Double Lie structures in Poisson and symplectic geometry

Samuel Francis Morgan

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Abstract

The notion of a symplectic groupoid first arose as an apparatus to solve the quantization problem for Poisson manifolds. It is well known that not every Poisson manifold can be globally realised by a symplectic groupoid. A key motivation for this thesis was Lu and Weinstein's construction of a (global) symplectic double groupoid for an arbitrary Poisson Lie group. We develop an extensive exposition of their results, and analyse some of the possible extensions of their construction. In particular, we produce a symplectic double groupoid for any pair of dual Poisson groupoids where the underlying Lie groupoid structures are of trivial type. Alongside these ideas, we also study the actions of double Lie structures. A detailed account of the actions of double Lie groupoids is given, and notions for the actions of $\mathcal{LA}$-groupoids are defined. As an application of these double actions, we consider an alternative approach to Xu's study of Poisson reduced spaces for actions of a symplectic groupoid. This approach is then extended to consider the Poisson reduced spaces for more general actions of Poisson groupoids.

Contents

Introduction

A brief history of the subject

Lie group theory was initially developed as the study of continuous symmetries of mathematical objects. From this viewpoint, the natural extension of Lie groups are Lie groupoids, which arise in the study of symmetric properties of bundle structures, and more complicated forms of symmetry. A groupoid can be thought of as a 'many-object' generalisation of a group. The study of groupoids was first initiated by Brandt [5], although he defined what are now referred to as transitive groupoids. The substantial scope of mathematics in which the usage of groupoids appears is often understated. Groupoids have arisen in algebraic geometry, as a means to study moduli spaces; in algebraic topology, whilst considering fundamental groups; and in analysis, to handle ergodic actions of groups.[1] Within differential geometry, the introduction of Lie groupoids can be traced back to C. Ehresmann [20]. His work was largely focused on studying categories with extra structure as a means to unify different aspects of mathematics.[2]

The development of a Lie theory for Lie groupoids that generalises the relation between Lie groups and Lie algebras was first outlined by J. Pradines in a series of papers [55, 56, 57, 58]. The proposed concept of a Lie algebroid provided a 'many-object' generalisation of a Lie algebra that proved fundamental in establishing the Lie theory for Lie groupoids. The construction of a Lie algebroid for a Lie groupoid follows in a similar fashion to the construction of a Lie algebra for a Lie group. However, not every Lie algebroid is integrable, so one does not have an analogue of Lie's Third Theorem. The first example of a (transitive) Lie algebroid which is not the Lie algebroid of any Lie groupoid was discovered by Almeida and Molino [3]. Obstructions to the integrability of transitive Lie algebroids were first described by Mackenzie in the monograph [42]. Over a decade later, obstructions to the integrability of arbitrary Lie algebroids were introduced by Crainic and Fernandes [13].

The emergence of the Lie theory of Lie groupoids in Poisson geometry began with the introduction of symplectic groupoids by Weinstein [62, 12]. They were also discovered independently by Karasev [29], and later by Zakrzewski [68, 69]. The question of whether every Poisson manifold admits a symplectic realisation was the chief motivation. Briefly, a symplectic realisation of a Poisson manifold P is a Poisson map $\phi\colon \Sigma \to P$ in which Σ is a symplectic manifold, and such that ϕ is a surjective submersion. A key observation of the Poisson bracket of a Poisson manifold P is that it gives rise to a Lie algebroid structure on the cotangent bundle T^*P. Moreover, when

[1] Brown [6] and Weinstein [64] give detailed surveys of the prevalence of groupoids within mathematics.

[2] A collection of his works with commentaries provided by A. Ehresmann can be found in [22].

this Lie algebroid T^*P is integrable, one gets a symplectic groupoid whose target projection provides a symplectic realisation of P. In situations where the cotangent bundle cannot be integrated, one only gets a *local* symplectic groupoid; however, Weinstein constructs a gluing argument that still produces a symplectic realisation of P [61, 12]. The same process was described by Karasev [29]. Other approaches were later provided by Cattaneo and Felder [10, 9], and Crainic and Fernandes [14, 15]. More recently, a direct global proof of the existence of a symplectic realisation was produced by Crainic and Mărcuţ [16].

Another vibrant topic of interest within Poisson geometry, which appeared during the late twentieth century, was that of Poisson Lie groups. In short, they are Lie groups with a Poisson structure such that the group multiplication is a Poisson map. They can be thought of as the classical limit of quantum groups. Poisson Lie groups were introduced by Drinfel'd [18, 19], and were well studied by Semenov-Tian-Shansky [59] and Lu and Weinstein [40].[3] Their infinitesimal counterparts are Lie bialgebras. Drinfel'd also described the correspondence between Poisson Lie groups and Lie bialgebras.[4] In the late 1980s, Lu and Weinstein [39, 37] showed that every Poisson Lie group G has a (global) symplectic groupoid Σ. Moreover, they showed that Σ also has a symplectic groupoid structure with base the dual Poisson Lie group G^*, and that this gives Σ the structure of a symplectic double groupoid. Mackenzie later showed, more generally, that the side groupoids of any symplectic double groupoid are dual Poisson groupoids [44].

A Poisson groupoid was another concept introduced by Weinstein [63] that unified the notions of a symplectic groupoid and a Poisson Lie group. Mackenzie and Xu were the first to uncover the infinitesimal analogues of Poisson groupoids – the so called Lie bialgebroids [49]. The standard formulation of Lie bialgebroids owes much to Y. Kosmann-Schwarzbach [30]. Mackenzie and Xu also later went on to prove the integrability of Lie bialgebroids [50].

For extensive details and references on the subject, we recommend the books of: Mackenzie [47]; Cannas da Silva and Weinstein [8]; Laurent-Gengoux, Pichereau and Vanhaecke [33]; Abraham and Marsden [1]; Vaisman [60]; Libermann and Marle [36]; Ortega and Ratiu [54]; McDuff and Salamon [51]; and Dufour and Zung [28].

An outline of the thesis

The thesis begins with a detailed overview of the theory of Lie groupoids and Lie algebroids. Within the categories of these objects, we give the relevant notions for morphisms, sub-objects and actions. Much of the material in the opening sections follows a similar formulation of the theory to that given in the comprehensive book of Mackenzie [47]. Our first chapter finishes with an introduction to double Lie structures. The notions of double Lie groupoids, $\mathcal{VB}$-groupoids, and $\mathcal{LA}$-groupoids are defined. We introduce the concept of a *weak* double Lie groupoid, which relaxes the usual surjectivity condition on the double source map, but helps unify the treatment of symplectic double groupoids within the literature. A detailed account of the cores of these double structures is also given.

