension of Noetherian ring that depends on its simple modules.

Corollary 2.16. *([19, Corollary 7.1.14].) If R is a Noetherian ring with* $\mathrm{gld}\ (R) < \infty$, *then* $\mathrm{gld}\ (R) = \sup\ \{\mathrm{pd}\,{}_RC \mid {}_RC \text{ simple}\}$.

2.1.6 The global dimension of the localization

Let R be any ring. A *multiplicatively closed set* (or *multiplicative set*) is a subset S of R such that $1 \in S$ and S is closed under multiplication.

Definition 2.17. *Let R be a commutative ring, P be a prime ideal of R and M be an R-module. Then the set $R\backslash P$ is a multiplicatively closed set. The* localization

$$R_P := (R\backslash P)^{-1}R = \{s^{-1}r \mid s \in R\backslash P, r \in R\}$$

of the ring R at $R\backslash P$ is a ring where addition and multiplication are given as follows: $s_1^{-1}r_1+s_2^{-1}r_2 = (s_1s_2)^{-1}(s_2r_1+s_1r_2)$ and $s_1^{-1}r_1 \cdot s_2^{-1}r_2 = (s_1s_2)^{-1}r_1r_2$. Two elements of the ring R_P are equal, $s_1^{-1}r_1 = s_2^{-1}r_2$ if and only if $s(s_2r_1 - s_1r_2) = 0$ for some element $s \in R\backslash P$. The localization of the R-module M at $R\backslash P$,

$$M_P := (R\backslash P)^{-1}M = \{s^{-1}m \mid s \in R\backslash P, m \in M\}$$

is an R_P-module where $s^{-1}r \cdot t^{-1}m = (st)^{-1}rm$.

Definition 2.18. *Let R be a ring and X be a non-empty subset of R. The set* $\mathrm{l.ann}_R(X) := \{r \in R \mid rX = 0\}$ *is called the* left annihilator *of X in R. Similarly, the set* $\mathrm{r.ann}_R(X) := \{r \in R \mid Xr = 0\}$ *is called the* right annihilator *of X in R. Clearly,* $\mathrm{l.ann}_R(X)$ *is a left ideal of R, and* $\mathrm{r.ann}_R(X)$ *is a right ideal of R. Let ${}_RM$ be an R-module and N be a non-empty subset of M, then* $\mathrm{ann}_R(N) := \{r \in R \mid rN = 0\}$ *is the* annihilator *of N in R; it is a left ideal of R. Clearly,* $\mathrm{ann}_R(N) = \bigcap_{n\in N} \mathrm{ann}_R(n)$. *If, in addition, the set N is a submodule of M, then* $\mathrm{ann}_R(N)$ *is an ideal of R. In particular,* $\mathrm{ann}_R(M)$ *is an ideal of R. It is the largest ideal I of R such that $IM = 0$.*

Definition 2.19. *Let R be a commutative ring. A proper ideal I of R is* semiprime *(or,* radical*) if I satisfies either of the following equivalent conditions:*

- *If $r^n \in I$ for some positive integer n and $r \in R$, then $r \in I$.*
- *If $r \in R$ but not in I, then all positive integer powers of r are not in I.*

The ring R is called semiprime *if the zero ideal is a semiprime ideal.*

The remarks below study the relationship between gld and Kdim of commutative Noetherian rings and their localizations.

Remark 2.20. *([19, p. 249].) If R is a commutative Noetherian ring of finite global dimension d, then*

1. *R is semiprime, and indeed, R is a finite direct sum of integral domains, and* gld $(R) = \text{Kdim}\,(R)$*;*

2. *if P is a prime ideal, then* gld $(R_P) \leq d$ *and* gld $(R_P) = \text{ht}\,(P) = \text{Kdim}\,(R_P)$*;*

3. *if M is a finitely generated R-module, then* $\text{pd}\,_R M = \sup\,\{\text{pd}\,_{R_P}(M_P)\}$*, over the maximal ideals P of R; and*

4. *if R is local with maximal ideal P, then*

   (a) $\text{pd}\,_R M = d$ *if and only if* $\text{ann}_R(m) = P$ *for some $m \in M$, and*

   (b) $\text{pd}\,_R M = 0$ *if and only if $_R M$ is free.*

2.1.7 Change of rings

In this section, we introduce the results regarding the global dimension, the weak global dimension of rings and the projective dimension of modules when we change from one ring to another. For instance, for two rings R, S with $R \subseteq S$, and for a ring S and its quotient ring S/I for some ideal I.

A module M_R is *faithfully flat* if it is flat and also for any $_R N$, $M \otimes_R N = 0$ implies $N = 0$ [19]. The following theorems show the relationship between the global dimensions of two rings R, S with $R \subseteq S$ under certain conditions; S is faithfully flat R-module (as in Theorem 2.21) or R is an R-bimodule direct summand of S (as in Theorem 2.22).

Theorem 2.21. *([19, Theorem 7.2.6].) Let R, S be rings with $R \subseteq S$, S_R faithfully flat and* lgd $(R) < \infty$*. If either*

1. *$_R S$ is projective, or*

2. *R is left Noetherian and $_R S$ is flat,*

then lgd $(R) \leq$ lgd (S).

Theorem 2.22. *([19, Theorem 7.2.8].) Let R, S be rings with $R \subseteq S$ such that R is an R-bimodule direct summand of S. Then*

1. lgd $(R) \leq$ lgd $(S) + \text{pd}\,_R S$*, and*

2. $\text{wd}\,(R) \leq \text{wd}\,(S) + \text{fd}\,_R S$.

Definition 2.23. *Let R be a ring. An element $r \in R$ is called a* left regular element *of R if the map $\cdot r : R \to R$, $x \mapsto xr$ is injective. The set of all left regular elements of R is denoted by ${}'\mathcal{C}_R$. Similarly, an element $r \in R$ is called a* right regular element *of R if the map $r\cdot : R \to R$, $x \mapsto rx$ is injective. The set of all right regular elements of R is denoted by $\mathcal{C}'_R$. The intersection $\mathcal{C}_R := {}'\mathcal{C}_R \cap \mathcal{C}'_R$ is called the set of regular elements of R, and any element of $\mathcal{C}_R$ is called a* regular element *of R (or a* non-zero divisor*). The sets $\mathcal{C}_R, {}'\mathcal{C}_R$ and $\mathcal{C}'_R$ are multiplicatively closed subsets of R. An element x of R is called a* normal *element of R if $xR = Rx$. So, for a normal element x of R, the set $xR = Rx$ is an ideal of R; it is the ideal of R generated by the element x, $(x) := RxR = Rx = xR$. Let M be an R-module. An element $r \in R$ is called* regular *on M if the map $r\cdot : M \to M$, $m \mapsto rm$ is injective.*

Now, we turn to another type of change of rings in the following results, specifically from ring S to the quotient ring S/I where $I = Sx$ with certain conditions on x.

Theorem 2.24. *([19, Theorem 7.3.5].) Let S be a ring and $I = Sx$ with x a regular normal nonunit.*

1. *If ${}_{S/I}M$ is nonzero and $\operatorname{pd}_{S/I}M = n < \infty$, then $\operatorname{pd}_S M = n + 1$.*
2. *If $\operatorname{lgd}(S/I) < \infty$, then $\operatorname{lgd}(S) \geq \operatorname{lgd}(S/I) + 1$.*

Definition 2.25. *For a ring R, the intersection of all its maximal left ideals is called the* Jacobson radical *of the ring R and is denoted by $\mathcal{J}(R)$. It is an ideal of R that is also equal to the intersection of all maximal right ideals of R. So,* Jacobson radical *ideal is a two-sided ideal of R.*

Proposition 2.26. *([19, Proposition 7.3.6].)*

1. *Let S be a ring and $I = Sx$ with x a regular normal nonunit, and suppose that x is regular on a module ${}_SM$. Then $\operatorname{pd}_{S/I}(M/IM) \leq \operatorname{pd}_S M$.*
2. *If, in addition, S is left Noetherian, ${}_SM$ is finitely generated, and $x \in \mathcal{J}(S)$, then $\operatorname{pd}_{S/I}(M/IM) = \operatorname{pd}_S M$.*

Theorem 2.27. *([19, Theorem 7.3.7].) Let S be a left Noetherian ring and x a regular normal element belonging to $\mathcal{J}(S)$. If $\operatorname{lgd}(S/I) < \infty$, then $\operatorname{lgd}(S) = \operatorname{lgd}(S/I) + 1$.*

Let R be a ring and σ an endomorphism of R. A *σ-derivation* on R is any additive map $\delta : R \to R$ such that

$$\delta(rs) = \sigma(r)\delta(s) + \delta(r)s \text{ for all } r, s \in R.$$

Definition 2.28. *([19, p. 15].) Let R be a ring and σ an automorphism of R.*

- *The* skew polynomial *ring $S = R[x, \sigma]$ of R is a ring generated by the ring R and a variable x that satisfy the defining relation*

$$rx = x\sigma(r) \text{ for all } r \in R.$$

- *Let δ be a σ-derivation on R. The* skew polynomial *ring $S = R[x; \sigma, \delta]$ of R is a ring generated by R and a variable x that satisfy the defining relation*

$$rx = x\sigma(r) + \delta(r) \text{ for all } r \in R.$$

- *The* skew Laurent polynomial *ring $S = R[x, x^{-1}; \sigma]$ is a ring of polynomials over R in x and x^{-1} subject to*

$$rx = x\sigma(r).$$

- *The* skew power series *ring, denoted by $R[[x; \sigma]]$, is the ring of power series $\sum_{i=0}^{\infty} x^i r_i$ subject only to the relation*

$$rx = x\sigma(r).$$

The following theorem is a very important result that we have used in our work. It gives the global dimension of the rings in the above definition according to the global dimension of the ring R.

Theorem 2.29. *([19, Theorem 7.5.3].) Let R be any ring, and let σ be an automorphism. Then*

1. $\operatorname{lgd}(R) \leq \operatorname{lgd}(R[x; \sigma, \delta]) \leq \operatorname{lgd}(R) + 1$ *if* $\operatorname{lgd}(R) < \infty$*;*
2. $\operatorname{lgd}(R) \leq \operatorname{lgd}(R[x, x^{-1}; \sigma]) \leq \operatorname{lgd}(R) + 1$*;*
3. $\operatorname{lgd}(R[x, \sigma]) = \operatorname{lgd}(R) + 1$*;*
4. $\operatorname{lgd}(R[x, x^{-1}]) = \operatorname{lgd}(R) + 1$*;*
5. $\operatorname{lgd}(R[[x; \sigma]]) = \operatorname{lgd}(R) + 1$ *if R is left Noetherian; and*

6. *if R is semisimple Artinian, then* $\operatorname{lgd}(R[x;\sigma,\delta]) = \operatorname{lgd}(R[x,x^{-1};\sigma]) = 1$.

The next corollary is a consequence of Theorem 2.29, since the global dimension (also, the weak global dimension) of any field K is zero [3].

Corollary 2.30. *([3, Corollary 4.3.8].) If K is a field, then the global dimension of the polynomial ring $K[x_1,\dots,x_n]$ is n.*

2.2 The Krull dimension

For a commutative ring R, the definition of the *(classical) Krull dimension* was given in the previous section: It is the supremum of the lengths of all chains of prime ideals of R. In general, even for Noetherian rings, the classical Krull dimension is not always finite. The following famous example by Nagata gives the case of a Noetherian ring with an infinite classical Krull dimension.

Example 2.31. *([27, Example A1].) Let $R = K[x_1,\dots,x_n,\dots]$ be the polynomial ring over a field K in countably infinitely many variables. Let $m_1, m_2, \dots$ be a sequence of natural numbers such that*

$$0 < m_i - m_{i-1} < m_{i+1} - m_i \ \textit{for all}\ i.$$

Let $\mathfrak{p}_i$ be the prime ideal of R generated by all the x_j such that $m_i \le j < m_{i+1}$, and let S be the intersection of complements of $\mathfrak{p}_i$ in R. Each $\mathfrak{p}_i$ is a prime ideal, and thus S is multiplicatively closed. Each $S^{-1}\mathfrak{p}_i$ has height equal to $m_{i+1} - m_i$, hence $\operatorname{Kdim}(S^{-1}R) = \infty$.

When the ring R is noncommutative, the classical Krull dimension extends naturally to it, but it does not give a good estimate because, for example, for a Noetherian ring, the Krull dimension is used to measure how close the ring is to being Artinian, as given in the following theorem.

Theorem 2.32. *([25, Theorem 8.5].) A ring R is Artinian $\Longleftrightarrow$ R is Noetherian and* $\operatorname{Kdim}(R) = 0$.

In the noncommutative case, there are many simple Noetherian rings that are far from being Artinian [19] (more details about the classical Krull dimension in 2.2.3). So, a new definition of the Krull dimension was introduced with advantages over the classical definition. The new definition is determined by a measure called *deviation* that will be presented in the next section.

2.2.1 Deviation of a poset

Definition 2.33. *([19, p. 186].) Given a and b belong to a partially ordered set (poset) A, and $a \geq b$, then we define $a/b = \{x \in A \,|\, a \geq x \geq b\}$. This is a subposet of A called the* factor *of a by b.*

We now define the *deviation* of a poset A, $\mathrm{dev}(A)$ for short. If A is trivial (i.e. the elements of A are incomparable), then $\mathrm{dev}(A) = -\infty$. If A is nontrivial but satisfies the descending chain condition (DCC), then $\mathrm{dev}(A) = 0$.

Definition 2.34. *([19, Definition 6.1.2].) For a general ordinal α, we define* $\mathrm{dev}(A) = \alpha$, *provided*

1. $\mathrm{dev}(A) \neq \beta < \alpha$; *and*

2. *in any descending chain of elements of A, all but finitely many factors have deviation less than α.*

Example 2.35. *([19, Example 6.1.3].) Consider the linearly ordered set $A = \{a_i \,|\, i \in \mathbb{N}\}$ in which $a_i > a_j$ if and only if $i < j$. The chain $a_1 > a_2 > \cdots > a_n > \cdots$ shows that* $\mathrm{dev}(A) \neq 0$, *but any factor a_i/a_j where $i < j$ has deviation* 0. *Hence,* $\mathrm{dev}(A) = 1$.

The next proposition gives an example for posets having a deviation.

Proposition 2.36. *([19, Proposition 6.1.8].) Any poset with an ascending chain condition (ACC) has a finite deviation.*

Properties of the deviation: ([19, pp. 186-190].)

1. If B is a subposet of A, then $\mathrm{dev}(B) \leq \mathrm{dev}(A)$.

2. If A is the disjoint union of posets A_i, with only the partial ordering inherited from these, then $\mathrm{dev}(A) = \sup\{\mathrm{dev}(A_i)\}$.

3. $\mathrm{dev}(A \times B) = \sup\{\mathrm{dev}(A), \mathrm{dev}(B)\}$.

2.2.2 Krull dimension of modules

Let M be a left R-module, the *lattice* of all R-submodules of M,

$$\mathcal{L}(M) = \{N \mid R\text{-module}, N \subseteq M\},$$

is a partially ordered set under inclusion. If we take the lattice of all R-submodules of the left module ${}_RR$, then $\mathcal{L}({}_RR)$ is the set of ideals of R. The *Krull dimension* of M, written $\mathcal{K}(M)$, is defined to be the deviation of $\mathcal{L}(M)$, the lattice of submodules of M. In particular, let $\Gamma(M)$ denote the set of all pairs (K, N) of submodules K and N of M with $N \subseteq K$. Define

$$\Gamma_0(M) = \{(K, N) \mid (K, N) \in \Gamma(M),\ K/N \text{ Artinian}\}.$$

$$\Gamma_\alpha(M) = \{(K, N) \mid (K, N) \in \Gamma(M),\ K \supseteq K_1 \supseteq \ldots \supseteq K_i \supseteq K_{i+1} \supseteq \ldots \supseteq N$$
$$\text{implies } (K_i, K_{i+1}) \in \bigcup_{\beta<\alpha} \Gamma_\beta(M) \text{ for almost all } i\},$$

for ordinals $\alpha > 0$. If there exists an ordinal α such that $\Gamma(M) = \Gamma_\alpha(M)$, then the left R-module M has a finite Krull dimension, and the smallest such ordinal is the Krull dimension, $\mathcal{K}(M)$, of the module M [13]. In general, such an ordinal does not always exist. However, If M is a Noetherian R-module, that is, M satisfies the ACC on submodules, then the Krull dimension $\mathcal{K}(M)$ is finite, whilst the converse is not necessarily true; modules with a Krull dimension need not be Noetherian [19]. The *left Krull dimension* of the ring R is defined to be the Krull dimension of the left module ${}_RR$, denoted by $\mathcal{K}({}_RR)$, or even $\mathcal{K}(R)$.

All the results about deviation apply to a module ${}_RM$ by viewing $\mathcal{L}(M)$ as a poset.

Properties of the Krull dimension: ([19, pp. 193-197].)

1. If ${}_RN$ is a submodule of ${}_RM$, then $\mathcal{K}(M) = \sup\{\mathcal{K}(N), \mathcal{K}(M/N)\}$.
2. If ${}_RM$ is finitely generated, then $\mathcal{K}(M) \leq \mathcal{K}({}_RR)$. Furthermore,

$$\mathcal{K}({}_RR) = \sup\{\mathcal{K}(M) \mid {}_RM \text{ finitely generated}\}.$$

3. Let M have a Krull dimension and also be the sum of submodules, each of which has a Krull dimension $\leq \alpha$. Then $\mathcal{K}(M) \leq \alpha$.

The following proposition gives a criterion for the polynomial module $M[x]$ over $R[x]$ to have a Krull dimension in terms of the left R-module M.

Proposition 2.37. *([19, Proposition 6.2.7].) ${}_{R[x]}M[x]$ has a Krull dimension if and only if ${}_RM$ is Noetherian.*

2.2.3 Krull dimension of rings

In this section, we introduce useful results about the Krull dimension of some rings, such as quotient rings, prime rings and skew polynomial rings. Also, we define the classical Krull dimension for ring R (not necessarily commutative) and compare it to the left Krull dimension $\mathcal{K}(R)$.

Lemma 2.38. *([19, Lemma 6.3.3].) If A is an ideal of R and ${}_RB \subseteq R$ and $\mathcal{K}(R/AB)$ exists, then*

$$\mathcal{K}(R/AB) = \sup\{\mathcal{K}(R/A), \mathcal{K}(R/B)\}.$$

Definition 2.39. *An element m of R-module M is called a* torsion element *of M if there exists a regular element $r \in R$ that annihilates m (i.e. $rm = 0$). A module M is called a* torsion module *if all of its elements are torsion elements.*

Definition 2.40. *A nonzero ring R is called a* prime ring *if for any two elements $a, b \in R$, $aRb = 0$ implies that either $a = 0$ or $b = 0$.*

The next proposition is concerned with the relationship between the Krull dimension of a prime ring and that of certain rings and modules.

Proposition 2.41. *([19, Proposition 6.3.11].) Let R be a prime ring with a left Krull dimension. Then*

1. *$\mathcal{K}(R) = \mathcal{K}(A)$ for each nonzero left ideal A;*
2. *$\mathcal{K}(R/B) < \mathcal{K}(R)$ for any nonzero ideal B of R; and*
3. *if ${}_RM$ is finitely generated, then $\mathcal{K}(M) < \mathcal{K}(R)$ if and only if M is a torsion module.*

Lemma 2.42. *([19, Lemma 6.3.12].) Let R, S be rings and ${}_SM_R$ a bimodule such that M_R is finitely generated, ${}_SM$ is faithful and $\mathcal{K}({}_SM)$ exists. Then $\mathcal{K}({}_SM) = \mathcal{K}({}_SS)$.*

The following proposition bounds the possible value for the Krull dimension of skew polynomial ring and skew Laurent polynomial ring and gives a more precise result under special conditions.

Proposition 2.43. *([19, Proposition 6.5.4].) Let R be a left Noetherian ring, σ an automorphism and δ a σ-derivation. Then*

1. $\mathcal{K}(R) \leq \mathcal{K}(R[x;\sigma,\delta]) \leq \mathcal{K}(R)+1$, *and in particular, if* $\delta = 0$, *then* $\mathcal{K}(R[x;\sigma]) = \mathcal{K}(R)+1$;
2. $\mathcal{K}(R) \leq \mathcal{K}(R[x,x^{-1};\sigma]) \leq \mathcal{K}(R)+1$, *and in particular, if* $\sigma = 1$, *then* $\mathcal{K}(R[x,x^{-1}]) = \mathcal{K}(R)+1$; *and*
3. *if R is left Artinian, then* $\mathcal{K}(R[x;\sigma,\delta]) = \mathcal{K}(R[x,x^{-1};\sigma]) = 1$.

For the poset $\mathrm{Spec}\,(R)$, the set of all prime ideals of R, let $\mathrm{Spec}_0\,(R)$ denote the set of all maximal ideals of R, and for ordinals $\alpha > 0$ let

$$\mathrm{Spec}_\alpha\,(R) = \{P \in \mathrm{Spec}\,(R) \mid P \subsetneq Q \in \mathrm{Spec}\,(R) \text{ implies } Q \in \bigcup_{\beta<\alpha} \mathrm{Spec}_\beta\,(R)\}.$$

The smallest ordinal α for which $\mathrm{Spec}\,(R) = Spec_\alpha\,(R)$ is called the classical Krull dimension of R, denoted by $\mathrm{Kdim}\,(R)$. In general, $\mathrm{Kdim}\,(R)$ need not always exist, and if such an ordinal α does not exist, then the Krull dimension does not exist either [13]. The necessary and sufficient condition for a ring R to have a finite classical Krull dimension is that R satisfies the maximum condition for prime ideals, that is every non-empty subset of $\mathrm{Spec}\,(R)$ has a maximal element [12]. If R is a commutative ring, this definition coincides with the definition given by terms of the lengths of chain of prime ideals. The classical Krull dimension of a left Noetherian ring R does not exceed the left Krull dimension of R, meaning $\mathrm{Kdim}\,(R) \leq \mathcal{K}(R)$ [13]. Moreover, If R is a ring that has a Krull dimension, then R has a classical Krull dimension and $\mathrm{Kdim}\,(R) \leq \mathcal{K}(R)$ [1]. In general, they are not equal; for example if K is a field of characteristic zero, then the first Weyl algebra $A_1(K)$ is a simple Noetherian ring, then $\mathrm{Kdim}\,(A_1(K)) = 0$, but $\mathcal{K}(A_1(K)) = 1$ [19]–[35]. Moreover, the equality is only possible for rings with many two-sided ideals [19].

Corollary 2.44. *([19, Corollary 6.4.8].) If R is a commutative Noetherian ring, then $\mathcal{K}(R) = Kdim\,(R)$.*

2.3 The dynamical Mordell-Lang conjecture

The dynamical Mordell-Lang conjecture is a dynamical analogue of the classical Mordell-Lang conjecture that was proved by Faltings and Vojta [22]. The Mordell-Lang conjecture concerns intersections of finitely generated subgroups and subvarieties in a semiabelian variety, whereas the dynamical conjecture predicts how the orbit of points in a quasiprojective variety under self maps should intersect subvarieties.

The motivation behind introducing the dynamical Mordell-Lang conjecture in our research is that in order to find the Krull dimension of GWAs (in Chapter 5), we will use Theorem 5.2, and therefore, we are interested in the intersection of the orbit of maximal ideal $\mathfrak{p}$ with a variety that is the zeros of the defining element a.

Before we state the dynamical Mordell-Lang conjecture, we will give an introduction to some basic and important concepts in algebraic geometry, such as algebraic sets, affine varieties, Zariski topology, projective varieties and quasiprojective varieties.

2.3.1 Algebraic sets

Let K be an algebraically closed field. The *n-dimensional affine space*, denoted by $\mathbb{A}^n_K$ or simply $\mathbb{A}^n$, over K is the set of all n-tuples of elements of K which is equipped with the Zariski topology, see below. Given any subset N of the polynomial ring $K[x_1, \ldots, x_n]$, we denote by $Z(N)$ the subset of the affine space $\mathbb{A}^n_K$ each of whose points is a zero for all the polynomials in N. Explicitly,

$$Z(N) = \{a \in \mathbb{A}^n_K \mid P(a) = 0 \text{ for all } P \in N\}.$$

If $\mathfrak{p}$ is the ideal of $K[x_1, \ldots, x_n]$ generated by the set N, then

$$Z(N) = Z(\mathfrak{p}).$$

Moreover, since the polynomial ring $K[x_1, \ldots, x_n]$ is Noetherian ring, any ideal is finitely generated, say $\mathfrak{p}$ is generated by the set $\{f_1, \ldots, f_m\}$. Hence, $Z(N)$ is the set of all common zeros of the polynomials $f_1, \ldots, f_m$. A subset X of $\mathbb{A}^n_K$ is an *algebraic set* if there exists a subset N of the polynomial ring $K[x_1, \ldots, x_n]$ such that $X = Z(N)$. For any subset X of $\mathbb{A}^n_K$, we define the ideal of X in

$K[x_1, \ldots, x_n]$ by the rule

$$I(X) = \{P \in K[x_1, \ldots, x_n] \mid P(a) = 0 \text{ for all } a \in X\}.$$

Now, we have two functions: Z which maps subsets of $K[x_1, \ldots, x_n]$ to algebraic sets, and I which maps subsets of $\mathbb{A}^n_K$ to ideals.

The following theorem gives an essential relationship between geometry and algebra by relating algebraic sets to ideals in polynomial rings.

Theorem 2.45. *(Hilbert's Nullstellensatz). ([10, Theorem 4.3].) For any ideal* $\mathfrak{p}$ *of* $K[x_1, \ldots, x_n]$*,*

$$I(Z(\mathfrak{p})) = \sqrt{\mathfrak{p}}$$

where $\sqrt{\mathfrak{p}} = \{P \in K[x_1, \ldots, x_n] \mid \exists\, n \in \mathbb{N},\ P^n \in \mathfrak{p}\}$ *denotes the radical of* $\mathfrak{p}$*.*

The Nullstellensatz indicates that the functions Z and I are order-reversing bijections between the set of algebraic sets of $\mathbb{A}^n_K$ and the set of radical ideals in $K[x_1, \ldots, x_n]$.

2.3.2 Zariski topology and affine varieties

The union of two algebraic sets is itself an algebraic set and the intersection of any family of algebraic sets is also an algebraic set. Also, the empty set and the whole space are algebraic sets ($\emptyset = Z(1)$ and $\mathbb{A}^n_K = Z(0)$) [29].

