Games on partial orders and other relational structures

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The candidate confirms that the work submitted is their own and that appropriate credit has been given where reference has been made to the work of others.

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Abstract

This thesis makes a contribution to the classification of certain specific relational structures under the relation of n-equivalence, where this means that Player II has a winning strategy in the n-move Ehrenfeucht-Fraïssé game played on the two structures. This provides a finer classification of structures than elementary equivalence, since two structures A and B are elementarily equivalent if and only if they are n-equivalent for all n. On each move of such a game, Player I picks a member of either A or B, and Player II responds with a member of the other structure. Player II wins the game if the map thereby produced from a substructure of A to a substructure of B is an isomorphism of induced substructures.

Certain ordered structures have been studied from this point of view in papers by Mostowski and Tarski, for ordinals [22], and Mwesigye and Truss, for ordinals [25], some scattered orders, and finite coloured linear orders [24]. Here we extend the known results on linear orders by classifying them all up to 3-equivalence (which had previously been done for 2-equivalence), of which there are 281, using the method of *characters*.

We also classify all partial orders up to 2-equivalence (there are 39), and discuss the difficulties of extending this to 3-equivalence, since the method of characters is not as effective as in the linear case. We classify (total) circular orders up to 3-equivalence, and relate the classification of partial circular orders to both these and to partial orders. A variety of related structures are discussed: trees, directed and undirected graphs, and unars (sets with a single unary function), which we categorise up to 2-equivalence.

In a *pebble game*, the players of an otherwise standard Ehrenfeucht-Fraïssé game are in addition provided with two identical sets of k distinguishable pebbles, and on each move they place a pebble on their chosen point. On each move, Player I may choose either to move a pebble to another point, or else use a new pebble, if any remain, and Player II must place the corresponding partner pebble. Such games correspond to logics in which there are only k variables, and moving a pebble corresponds to reusing the variable. Here we extend some work of Immerman and Kozen [14] on pebble games played on linear orders.

Contents

Chapter 1

Introduction

There are many kinds of games used throughout mathematical logic. Typically, two players take turns to select elements known as *moves* from some class of potential moves, and, after a predetermined finite number or infinite ordinal number of turns, one of the players wins and the other loses according to the preselected win conditions (which preclude a draw). The winner depends only on the moves chosen by each player, and not on any external elements like dice rolls or card draws, but in some games such as von Neumann games [29] we do allow the players to choose their moves probabilistically. We typically assume that the game has *perfect information*, which means that the full state of the game is known to both players, including the win conditions, the domain of possible moves, and the set of moves played up until the current moment in time. Games with *imperfect information*, such as Blackwell games [3] [19], have also been studied, where the players are unaware of some aspect of the state of the game.

Unlike many real world games, games in a mathematical sense are typically not played repeatedly for fun, but rather analysed to understand the strategies that either player can employ, and to determine under which circumstances either player can guarantee a win. Situations in which a player does, or does not, have a winning strategy correspond to certain properties of the structures on which the games are played, thanks to our choice of game, and the study of the games aims to elucidate these properties.

Since we require there to be only one winner, it is not possible for both players to simultaneously have a winning strategy, but it is not in general necessary that either of the players does. In fact, if we assume the Axiom of Choice, then there are games for which neither player has a winning strategy; we call these games *undetermined*. An alternative well known set theoretic notion, the Axiom of Determinacy [28], states that in every perfect information game on ω^ω, one of the players has a winning strategy (and so every such game is *determined*). This axiom is independent of ZF, though in view of the above remarks, it is necessarily incompatible with the Axiom of Choice. There are numerous similar determinacy axioms identifying classes of infinitary game for which the axiom asserts that there is always a winning strategy for some player, and these are also independent of ZF and often relate to various large cardinal notions. However, Gale and Stewart proved in 1953 that all games on ω^ω whose payoff sets (i.e. the set of sequences for which Player I wins) are either open or closed must be determined [8], and in 1975 Donald A. Martin strengthened this result to show that all Borel games are determined [18].

In games of finite length, all payoff sets must be both open and closed, and so one player or other must have a winning strategy. We may also prove this directly by backwards induction from the end of the game, as we show later in Lemma 3 for Ehrenfeucht-Fraïssé games. Note that some mathematical games have infinitely many moves, and others have finitely many. The game state is typically changed by every move, which ought to depend on the set of previous moves. If at some point in time there were no "next move" but an infinite descending sequence of upcoming moves, it is unclear how we could determine which moves are wise or even possible, since the state of the game at each move depends on all the previous moves! Even in a game of imperfect information, where each player may be unaware of the other's moves, each player would usually still be expected to recall their own previous moves. We therefore desire the moves of a game to be well-ordered, and label the pairs of moves by some ordinal α. If α is the order type of the pairs of moves of some game G, we say that G has *length* α. In practice, most work on infinite games uses games of length ω, such as the model-theoretic constructions in [12]. In a game of longer length $\alpha > \omega$, there exist moves (such as the ωth move) that occur after infinitely many previous moves, but in games of length ω, every move occurs at some finite index, and therefore depends on only finitely many predecessors. The games we will cover, Ehrenfeucht-Fraïssé games (and a variant thereof), have finitely many moves and must therefore be determined.

In 1944, John von Neumann and Oskar Morgenstern published their seminal book *Theory of Games and Economic Behavior* [29], establishing the field of game theory, which was studied extensively in the 1940s and 1950s. The analogy of games was used in a logical context in the 1950s by Leon Henkin, who was working on Tarski's notions of logical truth. First order logic permits only finitely many quantifiers, but we can conceive of formulae with infinitely many quantifier alternations, which correspond to meaningful statements that we would like to be able to represent. Henkin suggested using the analogy of a game between two players, both trying to win, who select elements for the universal or existential quantifiers respectively. Jaakko Hintikka [11] developed these ideas further, and described semantic games on first-order sentences such that the existential player has a winning strategy if and only if the sentence is true. This is closely related to the notion of games that we will be using.

In this thesis we use Ehrenfeucht-Fraïssé games, and a variant thereof, to classify partial orders, linear orders, and other relational structures.

Ehrenfeucht–Fraïssé games were formulated by Andrzej Ehrenfeucht [5] as a way of capturing the essence of Roland Fraïssé's application of back-and-forth methods in model theory [7]. In these games, two players play for a (fixed) finite number of moves on two structures, choosing a point in one of the structures at each move, and the second player always plays in the opposite structure to Player I's previous move. Player II is said to *win* a play of the game if the substructures induced by the points played in the two structures are isomorphic; otherwise, Player I wins. A *strategy* for a player is a rule telling her what to play next given the moves so far. It is a *winning strategy* if it results in a win for that player no matter which moves the other player makes. There is a natural notion of the *quantifier depth* of a formula (given in the next chapter), and one can verify that Player II has a winning strategy in a k-move game if and only if the two structures satisfy the same sentences of quantifier depth at most k. We deduce that if Player II has a winning strategy in the k-move game for every value of k, then the two structures are elementarily equivalent. On the other hand, if there is some k such that Player II does not have a winning strategy for a game of length k, then there must be a sentence of quantifier depth k that is true in one of the structures but false in the other.

If there exists a strategy for Player II such that the substructures induced by the points played in any game of length n in which she uses this strategy are isomorphic, then the two structures satisfy the same sentences of quantifier depth n. Therefore, Player II has winning strategies for games of all lengths if and only if the two structures are elementarily equivalent. On the other hand, if there exists k such that Player I has a winning strategy in the game of length k on two structures, then there is some formula ϕ of quantifier depth k such that one structure satisfies ϕ and the other $\neg\phi$.

We remark on two other well-known notions of "game" in the literature. The first of these is *Banach-Mazur* games, which are games of length ω first defined by Stanisław Mazur in the *Scottish Book* [20], a collaborative work between the mathematicians of Lwów. In these games, a subset $X \subseteq \mathbb{R}$ of the set of real numbers is given in advance, and the players play a sequence of non-trivial closed intervals, each a proper subset of the previous interval. If the intersection of this sequence is contained in X, then Player I wins; otherwise, Player II wins. This type of game is used in the analysis of a particular set-theoretical property of sets of reals, called the *Baire property*, since it can be easily shown that Player I has a winning strategy if and only if X is comeagre on some non-trivial interval, and Player II has a winning strategy if and only if X is meagre. From Banach-Mazur games originated other, related games corresponding to other topological notions, and games in this class are known as *topological games* [35].

The other rather all-embracing notion of "game" is given by Conway in his book *On numbers and games* [4] where he gives a very elegant and powerful definition of the two notions in the title, which are rather surprisingly closely related. There are many particular actual "games" in the colloquial sense which can be used to illustrate this definition, for instance, various versions of the game of Nim. While the games that we consider must necessarily come under his broad heading, they are much more specific to particular purposes (i.e. study of the finer logical properties of particular structures), so his study is not of direct relevance here.

The contents of the chapters are as follows. Chapter 2 introduces some preliminary notions in model theory and order theory. In Chapter 3 we survey some results on linear orders and classify them up to 3-equivalence. Chapter 4 contains a classification of the 2-equivalence classes of partial orders and various results illustrating the differences between the linear and partial cases. Chapter 5 deals with cyclic orders and the related notion of partial cyclic orders. Chapter 6 contains multiple short sections considering trees, graphs, directed graphs and unars. In Chapter 7 we consider a modified type of Ehrenfeucht-Fraïssé game, known as a *pebble game*, where previously played “pebbles” are re-used and their previous locations discarded. We shall finally conclude with a summary of our results and suggestions of directions for future work. A more detailed summary of the contents of each chapter follows.

In Chapter 2 we introduce some preliminary notions in model theory and order theory. The main definitions concern Ehrenfeucht-Fraïssé games, and the corresponding notion of n-equivalence, written $\equiv_n$. This gives a finer classification than elementary equivalence, since two structures are n-equivalent if and only if they satisfy the same sentences of quantifier depth at most n. It follows that if the language is finite and relational, then there are only finitely many $\equiv_n$-classes. One can therefore choose a representative of each class, often in a natural way such as one of minimal size or least order type, though an arbitrary choice can also be made.

The main result of Chapter 3 is a classification of all $\equiv_3$-classes of linear orders, of which there are 281. The ideas and methods are however of more interest than the actual number. The $\equiv_2$-classes of linear orders were given in [24] and are represented by $0, 1, 2, 3, \omega, \omega^*$ and $\mathbb{Z}$. The main tool used here is that of the *character* of a point. If we are seeking to pass from knowledge of $\equiv_n$-classes to $\equiv_{n+1}$-classes, we first assume that representatives have been chosen for the $\equiv_n$-classes, and write $[X]$ for the representative of X. Then the *n-character* of $a \in X$ is defined to be the ordered pair $([X^{<a}], [X^{>a}])$. The idea is that if a is played by one of the players in X, then the n-character keeps track of how that player will be able to play either to the left or right of a in subsequent moves. An easy lemma from [24] asserts that two ordered structures are $(n+1)$-equivalent if and only if they exhibit precisely the same n-characters. This means that, for instance, one of the main tasks in analysing linear orders up to 3-equivalence is to study sets of 2-characters, which can be taken to be ordered pairs of members of $\{0, 1, 2, 3, \omega, \omega^*, \mathbb{Z}\}$.

In Chapter 4 we seek to extend results from the linear case to general partial orders. As usual, the one move case is trivial, and games of length two are also straightforward, as an analogue of the linear order notion of the 2-character of a point a works quite smoothly. Here we have to identify three subsets A, B, C of the partial order, containing the points above, below, and incomparable with a respectively. Clearly $A > B$, $C \not\leq B$ and $C \not\geq A$. In two moves, the players have no time to exploit any relations between these sets, so two partial orders are 2-equivalent if and only if they exhibit the same 2-characters, as we show. This leads to a classification of all partial orders up to 2-equivalence - there are 39 classes. For 3-equivalence, things are considerably more complicated, and this is illustrated by counter-examples and discussions of uses of colours.

Chapter 5 concerns a different type of relation connected with orderings, namely cyclic orders and partial cyclic orders. We deduce a classification of the (total) cyclic orders up to 3-equivalence, which is related to a part of the classification of linear orders up to 2-equivalence, since we can obtain a linear order from a circular one by cutting it at a point. We then discuss partial cyclic orders, and show that some equivalence classes of these relate to cyclic orders and to partial orders, but others do not.

In Chapter 6 we consider a few different kinds of interesting relational structures. The first is trees. Although this is a special case of a partial order, the fact that it is simpler gives some hope of classifications for higher numbers of moves. In particular, we conjecture that two trees exhibiting the same $(n-1)$-characters should be n-equivalent, unlike for general partial orders. We prove this conjecture for a special case for $n = 3$, and also classify the trees up to 2-equivalence, which allows us to calculate a loose upper bound on the number of 3-equivalence classes. In this chapter we also make remarks about digraphs, and two particular special cases, namely undirected graphs and unars. By definition, a unar consists of a set together with a single unary function acting on the set. It therefore forms a dynamical system, and some previous work has studied Ehrenfeucht-Fraïssé games on unars from that point of view. Our principal contribution here is to classify unars up to 2-equivalence, of which there are 133 classes. Note that as the signature of a unar is not relational, we recast it as a binary relation, so that it may be viewed as a special kind of digraph. For notational ease, however, the function is still used, but it lacks official significance when it comes to equivalence in Ehrenfeucht-Fraïssé games.

The final chapter, Chapter 7, takes quite another direction, with a variant game known as a "pebble game". These correspond to logics in which there is a finite bound k on the number of variables used, in addition to the usual, unrelated, finite bound n on the number of moves. This affects the definition of the game by equipping each player with k pebbles which are deployed during the game to keep track of variable use. The classical result is that 3 pebbles suffice to distinguish linear orders up to elementary equivalence, and in no more moves than would be required in a standard Ehrenfeucht-Fraïssé game. We extend this by characterising the circumstances in which Player II wins a 2-pebble game, for any number n of moves. This result illustrates that the addition of the pebbles radically alters the players' strategies, and that a dearth of pebbles constricts the number of equivalence classes significantly - there are only quadratically many $(n, 2)$-equivalence classes of linear orders, but exponentially many n-equivalence classes of linear orders when there is no pebble restriction.

Chapter 2

Preliminaries

We begin by giving standard notation and introductory concepts in model theory and order theory. Readers familiar with these may wish to proceed to Chapter 3.

2.1 Notation

We list standard notation that we shall use throughout. Many of the notions mentioned will be formally introduced in what follows.

- $G_n(A, B)$, the Ehrenfeucht-Fraïssé game of length $n \in \mathbb{N}$ played on the structures A and B (which may be of any size)
- $G_n^k(A, B)$, the pebble game of length $n \in \mathbb{N}$ with $k \in \mathbb{N}$ pebbles played on the structures A and B (which may be of any size)
- $a^<$, where a is a member of some structure $(A, <)$, the set of points $b \in A$ such that $b < a$
- $a^>$, for $a \in A$, the set of points $b \in A$ such that $b > a$
- $a^\leq$, for $a \in A$, the set of points $b \in A$ such that $b \leq a$
- $a^\geq$, for $a \in A$, the set of points $b \in A$ such that $b \geq a$
- "tree", unless otherwise specified, a partial order P such that for all $a \in P$, $a^<$ is linearly ordered and such that for any $a, b \in P$ there exists $c \in P$ with $c \leq a, b$.

- α, an ordinal, or a set of that order type
- α^*, where α is an ordinal, a set of the reverse order type
- F, a colouring function from the set being coloured to some set of "colours"
- $[X]$, the canonical representative of the equivalence class of X under some equivalence relation
- $\bigsqcup_{i \in I} A_i = \bigcup_{i \in I} \{(x, i) : x \in A_i\}$, the disjoint union of the sets (A_i)
- $A \sqcup B$, the disjoint union of two sets, as defined above
- $\cong$, an isomorphism relation with $A \cong B$ if and only if there exists an isomorphism $\phi : A \to B$
- $*$, a placeholder entry in a component of a character to indicate that any of the possible values may hold in that component

2.2 Order theory

A *partial order* $(X, \leq)$ is a set X equipped with a relation $\leq$ such that:

- for all $a \in X$, $a \leq a$ (reflexivity)
- for all $a, b \in X$, if $a \leq b$ and $b \leq a$ then $a = b$ (antisymmetry)
- for all $a, b, c \in X$, if $a \leq b$ and $b \leq c$ then $a \leq c$ (transitivity)

We may define $<$ so that $a < b$ precisely when $a \leq b$ and $a \neq b$. Then $(X, <)$ is a *strict* partial order, while $(X, \leq)$ is a *non-strict* partial order. Since our language includes equality, each strict partial order naturally gives rise to a non-strict partial order, and vice versa, and the two formulations have equivalent properties. We may therefore use both $<$ and $\leq$, and we also define $>$ and $\geq$ in the obvious way: let $a > b$ if and only if $b < a$, and let $a \geq b$ if and only if $b \leq a$.

A *linear order* is a partial order $(X, \leq)$ that additionally satisfies the axiom of totality: for any $a, b \in X$, either $a \leq b$ or $b \leq a$. As with partial orders, we can have both strict and non-strict linear orders.

A linear order X is *dense* if whenever $x < y$, there exists some z such that $x < z < y$. If this property holds on some interval, we may say X is dense on that interval. If there exist x, y such that $x < y$ and there is no point z lying between them, then we say that x and y are *consecutive*.

If $(A, <_A)$ and $(B, <_B)$ are linear orders, then we may define the *concatenation* $A + B$ as follows: we take the underlying set to be $A \sqcup B$, the disjoint union of A and B, and we define the new relation $<$ as follows:

- for any $a \in A$ and $b \in B$, $a < b$
- for $a_1, a_2 \in A$, $a_1 < a_2$ if and only if $a_1 <_A a_2$
- for $b_1, b_2 \in B$, $b_1 < b_2$ if and only if $b_1 <_B b_2$

Lemma 1. *If A and B are linear orders, then $A + B$ is a linear order.*

Proof. We verify that $A + B$ satisfies the axioms of reflexivity, antisymmetry and transitivity given above. It is reflexive (when considered as a non-strict linear order), since $a \leq_A a$ for $a \in A$ and $b \leq_B b$ for $b \in B$ so certainly $x \leq x$ for all $x \in A + B$. For antisymmetry, take $x, y \in A + B$. If they both lie in A, or both in B, then we are done, as we know A and B to be antisymmetric. Otherwise, one point lies in each, suppose without loss of generality that $x \in A$ and $y \in B$, and we have $x < y$ but not $y < x$. In either case antisymmetry holds. For transitivity, suppose that $x, y, z \in A + B$, and suppose both $x < y$ and $y < z$. Again, if they are all in A, or all in B, we are done, so suppose not. B is upward closed, so if $x \in B$ then $y \in B$, and if $y \in B$ then $z \in B$. Therefore we must certainly have $z \in B$, and likewise we must have $x \in A$ if any of the three points are to lie in A. But everything in B lies above everything in A, so $x < z$. Finally, we verify totality. If $x, y \in A + B$ then either $x, y \in A$, which is total, or $x, y \in B$, which is total, or one lies in each. Again, we are done in the first two cases, so we consider the third, and without loss of generality take $x \in A$ and $y \in B$. But then $x < y$ is immediate and we are done. $\square$

If $(A, <_A)$ and $(B, <_B)$ are linear orders, then we may also define the *lexicographic product* $A \times B$ as follows. The underlying set is the Cartesian product $A \times B = \{(a, b) : a \in A \text{ and } b \in B\}$, and we take $(a_1, b_1) < (a_2, b_2)$ if and only if either $a_1 < a_2$ (regardless of b_1 and b_2), or both $a_1 = a_2$ and $b_1 < b_2$. We may think of this as taking A copies of B, since for any $a \in A$ the set $\{a\} \times B = \{(a, b) : b \in B\}$ has the same order type as B, and these copies are arranged in the order of the elements of A, with each copy lying either entirely above or entirely below any other distinct copy. For example, if A is a two-point linear order, then $A \times B = B + B$ for any linear order B. (Please note that some authors use the *anti-lexicographic* ordering, which is notated the other way around.)

Lemma 2. *If A and B are linear orders, then $A \times B$ is also a linear order.*

Proof. $A \times B$ is irreflexive as a strict linear order, since $(a, b) < (a, b)$ is false for any $(a, b) \in A \times B$. If $(a_1, b_1), (a_2, b_2)$ lie in $A \times B$, then by the above definition we can only have both $(a_1, b_1) < (a_2, b_2)$ and $(a_2, b_2) < (a_1, b_1)$ if either $a_1 < a_2$ and $a_2 < a_1$ both hold, which is impossible by antisymmetry of A, or if $a_1 = a_2$ and $b_1 < b_2 < b_1$, which is also impossible by antisymmetry of B. But then $(a_1, b_1) = (a_2, b_2)$ and we have antisymmetry. For transitivity, suppose that $(a_1, b_1) < (a_2, b_2)$ and $(a_2, b_2) < (a_3, b_3)$. Then $a_1 \leq a_2 \leq a_3$, so $a_1 \leq a_3$ by transitivity of A. If $a_1 < a_3$, then immediately $(a_1, b_1) < (a_3, b_3)$, so suppose $a_1 = a_3$. Then also $a_2 = a_1 = a_3$, so from the definition of lexicographic product it must be the case that $b_1 < b_2 < b_3$. Therefore $(a_1, b_1) < (a_3, b_3)$, since $a_1 = a_3$ and $b_1 < b_3$. Finally, we verify that this is total by noting that there are only nine possible cases for the relations between (a_1, b_1) and (a_2, b_2): either $a_1 < a_2$, $a_1 = a_2$ or $a_2 < a_1$, by linearity of A, and a similar trichotomy holds for b_1 and b_2. The above definition sets either $(a_1, b_1) < (a_2, b_2)$ or $(a_2, b_2) < (a_1, b_1)$ (or both, in the case where they are equal) for each of these nine possibilities, and therefore $<$ is total on $A \times B$. $\square$

A *colouring function* on a set X is a function $F : X \to C$, where C is some set, typically finite and small, whose elements are known as *colours*. As the name suggests, we can envision the values of C as colours applied to the relevant points of X, and we will often take $C = \{red, blue\}$ or $C = \{red, blue, green\}$, or simply $C = \{c_i : 1 \leq i \leq n\}$ for larger n. If the names of colours are chosen so as to encode relevant information, or if X is recoloured multiple times without losing the previous information, we may instead call F a *labelling function*.

2.3 Ehrenfeucht-Fraïssé games

The definitions and results of this section are well-known and may be found in standard model theory texts such as those by Hodges [13] and Marker [17].

An Ehrenfeucht-Fraïssé game is a logical game of finite length used as a model theoretic technique for comparing two relational structures [5]. We think of Ehrenfeucht-Fraïssé games as being played by two players, who take turns to play and who both wish to win the game. The number of moves is fixed in advance. The player to move first, Player I, wins if she can demonstrate that the two structures are different, in a sense to be made precise shortly, and the player who moves second, Player II, wins if she can prevent this.

The n move game played on the structures A and B is notated $G_n(A, B)$, and proceeds as follows:

- Player I makes a move by selecting any member of either A or B. If she selects an element of A, call this a_1; if she selects an element of B, this is b_1.
- Player II responds by selecting a member of the other structure, giving a pair $(a_1, b_1) \in A \times B$.
- This repeats $n - 1$ more times, for a total of n moves each.

Note that Player I is free to select from either A or B at any move, regardless of whether she previously played in A, B or both. She does not need to choose elements from the same structure every time.

When all n turns have been played, we have $(a_1, \ldots, a_n)$ from A and $(b_1, \ldots, b_n)$ from B, where a_i and b_i were selected on the ith move.

Player II wins if the map taking a_i to b_i for each i, and preserving the interpretations of any constant symbols, is an isomorphism of substructures. Otherwise, Player I wins.

A *strategy* for a player is a function from sequences of moves that may have already been played before their nth turn, to a possible nth move for that player. Player I's nth move may depend on her $(n-1)$th and previous moves and on Player II's $(n-1)$th and previous moves, while Player II's nth move may depend on Player I's nth and previous moves and on her own $(n-1)$th and previous moves. Since Player I always moves on odd turns, a strategy for her is a function $\sigma : \bigcup_{i=1}^{n}(A \times B)^{i-1} \to A \cup B$, while a strategy τ for Player II is a function $\tau : \bigcup_{i=1}^{n}(A \times B)^{i-1} \times (A \cup B) \to A \cup B$, subject to the restriction that $\tau(x) \in B$ if the last element of x lies in A, and $\tau(x) \in A$ if the last element of x lies in B (otherwise the move would not be valid).

A *winning strategy* for Player I is a strategy σ such that, for every strategy τ for Player II, Player I wins the game in which her moves are selected according to σ and Player II's are selected according to τ. Likewise, a winning strategy for Player II is one for which she is guaranteed to win if she plays according to it, regardless of the strategy employed by Player I.

Lemma 3. *In any Ehrenfeucht-Fraïssé game $G_n(A, B)$, either Player I has a winning strategy or Player II has a winning strategy (but not both). [5]*

Proof. Since their victory conditions are negations of each other, every completed game must be won by precisely one of Player I and Player II. Player I and Player II cannot both have winning strategies, since they could each play their winning strategy and thereby both win the same game, which is by assumption impossible.

We show that one of the players must have a winning strategy, by induction on n. In the 1-move game, either Player I has some move x for which every response by Player II results in a Player I win, or this is not the case, and every move x of Player I has some response y_x that could be played to give a Player II win. In the former case, playing x is a winning strategy for Player I; in the latter case, $\tau : x \mapsto y_x$ is a winning strategy for Player II.

Suppose now that games of length at most k are determined, and consider the game of length $(k+1)$. After the first moves x and y, the remaining subgame is of length k, so some player has a winning strategy. Possibly there exists some x such that if Player I plays it for her first move, then for any response y from Player II, Player I has a winning strategy σ in the remaining subgame of length k. If so, then playing x for the first move and then following σ on subsequent moves is a winning strategy for Player I. If not, then for every x that Player I may play as a first move, there exists some response y_x such that Player I does not have a winning strategy on the remaining subgame of length k. By the induction hypothesis, Player II therefore has a winning strategy τ on the remaining subgame. So playing y_x for her first move and then following τ is a winning strategy for Player II. In either case we have constructed a winning strategy for one of the players, so by induction, $G_n(A, B)$ is determined for any finite n. □

Informally, if Player I or Player II has a winning strategy in a given game, we may say she "wins" or "can win" that game. Since the Ehrenfeucht-Fraïssé games are really processes with which to compare structures, we are at all times concerned with whether there exist possible subsequent states in which either player has won, rather than envisioning the actions an actual person would take. We need not concern ourselves with human issues such as making mistakes, forgetting the strategy, intentionally letting the other player win, failing to specify a point clearly, having imperfect recall of all the preceding moves, and so on, and so we can relax the usual linguistic distinction between having the ability to win, and a way to win existing.

If Player II has a winning strategy in $G_n(A, B)$, we say that A and B are *n-equivalent*, and write $A \equiv_n B$.

Theorem 4. *As the notation suggests, $\equiv_n$ is an equivalence relation. [17]*

Proof. Consider $G_n(A, A)$, the Ehrenfeucht-Fraïssé game of length n played on two copies of A. Then of course there is an isomorphism ϕ between the two copies of A. Player II may adopt the strategy that if Player I plays a in the first copy of A, then Player II responds by playing $\phi(a)$ in the second copy of A, and likewise if Player I plays b in the second copy of A, then Player II responds by playing $\phi^{-1}(b)$. Since ϕ is an isomorphism, the resulting substructures will always be isomorphic, and so $A \equiv_n A$. This is therefore a winning strategy for Player II, and so $\equiv_n$ is reflexive.

Now consider $G_n(A, B)$ and $G_n(B, A)$. These are the Ehrenfeucht-Fraïssé games of length n played on A and B and on B and A, respectively. But in either game, Player I is free to play in either A or B on each move, and Player II must respond with a move in B or A respectively. So, if $A \equiv_n B$, then Player I has a winning strategy in $G_n(A, B)$, which is also a winning strategy in $G_n(B, A)$, since the movesets available to her and her opponent are identical, and so $B \equiv_n A$.

For transitivity, suppose that $X \equiv_n Y$ and $Y \equiv_n Z$, and consider $G_n(X, Z)$. If Player I makes a move x_1 in X, Player II can think of a "move" $\sigma(x_1) = y_1$ in Y consistent with her winning strategy σ in the auxiliary game $G_n(X, Y)$, and then find a move $\tau(y_1) = z_1$ in Z consistent with her winning strategy τ in $G_n(Y, Z)$. If Player I plays $z_1 \in Z$, then Player II chains her auxiliary strategies in the opposite direction and plays $x_1 = \sigma(\tau(z_1))$. When the game ends, both $(x_1, ..., x_n)$ and $(z_1, ..., z_n)$ will be isomorphic to $(y_1, ..., y_n)$, so must be isomorphic to each other, and so $X \equiv_n Z$. $\square$

As we shall see, n-equivalence is closely linked to the notion of *quantifier depth* $d(\phi)$ of a formula ϕ, which we define inductively as follows:

- if ϕ is atomic, then $d(\phi) = 0$
- $d(\neg\phi) = d(\phi)$
- $d(\phi \vee \psi) = d(\phi \wedge \psi) = d(\phi \to \psi) = max(d(\phi), d(\psi))$
- $d(\forall x\phi) = d(\exists x\phi) = d(\phi) + 1$

Note that this is not necessarily the same as the number of quantifiers in the prenex normal form of a formula. For example, $\phi := \exists x(\forall y(x < y) \vee \forall y(x > y))$ has quantifier depth two in our sense, but its prenex form is $\phi_1 := \exists x\forall y\forall z((x < y) \vee (x > z))$, which requires three quantifiers. Note that ϕ is *not* logically equivalent to the similar prenex formula of quantifier depth two $\psi := \exists x\forall y((x < y) \vee (x > y))$, since ψ holds in any linear order of size at least two, not just those with an endpoint.

A *language* is the set of symbols that may appear in our formulae. This may be of any size, but finite or countable languages are often used, as these suffice to describe many classes of structures. A language may contain symbols that represent functions, which are known as *function symbols*. A language that does not contain function symbols is known as a *relational* language, so called because it only has relation symbols (as well as constants, variables, connectives and quantifiers). We remark that it is sometimes possible to use the symbols of a relational language to formally define something that we think of informally as being a function (such as a colouring function, or the unary function of a unar in Section 6.4). In these cases the language is still a relational language, as it does not contain function symbols.

Lemma 5. *In a finite language with no function symbols, A and B are n-equivalent if and only if they satisfy the same formulae of quantifier depth at most n. [17]*

Proof. We prove both directions by an induction on the quantifier depth n. Within that, for each fixed quantifier depth we show that we may reduce to considering only sentences that are not a propositional combination of shorter sentences, since given the result for these we can use propositional connectives to extend the result to all other sentences of the same quantifier depth. We begin with the forward direction.

Suppose that ϕ is a sentence of quantifier depth n which is true in A but not in B (without loss of generality). We show that Player I has a winning strategy in the n-move game $G_n(A, B)$.

First, we reduce to the case in which ϕ is not a propositional combination of shorter sentences. It suffices to give a reduction for the cases where ϕ contains $\vee$ or $\neg$, since the other propositional connectives $\wedge$ and $\rightarrow$ are expressible in terms of these.

Suppose first that ϕ has the form $\psi \vee \chi$. Then either ψ or χ is true in A, but both are false in B. Let us assume without loss of generality that ψ is true in A. Since ψ is shorter than ϕ, true in A and false in B, we may use ψ to obtain a winning strategy for Player I.

Now suppose that $\phi = \neg\psi$. Then ψ is shorter than ϕ, and ψ is true in B but not in A (which we may relabel without loss of generality), so again the induction hypothesis gives a winning strategy for Player I.

By repeatedly applying the above two reductions we may therefore find, given that there exists some sentence of quantifier depth n satisfied by A and not by B, a sentence ϕ which is not a propositional combination of shorter sentences, has quantifier depth at most n, and is satisfied by A but not by B.

We now show by induction on n that Player I has a winning strategy. Suppose that ϕ is not a propositional combination of shorter sentences. In the base case $n = 0$, ϕ is atomic, so it has the form $R(c_1, \ldots, c_k)$, where R is a relation symbol and $c_1, \ldots, c_k$ are constant symbols. Then $R(c_1, \ldots, c_k)$ is true in A but false in B, so the structures on the interpretation $a_1, \ldots a_k$ of $c_1, \ldots, c_k$ in A and $b_1, \ldots, b_k$ of $c_1, \ldots, c_k$ in B are not isomorphic, and so Player I wins.

For the induction step, we again only need to consider one of the quantifiers, since $\forall x\varphi \leftrightarrow \neg\exists x\neg\varphi$. We therefore suppose without loss of generality that ϕ is $\exists x\psi(x)$.

Since ϕ is true in A, there must be some $a \in A$ for which $\psi(a)$ is true in A, but no $b \in B$ such that $\psi(b)$ is true in B. We augment the language with one more constant symbol c, which is interpreted in A as a. Let Player I play a on her first move. Then whichever $b \in B$ Player II plays in response, $\psi(b)$ will be false in B, so $\psi(c)$ is a sentence of quantifier depth at most $n - 1$ which is true in (A, a) but false in (B, b). Therefore, by the induction hypothesis, Player I has a winning strategy for the remaining $n - 1$ moves, completing a winning strategy in the whole game.

We now prove the other direction of the lemma: that A and B are n-equivalent if they do satisfy the same sentences of quantifier depth at most n.

For the base case $n = 0$, suppose that A and B satisfy the same quantifier-free sentences. Then they also satisfy the same atomic sentences, and so Player II wins $G_0(A, B)$ (without playing any moves).

Now suppose that A and B satisfy the same sentences of quantifier depth $n > 0$. We show that Player II has a winning strategy in $G_n(A, B)$. Suppose that Player I plays $a \in A$ on her first move. The language is finite, and remains so after one constant symbol denoting a is adjoined, so by a similar inductive analysis of formulae, we see that there are only finitely many sentences of the language of quantifier depth at most $n - 1$, up to relabelling and contractions. We may therefore form the conjunction of all the finitely many sentences (up to logical equivalence) satisfied by a in A. We write this conjunction as $\phi(c)$, where c is the constant symbol interpreted as a in A, and we note that ϕ also has quantifier depth at most $n-1$. Then $\exists x \phi(x)$ is certainly true in A, so it must also be true in B, and so Player II can choose a witness b to its truth in B. By construction, (A, a) and (B, b) satisfy the same sentences of quantifier depth at most $n-1$, so by the induction hypothesis, Player II has a winning strategy σ for the remaining $n-1$ moves. Playing b on her first move and then following σ therefore constitutes a winning strategy for Player II in the whole game $G_n(A, B)$.