The second chapter reviews the basics of Poisson geometry. The account we give

[3]Further references can be found in the book of Chari and Pressley [11].

[4]A thorough review of the theory of Lie bialgebras and Poisson Lie groups is given by Y. Kosmann-Schwarzbach [31] (see also [32]).

can be divided into two parts, the first of which gives a standard approach to the subject. Beyond the initial definitions, we introduce Poisson Lie groups and their Lie bialgebras, and revise the properties of coisotropic submanifolds. The second half of the chapter focuses on the relationship that Lie groupoids and Lie algebroids have with Poisson Geometry. We begin by showing how every Poisson manifold gives rise to a Lie algebroid structure on its cotangent bundle. Later the crucial concept of a Poisson groupoid is introduced, and we show how it simultaneously generalises the notions of a Poisson Lie group and a symplectic groupoid. We further show how every Poisson groupoid induces a Lie algebroid structure on the dual bundle of its Lie algebroid, and discuss the notion of duality. The chapter ends with a discourse on the role of double Lie structures within the Poisson and symplectic realms.

In the third chapter, we discuss Lu and Weinstein's construction of a symplectic double groupoid for a Poisson Lie group. We start with an analysis of the relations between a Poisson Lie group G, its dual G^*, and the corresponding Drinfel'd double Lie group. After this, we introduce two important Poisson structures on the Drinfel'd double Lie group, and describe their connection with the Poisson structures on the original Poisson Lie group and its dual. We then show how this gives rise to Poisson structures on the product manifolds $G \times G^*$ and $G^* \times G$, and to a symplectic structure on a specific pullback manifold Σ. In the second half of the chapter, we show that Σ has the structure of a symplectic double groupoid with side groupoids given by G and G^*.

In the fourth chapter, we discuss possible extensions of Lu and Weinstein's construction. We introduce the notion of a *Lie groupoid triple* which generalises the concept of a double Lie group given in [40]. We give two constructions of weak double Lie groupoids which extend Lu and Weinstein's double groupoid. Then we go on to show that in the case of a pair of dual Poisson groupoids which are trivial Lie groupoids, one of these constructions gives rise to a symplectic double groupoid.

In the fifth and final chapter, we study the actions of double Lie structures. We first review the notion of an action of a double Lie groupoid given by Brown and Mackenzie [7]. Then we proceed to introduce two concepts of action for $\mathcal{LA}$-groupoids. In addition, we show that an action of a double Lie groupoid gives rise to both types of action for an $\mathcal{LA}$-groupoid. In the final section, we discuss an application of these actions within Poisson geometry. We consider a construction of a symplectic groupoid for the Poisson reduced space of a free and proper Poisson groupoid action of a symplectic groupoid, given by Xu [67]. We show that there is an underlying action of a double Lie groupoid arising here. We then show that in the more general case of an arbitrary free and proper Poisson groupoid action, there is an underlying action of an $\mathcal{LA}$-groupoid.

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Chapter 1

Lie groupoids, Lie algebroids, and double structures

This first chapter is concerned with establishing the general theory of Lie groupoids, Lie algebroids, and double Lie structures. The opening sections on Lie groupoids and Lie algebroids review much of the theory detailed in [47, 27]. The final section on double Lie structures follows many of the same conventions as laid out in [43, 45].

§ 1.1 Lie groupoid theory

We begin by introducing the concept of a Lie groupoid which provides a 'many-object' generalisation of a Lie group. We then extend the notions of Lie group homomorphisms, Lie subgroups, and Lie group actions to morphisms of Lie groupoids, Lie subgroupoids, and Lie groupoid actions, respectively. We present many of the tools and key examples needed for later sections.

1.1.1 Lie groupoids

Let us first define the main objects of our study – groupoids. They give a natural generalisation of groups and arise in connection with the symmetries of objects which possess a bundle structure.

Definition 1.1. A *groupoid* is a pair of sets, G and M, equipped with the following maps:

- Two maps $\alpha, \beta \colon G \to M$, which we will call the *source* and *target projections*, respectively;
- A *partial multiplication* map $\kappa \colon G * G \to G$, $(h, g) \mapsto hg := \kappa(h, g)$, where $G * G = \{(h, g) \in G \times G \mid \alpha(h) = \beta(g)\}$;
- An *identity* map $1 \colon M \to G$, $x \mapsto 1_x$;
- And an *inversion* map $\iota \colon G \to G$, $g \mapsto g^{-1} := \iota(g)$;

which satisfy the following properties:

(i) $\alpha(1_x) = \beta(1_x) = x$ for all $x \in M$;

(ii) $\alpha(hg) = \alpha(g)$ and $\beta(hg) = \beta(h)$ for all $(h, g) \in G * G$;

(iii) $k(hg) = (kh)g$ for all $k, h, g \in G$ such that $(k, h), (h, g) \in G * G$;

(iv) $g1_{\alpha(g)} = g$ and $1_{\beta(g)}g = g$ for all $g \in G$;

(v) $(g, g^{-1}), (g^{-1}, g) \in G * G$, $g^{-1}g = 1_{\alpha(g)}$ and $gg^{-1} = 1_{\beta(g)}$ for all $g \in G$.

It is clear to see that when M is taken as a singleton set, the definition reduces to that of a group. A more concise definition is that a groupoid is a category in which all arrows are invertible. Interpreted this way, it is natural to refer to elements of G as *arrows*, and elements of M as *objects*. We also borrow the notation $G \rightrightarrows M$ from category theory to denote a groupoid. Here, the two arrows represent the source and target projections. When referring to a groupoid, we may say that 'G is a groupoid with base M', or simply that 'G is a groupoid on M'. The maps α, β, κ, 1 and ι are called the *structure maps* of the groupoid.

We list some basic properties for groupoids, which follow immediately from the definition.

Proposition 1.2. *Let G be a groupoid with base M, with structure maps labelled as above. Then the following hold:*

(i) α, β and κ are surjections;

(ii) The identity map 1 is an injection;

(iii) The inversion map ι is a bijection which is self-inverse;

(iv) We have the following cancellation laws:

- *For $(h, g), (k, g) \in G * G$, $hg = kg \implies h = k$;*
- *For $(g, h), (g, k) \in G * G$, $gh = gk \implies h = k$;*

(v) $\alpha \circ \iota = \beta$, $\beta \circ \iota = \alpha$ and $\iota \circ 1 = 1$. □

We often refer to elements of the form 1_x, for $x \in M$, as *identity* elements, and elements g^{-1}, for $g \in G$, as *inverses*. Analogous to a group, these elements have the expected uniqueness properties as a consequence of the cancellation laws given above. We will denote the set of identity elements by 1_M. We may sometimes use the alternative notation $G^{(2)}$ for the domain of the partial multiplication in place of the usual $G * G$.