We define the *Zariski topology* on $\mathbb{A}^n_K$ by taking the open subsets U to be the complements of the algebraic sets X ($U = \mathbb{A}^n_K \backslash X$). This satisfies the topology axioms and the algebraic sets are closed sets in *Zariski topology*, by definition [29]. The space $\mathbb{A}^n_K$ with Zariski topology is a *topological space.*

Definition 2.46. *A non-empty subset* X *of a topological space* Y *is* irreducible *if it cannot be written as the union of two proper closed subsets. An* affine algebraic variety*, or simply* affine variety *is an irreducible closed subset of* $\mathbb{A}^n_K$ *(with the induced topology). An open subset of an affine variety is called a* quasi-affine variety*. Let* V *and* W *be affine varieties, a map* $\varphi : V \to W$ *is called a* regular map *if it can be presented as a fraction of polynomials* f/g*, where* $g(v) \neq 0$ *for all* $v \in V$*. An affine variety* V *in* $\mathbb{A}^n_K$ *is an* algebraic group *if it has a group structure on it, where the maps* $V \times V \to V$*,* $(v, u) \mapsto vu$ *and* $V \times V \to V$*,* $v \mapsto v^{-1}$ *are regular maps.*

A topological space Y is called *Noetherian* if it satisfies the descending chain condition (DCC) for closed subsets. The topological space $\mathbb{A}_K^n$ is Noetherian since the polynomial ring $K[x_1, \ldots, x_n]$ is Noetherian [29]. If Y is a topological space, the *dimension* of Y, denoted by $\dim Y$, is the supremum of all integers n such that there exists a chain

$$Z_0 \subset Z_1 \subset \cdots \subset Z_n,$$

of distinct irreducible closed subsets of Y. The *dimension* of an affine or quasi-affine variety is defined to be its dimension as topological space. The dimension of the affine space $\mathbb{A}_K^n$ is n [29].

Zariski closure. Given any subset $X \subset \mathbb{A}_K^n$, we define the *Zariski closure* of X, denoted by $\overline{X}$, as the smallest closed set containing X, i.e.

$$\overline{X} = Z(I(X)).$$

A set X is called *dense* if its closure is equal to $\mathbb{A}_K^n$, In fact, every open set in Zariski topology is dense. A point $x \in K^n$ is called *closed point* if $\overline{\{x\}} = \{x\}$, i.e. x is closed point in the Zariski topology on $\mathbb{A}_K^n$ if $\{x\}$ is an algebraic set.

2.3.3 Projective and quasiprojective varieties

The polynomial ring $K[x_0, \ldots, x_n] = \bigoplus_{\alpha \in \mathbb{N}^{n+1}} Kx^\alpha$ is an $\mathbb{N}^{n+1}$-graded ring, i.e.

$$Kx^\alpha \cdot Kx^\beta \subseteq Kx^{\alpha+\beta} \quad \text{for all} \;\; \alpha, \beta \in \mathbb{N}^{n+1}$$

where for $\alpha = (\alpha_0, \ldots, \alpha_n) \in \mathbb{N}^{n+1}$, $x^\alpha = \prod_{i=0}^n x_i^{\alpha_i}$. The polynomial ring $K[x_0, \ldots, x_n] = \bigoplus_{i \in \mathbb{N}} K[x_0, \ldots, x_n]_i$ is also an $\mathbb{N}$-graded ring where

$$K[x_0, \ldots, x_n]_i = \bigoplus_{|\alpha|=i} Kx^\alpha \;\; \text{where} \;\; |\alpha| = \alpha_0 + \cdots + \alpha_n.$$

An element of $K[x_0, \ldots, x_n]_i$ is called a *homogeneous polynomial* of degree i. Clearly, for all $f(x_0, \ldots, x_n) \in K[x_0, \ldots, x_n]_i$ and $\lambda \in K$,

$$f(\lambda x_0, \ldots, \lambda x_n) = \lambda^i f(x_0, \ldots, x_n).$$

The definition of projective varieties is similar to affine varieties except that the projective varieties are defined in projective space. Let K be an algebraically closed field. The n-dimensional projective space over K, denoted by $\mathbb{P}^n_K$ or simply $\mathbb{P}^n$, is the set of equivalence classes of $n+1$-tuples $(a_0, \ldots, a_n)$ of elements of K such that not all $a_i = 0$, i.e. $\mathbb{P}^n$ is the set of equivalence classes of $\mathbb{A}^{n+1}_K \backslash \{0\}$, under the equivalence relation given by

$$(a_0, \ldots, a_n) \sim (\lambda a_0, \ldots, \lambda a_n) \text{ for all } \lambda \in K, \ \lambda \neq 0.$$

Definition 2.47. *Let K be an algebraically closed field. If T is any subset of homogeneous polynomials of the polynomial ring $K[x_0, \ldots, x_n]$, we define an* algebraic set*, denoted by $V(T)$, to be the subset of the projective space $\mathbb{P}^n_K$ that is the set of common zeros of T. Explicitly,*

$$V(T) = \{a \in \mathbb{P}^n \mid f(a) = 0 \textit{ for all } f \in T\}.$$

For a subset $X \subset \mathbb{P}^n$, the ideal

$$I(X) = \{f \in K[x_0, \ldots, x_n] \mid f(a) = 0 \textit{ for all } a \in X\}$$

is a homogeneous ideal *of the polynomial ring $K[x_0, \ldots, x_n]$.*

Zariski topology on the projective space $\mathbb{P}^n$.

Proposition 2.48. *([29, Proposition 2.1].) The union of two algebraic sets is an algebraic set. The intersection of any family of algebraic sets is an algebraic set. The empty set and the whole space are algebraic sets.*

Proposition 2.48 shows that the collection of algebraic sets in $\mathbb{P}^n$ satisfy the axioms of topological space for closed sets. This topology in which the open sets are the complements of algebraic sets is called the *Zariski topology* on $\mathbb{P}^n$. The notions of irreducible and the dimension of algebraic set in topological space on affine space that defined in 2.3.2 will also apply to topological space on projective space.

Definition 2.49. *A* projective algebraic variety *or* projective variety *is an irreducible algebraic set in the projective space $\mathbb{P}^n$, with the induced topology. A* quasiprojective variety *is an open subset of a projective variety.*

2.3.4 The dynamical Mordell-Lang conjecture

A *(discrete) dynamical system* is a pair (X, Φ) where X is a set and Φ is a *self-map* on X, i.e.

$$\Phi : X \to X, \ \ x \mapsto \Phi(x)$$

is a map on X. The goal of dynamics is to study the behaviour of points in the set X under applying Φ repeatedly. For any point $\alpha \in X$, we write

$$\Phi^n(\alpha) = \underbrace{\Phi \circ \Phi \circ \cdots \circ \Phi}_{n \text{ times}}(\alpha).$$

The number n is called the *number of iterations*. The *orbit* of α is the set of iterates of Φ to α, denoted by

$$\mathcal{O}_\Phi(\alpha) = \{\alpha, \Phi(\alpha), \Phi^2(\alpha), \Phi^3(\alpha), \ldots\},$$

where we assume $\Phi^0(\alpha) = \alpha$ is the identity map on X. The orbits in the *(discrete) dynamical system* are sequences. So, the orbits could be finite or infinite. If $\mathcal{O}_\Phi(\alpha)$ is finite then the point α is called *preperiodic*.

Definition 2.50. *An* arithmetic progression*, also known as an* arithmetic sequence*, is a sequence of the form*

$$\{an + b \mid n \in \mathbb{N}\},$$

with $a, b \in \mathbb{N}$ possibly with $a = 0$. Note that, if $a = 0$ then the set consists of a single element.

Dynamical Mordell-Lang conjecture [17]. Let X be a quasiprojective variety defined over $\mathbb{C}$, let Φ be any endomorphism of X, let $\alpha \in X$, and let $V \subseteq X$ be any subvariety. Then the set

$$S = \{n \in \mathbb{N} \mid \Phi^n(\alpha) \in V\} \tag{2.1}$$

is a union of finitely many arithmetic progressions.

2.3.4.1 Dynamical Mordell-Lang conjecture in characteristic 0

The dynamical Mordell-Lang conjecture in fields of characteristic zero has been proven for some cases under certain conditions (see [6], [7], [8], [18], [22]).

Definition 2.51. *([17, p. 22].) Let (X, Φ) be a discrete dynamical system. A point $p \in X$ is* periodic *of period n if there exists an integer $n \geq 1$ such that $\Phi^n(p) = p$. A curve C on X is* periodic *if $\Phi^n(C) \subseteq C$ for some $n \in \mathbb{N}$ and n called a* period *of C.*

If the dynamical Mordell-Lang conjecture is true and S (i.e $V \cap \mathcal{O}_\Phi(\alpha)$) is infinite, then V must contain the set

$$\Phi^r(\alpha), \Phi^{r+m}(\alpha), \ldots, \Phi^{r+km}(\alpha), \ldots$$

for some arithmetic progressions $\{r + km \mid k \in \mathbb{N}\}$, which is invariant under Φ^m. Then if we take the Zariski closure of these sets we will have a positive dimensional subvariety of V that is periodic under Φ [34].

The following theorem shows that the conjecture is true for maps of the form

$$\Phi : \mathbb{A}^2 \to \mathbb{A}^2, \quad \Phi(x, y) = (f(x), g(y)) \tag{2.2}$$

defined over field of characteristic zero where $f, g \in K[t]$ are polynomials and V is a line.

Theorem 2.52. *([7, Theorem 1.4].) Let K be a field of characteristic zero, let $f, g \in K[X]$, and let $x_0, y_0 \in K$. If the set*

$$\{(f^n(x_0), g^n(y_0)) \mid n \in \mathbb{N}\}$$

has infinite intersection with a line L in $\mathbb{A}^2$ defined over K, then L is periodic under the action of (f, g) on $\mathbb{A}^2$.

Definition 2.53. *A sequence of numbers $a = \{a_m\}$ is* complete *if every positive integer n is the sum of some subsequence of a, i.e. there exist integers $r_i = 0$ or 1 such that*

$$n = \sum_{i=1}^{\infty} r_i a_i.$$

A sequence that is infinite in both directions, i.e. that has neither a first nor a final element is called doubly infinite sequence.

The conjecture is also known in the case of Y is any affine variety and σ is an automorphism of Y as given in the next theorem.

Theorem 2.54. *([16, Theorem 1.3].) Let Y be an affine variety over a field K of characteristic zero, let q be a point in Y and let σ be an automorphism of Y. If X is a subvariety of Y, then the set $\{m \in \mathbb{Z} \mid \sigma^m(q) \in X\}$ is a union of a finite number of complete doubly-infinite arithmetic progressions and a finite set.*

2.3.4.2 Dynamical Mordell-Lang conjecture in characteristic p

If the field K has characteristic $p > 0$ then counterexamples have been found for the dynamical Mordell-Lang conjecture. Before we give a counterexample, we will define algebraic K-*torus* since the proven cases of the conjecture in characteristic p, so far, are for X is an algebraic K-*torus*.

For any field K, the *multiplicative group* of K, denoted by K^* or $K^\times$ or $\mathbb{G}_m(K)$, is the group of all nonzero elements of K under multiplication. An algebraic K-*torus* is an algebraic group that is isomorphic to some power of multiplicative group over $\overline{K}$, the algebraic closure of K. So, if K is an algebraically closed field, then algebraic K-*torus* is an algebraic group that is isomorphic to some finite product of copes of the multiplicative group of K. The following example shows that if $X = \mathbb{G}_m^3$ the set S in the dynamical Mordell-Lang conjecture cannot be written as a finite union of arithmetic progressions.

Example 2.55. *([9, Example 1.2].) Let* $\operatorname{char}(K) = p > 2$, $K = \mathbb{F}_p(t)$, $X = \mathbb{G}_m^3$,

$$\Phi : \mathbb{G}_m^3 \to \mathbb{G}_m^3, \quad \Phi(x, y, z) = (tx, (1+t)y, (1-t)z)$$

and $V \subset \mathbb{G}_m^3$ *be the hyperplane given by the equation*

$$y + z - 2x = 2,$$

and let $\alpha = (1, 1, 1)$. *Then the set S from the dynamical Mordell-Lang conjecture consists of all numbers of the form* $p^{n_1} + p^{n_2}$ *for* $n_1, n_2 \in \mathbb{N}$.

Therefore, the following conjecture in positive characteristic was proposed in ([17, Conjecture 13.2.0.1]).

Conjecture. ((Ghioca-Scanlon) Dynamical Mordell-Lang Conjecture in positive characteristic). Let X be a quasiprojective variety defined over a field K of characteristic $p > 0$. Let $\alpha \in X(K)$, let $V \subseteq X$ be a subvariety defined over K, and let

$$\Phi : X \to X$$

be an endomorphism defined over K. Then the set $S := S(X, \Phi, V, \alpha)$ of integers $n \in \mathbb{N}$ such that $\Phi^n(\alpha) \in V(K)$ is a union of finitely many arithmetic progressions along with finitely many sets of the form

$$\left\{ \sum_{j=1}^{m} c_j p^{k_j n_j} \mid n_j \in \mathbb{N} \text{ for each } j = 1, \dots, m \right\}, \tag{2.3}$$

for some $c_j \in \mathbb{Q}$, and some $k_j \in \mathbb{N}$.

If $k_j = 0$ for each $j = 1, \ldots, m$ in (2.3), then the corresponding set is a singleton. Therefor, the set S in the conjecture can consist of finitely many elements (since, the arithmetic progressions also can be a singleton).

The conjecture above has been proven for only few cases (see [17]). The following theorem shows that the conjecture is true in the case $X = \mathbb{G}_m^N$ is algebraic torus and V is irreducible curve.

Theorem 2.56. *([9, Theorem 1.3].) Let K be an algebraically closed field of characteristic $p > 0$, let $N \in \mathbb{N}$, let $V \subset X := \mathbb{G}_m^N$ be an irreducible curve and $\Phi : X \to X$ be a self map both defined over K, and let $\alpha \in \mathbb{G}_m^N(K)$. Then the set S of all $n \in \mathbb{N}$ such that $\Phi^n(\alpha) \in V(K)$ is either a finite union of arithmetic progressions, or a finite union of sets of the form*

$$\{ap^{kn} + b \,|\, n \in \mathbb{N}\},$$

for some $a, b \in \mathbb{Q}$ and $k \in \mathbb{N}$.

Also, the conjecture is proven in the case V is a surface in the following result.

Theorem 2.57. *([28, Theorem 1.2].) Let $\Phi : \mathbb{G}_m^N \to \mathbb{G}_m^N$ be a regular self map defined over an algebraically closed field K of characteristic $p > 0$, let $V \subset \mathbb{G}_m^N$ be a variety of dimension at most equal to 2, and let $\alpha \in \mathbb{G}_m^N(K)$. Then the set of $n \in \mathbb{N}$ such that $\Phi^n(\alpha) \in V(K)$ is a finite union of arithmetic progressions along with finitely many sets of the form*

$$\{c_0 + c_1 p^{k_1 n_1} + c_2 p^{k_2 n_2} \,|\, n_1, n_2 \in \mathbb{N}\},$$

for some $c_0, c_1, c_2 \in \mathbb{Q}$ and some $k_1, k_2 \in \mathbb{N}$.

2.3.4.3 A criterion for the set S to be infinite

As we mentioned before the set S, in the dynamical Mordell-Lang conjecture for both characteristic 0 or p, can be finite or infinite. Since we are interested in whether the base ring in Theorem 5.2 has a σ-unstable prime ideal $\mathfrak{p}$ for which there exist infinitely many integers i with $a \in \sigma^i(\mathfrak{p})$, so we concerned about the case S is infinite. Proposition 2.60 gives a criterion for S to be infinite, provided that S is a union of at most finitely many arithmetic progressions.

Definition 2.58. *Let V and W be varieties over K, and consider pairs (U, φ_U) where U is a dense open subset of V and φ_U is a regular map $U \to W$. Two such pairs (U, φ_U) and (U', φ'_U) are said to be* equivalent *if φ_U and φ'_U agree on $U \cap U'$. An equivalence class of pairs is called a* rational map $\varphi : V \dashrightarrow W$.

A rational map $\varphi : V \dashrightarrow W$ is a map not everywhere defined, sometimes the map can be extended to a larger open set in which φ is defined and the largest possible open set in which φ is defined is called the *domain* of φ [10]. For example, let $(U, \varphi_U) \sim (U', \varphi'_U)$ be equivalent rational maps, then one can extend (U, φ_U) to

$$U_0 := \bigcup_{U' \in \Lambda} U',$$

where Λ denotes the equivalent class. The points in $V - U_0$ is called *points of indeterminacy*. We denote by $I(\varphi) \subseteq V$ the *indeterminacy set* of φ.

Definition 2.59. *([22, Definition 4.1].) Let X be a smooth surface defined over an algebraically closed field, and $f : X \dashrightarrow X$ be a rational transformation. We say that the pair (X, f) satisfies the* DML property *if for any curve C on X and for any closed point $x \in X$ such that $f^n(x) \notin I(f)$ for all $n \geq 0$, the set $\{n \in \mathbb{N} \mid f^n(x) \in C\}$ is a union of at most finitely many arithmetic progressions.*

Proposition 2.60. *([22, Proposition 4.2].) Let X be a smooth surface defined over an algebraically closed field, $f : X \dashrightarrow X$ be a rational transformation and $I(f)$ be the indeterminacy set of the map f. The following statements are equivalent.*

1. *The pair (X, f) satisfies the DML property.*
2. *For any curve C on X and any closed point $x \in X$ such that $f^n(x) \notin I(f)$ for all $n \geq 0$ and the set $\{n \in \mathbb{N} \mid f^n(x) \in C\}$ is infinite, then x is preperiodic or C is periodic.*

Chapter 3

Generalized Weyl algebras and their properties

Generalized Weyl algebras are a class of algebras that generalizes the Weyl algebras. In this chapter, we present the definition of generalized Weyl algebra of degree n and its properties.

3.1 Definition of generalized Weyl algebra

Definition 3.1. *([42, Definition 1.1].) Let D be a ring, $\sigma = (\sigma_1, \dots, \sigma_n)$ be a set of commuting automorphism of D, i.e. $\sigma_i \sigma_j = \sigma_j \sigma_i$, and $a = (a_1, \dots, a_n)$ a set of nonzero elements of the centre $Z(D)$ of D, where $\sigma_i(a_j) = a_j$ for all $i \neq j$. The* **generalized Weyl algebra** *$A = D(\sigma, a)$ of degree n is a ring generated by the ring D and the $2n$ indeterminates $X_1^+, \dots, X_n^+$ and $X_1^-, \dots, X_n^-$ that are subject to the defining relations:*

$$X_i^- X_i^+ = a_i, \qquad X_i^+ X_i^- = \sigma_i(a_i),$$
$$X_i^{\pm} \alpha = \sigma_i^{\pm 1}(\alpha) X_i^{\pm} \quad \forall\, \alpha \in D,$$
$$[X_i^-, X_j^-] = [X_i^+, X_j^+] = [X_i^+, X_j^-] = 0, \quad \forall i \neq j.$$

The ring D is called the base ring *of the GWA. The sets σ and a are called the set of* defining automorphisms *and the set of* defining elements *of the GWA, respectively.*

The generalized Weyl algebras $A = D(\sigma, a)$ of degree 1 that defined in Chapter 1 is a case of GWA of degree n where $X = X_1^+$ and $Y = X_1^-$.

Example 3.2. *([42, Example 1.2].) The* Weyl algebra $A_n = A_n(K)$ *of degree* n *over a field* K *is algebra with* $2n$ *generators* $X_1, \dots, X_n, \partial_1, \dots, \partial_n$ *which satisfy the classical commutation relations*

$$[X_i, X_j] = [\partial_i, \partial_j] = [\partial_i, X_j] = 0 \quad for \quad i \neq j \quad and \quad [\partial_i, X_i] = 1.$$

The Weyl algebra A_n *is the GWA* $A = D(\sigma, a)$ *of degree* n *with base ring* $D = K[H_1, \dots, H_n]$ *is a polynomial ring in* n *variables, set of defining elements* $\{a_i = H_i \,|\, 1 \leq i \leq n\}$*, and set of defining automorphisms* $\{\sigma_i\}$ *such that*

$$\sigma_i(H_j) = \begin{cases} H_j - 1 & if \quad i = j, \\ H_j & if \quad i \neq j. \end{cases}$$

Furthermore, the map

$$X_i \to X_i^+, \quad \partial_i \to X_i^-, \quad \partial_i X_i \to H_i, \quad i = 1, \dots n,$$

is an isomorphism of the algebras.

3.2 Properties

In this section, we present the main properties of generalized Weyl algebras A of degree n.

3.2.1 Graded and tensor product of GWAs

The properties, in this section, are from [42] and [44].

Let $A = D(\sigma, a)$ be a GWA of degree n. For any vector $k = (k_1, \dots, k_n) \in \mathbb{Z}^n$ we put $v_k = v_{k_1}(1) \dots v_{k_n}(n)$ where for $1 \leq i \leq n$ and $m \geq 0$:

$$v_{\pm m}(i) = (X_i^{\pm})^m, \quad v_0(i) = 1.$$

In the case $n = 1$, we write v_m instead of $v_m(1)$ and for any integers n and m, we define the elements (n, m) and $\langle n, m \rangle$ of D:

$$v_n v_m = (n, m) v_{n+m} = v_{n+m} \langle n, m \rangle.$$

It is clear that $(n, m) = \sigma^{n+m}(\langle n, m \rangle)$. If $n > 0$ and $m > 0$ it follows from the defining relations that

$$n \geq m : (n, -m) = \sigma^n(a) \cdots \sigma^{n-m+1}(a), \quad (-n, m) = \sigma^{-n+1}(a) \cdots \sigma^{-n+m}(a),$$

$$n \leq m : (n, -m) = \sigma^n(a) \cdots \sigma(a), \quad (-n, m) = \sigma^{-n+1}(a) \cdots a,$$

in other cases $(n, m) = 1$.

It follows from the definition of the GWA that the generalized Weyl algebra A of degree n

$$A = \bigoplus_{k \in \mathbb{Z}^n} A_k$$

is a $\mathbb{Z}^n$-graded algebra ($A_k A_e \subset A_{k+e}$ for all $k, e \in \mathbb{Z}^n$), where $A_k = Dv_k$.

The category of generalized Weyl algebras is closed under the tensor product (over a base ring or field) of algebras:

$$A \otimes A' = D \otimes D'(\sigma \cup \sigma', a \cup a').$$

This is a very important property it allows us to build a GWA of degree n from GWAs of degree 1, as follows

let $\Lambda_i = D_i(\sigma_i, a_i)$ $(i = 1, \ldots, n)$ be generalized Weyl algebras of degree 1 over some field K such that each σ_i is a K-automorphism then by tensor product we will have a GWA of degree n over K,

$$\Lambda = \otimes_1^n \Lambda_i = (\otimes_1^n D_i)((\sigma_i), (a_i)).$$

For example, the n-th Weyl algebra A_n is the tensor product of the first Weyl algebra A_1 n times, that is

$$A_n = \underbrace{A_1 \otimes \ldots \otimes A_1}_{n \text{ times}}.$$

Also, the algebra opposite to a GWA $A = D(\sigma, a)$,

$$A^{op} = D^{op}(\sigma^{-1}, \sigma(a)), \quad \text{where} \quad \sigma^{-1} = (\sigma_i^{-1}),\ \sigma(a) = (\sigma_i(a_i)),$$

is again a GWA, and so there exists a symmetry between the left modules, ideals, etc., and their right analogs.

3.2.2 Noetherian property and integral domains

The generalized Weyl algebra $A = D(\sigma, a)$ has some inherited properties from its base ring D. The following proposition gives these properties.

Proposition 3.3. *([42, Proposition 1.3].) Let A be a GWA of degree n with base ring D. Then:*

1. *If D is a left (right) Noetherian ring, so is A.*
2. *If D is an integral domain and $a_i \neq 0$ for all $i = 1, \dots, n$, then A is an integral domain.*

3.2.3 The simplicity criterion for GWAs

Given any generalized Weyl algebra $A = D(\sigma, a)$. The cyclic group $G = \langle \sigma \rangle$ obviously acts on the set of maximal ideals of $\mathrm{Spec}\,(D)$ of the ring D [44],

$$(\sigma, \mathfrak{m}) \mapsto \sigma(\mathfrak{m}), \quad \mathfrak{m} \in \mathrm{Spec}\,(D).$$

So, for any maximal ideal $\mathfrak{m}$ of D the *orbit* $\mathcal{O}$ equals to

$$\mathcal{O}(\mathfrak{m}) = \{\sigma^i(\mathfrak{m}) \,|\, i \in \mathbb{Z}\}.$$

An ideal $\mathfrak{p}$ of D is called *semistable* (or *σ-semistable*) if $\sigma^i(\mathfrak{p}) = \mathfrak{p}$ for some $i \geq 1$, i.e. the orbit $\mathcal{O}(\mathfrak{p}) = \{\sigma^i(\mathfrak{p}) \,|\, i \in \mathbb{Z}\}$ is finite. Also, if I is any ideal of D then it is called *σ-stable* (or, *σ-invariant*) ideal if $\sigma(I) = I$ and the ring D is *σ-simple* if 0 and D are the only *σ-stable* ideals of D. The orbit $\mathcal{O}$ that contains an ideal $\mathfrak{p}$ such that $a \in \mathfrak{p}$ called *degenerate* orbit and such ideals are called *marked* ideals. The *degenerate* orbits that contain more than one marked ideal are called *strong degenerate* orbits. Let $\mathrm{Inn}(D) = \{\omega_u \,|\, u \in D^*\}$ be the group of inner automorphisms of D where $\omega_u(a) = uau^{-1}$.

The following theorem gives a criterion for generalized Weyl algebra $A = D(\sigma, a)$ of degree n to be simple algebra.

Theorem 3.4. *([39, Theorem 4.5].) Let $A = D(\sigma, a)$ be a generalized Weyl algebra of degree n, D has no zero divisors. Then A is simple if and only if the following hold*

1. *D has no proper σ-stable ideals;*
2. *the subgroup of the factor group $\mathrm{Aut}(D)/\mathrm{Inn}(D)$ generated by images of all σ_i'th is isomorphic to the free abelian group $\mathbb{Z}^n$;*
3. *$Da_i + D\sigma_i^m(a_i) = D$ for any $m \geq 1$ and $i = 1, \dots, n$.*

Let G be a group and $g \in G$. The *order* of g, denoted $\mathrm{or}\,(g)$, is the minimum non negative integer n such that $g^n = e$, that is

$$\mathrm{or}\,(g) = \min\{n \geq 1 \mid g^n = e\},$$

where e is the identity element. If $\mathrm{Aut}(G)$ is the group of automorphisms of G and $\sigma \in \mathrm{Aut}(G)$ then $\mathrm{or}\,(\sigma) = \min\{i \geq 1 \mid \sigma^i = \mathrm{id}\}$, where id is the identity map on G. If no such n exists, σ is said to have *infinite order*.

Let $A = D(\sigma, a)$ be a GWA of degree 1 where the base ring D is a commutative Noetherian ring. Then the following result gives a simplicity criterion for A.

Theorem 3.5. *([5, Theorem 6.1].) Let D be a commutative Noetherian ring with an automorphism σ and let $a \in D$. The ring $A = D(\sigma, a)$ is simple if and only if σ has infinite order, a is regular, D is σ-simple and, for all positive integers n,*

$$Da + D\sigma^n(a) = D.$$

3.3 The centre of $K[H, C](\sigma, a)$ with affine automorphism σ

In this section, firstly we find the ring of σ-invariants D^σ for $D = K[H, C]$ and σ is an affine automorphism of the polynomial algebra D. After that, we prove Theorem 1.4 that describe the centre $Z(A)$ of the algebra $A = K[H, C](\sigma, a)$.