□

Therefore, since every formula in our language has finite quantifier depth:

Corollary 6. *A and B are elementarily equivalent if and only if Player II wins $G_n(A, B)$ for all n.*

While infinitary versions of this game are sometimes used, as mentioned in the introduction, here we always take n to be a natural number. Note that the length of the game is chosen in advance, and that this is an important distinction. There exist structures where Player I can make a move such that, for any response by Player II, there exists a finite number n such that Player I can win if they continue playing for a total of n moves. This is *not* sufficient to show non-equivalence; n must be declared in advance, and Player II may use her knowledge of n to select a move such that Player I would only eventually win after $n+1$ moves or more. Then Player II would be the one with a winning strategy and the structures in question would be n-equivalent.

For example, consider $G_n(A, B)$ where $A = \omega + \mathbb{Z}$ and $B = \omega$. For her first move, Player I can play an element a of $\mathbb{Z} \subset A$, and Player II's response b must lie in B, and so must be at some finite index measured from the start of ω. If Player I were granted arbitrarily many moves from this situation, she could easily use the fact that b is at some finite index from the beginning, and a is not, to win the game. However, since Player II is aware prior to her move of the length of the game, she can pick b sufficiently large ($b > 2^n$ suffices, where n is the number of moves), such that Player I will not be able to distinguish the large but finite number of points in $B^{<b}$ from the infinite fragment of $A^{<a}$

in the $n-1$ remaining moves of the game. We therefore have $\omega + \mathbb{Z} \equiv_n \omega$ for any n, even though every point of ω has some finite index and none of the points in $\mathbb{Z} \subset A$ do. This corresponds to the fact that "not every point of A has a finite index" is not expressible as a sentence of first order logic, but "there is some point of A with index greater than n" is, for any particular finite value of n. Finite conjunctions, but not infinite conjunctions, of this are expressible.

Up to logical equivalence, there are only finitely many first-order formulae over our finite language of quantifier depth at most n, so there are only finitely many $\equiv_n$-equivalence classes. We may show this by induction:

Lemma 7. *In a finite relational language L with constants but no functions, there are only finitely many L-structures up to n-equivalence. [13]*

Proof. If $n = 0$ then all structures are equivalent as the game has length 0 and Player II immediately wins. This gives us a base case. Now assume the result for n, and consider $(n+1)$-equivalence: $A \equiv_{n+1} B$ if and only if for every $a \in A$ there exists $b \in B$, and vice versa, such that $(A, a) \equiv_n (B, b)$. The languages $L(A, a)$ and $L(B, b)$ must be finite, since they add a single extra constant symbol to $L(A)$ or $L(B)$, which are assumed to be finite. Thus $A \equiv_{n+1} B$ if and only if $\{(A, a) : a \in A\}$ and $\{(B, b) : b \in B\}$ are identical up to n-equivalence. By the induction hypothesis, there are finitely many n-equivalence classes for these, and so it follows that there are finitely many $\equiv_{n+1}$-classes. $\square$

We can see that this follows from the structure of the games: n-equivalence of A and B depends on whether the relations between $(a_1, \ldots, a_n)$ and those between $(b_1, \ldots, b_n)$ are the same, and to which of these n-tuple possibilities each k-tuple $(a_1, \ldots, a_k)$ (with $k < n$) along the way can be extended. We can therefore characterise structures by the set of relations of their k-tuples and on the n-tuples to which these extend, and there are finitely many such possibilities since n is finite and we are working over a finite relational language.

As there are only finitely many equivalence classes, we can find representatives of minimal size for those equivalence classes that contain finite members.

In general, we may play Ehrenfeucht-Fraïssé games on any relational structure, including those with high arities.

Lemma 8. *For a relational structure with relations of least arity k, not counting equality, there are $(n+1)$ many n-equivalence classes for each $n < k$.*

Proof. Suppose that the relation of least arity has arity k, not counting equality, and consider the game $G_n(A, B)$, where $n < k$ and A and B are arbitrary. Whichever moves are played, we obtain final tuples $(a_1, \ldots, a_n)$ and $(b_1, \ldots, b_n)$ of size n. However, there are no relations of arity n or below, so if $a_1, \ldots, a_n$ are distinct and likewise $b_1, \ldots, b_n$, then we must have $(a_1, \ldots, a_n) \equiv (b_1, \ldots, b_n)$. Therefore, any two structures containing at least n points are n-equivalent, as Player II can play such that $a_i = a_j$ if and only if $b_i = b_j$. Moreover, any two structures containing exactly $m < n$ points must be n-equivalent, since Player II can select a bijection of her choice and play the corresponding move at each turn, which is a winning strategy. However, if $|A| < n$ and $|A| < |B|$, then Player I can win by playing $(|A| + 1)$ distinct points in B, since Player II's responses in A must, by the pigeonhole principle, contain the same point twice. We therefore conclude that there are precisely $(n+1)$ n-equivalence classes, containing those structures with precisely $0, 1, \ldots, n$ points respectively. $\square$

Chapter 3

Linear orders

Many results on equivalence of linear orders are already known, for example by Mostowski and Tarski [22] and later by Bissell-Siders [2] [32] and by Mwesigye and Truss [24] [25] [26]. Rosenstein's text [30] on linear orders also gives an overview of some results. The results already obtained include classifications of the 2-equivalence classes of linear orders, the n-equivalence classes of finite orders and of ordinals, and some results on the scattered and coloured cases. When we require the linear orders on which we play to be well-ordered, we gain access to the corresponding notion of the minimal representative of a class, and existing work on ordinals has constructed bounds on the size of minimal representatives in both the monochromatic and coloured cases [26], as well as on particular subclasses of monochromatic scattered orders that can be constructed from concatenating copies of ω and ω^* [27]. The classification of finite linear orders up to n-equivalence is well-known: these have 2^n equivalence classes, of which $2^n - 1$ contain each linear order of size $0, 1, \ldots, 2^n - 2$ in its own equivalence class, and one contains all finite orders of size $2^n - 1$ or greater. The (entire) class of linear orders has heretofore been classified up to only 1- and 2-equivalence. There are two 1-equivalence classes of linear orders, of which one contains only the empty linear order and the other contains all non-empty ones, and seven 2-equivalence classes, which we shall give below. The main result of this section is to extend these existing results to a classification of all linear orders up to 3-equivalence.

3.1 Characters of linear orders

Theorem 9. *(Mwesigye-Truss [24]) For two linear orders A and B, $A \equiv_n B$ if and only if for every $a \in A$ there exists some $b \in B$, and vice versa, such that $\{x \in A : x < a\} \equiv_{n-1} \{y \in B : y < b\}$ and $\{x \in A : x > a\} \equiv_{n-1} \{y \in B : y > b\}$.*

Likewise, for two coloured linear orders A and B, $A \equiv_n B$ if and only if for every $a \in A$ there exists some $b \in B$ of the same colour as a, and vice versa, such that $\{x \in A : x < a\} \equiv_{n-1} \{y \in B : y < b\}$ and $\{x \in A : x > a\} \equiv_{n-1} \{y \in B : y > b\}$.

Bearing this in mind, we may construct a useful notion of character: let the *n-character* $\chi(a)$ of a point a in a linear order A be the pair $([A^{<a}], [A^{>a}])$, where $[X]$ denotes the chosen representative of the n-equivalence class of X. (Not all equivalence classes need have an obvious well-ordering, but there are only finitely many equivalence classes, so our ability to choose representatives is non-controversial.) We may therefore restate the above theorem using characters: two linear orders A and B are $(n+1)$-equivalent if and only if for every $a \in A$ there exists some $b \in B$, and vice versa, such that $\chi(a) = \chi(b)$, that is, if A and B realise the same n-characters.

By convention we take 0 and 1 as representatives of the 1-equivalence classes of linear orders, where 0 represents the class containing only the empty linear order and 1 represents the class containing all nonempty linear orders. For coloured linear orders with k colours there are 2^k distinct 1-equivalence classes, whose minimal members are those linear orders containing either 0 or 1 points of each colour. For example, the two-coloured linear orders have minimal representatives with a single blue point, a single red point, both a blue and a red point (in either order), and the empty linear order.

We remark that the 1-character of a point a is a binary pair $([A^{<a}], [A^{>a}])$, such that $[A^{<a}] = 1$ if there is a point b with $b < a$ and 0 otherwise, and likewise $[A^{>a}] = 1$ if there is a point $c > a$ and 0 otherwise. Therefore, the 2-equivalence classes of linear orders depend only on whether they realise endpoints (and non-endpoints, and singletons). We take $0, 1, 2, 3, \omega, \omega^*$ and $\mathbb{Z}$ as representatives of the seven 2-equivalence classes of linear orders [24]. Linear orders of size 0, 1 or 2 are the only ones in their class, and the other representatives are minimal members of theirs. For linear orders X of size ≥ 3 the following equivalences hold:

- $X \equiv_2 3$ if and only if X has both a least point and a greatest point
- $X \equiv_2 \omega$ if and only if X has a least point but no greatest point
- $X \equiv_2 \omega^*$ if and only if X has a greatest point but no least point
- $X \equiv_2 \mathbb{Z}$ if and only if X has neither a greatest nor a least point

These equivalences will be used extensively in the following section as we classify points according to the 2-equivalence classes of the segments of the line lying entirely above or below the point in question.

3.2 Linear orders up to 3-equivalence

To obtain our full classification of the linear orders up to 3-equivalence, we use the method of characters. Earlier we showed that n-equivalence of linear orders is determined by the $(n-1)$-characters realised by their points: here, every point has a 2-character, which is a pair of 2-equivalence classes of linear orders. We select representatives of each 2-equivalence class; arbitrary choices of representatives would entirely suffice, but we opt for "minimal" ones for elegance and convenience. $0, 1$ and 2 are the only members of their respective classes and so must be chosen, and we choose $3, \omega, \omega^*$ and $\mathbb{Z}$ for the other four. The first three of these are minimal with regard to the ordering or reverse ordering; all members of $\mathbb{Z}$'s equivalence class have no endpoints and so must be neither well ordered nor reverse well ordered, but every nonempty linear order without endpoints must embed $\mathbb{Z}$, so it is at least minimal in that sense.

We therefore take our 2-characters to be elements of $\{0, 1, 2, 3, \omega, \omega^*, \mathbb{Z}\}^2$. The 2-character of a point $x_0 \in X$ is the pair (L, R) such that $\{x \in X : x < x_0\} \equiv_2 L$ and $\{x \in X : x_0 < x\} \equiv_2 R$; by the prior classification of linear orders up to 2-equivalence, there is precisely one such pair in $\{0, 1, 2, 3, \omega, \omega^*, \mathbb{Z}\}^2$.

In our classification, we break down linear orders into a, usually large, middle section containing most of the general structure equipped with two relatively small end sections, containing the endpoint related structure.

We correspondingly define the notion of a large or small character: a point has a *small* character if one or both of the components of its character are $0, 1$ or 2. A character is *large* if it is not small, that is, if both of its components are $3, \omega, \omega^*$ or $\mathbb{Z}$. Either, neither, or both may arise, but points of small character may only be located at the ends of a linear order, while a typical linear order may consist mainly of many points of large character.

We begin by considering the subcase of linear orders without endpoints.

Theorem 10. *There are twelve* 3*-equivalence classes of linear orders without endpoints.*

Proof. We have the result from Theorem 9 that linear orders satisfy $A \equiv_n B$ if and only if for all $x \in A$ there exists $y \in B$, and vice versa, such that $A^{>x} \equiv_{n-1} B^{>y}$ and $A^{<x} \equiv_{n-1} B^{<y}$ [24]. Therefore, two linear orders are 3-equivalent if and only if they realise the same 2-characters, and a linear order without endpoints is not 3-equivalent to any linear order with an endpoint. The linear orders up to 2-equivalence are $0, 1, 2, 3, \omega, \omega^*$ and $\mathbb{Z}$. Of these, $0, 1, 2$ and 3 cannot occur in a linear order without endpoints, and ω cannot occur on the left side of a character nor ω^* on the right, as these would imply the existence of a least or greatest point respectively. The possible 2-characters are therefore $(\omega^*, \omega), (\omega^*, \mathbb{Z}), (\mathbb{Z}, \omega)$ and $(\mathbb{Z}, \mathbb{Z})$. The power set of these has sixteen elements; we show that four of these cannot be the set of 2-characters occurring in a linear order without endpoints, and exhibit linear orders realising the other twelve.

We remark that there is no clear canonical choice of minimal representatives, as we typically have a choice of the order in which to realise the large characters. For example, $\mathbb{Z} + \mathbb{Q}$ and $\mathbb{Q} + \mathbb{Z}$ represent the same 3-equivalence class, and while both are more natural than, say, $\mathbb{Q} + \mathbb{Z} + \mathbb{Z} + \mathbb{R} + \mathbb{Z} + \mathbb{R} + \mathbb{Q} + \mathbb{Q} + \mathbb{Z} + \mathbb{Q}$, which lies in the same equivalence class, there is no compelling reason to prefer one over the other. Even when we come to consider linear orders with endpoints in Theorem 11, which will have to begin or end with 1, 2, 3 or at least 4 points (in practice, 4 and ω suffice), the same situation arises with the points of large character. The representatives given here are chosen to be minimal in the sense of not containing a smaller example, and consist of finite concatenations of the reasonably canonical linear orders $1, \omega, \omega^*, \mathbb{Z}, \mathbb{Q}$ and $\mathbb{Q} \times \{0, 1\}$.

We begin by showing that the following sets of 2-characters are impossible: $\{(\omega^*, \mathbb{Z})\}$, $\{(\omega^*, \mathbb{Z}), (\mathbb{Z}, \mathbb{Z})\}, \{(\mathbb{Z}, \omega)\}$ and $\{(\mathbb{Z}, \omega), (\mathbb{Z}, \mathbb{Z})\}$. Note that any point with left character ω^* has a predecessor; this means that there exists a point with a successor, which must have right character ω. Likewise, any point with right character ω forces its successor to exist and to have left character ω^*. The four above mentioned sets of characters therefore cannot occur, since if $(\omega^*, \mathbb{Z})$ occurs then either $(\mathbb{Z}, \omega)$ or (ω^*, ω) must occur, and if $(\mathbb{Z}, \omega)$ occurs then either $(\omega^*, \mathbb{Z})$ or (ω^*, ω) must occur.

The other twelve sets of 2-characters are realisable, as we demonstrate below:

- $\emptyset$ is realised by the empty linear order.
- $\{(\omega^*, \omega)\}$ is realised by $\mathbb{Z}$.
- $\{(\mathbb{Z}, \mathbb{Z})\}$ is realised by $\mathbb{Q}$.
- $\{(\omega^*, \omega), (\omega^*, \mathbb{Z})\}$ is realised by $\omega^* + \mathbb{Z}$.
- $\{(\omega^*, \omega), (\mathbb{Z}, \omega)\}$ is realised by $\mathbb{Z} + \omega$.
- $\{(\omega^*, \omega), (\mathbb{Z}, \mathbb{Z})\}$ is realised by $\mathbb{Z} + \mathbb{Q}$.
- $\{(\omega^*, \mathbb{Z}), (\mathbb{Z}, \omega)\}$ is realised by $\mathbb{Q} \times \{0, 1\}$.
- $\{(\omega^*, \omega), (\omega^*, \mathbb{Z}), (\mathbb{Z}, \omega)\}$ is realised by $\omega^* + \mathbb{Z} + \omega$.
- $\{(\omega^*, \omega), (\omega^*, \mathbb{Z}), (\mathbb{Z}, \mathbb{Z})\}$ is realised by $\omega^* + \mathbb{Q}$.
- $\{(\omega^*, \omega), (\mathbb{Z}, \omega), (\mathbb{Z}, \mathbb{Z})\}$ is realised by $\mathbb{Q} + \omega$.
- $\{(\omega^*, \mathbb{Z}), (\mathbb{Z}, \omega), (\mathbb{Z}, \mathbb{Z})\}$ is realised by $\mathbb{Q} \times \{0, 1\} + \mathbb{Q}$.
- $\{(\omega^*, \omega), (\omega^*, \mathbb{Z}), (\mathbb{Z}, \omega), (\mathbb{Z}, \mathbb{Z})\}$ is realised by $\omega^* + \mathbb{Q} + \omega$.

□

We remark that for a linear order with one or both endpoints, there will similarly only be four possible 2-characters that can occur in the middle section. The 2-character of a point is determined by whether there are endpoints and/or middle points in the segments of the linear order lying before and after it. If we fix at least three points at the start and end of the linear order (whether this includes an endpoint or an unbounded chain of points), then only two possible left characters may be realised in between these two sets: if there is a left endpoint, then only ω and 3 are possible left characters; if there is no left endpoint, then only $\mathbb{Z}$ and ω^* may arise as left characters. In each case one left character occurs in points that have a predecessor (3 or ω^*), and one in those that do not have a predecessor (ω or $\mathbb{Z}$). There are similarly only two available right characters: either 3 and ω^*, or ω and $\mathbb{Z}$. We may therefore consider different possibilities for an initial and terminal set of size at least three, and for each, find which of the 16 subsets of the four possible characters may occur in between.

Theorem 11. *There are* 281 3*-equivalence classes of linear orders.*

Proof. We know that 3-equivalence depends on the set of 2-characters that are realised. We therefore show that there are 281 possible sets of 2-characters that may occur. Recall that the linear orders up to 2-equivalence are $0, 1, 2, 3, \omega, \omega^*$ and $\mathbb{Z}$, so the characters may be represented by ordered pairs from this list. Of these, $0, 1$ and 2 may occur on the left side of the character of at most one point, and on the right side of the character of at most one point. We call a character *small* if it contains $0, 1$ or 2 in at least one component, and we break down the possibilities according to the small characters realised. Two linear orders are 3-equivalent if and only if they realise precisely the same 2-characters, so by finding all possible sets of large character that may occur for each set of small characters, we obtain all equivalence classes without duplicates.

The large characters are pairs consisting only of $3, \omega, \omega^*$ and $\mathbb{Z}$. However, a linear order may not exhibit characters beginning with 3 or ω and also exhibit characters beginning with ω^* or $\mathbb{Z}$, since the first two characters imply that there is a least element and the latter two imply that there is not. Similarly, a linear order may not exhibit both characters ending with 3 or ω^* and characters ending with ω or $\mathbb{Z}$. A linear order may therefore realise at most four large characters (that is, those characters with neither component less than three); if it has no endpoints, these are $(\omega^*, \omega), (\omega^*, \mathbb{Z}), (\mathbb{Z}, \omega)$ and $(\mathbb{Z}, \mathbb{Z})$, as we showed above. If there is a left endpoint, then left components turn from ω^* to 3 and from $\mathbb{Z}$ to

ω, and if there is a right endpoint, then right components turn from ω to 3 and from $\mathbb{Z}$ to ω^*. Given the small characters occurring, there are therefore at most 16 possibilities for the set of large characters realised. In each case we determine how many of these may occur.

First, we consider the possible sets of characters containing only $0, 1$ and 2, that is, where both parts of the character are small. If $(0,0)$ occurs in a linear order A, then A is a single point. If $(0,1)$ or $(1,0)$ occurs, then they must both occur and A is a chain of length two. If $(0,2)$, $(1,1)$ or $(2,0)$ occurs, then they must all occur and A is a chain of length three. If $(1,2)$ or $(2,1)$ occurs, then so must the other of these as well as $(0,3)$ and $(3,0)$ and A is a chain of length four. If $(2,2)$ occurs, then so must $(0,3)$, $(1,3)$, $(3,1)$ and $(3,0)$, and A is a chain of length five.

Otherwise, no point has a character with both components less than 3, so we may consider the small characters at the right and left ends separately, as they now cannot overlap. A point with a small left character must now have large right character, and a point with a small right character must have a large left character.

Four cases remain: either there is a left endpoint, a right endpoint, both, or neither. These possibilities correspond to the existence or nonexistence of small characters. Left endpoints have small left character, as do the second and third points if these exist, but if there is no left endpoint then there is no point of small left character. Likewise, the final, penultimate and antepenultimate points have small right character if these exist, but if there is no right endpoint then there are no points with small right character.

Case 1 (no endpoints): If there is neither a left nor a right endpoint, then the linear order is 3-equivalent to one of the twelve linear orders without endpoints listed above in Theorem 10.

Case 2 (left endpoint): Suppose that there is a left endpoint but not a right endpoint. Then there are four possibilities for the set of small characters that can arise. There could be $(0,\mathbb{Z})$, in which case there can be no characters of the form $(1,*)$ or $(2,*)$. There could be $(0,\omega)$ and $(1,\mathbb{Z})$, in which case there must be no character of the form $(2,*)$. There could be $(0,\omega),(1,\omega)$ and $(2,\mathbb{Z})$, or there could be $(0,\omega),(1,\omega)$ and $(2,\omega)$. Note that this last possibility implies that there is a point with a character of the form $(3,*)$, since the point of character $(2, \omega)$ has a successor.

We consider the sets of large characters that may accompany these. Each large character is a member of $\{3, \omega, \omega^*, \mathbb{Z}\}^2$. Since there is a left endpoint, no character may have ω^* or $\mathbb{Z}$ as its left component; since there is no right endpoint, no character may have 3 or ω^* as its right component. The possible large characters in this case are therefore $(3, \omega), (3, \mathbb{Z}), (\omega, \omega)$ and $(\omega, \mathbb{Z})$.

Of the sixteen possible combinations of these, four are impossible in this case. We cannot have the empty set, since the overall linear order is infinite and so has infinitely many points of large character. The character set s$\{(3, \mathbb{Z})\}$ is impossible, as it requires every point to have a predecessor but no point to have a successor. Likewise, $\{(\omega, \omega)\}$ cannot arise, as every point would have a successor but no point would have a predecessor. The set $\{(\omega, \omega), (\omega, \mathbb{Z})\}$ is also impossible, since the point of character (ω, ω) has a successor but there is no character that could belong to its successor.

If the small characters are $\{(0, \mathbb{Z})\}$, $\{(0, \omega), (1, \mathbb{Z})\}$ or $\{(0, \omega), (1, \omega), (2, \mathbb{Z})\}$, then $\{(3, \mathbb{Z}), (\omega, \mathbb{Z})\}$ is also an impossible accompanying set of large characters, since any point of character $(3, \mathbb{Z})$ must have a predecessor. That predecessor must have character (x, ω) where $x \geq 2$, but neither the small character $(2, \omega)$ nor any large character of the form $(*, \omega)$ is realised, so no linear orders realising precisely these characters exists. If the small characters are $\{(0, \omega), (1, \omega), (2, \omega)\}$, however, $\{(3, \mathbb{Z}), (\omega, \mathbb{Z})\}$ is possible if $(3, \mathbb{Z})$ is realised only by the fourth point, for example in $4 + \mathbb{Q}$.

On the other hand, $\{(\omega, \mathbb{Z})\}$ is an impossible set of large characters when the small characters are $\{(0, \omega), (1, \omega), (2, \omega)\}$, since there must be a point of character $(3, *)$ to come next, but it is perfectly compatible with the other three sets of small characters. For example, $1 + \mathbb{Q}$, $2 + \mathbb{Q}$ and $3 + \mathbb{Q}$ realise $\{(0, \mathbb{Z}), (\omega, \mathbb{Z})\}$, $\{(0, \omega), (1, \mathbb{Z}), (\omega, \mathbb{Z})\}$ and $\{(0, \omega), (1, \omega), (2, \mathbb{Z}), (\omega, \mathbb{Z})\}$ respectively.

We have eliminated four sets of large character that are incompatible with any small character sets in this case, and a further two that may only arise with some of them. The other ten possible sets of large character are compatible with any of the four sets of small character that arise in this case. To show this, we exhibit examples. In general we may take similar representatives for the small character sets $\{(0, \mathbb{Z})\}$, $\{(0, \omega), (1, \mathbb{Z})\}$

and $\{(0,\omega),(1,\omega),(2,\mathbb{Z})\}$, but $\{(0,\omega),(1,\omega),(2,\omega)\}$ sometimes behaves differently due to the role played by points of character $(3,*)$. In fact, representatives from all 33 of the classes containing $\{(0,\mathbb{Z})\}$, $\{(0,\omega),(1,\mathbb{Z})\}$ or $\{(0,\omega),(1,\omega),(2,\mathbb{Z})\}$ may be obtained by prepending one, two or three points to the beginning of the eleven nonempty linear orders without endpoints given in Theorem 10 (of which one, $\{(\omega,\mathbb{Z})\}$, is considered above).

The following linear orders realise each of the ten remaining sets of large characters together with each of the four possible sets of small characters in this case:

- $\{(3,\omega)\}$ is realised by $n+\mathbb{Z}$ for $1 \leq n \leq 3$ and by ω for the $(2,\omega)$ case
- $\{(3,\omega),(3,\mathbb{Z})\}$ is realised by $n+\omega^*+\mathbb{Z}$ for $1 \leq n \leq 4$
- $\{(3,\omega),(\omega,\omega)\}$ is realised by $n+\mathbb{Z}+\omega$ for $1 \leq n \leq 3$ and by $\omega+\omega$ for the $(2,\omega)$ case
- $\{(3,\omega),(\omega,\mathbb{Z})\}$ is realised by $n+\mathbb{Z}+\mathbb{Q}$ for $1 \leq n \leq 3$ and by $\omega+1+\mathbb{Z}$ for the $(2,\omega)$ case
- $\{(3,\mathbb{Z}),(\omega,\omega)\}$ is realised by $n+\mathbb{Q}\times\{0,1\}$ for $1 \leq n \leq 4$
- $\{(3,\omega),(3,\mathbb{Z}),(\omega,\omega)\}$ is realised by $n+\omega^*+\mathbb{Z}+\omega$ for $1 \leq n \leq 4$
- $\{(3,\omega),(3,\mathbb{Z}),(\omega,\mathbb{Z})\}$ is realised by $n+\omega^*+\mathbb{Q}$ for $1 \leq n \leq 4$
- $\{(3,\omega),(\omega,\omega),(\omega,\mathbb{Z})\}$ is realised by $n+\mathbb{Q}+\omega$ for $1 \leq n \leq 3$ and by $\omega+\mathbb{Q}+\omega$ in the $(2,\omega)$ case
- $\{(3,\mathbb{Z}),(\omega,\omega),(\omega,\mathbb{Z})\}$ is realised by $n+\mathbb{Q}\times\{0,1\}+\mathbb{Q}$ for $1 \leq n \leq 4$
- $\{(3,\omega),(3,\mathbb{Z}),(\omega,\omega),(\omega,\mathbb{Z})\}$ is realised by $n+\omega^*+\mathbb{Q}+\omega$ for $1 \leq n \leq 4$.

So, there are eleven possible sets of large characters for each of the four possible small character sets, giving a total of 44 equivalence classes in this case.

Case 3 (right endpoint): Similarly to the previous case, if there is a right endpoint but not a left endpoint, then the possible sets of characters ending in $0, 1$, or 2 are $\{(\mathbb{Z},0)\}$, $\{(\omega^*,0),(\mathbb{Z},1)\}$, $\{(\omega^*,0),(\omega^*,1),(\mathbb{Z},2)\}$, and $\{(\omega^*,0),(\omega^*,1),(\omega^*,2)\}$. Again, the last possibility implies that there is a point with a character of the form $(*,3)$. By symmetry, we find that there are a further 44 classes in this case, whose members are the reverses of the orders in the classes arising in the previous case.

Case 4 (both endpoints): In the final case, there are both left and right endpoints, but we are not in one of the five classes above where the character of some point has both components less than 3. The possibilities for the small characters on the left are $\{(0,\omega^*)\}$, $\{(0,3),(1,\omega^*)\}$, $\{(0,3),(1,3),(2,\omega^*)\}$, and $\{(0,3),(1,3),(2,3)\}$. These sets are similar to those in the case with only a left endpoint, except that in this case the linear order has a final element. Similarly, the possibilities for the points of small character on the right hand side, given that there is a left endpoint, are $\{(\omega,0)\}$, $\{(3,0),(\omega,1)\}$, $\{(3,0),(3,1),(\omega,2)\}$, and $\{(3,0),(3,1),(3,2)\}$.

Considering the large characters, no character can begin with ω^* or $\mathbb{Z}$, and no character can end with ω or $\mathbb{Z}$. The possibilities are therefore $(3,3),(3,\omega^*),(\omega,3)$ and (ω,ω^*), and the sixteen potential large character sets are the power set of these.

We remark again that when the left hand points are $\{(0,3),(1,3),(2,3)\}$, there must be a next point with character $(3,*)$, and when the right hand points are $\{(3,0),(3,1),(3,2)\}$ there must be a point with character $(*,3)$. Naturally, if both $(2,3)$ and $(3,2)$ occur then both $(3,*)$ and $(*,3)$ must be realised by some middle point, and we could achieve this by either having just $(3,3)$, both $(3,\omega^*)$ and $(\omega,3)$, or by forgoing large characters altogether and having the point of character $(3,2)$ immediately follow the point of character $(2,3)$, in which case we find that the overall linear order is of size six.

As before, when $(2,3)$ and $(3,2)$ are both absent we may straightforwardly construct representatives by appending one, two or three points to the beginnings and ends of the eleven nonempty linear orders without endpoints listed previously. This gives representatives for the 99 classes where the points of small character are $\{(0,\omega^*)\}$, $\{(0,3),(1,\omega^*)\}$, or $\{(0,3),(1,3),(2,\omega^*)\}$ on the left and $\{(\omega,0)\}$, $\{(3,0),(\omega,1)\}$, or $\{(3,0),(3,1),(\omega,2)\}$ on the right. The other five sets of large character, $\emptyset$, $\{(3,\omega^*)\}$, $\{(\omega,3)\}$, $\{(\omega,3),(\omega,\omega^*)\}$ and $\{(3,\omega^*),(\omega,\omega^*)\}$, cannot occur with these small characters for the same reasons as before.

Suppose that $(2,3)$ is realised but not $(3,2)$. Then there are at most three points at the end of the linear order, but no point fourth from last. We may construct representatives of these classes by appending one, two or three points after the representatives for the case where there is a left endpoint but no right. In all cases taking $n \in \{1, 2, 3\}$:

- $\{(3,3)\}$ is realised by $\omega + n$
- $\{(3,3),(3,\omega^*)\}$ is realised by $4 + \mathbb{Z} + n$
- $\{(3,3),(\omega,3)\}$ is realised by $\omega + \omega + n$
- $\{(3,3),(\omega,\omega^*)\}$ is realised by $\omega + 1 + \mathbb{Z} + n$
- $\{(3,\omega^*),(\omega,3)\}$ is realised by $4 + \mathbb{Q} \times \{0,1\} + n$
- $\{(3,\omega^*),(\omega,\omega^*)\}$ is realised by $4 + \mathbb{Q} + n$
- $\{(3,3),(3,\omega^*),(\omega,3)\}$ is realised by $4 + \omega^* + \mathbb{Z} + \omega + n$
- $\{(3,3),(3,\omega^*),(\omega,\omega^*)\}$ is realised by $4 + \omega^* + \mathbb{Q} + n$
- $\{(3,3),(\omega,3),(\omega,\omega^*)\}$ is realised by $\omega + \omega + 1 + \mathbb{Z} + n$
- $\{(3,\omega^*),(\omega,3),(\omega,\omega^*)\}$ is realised by $4 + \mathbb{Q} \times \{0,1\} + \mathbb{Q} + n$
- $\{(3,3),(3,\omega^*),(\omega,3),(\omega,\omega^*)\}$ is realised by $4 + \omega^* + \mathbb{Q} + \omega + n$

The five sets of large characters which were impossible in the earlier case remain impossible now. We cannot have the empty set, since the overall linear order is infinite and so has infinitely many points of large character. The set $\{(3,\omega^*)\}$ is impossible, as it requires there to be points of large character all of which have a predecessor but none of which have a successor. The character sets $\{(\omega,3)\}$, $\{(\omega,3),(\omega,\omega^*)\}$ and $\{(\omega,\omega^*)\}$ are also impossible, since there must be a point of character $(3,*)$ in the fourth position, and we do not have $(3,2)$ (which would be impossible with these anyway), so this must be a large character point with character $(3,3)$ or $(3,\omega^*)$.

There are therefore 33 equivalence classes of linear orders with both endpoints that realise $(2,3)$ but not $(3,2)$. If $(3,2)$ is realised but not $(2,3)$, then the situation is the reverse of that above, and so we also have 33 equivalence classes in this case, whose members are the reverse orderings of the members of the previous case.

Finally, if both $(3,2)$ and $(2,3)$ are realised then there are eleven possibilities realising $\{(0,3),(1,3),(2,3),(3,2),(3,1),(3,0)\}$ as well as the following sets of large characters:

- $\emptyset$ is realised by the linear order with six elements
- $\{(3,3)\}$ is realised by the linear order with seven (or a larger finite number) elements
- $\{(3,3),(3,\omega^*)\}$ is realised by $4+\omega^*$
- $\{(3,3),(\omega,3)\}$ is realised by $\omega+4$
- $\{(3,3),(\omega,\omega^*)\}$ is realised by $\omega+\mathbb{Q}+\omega^*$
- $\{(3,\omega^*),(\omega,3)\}$ is realised by $4+\mathbb{Q}\times\{0,1\}+4$
- $\{(3,3),(3,\omega^*),(\omega,3)\}$ is realised by $4+\mathbb{Z}+4$
- $\{(3,3),(3,\omega^*),(\omega,\omega^*)\}$ is realised by $4+\omega^*+\mathbb{Q}+\omega^*$
- $\{(3,3),(\omega,3),(\omega,\omega^*)\}$ is realised by $\omega+\mathbb{Q}+\omega+4$
- $\{(3,\omega^*),(\omega,3),(\omega,\omega^*)\}$ is realised by $4+\mathbb{Q}+\mathbb{Q}\times\{0,1\}+4$
- $\{(3,3),(3,\omega^*),(\omega,3),(\omega,\omega^*)\}$ is realised by $\omega+2+\mathbb{Q}+4$

The character set $\{(3,\omega^*)\}$ is impossible, as it requires there to be points of large character all of which have a predecessor but none of which have a successor. Likewise, $\{(\omega,3)\}$ cannot arise, as every large character point would have a successor but no large character point would have a predecessor. The final three combinations may also not arise: $\{(3,\omega^*),(\omega,\omega^*)\}$ is impossible since there is no large character $(*,3)$ which can belong to the fourth from last point, $\{(\omega,3),(\omega,\omega^*)\}$ is impossible since there is no point of character $(3,*)$ to belong to the fourth point, and $\{(\omega,\omega^*)\}$ is impossible, since it fails both of these conditions.