Given $x \in M$, we call the subset $G_x := \alpha^{-1}(x)$ the *α–fibre* over x, and the subset $G^x := \beta^{-1}(x)$ the *β–fibre* over x. We will also use the notation $G_x^y := G_x \cap G^y$, for $x, y \in M$. In particular, we refer to the subset G_x^x as the *vertex group* at x – it is indeed a group under the operation given by restriction of the partial multiplication.

When studying geometric objects, often the groupoids we might be interested in possess additional smooth structure. This leads to the concept of a Lie groupoid, which generalises the notion of a Lie group within the realm of groupoids.

Definition 1.3. A *Lie groupoid* is a groupoid G with base M, such that G and M have smooth manifold structures that make the source and target projections submersions and the partial multiplication and identity maps smooth.

Since the source and target projections of a Lie groupoid G are submersions, they are transverse to one another, and so the domain of the partial multiplication map $G * G = (\alpha \times \beta)^{-1}(\Delta_M)$ inherits a smooth manifold structure that makes it a closed embedded submanifold of $G \times G$. It is also worth noting that since the source and target projections are smooth submersions, all α– and β–fibres are closed embedded submanifolds of G.

We collect some further properties of the structure maps of a Lie groupoid, which are straight-forward to verify.

Proposition 1.4. *Let G be a Lie groupoid with base manifold M, with structure maps labelled as above. Then the following hold:*

(i) The partial multiplication κ is a submersion;

(ii) The identity map 1 is an immersion;

(iii) The inversion map ι is a diffeomorphism. □

Just as in the theory of Lie groups, we also have notions of left- and right-translations for Lie groupoids. Given a Lie groupoid G on base M, and $g \in G$, we define the *left-translation* by g to be the map of β–fibres

$$L_g \colon G^{\alpha(g)} \to G^{\beta(g)}\,,\; h \mapsto gh.$$

In a similar manner, we define the *right-translation* by g, to be the map of α–fibres

$$R_g \colon G_{\beta(g)} \to G_{\alpha(g)}\,,\; h \mapsto hg.$$

Since the partial multiplication is smooth, it follows that these two maps are diffeomorphisms of the respective fibres.

We now cover some key examples, which will all appear again in later sections.

Example 1.5. Given a smooth manifold M, we can give the product manifold $M \times M$ the structure of a Lie groupoid with base M. We take the projections onto the first and second factors as the target and source projections, respectively. Given pairs $(z, y), (y, x) \in M \times M$, the partial multiplication is given by $(z, y)(y, x) = (z, x)$. For $x \in M$, the corresponding identity element is the pair (x, x). For $(z, y) \in M \times M$, the corresponding inverse is the pair (y, z). With these structure maps, we call $M \times M$ the *pair groupoid* on M. ⊠

Example 1.6. Let G and G' be Lie groupoids on M and M' respectively. There is a Lie groupoid structure on the product manifold $G \times G'$ with base $M \times M'$. The source and target projections are given by the product maps $\alpha \times \alpha'$ and $\beta \times \beta'$ respectively. The partial multiplication is given by $(h, h')(g, g') = (hg, h'g')$, for compatible pairs. The identity and inversion maps are just given by the product maps $1 \times 1'$ and $\iota \times \iota'$ respectively. We call $G \times G'$ the *Cartesian product groupoid* on base $M \times M'$. ⊠

Example 1.7. Given any Lie groupoid $G \rightrightarrows M$, by applying the tangent functor to each of the structure maps we get a Lie groupoid structure on TG with base TM. Note that, more precisely, the partial multiplication is given by the composite map

$$TG *_{TM} TG \xrightarrow{\cong} T(G *_M G) \xrightarrow{T(\kappa)} TG.$$

Here, we have used the property that the tangent functor preserves pullbacks. We call this Lie groupoid the *tangent prolongation groupoid* of $G \rightrightarrows M$. ⊠

Example 1.8. Suppose $\theta\colon G\times M\to M$, $(g,m)\mapsto g\cdot m$ is an action of a Lie group G on a smooth manifold M. We get an induced Lie groupoid structure on the product manifold $G\times M$ with base M. The source projection is given by $\alpha\colon G\times M\to M$, $(g,m)\mapsto m$, and the target projection by $\beta\colon G\times M\to M$, $(g,m)\mapsto g\cdot m$. The partial multiplication is defined by $(h,y)(g,x)=(hg,x)$ for compatible pairs. The identity map is given by $x\mapsto(1,x)$, and the inversion map by $(g,x)\mapsto(g^{-1},g\cdot x)$. We call this Lie groupoid the *action groupoid* of θ and denote it by $G\lessdot M$. ⊠

Example 1.9. For any Lie group G with Lie algebra $\mathfrak{g}$, we can define a groupoid structure on T^*G with base $\mathfrak{g}^*$ in the following way. We define source and target projections by

$$\alpha(\varphi)=\varphi\circ T(L_g),\quad \beta(\varphi)=\varphi\circ T(R_g),$$

for $\varphi\in T^*_gG$. The partial multiplication is defined by

$$\kappa(\psi,\varphi)=\psi\circ T(R_{g^{-1}})=\varphi\circ T(L_{h^{-1}}),$$

where $\varphi\in T^*_gG$, $\psi\in T^*_hG$, and $\alpha(\psi)=\beta(\varphi)$. The identity map is inclusion. ⊠

Given a groupoid G with base M, there is an induced equivalence relation defined on M, called the *transitivity relation*. For $m,m'\in M$, we write $m\sim m'$ whenever there exists $g\in G$ with $\beta(g)=m$ and $\alpha(g)=m'$. We call the corresponding equivalence classes *transitivity orbits* (or *transitivity components*). Furthermore, the space of all equivalence classes is denoted by $\tau(G)$ and called the *transitivity orbit space*.

In the study of Lie groups the class of simply connected Lie groups plays a vital role. We finish this section by defining a useful analogue for Lie groupoids.

Definition 1.10. A Lie groupoid G with base M is *α–simply connected* if the α–fibre G_x is simply connected for every $x\in M$.

1.1.2 Morphisms of Lie groupoids and Lie subgroupoids

We now turn our attention towards the morphisms and sub-objects within the category of Lie groupoids.

Definition 1.11. Let G and G' be groupoids on M and M', respectively, and let $F\colon G\to G'$ and $f\colon M\to M'$ be a pair of maps. We say that F is a *morphism of groupoids* over f if the following relations hold (on the domains for which they are well-defined):

(i) $\alpha'\circ F=f\circ\alpha$;

(ii) $\beta'\circ F=f\circ\beta$;

(iii) $\kappa'\circ(F\times F)=F\circ\kappa$.