The algebra of σ-invariants D^σ. Let A be an algebra over a field K, $\mathrm{Aut}_K(A)$ be the group of K-algebra automorphisms of A and G be a subgroup of $\mathrm{Aut}_K(A)$. The set of G-invariants

$$A^G = \{a \in A \mid g(a) = a \text{ for all } g \in G\}$$

is a subalgebra of A which is called the *algebra of G-invariants*. If H is a subgroup of $\mathrm{Aut}_K(A)$ such that $H \subseteq G$ then $A^H \supseteq A^G$.

Let K be a field. The set $K^* := K\backslash\{0\}$ is a multiplicative group. For each natural number $n \geq 1$, let $\mathcal{P}_n = \mathcal{P}_n(K)$ be the set of elements in K^* of order n. In general, the set $\mathcal{P}_n(K)$ can be an empty set: If $\mathbb{F}_2 = \{0, 1\}$ is the field of characteristic 2 then $\mathcal{P}_n = \emptyset$ for all $n \geq 2$. For each natural number $n \geq 1$, the set

$$\mathcal{M}_n := \{\lambda \in K \mid \lambda^n = 1\}$$

is a subgroup of K^*. If $n|m$ then $\mathcal{M}_n \subseteq \mathcal{M}_m$. The union $\mathcal{M}_\infty = \bigcup_{n\geq 1} \mathcal{M}_n$ is a subgroup of K^*, the group of roots of 1. Let Γ be an abelian group. It can be seen as a $\mathbb{Z}$-module. Then $\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma$ is a vector space over the field of rational numbers $\mathbb{Q}$ and $\mathrm{rk}\,(\Gamma) := \dim_{\mathbb{Q}}(\mathbb{Q} \otimes_{\mathbb{Z}} \Gamma)$ is called the *rank* of Γ. Clearly, $\mathrm{rk}(\Gamma) = 0$ if and only if every element of Γ has finite order. If

$$0 \to \Gamma_1 \to \Gamma \to \Gamma_2 \to 0$$

is a short exact sequence of abelian groups then $\mathrm{rk}\,(\Gamma) = \mathrm{rk}\,(\Gamma_1) + \mathrm{rk}\,(\Gamma_2)$.

Let σ be the automorphism of the polynomial algebra $D = K[H, C]$ given by the rule $\sigma(H) = \lambda H$ and $\sigma(C) = \mu C$ for some elements $\lambda, \mu \in K^*$. Let $G = \langle \lambda^{\pm 1}, \mu^{\pm 1} \rangle$ be the subgroup of K^* generated by the elements λ and μ. Let Γ be the kernel of the group epimorphism

$$\xi : \mathbb{Z}^2 \to G,\ (i, j) \mapsto \lambda^i \mu^j.$$

So, $\Gamma = \{(i, j) \in \mathbb{Z}^2 \mid \lambda^i \mu^j = 1\}$. Since $\sigma(H^i C^j) = \lambda^i \mu^j H^i C^j$,

$$D^\sigma = \oplus_{(i,j)\in\Gamma\cap\mathbb{N}^2} K H^i C^j. \tag{3.1}$$

If $\mathrm{rk}\,(G) = 2$ then $\Gamma = \{(0, 0)\}$ and $D^\sigma = K$.

If $\mathrm{rk}\,(G) = 1$ then $\mathrm{rk}\,(\Gamma) = 1$ and $\Gamma = \mathbb{Z}(s, t)$ for some nonzero element $(s, t) \in \mathbb{Z}^2$ such that either $s \geq 1$ or $t \geq 1$. Then

$$D^\sigma = \begin{cases} K[H^s C^t] & \text{if } \ s \geq 0,\ t \geq 0, \\ K & \text{otherwise.} \end{cases}$$

If $\mathrm{rk}\,(G) = 0$ then the group G is a finite group, and vice versa. Let $n = \mathrm{or}\,(\lambda)$ and $m = \mathrm{or}\,(\mu)$ be the orders of the elements λ and μ of the group G, i.e. $\lambda \in \mathcal{P}_n$ and $\mu \in \mathcal{P}_m$. Let $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\mathbb{Z}_m = \mathbb{Z}/m\mathbb{Z}$. The group Γ contains the subgroup $n\mathbb{Z} \oplus m\mathbb{Z}$. There is a short exact sequence of abelian groups

$$0 \to \overline{\Gamma} \to \mathbb{Z}_n \oplus \mathbb{Z}_m \to G \to 1 \ \text{ where } \ \overline{\Gamma} := \Gamma/(n\mathbb{Z} \oplus m\mathbb{Z}).$$

Notice that the factor monoid $\mathbb{N}_n := \mathbb{N}/n\mathbb{N} = \{\bar{i} = i + n\mathbb{N} \mid 0 \leq i \leq n - 1\}$ is a group isomorphic to $\mathbb{Z}_n$ via $\mathbb{N}_n \to \mathbb{Z}_n,\ \bar{i} \mapsto \bar{i}$. We identify these two groups, $\mathbb{N}_n = \mathbb{Z}_n$. We also identify the set $\mathbb{N}_n$ with a subset of $\mathbb{N}$ via the injection

$$\mathbb{N}_n \to \mathbb{N},\ \bar{i} \mapsto i \ \text{ where } \ i = 0, 1, \ldots, n - 1.$$

So, $\mathbb{N}_n \subseteq \mathbb{N}$. Then $\mathbb{N}_n \oplus \mathbb{N}_m \subset \mathbb{N}^2$, and so

$$\overline{\Gamma} \subseteq \mathbb{Z}_n \oplus \mathbb{Z}_m = \mathbb{N}_n \oplus \mathbb{N}_m \subseteq \mathbb{N}^2. \tag{3.2}$$

By (3.1) and (3.2),

$$D^\sigma = \oplus_{(i,j)\in\overline{\Gamma}} K[H^n, C^m] H^i C^j.$$

The statement 2 of Proposition 3.6 has been proven.

Proposition 3.6. *Let $D = K[H,C]$ be a polynomial algebra over a field K of characteristic p, σ be a K-algebra automorphism of D and λ, $\mu \in K^*$.*

1. *If $\sigma(H) = H - 1$ and $\sigma(C) = \lambda C$ then*

$$D^\sigma = \begin{cases} K[H_p, C^n] & \text{if } p > 0,\ \lambda \in \mathcal{P}_n, \\ K[H_p] & \text{if } p > 0,\ \lambda \notin \mathcal{M}_\infty, \\ K[C^n] & \text{if } p = 0,\ \lambda \in \mathcal{P}_n, \\ K & \text{if } p = 0,\ \lambda \notin \mathcal{M}_\infty, \end{cases}$$

and

$$\mathrm{or}\,(\sigma) = \begin{cases} \mathrm{lcm}(p,n) & \text{if } p > 0,\ \lambda \in \mathcal{P}_n, \\ \infty & \text{otherwise}, \end{cases}$$

where $H_n := H(H-1)\cdots(H-n+1)$ for $n \geq 1$.

2. *Suppose that $\sigma(H) = \lambda H$ and $\sigma(C) = \mu C$. We keep the notation as above ($G = \langle \lambda^\pm, \mu^\pm \rangle$, etc.). Then*

$$\mathrm{or}\,(\sigma) = \begin{cases} \mathrm{lcm}(n,m) & \text{if } \lambda \in \mathcal{P}_n,\ \mu \in \mathcal{P}_m, \\ \infty & \text{if either } \lambda \notin \mathcal{M}_\infty \text{ or } \mu \notin \mathcal{M}_\infty. \end{cases}$$

   (a) *If $\mathrm{rk}\,(G) = 0$ then $D^\sigma = \oplus_{(i,j)\in\overline{\Gamma}} K[H^n, C^m] H^i C^j$ where $\lambda \in \mathcal{P}_n$ and $\mu \in \mathcal{P}_m$.*

   (b) *If $\mathrm{rk}\,(G) = 1$ then $\Gamma = \mathbb{Z}(s,t)$ for some nonzero element $(s,t) \in \mathbb{Z}^2$ such that $s \geq 1$ or $t \geq 1$, and*

$$D^\sigma = \begin{cases} K[H^s C^t] & \text{if } s \geq 0,\ t \geq 0, \\ K & \text{otherwise}. \end{cases}$$

   (c) *If $\mathrm{rk}\,(G) = 2$ then $D^\sigma = K$.*

3. *If* $\sigma(H) = H - 1$ *and* $\sigma(C) = C + H$ *then*

$$D^\sigma = \begin{cases} K[H_2, \theta] & \text{if } p = 2, \\ K[H_p, \eta] & \text{if } p > 2, \\ K[\eta] & \text{if } p = 0, \end{cases} \quad \text{and} \quad \mathrm{or}\,(\sigma) = \begin{cases} 4 & \text{if } p = 2, \\ p & \text{if } p > 2, \\ \infty & \text{if } p = 0, \end{cases}$$

where $H_2 := H(H-1)$, $H_p := H(H-1)\cdots(H-p+1)$, $\theta = H^2(H+1) + C(C+1)$ *and* $\eta = C + \frac{H(H+1)}{2}$.

4. *If* $\sigma(H) = \lambda H + C$ *and* $\sigma(C) = \lambda C$ *for some* $\lambda \in K^*$ *then*

$$D^\sigma = \begin{cases} K[H_p, C] & \text{if } \lambda = 1,\, p > 0, \\ K[C^m] & \text{if } \lambda \in \mathcal{P}_m,\, m \geq 2,\, p > 0,\, p \nmid m, \\ K[C^m] & \text{if } \lambda \in \mathcal{P}_m,\, p = 0, \\ K & \text{otherwise,} \end{cases}$$

and

$$\mathrm{or}\,(\sigma) = \begin{cases} pm & \text{if } \lambda \in \mathcal{P}_m,\, p > 0,\, p \nmid m, \\ \infty & \text{otherwise,} \end{cases}$$

where $H_p = \prod_{i=0}^{p-1} \sigma^i(H) = H(H+C)(H+2C)\cdots(H+(p-1)C)$.

Remark. For the automorphism σ in statement 4, it is shown in the proof that there is no automorphism σ such that $\lambda \in \mathcal{P}_m$, $m \geq 1$, $p > 0$ and $p|m$.

Proof. 1. Notice that the vector space $D = \oplus_{i\geq 0} K[H]C^i$ is a direct sum of σ-invariant subspaces. Furthermore, for each $i \geq 0$, the vector space $K[H]C^i$ is the union $\bigcup_{j\geq 1} \ker(\sigma_i - \lambda^i)^j$ where σ_i is the restriction of σ to $K[H]C^i$. Therefore,

$$D^\sigma = \left(\oplus_{0\leq i\,|\,\lambda^i=1} K[H]C^i\right)^\sigma = \oplus_{0\leq i\,|\,\lambda^i=1} K[H]^\sigma C^i.$$

Since $K[H]^\sigma = K[H_p]$ if $p > 0$ and $K[H]^\sigma = K$ if $p = 0$, statement 1 follows.

2. Statement 2 has already been proven.

3. By induction on i, we have the equalities

$$\sigma^i(H) = H - i \quad \text{and} \quad \sigma^i(C) = C + iH - \frac{i(i-1)}{2} \quad \text{for } i \geq 1. \tag{3.3}$$

(i) *Case* $p = 2$: By (3.3), $\sigma^2(H) = H$ and $\sigma^2(C) = C + 1$. Hence, the order of the automorphism σ is 4. Clearly, $D^{\sigma^2} = K[H, \mathcal{E}]$ where $\mathcal{E} = C\sigma^2(C) = C(C+1)$.

Notice that $\theta = H_2H + \mathcal{E}$, and so $\sigma(\theta) = H_2(H+1) + \mathcal{E} + H_2 = \theta$. Therefore, $\theta \in D^\sigma$ and $D^{\sigma^2} = K[\theta][H]$. Now, $D^{\sigma^2} \supseteq D^\sigma = (D^{\sigma^2})^\sigma = K[\theta][H_2]$, as required.

(ii) *Case* $p > 2$: Clearly, $\eta \in D^\sigma$. Now, $D = K[C, H] = K[\eta] \otimes K[H]$. Therefore,

$$D^\sigma = K[\eta] \otimes K[H]^\sigma = K[\eta] \otimes K[H_p] = K[\eta, H_p].$$

By (3.3), the order of the automorphism σ is p.

(iii) *Case* $p = 0$: Clearly, $\eta \in D^\sigma$. Now, $D = K[\eta][H]$. In particular, every element $f \in D^\sigma$ is a unique sum $f = \sum_{i=0}^n \alpha_i H^i$ for some elements $\alpha_i \in K[\eta]$. We have to show that $f \in K[\eta]$. Now,

$$0 = \sigma(f) - f = \alpha_i i H^{n-1} + \cdots$$

where the three dots mean an element of degree $< n-1$ in H. This is possible if and only if $\alpha_1 = \cdots = \alpha_n = 0$, that is $f \in K[\eta]$, as required. By (3.3) the order of the automorphism σ is ∞.

4. By induction on i, we have the equalities

$$\sigma^i(H) = \lambda^i H + i\lambda^{i-1}C \;\text{ and }\; \sigma^i(C) = \lambda^i C \;\text{ for }\; i \geq 1. \tag{3.4}$$

Suppose that $l := \mathrm{or}\,(\sigma) < \infty$. By (3.4), $\sigma^l = \mathrm{id}$ if and only if $\lambda^l = 1$ and $l\lambda^{l-1} = 0$ if and only if $\lambda \in \mathcal{M}_l$, $p > 0$ and $p|l$. Let $m = \mathrm{or}\,(\lambda)$. Then $p \nmid m$ (otherwise, $m = pj$ for some $j \geq 1$, and so $0 = \lambda^m - 1 = (\lambda^j - 1)^p$ implies $\lambda^j = 1$ and so $j \geq \mathrm{or}(\lambda) = pj$, a contradiction). Then $l = im$ for some $i \geq 1$ (since $\lambda^l = 1$). Notice that $p|l$ if and only if $p|im$ if and only if $p|i$ (since $p \nmid m$). Then, by (3.4), $\sigma^{pm} = \mathrm{id}$, hence $l|pm$ (since $l = \mathrm{or}\,(\sigma)$) and also $pm|l$ (since $m|l$, $p|l$ and $p \nmid m$), and so $l = pm$. This proves that $\mathrm{or}(\sigma) < \infty$ if and only if $\mathrm{or}(\sigma) = pm$, $\lambda \in \mathcal{P}_m$, $p > 0$ and $p \nmid m$. Also, there is no automorphism σ such that $\lambda \in \mathcal{P}_m$, $m \geq 1$, $p > 0$ and $p|m$.

The polynomial algebra $D = \oplus_{i\geq 0} D_i$ is an $\mathbb{N}$-graded algebra where

$$D_i = \oplus_{s+t=i} KC^sH^t.$$

The automorphism σ respects the $\mathbb{N}$-grading ($\sigma(D_i) = D_i$ for all $i \geq 0$). Furthermore, every graded component D_i admits an ascending filtration by the degree of H that is $D_i = \bigcup_{j=0}^i D_{i,j}$ where $D_{i,j} = \oplus_{t=0}^j KC^{i-t}H^t$. The automorphism σ respects the filtration on each graded component D_i ($\sigma(D_{i,j}) = D_{i,j}$ for all $j = 0, \ldots, i$). The $(i+1) \times (i+1)$ matrix $M_i(\sigma)$ of the K-linear map $\sigma|_{D_i} : D_i \to D_i$, $d \mapsto \sigma(d)$ with respect to the basis $\{C^i, C^{i-1}H, \ldots, H^i\}$ is an *upper triangular* matrix where the diagonal elements are all equal to λ^i. Therefore,

$$D^\sigma = \begin{cases} \oplus_{i\geq 0} D^\sigma_{mi} & \text{if } \lambda \in \mathcal{P}_m,\ m \geq 1, \\ K & \text{if } \lambda \notin \mathcal{M}_\infty. \end{cases}$$

(i) *Case* $\lambda = 1$: In this case, $\sigma(H) = H + C$ and $\sigma(C) = C \in D^\sigma$. If $p = 0$ then $D^\sigma = K[C]$. Suppose that $p > 0$. Notice that

$$R := K[C, H_p] \subseteq D^\sigma,$$

the algebra D is a free R-module and the set $\{1, H, \ldots, H^{p-1}\}$ is a free R-basis. Suppose that $R \neq D^\sigma$, we seek a contradiction. Let us choose an element $f \in D^\sigma \backslash R$. Then $f = \sum_{i=0}^{p-1} \alpha_i H^i$ for some elements $\alpha_i \in R$. Let $n = \max\{i \mid \alpha_i \neq 0\}$. Then $n \geq 1$. We may assume that n is the least possible. Then $n > 1$ since otherwise $f = \alpha_0 + \alpha_1 H$ and $0 = \sigma(f) - f = \alpha_1 C \neq 0$, a contradiction. Now,

$$0 = \sigma(f) - f = n\alpha_n CH^{n-1} + \cdots$$

where the three dots mean a polynomial in $\sum_{i=0}^{n-2} RH^i$. We must have $n\alpha_n C = 0$, a contradiction since $1 < n < p$.

(ii) *Case* $\lambda \in \mathcal{P}_m$, $m \geq 2$, $p = 0$: Notice that $S := K[C^m] \subseteq D^\sigma = \oplus_{i \geq 0} D^\sigma_{mi}$. To prove that the reverse inclusion holds we have to show that for each $i \geq 1$, $D^\sigma_{mi} = KC^{mi}$ (this fact would imply the equality $D^\sigma = S$). Suppose that $S \neq D^\sigma$, we seek a contradiction. We can choose a polynomial

$$f = C^{mi-d}H^d + \cdots \in D^\sigma_{mi} \backslash KC^{mi} \text{ for some } i \geq 1 \text{ and } d \geq 1$$

where the three dots mean the smaller terms, i.e. of H-degree $< d$. Then

$$0 = \sigma(f) - f = \lambda^{mi-d}C^{mi-d}(\lambda H + C)^d - C^{mi-d}H^d + \cdots = d\lambda^{mi-1}C^{mi-d+1}H^{d-1} + \cdots,$$

and so $d\lambda^{mi-1} = 0$, a contradiction (since $d \geq 1$).

(iii) *Case* $\lambda \in \mathcal{P}_m$, $m \geq 2$, $p > 0$: We have to show that $D^\sigma_{mi} = KC^{mi}$ for all $i \geq 0$. Clearly, $D^\sigma_{mi} \supseteq KC^{mi}$ for all $i \geq 0$. Suppose that $D^\sigma_{mi} \neq KC^{mi}$ for some $i \geq 0$. The necessarily $i \geq 1$ (since $D^\sigma_0 = K$) and there exists a σ-invariant element, say g, of D^σ_{mi} of H-degree $n \geq 1$. Without loss of generality we may assume that

$$g = C^{mi-n}H^n + \gamma C^{mi-n+1}H^{n-1} + \cdots \text{ for some } \gamma \in K.$$

Then

$$0 = \sigma(g) - g = n\lambda^{-1}C^{mi-n+1}H^{n-1} + \cdots$$

where the three dots mean a polynomial in D_{mi} of H-degree $< n-1$. Hence, $n = 0$ (the equality holds in the field K), i.e. $p|n$, a contradiction (since $1 \leq n \leq p-1$). $\square$

Proof of Theorem 1.4. 1. The algebra $A = \oplus_{i \in \mathbb{Z}} A_i$ is a $\mathbb{Z}$-graded algebra where $A_i = Dv_i$ (for $i \geq 1$, $v_i = X^i$ and $v_{-i} = Y^i$; and $v_0 := 1$). Hence, the centre $Z = Z(A)$ is a $\mathbb{Z}$-graded subalgebra of A, $Z = \oplus_{i\in\mathbb{Z}} Z_i$ where $Z_i = Z \cap A_i$. Clearly, $Z_0 = D^\sigma$ (an element $d \in D$ belongs to Z if and only if

$$0 = Xd - dX = (\sigma(d) - d)X \ \text{ and } \ 0 = Yd - dY = (\sigma^{-1}(d) - d)Y$$

if and only if $\sigma^{\pm 1}(d) = d$ if and only if $d \in D^\sigma$). The algebra $A = A_+ + A_-$ is a sum of its two homogeneous subalgebras $A_+ = D[X;\sigma]$ and $A_- = D[Y;\sigma^{-1}]$. Notice that

$$Z(A_+) = \begin{cases} D^\sigma[X^n] & \text{if } n < \infty, \\ D^\sigma & \text{if } n = \infty, \end{cases} \qquad \text{and} \qquad Z(A_-) = \begin{cases} D^\sigma[Y^n] & \text{if } n < \infty, \\ D^\sigma & \text{if } n = \infty. \end{cases}$$

For all $i \geq 0$, $Z_i \subseteq Z(A_+)$ and $Z_{-i} \subseteq Z(A_-)$ since the algebras Z_i, A_+ and A_- are homogeneous. So,

$$Z \subseteq Z(A_+) + Z(A_-).$$

On the other hand, $Z(A_\pm) \subseteq Z$. This is obvious if $n = \infty$ ($D^\sigma \subseteq Z \subseteq Z(A_+) + Z(A_-) = D^\sigma$). If $n < \infty$ then it follows from

$$X^nY^n = (n, -n) = \sigma^n((-n, n)) = (-n, n) = Y^nX^n.$$

Therefore, $Z = Z(A_+) + Z(A_-)$ and the theorem follows.

2. By Proposition 3.6, the algebra D^σ is finitely generated. Hence, so is the algebra $Z(A)$, by statement 1.

3. Statement 3 follows from statement 1. $\square$

Chapter 4

The global dimension of $S^{-1}K[H,C](\sigma,a)$

In this chapter, we prove our main result, Theorem 1.5, and as an application of it and Theorem 4.2, we compute the global dimension of many classical algebras that are examples of GWAs in Theorem 1.5 (see section 4.2). Also, in section 4.4, the exact value of the global dimension of $S^{-1}K[H,C][X,Y;\sigma,a]$ is computed under the assumption that gld $(S^{-1}K[H,C][X,Y;\sigma,a]) < \infty$. Furthermore, we compute the global dimension of tensor product of GWAs in section 4.5. We conclude the chapter by finding the global dimension of some GWAs where the field K is not assumed to be algebraically closed.

4.1 The global dimension of GWAs $A = D(\sigma,a)$

The global dimension of GWAs $A = D(\sigma,a)$ of degree 1 were studied in a series of papers (see [42, 43, 44]). In this section, we present some known results that are used in the proofs. We start by presenting a result that gives the possible values for the left global dimension of A.

Theorem 4.1. *([44, Theorem 2.7].) Let $A = D(\sigma,a)$ be a GWA. Then* lgd $(A) =$ lgd (D)*,* lgd $(D)+1$ *or* ∞*.*

When the ring D is a commutative Noetherian ring the theorem below gives more accurate estimates for the global dimension of the generalized Weyl algebra $A = D(\sigma,a)$.

Theorem 4.2. *([44, Theorem 3.7].) Let D be a commutative Noetherian ring of global dimension $n < \infty$, $A = D(\sigma, a)$ be a GWA and the element a be a regular element of D. Suppose that* gld $(A) < \infty$. *Then* gld $(A) = n + 1$ *if and only if either there is a semistable maximal ideal of D of height n or there are maximal ideals $\mathfrak{p}$, $\mathfrak{q}$ of D of height n such that $\sigma^i(\mathfrak{p}) = \mathfrak{q}$ for some $i \neq 0 \in \mathbb{Z}$ and $a \in \mathfrak{p}, \mathfrak{q}$.*

The next result shows the global dimension of $A = D(\sigma, a)$ with D a commutative Dedekind domain.

Theorem 4.3. *([46, Theorem 1.6].) Let $A = D(\sigma, a)$ be a GWA where D is a commutative Dedekind domain and $Da = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_s^{n_s}$ is a product of distinct maximal ideals of D. Then*

$$\mathrm{gld}\ (A) = \begin{cases} \infty & \text{if } a = 0 \text{ or } n_i \geq 2 \text{ for some } i, \\ 2 & \text{if } a \neq 0,\ n_1 = \cdots = n_s = 1,\ s \geq 1 \\ & \text{or } a \text{ is invertible, and there exists an integer } k \geq 1 \\ & \text{such that either } \sigma^k(\mathfrak{p}_i) = \mathfrak{p}_j \text{ for some } i,\ j \\ & \text{or } \sigma^k(\mathfrak{q}) = \mathfrak{q} \text{ for some maximal ideal } \mathfrak{q} \text{ of } D, \\ 1 & \text{otherwise.} \end{cases}$$

If the base ring D is commutative Noetherian domain, the following result verify whether an algebra $A = D(\sigma, a)$ has infinite global dimension or not.

Theorem 4.4. *([44, Theorem 3.5].) Let $A = D(\sigma, a)$ be a GWA where D is a commutative Noetherian domain of finite global dimension and $a \neq 0$. Then* gld $(A) < \infty$ *if and only if* $\mathrm{pd}_A\,(A/A(X, \mathfrak{p})) < \infty$ *for all prime ideals $\mathfrak{p}$ of D such that $a \in \mathfrak{p}$.*

4.2 The global dimension of Witten's, Woronowicz's deformations and quantum algebras

The aim of this section is to apply Theorem 1.5 for computing the global dimension of many classical algebras given in Corollary 1.7. In this section, K is an algebraically closed field.

Examples of generalized Weyl algebras.

1. For q, $h = q - q^{-1} \in K = \mathbb{C}$ such that $q \neq \pm 1$, the algebra $U_q = U_q sl(2)$ is generated by X, Y, H_- and H_+ subject to the defining relations [44]:

$$H_+H_- = H_-H_+ = 1,\ \ XH_\pm = q^{\pm 1}H_\pm X,\ \ YH_\pm = q^{\mp 1}H_\pm Y,\ \ [X,Y] = \frac{H_+^2 - H_-^2}{h}.$$

It follows that the algebra U_q is a GWA,

$$U_q \simeq K[C,H,H^{-1}](\sigma, a = C + \big(H^2/(q^2-1) - H^{-2}/(q^{-2}-1)\big)/2h) \tag{4.1}$$

where $\sigma(H) = qH,\ \sigma(C) = C$.

Corollary 4.5. gld $(U_q sl(2)) = 3$.

Proof. Let $U = U_q sl(2)$.

(i) gld $(U) < \infty$: Since $\frac{\partial a}{\partial C} = 1$ (see (4.1)). Hence, gld $(U) < \infty$, by Theorem 1.5.

(ii) *If q is a root of unity then* gld $(U) = 3$: If $q^n = 1$ for some $n \geq 1$ then $\sigma^n = 1$. Since gld $(U) < \infty$, we must have gld $(U) = 3$, by Theorem 4.2.

(iii) *If q is not a root of unity then* gld $(U) = 3$: Since q is not a root of unity, every σ-orbit of a maximal ideal of the algebra $D = K[C,H,H^{-1}]$ is infinite. Furthermore, for all integers $i \geq 1$, $Da + D\sigma^i(a) = Da + D(\sigma^i(a) - a) = Da + D(\frac{q^{2i}-1}{q^2-1}H^2 - \frac{q^{-2i}-1}{q^{-2}-1}H^{-2}) \neq D$. Therefore, gld $(U) = 3$, by Theorem 4.2. $\square$

2. *Woronowicz's deformation* V is an algebra generated by elements V_0, V_- and V_+ subject to the defining relations, where $s \in K$ and $s^4 \neq 0, 1$ [4]:

$[V_0, V_+]_{s^2} := s^2 V_0 V_+ - s^{-2} V_+ V_0 = V_+,\ [V_-, V_0]_{s^2} = V_-,\ [V_+, V_-]_{1/s} := s^{-1}V_+V_- - sV_-V_+ = V_0$.

