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Stability conditions for the Kronecker quiver and quantum cohomology of the projective line.

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Abstract

We compute the spaces of stability conditions of the sequence of triangulated Calabi–Yau-n categories associated to the Kronecker quiver. By considering Frobenius structures, a relationship between these stability manifolds and the quantum cohomology of the projective line is explored.

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I'd like to dedicate this thesis to my grandpa Alan and aunt Sally. I can only imagine what they'd have had to say about it, but I suspect it'd have been quite a lot.

Contents

Introduction

The space of stability conditions is a complex manifold, constructed from the data of a triangulated category. The notion of a stability condition is motivated by ideas arising in string theory, and arrives in algebraic geometry via mirror symmetry. This motivation suggests that in some cases, the geometry of the space of stability conditions should be not just a complex manifold, but that it should additionally carry the structure of a Frobenius manifold.

The main focus of this work is the construction of the spaces of stability conditions associated to the Calabi–Yau-n triangulated categories of the Kronecker quiver, and the exploration of the connection between these spaces and the quantum cohomology of the projective line, which is a complex manifold with a known and well studied Frobenius structure.

Spaces of stability conditions on triangulated categories

Stability conditions on triangulated categories were first defined by Bridgeland [7], giving a purely mathematical interpretation of work in mathematical physics by Douglas [16]. The data needed to construct a stability condition is that of a triangulated category $\mathcal{D}$. Given such a category, the two questions that one asks are:

(i) Does a stability condition exist on $\mathcal{D}$?

(ii) What is the space of all stability conditions on $\mathcal{D}$ as a complex manifold?

The answer to the first question is related to the existence of a bounded t-structure on $\mathcal{D}$. Such structures were introduced by Beilinson, Bernstein, and Deligne [5] long before the modern study of mirror symmetry and string theory, and so this relationship underlines the purely mathematical, even categorical, nature of stability conditions. Hence any triangulated category on which the existence of a bounded t-structure is known is one for which we would like to compute the stability manifold. The first examples, that of the stability manifolds of the derived categories of curves, prove to be straightforward and the notion of stability conditions in this context can be related to the familiar concept of slope stability of coherent sheaves on curves. In the case of two-dimensional varieties the situation is more complicated and it becomes even more so as the dimension increases. The stability manifolds of K3 surfaces are computed by Bridgeland [8], and Li recently proved the existence of stability conditions on the quintic threefold [35].

String theory and mirror symmetry

That stability conditions lead to the construction of a complex manifold from the data of a triangulated category is the main result of [7]. The idea that further geometric structure exists on this space, namely that of a Frobenius manifold, is rooted in string theory.

One of the main mathematical objects of interest to physicists is the moduli space of $\mathcal{N} = 2$ superconformal field theories (SCFTs), as it is the existence of these theories that string theory predicts. To each $\mathcal{N} = 2$ SCFT, two topological conformal field theories (TCFTs) are associated. These are called the A-model and the B-model of the field theory. We do not define what these objects are, but instead consider how their properties and relationships to one another can be translated into mathematical language. The mathematical consequences of the relationship between these field theories gives rise to the field of mirror symmetry.

The central geometric objects in mirror symmetry are a class of varieties called Calabi–Yau threefolds. These are three-dimensional complex manifolds with the necessary geometric properties to occupy the 'extra' six real dimensions predicted by string theory. The translation from the language of algebraic geometry to that of physics and string theory is made via a construction called the non-linear σ-model, which associates to a Calabi–Yau threefold an $\mathcal{N} = 2$ SCFT. For a given Calabi–Yau threefold X and its corresponding field theory, the A and B models of string theory are expected to correspond to two A_∞-categories defined from the geometric properties of X: the derived Fukaya category, and the bounded derived category of coherent sheaves respectively.

From a purely mathematical perspective, a natural question to ask is what the relationship between these two categories is. Particularly, are they equivalent? In general, equivalence of these categories is not conjectured (although, in the elliptic curve formulation of this problem, these categories are in fact equivalent). The relationship, proposed by Kontsevich, requires the definition of another Calabi–Yau threefold, called the mirror of X. Two Calabi–Yau threefolds X and $\check{X}$ are called mirror to each other if the corresponding SCFTs are mapped to each other under an involution called the mirror map on the moduli space of superconformal field theories. On the level of associated TCFTs, the mirror map has the effect of switching the A and B models. In terms of A_∞-categories, this leads to the Homological Mirror Symmetry Conjecture of Kontsevich [33], which states that there is an equivalence of triangulated categories

$$\mathrm{D^b}\,\mathrm{Fuk}(X) \cong \mathrm{D^b}\,\mathrm{Coh}(\check{X}).$$

The category $\mathrm{D^b}\,\mathrm{Coh}(\check{X})$ is the derived category of coherent sheaves of the Calabi–Yau threefold $\check{X}$, with which algebraic geometers are familiar. On the other hand, the Fukaya category arises from the symplecto-geometric properties of the threefold. As such, homological mirror symmetry provides a bridge between the complex and symplectic geometry of a pair of mirror manifolds. Note that despite the philosophy of mirror Calabi–Yau pairs being rooted in the predictions of string theory, the relationship can be formulated concretely in mathematical language.

The Calabi–Yau threefolds X and $\check{X}$ are endowed with both complex and Kähler structures, and information about the moduli spaces of these structures is encoded in the moduli space of SCFTs. The stringy Kähler moduli space relates to the space of stability conditions constructed from the derived category of X, while the complex moduli space relates to the space of stability conditions on the Fukaya category. The space of stability conditions $\mathrm{Stab}(\mathrm{D^b}(X))$ is constructed in such a way that the stringy Kähler moduli space should embed into the quotient of the stability manifold by the actions of the complex numbers and the group of autoequivalences of $\mathrm{D^b}\,\mathrm{Coh}(X)$. Examples supporting this relationship between the stringy Kähler moduli space and stability conditions are considered by Hirano and Wemyss [24]. Understanding this embedding is one of the main motivations for studying the space of stability conditions. In order to do so, a better understanding of the geometric structure of the space of stability conditions is needed. Explicitly, it is expected in some cases that the space of stability conditions should not just be a complex manifold, but that it can be endowed with the additional geometric structure of a Frobenius manifold. In terms of mirror symmetry, the A-model Frobenius structure of a Calabi–Yau threefold X should be locally isomorphic to the B-model Frobenius structure of its mirror $\check{X}$ [17].

Frobenius manifolds

Frobenius manifolds were introduced by Dubrovin [17] and have been studied greatly due to their connection to the Witten–Dijkgraaf–Verlinde–Verlinde (WDVV) equations. Given a solution to the WDVV equations, a Frobenius manifold can be constructed, and conversely given a Frobenius manifold, the local structure determines solutions to the WDVV equations. The two most commonly studied examples of Frobenius manifolds are those arising from the quantum cohomology of projective spaces, and from the unfolding spaces of singularities. These unfolding spaces are special cases of the deformation spaces of Calabi–Yau threefolds and mirror symmetry conjectures that there is a relationship between these Frobenius structures and those arising as quantum cohomologies of projective varieties.

Previous work by Ikeda [27] for $\mathrm{A_N}$ quivers, and independently by Bridgeland, Qiu, and Sutherland [13] for the $\mathrm{A_2}$ case, has led to a connection between the Frobenius structure on the unfolding spaces associated to these quivers, and the spaces of stability conditions of the triangulated categories associated to them. The focus of this work is to extend these results to the Kronecker quiver, with the Frobenius structure on the space of stability conditions corresponding to that of the quantum cohomology of $\mathbb{P}^1$. Via mirror symmetry, this is the natural Frobenius structure on this space. For spaces of stability conditions where there is a priori no indication as to what the Frobenius structure may be, the question of how they can be endowed with Frobenius structures is the topic of ongoing work. Recent work by Ikeda and Qiu [28] develops the definition of a stability condition to that of an $\mathbb{X}$-stability condition. The extra data required for this definition seems to more naturally match up stability conditions with Frobenius structures.

We conclude this overview of the relationship between spaces of stability conditions and Frobenius manifolds by interpreting it in terms of TCFTs and mirror symmetry. The space of solutions to the WDVV equations is identified with the moduli space of topological conformal field theories. As an A_∞-category can be associated to a TCFT, it is thus possible to associate to this category a Frobenius manifold. Our aim is to understand the relationship between this Frobenius manifold and the complex manifold that arises as the space of stability conditions. If it can be shown that the space of stability conditions is itself a Frobenius manifold, then the next question is whether the space of stability conditions of a Calabi–Yau threefold is in fact the Frobenius manifold corresponding to the associated TCFT.

Calabi–Yau categories for quivers

Having motivated in terms of mirror symmetry why the study of Frobenius structures on spaces of stability conditions is of interest, let us refocus our attention on the explicit problem under consideration: The computation of spaces of stability conditions for the Kronecker quiver. As stability manifolds are constructed from triangulated categories, the first thing to do is to introduce the relevant triangulated categories. The Ginzburg algebras of a quiver Q are graded path algebras from which a sequence of triangulated categories, denoted $\mathrm{D_n}(Q)$ for $n \geq 2$ an integer, are constructed. The first question to address is why it is these triangulated categories that are studied, as our focus so far has been on the triangulated categories arising from the geometric properties of Calabi–Yau threefolds.

The categories for which we compute the spaces of stability conditions are examples of

Calabi–Yau-n categories. Such categories were introduced with the goal of creating new examples of triangulated categories which have similar features to the derived categories of Calabi–Yau threefolds. This is a necessary step, as in order for the relationship between the moduli spaces of complex and Kähler structures to be well defined, non-commutative varieties must be considered. A triangulated category $\mathcal{D}$ has the Calabi–Yau-n property for an integer $n \geq 2$ if for all objects X and Y in $\mathcal{D}$, there is an isomorphism

$$\mathrm{Hom}^*_{\mathcal{D}}(X, Y) \cong \mathrm{Hom}^*_{\mathcal{D}}(Y, X[n])^\vee.$$

This duality is the defining feature of Calabi–Yau-n categories. Some of the most natural examples of these categories are the derived categories of Calabi–Yau n-folds, and the Ginzburg algebras of a quiver are constructed in such a way that their derived categories also exhibit this property [21].

Main result

It is the spaces of stability conditions for this sequence of categories $\mathrm{D_n}(\mathrm{K}_2)$ for K_2 the Kronecker quiver that we compute. Another point of interest of this problem is the relationship between the Kronecker quiver and the projective line. It is well known that there is an equivalence between the triangulated categories $\mathrm{D^b}(\mathrm{K}_2)$ and $\mathrm{D^b}(\mathbb{P}^1)$. The quantum cohomology of a variety is an invariant which encodes information about the space's symplectic geometry. Our study of the spaces of stability conditions of the Kronecker quiver reveals that geometric features of the quantum cohomology of $\mathbb{P}^1$ appear naturally in the construction of these stability manifolds.

Theorem. *For $3 \leq n < \infty$, there is an isomorphism of complex manifolds*

$$\Psi_n \colon (\mathbb{C}_{(a)} \times \mathbb{C}^*_{(b)}) \setminus \Delta \xrightarrow{\sim} \mathrm{Stab}_*(\mathrm{D_n}(\mathrm{K}_2))/G_n$$

where $\Delta := \{(a, b) \colon a^2 = b\}$, $\mathrm{Stab}_*(\mathrm{D_n}(\mathrm{K}_2))$ is a connected component of the stability manifold, and G_n is a subquotient group of the group of autoequivalences of $\mathrm{D_n}(\mathrm{K}_2)$. Under this isomorphism, the central charge map corresponds to the twisted period mapping of the quantum cohomology of the projective line

$$(a, b) \mapsto \left(\int_{\xi_1} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\frac{n-2}{2}} \frac{dz}{z}, \int_{\xi_2} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\frac{n-2}{2}} \frac{dz}{z} \right).$$

The group G_n is an index $n-2$ subgroup of the subquotient group of autoequivalences which preserve the connected component $\mathrm{Stab}_*(\mathrm{D_n}(\mathrm{K}_2))$, modulo those which act trivially. In particular, for the category $\mathrm{D}_3(\mathrm{K}_2)$, it is exactly this subquotient group.

The space of stability conditions of the projective line was calculated as $\mathrm{Stab}_*(\mathbb{P}^1) \cong \mathbb{C}^2$ by Okada [37]. Extending the methods which are used to compute the spaces of stability conditions for the Calabi–Yau-n categories of the Kronecker quiver would reproduce this result at the level of complex manifolds for the category $\mathrm{D}_\infty(\mathrm{K}_2) := \mathrm{D}^{\mathrm{b}}(\mathrm{Rep}(\mathrm{K}_2))$.

Conjecture. *There is an isomorphism of complex manifolds*

$$\Psi_\infty \colon \mathbb{C}^2_{(a,b)} \xrightarrow{\sim} \mathrm{Stab}_*(\mathrm{D}_\infty(\mathrm{K}_2))$$

under which the central charge map corresponds to the map

$$(a,b) \mapsto \left(\int_{\xi_1} e^{(\sqrt{b}z+2a+\sqrt{b}z^{-1})} \frac{dz}{z}, \int_{\xi_2} e^{(\sqrt{b}z+2a+\sqrt{b}z^{-1})} \frac{dz}{z} \right).$$

A natural question to ask is whether there is a relationship between the Kronecker quiver and projective line beyond the equivalence $\mathrm{D}^{\mathrm{b}}(\mathrm{K}_2) \cong \mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)$, given the relationship between the stability manifolds of triangulated categories associated to the Kronecker quiver and the quantum cohomology of $\mathbb{P}^1$. We have mentioned that it is conjectured that the two main classes of Frobenius manifolds are related via mirror symmetry. Some of the geometric ideas underlying this conjecture can be highlighted by the above theorem: Given the projective variety $\mathbb{P}^1$, its quantum cohomology is computed, defining a Frobenius manifold. Part of the data of the mirror of $\mathbb{P}^1$ is the Landau–Ginzburg potential, which is the function $p \colon \mathbb{C} \longrightarrow \mathbb{C}$ defined by $p(z) = \sqrt{b}z+2a+\sqrt{b}z^{-1}$ for $a \in \mathbb{C}$ and $b \in \mathbb{C}^*$. The unfolding space of this function also carries a Frobenius structure. This construction can be generalised to a relationship between the Frobenius structures of the quantum cohomology of a projective Fano variety and the unfolding space of the Landau–Ginzburg potential of its mirror. The Landau–Ginzburg potential of the mirror of the projective line parametrises the space of quadratic differentials on the marked surface associated to the Kronecker quiver, demonstrating another aspect of the relationship between these two objects.

Structure

The first two chapters introduce the necessary background material for the rest of the thesis. Chapter 1 contains the categorical background for the computations of the spaces of stability conditions on the Calabi–Yau-n categories of the Kronecker quiver; the definition and properties of Frobenius manifolds, specifically the example of the quantum cohomology of $\mathbb{P}^1$, are discussed in Chapter 2.

The remaining chapters lead towards the computation of the stability manifolds in the main theorem as follows: Chapter 3 details the construction of the exchange graphs of the sequence of triangulated categories associated to the Kronecker quiver. This information is utilised to compute the groups of autoequivalences of these categories. The focus of Chapter 4 is the construction of a fundamental domain for the action of the group of autoequivalences on the space of stability conditions. In Chapter 5 the Schwarz Triangle Theorem is applied to show that this region is biholomorphic to the upper half plane. The calculations of these three chapters culminate in the description of the spaces of stability conditions for the Kronecker quiver in Chapter 6.

Chapter 1

Stability conditions for quivers

This chapter introduces key definitions and concepts for the computation of the spaces of stability conditions of triangulated categories associated to quivers.

It begins with a review of some properties of the derived category of the Kronecker quiver, then details the construction of the sequence of triangulated categories for which we construct the spaces of stability conditions. Tilting theory is then developed in order to define the exchange graph of a triangulated category, and then some important autoequivalences are introduced. The final section defines stability conditions and recalls some of the key features of this theory.

1.1 Preliminaries on the Kronecker quiver

This section fixes notation and reviews material on the representations of the Kronecker quiver and some properties of its derived category.

Denote by $Q = \{Q_0, Q_1\}$ the quiver with set of vertices Q_0 and set of arrows Q_1. For an arrow $a \in Q_1$, denote by $s(a)$ the vertex at which it starts and $t(a)$ the vertex at which it terminates.

Definition 1.1.1. Let $Q = \{Q_0, Q_1\}$ be a quiver.

(i) A representation of Q is an assignment of a vector space V_i to each vertex $i \in Q_0$, and linear maps $\mu\colon V_i \longrightarrow V_j$ for each arrow $a\colon i \longrightarrow j$ in Q_1.

(ii) A *finite dimensional representation* of Q is a representation in which every V_i is finite dimensional.

(iii) The *dimension vector* of a representation is the vector $\underline{m} = (\dim V_i)_{i \in Q_0}$.

Let K_2 be the 2-arrow Kronecker quiver $\bullet_1 \rightrightarrows \bullet_2$. A finite dimensional representation of K_2 has dimension vector $\underline{m} = (m_1, m_2)$ where $m_i = \dim V_i$. Let $\mu_1, \mu_2 \colon \mathbb{C}^{m_1} \longrightarrow \mathbb{C}^{m_2}$ be the linear maps corresponding to each of the arrows of K_2. The Kronecker quiver has two simple representations

$$\mathbb{C} \overset{0}{\underset{0}{\rightrightarrows}} 0 \qquad\qquad 0 \overset{0}{\underset{0}{\rightrightarrows}} \mathbb{C}$$

which we denote by S_1 and S_2 respectively. Note that these representations have dimension vectors $(1, 0)$ and $(0, 1)$.

For an integer $m \geq 1$, consider the following three classes of indecomposable representations of the quiver K_2.

(i) Denote by $E_{m,m+1}$ the indecomposable representation with dimension vector $(m, m+1)$. The linear maps μ_1 and μ_2 are the $(m+1) \times m$ matrices

$$\mu_1 = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ m \\ m{+}1 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \dots & m \end{array} \\ \begin{bmatrix} 1 & 0 & \dots & 0 \\ 0 & 1 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix} \end{array} \qquad \mu_2 = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ m \\ m{+}1 \end{array} \begin{array}{c} \begin{array}{cccc} 1 & 2 & \dots & m \end{array} \\ \begin{bmatrix} 0 & 0 & \dots & 0 \\ 0 & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \\ 1 & \dots & \dots & 1 \end{bmatrix} \end{array}.$$

For example, $E_{1,2}$ is the indecomposable representation $\mathbb{C} \overset{\mu_1}{\underset{\mu_2}{\rightrightarrows}} \mathbb{C}^2$, with the linear maps μ_1 and μ_2 the matrices

$$\mu_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad\qquad \mu_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

The simple representation S_2 can be viewed as a special $m = 0$ case of this classification of indecomposable representations.

(ii) Denote by $E_{m+1,m}$ the representations with dimension vector $(m+1, m)$. The linear maps μ_1 and μ_2 are the $m \times (m+1)$ matrices

$$\mu_1 = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & \dots & m & m{+}1 \end{array} \\ \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \\ 0 & 1 & \dots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & 0 \end{bmatrix} \end{array} \qquad \mu_2 = \begin{array}{c} \\ 1 \\ 2 \\ \vdots \\ m \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & \dots & m & m{+}1 \end{array} \\ \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & \dots & 0 & 1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & \dots & 0 & 1 \end{bmatrix} \end{array}.$$

For example, $E_{2,1}$ is the indecomposable representation $\mathbb{C}^2 \underset{\mu_2}{\overset{\mu_1}{\rightrightarrows}} \mathbb{C}$, with the linear maps μ_1 and μ_2 the matrices

$$\mu_1 = \begin{bmatrix} 1 & 0 \end{bmatrix} \qquad\qquad \mu_2 = \begin{bmatrix} 0 & 1 \end{bmatrix}.$$

The simple representation S_1 can be viewed as a special $m = 0$ case of this classification of indecomposable representations.

(iii) For each $\lambda \in \mathbb{C}$, let $E^{\lambda}_{m,m}$ be the indecomposable representation with dimension vector (m, m). Let μ_1 be the identity matrix and μ_2 the diagonal matrix with entries λ. Additionally, let $E^{\infty}_{m,m}$ be the indecomposable representation for which $\mu_1 = 0$ and $\mu_2 = \mathrm{id}$.

Denote by $\mathrm{Rep}(Q)$ the category of representations of Q. This is an abelian category, in which each indecomposable representation E_{m_1,m_2} fits into a short exact sequence of the form $0 \longrightarrow S_2^{\oplus m_2} \longrightarrow E_{m_1,m_2} \longrightarrow S_1^{\oplus m_1} \longrightarrow 0$, i.e. there is a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 0 \arrow[r, "f_1"] \arrow[d, shift left] \arrow[d, shift right] & \mathbb{C}^{m_1} \arrow[r, "g_1"] \arrow[d, shift left] \arrow[d, shift right] & \mathbb{C}^{m_1} \arrow[r] \arrow[d, shift left] \arrow[d, shift right] & 0 \arrow[d, shift left] \arrow[d, shift right] \\
0 \arrow[r] & \mathbb{C}^{m_2} \arrow[r, "f_2"] & \mathbb{C}^{m_2} \arrow[r, "g_2"] & 0 \arrow[r] & 0
\end{tikzcd}$$

of vector spaces and linear maps.

By studying the indecomposable representations of a quiver Q, we can construct another quiver, the Auslander–Reiten quiver.

Definition 1.1.2. [2] The *Auslander–Reiten quiver* of a quiver Q is the quiver with

(i) vertices the indecomposable representations of Q;

(ii) arrows the irreducible morphisms between representations, i.e. an arrow exists between the vertices E_1 and E_2 if there exists a homomorphism $f\colon E_1 \longrightarrow E_2$ which is not an isomorphism and for any factorisation $f = f_2 \circ f_1$, either f_1 is a split monomorphism or f_2 is a split epimorphism.

The construction of Auslander–Reiten quivers is a key part of the study of the representation theory of the path algebras of quivers. We restrict to the statement of the following theorem as we only use the Auslander–Reiten quiver as an aid to visualise the classification of different types of indecomposable representations of the Kronecker quiver.

Theorem 1.1.3. *[2, Sec. 3.5] [19] The Auslander–Reiten quiver of the Kronecker quiver comprises of three connected components.*

(i) The preprojective component, which has vertices the projective indecomposable representations $E_{m,m+1}$ for $m \geq 0$. This is the component to the left of Figure 1.1.

(ii) The preinjective component, which has vertices the injective indecomposable representations $E_{m+1,m}$, for $m \geq 0$. This is the component to the right of Figure 1.1.

(iii) For each $\lambda \in \mathbb{C} \cup \{\infty\}$, there is a connected component of indecomposable representations with dimension vector (m, m) for $m \geq 1$. These are the central components in Figure 1.1.

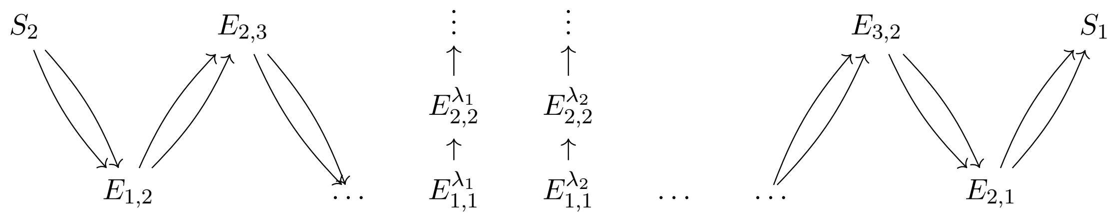


Figure 1.1: The Auslander–Reiten quiver of the Kronecker quiver.

Definition 1.1.4. Let Q be a quiver. The *derived category of Q* is defined to be $\mathrm{D^b}(Q) := \mathrm{D^b}\operatorname{Rep}(Q)$.

We fix some notation regarding the spaces of morphisms in a triangulated category.

Proposition 1.1.5. *[25, Prop. 2.56] Let $\mathcal{A}$ be an abelian category and X,Y objects in $\mathcal{A}$. Then*

$$\operatorname{Ext}^i_A(X, Y) \cong \operatorname{Hom}_{\mathcal{D}^b(\mathcal{A})}(X, Y[i]).$$

On a triangulated category $\mathcal{D}$, we introduce the following notation for the morphism spaces

$$\operatorname{Hom}^i_{\mathcal{D}}(X, Y) := \operatorname{Hom}_{\mathcal{D}}(X, Y[i]).$$

From these spaces the Hom-space between two objects X and Y is defined as

$$\operatorname{Hom}^*_{\mathcal{D}}(X, Y) := \bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}^i_{\mathcal{D}}(X, Y).$$

Theorem 1.1.6. *[4] The derived category of the Kronecker quiver is equivalent to the derived category of coherent sheaves on the projective line.*

Remark 1.1.7. There is no equivalence between the abelian categories $\operatorname{Coh}(\mathbb{P}^1)$ and $\operatorname{Rep}(K_2)$. An equivalence of categories maps simple objects to simple objects. There are only two simple representations of the Kronecker quiver, while there are infinitely many simple objects in $\operatorname{Coh}(\mathbb{P}^1)$, namely the skyscraper sheaves.

Example 1.1.8. Let $F\colon \mathrm{Coh}(\mathbb{P}^1) \longrightarrow \mathrm{Rep}(\mathrm{K}_2)$ be the functor defined by

$$F(E) = \Big(\mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}(1), E) \rightrightarrows \mathrm{Hom}_{\mathbb{P}^1}(\mathcal{O}, E)\Big).$$

Any object in $\mathrm{D^b}(\mathbb{P}^1)$ is the direct sum of shifts of coherent sheaves on $\mathbb{P}^1$ [25, Cor. 3.15]. Hence it is enough to know how this functor acts on line bundles and skyscraper sheaves to compute the image in $\mathrm{D^b}(\mathrm{K}_2)$ of an object in $\mathrm{D^b}(\mathbb{P}^1)$. For $n \in \mathbb{Z}$, the twisting sheaf $\mathcal{O}(n)$ on $\mathbb{P}^1$ is mapped to the indecomposable representation

$$\mathcal{O}(n) \longmapsto \Big(\mathrm{Hom}^*_{\mathrm{D^b}(\mathbb{P}^1)}(\mathcal{O}(1), \mathcal{O}(n)) \rightrightarrows \mathrm{Hom}^*_{\mathrm{D^b}(\mathbb{P}^1)}(\mathcal{O}, \mathcal{O}(n))\Big).$$

In particular

$$\mathcal{O}(-1) \longmapsto S_1[-1] \qquad\qquad \mathcal{O} \longmapsto S_2 \qquad\qquad \mathcal{O}(1) \longmapsto E_{1,2}.$$

Finally $\mathcal{O}_{mx} \mapsto E^{\lambda}_{m,m}$ with λ determined by the point $x \in \mathbb{P}^1$.

1.2 Construction of the sequence of triangulated categories

The derived category of $\mathrm{Rep}(\mathrm{K}_2)$ is the first example of a triangulated category associated to the Kronecker quiver. In order to explore the relationship between stability conditions and Frobenius manifolds, our aim is to construct a sequence of spaces of stability conditions for this quiver, which are indexed by integers n with $n \geq 2$. This section introduces the categories from which such a sequence of stability manifolds is constructed. In addition to being triangulated, these categories are classes of examples of both differential graded categories, and Calabi–Yau categories. We begin with some background on differential graded categories, then introduce Calabi–Yau-n categories. Finally the examples of differential graded Calabi–Yau-n categories related to the Kronecker quiver are introduced.

Differential graded categories

The categories involved in this section are differential graded (dg) categories. We state the dg versions of some familiar algebraic definitions, namely algebras, modules, and categories, and consider the construction of the derived category of a dg, as opposed to abelian, category. More thorough discussion of this material is presented in Keller's survey paper [29], or the Stacks Project [41, Chap. 22]. In the following, k is a commutative ring, and all gradings are over the integers.

Definition 1.2.1. (i) A *dg algebra* A is a graded k-algebra with a differential d_A such that for all elements $a \in A^i$ and all $b \in A$, the differential satisfies the Leibniz rule

$$d_A(ab) = d_A(a)b + (-1)^i a d_A(b).$$

(ii) A *dg A-module* M is a graded A-module with a differential d_M which satisfies

$$d_M(ma) = d_M(m)a + (-1)^i m d_A(a)$$

for all $m \in M^i$ and all $a \in A$.

(iii) A *morphism of dg A-modules of degree* p is an A-linear morphism $f\colon M \longrightarrow N$ such that $f(M^i) \subset N^{i+p}$.

(iv) Let A be a dg k-algebra and M, N dg A-modules. Define the *morphism complex* of M and N, denoted $\underline{\mathrm{Hom}}_A(M, N)$, as follows. For each $i \in \mathbb{Z}$, consider the product

$$\prod_{j \in \mathbb{Z}} \mathrm{Hom}_k(M^j, N^{j+i}).$$

Then $\underline{\mathrm{Hom}}^i_A(M, N)$ is the subspace of the product with elements morphisms f satisfying $f(ma) = f(m)a$ for all $m \in M$ and $a \in A$. A differential d is defined on $\underline{\mathrm{Hom}}_A(M, N)$ by $d(f) = f \circ d_M - (-1)^{\deg(f)} d_N \circ f$.

(v) A morphism $f \in \underline{\mathrm{Hom}}_A(M, N)$ is *closed* if $d(f) = 0$,

Definition 1.2.2. [29, Sec. 2] A *dg category* $\mathcal{A}$ is a k-linear category in which for all objects L, M, N:

(i) the spaces $\mathrm{Hom}_{\mathcal{A}}(M, N)$ are dg k-modules;

(ii) the composition of morphisms $\mathrm{Hom}_{\mathcal{A}}(N, L) \otimes \mathrm{Hom}_{\mathcal{A}}(M, N) \longrightarrow \mathrm{Hom}_{\mathcal{A}}(M, L)$ is a morphism of dg k-modules.

The data of a dg algebra A is equivalent to the data of a dg category with one object. We restrict to constructing the derived category of such a dg category as it is this situation which is relevant for our work.

Definition 1.2.3. Let A be a dg algebra and define $\mathcal{C}(A)$ to be the category with objects right *dg* A-modules, and morphisms the closed elements of degree zero in the morphism complex.

As in the construction of the derived category of an abelian category, the next step is to consider the homotopy category. Denote the homotopy category of the dg algebra A by $\mathcal{H}(A)$.

The objects of $\mathcal{H}(A)$ are again dg A-modules and its morphisms are the closed elements in $\underline{\mathrm{Hom}}^0_A(M,N)$, up to homotopy. The homotopy category is triangulated: Exact triangles are those arising form short exact sequences of dg A-modules, and the shift functor $[n]$ for an integer n on an A-module M is defined to be $M[n]$ with graded components $M[n]^p = M^{p+n}$ and the differential $d_{M[n]} = (-1)^n d_M$.