Alternatively, we can say that (F,f) is a morphism of groupoids.

When G and G' are groupoids on the same base M, and f is just the identity map on M, we say that F is a *morphism over M*.

If G and G' are Lie groupoids, and F and f are smooth maps, we say that F is a *morphism of Lie groupoids* over f.

F is said to be an *isomorphism of Lie groupoids* if F is also a diffeomorphism.

One might ask why we do not require similar commutativity relations for F and f with the identity and inversion maps. In fact, the assumptions in the definition immediately imply such relations.

Proposition 1.12. *Let $F\colon G \to G'$ be a morphism of groupoids over $f\colon M \to M'$. Then the following relations hold:*

(iv) $F \circ 1 = 1' \circ f$;

(v) $\iota' \circ F = F \circ \iota$. □

Remark 1.13. Given a morphism of groupoids $F\colon G \to G'$, the properties *(i)–(v)* can be rewritten in the following equivalent way:

(i) $\alpha'(F(g)) = f(\alpha(g))$ for all $g \in G$;

(ii) $\beta'(F(g)) = f(\beta(g))$ for all $g \in G$;

(iii) $F(hg) = F(g)F(h)$ for all $(h, g) \in G * G$;

(iv) $F(1_m) = 1'_{f(m)}$ for all $m \in M$;

(v) $F(g)^{-1} = F(g^{-1})$ for all $g \in G$. □

The following are the simplest examples of morphisms of Lie groupoids that one can manufacture.

Example 1.14. Let $f\colon M \to M'$ be a smooth map of manifolds. The product map $f \times f\colon M \times M \to M' \times M'$ defines a morphism of Lie groupoids over f between the pair groupoids $M \times M$ and $M' \times M'$, as defined in Example 1.5. ⊠

Example 1.15. Let G be a Lie groupoid with base manifold M. The map given by $\chi = (\beta, \alpha)\colon G \to M \times M$ is a morphism of Lie groupoids over M. Here $M \times M$ is the pair groupoid on base M. This morphism is often referred to as the *anchor* of G. ⊠

This notion of anchor gives us a method to classify certain types of Lie groupoid.

Definition 1.16. Let G be a Lie groupoid with base M. Then G is called *locally trivial* if the anchor $\chi\colon G \to M \times M$ is a surjective submersion.

Definition 1.17. Let G be a groupoid with base M. Then G is called *totally intransitive* if the source and target projections are identical. This is equivalent to the image of the anchor $\chi\colon G \to M \times M$ being equal to the diagonal Δ_M.

We also can extend the notion of a kernel of a group homomorphism to the kernel of a morphisms of groupoids.

Definition 1.18. Let $F\colon G \to G'$ be a morphism of groupoids over $f\colon M \to M'$. The *kernel* of (F, f) is the subset of G given by

$$\ker(F, f) = \{g \in G \mid F(g) \in 1_{M'}\}.$$

Let us now define some classes of morphisms of Lie groupoids which will arise in later constructions. Consider a morphism of Lie groupoids $F\colon G' \to G$ over a smooth map

$f\colon M' \to M$. We denote the pullback manifold of f and α by $f^!G$. The pullback diagram is given by

$$\begin{array}{ccc} f^!G & \longrightarrow & G \\ \big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ M' & \xrightarrow{\;f\;} & M. \end{array} \tag{1.1}$$

We use the notation $F^!\colon G' \to f^!G$ for the map defined by $g' \mapsto (\alpha'(g'), F(g'))$. This is a well-defined map because F is a morphism of Lie groupoids.

Definition 1.19. A morphism of Lie groupoids $F\colon G' \to G$ over $f\colon M' \to M$ is a *partial fibration* if the map $F^!\colon G' \to f^!G$ is a submersion.

F is called a *fibration* when $F^!$ is a surjective submersion, and an *action morphism* when $F^!$ is a diffeomorphism.

The concepts of a subgroup and a normal group have analogues within the theory of groupoids. We briefly introduce these notions here.

Definition 1.20. Let G be a groupoid with base M. A groupoid G' with base M' is a *subgroupoid of* G if there exists injections $\widetilde{i}\colon G' \to G$ and $i\colon M' \to M$, such that $(\widetilde{i}, i)$ is a morphism of groupoids.

If in addition, $M' = M$ and $i = \mathrm{id}_M$, we call G' a *wide* subgroupoid of G.

When $G \rightrightarrows M$ and $G' \rightrightarrows M'$ are Lie groupoids and the injections $\widetilde{i}\colon G' \to G$ and $i\colon M' \to M$ are immersions, we say that G' is a *Lie subgroupoid of* G.

We call G' an *embedded Lie subgroupoid of* G when we have the further condition that $\widetilde{i}$ and i are smooth embeddings.

Example 1.21. Let G be a groupoid with base M. Consider the subset of G given by

$$IG = \{g \in G \mid \alpha(g) = \beta(g)\}.$$

Then the structure maps of G restrict to give IG a groupoid structure with base M. It follows that IG is a subgroupoid of G via inclusion. We call IG the *inner subgroupoid of* G. Note that IG is not necessarily a Lie subgroupoid when G is a Lie groupoid.[1]

Definition 1.22. Let G be a groupoid with base M, and N a totally intransitive wide subgroupoid of G. Denote the source and target projection of N by $q\colon N \to M$. Then N is a *normal subgroupoid* of G if for all $g \in G$ and $n \in N$ satisfying $\alpha(g) = q(n)$, we have $gng^{-1} \in N$.

Example 1.23. Let G and G' be groupoids on the same base M. Suppose that $F\colon G \to G'$ is a morphism of groupoids over M. Then the kernel of (F, id_M) is a normal subgroupoid of $G \rightrightarrows M$. Moreover, it can be shown that every normal subgroupoid is the kernel of a morphism of groupoids over a fixed base. ⊠

[1] One situation in which IG has a Lie subgroupoid structure occurs when G is taken to be a locally trivial Lie groupoid (see [47, Proposition 1.3.9]).

1.1.3 Actions of Lie groupoids

The notion of an action of a Lie group can be extended to an action of a Lie groupoid.

Definition 1.24. Let G be a groupoid on base M, and $f\colon M' \to M$ a map. A *groupoid action* of G on f is a map $\theta\colon G * M' \to M'$, $(g, m') \mapsto g \cdot m'$, satisfying

(i) $f(g \cdot m') = \beta(g)$, for all $(g, m') \in G * M'$;

(ii) $h \cdot (g \cdot m') = (hg) \cdot m'$, for all $(h, g) \in G * G$, and all $m' \in M'$, such that $(g, m') \in G * M'$;

(iii) $1_{f(m')} \cdot m' = m'$, for all $m' \in M'$;

where $G * M' = (\alpha \times f)^{-1}(\Delta_M)$.