The algebra V is a GWA,

$$V \simeq K[u,v](\sigma, a = v),\ \ V_+ \leftrightarrow x,\ \ V_- \leftrightarrow y,\ \ V_0 \leftrightarrow u,\ \ V_-V_+ \leftrightarrow v$$

where $\sigma : u \rightarrow s^2(s^2u - 1),\ v \rightarrow s^2 v + su$, is the automorphism of the polynomial ring $K[u,v]$ in two variables u and v. Let

$$H = u + s^2/(1-s^4)\ \text{ and }\ Z = v + u/s(1-s^2) + s^3/(1-s^2)(1-s^4).$$

Then $\sigma(H) = s^4 H$, $\sigma(Z) = s^2 Z$ and $K[u,v] = K[H,Z]$. So,

$$V \simeq K[H,Z](\sigma, a = Z + \alpha H + \beta),\ \ V_+ \leftrightarrow x,\ \ V_- \leftrightarrow y,\ \ V_0 \leftrightarrow H - s^2/(1-s^4) \tag{4.2}$$

where $\sigma : H \rightarrow s^4 H,\ Z \rightarrow s^2 Z$; $\alpha = -1/s(1-s^2)$ and $\beta = s/(1-s^4)$.

Corollary 4.6. *The global dimension of Woronowicz's deformation is* 3.

Proof. By (4.2), $\frac{\partial a}{\partial Z} = 1$. Hence, gld $(V) < \infty$, by Theorem 1.5. The maximal ideal (H, Z) of the polynomial algebra $K[H, Z]$ is σ-invariant. Hence, gld $(V) = 3$, by Theorem 4.2. □

3. *Witten's first deformation* E is an algebra generated by elements E_0, E_- and E_+ subject to the defining relations [4]:

$$[E_0, E_+]_p := pE_0E_+ - p^{-1}E_+E_0 = E_+, \ \ [E_-, E_0]_p = E_-, \ \ [E_+, E_-] = E_0 - (p-1/p)E_0^2$$

where $p \in K$ and $p^4 \neq 0, 1$. The element $C = E_-E_+ + \frac{E_0(E_0+p)}{p(p^2+1)}$ is central in E. Witten's first deformation is a GWA,

$$E \simeq K[C, H](\sigma, a = C - H(H+1)/(p+p^{-1})), \ \ E_+ \leftrightarrow x, \ \ E_- \leftrightarrow y, \ \ E_0 \leftrightarrow pH$$

where $\sigma : C \to C$, $H \to p^2(H-1)$.

Let $\lambda = p^2/(p^2-1)$ and $H' = H - \lambda$. Then $K[C, H] = K[C, H']$ and

$$E \simeq K[C, H'](\sigma, a = C - \frac{(H'+\lambda)(H'+\lambda+1)}{p+p^{-1}}) \tag{4.3}$$

where $\sigma(C) = C$ and $\sigma(H') = p^2H'$.

Corollary 4.7. *The global dimension of Witten's first deformation is* 3.

Proof. By (4.3), $\frac{\partial a}{\partial C} = 1$. Hence, gld $(E) < \infty$, by Theorem 1.5. The maximal ideal (C, H') of $K[C, H']$ is σ-invariant. Hence, gld $(E) = 3$, by Theorem 4.2. □

4. *Witten's second deformation* W is an algebra generated by W_0, W_- and W_+ subject to the defining relations [4]:

$$[W_0, W_+]_r = W_+, \ \ [W_-, W_0]_r = W_-, \ \ [W_+, W_-]_{1/r^2} = W_0$$

where $r \in K$ and $r^4 \neq 0, 1$. The algebra W is a GWA [44]:

$$W \simeq K[H, C](\sigma, a = C - \alpha), \ \ W_+ \leftrightarrow X, \ \ W_- \leftrightarrow Y, \ \ W_0 \leftrightarrow H - \frac{r}{1-r^2} \tag{4.4}$$

where $\sigma(C) = r^4C$, $\sigma(H) = r^2H$ and $\alpha = (H - \frac{r}{1-r^2})(H - \frac{r^3}{1-r^2})/r^2(r+r^{-1})$.

Corollary 4.8. *The global dimension of Witten's second deformation is* 3.

Proof. By (4.4), $\frac{\partial a}{\partial C} = 1$, and so gld $(W) < \infty$, by Theorem 1.5. The maximal ideal (H, C) of $K[H, C]$ is σ-invariant. Hence, gld $(W) = 3$, by Theorem 4.2. □

5. *The quantum group*

$$\mathcal{O}_{q^2}(\mathrm{so}(K,3)) = K[H]\langle \sigma; b = (q - q^{-1})H, \rho = 1\rangle,$$

$\sigma(H) = q^2H$, $q \in K$ [32], is isomorphic to the GWA of degree 1:

$$\mathcal{O}_{q^2}(\mathrm{so}(K,3)) \simeq K[H,C](\sigma, a = C + H^2/q(1+q^2)),\ \sigma(H) = q^2H,\ \sigma(C) = C.$$

Corollary 4.9. *The global dimension of the quantum group $\mathcal{O}_{q^2}(\mathrm{so}(K,3))$ is* 3.

Proof. Since $\frac{\partial a}{\partial C} = 1$, gld $(\mathcal{O}_{q^2}(\mathrm{so}(K,3))) < \infty$, by Theorem 1.5. The maximal ideal (H, C) of $K[H, C]$ is σ-invariant. Hence, gld $(\mathcal{O}_{q^2}(\mathrm{so}(K,3))) = 3$, by Theorem 4.2. □

6. Let $\mathcal{N} = KX \oplus KY \oplus KZ$ be the 3-dimensional *Heisenberg Lie algebra* where $[X, Y] = Z$ is a central element of $\mathcal{N}$ and $\mathcal{H} = U(\mathcal{N})$ be its universal enveloping algebra. The algebra $\mathcal{H}$ is a GWA.

$$\mathcal{H} \simeq K[H,Z][X,Y;\sigma,H] \tag{4.5}$$

where $\sigma(H) = H + Z$. (Since $XH = XYX = (YX + [X,Y])X = (H + Z)X = \sigma(H)X$).

Corollary 4.10. gld $(\mathcal{H}) = 3$.

Proof. By (4.5), $\frac{\partial a}{\partial H} = 1$, and so gld $(\mathcal{H}) < \infty$, by Theorem 1.5. The maximal ideal (H, Z) of $K[H, Z]$ is σ-invariant. Hence, gld $(\mathcal{H}) = 3$, by Theorem 4.2. □

7. The *quantum Heisenberg algebra* ([11, 24, 2]):

$$\mathcal{H}_q = K\langle X, Y, H \,|\, XH = q^2HX, YH = q^{-2}HY, XY - q^{-2}YX = q^{-1}H\rangle,$$

$q \in K$, $q^4 \neq 0, 1$ is a GWA [46]:

$$\mathcal{H}_q \simeq K[H,C](\sigma, a = q^2(C - \frac{H}{q(1-q^4)})) \tag{4.6}$$

where $\sigma(H) = q^2H$ and $\sigma(C) = q^{-2}C$.

Corollary 4.11. gld $(\mathcal{H}_q) = 3$.

Proof. By (4.6), $\frac{\partial a}{\partial C} = q^2 \neq 0$, and so gld $(\mathcal{H}_q) < \infty$, by Theorem 1.5. The maximal ideal (H, C) of $K[H, C]$ is σ-invariant. Hence, gld $(\mathcal{H}_q) = 3$, by Theorem 4.2. $\square$

8. Recall that the algebra

$$Usl(2) = K\langle X, Y, H \,|\, [H, X] = X, [H, Y] = -Y, [X, Y] = 2H\rangle$$

is the universal enveloping algebra of the Lie algebra $sl(2)$ over a field of *characteristic zero* K [41]. Let us consider its deformation ([30, 40, 43]):

$$\Lambda(b) = K\langle X, Y, H \,|\, [H, X] = X,\ [H, Y] = -Y,\ [X, Y] = H\rangle \tag{4.7}$$

where $b \in K[H]$. The algebra $\Lambda(b)$ is a GWA (see [46, Example 3]):

$$\Lambda(b) \simeq K[H, C][X, Y; \sigma, a = C - \alpha] \tag{4.8}$$

where $\sigma(H) = H - 1$, $\sigma(C) = C$ and $\alpha \in K[H]$ is a solution of the equation $\alpha - \sigma(\alpha) = b$ (which exists as the map $1 - \sigma : K[H] \to K[H]$ is a locally nilpotent map that is $K[H] = \cup_{i \geq 1} \ker(1 - \sigma)^i$).

Corollary 4.12. *1.* gld $(\Lambda(0)) = 3$ *and if* $b \neq 0$ *then*

$$\text{gld } (\Lambda(b)) = \begin{cases} 3 & \text{if } \alpha(\mu) = \alpha(\mu + i) \text{ for some } \mu \in K \text{ and } i \in \mathbb{N}\backslash\{0\}, \\ 2 & \text{otherwise.} \end{cases}$$

2. gld $(Usl(2)) = 3$.

Proof. 1. By (4.8), $\frac{\partial a}{\partial C} = 1$, and so gld $(\Lambda(b)) < \infty$, by Theorem 1.5. If $b = 0$ then $a = C$ (since $0 - \sigma(0) = 0$). Then the maximal ideals (H, C) and $\sigma(H, C) = (H - 1, C)$ both contain the element a and are distinct. By Theorem 4.2, gld $(\Lambda(0)) = 3$. Suppose that $b \neq 0$. Since char $(K) = 0$, for every maximal ideal its σ-orbit is infinite. Since $a = C - \alpha$, gld $(A) = 3$ if and only if there is a maximal ideal $(C - \lambda, H - \mu)$ of $K[H, C]$ that contains a (i.e. $\lambda = \alpha(\mu)$) such that the maximal ideal

$$\sigma^i(C - \lambda, H - \mu) = (C - \lambda, H - i - \mu)$$

also contains the element a (i.e. $\lambda = \alpha(\mu + i)$) for some natural number $i \geq 1$ if and only if $\alpha(\mu) = \alpha(\mu + i)$ for some $\mu \in K$ and $i \in \mathbb{N}\backslash\{0\}$. Now, statement 1 follows from Theorem 4.2.

2. Since $b = 2H$, the element α can be chosen as $\alpha = H(H+1)$ (since $\alpha - \sigma(\alpha) = H(H+1) - (H-1)H = 2H$). Hence, gld $(Usl(2)) = 3$, by statement 1, since $\alpha(-1) = \alpha(0) = 0$. $\square$

9. In [35], it was shown that the the algebra $\mathcal{O}_q(\mathrm{SL}_2(K))$ (the coordinate ring of the quantum $\mathrm{SL}_2(K)$ where $q \in K$ is such that $q^4 \neq 0, 1$) is the GWA,

$$\mathcal{O}_q(\mathrm{SL}_2(K)) \simeq K[H,C][X,Y;\sigma, a = 1 + q^{-2}HC]$$

where $\sigma(H) = q^2H$ and $\sigma(C) = q^2C$.

Corollary 4.13. gld $(\mathcal{O}_q(\mathrm{SL}_2(K))) = 3$.

Proof. Since $a = 1+q^{-2}HC$ and $\mathrm{grad}(a) = q^{-2}(C,H)$, we have that gld $(\mathcal{O}_q(\mathrm{SL}_2(K))) < \infty$, by Theorem 1.5. The maximal ideal (H,C) of $K[H,C]$ is σ-invariant. Hence, gld $(\mathcal{O}_q(\mathrm{SL}_2(K))) = 3$, by Theorem 4.2. $\square$

4.3 Proof of Theorem 1.5

The aim of this section is to prove Theorem 1.5. The strategy of proving Theorem 1.5 is roughly as follows. A small class of left ideals of the GWA A is considered that has property that gld $(A) < \infty$ if and only if their projective dimensions are finite (Theorem 4.4). For every such left ideal an explicit projective resolution is produced (Theorem 4.23). A criterion (Theorem 4.24) is given for all such left ideal to have finite projective dimension from which Theorem 1.5 follows (using some other results). In order to produce the projective resolution in Theorem 4.23, the 4×4 matrices e and d are used, see (4.14). We study their properties first (Proposition 4.21 and Lemma 4.22).

Proposition 4.14. *Let $\mathcal{D}$ be an arbitrary ring and $A = \mathcal{D}[X,Y;\sigma,0]$ be a GWA. Then* $\mathrm{l.gld}\,(A) = \mathrm{r.gld}\,(A) = \infty$.

Proof. Let us show that $\mathrm{l.gld}\,(A) = \infty$. Using the $\mathbb{Z}$-grading of the GWA A, we obtain the following short exact sequences of left A-modules:

$$0 \to AX \xrightarrow{\alpha} A \xrightarrow{\varphi} AY \to 0, \quad \alpha(u) = u, \ \ \varphi(v) = vY, \tag{4.9}$$

$$0 \to AY \xrightarrow{\beta} A \xrightarrow{\psi} AX \to 0, \quad \beta(u) = u, \ \ \psi(v) = vX. \tag{4.10}$$

Notice that $AX = \oplus_{i\geq 1}\mathcal{D}X^i$ and $AY = \oplus_{i\geq 1}\mathcal{D}Y^i$ (since $a = 0$). The short exact sequence (4.9) does not split since otherwise we would have a splitting homomorphism $\chi : A \to AX$, $\chi\alpha = \mathrm{id}$ where id is the identity map on AX. Then $X = \chi\alpha(X) = \chi(X) = X\chi(1) \in X \cdot AX = \oplus_{i\geq 2}\mathcal{D}X^i$, a contradiction.

By a similar reason, the short exact sequence (4.10) does not split. So, the left A-modules AX and AY are not projective. Therefore, $\mathrm{l.gld}\,(A) = \infty$. Since the opposite algebra A^{op} to the GWA A is the GWA $\mathcal{D}^{op}[Y, X; \sigma, 0]$, we have that $\mathrm{r.gld}\,(A) = \mathrm{l.gld}\,(A^{op}) = \mathrm{l.gld}\,(\mathcal{D}^{op}[Y, X; \sigma, 0]) = \infty$. $\square$

Lemma 4.15. *([46, Lemma 7.3].) Let $A = D[X, Y; \sigma, a]$ be a GWA where D is a commutative ring, a is a regular element which is not a unit and $\mathfrak{p}$ be an ideal of D containing a.*

1. *The following sequences of A-modules are exact:*

$$0 \to A(\sigma(\mathfrak{p}), Y) \xrightarrow{\varphi} A\mathfrak{p} \oplus A \xrightarrow{\psi} A(\mathfrak{p}, X) \to 0 \tag{4.11}$$

where $\varphi(\omega) = (\omega X, \omega)$ and $\psi(u, v) = u - vX$,

$$0 \to A(\mathfrak{p}, X) \xrightarrow{\varphi} A\sigma(\mathfrak{p}) \oplus A \xrightarrow{\psi} A(\sigma(\mathfrak{p}), Y) \to 0 \tag{4.12}$$

where $\varphi(\omega) = (\omega Y, \omega)$ and $\psi(u, v) = u - vY$.

2. *The sequence (4.11) (or (4.12)) splits if and only if $\mathfrak{p}(1 - p_0) \subseteq Da$ for some element $p_0 \in \mathfrak{p}$.*

3. *If, in addition, D is a Dedekind domain and $\mathfrak{p}$ is a maximal ideal of D then the sequence (4.11) (or (4.12)) does not split if and only if $a \in \mathfrak{p}^2$.*

Till the end of the section *let $D = S^{-1}K[H, C]$ be a localization of the polynomial algebra $K[H, C]$ at a multiplicative set S.* If $S = \{1\}$ then $D = K[H, C]$. Let D^* be the group of units of D. The polynomial algebra $K[H, C]$ is a *unique factorization domain* (UFD). Hence, so is the algebra D. A set of ideals $\{\mathfrak{a}_i \,|\, i \in I\}$ of a ring R is called a *co-maximal* set of ideals if $\mathfrak{a}_i + \mathfrak{a}_j = R$ for all $i \neq j$.

Definition 4.16. *We say that an element $a \in D\backslash D^*$ is* co-maximal *if the ideal $Da = \mathfrak{p}_1 \cdots \mathfrak{p}_s$ is a product of co-maximal height 1 ideals $\mathfrak{p}_1, \dots, \mathfrak{p}_s$, i.e. either $s = 1$ or otherwise, $\mathfrak{p}_i + \mathfrak{p}_j = D$ for all $i \neq j$.*

Proposition 4.17. *Let an algebra D be a localization of the polynomial algebra $K[H,C]$ of Krull dimension 2. Suppose that $a \notin D^*$. Then $Da = \mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_s^{n_s}$ $(s \geq 1)$ is a unique product of height 1 ideals $\mathfrak{p}_i$ of the algebra D (where n_i is the multiplicity of $\mathfrak{p}_i$) and the following statements are equivalent.*

1. $\mathrm{pd}_A\, A(\mathfrak{p}_i, X) < \infty$.
2. *The left ideal $A(\mathfrak{p}_i, X)$ of the algebra A is a projective left A-module.*
3. *The short exact sequence (4.11) splits where $\mathfrak{p} = \mathfrak{p}_i$.*
4. *$n_i = 1$ and $\mathfrak{p}_i + \mathfrak{p}_j = D$ for all $j \neq i$.*

Proof. $(1 \Leftrightarrow 2 \Leftrightarrow 3)$ Let $\mathfrak{p} = \mathfrak{p}_i$. The polynomial algebra $K[H,C]$ is a unique factorization domain, hence so is the algebra D. So, $\mathfrak{p} = Dp$ for an element $p \in \mathfrak{p}$. Therefore, $A\mathfrak{p} \simeq A \simeq A\sigma(\mathfrak{p})$, as left A-modules. In particular, the middle terms of the short exact sequences (4.11) and (4.12) are free/projective A-modules. Now $1 \Leftrightarrow 2 \Leftrightarrow 3$, by Lemma 4.15.(1).

$(3 \Leftrightarrow 4)$ By Lemma 4.15.(2), the short exact sequence (4.11) splits for $\mathfrak{p} = \mathfrak{p}_i = Dp$ if and only if $1 - p_0 \in Dap^{-1}$ for some element $p_0 \in \mathfrak{p}$ if and only if the element $ap^{-1} + \mathfrak{p}$ of the factor ring $D/\mathfrak{p}$ is a unit if and only if $n_i = 1$ and all the elements $p_j + \mathfrak{p}$ $(j \neq i)$ are units of $D/\mathfrak{p}$ (where $\mathfrak{p}_j = Dp_j$ for some element $p_j \in \mathfrak{p}_j$) if and only if $n_i = 1$ and $\mathfrak{p}_i + \mathfrak{p}_j = D$ for all $j \neq i$. □

The next corollary is a 'dual' version of Proposition 4.17.

Corollary 4.18. *Suppose that D and a be as in Proposition 4.17. Then $Da = \mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_s^{n_s}$ $(s \geq 1)$ is a unique product of height 1 ideals $\mathfrak{p}_i$ of the algebra D and the following statements are equivalent.*

1. $\mathrm{pd}_A\, A(\sigma(\mathfrak{p}_i), Y) < \infty$.
2. *The left ideal $A(\sigma(\mathfrak{p}_i), Y)$ of the algebra A is a projective left A-module.*
3. *The short exact sequence (4.12) splits where $\mathfrak{p} = \mathfrak{p}_i$.*
4. *$n_i = 1$ and $\mathfrak{p}_i + \mathfrak{p}_j = D$ for all $j \neq i$.*

Proof. $(1 \Leftrightarrow 2 \Leftrightarrow 3)$ Let $\mathfrak{p} = \mathfrak{p}_i$. The algebra D is a unique factorization domain. So, $\mathfrak{p} = Dp$ for an element $p \in \mathfrak{p}$. Therefore, $A\mathfrak{p} \simeq A \simeq A\sigma(\mathfrak{p})$, as left A-modules. In particular, the middle terms of the short exact sequences (4.11) and (4.12) are free/projective A-modules. Now $1 \Leftrightarrow 2 \Leftrightarrow 3$, by Lemma 4.15.(1).

($3 \Leftrightarrow 4$) By Lemma 4.15.(1), the short exact sequence (4.12) splits if and only if the short exact sequence (4.11) does. Now, the equivalence ($3 \Leftrightarrow 4$) follows from the equivalence ($3 \Leftrightarrow 4$) of Proposition 4.17. $\square$

Corollary 4.19. *Suppose that the element $a \in D\backslash D^*$ is co-maximal and an element $d \in D\backslash D^*$ is a divisor of a. Let $\mathfrak{p} = Dd$. Then the left ideals $A(\mathfrak{p}, X)$ and $A(\sigma(\mathfrak{p}), Y)$ of the GWA $A = D(\sigma, a)$ are projective A-modules.*

Proof. Since the element a is co-maximal, the ring D is a unique factorization domain and $d|a$, the ideal $\mathfrak{p}$ is a product of co-maximal height 1 prime ideals, say $\mathfrak{p} = \mathfrak{p}_1 \cdots \mathfrak{p}_t$. Then $D/\mathfrak{p} = D/\mathfrak{p}_1 \cdots \mathfrak{p}_t = D/\cap_{i=1}^t \mathfrak{p}_i \simeq \prod_{i=1}^t D/\mathfrak{p}_i$ since the ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_t$ are co-maximal. Let $\mathfrak{p}_1 = Dp_1, \ldots, \mathfrak{p}_t = Dp_t$ for some elements $p_1 \in \mathfrak{p}_1, \ldots, p_t \in \mathfrak{p}_t$. Since a is co-maximal and $d|a$, we have that the image of the element $b = ad^{-1} \in D$ in the factor ring $D/\mathfrak{p}$ is a unit. Therefore, $cb = 1 - p_0$ for some elements $c \in D$ and $p_0 \in \mathfrak{p}$, i.e. $d(1 - p_0) = ca \in Da$, and so $\mathfrak{p}(1 - p_0) \subseteq Da$. By Lemma 4.15.(2), the short exact sequence (4.11) splits. Therefore, the left ideals $A(\mathfrak{p}, X)$ and $A(\sigma(\mathfrak{p}), Y)$ are projective A-modules. $\square$

Lemma 4.20. *Let $a \in D\backslash D^*$ be a co-maximal element and $\mathfrak{m} = (H - \alpha, C - \beta)$ be a maximal ideal of D such that $a \in \mathfrak{m}$ where $\alpha, \beta \in K$. Then $a = a_1(H - \alpha) + a_2(C - \beta)$ for some elements $a_1, a_2 \in D$ and for arbitrary choice of the elements a_1 and a_2 either $(C - \beta) \nmid a_1$ or $(H - \alpha) \nmid a_2$.*

Proof. Since $\mathfrak{m} = (H - \alpha, C - \beta)$ is a maximal ideal the elements $H - \alpha$ and $C - \beta$ are not units. Suppose for some choice of the elements a_1 and a_2, $(C - \beta)|a_1$ and $(H - \alpha)|a_2$, we seek a contradiction. Then $a = (H - \alpha)(C - \beta)b$ for some element b. The element a is co-maximal. So, $D = D(H - \alpha) + D(C - \beta) = \mathfrak{m} \neq D$, a contradiction. $\square$

The matrices e and d. Suppose that $a \in D\backslash D^*$, $\mathfrak{m} = (H - \alpha, C - \beta)$ is a maximal ideal of D such that $a \in \mathfrak{m}$ where α, $\beta \in K$, i.e. $a = a_1(H - \alpha) + a_2(C - \beta)$ for some elements $a_1, a_2 \in D$. We denote by $M_n(D)$ the algebra of all $n \times n$ matrices with entries in D. The automorphism σ of the algebra D can be extended to an automorphism of the algebra $M_n(D)$ by the rule: For a matrix $(a_{ij}) \in M_n(D)$, $\sigma((a_{ij})) = (\sigma(a_{ij}))$. Let

$$\mathcal{A} = \begin{pmatrix} a_1 & a_2 \\ C - \beta & -(H - \alpha) \end{pmatrix} \in M_2(D).$$

Then $\det(\mathcal{A}) = -(a_1(H-\alpha)+a_2(C-\beta)) = -a$ and

$$\mathcal{A}^{-1} = -\frac{1}{a}\begin{pmatrix} -(H-\alpha) & -a_2 \\ -(C-\beta) & a_1 \end{pmatrix} \in M_2(D_a)$$

where $D_a = S_a^{-1}D$ and $S_a = \{a^i \mid i \geq 0\}$.

The matrix $\tilde{\mathcal{A}} = -\det(\mathcal{A})\,\mathcal{A}^{-1} = \begin{pmatrix} H-\alpha & a_2 \\ C-\beta & -a_1 \end{pmatrix}$ satisfies the property that

$$\mathcal{A}\tilde{\mathcal{A}} = \tilde{\mathcal{A}}\mathcal{A} = a \quad \text{where} \quad a = a\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.13}$$

Let us consider the following matrices of $M_4(A)$:

$$\begin{aligned} e &= \begin{pmatrix} a_1 & a_2 & -Y & 0 \\ C-\beta & -(H-\alpha) & 0 & -Y \\ X & 0 & -\sigma(H-\alpha) & -\sigma(a_2) \\ 0 & X & -\sigma(C-\beta) & \sigma(a_1) \end{pmatrix} \quad \text{and} \\ d &= \begin{pmatrix} H-\alpha & a_2 & -Y & 0 \\ C-\beta & -a_1 & 0 & -Y \\ X & 0 & -\sigma(a_1) & -\sigma(a_2) \\ 0 & X & -\sigma(C-\beta) & \sigma(H-\alpha) \end{pmatrix}. \end{aligned} \tag{4.14}$$

The matrices e and d can be written as 2×2 matrices with entries in 2×2 matrices as follows

$$e = \begin{pmatrix} \mathcal{A} & -Y \\ X & -\sigma(\tilde{\mathcal{A}}) \end{pmatrix} \text{ and } d = \begin{pmatrix} \tilde{\mathcal{A}} & -Y \\ X & -\sigma(\mathcal{A}) \end{pmatrix}$$

where $X = \begin{pmatrix} X & 0 \\ 0 & X \end{pmatrix}$ and $Y = \begin{pmatrix} Y & 0 \\ 0 & Y \end{pmatrix}$. Let $A^4 = \{(u_1,u_2,u_3,u_4) \mid u_i \in A\}$ be a free left A-module of rank 4 and the set $\{e_1 = (1,0,0,0),\dots,e_4 = (0,0,0,1)\}$ be its canonical basis. The matrices e and d determine A-homomorphisms of A^4 by the rule

$$e : A^4 \to A^4, \quad u = (u_1,\dots,u_4) \mapsto ue,$$

$$d : A^4 \to A^4, \quad u = (u_1,\dots,u_4) \mapsto ud.$$

In particular, $\ker(e) = \{u \in A^4 \mid ue = 0\}$, $\operatorname{im}(e) = A^4 e$, $\ker(d) = \{u \in A^4 \mid ud = 0\}$ and $\operatorname{im}(d) = A^4 d$.

Proposition 4.21. *1. $ed = 0$ and $de = 0$.*

2. $\ker(e) = \mathrm{im}(d)$ *and* $\ker(d) = \mathrm{im}(e)$.