In total we have five equivalence classes where some character is small in both components; 12 classes with no small characters at all; 44 equivalence classes where there are small characters on the left but not the right; 44 classes with small characters on the right but not the left, and 176 classes with small characters on the left and small characters on the right, but no character which is small in both. We have shown that no other sets of characters are possible, and exhibited members of each of these equivalence classes, therefore there are 281 3-equivalence classes of linear orders. □

3.3 Bounds on 4-equivalence

We remark that Theorem 11 allows us a crude upper bound on the number of 4-equivalence classes of linear orders. Since n-equivalence is determined by the set of $(n-1)$-characters that are exhibited, there can be at most 2^{78961} 4-equivalence classes. Unfortunately this bound has 23770 digits so is unlikely to be close to the actual figure. We can obtain analogously loose bounds for general n: if $B_3 = 281$, then $B_n = 2^{(B_{n-1}^2)}$ is an upper bound on the number of n-equivalence classes of linear orders. These bounds are rather large, though computable.

In order to directly calculate the 4-equivalence classes of linear orders, we would wish to again apply the method of characters. The 3-character of a point a is a pair of elements (X, Y) from the above set (or another choice of representatives), such that $A^< \equiv_3 X$ and $A^> \equiv_3 Y$. In an analogous way to the above proof, we can determine which combinations of 3-characters may co-occur - however, since we are starting with 281^2 possible characters rather than 7^2, the list of permissible combinations would be substantially longer.

We may use an alternative method to produce better upper bounds. Any linear order may be broken down into three (not necessarily nonempty) components: an initial well-ordered section, a middle section without endpoints, and a terminal reverse-well-ordered section. We may therefore construct an upper bound on the number of n-equivalence classes of linear orders by determining the number of linear orders without endpoints up to n-equivalence, and optionally modifying them with an ordinal at the beginning and a reversed ordinal at the end.

We note that the bounds thus obtained are significantly sharper than by the above method of counting elements in the power set. Given that the number of 2-equivalence classes of linear orders is seven, we could have calculated that there are at most $2^{49} = 562949953421312$ 3-equivalence classes of linear orders.

Using ordinals, we may achieve a much more reasonable bound. Any linear order is 3-equivalent to one of the form $\alpha + X + \beta^*$ where α and β are ordinals and X is a linear order without endpoints. We may take α and β to be minimal. The ordinals up to 3-equivalence have been classified by Feresiano Mwesigye [23]. There are seventeen equivalence classes, and since the class of ordinals is well ordered there exist minimal representatives, which are $0, 1, 2, 3, 4, 5, 6, 7, \omega, \omega+1, \omega+2, \omega+3, \omega+4, \omega+\omega, \omega+\omega+1, \omega+\omega+2$ and $\omega+\omega+3$.

We may therefore construct an upper bound on the number of 3-equivalence classes of linear orders: every linear order is 3-equivalent to a linear order of one of the form $L + M + U$, where L is a member of this set of 16 ordinals, M is one of the twelve members of the list of linear orders without endpoints given in Theorem 10, and U is the reverse of a 3-minimal ordinal. This gives us an upper bound of 3072 linear orders up to 3-equivalence, much closer to the true value of 281.

To perform the analogous calculation for 4-equivalence, we need to know the ordinals up to 4-equivalence, and the linear orders without endpoints up to 4-equivalence.

The ordinals up to 4-equivalence have been classified by Mwesigye and Truss [25] [23]. There are 63 4-equivalence classes, and since we are considering only ordinals we are able to easily select a minimal representative for each using the usual ordering. From the above papers these are:

- $0, 1, 2, \ldots, 15$
- $\omega, \omega+1, \omega+2, \ldots, \omega+12$
- $\omega \cdot 2, \omega \cdot 2+1, \omega \cdot 2+2, \ldots, \omega \cdot 2+12$
- $\omega \cdot 3, \omega \cdot 3+1, \omega \cdot 3+2, \ldots, \omega \cdot 3+12$
- $\omega \cdot 4, \omega \cdot 4+1, \omega \cdot 4+2, \omega \cdot 4+3$
- $\omega^2, \omega^2+1, \omega^2+2, \omega^2+3$

There are therefore at most $63^2 = 3969$ times as many 4-equivalence classes of linear orders as there are 4-equivalence classes of linear orders without endpoints.

Chapter 4

Partial orders

A natural extension of the linear case is that of partial orders. We shall see that things are considerably more complicated here, but we can at least make progress in some special cases. We also consider coloured partial orders, though here things become a lot harder than for linear orders, for which the coloured version obeys the coloured variation of Theorem 9 and may thus be broken down into smaller pieces just as in the monochromatic case.

The main initial result gives a complete listing of all 2-equivalence classes of partial orders. Bearing in mind that there are only 7 linear orders up to 2-equivalence, the fact that our list has 39 illustrates the increased complexity, and this discrepancy is expected to increase greatly for 3-equivalence and above. The natural technique in the linear case of the use of characters can be adopted here, but with a lot less success. For a start we require three sets relating to a point rather than two: those points above, below, and incomparable to it. Moreover, we would also require the sets containing all points either above or incomparable to our given point to be equivalent, and likewise for the sets of points lying either below or incomparable. Yet even this does not suffice, as we shall show, and a more satisfying notion of character is not apparent. From the structure of an Ehrenfeucht-Fraïssé game, we are able to extract a notion of character that does entail all the necessary information, but this is unwieldy and may be unlikely to yield many fruitful results aside from large upper bounds on the number of n-equivalence classes.

4.1 Characterising 2-equivalence

As with linear orders, there are only two 1-equivalence classes of partial orders: the class containing the empty partial order and the class containing the non-empty partial orders. By analogy with 1-characters on linear orders, we may define a notion of 1-character for partial orders, which still takes the value of 0 or 1 in each component corresponding to the 1-equivalence class arising there, but requires three components rather than two.

Theorem 12. *Let the* 1-character *of a point x in a partial order be the tuple $(\chi_a(x), \chi_b(x), \chi_i(x))$, where*

- $\chi_a(x) = 1$ *if* $\{y : y > x\}$ *is nonempty,* 0 *otherwise;*
- $\chi_b(x) = 1$ *if* $\{y : y < x\}$ *is nonempty,* 0 *otherwise;*
- $\chi_i(x) = 1$ *if* $\{y : y \not< x, y \not> x\}$ *is nonempty,* 0 *otherwise.*

Then the partial orders P and Q are 2*-equivalent if and only if they realise the same* 1*-characters.*

Proof. Suppose that P and Q do not realise the same 1-characters. Then there is some character χ in one of them, suppose without loss of generality in P, that does not occur in Q. Let Player I pick a point x of character χ. Player II must then pick a point in Q, but there is no point of character χ, so she must pick a point y of character $\chi_1 \neq \chi$. Player I compares χ and χ_1 and identifies a coordinate that is 1 in one of them and 0 in the other. This being the first, second or third coordinate respectively indicates to her that there exists a point above, below or incomparable with x but not with y (or vice versa). She therefore picks a point with the corresponding relationship with x, such that no point has this relationship with y (or vice versa). Player II is then unable to pick a point with this relationship with y (or x) and so Player I wins.

Conversely, suppose that P and Q realise the same 1-characters. Then whichever x Player I plays, Player II can play a point y in the other partial order with identical character. Since the characters are equal, there are points above x if and only if there are points above y, points below x if and only if there are points below y, and points incomparable with x if and only if there are points incomparable with y. Player I may then select a second point of her choosing, but whatever its relationship to the point played in the first round, there must exist a point of the same relationship to the other point played in the first round, which Player II can play in response. Therefore Player II wins and P and Q are 2-equivalent. $\square$

While the name "1-character" arises by analogy with the n-characters of linear orders, we have not defined n-characters for partial orders in general. This is because we do not have the same inductive condition as with linear orders, and so categorising points according to which $(n-1)$-equivalence class the sets of points above/below/incomparable to them belong to is not actually very useful, as we shall see later.

The comparatively straightforward n-equivalence categorisation of partial orders when $n = 2$ is more amenable. During the two-move game, only two points in each structure are selected, which must be either ascending, descending or incomparable to each other. Ternary properties such as betweenness cannot be expressed and so cannot affect the 2-equivalence classification. When we consider the three-move game, we must consider the relations that each possible choice of third point can have to both of the preceding points, and so the properties required for equivalence will become correspondingly more complex.

We remark that the analogous result holds for coloured partial orders if we append the colour of each point to its character, and modify χ_a, χ_b and χ_c to record the sets of the colours of points lying above, below and incomparable to it respectively.

4.2 Monochromatic partial orders up to 2-equivalence

We classify the partial orders $(X, <)$ up to 2-equivalence, using characters. Recall that a partial order is 2-equivalent to another partial order if and only if the sets of 1-characters (defined in Theorem 12) that occur are identical. There are eight possible 1-characters, as they are triples of 1-equivalence classes, so there are at most 2^8 combinations of characters that could occur in a partial order. Of course, the antisymmetry and transitivity of aboveness as well as the symmetry of incomparability mean that not all of these combinations can occur.

Theorem 13. *There are 39 monochromatic partial orders up to 2-equivalence.*

Proof. We first dispose of two trivial cases: if no 1-characters at all are realised, then the partial order must be $\emptyset$, and if some point realises $(0,0,0)$, then there can only be one point. From now on we therefore suppose that there are at least two points in our partial order X, and that the character $(0,0,0)$ does not occur.

We break the possibilities down into cases according to the existence of greatest and least points. Here a *greatest* point is a point that lies above all other points, in contrast to a *maximal* point, which is merely required to have no points lying above it. When these characters occur, $(1,0,0)$ points are least, $(0,1,0)$ points are greatest, $(1,0,1)$ points are minimal but not least, $(0,1,1)$ points are maximal but not greatest, $(0,0,0)$ points are both greatest and least, $(0,0,1)$ points are both minimal and maximal but not least or greatest, and $(1,1,0)$ and $(1,1,1)$ points are neither maximal nor minimal.

We observe that if there are at least two points in total, then there is a greatest point if and only if $(0,1,0)$ is realised, and there is a least point if and only if $(1,0,0)$ is realised. If there is a $(0,0,1)$ point, then it has nothing above or below it, so it must be both maximal and minimal (but neither greatest nor least). We deduce that there is a maximal point which is not greatest if and only if $(0,1,1)$ or $(0,0,1)$ is realised, and similarly there is a minimal point which is not least if and only if $(1,0,1)$ or $(0,0,1)$ is realised. This helps us keep track of the nine various cases which arise.

Case 1: There are both greatest and least points. Thus characters $(0,1,0)$ and $(1,0,0)$ both occur. Since all other points lie between these two, the only other possible characters are $(1,1,0)$ and $(1,1,1)$. This gives rise to four possibilities, which can all occur, as follows:

3. $\{(0,1,0),(1,0,0)\}$, a chain of two elements (which is the only example realising this set of characters).

4. $\{(0,1,0),(1,0,0),(1,1,0)\}$, a chain of size three, or any larger chain with both top and bottom points (so here there are many examples, both finite and infinite).

5. $\{(0,1,0),(1,0,0),(1,1,1)\}$, of which the smallest example has four elements in a diamond shape, though there are many others (finite and infinite).

6. $\{(0,1,0),(1,0,0),(1,1,0),(1,1,1)\}$, obtained by combining the two structures in 4 and

5, for instance, a diamond with an extra point added at the top.

Case 2: There is a greatest element, and also a minimal one, but no least element. Here $(0,0,1)$ cannot arise, since such a point would have to be incomparable with everything else, which contradicts there being a greatest in this case. We must have characters $(0,1,0)$, at the greatest element, and $(1,0,1)$, at a minimal element, but not $(0,1,1)$ or $(1,0,0)$ in this case. The remaining characters which may arise are $(1,1,0)$ and $(1,1,1)$ and this gives rise to four possibilities as follows:

7. $\{(0,1,0),(1,0,1)\}$, for instance, a Λ shape or fan with a single top point and multiple bottom points but none in the middle.

8. $\{(0,1,0),(1,0,1),(1,1,1)\}$, for instance, a Λ shape with a point added below one of its minimal elements.

9. $\{(0,1,0),(1,0,1),(1,1,0)\}$, for instance, a Λ shape with a point added at the top.

10. $\{(0,1,0),(1,0,1),(1,1,0),(1,1,1)\}$, such as a Λ shape with points added at both top and bottom.

Case 3: There is a least element, and also a maximal one, but no greatest. This is case 2 inverted, so there are four more equivalence classes whose members are the inversions of those in the previous case.

11. $\{(1,0,0),(0,1,1)\}$, for instance, three points in a V shape.

12. $\{(1,0,0),(0,1,1),(1,1,1)\}$, for instance, a V shape with a point added above one of its maximal elements.

13. $\{(1,0,0),(0,1,1),(1,1,0)\}$, for example, a V shape with a point added at the bottom.

14. $\{(1,0,0),(0,1,1),(1,1,0),(1,1,1)\}$, for example, a V shape with points added at the top and bottom.

Case 4: There is a greatest element, but no minimal one. Here we have character $(0,1,0)$ but none of $(0,0,1)$, $(0,1,1)$, $(1,0,0)$, and $(1,0,1)$, as the first two fail to be below the greatest element and the latter two would be minimal. The remaining possible characters are $(1,1,0)$ and $(1,1,1)$, and since we require some additional point at least one of these must arise, so there are three more options:

15. $\{(0,1,0),(1,1,0)\}$, for instance, ω^*.

16. $\{(0,1,0),(1,1,1)\}$, for instance, two otherwise disjoint copies of ω^*, with their top

points identified.

17. $\{(0,1,0),(1,1,0),(1,1,1)\}$, of which an example may be obtained by adding one more point at the top of our example for 16.

Notice that this is the first case in which all examples must be infinite; all previous equivalence classes had finite members.

Case 5: There is a least element, but no maximal one. This is Case 4 inverted, so there are three more examples, which are also necessarily infinite.

18. $\{(1,0,0),(1,1,0)\}$, for instance, ω.

19. $\{(1,0,0),(1,1,1)\}$, for instance, two otherwise disjoint copies of ω, with their bottom points identified.

20. $\{(1,0,0),(1,1,0),(1,1,1)\}$, of which an example may be obtained by adding a point at the bottom of our example for 19.

Case 6: There are maximal and minimal elements, but no greatest or least element. Thus $(0,1,0)$ and $(1,0,0)$ do not occur, and either $(0,0,1)$ occurs, or $(0,1,1)$ and $(1,0,1)$ both do. Supposing first that $(0,0,1)$ does not occur, we obtain the following possibilities:

21. $\{(0,1,1),(1,0,1)\}$, for instance, a partial order of size four which is the disjoint union of two chains of length two.

22. $\{(0,1,1),(1,0,1),(1,1,0)\}$, for instance, an X shaped partial order with two maximal elements, two minimal elements, and a point in between.

23. $\{(0,1,1),(1,0,1),(1,1,1)\}$, for instance, the disjoint union of a chain of length three and a chain of length two.

24. $\{(0,1,1),(1,0,1),(1,1,0),(1,1,1)\}$, for instance, obtained from 22 by adding a new point above one of the maximal ones. Note that the minimal representative of this equivalence class has size six; a smaller example is impossible since there must be at least two points realising each of the characters $(1,0,1)$ and $(0,1,1)$, and of course $(1,1,0)$ and $(1,1,1)$ must also occur at at least one point. This is the largest minimal representative of any 2-equivalence class of finite partial orders.

Next suppose that $(0,0,1)$ does occur. This point is incomparable with all others, so $(1,1,0)$ does not occur (as well as the other $(*,*,0)$ characters, which we have already ruled out). The remaining characters which may occur are $(1,0,1)$, $(0,1,1)$, and $(1,1,1)$.

We cannot have just $\{(0,0,1),(1,0,1)\}$, since the minimal $(1,0,1)$ point must have some point with a different character above it; likewise, we cannot have just $\{(0,0,1),(0,1,1)\}$. This leaves us with six possibilities as follows:

25. $\{(0,0,1)\}$, which arises in any antichain of any size at least two.

26. $\{(0,0,1),(1,1,1)\}$, for instance, $\mathbb{Z}$ together with one isolated point. There are no finite examples.

27. $\{(0,0,1),(1,0,1),(0,1,1)\}$, for instance, a two-element chain and one isolated point.

28. $\{(0,0,1),(1,0,1),(1,1,1)\}$, for instance, the disjoint union of ω and one isolated point. There are no finite examples.

29. $\{(0,0,1),(0,1,1),(1,1,1)\}$, for instance, the disjoint union of ω^* and an isolated point. Again, this cannot arise in a finite partial order.

30. $\{(0,0,1),(1,0,1),(0,1,1),(1,1,1)\}$, for instance the disjoint union of a chain of three elements and an isolated point.

We note that seven of the equivalence classes arising in Case 6 have finite representatives.

Case 7: There is a maximal element but no greatest or minimal one. Hence $(1,0,0)$, $(0,1,0)$, $(1,0,1)$ and $(0,0,1)$ do not arise, but $(0,1,1)$ does. At least one of $(1,1,0)$ and $(1,1,1)$ must also arise, since not all elements are maximal (or they would also all be minimal). This gives rise to the following three options:

31. $\{(0,1,1),(1,1,0)\}$, for instance ω^* with two extra maximal points added at the top.

32. $\{(0,1,1),(1,1,1)\}$, for instance two disjoint copies of ω^*.

33. $\{(0,1,1),(1,1,0),(1,1,1)\}$, which may be obtained from 31 by adding an extra maximal point above one of the already top points.

Case 8: There is a minimal element but no least or maximal one. This is Case 7 inverted, so there are just three more possibilities:

34. $\{(1,0,1),(1,1,0)\}$, for instance ω with two extra minimal points added at the bottom.

35. $\{(1,0,1),(1,1,1)\}$, for instance two disjoint copies of ω.

36. $\{(1,0,1),(1,1,0),(1,1,1)\}$, which may be obtained from 34 by adding an extra minimal point below one of the already bottom points.

Case 9: There is no maximal or minimal element. Then the only possible characters are $(1,1,0)$ and $(1,1,1)$. Since there is at least one point, the options are therefore as follows:

37. $\{(1,1,0)\}$, for instance $\mathbb{Z}$.

38. $\{(1,1,1)\}$, for instance two disjoint copies of $\mathbb{Z}$.

39. $\{(1,1,0),(1,1,1)\}$, for instance two copies of $\mathbb{Z}$, disjoint except that the zero elements of each are identified.

This covers all possible cases, and so there are precisely 39 equivalence classes. $\square$

Of these 39 equivalence classes, 21 contain finite partial orders: the trivial two, all those in Cases 1-3, and seven of those from Case 6. From our above observations we may deduce the following:

Corollary 14. *If a partial order is* 2*-equivalent to a finite partial order, then it is* 2*-equivalent to one of size at most six.*

We have given representatives of size ≤ 6 for each of the 21 equivalence classes with finite members, and shown that one class (numbered 24) contains no partial orders of size 5 or smaller, so we deduce that the largest minimal representative of the 2-equivalence classes of finite partial orders has size six. In fact, the representatives listed above are each minimal representatives of their respective classes, as may be easily verified by counting the number of points required to realise each. In general, there must be at least as many points in the partial order as there are distinct characters; any point with a character of the form $(*,*,1)$ must have some other point of with character of the form $(*,*,1)$ to which it can be incomparable; and the characters $(1,0,1)$ and $(0,1,1)$ cannot occur only once in a finite partial order unless $(0,0,1)$ is also present (though they may of course occur zero times). Minimal representatives of each class for which finite members exist are shown in Figure 4.1 (for those arising in Cases $1-3$), and Figure 4.2 (for those arising in Case 6).

Almost all of the 2-equivalence classes of partial orders contain infinitely many members. The equivalence class for the empty partial order and the equivalence class containing the one point partial order are both trivially finite, as no other partial order is 2-equivalent to either of these. The equivalence class of the partial order with two comparable points

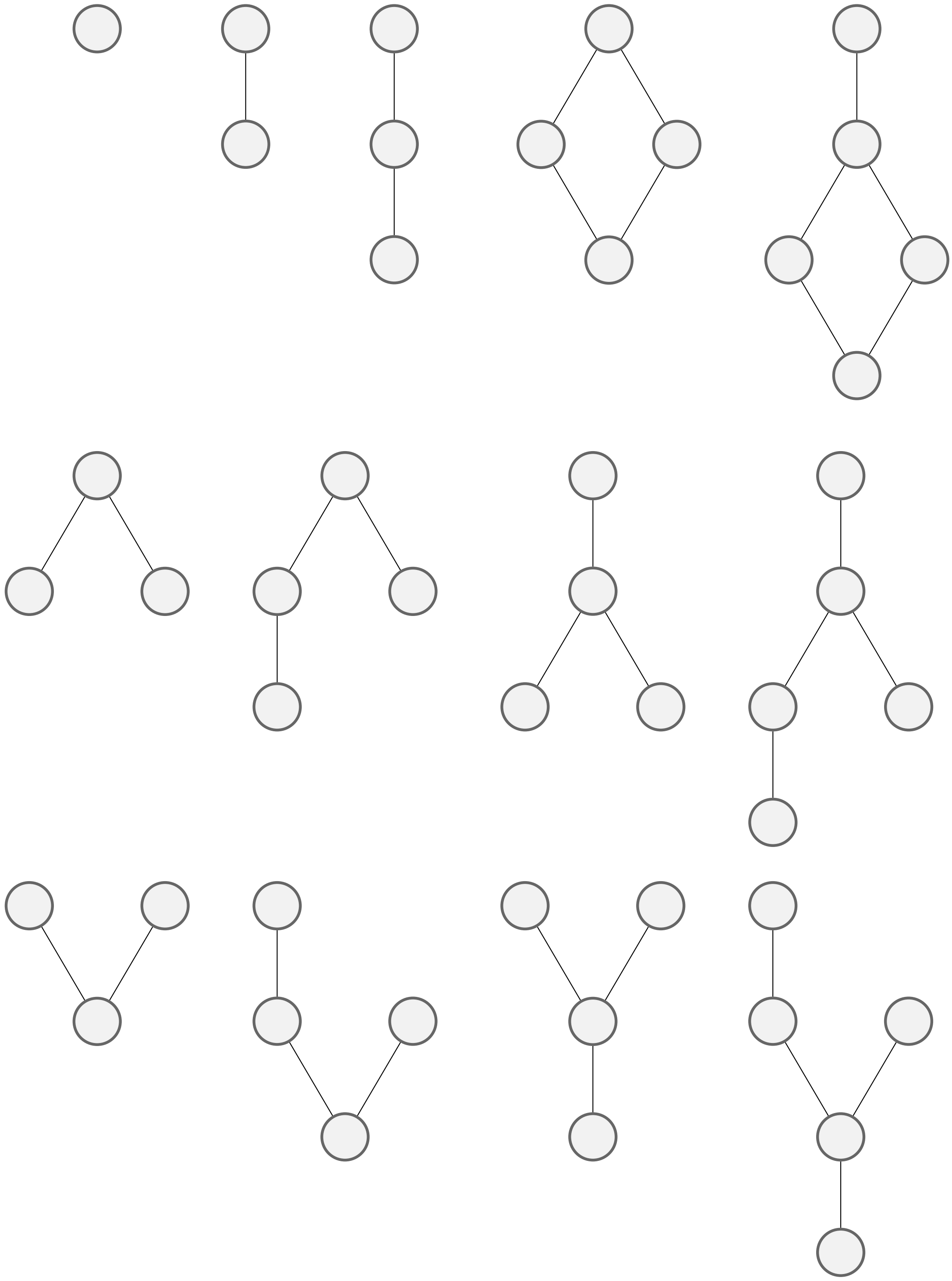

Figure 4.1: Finite minimal representatives of equivalence classes 1 to 14, in that order. Since the representative of the first equivalence class is the empty partial order, there are only thirteen visible diagrams: those of classes $2-6$ in the top row, $7-10$ in the middle row, and $11-14$ in the bottom row.

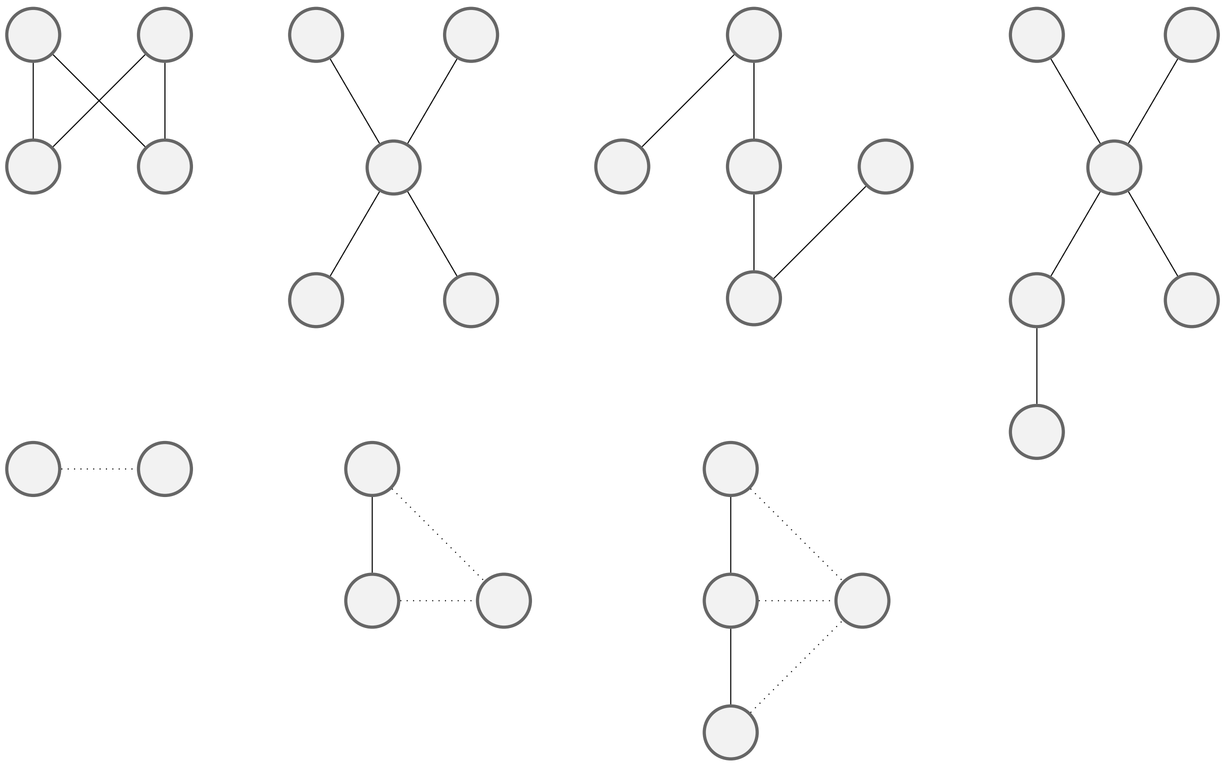

Figure 4.2: Finite minimal representatives of equivalence classes $21, 22, 23, 24, 25, 27$ and 30, in that order, with $21 - 24$ in the top row and $25, 27$ and 30 in the bottom row. Since the minimal representatives (and all other members) of classes $25, 27$ and 30 have multiple components, incomparable pairs of points in these are connected with a dotted line.

also has only one member, as its character $\{(0,1,0),(1,0,0)\}$ requires that there be a point above everything and a point below everything, but nothing with points both above and below it. There must therefore be one top point, one bottom point, and no other points.

Any partial order other than these three can be modified to give another partial order in its equivalence class. It must either have middle points with a character of the form $(1,1,*)$, which can be replaced by a chain to give a new partial order realising precisely the same characters as the original, or else it has incomparable points with a character of the form $(*,*,1)$, which can be replaced by an antichain to give a new partial order realising exactly the same characters. These chains and antichains may be of any size, so there are infinitely many distinct partial orders that may be obtained by this method, and therefore 36 of the 39 equivalence classes of partial orders (with the above three classes as the exceptions) contain infinitely many members.

There are 18 infinite-only equivalence classes, three of which realise $(0,0,1)$ alongside some other points that we can take to be an infinite linear order with a top, bottom or neither, and fifteen of which fit into a top/middle/bottom classification as follows.

We say that a point x is *split* if there is some $y \neq x$ with $y \not< x$, $x \not< y$, that is, some point y which is incomparable to x. Then a greatest point that lies above all other points is an unsplit top point, a maximal point that does not lie above all other points is a split top point, and likewise for least and non-least minimal points. Similarly, middle points of character $(1,1,1)$ are split and $(1,1,0)$ are unsplit.

We obtain the infinite-only equivalence classes by combining the possibilities for the combinations of split and unsplit points that can occur at the top, middle and bottom of the partial order. The infinite-only equivalence classes contain infinite chains that are split, unsplit or both; with either no top and no bottom, a split or unsplit top but no bottom, or a split or unsplit bottom but no top.

This gives $3*(1+2+2) = 15$ possibilities, which are realisable by combining a suitable choice of top, middle and bottom sets. For example, we may take a single point for an unsplit top or bottom set, two incomparable points for a split top or bottom set, $\mathbb{Z}$ for an unsplit middle set, two disjoint copies of $\mathbb{Z}$ for a split middle set, and $\mathbb{Z}$ with one point replaced by two incomparable points (or two copies of $\mathbb{Z}$ with a point in each identified, or many other possibilities) for the middle set with both split and unsplit points.

As noted above, the remaining three possibilities arise when there is a point of character $(0,0,1)$. This universally unrelated point makes everything else split, but the fact that this point has character $(0,0,1)$ distinguishes this case from the situation included above where there is a split chain with a split top $(0,1,1)$ and a split bottom $(1,0,1)$.

The 21 finite cases can also be understood in this framework: they must have both maximal and minimal elements, being finite, so there are $4*2*2=16$ possibilities with a chain that is split $(1,1,1)$, unsplit $(1,1,0)$, both, or neither; with a top that is split $(0,1,1)$ or unsplit $(0,1,0)$ (but not neither, as this is the finite case); and a bottom that is split $(1,0,1)$ or unsplit $(1,0,0)$, plus three possibilities with a universally unrelated point $(0,0,1)$ along with a linear order (a single point $(0,0,1)$, two points $(1,0,1)$ and $(0,1,1)$, or more than two points $(1,0,1)$, $(0,1,1)$ and $(1,1,1)$), as well as the two degenerate classes containing only the single point partial order $(0,0,0)$ and the empty partial order.

Remark. *Not all partial orders contain a minimal representative of their equivalence class.*

Every linear order contains a minimal representative of its 2-equivalence class. This is implicit in [24], lemma 3.2. However, the same does not hold for partial orders. There exist monochromatic finite partial orders P such that no proper subset of P is 2-equivalent to P, but such that there exists another partial order of size $< |P|$ which is 2-equivalent to P.

For example, let $P = \{a_1, a_2, b_1, b_2, c_1, c_2\}$ with $a_i > b_j > c_k$ for all i, j, k and no other relations. A diagram of P is given in Figure 4.3. Then P has points with characters $(0,1,1)$, $(1,1,1)$ and $(1,0,1)$. However, any proper subset of P would not realise all these characters. The points are on three tiers with two points on each tier, so removing any points would either remove a tier or leave a point alone on its tier, either of which would make this set of characters impossible. Removing a tier would prevent $(1,1,1)$ from arising, as a point of this character has points both above and below it, and having a tier with only one point would give that point a character of the form $(*,*,0)$, which is not present in P. However, $Q = \{a, b, c, d, e\}$ with $a < b > c > d < e$ realises exactly the same characters

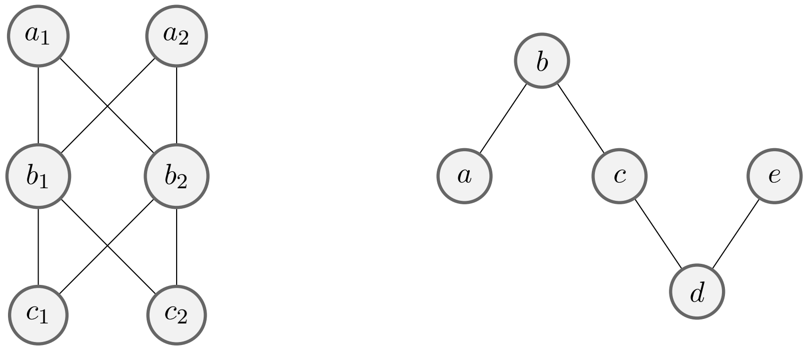

Figure 4.3: P (left) and Q (right) are 2-equivalent, but P has no equivalent substructure of at most the size of Q.

and $|Q| = 5 < 6 = |P|$.

4.3 Coloured partial orders

A *coloured linear order* is a tuple $(A, <, F)$ consisting of a set A, a total order $<$ on A, and a function known as a *colouring function* $F : A \to C$, where C is some set of "colours". Likewise, a *coloured partial order* is a tuple $(A, <, F)$ where $<$ is a partial order on A and F is a colouring function. A monochromatic linear order may be obtained by taking C to be a singleton. It is often of interest to take C to be finite; if $|C| = n$ then we refer to F as an *n-colouring* of A and say that A is *n-coloured.*

Since there are 39 2-equivalence classes of monochromatic partial orders, there must be at least 39^n 2-equivalence classes of n-coloured partial orders, because we can combine arbitrary partial orders of each colour with no inter-colour relations to give partial orders that are certainly pairwise non-2-equivalent if their substructures of any colour are not 2-equivalent. It seems likely that there are many more, however, as two partial orders of different colours can be combined in multiple possible ways. In general, the 1-character of a point x is the tuple (colour of x, set of colours of points above x, set of colours of points below x, set of colours of points incomparable with x). If there are n permissible colours, this gives $n * 2^{3n}$ possible 1-characters. Since structures are 2-equivalent if and only if they realise the same 1-characters, this gives an upper bound of $2^{n2^{3n}}$ possible equivalence classes. Of course the true number is much smaller due to various constraints - for example, a point of any colour with character $(0, 0, 0)$ would imply that no other points existed, so this upper bound may be shrunk by a factor of almost 2^n immediately.

There are a large number of ways of combining two differently coloured partial orders. Even adding a single red point to various blue partial orders yields perhaps ten or fourteen 2-equivalence-distinct results, depending on which partial order was started with. Moreover, one might find that different possibilities arise when adding a new colour of point to two different members of the same equivalence class. Even adding a red point to something as simple as a three-point or four-point blue linear order gives an additional possibility in the 4-point case, yet by themselves the three-point and the four-point linear order lie in the same 2-equivalence class. This removes any hope of finding the number of 2-colour 2-move equivalence classes by simply enumerating the possible equivalence classes arising from each of the 39^2 pairs of equivalence classes of the red and blue substructures.

If two coloured partial orders are n-equivalent, then the induced substructures consisting of the points of each colour must be n-equivalent. If two coloured partial orders are n-equivalent, then the reducts formed by discarding the colour information must also be n-equivalent. If two partial orders are n-equivalent, then there exists a recolouring such that the recoloured versions of each are not n-equivalent. There must also exist a recolouring such that the recoloured versions are n-equivalent, such as colouring all points the same colour.