Given a groupoid $G \rightrightarrows M$, and a groupoid action of G on a map $f\colon M' \to M$, we get an induced groupoid structure on the set $G * M'$ with base M'. This is similar to the construction in Example 1.8 for an action of a Lie group. The source and target projections are defined by

$$\alpha_{\triangleleft}\colon G * M' \to M'\,,\ (g, m') \mapsto m'; \quad \beta_{\triangleleft}\colon G * M' \to M'\,,\ (g, m') \mapsto g \cdot m';$$

and the partial multiplication is defined by

$$\kappa_{\triangleleft}\colon G * M'^{(2)} \to G * M'\,,\ (h, n')(g, m') = (hg, m').$$

The identity map is given by

$$1^{\triangleleft}\colon M' \to G * M'\,,\ m' \mapsto (1_{f(m')}, m').$$

and the inversion map by

$$\iota_{\triangleleft}\colon G * M' \to G * M'\,,\ (g, m') \mapsto (g^{-1}, g \cdot m').$$

With this groupoid structure, we denote $G * M'$ by $G \triangleleft M'$ or $G \triangleleft f$. We refer to a groupoid of this form as an *action groupoid*.

Definition 1.25. Let G be a Lie groupoid on base M, and $f\colon M' \to M$ a smooth map. A *Lie groupoid action of* G *on* f is a groupoid action $\theta\colon G * M' \to M'$ which is also a smooth map.

Given a Lie groupoid action of a Lie groupoid $G \rightrightarrows M$ on a smooth map $f\colon M' \to M$, the pullback $G * M' = (\alpha \times f)^{-1}(\Delta_M)$ is an embedded submanifold of the product $G \times M'$. It is not hard to verify that the corresponding action groupoid $G \triangleleft M' \rightrightarrows M'$ is in fact a Lie groupoid. Furthermore, the projection $f_!\colon G \triangleleft M' \to G$, $(g, m') \mapsto g$ is an action morphism over $f\colon M' \to M$.

Example 1.26. Let θ be a Lie groupoid action of a Lie groupoid $G \rightrightarrows M$ on a smooth map $f\colon M' \to M$. Consider the tangent prolongation groupoid $TG \rightrightarrows TM$ of Example 1.7, and the differential of f, $T(f)\colon TM' \to TM$. The differential of θ produces a smooth map $T(\theta)\colon TG * TM' \to TM'$. Here, we are using the property that the tangent functor preserves pullbacks to identify $T(G * M')$ with $TG * TM'$. It is straightforward to verify that $T(\theta)$ defines a Lie groupoid action of $TG \rightrightarrows TM$ on $T(f)$. We call this the *tangent action* of G on f. ⊠

Example 1.27. Let G be a Lie group with Lie algebra $\mathfrak{g}$, and M a smooth manifold. Consider a Lie group action $\theta\colon G \times M \to M$. We get a Lie algebra action $\mathfrak{g} \to \mathfrak{X}(M)$, $X \mapsto X^\dagger$ where

$$X^\dagger(m) = T_e(\theta_m)(X).$$

Dualizing this infinitesimal action produces a map $\mathfrak{p}\colon T^*M \to \mathfrak{g}^*$ given by

$$\langle \mathfrak{p}(\varphi), X\rangle = \langle \varphi, X^\dagger(m)\rangle,$$

for $\varphi \in T^*_mM$, $X \in \mathfrak{g}$. We call $\mathfrak{p}$ the *pith* of the action.

We claim that θ lifts to a Lie groupoid action $\widetilde{\theta}\colon T^*G * T^*M \to T^*M$ of the Lie groupoid $T^*G \rightrightarrows \mathfrak{g}^*$ (see Example 1.9) on the pith $\mathfrak{p}$, by

$$\psi \cdot \varphi = \varphi \circ T(\theta_{g^{-1}}),$$

for $\psi \in T^*_gG$ and $\varphi \in T^*M$ with $\alpha(\psi) = \mathfrak{p}(\varphi)$. Before proving this claim, we make the following observation. For any $g, h \in G$ and $m \in M$,

$$\theta_{g^{-1}} \circ \theta_{g\cdot m}(h) = g^{-1} \cdot (h \cdot (g \cdot m)) = (g^{-1}hg) \cdot m = \theta_m \circ C_{g^{-1}}(h),$$

where $C_{g^{-1}}\colon G \to G$ represents conjugation by g^{-1}. Hence, for any $X \in \mathfrak{g}$, $\psi \in T^*_gG$, and $\varphi \in T^*_mM$ with $\alpha(\psi) = \mathfrak{p}(\varphi)$, we have

$$\begin{aligned}
\langle \mathfrak{p}(\psi\cdot\varphi), X\rangle &= \langle \psi\cdot\varphi, X^\dagger(g\cdot m)\rangle \\
&= \langle \varphi \circ T(\theta_{g^{-1}}), T_e(\theta_{g\cdot m})(X)\rangle \\
&= \langle \varphi, T_e(\theta_{g^{-1}} \circ \theta_{g\cdot m})(X)\rangle \\
&= \langle \varphi, T_e(\theta_m \circ C_{g^{-1}})(X)\rangle \\
&= \langle \varphi, (\mathrm{Ad}_{g^{-1}}(X))^\dagger(m)\rangle \\
&= \langle \mathfrak{p}(\varphi), \mathrm{Ad}_{g^{-1}}(X)\rangle \\
&= \langle \mathrm{Ad}^*_g(\mathfrak{p}(\varphi)), X\rangle.
\end{aligned}$$

Therefore, we have the relation $\mathfrak{p}(\psi\cdot\varphi) = \mathrm{Ad}^*_g(\mathfrak{p}(\varphi))$. Let us now show that $\widetilde{\theta}$ defines a Lie groupoid action on the pith $\mathfrak{p}$. Firstly note that, for any $X \in \mathfrak{g}$, $\psi \in T^*_gG$, and $\varphi \in T^*_mM$ with $\alpha(\psi) = \mathfrak{p}(\varphi)$, we have

$$\begin{aligned}
\langle \mathfrak{p}(\psi\cdot\varphi), X\rangle &= \langle \mathfrak{p}(\varphi), \mathrm{Ad}_{g^{-1}}(X)\rangle \\
&= \langle \alpha(\psi), \mathrm{Ad}_{g^{-1}}(X)\rangle \\
&= \langle \psi \circ T(L_g), \mathrm{Ad}_{g^{-1}}(X)\rangle \\
&= \langle \psi \circ T(R_g), X\rangle \\
&= \langle \beta(\psi), X\rangle.
\end{aligned}$$