Proof. 1.

$$\begin{aligned} ed = \begin{pmatrix} \mathcal{A} & -Y \\ X & -\sigma(\tilde{\mathcal{A}}) \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{A}} & -Y \\ X & -\sigma(\mathcal{A}) \end{pmatrix} &= \begin{pmatrix} \mathcal{A}\tilde{\mathcal{A}} - YX & -\mathcal{A}Y + Y\sigma(\mathcal{A}) \\ X\tilde{\mathcal{A}} - \sigma(\tilde{\mathcal{A}})X & -XY + \sigma(\tilde{\mathcal{A}}\mathcal{A}) \end{pmatrix} \\ &= \begin{pmatrix} a - a & 0 \\ 0 & -\sigma(a) + \sigma(a) \end{pmatrix} = 0, \end{aligned}$$

$$\begin{aligned} de = \begin{pmatrix} \tilde{\mathcal{A}} & -Y \\ X & -\sigma(\mathcal{A}) \end{pmatrix} \begin{pmatrix} \mathcal{A} & -Y \\ X & -\sigma(\tilde{\mathcal{A}}) \end{pmatrix} &= \begin{pmatrix} \tilde{\mathcal{A}}\mathcal{A} - YX & -\tilde{\mathcal{A}}Y + Y\sigma(\tilde{\mathcal{A}}) \\ X\mathcal{A} - \sigma(\mathcal{A})X & -XY + \sigma(\mathcal{A}\tilde{\mathcal{A}}) \end{pmatrix} \\ &= \begin{pmatrix} a - a & 0 \\ 0 & -\sigma(a) + \sigma(a) \end{pmatrix} = 0. \end{aligned}$$

2. (i) $\ker(e) = \mathrm{im}(d)$: $\ker(e) = \{(w_1, w_2) \in A^2 \times A^2 \mid (w_1, w_2)e = 0\}$. Notice that $0 = (w_1, w_2)e = (w_1, w_2) \begin{pmatrix} \mathcal{A} & -Y \\ X & -\sigma(\tilde{\mathcal{A}}) \end{pmatrix} \Leftrightarrow \begin{cases} w_1\,\mathcal{A} + w_2\,X = 0 \\ w_1\,Y + w_2\,\sigma(\tilde{\mathcal{A}}) = 0 \end{cases} \Leftrightarrow w_1\,\mathcal{A} + w_2\,X = 0,$

since the second equation is redundant. Indeed, the map

$$\cdot\,\sigma(\mathcal{A}) : A^2 \to A^2,\ w \mapsto w\sigma(\mathcal{A})$$

is a monomorphism since $\sigma(\mathcal{A})\,\sigma(\tilde{\mathcal{A}}) = \sigma(\mathcal{A}\tilde{\mathcal{A}}) = \sigma(a)E$ where $E = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$. So, by applying the map $\cdot\,\sigma(\mathcal{A})$ to the second equation we have that

$$\begin{aligned} 0 &= w_1\,Y\sigma(\mathcal{A}) + w_2\,\sigma(\mathcal{A}\tilde{\mathcal{A}}) \\ &= w_1\,\mathcal{A}Y + w_2\,\sigma(a)E \\ &= w_1\,\mathcal{A}Y + w_2\,XY \\ &= (w_1\,\mathcal{A} + w_2\,X)Y \\ &\Leftrightarrow\ w_1\,\mathcal{A} + w_2\,X = 0 \end{aligned}$$

since the map $\cdot Y : A^2 \to A^2$, $w \mapsto wY$ is a monomorphism. Therefore,

$$\ker(e) = \{(w_1, w_2) \in A^2 \times A^2 \mid w_1\,\mathcal{A} + w_2\,X = 0\}.$$

The algebra A contains subalgebras $A_- = \oplus_{i\geq 0} DY^i$ and $A_+ = \oplus_{i\geq 0} DX^i$. Clearly, $A = A_- \oplus A_+X = A_-Y \oplus A_+$. So, the elements w_1, $w_2 \in A^2$ can be uniquely written as follows: $w_1 = w_{1,+}\,X + w_{1,-}$ and $w_2 = w_{2,+} + w_{2,-}\,Y$ for some elements $w_{1,+},\, w_{2,+} \in A_+$ and $w_{1,-},\, w_{2,-} \in A_-$. Now,

$$\begin{aligned} 0 &= w_1\,\mathcal{A} + w_2\,X = (w_{1,+}\,X + w_{1,-})\mathcal{A} + (w_{2,+} + w_{2,-}\,Y)X \\ &= (w_{1,+}\,\sigma(\mathcal{A}) + w_{2,+})X + (w_{1,-} + w_{2,-}\,\tilde{\mathcal{A}})\mathcal{A}. \end{aligned}$$

Hence, using the $\mathbb{Z}$-grading of the GWA A we see that $(w_{1,+}\,\sigma(\mathcal{A}) + w_{2,+})X = 0$ and $(w_{1,-}+w_{2,-}\,\tilde{\mathcal{A}})\mathcal{A} = 0$. Equivalently, $w_{1,+}\,\sigma(\mathcal{A})+w_{2,+} = 0$ and $w_{1,-}+w_{2,-}\,\tilde{\mathcal{A}} = 0$ (since the maps $\cdot X$ and $\cdot\mathcal{A}$ are injections). Finally,

$$\begin{aligned} (w_1, w_2) &= (w_{1,+}\,X - w_{2,-}\,\tilde{\mathcal{A}}, -w_{1,+}\,\sigma(\mathcal{A}) + w_{2,-}\,Y) \\ &= (w_{1,+}\,X, -w_{1,+}\,\sigma(\mathcal{A})) - (w_{2,-}\,\tilde{\mathcal{A}}, w_{2,-}\,(-Y)). \end{aligned}$$

This means that the element (w_1, w_2) of $\ker(e)$ is an element of the A-submodule of A^4 generated by the 4 rows of the 4×4 matrix $\begin{pmatrix} \tilde{\mathcal{A}} & -Y \\ X & -\sigma(\mathcal{A}) \end{pmatrix} = d$. Therefore, $(w_1, w_2) \in \mathrm{im}(d)$, i.e. $\ker(e) \subseteq \mathrm{im}(d)$. By statement 1, $ed = 0$, and so we have $\mathrm{im}(d) \subseteq \ker(e)$. Therefore, $\ker(e) = \mathrm{im}(d)$.

(ii) $\ker(d) = \mathrm{im}(e)$: The statement (ii) follows from the statement (i) in view of $(\mathcal{A}, \tilde{\mathcal{A}})$-symmetry: The matrix d is obtained from the matrix e by replacing $\mathcal{A}$ by $\tilde{\mathcal{A}}$, and vice versa. The facts used in the proof of statement (i), which are the equality (4.13) and statement 1, are also $(\mathcal{A}, \tilde{\mathcal{A}})$-symmetrical. □

Lemma 4.22. *We keep the notations of Proposition 4.21. In the equalities below 1 stands for* $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ *and the three matrices in the middle in statements 1 and 2 are equal to the three matrices on the RHS, respectively.*

1. $e = e_-\,e_0\,e_+ = \begin{pmatrix} \mathcal{A} & 0 \\ X & X \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & -\mathcal{A}^{-1}Y \\ 0 & Y \end{pmatrix}.$

2. $d = d_-\,d_0\,d_+ = \begin{pmatrix} Y & Y \\ 0 & \sigma(\mathcal{A}) \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} X & 0 \\ \sigma(\mathcal{A}^{-1})X & -1 \end{pmatrix}.$

3. $d_+\,e_- = \begin{pmatrix} X\mathcal{A} & 0 \\ 0 & -X \end{pmatrix}$ *and* $e_+\,d_- = \begin{pmatrix} Y & 0 \\ 0 & Y\sigma(\mathcal{A}) \end{pmatrix}.$

Proof. Straightforward. □

Let R be a ring and M be an R-module. A projective resolution of the R-module M of the type

$$\cdots \xrightarrow{f_0} P_{n-1} \xrightarrow{f_{n-1}} \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} P_{n-1} \xrightarrow{f_{n-1}} P_{n-2} \to \cdots \to P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M \to 0$$

is called an *n-periodic projective resolution* or a *projective resolution of period n*. A projective resolution of the R-module M is called an *eventually n-periodic projective resolution* if it becomes n-periodic at certain point. An A-module M admits an n-periodic projective resolution if and only if there is an exact sequence of the type

$$0 \to M \to P_{n-1} \to \cdots \to P_0 \to M \to 0$$

where $P_0, \ldots, P_{n-1}$ are projective A-modules.

A projective resolution for the left ideal $A(\sigma(\mathfrak{m}), Y)$. The next theorem gives such a resolution.

Theorem 4.23. *Let an algebra D be a localization of the polynomial algebra $K[H, C]$ of Krull dimension 2. Suppose that the element $a \in D \backslash D^*$ is co-maximal, $\mathfrak{m} = (H - \alpha, C - \beta)$ is a maximal ideal of D such that $a \in \mathfrak{m}$ where $\alpha,\beta \in K$, i.e. $a = a_1(H - \alpha) + a_2(C - \beta)$ for some elements a_1, $a_2 \in D$. Then*

$$\cdots \to A^4 \xrightarrow{d} A^4 \xrightarrow{e} A^4 \xrightarrow{d} A^4 \xrightarrow{e} A^4 \xrightarrow{f} A^3 \xrightarrow{g} A(\sigma(\mathfrak{m}), Y) \to 0 \qquad (4.15)$$

is an eventually 2-periodic projective resolution of the left ideal $A(\sigma(\mathfrak{m}), Y)$ of the GWA A where the maps/matrices e and d are as in Proposition 4.21, $f(u_1, u_2, u_3, u_4) = (u_1 Y + u_3\, \sigma(a_1) + u_4\, \sigma(C - \beta), u_2 Y + u_3\, \sigma(a_2) - u_4\, \sigma(H - \alpha), u_1(H-\alpha)+u_2(C-\beta)+u_3 X)$ and $g(v_1, v_2, v_3) = v_1\, \sigma(H-\alpha)+v_2\, \sigma(C-\beta)-v_3 Y$.

Proof. (i) *g is an epimorphism* (since $A(\sigma(\mathfrak{m}), Y) = A\, \sigma(H-\alpha)+A\, \sigma(C-\beta)+AY$).

(ii) $gf = 0$: Let the set of elements $e_1 = (1, 0, 0, 0), \ldots, e_4 = (0, 0, 0, 1)$ be the canonical basis of the free A-module $A^4 = \oplus_{i=1}^4 Ae_i$. Then

$$\begin{aligned}
gf(e_1) &= Y\sigma(H - \alpha) - (H - \alpha)Y = 0,\\
gf(e_2) &= Y\sigma(C - \beta) - (C - \beta)Y = 0,\\
gf(e_3) &= \sigma(a_1)\, \sigma(H - \alpha) + \sigma(a_2)\, \sigma(C - \beta) - XY\\
&= \sigma(a_1(H - \alpha) + a_2(C - \beta)) - \sigma(a)\\
&= \sigma(a) - \sigma(a) = 0,\\
gf(e_4) &= \sigma(C - \beta)\, \sigma(H - \alpha) - \sigma(H - \alpha)\, \sigma(C - \beta) = 0.
\end{aligned}$$

(iii) $\ker(g) \subseteq \operatorname{im}(f)$: Recall that the elements $e_1, \ldots, e_4$ are the canonical free generators of the left free A-module A^4. Consider their images under the homomorphism f,

$$\begin{aligned} e_1' &= f(e_1) = (Y, 0, H-\alpha), \\ e_2' &= f(e_2) = (0, Y, C-\beta), \\ e_3' &= f(e_3) = (\sigma(a_1), \sigma(a_2), X), \\ e_4' &= f(e_4) = (\sigma(C-\beta), -\sigma(H-\alpha), 0). \end{aligned}$$

By Lemma 4.20, either $(C-\beta) \nmid a_1$ or $(H-\alpha) \nmid a_2$. Up to interchanging the elements H and C, we may assume that $(C-\beta) \nmid a_1$. In this case, $a_1 \neq 0$. Furthermore, we can choose the element a_1 from the set $D^*K[H]\backslash\{0\} := \{dp \,|\, d \in D^*,\, p \in K[H]\backslash\{0\}\}$. Let an element $v = (v_1, v_2, v_3) \in A^3$ belong to $\ker(g)$,

$$g(v) = v_1\, \sigma(H-\alpha) + v_2\, \sigma(C-\beta) - v_3\, Y = 0. \tag{4.16}$$

Notice that $A = AY + \sum_{i\geq 0} X^i D$. Then the element v_1 (respectively, v_2) up to addition of an element of Ae_1' (respectively, Ae_2') is a sum $v_1 = \sum_{i\geq 0} X^i\sigma(\alpha_i)$ (respectively, $v_2 = \sum_{i\geq 0} X^i\sigma(\beta_i)$) for some elements $\alpha_i \in D$ (respectively, $\beta_i \in D$). Furthermore, up to addition of an element of the left ideal $Ae_3' + Ae_4'$ we can assume that all $\alpha_i \in K[H]$ satisfy $\deg_H(\alpha_i) < d_1$ where d_1 is the dimension of the factor algebra

$$D/(D\,\sigma(a_1) + D\,\sigma(C-\beta)) = \sigma(D/(Da_1 + D(C-\beta)) = \sigma(\oplus_{i=0}^{d_1-1} KH^i)$$

and $\sigma(\alpha_i)$ is defined up to $D\,\sigma(a_1) + D\,\sigma(C-\beta)$. By (4.16) and $v_1, v_2 \in \sum_{i\geq 0} X^i D$, we have $v_3 \in \sum_{i\geq 1} X^i D$, i.e. $v_3 = \sum_{i\geq 0} X^{i+1}\gamma_i$ for some elements $\gamma_i \in D$, and

$$\sigma(\alpha_i)\,\sigma(H-\alpha) + \sigma(\beta_i)\,\sigma(C-\beta) = \sigma(a)\,\sigma(\gamma_i) \quad \text{for} \quad i \geq 0. \tag{4.17}$$

Replacing the element a in (4.17) by the sum $a = a_1(H-\alpha) + a_2(C-\beta)$ and then taking the result modulo $D\sigma(C-\beta)$ we obtain the equality

$$(\sigma(\alpha_i) - \sigma(a_1)\,\sigma(\gamma_i))\,\sigma(H-\alpha) \equiv 0 \mod D\sigma(C-\beta)$$

in the domain $D/D\sigma(C-\beta)$. Hence, the first factor must be zero. This means that

$$\sigma(\alpha_i) \in D\,\sigma(a_1) + D\,\sigma(C-\beta).$$

Hence, $v_1 \in Ae_3' + Ae_4' \in \operatorname{im}(f)$. So, we can assume that $v_1 = 0$ (all $\alpha_i = 0$). Then the equalities (4.17) take the form after applying σ^{-1}:

$$\beta_i(C-\beta) = a\gamma_i \quad \text{for} \quad i \geq 0. \tag{4.18}$$

Since the element a is co-maximal and $a_1 \in D^*K[H]\backslash\{0\}$, $(C-\beta) \nmid a$ (since otherwise we would have $(C-\beta)|a_1(H-\alpha)$, a contradiction as $(C-\beta) \nmid a_1$ and $(C-\beta) \nmid (H-\alpha)$). By (4.18), $(C-\beta)|\gamma_i$ for $i \geq 0$, and so $\beta_i = \gamma_i' a = \gamma_i' YX$ where $\gamma_i' = (C-\beta)^{-1}\gamma_i \in D$. Now,

$$\begin{aligned} v = (0, v_2, v_3) &= (0, \sum_{i\geq 0} X^i \sigma(\gamma_i')\, \sigma(a), \sum_{i\geq 0} X^{i+1}\gamma_i'(C-\beta)) \\ &= \left(\sum_{i\geq 0} X^i\sigma(\gamma_i')\right)(0, \sigma(a), \sigma(C-\beta)X) \in \mathrm{im}(f) \end{aligned}$$

since

$$\begin{aligned} (0, \sigma(a), \sigma(C-\beta)X) &= \sigma(C-\beta)\, e_3' - (\sigma(C-\beta)\,\sigma(a_1), \sigma(C-\beta)\,\sigma(a_2) - \sigma(a), 0) \\ &= \sigma(C-\beta)\, e_3' - \sigma(a_1)\, e_4' \in \mathrm{im}(f) \end{aligned}$$

as $\sigma(C-\beta)\,\sigma(a_2) - \sigma(a) = -\sigma(H-\beta)\,\sigma(a_1)$. The proof of statement (iii) is complete.

(iv) $\ker(f) = A\theta_1 + A\theta_2 + A\eta_1 + A\eta_2 = \mathrm{im}(e)$ *where* $\theta_1 = (a_1, a_2, -Y, 0)$, $\theta_2 = (C-\beta, -(H-\alpha), 0, -Y)$, $\eta_1 = (X, 0, -\sigma(H-\alpha), -\sigma(a_2))$ *and* $\eta_2 = (0, X, -\sigma(C-\beta), \sigma(a_1))$ *are the rows of the matrix* e: It can be easily verified that the elements $\theta_1, \theta_2, \eta_1$ and η_2 belong to $\ker(f)$. An element $u = (u_1, u_2, u_3, u_4) \in A^4$ belongs to $\ker(f)$ if and only if

$$\begin{cases} u_1\, Y = -u_3\,\sigma(a_1) - u_4\,\sigma(C-\beta), \\ u_2\, Y = -u_3\,\sigma(a_2) + u_4\,\sigma(H-\alpha), \\ u_1(H-\alpha) + u_2(C-\beta) + u_3\, X = 0. \end{cases}$$

Then

$$\begin{aligned} u_1\, a &= u_1\, YX = -u_3\,\sigma(a_1)X - u_4\,\sigma(C-\beta)X = -u_3\, Xa_1 - u_4\, X(C-\beta), \\ u_2\, a &= u_2\, YX = -u_3\,\sigma(a_2)X + u_4\,\sigma(H-\alpha)X = -u_3\, Xa_2 + u_4\, X(H-\alpha). \end{aligned}$$

Hence, $u_1 = -u_3\, Xa_1a^{-1} - u_4\, X(C-\beta)\, a^{-1}$ and $u_2 = -u_3\, Xa_2a^{-1} + u_4\, X(H-\alpha)\, a^{-1}$. Now, the third equation of the system is redundant:

$$\begin{aligned} u_1(H-\alpha) + u_2(C-\beta) + u_3\, X &= -u_3X((a_1(H-\alpha) + a_2(C-\beta))\, a^{-1} - 1) \\ &\quad -u_4\, X((C-\beta)(H-\alpha) - (H-\alpha)(C-\beta))\, a^{-1} \\ &= -u_3X(aa^{-1} - 1) = 0. \end{aligned}$$

So, the third equation can be dropped. The elements u_3 and u_4 are unique sums

$$u_3 = \sum_{i\geq 1}\gamma_{-i}Y^i + \sum_{i\geq 0} X^i\,\sigma(\gamma_i),\ u_4 = \sum_{i\geq 1}\delta_{-i}Y^i + \sum_{i\geq 0} X^i\,\sigma(\delta_i)$$

where $\gamma_i, \delta_i \in D$ and $i \in \mathbb{Z}$. Now,

$$\begin{aligned}
u_1 &= -u_3\, Xa_1a^{-1} - u_4\, X(C-\beta)\, a^{-1} \\
&= -\sum_{i\geq 1}(\gamma_{-i}Y^{i-1}a_1 + \delta_{-i}Y^{i-1}(C-\beta)) - \sum_{i\geq 0} X^{i+1}(\gamma_i a_1 + \delta_i(C-\beta))\, a^{-1}, \\
u_2 &= -u_3\, Xa_2a^{-1} + u_4\, X(H-\alpha)\, a^{-1} \\
&= -\sum_{i\geq 1}(\gamma_{-i}Y^{i-1}a_2 - \delta_{-i}Y^{i-1}(H-\alpha)) - \sum_{i\geq 0} X^{i+1}(\gamma_i a_2 - \delta_i(H-\alpha))\, a^{-1}.
\end{aligned}$$

The conditions that $u_1 \in A$ and $u_2 \in A$ are equivalent to the following conditions: For all $i \geq 0$,

$$\gamma_i a_1 + \delta_i(C-\beta) \in Da, \tag{4.19}$$

$$\gamma_i a_2 - \delta_i(H-\alpha) \in Da. \tag{4.20}$$

By multiplying the first inclusion by $H-\alpha$ and the second one by $C-\beta$ and taking their sum we obtain that

$$a\mathfrak{m} \ni \gamma_i(a_1(H-\alpha) + a_2(C-\beta)) = \gamma_i a,$$

and so $\gamma_i \in \mathfrak{m}$. So,

$$\gamma_i = \gamma_i'(H-\alpha) + \gamma_i''(C-\beta) \quad \text{for some elements} \quad \gamma_i',\ \gamma_i'' \in D. \tag{4.21}$$

(iv-1) *For all $i \geq 0$, $\delta_i = \gamma_i' a_2 - \gamma_i'' a_1$*: Let $\Delta_i := \delta_i - (\gamma_i' a_2 - \gamma_i'' a_1)$. Suppose that $\Delta_i \neq 0$, we seek a contradiction. Notice that $(C-\beta)\, a_2 = a - a_1(H-\alpha) \equiv -a_1(H-\alpha) \mod Da$. By taking (4.20) modulo Da and using (4.21), we obtain the equality

$$0 \equiv (\gamma_i'(H-\alpha) + \gamma_i''(C-\beta))\, a_2 - \delta_i(H-\alpha) = -\Delta_i(H-\alpha).$$

Notice that $(H-\alpha)\, a_1 = a - a_2(C-\beta) \equiv -a_2(C-\beta) \mod Da$. Similarly, by taking (4.19) modulo Da and using (4.21), we obtain the equality

$$0 \equiv (\gamma_i'(H-\alpha) + \gamma_i''(C-\beta))\, a_1 + \delta_i(C-\beta) = \Delta_i(C-\beta).$$

So, we have $\Delta_i(H-\alpha) = p_i a$, $\Delta_i(C-\beta) = q_i a$ for some nonzero elements p_i and q_i (since $\Delta_i \neq 0$). Since $a = a_1(H-\alpha) + a_2(C-\beta)$, the first (respectively, second) equality yields $(H-\alpha)|a_2$ (respectively, $(C-\beta)|a_1$). This means that $a = (H-\alpha)(C-\beta)\, a'$ for some $a' \in D\backslash\{0\}$. The element a is co-maximal. So, $D = D(H-\alpha) + D(C-\beta) = \mathfrak{m}$, a contradiction. Therefore, $\Delta_i = 0$, the proof of statement (iv-1) is complete.

(iv-2) *For all* $i \geq 1$, $(\gamma_i a_2 - \delta_i(H-\alpha))\, a^{-1} = \gamma_i''$ *and* $(\gamma_i a_1 + \delta_i(C-\beta))\, a^{-1} = \gamma_i'$:

$$\begin{aligned}
\gamma_i a_2 - \delta_i(H-\alpha) &= (\gamma_i'(H-\alpha) + \gamma_i''(C-\beta))a_2 - (\gamma_i' a_2 - \gamma_i'' a_1)(H-\alpha) \\
&= \gamma_i''(a_1(H-\alpha) + a_2(C-\beta)) \\
&= \gamma_i'' a, \\
\gamma_i a_1 + \delta_i(C-\beta) &= (\gamma_i'(H-\alpha) + \gamma_i''(C-\beta))a_1 + (\gamma_i' a_2 - \gamma_i'' a_1)(C-\beta) \\
&= \gamma_i'(a_1(H-\alpha) + a_2(C-\beta)) = \gamma_i' a.
\end{aligned}$$

By statements (iv-1) and (iv-2),

$u = (u_1, u_2, u_3, u_4) = -(\sum_{i\geq 1} \gamma_{-i} Y^{i-1})\,\theta_1 - (\sum_{i\geq 1} \delta_{-i} Y^{i-1})\,\theta_2 - (\sum_{i\geq 0} X^i \sigma(\gamma_i'))\,\eta_1 - (\sum_{i\geq 0} X^i \sigma(\gamma_i''))\,\eta_2$.

This finishes the proof of statement (iv). Now, the fact that (4.15) is an eventually 2-periodic resolution follows from Proposition 4.21. $\square$

Criterion for the left ideals $A(\mathfrak{m}, X)$ and $A(\sigma(\mathfrak{m}), Y)$ to have finite projective dimension. Below such a criterion is given.

Theorem 4.24. *Let D be as in Theorem 4.23. Suppose that the element $a \in D\backslash D^*$ is co-maximal, $\mathfrak{m} = (H-\alpha, C-\beta)$ is a maximal ideal of D such that $a \in \mathfrak{m}$ where α, $\beta \in K$, i.e. $a = a_1(H-\alpha) + a_2(C-\beta)$ for some elements a_1, $a_2 \in D$. The following statements are equivalent.*

1. $\mathrm{pd}_A\, A(\sigma(\mathfrak{m}), Y) < \infty$.
2. $\mathrm{pd}_A\, A(\sigma(\mathfrak{m}), Y) = 1$.
3. $a \notin \mathfrak{m}^2$.
4. *The A-submodule $A^4 e$ of A^4 is a projective A-module where e is as in Theorem 4.23.*
5. *The A-submodule $A^4 d$ of A^4 is a projective A-module where d is as in Theorem 4.23.*
6. $\mathrm{pd}_A\, A(\mathfrak{m}, X) < \infty$.
7. $\mathrm{pd}_A\, A(\mathfrak{m}, X) = 1$.

Proof. ($2 \Rightarrow 1$, $7 \Rightarrow 6$) The implications are obvious.

($1 \Leftrightarrow 4 \Leftrightarrow 5$) The implications follow at once from the exact sequence (4.15).

($2 \Leftarrow 1 \Leftrightarrow 6 \Rightarrow 7$) The ring D is a Noetherian ring of global dimension 2 and $\mathrm{pd}_D\, \mathfrak{m} = 1$. Then $\mathrm{pd}_A\,(A\mathfrak{m}) = \mathrm{pd}_A\,(A \otimes_D \mathfrak{m}) = \mathrm{pd}_D\,(\mathfrak{m}) = 1$, by [44, Corollary 2.3]. Therefore, $\mathrm{pd}_A\,(A\mathfrak{m} \oplus A) = 1$. Similarly, $\mathrm{pd}_A\,(A\sigma(\mathfrak{m}) \oplus A) = 1$. Now, by the short exact sequence (4.11) and (4.12) either the left ideals $A(\mathfrak{m}, X)$ and $A(\sigma(\mathfrak{m}), Y)$ have both infinite projective dimension as left A-modules or otherwise they both have the same projective dimension as the left A-modules $A\mathfrak{m} \oplus A$ and $A\sigma(\mathfrak{m}) \oplus A$, i.e. 1.

($5 \Rightarrow 3$) In view of the exact sequence (4.15), there is a short exact sequence

$$0 \to A^4/A^4 d \xrightarrow{e} A^4 \xrightarrow{d} A^4 d \to 0. \tag{4.22}$$

By the assumption the submodule $A^4 d$ of A^4 is projective. Hence, the short exact sequence (4.22) splits. In particular, the monomorphism $A^4/A^4 d \xrightarrow{e} A^4$ admits splitting. This means that there is a matrix $e' \in M_4(A)$ such that

$$ee' + d'd = 1 \tag{4.23}$$

for some matrix $d' \in M_4(A)$ where 1 is the identity element of the algebra $M_4(A)$.