Definition 1.2.4. The derived category $\mathcal{D}(A)$ of a dg algebra A is the category with objects dg A-modules and morphisms defined by formally inverting quasi-isomorphisms in the homotopy category $\mathcal{H}(A)$.

Importantly, this is a triangulated category. The triangulated structure is induced from the triangulated structure on the homotopy category.

Calabi–Yau categories

Calabi–Yau algebras were introduced by Ginzburg [21]. The philosophy behind these algebras is the aim of constructing triangulated categories which can be viewed as carrying the key geometric features of Calabi–Yau varieties. The geometric features of Calabi–Yau varieties that we want to imitate are:

(i) smoothness- the smooth geometry of a Calabi–Yau variety is replaced with the condition of homological smoothness of the algebra;

(ii) compactness- this is a requirement that the cohomology of the algebra is finitely generated;

(iii) the triviality of the canonical bundle- this is related to the shift functor behaving as a Serre functor, which is a homological condition on the algebra.

The derived categories of such algebras share one of the most important features of the derived categories of Calabi–Yau n-folds.

Definition 1.2.5. A triangulated category $\mathcal{D}$ is *Calabi–Yau-n* if the shift functor $[n]$ is a Serre functor, that is for all objects $X, Y \in \mathcal{D}$, there is an isomorphism

$$\mathrm{Hom}^*_{\mathcal{D}}(X,Y) \cong \mathrm{Hom}^*_{\mathcal{D}}(Y, X[n])^\vee$$

which is functorial in both variables.

Ginzburg algebras of a quiver

The rest of the section focuses on the dg Calabi–Yau-n categories associated to the Kronecker quiver, which are defined from its Ginzburg algebras. Such algebras can be constructed for any quiver. We detail the construction for an acyclic quiver.

Definition 1.2.6. [31, Def. 2.6] Let $Q = (Q_0, Q_1)$ be an acyclic quiver and fix an integer $n \geq 2$.

(i) The *Ginzburg quiver* of Q is the graded quiver $\bar{Q}$ constructed as follows.

   (1) The set of vertices $\bar{Q}_0$ of $\bar{Q}$ is Q_0.

   (2) The set of arrows $\bar{Q}_1$ of $\bar{Q}$ contains:

      (a) in degree 0, the set of arrows Q_1 of Q;

      (b) in degree $2-n$, an arrow $a^*\colon j \longrightarrow i$ for each arrow $a\colon i \longrightarrow j$ in Q_1;

      (c) in degree $1-n$, a loop $e_i^*\colon i \longrightarrow i$ for each vertex i in Q_0.

(ii) The *complete Ginzburg dg algebra* $\Gamma\bar{Q}$ of Q is the completion of the path algebra of $\bar{Q}$ with respect to the ideal generated by its arrows, with the degree 1 differential $d\colon k\bar{Q} \longrightarrow k\bar{Q}$ which satisfies the usual Leibniz condition

$$d(a \cdot b) = d(a) \cdot b + (-1)^{\deg(a)} a \cdot d(b)$$

and is defined on arrows by

   (1) $da = da^* = 0$ for arrows a, a^* of $\bar{Q}$;

   (2) $de_i^* = e_i\Big(\sum_a (aa^* - a^*a)\Big)e_i$ for the paths e_i^* at each vertex i of $\bar{Q}$.

Example 1.2.7. Let $\mathrm{K}_2 = \bullet_1 \rightrightarrows \bullet_2$ be the Kronecker quiver. For $n \geq 2$, Figure 1.2 shows the graded Ginzburg quiver of K_2.

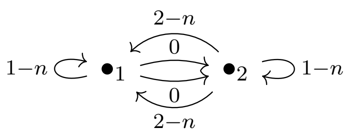


Figure 1.2: The Ginzburg quiver of K_2.

The categories for which we construct the spaces of stability conditions are the derived categories of the Ginzburg algebras of the quiver in Figure 1.2. The following theorem highlights one of the critical features of these categories. The other important property is the existence of a distinguished abelian subcategory. The techniques needed to define this are discussed in Section 1.3.

Theorem 1.2.8. *[31] Let Q be an acyclic quiver and $\Gamma\bar{Q}$ its dg Ginzburg algebra with respect to a fixed integer $n \geq 2$. Then the derived category of $\Gamma\bar{Q}$ is a Calabi–Yau-n category.*

Definition 1.2.9. Let Q be an acyclic quiver and fix an integer $n \geq 2$. Define *the Calabi–Yau-n category* $\mathrm{D_n}(Q)$ *of* Q as the derived category of the dg algebra $\Gamma\bar{Q}$.

When it is clear what the quiver in question is, we simplify notation by writing $\mathrm{D_n}$. Denote by D_∞ the bounded derived category $\mathrm{D^b}(Q)$ of a quiver Q.

The correspondence between modules over the path algebra of a quiver and representations of the quiver can be extended to a one-to-one correspondence between graded modules over the path algebra of the Ginzburg quiver and graded representations of the quiver. Hence for a dg module M over the Ginzburg algebra of a quiver $Q = \{Q_0, Q_1\}$, there is a decomposition of the form

$$M = \bigoplus_{i \in Q_0} \bigoplus_{j \in \mathbb{Z}} M_i^j.$$

The arrows generating the path algebra define the linear maps between these vector spaces

$$a\colon M_i^j \longrightarrow M_{i'}^j \qquad e_i^*\colon M_i^j \longrightarrow M_i^{j+(n-1)} \qquad a^*\colon M_{i'}^j \longrightarrow M_i^{j+(n-2)}$$

for vertices $i, i' \in Q_0$, arrows $a\colon i \longrightarrow i'$, and $j \in \mathbb{Z}$.

1.3 Exchange graphs

This section introduces the techniques necessary to construct the exchange graphs associated to the Kronecker quiver. To define the exchange graph of a quiver, we begin by defining the heart of a bounded t-structure. Throughout this section, $\mathcal{D}$ denotes a triangulated category.

An important construction in the theory of torsion theories, t-structures and tilting is that of the left and right orthogonal complement of a full subcategory. For a subcategory $\mathcal{D}' \subset \mathcal{D}$, these categories are defined respectively as

$$^{\perp}\mathcal{D}' := \{E \in \mathcal{D}\colon\ \mathrm{Hom}_{\mathcal{D}}(E, X) = 0 \text{ for all } X \in \mathcal{D}'\}$$
$$\mathcal{D}'^{\perp} := \{E \in \mathcal{D}\colon\ \mathrm{Hom}_{\mathcal{D}}(X, E) = 0 \text{ for all } X \in \mathcal{D}'\}.$$

Definition 1.3.1. [20, Def. IV.4.2] Let $\mathcal{D}$ be a triangulated category.

(i) A *t-structure* on $\mathcal{D}$ is given by a pair of full additive subcategories $(\mathcal{D}^{\leq 0}, \mathcal{D}^{>0})$ which satisfy the following conditions:

a) $\mathcal{D}^{>0} = \mathcal{D}^{\leq 0 \perp}$;

b) for every object $E \in \mathcal{D}$ there exists an exact triangle $X \longrightarrow E \longrightarrow Y$ with $X \in \mathcal{D}^{\leq 0}$ and $Y \in \mathcal{D}^{>0}$;

c) $\mathcal{D}^{\leq -1} := \mathcal{D}^{\leq 0}[1] \subset \mathcal{D}^{\leq 0}$.

(ii) The subcategory $\mathcal{D}^{\leq 0}$ is the *aisle* of the t-structure and $\mathcal{D}^{>0}$ is the *coaisle*.

(iii) The t-structure is *bounded* if every object $E \in \mathcal{D}$ is contained in $\mathcal{D}^{\leq n} \cap \mathcal{D}^{\geq n}$ for $n \gg 0$.

Example 1.3.2. Let $\mathcal{A}$ be an abelian category and fix $\mathcal{D} = \mathrm{D}^{\mathrm{b}}(\mathcal{A})$. An example of a bounded t-structure on $\mathcal{D}$ is given by

$$\mathcal{D}^{\leq 0} := \{E \in \mathcal{D} \colon H^i(E) = 0 \text{ for } i > 0\}$$
$$\mathcal{D}^{>0} := \{E \in \mathcal{D} \colon H^i(E) = 0 \text{ for } i \leq 0\}$$

where H^i are the cohomology functors. This is called the *standard t-structure* on a derived category.

Definition 1.3.3. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{>0})$ be a t-structure on a triangulated category $\mathcal{D}$.

(i) [20, Sec. IV.4] The *heart* of the t-structure is the full subcategory

$$\mathcal{A}_{(\mathcal{D}^{\leq 0}, \mathcal{D}^{>0})} := \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}.$$

(ii) The heart of a t-structure is *of finite length* if it is both artinian and noetherian.

Proposition 1.3.4. *[5, Thm. 1.3.6] The heart of a bounded t-structure on a triangulated category $\mathcal{D}$ is an abelian subcategory.*

Example 1.3.5. The heart of the standard t-structure on $\mathrm{D}^{\mathrm{b}}(\mathcal{A})$ is $\mathcal{A}$, as it is exactly the complexes with cohomology vanishing in non-zero degrees.

The following lemma characterises abelian categories that are the hearts of bounded t-structures.

Lemma 1.3.6. *[26, Rem. 1.16] A full additive subcategory $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure on $\mathcal{D}$ if and only if the following two conditions are satisfied:*

(i) for $k_1, k_2 \in \mathbb{Z}$ with $k_1 > k_2$, $\mathrm{Hom}_{\mathcal{D}}(A_1, A_2) = 0$ for $A_i \in \mathcal{A}[k_i]$;

(ii) for every object $E \in \mathcal{D}$ there exists a sequence of exact triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

and a sequence of integers $k_1 > k_2 > \ldots > k_n$ such that $A_i \in \mathcal{A}[k_i]$.

Remark 1.3.7. Suppose that $\mathcal{A} \subset \mathcal{D}$ is the heart of a bounded t-structure. The decomposition of objects as given in condition (ii) of Lemma 1.3.6 defines cohomology functors on $\mathcal{D}$ with respect to $\mathcal{A}$, which are denoted by $H_{\mathcal{A}}$.

Given a triangulated category $\mathcal{D}$, it is natural to ask in what situations a bounded t-structure exists on $\mathcal{D}$. For the derived categories of abelian categories, such structures can always be defined by considering the standard t-structure. More generally, there are certain properties that triangulated categories endowed with t-structures exhibit (as discussed for example in [34]). The importance of this question becomes clear in the following section when we discuss the correspondence between t-structures and stability conditions. Although the triangulated categories that we study are not the derived categories of abelian categories, they are endowed with a t-structure.

Proposition 1.3.8. *[1, Sec.2] [27, Prop. 4.4] For $3 \leq n < \infty$, let $\mathrm{D_n}(Q)$ be the derived categories of the Ginzburg algebras of a finite acyclic quiver Q. Then there is a bounded t-structure on these categories with heart equivalent to* $\mathrm{Rep}(Q)$.

Proof. Define a t-structure on $\mathrm{D_n} = \mathrm{D_n}(Q)$ by

$$\begin{aligned}
{\mathrm{D_n}}^{\leq 0} &:= \{E \in \mathrm{D_n} \colon H^i(E) = 0 \text{ for } i > 0\} \\
{\mathrm{D_n}}^{> 0} &:= \{E \in \mathrm{D_n} \colon H^i(E) = 0 \text{ for } i \leq 0\}.
\end{aligned}$$

This t-structure is bounded as the path algebra of the graded quiver from which these categories are constructed is concentrated in non-positive degrees. The objects in the heart ${\mathrm{D_n}}^{\leq 0} \cap {\mathrm{D_n}}^{\geq 0}$ are those with non-vanishing cohomology only in degree zero. As this is a Calabi–Yau-n category, we must have $n \geq 3$ in order for this heart to be equivalent to $\mathrm{Rep}(Q)$, as the Calabi–Yau-2 condition gives an isomorphism $\mathrm{Ext}^1(S_i, S_j) \cong \mathrm{Ext}^1(S_j, S_i)$, contradicting the assumption that the quiver is acyclic. $\square$

This t-structure is the standard t-structure on a Calabi–Yau-n category and its heart is the *standard heart* of the category $\mathrm{D_n}(\mathrm{K_2})$.

Remark 1.3.9. In the case of the category D_2, the situation is different. Let S_i and S_j be simple representations of an acyclic quiver Q such that the group $\mathrm{Ext}^1(S_i, S_j)$ is non-trivial. In the triangulated category D_2, the Ext-groups $\mathrm{Ext}^1(S_i, S_j)$ and $\mathrm{Ext}^1(S_j, S_i)$ are isomorphic by the Calabi–Yau-2 property. However, in the abelian category $\mathrm{Rep}(Q)$, if one of these groups does not vanish, the other does vanish as the quiver is acyclic. To deal with this, we introduce another abelian category constructed from a quiver, the category of representations of the preprojective algebra.

Definition 1.3.10. [39] The *preprojective algebra* $\mathcal{P}(Q)$ of a quiver $Q = \{Q_0, Q_1\}$ is the completion of the quotient of the path algebra $\mathbb{C}\tilde{Q}/(\rho)$ where $\tilde{Q}$ is the quiver Q with additional arrows $a^*\colon j \longrightarrow i$ for each $a\colon i \longrightarrow j$ in Q_1, and (ρ) is the ideal generated by the elements

$$\rho = \sum_{a \in \bar{Q}_1} (a^* a - a a^*).$$

In the category of representations of this algebra $\mathrm{Ext}^1(S_i, S_j) \cong \mathrm{Ext}^1(S_j, S_i)$. Hence the standard heart $\mathcal{A}$ in the category D_2 is the category of finite dimensional representations of the preprojective algebra of Q.

We focus on the categories $\mathrm{D_n}$ for $n \geq 3$, as for the Calabi–Yau-2 category of the the Kronecker quiver, the space of stability conditions is computed in [10].

A related concept to that of a t-structure on a triangulated category in that of a torsion theory on an abelian category.

Definition 1.3.11. Let $\mathcal{A}$ be an abelian category. A *torsion theory* on $\mathcal{A}$ is a pair of full additive subcategories $(\mathcal{T}, \mathcal{F})$ satisfying the following conditions:

(i) $\mathcal{F} = \mathcal{T}^{\perp}$;

(ii) for all objects $E \in \mathcal{A}$, there is a short exact sequence $0 \longrightarrow T \longrightarrow E \longrightarrow F \longrightarrow 0$ with $T \in \mathcal{T}$ and $F \in \mathcal{F}$.

We call the subcategory $\mathcal{T}$ the torsion part of the torsion theory and $\mathcal{F}$ the torsion free part.

Given a torsion theory on an abelian category $\mathcal{A}$, the tilting operation can be performed. The tilt of $\mathcal{A}$ with respect to a given torsion theory is itself an abelian category. In this way different abelian subcategories of a triangulated category can be studied.

Definition 1.3.12. [23] Let $\mathcal{D}$ be a triangulated category. The *backward tilt* of an abelian category $\mathcal{A} \subset \mathcal{D}$ at a torsion theory $(\mathcal{T}, \mathcal{F})$ is the category

$$\mathcal{A}^{\flat}_{(\mathcal{T},\mathcal{F})} := \{E \in \mathcal{D}\colon H^0_{\mathcal{A}}(E) \in \mathcal{T}, H^{-1}_{\mathcal{A}}(E) \in \mathcal{F} \text{ and } H^i_{\mathcal{A}}(E) = 0 \text{ for } i \notin \{0, -1\}\}.$$

The *forward tilt* of an abelian category $\mathcal{A} \subset \mathcal{D}$ at a torsion theory $(\mathcal{T}, \mathcal{F})$ is the category

$$\mathcal{A}^{\sharp}_{(\mathcal{T},\mathcal{F})} := \{E \in \mathcal{D} \colon H^1_{\mathcal{A}}(E) \in \mathcal{T}, H^0_{\mathcal{A}}(E) \in \mathcal{F} \text{ and } H^i_{\mathcal{A}}(E) = 0 \text{ for } i \notin \{0,1\}\}.$$

Clearly the backward and forward tilts of an abelian category $\mathcal{A}$ at a torsion theory $(\mathcal{T}, \mathcal{F})$ are related by $\mathcal{A}^{\flat} = \mathcal{A}^{\sharp}[1]$.

Proposition 1.3.13. *[26, Prop. 1.20] If $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ on which a torsion theory is defined, then the tilt of $\mathcal{A}$ at this torsion theory is the heart of a bounded t-structure on $\mathcal{D}$.*

Proof. Let $(\mathcal{D}^{\leq 0}, \mathcal{D}^{>0})$ be the bounded t-structure on $\mathcal{D}$ such that $\mathcal{A} = \mathcal{D}^{\leq 0} \cap \mathcal{D}^{\geq 0}$. Let $(\mathcal{T}, \mathcal{F})$ be a torsion theory on $\mathcal{A}$ and $\mathcal{A}^{\flat}_{(\mathcal{T},\mathcal{F})}$ its backward tilt at the torsion theory. Define another bounded t-structure on $\mathcal{D}$ by

$$\mathcal{D}'^{\leq 0} := \{E \in \mathcal{D} \colon H^0_{\mathcal{A}}(E) \in \mathcal{T} \text{ and } H^i(E) = 0 \text{ for } i > 0\}$$
$$\mathcal{D}'^{>0} := \{E \in \mathcal{D} \colon H^{-1}_{\mathcal{A}}(E) \in \mathcal{F} \text{ and } H^i(E) = 0 \text{ for } i < 0\}.$$

Then $\mathcal{D}'^{\leq 0} \cap \mathcal{D}'^{\geq 0} = \mathcal{A}^{\flat}_{(\mathcal{T},\mathcal{F})}$. □

We are concerned with a particular example of torsion theories, those defined by simple objects, and constructed as follows. Let $\mathcal{D}$ be a triangulated category and $\mathcal{A} \subset \mathcal{D}$ an abelian subcategory of finite type. For simple objects $S_1, \dots S_n \in \mathcal{A}$, let $\langle S_1, \dots S_n \rangle$ denote the full extension closed subcategory of $\mathcal{D}$ containing objects whose simple factors are isomorphic to $S_1, \dots S_n$. The left and right orthogonal complements of an object S are defined respectively as

$$^{\perp}S := \{X \in \mathcal{A} \colon \operatorname{Hom}_{\mathcal{A}}(X, S) = 0\}$$
$$S^{\perp} := \{X \in \mathcal{A} \colon \operatorname{Hom}_{\mathcal{A}}(S, X) = 0\}.$$

Definition 1.3.14. Let $S \in \mathcal{A}$ be a simple object in a finite length abelian category.

(i) Define a torsion theory on $\mathcal{A}$ by $\mathcal{T} = {}^{\perp}S$ and $\mathcal{F} = S^{\perp}$. The *left tilt of $\mathcal{A}$ at the simple S*, $L_S(\mathcal{A})$, is the backward tilt of the category $\mathcal{A}$ with respect to the torsion theory $({}^{\perp}S, \langle S \rangle)$.

(ii) Define a torsion theory on $\mathcal{A}$ by $\mathcal{T} = \langle S \rangle$ and $\mathcal{F} = \langle S \rangle$. The *right tilt of $\mathcal{A}$ at the simple S*, $R_S(\mathcal{A})$, is the forward tilt of the category $\mathcal{A}$ with respect to the torsion theory $(\langle S \rangle, S^{\perp})$.

This choice of backward and forward tilting depending on whether we consider the left or right tilt is designed to simplify the description of how autoequivalences act on the exchange graph.

Lemma 1.3.15. *[32, Prop. 4.4] Let $\mathcal{A}$ be a finite length heart of a bounded t-structure on a Calabi–Yau-n triangulated category $\mathrm{D_n}$, and let S be a simple object in $\mathcal{A}$.*

(i) The left and right tilts of $\mathcal{A}$ at S are the categories

$$L_S(\mathcal{A}) = \langle S[1], {}^{\perp}S\rangle \qquad\qquad R_S(\mathcal{A}) = \langle S^{\perp}, S[-1]\rangle.$$

The two tilts are related by $R_{S[1]}L_S\mathcal{A} = \mathcal{A}$.

(ii) The simple tilt of a finite length heart is of finite length.

Definition 1.3.16. The *exchange graph* $\mathrm{EG}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ is the graph with:

(i) vertices given by the finite length hearts of bounded t-structures;

(ii) edges between vertices given by simple tilts between such hearts.

Denote by $\mathrm{EG}_*(\mathcal{D})$ the connected component of the exchange graph containing the standard heart of $\mathcal{D}$.

1.4 Autoequivalences of triangulated categories

An important feature of the space of stability conditions of a triangulated category is that it carries an action of the group of autoequivalences of the category. This group also acts on the exchange graph. These actions and their relationship to each other are discussed in the final section of the chapter. This section recalls some facts about autoequivalences of a triangulated category, in particular the spherical twists functors. Given a triangulated category $\mathcal{D}$, denote its group of autoequivalences by $\mathrm{Aut}(\mathcal{D})$.

Spherical twists provide an important class of examples of autoequivalences on Calabi–Yau varieties. They were introduced by Seidel and Thomas [40].

Definition 1.4.1. Let $\mathcal{D}$ be a $\mathbb{C}$-linear Calabi–Yau-n triangulated category. An object $E \in \mathcal{D}$ is *spherical* if

$$\mathrm{Hom}_{\mathcal{D}}(E, E[i]) = \begin{cases} \mathbb{C} & \text{if } i \in \{0, n\}, \\ 0 & \text{if } i \notin \{0, n\}. \end{cases}$$

Example 1.4.2. Let $\mathrm{D_n}$ be the Calabi–Yau-n triangulated category associated to the Kronecker quiver as constructed in Section 1.2. Then the simple objects S_1 and S_2 are spherical [30, Lem. 4.4], that is

$$\mathrm{Hom}_{\mathrm{D_n}}(S_i, S_i[i]) = \begin{cases} \mathbb{C} & \text{if } i \in \{0, n\}, \\ 0 & \text{if } i \notin \{0, n\}. \end{cases}$$

Spherical twists are functors which can be constructed from spherical objects using the mapping cone construction.

Proposition 1.4.3. *[40, Prop.2.10] Let Q be an acyclic quiver, and let E be a spherical object in the Calabi–Yau-n triangulated category $\mathrm{D_n}(Q)$. Then there exists an autoequivalence $\mathrm{Tw_E} \in \mathrm{Aut}(\mathrm{D_n}(Q))$ called the spherical twist at E for which*

$$\mathrm{Tw_E}(X) \cong \mathrm{Cone}(\mathrm{Hom}^*_{\mathrm{D_n}(Q)}(E, X) \otimes E \xrightarrow{ev} X)$$

for all objects X in $\mathrm{D_n}(Q)$, where ev is the evaluation morphism.

Remark 1.4.4. Spherical twists are defined on a broader class of triangulated categories, but not on an arbitrary triangulated category. Proposition 1.4.3 provides a sufficient definition to allow us to carry out computations in the categorical setting that we are interested in.

In order to compute spherical twists, we need to know more about the Hom-spaces between indecomposable representations

Lemma 1.4.5. *[30, Lemma 4.4] For $3 \leq n < \infty$, let $\mathrm{D_n}$ be the Calabi–Yau-n triangulated category associated to the Kronecker quiver. Let $m \geq 1$.*

(i) For S_1 and S_2 the simple representations of the Kronecker quiver,

$$\mathrm{Hom}^i_{\mathrm{D_n}}(S_1, S_2) \cong \begin{cases} \mathbb{C}^2 & \text{if } i = 1, \\ 0 & \text{if } i \neq 1. \end{cases}$$

(ii) For $E_{m-1,m}$ and $E_{m,m+1}$ the indecomposable representations of the Kronecker quiver with dimension vectors $(m-1, m)$ and $(m, m+1)$,

$$\mathrm{Hom}^i_{\mathrm{D_n}}(E_{m-1,m}, E_{m,m+1}) \cong \begin{cases} \mathbb{C}^2 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

(iii) For $E_{m+1,m}$ and $E_{m,m-1}$ the indecomposable representations of the Kronecker quiver with dimension vectors $(m+1, m)$ and $(m, m-1)$,

$$\mathrm{Hom}^i_{\mathrm{D_n}}(E_{m+1,m}, E_{m,m-1}) \cong \begin{cases} \mathbb{C}^2 & \text{if } i = 0, \\ 0 & \text{if } i \neq 0. \end{cases}$$

Example 1.4.6. In the $n = 3$ case, we can compute the space $\mathrm{Hom}^*_{\mathrm{D_3}}(S_1, S_2)$ from the morphisms between quiver representations and the Calabi–Yau-3 property. To compute the space $\mathrm{Hom}^0_{\mathrm{D_3}}(S_1, S_2)$, observe that a morphism f between the representations S_1 and S_2 is a pair of linear maps $f = (f_1, f_2)$ such that the diagram

$$\begin{tikzcd}
\mathbb{C} \arrow[r, "f_1"] \arrow[d, Rightarrow] & 0 \arrow[d, Rightarrow] \\
0 \arrow[r, "f_2"] & \mathbb{C}
\end{tikzcd}$$

commutes. Clearly $(f_1, f_2) = (0, 0)$. Therefore $\mathrm{Hom}^0_{\mathrm{D_3}}(S_1, S_2) = 0$.

For $\mathrm{Hom}^1_{\mathrm{D_3}}(S_1, S_2) = \mathrm{Hom}^0_{\mathrm{D_3}}(S_1, S_2[1])$, observe that as there is a short exact sequence of indecomposable representations $0 \longrightarrow S_2 \longrightarrow E_{1,1} \longrightarrow S_1 \longrightarrow 0$, morphisms in the space $\mathrm{Hom}^1_{\mathrm{D_3}}(S_1, S_2)$ are pairs (f, g) such that the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, Rightarrow] & 0 \arrow[r, "f_1"] \arrow[d, Rightarrow] & \mathbb{C} \arrow[r, "g_1"] \arrow[d, Rightarrow] & \mathbb{C} \arrow[r] \arrow[d, Rightarrow] & 0 \arrow[d, Rightarrow] \\
0 \arrow[r] & \mathbb{C} \arrow[r, "f_2"] & \mathbb{C} \arrow[r, "g_2"] & 0 \arrow[r] & 0
\end{tikzcd}$$

commutes. Hence $\mathrm{Hom}^1_{\mathrm{D_3}}(S_1, S_2) \cong \mathbb{C}^2$.

Using the Calabi–Yau-3 property,

$$\mathrm{Hom}^2_{\mathrm{D_3}}(S_1, S_2) \cong \mathrm{Hom}^1_{\mathrm{D_3}}(S_2, S_1) = 0$$
$$\mathrm{Hom}^3_{\mathrm{D_3}}(S_1, S_2) \cong \mathrm{Hom}^0_{\mathrm{D_3}}(S_2, S_1) = 0.$$

Therefore $\mathrm{Hom}^*_{\mathrm{D_3}}(S_1, S_2) \cong \mathbb{C}^2[-1]$.

Corollary 1.4.7. *In the Calabi–Yau-n triangulated category* $\mathrm{D_n}$ *associated to the Kronecker quiver*

$$\mathrm{Hom}^i_{\mathrm{D_n}}(S_2, S_1) = \begin{cases} \mathbb{C}^2 & \text{if } i = n-1, \\ 0 & \text{otherwise} . \end{cases}$$

Example 1.4.8. We compute some spherical twists that we will use later.

(i) For the spherical object S_1 in $\mathrm{D_n}$

$$\begin{aligned}\mathrm{Tw}_{\mathrm{S_1}}(S_2) &\cong \mathrm{Cone}(\mathrm{Hom}^*_{\mathrm{D_n}}(S_1, S_2) \otimes S_1 \xrightarrow{\mathrm{ev}} S_2) \\ &\cong \mathrm{Cone}(\mathbb{C}^2[-1] \otimes S_1 \xrightarrow{\mathrm{ev}} S_2) \\ &\cong \mathrm{Cone}(S_1[-1]^{\oplus 2} \xrightarrow{\mathrm{ev}} S_2).\end{aligned}$$

This is equivalent to a triangle $S_2 \longrightarrow \mathrm{Tw}_{\mathrm{S_1}}(S_2) \longrightarrow S_1^{\oplus 2}$ in $\mathrm{D_n}$. As the cohomology of S_1 and S_2 vanishes apart from in degree zero, the long exact cohomology sequence of this triangle reduces to

$$0 \longrightarrow S_2 \longrightarrow H^0(\mathrm{Tw}_{\mathrm{S_1}}(S_2)) \longrightarrow S_1^{\oplus 2} \longrightarrow 0.$$

Therefore $\mathrm{Tw}_{\mathrm{S}_1}(S_2)$ has dimension vector $(2,1)$, and as S_2 is indecomposable, its image under the autoequivalence is indecomposable. Given the classification of the indecomposable representations of the Kronecker quiver in Section 1.1, we conclude that $\mathrm{Tw}_{\mathrm{S}_1}(S_2) \cong E_{2,1}$.

(ii) For the spherical object S_2 in $\mathrm{D_n}$

$$\begin{aligned}\mathrm{Tw}_{\mathrm{S}_2}(E_{1,2}) &\cong \mathrm{Cone}(\mathrm{Hom}^*_{\mathrm{D_n}}(S_2, E_{1,2}) \otimes S_2 \xrightarrow{\mathrm{ev}} E_{1,2}) \\ &\cong \mathrm{Cone}(\mathbb{C}^2 \otimes S_2 \xrightarrow{\mathrm{ev}} E_{1,2}) \\ &\cong \mathrm{Cone}(S_2^{\oplus 2} \xrightarrow{\mathrm{ev}} E_{1,2}).\end{aligned}$$

This is equivalent to a triangle $S_2^{\oplus 2} \xrightarrow{\mathrm{ev}} E_{1,2} \longrightarrow \mathrm{Tw}_{\mathrm{S}_2}(E_{1,2})$ in $\mathrm{D_n}$ with cohomology

$$0 \longrightarrow H^{-1}(\mathrm{Tw}_{\mathrm{S}_2}(E_{1,2})) \longrightarrow S_2^{\oplus 2} \xrightarrow{\mathrm{ev}} E_{1,2} \longrightarrow H^0(\mathrm{Tw}_{\mathrm{S}_2}(E_{1,2})) \longrightarrow 0.$$

As the map $S_2^{\oplus 2} \xrightarrow{\mathrm{ev}} E_{1,2}$ is injective, $H^{-1}(\mathrm{Tw}_{\mathrm{S}_2}(E_{1,2})) = 0$. Therefore the cohomology of the object $\mathrm{Tw}_{\mathrm{S}_2}(E_{1,2})$ vanishes for $i \neq 0$, and $\mathrm{Tw}_{\mathrm{S}_2}(E_{1,2}) \cong H^0(\mathrm{Tw}_{\mathrm{S}_2}(E_{1,2})) = S_1$.