Recolourings that increase the number of colours need not in general preserve equivalence. Consider applying a colouring to the monochromatic 3-point and 4-point linear orders. For the coloured versions to remain equivalent, the top points must be the same colour, the bottom points must be the same colour, and the middle points must be the same colour. Then either all points are the same colour, in which case we have not successfully performed a nontrivial recolouring, or the upper (or, respectively, lower) middle point has two different colours below (respectively, above) it in the 4-point linear order, and so the recoloured linear orders are not 2-equivalent.

These examples are monochromatic, but we can use a similar construction to show that m to $m + 1$ recolourings need not preserve equivalence for any m. By colouring the

3- and 4- point linear orders with colour c_1 and adding one point of each colour $c_2, ..., c_m$ in the same order above (or below) them, we similarly find an example of two m-coloured linear orders and a recolouring with $m+1$ colours such that the original linear orders are 2-equivalent but the recoloured linear orders are not.

4.4 Notions of n-character for partial orders

A number of interesting properties may be characterised in three moves but not in two, including:

- two points having a common upper bound
- two points having a common lower bound
- existence of an antichain of size at least three
- existence of a chain of size at least three
- density
- branching - a point having at least two incomparable points above it (or below)
- "extensibility" - x has a point above (or below) it that is incomparable to y

The required characters are no longer as straightforward in a game of length three as they were in the two move case. In a game of length two, it only matters whether each component is empty or nonempty; for a game of length three, the internal structure of each component matters, as well as the relations between them.

In order to sensibly work on the classification of partial orders up to 3-equivalence or higher, or for n-equivalence in general, we would need a more advanced notion of character than the restricted notion of 1-character used in Theorem 13.

We introduce a notion of character analogous to that of linear orders. We consider the partition of the partial order P: $P_x = \{x\} \cup A_x \cup B_x \cup C_x$ where

$A_x = \{y \in P : y > x\}$,

$B_x = \{y \in P : y < x\}$,

$C_x = \{y \in P : y \neq x, y \not< x, y \not> x\}$.

As well as being an initial segment of the alphabet, note that this choice of letters lets A, B and C refer to the points which are Above, Below, and inComparable with x, respectively.

For a linear order, it is true that X is $(n+1)$-equivalent to Y if and only if for all $x \in X$ there exists $y \in Y$ with $A_x \equiv_n A_y$, $B_x \equiv_n B_y$, and vice versa. Here however we cannot merely consider n-equivalence of each part separately: the relationships between elements of different components still matter, and unlike in the linear case these are not necessarily determined by which components they lie in. We still know that $a > b$ for $a \in A_x$, $b \in B_x$, since $a > x > b$. However, points in A_x may be either above or incomparable with points in C_x (but not below, or we would have $c > a > x$, and by definition $c \not> x$), and likewise, points in B_x may be either below or incomparable with points in C_x. To account for this, we must also require n-equivalence of $A_x \cup C_x$ and $B_x \cup C_x$, however: there exist partial orders P and Q and $n \in \mathbb{N}$ such that for all $x \in P$ there exists $y \in Q$ such that $A_x \equiv_n A_y, B_x \equiv_n B_y, C_x \equiv_n C_y, A_x \cup C_x \equiv_n A_y \cup C_y$, and $B_x \cup C_x \equiv_n B_y \cup C_y$, and for all $y \in Q$ there exists $x \in P$ such that the above equivalences hold, but P and Q are not $(n+1)$-equivalent.

In fact, such partial orders even exist for $n = 2$.

Theorem 15. *There exist partial orders P, Q such that for all $p \in P$ there exists $q \in Q$ and vice versa such that $A_p \equiv_2 A_q, B_p \equiv_2 B_q, C_p \equiv_2 C_q$, $A_p \cup C_p \equiv_2 A_q \cup C_q$, and $B_p \cup C_p \equiv_2 B_q \cup C_q$, but P and Q are not 3-equivalent.*

Proof. Let P be a partial order consisting of countably many points $\{a_n : n \in \mathbb{Z}\} \cup \{b_n : n \in \mathbb{Z}\} \cup \{c_n : n \in \mathbb{Z}\} \cup \{d\}$.

For n even, let $a_n > c_n$, $a_{n+1} > c_n$, $a_n > c_{n+1}$, $a_{n+1} > c_{n+1}$. For n odd, let $b_n < c_n$, $b_{n+1} < c_n$, $b_n < c_{n+1}$, $b_{n+1} < c_{n+1}$, and let the transitive closure of these relations be the only relations on P. (See Figure 4.4 for a diagram of a finite quotient set of P, obtained by identifying n with $n + 6$ for each $n \in \mathbb{N}$.)

Let Q also consist of countably many points $\{x_n : n \in \mathbb{Z}\} \cup \{y_n : n \in \mathbb{Z}\} \cup \{z_n : n \in \mathbb{Z}\} \cup \{w\}$. Let the relations in Q be those induced by the following: for each $n \in \mathbb{Z}$,

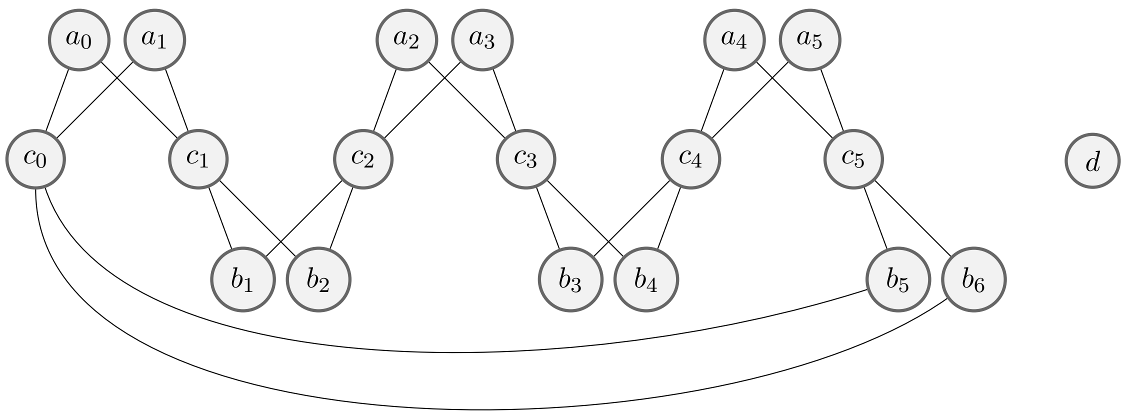

Figure 4.4: A finite quotient set of P obtained by setting $a_n = a_{n+6}$, $b_n = b_{n+6}$, $c_n = c_{n+6}$. Note that b_5 and b_6 lie below c_0; these lines are depicted as dipping slightly downwards only to avoid overlap.

$x_n > z_n$, $x_{n+1} > z_n$, $y_n < z_n$ and $y_{n+1} < z_n$. (See Figure 4.5 for a diagram of a finite quotient set of Q, obtained by identifying n with $n+4$ for each $n \in \mathbb{N}$.)

We show that for each $p \in P$ there exists $q \in Q$, and vice versa, such that $A_p \equiv_2 A_q, B_p \equiv_2 B_q, C_p \equiv_2 C_q$, $A_p \cup C_p \equiv_2 A_q \cup C_q$, and $B_p \cup C_p \equiv_2 B_q \cup C_q$. We also show that P and Q are not 3-equivalent.

First, we show that the points of P and Q have counterparts in the other structure such that each pair of induced components are 2-equivalent. We recall that two sets are 2-equivalent if and only if they realise the same 1-characters.

For a point $a_i \in P$, where $i \in \mathbb{Z}$, we have $A_{a_i} = \emptyset$ which realises no 1-characters. The set B_{a_i} contains the points $\{c_{2\lfloor i/2\rfloor}, c_{2\lfloor i/2\rfloor+1}, b_{2\lfloor i/2\rfloor-1}, b_{2\lfloor i/2\rfloor}, b_{2\lfloor i/2\rfloor+1}, b_{2\lfloor i/2\rfloor+2}\}$ with relations $b_{2\lfloor i/2\rfloor-1}, b_{2\lfloor i/2\rfloor} < c_{2\lfloor i/2\rfloor}$ and $b_{2\lfloor i/2\rfloor+1}, b_{2\lfloor i/2\rfloor+2} < c_{2\lfloor i/2\rfloor+1}$, so B_{a_i} is the union of two disjoint Λ shapes. The set of 1-characters realised by this is $\{(1,0,1),(0,1,1)\}$. The set C_{a_i} is the remainder of P, which realises the 1-characters $\{(1,0,1),(1,1,1),(0,1,1),(0,0,1)\}$. We have $A_{a_i} \cup C_{a_i} = C_{a_i}$, so it also realises precisely $\{(1,0,1),(1,1,1),(0,1,1),(0,0,1)\}$, and so does $B_{a_i} \cup C_{a_i}$.

Now consider $x_j \in Q$. Now $A_{x_j} = \emptyset$, B_{x_j} is an M shape which realises precisely $\{(1,0,1),(0,1,1)\}$. The set C_{x_j} again realises only $\{(1,0,1),(1,1,1),(0,1,1),(0,0,1)\}$, as

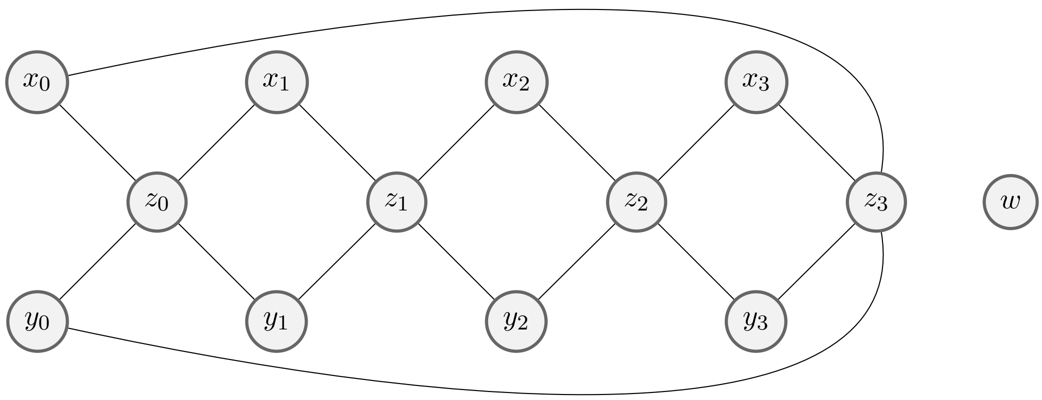

Figure 4.5: A finite quotient set of Q obtained by setting $x_n = x_{n+4}$, $y_n = y_{n+4}$, $z_n = z_{n+4}$.

do $A_{x_j} \cup C_{x_j}$ and $B_{x_j} \cup C_{x_j}$. Therefore, the a_is and x_js give components that realise precisely the same 1-characters, and so each corresponding pair of these components are 2-equivalent.

Since both P and Q are invariant under inverting $<$, we observe that the components A_{b_i}, B_{b_i} and C_{b_i} for $b_i \in P$ are just the components for a_i flipped upside down, and likewise the components for $y_j \in Q$ are the inverted components for x_j. Therefore, we also have $A_{b_i} \equiv_2 A_{y_j}, B_{b_i} \equiv_2 B_{y_j}, C_{b_i} \equiv_2 C_{y_j}$, $A_{b_i} \cup C_{b_i} \equiv_2 A_{y_j} \cup C_{y_j}$ and $B_{b_i} \cup C_{b_1} \equiv_2 B_{y_j} \cup C_{y_j}$, for any i, j.

For the central points $c_i \in P$, A_{c_i} and B_{c_i} are both antichains of size two, and C_{c_i} is large but realises only $\{(1,0,1),(1,1,1),(0,1,1),(0,0,1)\}$. So must $A_{c_i} \cup C_{c_i}$ and $B_{c_i} \cup C_{c_i}$, since every point is incomparable to d and so no characters of the form $(*,*,0)$ may be realised. For each $z_j \in Q$, A_{z_j} and B_{z_j} are also two point antichains, and C_{z_j}, $A_{z_j} \cup C_{z_j}$ and $B_{z_j} \cup C_{z_j}$ also realise all four of the possible characters $\{(1,0,1),(1,1,1),(0,1,1),(0,0,1)\}$, so again we find that the components produced from breaking down P and Q according to the relationship with c_i and z_j respectively form 2-equivalent pairs.

Finally, we consider the lone points d and w. (These were included to compensate for the fact that $C_{a_{2n}}$ for example contains an isolated point a_{2n+1} which is unrelated to anything else, but C_{x_m} would not contain any isolated points if not for w.) Everything is incomparable to these points, so we have $A = B = \emptyset$ and $C = A \cup C = B \cup C$ for each of them. We verify by inspection that C_d and C_w both realise precisely the characters in $\{(1,0,1),(1,1,0),(0,1,1)\}$, and so therefore do their unions with the empty A and B sets.

This enumeration has exhausted all the classes of points of both P and Q, and so the above correspondences between the (a_i) and the (x_i), the (b_i) and the (y_i), the (c_i) and the (z_i) and between d and w demonstrate that for all $p \in P$ there exists $q \in Q$, and vice versa, with $A_p \equiv_2 A_q, B_p \equiv_2 B_q, C_p \equiv_2 C_q$, $A_p \cup C_p \equiv_2 A_q \cup C_q$, and $B_p \cup C_p \equiv_2 B_q \cup C_q$. It remains only to show that P and Q are not 3-equivalent.

We show this by giving a winning strategy for Player I in $G_3(P,Q)$. Player I begins by playing z_0 in Q. Player II must respond by playing a point with something both above it and below it (otherwise, Player I wins on her next move by playing above/below z_0 accordingly), so she must play c_i for some i. Player I then plays z_1. There exists a point above both z_0 and z_1, and there exists another point below them both; however, Player II cannot achieve this setup in P.

Suppose that i is odd. If Player II plays c_{i+1}, then Player I wins by playing x_1, since x_1 lies above both z_0 and z_1 but no point in P lies above both c_i and c_{i+1}. Should Player II play a_{i-1}, a_i, a_{i+1} or a_{i+2}, then Player I wins by playing x_2, which lies above z_1, as no point lies above a_{i-1}, a_i, a_{i+1} or a_{i+2}. For any move other than $c_{i+1}, a_{i-1}, a_i, a_{i+1}$ and a_{i+2}, Player I wins by playing y_1, since y_1 lies below both z_0 and z_1 but Player II's second move does not have a lower bound in common with c_i.

If i is even, then Player I's strategy is analogous: If Player II plays c_{i+1}, then Player I wins by playing y_1, which lies below both z_0 and z_1; if she plays b_{i-1}, b_i, b_{i+1} or b_{i+2}, then Player I wins by playing y_2, which lies below z_1; and for any other move Player I wins by playing x_1.

Therefore, P and Q are not 3-equivalent and the proof is complete. □

The above examples are infinite but finite examples also exist. By choosing m and identifying a_n with a_{n+m}, b_n with b_{n+m}, and likewise for c_n, x_n, y_n and z_n, we may obtain crown-like partial orders which have finitely many points but have similar structures to P and Q as defined above. In particular, taking $m = 6$ gives the smallest example of this form. The values $m = 4, 5$ also work for Q but not for P, for which $m = 4$ is too small and odd values are incompatible with the desired alternating zigzag. The 19 point finite quotient of P and 13 point finite quotient of Q are therefore also 2-equivalent in all components but not 3-equivalent. These are depicted in Figures 4.4 and 4.5.

4.5 Iterated colouring system

Although we cannot break down our partial order into much smaller pieces as with linear orders, we nevertheless wish to investigate higher numbers of moves. We introduce an iterated colouring scheme to encode the information that would be necessary to classify n-equivalence of partial orders. This may also be thought of as a sequence of labels. After each move $x_i \in P$, we remove x_i and colour/label the remaining points of $P\backslash\{x_i\}$ according to their relationship with x_i. For the three-move situation, this need only be done once (and is therefore most obviously analogous to colouring); for more than three moves, iterated colouring is required.

For each $x \in P$ and $y \in P\backslash\{x\}$, let a colouring function c_x colour the points according to their relationship to x. Let $c_x(y) := c_0^{(x)}$ if $y > x$, $c_x(y) := c_1^{(x)}$ if $x > y$ and $c_x(y) := c_2^{(x)}$ otherwise. This is the x-induced colouring on $P\backslash\{x\}$. For three moves, the colouring need only be done once, as shown in Theorem 12, so there are only three colours $c_0^{(x)}, c_1^{(x)}$ and $c_2^{(x)}$, which we take to be blue, red and green respectively.

Let A_x, B_x and C_x be the points coloured blue, red and green respectively. (Equivalently, let them be the sets of points in $P\backslash\{x\}$ that are above, below and incomparable to x, respectively.) We consider the tuple $(P\backslash\{x\}, A_x, B_x, C_x)$.

Theorem 16. *$P \equiv_n Q$ if and only if for all $x \in P$ there exists some $y \in Q$, and vice versa, such that $(P\backslash\{x\}, A_x, B_x, C_x) \equiv_{n-1} (Q\backslash\{y\}, A_y, B_y, C_y)$*

Proof. First, suppose that $P \equiv_n Q$. Then Player II has a winning strategy in $G_n(P, Q)$. So for any first move x that Player I may make, without loss of generality let $x \in P$, there exists some $y \in Q$ such that if Player II plays y, she can proceed to win the game $G_{n-1}(P, Q)$. Therefore, Player II certainly has a winning strategy on $G_{n-1}(P\backslash\{x\}, Q\backslash\{y\})$. As the game continues, Player I may choose to play a subsequent move in A_x, which would lie above x in P, and since Player II's strategy wins it must provide a move in A_y, as her response must lie above y in Q. Likewise, the moves produced by the winning strategy in response to moves in A_y, B_x, B_y, C_x and C_y must lie in the corresponding component of the other structure. So Player II's original winning strategy in $G_n(P, Q)$ gives her a winning strategy in $G_{n-1}((P\backslash\{x\}, A_x, B_x, C_x), (Q\backslash\{y\}, A_y, B_y, C_y))$ and we are done.

Conversely, suppose that for all $x \in P$ there exists some $y \in Q$, and vice versa, such that $(P\backslash\{x\}, A_x, B_x, C_x) \equiv_{n-1} (Q\backslash\{y\}, A_y, B_y, C_y)$. We construct a winning strategy for Player II in $G_n(P, Q)$. Without loss of generality let Player I's first move be $x \in P$, then by assumption there exists some $y \in Q$ such that $(P\backslash\{x\}, A_x, B_x, C_x) \equiv_{n-1} (Q\backslash\{y\}, A_y, B_y, C_y)$; we take this y as the first move of Player II's strategy. Now, assuming that she does not repeat a move (which is not to her advantage, as Player II can just repeat her corresponding move), each of Player I's subsequent moves x_i in $G_n(P, Q)$ must lie in either A_x, A_y, B_x, B_y, C_x or C_y, to which Player II's winning strategy in $G_{n-1}((P\backslash\{x\}, A_x, B_x, C_x), (Q\backslash\{y\}, A_y, B_y, C_y))$ will give a response y_i in A_y, A_x, B_y, B_x, C_y or C_x respectively. But since x_i and y_i are always in corresponding components (relative to x and y) of opposite structures, the moves given by this strategy also form a winning strategy for Player II in $G_n(P, Q)$, and so $P \equiv_n Q$. □

We remark briefly on the distinction between this colouring scheme and the setup for our earlier result where we showed that the existence of, for all $p \in A$, some $q \in B$, and vice versa, such that $A_p \equiv_{n-1} A_q, B_p \equiv_{n-1} B_q, C_p \equiv_{n-1} C_q$, $A_p \cup C_p \equiv_{n-1} A_q \cup C_q$, and $B_p \cup C_p \equiv_{n-1} B_q \cup C_q$, does not guarantee n-equivalence. In the previous scenario, we did not distinguish the correct location of A within $A \cup C$ nor B within $B \cup C$, and so although substrategies on each of these subsets existed, they did not need to be compatible. The inequivalent example given in that section used a "misalignment" between the A and B sets and the wider structure: in one structure, each pair of consecutive middle layer points shared both an upper and a lower bound, while in the other structure each pair of consecutive middle layer points shared either two upper or two lower bounds, but never one of each. With this colouring scheme, this sort of "misalignment" is strictly prohibited - each x's counterpart y must not only have similar sets of points above (A_y) and below (B_y) it, but these must fit into the wider structure in a similar way, since we now require that a copy of $P\backslash\{x\}$ with A_x and B_x marked out with distinct colours be $(n-1)$-equivalent to the copy of $Q\backslash\{y\}$ with A_y and B_y distinguished by colour. Our earlier example would fail this strengthened 2-equivalence condition, since given x there does not always exist y such that $(P\backslash\{x\}, A_x, B_x, C_x) \equiv_2 (Q\backslash\{y\}, A_y, B_y, C_y)$.

This result allows us to analyse n-move games on monochromatic partial orders in terms of $(n-1)$-move games on 3-coloured partial orders. However, it is more broadly applicable - we can apply an analogous method to n-move games on multicoloured partial orders. If there are m colours colouring P initially, then we require at most $3m$ colours for $(P\backslash\{x\}, A_x, B_x, C_x)$, to distinguish the points of colour c_i which lie above, below or incomparable to x, for each i. In practice, we let these $3m$ colours be tuples (c, j), where c is one of the original m colours and $j \in \{1, 2, 3\}$. By iterating this process, we can therefore turn an n move game on an m-coloured partial order into a 1-move game on a $3^{n-1}m$-coloured partial order, where the colouring depends on the $n-1$ choices of x with respect to which we colour.

While this notion of character does contain all the necessary information to determine n-equivalence (unlike the more reductive linear-like notion discussed in Section 4.4), it may unfortunately be too unwieldy to be of much practical use, since the partial orders to which we reduce are almost as large as the partial order from which we started, and have complicated labels. The equivalence may be useful to know for future work, however, and in any case the apparent unavailability of an inductive equivalence condition depending on (much) smaller subsets illustrates the relative complexity of classifying partial orders when compared to linear orders.

Chapter 5

Cyclic orders

5.1 Cyclic orders

A *cyclic order*, or *circular order*, is a set X equipped with a ternary relation R, where we think of $R(a, b, c)$ as holding when, starting from a and proceeding anticlockwise around the circle on which the set is arranged, one reaches b before one reaches c. Formally, this relation satisfies the following axioms:

- If $R(a, b, c)$, then $R(b, c, a)$.
- If $R(a, b, c)$, then $\neg R(c, b, a)$.
- If $R(a, b, c)$ and $R(a, c, d)$, then $R(a, b, d)$.
- If a, b, and c are distinct, then either $R(a, b, c)$ or $R(a, c, b)$.

Conventionally, we take $R(a, b, c)$ to hold when, going anticlockwise from a, one reaches b before c, but so long as one is consistent the choice between clockwise and anticlockwise is unimportant. Choosing the clockwise direction would also satisfy the above axioms and gives a complementary structure to that obtained with the anticlockwise sense: by the final bullet point above, either $R(a, b, c)$ or $R(a, c, b)$ holds for each choice of distinct a, b, c, so $R(a, b, c)$ holds in the clockwise sense if and only if $R(a, c, b)$ holds in the anticlockwise sense. We remark that we could also have opted to use a non-strict, irreflexive relation R', in which $R'(a, a, b)$ and $R'(a, b, b)$ hold for any a, b. As with $<$ and $\leq$ in partial orders, it does not matter whether we use R' or R, since our language contains equality, and so we may reconstruct either the strict or non-strict relation from the other using the equivalences $R(a,b,c) \leftrightarrow R'(a,b,c) \wedge (a \neq b \neq c)$ and $R'(a,b,c) \leftrightarrow R(a,b,c) \vee (a = b) \vee (b = c)$.

Given a linear order L, we may create a related cyclic order by setting $R(a,b,c)$ if and only if $a < b < c$ or $b < c < a$ or $c < a < b$, for each $a, b, c \in L$. We call this *rolling* L, and may think of it as "bending L into a circle", since if one were to pick up a line segment containing L and curl it back on itself to form a circle, the ordering on this circle would have the new cyclic order. Conversely, given a cyclic order C, we may construct a linear order on $C\backslash\{x\}$ for any point $x \in C$ by setting $a < b$ if and only if $R(x,a,b)$ in C. This linear order is called the *cut* of C at x, and the ordering may be denoted by $<_x$.

This is not the only natural way to turn a cyclic order into a linear order. For example, we have a natural notion of density: a cyclic order C is *dense* if for any $a, b \in C$ there exists some $c \in C$ with $R(a,c,b)$. Then if C is not dense, we could also split C between two consecutive elements a and b, by adding some c such that $R(a,c,b)$ and then taking the cut of C at c. Likewise, we could add a new point in some other natural way, such as a point corresponding to a notion of a limit point or a completion point under some metric, and cut there. However, for the purposes of Ehrenfeucht-Fraïssé games, which are of finite length and involve finitely many actual points, we are interested in the possibilities that arise on either side of an actual point in a cyclic order, and so all of our cuts will occur at points of the cyclic order.

Since the relation of least arity here is ternary, all nonempty cyclic orders must be 1-equivalent and all cyclic orders with at least two points must be 2-equivalent, as we showed earlier in Lemma 8.

Theorem 17. *Let $a, b \in \mathbb{N}$ and let C_a and C_b be cyclic orders with $|C_a| = a$ and $|C_b| = b$. Then $C_a \equiv_n C_b$ if and only if the linear order of size $a-1$ is $(n-1)$-equivalent to the linear order of size $b-1$.*

Proof. Suppose $a - 1 \equiv_{n-1} b - 1$ as linear orders and consider $G_n(C_a, C_b)$. On her first move, Player I plays some element $x_1 \in C_a$, and Player II may respond with any element of C_b. As C_b is entirely symmetric, it does not matter which point Player II chooses; select an arbitrary move in C_b and call it y_1.

The remainder of the game is to be played on $C_a \backslash \{x_1\}$ and $C_b \backslash \{y_1\}$. Player II has a winning strategy if these moves can be selected such that $R(x_1, x_i, x_j)$ if and only if $R(y_1, y_i, y_j)$ for each $1 < i, j \leq n$ (which implies, by transitivity of R, that $R(x_i, x_j, x_k)$ if and only if $R(y_i, y_j, y_k)$ for all $i, j, k \leq n$). This is possible if the moves can be selected such that $x_i < x_j$ if and only if $y_i < y_j$, using the linear orders induced on $C_a \backslash \{x_1\}$ and $C_b \backslash \{y_1\}$ by setting $c < d$ if and only if $R(x_1, c, d)$ in C_a and $z < w$ if and only if $R(y_1, z, w)$ in C_b respectively. Player II has a strategy to accomplish this if $a - 1 \equiv_{n-1} b - 1$, so $a - 1 \equiv_{n-1} b - 1$ implies that Player II also has a winning strategy on $G_n(C_a, C_b)$.

Conversely, suppose that $a - 1 \not\equiv_{n-1} b - 1$ as linear orders. Then Player I has a winning strategy in $G_{n-1}(a - 1, b - 1)$. In $G_n(C_a, C_b)$, let Player I select an arbitrary element as her first move, and Player II will naturally respond in the other cyclic order. Let the moves played so far be $x \in C_a$ and $y \in C_b$. Then, Player I considers the cut of C_a at x and the cut of C_b at y. These are linear orders of size $(a - 1)$ and $(b - 1)$ respectively. Player I has a winning strategy on $G_{n-1}(a - 1, b - 1)$, and she plays according to this. Player II's moves in $G_n(C_a, C_b)$ must respect the cyclic relations with x or y, and so must respect the linear orders of these cuts, so her moves must be valid moves in $G_{n-1}(a - 1, b - 1)$. Player I's winning strategy on $G_{n-1}(a - 1, b - 1)$ therefore gives the remainder of a winning strategy on $G_n(C_a, C_b)$ and so these structures are inequivalent. $\square$

Corollary 18. *Let C_1 and C_2 be cyclic orders of finite size a and b respectively. Then $C_1 \equiv_n C_2$ if and only if either $|a| = |b| < 2^{n-1}$ or $|a|, |b| \geq 2^{n-1}$*

Proof. We know from [25] that for finite linear orders L_1, L_2 of finite size a and b respectively, $L_1 \equiv_n L_2$ if and only if either $a = b < 2^n - 1$ or $a, b \geq 2^n - 1$. Since C_1 and C_2 are n-equivalent if and only if the finite linear orders obtained by cutting them are $(n - 1)$-equivalent, this corollary follows. $\square$

In fact, an analogous result to Theorem 17 also holds for infinite cyclic orders:

Theorem 19. *Let C_1 and C_2 be cyclic orders. Then $C_1 \equiv_n C_2$ if and only if for all $x \in C_1$ there exists $y \in C_2$ such that $(C_1 \backslash \{x\}, <_x) \equiv_{n-1} (C_2 \backslash \{y\}, <_y)$ as linear orders, and for all $y \in C_2$ there exists $x \in C_1$ such that the same condition holds.*

Proof. Suppose first that there is some $x_1 \in C_1$ (without loss of generality) such that for no $y \in C_2$ does $(C_1\backslash\{x_1\}, <_{x_1}) \equiv_{n-1} (C_2\backslash\{y\}, <_y)$, and consider $G_n(C_1, C_2)$. Player I may play x_1 as her first move, and whatever Player II's response y_1, we must have $(C_1\backslash\{x_1\}, <_{x_1}) \not\equiv_{n-1} (C_2\backslash\{y_1\}, <_{y_1})$. Therefore Player I has a winning strategy σ in $G_{n-1}((C_1\backslash\{x_1\}, <_{x_1}), (C_2\backslash\{y_1\}, <_{y_1}))$. Player I may play the moves provided by σ for her remaining moves of $G_n(C_1, C_2)$. At the end of the game, the selected points $x_1, \ldots, x_n \in C_1$ and $y_1, \ldots, y_n \in C_2$ must have provided a win for Player I in the subgame $G_{n-1}((C_1\backslash\{x_1\}, <_{x_1}), (C_2\backslash\{y_1\}, <_{y_1}))$, and so there must exist $i, j > 1$ such that $x_i <_{x_1} x_j$ but $y_j <_{x_1} y_i$, or vice versa. But then $R(x_i, x_j, x_1)$ holds but $R(y_i, y_j, y_1)$ does not, so Player I has won $G_n(C_1, C_2)$ using this strategy.

Conversely, suppose that $C_1 \not\equiv_n C_2$. Then there exists some strategy σ for Player I such that she will win $G_n(C_1, C_2)$, and we must have $n \geq 2$ (since all nonempty cyclic orders are 1-equivalent). Let the first move of σ be x_1 and without loss of generality let $x_1 \in C_1$. Player II's response must lie in C_2, call it y_1. There must exist subsequent moves, which must lie in $C_1\backslash\{x_1\}$ and $C_2\backslash\{y_1\}$ (always choosing the same point is not a winning strategy for Player I). If $n = 2$, then we are in the case where exactly one of C_1 or C_2 has size 1, so the conclusion holds, since either $C_1\backslash\{x_1\}$ or $C_2\backslash\{y_1\}$ is empty and so they are not 1-equivalent.

If $n > 2$, let the subsequent moves be $x_2, \ldots, x_n \in C_1$ and $y_2, \ldots, y_n \in C_2$. We know that $(x_1, \ldots, x_n) \not\cong (y_1, \ldots, y_n)$, since Player I played her winning strategy. Therefore, there exist distinct i, j, k such that $R(x_i, x_j, x_k)$ but $R(y_j, y_i, y_k)$. Possibly we have $i = 1$ or $j = 1$ or $k = 1$ already, in which case we proceed to the next paragraph. If $i, j, k \neq 1$, then x_1 must lie in one of these three intervals, so one of these sets of relations holds: $\{R(x_i, x_1, x_j),\ R(x_1, x_j, x_k),\ R(x_k, x_i, x_1)\}$, $\{R(x_j, x_1, x_k),\ R(x_1, x_k, x_i),\ R(x_i, x_j, x_1)\}$ or $\{R(x_k, x_1, x_i), R(x_1, x_i, x_j), R(x_j, x_k, x_1)\}$. But similarly, either $\{R(y_j, y_1, y_i), R(y_1, y_i, y_k), R(y_k, y_j, y_1)\}$, $\{R(y_i, y_1, y_k),\ R(y_1, y_k, y_j),\ R(y_j, y_i, y_1)\}$ or $\{R(y_k, y_1, y_j),\ R(y_1, y_j, y_i), R(y_i, y_k, y_1)\}$ hold. In any of the nine possible combinations, we find some $a, b \in \{i, j, k\}$ such that $R(x_1, x_a, x_b)$ but $R(y_1, y_b, y_a)$.

Therefore, $x_a <_{x_1} x_b$ but $y_b <_{y_1} y_a$, and so $(C_1\backslash\{x_1\}, <_{x_1}) \not\equiv_{n-1} (C_2\backslash\{y_1\}, <_{y_1})$. Since we made no assumptions about y_1, we conclude that x_1 witnesses that it is false that for all $x \in C_1$ there exists $y \in C_2$, and vice versa, such that $(C_1 \backslash \{x\}, <_x) \equiv_{n-1} (C_2 \backslash \{y\}, <_y)$, thus completing the proof.

□

5.2 3-equivalence of cyclic orders

We may therefore classify the cyclic orders up to 3-equivalence using existing results about 2-equivalence of linear orders.

Theorem 20. *There are 15 circular orders up to* 3*-equivalence.*

Proof. By the above result, two circular orders (X, R_X) and (Y, R_Y) are 3-equivalent if and only if for every $x \in X$ there exists $y \in Y$, and vice versa, such that $X \backslash \{x\} \equiv_2 Y \backslash \{y\}$, where each has the linear ordering induced on it from the cut. We may therefore characterise a circular order (X, R) up to 3-equivalence by the set of 2-equivalence classes of the linear orders $(X \backslash \{x\}, <_x)$ for $x \in X$, which we view as the characters of the points of X.

Recall that the linear orders up to 2-equivalence are $0, 1, 2, 3, \omega, \omega^*$, and $\mathbb{Z}$. If $(X \backslash \{x\}, <_x) \equiv_2 0$ then $X = \{x\}$; similarly if $(X \backslash \{x\}, <_x) \equiv_2 1$ then X is the two-element circular order and if $(X \backslash \{x\}, <_x) \equiv_2 2$ then X is the three-element circular order. In each of these cases, no distinct characters can occur in X, and so the possible sets of characters are just the singletons $\{0\}, \{1\}$ and $\{2\}$. This gives us three equivalence classes of cyclic orders.

The remaining classes arise from considering the remaining possible characters: $3, \omega, \omega^*$ and $\mathbb{Z}$. Of the sixteen subsets of these, 12 can arise; we rule four out, and exhibit circular orders corresponding to each of the others.