Next, note that for $\psi_2 \in T^*_hG$, $\psi_1 \in T^*_gG$, and $\varphi \in T^*_mM$, with $\alpha(\psi_2) = \beta(\psi_1)$ and $\alpha(\psi_1) = \mathfrak{p}(\varphi)$, we have

$$\begin{aligned}
\psi_2 \cdot (\psi_1 \cdot \varphi) &= \psi_2 \cdot (\varphi \circ T(\theta_{g^{-1}})) \\
&= \varphi \circ T(\theta_{g^{-1}}) \circ T(\theta_{h^{-1}}) \\
&= \varphi \circ T(\theta_{(hg)^{-1}}) \\
&= \kappa(\psi_2, \psi_1) \cdot \varphi.
\end{aligned}$$

The last line follows since $\kappa(\psi_2, \psi_1) \in T^*_{hg}G$. Lastly, observe that for any $\varphi \in T^*M$,

$$1_{\mathfrak{p}(\varphi)} \cdot \varphi = \mathfrak{p}(\varphi) \cdot \varphi = \varphi \circ T(\theta_{e^{-1}}) = \varphi.$$

Thus, $\widetilde{\theta}$ forms a Lie groupoid action of $T^*G \rightrightarrows \mathfrak{g}^*$ on $\mathfrak{p}$. ⊠

The notion of an action for groupoids gives us a natural way to generalise semidirect products from groups to groupoids.

Consider a groupoid G on base M, and a totally intransitive groupoid H also on base M, with source (and target) projection denoted by $q\colon H \to M$. Suppose we have a groupoid action of G on q, denoted $\theta\colon G * H \to H$, such that the map

$$\theta_g\colon H_{\alpha(g)} \to H_{\beta(g)}\,,\ h \mapsto h \cdot g,$$

is an isomorphism of groups, for every $g \in G$. Then we can define a groupoid structure on $G * H$ with base M that generalises the group structure of a semidirect product. We define the source and target projections by

$$\alpha'(g,h) = \alpha(g)\,, \quad \beta'(g,h) = \beta(g),$$

for all $(g,h) \in G * H$. The partial multiplication is defined by

$$(g_2,h_2)(g_1,h_1) = (g_2g_1, (g_1^{-1} \cdot h_2)h_1),$$

for compatible pairs $(g_2,h_2),(g_1,h_1) \in G * H$. For any $m \in M$, the corresponding identity element is given by

$$1'_m = (1^G_m, 1^H_m),$$

and for any $(g,h) \in G * H$, the corresponding inverse is given by

$$(g,h)^{-1} = (g^{-1}, g \cdot (h^{-1})).$$

To check that this gives a well-defined groupoid is routine. Let us verify the associativity of the partial multiplication; the remaining checks we leave to the reader. For pairs $(g_3,h_3),(g_2,h_2),(g_1,h_1) \in G * H$, satisfying $\alpha(g_3) = \beta(g_2)$ and $\alpha(g_2) = \beta(g_1)$, we have

$$\begin{aligned}
(g_3,h_3)\big((g_2,h_2)(g_1,h_1)\big) &= (g_3,h_3)(g_2g_1, (g_1^{-1} \cdot h_2)h_1)\\
&= (g_3g_2g_1, ((g_2g_1)^{-1} \cdot h_3)(g_1^{-1} \cdot h_2)h_1)\\
&= (g_3g_2g_1, (g_1^{-1} \cdot (g_2^{-1} \cdot h_3))(g_1^{-1} \cdot h_2)h_1)\\
&= (g_3g_2g_1, (g_1^{-1} \cdot ((g_2^{-1} \cdot h_3)h_2))h_1)\\
&= (g_3g_2, (g_2^{-1} \cdot h_3)h_2)(g_1,h_1)\\
&= \big((g_3,h_3)(g_2,h_2)\big)(g_1,h_1).
\end{aligned}$$

Note that in the fourth line of this computation we have used the fact that the map $\theta_{g_1^{-1}}\colon H_{\beta(g_1)} \to H_{\alpha(g_1)}$ is an isomorphism of groups.

We denote this groupoid by $G \ltimes H \rightrightarrows M$; it is called the *semidirect product groupoid.*

Example 1.28. Let G be a groupoid on base M, and N a normal subgroupoid. Then, we have a groupoid action $\theta\colon G * N \to N$, defined by

$$(g,n) \mapsto g \cdot n := gng^{-1}.$$

Since the action is just by conjugation, it should be clear that, for any $g \in G$, the map $\theta_g\colon N_{\alpha(g)} \to N_{\beta(g)}$ is an isomorphism of groups. Hence, we can form the semidirect product groupoid $G \ltimes N \rightrightarrows M$. ⊠

Given a semidirect product groupoid $G \ltimes H \rightrightarrows M$, we can make a slight modification that induces another groupoid structure on $G * H$ with base M. We first redefine the partial multiplication by

$$(g_2, h_2)(g_1, h_1) = (g_2 g_1, h_1(g_1^{-1} \cdot h_2)), \tag{1.2}$$

for compatible pairs $(g_2, h_2), (g_1, h_1) \in G * H$. The remaining structure maps we keep as before. To check that this again defines a groupoid is straightforward and only differs slightly from the original case. We call this the *opposite semidirect product groupoid* and denote it by $G \bar{\ltimes} H \rightrightarrows M$.

The construction of semidirect products can also be extended to the case of Lie groupoids.

Proposition 1.29. *Let G and H be Lie groupoids on a base manifold M. Suppose that H is totally intransitive with source and target projection denoted by $q\colon H \to M$. If $\theta\colon G * H \to H$ is a Lie groupoid action of G on q, such that*

$$\theta_g\colon H_{\alpha(g)} \to H_{\beta(g)}\,,\ h \mapsto h \cdot g,$$

is an isomorphism of Lie groups, for every $g \in G$, then the semidirect product groupoid $G \ltimes H \rightrightarrows M$ is a Lie groupoid.

Proof. Since the source projection of G is a submersion, it follows that the space $G * H = (\alpha \times q)^{-1}(\Delta_M)$ is an embedded submanifold of the product manifold $G \times H$. It is not hard to confirm that the structure maps of $G \ltimes H$ are smooth. It remains only to show that the source and target projections are submersions.