(iii) The outputs of the previous two spherical twists computations are objects in the standard heart. However this is not always the case. Consider

$$\begin{aligned}\mathrm{Tw}_{\mathrm{S}_1}(E_{2,1}) &\cong \mathrm{Cone}(\mathrm{Hom}^*_{\mathrm{D_n}}(S_1, E_{2,1}) \otimes S_1 \xrightarrow{\mathrm{ev}} E_{2,1}) \\ &\cong \mathrm{Cone}(\mathbb{C}^2[1-n] \otimes S_1 \xrightarrow{\mathrm{ev}} E_{2,1}) \\ &\cong \mathrm{Cone}(S_1[1-n]^{\oplus 2} \xrightarrow{\mathrm{ev}} E_{2,1}).\end{aligned}$$

The long exact cohomology sequence of the resulting triangle in $\mathrm{D_n}$ reduces to

$$\begin{aligned}&0 \longrightarrow H^0(E_{2,1}) \xrightarrow{\cong} H^0(\mathrm{Tw}_{\mathrm{S}_1}(E_{2,1})) \longrightarrow 0 \\ &0 \longrightarrow H^{n-2}(\mathrm{Tw}_{\mathrm{S}_1}(E_{2,1})) \xrightarrow{\cong} H^{n-1}(S_1[1-n]^{\oplus 2}) \longrightarrow 0.\end{aligned}$$

Hence $\mathrm{Tw}_{\mathrm{S}_1}(E_{2,1})$ is isomorphic to an object in $\mathrm{D_n}$ with non-zero cohomology in degree zero and $n-2$. This is not an object in the standard heart on $\mathrm{D_n}$.

(iv) Finally, applying a spherical twist to the spherical object itself

$$\begin{aligned}\mathrm{Tw}_{S_i}(S_i) &\cong \mathrm{Cone}(\mathrm{Hom}^*_{\mathrm{D_n}}(S_i, S_i) \otimes S_i \xrightarrow{\mathrm{ev}} S_i) \\ &\cong \mathrm{Cone}((\mathbb{C} \oplus \mathbb{C}[-n]) \otimes S_i \xrightarrow{\mathrm{ev}} S_i) \\ &\cong \mathrm{Cone}(S_i \oplus S_i[-n] \xrightarrow{\mathrm{ev}} S_i).\end{aligned}$$

The long exact cohomology sequence of the resulting triangle reduces to the sequences

$$\begin{aligned}&0 \longrightarrow H^{-1}(\mathrm{Tw}_{S_i}(S_i)) \longrightarrow H^0(S_i \oplus S_i[-n]) \xrightarrow{\cong} H^0(S_i) \longrightarrow H^0(\mathrm{Tw}_{S_i}(S_i)) \longrightarrow 0 \\ &0 \longrightarrow H^{n-1}(\mathrm{Tw}_{S_i}(S_i)) \xrightarrow{\cong} H^n(S_i \oplus S_i[-n]) \longrightarrow 0.\end{aligned}$$

Therefore $\mathrm{Tw}_{S_i}(S_i) = S_i[1-n]$. This is true in any triangulated category on which spherical twists are defined.

The final result that we need regarding spherical twists is how they interact with other functors.

Lemma 1.4.9. *[25, Lem. 8.21] Let E be a spherical object in a triangulated category $\mathcal{D}$ and φ an autoequivalence of $\mathcal{D}$. Then $\varphi \circ \mathrm{Tw_E} \cong \mathrm{Tw}_{\varphi(E)} \circ \varphi$.*

1.5 Stability conditions

We now give a brief introduction to stability conditions and indicate the relevance of studying the exchange graph to computing the space of stability conditions. All definitions and further details can be found in [7]. Again, in this section, $\mathcal{D}$ denotes a triangulated category and $\mathrm{K}(\mathcal{D})$ is its Grothendieck group.

Definition 1.5.1. A *stability condition* on $\mathcal{D}$ is a pair $\sigma = (Z, \mathcal{P})$ where $Z \colon \mathrm{K}(\mathcal{D}) \longrightarrow \mathbb{C}$ is a group homomorphism, and $\mathcal{P} = \cup_{\phi \in \mathbb{R}} \mathcal{P}(\phi)$ is a collection of full additive subcategories of $\mathcal{D}$, called a slicing, which satisfy the following axioms:

(i) if $E \in \mathcal{P}(\phi)$ then $Z(E) = m_E e^{\pi i \phi}$ for $m_E \in \mathbb{R}_{>0}$;

(ii) for $E_i \in \mathcal{P}(\phi_i)$, $\mathrm{Hom}_{\mathcal{D}}(E_i, E_j) = 0$ for $\phi_i > \phi_j$;

(iii) $\mathcal{P}(\phi)[1] = \mathcal{P}(\phi + 1)$;

(iv) for each object $E \in \mathcal{D}$ there exists a sequence of exact triangles

$$\begin{tikzcd}
0 = E_0 \arrow[rr] & & E_1 \arrow[rr] \arrow[dl] & & E_2 \arrow[r] \arrow[dl] & \cdots \arrow[r] & E_{n-1} \arrow[rr] & & E_n = E \arrow[dl] \\
& A_1 \arrow[ul, dashed] & & A_2 \arrow[ul, dashed] & & & & A_n \arrow[ul, dashed] &
\end{tikzcd}$$

such that $A_i \in \mathcal{P}(\phi_i)$ and $\phi_1 > \ldots > \phi_n$.

Given a stability condition σ, for the sequence of triangles in condition (iv) of Definition 1.5.1 we define $\phi_\sigma^+(E) := \phi_1$ and $\phi_\sigma^-(E) := \phi_n$, and call $w_\sigma(E) := \phi_\sigma^+(E) - \phi_\sigma^-(E)$ the *width* of E with respect to σ. The objects A_i are the *Harder–Narasimhan factors* of E with respect to σ.

Lemma 1.5.2. *[7, Lem. 5.2] The subcategories $\mathcal{P}(\phi)$ are abelian.*

Definition 1.5.3. Let $E \in \mathcal{D}$ and $\sigma = (Z, \mathcal{P})$ be a stability condition on $\mathcal{D}$. If

(i) $E \in \mathcal{P}(\phi) \subset \mathcal{D}$, then E is *semistable with phase* ϕ with respect to σ;

(ii) E is semistable and simple in $\mathcal{P}(\phi)$, then E is *stable with phase* ϕ with respect to σ;

(iii) E is not in $\mathcal{P}(\phi)$ for any $\phi \in \mathbb{R}$, then E is *unstable* with respect to σ.

Note that semistable and stable objects have width zero, whilst unstable objects have strictly positive width.

If I is an interval in $\mathbb{R}$ with length less than one, let $\mathcal{P}(I)$ be the extension closed subcategory of objects which have Harder–Narasimhan factors of phase $\phi \in I$ with respect to σ.

There is an equivalent formulation of stability conditions in terms of the hearts of bounded t-structures.

Definition 1.5.4. A *stability condition* on $\mathcal{D}$ is a pair $(\mathcal{A}, Z)$ where $\mathcal{A}$ is the heart of a bounded t-structure on $\mathcal{D}$ and $Z\colon \mathrm{K}(\mathcal{A}) \longrightarrow \mathbb{C}$ is a group homomorphism such that:

(i) for non-zero objects $E \in \mathcal{A}$, $Z(E) = m_E e^{i\pi\phi(E)}$ lies in the subset

$$\bar{\mathcal{H}} := \{z \in \mathbb{C}\colon \operatorname{Im}(z) > 0 \text{ or } z \in \mathbb{R}_{<0}\};$$

(ii) for all objects E in $\mathcal{A}$ there exists a filtration

$$0 = E_0 \subset E_1 \subset E_2 \subset \cdots \subset E_{n-1} \subset E_n = E$$

with the quotients $A_i := E_i/E_{i-1}$ satisfying $\phi(A_1) > \phi(A_2) > \ldots > \phi(A_n)$. This is called the *Harder–Narasimhan property* of $\mathcal{A}$ with respect to the stability function Z.

Lemma 1.5.5. *[7, Prop. 5.3] Definitions 1.5.1 and 1.5.4 are equivalent.*

Proof. Let $\sigma = (Z, \mathcal{P})$ be a stability condition on a triangulated category $\mathcal{D}$ in the sense of Definition 1.5.1. Then a bounded t-structure on $\mathcal{D}$ is given by $\mathcal{D}^{\leq 0} := \mathcal{P}(>0)$ and $\mathcal{D}^{>0} := \mathcal{P}(\leq 0)$. If we restrict Z from $\mathrm{K}(\mathcal{D})$ to $\mathrm{K}(\mathcal{A})$ for the heart $\mathcal{A} = \mathcal{P}((0,1])$, then we have a stability condition in the sense of Definition 1.5.4.

Conversely, let $\sigma = (\mathcal{A}, Z)$ be a stability condition characterised by the heart of a bounded t-structure as in Definition 1.5.4. Define

$$\mathcal{P}(\phi) := \{E \in \mathcal{A}\colon E \text{ is semistable of phase } \phi\}$$

for $\phi \in (0,1]$. Using condition (iii) in Definition 1.5.1, we define the categories $\mathcal{P}(\phi)$ for all $\phi \in \mathbb{R}$. Thus we have a slicing on $\mathcal{D}$ which is compatible with the stability function Z. $\square$

Definition 1.5.6. A stability condition $\sigma = (Z, \mathcal{P})$ is *supported on a heart* $\mathcal{A}$ when $(\mathcal{A}, Z)$ is the pair related to σ by Lemma 1.5.5.

This relationship between stability conditions and hearts of bounded t-structures allows us to view the exchange graph as a combinatorial description of the space of stability conditions.

The remainder of this section details how a complex manifold can be constructed from the set of stability conditions and introduces some of the properties of this manifold that we use later.

In order to define the space of stability conditions of a triangulated category $\mathcal{D}$, the first detail to consider is whether the Grothendieck group $\mathrm{K}(\mathcal{D})$ is of finite rank. For the triangulated categories $\mathrm{D_n}(\mathrm{K}_2)$ which we study this is indeed the case as the group $\mathrm{K}(\mathrm{D_n}(\mathrm{K}_2))$ is generated by the simple objects S_1 and S_2. If $\mathrm{K}(\mathcal{D})$ is not of finite rank, it is necessary to introduce a finite dimensional lattice Λ and a map $\mathrm{K}(\mathcal{D}) \longrightarrow \Lambda$, and to consider only stability conditions with central charges factoring through this lattice. Fix a norm $\|\cdot\|$ on $\Lambda \otimes \mathbb{R}$.

Definition 1.5.7. A stability condition $\sigma = (Z, \mathcal{P})$ has the *support property* if

$$\inf \left\{ \frac{|Z(E)|}{\|[E]\|} : E \text{ is } \sigma\text{-semistable} \right\} > 0.$$

The norm induces a metric on the dual lattice which is the set of central charges $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$. A generalised metric on the set of stability conditions with the support property is defined by

$$d(\sigma_1, \sigma_2) = \sup_{0 \neq E \in \mathcal{D}} \left\{ |\phi^+_{\sigma_2}(E) - \phi^+_{\sigma_1}(E)|, |\phi^-_{\sigma_2}(E) - \phi^-_{\sigma_1}(E)|, \|Z_{\sigma_1} - Z_{\sigma_2}\| \right\} \in [0, \infty]. \qquad (1.5.1)$$

Definition 1.5.8. The *space of stability conditions* $\mathrm{Stab}(\mathcal{D})$ of a triangulated category $\mathcal{D}$ is the set of stability conditions satisfying the support property, together with the topology induced by the metric in 1.5.1.

Theorem 1.5.9. *[3] Each connected component of the space of stability conditions on a triangulated category $\mathcal{D}$ is locally homeomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\Lambda, \mathbb{C})$. The homeomorphism sends a stability condition $\sigma = (Z, \mathcal{P})$ to its central charge Z.*

Remark 1.5.10.
(i) If $\mathrm{K}(\mathcal{D})$ has rank $n < \infty$, then $\mathrm{Stab}(\mathcal{D})$ is locally homeomorphic to $\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}(\mathcal{D}), \mathbb{C})$ and has complex dimension n.

(ii) This complex manifold is endowed with a wall and chamber structure. Heuristically, this means that if an object in $\mathcal{D}$ is stable with respect to a stability condition σ, then it is stable for all stability conditions in an open neighbourhood of σ, and becomes unstable for stability conditions on the other side of the walls, which are codimension one submanifolds.

Proposition 1.5.11. *(i) For a fixed object $E \in \mathcal{D}$, the functions $\phi^+_\sigma(E)$ and $\phi^-_\sigma(E)$ are continuous with respect to stability conditions $\sigma \in \mathrm{Stab}(\mathcal{D})$.*

(ii) If an object E is stable with respect to a stability condition σ, then it is stable for stability conditions in an open neighbourhood of σ.

(iii) If an object is unstable with respect to a stability condition σ then it remains unstable in an open neighbourhood of σ. Equivalently, a semistable object with respect to σ remains semistable in a closed neighbourhood of σ.

Proof. (i) Follows from the definition of the metric.

(ii) As in [12, Prop. 7.6].

(iii) Let $\epsilon > 0$. Suppose that X is unstable in σ_0 with width $w_{\sigma_0}(X) = r$ and stable in $\sigma \in B_\epsilon(\sigma_0)$, so $w_\sigma(X) = 0$.

Then $d(\sigma, \sigma_0) > 0$, and

$$\begin{aligned}|w_\sigma(X) - w_{\sigma_0}(X)| &= |(\phi^+_\sigma(X) - \phi^-_\sigma(X)) - (\phi^+_{\sigma_0}(X) - \phi^-_{\sigma_0}(X))| \\ &= |(\phi^+_\sigma(X) - \phi^+_{\sigma_0}(X)) - (\phi^-_\sigma(X) - \phi^-_{\sigma_0}(X))| \\ &\leq |\phi^+_\sigma(X) - \phi^+_{\sigma_0}(X)| + |\phi^-_\sigma(X) - \phi^-_{\sigma_0}(X)| \\ &\leq 2\epsilon.\end{aligned}$$

Now $|w_\sigma(X) - w_{\sigma_0}(X)| = r$. Let $\epsilon = \frac{r}{4}$. Then X cannot become stable for $\sigma \in B_{\frac{r}{4}}(\sigma_0)$.

□

Lemma 1.5.12. *If $d(\sigma_1, \sigma_2) = \epsilon$, then objects X which are semistable in σ_1 have width $w_{\sigma_2}(X) \leq 2\epsilon$.*

Proof. If $d(\sigma_1, \sigma_2) = \epsilon$ and $w_{\sigma_1}(X) = 0$, then

$$\begin{aligned}|w_{\sigma_2}(X)| = |w_{\sigma_1}(X) - w_{\sigma_2}(X)| &= |(\phi^+_{\sigma_1}(X) - \phi^-_{\sigma_1}(X)) - (\phi^+_{\sigma_2}(X) - \phi^-_{\sigma_2}(X))| \\ &= |(\phi^+_{\sigma_1}(X) - \phi^+_{\sigma_2}(X)) - (\phi^-_{\sigma_1}(X) - \phi^-_{\sigma_2}(X))| \\ &\leq |\phi^+_{\sigma_1}(X) - \phi^+_{\sigma_2}(X)| + |\phi^-_{\sigma_1}(X) - \phi^-_{\sigma_2}(X)| \\ &\leq 2\epsilon.\end{aligned}$$

□

The final properties of the space of stability conditions which we consider are the actions of the group of autoequivalences and the additive group of complex numbers with which it is endowed.

Lemma 1.5.13. *[7, Lem. 8.2] For a triangulated category $\mathcal{D}$, there is an action of the group of autoequivalences* $\mathrm{Aut}(\mathcal{D})$ *on the space of stability conditions* $\mathrm{Stab}(\mathcal{D})$.

Proof. The action is defined as follows. For each autoequivalence $\Psi \in \mathrm{Aut}(\mathcal{D})$, there is an automorphism $\tilde{\Psi}\colon \mathrm{K}(\mathcal{D}) \longrightarrow \mathrm{K}(\mathcal{D})$ of the Grothendieck group. Then $\Psi \cdot \sigma$ has central charge $Z \circ \tilde{\Psi}^{-1}$ and the subcategories $\Psi(\mathcal{P}(\phi))$ give the slicing. ☐

The action of the group of autoequivalences on the stability manifold can uncover new information about the triangulated category. Depending on the triangulated category, the group of autoequivalences may be well understood in which case this action is a useful tool to understand the space of stability conditions. Conversely for categories where a full description of the group of autoequivalences is not yet known, understanding the space of stability conditions may provide insights to some of the properties of this group.

Lemma 1.5.14. *[9] The action of the additive group of the complex numbers on the space of stability conditions is free.*

Proof. Let $\sigma = (Z, \mathcal{P})$ be a stability condition on $\mathcal{D}$ with $Z(E) = m_E e^{i\pi\phi}$ for all $E \in \mathcal{D}$. Using the isomorphism $\mathbb{C} \cong \mathbb{R}^2$, write $z \in \mathbb{C}$ as $z = x + iy$. The action of $\mathbb{C}$ on $\mathrm{Stab}(\mathcal{D})$ is defined by $z \cdot \sigma = (Z', \mathcal{P}')$ where $Z'(E) = e^{-i\pi z} m_E e^{i\pi\phi}$ and $\mathcal{P}'(\phi) = \mathcal{P}(\phi - x)$. Suppose that $z \cdot \sigma = \sigma$. As

$$\begin{aligned} Z'(E) &= e^{-i\pi z} m_E e^{i\pi\phi} \\ &= e^{\pi y} m_E e^{i\pi(\phi - x)} \end{aligned}$$

with $e^{\pi y} \in \mathbb{R}$, the central charges satisfy $Z(E) = Z'(E)$ if and only if $y = 0$ and $x = \phi + 2k$ for $k \in \mathbb{Z}$. Turning our attention to the action on the slicings, $\mathcal{P}(\phi) = \mathcal{P}'(\phi)$ if and only if $x = 0$. Hence $z = 0$ and the action is free. ☐

Remark 1.5.15. (i) The effect of this action is to scale the mass of the central charge by a factor of $e^{\pi y}$ and change the phase by $-x$.

(ii) Restricting the action to that of the subgroup $\mathbb{Z} \subset \mathbb{C}$ results in an action which coincides with that of the subgroup of autoequivalences generated by the shift functor.

Chapter 2

Frobenius manifolds

Frobenius manifolds are complex manifolds whose tangent bundles have the structure of Frobenius algebras, together with additional geometric data. They were introduced by Dubrovin [17], and are related to the WDVV equations: A Frobenius manifold corresponds to a system of solutions to these equations.

This chapter gives the background material necessary to appreciate the richness of their geometry, and considers the particular example of the Frobenius manifold structure on the quantum cohomology of the projective line, which is relevant to the study of the spaces of stability conditions for the Kronecker quiver. The relationship between these stability manifolds and Frobenius structures is developed in the final section of the chapter.

2.1 The geometry of Frobenius manifolds

This section is primarily a collection of key definitions and results on Frobenius manifolds. This material is originally presented by Dubrovin [18]. Particular focus is given to the various coordinate systems that are defined on complex manifolds endowed with Frobenius structures.

We begin with the definition of Frobenius algebras. These structures are important mathematical objects in their own right, and are of particular interest in the study of topological quantum field theories.

Definition 2.1.1. [18, Sec. 2.1] A *Frobenius algebra* over a field k is a pair $(A, \langle\ ,\ \rangle)$ such that:

(i) A is a finite dimensional, associative, commutative, unital algebra over k;

(ii) $\langle\ ,\ \rangle$ is a k-bilinear, symmetric, non-degenerate form on A satisfying

$$\langle a \cdot b, c\rangle = \langle a, b \cdot c\rangle$$

for a, b, $c \in A$.

In the following, $\mathcal{T}_M$ denotes the holomorphic tangent bundle of a complex manifold M, and $\mathcal{O}_M$ is the ring of holomorphic functions on M.

Definition 2.1.2. [11, Def. 2.1] A *Frobenius manifold* is a quadruple of data $(M, g, \star, E)$ where:

(i) M is an N-dimensional complex manifold;

(ii) g is a holomorphic map $g\colon \mathcal{T}_M \otimes \mathcal{T}_M \longrightarrow \mathcal{O}_M$, called the metric, which induces a symmetric, non-degenerate, bilinear form on the tangent spaces;

(iii) $\star$ is a holomorphic map $\star\colon \mathcal{T}_M \otimes \mathcal{T}_M \longrightarrow \mathcal{T}_M$, called the product, which induces the structure of a commutative and associative unital algebra on the tangent spaces;

(iv) E is a holomorphic section $E\colon \mathcal{O}_M \longrightarrow \mathcal{T}_M$, called the Euler vector field;

such that the following axioms are satisfied.

(FM1) The Levi-Civita connection defined by the metric g is flat.

(FM2) There exists a flat vector field $e\colon \mathcal{O}_M \longrightarrow \mathcal{T}_M$, which is a unit for the product $\star$.

(FM3) In local coordinates $(t_1, \ldots, t_N)$ for the metric g, there exists a locally defined function $F(t_1, \ldots, t_N)$ satisfying

$$g\left(\frac{\partial}{\partial t_i} \star \frac{\partial}{\partial t_j}, \frac{\partial}{\partial t_k}\right) = \frac{\partial^3 F}{\partial t_i \partial t_j \partial t_k} = g\left(\frac{\partial}{\partial t_i}, \frac{\partial}{\partial t_j} \star \frac{\partial}{\partial t_k}\right). \tag{2.1.1}$$

The function F is called the potential of the Frobenius manifold.

(FM4) The Euler vector field E is linear and satisfies

$$\begin{aligned} &Eg(X,Y) - g([E,X],Y) - g(X,[E,Y]) = (2-d)g(X,Y) \\ &[E, X \star Y] - [E,X] \star Y - X \star [E,Y] = X \star Y \end{aligned}$$

with respect to the metric g, product $\star$ and Lie bracket $[,]$. The complex number d is the charge of the Frobenius manifold.

Remark 2.1.3. (i) The structures induced on the tangent space by the metric and product mean that the tangent spaces of a Frobenius manifold are Frobenius algebras.

(ii) At the start of this chapter we mentioned the relationship between Frobenius manifolds and the solutions to the WDVV equations. The associativity conditions on the third derivatives of the potential of the Frobenius manifold, 2.1.1, are precisely the WDVV associativity equations. Further discussion of the relationship between these equations and Frobenius manifolds can be found in [17].

(iii) We consider the role played by the charge d of the manifold. Using the charge d, the additional structure of a graded Frobenius algebra is defined on the space of vector fields. In the example of quantum cohomology that we consider in the next section, the charge corresponds to the dimension of the Frobenius manifold, and the grading on the space of vector fields aligns with the usual grading on cohomology.

In addition to the local flat coordinates $(t_1, \ldots, t_N)$ defined from the metric g, there are other ways to define coordinate systems on a Frobenius manifold. The rest of this section considers some of theses.

Deformed flat coordinates

The second system of coordinates that we construct are the deformed flat coordinates, which are functions defined on $M \times \mathbb{C}^*$.

Definition 2.1.4. Let M be a Frobenius manifold of charge d. Define on the tangent bundle an antisymmetric operator

$$\mathcal{V} := \frac{2-d}{2} - \nabla E.$$

Definition 2.1.5. The *deformed flat connection* of a Frobenius manifold M is the connection $\tilde{\nabla}$ defined on $M \times \mathbb{C}^*$ satisfying

$$\begin{aligned}
\tilde{\nabla}_X Y &= \nabla_X Y + zX \star Y \\
\tilde{\nabla}_{\frac{d}{dz}} Y &= \partial_z Y + E \star Y - \frac{1}{z}\mathcal{V}Y \\
\tilde{\nabla}_X \frac{d}{dz} &= \tilde{\nabla}_{\frac{d}{dz}} \frac{d}{dz} = 0
\end{aligned}$$

for $X, Y \in \mathcal{T}_{M,t}$ and $z \in \mathbb{C}^*$.

Definition 2.1.6. A function f on $M \times \mathbb{C}^*$ is $\tilde{\nabla}$-*flat* if $\tilde{\nabla} df = 0$.

In the terms of the local flat coordinates $t_1, \dots t_N$ of M and parameter $z \in \mathbb{C}^*$, the flatness of a function $f(t,z)$ on $M \times \mathbb{C}^*$ is equivalent to

$$\begin{aligned} \partial_{t_i}\zeta &= z\zeta C_i(t) \\ \partial_z \zeta &= \zeta\left(\mathcal{U}(t) - \frac{\mathcal{V}}{z}\right) \end{aligned} \tag{2.1.2}$$

where $\zeta = (\zeta_1, \dots, \zeta_N)$ is the vector with i-th entry the partial derivative of f with respect to the local flat coordinate t_i of M, $C_i(t)$ the multiplication matrix of $\frac{\partial}{\partial t_i}$ and the operator $\mathcal{U}(t)\colon \mathcal{T}_{M,t} \longrightarrow \mathcal{T}_{M,t}$ is defined as multiplication with the Euler vector field at t. These equations are obtained from the first and second equations in Definition 2.1.5.

As the flat connection ∇ defined the flat coordinates of M, the deformed flat connection $\tilde{\nabla}$ defines a local system of coordinates on $M \times \mathbb{C}^*$.

Definition 2.1.7. The *deformed flat coordinates* $\tilde{t_1}(t,z) \dots \tilde{t_N}(t,z)$ of the Frobenius manifold M are an independent system of $\tilde{\nabla}$-flat functions defined locally on $M \times \mathbb{C}^*$.

Canonical coordinates

We have now defined two systems of coordinates on a Frobenius manifold, both of which arise from the existence of flat connections on the manifold. One can define a further two coordinate systems by restricting to a certain class of Frobenius manifolds.

Definition 2.1.8. (i) An associative algebra is *semisimple* if its Jacobson radical is trivial.

(ii) [18, Sec. 2.2] A Frobenius manifold M is *semisimple* if there exists an open dense subset of M for which the Frobenius algebra structure of the tangent spaces is semisimple.

We restrict to points $t \in M_{s,s}$, defined to be the open dense subset of M on which the operator $\mathcal{U}(t)\colon \mathcal{T}_{M,t} \longrightarrow \mathcal{T}_{M,t}$ has simple eigenvalues.

Proposition 2.1.9. *[18, Thm. 2.2] The eigenvalues of the operator $\mathcal{U}(t)$, which are denoted by $u_1(t), \dots u_N(t)$, are a system of local coordinates on $M_{s,s} \subset M$.*

Definition 2.1.10. The system of local coordinates on $M_{s,s}$ defined by the eigenvalues of the operator $\mathcal{U}(t)$ are the *canonical coordinates* of the Frobenius manifold M.

Twisted periods

To construct the fourth and final coordinate system on a Frobenius manifold, we consider the dual almost Frobenius manifold. As the name suggests, this is a dual construction on a Frobenius manifold which satisfies almost all of the axioms of a Frobenius manifold. The underlying manifold of the dual almost Frobenius manifold to M is an open subset of the original manifold, defined using the following features of M.

Definition 2.1.11. The *intersection form* $(\ , \ )$ of a Frobenius manifold M is the bilinear form defined on the cotangent bundle $\mathcal{T}^*_M$ by the metric g.

Definition 2.1.12. The *discriminant* Λ of a Frobenius manifold M is the subset of M for which the determinant of the intersection form is zero, i.e.

$$\Lambda := \{t \in M \colon \det[(\ , \ )] = 0\}.$$

Define $M_* := M \setminus \Lambda$ and let $\tilde{\nabla}_*$ be the deformed flat connection on $M_* \times \mathbb{C}$. A function $\tilde{p}(t, \nu)$ on $M_* \times \mathbb{C}$ is $\tilde{\nabla}_*$-flat if

$$\partial_i \zeta \cdot \mathcal{U}(t) = \zeta \cdot \Big(\mathcal{V} + \nu - \frac{1}{2}\Big) C_i(t)$$

for $t_1, \ldots t_N$ the local flat coordinates of M and the vector $\zeta = \zeta(\tilde{p})$ defined as in 2.1.2, the condition for the flatness of a function on $M \times \mathbb{C}^*$.

Definition 2.1.13. The *twisted periods* $\tilde{p}_1(t, \nu), \ldots, \tilde{p}_N(t, \nu)$ of a Frobenius manifold M are an independent system of $\tilde{\nabla}_*$-flat functions defined locally on $M_* \times \mathbb{C}$.

Remark 2.1.14. The subset M_* can itself be endowed with some of the features of a Frobenius manifold. It is called the *dual almost Frobenius manifold of M*. In particular, the twisted periods of the Frobenius manifold M are the deformed flat coordinates of its dual almost Frobenius structure.

It is these twisted periods that are of particular importance in our study of the spaces of stability conditions of the Kronecker quiver. The theory of this relationship is discussed in the final section of this chapter.

2.2 Quantum cohomology of $\mathbb{P}^1$

This section introduces the Frobenius manifold that plays a role in the study of the spaces of stability conditions of the Kronecker quiver, namely the quantum cohomology of $\mathbb{P}^1$.

The quantum cohomology of a projective variety X, which we denote by $QH^*(X)$, provides information about its symplectic geometry. In fact two versions of quantum cohomology can be defined. The big quantum cohomology is defined in terms of Gromov–Witten invariants of X, and the small quantum cohomology in terms of classes in the cohomology group $H^2(X)$. Our interest is in the Frobenius structure given by quantum cohomology, instead of the symplectic geometry it encodes, so we restrict to defining the quantum cohomology as a Frobenius manifold, instead of as a symplectic invariant. Discussion of quantum cohomology of N-dimensional complex projective spaces can be found in [18, Sec. 5.4].

In order to study the Frobenius manifold defined by the big quantum cohomology of $\mathbb{P}^1$, recall from Definition 2.1.2 that the data of a Frobenius manifold is given by the quadruple of $(M, g, \star, E)$: the complex manifold, the metric, the product, and the Euler vector field. For the quantum cohomology of $\mathbb{P}^1$, the underlying complex manifold M is $\mathbb{C} \times \mathbb{C}^*$. We denote the local flat coordinates by (s, t), and $\frac{\partial}{\partial s}$ is the unit in the Frobenius algebras.