The matrix algebra

$$M_4(A) = M_4(K) \otimes_K A = M_4(D)[X, Y; \sigma, a] \tag{4.24}$$

is a GWA where the new automorphism $\sigma \in \mathrm{Aut}_K(M_4(D))$ is a unique extension of the old automorphism $\sigma \in \mathrm{Aut}_K(D)$ that acts as the identity map on the subalgebra of scalar matrices $M_4(K)$ in $M_4(D) = M_4(K) \otimes_K D$. Using the $\mathbb{Z}$-grading of the GWA $M_4(A)$ and, in particular, the fact that the identity matrix 1 of $M_4(K)$ belongs to the zero component $M_4(D)$ of the GWA $M_4(A)$, we see that the equality (4.23) yields the equality

$$\begin{pmatrix} \mathcal{A} & -Y \\ X & -\sigma(\tilde{\mathcal{A}}) \end{pmatrix} \begin{pmatrix} e'_{11} & * \\ Xe'_{21} & * \end{pmatrix} + \begin{pmatrix} d'_{11} & d'_{12}Y \\ * & * \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{A}} & -Y \\ X & -\sigma(\mathcal{A}) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

for some matrices e'_{11}, e'_{21}, d'_{11}, $d'_{12} \in M_2(D)$ where $1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \in M_2(D)$. By equating the $(1,1)$-elements of the matrices on both sides of the equality we obtain the equality in the matrix algebra $M_2(D)$:

$$\mathcal{A}\, e'_{11} - a\, e'_{21} + d'_{11}\, \tilde{\mathcal{A}} + d'_{12}\, a = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{4.25}$$

The algebra $M_2(D)$ contains the ideal $M_2(\mathfrak{m})$. Bearing in mind that

$$a \in \mathfrak{m}, \;\; \mathcal{A} = \begin{pmatrix} a_1 & a_2 \\ C - \beta & -(H - \alpha) \end{pmatrix} \text{ and } \tilde{\mathcal{A}} = \begin{pmatrix} H - \alpha & a_2 \\ C - \beta & -a_1 \end{pmatrix}$$

and taking the equality (4.25) modulo the ideal $M_2(\mathfrak{m})$ we obtain the equality in the matrix algebra $M_2(D)/M_2(\mathfrak{m}) \simeq M_2(D/\mathfrak{m}) \simeq M_2(K)$:

$$\begin{pmatrix} a_1 & a_2 \\ 0 & 0 \end{pmatrix} e'_{11} + d'_{11} \begin{pmatrix} 0 & a_2 \\ 0 & -a_1 \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \mod M_2(\mathfrak{m}).$$

By equating the $(1,1)$-elements of the matrices on both sides of the equality we see that

$$1 \in Da_1 + Da_2 + \mathfrak{m}. \tag{4.26}$$

Equivalently, $Da_1 + Da_2 \nsubseteq \mathfrak{m}$. Since $a = a_1(H-\alpha) + a_2(C-\beta) \in \mathfrak{m}$, we conclude that $a \notin \mathfrak{m}^2$ (since

$$\overline{a} = \overline{a_1}\overline{(H-\alpha)} + \overline{a_2}\overline{(C-\beta)} \in \mathfrak{m}/\mathfrak{m}^2 = K\overline{(H-\alpha)} \oplus K\overline{(C-\beta)}$$

and either $\overline{a_1} \in K\backslash\{0\}$ or $\overline{a_2} \in K\backslash\{0\}$, hence $\overline{a} \neq 0$ where the bar means modulo $\mathfrak{m}^2$).

$(3 \Rightarrow 5)$ By the assumption, $a \in \mathfrak{m}\backslash\mathfrak{m}^2$ which is equivalent to (4.26). This means that there are elements $a'_1, \ldots, a'_4 \in D$ such that

$$a_1\, a'_1 + a_2\, a'_2 + a'_3\, (H-\alpha) + a'_4\, (C-\beta) = 1. \tag{4.27}$$

We have to show that the short exact sequence (4.22) splits, i.e. the equality (4.23) holds for some elements e', $d' \in M_4(A)$.

CLAIM: *Let* $e' = \begin{pmatrix} e'_{11} & Y\sigma(\mathcal{A}) \\ X\mathcal{A} & \sigma(e'_{22}) \end{pmatrix}$, $d' = \begin{pmatrix} -e'_{22} & \mathcal{A}Y \\ \sigma(\mathcal{A})X & -\sigma(e'_{11}) \end{pmatrix}$, $e'_{11} = \begin{pmatrix} a'_1 & a'_4 \\ a'_2 & -a'_3 \end{pmatrix}$ *and* $e'_{22} = \begin{pmatrix} -a'_3 & -a'_4 \\ -a'_2 & a'_1 \end{pmatrix}$. *Then* $ee' + d'd = 1$ *in* $M_4(A)$.

The equality $ee' + d'd = 1$ follows by direct computation and using the equalities $YX = a$, $XY = \sigma(a)$, $\tilde{\mathcal{A}}\mathcal{A} = \mathcal{A}\tilde{\mathcal{A}}$, $\mathcal{A}e'_{11} - e'_{22}\tilde{\mathcal{A}} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ and $e'_{11}\mathcal{A} - \tilde{\mathcal{A}}e'_{22} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$:

$$\begin{aligned} ee' + d'd &= \begin{pmatrix} \mathcal{A} & -Y \\ X & -\sigma(\tilde{\mathcal{A}}) \end{pmatrix} \begin{pmatrix} e'_{11} & Y\sigma(\mathcal{A}) \\ X\mathcal{A} & \sigma(e'_{22}) \end{pmatrix} + \begin{pmatrix} -e'_{22} & \mathcal{A}Y \\ \sigma(\mathcal{A})X & -\sigma(e'_{11}) \end{pmatrix} \begin{pmatrix} \tilde{\mathcal{A}} & -Y \\ X & -\sigma(\mathcal{A}) \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{A}e'_{11} - a\mathcal{A} - e'_{22}\tilde{\mathcal{A}} + \mathcal{A}a & (\mathcal{A}^2 - e'_{22} + e'_{22} - \mathcal{A}^2)Y \\ X(e'_{11} - \tilde{\mathcal{A}}\mathcal{A} + \mathcal{A}\tilde{\mathcal{A}} - e'_{11}) & \sigma(a\mathcal{A} - \tilde{\mathcal{A}}e'_{22} - \mathcal{A}a + e'_{11}\mathcal{A}) \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & \sigma(1) \end{pmatrix} = 1. \end{aligned}$$

In more detail,

$$\begin{aligned}
\mathcal{A}e'_{11} - e'_{22}\tilde{\mathcal{A}} &= \begin{pmatrix} a_1 & a_2 \\ C-\beta & -(H-\alpha) \end{pmatrix} \begin{pmatrix} a'_1 & a'_4 \\ a'_2 & -a'_3 \end{pmatrix} - \begin{pmatrix} -a'_3 & -a'_4 \\ -a'_2 & a'_1 \end{pmatrix} \begin{pmatrix} H-\alpha & a_2 \\ C-\beta & -a_1 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{L} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}
\end{aligned}$$

where $\mathcal{L} := a_1\, a'_1 + a_2\, a'_2 + a'_3\,(H-\alpha) + a'_4\,(C-\beta) = 1$, by (4.27). Similarly,

$$\begin{aligned}
e'_{11}\mathcal{A} - \tilde{\mathcal{A}}e'_{22} &= \begin{pmatrix} a'_1 & a'_4 \\ a'_2 & -a'_3 \end{pmatrix} \begin{pmatrix} a_1 & a_2 \\ C-\beta & -(H-\alpha) \end{pmatrix} - \begin{pmatrix} H-\alpha & a_2 \\ C-\beta & -a_1 \end{pmatrix} \begin{pmatrix} -a'_3 & -a'_4 \\ -a'_2 & a'_1 \end{pmatrix} \\
&= \begin{pmatrix} \mathcal{L} & 0 \\ 0 & \mathcal{L} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.
\end{aligned}$$

The proof of the theorem is complete. $\square$

For a commutative ring D, we denote by $\mathrm{Max}\,(D)$ its *maximal spectrum*, the set of maximal ideals of D. For an element $d \in D$, let $\mathcal{V}(d) := \{\mathfrak{m} \in \mathrm{Max}\,(D) \mid d \in \mathfrak{m}\}$.

Lemma 4.25. *Let an algebra D be a localization of the polynomial algebra $K[H,C]$ of Krull dimension 2, $a \in D \backslash D^*$ and K be an algebraically closed field. Then the following two statements are equivalent.*

1. *For all $\mathfrak{m} \in \mathcal{V}(a)$, $a \notin \mathfrak{m}^2$.*
2. *For all $\mathfrak{m} \in \mathcal{V}(a)$, $D\,\frac{\partial a}{\partial H} + D\,\frac{\partial a}{\partial C} \not\subseteq \mathfrak{m}$.*

Proof. Fix a unit $u \in D^* \cap K[H,C]$ such that $a' := ua \in K[H,C]$. Then $\mathcal{V}(a) = \mathcal{V}(a')$ and for every $\mathfrak{m} \in \mathcal{V}(a) = \mathcal{V}(a')$, $a \in \mathfrak{m}$ (respectively, $a \in \mathfrak{m}^2$) if and only if $a' \in \mathfrak{m}$ (respectively, $a' \in \mathfrak{m}^2$). Since the field K is algebraically closed, every maximal ideal $\mathfrak{m}$ is equal to $\mathfrak{m} = (H-\alpha, C-\beta)$ for some elements α, $\beta \in K$. Let $\mathfrak{m} = (H-\alpha, C-\beta) \in \mathcal{V}(a)$. Now, the lemma follows from

$$a' \equiv \frac{\partial a'}{\partial H}(H-\alpha) + \frac{\partial a'}{\partial C}(C-\beta) \equiv u\,(\frac{\partial a}{\partial H}(H-\alpha) + \frac{\partial a}{\partial C}(C-\beta)) \mod \mathfrak{m}. \quad \square$$

Every element $a \in D\backslash D^*$ determines an algebraic curve $\mathcal{V}(a) = \{\mathfrak{m} \in \mathrm{Max}\,(D) \mid a \in \mathfrak{m}\}$. It is given by the equation $a = 0$ (in $\mathrm{Max}\,(D) \subseteq K^2$ via $\mathfrak{m} = (H-\alpha, C-\beta) \leftrightarrow (\alpha, \beta)$).

Definition 4.26. *The algebraic curve $a = 0$ (in $\mathrm{Max}\,(D) \subseteq K^2$ where $a \in D\backslash D^*$) is called a* smooth algebraic curve *if it satisfies the second condition of Lemma 4.25 which means that the dimension of the tangent space at every point of the algebraic curve is a 1-dimensional vector space over the field K.*

Lemma 4.27. *Let D and a be as in Lemma 4.25. If the algebraic curve $a = 0$ is smooth then the element a is co-maximal.*

Proof. The element $a = p_1 \cdots p_s$ is a product of irreducible elements of D. If $s = 1$ there is nothing to prove. Suppose that $s \geq 2$. We have to show that $Dp_i + Dp_j = D$ for all $i \neq j$. Suppose this is not the case for some $i \neq j$. Then there is a maximal ideal $\mathfrak{m}$ of D such that $p_i \in \mathfrak{m}$ and $p_j \in \mathfrak{m}$. Clearly, $\frac{\partial a}{\partial H} \in \mathfrak{m}$ and $\frac{\partial a}{\partial C} \in \mathfrak{m}$. This contradicts to the assumption that the algebraic curve $a = 0$ is smooth. $\square$

Proof of Theorem 1.5. If $a = 0$ then gld $(A) = \infty$, by Proposition 4.14. If $a \in D^*$ then $A \simeq D[X, X^{-1}; \sigma]$ and gld $(A) \leq$ gld $(D) + 1 < \infty$, by [19, Theorem 7.5.3]. If $a \in D\backslash D^*$ and $a \neq 0$ then by [44, Theorem 3.5], gld $(A) < \infty$ if and only if $\mathrm{pd}_A\, A(\mathfrak{p}, X) < \infty$ for all prime ideals $\mathfrak{p}$ of the algebra D that contain the element a if and only if the element a is co-maximal (Proposition 4.17) and $a \notin \mathfrak{m}^2$ for all maximal ideals $\mathfrak{m}$ of D that contain a (Theorem 4.24) if and only if a is co-maximal and the algebraic curve $a = 0$ is smooth (Lemma 4.25) if and only if the algebraic curve $a = 0$ is smooth (Lemma 4.27). $\square$

Proof of Theorem 1.9. By Lemma 1.8, the algebra $E = D[X, Y; \sigma, a = H]$ is a GWA where $D = \mathcal{D}[H]$ and $\sigma(H) = \rho H + b$.

(i) *For all natural numbers $n \geq 1$ $\sigma^n(H) = \rho^n H + \xi_n$ where $\xi_n = \sum_{i=0}^{n-1} \sigma^i(\rho^{n-i-1} b)$*:

The statement is proven by induction on n. The initial case $n = 1$ is obvious ($\sigma(H) = \rho H + b$). So, let $n > 1$ and we assume that the equality holds for all $n' < n$. Now,

$$\sigma^n(H) = \sigma(\rho^{n-1} H + \xi_{n-1}) = \rho^n H + \rho^{n-1} b + \sigma(\xi_{n-1}) = \rho^n H + \xi_n,$$

as required.

(ii) If $\mathcal{D}$ is a field then the algebra $D = \mathcal{D}[H]$ is a Dedekind domain and statement 1 follows from Theorem 4.3. Suppose that the algebra $\mathcal{D}$ is not a field. Then the algebra $\mathcal{D} = \mathcal{D}[H] = S^{-1}K[C, H]$ has Krull dimension 2.

(iii) gld $(E) = 2$ *or* 3: Since $\mathrm{grad}(a) = \mathrm{grad}(H) = (1, 0)$, gld $(E) < \infty$, by Theorem 1.5, and so gld $(E) = 2$ or 3, by Theorem 4.1.

(iv) gld $(E) = 3$ *if and only if there are natural number* $n \geq 1$ *and an element* $\beta \in K$ *such that* $\mathcal{D}\xi_n + \mathcal{D}(\sigma^n(C) - \beta) \neq \mathcal{D}$ *(where* K *is an algebraically closed field)*: By Theorem 4.2, gld $(E) = 3$ if and only if there is a maximal ideal $\mathfrak{m} = (H, C - \beta)$ (where $\beta \in K$) such that $H \in \sigma^n(\mathfrak{m})$ for some natural number $n \geq 1$. By the statement (i), $\sigma^n(\mathfrak{m}) = (\rho^n H + \xi_n, \sigma^n(C) - \beta)$. Hence, $H \in \sigma^n(\mathfrak{m})$ if and only if $\mathcal{D}\xi_n + \mathcal{D}(\sigma^n(C) - \beta) \neq \mathcal{D}$. The proof of Theorem 1.9 is complete. $\square$

4.4 The global dimension of $S^{-1}K[H,C](\sigma,a)$ with affine automorphism σ

In this section, $A := K[H,C][X,Y;\sigma,a]$ is a generalized Weyl algebra where $D := K[H,C]$ is a polynomial algebra in two variables over an algebraically closed field K, σ is an *affine automorphism* of D and S is a multiplicative subset of $K[H,C]\backslash\{0\}$ such that $\sigma(S) = S$. The algebra $\mathcal{A} = S^{-1}A = \mathcal{D}(\sigma,a)$ is a GWA where $\mathcal{D} = S^{-1}D$ and the automorphism σ is defined by the rule $\sigma(s^{-1}d) = \sigma(s)^{-1}\sigma(d)$. As we already mentioned in Chapter 1, we will assume that the Krull dimension of $\mathcal{D}$ is 2.

The aim of the section is to calculate an explicit value of the global dimension of A (respectively, $\mathcal{A} = S^{-1}D(\sigma,a)$) in terms of the roots of the polynomial $a \in K[H,C]$ (respectively, $a \in S^{-1}D$) accordingly to the four cases of the affine automorphism σ.

Proposition 4.28. *Let* K *be an algebraically closed field and* $A = K[H,C](\sigma,a)$ *be a GWA. Then* gld $(A) < \infty$ *if and only if either* $a \in K^*$ *or* $a \notin K$ *and* $a = p_1 \cdots p_s$ *is a product of irreducible polynomials such that* $(p_i) + (p_j) = (1)$ *for all* $i \neq j$ *and the algebraic curves* $p_i = 0$ *are smooth (i.e. the algebraic curve* $a = 0$ *is a disjoint union of smooth algebraic curves* $p_i = 0$*).*

Proof. The proposition is a particular case of Theorem 1.5. $\square$

In the following theorems we assume that gld $(A) < \infty$ and we find the values of gld (A) for the four types of automorphism σ of D. Now, If σ of type 1 Theorem 1.6 gives the explicit value of the global dimension of GWA A.

Proof of Theorem 1.6. By Theorem 4.1, gld $(A) = 2$, 3 (since we assume that gld $(A) < \infty$).

1. If char $(K) = p$ then the maximal ideal (C, H) of the algebra D is σ^p-invariant. Hence, gld $(A) = 3$, by Theorem 4.2, since gld $(A) < \infty$. If char $(K) = 0$ then for every maximal ideal $\mathfrak{m}$ of D the orbit $\mathcal{O}(\mathfrak{m}) = \{\sigma^i(\mathfrak{m}) \mid i \in \mathbb{Z}\}$ is infinite. Recall that we assume that gld $(A) < \infty$. Then, by Theorem 4.2, gld $(A) = 3$ if and only if there exist elements α, $\beta \in K$ and $i \in \mathbb{N}\backslash\{0\}$ such that $a(\alpha, \beta) = 0$ and $a(\alpha + i, \lambda^{-i}\beta) = 0$.

2. By Theorem 4.1, gld $(A) = 2, 3$ (since we assume that gld $(A) < \infty$). Now, statement 2 follows from statement 1.

3. Recall that the Krull dimension of the ring $\mathcal{D}$ is 2. Notice that there is a maximal ideal $\mathfrak{m} = (H - \alpha, C - \beta)$ of $\mathcal{D}$ (where $\alpha, \beta \in K$) such that its σ-orbit is finite if and only if either char$(K) \neq 0$ and λ is a root of unity or char$(K) \neq 0$, λ is not a root of unity and $(H - \alpha, C) \neq \mathcal{D}$. By Theorem 4.1, gld $(\mathcal{A}) = 2, 3$ (since gld $(\mathcal{A}) < \infty$). Now, statement 3 follows from Theorem 4.2. $\square$

Theorem 4.29. *Let $A = D(\sigma, a)$ be a GWA such that $D = K[H, C]$, K is an algebraically closed field, $\sigma(H) = \lambda H$ and $\sigma(C) = \mu C$ where λ, $\mu \in K^*$.*

1. *Suppose that* gld $(A) < \infty$. *Then* gld $(A) = 3$.

2. *Suppose that S is a multiplicative subset of D such that $\sigma(S) = S$ and the algebra $\mathcal{D} = S^{-1}D$ has Krull dimension 2. Let $\mathcal{A} = \mathcal{D}(\sigma, a)$ where $a \in \mathcal{D}$. Suppose that* gld $(\mathcal{A}) < \infty$. *Then* gld $(\mathcal{A}) = 2, 3$. *Furthermore,* gld $(\mathcal{A}) = 3$ *if and only if either λ, μ are roots of unity or λ is not a root of unity, μ is a root of unity and $(H, C - \beta) \neq \mathcal{D}$ for some $\beta \in K$ or λ is a root of unity, μ is not a root of unity and $(H - \alpha, C) \neq \mathcal{D}$ for some $\alpha \in K$ or λ, μ are not roots of unity and $(H, C) \neq \mathcal{D}$ or there is a maximal ideal $\mathfrak{m} = (H - \alpha, C - \beta)$ of $\mathcal{D}$ (where $\alpha, \beta \in K$) such that $a \in \mathfrak{m}$ and $a \in \sigma^i(\mathfrak{m}) = (H - \lambda^{-i}\alpha, C - \mu^{-i}\beta)$ for some $i \geq 1$.*

Proof. 1. By Theorem 4.2, gld $(A) = 3$ since the maximal ideal (H, C) of the polynomial algebra D is σ-invariant.

2. Recall that the Krull dimension of the ring $\mathcal{D}$ is 2. By Theorem 4.1, gld $(\mathcal{A}) = 2, 3$ (since we assume that gld $(\mathcal{A}) < \infty$).

In the last sentence of statement 2, the first four conditions are equivalent to the fact that there is a maximal ideal of $\mathcal{D}$ of height 2 such that its σ-orbit is finite. Now, the 'if and only if' statement follows from Theorem 4.2. $\square$

Theorem 4.30. *Let $A = D(\sigma, a)$ be a GWA such that $D = K[H,C]$, K is an algebraically closed field, $\sigma(H) = H - 1$ and $\sigma(C) = C + H$. Suppose that* gld $(A) < \infty$. *Then* gld $(A) = 2, 3$ *and*

1. gld $(A) = 3$ *if and only if either* char $(K) \neq 0$ *or* char $(K) = 0$ *and there exist elements α, $\beta \in K$ and $i \in \mathbb{N}\backslash\{0\}$ such that $a(\alpha, \beta) = 0$ and $a(\alpha + i, \beta - i\alpha - \frac{i(i+1)}{2}) = 0$.*

2. gld $(A) = 2$ *if and only if* char $(K) = 0$ *and if $a(\alpha, \beta) = 0$ for some α, $\beta \in K$ then $a(\alpha + i, \beta - i\alpha - \frac{i(i+1)}{2}) \neq 0$ for all $i \in \mathbb{N}\backslash\{0\}$.*

3. *Suppose that S is a multiplicative subset of D such that $\sigma(S) = S$ and the algebra $\mathcal{D} = S^{-1}D$ has Krull dimension 2. Let $\mathcal{A} = \mathcal{D}(\sigma, a)$ where $a \in \mathcal{D}$. Suppose that* gld $(\mathcal{A}) < \infty$. *Then* gld $(\mathcal{A}) = 2, 3$. *Furthermore,* gld $(\mathcal{A}) = 3$ *if and only if either* char$(K) \neq 0$ *or* char$(K) = 0$ *and there is a maximal ideal $\mathfrak{m} = (H - \alpha, C - \beta)$ of $\mathcal{D}$ (where $\alpha, \beta \in K$) such that $a \in \mathfrak{m}$ and $a \in \sigma^i(\mathfrak{m}) = (H - \alpha - i, C - \beta + i\alpha + \frac{i(i+1)}{2})$ for some $i \geq 1$.*

Proof. By Theorem 4.1, gld $(A) = 2$, 3 (since we assume that gld $(A) < \infty$).

1. If char $(K) \neq 0$ then, by Proposition 3.6 (3), the order of the automorphism σ is finite. Hence, gld $(A) = 3$, by Theorem 4.2, since gld $(A) < \infty$. If char $(K) = 0$ then for every maximal ideal $\mathfrak{m}$ of D its orbit $\mathcal{O}(\mathfrak{m}) = \{\sigma^i(\mathfrak{m}) \,|\, i \in \mathbb{Z}\}$ is infinite. By Theorem 4.2, gld $(A) = 3$ if and only if there exists a maximal ideal $\mathfrak{m} = (H - \alpha, C - \beta)$ of D (where $\alpha, \beta \in K$) such that $a \in \mathfrak{m}$ and $a \in \sigma^i(\mathfrak{m})$ for some $i \geq 1$ if and only if $a(\alpha, \beta) = 0$ and $a(\alpha + i, \beta - i\alpha - \frac{i(i+1)}{2}) = 0$ since, by (3.3),

$$\sigma^i(\mathfrak{m}) = (H - \alpha - i, C + iH - \frac{i(i-1)}{2} - \beta) = (H - (\alpha + i), C - (\beta - i\alpha - \frac{i(i+1)}{2})).$$

2. Since gld $(A) = 2$, 3. Now, statement 2 follows from statement 1.

3. Recall that the Krull dimension of $\mathcal{D}$ is 2. By Theorem 4.1, gld $(\mathcal{A}) = 2, 3$ (since gld $(\mathcal{A}) < \infty$). Now, repeat the proof of statement 1 by replacing the ring D by $\mathcal{D}$. □

Theorem 4.31. *Let $A = D(\sigma, a)$ be a GWA such that $D = K[H,C]$, K is an algebraically closed field, $\sigma(H) = \lambda H + C$ and $\sigma(C) = \lambda C$ where $\lambda \in K^*$.*

1. *Suppose that* gld $(A) < \infty$. *Then* gld $(A) = 3$.

2. *Suppose that S is a multiplicative subset of D such that $\sigma(S) = S$ and the algebra $\mathcal{D} = S^{-1}D$ has Krull dimension 2. Let $\mathcal{A} = \mathcal{D}(\sigma, a)$ where $a \in \mathcal{D}$. Suppose that* gld $(\mathcal{A}) < \infty$. *Then* gld $(\mathcal{A}) = 2, 3$. *Furthermore,* gld $(\mathcal{A}) = 3$ *if and only if either λ is a root of unity,* $\mathrm{char}(K) \neq 0$ *and $(H-\alpha, C-\beta) \neq \mathcal{D}$ for some $\alpha, \beta \in K$ such that $\beta \neq 0$ or $(H, C) \neq \mathcal{D}$ or λ is a root of unity and $(H-\alpha, C) \neq \mathcal{D}$ for some $\alpha \in K$ or there is a maximal ideal $\mathfrak{m} = (H-\alpha, C-\beta)$ of $\mathcal{D}$ (where $\alpha, \beta \in K$) such that $a \in \mathfrak{m}$ and $a \in \sigma^i(\mathfrak{m}) = (H - (\lambda^{-i}\alpha - i\lambda^{-i-1}\beta), C - \lambda^{-i}\beta)$ for some $i \geq 1$.*

Proof. 1. The maximal ideal $\mathfrak{m} = (H, C)$ is σ-invariant. Hence, gld $(A) = 3$, by Theorem 4.2, since gld $(A) < \infty$.

2. Recall that the Krull dimension of $\mathcal{D}$ is 2. By Theorem 4.1, gld $(\mathcal{A}) = 2, 3$ (since gld $(\mathcal{A}) < \infty$). Given a maximal ideal $\mathfrak{m} = (H-\alpha, C-\beta)$ of the algebra $\mathcal{D}$ (where $\alpha, \beta \in K$). Then, by (3.4), for all $i \in \mathbb{Z}$,

$$\sigma^i(\mathfrak{m}) = (H + i\lambda^{-1}C - \lambda^{-i}\alpha, C - \lambda^{-i}\beta) = (H - (\lambda^{-i}\alpha - i\lambda^{-i-1}\beta), C - \lambda^{-i}\beta). \quad (4.28)$$

Then $\sigma^i(\mathfrak{m}) = \mathfrak{m}$ for some $i \geq 1$ if and only if $(\lambda^{-i} - 1)\beta = 0$ and $(\lambda^{-i} - 1)\alpha = i\lambda^{-i-1}\beta$ if and only if either $\beta \neq 0$, $\lambda^i = 1$ and $\mathrm{char}(K) \neq 0$ or $\beta = 0$, $\alpha = 0$ or $\beta = 0$, $\lambda^i = 1$ if and only if either λ is a root of unity, $\mathrm{char}(K) \neq 0$ and $(H-\alpha, C-\beta) \neq \mathcal{D}$ for some $\alpha, \beta \in K$ such that $\beta \neq 0$ or $(H, C) \neq \mathcal{D}$ or λ is a root of unity and $(H-\alpha, C) \neq \mathcal{D}$ for some $\alpha \in K$. Now, statement 2 follows from Theorem 4.2. □

4.5 The global dimension of tensor products of GWAs

The question about the global dimension of the tensor product of algebras of finite global dimensions is one of the difficult questions concerning the dimension of the tensor product. It is well known that for all left Noetherian algebras A and B [26],

$$\mathrm{lgd}(A \otimes B) \geq \mathrm{lgd}\, A + \mathrm{lgd}\, B.$$

The equality is satisfied with some restricted conditions on the algebras A and B.