For an arbitrary pair $(g, h) \in G * H$, consider the tangent map of the source projection $T_{(g,h)}(\alpha')\colon T_{(g,h)}(G * H) \to T_{\alpha(g)}M$. Take any $Z \in T_{\alpha(g)}M = T_{q(h)}M$. Then since the source projections α and q are submersions, there exists $X \in T_g G$ and $Y \in T_h H$ such that

$$T(\alpha)(X) = Z = T(q)(Y).$$

Hence, $(X, Y) \in T_{(g,h)}(G * H)$, and $T(\alpha')(X, Y) = T(\alpha)(X) = Z$. Thus, the tangent map $T_{(g,h)}(\alpha')$ is surjective, and so α' is a submersion. We can prove that the target projection is also a submersion in an analogous way. □

The groupoid actions we have been discussing are more accurately referred to as *left* groupoid actions. We will continue to omit the term 'left' whenever it is clear from the context. We finish this section by discussing the notion of a *right* groupoid action.

Definition 1.30. Let G be a groupoid on base M, and $f\colon M' \to M$ a map. A *right groupoid action* of G on f is a map $\theta\colon M' * G \to M'$, $(m', g) \mapsto m' \cdot g$, satisfying

(i) $f(m' \cdot g) = \alpha(g)$, for all $(m', g) \in M' * G$;

(ii) $(m' \cdot h) \cdot g = m' \cdot (hg)$, for all $(h, g) \in G * G$, and all $m' \in M'$, such that $(m', g) \in M' * G$;

(iii) $m' \cdot 1_{f(m')} = m'$, for all $m' \in M'$;

where $G * M' = (\beta \times f)^{-1}(\Delta_M)$.

Moreover, if $G \rightrightarrows M$ is a Lie groupoid and θ and f are smooth maps, then we call θ a *right Lie groupoid action* of G on f.

1.1.4 Pullback groupoids

Given a Lie groupoid G with base manifold M, and a smooth map $f\colon M' \to M$, we have an induced groupoid structure on the set

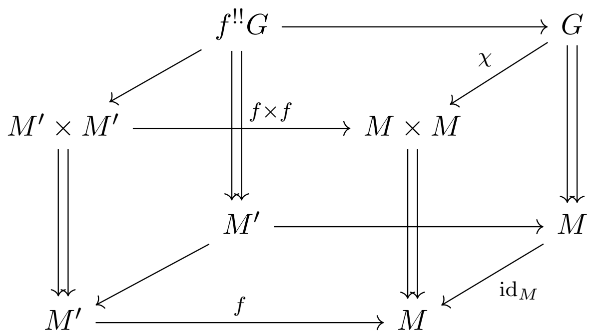
$$M' * G * M' = \{(y', g, x') \in M' \times G \times M' \mid f(y') = \beta(g),\ \alpha(g) = f(x')\}$$

with base M'. The target and source projections are defined by the projections onto the first and third factors, respectively. For compatible elements, the partial multiplication is given by

$$(z', h, y')(y', g, x') = (z', hg, x').$$

For $x' \in M'$, the corresponding identity element is $(x', 1_{f(x')}, x')$; and for an element $(y', g, x') \in M' * G * M'$, the corresponding inverse is given by (x', g^{-1}, y'). With these structure maps, we denote $M' * G * M'$ by $f^{!!}G$ and call it the *pullback groupoid* of G over f.

The groupoid structure takes this name, primarily, because it can be obtained by taking the pullback in the category of groupoids of the morphisms $(f \times f, f)$ and (χ, id_M). Here, $f \times f\colon M' \times M' \to M \times M$ is the morphism of pair groupoids described in Example 1.14, and $\chi\colon G \to M \times M$ is the anchor of G defined in Example 1.15. The pullback diagram is given by:



(1.3)

Note that $f^{!!}G$ is not necessarily a Lie groupoid. As a set, we can view $M' * G * M'$ as the pullback of $f \times f$ and $\chi = (\beta, \alpha)$, which has a manifold structure if these smooth maps are transversal. However, this transversality condition is not always sufficient as the source and target projections may not be submersions. This can be rectified if we take the further condition that the map $f^{!}G \to M$, $(g, x') \mapsto \beta(g)$ is a submersion (where $f^{!}G$ is the pullback manifold of f and α). One situation in which these two conditions are met is when f is a submersion.

When G is taken to be just a Lie group, then the corresponding pullback groupoid that we get is referred to as a *trivial Lie groupoid*. Moreover, if G is the trivial group, then we are just reduced to a pair groupoid as defined in Example 1.5. A more interesting example will be described later in Example 1.68.

§ 1.2 Lie algebroid theory

In this section, we provide a concise treatment of the infinitesimal objects associated to Lie groupoids. Entitled *Lie algebroids*, these objects were first introduced by

Pradines [57], whose objective was to describe a Lie theory for Lie groupoids. The main results and terminology in this section follow closely the work of Higgins and Mackenzie [27].

1.2.1 Lie algebroids

We begin with the abstract notion of a Lie algebroid. In the next section, we will see how these arise as infinitesimal invariants of Lie groupoids.

Definition 1.31. A *Lie algebroid* is a smooth vector bundle $A \to M$ with a Lie algebra structure on its module of sections $\Gamma(A)$, equipped with a vector bundle morphism $a\colon A \to TM$, that satisfies the following 'Leibniz rule',

$$[X, fY] = f[X,Y] + a(X)(f)Y, \tag{1.4}$$

for all $X, Y \in \Gamma(A)$ $f \in \mathcal{C}^\infty(M)$. We call a the *anchor map* of the Lie algebroid.

We refer to M as the *base* of the Lie algebroid, and we will often use the phrase 'A is a Lie algebroid on M'. In the literature, the following consequence is often included within the definition.

Lemma 1.32. *Let A be a Lie algebroid on M. Then the anchor map, considered as a map of sections $a\colon \Gamma(A) \to \mathfrak{X}(M)$, is a Lie algebra homomorphism.*

Proof. Let $X, Y \in \Gamma(A)$. For any $f \in \mathcal{C}^\infty(M)$ and $Z \in \Gamma(A)$, we have

$$\begin{aligned}[X,[Y,fZ]] &= [X, f[Y,Z] + a(Y)(f)Z] \\ &= f[X,[Y,Z]] + a(X)(f)[Y,Z] + a(Y)(f)[X,Z] + a(X)(a(Y)(f))Z.\end{aligned}$$

On the other hand, the Jacobi identity leads to the following,

$$\begin{aligned}[X,[Y,fZ]] &= [Y,[X,fZ]] + [[X,Y],fZ] \\ &= f[Y,[X,Z]] + a(Y)(f)[X,Z] + a(X)(f)[Y,Z] + a(Y)(a(X)(f))Z \\ &\quad + f[[X,Y],Z] + a([X,Y])(f)Z.\end{aligned}$$

Equating these two expressions, and utilising the Jacobi identity again, we get the equation

$$(a(X) \circ a(Y) - a(Y) \circ a(X) - a([X,Y]))(f)Z = 0.$$

Since this holds for every given f and Z, we deduce that $a([X,Y]) = [a(X), a(Y)]$. □

It should be clear that a Lie algebra $\mathfrak{g}$ is a Lie algebroid by considering $\mathfrak{g}$ as a vector bundle on a singleton set, and then identifying its module of sections $\Gamma(\mathfrak{g})$ with itself $\mathfrak{g}$. Its anchor map a is the zero map, and by identifying smooth functions on this singleton set with $\mathbb{R}$, property (1.4) is trivial. The next simplest example of a Lie algebroid is the tangent bundle of a smooth manifold with the identity map serving as its anchor. In Example 1.8 we saw that an action of a Lie group gave rise to a Lie groupoid; there is an analogous result for an action of a Lie algebra.