We consider the Landau–Ginzburg potential associated to $\mathbb{P}^1$ in order to study the Frobenius structure on $QH^*(\mathbb{P}^1)$. Define the function $p \colon \mathbb{C} \longrightarrow \mathbb{C}$ by

$$p(z) = e^{\frac{t}{2}} z + s + e^{\frac{t}{2}} z^{-1}.$$

The canonical coordinates of the Frobenius structure are defined by the critical values of the function p. Observe that if

$$\frac{dp}{dz} = e^{\frac{t}{2}} - \frac{e^{\frac{t}{2}}}{z^2} = 0$$

then the function p has critical values at $z = \pm 1$. The canonical coordinates are then defined to be

$$u_1 := p(1) = s + 2e^{\frac{t}{2}} \qquad\qquad u_2 := p(-1) = s - 2e^{\frac{t}{2}}.$$

The multiplication $\star$ of the Frobenius algebras of the tangent spaces satisfy

$$\begin{aligned}
\frac{\partial}{\partial s} \star \frac{\partial}{\partial s} &= \frac{\partial}{\partial s} \\
\frac{\partial}{\partial t} \star \frac{\partial}{\partial t} &= e^t \frac{\partial}{\partial s} \\
\frac{\partial}{\partial s} \star \frac{\partial}{\partial t} &= \frac{\partial}{\partial t} \star \frac{\partial}{\partial s} = e^t \frac{\partial}{\partial t}.
\end{aligned}$$

In terms of the flat coordinates, the Euler vector field E is given by

$$\begin{aligned}
E &= u_1 \frac{\partial}{\partial u_1} + u_2 \frac{\partial}{\partial u_2} \\
&= (s + 2e^{\frac{t}{2}}) \left(\frac{1}{2} \frac{\partial}{\partial s} + \frac{1}{2e^{\frac{t}{2}}} \frac{\partial}{\partial t} \right) + (s - 2e^{\frac{t}{2}}) \left(\frac{1}{2} \frac{\partial}{\partial s} - \frac{1}{2e^{\frac{t}{2}}} \frac{\partial}{\partial t} \right) \\
&= s \frac{\partial}{\partial s} + 2 \frac{\partial}{\partial t}.
\end{aligned}$$

With this data, we consider the features of the quantum cohomology of $\mathbb{P}^1$ that are of most interest in the construction of the spaces of stability conditions: the twisted periods.

Proposition 2.2.1. *[18, Prop. 5.13] Let s and t be the flat coordinates of the big quantum cohomology of $\mathbb{P}^1$. The twisted periods $\tilde{p} = \tilde{p}(t,\nu)$ are the solutions to the hypergeometric equation*

$$\partial_t^2 \tilde{p} - \frac{e^t}{s^2}\Big[(-2\partial_t + \nu)(-2\partial_t + \nu - 1)\Big]\tilde{p} = 0$$

which satisfy the quasi-homogeneity condition

$$s\partial_s \tilde{p} + 2\partial_t \tilde{p} = \nu \tilde{p}.$$

2.3 Frobenius structures on spaces of stability conditions

This section discusses the motivation for studying the connection between Frobenius manifolds and spaces of stability conditions. In the introduction we considered heuristically why there should be some connection between these manifolds. Here we make this idea more precise and particularly highlight the relationship between the twisted periods of a Frobenius manifold and the central charges of stability conditions.

The two main classes of examples of Frobenius manifolds are given by the unfolding spaces of singularities and quantum cohomologies. Moreover mirror symmetry predicts that there is a correspondence between these classes. For a smooth projective Fano variety X, the mirror map should exchange the Frobenius manifold of its quantum cohomology with the Frobenius manifold of the unfolding space of the Landau–Ginzburg potential of its mirror. In the situation that we study, the Fano variety in question is $\mathbb{P}^1$, with mirror Landau–Ginzburg potential given by the function $p(z) = \sqrt{b}z + 2a + \sqrt{b}z^{-1}$. In Chapter 5, the role that this function plays in the computations of spaces of stability conditions is discussed. However, at that this point we have developed enough machinery to discuss more concretely part of the connection of Frobenius manifolds to stability conditions, namely the relationship between the central charges and twisted periods.

The central charge map projects a stability condition $\sigma = (Z, \mathcal{P}) \in \mathrm{Stab}(\mathcal{D})$ to its central charge $Z \in \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}(\mathcal{D}), \mathbb{C})$. This induces a map $\mathcal{Z}\colon \mathrm{Stab}(\mathcal{D}) \longrightarrow \mathbb{C}^N$, where N is the rank of the Grothendieck group of $\mathcal{D}$. On the Frobenius manifold side of the theory, the twisted periods also define a map from a complex manifold of dimension N to $\mathbb{C}^N$.

Definition 2.3.1. The *twisted period mapping* of a Frobenius manifold M is a map

$$\tilde{p}_1(t,\nu), \ldots, \tilde{p}_N(t,\nu)\colon \widetilde{M_*} \longrightarrow \mathbb{C}^N$$

where the functions $\tilde{p}_1(t,\nu), \ldots, \tilde{p}_N(t,\nu)$ are the twisted periods of M, and $\widetilde{M_*}$ is the universal cover of M_*.

Conjecture 2.3.2 (Bridgeland)**.** *Let Q be an acyclic quiver with N vertices. Then there exists an N-dimensional Frobenius manifold M such that the space of stability conditions of the Calabi–Yau-n category associated to Q is isomorphic to the universal cover of M_*. Under this isomorphism the central charges of stability conditions on each Calabi–Yau-n category are the twisted periods of M with parameter of deformation $\nu = \frac{n-2}{2}$.*

Figure 2.1 summarises the relationship between the main objects in Conjecture 2.3.2.

$$\begin{tikzcd}
 & & \text{acyclic quiver with } N \text{ vertices} \arrow[d, dashed] \\
\text{N-dimensional Frobenius manifold } M \arrow[d, dashed] & & \text{sequence of Calabi–Yau-}n\text{ categories } \mathrm{D_n} \text{ indexed by integers} \arrow[d, dashed] \\
\widetilde{M_*} \arrow[rr, "\cong"] \arrow[dr, "{\tilde{p}_1(t,\frac{n-2}{2}),\ldots,\tilde{p}_N(t,\frac{n-2}{2})}"'] & & \mathrm{Stab}_*(\mathrm{D_n}) \arrow[dl, "{Z(S_1),\ldots,Z(S_N)}"] \\
 & \mathbb{C}^N &
\end{tikzcd}$$

Figure 2.1: Summary of Conjecture 2.3.2.

In the following chapters we prove this conjecture for the Kronecker quiver. It is proved for the $\mathrm{A_2}$ quiver by Bridgeland, Qiu and Sutherland [13] and for type $\mathrm{A_N}$ Dynkin quivers by Ikeda [27].

Chapter 3

Exchange graphs and autoequivalences

This chapter contains the first steps in the computation of the spaces of stability conditions of the Kronecker quiver. The first section details the construction of the exchange graphs of the categories $\mathrm{D_n}(\mathrm{K}_2)$ for $3 \leq n \leq \infty$ obtained from simple tilts of the standard heart $\mathrm{Rep}(\mathrm{K}_2) \subset \mathrm{D_n}(\mathrm{K}_2)$, and the definition of the autoequivalences whose action corresponds to these tilts. The latter two sections detail the group of autoequivalences for the Calabi–Yau-n categories and for the derived category of the Kronecker quiver respectively.

3.1 The relationship between tilts and autoequivalences

In this section we construct the exchange graphs of the categories $\mathrm{D_n}(\mathrm{K}_2)$ and detail how the groups of autoequivalences of the categories act on them.

We begin by fixing some notation. As we are only concerned with the Kronecker quiver, we denote the associated Calabi–Yau-n and derived categories simply by $\mathrm{D_n}$, respectively D_∞. Recall from Section 1.3 that the triangulated category $\mathrm{D_n}$ has a standard heart $\mathcal{A}_0$ which is the smallest extension-closed subcategory of $\mathrm{D_n}$ containing the simple representations S_1 and S_2. Furthermore $\mathcal{A}_0$ is equivalent to $\mathrm{Rep}(\mathrm{K}_2)$ for $3 \leq n \leq \infty$, and to the category of representations of the preprojective algebra of K_2 for $n = 2$. We consider only the connected component of the exchange graph $\mathrm{EG}_*(\mathrm{D_n})$ which contains $\mathcal{A}_0$. In the study of spaces of stability conditions, it is often not known whether the space is connected. Restricting to this distinguished component of the exchange graph is analogous to studying the connected component $\mathrm{Stab}_*(\mathrm{D_n})$ of the space of stability conditions containing stability conditions supported on the standard heart. All hearts in this component can be reached by a sequence of simple tilts of $\mathcal{A}_0$. The aim of this section is to deduce which of these hearts are equivalent to $\mathcal{A}_0$ in the following sense.

Definition 3.1.1. A heart in the exchange graph which is in the same orbit of $\mathrm{Aut}_*(\mathrm{D_n})$ as $\mathcal{A}_0$ is called *full*.

An autoequivalence in $\mathrm{Aut}(\mathrm{D_n})$ is *reachable* if it preserves the connected component of the exchange graph of $\mathrm{D_n}$ containing the standard heart. Denote by $\mathrm{Aut}_*(\mathrm{D_n})$ the subquotient group of the group of autoequivalences $\mathrm{Aut}(\mathrm{D_n})$ which preserve $\mathrm{EG}_*(\mathrm{D_n})$, modulo those which act trivially. Therefore if an autoequivalence φ preserves the simple objects S_1 and S_2, it is the identity in the subquotient group $\mathrm{Aut}_*(\mathrm{D_n})$. This is equivalent to defining $\mathrm{Aut}_*(\mathrm{D_n})$ as the subquotient group which preserves the connected component $\mathrm{Stab}_*(\mathrm{D_n})$ of the space of stability conditions, modulo those which act trivially.

The vertices of the exchange graph are the finite length hearts of bounded t-structures on $\mathrm{D_n}$ obtained by simple tilts of the abelian category $\langle S_1, S_2\rangle$ and hence are of the form $\langle X_1, X_2\rangle$ where the objects X_i are simple in the tilted heart. If there exists an autoequivalence of the triangulated category $\mathrm{D_n}$ under which one finite length heart is sent to another, then it maps simple objects to simple objects and preserves the morphisms between them. Importantly, if two such Hom-spaces are not isomorphic, then there is not an autoequivalence between the corresponding hearts.

The group of autoequivalences acts on the exchange graph via a right action.

Lemma 3.1.2. *Let $\mathcal{A}_0 \subset \mathrm{D_n}$ be the standard heart of the Calabi–Yau-n category of the Kronecker quiver. For $2 \le n \le \infty$, the simple tilts of $\mathcal{A}_0$ are*

$$L_{S_2}\mathcal{A}_0 = \langle E_{1,2}, S_2[1]\rangle \qquad\qquad R_{S_1}\mathcal{A}_0 = \langle S_1[-1], E_{2,1}\rangle.$$

For $3 \le n \le \infty$, the simple tilts of $\mathcal{A}_0$ are

$$L_{S_1}\mathcal{A}_0 = \langle S_1[1], S_2\rangle \qquad\qquad R_{S_2}\mathcal{A}_0 = \langle S_1, S_2[-1]\rangle.$$

Proof. We use the description of the left and right tilts of the heart of a bounded t-structure given in Lemma 1.3.15.

First we compute the heart $L_{S_2}\mathcal{A}_0$, for $2 \le n \le \infty$. Consider the torsion theory $(\mathcal{T}, \mathcal{F})$ defining the left tilt of $\mathcal{A}_0$ at S_2, which is given by

$$\mathcal{T} = {}^{\perp}S_2 = \{E \in \mathcal{A}_0 \colon \mathrm{Hom}(E, S_2) = 0\} \qquad\qquad \mathcal{F} = \langle S_2\rangle.$$

Then

$$L_{S_2}\mathcal{A}_0 = \{E \in \mathrm{D_n} \colon H^0(E) \in {}^{\perp}S_2, H^{-1}(E) \in \langle S_2\rangle \text{ and } H^i(E) = 0 \text{ for } i \notin \{0, -1\}\}.$$

The objects in the category $\langle S_2[1]\rangle$ satisfy $H^{-1}(E) \in \mathcal{F}$ and $H^i(E) = 0$ otherwise. The objects with $H^0(E) \in \mathcal{T}$ and otherwise vanishing are exactly the objects of $\mathcal{T}$. Finally the category may contain two term complexes.

As $\mathrm{Hom}_{\mathrm{D_n}}(E_{1,2}, S_2) = 0$, the indecomposable object $E_{1,2}$ lies in the left orthogonal complement of S_2. It remains to check that $S_2[1]$ and $E_{1,2}$ are simple objects in $L_{S_2}\mathcal{A}_0$.

First consider $S_2[1]$ and any short exact sequence of the form $0 \longrightarrow X \longrightarrow S_2[1] \longrightarrow Y \longrightarrow 0$ in the abelian category $L_{S_2}\mathcal{A}_0$. The long exact cohomology sequence reduces to

$$0 \longrightarrow H^{-1}(X) \longrightarrow H^{-1}(S_2[1]) \longrightarrow H^{-1}(Y) \longrightarrow H^0(X) \longrightarrow H^0(S_2[1]) \longrightarrow H^0(Y) \longrightarrow 0$$

from which we deduce that $H^0(Y) = 0$ as $H^0(S_2[1]) = 0$. This leaves the four term exact sequence

$$0 \longrightarrow H^{-1}(X) \longrightarrow S_2 \longrightarrow H^{-1}(Y) \longrightarrow H^0(X) \longrightarrow 0. \tag{3.1.1}$$

Define $M := \ker(H^{-1}(Y) \longrightarrow H^0(X))$. Splitting the sequence 3.1.1 at M, we obtain the exact sequences

$$0 \longrightarrow H^{-1}(X) \longrightarrow S_2 \longrightarrow M \longrightarrow 0$$
$$0 \longrightarrow M \longrightarrow H^{-1}(Y) \longrightarrow H^0(X) \longrightarrow 0.$$

If $M = 0$, then $H^{-1}(X) = S_2$ and $H^{-1}(Y) \cong H^0(X)$. Hence $H^{-1}(Y) = 0 = H^0(X)$ as $H^{-1}(Y) \in \langle S_2\rangle$ and $H^0(X) \in {}^{\perp}S_2$. Therefore $X = S_2[1]$ and $Y = 0$. Now suppose that $M \neq 0$. Then $H^{-1}(X) = 0$ and $M = S_2$. As $H^0(X) \in {}^{\perp}S_2$, $X = 0$ and therefore $Y = S_2[1]$. Thus $S_2[1]$ is simple in $L_{S_2}\mathcal{A}_0$.

Similarly for $E_{1,2} \in L_{S_2}\mathcal{A}_0$, any short exact sequence of the form $0 \longrightarrow X \longrightarrow E_{1,2} \longrightarrow Y \longrightarrow 0$ gives rise to the long exact cohomology sequence

$$0 \longrightarrow H^{-1}(X) \longrightarrow H^{-1}(E_{1,2}) \longrightarrow H^{-1}(Y) \longrightarrow H^0(X) \longrightarrow H^0(E_{1,2}) \longrightarrow H^0(Y) \longrightarrow 0.$$

As $H^{-1}(E_{1,2}) = 0$, also $H^{-1}(X) = 0$ and the sequence reduces to

$$0 \longrightarrow H^{-1}(Y) \longrightarrow H^0(X) \longrightarrow E_{1,2} \longrightarrow H^0(Y) \longrightarrow 0.$$

Define $M := \ker(E_{1,2} \longrightarrow H^0(Y))$. Then the sequence splits to the short exact sequences

$$0 \longrightarrow H^{-1}(Y) \longrightarrow H^0(X) \longrightarrow M \longrightarrow 0$$
$$0 \longrightarrow M \longrightarrow E_{1,2} \longrightarrow H^0(Y) \longrightarrow 0.$$

Again if $M = 0$, then $X = 0$ and $Y = E_{1,2}$. Now suppose that $M \neq 0$. Then M is a subobject of $E_{1,2}$ and so is either S_2 or $E_{1,2}$. In fact $M \neq S_2$ as $\mathrm{Hom}(H^0(X), S_2) = 0$. Suppose that

$M = E_{1,2}$. As $H^{-1}(Y) \in \langle S_2 \rangle$, and $\mathrm{Ext}(E_{1,2}, S_2) = 0$, we must have $H^0(Y) = 0$ as otherwise the sequence splits and there is a morphism from $H^0(X) \in {}^\perp S_2$ to $H^{-1}(Y) \in \langle S_2 \rangle$. Therefore $H^0(Y) = 0 = H^{-1}(Y)$ and $X = E_{1,2}$.

The heart $R_{S_1}\mathcal{A}_0$ is computed analogously.

Now let $3 \leq n \leq \infty$. We compute $L_{S_1}\mathcal{A}_0$. The torsion theory $(\mathcal{T}, \mathcal{F})$ defining the left tilt at S_1 is

$$\mathcal{T} = {}^\perp S_1 = \{E \in \mathcal{A}_0 \colon \mathrm{Hom}(E, S_1) = 0\} \qquad\qquad \mathcal{F} = \langle S_1 \rangle.$$

The heart of this torsion theory is

$$L_{S_1}\mathcal{A}_0 = \{E \in \mathrm{D_n} \colon H^0(E) \in {}^\perp S_1, H^{-1}(E) \in \langle S_1 \rangle \text{ and } H^i(E) = 0 \text{ for } i \notin \{0, -1\}\}.$$

As $\mathrm{Hom}(S_2, S_1) = 0$, to see that $L_{S_1}\mathcal{A}_0 = \langle S_1[1], S_2 \rangle$, we have to check that $S_1[1]$ and S_2 are simple in this category. We use the same technique as in the computation for the left tilt at S_2.

Consider a short exact sequence of the form $0 \longrightarrow X \longrightarrow S_2 \longrightarrow Y \longrightarrow 0$ in $L_{S_1}\mathcal{A}_0$. The long exact cohomology sequence reduces to

$$0 \longrightarrow H^{-1}(Y) \longrightarrow H^0(X) \longrightarrow S_2 \longrightarrow H^0(Y) \longrightarrow 0.$$

We split this sequence, and with $M := \ker(S_2 \longrightarrow H^0(Y))$, obtain the exact sequences

$$0 \longrightarrow H^{-1}(Y) \longrightarrow H^0(X) \longrightarrow M \longrightarrow 0$$
$$0 \longrightarrow M \longrightarrow S_2 \longrightarrow H^0(Y) \longrightarrow 0.$$

Suppose that $M = S_2$. However, as $H^{-1}(Y) \in \langle S_1 \rangle$ and $\mathrm{Ext}^1(S_2, S_1) = 0$ in the standard heart, the first short exact sequence would split, contradicting the assumption that $H^0(X) \in {}^\perp S_1$. Therefore $M = 0$, and S_2 is simple.

Similarly, for a short exact sequence $0 \longrightarrow X \longrightarrow S_1[1] \longrightarrow Y \longrightarrow 0$, the long exact cohomology sequence reduces to

$$0 \longrightarrow H^{-1}(X) \longrightarrow S_1 \longrightarrow H^{-1}(Y) \longrightarrow H^0(X) \longrightarrow 0.$$

Define $M := \ker(H^{-1}(Y) \longrightarrow H^0(X))$ and consider the short exact sequences

$$0 \longrightarrow H^{-1}(X) \longrightarrow S_1 \longrightarrow M \longrightarrow 0$$
$$0 \longrightarrow M \longrightarrow H^{-1}(Y) \longrightarrow H^0(X) \longrightarrow 0$$

obtained from splitting the long exact cohomology sequence at M. Again due to the vanishing of the group $\mathrm{Ext}^1(S_2, S_1)$ in the standard heart, $M = 0$ and $S_1[1]$ is simple in $L_{S_1}\mathcal{A}_0$.

Again the right tilt $R_{S_2}\mathcal{A}_0$ is computed analogously. □

Remark 3.1.3. Recall from Remark 1.3.9, that the standard heart in the category $\mathrm{D_2}$ is the category of representations of the preprojective algebra of $\mathrm{K_2}$, and that in particular this category differs from $\mathrm{Rep(K_2)}$ as $\mathrm{Ext}^1(S_2, S_1) = \mathbb{C}^2$. Therefore in the calculation of $L_{S_1}\mathcal{A}_0$, if $n = 2$ then $\mathrm{Ext}^1(S_2, S_1) = \mathbb{C}^2$ and therefore the left tilt at S_1 and right tilt at S_2 are different in this category. For this, reason most of our results concerning the exchange graphs are restricted to $n \geq 3$.

Proposition 3.1.4. *Let $3 \leq n < \infty$. Let S_1 and S_2 be the simple representations of the Kronecker quiver viewed as objects in the category $\mathrm{D_n}$. Then there exists an autoequivalence $\Upsilon \in \mathrm{Aut}_*(\mathrm{D_n})$ such that $\Upsilon(S_1) = S_2$ and $\Upsilon(S_2) = S_1[n-2]$.*

Proof. Recall from the construction of the derived category of a dg algebra in Section 1.2 that a dg module over the Ginzburg algebra of a quiver can be decomposed with respect to the idempotents of the path algebra. Hence an object E in the category $\mathrm{D_n}$ can be decomposed as

$$E = \bigoplus_{j\in\mathbb{Z}} E_1^j \oplus \bigoplus_{j\in\mathbb{Z}} E_2^j$$

with E_i^j vector spaces. With the decompositions of two objects X and Y in $\mathrm{D_n}$ given by

$$X = \bigoplus_{j\in\mathbb{Z}} X_1^j \oplus \bigoplus_{j\in\mathbb{Z}} X_2^j \qquad\qquad Y = \bigoplus_{j\in\mathbb{Z}} Y_1^j \oplus \bigoplus_{j\in\mathbb{Z}} Y_2^j$$

a morphism $f\colon X \longrightarrow Y$ is defined on each of the vector spaces in the components of this decomposition, $f_i^j\colon X_i^j \longrightarrow Y_i^j$. We construct a functor of dg modules $(\tilde{\ })\colon \mathcal{C}(\mathcal{A}) \longrightarrow \mathcal{C}(\mathcal{A})$ as follows. Given a dg-module E, let $\tilde{E}_1^j = E_2^{j+n-2}$ and $\tilde{E}_2^j = E_1^j$ so that on vector spaces $\tilde{E}$ is defined as

$$\tilde{E} = \bigoplus_{j\in\mathbb{Z}} \tilde{E}_1^j \oplus \bigoplus_{j\in\mathbb{Z}} \tilde{E}_2^j.$$

On the linear maps defined by the arrows in the graded quiver, let

$$\begin{aligned} &\tilde{e}_1 = e_2 &\qquad &\tilde{e}_1^* = e_2^* &\qquad &\tilde{a}_i = a_i^* \\ &\tilde{e}_2 = e_1 &\qquad &\tilde{e}_2^* = e_1^* &\qquad &\tilde{a}_i^* = a_i \text{ for } i \in \{1,2\}. \end{aligned}$$

Define this functor on morphisms as $\tilde{f}\colon \tilde{X} \longrightarrow \tilde{Y}$ using the definition of f on the decomposition of the dg modules X and Y.

We denote the resulting functor on $\mathrm{D_n}$ by Υ. This functor is in fact an autoequivalence on $\mathrm{D_n}$ as it has an inverse. The inverse autoequivalence Υ^{-1} is constructed similarly. □

Remark 3.1.5. Under the autoequivalence Υ, $S_1 \mapsto S_2$ and $S_2 \mapsto S_1[n-2]$. As the abelian category $\mathcal{A}_0$ is the extension closure of S_1 and S_2, at the level of abelian subcategories of $\mathrm{D_n}$, Υ induces an equivalence between $\mathcal{A}_0$ and the category $\langle S_1[n-2], S_2\rangle$.

Lemma 3.1.6. *Let $3 \leq n < \infty$. Then there is a sequence of left tilts of the standard heart $\mathcal{A}_0$ at shifts of S_1 given by*

$$\langle S_1, S_2\rangle \xrightarrow{L_{S_1}} \langle S_1[1], S_2\rangle \xrightarrow{L_{S_1[1]}} \langle S_1[2], S_2\rangle \xrightarrow{L_{S_1[2]}} \dots \xrightarrow{L_{S_1[n-3]}} \langle S_1[n-2], S_2\rangle$$

such that $\langle S_1[n-2], S_2\rangle = \Upsilon\mathcal{A}_0 \cong \mathcal{A}_0$. For $n > 3$, the intermediate hearts are not equivalent to $\mathcal{A}_0$.

Proof. We first show that successive left tilts at shifts of S_1 do indeed give rise to this sequence of hearts.

From Lemma 3.1.2, $L_{S_1}(\mathcal{A}_0) = \langle S_1[1], S_2\rangle$. Tilting the category $\langle S_1[k], S_2\rangle$ at the torsion theory defined by $\mathcal{T} = {}^{\perp}S_1[k]$ and $\mathcal{F} = \langle S_1[k]\rangle$ defines the heart

$$L_{S_1[k]}\langle S_1[k], S_2\rangle = \langle S_1[k+1], {}^{\perp}S_1[k]\rangle.$$

Now by similar arguments to the computation of $L_{S_1}(\mathcal{A}_0)$ in Lemma 3.1.2, ${}^{\perp}S_1[k]$ can be replaced by S_2. Notice that $\langle S_1[n-2], S_2\rangle = \Upsilon\mathcal{A}_0$ as defined in Proposition 3.1.4.

To check that the other hearts in this sequence of tilts are not equivalent to $\mathcal{A}_0$ we examine the Hom-spaces of their simple objects. For $1 \leq k \leq n-3$, if an autoequivalence $\varphi \in \mathrm{Aut}_*(\mathrm{D_n})$ exists such that $\varphi(\mathcal{A}_0) = \langle S_1[k], S_2\rangle$, then it must act in one of the following two ways.

(i) Suppose $\varphi(S_1) = S_1[k]$ and $\varphi(S_2) = S_2$. Then

$$\mathrm{Hom}^1_{\mathrm{D_n}}(S_1, S_2) = \mathrm{Hom}^1_{\mathrm{D_n}}(S_1[k], S_2).$$

However $\mathrm{Hom}^1_{\mathrm{D_n}}(S_1, S_2) = \mathbb{C}^2$ and $\mathrm{Hom}^1_{\mathrm{D_n}}(S_1[k], S_2) = 0$ by Lemma 1.4.5, and so there is no such autoequivalence.

(ii) If $\varphi(S_1) = S_2$ and $\varphi(S_2) = S_1[k]$, then

$$\mathrm{Hom}^1_{\mathrm{D_n}}(S_1, S_2) = \mathrm{Hom}^1_{\mathrm{D_n}}(S_2, S_1[k]).$$

As in the previous case no such autoequivalence exists as $\mathrm{Hom}^1_{\mathrm{D_n}}(S_1, S_2) = \mathbb{C}^2$ and $\mathrm{Hom}^1_{\mathrm{D_n}}(S_2, S_1[k]) = 0$.

□

Lemma 3.1.7. *There is no full heart which is obtained by left tilts at shifts of S_1 in the category D_∞.*

Proof. The category D_∞ does not have the Calabi–Yau property. In this category

$$\mathrm{Hom}^*_{\mathrm{D}_\infty}(S_2, S_1) = 0$$

and hence there is no equivalence under which $S_1 \mapsto S_2$ and $S_2 \mapsto S_1[k]$ for some positive integer k. □

We denote by

$$\mathcal{A}_k := \langle S_1[k], S_2 \rangle$$

the hearts resulting from the sequence of tilts in Lemma 3.1.6.

Lemma 3.1.8. *The right tilt of $\mathcal{A}_0$ at S_2 is equivalent to the non-full heart $\mathcal{A}_{n-3}$.*

Proof. By Lemma 3.1.2, $R_{S_2}\mathcal{A}_0 = \langle S_1, S_2[-1] \rangle$. Under the action of the autoequivalence Υ^{-1} as defined in Proposition 3.1.4, the non-full heart $\mathcal{A}_{n-3} = \langle S_1[n-3], S_2 \rangle$ is sent to the heart $\Upsilon^{-1}\mathcal{A}_{n-3} = \langle S_1, S_2[-1] \rangle$. Therefore

$$R_{S_2}\mathcal{A}_0 \cong \Upsilon^{-1}\mathcal{A}_{n-3}.$$

□

Lemma 3.1.9. *For $0 \le k < l \le n-3$, let $\mathcal{A}_k = \langle S_1[k], S_2 \rangle$ and $\mathcal{A}_l = \langle S_1[l], S_2 \rangle$ be the abelian subcategories of $\mathrm{D_n}$ realised as tilts of the standard heart $\mathcal{A}_0$. Then there is no autoequivalence $\varphi \in \mathrm{Aut}_*(\mathrm{D_n})$ such that $\varphi(\mathcal{A}_k) \cong \mathcal{A}_l$ unless $k + l = n - 2$.*

Proof. We consider the Hom-spaces in these categories, which are

$$\mathrm{Hom}^i_{\mathrm{D_n}}(S_1[k], S_2) = \mathrm{Hom}^{i-k}_{\mathrm{D_n}}(S_1, S_2) = \begin{cases} \mathbb{C}^2 & \text{if } i = k+1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\mathrm{Hom}^i_{\mathrm{D_n}}(S_2, S_1[l]) = \mathrm{Hom}^{i+l}_{\mathrm{D_n}}(S_2, S_1) = \begin{cases} \mathbb{C}^2 & \text{if } i = n-l-1, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore $\mathrm{Hom}^*_{\mathrm{D_n}}(S_1[k], S_2) \cong \mathrm{Hom}^*_{\mathrm{D_n}}(S_2, S_1[l])$ if and only if $k + l = n - 2$, while

$$\mathrm{Hom}^*_{\mathrm{D_n}}(S_1[k], S_2) \ncong \mathrm{Hom}^*_{\mathrm{D_n}}(S_1[l], S_2)$$

unless $k = l$, contradicting the conditions on k and l.

Consider the case where $k + l = n - 2$. Under the autoequivalence $\Upsilon[-k] \in \mathrm{Aut}_*(\mathrm{D_n})$ the simple objects in the category $\mathcal{A}_k$ are mapped to

$$S_1[k] \mapsto S_2 \qquad\qquad S_2 \mapsto S_1[n-2-k] = S_1[l].$$

These are the simple objects in the category $\mathcal{A}_l$. □

Remark 3.1.10. Following on from Lemma 3.1.7, for $\mathcal{A}_k$ abelian subcategories of D_∞, due to the lack of Serre functor on this category, $\mathcal{A}_k \ncong \mathcal{A}_l$ for $k \neq l$.