We may define notions of predecessor and successor analogously to linear orders: a is the *immediate predecessor* of b if $a \neq b$ and there is no $c \in C$ such that $R(a, c, b)$, and also b is the *immediate successor* of a in this case. Having already dispensed of the case of cyclic orders of size less than three, we may assume that a point's immediate predecessor and immediate successor (if it has them) are distinct. We remark that 3 is realised if and only if there is some point with both an immediate predecessor and successor, ω is realised if and only if there is some point with an immediate successor but no immediate predecessor, ω^* is realised if and only if there is some point with an immediate predecessor but no immediate successor, and $\mathbb{Z}$ is realised if and only if there is some point with neither an immediate predecessor nor a successor, since the predecessor and successor if present become endpoints of the remaining linear order. Therefore, $\{\omega\}$ cannot occur, since if $(X \backslash \{x\}, <_x)$ has order type ω then x has a successor y and no predecessor, but then y has a predecessor, x, and so y cannot have character ω. Similarly, $\{\omega, \mathbb{Z}\}$ cannot occur, as the point of character ω has a successor which has a predecessor, and points of characters ω and $\mathbb{Z}$ have no predecessor. By reversing the orderings for these, we also see that $\{\omega^*\}$ and $\{\omega^*, \mathbb{Z}\}$ cannot arise.

The other twelve characters are realisable:

- $\emptyset$ is realised by the empty cyclic order
- $\{3\}$ is realised by a cyclic order of finite size $n \geq 4$
- $\{\mathbb{Z}\}$ is realised by $\mathbb{Q}$ "rolled into a circle" (a countable dense cyclic order)
- $\{3, \omega\}$ is realised by ω rolled into a circle
- $\{3, \omega^*\}$ is realised by ω^* rolled into a circle
- $\{3, \mathbb{Z}\}$ is realised by $\mathbb{Z} + \mathbb{Q}$ rolled into a circle
- $\{\omega, \omega^*\}$ is realised by $\mathbb{Q} \times \{0, 1\}$ rolled into a circle
- $\{3, \omega, \omega^*\}$ is realised by $\mathbb{Q} \times \{0, 1, 2\}$ or $\mathbb{Z} + 2$ rolled into a circle
- $\{3, \omega, \mathbb{Z}\}$ is realised by $\omega + \mathbb{Q}$ rolled into a circle
- $\{3, \omega^*, \mathbb{Z}\}$ is realised by $\omega^* + \mathbb{Q}$ rolled into a circle
- $\{\omega, \omega^*, \mathbb{Z}\}$ is realised by $\mathbb{Q} + 2$ rolled into a circle
- $\{3, \omega, \omega^*, \mathbb{Z}\}$ is realised by $\mathbb{Q} + 3$ rolled into a circle

We remark that this list directly corresponds to the list of twelve possible linear orders without endpoints up to 3-equivalence. This is because both depend only on which points have successors and/or predecessors, since cyclic orders have no endpoints and the issue of endpoints has been deliberately excluded from consideration of the linear orders without endpoints. In cyclic orders, a character of 3 indicates a point with both a successor and a predecessor, corresponding to (ω^*, ω) in the linear orders without endpoints. Similarly, ω and $(\mathbb{Z}, \omega)$ correspond to points with a successor but no predecessor, ω^* and $(\omega^*, \mathbb{Z})$ correspond to points with a predecessor but no successor, and $\mathbb{Z}$ and $(\mathbb{Z}, \mathbb{Z})$ correspond to points with neither a predecessor nor a successor. Having determined that one cannot have a point with a predecessor if there are no points with a successor, or vice versa, the remaining twelve possibilities for predecessor/successor combinations give the twelve possibilities in each case. In fact, the twelve representatives given for linear orders without endpoints give representatives for the cyclic orders when rolled. $\square$

Deducing the n-equivalence classes of cyclic orders from the $(n-1)$-equivalence classes of linear orders is not in general quite so straightforward. For 4-equivalence, rather than considering only the predecessor and successor of a point, we would need to consider the small characters (in the sense of Chapter 3) arising to its immediate left and right, as well as considering the large characters arising in points elsewhere, which themselves depend both on their predecessors/successors and on the endpoints of the remaining linear order. Likewise, for n-equivalence, we need to consider all the $(n-1)$-characters that may arise in the linear order obtained by each cut, which means determining the precise behaviour near the newly designated endpoints (or cofinal chains) in increasing detail. Naturally, the other points arising near a potential endpoint are also themselves potential candidates for a cut, and so the relationships between the characters that may co-occur are intertwined in a way that does not occur for linear orders where behaviour between the endpoints need only be considered within the context of that fixed choice of splitting point.

5.3 Partial cyclic orders

We may define partial cyclic orders, which differ from cyclic orders in an analogous way to how partial orders differ from linear orders. Precisely, a *partial cyclic order* is a set X equipped with a ternary relation R such that

- If $R(a, b, c)$, then $R(b, c, a)$.
- If $R(a, b, c)$, then $\neg R(c, b, a)$.
- If $R(a, b, c)$ and $R(a, c, d)$, then $R(a, b, d)$.

Note that this is a proper subset of the set of axioms for a cyclic order. It is immediate that all cyclic orders are partial cyclic orders. In a cyclic order, we additionally require that for all distinct a, b, and c, either $R(a, b, c)$ or $R(a, c, b)$. This is the axiom of totality, and just as the linear orders are precisely the partial orders that are total, the cyclic orders are precisely the partial cyclic orders that are total.

If a partial cyclic order X may be extended to a (total) cyclic order Y, then we say that X is *totalisable*.

Lemma 21. *Not every partial cyclic order may be extended to a cyclic order.*

Proof. A counterexample is given by Megiddo in [21]:

Let $X = \{a, b, c, d, e, f, g, h, i, j, k, l, m\}$ and let the following relations hold on X: $R(a, c, d)$, $R(b, d, e)$, $R(c, e, f)$, $R(d, f, g)$, $R(e, g, h)$, $R(f, h, a)$, $R(g, a, c)$, $R(h, c, b)$, $R(a, b, i)$, $R(c, i, j)$, $R(b, j, k)$, $R(i, k, l)$, $R(j, l, m)$, $R(k, m, a)$, $R(l, a, b)$, $R(m, b, c)$, $R(h, c, m)$, $R(b, h, m)$, as well as their cyclic permutations (e.g., $R(a, c, d)$ also entails $R(c, d, a)$ and $R(d, a, c)$, and likewise for all other triples of related elements). This satisfies one of our three axioms for a partial cyclic order by definition: if $R(a, b, c)$, then $R(b, c, a)$. For transitivity we require that if $R(a, b, c)$ and $R(a, c, d)$, then $R(a, b, d)$; this may be seen by noting that the only pairs of elements that appear together in more than one cycle are the pairs in $\{a, b, c\}^2$ and in $\{b, c, h, m\}^2$, and we can verify that each transitive consequence is included above. We also require X to satisfy the asymmetry axiom: if $R(a, b, c)$, then not $R(c, b, a)$; since the above list is transitively closed, this can be easily verified by noting that no two triples contain the same three elements in orders of opposite chirality, and so (X, R) is a partial cyclic order.

However, X cannot be extended to a cyclic order. Any cyclic order X' extending X must be total, and so it must realise either the relation $R(a, b, c)$ or $R(a, c, b)$. However, if the relation $R(a, b, c)$ is added to the set, then since $R(a, c, d)$ is in our set we must also add $R(b, c, d)$, and since $R(b, d, e)$ also holds we must have $R(c, d, e)$, and so on. After applying this chain of reasoning to each of $R(a, c, d)$, $R(b, d, e)$, $R(c, e, f)$, $R(d, f, g)$, $R(e, g, h)$, $R(f, h, a)$, $R(g, a, c)$, $R(h, c, b)$ in order, we deduce that $R(a, c, b)$ holds on X'. But then X' is not a cyclic order, since both $R(a, b, c)$ and $R(a, c, b)$ hold.

Suppose instead that X' contains the relation $R(a, c, b)$. Using another sequence of relationships on X, $R(a, b, i)$, $R(c, i, j)$, $R(b, j, k)$, $R(i, k, l)$, $R(j, l, m)$, $R(k, m, a)$, $R(l, a, b)$, $R(m, b, c)$, we get a similar chain of deductions: $R(a, c, b)$ and $R(a, b, i)$ implies $R(c, b, i)$; $R(c, b, i)$ and $R(c, i, j)$ imply $R(b, i, j)$, and so on, and finally we deduce $R(a, b, c)$.

It is therefore impossible for a total extension of X to contain precisely one of $R(a, b, c)$ and $R(a, c, b)$, and so there is no cyclic order extending X. $\square$

We recall that any linear order corresponds to a unique (up to choice of direction) cyclic order, and that a cyclic order likewise induces a linear order for each possible choice of point, gap, or completion point to cut at. There are likewise interesting relationships between (non-cyclic) partial orders and partial cyclic orders.

5.4 Partial orders and partial cyclic orders

Theorem 22. *Let $(P, <)$ be a partial order, and let us define R as follows: $R(a, b, c)$ if and only if $a < b < c$ or $b < c < a$ or $c < a < b$ for each $a, b, c \in L$. Then (P, R) is a partial cyclic order.*

Proof. We verify the three axioms listed above: cyclicity, asymmetry and transitivity.

Suppose that $R(a, b, c)$ holds in (P, R). Then either $a < b < c$ or $b < c < a$ or $c < a < b$ in $(P, <)$. But of course this is precisely the same as $b < c < a$ or $c < a < b$ or $a < b < c$ holding in $(P, <)$, so also $R(b, c, a)$.

For asymmetry, again suppose that $R(a,b,c)$ holds in (P,R), which implies that $a < b < c$ or $b < c < a$ or $c < a < b$ in $(P,<)$. Because $(P,<)$ is a partial order, $<$ is asymmetric, so we do not have $a < c < b$ or $b < a < c$ or $c < b < a$ in $(P,<)$, since any of these would give a contradiction. We therefore deduce that $R(c,b,a)$ cannot hold in (P,R).

Finally, we assume that both $R(a,b,c)$ and $R(a,c,d)$ hold, and hope to show $R(a,b,d)$. By definition, $R(a,b,c)$ implies that $a < b < c$ or $b < c < a$ or $c < a < b$ in $(P,<)$, and similarly $R(a,c,d)$ implies that $a < c < d$ or $c < d < a$ or $d < a < c$ in $(P,<)$. The relations $a < b < c$ and $a < c < d$ together imply $a < b < d$; $a < b < c$ and $d < a < c$ together imply $d < a < b$; $b < c < a$ and $c < d < a$ together imply $b < d < a$; and $c < a < b$ and $c < d < a$ together imply $d < a < b$. The other five combinations are impossible, since we would have both $a < c$ and $c < a$. Therefore, whichever possible combination holds, we have $R(a,b,d)$ in (P,R) and so R is transitive. □

We remark that this is analogous to our earlier process of rolling for converting linear orders into cyclic orders. Given an arbitrary partial order, we can perform the above rolling operation on it to give a partial cyclic order, and the converse question naturally arises - given an arbitrary partial cyclic order, can we always obtain a corresponding partial order? We can already answer this in the negative.

Lemma 23. *Every partial cyclic order obtained by rolling a partial order must be extensible to a total cyclic order.*

Proof. It is well known that every partial order may be extended to a total order; this is the Szpilrajn extension theorem [34]. Let P be a partial order, and let $\prec$ be some linear order on P extending $<$. Let $(P, R_<)$ be obtained by letting, for each $a,b,c \in P$, $R_<(a,b,c)$ if and only if $a < b < c$ or $b < c < a$ or $c < a < b$ in P. As shown in Theorem 22, $(P, R_<)$ is a partial cyclic order. Similarly, let $(P, R_\prec)$ be obtained by letting, for each $a,b,c \in P$, $R_\prec(a,b,c)$ if and only if $a \prec b \prec c$ or $b \prec c \prec a$ or $c \prec a \prec b$. Then $(P, R_\prec)$ is a (total) cyclic order, as we verify below.

First, we note that $(P,\prec)$ is itself a partial order (specifically, a total one), so $(P, R_\prec)$ must be a partial cyclic order by the previous result in Theorem 22. Therefore, $(P, R_\prec)$

must satisfy the axioms of cyclicity, antisymmetry and transitivity, and it remains only to verify that $(P, R_\prec)$ is total. Let $a, b, c \in P$ be distinct. By linearity of $(P, \prec)$, either $a \prec b \prec c$, $b \prec c \prec a$, $c \prec a \prec b$, $a \prec c \prec b$, $b \prec a \prec c$ or $c \prec b \prec a$ must hold, and therefore either $R_\prec(a, b, c)$ or $R_\prec(a, c, b)$ must hold.

We have established that $(P, R_\prec)$ is total; now we show that it extends $(P, R_<)$. Suppose that $R_<(a, b, c)$ holds in the partial cyclic order $(P, R_<)$, for some $a, b, c \in P$. Then either $a < b < c$ or $b < c < a$ or $c < a < b$ in the partial order $(P, <)$. By assumption, $\prec$ extends $<$, so whichever of these is true, it must certainly also hold in $(P, \prec)$. But then when we derive $(P, R_\prec)$ from $(P, \prec)$, we must deduce that $R_\prec(a, b, c)$ holds in $(P, R_\prec)$. Therefore $(P, R_\prec)$ extends the partial order $(P, R_<)$. $\square$

Therefore any partial cyclic order which does not extend to a total cyclic order, such as the one exhibited above in the proof of Lemma 23, cannot be obtained by rolling a partial order.

5.5 Totalisable partial cyclic orders

Stehr showed in [33] that his generalised notion of cyclic orders, which is a multi-arity generalisation of the notion of partial cyclic orders considered here, are totalisable precisely when they admit a *clock representation* - that is, when they can be arranged on a plane such that all cycles (here, the triplets representing the partial cyclic relation R) are oriented clockwise around a common centre. In [9], Haar gives two further equivalent conditions. One uses separating sets, thereby giving a useful notion of cut: a partial cyclic order P is totalisable if and only if it may be extended to a (partially cyclically ordered) superstructure P' in which every *cut*, a maximal subset of P' such that no pair extends to a related triple, intersects every *cycle*, a finite sequence of edges whose endpoints match up in the obvious way. An *edge* in this context is defined precisely in [9], but may be thought of as an interval in some component cycle of P' within which the points may be linearly ordered.

The other equivalence given is to a "winding" of a periodic partial order, and the proof of the equivalence gives a method to construct a periodic partial order P and a group G

of automorphisms of P from a partial cyclic order Q, such that the orbits of P under G form Q. This is different to our previous notion of rolling a partial order, and is in fact more general. Instead of rolling an arbitrary partial order P as above, we may instead wind the periodic partial order $\mathbb{Z} \times P$, to give the same partial cyclic order. However, partial cyclic orders obtained by the method of rolling must have a "gap" between one copy of P and the next, such that all points in a lower copy of P are below all points in the higher copy. This constrains the relations that may arise. In contrast, winding a periodic partial order allows the copies to be more intertwined, and there may not exist such a gap. For example, let $P = \{a, b, c, d, e, f\}$, with the following relations: $R(d, e, f)$, $R(a, e, f)$, $R(b, f, d)$, $R(c, d, e)$, and their cyclic and transitive closure. Then P is a partial cyclic order, and moreover it is totalisable, as we know immediately from the fact that it admits a clock representation (depicted in Figure 5.1), or by verifying that, for example, the total cyclic order consisting of a, d, b, e, c, f in that order extends P. We may obtain P by winding the partial order Q, shown in Figure 5.2, consisting of $\bigcup_{i\in\mathbb{Z}}\{a_i, b_i, c_i, d_i, e_i, f_i\}$ with the relations $d_n < e_n$, $e_n < f_n$, $f_n < d_{n+1}$, $a_n < e_n$, $b_n < f_n$, $c_n < d_{n+1}$, $d_n < b_n$, $e_n < c_n$, $f_n < a_{n+1}$, for each $n \in \mathbb{Z}$. However, we cannot find any partial order which may be rolled to give P, as its set of maximal points would be a maximal antichain A which could be extended to related triples by precisely the same pairs, and no such subset of P exists.

Relating the equivalence classes of partial cyclic orders to those of partial orders is therefore not as straightforward as in the total cyclic order to linear order case. However, the known equivalence to periodic partial orders does give a potential route to classifying partial cyclic orders. Moreover, the method of rolling inequivalent partial orders to give inequivalent partial cyclic orders (up to new equivalencies caused by no longer distinguishing maximality and minimality, of course) would still suffice to provide lower bounds on the number of n-equivalence classes of partial cyclic orders.

Considering the n-equivalence classes for small n, we see from Lemma 8 that all nonempty partial cyclic orders are 1-equivalent, and all partial cyclic orders with at least 2 points are 2-equivalent, just as with cyclic orders, because the cycle relation is ternary. Therefore, 3-equivalence of partial cyclic orders is the first interesting length of game. Re-

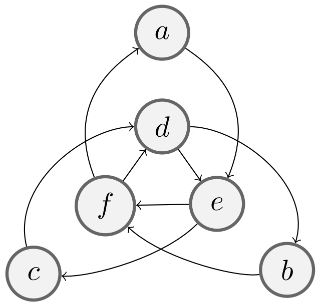

Figure 5.1: A totalisable partial cyclic order that cannot arise from rolling a partial order.

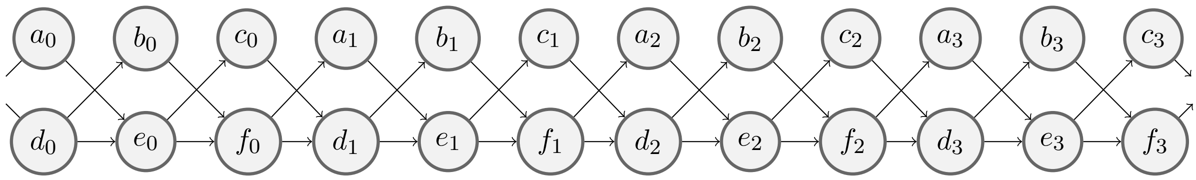

Figure 5.2: A (finite fragment of a) periodic partial order that may be wound into the partial cyclic order in Figure 5.1 above.

call that there are as many 3-equivalence classes of total cyclic orders as of 3-equivalence classes of linear orders without endpoints; as discussed above, the correspondence is not so direct for partial cyclic orders, but it may still give us a lower bound.

We note that there are 7 2-equivalence classes and 281 3-equivalence classes of linear orders, which give rise to 15 2-equivalence classes of cyclic orders. For partial orders, we know there to be 39 2-equivalence classes, and the number of 3-equivalence classes is expected to be rather larger. We therefore suggest that it may be rather impractical to determine the 3-equivalence classes of partial orders and thereby obtain this lower bound on the number of 3-equivalence classes of totalisable partial cyclic orders, but perhaps it is tractable by identifying a sufficing subset of the 3-equivalence classes of partial orders. Recall that the linear orders without endpoints up to 3-equivalence already gave enough information to determine the cyclic orders up to 3-equivalence, and the considerable additional complexity arising from the endpoints of linear orders was irrelevant to the cyclic case. Similarly, the 3-equivalence classes of partial orders will depend on the local structure near the endpoints, though the "endpoints" may be considerably more complex than in the linear case, and it may be possible to extract only the information about global structure necessary to provide distinct 3-equivalence classes of partial cyclic orders. Since this would only give a lower bound, however, we suggest that the more promising route may be to calculate the exact number of 3-equivalence classes of totalisable partial cyclic orders via study of 3-equivalence classes of periodic partial cyclic orders.

Chapter 6

Other structures

6.1 Trees

One may encounter multiple definitions of the word “tree” in mathematics. Here we take a rather broad definition: a *tree* is a partial order T such that for each $x \in T$, its set of predecessors $\{y : y < x\}$ is linearly ordered, and such that for any $a, b \in T$, there exists $c \in T$ with $c \leq a, b$. This may also be called a *semilinear order*. Note that this differs from another common set theoretic definition of “tree”, in which the set of predecessors must be well-ordered, as well as from graph theoretical notions of trees as cycle-free connected graphs. If there exists some $r \in T$ such that for all $t \in T$, $r \leq t$, then we say that r is the *root* of the tree T. A subset $B \subset T$ is a *branch* if B is linearly ordered and maximal. The disjoint union of a set of trees is a *forest*, which is also what we would obtain if we removed the requirement for every two elements to have a lower bound. This is the motivation for requiring trees to have all lower bounds - forests would be broken down into their constituent trees for analysis anyway. Note that we do not assume that the trees are necessarily finite, that there is a root, or that the branches are well ordered.

Since any two points which are incomparable themselves cannot have a common upper bound, splitting is “permanent”, and the situation is more constrained than for general partial orders. There are far fewer possible combinations of characters, so there are fewer 2-equivalence classes that can actually arise.

In terms of our components (A_x, B_x, C_x) as in Chapter 4, A_x is never above anything in C_x, and $B_x \cup C_x$ is also more constrained - B_x is a linear order, and everything in C_x lies above some member of B_x.

6.1.1 2-equivalence classes of trees

Theorem 24. *There are* 18 2*-equivalence classes of trees (where* tree *is in the sense given above, that is, a semilinear order with pairwise common lower bounds).*

Proof. Trees may exhibit the following characters:

- a root below everything, of character $(1, 0, 0)$
- points before any branching but above the root if there is one, of character $(1, 1, 0)$
- middle points after at least one branching point, of character $(1, 1, 1)$
- top points after branching, of character $(0, 1, 1)$

There are also the degenerate cases where the tree is empty; some point has character $(0, 0, 0)$ and so that point is the entire tree; and five further cases where there is no branching: either there is a root of character $(1, 0, 0)$ with a single point above it of character $(0, 1, 0)$; a $(1, 0, 0)$ root with a string of middle points of character $(1, 1, 0)$ and then a top point of character $(0, 1, 0)$, a $(1, 0, 0)$ root with an infinite ascending chain all of character $(1, 1, 0)$, a chain of order type ω^* with a top point of character $(0, 1, 0)$ and middle points of character $(1, 1, 0)$ but no root, or a chain of order type $\mathbb{Z}$ containing only points of character $(1, 1, 0)$.

This gives a total of seven non-branching trees corresponding precisely to the linear orders up to 2-equivalence.

The character $(0, 0, 0)$ only occurs in the degenerate single point case, and the character $(0, 1, 0)$ only occurs if there is no branching, so its presence implies that we are in one of the cases above. The characters $(0, 0, 1)$ and $(1, 0, 1)$ cannot occur in trees, as a point x with either of these characters is incomparable to some other point y, but there are no points below x and so x and y fail to have a common lower bound.

The remaining possible characters are $(1,0,0),(0,1,1),(1,1,0)$ and $(1,1,1)$. We consider first the trees with a root, which must have character $(1,0,0)$. The character set $\{(1,0,0)\}$ cannot arise, as there is no point to go above a point of character $(1,0,0)$, and $\{(1,0,0),(1,1,0)\}$ was already listed when we considered the linear orders above.

The remaining six possibilities for sets of characters that can occur if there is a root and there is branching are:

8. $\{(1,0,0),(0,1,1)\}$, a fan consisting of a root with arbitrarily many (but at least two) pairwise incomparable points above it.

9. $\{(1,0,0),(0,1,1),(1,1,0)\}$, any linear order of size at least two that has a least and a greatest point, plus a fan coming out of the greatest point. (The greatest point is necessary to give a common lower bound to the leaves above.)

10. $\{(1,0,0),(0,1,1),(1,1,1)\}$, a tree with a root, but no other points below all branching. Either there is more than one point immediately above the root, or else there is a branch along which the root has no immediate successor and there are branching points arbitrarily close to the root. A linear order of order type $1+\omega*$ with leaves attached to the even points would fulfil this, for example. At least one branch must be of length at least three (otherwise, we are in the fan case), and at least one branch is finite.

11. $\{(1,0,0),(0,1,1),(1,1,0),(1,1,1)\}$, a tree with at least two points below all branching - there is a root, and at least one point above the root which lies either below or above all other points. Note that there need not be some non-root point lying below every other non-root point - we could have a linear order where the root has no successor, such as $1+\omega^*$. There is at least one chain of length two lying above a branching point (or else we are in the linear order then fan case), and there is at least one leaf.

12. $\{(1,0,0),(1,1,1)\}$, a rooted tree that immediately starts branching, and has no leaves.

13. $\{(1,0,0),(1,1,0),(1,1,1)\}$, a rooted tree that does not immediately start branching, but does after at least one non-root point, and has no leaves.

Finally, we determine the equivalence classes in which there is branching, but no root. The sets of characters in this case are subsets of $\{(0,1,1),(1,1,0),(1,1,1)\}$, though $\{(0,1,1)\}$ and $\{(1,1,0)\}$ are impossible. The former requires some point to have a point below it but no point to have a point above it, and the latter has no branching and corresponds to one of the linear cases already given. Excluding these two sets and the empty set, the other five possible subsets of these characters are:

14. $\{(1,1,1)\}$, a bottomless tree in which no branches have endpoints, and there is branching below any given point. For example, $\mathbb{Z}$ with a copy of ω extending from each point $x \in \mathbb{Z}$.

15. $\{(0,1,1),(1,1,1)\}$, a bottomless tree in which some branches have endpoints, and there is branching below any given point. For example, $\mathbb{Z}$ with a single leaf extending from each point.

16. $\{(1,1,0),(1,1,1)\}$, a bottomless tree in which no branches have endpoints but there are points below all branching, for example, $\mathbb{Z}$ with a copy of ω extending from each point $x \in \mathbb{N}$.

17. $\{(0,1,1),(1,1,0),(1,1,1)\}$, a bottomless tree in which some branches have endpoints and there are points below all branching, for example, $\mathbb{Z}$ with a leaf extending from each point $x \in \mathbb{N}$.

18. $\{(0,1,1),(1,1,0)\}$, a bottomless tree containing only points below all branching and endpoints of branches. This tree therefore consists of a linear order with no least point, with at least two points above the linear order, all of which are mutually incomparable. For example, ω^* equipped with two incomparable points above it.

This gives a total of 18 2-equivalence classes of trees, of which seven are the linear orders. $\square$

We remark that the 2-equivalence classes of trees form a subset of the 2-equivalence classes of partial orders given in Theorem 13.

Excluding the seven equivalence classes where the tree is linearly ordered, we see that there are only eleven nontrivial 2-equivalence classes for trees. The first four have finite representatives; the rest do not, since they either have all their branches unbounded above or they have no least point. The minimal representatives of the finite classes have sizes three, four, four and five respectively, as may easily be constructed - a V shape, a Y shape, and each of these with an extra point added to the top of one branch. These are depicted in the bottom row of Figure 4.1.

If we consider the tree equivalence classes as equivalence classes of partial orders, then there are other, non-tree, partial orders in most of the equivalence classes, with three exceptions: every partial order realising precisely $\{(1,0,0),(0,1,1)\}$, $\{(1,0,0),(0,1,1),(1,1,0)\}$ or $\{(0,1,1),(1,1,0)\}$ is a tree. Every other equivalence class either only contains linear orders, or contains trees with a point of character $\{(1,1,1)\}$. However, in any equivalence class realising $\{(1,1,1)\}$, there are both trees and non-tree partial orders, since given a tree T and a point $x \in T$ of character $(1,1,1)$ we may create T' from T by substituting x with a diamond consisting of x_1, x_2, x_3, x_4 with $x_1 < x_2 < x_4$ and $x_1 < x_3 < x_4$ and x_2 and x_3 incomparable. A diagram of T and T' is given in Figure 6.1. We select witnesses $a, b, c \in T$ such that $a > x$, $b < x$ and c is incomparable with x, which must all exist, by x's character. But when we perform the diamond substitution, we have $a > x_1, x_2, x_3, x_4$, $b > x_1, x_2, x_3, x_4$ and c is incomparable with x_1, x_2, x_3, x_4. So x_1, x_2, x_3, x_4 also have character $(1,1,1)$; moreover, they have the same relationship to each other point of the tree as x did, so the characters of all other points are unchanged, and so the set of characters realised in T' is identical to that realised in T. However, T' is certainly not a tree, as x_2 and x_3 are incomparable and so x_4's predecessors are not linearly ordered.

This shows that we cannot characterise being a tree in just 2 moves; in fact, the property of being a tree involves in its most natural form the quantifier depth 3 sentence $\forall x \forall y \forall z (y \leq x \wedge z \leq x \rightarrow (y \leq z \vee z \leq y))$, which asserts that the predecessors of each point are linearly ordered.

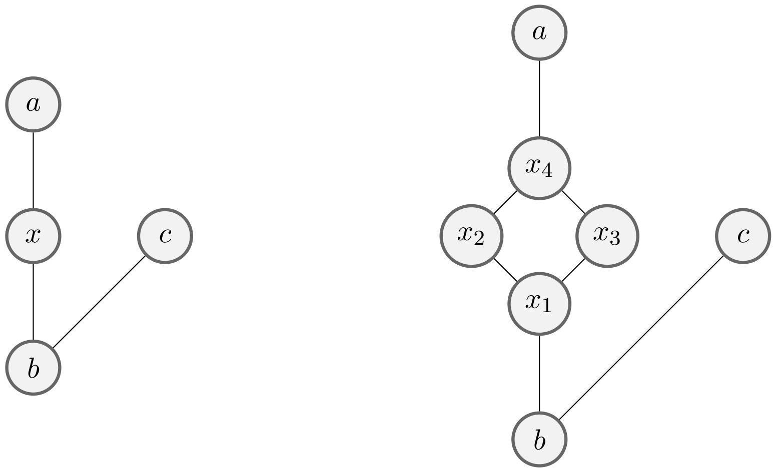

Figure 6.1: We may replace a point x of character $(1, 1, 1)$ with x_1, x_2, x_3, x_4 as shown to create a non-tree which is 2-equivalent to the original tree.

6.1.2 n-equivalence of trees

We hoped to characterise n-equivalence of trees in terms of $(n-1)$-equivalence of the components A, B, C, and $B \cup C$, as was not possible for more general partial orders. This seemed plausible, as the absence of cycles prevented many undesirable configurations from occurring. Trees are significantly more constrained than partial orders, since there is only branching in one direction, and in the other direction the behaviour is linear-like.

However, neither a proof nor a counterexample was forthcoming. In order for a proof to work, we would need to show that strategies on B, C, and $B \cup C$ may always be made compatible. A counterexample, on the other hand, would need to find two trees with incompatible strategies, for example two similar possibilities for B lay in different places in $B \cup C$.

Conjecture 25. *Two trees T_1 and T_2 are n-equivalent if and only if for all $x \in T_1$ there exists $y \in T_2$, and vice versa, with $(A_x, B_x, C_x, B_x \cup C_x) \equiv_{n-1} (A_y, B_y, C_y, B_y \cup C_y)$.*

The forwards direction of this conjecture is clear, as any winning strategy for Player II in $G_n(T_1, T_2)$ allows us to find such a y for any x by presenting x as Player I's move and taking y to be the strategy's response.

We prove the backwards direction in a special case.

Let the *height* of a tree be the size of a maximal chain. We note that some authors may use this to refer either to the number of vertices or the number of edges in a maximal chain; we choose the former usage.

For height 3 trees, it is in fact the case that 2-equivalence of the components A, B, C, $B \cup C$ implies 3-equivalence overall. We may prove this by using the componentwise 2-equivalence to find a matching second layer point and hence a matching branch in the other tree.

Theorem 26. *Two trees X and Y of height at most 3 are 3-equivalent if and only if for all $x \in X$ there exists $y \in Y$, and vice versa, with $(A_x, B_x, C_x, B_x \cup C_x) \equiv_2 (A_y, B_y, C_y, B_y \cup C_y)$.*

Proof. The forward direction is immediate, as with the more general conjecture above. We prove the backward direction by considering in turn each level of the tree at which x may be located, and giving a winning strategy for Player II in each case.

First, suppose that x is in the bottom layer of X. Then x is the root of X, and every other point in X lies above x. If Player II responds with some point z that is not on the bottom layer of Y, then Player I may immediately win by playing below z, so Player II's first move strategy is to play the root y of Y. Since x and y are the roots of their respective trees, we have $A_x = X \backslash \{x\}$ and $A_y = Y \backslash \{y\}$. But by assumption, $A_x \equiv_2 A_y$, so Player II has a winning strategy σ in $G_2(A_x, A_y)$. Player II may therefore play according to σ for her last two moves, completing her winning strategy in $G_3(X, Y)$.

Now suppose that x is in the middle layer of X, that is, that there is precisely one point below x. By assumption, there exists y such that $(A_x, B_x, C_x, B_x \cup C_x) \equiv_2 (A_y, B_y, C_y, B_y \cup C_y)$, so let it be Player II's strategy to play this y on her first move. B_x is a single point and $B_x \equiv_2 B_y$, so y must also be in the middle layer of Y. Player II may use the equivalences to give a winning strategy. $A_x \equiv_2 A_y$, so if Player I plays a point (or two points) in A_x or A_y then Player II may also play in A_y or A_x (and her strategy is to do so). B_x and B_y contain only the singleton points at the roots of the trees, and if Player I plays one root then Player II's strategy is to play the other root. All other points are in C_x and C_y, and if these are played then Player II should respond according to her winning strategy σ in $G_2(C_x, C_y)$. By playing according to these strategies, we see that whatever Player I plays, Player II will win. In any substructure that may result from any strategy of Player I, points in A_x will lie above x and above the point of B_x, if played, but will be incomparable to the points of C_x. The point in B_x will lie below all other points, and points in C_x will lie above B_x and be incomparable to x and any played points in A_x. All of these relations hold for A_y, B_y and C_y as well. If two points in C_x are played, these may or may not be related, but since Player II is playing according to σ, the relations obtained will be the same in both structures. Player II therefore wins if x is on the middle layer of X.

Finally, suppose that x is in the top layer of X, that is, that there are precisely two points below x. By assumption, there exists y such that $(A_x, B_x, C_x, B_x \cup C_x) \equiv_2 (A_y, B_y, C_y, B_y \cup C_y)$, but we must in this case be a little careful about which y we take. Consider the point z immediately below x. There is a unique such z, since X is a tree, and z must be on the middle layer of X. Possibly, z could have exactly one point above it (this point being x), or it could have more than one point lying above it. However, since the equivalence assumption holds for all points, there must exist some $w \in Y$ such that $(A_z, B_z, C_z, B_z \cup C_z) \equiv_2 (A_w, B_w, C_w, B_w \cup C_w)$. In particular, this implies that w is also on the middle layer of Y, and that $A_w \equiv_2 A_z$. Therefore, w has only one point above it if z has only one point above it, and w has multiple points above it if z does so. (There may also exist middle layer points in Y with zero points above them, but w cannot be these in either case.)