Definition 4.32. *([46, p. 2].) An algebra A is called a* **tensor homological minimal** *(THM) algebra with respect to a class of algebras Ω if*

$$\mathrm{lgd}\ (A \otimes B) = \mathrm{lgd}\ (A) + \mathrm{lgd}\ (B) \quad \text{for all} \quad B \in \Omega.$$

Example 4.33. *([44, p. 82].) The polynomial ring $P_n = K[x_1, \dots, x_n]$ is THM with respect to any class of algebras, since*

$$\operatorname{lgd}(P_n \otimes B) = n + \operatorname{lgd} B = \operatorname{lgd} P_n + \operatorname{lgd} B,$$

for any algebra B.

Let $\mathcal{N}$ be the class of *countable Noetherian algebras* (i.e. algebras that have countable basis). In [44, Corollary 4.5], it was proved that the GWA $A = K[x_1, \dots, x_n](\sigma, a)$ of degree 1 where K is an algebraically closed uncountable field is THM with respect to $\mathcal{N}$ (as well as the tensor product of these algebras).

Theorem 4.34. *Let K be an algebraically uncountable closed field, $\Lambda = \otimes_{i=1}^{n} \Lambda_i$ be a tensor product of finitely generated GWAs in Theorem 1.5 (eg, all algebras in Corollary 1.7). Then the algebra Λ is a THM algebra with respect to the class $\mathcal{N}$. In particular,* $\operatorname{lgd}(\Lambda) = \sum_{i=1}^{n} \operatorname{gld}(\Lambda_i)$.

Proof. By [44, Corollary 4.5], the algebras Λ_i are THM with respect to the class of algebras $\mathcal{N}$ and the result follows. □

4.6 Special cases of GWAs

In this section, we study some special cases of GWAs where the field K is not assumed to be algebraically closed. The computation of the global dimension in these cases is based on different ideas that can be useful in other situations.

Proposition 4.35. *Let $A = K[H,C][X,Y;\sigma,a]$ be a GWA of type 1 ($\sigma(H) = H-1$, $\sigma(C) = \lambda C$) such that $a \in K[H]$. Let $K[H]a = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_s^{n_s}$ be a product of maximal ideals of $K[H]$ provided $a \in K[H] \backslash K$. Then*

1. *The algebra A is a skew polynomial ring $A'[C,\tau]$ where $A' = K[H][X,Y;\sigma,a]$ is a GWA with $\sigma(H) = H-1$; $\tau \in \operatorname{Aut}_K(A')$ where $\tau(X) = \lambda^{-1}X$, $\tau(Y) = \lambda Y$ and $\tau(H) = H$.*

2. gld $(A) =$ gld $(A') + 1$ *where*

$$\text{gld }(A') = \begin{cases} \infty & \text{if } a = 0 \text{ or } n_i \geq 2 \text{ for some } i, \\ 2 & \text{if } a \neq 0,\ n_1 = \cdots = n_s = 1,\ s \geq 1 \\ & \text{or } a \text{ is invertible, and there exists an integer } k \geq 1 \\ & \text{such that either } \sigma^k(\mathfrak{p}_i) = \mathfrak{p}_j \text{ for some } i,\, j \\ & \text{or } \sigma^k(\mathfrak{q}) = \mathfrak{q} \text{ for some maximal ideal } \mathfrak{q} \text{ of } K[H], \\ 1 & \text{otherwise.} \end{cases}$$

Proof. 1. Using the $\mathbb{Z}$-grading of the GWA A, we see that as a vector space the algebra A is the tensor product of its subalgebras $A' \otimes K[C]$. Therefore, the algebra A is the skew polynomial algebra $A'[C, \tau]$ since $CX = \lambda^{-1}XC$, $CY = \lambda YC$ and $CH = HC$.

2. By [19, Theorem 7.5.(iii)] and statement 1,

$$\text{gld }(A) = \text{gld }(A'[C, \tau]) = \text{gld }(A') + 1.$$

Finally, by [46, Theorem 1.6], the expression for gld (A') in statement 2 follows. $\square$

Let $A = K[H, C][X, Y; \sigma, a]$ be a GWA of type 1 ($\sigma(H) = H - 1$, $\sigma(C) = \lambda C$) such that $a \in K[C]$. Then the algebra A is a skew polynomial ring $A'[H; \tau := \text{id}_{A'}, \delta]$ of the GWA $A' = K[C][X, Y; \sigma, a]$ with $\sigma(C) = \lambda C$; and δ is a τ-derivative of algebra A' given by the rule $\delta(X) = X$, $\delta(Y) = -Y$ and $\delta(C) = 0$.

Proposition 4.36 and Proposition 4.37 are special cases of generalized Weyl algebras of type 2 and the global dimension of these algebras can be found by applying some known results about GWAs and skew polynomial rings.

Proposition 4.36. *Let $A = K[H, C][X, Y; \sigma, a]$ be a GWA of type 2 ($\sigma(H) = \lambda H$, $\sigma(C) = \mu C$) such that $a \in K[H]$ and $K[H]a = \mathfrak{p}_1^{n_1} \cdots \mathfrak{p}_s^{n_s}$ be a product of maximal ideals of $K[H]$ provided $a \in K[H] \backslash K$. Then*

1. *The algebra A is a skew polynomial ring $A'[C, \tau]$ where $A' = K[H][X, Y; \sigma, a]$ is a GWA with $\sigma(H) = \lambda H$; $\tau \in \text{Aut}_K(A')$ where $\tau(X) = \mu^{-1}X$, $\tau(Y) = \mu Y$ and $\tau(H) = H$.*

2. gld $(A)=$ gld $(A')+1$ *where*

$$\text{gld }(A')=\begin{cases}\infty & \text{if } a=0 \text{ or } n_i\geq 2 \text{ for some } i,\\ 2 & \text{if } a\neq 0,\ n_1=\cdots=n_s=1,\ s\geq 1\\ & \text{or } a \text{ is invertible, and there exists an integer } k\geq 1\\ & \text{such that either } \sigma^k(\mathfrak{p}_i)=\mathfrak{p}_j \text{ for some } i,\ j\\ & \text{or } \sigma^k(\mathfrak{q})=\mathfrak{q} \text{ for some maximal ideal } \mathfrak{q} \text{ of } K[H],\\ 1 & \text{otherwise.}\end{cases}$$

Proof. 1. Using the $\mathbb{Z}$-grading of the GWA A, we see that as a vector space the algebra A is the tensor product of its subalgebras $A'\otimes K[C]$ where A' is the GWA as in statement 1. Therefore, the algebra A is the skew polynomial algebra $A'[C,\tau]$ since $CX=\mu^{-1}XC$, $CY=\mu YC$ and $CH=HC$.

2. By [19, Theorem 7.5.(iii)] and statement 1,

$$\text{gld }(A)=\text{gld }(A'[C,\tau])=\text{gld }(A')+1.$$

Finally, by [46, Theorem 1.6], the expression for gld (A') in statement 2 follows. □

Proposition 4.37. *Let $A=K[H,C][X,Y;\sigma,a]$ be a GWA of type 2 ($\sigma(H)=\lambda H$, $\sigma(C)=\mu C$) such that $a\in K[C]$ and $K[C]a=\mathfrak{p}_1^{n_1}\cdots\mathfrak{p}_s^{n_s}$ be a product of maximal ideals of $K[C]$ provided $a\in K[C]\backslash K$. Then*

1. *The algebra A is a skew polynomial ring $A'[H,\tau]$ where $A'=K[C][X,Y;\sigma,a]$ is a GWA with $\sigma(C)=\mu C$; $\tau\in \mathrm{Aut}_K(A')$ where $\tau(X)=\lambda^{-1}X$, $\tau(Y)=\lambda Y$ and $\tau(C)=C$.*

2. gld $(A)=$ gld $(A')+1$ *where*

$$\text{gld }(A')=\begin{cases}\infty & \text{if } a=0 \text{ or } n_i\geq 2 \text{ for some } i,\\ 2 & \text{if } a\neq 0,\ n_1=\cdots=n_s=1,\ s\geq 1\\ & \text{or } a \text{ is invertible, and there exists an integer } k\geq 1\\ & \text{such that either } \sigma^k(\mathfrak{p}_i)=\mathfrak{p}_j \text{ for some } i,\ j\\ & \text{or } \sigma^k(\mathfrak{q})=\mathfrak{q} \text{ for some maximal ideal } \mathfrak{q} \text{ of } K[C],\\ 1 & \text{otherwise.}\end{cases}$$

Proof. 1. Using the $\mathbb{Z}$-grading of the GWA A, we see that as a vector space the algebra A is the tensor product of its subalgebras $A'\otimes K[H]$ where A' is the

GWA as in statement 1. Therefore, the algebra A is the skew polynomial algebra $A'[H,\tau]$ since $HX = \lambda^{-1}XH$, $HY = \lambda YH$ and $HC = CH$.

2. By [19, Theorem 7.5.(iii)] and statement 1,

$$\text{gld }(A) = \text{gld }(A'[H,\tau]) = \text{gld }(A') + 1.$$

Finally, by [46, Theorem 1.6], the expression for gld (A') in statement 2 follows. □

Let $A = K[H,C][X,Y;\sigma,a]$ be a GWA of type 3 ($\sigma(H) = H-1$, $\sigma(C) = C+H$) such that $a \in K[H]$. Then the algebra A is an Ore extension $A'[C;\tau = \text{id}_{A'},\delta]$ of the GWA $A' = K[H][X,Y;\sigma,a]$ with $\sigma(H) = H-1$; and δ is a τ-derivative of algebra A' given by the rule $\delta(X) = -XH$, $\delta(Y) = YH$ and $\delta(H) = 0$.

Theorem 4.38. *Let $A = K[H,C][X,Y;\sigma,a]$ be a GWA of type 4 ($\sigma(H) = \lambda H + C, \sigma(C) = \lambda C$) such that $a \in K^*C$. Then*

1. gld $(A) = 3$.
2. *The algebra A is a skew polynomial ring $A'[H;\tau,\delta]$ where $A' = K[C][X,Y;\sigma,a]$ is a GWA with $\sigma(C) = \lambda C$; $\tau \in \text{Aut}_K(A')$ where $\tau(X) = \lambda^{-1}X$, $\tau(Y) = \lambda Y$ and $\tau(C) = C$; and δ is a τ-derivative of the algebra A' given by the rule $\delta(X) = -\lambda^{-1}CX$, $\delta(Y) = YC$ and $\delta(C) = 0$.*
3. gld $(A') = 2$.

Proof. 3. By [46, Theorem 1.6], gld (A') = gld $(K[C]) + 1 = 2$ since a is an irreducible element of $K[C]$ and $\sigma(K[C]a) = K[C]a$.

2. Using the $\mathbb{Z}$-grading of the GWA A, we see that as a vector space the algebra A is equal to the tensor product of its subalgebras $A' \otimes K[H]$ where A' is the GWA $K[C][X,Y;\sigma,a]$. Therefore, the algebra A is the skew polynomial ring $A'[H;\tau,\delta]$ since $XH = (\lambda H + C)X$, $HY = Y(\lambda H + C)$ and $HC = CH$ (equivalently, $HX - \lambda^{-1}XH = -\lambda^{-1}CX$, $HY - \lambda YH = YC$ and $HC - CH = 0$).

1. By [19, Theorem 7.5.(3).(i)] and statements 2, 3, we have gld $(A) \leq$ gld $(A') + 1 = 2 + 1 = 3 < \infty$. Notice that the maximal ideal $\mathfrak{m} = (C,H)$ of the algebra $K[C,H]$ is σ-invariant of height 2. Therefore, gld (A) = gld $(K[H,C]) + 1 = 2 + 1 = 3$, by Theorem 4.2. □

Chapter 5

The Krull dimension of $S^{-1}K[H, C](\sigma, a)$

This chapter contains the main work of finding the Krull dimension of generalized Weyl algebras $S^{-1}K[H, C][X, Y; \sigma, a]$ and the proofs of Theorem 1.10 and Theorem 1.11.

5.1 The Krull dimension of GWAs $A = D(\sigma, a)$

In this section, we present some of the known results about the Krull dimension of the generalized Weyl algebra that will be used in our proofs. The Krull dimension of generalized Weyl algebra A of degree 1 was studied in two cases, when the base ring is a noncommutative ring, in [37]; and when it is commutative ring, in [35]. Because our research has a focus on GWAs A when the base ring D is a commutative Noetherian ring, the following proposition specifies the two possible values for $\mathcal{K}(A)$.

Proposition 5.1. *([35, Proposition 2.2].) Let D be a left Noetherian ring. Then*

$$\mathcal{K}(D) \leq \mathcal{K}(A) \leq \mathcal{K}(D) + 1.$$

The next theorem gives more accurate result for $\mathcal{K}(A)$, provided that $\mathcal{K}(D) < \infty$.

Theorem 5.2. *([35, Theorem 1.2].) Let D be a commutative Noetherian ring with $\mathcal{K}(D) < \infty$ and $A = D(\sigma, a)$ be a generalized Weyl algebra. Then*

$\mathcal{K}(A) = \sup\{\mathcal{K}(D), \operatorname{ht}(\mathfrak{p}) + 1, \operatorname{ht}(\mathfrak{q}) + 1 \mid \mathfrak{p}$ *is a σ-unstable prime ideal of D*

for which there exist infinitely many integers i with $a \in \sigma^i(\mathfrak{p})$;
$\mathfrak{q}$ is a σ-semistable prime ideal of $D\}$.

Let D be a commutative Noetherian ring with gld $(D) = n < \infty$ and a be a regular element. Suppose that gld $(A) < \infty$. Then Theorem 4.2 says that

$$\begin{aligned}\text{gld }(A) = \sup\{&\text{gld }(D),\ \text{ht}\,(\mathfrak{p})+1,\ \text{ht}\,(\mathfrak{q})+1 \,|\, \mathfrak{p} \text{ is a } \sigma\text{-unstable prime ideal of } D\\ &\text{for which there exist distinct integers } i \text{ and } j \text{ with } a \in \sigma^i(\mathfrak{p}) \text{ and}\\ &a \in \sigma^j(\mathfrak{p});\ \mathfrak{q} \text{ is a } \sigma\text{-semistable prime ideal of } D\}.\end{aligned}$$

This formula of the global dimension and the formula of the Krull dimension given in Theorem 5.2, give a relationship between these two dimensions as follows [35],

$$\mathcal{K}\,(A) \leq \text{gld }(A).$$

5.2 Maximal ideals of $\mathcal{D} = S^{-1}K[H,C]$

Let $D = K[H,C]$ where K is an algebraically closed field. Suppose that $\mathcal{D} = S^{-1}D$ has Krull dimension 2. Recall that the maximal ideals of D are the prime ideals of D of height 2. Recall that $\text{Max}\,(D) = \{(H-\alpha, C-\beta) \,|\, \alpha,\, \beta \in K\}$ since the field K is algebraically closed. So, the set $\text{Max}\,(\mathcal{D}, \text{ht} = 2)$ of maximal ideals of $\mathcal{D}$ of height 2 is equal to

$$\{S^{-1}(H-\alpha, C-\beta) \,|\, (H-\alpha, C-\beta) \cap S = \emptyset \ \text{ where } \ \alpha, \beta \in K\}.$$

The algebra D is a *unique factorization* domain (UFD), hence so is $\mathcal{D}$. The group D^* of units of D is $K^* = K\backslash\{0\}$ but the group $\mathcal{D}^*$ of units of $\mathcal{D}$ is generated by all the irreducible divisors of elements of S since $\mathcal{D}$ is a UFD.

Recall that a nonzero element p of a commutative domain R is called an *irreducible element* of R if p is not a unit of R and $p = qr$ for some $q,\, r \in R$ implies that either q or r is a unit. The set of irreducible elements of R is denoted by $\text{Irr}\,(R)$. Given $p \in R$. Then $p \in \text{Irr}\,(R)$ if and only if $up \in \text{Irr}\,(R)$ for all $u \in R^*$.

For each height 2, maximal ideal $\mathfrak{m} = S^{-1}(H-\alpha, C-\beta)$ of $\mathcal{D}$,

$$\mathcal{D}/\mathfrak{m} \simeq S^{-1}(D/(H-\alpha, C-\beta)) \simeq K.$$

The canonical epimorphism

$$\pi_{\mathfrak{m}} : \mathcal{D} \longrightarrow \mathcal{D}/\mathfrak{m} = K, \qquad s^{-1}p \mapsto s^{-1}p + \mathfrak{m} \tag{5.1}$$

is the evaluation map at the point (α, β),

$$\pi_{\mathfrak{m}}(s^{-1}(H,C)p(H,C)) = s^{-1}(\alpha,\beta)p(\alpha,\beta). \tag{5.2}$$

Let $\mathcal{D} = S^{-1}D$ has Krull dimension 2. Recall that $\mathrm{Max}\,(\mathcal{D}, \mathrm{ht} = 2)$ is the set of maximal ideals of $\mathcal{D}$ of height 2 and let $\mathrm{Max}\,(\mathcal{D}, \mathrm{ht} = 1)$ be the set of maximal ideals of $\mathcal{D}$ of height 1.

In general, under localization of the polynomial algebra $D = K[H, C]$, the maximal ideals have different heights (see statements 3 and 4 of Proposition 5.3).

Proposition 5.3. *Let $D = K[H, C]$ where K is an algebraically closed field, $K = K_H \sqcup K_C$ is a disjoint union of non-empty subsets, S is a multiplicative submonoid of $(D\backslash\{0\}, \cdot)$ generated by the elements $\{H - \alpha,\, C - \beta \,|\, \alpha \in K_H,\, \beta \in K_C\}$ and $\mathcal{D} = S^{-1}D$.*

1. *$\mathcal{D}^* = \{K^* s^{-1}t \,|\, s,\, t \in S\}$ and $\mathcal{D}^* \cap D = K^*S \simeq K^* \times S$ a direct product of monoids.*

2. *Let $\alpha' \in K_C$ and $\beta' \in K_H$. Then $\mathcal{D}/(H - \alpha') \simeq S_2^{-1}K[C]$ and $\mathcal{D}/(C - \beta') \simeq S_1^{-1}K[H]$ where S_1 (respectively, S_2) is a multiplicative submonoid of $(K[H]\backslash\{0\}, \cdot)$ (respectively, $(K[C]\backslash\{0\}, \cdot)$) generated by the set $\{H - \alpha \,|\, \alpha \in K_H\}$ (respectively, $\{C - \beta \,|\, \beta \in K_C\}$). Furthermore, $(S_2^{-1}K[C])^* = \{K^* s^{-1}t \,|\, s,\, t \in S_2\}$ and $(S_1^{-1}K[H])^* = \{K^* s^{-1}t \,|\, s,\, t \in S_1\}$.*

3. $\mathrm{Max}\,(\mathcal{D}, \mathrm{ht} = 2) = \{S^{-1}(H - \beta, C - \alpha) \,|\, \alpha \in K_H,\, \beta \in K_C\}$*, the set of maximal ideals of $\mathcal{D}$ of height 2.*

4. $\mathrm{Max}\,(\mathcal{D}, \mathrm{ht} = 1) = \{\mathcal{D}p \,|\, p = p(H, C)$ *is an irreducible element of D which is not of the type $\{K^*(H - \alpha),\, K^*(C - \beta) \,|\, \alpha \in K_H,\, \beta \in K_C\}$ and such that $p(\beta, \alpha) \neq 0$ for all $\alpha \in K_H,\, \beta \in K_C\}$.*

5. *$\mathcal{D}(H - C) \in \mathrm{Max}\,(\mathcal{D}, \mathrm{ht} = 1)$.*

Proof. 1. Notice that $L := \{K^* s^{-1}t \,|\, s,\, t \in S\} \subseteq \mathcal{D}^*$. Conversely, given an element $s^{-1}d \in \mathcal{D}^*$, where $s \in S$ and $d \in D$. Equivalently, $d \in \mathcal{D}^*$. That is $s_1^{-1}d_1 d = 1$ for some $s_1 \in S$ and $d_1 \in D$, and so $d_1 d = s_1 \in S$. The ring D is a unique factorization domain and all the generators of the monoid S are irreducible elements of D, hence $d \in S$. Then $s^{-1}d \in L$, and so $\mathcal{D}^* = L$, as required. Now, $\mathcal{D}^* \cap D = K^*S = K^* \times S$.

2. Recall that $\alpha' \in K_C$. Let

$$\pi : D \longrightarrow \overline{D} = D/(H - \alpha') \simeq K[C], \qquad d \mapsto \overline{d} = d + (H - \alpha').$$

Then $S_2 = \pi(S)$. Now, $\mathcal{D}/(H - \alpha') = \mathcal{D}/\mathcal{D}(H - \alpha') = S^{-1}D/S^{-1}(H - \alpha') \simeq S^{-1}(D/(H - \alpha')) \simeq \overline{S}^{-1}\overline{D} \simeq S_2^{-1}K[C]$.

The ring $K[C]$ is a unique factorization domain and the multiplicative monoid S_2 of $(K[C]\backslash\{0\}, \cdot)$ is generated by irreducible elements $\{C - \beta \,|\, \beta \in K_C\}$ of $K[C]$. Therefore, $(S_2^{-1}K[C])^* = \{K^*s^{-1}t \,|\, s,\, t \in S_2\}$.

By the (H, C)-symmetry, we have the analogue results for the factor ring $\mathcal{D}/(C - \beta')$ where $\beta' \in K_H$.

3. Let R be the RHS of the equality in statement 3.

(i) $\mathrm{Max}\,(\mathcal{D}, \mathrm{ht} = 2) \supseteq R$: Given $S^{-1}(H - \beta, C - \alpha) \in R$ (where $\beta \in K_C$, $\alpha \in K_H$). By statement 2, the ideals $\mathcal{D}(H - \beta)$ and $\mathcal{D}(H - \beta, C - \alpha)$ are distinct prime ideals since $\mathcal{D}/\mathcal{D}(H - \beta) \simeq S_2^{-1}K[C]$ and

$$\mathcal{D}/\mathcal{D}(H - \beta, C - \alpha) \simeq S_2^{-1}K[C]/S_2^{-1}K[C](C - \alpha) \simeq K$$

are domains. Hence,

$$0 \subset \mathcal{D}(H - \beta) \subset \mathcal{D}(H - \beta, C - \alpha)$$

is a chain of distinct prime ideals of $\mathcal{D}$, and so $\mathrm{ht}\,\mathcal{D}(H - \beta, C - \alpha) \geq 2$. On the other hand,

$$\mathrm{ht}\,\mathcal{D}(H - \beta, C - \alpha) = \mathrm{ht}\,S^{-1}(H - \beta, C - \alpha) \leq \mathrm{ht}\,D(H - \beta, C - \alpha) = 2.$$

Therefore, $\mathrm{ht}\,\mathcal{D}(H - \beta, C - \alpha) = 2$, as required.

(ii) $\mathrm{Max}\,(\mathcal{D}, \mathrm{ht} = 2) \subseteq R$: Given $\mathfrak{m} \in \mathrm{Max}\,(\mathcal{D}, \mathrm{ht} = 2)$. Then $\mathfrak{m} = S^{-1}(H - \alpha, C - \beta)$ for some $\alpha,\, \beta \in K$. Then $\alpha \in K_C$ (respectively, $\beta \in K_H$) since otherwise the element $H - \alpha \in S$ (respectively, $C - \beta \in S$) is a unit of $\mathcal{D}$, and the statement (ii) follows.

4. Notice that an ideal of $\mathcal{D}$ belongs to $\mathrm{Max}\,(\mathcal{D}, \mathrm{ht} = 1)$ if and only if it is of type $\mathcal{D}p$ where p is an irreducible element of D which is not a unit of $\mathcal{D}$ (i.e. $p \notin \{K^*(H - \alpha),\, K^*(C - \beta) \,|\, \alpha \in K_H,\, \beta \in K_C\}$, by statement 1) and is not properly contained in a maximal ideal of $\mathcal{D}$ of height 2 (i.e. $p(\beta, \alpha) \neq 0$ for all $\alpha \in K_H$, $\beta \in K_C$, by statement 3, and since $\mathrm{ht}\,(\mathcal{D}p) = 1$ for all irreducible elements p of D which are not units of $\mathcal{D}$).

5. The element $p = H - C$ satisfies the conditions of statement 4: $p = p(H,C)$ is an irreducible element of D which is not of type

$$\{K^*(H-\alpha),\, K^*(C-\beta)\,|\,\alpha\in K_H,\, \beta\in K_C\}$$

and such that $p(\beta,\alpha) = \beta - \alpha \neq 0$ for all $\alpha \in K_H$ and $\beta \in K_C$ (since otherwise, $\alpha = \beta \in K_H \cap K_C = \emptyset$, a contradiction). $\square$

The next lemma is a criteria for the ideals $\mathcal{D}(H-\alpha)$ and $\mathcal{D}(C-\beta)$ not to be a maximal ideal of $\mathcal{D}$ where $\alpha,\, \beta \in K$.

Lemma 5.4. *Let $D = K[H,C]$ where K is an algebraically closed field and $\mathcal{D} = S^{-1}D$ has Krull dimension 2.*

1. *Suppose that $H - \alpha \notin \mathcal{D}^*$ for some $\alpha \in K$ (i.e. $(H-\alpha) \nmid s$ for all $s \in S$). Let $\overline{S}_{H-\alpha} = \{s + (H-\alpha) \in D/(H-\alpha) \simeq K[C]\,|\, s \in S\}$. Then $\mathcal{D}(H-\alpha) \notin \mathrm{Max}\,(\mathcal{D})$ if and only if $\overline{S}_{H-\alpha}^{-1}K[C]$ is not a field if and only if there is an element $\beta \in K$ such that $s(\alpha,\beta) \neq 0$ for all elements $s = s(H,C) \in S$.*
2. *Suppose that $C - \beta \notin \mathcal{D}^*$ for some $\beta \in K$ (i.e. $(C-\beta) \nmid s$ for all $s \in S$). Let $\overline{S}_{C-\beta} = \{s + (C-\beta) \in D/(C-\beta) \simeq K[H]\,|\, s \in S\}$. Then $\mathcal{D}(C-\beta) \notin \mathrm{Max}\,(\mathcal{D})$ if and only if $\overline{S}_{C-\beta}^{-1}K[H]$ is not a field if and only if there is an element $\alpha \in K$ such that $s(\alpha,\beta) \neq 0$ for all elements $s = s(H,C) \in S$.*

Proof. 1. By the assumption, $H - \alpha \notin \mathcal{D}^*$. Hence, $\mathcal{D}(H-\alpha)$ is a height 1 prime ideal of $\mathcal{D}$. Then the ideal is not maximal if and only if

$$\mathcal{D}(H-\alpha) \subsetneqq \mathcal{D}(H-\alpha, C-\beta)$$

for some $\beta \in K$ if and only if the algebra $\mathcal{D}/\mathcal{D}(H-\alpha) \simeq \overline{S}_{H-\alpha}^{-1}K[C]$ is not a field if and only if there is an element $\beta \in K$ such that $s(\alpha,\beta) \neq 0$ for all elements $s \in S$ (see (5.2)).