Lemma 3.1.11. *Let $\mathcal{A}$ be the finite length heart of a bounded t-structure on a triangulated category $\mathcal{D}$, and consider an autoequivalence $\varphi \in \mathrm{Aut}_*(\mathcal{D})$. Let $S \in \mathcal{A}$ be a simple object. Then*

$$L_{\varphi(S)}(\varphi(\mathcal{A})) = \varphi(L_S\mathcal{A}).$$

Analogously, $R_{\varphi(S)}(\varphi(\mathcal{A})) = \varphi(R_S\mathcal{A})$.

Proof. The tilting construction gives us the following two abelian categories

$$\varphi(L_S\mathcal{A}) = \langle \varphi(S[1]), \varphi({}^\perp S)\rangle$$
$$L_{\varphi(S)}\varphi(\mathcal{A}) = \langle \varphi(S)[1], {}^\perp\varphi(S)\rangle.$$

As the shift functor commutes with other autoequivalences, $\varphi(S[1]) = \varphi(S)[1]$ and hence it remains to check that $\varphi({}^\perp S) = {}^\perp\varphi(S)$.

Let $E \in {}^\perp\varphi(S)$. Then

$$\mathrm{Hom}^0_{\mathrm{D_n}}(E, \varphi(S)) = \mathrm{Hom}^0_{\mathrm{D_n}}(\varphi^{-1}(E), S) = 0$$

and $\varphi^{-1}(E) \in {}^\perp S$ and $E \in \varphi({}^\perp S)$.

Conversely if $E \in \varphi({}^\perp S)$, then $\varphi^{-1}(E) \in {}^\perp S$, and $E \in {}^\perp\varphi(S)$. □

We use the existence of the autoequivalence Υ together with the spherical twists $\mathrm{Tw_{S_1}}$ and $\mathrm{Tw_{S_2}}$ to define another autoequivalence acting on the exchange graph. Recall from Lemma 1.4.9 the relation $\varphi \circ \mathrm{Tw_E} \cong \mathrm{Tw}_{\varphi(E)} \circ \varphi$ for E a spherical object in $\mathrm{D_n}$ and φ an autoequivalence in $\mathrm{Aut}_*(\mathrm{D_n})$.

Proposition 3.1.12. *Let $3 \leq n < \infty$. The simple tilts of the standard heart $L_{S_2}\mathcal{A}_0$ and $R_{S_1}\mathcal{A}_0$ are equivalent to $\mathcal{A}_0$ via the autoequivalences $\Sigma := {\mathrm{Tw_{S_2}}}^{-1} \circ \Upsilon^{-1}$ and $\Sigma^{-1} = \mathrm{Tw_{S_1}} \circ \Upsilon$ respectively.*

Proof. Consider the action of the autoequivalence $\Sigma := {\mathrm{Tw_{S_2}}}^{-1} \circ \Upsilon^{-1}$ on the simple objects of $\mathcal{A}_0$

$$S_1 \mapsto S_2[2-n] \mapsto S_2[1]$$
$$S_2 \mapsto S_1 \mapsto E_{1,2}.$$

The category with simple objects $S_2[1]$ and $E_{1,2}$ is the tilt $L_{S_2}\mathcal{A}_0$.

Similarly under $\Sigma^{-1} = \mathrm{Tw}_{\mathrm{S}_1} \circ \Upsilon$ the simple objects are mapped to

$$S_1 \mapsto S_2 \mapsto E_{2,1}$$
$$S_2 \mapsto S_1[n-2] \mapsto S_1[-1]$$

and hence $\Sigma^{-1}\mathcal{A}_0 = R_{S_1}\mathcal{A}_0$. □

We observe that successive tilts of these hearts are also equivalent to the standard heart. For example,

$$\begin{aligned} L_{\Sigma(S_2)}\Sigma\mathcal{A}_0 &= L_{E_{1,2}}\langle S_2[1], E_{1,2}\rangle \\ &= \langle E_{1,2}[1], {}^{\perp}E_{1,2}\rangle \\ &= \langle E_{1,2}[1], E_{2,3}\rangle \end{aligned}$$

as $\mathrm{Hom}_{\mathrm{D_n}}(E_{2,3}, E_{1,2}) = 0$ and therefore the indecomposable representation $E_{2,3}$ lies in the set ${}^{\perp}E_{1,2}$. Analogous calculations to Lemma 3.1.2 show that $E_{2,3}$ is a simple object in $L_{\Sigma(S_2)}\Sigma\mathcal{A}_0$. Appling Lemma 3.1.11,

$$\begin{aligned} L_{\Sigma(S_2)}\Sigma\mathcal{A}_0 &= \Sigma\langle S_2[1], E_{1,2}\rangle \\ &= \langle E_{1,2}[1], E_{2,3}\rangle \end{aligned}$$

and therefore $\Sigma(E_{1,2}) = E_{2,3}$. Repeating this process, the action of Σ on the objects of $\mathrm{D_n}$ which correspond to indecomposable representations of the Kronecker quiver is given by

$$E_{m-1,m} \mapsto E_{m,m+1} \qquad E_{m+1,m} \mapsto E_{m,m-1} \qquad S_1 \mapsto S_2[1]$$

for $m \in \mathbb{Z}_{>0}$.

The autoequivalence $\Sigma \in \mathrm{Aut}_*(\mathrm{D_n})$ is defined as the composition of the autoequivalence Υ and spherical twists. Neither of these autoequivalences are defined on the category D_∞. From Lemma 3.1.7 there is no autoequivalence of the category D_∞ that corresponds to the sequence of simple tilts of the standard heart $\mathcal{A}_0$ left at S_1 and right at S_2. However, we can define an autoequivalence on D_∞ that behaves analogously to Σ on the Calabi–Yau-n categories.

Lemma 3.1.13. *There exists an autoequivalence in* $\mathrm{Aut}_*(\mathrm{D}_\infty)$ *which behaves in the same way on the indecomposable representations as the autoequivalence* $\Sigma \in \mathrm{Aut}_*(\mathrm{D_n})$.

Proof. Consider the autoequivalence on $\mathrm{D^b}(\mathbb{P}^1)$ that is defined by twisting a coherent sheaf by the line bundle $\mathcal{O}(1)$. Recall the equivalence between $\mathrm{D^b}(\mathbb{P}^1)$ and $\mathrm{D}_\infty(\mathrm{K}_2)$ defined in Example 1.1.8, under which the twisting sheaf $\mathcal{O}(n)$ maps to

$$\mathcal{O}(n) \longmapsto \Big(\mathrm{Hom}^*_{\mathrm{D^b}(\mathbb{P}^1)}(\mathcal{O}(1), \mathcal{O}(n)) \rightrightarrows \mathrm{Hom}^*_{\mathrm{D^b}(\mathbb{P}^1)}(\mathcal{O}, \mathcal{O}(n))\Big)$$
$$\mathcal{O}(n)\otimes\mathcal{O}(1) \longmapsto \Big(\mathrm{Hom}^*_{\mathrm{D^b}(\mathbb{P}^1)}(\mathcal{O}(1), \mathcal{O}(n)\otimes\mathcal{O}(1)) \rightrightarrows \mathrm{Hom}^*_{\mathrm{D^b}(\mathbb{P}^1)}(\mathcal{O}, \mathcal{O}(n)\otimes\mathcal{O}(1))\Big).$$

Then in order for the diagram

$$\begin{array}{ccc} \mathrm{D^b}(\mathbb{P}^1) & \xrightarrow{\otimes\mathcal{O}(1)} & \mathrm{D^b}(\mathbb{P}^1) \\ \downarrow & & \downarrow \\ \mathrm{D}_\infty(\mathrm{K}_2) & \xrightarrow{\Sigma_\infty} & \mathrm{D}_\infty(\mathrm{K}_2) \end{array}$$

to commute, the autoequivalence Σ_∞ acts on the indecomposable representations of K_2 in the same way as the autoequivalence Σ defined on the categories $\mathrm{D_n}$. Explicitly

$$E_{m-1,m} \mapsto E_{m,m+1} \qquad E_{m+1,m} \mapsto E_{m,m-1} \qquad S_1 \mapsto S_2[1].$$

□

We denote this autoequivalence on $\mathrm{Aut}_*(\mathrm{D}_\infty)$ by Σ.

An explicit description of the group $\mathrm{Aut}_*(\mathrm{D_n})$ in terms of generators and relations is given in the following section. We focus first on its action on the exchange graph.

Proposition 3.1.14. *(i) For $3 \leq n < \infty$, the right action of $\mathrm{Aut}_*(\mathrm{D_n})$ on $\mathrm{EG}_*(\mathrm{D_n})$ has $\lceil \frac{n-2}{2} \rceil$ orbits.*

(ii) The right action of $\mathrm{Aut}_*(\mathrm{D}_\infty)$ on $\mathrm{EG}_*(\mathrm{D}_\infty)$ has infinitely many orbits.

Proof. To show that there are at least $\lceil \frac{n-2}{2} \rceil$ orbits, we first observe that the action is transitive on the full hearts of $\mathrm{EG}_*(\mathrm{D_n})$ by definition, as a heart $\mathcal{B}$ in $\mathrm{EG}_*(\mathrm{D_n})$ is full precisely if there is an autoequivalence $\varphi \in \mathrm{Aut}_*(\mathrm{D_n})$ such that $\mathcal{B} = \varphi\mathcal{A}_0$. Therefore all the full hearts are in the same orbit of $\mathrm{Aut}_*(\mathrm{D_n})$ on $\mathrm{EG}_*(\mathrm{D_n})$.

By Lemma 3.1.9, there must be at least $\lceil \frac{n-2}{2} \rceil$ orbits as for $k \neq l$ the hearts $\mathcal{A}_k$ and A_l are only equivalent to each other if $k + l = n - 2$.

Conversely a heart in the connected component of the exchange graph must be realisable as a sequence of tilts of the standard heart $\mathcal{A}_0 = \langle S_1, S_2 \rangle$. Hence to show that all hearts are equivalent to one of the $\mathcal{A}_k$, and therefore that there are at most $n-2$ orbits, we consider the possible behaviour under the four tilting directions of each of the hearts $\mathcal{A}_k$ for $0 \leq k \leq n-3$.

By the above discussion, we know that the four simple tilts of $\mathcal{A}_0$ are $\Sigma(\mathcal{A}_0)$, $\Sigma^{-1}(\mathcal{A}_0)$, $\mathcal{A}_1$ and $\Upsilon^{-1}(\mathcal{A}_{n-3})$.

For $1 \leq k \leq n-3$, the tilts of $\mathcal{A}_k = \langle S_1[k], S_2 \rangle$ are

$$L_{S_1[k]}\mathcal{A}_k = \langle S_1[k+1], S_2 \rangle = \mathcal{A}_{k+1} \qquad R_{S_1[k]}\mathcal{A}_k = \langle S_2, S_1[k-1] \rangle = \mathcal{A}_{k-1}$$
$$L_{S_2}\mathcal{A}_k = \langle S_2[1], S_1[k] \rangle = \mathcal{A}_{k-1}[1] \qquad R_{S_2}\mathcal{A}_k = \langle S_1[k], S_2[-1] \rangle = \mathcal{A}_{k+1}[-1].$$

These are computed analogously to the tilts in Lemma 3.1.6.

Hence there is no sequence of tilts of $\mathcal{A}_0$ which gives rise to a heart which is not equivalent to one of the hearts $\mathcal{A}_k$ for $0 \leq k \leq n-2$, and so every heart lies in the same orbit of $\mathrm{Aut}_*(\mathrm{D_n})$ as one of these hearts.

For the autoequivalence action on $\mathrm{EG}_*(\mathrm{D}_\infty)$, there is no upper bound on the shift $[k]$, and hearts $\mathcal{A}_k$ are not equivalent to each other as discussed in Remark 3.1.10. □

Lemma 3.1.15. *Let $3 \leq n \leq \infty$.*

(i) The action of $\mathrm{Aut}_*(\mathrm{D_n})$ on $\mathrm{EG}_*(\mathrm{D_n})$ is free if n is odd or $n = \infty$.

(ii) If n is even, then the hearts $\mathcal{A}_k$ where $k = \frac{n-2}{2}$ are fixed by $\Upsilon[-k]$.

Proof. Let $\mathcal{B}$ be a vertex of $\mathrm{EG}_*(\mathrm{D_n})$ and $\varphi \in \mathrm{Aut}_*(\mathrm{D_n})$ an autoequivalence such that $\varphi(\mathcal{B}) = \mathcal{B}$. As autoequivalences which act trivially on the exchange graph are identified with the identity in $\mathrm{Aut}_*(\mathrm{D_n})$, to check that points are fixed it suffices to check that φ fixes the simple objects. As $\mathcal{B}$ is a simple tilt of $\mathcal{A}_0$, it has two simple objects. Denote by X_1, X_2 the simple objects such that $\langle X_1, X_2 \rangle = \mathcal{B}$. Either $\varphi(X_i) = X_i$ giving the desired result, or φ swaps the simple objects.

It is enough to study this action on the hearts $\mathcal{A}_k = \langle S_1[k], S_2 \rangle$, as all other hearts in the exchange graph are equivalent to such a heart by Proposition 3.1.14. Consider the Hom-spaces of these simple objects. If an autoequivalence exists which switches these simples, then it also preserves the homomorphisms between them and hence the corresponding Hom-spaces are isomorphic.

As $\mathrm{Hom}^*_{\mathrm{D}_\infty}(S_2, S_1) = 0$, but $\mathrm{Hom}^*_{\mathrm{D}_\infty}(S_1, S_2) = \mathbb{C}^2[-1]$, there is no such autoequivalence on D_∞ and hence the action is free.

Let $3 \leq n < \infty$. From Lemma 1.4.5, the graded components of the Hom-spaces of the heart $\mathcal{A}_k$ are

$$\mathrm{Hom}^i_{\mathrm{D_n}}(S_1[k], S_2) = \begin{cases} \mathbb{C}^2 & \text{if } i = k+1, \\ 0 & \text{otherwise} \end{cases}$$

and $\mathrm{Hom}^i_{\mathrm{D_n}}(S_2, S_1[k]) = \mathbb{C}^2$ if $i + k = n - 1$ and is zero otherwise. Hence the $\mathbb{C}^2$ components of these algebras have the same grading when $k = \frac{n-2}{2}$. As there is no such category when n is odd, there are no fixed points in these cases. If n is even, then the heart $\mathcal{A}_k$ is fixed by the autoequivalence $\Upsilon[-k]$. □

Example 3.1.16. The exchange graph has infinitely many vertices. Figure 3.1 shows the first few tilts of the standard heart for the category D_3. In this case, all the hearts are full.

$$
\begin{tikzcd}
 & & \langle S_1[1], S_2[1]\rangle \arrow[d, shift left, "R_{S_2[1]}"] & & \\
 & & \langle S_2, S_1[1]\rangle \arrow[u, shift left, "L_{S_2}"] \arrow[d, shift left, "R_{S_1[1]}"] & & \\
\langle E_{3,2}, E_{2,1}[-1]\rangle \arrow[r, shift right, "L_{E_{2,1}[-1]}"'] & \langle E_{2,1}, S_1[-1]\rangle \arrow[l, shift right, "R_{E_{2,1}}"'] \arrow[r, shift right, "L_{S_1[-1]}"'] & \langle S_1, S_2\rangle \arrow[l, shift right, "R_{S_1}"'] \arrow[u, shift left, "L_{S_1}"] \arrow[r, shift right, "L_{S_2}"'] \arrow[d, shift left, "R_{S_2}"] & \langle S_2[1], E_{1,2}\rangle \arrow[l, shift right, "R_{S_2[1]}"'] \arrow[r, shift right, "L_{E_{1,2}}"'] & \langle E_{1,2}[1], E_{2,3}\rangle \arrow[l, shift right, "R_{E_{1,2}[1]}"'] \\
 & & \langle S_2[-1], S_1\rangle \arrow[u, shift left, "L_{S_2[-1]}"] \arrow[d, shift left, "R_{S_1}"] & & \\
 & & \langle S_1[-1], S_2[-1]\rangle \arrow[u, shift left, "L_{S_1[-1]}"] & &
\end{tikzcd}
$$

Figure 3.1: Tilts of the standard heart corresponding to the autoequivalences Υ, Υ^2, Σ, Σ^2 and their inverses.

3.2 The group of autoequivalences of $\mathrm{D_n}$

In this section we develop the work in the previous section to obtain an explicit description of the group $\mathrm{Aut}_*(\mathrm{D_n})$ for $3 \leq n < \infty$ in terms of generators and relations.

In [27] and [13], the autoequivalence groups of the Calabi–Yau-n categories constructed from Dynkin quivers of type $\mathrm{A_N}$ are described purely in terms of spherical twists and the shift functor. For the Kronecker quiver the generators of the group are more complicated. However we are able to relate shifts and spherical twists to the autoequivalences Σ and Υ defined in the Section 3.1.

We recall from Example 1.4.8 some of the actions of spherical twists on indecomposable representations

$$\mathrm{Tw}_{\mathrm{S_1}}(S_2) = E_{2,1} \qquad \mathrm{Tw}_{\mathrm{S_2}}(E_{1,2}) = S_1 \qquad \mathrm{Tw}_{S_i}(S_i) = S_i[1-n].$$

Lemma 3.2.1. *The following relations hold in* $\mathrm{Aut}_*(\mathrm{D_n})$

$$\Upsilon^2 = [n-2] \qquad \Sigma^2 = \mathrm{Tw}_{S_2}^{-1} \circ \mathrm{Tw}_{S_1}^{-1}[2-n].$$

Proof. On the simple objects S_1 and S_2, the action of Υ^2 is

$$S_2 \mapsto S_1[n-2] \mapsto S_2[n-2] \qquad S_1 \mapsto S_2 \mapsto S_1[n-2].$$

Therefore $\Upsilon^2[2-n]$ is the identity in $\mathrm{Aut}_*(\mathrm{D_n})$ and so the relation $\Upsilon^2 = [n-2]$ holds in this subquotient group.

For the second equality, observe that the autoequivalence $\mathrm{Tw}_{S_2}^{-1} \circ \mathrm{Tw}_{S_1}^{-1}[2-n]$ maps the simple objects of the hearts $\Sigma^{-1}\mathcal{A}_0$, to those of $\Sigma\mathcal{A}_0$

$$E_{2,1} \mapsto S_2 \mapsto S_2[n-1] \mapsto S_2[1]$$
$$S_1[-1] \mapsto S_1[n-2] \mapsto E_{1,2}[n-2] \mapsto E_{1,2}$$

and that Σ^2 also maps these simple objects to each other: $E_{2,1} \mapsto S_1 \mapsto S_2[1]$, and $S_1[-1] \mapsto S_2 \mapsto E_{1,2}$. □

Proposition 3.2.2. *(i) The subgroup of* $\mathrm{Aut}_*(\mathrm{D_n})$ *generated by* Σ *and* Υ *is free.*

(ii) The shift functor $[m]$ *is not in the group* $\langle \Sigma, \Upsilon \rangle$ *unless* m *is an integer multiple of* $n-2$.

Proof. We consider the action of the autoequivalences on the Grothendieck group $\mathrm{K}(\mathrm{D_n})$, and consider the cases for n odd and even separately as the action of the autoequivalence Υ on $\mathrm{K}(\mathrm{D_n})$ depends on n. Let $S_1 + S_2$ and $S_1 - S_2$ be a basis for $\mathrm{K}(\mathrm{D_n}) \otimes \mathbb{Q}$.

Let n be odd. Then the autoequivalences Σ and Υ are identified with the matrices

$$\Sigma = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix} \qquad \Upsilon = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

Observe that $\Sigma, \Upsilon \in \mathrm{SL}_2(\mathbb{Z})$. We consider them as elements in the quotient $\mathrm{PSL}_2(\mathbb{Z})$. Let

$$U = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad V = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$$

denote the generators of $\mathrm{PSL}_2(\mathbb{Z})$. The matrices Σ and Υ generate a proper subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ as $\Upsilon = V$ and $\Sigma = U^{-2}$. Moreover this is an index two subgroup of $\mathrm{PSL}_2(\mathbb{Z})$ [38, Table 3]. Note that $\Upsilon^2 = -\mathrm{id}$. Any relation between the generators of this group is of the form $\Upsilon\Sigma^{a_1} \dots \Upsilon\Sigma^{a_r}$ where $a_i \in \mathbb{Z}$. As $\Upsilon\Sigma^{a_i} \neq \mathrm{id} \neq \Sigma^{a_i}\Upsilon$, it is enough to check that there are no relations between $\Upsilon\Sigma\Upsilon$ and Σ^{-1}. Consulting Table 4 in [38], we see that these two matrices are the generators of a normal index six subgroup of $\mathrm{PSL}_2(\mathbb{Z})$, and that this group is free [38, Thm. 1.7.4]. Thus the group $\langle \Sigma, \Upsilon \rangle$ has a free subgroup $\langle \Upsilon\Sigma\Upsilon, \Sigma \rangle$ and is itself free.

Now let n be even. Then the autoequivalence Υ is identified with the matrix

$$\Upsilon = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

in $\mathrm{GL}_2(\mathbb{Z})$. This case is treated separately as this matrix is not in $\mathrm{SL}_2(\mathbb{Z})$. In this case $\Upsilon^2 = \mathrm{id}$. In the subgroup of $\mathrm{GL}_2(\mathbb{Z})$, generated by Υ and Σ, there is a relation $\Upsilon\Sigma\Upsilon\Sigma = \mathrm{id}$. Therefore this autoequivalence act trivially on the Grothendieck group, but by computing the action of this autoequivalence on the simple representations of the quiver, this relation does not hold on $\mathrm{D_n}$.

Clearly shifts which are multiples of $n-2$ are in the subgroup $\langle \Sigma, \Upsilon \rangle$ as $\Upsilon^2 = [n-2]$. However as there are no other relations between Σ and Υ, the shift functor [1] is not in this group. $\square$

Lemma 3.2.3. *The spherical twists* $\mathrm{Tw}_{\mathrm{S}_1}$ *and* $\mathrm{Tw}_{\mathrm{S}_2}$ *are in the group generated by* Υ *and* Σ*, and are reachable.*

Proof. This is immediate from the definition of the autoequivalence Σ in Proposition 3.1.12. Explicitly the relations generating the spherical twists in $\langle \Upsilon, \Sigma \rangle$ are

$$\mathrm{Tw}_{\mathrm{S}_1} = \Sigma^{-1} \circ \Upsilon^{-1} \qquad \mathrm{Tw}_{\mathrm{S}_2} = \Upsilon^{-1} \circ \Sigma^{-1}.$$

$\square$

Lemma 3.2.4. *The shift functor is reachable.*

Proof. This follows from Proposition 3.1.14 as $\mathcal{A}_{k-1}[1] = L_{S_2}\mathcal{A}_k$ for $1 \leq k \leq n-3$. $\square$

Proposition 3.2.5. *The group* $\mathrm{Aut}_*(\mathrm{D_n})$ *is generated by the autoequivalences* Σ*,* Υ*, and* [1]*. The shift functor commutes with both* Σ *and* Υ *and the only other relation between the generators is* $\Upsilon^2 = [n-2]$*.*

Proof. Let $\mathcal{B}$ be a vertex of the exchange graph obtained by tilting the standard heart $\mathcal{A}_0$ m times. Such a heart can be written as a sequence of tilts

$$\mathcal{B} = T_{X_m} T_{X_{m-1}} \ldots T_{X_1} \mathcal{A}_0$$

where T_{X_i} is either a left or right tilt at the simple object X_i in the heart $T_{X_{i-1}} \ldots T_{X_1} \mathcal{A}_0$. In particular X_1 is either S_1 or S_2. Our aim is to replace this expression for $\mathcal{B}$ in terms of simple tilts of the standard heart with an expression in the autoequivalences Σ, Υ, and [1], and the hearts $\mathcal{A}_k$.

From Lemma 3.1.6 and Proposition 3.1.12, $T_{X_1}\mathcal{A}_0$ is one of the hearts $\Sigma\mathcal{A}_0$, $\Sigma^{-1}\mathcal{A}_0$, $\mathcal{A}_1$ or $\mathcal{A}_{n-3}[n-2]$. Therefore this term in the expression for $\mathcal{B}$ can be replaced with the appropriate heart. Without loss of generality let T_{X_1} be a left tilt. Then the heart $\mathcal{B}$ can be expressed as either $\mathcal{B} = T_{X_m} T_{X_{m-1}} \ldots T_{X_2} \Sigma \mathcal{A}_0$ or $\mathcal{B} = T_{X_m} T_{X_{m-1}} \ldots T_{X_2} \mathcal{A}_1$, corresponding to tilts at S_2 and S_1 respectively.

If $\mathcal{B} = T_{X_m} T_{X_{m-1}} \dots T_{X_2} \Sigma \mathcal{A}_0$, then the options for the tilt $T_{X_2} \Sigma \mathcal{A}_0$ are equivalent to those for T_{X_1} as discussed above. That is, the heart $T_{X_2} \Sigma \mathcal{A}_0$ is one of

$$L_{\Sigma(S_1)} \Sigma \mathcal{A}_0 = L_{S_2[1]} \Sigma \mathcal{A}_0 = \Sigma \mathcal{A}_1 \qquad L_{\Sigma(S_2)} \Sigma \mathcal{A}_0 = L_{E_{1,2}} \Sigma \mathcal{A}_0 = \Sigma^2 \mathcal{A}_0$$
$$R_{\Sigma(S_1)} \Sigma \mathcal{A}_0 = R_{S_2[1]} \Sigma \mathcal{A}_0 = \mathcal{A}_0 \qquad R_{\Sigma(S_2)} \Sigma \mathcal{A}_0 = R_{E_{1,2}} \Sigma \mathcal{A}_0 = \Sigma \mathcal{A}_{n-3}[n-2].$$

On the other hand if $\mathcal{B} = T_{X_m} T_{X_{m-1}} \dots T_{X_2} \mathcal{A}_1$, from the calculations in Proposition 3.1.14, $T_{X_2} \mathcal{A}_1$ is either $\mathcal{A}_0$, $\mathcal{A}_2$, $\mathcal{A}_0[1]$ or $\mathcal{A}_2[-1]$. Once again we can replace the term $T_{X_2} \mathcal{A}_1$ in the expression for $\mathcal{B}$. Again we assume that T_{X_2} is a left tilt. Therefore the heart $\mathcal{B}$ can be expressed either as $\mathcal{B} = T_{X_m} T_{X_{m-1}} \dots T_{X_3} \mathcal{A}_2$ or $\mathcal{B} = T_{X_m} T_{X_{m-1}} \dots T_{X_3} \mathcal{A}_0[1]$. The explicit description that we have of each of the tilts allows us to repeat this process m times. At each step we replace the tilted heart with an expression in $\Sigma, \Upsilon, [1]$ and $\mathcal{A}_k$. Therefore $\mathcal{B} = \Sigma^{a_1} \Upsilon \dots \Upsilon \Sigma^{a_r} \mathcal{A}_k[m]$ with $a_i, m \in \mathbb{Z}$

Now if $\varphi \in \mathrm{Aut}_*(\mathrm{D_n})$, such that $\mathcal{B} = \varphi \mathcal{A}_l$ for some $0 \leq l \leq n-3$, then

$$\varphi^{-1} \circ \Sigma^{a_1} \Upsilon \dots \Upsilon \Sigma^{a_r} \mathcal{A}_k[m] = \mathcal{A}_l.$$

By Lemma 3.1.9, either $k = l$, $k + l = n - 2$ with $\varphi^{-1} \circ \Sigma^{a_1} \Upsilon \dots \Upsilon \Sigma^{a_r} = \Upsilon$ and $m = -k$, or $k = l + n - 2$, in which case $\mathcal{A}_k = \mathcal{A}_l[n-2]$.

It is clear that the shift functor commutes with Σ and Υ. That there are no other relations between the generators other than $\Upsilon^2 = [n-2]$ is shown in Proposition 3.2.2. $\square$

Corollary 3.2.6. *The action of* $\mathrm{Aut}_*(\mathrm{D}_3)$ *on* $\mathrm{EG}_*(\mathrm{D}_3)$ *is free and transitive:* $\mathrm{EG}_*(\mathrm{D}_3)$ *is the Cayley graph of* $\mathrm{Aut}_*(\mathrm{D}_3)$ *with respect to the generators* Σ *and* Υ*.*

3.3 The group of autoequivalences of D_∞

We conclude this chapter by considering which of the results in the previous section can be applied to the group of autoequivalences of the category D_∞. Due to the equivalence of triangulated categories between $\mathrm{D}_\infty(\mathrm{K}_2)$ and $\mathrm{D^b}(\mathbb{P}^1)$ discussed in Example 1.1.8, we can add some geometric interpretation to the results for the Calabi–Yau-n categories.

Bondal and Orlov describe the group of autoequivalences of a projective variety with ample (anti-)canonical bundle [6, Thm. 3.1]. For the variety $\mathbb{P}^1$ this is

$$\begin{aligned} \mathrm{Aut}(\mathrm{D^b}(\mathbb{P}^1)) &\cong \langle [1] \rangle \times \mathrm{Aut}(\mathbb{P}^1) \ltimes \mathrm{Pic}(\mathbb{P}^1) \\ &\cong \mathbb{Z} \times \mathrm{PSL}(2, \mathbb{C}) \ltimes \mathbb{Z}. \end{aligned}$$

Proposition 3.3.1. *The group of autoequivalences* $\mathrm{Aut}_*(\mathrm{D}_\infty)$ *is isomorphic to* $\mathbb{Z} \times \mathbb{Z}$*.*

Proof. One copy of the integers is generated by the shift functor. The proof of Lemma 3.2.4, that the shift functor lies in $\mathrm{Aut}_*(\mathrm{D_n})$ for $n < \infty$, can be extended to the category D_∞. The other copy of the integers is generated by the autoequivalence Σ, defined on D_∞ in Lemma 3.1.13. The shift functor and Σ commute.

To see that these autoequivalences generate the group $\mathrm{Aut}_*(\mathrm{D_n})$, we use the same method as in Proposition 3.2.5. The difference to the case $n < \infty$ is of course that the autoequivalence Υ does not exist on the category D_∞. Therefore if we consider a vertex $\mathcal{B}$ expressed as a sequence of tilts of the standard heart, $T_{X_m} T_{X_{m-1}} \dots T_{X_1} \mathcal{A}_0$ and perform the iterative process of considering the possible tilts at each stage, the expression for $\mathcal{B}$ in terms of autoequivalences is $\mathcal{B} = \Sigma^a \mathcal{A}_k[m]$ for $a, m \in \mathbb{Z}$. If $\varphi \in \mathrm{Aut}_*(\mathrm{D_n})$ is an autoequivalence such that $\mathcal{B} = \varphi \mathcal{A}_l$ for some l, then $\varphi^{-1} \circ \Sigma^a \mathcal{A}_k[m] = \mathcal{A}_l$. However following Remark 3.1.10 there is no such autoequivalence in $\mathrm{Aut}_*(\mathrm{D}_\infty)$ unless $k = l$. $\square$

Corollary 3.3.2. *The group of Möbius transformations acts trivially on the space of stability conditions of* $\mathbb{P}^1$.