Player II's first move is to pick a point y lying above w. We have now ensured that there is branching at the point below x if and only if there is branching at the point below y, and we shall find that this condition is sufficient. We run through the possibilities for Player I's remaining moves. Player I cannot play in A_x or A_y, as these are empty. If Player I plays in B_x on her second move, then she must play either w or the root of the tree. If she plays w, then Player II responds by playing z; if she plays the root of X, then Player II plays the root of Y. Player II plays analogously with the labels swapped if Player I plays in B_y. Now Player II will win, because there is branching at the root in X if and only if there is branching at the root in Y, and likewise for z and w, so she has a corresponding response to every possible third move by Player I.

If Player I's second move is in C_x or C_y, assume that she plays in C_x (as we may relabel if necessary). Points in C_x are of four types: those in the middle layer with a point above them, those in the middle layer with no point above them, those in the top layer lying above z, and those in the top layer not lying above z. In each of these cases, Player II determines a point satisfying the same property in C_y and plays that.

If Player I plays a point $p \in C_x$ in the middle layer with a point above it, then A_p and B_p are single points, so by assumption there is some point $q \in Y$ for which A_q and B_q are singletons. Of course w is such a point of Y, but it cannot be the only one, or C_y would contain no chains of length 2 and we would not have $C_x \equiv_2 C_y$, so there must exist some $q \neq w$ such that A_q and B_q are singletons; let Player II play this q. Note that there is a point below both p and x, a point above p that is incomparable with x, a point below x that is incomparable with p, and possibly points incomparable with both p and x. There is also a point below both q and y, a point above q that is incomparable with y, a point below y that is incomparable with q, and there are points incomparable with both q and y if and only if there are points incomparable with both p and x, since $C_x \equiv_n C_y$. Therefore, whichever point Player I chooses as her third move, Player II has a winning response.

If Player I plays $p \in X$ in the middle layer with no point above it, then A_p is empty and B_p is a single point, so by assumption there is some point $q \in Y$ with A_q empty and B_q a singleton, and Player II plays q. In this case, there is a point below both p and x (and a point below both q and y), a point below x and incomparable with p (and a point below y and incomparable with q), and possibly points incomparable with both p and x, which arise if and only if there are also points in Y incomparable with both q and y. Therefore, Player II again has a winning response to every possible third move by Player II.

If Player I plays a point $p \neq x$ in the top layer lying above z, then z has at least 2 points above it, so w also does, and so Player II plays any $q \neq y$ with $q > w$. In this case, there are points below both p and x, points below both q and y, and possibly points incomparable with both p and x, which arise if and only if there are points incomparable with both q and y, so again Player II has a winning strategy.

If Player I plays a point in the top layer of X not lying above z, then there must also exist a point q in the top layer of Y that does not lie above w, since $C_w \equiv_2 C_z$ and C_z contains a chain of length 2. Player II plays this q. In this case, there is a point below both p and x, a point below p that is incomparable to x, a point below x that is incomparable to p, and possibly points incomparable to both, and likewise for q and y. As before, the equivalent conditions also hold on q and y, and so Player II also wins in this case.

We have given a winning response for Player II for any possible moves by Player I, and so $X \equiv_3 Y$. □

In the above proof, the key idea is that we must pass to a point on the middle level with the desired ramification, and that the correspondence of these mid level points gives a winning strategy for Player II in the game of length 3. We may therefore determine the trees of height at most 3 up to 3-equivalence by determining which types of points may exist on the middle level.

Theorem 27. *There are* 24 *trees of height at most* 3*, up to* 3*-equivalence.*

Proof. Every tree of height at most 3 has a single point on the bottom layer, unless it is the empty tree. The middle layer may contain various points with, potentially, various numbers of points above them. As in the proof above, there are three possibilities here for a point in the middle layer: either a middle layer point has no points above it, in which case we say it is of type 0, or it has precisely one point above it, in which case we say it is of type 1, or it has two or more distinct points above it, in which case we say it is of type 2. The points arising in the top layer are determined, up to 3-equivalence, by the types of the middle points, except in the case where there is a single middle point of type 2, when there may be either precisely 2 or ≥ 3 top points.

We determine all combinations of types of middle points that may arise, up to 3-equivalence.

Considering first the number of type 0 middle points, we see that if a height ≤ 3 tree T has at least two type 0 points , then Player I may play two type 0 points on her first two moves. If Player II's responses are not both of type 0, then Player I may win by playing above one of them . We may therefore distinguish whether there are zero, one or at least two type 0 points in a height ≤ 3 tree.

Middle points of types 1 and 2 both have one or more points above them, and the only way to distinguish these from one another is by playing those points. There are certainly enough moves in the game to distinguish the existence of one middle point of type 2, by playing it and the two points above it, or one middle point of type 1, by playing it and then either the point above it, if the response has type 0, or the two points above the response, if the response has type 2.

We can also determine whether a height ≤ 3 tree has at least two points in total of types 1 and 2, because if Player I plays two such points, Player II must be able to respond with two middle points that are not maximal.

Finally, we can also determine whether a tree has at least three middle points in total, since Player I can play a middle point on the first two moves, which must be responded to by two middle points of the other tree, and then play some point incomparable to both previous middle points on her third move.

We now show that any height ≤ 3 trees that agree on each of these five conditions are 3-equivalent. Let T_1 and T_2 be trees which satisfy these conditions: they either both have 0, both have 1, or both have at least 2 points of type 0; they each have a point of types 1 and 2 if and only if the other also has a point of this type; if one of them has a single type 2 middle point with no other middle points and precisely two points above it, then they both do; they only have either one or zero total middle points of types 1 and 2 if the other tree also has the same number; and they only have fewer than three middle points in total if the other tree also has the same number. Then Player II has a winning strategy.

Without loss of generality, assume that Player I plays in T_1 on her first move. Then on her first move, Player II plays the root of T_2 if Player I played the root of T_1, a middle point of type n if Player I played a middle point of type n, and a top point that lies above a point of type n if Player I played a top point above a middle point of type n. This is possible because any middle point type present at least once in T_1 must also be present in T_2.

We now give a second move strategy for Player II such that Player II will also have a winning third move. On the second move, if Player I plays above a middle point already chosen, then Player II plays above the middle point already chosen in the other tree (which must be possible, because the middle points played are of the same type). We verify that Player II has a winning third move, since the roots are below both moves, there are no points above both moves, and there are points incomparable to both moves or above the middle point but incomparable to the top one if and only if these are also present in the other tree. Likewise, if Player I plays a top point on her first move and a middle point on her second move, then Player II also plays below the first move, which is again guaranteed to be possible by her first move strategy, and the reasoning for her third move is similar. If Player I plays the root, Player II again plays the root, and here all possible third moves are above the second move and the same options arise in each tree for the relations to the first move.

If Player I plays a top point with the same middle point predecessor as the first move, then Player II plays a top point with the same middle point predecessor as the first move in the other tree. This must be possible since the first move was above a point of type 2 if and only if the first move in the other tree was also above a point of type 2. Here all possible third moves are either below both of them (which must be possible) or incomparable to both of them, which is possible in one tree if and only if it is possible in the other, due to the conditions on middle points and the condition on trees with a single, type 2, middle point.

If Player I plays any top point which is neither above the first move nor above the same middle point as it, then Player II plays a top point in the other tree which is also neither above nor sharing a predecessor with the first move. Such a point must exist, because in this case there are at least two middle points of types 1 and 2 in both structures. Then the possible third moves are the point below this, which is incomparable with the first move, the root, points above/below the first move if present, and any other incomparable moves, which must be present in one tree if in the other.

If Player I plays a middle point of type 1 or 2 which is not below the first move, then Player II plays a point of type 1 or 2 in the other structure, which must exist by the same reasoning. The third move may lie above or below this, and the options must similarly be the same in both trees.

Finally, if Player I plays a middle point of type 0, then Player II plays a middle point of type 0 in the other structure, which again must exist by the condition on the counts of type 0 points. The third move may only be below this if it is the root, otherwise it will be incomparable to the second move and so the first move strategy guarantees that the same options arise.

We have already determined that trees of height at most 3 are inequivalent to one another if they differ in any of these conditions, and so we may now list multisets of characters of the middle points of trees which correspond to every 3-equivalence class of trees of height ≤ 3, and exhibit the corresponding trees.

That this list is complete may be seen by considering the tuple (n_0, n_1, n_2), where n_i is the number of middle points of type i. As discussed above, except for the two classes arising from $(n_0, n_1, n_2) = (0, 0, 1)$, we may distinguish precisely the cases where n_0 is 0, 1 or ≥ 2, n_1 is 0 or ≥ 1, n_2 is 0 or ≥ 1, $n_1 + n_2$ is $0, 1$, or ≥ 2 and $n_0 + n_1 + n_2$ is $0, 1, 2$ or ≥ 3.

There are two equivalence classes in which no middle points arise: that where the tree also has no root (the empty tree), and that in which the tree does have a root (the tree of size 1). For all other equivalence classes, the following multisets of types of middle points give a representative:

$\{0\}$, $\{0, 0\}$, $\{0, 0, 0\}$, $\{1\}$, $\{1, 1\}$, $\{1, 1, 1\}$, $\{1, 0\}$, $\{1, 1, 0\}$, $\{1, 0, 0\}$, $\{1, 1, 0, 0\}$, $\{2\}$ (which gives rise to two equivalence classes), $\{2, 2\}$, $\{2, 2, 2\}$, $\{2, 0\}$, $\{2, 2, 0\}$, $\{2, 0, 0\}$, $\{2, 2, 0, 0\}$, $\{2, 1\}$, $\{2, 1, 1\}$, $\{2, 1, 0\}$, and $\{2, 1, 0, 0\}$.

For each multiset, the corresponding tree may be constructed by taking a point for the root and adding above it a middle point of each type listed, with the corresponding number of top points above each (usually the number of the type, but the tree for the second equivalence class of $\{2\}$ has three (or more) top points). These are shown in Figure 6.2 and Figure 6.3.

We have shown that two trees of height at most 3 are equivalent if and only if they realise the same multiset of types (or are one of two $\{2\}$ trees, or the trees of size zero or one), and that there exist trees realising each of these multisets of types (encoded above, and depicted in Figures 6.2 and 6.3). Therefore, there are a total of 24 3-equivalence classes of trees of height at most 3. $\square$

Assuming Conjecture 25, we can use the 2-equivalence classes of trees to bound the number of 3-equivalence classes of trees of any height. Since every linear order is a (somewhat degenerate) tree, we must have at least 281 3-equivalence classes of trees altogether, by Theorem 11. The true number is likely to be much higher than this, and yet much lower than the following upper bound.

If Conjecture 25 holds, then 3-equivalence of trees depends precisely on the 2-equivalence classes of $(A_x, B_x, C_x, B_x \cup C_x)$ that arise. Of these, A_x is a forest, B_x is a linear order, C_x is a forest, and $B_x \cup C_x$ is a tree. There are 18 2-equivalence classes of trees, and 7 2-equivalence classes of linear orders. We did not calculate the 2-equivalence classes of forests, but since there are 8 possible 2-characters that they can realise, there can be at most 2^8 2-equivalence classes of forests. (The upper bound would still be very loose even with a tighter bound for the number of forests.)

There are therefore at most $2^8 * 7 * 2^8 * 18 = 8257536$ possible tuples $(A_x, B_x, C_x, B_x \cup C_x)$ up to 2-equivalence. This gives an upper bound of $2^{8257536}$ 3-equivalence classes. Even with a more conservative upper bound m for the number of forests, we would get an upper bound of 2^{126m^2}, and since every tree is a forest we would certainly have $m \geq 18$. We may likewise obtain a large bound for the number of n-equivalence classes by repeating this method.

We suggest that for the case of 3-equivalence, it may be possible to determine all combinations of 2-equivalence classes that may actually arise. In particular, this may be practical for the subclass of finite trees, since there are only eight 2-equivalence classes of finite trees, of which four are the finite linear orders of lengths $0, 1, 2$ and at least 3. These would still give quite a large upper bound with the analogous calculations to those above, but many of the combinations counted in that loose bound would actually be impossible, and so the number should whittle down to something closer to the mere 24 3-equivalence classes in the height 3 case.

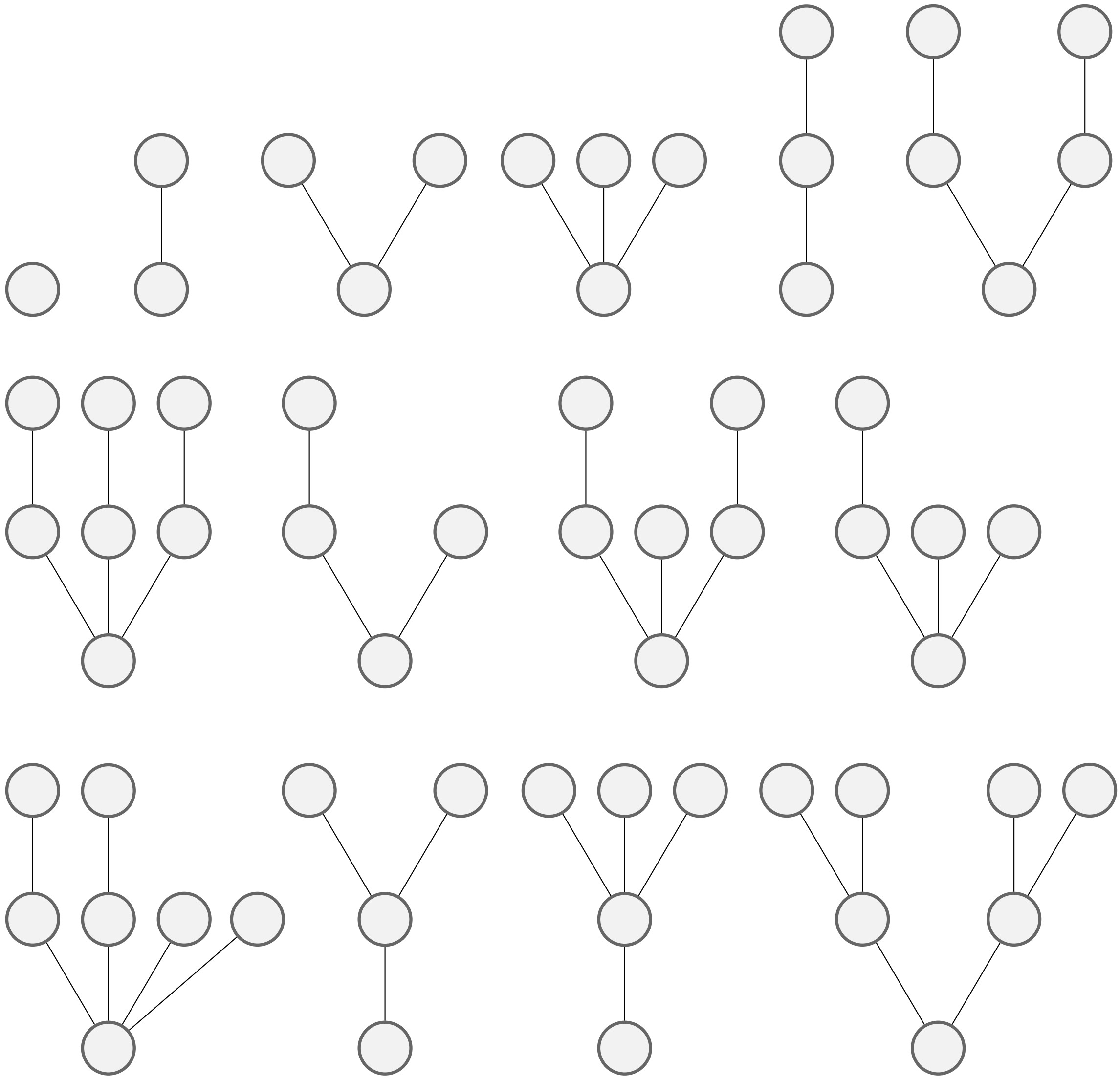

Figure 6.2: The trees corresponding to the middle point types $\emptyset$, $\{0\}$, $\{0,0\}$, $\{0,0,0\}$, $\{1\}$ and $\{1,1\}$, on the top row; $\{1,1,1\}$, $\{1,0\}$, $\{1,1,0\}$ and $\{1,0,0\}$, on the middle row; $\{1,1,0,0\}$, $\{2\}$, $\{2\}$ and $\{2,2\}$, on the bottom row.

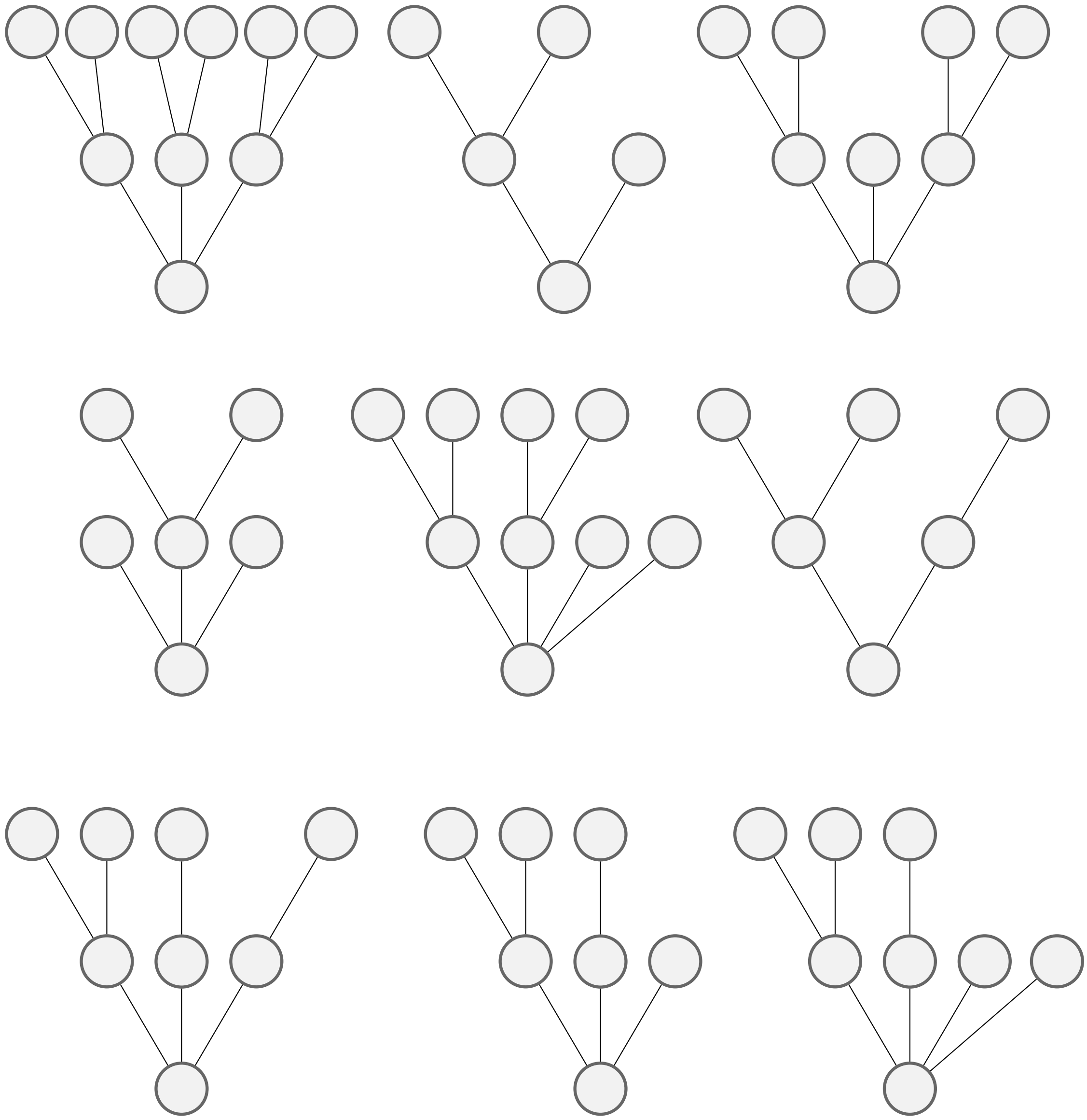

Figure 6.3: Trees corresponding to the remaining multisets of middle point types. Top row: $\{2,2,2\}$, $\{2,0\}$ and $\{2,2,0\}$. Middle row: $\{2,0,0\}$, $\{2,2,0,0\}$ and $\{2,1\}$. Bottom row: $\{2,1,1\}$, $\{2,1,0\}$ and $\{2,1,0,0\}$.

6.2 Graphs

A *graph* is a set of points X and a subset $E \subseteq X^2$ of *edges* between points in X. In order for E to correspond to the edges of an undirected graph we require it to be symmetric (if not, we just take the symmetric completion): $(x, y) \in E$ if and only if $(y, x) \in E$. This may also be thought of as a relation R where xRy if and only if there is an edge between x and y. The relation R is symmetric and irreflexive, as we do not permit xRx.

Considering graphs up to 1-equivalence, we find there are only two equivalence classes - one containing the empty graph, and one containing all nonempty graphs. Since the only relation is binary, we certainly have $G \equiv_1 H$ when G and H are nonempty graphs.

In the two move game we have two possible relationships between the pair (x, y): either xRy or $\neg xRy$. For the purpose of 2-equivalence, we can assign each point x a character (a, b) where $a = 1$ if there exists y such that $(x, y) \in E$ and $a = 0$ otherwise, and $b = 1$ if there exists y such that $(x, y) \notin E$ and $b = 0$ otherwise. Note that no graph can realise both $(1, 0)$, which is realised only at points adjacent to all other points (and there is at least one other point), and $(0, 1)$, which is realised only at points such that there are other points and it is not adjacent to any of them.

Lemma 28. *There are seven* 2*-equivalence classes of graphs.*

- the empty graph, if no characters are realised
- the graph with one point, if the character $(0, 0)$ is realised
- the complete graph on two or more vertices, if only the character $(1, 0)$ is realised
- the empty graph on two or more vertices, if only the character $(0, 1)$ is realised
- graphs realising only $(1, 1)$, where every point has both points adjacent to it and points not adjacent to it, for example $\{a, b, c, d\}$ with aRb and cRd only, or the random (Rado) graph [6]
- graphs realising $\{(1, 0), (1, 1)\}$, where there is at least one point adjacent to all other points, and at least one pair of points with no edge between them, for example a fan such as $\{a, b, c\}$ with aRb and aRc only

- graphs realising $\{(0,1),(1,1)\}$, whose complement is in the class above: here there is at least one point with no edges, and at least one pair of points with an edge, for example $\{a,b,c\}$ with bRc only

Graphs need not be transitive, or satisfy any other particular constraint involving more than two points. So, when constructing a graph, we can choose which relations hold almost entirely freely, with the caveat that the edge relation must be symmetric. If we add a new point x to a graph containing m points, then we can choose any of the 2^m possible sets of edges to x and still get a valid supergraph.

Consider the three move game, played on graphs with at least three points. A point is selected for the first move, which may have points adjacent to it, points not adjacent to it, or both. After the second move, we have a pair of points joined by an edge (or not joined by an edge), and the other points in the graph are in one of four states: adjacent to the first point but not the second, adjacent to the second point but not the first, adjacent to both or adjacent to neither. Likewise, in longer games there will be 2^n possible states of other points after the nth move.

Note that we cannot just multiply the number of possible states at each move to give the number of equivalence classes, since if we make the first move x such that all other points are adjacent to x, then we certainly shall not be able to play points that are not adjacent to x on subsequent moves.

However, we may determine the 3-equivalence classes, and more generally the n-equivalence classes, by using the $(n-1)$-equivalence classes as follows.

We may use a similar method of character to those we used for linear and partial orders. Instead of partitioning according to an order relation, we may partition a graph G into G^x and $G^{\bar{x}}$, where $G^x = \{y \in G : (x,y) \in E\}$ and $G^{\bar{x}} = \{y \in G : (x,y) \notin E\}$. Note that these partitions are not directly analogous to those in a linear order - for a linear order A, if $x \in A^{<y}$ then $y \in A^{>x}$ asymmetrically, but for a graph G, if $x \in G^y$ then $y \in G^x$ symmetrically.

We may then apply an iterated colouring scheme similar to the one proposed for partial orders in Section 4.5. If the first moves chosen are $x \in G_1$ and $y \in G_2$, then we colour

$G_1 \backslash \{x\}$ and $G_2 \backslash \{y\}$ according to the edge relation: let G^x be coloured red and $G^{\bar{x}}$ blue. Then $G_1 \equiv_n G_2$ if and only if for all $x \in G_1$ there exists $y \in G_2$, and vice versa, such that $(G_1, G_1^x, G_1^{\bar{x}}) \equiv_{n-1} (G_2, G_2^y, G_2^{\bar{y}})$.

Note that we cannot simply consider the $(n-1)$-characters of G^x and $G^{\bar{x}}$ separately to determine n-equivalence, because the edges between the points of G^x and those of $G^{\bar{x}}$ matter too, at least for games of length three or longer. For example, in the three move game, given x and y, we can distinguish structures in which $\exists z(zRx \wedge zRy)$, as well as $\exists z(\neg zRx \wedge zRy)$, $\exists z(zRx \wedge \neg zRy)$ and $\exists z(\neg zRx \wedge \neg zRy)$. Any useful notion of character must therefore also be able to distinguish these properties, but if xRy then the $(n-2)$-characters of $(G^x)^y$ and $(G^x)^{\bar{y}}$ would provide no information about whether any points w with $\neg wRx$ satisfy wRy, and likewise if $\neg xRy$ then the character would not distinguish whether or not any points v with vRx satisfy vRy.

We also remark that the local nature of the graph relation does not enable us to reduce any infinite graph to an n-equivalent finite graph. For example:

Lemma 29. *Let G be a graph with vertex set $\mathbb{N}$, with an edge between n and $n+1$ for each $n \in \mathbb{N}$, and no other edges. Then G is not 4-equivalent to any finite graph.*

Proof. Any finite graph H contains an integer number of edges (possibly 0), each of which runs between two vertices. Therefore, the sum of the degrees of the vertices in H is even, and so H must contain an even number of vertices of odd degree.

Suppose first that H contains no vertices of odd degree. Then Player I may win $G_4(G,H)$ (or even $G_3(G,H)$) by playing $0, 1 \in G$ on her first two moves. Let Player II's corresponding moves in H be h_0, h_1. By assumption, h_0 has even degree, so there must exist some $x \neq h_1$ adjacent to h_0. For her third move, Player I plays x, which wins because Player II can find no additional vertex adjacent to 0 in G.

Now suppose that H has more than one vertex of degree 1. Then Player I wins $G_4(G,H)$ by the following strategy. First, she plays two distinct vertices of H that have degree 1. Player II's distinct responses in G must include a vertex of degree 2. Player I then plays both neighbours of the vertex of degree 2, and Player II is unable to play two distinct neighbours of the corresponding degree 1 vertex in H, so Player I wins.

Finally, suppose that H contains a vertex of degree > 2. Then Player I wins $G_4(G, H)$ by playing that vertex and three distinct neighbours. No vertex in G has more than two neighbours, and so Player II's response must lose.

In any case, H is not 4-equivalent to G. All finite graphs must fall into one of these three cases, and so we conclude that G is not 4-equivalent to any finite graph.

□

6.2.1 Hypergraphs

Suppose we play on a hypergraph rather than a graph. A *hypergraph* G is a pair (X, E) where $E \subseteq \mathcal{P}(X)$. We call X the *vertex set* of G and E the *hyperedge* set. If every member of E has size m, we call the hypergraph an *m-graph*, and the (hyper)edge relation is defined on sets of size m. Note that we recover the usual notion of graph when $m = 2$. Taking $m = 0$ gives a set with no relations, and so there must be $n+1$ many n-equivalence classes - those of size $0, 1, \ldots, n-1$, and those of size at least n. Taking $m = 1$ gives a slightly less trivial case - we have a set with a single unary relation, so there are two classes of points: those for which the relation holds, and those for which it does not. If G and H had different amounts of one type of point, and at least one of these was a finite number smaller than n, then Player I would be able to win $G_n(G, H)$ by exhaustively choosing the points of that type. Therefore there are at least $(n+1)^2$ n-equivalence classes, since the number of type 1 and type 2 points may each be $0, 1, \ldots, n-1$ or $\geq n$. We may easily verify that this condition is sufficient and so there are precisely $(n+1)^2$ n-equivalence classes of 1-graphs.

Having dispensed with the lower values, we consider $m \geq 3$ ($m = 2$ being the standard graphs discussed above). As usual, games shorter than the least arity of a relation have only the trivial counting equivalence classes, so there are $(n+1)$ n-equivalence classes of m-graphs for $n < m$. When $n \geq m$, we run into new considerations we did not have for $m = 2$. For 2-graphs, we may partition the points of $G \backslash \{x\}$ into those with an edge to x and those without. When $m > 2$, the partition is no longer determined solely by x and the singular other point under consideration. Rather, there are three types of second move y after a first move x: those for which $(x, y, a_3, \ldots, a_n)$ is a hyperedge for every string of subsequent moves $a_3, \dots, a_n$, those for which no sequence of subsequent moves $b_3, \dots, b_n$ makes $(x, y, b_3, \dots, b_n)$ a hyperedge, and those for which there are both $a_3, \dots, a_n$ such that $(x, y, a_3, \dots, a_n)$ is a hyperedge and $b_3, \dots, b_n$ such that $(x, y, b_3, \dots, b_n)$ is not a hyperedge. Naturally this last class breaks down further into possibilities at each move.

6.3 Directed graphs

We have primarily been trying to generalise work on linear orders to partial orders, or at least to some intermediate class like semilinear orders (trees). We look briefly in the direction of yet more generality.

Partial orders are characterised by three properties:

- reflexivity - we always have $x \leq x$, for all x
- antisymmetry - if $x \leq y$ and $y \leq x$ then $x = y$
- transitivity - if $x \leq y$ and $y \leq z$ then $x \leq z$.

In the maximal natural generalisation we would just have a set of points with a binary (or otherwise) relation upon them, and none of these properties would necessarily hold. Dropping reflexivity does not give us anything interesting - we already have strict and non-strict partial orders in which either all points do or all points do not relate to themselves, and considering a structure in which some points related to themselves and some did not would just create two different types of point, which could be done more elegantly by just using coloured partial orders. We could drop antisymmetry, which would give a preorder, or we could explore dropping transitivity.

A strict partial order is just the transitive closure of a directed acyclic graph, as we may see by considering each edge of its Hasse diagram as an arrow from the lower to the higher element. So, we may consider antisymmetric directed graphs as a generalisation of partial orders. Our characters (a, b, c) now indicate whether there are points with arrows to x, with arrows from x, and with neither an edge to or an edge from x, respectively. We find the 2-equivalence classes.

Clearly every partial order 2-equivalence class is contained in a distinct directed graph 2-equivalence class. With graph equivalence classes, many of the same constraints still hold. We still cannot have a point of character $(0, 0, 0)$ with anything else, or $(1, 0, 0)$ with any character of the form $(*, 0, *)$, or $(0, 1, 0)$ with $(0, *, *)$. These imply that we again cannot have both a greatest element and a non-greatest maximal point (as these would have characters $(0, 1, 1)$ and $(0, 1, 0)$), or both a least element and a non-least minimal point (of characters $(1, 0, 1)$ and $(1, 0, 0)$).

Recall that a greatest point in this new context is a point with at least one edge to it and no edges from it, and a maximal one is where every other point has an edge to it. Least points are similarly ones with arrows from them but none to them, and points with arrows both to and from them are middle points. There are only 39 equivalence classes satisfying these constraints. As there were 39 equivalence classes for partial orders, these must be the same.

For three moves, there are more equivalence classes in games on directed graphs than on partial orders, since there are more configurations that can arise between three points: for example, we can have $A \to B \to C \to A$ in a directed graph but not $A < B < C < A$ in a partial order. We can also more generally have $A \to B \to C$ but $A \not\to C$, which cannot arise in a partial order. This corresponds to the notion of transitivity, which is a three-point property, so its absence makes no difference when in the game of length two. For example, the 3-cycle above is 2-equivalent to a transitive directed graph of order type $\mathbb{Z}$, since both realise only the character $(1, 1, 0)$. However, the latter is transitive, and so corresponds to a partial order, but the former does not.

Lemma 30. *Let P be a partial order and D a directed graph. If $P \equiv_n D$ for some $n \geq 3$, then D is a partial order.*

Proof. Consider the sentence of quantifier depth 3 $\phi = \exists x \exists y \exists z (xRy \wedge yRz \wedge \neg xRz)$. By assumption, P and D are 3-equivalent, and so by Lemma 5, they satisfy precisely the same sentences of quantifier depth 3. In P, ϕ is false, and therefore ϕ must also be false in D. But ϕ is the negation of the property of transitivity, so this means that D must be transitive, and so D is a partial order. $\square$

For values of $n \geq 3$, therefore, every n-equivalence class of transitive digraphs is also an n-equivalence class of partial orders, with precisely the same members. There are also n-equivalence classes containing non-transitive digraphs, such as the n-equivalence class of the triangular digraph given above, and these do not correspond to equivalence classes of partial orders.

If we permit our directed graphs to have edges in both directions, thereby also sacrificing the two-point property of antisymmetry, we could have $A \to B \to A$, whereas $A < B < A$ is of course impossible. In this case our notion of character would be a poor choice. The current notion has three components, for points such that xRy, yRx, or neither, where the relation R is either aboveness or directed edges. Permitting both xRy and yRx, that is, breaking antisymmetry, would necessitate a fourth character component for points such that both xRy and yRx hold, in addition to our original triplet of the equivalence classes of the sets of points such that xRy but not yRx, yRx but not xRy, and neither, hold respectively. As partial orders are antisymmetric, all points of all partial orders would have fourth components of zero in this notion of character, but points in some digraphs do not if $y \to x \to y$ is permitted.

6.4 Unars

A *unar* is a pair (X, f) consisting of a set X and a unary function $f : X \to X$, whence the name [16] [15].

One can obtain many structures of interest to mathematicians by adding more restrictions on the properties of X and f, but here we consider unars in general, as well as the following modification: an *ordered unar* $(X, f, <)$ consists of a set X which is equipped with both a unary function f, and a total order $<$. Some analysis of these from a dynamical systems perspective may be found in [31]. The (unordered) unar (X, f) and the linear order $(X, <)$ are both reducts of the ordered unar $(X, f, <)$.

As with colouring functions, we actually view the function as a relation, so that unars are relational structures and we can apply the theory of Ehrenfeucht-Fraïssé games. Given a unary function f on X, let R be a relation such that xRy if and only if $f(x) = y$. Then

(X, R) and $(X, R, <)$ are relational structures, and it is these that we consider. We may easily recover the function f from this definition, since for every x there is a unique y such that xRy. By defining a function f to take these y values at their respective values of x, we recover the desired function $f : X \to X$, which we refer to in practice instead of R.