2. Statement 2 follows from statement 1 by the (H,C)-symmetry. $\square$

5.3 The Krull dimension of algebras $S^{-1}K[H,C](\sigma,a)$

As we mentioned in Chapter 1, if $\mathcal{A} = \mathcal{D}(\sigma,a)$ be a GWA of degree 1 where $\mathcal{D} = S^{-1}D$ is a localization of the polynomial algebra $D = K[H,C]$ over an algebraically closed field K at a multiplicative set S, generally there are three options for the algebra $\mathcal{D} = S^{-1}K[H,C]$: a field, a Dedekind domain or to have Krull dimension 2.

In the following proposition, we find the Krull dimension of $\mathcal{A}$ if $\mathcal{D}$ is a field or if it is a Dedekind domain.

Proposition 5.5. *Let $\mathcal{A} = \mathcal{D}(\sigma, a)$ be a GWA where $\mathcal{D} = S^{-1}D$ is a localization of the polynomial algebra $D = K[H,C]$ over a field K at a multiplicative set S.*

1. *If $a = 0$ then $\mathcal{K}(\mathcal{A}) = \mathcal{K}(\mathcal{D}) + 1$.*
2. *If $\mathcal{D}$ is a field then $\mathcal{K}(\mathcal{A}) = 1$.*
3. *If $\mathcal{K}(\mathcal{D}) = 1$ (i.e. $\mathcal{D}$ is a Dedekind domain) then*

$$\mathcal{K}(\mathcal{A}) = \begin{cases} 1 & \textit{if all maximal ideals of } \mathcal{D} \textit{ are } \sigma\textit{-unstable,} \\ 2 & \textit{otherwise.} \end{cases}$$

Proof. 1. If $a = 0$ then $\mathcal{K}(\mathcal{A}) = \mathcal{K}(\mathcal{D}) + 1$, by Theorem 5.2.

2. By Theorem 5.2, $\mathcal{K}(\mathcal{A}) = 1$ since 0 is a σ-semistable prime ideal of $\mathcal{D}$.

3. Since $\mathcal{D}$ is a Dedekind domain, for every nonzero element $a \in \mathcal{D}$ there are only finitely many maximal ideals of $\mathcal{D}$ that contain it. Now, statement 3 follows from Theorem 5.2. $\square$

Thus, the remaining case of $\mathcal{D}$ is that it has Krull dimension 2 that will discuss in Section 5.3.1 and Section 5.3.2.

5.3.1 The Krull dimension of $K[H,C](\sigma, a)$ i.e. $S = \{1\}$

Let R be a commutative ring, σ be an automorphism of R, and $a \in R$. Recall that $\mathcal{V}(a) = \{\mathfrak{m} \in \mathrm{Max}(R) \,|\, a \in \mathfrak{m}\}$ and let $V(a) := \{\mathfrak{p} \in \mathrm{Spec}\,(R) \,|\, a \in \mathfrak{p}\}$.

Let G be a group that acts on a set S and V be a subset of S. A typical example is $S = \mathrm{Spec}\,(A)$ where A is a commutative algebra, G is a group of automorphisms of A, $\mathfrak{a}$ is an ideal of A and $V(\mathfrak{a}) = \{\mathfrak{p} \in \mathrm{Spec}\,(A) \,|\, \mathfrak{a} \subseteq \mathfrak{p}\}$ is a subset of S. We are interested in question when $|\mathcal{O} \cap V| < \infty$ for all G-orbits $\mathcal{O}$.

Let H be a subgroup of G. Then $H\backslash G = \{Hg \,|\, g \in G\}$ be the set of right H-cosets.

Lemma 5.6. *Let G, S and V be as above and $\mathcal{O}$ be a G-orbit. Suppose that H is a subgroup of G such that the set of right H-cosets $H\backslash G$ is finite. Then $|V\cap\mathcal{O}|<\infty$ for all G-orbits in S if and only if $|V\cap\mathcal{O}'|<\infty$ for all H-orbits $\mathcal{O}'$ in S. In particular, $|V\cap\mathcal{O}|<\infty$ for a G-orbit $\mathcal{O}$ if and only if $|V\cap Hx|<\infty$ for all $x\in\mathcal{O}$.*

Proof. Let $H\backslash G=\{Hg_1,\dots,Hg_n\}$ and $s\in S$. Then $Gs=\bigcup_{i=1}^n Hg_is$ is a union of H-orbits. Hence, $V\bigcap Gs=\bigcup_{i=1}^n(V\cap Hg_is)$ and the statement follows. □

Lemma 5.7. *Let $D=K[H,C]$ and σ be an affine automorphism of D. If $\sigma^n(p)=bp$ for some $0\neq b\in D$ then $b\in K^*$.*

Proof. Since σ is an affine automorphism, it preserves the total degree. Then $\deg p=\deg\sigma^n(p)=\deg bp=\deg p+\deg b$, and so $\deg b=0$ since $b\neq 0$, and so we must have $b\in K^*$. □

The following proposition gives a criterion for $|\mathcal{V}(a)\cap\mathcal{O}|=\infty$ in the case σ is of type 1 of the affine automorphisms of the polynomial algebra $K[H,C]$.

Proposition 5.8. *Let $D=K[H,C]$ be a polynomial algebra over an algebraically closed field K, $\sigma\in\mathrm{Aut}_K(D)$, $\sigma(H)=H-1$ and $\sigma(C)=\lambda C$. Let $a\in D\backslash K$, $\mathfrak{m}=(H-\alpha,C-\beta)$ be a maximal ideal of D where α, $\beta\in K$ and $\mathcal{O}=\mathcal{O}(\mathfrak{m})$ be the σ-orbit of $\mathfrak{m}$. Then*

1. *If λ is a root of unity and* $\mathrm{char}\,(K)=0$ *then $|\mathcal{V}(a)\cap\mathcal{O}|=\infty$ if and only if $a\in(C-\beta)$.*
2. *If λ is not a root of unity,* $\mathrm{char}\,(K)=p>0$ *and $|\mathcal{O}|=\infty$ (i.e. $\beta\neq 0$) then $|\mathcal{V}(a)\cap\mathcal{O}|=\infty$ if and only if $a\in(H-\alpha)$.*
3. *If λ is not a root of unity and* $\mathrm{char}\,(K)=0$ *then $|\mathcal{V}(a)\cap\mathcal{O}|=\infty$ if and only if $a\in(C)$ and $\mathfrak{m}=(H-\alpha,C)$.*

Proof. 1. Notice that $|\mathcal{O}|=\infty$. Suppose that $\lambda=1$ and $|\mathcal{V}(a)\cap\mathcal{O}|=\infty$. That is $\mathcal{V}(a)\cap\mathcal{O}=\{\sigma^j(\mathfrak{m})\,|\,j\in J\}$ and J is an infinite subset of $\mathbb{Z}$. Notice that $\sigma^j(\mathfrak{m})=(H-\alpha-j,C-\beta)$.

CLAIM: $I:=\bigcap_{j\in J}(H-\alpha-j,C-\beta)=(C-\beta)$.

Clearly, $I\supseteq(C-\beta)$. We have to show that $I\subseteq(C-\beta)$. Suppose that this is not true, i.e. there is a polynomial $p\in I\backslash(C-\beta)$, we seek a contradiction.

Then $p = \sum_{i=0}^{n} p_i(C)H^i$ for some polynomials $p_i = p_i(C) \in K[C]$ such that not all scalars $\lambda_i := p_i(\beta)$ are equal to zero. Let $\overline{p}(H) = p(H, \beta) = \sum_{i=0}^{n} \lambda_i H^i$. Then the infinite set $\{\alpha + j \mid j \in J\}$ consists of roots of the nonzero polynomial $\overline{p}(H) \in K[H]$, a contradiction.

Since λ is a root of unity, $\lambda^n = 1$ for some $n \geq 1$. The group $G = \langle \sigma \rangle$ contains the subgroup $H = \langle \sigma^n \rangle$. Clearly, $|H \backslash G| < \infty$. Changing the variables (H, C) to $(H' := n^{-1}H, C)$ we see that $\sigma^n(H') = H' - 1$ and $\sigma^n(C) = \lambda^n C = C$. By the case $\lambda = 1$, $|\mathcal{V}(a) \cap \mathcal{O}'| = \infty$ for some H-orbit $\mathcal{O}'$ such that $\mathcal{O}' \subseteq \mathcal{O}$ if and only if $a \in (C - \beta)D$ for some $\beta \in K$. Now, by Lemma 5.6, $|\mathcal{V}(a) \cap \mathcal{O}| = \infty$ if and only if $a \in (C - \beta)D$.

2. $(\Rightarrow)$ In view of Lemma 5.6, it suffices to prove the statement for the automorphism σ^p rather than for σ. Notice that $\sigma^p(H) = H$ and $\sigma^p(C) = \lambda^p C$ and λ^p is not a root of unity. Suppose that $|\mathcal{V}(a) \cap \mathcal{O}'| = \infty$ for some σ^p-orbit $\mathcal{O}'$ of the maximal ideal $\mathfrak{m} = (H - \alpha, C - \beta)$ of the ring D. Notice that $\sigma^{pj}(\mathfrak{m}) = (H - \alpha, C - \lambda^{-pj}\beta)$ for all $j \in \mathbb{Z}$. That is $\mathcal{V}(a) \cap \mathcal{O}' = \{\sigma^{pj}(\mathfrak{m}) \mid j \in J\}$ where J is an infinite subset of $\mathbb{Z}$. Recall that $\beta \neq 0$.

CLAIM: $I := \bigcap_{j \in J}(H - \alpha, C - \lambda^{-pj}\beta) = (H - \alpha)$.

Clearly, $I \supseteq (H - \alpha)$. We have to show that $I \subseteq (H - \alpha)$. Suppose that this is not true, i.e. there is a polynomial $q \in I \backslash (H - \alpha)$, we seek a contradiction. Then $q = \sum_{i=0}^{m} q_i(H)C^i$ for some polynomials $q_i(H) \in K[H]$ such that not all scalars $\mu_i := q_i(\alpha)$ are equal to zero. Let $\overline{q}(C) = q(\alpha, C) = \sum_{i=0}^{m} \mu_i C^i$. Then the infinite set $\{\lambda^{-pj}\beta \mid j \in J\}$ consists of roots of the nonzero polynomial $\overline{q}(C) \in K[C]$, a contradiction.

$(\Leftarrow)$ The implication is obvious.

3. $(\Rightarrow)$ If $|\mathcal{V}(a) \cap \mathcal{O}| = \infty$ then, by Theorem 2.54 and Proposition 2.60, there is an irreducible divisor, say p, of the element a such that $\sigma^n(p) \in (p)$ and $p \in \mathfrak{m}$, i.e. $\sigma^n(p) = \nu p$ for some $n \geq 1$ and $\nu \in K^*$, by Lemma 5.7. The polynomial algebra $D = K[H] \otimes K[C]$ is a tensor product of polynomial σ^n-invariant subalgebras $K[H]$ and $K[C]$ of D. Furthermore, the automorphism σ^n acts on $K[H]$ locally nilpotently but on $K[C]$ it acts semi-simply, ($K[C] = \oplus_{i \geq 0} KC^i$ and $\sigma^n(C^i) = \lambda^{ni}C^i$ for all $i \geq 0$). Therefore, $D = \oplus_{i \geq 0} D^{\lambda^{ni}}$ where $D^{\lambda^{ni}} = \bigcup_{j \geq 0} \ker_D(\sigma^n - \lambda^{ni})^j = K[H]C^i$. Hence, an eigenvector of σ^n is equal to γC^i for some $\gamma \in K^*$ and $i \geq 0$, and vice versa. Hence, $p \in K^*C$ (since p is irreducible). Therefore, $a \in (C)$ and $C \in \mathfrak{m}$ (i.e., $\mathfrak{m} = (H - \alpha, C)$), as required.

$(\Leftarrow)$ This implication is obvious since $\sigma(C) = \lambda C \in (C)$. $\square$

Now, we prove Theorem 1.10 that gives the values of the Krull dimension of

$A = K[H,C](\sigma, a)$ where σ is of type 1, that is $\sigma(H) = H - 1$ and $\sigma(C) = \lambda C$ for some $\lambda \in K^*$.

Proof of Theorem 1.10. By Theorem 5.2, $\mathcal{K}(A) = 2$ or 3.

If $\mathrm{char}(K) = p > 0$ then the maximal ideal $\mathfrak{m} = (C, H)$ of the algebra D is σ^p-stable and $\mathrm{ht}(\mathfrak{m}) = 2 = \mathcal{K}(D)$. Hence, $\mathcal{K}(A) = 3$, by Theorem 5.2.

If $\mathrm{char}(K) = 0$ and λ is a root of unity then all maximal ideals of the algebra $K[H, C]$ are σ-unstable. Hence, by Theorem 5.2, $\mathcal{K}(A) = 3$ if and only if there is a σ-orbit $\mathcal{O} = \mathcal{O}(\mathfrak{m})$ of a maximal ideal $\mathfrak{m} = (H - \alpha, C - \beta)$ of $K[H, C]$ (where α, $\beta \in K$) such that $|\mathcal{V}(a) \cap \mathcal{O}| = \infty$. Now, by Theorem 5.2 and Proposition 5.8.(1), $\mathcal{K}(A) = 3$ if and only if $a \in (C - \beta)$ for some $\beta \in K$.

Suppose that $\mathrm{char}(K) = 0$ and λ is not a root of unity. Then all maximal ideals of the algebra $K[H, C]$ are σ-unstable. Hence, by Theorem 5.2, $\mathcal{K}(A) = 3$ if and only if there is a σ-orbit $\mathcal{O} = \mathcal{O}(\mathfrak{m})$ of a maximal ideal $\mathfrak{m} = (H - \alpha, C - \beta)$ of $K[H, C]$ (where α, $\beta \in K$) such that $|\mathcal{V}(a) \cap \mathcal{O}| = \infty$. Now, by Theorem 5.2 and Proposition 5.8.(3), $\mathcal{K}(A) = 3$ if and only if $a \in (C)$. $\square$

Theorem 5.9. *Let $A = D(\sigma, a)$ be a GWA such that $D = K[H, C]$, $\sigma(H) = \lambda H$ and $\sigma(C) = \mu C$ where λ, $\mu \in K^*$. Then $\mathcal{K}(A) = 3$.*

Proof. By Theorem 5.2, $\mathcal{K}(A) = 3$ since the maximal ideal $\mathfrak{m} = (H, C)$ of the polynomial algebra D is σ-stable and $\mathrm{ht}(\mathfrak{m}) = 2 = \mathcal{K}(D)$. $\square$

Theorem 5.10. *Let $A = D(\sigma, a)$ be a GWA such that $D = K[H, C]$, K is an algebraically closed field, $\sigma(H) = H - 1$ and $\sigma(C) = C + H$. Then*

1. $\mathcal{K}(A) = 3$ *if and only if either* $\mathrm{char}(K) \neq 0$ *or* $\mathrm{char}(K) = 0$ *and the element a is divisible by the polynomial $\eta - \mu$ for some $\mu \in K$ where $\eta = C + \frac{H(H+1)}{2}$.*
2. $\mathcal{K}(A) = 2$ *otherwise.*

Proof. 1. If $\mathrm{char}(K) \neq 0$ then the maximal ideal $\mathfrak{m} = (C, H)$ of the algebra D is σ-semistable and $\mathrm{ht}(\mathfrak{m}) = 2 = \mathrm{Kdim}(D)$. Hence, $\mathrm{Kdim}(A) = 3$, by Theorem 5.2. Suppose that $\mathrm{char}(K) = 0$. By Proposition 3.6.(3), $D = K[\eta] \otimes K[H]$.

(i) *If $\sigma^n(d) = \gamma d$ for some $n \geq 1$ and γ then $\gamma = 1$:*

Since $\sigma^n(\eta) = \eta$ and $\sigma^n(H) = H - n$. The K-liner map $\sigma^n - 1 : D \to D$, $u \mapsto \sigma^n(u) - u$ is a locally nilpotent map. Therefore, 0 is the only eigenvalue for $\sigma^n - 1$, and the statement (i) follows.

(ii) *If* $|\mathcal{O}(\mathfrak{m}) \cap \mathcal{V}(a)| = \infty$ *for some* $\mathfrak{m} \in \mathrm{Max}(D)$ *iff* a *is divisible by* $\eta - \mu$ *for some* $\mu \in K$:

($\Rightarrow$) Let $a = p_1 \dots p_s$ be a product of irreducible polynomials in D. Then $|\mathcal{O}(\mathfrak{m}) \cap \mathcal{V}(p_i)| = \infty$ for some i. By Theorem 2.54 and Proposition 2.60, $\sigma^n(p_i) = \gamma p_i$ for some $n \geq 1$ and $\gamma \in K^*$. By the statement (i), $\gamma = 1$, i.e., $p_i \in D^{\sigma^n} = D^\sigma = K[\eta]$ (since $\mathrm{char}\,(K) = 0$ and $\eta = C + \frac{H(H+1)}{2}$). Since K is an algebraically closed field and p_i is an irreducible polynomial of D, we must have $p_i = \nu(\eta - \mu)$ for some $\nu \in K^*$ and $\mu \in K$. So, the element a is divisible by $\eta - \mu$.

($\Leftarrow$) This implication is obvious since then $\mathfrak{m} \in \mathcal{V}(\eta - \mu)$. Then $|\mathcal{O}(\mathfrak{m}) \cap \mathcal{V}(a)| \geq |\mathcal{O}(\mathfrak{m}) \cap \mathcal{V}(\eta - \mu)| = |\mathbb{Z}| = \infty$.

(iii) *If* $\mathrm{char}\,(K) = 0$ *then* $\mathrm{Kdim}\,(A) = 3$ *iff the element* a *is divisible by* $\eta - \mu$ *for some* $\mu \in K$:

Since $\sigma(H) = H - 1$, $|\mathcal{O}(\mathfrak{m})| = \infty$ for all $\mathfrak{m} \in \mathrm{Max}(D)$. Then the statement (iii) follows from the statement (ii) and Theorem 5.2.

2. By Theorem 5.2, $\mathrm{Kdim}\,(A) = 2$ or 3. Now, statement 2 follows from statement 1. $\square$

Theorem 5.11. *Let* $A = D(\sigma, a)$ *be a GWA such that* $D = K[H, C]$, $\sigma(H) = \lambda H + C$ *and* $\sigma(C) = \lambda C$ *where* $\lambda \in K^*$. *Then* $\mathcal{K}\,(A) = 3$.

Proof. The maximal ideal $\mathfrak{m} = (H, C)$ is σ-stable and $\mathrm{ht}\,(\mathfrak{m}) = 2 = \mathcal{K}\,(D)$. Hence, $\mathcal{K}\,(A) = 3$, by Theorem 5.2. $\square$

5.3.2 The Krull dimension of some algebras $S^{-1}K[H, C](\sigma, a)$

Let $A = D[X, Y; \sigma, a]$ be a GWA. Suppose that $a = ub$ for some unit u of D. Then

$$A = D[X, u^{-1}Y; \sigma, b]. \tag{5.3}$$

So, if $a \in \mathcal{D} = S^{-1}D$ we can assume that $a \in D$ without less of generality.

Proof of Theorem 1.11. By (5.3), without less of generality we may assume that $a \in D$. Recall that $\mathcal{K}\,(\mathcal{D}) = 2$. Then, by Theorem 5.2, $\mathcal{K}\,(\mathcal{A}) = 2$ or 3.

Notice that there is a maximal ideal $\mathfrak{m} = (H - \alpha, C - \beta)$ of $\mathcal{D}$ (where α, $\beta \in K$) such that its σ-orbit is finite if and only if either $\mathrm{char}\,(K) \neq 0$ and λ is a root of unity or $\mathrm{char}\,(K) \neq 0$, λ is not a root of unity and $(H - \alpha, C) \neq \mathcal{D}$ (this follows from the equality $\sigma^i(\mathfrak{m}) = (H - \alpha - i, C - \lambda^{-i}\beta)$ for all $i \in \mathbb{Z}$).

Suppose that $\mathrm{char}\,(K) \neq 0$, λ is not a root of unity, $(H-\alpha, C) = \mathcal{D}$ for all $\alpha \in K$. In this case every σ-orbit is infinite. Then, by Proposition 5.8.(2), $|\mathcal{V}_{\mathcal{D}}(a) \cap \mathcal{O}| = \infty$ for the σ-orbit of a maximal ideal $\mathfrak{m} = (H-\alpha, C-\beta)$ of $\mathcal{D}$ (where α, $\beta \in K$) if and only if $a \in (H-\alpha)$ and the prime ideal $(H-\alpha)$ is not a maximal ideal of $\mathcal{D}$.

Suppose that $\mathrm{char}\,(K) = 0$ and λ is a root of unity. Notice that $\sigma(D) = D$, $\mathrm{Spec}\,(\mathcal{D}) \subseteq \mathrm{Spec}\,(D)$ (via $\mathfrak{m} \mapsto \mathfrak{m} \cap D$) and every σ-orbit in $\mathrm{Spec}\,(\mathcal{D})$ is also a σ-orbit in $\mathrm{Spec}\,(D)$. Then $|\mathcal{V}_{\mathcal{D}}(a) \cap \mathcal{O}| = \infty$ for the σ-orbit of a maximal ideal $\mathfrak{m} = (H-\alpha, C-\beta)$ of $\mathcal{D}$ (where α, $\beta \in K$) if and only if $|\mathcal{V}_{D}(a) \cap \mathcal{O}| = \infty$ for the σ-orbit of the maximal ideal $\mathfrak{m} \cap D = (H-\alpha, C-\beta)$ of D if and only if $a \in \mathcal{D}(C-\beta)$, by Proposition 5.8.(1). Now, the statement 4 follows.

Suppose that $\mathrm{char}\,(K) = 0$ and λ is not a root of unity. Then $|\mathcal{V}_{\mathcal{D}}(a) \cap \mathcal{O}| = \infty$ for the σ-orbit of a maximal ideal $\mathfrak{m} = (H-\alpha, C-\beta)$ of $\mathcal{D}$ (where α, $\beta \in K$) if and only if $|\mathcal{V}_{D}(a) \cap \mathcal{O}| = \infty$ for the σ-orbit of the maximal ideal $\mathfrak{m} \cap D = (H-\alpha, C-\beta)$ of D if and only if $a \in DC$ and $\beta = 0$, by Proposition 5.8.(3), and the statement 5 follows.

Now, the theorem follows from Theorem 5.2. □

Theorem 5.12. *Let S be a multiplicative subset of $D = K[H,C]$ such that K is an algebraically closed field, $\sigma(H) = H - 1$, $\sigma(C) = C + H$, $\sigma(S) = S$, and $\mathcal{K}\,(\mathcal{D}) = 2$ where $\mathcal{D} = S^{-1}D$, and $\mathcal{A} = \mathcal{D}(\sigma, a)$ is a GWA where $0 \neq a \in \mathcal{D}$. Then $\mathcal{K}\,(\mathcal{A}) = 2$ or 3; and $\mathcal{K}\,(\mathcal{A}) = 3$ if and only if either* $\mathrm{char}\,(K) \neq 0$ *or* $\mathrm{char}\,(K) = 0$ *and a divided by $\eta - \mu$ for some $\mu \in K$.*

Proof. Recall that $\mathcal{K}\,(\mathcal{D}) = 2$. Then, by Theorem 5.2, $\mathcal{K}\,(\mathcal{A}) = 2$ or 3. If $\mathrm{char}\,(K) \neq 0$, then by Proposition 3.6.(3), the order of the automorphism σ is finite. Hence, $\mathcal{K}\,(\mathcal{A}) = 3$.

By (5.3), without less of generality we may assume that $a \in D$. Suppose that $\mathrm{char}\,(K) = 0$. Notice that $\sigma(D) = D$, $\mathrm{Spec}\,(\mathcal{D}) \subseteq \mathrm{Spec}\,(D)$ (via $\mathfrak{m} \mapsto \mathfrak{m} \cap D$) and every σ-orbit in $\mathrm{Spec}\,(\mathcal{D})$ is also a σ-orbit in $\mathrm{Spec}\,(D)$. Then $|\mathcal{V}_{\mathcal{D}}(a) \cap \mathcal{O}| = \infty$ for the σ-orbit of a maximal ideal $\mathfrak{m} = (H-\alpha, C-\beta)$ of $\mathcal{D}$ (where α, $\beta \in K$) if and only if $|\mathcal{V}_{D}(a) \cap \mathcal{O}| = \infty$ for the σ-orbit of the maximal ideal $\mathfrak{m} \cap D = (H-\alpha, C-\beta)$ of D if and only if a divided by $\eta - \mu$ for some $\mu \in K$, by Theorem 5.10.(1). Now, the theorem follows from Theorem 5.2. □

5.4 The Krull dimension of tensor products of GWAs

For any field K it is well known that if A and B are left Noetherian K-algebras such that $\mathcal{K}(A \otimes_K B)$ exists then [36],

$$\mathcal{K}(A \otimes_K B) \geq \mathcal{K}(A) + \mathcal{K}(B).$$

In general, the equality does not hold, for example if $K(X)$ is a field of rational functions in n indeterminates $K(x_1, \ldots, x_n)$ then the Krull dimension

$$\mathcal{K}(K(X) \otimes K(X)) = n,$$

whereas the sum of their Krull dimensions $\mathcal{K}(K(X)) + \mathcal{K}(K(X)) = 0$ [36]. However, the equality can be satisfied with some conditions on these algebras, the following definition gives an example of algebra where the equality is achieved.

Definition 5.13. *([36, p. 100].) A K-algebra A is said to be* tensor Krull minimal (TKM) *with respect to a class of K-algebras Ω if*

$$\mathcal{K}(A \otimes B) = \mathcal{K}(A) + \mathcal{K}(B),$$

foe each $B \in \Omega$.

Bavula and Lenagan proved that the generalized Weyl algebras over affine commutative K-algebras where K is an uncountable algebraically closed field, and their tensor products, are TKM with respect to the class of countably generated left Noetherian K-algebras. In particular,

$$\mathcal{K}(\bigotimes_{i=1}^{n} A_i) = \sum_{i=1}^{n} \mathcal{K}(A_i),$$

where $A_i = D_i(\sigma_i, a_i)$ are generalized Weyl algebras with each D_i is an affine commutative algebra over an uncountable algebraically closed field K [36, Lemma 2.1, Theorem 2.2]. This result can be used to compute the Krull dimension of GWA of degree n that is a tensor product of GWAs of degree 1 in Section 5.3.1 and Section 5.3.2.

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