Chapter 4

A fundamental domain

This chapter introduces the space of projective stability conditions. We construct a fundamental domain for the corresponding action of the group of projective autoequivalences on this space, which we then prove to be biholomorphic to a subset of $\mathbb{C}$. The first section defines the notion of a projective stability condition, and the quotient of the group of autoequivalences which acts on this space. The remainder of the chapter gives an explicit description of the fundamental domain.

Throughout this chapter $\mathrm{D_n}$, for $3 \leq n < \infty$, and D_∞ are the triangulated categories constructed from the Kronecker quiver as in Section 1.2. The difference in the group of autoequivalences between the Calabi–Yau-n categories and the category D_∞ leads to differences in the construction of their spaces of stability conditions. As in Chapter 3, the categories to which each result applies are specified.

4.1 Projective stability conditions

We begin by defining the space and group for which we construct a fundamental domain.

In order to define the space of projective stability conditions, recall from Lemma 1.5.14 that the space of stability conditions of a triangulated category $\mathcal{D}$ is endowed with a free action of the additive group of complex numbers. As stated in Remark 1.5.15, the effect of this action on the central charge of an object is to rescale its mass and shift the phase. We consider the quotient of the stability manifold under this action.

Definition 4.1.1. The *space of projective stability conditions* $\mathbb{P}\operatorname{Stab}_*(\mathcal{D})$ is the quotient arising from the free group action of $\mathbb{C}$ on $\operatorname{Stab}_*(\mathcal{D})$.

Denote by $\bar{\sigma} \in \mathbb{P}\,\mathrm{Stab}(\mathcal{D})$ the equivalence class of a stability condition $\sigma \in \mathrm{Stab}(\mathcal{D})$ under this action.

Proposition 4.1.2. *The quotient* $\mathbb{P}\,\mathrm{Stab}_*(\mathcal{D})$ *is a complex manifold.*

Proof. The quotient of a complex manifold by a Lie group with a free action is a manifold [15, 4.2]. □

Definition 4.1.3. The *group of projective autoequivalences* is the quotient group

$$\mathbb{P}\,\mathrm{Aut}_*(\mathcal{D}) := \mathrm{Aut}_*(\mathcal{D})/\langle[1]\rangle.$$

Recall from Remark 1.5.15 that the subgroup of $\mathrm{Aut}_*(\mathcal{D})$ generated by the shift functor is identified with $\mathbb{Z} \subset \mathbb{C}$. The action of $\mathrm{Aut}_*(\mathcal{D})$ on $\mathrm{Stab}_*(\mathcal{D})$ descends to an action of $\mathbb{P}\,\mathrm{Aut}_*(\mathcal{D})$ on $\mathbb{P}\,\mathrm{Stab}_*(\mathcal{D})$, as the shift functor [1] acts as $1 \in \mathbb{C}$.

For the space of projective stability conditions of the categories $\mathrm{D_n}$, we can choose a basis for the Grothendieck group $\mathrm{K(D_n)}$ such that

$$\mathbb{P}\,\mathrm{Hom}(\mathrm{K(D_n)}, \mathbb{C}) \cong \mathbb{P}^1.$$

Furthermore, that $\mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$ is a one-dimensional complex manifold follows from Remark 1.5.10, as the Grothendieck group of $\mathrm{D_n}$ is two-dimensional.

4.2 Construction of a fundamental domain

This section details the construction of a fundamental domain for the action of the group of autoequivalences $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$ on the space of projective stability conditions. We consider the categories $\mathrm{D_n}$ for $3 \leq n \leq \infty$.

Definition 4.2.1. Let G be a group acting on a topological space X. An open subset $U \subset X$ is a *fundamental domain* for the action of G on X if

$$\bigcup_{g \in G} g(\bar{U}) = X$$

and for distinct $g_1, g_2 \in G$ the intersection $g_1(U) \cap g_2(U)$ is empty.

For a stability condition $\sigma = (Z, P)$ let $\mathcal{S}(\sigma)$ denote the set of isomorphism classes of indecomposable objects which are semistable with respect to σ. Let $\mathbb{P}\mathcal{S}(\sigma)$ denote indecomposable semistable objects of σ up to shift.

Definition 4.2.2. We define $U_n := U_n^{<0} \cup U_n^{\geq 0} \subset \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$ where

(a) $U_n^{<0}$ is the set of projective stability conditions with indecomposable semistable objects the set

$$\mathbb{P}\mathcal{S}(\sigma) = \{S_1, S_2\}$$

with phases satisfying

$$0 < \phi(S_2) - \phi(S_1) < \frac{n-2}{2};$$

(b) $U_n^{\geq 0}$ is the set of stability conditions for which $\mathbb{P}\mathcal{S}(\sigma)$ is the set of all indecomposable representations of the Kronecker quiver, up to isomorphism and shift, and for which the two phase conditions

$$0 \leq \phi(S_1) - \phi(S_2) < \phi(S_2[1]) - \phi(E_{1,2})$$
$$0 \leq \phi(S_1) - \phi(S_2) < \phi(E_{2,1}[1]) - \phi(S_1)$$

are satisfied.

Definition 4.2.3. Define the map

$$\bar{\mathcal{Z}} \colon \mathbb{P}\,\mathrm{Stab}(\mathrm{D_n}) \longrightarrow \mathbb{P}\,\mathrm{Hom}(\mathrm{K}(\mathrm{D_n}), \mathbb{C})$$
$$\bar{\sigma} = (Z, \mathcal{P}) \longmapsto \bar{\mathcal{Z}}(\bar{\sigma}) := \frac{Z_{\bar{\sigma}}(S_1)}{Z_{\bar{\sigma}}(S_2)}$$

to be the *projective central charge map.*

We now discuss the geometric significance of the inequalities defining this region U_n. The Calabi–Yau-3 case is illustrated in Figure 4.2. Let X_n denote the image of U_n under $\bar{\mathcal{Z}}$ and define $z_{\bar{\sigma}} := \bar{\mathcal{Z}}(\bar{\sigma})$. The inequalities on the phases of the semistable objects for $\bar{\sigma} \in U_n$ are geometrically interpreted as follows

- On the boundary of $U_n^{<0}$, $\phi(S_2) - \phi(S_1) = \frac{n-2}{2}$ is the ray at an angle of $\frac{(2-n)\pi}{2}$ from the origin.

- On the boundary between $U_n^{<0}$ and $U_n^{\geq 0}$, the condition $\phi(S_2) - \phi(S_1) = 0$ corresponds to the positive real axis.

- On the boundary of $U_n^{\geq 0}$, there are two inequalities to consider. The inequality

$$0 \leq \phi(S_1) - \phi(S_2) < \phi(S_2[1]) - \phi(E_{1,2})$$

  on the phases of the central charges of semistable objects in $U_n^{\geq 0}$ is illustrated in Figure 4.1, with the angles α and γ satisfying the condition $\gamma > \alpha$. For stability conditions on the

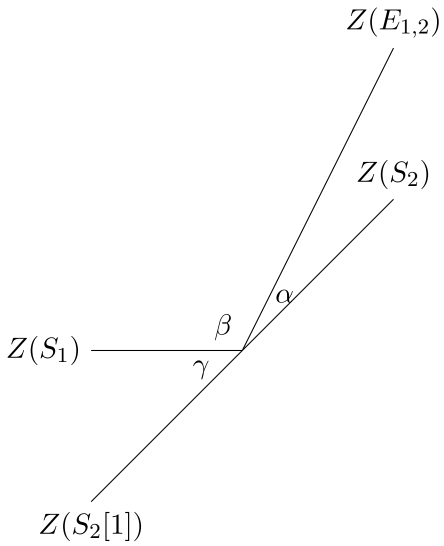


Figure 4.1: Central charges in $U_n^{\geq 0}$.

boundary of this region, the phase condition is the equality

$$\phi(S_1) - \phi(S_2) = \phi(S_2[1]) - \phi(E_{1,2}).$$

For this equality to hold in $U_n^{\geq 0}$, the angles in Figure 4.1 must satisfy $\alpha = \gamma$. This can only occur if $|Z(S_1)| = |Z(E_{1,2})|$. Under the projective central charge map, this equality is the line $|z_{\bar{\sigma}} + 2| = |z_{\bar{\sigma}}|$.

- On the boundary, the other inequality defining $U_n^{\geq 0}$ becomes the equality $\phi(S_1) - \phi(S_2) = \phi(E_{2,1}[1]) - \phi(S_1)$. By similar arguments to the other inequality, this is the circle $|z_{\bar{\sigma}} + \frac{1}{2}| = \frac{1}{2}$.

In the $n = 3$ case, we can very clearly picture what is happening to projective stability conditions under the central charge map. Note that the central charge map is injective in this case.

- Projective stability conditions lying in $U_3^{<0}$ are mapped by the central charge map to the quadrant

$$\{z \in \mathbb{C}\colon \operatorname{Im}(z) < 0 \text{ and } \operatorname{Re}(z) > 0\}.$$

  This is the grey region below the real axis in Figure 4.2.

- The subset $U_3^{\geq 0}$ is mapped to the upper half plane outside of the circle $|z + \frac{1}{2}| = \frac{1}{2}$ and to the right of the line $|z + 2| = |z|$. This is the grey region above the real axis in Figure 4.2.

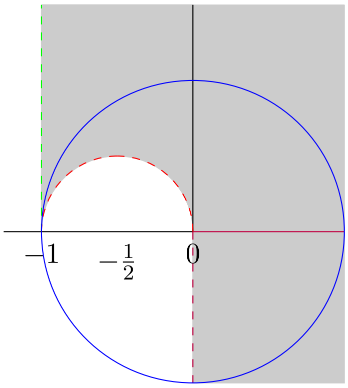


Figure 4.2: The region X_3.

For $n = 4$, the map is again injective, as the only difference to the $n = 3$ case is that the region is bounded by the negative real axis, instead of the negative imaginary axis. However for $n > 4$, the map $\bar{\mathcal{Z}}$ is not injective. Stability conditions in $U_n^{<0}$ satisfy the phase condition $0 < \phi(S_2) - \phi(S_1) < \frac{n-2}{2}$ and so the boundary of X_n is the ray at an angle of $\frac{(2-n)\pi}{2} > -\pi$ from the origin. Therefore there will be stability conditions in $U_n^{<0}$ mapped to the same point in $\mathbb{C}$ as stability conditions in $U_n^{\geq 0}$. In particular, for projective stability conditions on the derived category D_∞, the image in $\mathbb{C}$ of U_∞ under the central charge map is unbounded on this side.

In order to construct a bijection between the subset $U_n \subset \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$ and a subset of $\mathbb{C}$ we introduce the complex logarithm. By viewing the region X_n as a subset of $\mathbb{C}$ with branch cuts, we can compose the central charge map with the map $z \mapsto \frac{1}{i\pi}\log(z)$ to get a bijective function from U_n to a subset of $\mathbb{C}$ for all n. The image of this map is the grey area illustrated in Figure 4.3, which we denote by C_n.

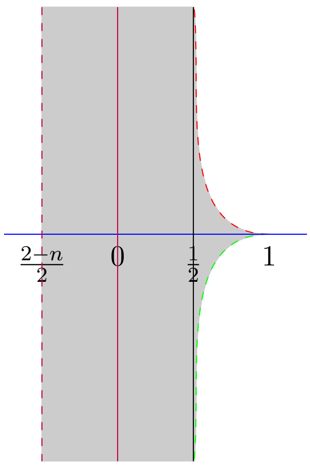


Figure 4.3: The region C_n.

Explicitly, the boundaries of $C_n \subset \mathbb{C}$ on the right are:

- below the real axis, the image of the line $|z+2| = |z|$ between -1 and ∞;
- above the real axis, the image of the circle $|z+\frac{1}{2}| = \frac{1}{2}$ between -1 and 0.

For $3 \leq n < \infty$ it is bounded on the left by the line $\mathrm{Re}(z) = \frac{2-n}{2}$. For $n = \infty$ it is unbounded on the left. The unit circle is mapped to the real axis and in particular $-1 \mapsto 1$. The real axis in X_n, which is the boundary $U_n^{=0}$ between $U_n^{\geq 0}$ and $U_n^{<0}$ is mapped to the imaginary axis. Furthermore, points above the real axis correspond to stability conditions with $|Z(S_2)| > |Z(S_1)|$, and points below the real axis to stability conditions with $|Z(S_1)| > |Z(S_2)|$. For z on the real axis, we have $z = \phi(S_1) - \phi(S_2)$, and more generally $\mathrm{Re}(z) = \phi(S_1) - \phi(S_2)$. In particular the imaginary axis corresponds to stability conditions with $\phi(S_1) = \phi(S_2)$.

The line $\mathrm{Re}(z) = \frac{1}{2}$ is an asymptote of the boundary lines on the right of the region. Near the boundary below the real axis, as $\phi(S_2[1]) - \phi(E_{1,2}) \to \phi(S_1) - \phi(S_2) = \frac{1}{2}$, $|Z(E_{1,2})| \to |Z(S_1)|$ however the masses of these objects cannot be equal as this indicates that S_1 becomes unstable with $Z(S_1) = 0$. Similarly for points corresponding to stability conditions above the real axis, as $\phi(E_{2,1}[1]) - \phi(S_2[1]) \to \phi(S_1) - \phi(S_2)$, $|Z(E_{2,1})| \to |Z(S_2)|$.

Lemma 4.2.4. *There is no stability condition σ with $\bar{\sigma} \in \bar{U}_n$ for which $\frac{1}{i\pi}\log(\bar{\mathcal{Z}}(\bar{\sigma})) = 1$.*

Proof. Consider a projective stability condition $\bar{\sigma} \in \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$ such that $\frac{1}{i\pi}\log(\bar{\mathcal{Z}}(\bar{\sigma})) = 1$. As this is a point on the real axis, from the discussion above, the central charges and phases of this stability condition satisfy $|Z(S_1)| = |Z(S_2)|$ and $\phi(S_1) - \phi(S_2) = 1$ respectively. Consequently for such a stability condition, the indecomposable object $E_{1,1}$ is not semistable as $|Z(E_{1,1})| = 0$.

Observe also that $\bar{\sigma}$ is on the boundary of the closure of $U_n^{\geq 0}$. Therefore if an object X is semistable with respect to a stability conditions in $U_n^{\geq 0}$, it is semistable with respect to $\bar{\sigma}$, as by Proposition 1.5.11, semistability is a closed property. However, in Definition 4.2.2, we specify that the object $E_{1,1}$ is semistable with respect to stability conditions in $U_n^{\geq 0}$. Hence such a stability condition does not lie in the closure of U_n. $\square$

A disadvantage of considering projective stability conditions is that the one-to-one correspondence with hearts of bounded t-structures is lost. The following lemma characterises the hearts on which lifts to $\mathrm{Stab}_*(\mathrm{D_n})$ of projective stability conditions in U_n are supported.

Lemma 4.2.5. *Let $\bar{\sigma} \in U_n$ be a projective stability condition.*

(i) If $\bar{\sigma} \in U_n^{<0}$, it can be lifted to a stability condition $\sigma \in \mathrm{Stab}_*(\mathrm{D_n})$ which is supported on either a shift of the standard heart $\mathcal{A}_0 \cong \mathrm{Rep}(\mathrm{K}_2)$ or a shift of one of the non-full hearts $\mathcal{A}_1 \ldots \mathcal{A}_l$ where $l = \lfloor \frac{n-3}{2} \rfloor$.

(ii) *If $\bar{\sigma} \in U_n^{\geq 0}$, it can be lifted to a stability condition $\sigma \in \mathrm{Stab}_*(\mathrm{D_n})$ which is supported on the standard heart $\mathcal{A}_0 \cong \mathrm{Rep}(\mathrm{K}_2)$ up to shift.*

Proof. (i) Let $\bar{\sigma} \in U_n^{<0}$. Then there exists $k \in \mathbb{Z}$ with $0 \leq k \leq n-3$ such that

$$\frac{k}{2} \leq \phi(S_2) - \phi(S_1) < \frac{k+1}{2}$$

with the first equality strict when $k = 0$.

This inequality is equivalent to

$$0 \leq \phi(S_2) - \phi(S_1\left[\frac{k}{2}\right]) < \frac{1}{2} \qquad\qquad \frac{1}{2} \leq \phi(S_2) - \phi(S_1\left[\frac{k-1}{2}\right]) < 1$$

for k even or odd respectively.

Define $l := \lfloor \frac{k}{2} \rfloor$. Then $0 \leq \phi(S_2) - \phi(S_1[l]) < 1$. Hence S_2 and $S_1[l]$ are objects in the heart $\mathcal{P}((0,1])[m]$ for some integer m. As these categories are closed under extension, the category containing them is $\langle S_2, S_1[l] \rangle \cong \mathcal{A}_l$. Hence $\bar{\sigma}$ lifts to stability condition σ which is supported on the heart $\mathcal{A}_l$, up to shift.

(ii) Let $\bar{\sigma} \in U_n^{\geq 0}$. Then

$$0 \leq \phi(S_1) - \phi(S_2) < \phi(S_2[1]) - \phi(E_{1,2}) = \phi(S_2) - \phi(E_{1,2}) + 1.$$

As both S_2 and $E_{1,2}$ are semistable and not of equal phase, and $\mathrm{Hom}^0_{\mathrm{D_n}}(S_2, E_{1,2}) \neq 0$ by Lemma 1.4.5, $\phi(S_2) - \phi(E_{1,2}) \leq 0$ and therefore

$$\phi(S_2) - \phi(E_{1,2}) + 1 \leq 1.$$

As $0 \leq \phi(S_1) - \phi(S_2) < 1$, the objects S_1 and S_2 lie in a shift of the heart $\mathcal{P}((0,1])$ and $\bar{\sigma}$ is supported on $\mathcal{A}_0 = \langle S_1, S_2 \rangle$.

□

Remark 4.2.6. The converse of Lemma 4.2.5 is not true as stability conditions with one of the following two properties may exist.

(i) For a stability condition $\sigma \in \mathrm{Stab}(\mathrm{D}_3)$ supported on the standard heart $\mathcal{A}_0$ with indecomposable semistable objects S_1 and S_2 and $\phi(S_2) - \phi(S_1) \in [\frac{1}{2}, 1]$, the corresponding projective stability condition is not in U_3.

(ii) If $\sigma \in \mathrm{Stab}_*(\mathrm{D_n})$ is supported on the standard heart with all indecomposable representations semistable with phases satisfying either of the inequalities

$$\phi(S_1) - \phi(S_2) \geq \phi(S_2[1]) - \phi(E_{1,2}) > 0$$

$$\phi(S_1) - \phi(S_2) \geq \phi(E_{2,1}[1]) - \phi(S_1) > 0$$

then $\bar{\sigma} \notin U_n$.

The next step is the characterisation of the objects of $\mathrm{D_n}$ which are unstable with respect to projective stability conditions in U_n.

Lemma 4.2.7. *Let σ be a stability condition on the category $\mathrm{D_n}$ such that $\bar{\sigma} \in U_n^{<0}$. If an indecomposable object $X \in \mathrm{D_n}$ is unstable with respect to σ and has width $w_\sigma(X) < \frac{1}{2}$, then X is isomorphic to a shift of an indecomposable representation of the Kronecker quiver.*

Proof. Fix $\sigma \in \mathrm{Stab}_*(\mathrm{D_n})$ such that $\bar{\sigma} \in U_n^{<0}$. Let $X \in \mathrm{D_n}$ be an indecomposable object which is unstable with respect to σ, and consider its Harder–Narasimhan decomposition of X with respect to σ. As $\bar{\sigma} \in U_n^{<0}$, the only indecomposable semistable objects with respect to σ are shifts of the two simple objects. Therefore the Harder–Narasimhan factors of X are of the form $S_i[l]^{\oplus k_i}$ where $l \in \mathbb{Z}$ and $k_i \in \mathbb{Z}_{\geq 0}$.

This object has exactly two Harder–Narasimhan factors. If the decomposition were to have more than two factors, it would include shifts of factors already present. Then the phase condition $\phi(S_i[1]) = \phi(S_i) + 1$ would imply that $w_\sigma(X) \geq 1$. It has at least two factors as the one factor filtration is the exact triangle

$$0 \longrightarrow X \longrightarrow S_i[l]^{\oplus k_i} \longrightarrow 0$$

and in this case X is isomorphic to $S_i[l]^{\oplus k_i}$ and therefore stable if $k_i = 0$, or decomposable for $k_i > 0$, contradicting the conditions on X.

The ordering of these factors is then determined by the morphisms between them. Recall from Lemma 1.4.5, the only non-zero Hom-spaces between the two simple objects are

$$\mathrm{Hom}^1_{\mathrm{D_n}}(S_1, S_2) \cong \mathrm{Hom}^{n-1}_{\mathrm{D_n}}(S_2, S_1) \cong \mathbb{C}^2.$$

Therefore for an indecomposable object X there is a sequence of exact triangles of the form

$$\begin{tikzcd}
0 \arrow[rr] & & S_2^{\oplus k_2} \arrow[rr] \arrow[dl] & & X \arrow[dl] \\
& S_2^{\oplus k_2} \arrow[ul, dashed] & & S_1^{\oplus k_1} \arrow[ul, dashed] &
\end{tikzcd}$$

up to shift of the simples by an integer l. Therefore X is the indecomposable representation with dimension vector (k_1, k_2) where either $k_1 = k_2$, $k_1 = k_2 + 1$ or $k_1 = k_2 - 1$. $\square$

Lemma 4.2.8. *The region U_n is open in $\mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$.*

Proof. Consider the projection map $\Pi\colon \mathrm{Stab}_*(\mathrm{D_n}) \longrightarrow \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$. That the region $U_n \subset \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$ is open is equivalent to the preimage $\Pi^{-1}(U_n)$ being an open subset of $\mathrm{Stab}_*(\mathrm{D_n})$. Hence it suffices to show that for $\sigma_0 \in \mathrm{Stab}_*(\mathrm{D_n})$ such that $\bar{\sigma}_0 := \Pi(\sigma_0) \in U_n$, there is an open ball of stability conditions around σ_0 contained in $\Pi^{-1}(U_n)$.

For $\sigma_0 \in \mathrm{Stab}_*(\mathrm{D_n})$ with $\bar{\sigma} \in U_n$, there are three cases to consider:

(i) $\bar{\sigma}_0 \in U_n^{<0}$;

(ii) $\bar{\sigma}_0 \in U_n^{\geq 0}$ and $0 < \phi(S_1) - \phi(S_2)$;

(iii) $\bar{\sigma}_0 \in U_n^{\geq 0}$ and $0 = \phi(S_1) - \phi(S_2)$.

By Proposition 1.5.11, if X is an object which is stable with respect to σ_0, then in an open neighbourhood of σ_0 it remains stable.

Suppose that X is an object which is semistable with respect to σ_0, with $\bar{\sigma}_0 \in U_n^{<0}$. These objects remain semistable, and the ball $B_\epsilon(\sigma_0)$ is contained in the preimage of $U_n^{<0}$. An analogous argument holds for stability conditions satisfying condition (ii). Finally suppose that σ_0 is a stability condition satisfying (iii). Then all indecomposable representations are semistable and with equal phase. As the phase function is continuous, for stability conditions in a neighbourhood of σ_0, either $\phi(S_1) - \phi(S_2) = \epsilon > 0$ or $\phi(S_2) - \phi(S_1) = \epsilon > 0$. Again there is an open ball around σ_0 which is contained in the preimage of U_n.

The last condition on the stability of the objects that must be checked is that objects which are unstable with respect to σ_0 do not become semistable within a neighbourhood of σ_0. Although the property of an object being unstable is also open, there are infinitely many objects in $\mathrm{D_n}$ to consider. Hence it is necessary to consider each of the above three cases separately.

(i) Let $\bar{\sigma} \in U_n^{<0}$. The only indecomposable semistable objects are shifts of the simples. To ensure that no other objects become semistable in the neighbourhood of σ_0, we check that

$$\inf_{X \text{ unstable in } \sigma_0} \{w_{\sigma_0}(X)\} > 0.$$

The only semistable factors possible in the Harder–Narasimhan decomposition of X are direct sums of S_1 and S_2 up to shift. Hence $w_{\sigma_0}(X) = \phi(S_2) - \phi(S_1) + k$ for some integer k. The infimum of the widths of unstable objects is the width of the objects X for which $\phi(S_2) - \phi(S_1) + k \in (0, 1]$, and as $\phi(S_2) - \phi(S_1)$ is fixed for σ_0, there is exactly one such value.

(ii) For such σ_0, recall from Lemma 4.2.5, these stability conditions are supported on a shift of the standard heart $\mathcal{A}_0$.

Let X be unstable with respect to a stability condition $\sigma_0 \in \mathrm{Stab}_*(\mathrm{D_n})$ such that $\bar{\sigma} \in U_n^{\geq 0}$. We restrict to unstable objects X with width $w_{\sigma_0}(X) \leq 1$. If $w_{\sigma_0}(X) > 1$, then the infimum is clearly positive, and the argument from the previous case holds.

If $0 < w_{\sigma_0}(X) \leq 1$, then (up to shift) the phase of each of the semistable factors A_i must lie in the interval $\phi(S_2) < \phi(A_i) < \phi(S_1)$, as there are no semistable factors with phase between $\phi(S_1)$ and $\phi(S_2[1])$. In particular this means that semistable factors all lie in the standard heart. Hence the unstable object X is an object in this category. If X is decomposable, as the indecomposable objects of $\mathrm{Rep}(\mathrm{K_2})$ are semistable with respect to σ_0, let X be semistable with respect to a stability condition σ in a neighbourhood of σ_0. As $w_\sigma(X) = 0$, $\phi(S_2) = \phi(S_1)$. For such σ, $\bar{\sigma}$ lies on the boundary between $U_n^{<0}$ and $U_n^{\geq 0}$, in which case the image of the ball $B_\epsilon(\sigma_0)$ is contained in U_n.

(iii) Now suppose that $\phi(S_1) = \phi(S_2)$ at σ_0. As the phase function is continuous, in an neighbourhood of σ_0, either $0 \leq \phi(S_1) - \phi(S_2)$ or $0 < \phi(S_2) - \phi(S_1)$, in which case we are in one of the two conditions dealt with above.

□

Proposition 4.2.9. *The open subset U_n of $\mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$ is a fundamental domain for the group action of $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$ on $\mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$.*

Proof. We first check that the intersection $U_n \cap \psi U_n$ is empty for some autoequivalence ψ in $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$ which is not the identity in $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$. Suppose that there exists a projective stability condition $\bar{\sigma}$ in the intersection $U_n \cap \psi U_n$. We consider the two cases $\bar{\sigma} \in U_n^{<0} \cap \psi(U_n)$ and $\bar{\sigma} \in U_n^{\geq 0} \cap \psi(U_n)$ separately.

(i) Suppose that $\bar{\sigma} \in U_n^{<0} \cap \psi(U_n)$. Therefore $\mathbb{P}\mathcal{S}(\sigma) = \{S_1, S_2\}$. As $\bar{\sigma}$ is also in ψU_n for some autoequivalence ψ, this set can also be written as $\mathbb{P}\mathcal{S}(\sigma) = \{\psi(S_1), \psi(S_2)\}$. Hence under the autoequivalence ψ, the set of shifts of simple objects is mapped to itself. Therefore ψ acts by either swapping or fixing the simples. If it fixes the simples, then it is the identity in $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$. If ψ swaps the simples, then its composition with $\bar{\Upsilon}$ acts as the identity in $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$. Given the relations on the generators of $\mathrm{Aut}_*(\mathrm{D_n})$ discussed in Proposition 3.2.2, either ψ is $\bar{\Upsilon}^{-1}$ and the composition with Υ acts trivially, or $\psi = \bar{\Upsilon}$. Then $\bar{\Upsilon} \circ \psi = \bar{\Upsilon}^2$ acts trivially in $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$.

If $\bar{\sigma}$ lies in $\bar{\Upsilon}(U_n^{<0})$ or $\bar{\Upsilon}^{-1}(U_n^{<0})$ then the phases of the semistable objects satisfy

$$0 < n - 2 + \phi(S_1) - \phi(S_2) < \frac{n-2}{2}$$

which is equivalent to

$$\phi(S_2) - \phi(S_1) > \frac{n-2}{2}.$$

This contradicts the phase condition on stability conditions in $U_n^{<0}$, and hence the intersection is empty

(ii) Let $\bar{\sigma} \in U_n^{\geq 0} \cap \psi U_n$. Then ψ maps the set of shifts of indecomposable representations of the quiver K_2 to itself. Therefore $\psi = \bar{\Sigma}^k$ for some $k \in \mathbb{Z}$, as $\bar{\Upsilon}$ and $\bar{\Sigma}$ generate $\mathbb{P}\operatorname{Aut}_*(\mathrm{D_n})$, and $\bar{\Upsilon}$ does not preserve this set of semistable objects.

However the phases of the semistable objects with respect to $\bar{\sigma}$ a stability condition lying in this intersection satisfies the inequalities

$$\begin{aligned}
&0 \leq \phi(S_1) - \phi(S_2) < \phi(S_2[1]) - \phi(E_{1,2})\\
&0 \leq \phi(S_1) - \phi(S_2) < \phi(E_{2,1}[1]) - \phi(S_1)\\
&0 \leq \phi(\Sigma^k S_1) - \phi(\Sigma^k S_2) < \phi(\Sigma^k S_2[1]) - \phi(\Sigma^k E_{1,2})\\
&0 \leq \phi(\Sigma^k S_1) - \phi(\Sigma^k S_2) < \phi(\Sigma^k E_{2,1}[1]) - \phi(\Sigma^k S_1).
\end{aligned}$$

This is only the case when $k = 0$, and then ψ is trivial.

We now check that the subset

$$\bigcup_{\psi \in \mathbb{P}\operatorname{Aut}_*(\mathrm{D_n})} \psi(\bar{U}_n) \subset \mathbb{P}\operatorname{Stab}_*(\mathrm{D_n})$$

is open and closed, and hence the inclusion is in fact an equality.

As U_n is an open subspace of $\mathbb{P}\operatorname{Stab}_*(\mathrm{D_n})$ by Lemma 4.2.8, its image $\psi(U_n)$ is also an open subset of $\mathbb{P}\operatorname{Stab}_*(\mathrm{D_n})$ for $\psi \in \mathbb{P}\operatorname{Aut}_*(\mathrm{D_n})$. Therefore it remains to examine the boundaries of its closure $\bar{U}_n$. The method used to do this is to determine the indecomposable semistable objects and their phase conditions for a projective stability condition $\bar{\sigma}$ on each of the boundary components. From this, we see that around a projective stability condition $\bar{\sigma} \in \partial\bar{U}_n$, there is an open ball $B_\epsilon(\bar{\sigma})$ contained in the union $\bar{U}_n \cup \psi(\bar{U}_n)$.