A unar may be represented as a kind of directed graph, in which loops are permitted, and edges in both directions between two points, but not multiple edges in the same direction between the same two points. Let X be the vertex set, and for each $x \in X$, let there be a directed edge e from x to $f(x)$, and no other edges. Then (X, E), where E is the set of directed edges on X, is a directed graph. In a directed graph corresponding to a unar, every vertex x has out-degree exactly 1, since $f(x)$ has only one value.

6.4.1 Structures of unars

In a unar (X, f), a *cycle* is a finite set of points $\{x_1, \ldots, x_n\} \subseteq X$ such that $f(x_n) = x_1$ and for $i < n$, $f(x_i) = x_{i+1}$. We do not count an infinite chain of points $\{x_m : m \in \mathbb{Z}\}$ with $f(x_i) = x_{i+1}$ as a cycle, though it resembles them in some ways.

Lemma 31. *Every connected component in the directed graph of a unar X contains at most one cycle.*

Proof. A proof is given in [10], which studies unars under the name of *functional digraphs*; we also include a proof for completeness, as it is reasonably brief and instructive to do so.

Suppose that X contains two distinct cycles $(x, f(x), \ldots, f^{n-1}(x))$ and $(y, f(y), \ldots, f^{m-1}(y))$, where $f^n(x) = x$ and $f^m(y) = y$. Since every vertex has out-degree 1, these cycles must be disjoint. If they were to occur in the same connected component, then there would exist some path between $f^i(x)$ and $f^j(y)$ for some i, j, disregarding the direction of arrows. Let $\{x_1, \ldots, x_k\}$ be the points of such a path, chosen such that k is minimal and there is an edge (in either direction) between $f^i(x)$ and x_1, there is an edge between x_a and x_{a+1} for each a, and there is an edge between x_k and $f^j(y)$. This path contains $k + 2$ points, so it must have $k + 1$ edges. However, there is no edge from $f^i(x)$ or $f^j(x)$ in the path, since $f(f^i(x)) = f^{i+1}(x)$ and $f(f^j(y)) = f^{j+1}(y)$ are both members of the cycles and so cannot be on the path by minimality of k. We therefore only have, at most, the edges from $\{x_1, \ldots, x_k\}$ in the path, which gives at most k edges, and so a connecting path cannot exist. $\square$

A unar may therefore be split into connected components, which are of two kinds. Some components may contain a (unique, finite) cycle, and each point of the cycle may also have predecessors outside the cycle. For any point that does not lie in this cycle, there is a unique path under the action of f which intersects the cycle in a unique vertex. The set of all vertices whose path intersects the cycle at a particular point cannot itself contain a cycle by Lemma 31, and so each point of the cycle is the root of a tree (possibly empty) with all edges given direction pointing towards the root (an *anti-arborescence*). Alternatively, a component may contain no finite cycle, in which case it contains some infinite sequence $\{x_n : n \in \omega\}$ with an edge from x_n to x_{n+1} for each n. These edges are the only ones coming from the points of the infinite sequence, but there are again trees (which may each be empty, finite or infinite) "leading into" each x_i, with every other point in the component leading into some x_i. In a cycle-free component, the infinite sequence (x_i) need not be unique, but any two infinite sequences (x_i) and (y_j) in the same component must agree beyond some point, that is, given (x_i) and (y_j), we may find M and N such that $x_{M+n} = y_{N+n}$ for all n.

Note that here we take "tree" in the graph-theoretical sense of a tree as a set of points equipped with edges between some subset of the set of pairs of them, in contrast to our earlier usage of "tree" as a particular sort of partial order in Section 6.1. Earlier, we were quite happy to declare a set of order type $\omega + 1$ a tree; this is explicitly disallowed here since the final point has no predecessor and so such an ordering cannot be meaningfully constructed using a chain of edges. Instead, a directed tree in this sense is a connected directed acyclic graph.

There is a relationship between trees as discussed in Section 6.1 and unars, but not a comprehensive correspondence. Discrete trees with their edges directed towards the root give a subclass of the class of unars, but the existence of loops renders unars quite different overall. One could categorise unars using the discrete trees feeding into each of the points of each of its cycles, but this would be rather complicated in practice.

We say y is *reachable from* x if there exists $n \geq 0$ such that $f^n(x) = y$. If y is not in a cycle, then there will be at most one such n, but if y is in a cycle of length k then of course $f^{n+mk}(x) = y$ for any $m \in \mathbb{N}$ and there are infinitely many such values. If y is reachable from x and x is reachable from y but $y \neq x$, then there exists a cycle containing both y and x.

For $n \geq 3$, there exist unars that are not n-equivalent to any finite unar, such as injective unars that contain some point x with no predecessor. These cannot be 3-equivalent to any finite unar, because any finite unar must either fail to be injective or must have predecessors for all points. In the former case, Player I can win by identifying two distinct points x and y in the finite unar with $f(x) = f(y)$, and playing x, y and $f(y)$ on her first three moves; in the latter case, Player I can win by playing x and then any predecessor of Player II's response.

Unlike the other monochromatic structures we have considered, not all nonempty unars are 1-equivalent. We observe that there are two distinct types that may each be realised: $\{x \in X : f(x) = x\}$ and $\{x \in X : f(x) \neq x\}$ may each be empty or nonempty, and so there are four 1-equivalence classes of unars (of which one only contains the empty set, leaving three 1-equivalence classes of nonempty unars: those that realise only the first type, only the second, and those that realise both). By a similar argument, there are four 1-equivalence classes of ordered unars; this is to be expected since $<$ is a binary relation and so cannot be of relevance to equivalence in games of length less than two. For games of length n in general, however, we expect there to be many more equivalence classes of ordered unars, due to the interplay between the two relations.

6.4.2 2-equivalence of unars

We classify the 2-equivalence classes of unordered unars. To do this, we determine the possible sets of relationships that can hold on pairs of points in a unar. Given a point $x \in X$, there must exist precisely one point w such that $f(x) = w$, and precisely one of the following must hold:

- $f(x) = x$

- $f(x) \neq x$, but $f(f(x)) = x$
- $f(x) \neq x$ and $f(f(x)) = f(x)$
- $x \neq f(x)$, $f(x) \neq f(f(x))$ and $f(f(x)) \neq x$

There may possibly also exist points $y \neq x, f(x)$. Each of the following sets may be empty or nonempty, but they are disjoint and together partition $X \backslash \{f(x), x\}$.

- $\{y : x \neq y \neq f(x) \text{ and } f(y) = x\}$
- $\{y : x \neq y \neq f(x) \text{ and } f(y) = y\}$
- $\{y : x \neq y \neq f(x) \text{ and } y \neq f(y) \neq x\}$

Note that we do not need to separately consider $\{y : x \neq y \neq f(x) \text{ and } f(y) = f(x)\}$, because we cannot distinguish whether $f(y) = f(x)$ in any length of game if some y and x are played with $f(x), f(y) \neq x, y$, unless $f(x)$ or $f(y)$ is also played at some point in the game. Recall that $f(x)$ is the value of z such that $R(x, z)$ holds, and we may assume that z is unique in this because we are considering games on unars (X, R) where R is a relation with this property, not an arbitrary relation. Therefore, $f(x) = f(y)$ if and only if $\exists z(R(x, z) \wedge R(y, z))$. For the purposes of determining the winner of the game, however, only the points played in the game affect the outcome, and so even if there were points x_1, x_2, y_1, y_2 with $f(x_1) = f(y_1) = z$ and $f(x_2) \neq f(y_2)$ but the structures were otherwise similar, we would still have $(x_1, x_2) \cong (y_1, y_2)$ in the substructures played in the game, and so Player I would not be able to win a game using only this difference unless z were also played. Since we are currently considering the two move case, we cannot have any games in which x, y and z are all played, and so we do not need to distinguish the values of y with $f(y) = f(x)$ from other members of $\{y : x \neq y \neq f(x) \text{ and } y \neq f(y) \neq x\}$. This will be more precisely demonstrated in the proof of Lemma 32.

The above lists give at most $4 * 2^3 = 32$ possible characters that can arise at a point $x \in X$ (as we shall see, all of these but one are possible, though not simultaneously). Let the *unar-character* of x be the tuple (n, i, j, k), where $n \in \{1, 2, 3, 4\}$ indicates which of the four possible conditions on $f(x)$ holds, in the order listed above, and $i, j, k \in \{0, 1\}$

indicate whether each of the latter three sets, respectively, is empty or inhabited. We show that this is the appropriate notion of character to characterise 2-equivalence.

Lemma 32. *Let (X, f) and (Y, g) be unars. Then $(X, f) \equiv_2 (Y, g)$ if and only if for all $x \in X$ there exists $y \in Y$, and vice versa, such that x and y have the same character.*

Proof. Suppose there is $a_1 \in X$ such that no member of Y has the same character. We find a winning strategy for Player I. On her first move she plays a_1, and suppose that $b_1 \in Y$ is Player II's response. By assumption, the characters of a_1 and b_1 must differ in at least one component.

In each case we describe Player I's second move and assume for a contradiction that Player II can play in such a way that $(a_1, a_2) \cong (b_1, b_2)$.

First, suppose that the characters of a_1 and b_1 differ in the first component.

If the first component of character of exactly one of a_1 and b_1 is 1, we assume without loss of generality that $f(a_1) \neq a_1$ and $g(b_1) = b_1$. On her second move, Player I plays $a_2 = f(a_1)$, and we let b_2 be Player II's response, which we suppose for a contradiction does not lose. Since $f(a_1) \neq a_1$, it follows that $b_2 \neq b_1$, and since $R_X(a_1, a_2)$, also $R_Y(b_1, b_2)$, where R_X and R_Y are the relations on X and Y from which the functions f and g respectively are constructed. Therefore $g(b_1) = b_2 \neq b_1$, which is a contradiction.

Next suppose (again without loss of generality) that $f(f(a_1)) = a_1$ and $g(g(b_1)) \neq b_1$. Now Player I plays $a_2 = f(a_1)$ (which may equal a_1, though this would have been covered in the previous case). In this case, we have both $R_X(a_1, a_2)$ and $R_X(a_2, a_1)$, so it follows that $R_Y(b_1, b_2)$ and $R_Y(b_2, b_1)$. Therefore $g(g(b_1)) = g(b_2) = b_1$, again a contradiction.

If instead $f(f(a_1)) = f(a_1)$ but $g(g(b_1)) \neq g(b_1)$, Player I again plays $a_2 = f(a_1)$. This time, $R_X(a_1, a_2)$ and $R_X(a_2, a_2)$, from which it follows that $R_Y(b_1, b_2)$ and $R_Y(b_2, b_2)$. Therefore $g(b_1) = b_2$ and $g(b_2) = b_2$, which give $g(g(b_1)) = g(b_1)$, a contradiction.

Finally, if $a_1 \neq f(a_1) \neq f(f(a_1)) \neq a_1$, then we do not have $b_1 \neq g(b_1) \neq g(g(b_1)) \neq b_1$, so we must have at least one equality, and we can use the previous lines to get a contradiction.

Now suppose that the characters of a_1 and b_1 agree on the first component, but differ elsewhere. By swapping X and Y if necessary, we may suppose that there is $a_2 \neq a_1, f(a_1)$

in X such that $f(a_2) = a_1$ (or $f(a_2) = a_2$, or $f(a_2)$ is unequal to a_1, a_2 in the other cases, according to whether the second, third or fourth component of the character differs respectively), but that there is no such member of Y corresponding to b_1. Player I now plays this a_2, and again we suppose that b_2 is Player II's response, and that $(a_1, a_2) \cong (b_1, b_2)$.

Since $a_1 \neq a_2$, also $b_1 \neq b_2$. We also have $\neg R_X(a_1, a_2)$ so it follows that $\neg R_Y(b_1, b_2)$, and so $b_2 \neq g(b_1)$. If $f(a_2) = a_1$ then $R_X(a_2, a_1)$ and therefore $R_Y(b_2, b_1)$, giving $g(b_2) = b_1$, which is contrary to the characters differing on this component, and similarly if $f(a_2) = a_2$. Finally, if $f(a_2) \neq a_1, a_2$, then both $\neg R_X(a_1, a_2)$ and $\neg R_X(a_2, a_2)$ hold, from which it follows that $\neg R_Y(b_1, b_2)$ and $\neg R_Y(b_2, b_2)$ also hold, which again contradicts the characters differing here.

Conversely, suppose that X and Y exhibit the same characters. Then it is Player II who has a winning strategy. Whatever Player I's first move, she may respond with a point in the other structure which has the same character.

Let these two moves played be $a_1 \in X$ and $b_1 \in Y$. By assumption these have the same character, and in particular identical first components, so $f(a_1) = a_1 \Leftrightarrow g(b_1) = b_1$, and similarly for the other possible values, 2, 3 and 4, of the first component. Player I's second move will give a pair of points in one of the structures which Player II wishes to duplicate in the other structure. Suppose without loss of generality that Player I plays $a_2 \in X$ on her second move. Since a_1 and b_1 were chosen to have the same characters, there must be a point $b_2 \in Y$ witnessing the relevant component of b_1's character, and this will give a winning strategy for Player II. We show how $b_2 \in Y$ can be chosen such that (a_1, a_2) and (b_1, b_2) are isomorphic, by checking through the different options.

First, if $a_2 = a_1$ then Player II chooses b_2 to equal b_1, and the fact that $(a_1, a_2) \cong (b_1, b_2)$ follows from $f(a_1) = a_1 \Leftrightarrow g(b_1) = b_1$. Similarly, if $a_2 = f(a_1) \neq a_1$, then b_2 is chosen to be $g(b_1)$, and the equality of the first components of a_1 and b_1's characters ensures that correct relations are preserved under $(a_1, a_2) \cong (b_1, b_2)$.

Otherwise, $a_2 \neq a_1, f(a_1)$, and so a_2 inhabits one of the sets used in defining the final three components of the character of a_1. Since a_1 and b_1 were chosen to have the same characters, it follows that there is a point $b_2 \in Y$ witnessing the corresponding component of b_1's character. To check that $(a_1, a_2) \cong (b_1, b_2)$, we note that $R_X(a_1, a_1) \Leftrightarrow f(a_1) = a_1 \Leftrightarrow g(b_1) = b_1 \Leftrightarrow R_Y(b_1, b_1)$; the remaining 3 character components cover the correspondences for $R_X(a_2, a_1)$, $R_X(a_2, a_2)$, and their negations, respectively. $\square$

This is a similar method to that which we applied to linear and partial orders earlier, though the details of the appropriate character to use are different.

Since there are 32 possible characters and the set of these that arises precisely determines equivalence, there are at most 2^{32} 2-equivalence classes of unordered unars. In fact there are only 133, as we shall show.

To determine which combinations of characters may occur, we investigate which configurations of points must occur in the immediate vicinity of a point of each character. Let a character with first component k be a *type* k character, and we call a point with such a character a *type* k point. By Lemma 31 above, each component of a unar contains at most one cycle, with optional anti-arborations directed towards each point of the cycle. Many of the characters determine the length of the cycle which that point lies in or is next to. Let the unar-character of a point x be (n, m_1, m_2, m_3). If $n = 1$, then $f(x) = x$ and x is in a cycle of length 1. Depending on the values of the m_i, there may be one or more points leading into x, and/or points elsewhere of either or both of two different types. If $m_1 = 1$, then there is some point $y \neq x$ with $f(y) = x$, if $m_2 = 1$, then there is some point $z \neq x$ with $z = f(z)$, and if $m_3 = 1$, then there is some point $w \neq x$ with $w \neq f(w) \neq x$.

If $n = 3$, then $f(x) \neq x$ and $f(f(x)) = f(x)$, and x is immediately before a cycle of length 1. This of course implies the existence of a point $f(x)$ of character $(1, 1, *, *)$. Again, there may be further points pointing to x and/or points elsewhere in the structure of either 1-type, depending on the values of m_1, m_2, m_3.

If $n = 2$, then $f(x) \neq x$, but $f(f(x)) = x$, and x is in a cycle of length 2, with $f(x)$ as the other point. As before, there may possibly be points pointing to x, and possible points elsewhere of either type, according to m_1, m_2, m_3.

Finally, if $n = 4$, then $x \neq f(x) \neq f(f(x)) \neq x$. So x is at the start of a chain of length at least three, which may lie either in a finite cycle of length 3 or more, in an infinite chain, or far enough from the end in a component ending in a 1- or 2-cycle. However, these possibilities are not always interchangeable. For example, a unar consisting only of a 3-cycle is not 2-equivalent to a unar consisting only of a 4-cycle, since the latter has a point x such that there exists a y with $f(x) \neq y \neq x \neq f(y)$ and the former does not, but if we append another component Z containing some $z \neq f(z)$ to the structure we would find that the disjoint union of Z with a 3-cycle is now 2-equivalent to the disjoint union of Z and a 4-cycle. In terms of the unar-characteristic, the distinction is, as ever, that there are those points with $x \neq f(x) \neq f(f(x)) \neq x$ where $\{y : y \neq f(x) \text{ and } f(y) = x\}$ is inhabited or empty, and either with or without the two additional 1-types of point occurring in the structure. We also note that a point x of character $(4, 0, 0, 0)$ is impossible in a unar. The totality of f requires that $f(f(f(x)))$ exist, but $f(f(f(x))) \neq x$, as x's character has second component 0 so x must have no predecessors. If $f(f(f(x))) = f(f(x))$, then x's character would have third component 1 to reflect this, and if $f(f(f(x))) \neq x, f(f(x))$, then x's character would have fourth component 1. It is therefore impossible for a point to have character $(4, 0, 0, 0)$, as this leaves no possible values for $f(f(f(x)))$ to take.

In all of these cases, any character of the form $(*, 1, *, *)$, $(*, *, 1, *)$ or $(*, *, *, 1)$, or for that matter any character of type 2, 3, or 4, implies the existence of points elsewhere, which must of course have (validly co-occurring, but not necessarily distinct) characters of their own.

Diagrams of points of each type may be found in Figure 6.4. The reader is encouraged to recall or refer back to this during the proof of Theorem 33. The first component of the character gives the "type" of point (describing the behaviour of f and f^2), the second indicates whether it has predecessors, the third indicates whether there are 1-loops elsewhere and the fourth indicates whether there are non-1-loop points elsewhere.

The first two components of the character determine the shape of the unar locally, while the latter two relate to more global behaviour.

Note that the third component of the character of most of the points must be identical, in the following sense. If some point $x \in X$ has character $(*, *, 1, *)$, then there is some $y \in X$ with $f(y) = y$ and $f(x) \neq y$. But then y will also have this relationship to any other point $z \neq y$, unless $f(z) = y$, and so any other point z meeting this condition must also have character $(*, *, 1, *)$. Moreover, if there are two or more 1-loops, say $f(y_1) = y_1$

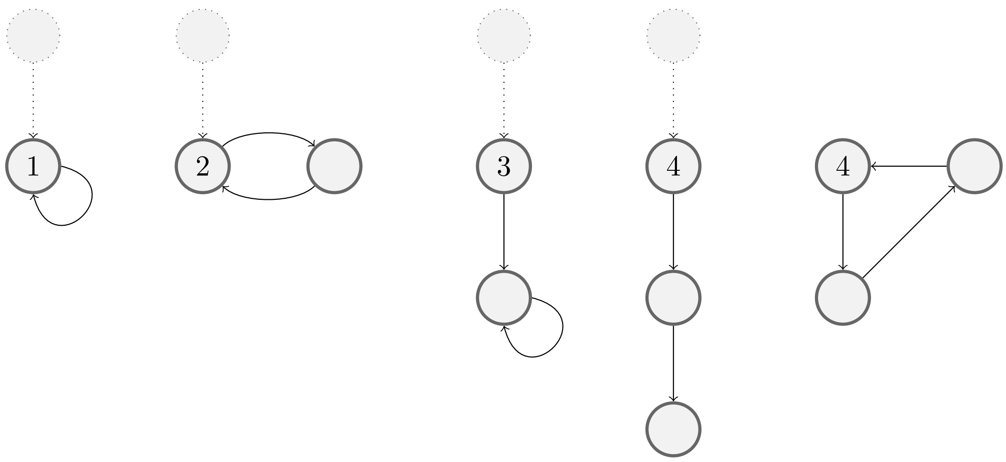

Figure 6.4: The minimal unars realising points of types 1, 2, 3, 4 and 4 again respectively, as labelled. The dotted lines and points are present if and only if the second component of the character of the indicated point is 1. Points of type 4 may also arise in a triangle as shown in the fifth diagram, in which case they always have second character component 1.

and $f(y_2) = y_2$, then no point can be equal to or a predecessor of both of these, and so every point in the structure will have character $(*, *, 1, *)$, including y_1 and y_2.

Likewise, a witness to the fourth component of some character being 1 would typically witness this for most of the points of the unar. If some point $x \in X$ has character $(*, *, *, 1)$, then there is some $y \in X$ with $y \neq f(y), x, f(x)$. Again, y will also have this relationship to any other point z unless $z = y$ or $f(z) = y$, and so any such point z will have character $(*, *, *, 1)$. Unlike for the third component, it is possible to have two points $z_1 \neq z_2$ with $z_1 \neq f(z_1)$ and $z_2 \neq f(z_2)$ and still have a character of the form $(*, *, *, 0)$ arise, if these are the only two such points and $f(z_1) = z_2$ (or vice versa), in which case z_1 has character $(4, 0, 1, 0)$. If there are three or more such points, however, no point can be equal to or a predecessor of all of them simultaneously, and so for every point a there will exist some $b \neq a, f(a)$ with $f(b) \neq b$ witnessing that the fourth component of a's character is 1.

We consider first the sets of pairs (x, y) for which a character of the form (x, y, z, w) arises; and then determine what the possible values of z and w in each character are.

There are 4 possible values of x and 2 of y, which gives at most $2^8 = 256$ possible sets of (x, y) arising; in practice, some of these sets cannot arise, and most of them give only one possibility for z and w due to the above described global behaviour of these components, so the task is not quite as daunting as it may at first appear.

Theorem 33. *There are* 133 2*-equivalence classes of unars.*

Proof. We determine the possible sets of characters that may arise, each of which corresponds to an equivalence class as shown in Lemma 32. The structure of this proof is as follows: first, we systematically consider all sets of characters that include a type 3 character, in Cases $1a - 1e$. Then, we consider sets of characters including a type 1 character but no type 3 character in Cases $2a - 2c$, and finally, we consider sets of characters including only type 2 and 4 characters in Case 3. Note that because the presence of 1-loops changes the third component of the character of all other points, Cases $1a$ and $1b$ are analogous to Case $1c$, Case $1d$ is analogous to Case $1e$, and Case $2c$ is analogous to Case 3. The relationship between the fourth components of characters is slightly less straightforward and so this correspondence is less obvious.

If there is any point x of character $(3, *, *, *)$, then we have both $x \neq f(x)$ and $f(x) = f(f(x))$, so certainly any point $y \neq x, f(x)$, with $f(y) \neq x, f(x)$ must have character $(*, *, 1, 1)$. Note that we do not assume that the type 3 character arising in each case is the only type 3 character present; rather, this condition precludes most 3-characters from co-occurring, with the exceptions of $\{(3, 0, 0, 1), (3, 1, 0, 1)\}$ and $\{(3, 0, 1, 1), (3, 1, 1, 1)\}$ which arise together in Cases $1d$ and $1e$ respectively. We consider each possibility in turn.

Case 1a If x has character $(3, 0, 0, 0)$, then there are no other points than x and $f(x)$, so there is precisely one equivalence class of unar satisfying this. The unar with set $\{x, y\}$ and function f such that $f(x) = y$, $f(y) = y$ is the sole example of this case, and the characters arising here are $\{(3, 0, 0, 0), (1, 1, 0, 0)\}$.

Case 1b If x has character $(3, 1, 0, 0)$, then again there are no other points than x, $f(x)$, and x's predecessors, so this gives two equivalence classes. If x has precisely one predecessor, then the characters $\{(3, 1, 0, 0), (1, 1, 0, 1), (4, 0, 1, 0)\}$ are realised, for example in the unar on $\{x, y, z\}$ with $f(x) = y$, $f(y) = y$, $f(z) = x$. If x has more than one predecessor, then the characters realised are $\{(3,1,0,0),(1,1,0,1),(4,0,1,1)\}$, and an example of this is obtainable by duplicating the point z in the previous example.

Case 1c If x has character $(3,0,1,0)$ or $(3,1,1,0)$, then all other points $z \neq x, f(x)$ (and $f(z) \neq x$) satisfy $f(z) = z$, and there must exist at least one such point. These z cannot have a predecessor, or that predecessor would fail this condition, so their character must be $(1,0,1,1)$. Therefore, $\{(3,0,1,0),(1,1,1,0),(1,0,1,1)\}$, $\{(3,1,1,0),(1,1,1,1),(4,0,1,0)\}$ and $\{(3,1,1,0),(1,1,1,1),(4,0,1,1)\}$ are the only possible sets in this case, and these are each realisable by appending another point w with $f(w) = w$ to the unars from the previous three cases.

Case 1d If x has character $(3,0,0,1)$ or $(3,1,0,1)$, then every other point $z \neq x, f(x)$ must have $z \neq f(z)$. The point $f(x)$ is fixed by f, but is the only such point, so any other point z must either have character $(*,*,1,*)$ or satisfy $f(z) = f(x)$. Likewise, any point not immediately preceding x must have character $(*,*,*,1)$. Points of type 1 may not occur, apart from $f(x)$ which must exist and has character $(1,1,0,*)$.

If $f(x)$ has character $(1,1,0,0)$, then every point z has $f(z) = f(x)$ and the set of characters arising is $\{(3,0,0,1),(1,1,0,0)\}$.

Otherwise, $f(x)$ has character $(1,1,0,1)$, for which there are multiple options which we now enumerate. First we consider any points y of type 2, if these exist. These satisfy $f(f(y)) = y$, so they cannot have $f(y) = x$ or $f(y) = f(x)$. Their character, if present, must therefore be of the form $(2,*,1,1)$.

Now, we consider whether any points z of type 3 exist. All of the previously considered $(3,*,*,*)$ characters are incompatible with x and hence cannot arise, since x's possible characters $(3,0,0,1)$ and $(3,1,0,1)$ did not arise as a possibility in any of the preceding cases. Points z of character $(3,*,1,1)$ are also impossible, as they require at least two fixed points of f. Therefore, z must have character $(3,*,0,1)$.

Finally, considering type 4 points, we note that $(4,0,1,0)$ is an impossible character, as it precludes x's character from having final component 1, so type 4 points w must have character $(4,*,1,1)$ or $(4,1,1,0)$. If w has character $(4,1,1,0)$, then x has character $(3,1,0,1)$ and the whole unar is determined. We must have $f(w) = x$, since the final component of the character of w is 0, and there must further exist a such that

$f(a) = w$, and no other points. The set of characters arising in this case is therefore $\{(3,1,0,1),(1,1,0,1),(4,1,1,0),(4,0,1,1)\}$.

This leaves only cases in which the character $(1,1,0,1)$ is present. As well as $(1,1,0,1)$, at least one of $(3,0,0,1)$ and $(3,1,0,1)$ must arise, and optionally $(2,0,1,1)$, $(2,1,1,1)$, $(4,0,1,1)$ and $(4,1,1,1)$. We now go through the equivalence classes arising from these.

The character $(4,0,1,1)$ only arises before a point of character $(*,1,*,*)$, so can only occur before either $(4,1,1,1)$, $(2,1,1,1)$, or $(3,1,0,1)$. Points of character $(3,1,0,1)$ and $(2,1,1,1)$ must have predecessors of character $(4,0,1,1)$ or $(4,1,1,1)$. (Of course, $(4,1,1,1)$ requires both a predecessor and a successor, but these can also be of character $(4,1,1,1)$ and so no additional characters are necessary when this arises.)

Subject to these restrictions, examples may be easily constructed of all remaining combinations, by adding in the desired points. Points of character $(3,0,0,1)$ may be added before $f(x)$, as can $(3,1,0,1)$ with their predecessor points. The character $(2,0,1,1)$ may be added by adding a 2-cycle, $(2,1,1,1)$ by adding a 2-cycle where both points also have another predecessor, $(4,0,1,1)$ by prepending it to whichever point it precedes, and $(4,1,1,1)$ by appending a triangle to the unar. As there are many combinations, and similar situations will arise later in the proof, we shall not explicitly describe examples for each, but this method allows one to easily construct a member of any particular equivalence class in this subcase and demonstrates that the above conditions on existence of predecessors of $(4,0,1,1)$ and successors of $(3,1,0,1)$ and $(2,1,1,1)$ are sufficient as well as necessary.

There are therefore 6 classes of unars without either $(3,1,0,1)$ or $(4,0,1,1)$, 6 classes of unars without $(3,1,0,1)$ but with $(4,0,1,1)$, 8 classes of unars with $(3,1,0,1)$ but without $(4,0,1,1)$, and 16 classes of unars with both $(3,1,0,1)$ and $(4,0,1,1)$. We also recall that we already identified two other possible sets of characters earlier in this case: $\{(3,0,0,1),(1,1,0,0)\}$ and $\{(3,1,0,1),(1,1,0,1),(4,1,1,0),(4,0,1,1)\}$. This gives a total of $2+6+6+8+16 = 38$ equivalence classes containing a character of the form $(3,*,0,1)$.

Case 1e In the final type 3 case, let x have character $(3,*,1,1)$, which requires there to exist both another $z \neq f(x)$ with $f(z) = z$, and some other point $y \neq x$ with $f(y) \neq y$. If x has this character, then $f(x)$ has character $(1,1,1,*)$.

Suppose first that $f(x)$ has character $(1,1,1,0)$. Then every y has either $f(y) = y$ or $f(y) = f(x)$, and there must exist at least one of each, by the third and fourth components of x's character. Therefore x must have character $(3,0,1,1)$, as it has no predecessor, and the set of characters realised is $\{(3,0,1,1),(1,1,1,0),(1,0,1,1)\}$.

We suppose for the remainder of this case that $f(x)$ has character $(1,1,1,1)$. Then there is some y with $f(y) \neq y, f(x)$, as well as some z with $f(x) \neq z = f(z)$. Then every character present must be of the form $(*,*,1,*)$, since no point w can give $\{z, f(x)\} \subseteq \{w, f(w)\} \cup \{a : f(a) = w\}$. Similarly, the only way a point w can have a character with fourth component 0 is if every point y with $f(y) \neq y$ is equal to, a predecessor of, or a successor of w. Having already established that x and $f(x)$'s characters have fourth component 1, this implies that w is a predecessor of x, and in order for x's fourth character to be correct, all other such points must precede w. This gives us one equivalence class of unars, realising the characters $\{(3,1,1,1),(1,1,1,1),(4,1,1,0),(4,0,1,1),(1,0,1,1)\}$. Otherwise, all characters must have the form $(*,*,1,1)$, and as in the previous case, we have almost complete freedom as to which of these arise. The character $(1,1,1,1)$ must exist by assumption, at least one of $(3,1,1,1)$ and $(3,0,1,1)$ must arise, and $(2,0,1,1)$, $(2,1,1,1)$, $(4,0,1,1)$ and $(4,1,1,1)$ are optionally present.

If $(4,0,1,1)$ is present, then it must arise at a point preceding a point of character $(2,1,1,1)$, $(3,1,1,1)$ or $(4,1,1,1)$, and $(2,1,1,1)$ and $(3,1,1,1)$ require predecessors of character $(4,0,1,1)$ or $(4,1,1,1)$. The combinations are analogous to the previous case, and so there are again 36 equivalence classes of this form, giving a total of 38 for Case 1e and a grand total of 82 in Case 1.

Having categorised the 82 equivalence classes containing a point of type 3, we now consider those with a point of type 1 but no point of type 3. If a point of type 1 had a predecessor, then that predecessor would be of type 3, so in this case they cannot have a predecessor and so their character must be of the form $(1,0,*,*)$.

Case 2a If x has character $(1,0,0,0)$ then x is the only point present and the characters realised are $\{(1,0,0,0)\}$.

Case 2b If x has character $(1,0,1,0)$ then there are only type 1 points present, but at least two such, and the only characters realised are $\{(1,0,1,0)\}$.

Case 2c If x has character $(1,0,*,1)$ then there are other points present of type 2 or 4. As in the previous two cases, if the third component of x's character is 0 then it is the only type 1 point present, and if it is a 1 then there is at least one other type 1 point present, but in either case only one type 1 character may arise. The other possible characters are $(2,*,1,*)$ and $(4,*,1,*)$.

The character $(4,0,1,0)$ cannot arise in this case, since if z has this character then $f(f(z))$ may be mapped nowhere but back to itself, and then $f(z)$ is of type 3 which is forbidden in this case. However, $(4,1,1,0)$ may arise, in which case $f(f(f(z))) = z$ (which was impermissible for $(4,0,1,0)$, which arises only at points with no predecessors). This gives rise to four classes, depending on the third component of x and whether z has another predecessor outside the triangle: $\{(1,0,0,1),(4,1,1,0)\}$, $\{(1,0,1,1),(4,1,1,0)\}$, $\{(1,0,0,1),(4,1,1,0),(4,0,1,1),(4,1,1,1)\}$ and $\{(1,0,1,1),(4,1,1,0),(4,0,1,1),(4,1,1,1)\}$. If $(2,0,1,0)$ is realised at a point z, then z is in a 2-loop with another point also of character $(2,0,1,0)$, and no other points may arise, so the only sets of characters in which this may occur are $\{(1,0,0,1),(2,0,1,0)\}$ and $\{(1,0,1,1),(2,0,1,0)\}$. The character $(2,1,1,0)$ is similar but has another predecessor, which must be of character $(4,0,1,1)$, so the possible characters here are $\{(1,0,0,1),(2,1,1,0),(2,0,1,1),(4,0,1,1)\}$ and $\{(1,0,1,1),(2,1,1,0),(2,0,1,1),(4,0,1,1)\}$.

Having dealt with the characters with fourth component 0, the remaining equivalence classes in this case realise only some subset of $(2,0,1,1)$, $(2,1,1,1)$, $(4,0,1,1)$, $(4,1,1,1)$, and of course precisely one of $(1,0,0,1)$ and $(1,0,1,1)$. The by now familiar constraints apply - if $(4,0,1,1)$ appears, then it has a successor of character $(2,1,1,1)$ or $(4,1,1,1)$, and if $(2,1,1,1)$ appears, then it has a predecessor of character $(4,0,1,1)$ or $(4,1,1,1)$. So for each of our two choices of type 1 character, there are six possibilities omitting $(4,0,1,1)$, as in this case $(2,1,1,1)$ may only occur if $(4,1,1,1)$ occurs, and six realising $(4,0,1,1)$, as in this case either $(2,1,1,1)$ or $(4,1,1,1)$ must also occur. As usual, examples of each of these may easily be constructed by combining simple components realising the desired characters. Adding these 24 to the eight classes already identified gives 32 cases for Case 2c and a total of 34 for Case 2.

Case 3 Finally, we assume that there are no points of either type 1 or type 3, and thus all points are either of type 2 or 4. The attentive reader will recall, and the exhausted reader will be relieved to note, that this case is analogous to Case 2. Appending a point z with $f(z) = z$ to any Case 3 equivalence class will yield a corresponding Case 2 equivalence class, adding a second point y with $f(y) = y$ will give a different Case 2 class, and removing all type 1 points from a Case 2 class will likewise give a Case 3 equivalence class. The characters of the points of types 2 and 4 will be unchanged in their first, second and fourth components, but their third component is 0 in Case 3 and 1 in Case 2. Therefore, all 32 of the equivalence classes in Case 2c have a corresponding equivalence class omitting the point of character $(1, 0, 0, 1)$ or $(1, 0, 1, 1)$ and changing the third component of all other characters to 0. This map is two-to-one, so there are precisely 16 equivalence classes of unars realising at least one character of type 2 or 4:

$\{(4, 1, 0, 0)\}$, $\{(4, 1, 0, 0), (4, 0, 0, 1), (4, 1, 0, 1)\}$, $\{(2, 0, 0, 0)\}$, $\{(2, 1, 0, 0), (2, 0, 0, 1), (4, 0, 0, 1)\}$, and the twelve valid combinations of $(2, 0, 0, 1)$, $(2, 1, 0, 1)$, $(4, 0, 0, 1)$, and $(4, 1, 0, 1)$.

These are all the equivalence classes where all points have type 2 or 4 and there is at least one point present; to complete the proof, we must also count the equivalence class of the unar realising no characters of these types - the empty unar, whose counterparts under the above map are the unars containing *only* type 1 points, in Cases 2a and 2b. This gives a total of 17 equivalence classes in Case 3.

We have identified the 82 equivalence classes realising a type 3 point, 34 classes realising a type 1 point but no type 3 point, and 17 classes realising neither, for a total of 133 equivalence classes. $\square$

6.4.3 n-equivalence of unars

We remark that in unars, only local behaviour may be distinguished, in contrast to structures with an order relation or cyclic order relation. In a partial order, for instance, $<$ is transitive, and we may immediately determine that $x < y$ (where this holds) even when there are several or even infinitely many points in between x and y. In a unar, however, if $f^m(x) = y$ for some m large relative to the number of moves n in the game ($m > 2^n$, for example), then x does lie "upstream" of y but this is not demonstrable within the time limit of the game. No single first-order sentence expresses upstreamness in general, although the disjunction of the infinite family of formulae $f^m(x) = y$ for $m \in \mathbb{N}$ would express this. With a finite game, however, we are only able to verify this for finitely many m. Due to this behaviour, finiteness is not a distinguishable property. For example, an infinite unar consisting of a chain $\{f^k(x) : k \in \mathbb{Z}\}$ with $f^i(x) \neq f^j(x)$ whenever $i \neq j$, is n-equivalent to any sufficiently large finite cycle, such as the cycle $\{f^k(x) : 0 \leq k \leq 2^n\}$ with $f^{2^n}(x) = x$, which has length 2^n.

Lemma 34. *In unars, a cycle of length 2^{n-1} is n-equivalent to any longer cycle.*

Proof. By Corollary 18, we can distinguish finite cycles in cyclic orders of lengths $1, \ldots,$ $2^{n-1} - 1$, with cycles of length 2^{n-1} indistinguishable from longer cycles. We may translate this result to unars by defining a cyclic relation R on a cycle C contained in a unar (X, f) such that $R(x, y, z)$ holds for $x, y, z \in C$ if and only if $\exists a(f^a(x) = y)$ and $(\forall b < a)(f^b(x) \neq z)$. Since z is in C we will of course have $f^c(x) = z$ for some $c > a$. The relation R holds when x, y, z occur in that order when following the arrows in the appropriate direction. Note the positive direction here is the direction of f (as opposed to the direction of f^{-1}), rather than anticlockwise in any particular planar embedding of the directed graph of (X, f). However, this still presents a cyclic order, to which we may apply Corollary 18 and discover that we may distinguish only the cases where C has size $1, \ldots, 2^{n-1} - 1$, and that cycles of length $\geq 2^{n-1}$ are mutually indistinguishable.

Therefore, only cycles of length 2^{n-1} or shorter may be distinguishable in unars. □

It is possible to distinguish shorter cycles, however, and in fact we may distinguish the number of small cycles with some granularity.

Recall that every unar consists of a union of finite cycles and infinite chains, with anti-arborescences (downwards-pointing directed graph trees), attached to some of the points of the cycles and chains. General equivalence of unars depends both on the lengths of cycles and on the existence and structure of points in the anti-arborescences leading into these cycles and chains.

6.4.4 Injective unars

In the special case of injective unars, which have been studied in [1], these anti-arborescences are just single points, and the whole unar is simply the union of finite cycles and infinite chains, where the infinite chains may either have order type $\mathbb{Z}$, in which case they are infinite in both directions, or $\mathbb{N}$, in which case they have a first element and are only infinite in one direction.

We may therefore characterise the injective unars up to n-equivalence by the number of $1, \dots, 2^n - 1$ cycles up to some number of maximum discernible depending on length and n, and the number of copies of $\mathbb{N}$ (that is, infinite chains with an endpoint at the beginning), again up to some maximum depending on n. Note that components of order type $\mathbb{Z}$ are indistinguishable from finite cycles that are sufficiently large compared to n, and so do not need to be considered separately.

One possible type of component in injective unars is a chain of order type $\mathbb{N}$, that is, chains $\{x_i : i < \omega\}$ with $f(x_i) = x_{i+1}$ and where x_0 has no predecessors. These are easy to distinguish in injective unars (unlike in unars in general), because the x_0 points at the start of these chains are the only points without predecessors.

Lemma 35. *If the injective unars X and Y are n-equivalent, then either X and Y have the same number of chains of order type $\mathbb{N}$, or they both have at least $n-1$ chains of order type $\mathbb{N}$.*

Proof. We prove the contrapositive. Suppose that X has more chains of order type $\mathbb{N}$ than Y, and that Y has fewer than $n-1$ chains of order type $\mathbb{N}$. Then Player I may play a distinct point $x_i \in X$ without a predecessor on every move $i \leq n-1$. By assumption, there are fewer than $n-1$ chains of order type $\mathbb{N}$ in Y, so Player II cannot play $n-1$ distinct points without predecessors. On her nth move, Player I may win by playing a predecessor of one of Player II's previous moves, and so X and Y are not n-equivalent. □

The other possible type of component is a finite cycle. In the extreme case of 1-cycles, an n-move game allows us to discern whether there are $1, \dots, n-1$ many 1-cycles present, or whether there are at least n. For 2-cycles, we can certainly verify that there are at least

k many 2-cycles in $2k$ moves, by playing both of the points in each of k distinct 2-cycles. In fact, $k+1$ moves suffices, as we shall see.

Lemma 36. *If the injective unars X and Y are n-equivalent, then for every cycle length $m \leq n$, either X and Y have the same number of m-cycles, or they both have at least $(n-m+1)$ many m-cycles.*

Proof. We prove the contrapositive. Suppose that X has more m-cycles than Y, and that Y has fewer than $(n-m+1)$ m-cycles. Then Player I has a winning strategy in $G_n(X, Y)$ as follows: on her ith move for $i \leq (n-m+1)$, Player I plays a point $x_i \in X$ that lies in a distinct m-cycle to all previous x_j. Since, by assumption, there are not this many m-cycles in Y, Player II must at some point respond with a point $y_j \in Y$ that either does not lie in an m-cycle, or lies in the same m-cycle as a point already played.

Player I then plays $f(x_j), f^2(x_j), \ldots, f^{m-1}(x_j)$ on her remaining moves. If y_j does not lie in an m-cycle, then Player II is unable to respond with $z_1, \ldots, z_{m-1}$ such that $f(y_j) = z_1$, $f(z_j) = z_{j+1}$ for each $j < m-1$, and $f(z_{m-1}) = y_j$. If y_j lies in the same m-cycle as a point y_a already played on the ath move, for some $a < j$, then for some $b < m$ we have $f^b(y_j) = y_a$, but $f^b(x_j) \neq x_a$, because x_j and x_a lie in different cycles. Either way, Player I wins, and so X and Y are not n-equivalent. $\square$

We remark that the above conditions are not sufficient, that is, pairs of unars satisfying weaker conditions may be equivalent. For example, if Y is the union of precisely $n-1$ many 2-cycles, and X is the union of n many 2-cycles, then Player I may win $G_n(X, Y)$ by playing a point in every 2-cycle. This happens because there are no other points in Y.

Another, more general issue with the strategy above is that it is not optimally fast, as Player I unnecessarily wastes moves. To give a bound on the number of moves required, it assumes that Player I may distinguish an n-cycle from any other length of cycle in n moves, but this distinction may in general require fewer moves. For example, if Player I plays n unrelated points in a cycle of size $2n$, and Player II's n unrelated responses do not also lie in a $2n$-cycle, then these responses cannot form a cycle of points exactly distance 2 apart, and so Player I may win on the $n+1$th move by playing the point $f(x)$ between some already chosen points x and $f(f(x))$ such that their counterparts y and z in the other structure do not have $f(f(y)) = z$. This allows Player I to distinguish $2n$-cycles in only $n+1$ moves, rather than $2n$ moves. Of course, Player I may instead play every third point of a cycle rather than every second, in which case she would play the two intermediate points in whichever interval was not of size three in the other structure. Similarly, she could play every kth point and then play the at most k points between the appropriate pair, for a total of $\lceil \frac{m}{k} \rceil + k$ moves to distinguish an m-cycle from an $(m-1)$-cycle. This strategy allows her to distinguish cycles of size quadratic in n within n moves, but again we can do better by avoiding playing all of the k consecutive points within the interval. A better strategy would be to repeatedly bisect the cycle to give an exponential relationship between number of moves and size of cycle distinguishable, in a way roughly analogous to an interval bisection strategy on finite linear orders, though the precise details differ due to the difference in the relation under consideration.

Chapter 7

Pebble games

7.1 Pebble games

As well as considering ordinary Ehrenfeucht-Fraïssé games of various lengths, we can also restrict the number of variables that may be bound at one time. The notion of investigating which theories may be distinguished using a limited number of bound variables, which may be re-assigned, was investigated by Immerman and Kozen [14]. In this system, we permit reuse of variables in a formula, such as in the formula $\exists x(\exists y : y > x \wedge (\exists x : x > y))$, which asserts the existence of a chain of three points, but manages to do so using only two distinct variables - x and y. The x has been reused, or reassigned, when we could more typically have used a new variable and written the formula as $\exists x(\exists y : y > x \wedge (\exists z : z > y))$.

Determining equivalence up to a limited number of reusable variables corresponds to playing a modified Ehrenfeucht-Fraïssé game, known as a *pebble game*.

In the pebble game $G_n^k(A, B)$, each player is issued k pebbles, one of each colour $c_0, ..., c_{k-1}$, and the game lasts for n turns. The players take turns to place pebbles on points of A and B as in the normal Ehrenfeucht-Fraïssé game of length n on A and B. On each turn, Player I may either place a pebble of a new colour, if any colours remain to be played, or pick up a pebble that has already been played and move it to the new point. If she moves a pebble that has previously been played, then its previous position is no longer distinguished. The substructures whose equivalence is to be considered are those marked by the pebbles, not a complete history of the pebbles' previous positions. Moving the pebble to a new point in this way corresponds to reusing a variable in a formula. Player II must respond to each Player I move by moving the pebble of the same colour, and the remainder of the rules are as one would expect for an Ehrenfeucht-Fraïssé game. If Player II has a winning strategy in $G_n^k(A, B)$, then we say that A and B are *(n,k)-equivalent*, which we may also write $A \equiv_n^k B$.

Lemma 37. *For all $n, k \in \mathbb{N}$, $\equiv_n^k$ is an equivalence relation.*

Proof. We may see that it is reflexive, since $A \cong A$ so Player I certainly cannot have a winning strategy in $G_n^k(A, A)$. Symmetry is also clear from the rules of the Ehrenfeucht-Fraïssé game, as the games $G_n^k(A, B)$ and $G_n^k(B, A)$ are identical. To show transitivity, suppose that $X \equiv_n^k Y$ and $Y \equiv_n^k Z$, and consider $G_n^k(X, Z)$. If Player I makes a move x in X, Player II can think of a "move" $y \in Y$ according to her winning strategy σ in $G_n^k(X, Y)$, and then use y to find a move $z \in Z$ according to her winning strategy τ in $G_n^k(Y, Z)$. Or, if Player I makes a move $z \in Z$, then Player II finds $y \in Y$ according to τ and then $x \in X$ according to σ. Because these moves come from composing the winning strategies σ and τ which preserve relations, they must have the correct relation to all previous moves, and must occur with the correctly coloured pebble. Repeating this process each move therefore gives Player II a winning strategy on $G_n^k(X, Z)$, and so $\equiv_n^k$ is transitive. $\square$

We remark that there must be finitely many $\equiv_n^k$-equivalence classes, since there are only finitely many inequivalent first order formulae of quantifier depth at most n that can be expressed using at most k distinct variables, and also since these equivalence classes are a coarser partition than the $\equiv_n$ equivalence classes, of which we already know there are finitely many by Lemma 7.

We note a few relationships between pebble games and standard Ehrenfeucht-Fraïssé games. If $n \leq k$ then Player I could choose to use a new colour of pebble for every move. It never harms Player I to keep all previously played pebbles in position, which is what happens when she uses a new colour every move, so she has a winning strategy in $G_n^{n+m}(A, B)$ for $m \geq 0$ if and only if she has one in the usual Ehrenfeucht-Fraïssé game $G_n(A, B)$. If $n > k$, then certainly any winning strategy for Player I in $G_n^k(A, B)$ would still induce a winning strategy in $G_n(A, B)$, but the converse need not (and does not in general) hold. That is, it is possible to have A, B such that $A \equiv_n^k B$ but $A \not\equiv_n B$; however, $A \equiv_n B$ implies that $A \equiv_n^k B$ for all k.

We also note that if $j < k$, then $A \equiv_n^k B$ implies $A \equiv_n^j B$, since any sequence of moves that Player I might play in $G_n^j(A, B)$ could also be played in $G_n^k(A, B)$, and so Player II's winning strategy in $G_n^k(A, B)$ gives her a valid winning sequence of moves to play in response. If we consider the subset of partial orders consisting of only the m-element antichains for all $m \in \mathbb{N}$ (including 0), we note that there are $(k+1)$ many $\equiv_n^k$-equivalence classes: one containing the empty antichain, $(k - 1)$ classes each containing only one nonempty antichain with size less than k, and one class containing all antichains of size $\geq k$. Therefore, for $j < k$, $\equiv_n^k$ gives a strictly finer partition of the partial orders into equivalence classes than $\equiv_n^j$ does.

7.2 Pebble games on linear orders

Immerman and Kozen show that three pebbles suffice to distinguish linear orders up to elementary equivalence [14]. This also holds for coloured linear orders.

Theorem 38. *Let A and B be coloured linear orders. Then $A \equiv_n B$ if and only if $A \equiv_n^3 B$.*

Proof. If $A \not\equiv_n^3 B$ then Player I's winning strategy in $G_n^3(A, B)$ induces a winning strategy in $G_n(A, B)$ by disregarding the pebbles, so $A \not\equiv_n B$.

Suppose that $A \not\equiv_n B$. Then Player I has a winning strategy in $G_n(A, B)$. We give a related strategy for Player I in the pebble game $G_n^3(A, B)$, in which the same points are chosen but only three colours of pebble are used. By a judicious choice of which pebble to move at each of her moves, Player I may restrict play at each move of $G_n^3(A, B)$ to the same sequence of intervals as in her winning strategy on $G_n(A, B)$, and therefore win in the same number of moves.

By Theorem 9, there exists $a_1 \in A$ (without loss of generality) such that for all $b_1 \in B$ with $F(a_1) = F(b_1)$, either $a_1^< \not\equiv_{n-1} b_1^<$ or $a_1^> \not\equiv_{n-1} b_1^>$; the first move of Player I's strategy is to place a red pebble on a_1. Whatever Player II's response b_1, either $a_1^< \not\equiv_{n-1} b_1^<$ or

$a_1^> \not\equiv_{n-1} b_1^>$, without loss of generality suppose the former. Then Player I may again find some $a_2 < a_1$ such that for all $b_2 < b_1$, either $\{a \in A : a < a_2\} \not\equiv_{n-2} \{b \in B : b < b_2\}$ or $\{a \in A : a_2 < a < a_1\} \not\equiv_{n-2} \{b \in B : b_2 < b < b_1\}$; this time she places a blue pebble, and on her third move she places the green pebble.

For $k > 3$, there will always be a valid winning-strategy kth move for Player I in the non pebble game $G_n(A, B)$; we need only to show that the pebble moved may be chosen to maintain the restriction to the same subinterval as the same move of the non-pebble game. If the two pairs of pebbles that did not move on the $(k-1)$th turn were at $p_a, q_a \in A$ and $p_b, q_b \in B$ with $p_a < q_a$ and $p_b < q_b$, then the $(k-1)$th move a_{k-1} must be in either $\{a \in A : a < p_a\}$, $\{a \in A : p_a < a < q_a\}$, or in $\{a \in A : q_a < a\}$. In the former case, we move the pebble on q_a on the kth turn; in the latter, we move the pebble on p_a. If Player I's $(k-1)$th move was to place a pebble on $r \in \{a \in A : p_a < a < q_a\}$, then we may determine the pebble to move on Player I's kth move based on Player II's $(k-1)$th move: any non-losing move b for Player I must lie between p_b and q_b, and either $[p_b, b] \not\equiv_{n-k+1} [p_a, a]$ or $[b, q_b] \not\equiv_{n-k+1} [a, q_a]$. In the former case, we wish to continue the game on the subintervals $[p_a, a]$ and $[p_b, b]$, so Player I moves the pebble on q_a or q_b for her kth move; in the latter case, she moves the pebble on p_a or p_b.

Player I is therefore able to select a pebble to move at each move of $G_n^3(A, B)$ such that subsequent moves are restricted to the same intervals she uses to win in $G_n(A, B)$. Any valid sequence of moves for Player II will therefore also be valid in $G_n(A, B)$, so since Player II loses $G_n(A, B)$ she also has no winning strategy in $G_n^3(A, B)$, and so $A \not\equiv_n^3 B$. $\square$

Given only one pebble, which would have to be moved every turn, it is clear that any two nonempty linear orders would be equivalent in a game of any length. For two pebbles, there is an interesting intermediate level of detail.

Let the *upper point number* u_A of a linear order A be 0 if A has no greatest element. If A has a greatest element a, and $A \backslash \{a\}$ has upper point number u, let the upper point number of A be $u+1$. Let the *lower point number* l_A of a linear order be the corresponding notion concerning least elements. These point numbers are the size of the largest finite initial and terminal segments. If there is no nonempty finite initial or terminal segment, we have $u_A = 0$ or $l_A = 0$. If there are initial or terminal segments of arbitrary finite size, we take $u_A = \infty$ or $l_A = \infty$. We remark that, since the games have finite length, it is not necessary to distinguish between cofinalities of different infinite sizes.

In the game $G_n^2(A, B)$, we care about the existence or nonexistence of the top $n-1$ and bottom $n-1$ elements, where n is the length of the game. Let the *upper points* of A be $U_A = \{a \in A : |\{b : b > a\}| < n-1\}$, and similarly let the *lower points* $L_A = \{a \in A : |\{b : b < a\}| < n-1\}$. If $u_A \geq n-1$ then U_A will have size $n-1$; otherwise it will be smaller. If A has no largest element, then $U_A = \emptyset$. Likewise, if $l_A \geq n-1$ then L_A has size $n-1$, and if A has no least element, then $L_A = \emptyset$.

Theorem 39. *Let A and B be linear orders, and n a positive integer. Then Player II wins the two-pebble game $G_n^2(A, B)$ if and only if the following hold: $min(u_A, n-1) = min(u_B, n-1)$; $min(l_A, n-1) = min(l_B, n-1)$; $|U_A \cap L_A| = |U_B \cap L_B|$; and ($A = L_A \cup U_A$ if and only if $B = L_B \cup U_B$).*

Proof. We assume the above conditions hold and construct a winning strategy for Player II in $G_n^2(A, B)$, in the above cases. It is immediate from the equalities above that, for any $k \leq n-2$, there is a point $x \in A$ with precisely k points above it in A if and only if there is a point $y \in B$ with precisely k points above it in B, and likewise for belowness.

Moreover, a point $x \in A$ can only be in both the first $n-1$ points and the last $n-1$ points if $|U_A \cap L_A| > 0$. This implies that $|A| < 2n-2$ and in fact gives us a unique finite linear order for A: $u_A = l_A = m < n-1$ implies that $|A| = m$, and $u_A \geq n-1, l_A \geq n-1$ and $|U_A \cap L_A| > 0$ implies that A is the linear order of size $2n-2-|U_A \cap L_A|$. The above equalities imply that B must be a linear order of the same size as A, and so we also deduce that for any $k_1, k_2 < n-2$, there is some point $x \in A$ with precisely k_1 points above x and precisely k_2 points below it if and only if there is some point $y \in B$ with precisely k_1 points above y and precisely k_2 points below it.

We show that Player II can, on the mth move, match Player I's moves in the first or last $(n-m-1)$ elements, in the sense that Player II plays a point which is kth from the top or bottom for some $k \leq (n-m-1)$ only in response to a Player I move kth from the top or bottom.

If Player I plays a move x which is kth from the top of A for some $k \leq (n - m - 1)$, Player II's strategy is to respond with the point $y \in B$ which is kth from the top of B. Similarly, if Player I plays some x which is kth from the bottom of one structure, then Player II should respond by playing the point kth from the bottom of the other structure. Note that if Player I's move is both k_1th from the top and k_2th from the bottom of A, then the points k_1th from the top and k_2th from the bottom of B will be the same point, so these requirements are mutually satisfiable. We observe that these moves give the correct ordering with the other pair of pebbles whether the other pebbles are jth from the end for $j < k$ or whether they are further in the middle.

If Player I plays a move x on her mth turn which is not kth from the top or bottom of A for some $k \leq (n - m - 1)$, then Player II responds by playing a point which is also at least $(n-m-1)$th from the top or bottom, and which respects the ordering with the other pebble. We show that this is always possible. On the first move, our condition $A = U_A \cap L_A$ if and only if $B = U_B \cap L_B$ guarantees that there will be a point $a \in A \backslash (U_A \cap L_A)$ if and only if there is a point $b \in B \backslash (U_B \cap L_B)$, and the other pair of pebbles have not yet been placed, so a and b will be valid Player II moves in response to any Player I moves in $B \backslash (U_B \cap L_B)$ and $A \backslash (U_A \cap L_A)$, respectively.

On subsequent moves, either Player I has only moved the pebble of the first colour, in which case Player II responds as before, or Player I places a pebble of the second colour while the pebbles of the first colour are still in play. Let the pebbles coloured c_1 be on a_1 and b_1, and let Player I play her pebble coloured c_2 on her mth move. Since the c_1 coloured pebbles were placed on the $(m - 1)$th move or earlier, they were not played in the $(n - m)$ highest or lowest points. This implies that there are at least $(n - m)$ points above a_1, $(n - m)$ points below a_1, $(n - m)$ points above b_1, and $(n - m)$ points below b_1, of which at most $(n - m - 1)$ points from each of these sets are forbidden to be played as Player II's mth move. Therefore, whether Player I has moved the c_1 coloured pebble above or below a_1 or b_1, there exists a point above or below b_1 or a_1 respectively to which Player II may (and should) move her c_1 coloured pebble. This move gives the correct ordering between pebbles and respects the desired exclusion zones.

Choosing Player II's moves as described above always gives corresponding pebble orderings between the two structures, and so this gives a winning strategy for Player II whenever these equalities hold.

Now, we provide a winning strategy for Player I in case the four requisite equalities do not hold. Suppose that $min(u_A, n-1) \neq min(u_B, n-1)$. Then $u_A \neq u_B$ and one of these is smaller than $n-1$, without loss of generality let $u_A < u_B$ and $u_A < n-1$. But then there is a point $b \in B$ which is (u_A+1)th from the top of B, but no point in A which is precisely (u_A+1)th from the top of A (by definition of u_A). Player I's strategy is to play b for the first move. If Player II responds by playing a point $a \in U_A$, then Player I plays the points which lie (u_A+2-m)th from the top of B as her mth moves for $m \geq 2$, alternating which colour pebble she moves. After u_A+1 moves this will have given an increasing sequence of points in B of size u_A+1, so in order not to lose, Player II would have had to play an increasing sequence of points in A of size u_A+1 beginning with a, which is impossible.

If Player II instead plays a point in $a \notin U_A$, then Player I selects an increasing sequence of points above a of length $n-1$ and plays those in increasing order, again moving one colour pebble on even moves and the other on odd moves. In response, Player II is forced to play an increasing sequence of points above b of length $n-1$; however, there are only $u_A < n-1$ points above b in B, and so Player II loses again.

The case where $min(l_A, n-1) \neq min(l_B, n-1)$ is exactly analogous to that where $min(u_A, n-1) \neq min(u_B, n-1)$, with the orderings reversed.

For $|U_A \cap L_A| \neq |U_B \cap L_B|$, without loss of generality let $|U_A \cap L_A| > |U_B \cap L_B|$. Then there must exist some $a \in U_A \cap L_A$ such that for some $k_1, k_2 \leq n-1$, a is k_1th from the top of A, a is k_2th from the bottom of A, and there is no $b \in B$ which lies both k_1th from the top of B and k_2th from the bottom of B. (This is clear from a simple counting argument - each element in $U_A \cap L_A$ has a distinct (k_1, k_2), and so each element in the smaller set $U_B \cap L_B$ may share its (k_1, k_2) with at most one element of $U_A \cap L_A$.) Player I's strategy is therefore to play a on her first move, to which Player II's response b must either fail to be k_1th from the top of B or to be k_2th from the bottom of B, without loss of generality assume the former. The rest proceeds as in the previous case - if $|\{c \in B : c > b\}| < (k_1 - 1)$, then Player I plays consecutive points above a until Player

II loses; if $|\{c \in B : c > b\}| > (k_1 - 1)$, then Player I picks an increasing sequence of length k_1 above b and plays that until Player II loses.

The final possible equality failure is that $A = L_A \cup U_A$ but $B \neq L_B \cup U_B$ (or vice versa). In this case, Player I plays some $b \in B \backslash (L_B \cup U_B)$. Player II's response must lie in A, so must be in either L_A or in U_A, so it must lie at most $(n-1)$th from the top or bottom of A (without loss of generality, assume the top). Since $b \notin U_B$, Player I may select an increasing sequence in B of length $n-1$ and play this in increasing order on subsequent moves, alternating which pebble is moved, and Player II cannot play an increasing sequence in A of the same length, so this strategy wins for Player I.

$\square$

Corollary 40. *There are $n^2 + 2n - 1$ $(n, 2)$-equivalence classes of linear orders.*

Proof. By Theorem 39 above, the $(n, 2)$-equivalence class of A depends only on $|U_A|$, $|L_A|$, $|U_A \cap L_A|$ and whether $A \backslash (U_A \cup L_A)$ is empty. We may therefore enumerate the possibilities using four cases:

Case 1: $U_A \cap L_A = \emptyset$, $A \backslash (U_A \cup L_A) \neq \emptyset$, $|U_A| = x$, $|L_A| = y$. There are n^2 equivalence classes of this form, since $0 \leq x, y \leq n-1$ and each distinct (x, y) gives a distinct equivalence class. One choice of representatives for these equivalence classes is $y + \mathbb{Z} + x$, where $x, y \leq n - 1$.

Case 2: $U_A \cap L_A = \emptyset$ and $A \backslash (U_A \cup L_A) = \emptyset$. Then $A = U_A + L_A$, so either $U_A = L_A = A = \emptyset$ or $|U_A| = n-1, |L_A| = n-1$ and $A = 2n - 2$.

Case 3: $|U_A \cap L_A| > 0$ and $|U_A| < n - 1$. Then $U_A = L_A = A$ and A is just a small finite nonempty linear order of size at most $n-2$, so there are $n-2$ equivalence classes in this case.

Case 4: $|U_A \cap L_A| > 0$ and $|U_A| = n - 1$. Then also $|L_A| = n - 1$, and $A = U_A \cup L_A$ is determined by how much they overlap. $A \in \{n-1, n, \ldots, 2n-3\}$, so there are $n-1$ possibilities in this case.

This gives a total of $n^2 + 2n - 1$ $(n, 2)$-equivalence classes. $\square$

7.3 Pebble games on partial orders of bounded width

Pebble games are so effective on linear orders because placing at a pebble at one point has a powerful effect - the pebble lies either above or below every other point. It is therefore possible to specify an interval with only two pebbles, which is directly relevant to the above proof. When playing pebble games on partial orders, this is no longer the case, and pebble games may be much weaker than normal Ehrenfeucht-Fraïssé games if there are too few pebbles to pin down play to an area on which Player I can win. One crucial factor here is the *width* of a partial order - the size of the largest antichain. Partial orders of width 1 are linearly ordered.

Given a non maximal point x in a partial order P of width w, we can construct antichains $X = \{x_1, ..., x_m\}$ such that $x_i > x$ for each $i \leq m$, $x_i \not< x_j$ for each $i, j \leq m$, and $m \leq w$, by picking arbitrary incomparable points of $P^{>x}$ and adding them to the set. When the antichain is maximal (in the sense that no other points may be added to it to give an antichain, rather than maximal in the sense of the ordering), it constitutes a set X of size $\leq w$ such that for all $y \in P$, either $y \in X$, or there exists some $z \in X$ with either $z > y$ or $y > z$. This set lies above x, and we may likewise construct in a similar way a maximal antichain lying below any non minimal point in P.

If the number of pebbles is too few relative to the width of a partial order, then we will be unable to construct maximal antichains, and so Player I may be unable to narrow down play to ever smaller subintervals. However, even having small enough width does not guarantee that two partial orders will be equivalent in a pebble game.

Theorem 41. *There exist partial orders P, Q of width 2 such that $P \not\equiv_3 Q$ but $P \equiv_n^2 Q$ for all n.*

Proof. Let $P = \{a_i : i \in \mathbb{Z}\} \cup \{b_i : i \in \mathbb{Z}\}$, with the relations $a_i < a_j$ and $b_i < b_j$ for any $i < j$. Let $Q = \{c_i : i \in \mathbb{Z}\} \cup \{d_i : i \in \mathbb{Z}\}$, with the relations $c_i < c_j$, $d_i < d_j$, $c_i < d_j$ and $d_i < c_j$ for any $i < j$.

Then P and Q both have width 2, since any antichain of P may contain at most one a_i and at most one b_j, and likewise for Q.

We give a winning strategy for Player I in $G_3(P,Q)$, the three-move game without pebbles. Player I should play c_0, c_1 and d_1. These form a V shape with $c_0 < c_1$, $c_0 < d_1$ and $c_1 \not< d_1 \not< c_1$. No such triplet of points exists in P, so any strategy for Player II loses. Therefore $P \not\equiv_3 Q$.

However, there exists a winning strategy for Player II in $G^2_n(P,Q)$. Let her strategy be as follows: for her first move, she plays any point, say a_0 or c_0. On subsequent moves, suppose without loss of generality that the blue pebbles lie on a_i and c_j, for some i and j, and that Player I places the red pebble.

If Player I places the red pebble on a_k for some $k > i$, then Player II plays her red pebble on c_{j+1}. Then both red pebbles are above their respective blue pebbles. Likewise, if she places the red pebble on c_k for some $k > j$, then Player II plays hers on a_{i+1}.

If Player I places the red pebble on a_k (or c_k) for some $k < i$, then Player II plays her red pebble on c_{j-1} (or a_{i-1}), and both red pebbles are below their respective blue pebbles.

If Player I places the red pebble on any b_k, then Player II places her red pebble on d_j, the point of Q which is incomparable to c_j, and both pairs of pebbles are incomparable.

If Player I plays d_j, the one point of Q which is incomparable to c_j, then Player II plays b_i, which is incomparable to a_i.

If Player I plays d_k for some $k > j$, then Player II plays a_{i+1}, and both red pebbles are above their respective blue pebbles, and if Player I plays d_k for some $k < j$, then Player II plays a_{i-1}, and both red pebbles are above their respective blue pebbles.

Since each sequence is indexed by $\mathbb{Z}$, Player II will never run out of points to play, and so she can use the above strategy to ensure that the 2-element substructures indicated by the pebbles are isomorphic after each of her moves. Player II therefore wins $G^2_n(P,Q)$ with this strategy, and so $P \equiv^2_n Q$. $\square$

Chapter 8

Conclusion

We have seen that it is possible to determine the equivalence classes of a number of relational structures by determining an appropriate notion of character. The success of this method depends on the structures in question, and the extent to which matters are complicated by the interaction of local and global properties. The cases of graphs, considered in Section 6.2, and linear orders in Chapter 3, represent in some sense the two extremes of this spectrum. In the former case, the relations that a point may hold to the other points of the structure may be chosen entirely independently, giving $2^{|G|}$ ways to append a point to a graph G, but in the latter case, most sets of relations are impossible, and due to the linearity constraint there are only at most $|L| + 1$ sets of relations a new point may hold to the existing points L.

Partial orders and trees lie somewhere between these two, due to the intermediate strength of the constraints on relations. Trees of course behave slightly more like linear orders than partial orders do, since the branching is in only one direction and the other is rather linear.

Directed graphs are rather similar to graphs, though arrows on the edges permit more possibilities. A new point has an entirely free choice of how to relate to the existing points. Unars are different both in that the injectivity provides a much greater restriction on these relations, and in that a point x may have $f(x) = x$. These result in the existence of several mutually exclusive small configurations that may hold in the immediate vicinity of any point, allowing us to categorise them into types.

8.1 Future work

In theory there is a lot of possible future work determining further equivalence classes for these structures, but some may be more tractable than others. Some suggestions for future work include:

- Proving or disproving Conjecture 25 on the n-equivalence of trees, or at least extending our partial result Theorem 26 to some wider subclass
- Determining the 3-equivalence classes of finite trees
- Classifying the injective unars up to n-equivalence, using results like Lemmas 36 and 35 and considering the combinations of components
- Classifying the totalisable partial cyclic orders, as discussed in Section 5.5
- Considering coloured versions of the structures featured here
- Extending the work on pebble games to other structures
- Investigating the equivalence conditions of structures not explored here, such as preorders, hypergraphs, tournaments, betweenness relations, B/C/D relations, particular kinds of partial order or graph, and other relational structures

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