We fist consider a projective stability condition $\bar{\sigma}$ on the boundary of the closure of U_n corresponding to the equality $\phi(S_2) - \phi(S_1) = \frac{n-2}{2}$ on phase conditions. As being semistable is a closed property by Proposition 1.5.11, $\mathbb{P}S(\sigma) = \{S_1, S_2\}$. Consider an open ball $B_\epsilon(\bar{\sigma})$ with $0 < \epsilon \ll 1$ and suppose that $\bar{\sigma}'$ lies in $B_\epsilon(\bar{\sigma}) \setminus \bar{U}_n$. As the phase function is continuous by Proposition 1.5.11, such a stability condition satisfies the phase condition

$$\begin{aligned}
&0 < \phi(S_1[n-2]) - \phi(S_2) < \frac{n-2}{2}\\
&\phi(S_2) - \phi(S_1) > \frac{n-2}{2} > 0.
\end{aligned}$$

However this is the phase condition satisfied by projective stability conditions in $\bar{\Upsilon} U_n$, and so $B_\epsilon(\bar{\sigma}) \subset \bar{U}_n \cup \bar{\Upsilon}\bar{U}_n$, and also $\bar{\Upsilon}$ preserves the set $\mathbb{P}S(\sigma) = \{S_1, S_2\}$.

The situation for projective stability conditions $\bar{\sigma}$ on the other boundary of $\bar{U}_n$ is similar. Suppose that the phases of the semistable objects of $\bar{\sigma} \in \bar{U}_n$ satisfy

$$0 \le \phi(S_1) - \phi(S_2) < \phi(S_2[1]) - \phi(E_{1,2})$$
$$0 \le \phi(S_1) - \phi(S_2) = \phi(E_{2,1}[1]) - \phi(S_1)$$

Again as these objects are semistable in $U_n^{\geq 0}$, by Proposition 1.5.11, they are semistable in the closure of $U_n^{\geq 0}$. For $\bar{\sigma}' \in B_\epsilon(\bar{\sigma}) \setminus \bar{U}_n$, the continuity of the phase function implies that the semistable objects of $\bar{\sigma}'$ satisfy the inequality

$$0 \le \phi(S_1) - \phi(S_2) < \phi(S_2[1]) - \phi(E_{1,2})$$
$$0 < \phi(S_2[1]) - \phi(E_{1,2}) < \phi(S_1) - \phi(S_2).$$

This is equivalent to the phase condition satisfied by projective stability conditions in $\bar{\Sigma} U_n$. The objects of $\mathbb{P}\mathcal{S}(\sigma)$ are the indecomposable representations of K_2. This set of semistable objects remains the same under the autoequivalence Σ. Therefore for projective stability conditions $\bar{\sigma}$ on this boundary we have $B_\epsilon(\bar{\sigma}) \subset \bar{U}_n \cup \bar{\Sigma}(\bar{U}_n)$.

In the final case, suppose that $\bar{\sigma} \in \bar{U}_n$ such that the phase condition

$$0 \le \phi(S_1) - \phi(S_2) = \phi(S_2[1]) - \phi(E_{1,2})$$
$$0 \le \phi(S_1) - \phi(S_2) < \phi(E_{2,1}[1]) - \phi(S_1)$$

is satisfied. Then an analogous argument to the previous case applies, with the autoequivalence $\bar{\Sigma}$ replaced by $\bar{\Sigma}^{-1}$.

To check that this region is closed, suppose that there exists a projective stability condition

$$\bar{\sigma} \in \bigcap_{i \in I} \psi_i \bar{U}_n$$

over some set I. Considering the preimage of this intersection under some autoequivalence ψ_i, $\bar{\sigma}$ lies on the boundary of U_n. Then this intersection is finite as each boundary component of U_n corresponds to exactly one of the autoequivalences Υ, Σ or Σ^{-1}. □

Remark 4.2.10. In Figure 4.4 the relationship between the autoequivalences in $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$ and the boundaries of region $C_n \cong U_n$ is illustrated. We draw particular attention to the relationship of the angles in C_n with the order of the autoequivalences in $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$.

As $\bar{\Upsilon}^2 = \mathrm{id} \in \mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$, Figure 4.4 shows that there is a $\mathbb{Z}/2\mathbb{Z}$ action about the point $\frac{2-n}{2}$ on the real axis. The other interesting behaviour occurs as $z \to 1$. By Lemma 4.2.4, this point

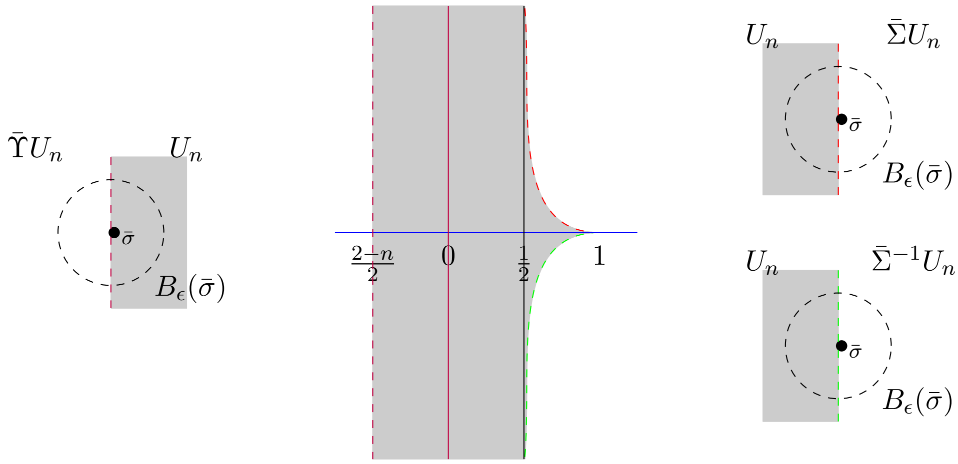


Figure 4.4: Stability conditions on the boundary of U_n.

is not in the closure of U_n. The two boundaries on the right determined by the equalities

$$0 \leq \phi(S_1) - \phi(S_2) = \phi(S_2[1]) - \phi(E_{1,2})$$
$$0 \leq \phi(S_1) - \phi(S_2) = \phi(E_{2,1}[1]) - \phi(S_1)$$

correspond to the action of the autoequivalences $\bar{\Sigma}^{-1}$ and $\bar{\Sigma}$ respectively. That they approach each other with angle zero encodes that the autoequivalence Σ is of infinite order in $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n})$.

Proposition 4.2.11. *The map* $g\colon \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n}) \longrightarrow \mathbb{C}$ *defined by* $g(\bar{\sigma}) := \frac{1}{i\pi}\log\left(\frac{Z_{\bar{\sigma}}(S_2)}{Z_{\bar{\sigma}}(S_1)}\right)$ *defines a biholomorphism between the fundamental domain* U_n *and* C_n.

Proof. This map is injective by construction, as although the projective central charge map $\bar{\mathcal{Z}}$ is not injective for $n > 4$, its composition with the complex logarithm defines an injective map from U_n to C_n.

That the image of $\bar{\sigma} \in U_n$ under g lies in C_n is immediate from the description and definition of C_n given in the preceding construction and illustrated in Figure 4.3.

Conversely, for a point $z = x+iy \in C_n$, $x = \phi(S_1)-\phi(S_2)$ determines the phases of a stability condition in $\mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$, whilst the ratio of the central charges $|Z(S_1)|$ to $|Z(S_2)|$ determines $|z|$. The boundaries of the region C_n ensure that the phases of this central charge satisfy the inequalities necessary for it to lie in U_n. □

Chapter 5

The Schwarz Triangle Theorem and stability conditions

In this chapter, we consider how to construct a biholomorphism between the upper half plane and fundamental domain U_n constructed in Chapter 4. The key step to do this is to view $C_n \cong U_n$ as a curvilinear triangle and apply the Schwarz Triangle Theorem

The first section of this chapter is a review of the necessary complex analysis and discussion of the Schwarz Triangle Theorem. In the second section the biholomorphism between the upper half plane and the fundamental domain is constructed. Finally we consider how these results support the existence of Frobenius structures on stability manifolds through the relationship of central charges to twisted periods.

This material applies only to the Calabi–Yau-n categories of the Kronecker quiver, however we indicate adaptions that need to be made to extend it to the derived category D_∞.

5.1 The Schwarz Triangle Theorem

The Schwarz Triangle Theorem details how to construct an explicit biholomorphic map from a curvilinear triangle in $\mathbb{C}$ to the upper half plane, which we denote by $\mathcal{H} := \{z \in \mathbb{C} \colon \operatorname{Im}(z) > 0\}$. We denote the closed upper half plane by $\bar{\mathcal{H}}$ and the lower half plane by $\mathcal{H}^-$. The map is given by the ratio of solutions to a hypergeometric equation determined by the angles of the triangle. This section reviews the necessary properties of the Schwarzian derivative to prove this theorem. The reference for the following is Nehari's book [36, Chap. V].

Definition 5.1.1. The *Schwarzian derivative* S of a holomorphic function f is a differential operator such that

$$(Sf)(z) := \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2$$

is a meromorphic function.

Lemma 5.1.2. *[36, Chap. V, Sec. 7]*

(i) The Schwarzian derivative of a holomorphic function f is identically zero if and only if f is a Möbius transformation.

(ii) For f and g holomorphic functions, the Schwarzian derivative satisfies the following chain rule

$$(S(f \circ g)) = ((Sf) \circ g) \cdot g'^2 + (Sg).$$

(iii) If f is a Möbius transformation, then $(S(f \circ g)) = (Sg)$.

Proof. (i) Let $f(z) = \frac{az+b}{cz+d}$ be a Möbius transformation. Then

$$\frac{f''(z)}{f'(z)} = \frac{-2c}{cz+d}$$

and

$$(Sf)(z) = \frac{2c^2}{(cz+d)^2} - \frac{2c^2}{(cz+d)^2} = 0.$$

Conversely suppose that the Schwarzian derivative of f vanishes identically, and therefore

$$\left(\frac{f''}{f'}\right)' = \frac{1}{2}\left(\frac{f''}{f'}\right)^2.$$

Integrating this we see that f is indeed a Möbius transformation.

(ii) This is a direct computation.

(iii) This is a corollary of parts (i) and (ii).

□

Definition 5.1.3. A *hypergeometric differential equation* is a linear second order differential equation of the form

$$z(1-z)y''(z) + \Big(\rho_3 - (\rho_1 + \rho_2 + 1)z\Big)y'(z) - \rho_1\rho_2 y(z) = 0$$

where $\rho_1, \rho_2, \rho_3 \in \mathbb{C}$ with $\rho_3 > 0$ and $z \in \mathbb{C}$.

Lemma 5.1.4. *[36, Chap. V, Sec. 7] The hypergeometric equation in Definition 5.1.3 can be written in the form*

$$u''(z) + Q(\rho_1, \rho_2, \rho_3; z)u(z) = 0$$

where $Q(\rho_1, \rho_2, \rho_3; z)$ is a Laurent polynomial. This is called the Q-form of the hypergeometric equation.

Proof. Any second order differential equation of the form $y(z)'' + A(z)y(z)' + B(z)y(z) = 0$ can be expressed in the form $u''(z) + Q(z)u(z) = 0$ by using the substitution $y(z) = \exp(-\frac{1}{2}\int A dz)u(z)$. This gives a second order differential equation

$$u''(z) + \Big(B(z) - \frac{1}{2}A'(z) - \frac{1}{4}A(z)^2\Big)u(z) = 0.$$

For the hypergeometric equation in Definition 5.1.3, the functions $A(z)$ and $B(z)$ also depend on ρ_1, ρ_2 and ρ_3 and are $A(z) = \frac{\rho_3 - (\rho_1 + \rho_2 + 1)z}{z(1-z)}$ and $B(z) = \frac{-\rho_1\rho_2}{z(1-z)}$. □

The following lemma characterises the way that the solutions of this equation interact with the Schwarzian derivative.

Proposition 5.1.5. *[36, Chap. V, Sec. 7] Let $u_1(z)$ and $u_2(z)$ be linearly independent solutions to the second order differential equation $u''(z) + Q(z)u(z) = 0$. Then the Schwarzian derivative of the ratio of their solutions, denoted $f(z)$, satisfies*

$$(Sf)(z) = 2Q(z).$$

Proof. Instead of directly computing the Schwarzian derivative of the function $f = \frac{u_1}{u_2}$ by using the quotient rule, which quickly becomes unpleasant, we find the ratio $\frac{f''}{f'}$ by making the substitution $u_1 = fu_2$ in the equation $u_1'' + Qu_1 = 0$. As $u_2'' + Qu_2 = 0$, we obtain $f''u_2 + 2f'u_2' = 0$, and hence $\frac{f''}{f'} = \frac{-2u_2'}{u_2}$. Inputting this in to the equation for the Schwarzian derivative gives us the desired result. □

Theorem 5.1.6 (Riemann Mapping Theorem)**.** *[14, Chap. VII, 4.2] Let U be a non-empty simply connected open subset of $\mathbb{C}$. Then there exists a unique biholomorphic function from U to the open unit disc.*

Theorem 5.1.7 (Schwarz Reflection Principle)**.** *[14, Chap. IX, 1.2] Let f be a a continuous function which is holomorphic on the upper half plane, such that $f(z)$ is real valued for $z \in \mathbb{R}$. Then f can be extended to a function F which is holomorphic on the whole complex plane, and is defined as*

$$F(z) = \begin{cases} f(z) \text{ for } z \in \bar{\mathcal{H}} \\ \overline{f(\bar{z})} \text{ for } z \in \mathcal{H}^-. \end{cases}$$

The motivation for the method used to construct a biholomorphism between the fundamental domain defined in Chapter 4 and the upper half plane comes from the Schwarz Triangle Theorem.

Theorem 5.1.8 (Schwarz Triangle Theorem)**.** *[36, pg. 206] Let $C \subset \mathbb{C}$ be a curvilinear triangle with angles $\pi\alpha$, $\pi\beta$, and $\pi\gamma$, where $0 < \alpha, \beta, \gamma < 1$. There is a bijection $\mathcal{W}\colon \mathcal{H} \longrightarrow C$ mapping the upper half plane to the interior of C of the form $\mathcal{W}(z) = \frac{y_1(z)}{y_2(z)}$, where $y_1(z)$ and $y_2(z)$ are linearly independent solution of the hypergeometric equation*

$$z(1-z)y''(z) + [\rho_3 - (\rho_1 + \rho_2 + 1)z]y'(z) - \rho_1\rho_2 y(z) = 0$$

with ρ_1, ρ_2, ρ_3 determined by α, β and γ as

$$\rho_1 = \frac{1}{2}\Big(1 - \alpha + \beta - \gamma\Big) \qquad \rho_2 = \frac{1}{2}\Big(1 - \alpha - \beta - \gamma\Big) \qquad \rho_3 = 1 - \alpha.$$

5.2 The fundamental domain as a curvilinear triangle

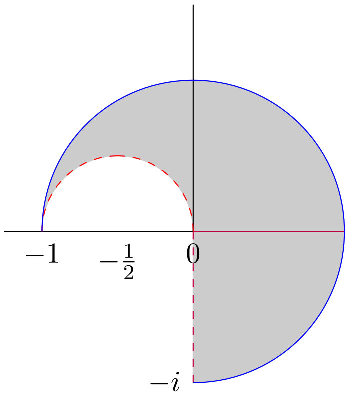
In this section we apply the Schwarz Triangle Theorem to the fundamental domain $U_n \subset \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$ constructed in Chapter 4.

The first step to doing this is to reinterpret U_n as a curvilinear triangle. We begin by splitting the fundamental domain into two parts. Let the elements of $U_n^+ \subset U_n$ be those projective stability conditions in U_n for which $|Z(S_1)| > |Z(S_2)|$, and let $U_n^- \subset U_n$ consist of projective stability conditions for which $|Z(S_1)| < |Z(S_2)|$. The stability conditions with $|Z(S_2)| = |Z(S_1)|$ correspond to points on the shared boundary of these regions. By X_n^+ and C_n^+ we denote the image of U_n^+ in X_n and C_n respectively. This splits C_n into the following two subsets.

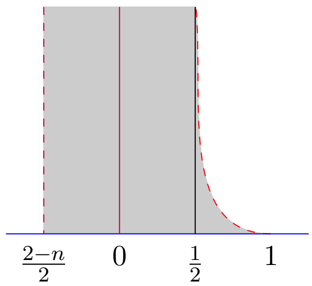
$$C_n^+ := \{z \in C_n\colon\ \mathrm{Im}(z) > 0\} \qquad\qquad C_n^- := \{z \in C_n\colon\ \mathrm{Im}(z) < 0\}.$$

The region X_n^+, the image of C_n^+ under the map $z \mapsto \exp(i\pi z)$ is a curvilinear triangle with the angles $0, \frac{\pi}{2}$ and $\frac{(n-1)\pi}{2}$. As in the construction of the fundamental domain, the example for the Calabi–Yau-3 category is easily illustrated. Figure 5.1 shows the curvilinear triangle X_3^+. The interior of the triangle is the grey region, with angle 0 at the vertex at -1, $\frac{\pi}{2}$ at $-i$ and $\frac{(n-1)\pi}{2} = \pi$ at the origin.

We now introduce the functions which play a key role in the construction of a biholomorphism between the fundamental domain and the upper half plane. The key theory here is the relationship between stability conditions and quadratic differentials as considered by Bridgeland and Smith [12]. We summarise the necessary constructions for this work.



(a) The region X_3^+.



(b) The region C_3^+.

Figure 5.1: Images of the fundamental domain U_3.

Definition 5.2.1. A *quadratic differential* ϕ *on a Riemann surface* S is a meromorphic section of the cotangent bundle $\omega_S^{\otimes 2}$.

Consider a quadratic differential ϕ on a Riemann surface S. The spectral cover of S with respect to the quadratic differential ϕ is a double cover branched at the zeroes and simple poles of the quadratic differential. We denote this by $\pi\colon \hat{S}_\phi \longrightarrow S$. The pullback of the quadratic differential ϕ along the covering map can be written as $\pi^*(\phi) = \psi \otimes \psi$ where ψ is a meromorphic 1-form on $\hat{S}_\phi$. The hat homology $\hat{\mathrm{H}}(\phi)$ of the differential ϕ is defined as the anti-invariant part of the homology group $\mathrm{H}_1(\hat{S}_\phi \setminus \pi^* \,\mathrm{Crit})$ with respect to the covering involution,where Crit is the set of simple poles of ϕ. Of importance are the paths ξ_i on the spectral cover of S with respect to ϕ which generate the hat homology group $\hat{\mathrm{H}}(\phi)$.

Definition 5.2.2. The *period integrals* of a quadratic differential ϕ are the functions

$$Z_\phi(\xi_i) = \int_{\xi_i} \psi$$

where ξ_i is a generating homology class of the hat homology $\hat{H}(\phi)$.

By blowing up the spectral cover at the critical points, we obtain a surface with marked points on the boundary. Such a surface can be triangulated. By associating vertices to edges of the triangulation and arrows to internal triangles, we construct a quiver from a quadratic differential.

The main result of [12] is the construction of an isomorphism between the stability conditions of the Calabi–Yau-3 category of a quiver, and the space of quadratic differentials on the associated marked surface.

Example 5.2.3. For the marked surface associated to the Kronecker quiver, the space of quadratic differentials is parametrised differentials of the form

$$\phi(z) = (\sqrt{b}z + 2a + \sqrt{b}z^{-1})\frac{dz^{\otimes 2}}{z^2}$$

on $\mathbb{P}^1$ for $a \in \mathbb{C}$ and $b \in \mathbb{C}^*$ [12, Ex. 12.5]. We refer to this as the quadratic differential associated to the Kronecker quiver. The spectral cover of $\mathbb{P}^1$ with respect to the quadratic differential $\phi(z)$ is shown in Figure 5.2.

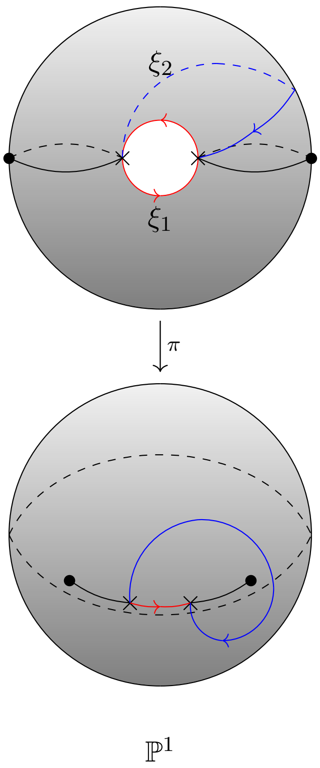


Figure 5.2: The spectral cover of $\mathbb{P}^1$ with respect to the quadratic differential ϕ.

The period integrals of quadratic differentials associated to the Kronecker quiver are the integrals

$$Z_\phi(\xi_i) = \int_{\xi_i} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\frac{1}{2}}\frac{dz}{z}$$

where ξ_1 and ξ_2 are the generators of the hat homology as shown in Figure 5.2.

The triangulation of the associated marked surface is shown in Figure 5.3.

The relationship between stability conditions and quadratic differentials, and in particular between central charges and period integrals, is considered by Bridgeland and Smith in [12]. Work of Ikeda [27] and Wang [42] utilises this relationship to construct spaces of stability conditions for classes of quivers. In particular in [27], in the case of $\mathrm{A_N}$ quivers the relationship is developed

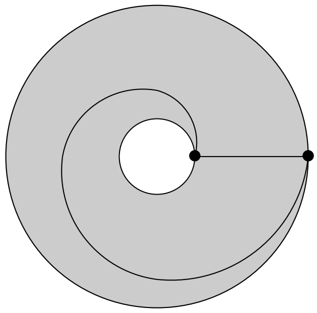


Figure 5.3: The marked surface associated to the Kronecker quiver.

for $n > 3$ by considering quadratic differentials of the form

$$\phi(z) = p_N(z)^{n-2} dz^{\otimes 2}$$

for $p_N(z)$ the polynomial associated to the $\mathrm{A_N}$ quiver. Therefore we can generalise the above construction for the Calabi–Yau-3 case to other n by considering period integrals of the form

$$Z_\phi(\xi_i) = \int_{\xi_i} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\frac{n-2}{2}} \frac{dz}{z}.$$

Proposition 5.2.4. *For $a \in \mathbb{C}$ and $b \in \mathbb{C}^*$, let $w = \frac{a^2}{b}$. Let ξ_1 and ξ_2 be the paths connecting the zeroes of the function $p(z) = \sqrt{b}z + 2a + \sqrt{b}z^{-1}$ on the spectral cover of $\mathbb{P}^1$ with respect to the quadratic differential associated to the Kronecker quiver. Then*

$$Z_{\xi_i}(w) = \int_{\xi_i} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\nu} \frac{dz}{z}$$

are solutions to the hypergeometric equation

$$z(1-z)\partial_z^2 + (\rho_3 + (1 + \rho_1 + \rho_2)z)\partial_z - \rho_1\rho_2 = 0$$

with $\rho_1, \rho_2 = -\nu$, and $\rho_3 = \frac{1}{2} - \nu$.

Proof. The set-up for this result is as in Example 5.2.3. In particular, z is a point on the spectral cover of $\mathbb{P}^1$ with respect to the quadratic differential ϕ associated to the Kronecker quiver, and ξ_1 and ξ_2 are the paths on $\hat{\mathbb{P}}^1_\phi$ which are a basis for the hat homology group $\hat{\mathrm{H}}(\phi)$. For $h \in \mathbb{R}_{>0}$, $\nu \in \mathbb{C}$, and $b \in \mathbb{C}^*$, define the function

$$f_b(h) := h^{-(\nu+1)} \int_{\xi_i} e^{h(\sqrt{b}z + \sqrt{b}z^{-1})} \frac{dz}{z}.$$

In order to simplify our calculation, we rescale z by h. Denote this rescaling by $z' = hz$. Then

$$f_b(h) = h^{-(\nu+1)} \int_{\xi_i} e^{p(z')} \frac{dz'}{z'}$$

for $p(z') = \sqrt{b}z' + h^2\sqrt{b}z'^{-1}$.

By computing the partial derivative of $f_b(h)$ with respect to h, and calculating $\partial_h f_b(h)$, $h\partial_h f_b(h)$, and $\partial_h h\partial_h h f_b(h)$, the equation

$$(h\partial_h + \nu + 1)^2 f_b(h) = 4h^{-(\nu+1)}h^2\sqrt{b}\int_{\xi_i}(z'^{-1} + h^2\sqrt{b}z'^{-2})e^{p(z')}\frac{dz'}{z'}$$

is obtained. Rearranging this and using the equality

$$\frac{d}{dz'}\left(\frac{e^{p(z')}}{z'}\right) = -(z'^{-2} + h^2\sqrt{b}z'^{-3} - \sqrt{b}z'^{-1})e^{p(z')},$$

we obtain a second order differential equation with $f_b(h)$ as a solution:

$$\begin{aligned}((h\partial_h + \nu + 1)^2 - 4h^2 b)f_b(h) &= 4h^{-(\nu+1)}h^2\sqrt{b}\int_{\xi_i}(z'^{-1} + h^2\sqrt{b}z'^{-2})e^{p(z')}\frac{dz'}{z'} - 4h^2 b f_b(h)\\ &= 4h^{-(\nu+1)}h^2\sqrt{b}\int_{\xi_i}(z'^{-1} + h^2\sqrt{b}z'^{-2} - \sqrt{b})e^{p(z')}\frac{dz'}{z'}\\ &= 4h^{-(\nu+1)}h^2\sqrt{b}\int_{\xi_i}(z'^{-2} + h^2\sqrt{b}z'^{-3} - \sqrt{b}z'^{-1})e^{p(z')}dz'\\ &= 0.\end{aligned} \tag{5.2.1}$$

The next step is to change this differential equation to a hypergeometric equation in z. Let $a \in \mathbb{C}$ and denote by $\mathcal{L}\{f_b(h)\}(2a)$ the Laplace transform of the function $f_b(h)$, then

$$\begin{aligned}\mathcal{L}\{f_b(h)\}(2a) &= \int_0^\infty e^{h(2a)}f_b(h)dh\\ &= \int_0^\infty e^{h(2a)}h^{-(\nu+1)}\int_{\xi_i}e^{h(\sqrt{b}z+\sqrt{b}z^{-1})}\frac{dz}{z}dh\\ &= \int_0^\infty\int_{\xi_i}h^{-(\nu+1)}e^{h(\sqrt{b}z+2a+\sqrt{b}z^{-1})}\frac{dz}{z}dh\\ &= \int_{\xi_i}(-(\nu+1))!(\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\nu}\frac{dz}{z}\end{aligned} \tag{5.2.2}$$

(using the identity $\int_0^\infty h^{-(\nu+1)}e^{h(y+\lambda)}dh = (-(\nu+1))!(y+\lambda)^{\nu}$). Observe the following relationship between the Laplace transform in 5.2.2 and the period integrals of the quadratic differential associated to the Kronecker quiver

$$\int_{\xi_i}(\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\nu}\frac{dz}{z} = \frac{\mathcal{L}\{f_b(h)\}(2a)}{-(\nu+1)!}$$

Additionally, under the Laplace transform, $h \mapsto \partial_{2a} = \frac{\partial_a}{2}$ and $h\partial_h \mapsto -a\partial_a - 1$. Hence the equation

$$((h\partial_h + \nu + 1)^2 - 4h^2 b)f_b(h) = 0$$

obtained above in calculation 5.2.1, becomes

$$((a\partial_a - \nu)^2 - b\partial_a^2)\int_{\xi_i}(\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\nu}\frac{dz}{z} = 0. \tag{5.2.3}$$

The final step to show that the period integrals are solutions to a hypergeometric equation in z, is to remove the parameters a and b. To do this, recall that z is a point on the spectral cover $\hat{\mathbb{P}}^1_\phi$. The covering map sends a point $z \in \hat{\mathbb{P}}^1_\phi$ to $w = (2z-1)^2 \in \mathbb{P}^1$. As $a \in \mathbb{C}$ and $b \in \mathbb{C}^*$, by setting $w = \frac{a^2}{b}$, we restrict to w a point in $\mathbb{C}$. Therefore by making the substitutions $a = d(2z-1)$ with $\frac{b}{d^2} = 1$, equation 5.2.3 become the hypergeometric equation

$$z(1-z)\partial_z^2 + (\rho_3 + (1+\rho_1+\rho_2)z)\partial_z) - \rho_1\rho_2 = 0 \tag{5.2.4}$$

with $\rho_1 = \rho_2 = -\nu$ and $\rho_3 = -\nu + \frac{1}{2}$. □

Recall that the Schwarz Triangle Theorem, Theorem 5.1.8, states that a biholomorphism between a curvilinear triangle C in $\mathbb{C}$ and the upper half plane is defined by the ratio of linearly independent solutions to a hypergeometric equation determined by the angles of C. The proof of the following proposition is a modified proof of the Schwarz Triangle Theorem for the curvilinear triangle X_n^+. This modification is necessary as the angles of X_n^+ do not satisfy the requirements of the Schwarz Triangle Theorem.

Proposition 5.2.5. *For $n \geq 3$, there is a biholomorphism $f_n^+ \colon \mathcal{H} \longrightarrow C_n^+$ defined by*

$$f_n^+(w) = \frac{1}{i\pi}\log\left(\frac{Z_{\xi_1}(w)}{Z_{\xi_2}(w)}\right)$$

where

$$Z_{\xi_i}(w) = \int_{\xi_i} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^\nu \frac{dz}{z}$$

are the solutions to the hypergeometric equation in Proposition 5.2.4, and for $a \in \mathbb{C}$, $b \in \mathbb{C}^*$, $w = \frac{a^2}{b}$.

Proof. That a biholomorphic map $f_n^+ \colon \mathcal{H} \longrightarrow C_n^+$ exists is a consequence of the Riemann Mapping Theorem, 5.1.6. Under this map, the image of the strict upper half plane is the interior of the region C_n^+.

Let us consider the map $\mathcal{W}_n \colon \mathcal{H} \longrightarrow X_n^+$ defined by

$$\mathcal{W}_n(w) := \exp(i\pi f_n^+(w)).$$

Denote by z_0, z_1 and z_∞ the points on the real axis such that $\mathcal{W}_n(z_0)$, $\mathcal{W}_n(z_1)$ and $\mathcal{W}_n(z_\infty)$ are the vertices of X_n^+ with angles $\frac{\pi}{2}$, $\frac{(n-1)\pi}{2}$, and zero respectively. The intervals of the real axis between these points are mapped to the corresponding edges of X_n^+. The function $\mathcal{W}_n$ is holomorphic on the upper half plane and on the real axis except for at the points z_0, z_1, z_∞. Therefore its Schwarzian derivative $(S\mathcal{W}_n)$ is holomorphic on this domain.

Now consider a Möbius transformation which maps a curvilinear edge of X_n^+ to the real axis. Then the composition of $\mathcal{W}_n$ with this transformation is real valued on the real axis, and therefore its Schwarzian derivative also has this property. As one of the key properties of the Schwarzian derivative is that it is invariant under Möbius transformations, c.f. Lemma 5.1.2, the Schwarzian derivative $(S\mathcal{W}_n)$ is real valued on the real axis apart from at the points z_0, z_1 and z_∞.

The singularities of $\mathcal{W}_n$ coincide with the singularities of $(S\mathcal{W}_n) = (S(G \circ \mathcal{W}_n))$. We now consider the behaviour of the Schwarzian derivative of $\mathcal{W}_n$ at the singular points z_0, z_1 and z_∞. Consider a Möbius transformation G_α which maps two edges of a curvilinear triangle meeting at angle $\alpha\pi$ to two straight lines meeting at the origin with angle $\alpha\pi$. Then the composition $G \circ \mathcal{W}_n$ maps the upper half plane to two straight lines meeting at the origin with angle $\alpha\pi$, and such a map is of the form $g(z) := (z - z_\alpha)^\alpha h(z)$, where h is a function which is holomorphic and non-zero at $z = z_\alpha$, and real valued for $z \in \mathbb{R}$. For the angles of the curvilinear triangle X_n^+, the functions denoted by g in this construction are

$$g_0(z) := (z - z_0)^{\frac{1}{2}} h_0(z) \qquad g_1(z) := (z - z_1)^{\frac{(n-1)}{2}} h_1(z) \qquad g_\infty(z) := \frac{1}{\log(z - z_\infty)} h_\infty(z).$$

Each function h_μ for $\mu \in \{0, 1, \infty\}$, is holomorphic at z_μ and real valued for $z \in \mathbb{R}$. The map g_0 maps $\mathcal{H}$ to the quadrant bounded by the positive real and positive imaginary axes. In particular the image of $(z_0, \infty) \subset \mathbb{R}$ is the positive real axis, and the image of $(-\infty, z_0) \subset \mathbb{R}$ the ray from the origin at angle $\frac{\pi}{2}$. Similarly the map g_1 maps the real axis to the segment bounded by two rays meeting at the origin at angle $\frac{(n-1)\pi}{2}$. Finally the map g_∞ maps $(z_\infty, \infty) \subset \mathbb{R}$ to the negative real axis, and $(-\infty, z_\infty)$ to the circle $|z + \frac{i}{2\pi}| = \frac{1}{2\pi}$. Importantly this circle meets the real axis at zero with angle zero. Figure 5.4 summarises this construction for the angle $\frac{\pi}{2}$ in the case $n = 3$.

Now for each $\mu \in \{0, 1, \infty\}$, note that $(S\mathcal{W}_n) = (S(G \circ \mathcal{W}_n)) = (Sg_\mu)$. By considering the poles of the Schwarzian derivatives (Sg_μ) and identifying the points on the real axis which map to the vertices with $z_0 = 0$, $z_1 = 1$ and $z_\infty = \infty$, we obtain the function

$$(S(G \circ \mathcal{W}_n))(z) = S(\mathcal{W}_n)(z) = \frac{1 - (\frac{1}{2})^2}{2z^2} + \frac{1 - (\frac{n-1}{2})^2}{2(z-1)^2} + \frac{(\frac{1}{2}) + (\frac{n-1}{2})^2 - 1}{2z(z-1)}.$$

Using Proposition 5.1.5, $\mathcal{W}_n$ is the ratio of linearly independent solutions $y(z)$ of the hypergeometric equation

$$z(1-z)y''(z) + \Big(\rho_3 - (\rho_1 + \rho_2 + 1)z\Big)y'(z) - \rho_1\rho_2 y(z) = 0$$

in which the coefficients ρ_i are functions of 0, $\frac{1}{2}$, and $\frac{n-1}{2}$. However the hypergeometric equation in Proposition 5.2.4 corresponds to a triangle with two vertices of angle $\frac{(n-1)\pi}{2}$ and one of

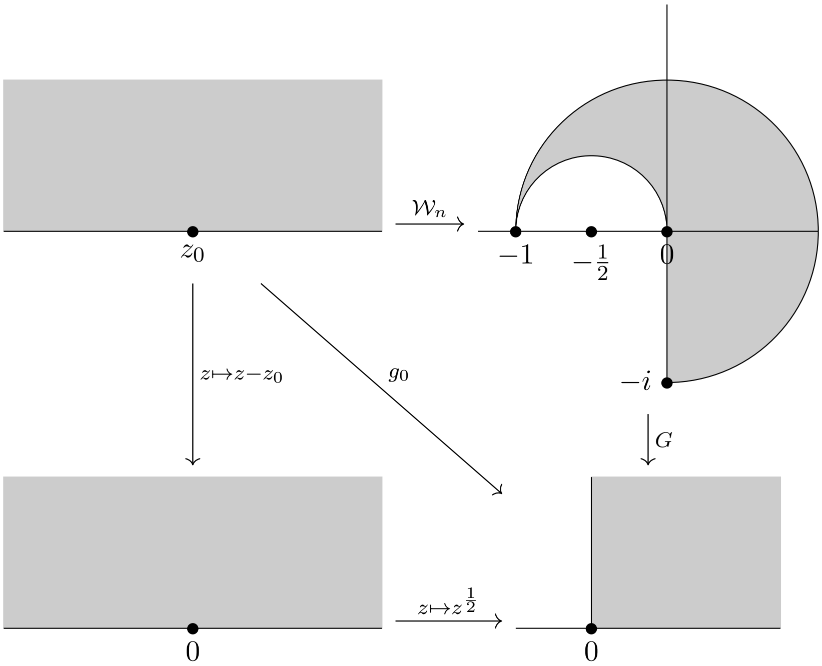


Figure 5.4: Mapping the upper half plane to a curvilinear triangle.

angle zero on the spectral cover of $\mathbb{P}^1$ with respect to the quadratic differential associated to the Kronecker quiver. Under the covering map $w = (2z-1)^2$, this triangle maps to to the curvilinear triangle X_n^+ in $\mathbb{P}^1$. Therefore by applying the construction above to the triangle with vertices $\frac{(n-1)\pi}{2}$, 0, and $\frac{(n-1)\pi}{2}$, and fixing $\nu = \frac{n-2}{2}$, the map $\mathcal{W}_n$ is the ratio of solutions to the hypergeometric equation

$$z(1-z)\partial_z^2 + (\frac{1}{2} - \nu + (1-2\nu)z)\partial_z) - \nu^2 = 0.$$

From Proposition 5.2.4, these solutions are the period integrals

$$Z_{\xi_i}(w) = \int_{\xi_i} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\frac{n-2}{2}} \frac{dz}{z}.$$

The map $\mathcal{W}_n \colon \mathcal{H} \longrightarrow X_n^+$ that we have constructed is not bijective, as the behaviour of the function $g_1(z)$ is to wrap the upper half plane around the point $1 \in \mathbb{R}$. As in the construction of the fundamental domain U_n in Section 4.2, the function $\mathcal{W}_n$ is composed with the complex logarithm in order to obtain a bijective function $f_n^+ \colon H \longrightarrow \mathcal{C}_n^+$ defined as

$$f_n^+(w) = \frac{1}{i\pi} \log \left(\frac{\int_{\xi_1} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\frac{n-2}{2}} \frac{dz}{z}}{\int_{\xi_2} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\frac{n-2}{2}} \frac{dz}{z}} \right)$$

where $w = \frac{a^2}{b}$. □

Remark 5.2.6. As the region C_∞ is not bounded on the left, X_∞ is not a curvilinear triangle. There is a biholomorphism $f_\infty \colon \mathcal{H} \longrightarrow C_\infty$, but it remains to be shown that this map is given by the ratio

$$\frac{\int_{\xi_1} e^{(\sqrt{b}z+2a+\sqrt{b}z^{-1})}\frac{dz}{z}}{\int_{\xi_2} e^{(\sqrt{b}z+2a+\sqrt{b}z^{-1})}\frac{dz}{z}}$$

and that these are the deformed flat coordinates of quantum cohomology.

Note that this ratio does not depend on a, and that with the factor e^{2a} removed, these are the functions $f_b(h)$ from Proposition 5.2.4 with $h = 1$.

5.3 Twisted periods and central charges

In Section 2.3 we discussed the conjectured relationship between the twisted periods of a Frobenius manifold and the central charges of stability conditions. We conclude this chapter by discussing how our result supports this idea.

Corollary 5.3.1. *The solutions to the hypergeometric equation associated to the fundamental domain U_n are the twisted periods of the quantum cohomology of $\mathbb{P}^1$.*

Proof. In Proposition 2.2.1 we stated that the solutions to the hypergeometric equation

$$\partial_t^2 \tilde{p} - \frac{e^t}{s^2}\Big[(-2\partial_t + \nu)(-2\partial_t + \nu - 1)\Big]\tilde{p} = 0$$

which satisfy the quasi-homogeneity condition

$$s\partial_s \tilde{p} + 2\partial_t \tilde{p} = \nu \tilde{p}$$

are the twisted periods of the quantum cohomology of $\mathbb{P}^1$ with (s,t) the flat coordinates of the Frobenius structure. Let (a,b) be the coordinates defined in Proposition 5.2.4. Then via the change of coordinates $2a = s$ and $b = e^t$, the hypergeometric equation satisfied by the twisted periods of the quantum cohomology of $\mathbb{P}^1$ is the same as the hypergeometric equation 5.2.4 in Proposition 5.2.4 which is used to construct the biholomorphic map from the upper half plane to the fundamental domain in Proposition 5.2.5. $\square$

$$\begin{tikzcd}
U_n^+ \arrow[r, "\bar{\mathcal{Z}}"] \arrow[dr, "g"'] & X_n^+ & \mathcal{H} \arrow[l, "\mathcal{W}_n"'] \arrow[dl, "f_n^+"] \\
& C_n^+ \arrow[u, "e^{(i\pi -)}"'] &
\end{tikzcd}$$

Figure 5.5: Summary of Chapters 4 and 5.

Figure 5.5 summarises the regions and maps considered in Chapters 4 and 5. The map $\bar{\mathcal{Z}}$ is the ratio of central charges, and the map $\mathcal{W}_n$ is the ratio of the period integrals which coincide with the twisted periods of $\mathbb{P}^1$. Via the composition $f_n^{+-1} \circ g$, each half of the fundamental domain U_n is biholomorphic to the upper half plane $\mathcal{H}$. For projective stability conditions in U_n^-, analogous constructions to Proposition 5.2.5 show that this half of the fundamental domain is also biholomorphic to the upper half plane. Under this biholomorphism, the central charges correspond to the twisted periods.

Chapter 6

Spaces of stability conditions

We now collate the results of the previous chapters to give a complete description of the spaces of stability conditions of the Calabi–Yau-n categories of the Kronecker quiver. This is done in two steps: The extension of the biholomorphism in Chapter 5 to the rest of the space of projective stability conditions, and the lifting of this isomorphism to the full space of stability conditions. In the final section we consider how we expect these results to extend to the category D_∞.

6.1 The spaces of projective stability conditions

The penultimate step to computing the spaces of stability conditions is to use the fundamental domain U_n to construct the space of projective stability conditions, up to the action of the autoequivalences.

Definition 6.1.1. [22, Ex. 13.1] For $n \geq 1$, *weighted projective n-space* $\mathbb{P}(a_0, \ldots, a_n)$ with $a_i \in \mathbb{Z}_{>0}$ is the variety defined from $\operatorname{Proj} \mathbb{C}[x_0, \ldots x_n]$ where the degree of x_i is a_i.

Proposition 6.1.2. *For $3 \leq n < \infty$, there is an isomorphism of orbifolds*

$$\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\} \cong \mathbb{P}\operatorname{Stab}_*(\mathrm{D_n})/\mathbb{P}\operatorname{Aut}_*(\mathrm{D_n})$$

where $\mathbb{P}(1,2)$ is the weighted projective line, and Δ and μ_∞ are the points $\Delta = [1:1]$ and $\mu_\infty = [1:0]$.

Proof. In Proposition 5.2.5, a biholomorphic map $f_n^+ \colon \mathcal{H} \longrightarrow C_n^+$ is defined by

$$f_n^+(w) = \frac{1}{i\pi} \log\left(\frac{Z_{\xi_1}(w)}{Z_{\xi_2}(w)}\right)$$

where

$$Z_{\xi_i}(w) = \int_{\xi_i} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^\nu \frac{dz}{z}$$

and the coordinate in $\mathcal{H}$ is defined by $w = \frac{a^2}{b}$. Observe that rescaling (a, b) with weights $(1, 2)$ does not change the coordinate $w = \frac{a^2}{b} \in \mathcal{H}$. Therefore we view the upper half plane $\mathcal{H}$ as a subset of weighted projective space $\mathbb{P}(1, 2)$ with coordinates $[a : b]$.

The first step is to extend the domain of the inverse map $f_n^{+-1} \colon C_n^+ \longrightarrow \mathcal{H}$, as constructed in Proposition 5.2.5, to a map $f_n^{-1} \colon C_n \longrightarrow \mathbb{P}^1$. Figure 6.1 illustrates this construction. The map

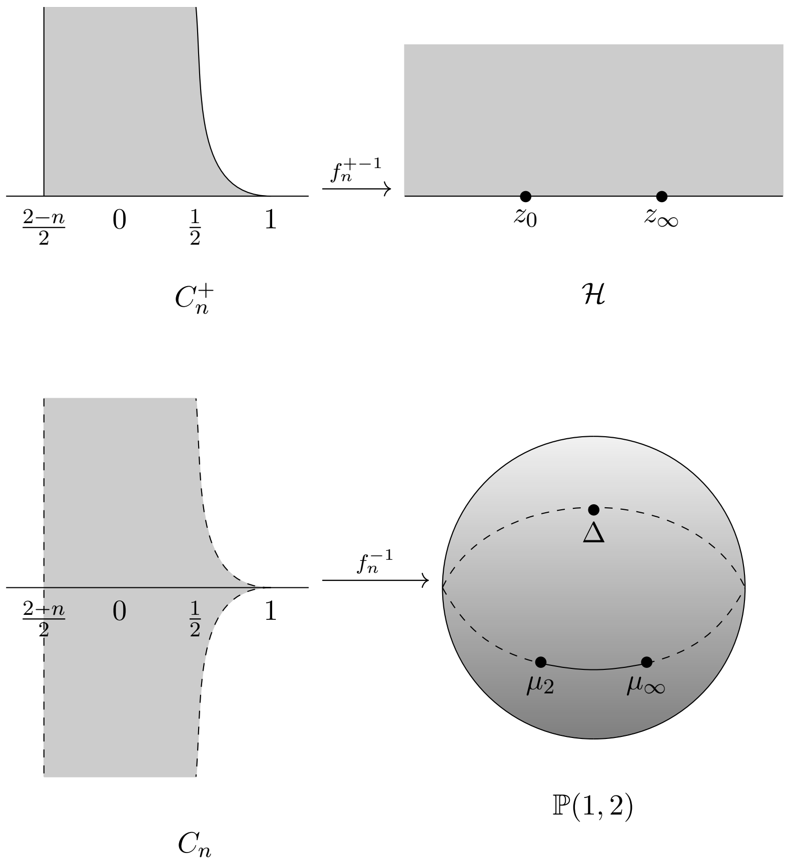


Figure 6.1: Extending the biholomorphism f_n^+ to f_n.

f_n^{+-1} is biholomorphic on the region C_n^+ and sends the interval $\left(\frac{2-n}{2}, 1\right) \subset \mathbb{R}$ to the real axis. Therefore the Schwarz Reflection Principle, Theorem 5.1.7, can be applied to the map f_n^{+-1}, giving

$$f_n^{-1}(z) = \begin{cases} f_n^{+-1}(z) \text{ for } z \in C_n^+ \\ \overline{f_n^{+-1}(\bar{z})} \text{ for } z \in C_n^-. \end{cases}$$

The effect of the map f_n is to glue two copies of the upper half plane $\mathcal{H} \subset \mathbb{P}(1, 2)$ along the points $\mu_2 := [0 : 1]$ and $\mu_\infty := [1 : 0]$. These copies of $\mathcal{H}$ arise as the image of C_n^+ and C_n^- glued along the real axis between $\frac{2-n}{2}$ and 1. The image of the real axis outside of the interval $\left(\frac{2-n}{2}, 1\right)$ is not real, and so f_n^{+-1} cannot be extended continuously at such points using the Schwarz Reflection

Principle. Therefore the image of f_n^{-1} is

$$\mathbb{P}(1,2) \setminus [\mu_2, \Delta] \cup [\Delta, \mu_\infty]$$

where $\Delta := [1:1]$ is the discriminant of the function $p(z) = \sqrt{b}z + 2a + \sqrt{b}z^{-1}$.

Having constructed a map $f_n^{-1} \colon C_n \longrightarrow \mathbb{P}(1,2) \setminus [\mu_2, \Delta] \cup [\Delta, \mu_\infty]$, we are able to compose f_n^{-1} with the biholomorphism $g \colon U_n \longrightarrow C_n$ from Proposition 4.2.11. Thus we obtain a biholomorphism

$$f_n^{-1} \circ g \colon U_n \longrightarrow \mathbb{P}(1,2) \setminus [\mu_2, \Delta] \cup [\Delta, \mu_\infty].$$

Our aim is to obtain a description of the quotient $\mathbb{P}\operatorname{Stab}_*(\mathrm{D_n})/\mathbb{P}\operatorname{Aut}_*(\mathrm{D_n})$. To do this we extend the biholomorphism $f_n^{-1} \circ g$ to the boundaries of U_n. In order to define the map $f_n^{-1} \circ g$ on the left and right boundaries of U_n, recall from Proposition 4.2.9 that these boundaries correspond to the action of the generators $\bar{\Sigma}$ and $\bar{\Upsilon}$ of $\mathbb{P}\operatorname{Aut}_*(\mathrm{D_n})$ on the fundamental domain. Recall also from Remark 4.2.10 that the autoequivalence $\bar{\Upsilon}$ induces a $\mathbb{Z}/2\mathbb{Z}$ action about the point $\frac{2-n}{2}$. Therefore μ_2 is an orbifold point with monodromy group $\mathbb{Z}/2\mathbb{Z}$. There is also an action about the point μ_∞ corresponding to the action of the autoequivalence $\bar{\Sigma}$.

We begin by relating the group $\mathbb{P}\operatorname{Aut}_*(\mathrm{D_n})$ to the orbifold fundamental group of $\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\}$ which is $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$. Therefore there is a group isomorphism

$$\theta \colon \mathbb{Z} * \mathbb{Z}/2\mathbb{Z} \longrightarrow \mathbb{P}\operatorname{Aut}_*(\mathrm{D_n})$$

defined by sending the generators of $\mathbb{Z}$ and $\mathbb{Z}/2\mathbb{Z}$ to the autoequivalences $\bar{\Sigma}$ and $\bar{\Upsilon}$ respectively.

Consider the universal cover $\widetilde{\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\}} \xrightarrow{\tilde{p}} \mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\}$ on which the group $\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}$ acts. As the region $\mathbb{P}(1,2)\setminus[\mu_2,\Delta]\cup[\Delta,\mu_\infty]$ is an open dense subset of $\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\}$, it is a fundamental domain for this action. We want to construct a map

$$\widetilde{\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\}} \longrightarrow \mathbb{P}\operatorname{Stab}_*(\mathrm{D_n})$$

which is equivariant with respect to θ. The preimage of the upper half plane $\mathcal{H}$ is an open subset of the universal cover $\widetilde{\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\}}$, and U_n is a subset of $\mathbb{P}\operatorname{Stab}_*(\mathrm{D_n})$. Additionally, we know how the fundamental group $\mathbb{Z}*\mathbb{Z}/2\mathbb{Z}$ acts on $\tilde{p}^{-1}(\mathcal{H}) \subset \widetilde{\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\}}$ and $U_n \subset \mathbb{P}\operatorname{Stab}_*(\mathrm{D_n})$ and can hence extend the biholomorphism $f_n^{-1} \circ g$ between them equivariantly to the whole spaces $\widetilde{\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\}}$ and $\mathbb{P}\operatorname{Stab}_*(\mathrm{D_n})$.

Taking the quotient of both these spaces by the $\mathbb{Z} * \mathbb{Z}/2\mathbb{Z}$ action gives an isomorphism

$$\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\} \cong \mathbb{P}\operatorname{Stab}_*(\mathrm{D_n})/\mathbb{P}\operatorname{Aut}_*(\mathrm{D_n})$$

of complex orbifolds. $\square$

6.2 The spaces of stability conditions

This section completes the computation of the spaces of stability conditions by lifting the results in the previous section to $\mathrm{Stab}_*(\mathrm{D_n})$.

The techniques used thus far are analogous to those employed in [13], where the final manifold computed is the quotient $\mathrm{Stab}_*(\mathrm{D_n})/\,\mathrm{Sph}_*(\mathrm{D_n})$. As the Kronecker quiver is not of type $\mathrm{A_N}$, the group of autoequivalences is not generated solely in terms of spherical twists and the shift functor. Hence to consider the quotient $\mathrm{Stab}_*(\mathrm{D_n})/\,\mathrm{Sph}_*(\mathrm{D_n})$ is not analogous to computing this quotient in the $\mathrm{A_N}$ case. In particular, this quotient would not take account of the action of the autoequivalences Σ and Υ. We first consider what the appropriate quotient to consider is.

The motivation for studying the space of stability conditions modulo the action of spherical twists arises from the relationship between the spherical twists and the braid group in the $\mathrm{A_N}$ case, as the braid group is the mapping class group of the surface associated to the $\mathrm{A_N}$ quiver. In these cases, it is the spherical twists together with the shift functor that generate the group $\mathrm{Aut}_*(\mathrm{D_n}(\mathrm{A_N}))$, and there is an isomorphism between $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n}(\mathrm{A_N}))$ and $\mathbb{P}\,\mathrm{Sph}_*(\mathrm{D_n}(\mathrm{A_N}))$.

Define the subgroup $G_n := \langle \Sigma, \Upsilon \rangle$ of $\mathrm{Aut}_*(\mathrm{D_n})$. Recall from Section 3.2 that this group contains $\mathrm{Tw}_{\mathrm{S}_1}$, $\mathrm{Tw}_{\mathrm{S}_2}$, and the shift functors $[k(n-2)]$, for $k \in \mathbb{Z}$.

Lemma 6.2.1. *There is an isomorphism*

$$\Phi_n \colon \mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\} \xrightarrow{\sim} \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})/\mathbb{P}G_n.$$

Proof. The isomorphism on the orbifolds is a consequence of the isomorphism between the groups $\mathbb{P}\,\mathrm{Aut}_*(\mathrm{D_n}) = \mathbb{P}G_n$. □

Theorem 6.2.2. *For $3 \leq n < \infty$, there is an isomorphism of complex manifolds*

$$\Psi_n \colon (\mathbb{C}_{(a)} \times \mathbb{C}^*_{(b)}) \setminus \Delta \xrightarrow{\sim} \mathrm{Stab}_*(\mathrm{D_n})/G_n.$$

Under this isomorphism, the central charge map corresponds to the twisted period mapping of the quantum cohomology of the projective line

$$(a,b) \mapsto \left(\int_{\xi_1} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\frac{n-2}{2}} \frac{dz}{z}, \int_{\xi_2} (\sqrt{b}z + 2a + \sqrt{b}z^{-1})^{\frac{n-2}{2}} \frac{dz}{z} \right)$$

where ξ_i are the paths on the spectral cover of $\mathbb{P}^1$ as defined in Example 5.2.3.

Proof. We want to construct an isomorphism Ψ_n which completes the commutative diagram

$$\begin{tikzcd}
(\mathbb{C}_{(a)} \times \mathbb{C}^*_{(b)}) \setminus \Delta \arrow[r, dashed, "\Psi_n"] \arrow[d, "\pi_1"] & \mathrm{Stab}_*(\mathrm{D_n})/G_n \arrow[d, "\pi_2"] \\
\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\} \arrow[r, "\Phi_n"] & \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})/\mathbb{P}G_n
\end{tikzcd} \tag{6.2.1}$$

in which Φ_n is the isomorphism of orbifolds constructed in Proposition 6.1.2, and the projections π_1 and π_2 are $\mathbb{C}^*$-bundles.

In Proposition 6.1.2, Φ_n is constructed using the biholomorphism between $\mathcal{H}$ and $U_n^+ \subset \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})$ which is determined by the ratio of the period integrals, c.f. Propositions 4.2.11 and 5.2.5. In Corollary 5.3.1, these period integrals are shown to be the twisted periods of quantum cohomology, and therefore their ratio defines a projective version of the twisted period mapping, Definition 2.3.1, from $\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\}$ to $\mathbb{P}\,\mathrm{Hom}(\mathrm{K}(\mathrm{D_n}), \mathbb{C})$. Hence under the isomorphism Φ_n, the ratio of the period integrals corresponds to the projective central charge map $\bar{\mathcal{Z}}(\bar{\sigma}) = \frac{Z_{\bar{\sigma}}(S_1)}{Z_{\bar{\sigma}}(S_2)}$ as defined in Definition 4.2.3, and the diagram

$$\begin{tikzcd}
\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\} \arrow[rr, "\Phi_n"] \arrow[dr, "{[Z_{\xi_1}:Z_{\xi_2}]}"'] & & \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})/\mathbb{P}G_n \arrow[dl, "\bar{\mathcal{Z}}"] \\
& \mathbb{P}\,\mathrm{Hom}(\mathrm{K}(\mathrm{D_n}), \mathbb{C}) &
\end{tikzcd} \tag{6.2.2}$$

commutes.

Combining diagrams 6.2.1 and 6.2.2, the isomorphism Ψ_n that we want to construct fits in to the commutative diagram

$$\begin{tikzcd}
(\mathbb{C}_{(a)} \times \mathbb{C}^*_{(b)}) \setminus \Delta \arrow[r, dashed, "\Psi_n"] \arrow[d, "\pi_1"] \arrow[rr, bend left, "{(Z_{\xi_1}, Z_{\xi_2})}"] & \mathrm{Stab}_*(\mathrm{D_n})/G_n \arrow[r, "Z"] \arrow[d, "\pi_2"] & \mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}(\mathrm{D_n}), \mathbb{C}) \setminus \{0\} \arrow[d, "\pi_3"] \\
\mathbb{P}(1,2) \setminus \{\Delta, \mu_\infty\} \arrow[r, "\Phi_n"] \arrow[rr, bend right, "{[Z_{\xi_1}:Z_{\xi_2}]}"'] & \mathbb{P}\,\mathrm{Stab}_*(\mathrm{D_n})/\mathbb{P}G_n \arrow[r, "\bar{\mathcal{Z}}"] & \mathbb{P}\,\mathrm{Hom}_{\mathbb{Z}}(\mathrm{K}(\mathrm{D_n}), \mathbb{C})
\end{tikzcd}$$

where $Z(\sigma) = (Z(S_1), Z(S_2))$ is the central charge map as introduced in Theorem 1.5.9, and the projection π_3 is a $\mathbb{C}^*$-bundle. Therefore an isomorphism Ψ_n of complex manifolds exists, and as the central charge map is determined by the simple objects in the category $\mathrm{D_n}$, it is determined uniquely by the lift of the map $[Z_{\xi_1} : Z_{\xi_2}]$ to (Z_{ξ_1}, Z_{ξ_2}). $\square$

The interest in this result lies not just in providing a description of the spaces of stability conditions of the Calabi–Yau-n categories of the Kronecker quiver, but in doing so by explicitly using the period integrals of the Frobenius structure of the quantum cohomology of $\mathbb{P}^1$.

We conclude by comparing this result to Conjecture 2.3.2.

Corollary 6.2.3. *The space of stability conditions of the Calabi–Yau-n category of the Kronecker quiver is the covering space of the big quantum cohomology of $\mathbb{P}^1$.*

Proof. This follows from the fact that the fundamental group of $(\mathbb{C}_{(a)} \times \mathbb{C}^*_{(b)}) \setminus \Delta$ is $G_n \cong \mathbb{Z} \star \mathbb{Z}/2\mathbb{Z}$. □

6.3 Stability conditions on D_∞

In order to apply this approach to the space of stability conditions of D_∞, it is necessary to have an equivalent definition of the biholomorphic map from the fundamental domain to the upper half plane as discussed in Remark 5.2.6. Given the role played in this biholomorphism by the twisted periods of quantum cohomology and the results in the A_2 case, we formulate the following conjecture for the space of stability conditions $\mathrm{Stab}_*(\mathrm{D}_\infty)$. In the case of the A_2 quiver, under this isomorphism the central charges correspond to the deformed flat coordinates of the Frobenius structure on the unfolding space of the A_2-singularity. For the space of stability conditions of the category $\mathrm{D}_\infty(\mathrm{K}_2)$, the central charge map should correspond to the map given by the ratio of the integrals

$$\int_{\xi_i} e^{\sqrt{b}z+2a+\sqrt{b}z^{-1})}\frac{dz}{z}$$

over ξ_1 and ξ_2. Note in particular that the ratio of these integrals depends only on the parameter b.

Conjecture 6.3.1. *There is an isomorphism of complex manifolds*

$$\Psi_\infty \colon \mathbb{C}^2_{(a,b)} \xrightarrow{\sim} \mathrm{Stab}_*(\mathrm{D}_\infty)$$

under which the central charge map corresponds to the map

$$(a,b) \mapsto \left(\int_{\xi_1} e^{(\sqrt{b}z+2a+\sqrt{b}z^{-1})}\frac{dz}{z}, \int_{\xi_2} e^{(\sqrt{b}z+2a+\sqrt{b}z^{-1})}\frac{dz}{z}\right).$$

That the space of stability conditions of D_∞ is $\mathbb{C}^2$ as a complex manifold is known [37]. Therefore as in the case of the Calabi–Yau-n categories, the importance of this conjecture is that an isomorphism between these manifolds can be constructed under which the central charges of stability conditions on $\mathrm{D}_\infty \cong \mathrm{D}^{\mathrm{b}}(\mathbb{P}^1)$ are given by the deformed flat coordinates of the quantum cohomology of $\mathbb{P}^1$ with deformation parameter one.

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