Example 1.33. Suppose that $\hat{\theta}\colon \mathfrak{g} \to \mathfrak{X}(M)$, $X \mapsto X^\dagger$ is an action of a Lie algebra $\mathfrak{g}$ on a smooth manifold M. There is an induced Lie algebroid structure on the trivial vector bundle $\mathfrak{g} \times M$ on M. The anchor map $a\colon \mathfrak{g} \times M \to TM$ is given by $a(X, m) = X^\dagger(m)$. Note that we can identify sections of this vector bundle with smooth vector valued functions $V\colon M \to \mathfrak{g}$. For constant maps $V, W\colon M \to \mathfrak{g}$ we define the bracket by $[V,W](m) = [V(m), W(m)]$ for $m \in M$, and one can check that in fact this is enough to determine the bracket everywhere. In general, for smooth maps $V, W\colon M \to \mathfrak{g}$, and $m \in M$, we have

$$[V, M](m) = [V(m), W(m)] + (\mathcal{L}_{V(m)^\dagger} W - \mathcal{L}_{W(m)^\dagger} V)(m).$$

We call this the *action Lie algebroid* of $\hat{\theta}$ and denote it by $\mathfrak{g} \lessdot M$. ⊠

Definition 1.34. Let A be a Lie algebroid on M with anchor map $a\colon A \to TM$. We say that:

(i) A is *transitive* if a is a fibrewise surjection;

(ii) A is *regular* if a has constant rank;

(iii) A is *totally intransitive* if a is the zero map.

1.2.2 The Lie algebroid of a Lie groupoid

In the study of Lie groups, an important construction is that of the Lie algebra of a Lie group. In this section, we focus on the analogous construction of the Lie algebroid of a Lie groupoid.

For a Lie group, one preference is to consider its space of right-invariant vector fields. This space inherits the standard bracket of vector fields which makes it a Lie algebra. As a vector space, it is often identified with the tangent space at the identity element of the group because it can be shown that every right-invariant vector field is determined by only its value at the identity.

In the extension to the general case, given a Lie groupoid G on base M, we construct a Lie algebroid on M. The underlying vector bundle of this Lie algebroid is denoted by AG. We extend the notion of right-invariance to vector fields on Lie groupoids. Then it becomes natural to identify the sections of AG with the right-invariant vector fields of G. Once again, the Lie bracket is inherited from the standard bracket of vector fields on G. The anchor map for AG is in some sense just a restriction of the differential of the target projection.

Let us now examine this construction in more detail. We fix a Lie groupoid G on base M with structure maps denoted in the usual way by α, β, κ, 1 and ι.

Definition 1.35. The vector bundle $AG \to M$ is the pullback bundle of $T^\alpha G \to G$ by the identity map $1\colon M \to G$,

$$\begin{array}{ccc} AG & \longrightarrow & T^\alpha G \\ \downarrow & & \downarrow \\ M & \xrightarrow{\;1\;} & G. \end{array} \tag{1.5}$$

Since the identity map is an embedding, for each $x \in M$ the fibre $A_x G$ can be naturally identified with the tangent space $T_{1_x}(G_x)$. This identification will be used throughout without further comment.

Definition 1.36. The *anchor map* $a_G\colon AG \to TM$ is the composite of the vector bundle morphisms

$$\begin{array}{ccccccc} AG & \longrightarrow & T^{\alpha}G & \hookrightarrow & TG & \xrightarrow{T(\beta)} & TM \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ M & \xrightarrow{1} & G & = & G & \xrightarrow{\beta} & M. \end{array} \tag{1.6}$$

It is important to note that a_G is a bundle morphism over M since the base map $\beta \circ 1$ is equal to the identity map id_M.

It remains to equip the module of sections $\Gamma(AG)$ with an appropriate Lie bracket. To do this, we will consider an alternative description of these sections. In this endeavour, we first give a definition of right-invariance for vector fields on the Lie groupoid G.

Definition 1.37. Let $X \in \mathfrak{X}(G)$. We say that:

(i) X is a *vertical* vector field on G if $X(g) \in T^{\alpha}_g G$, for all $g \in G$;

(ii) X is a *right-invariant* vector field on G if it is vertical and satisfies the condition that $X(hg) = T_h(R_g)(X(h))$, for all $(h, g) \in G * G$.

Note that right-invariant vector fields on G are determined solely by the values they take on the identity elements of the groupoid. This follows because any vertical vector field $X \in \mathfrak{X}(G)$ is right-invariant if and only if $X(g) = T_{1_x}(R_g)(X(1_x))$, for all $g \in G^x$, and $x \in M$.

We denote the set of right-invariant vector fields on G by $\mathfrak{X}^R(G)$. It is a $\mathcal{C}^\infty(G)$–submodule of both $\mathfrak{X}(G)$ and $\Gamma(T^{\alpha}G)$. It is also a $\mathcal{C}^\infty(M)$–module under the scalar multiplication $fX = (f \circ \beta)X$, where $f \in \mathcal{C}^\infty(M)$, $X \in \mathfrak{X}^R(G)$. Clearly, $\mathfrak{X}^R(G)$ is a vector subspace of $\mathfrak{X}(G)$, moreover we have the following result.

Lemma 1.38. *$\mathfrak{X}^R(G)$ is a Lie subalgebra of $\mathfrak{X}(G)$.*

Proof. Let $X, Y \in \mathfrak{X}^R(G)$. We need to show that $[X, Y] \in \mathfrak{X}^R(G)$.

Since X and Y are vertical vector fields on G, X and Y are both α–related to the zero section of TM. It follows by the naturality of the Lie bracket that $[X, Y]$ is also α–related to the zero section, and so $[X, Y]$ is a vertical vector field.

The condition that X and Y are right-invariant implies that for any $g \in G^y_x$, $x, y \in M$, the restrictions $X|_{G_y}$ and $Y|_{G_y}$ are R_g-related to $X|_{G_x}$ and $Y|_{G_x}$, respectively. Hence, $[X, Y]|_{G_y}$ is R_g-related to $[X, Y]|_{G_x}$, again using the naturality of the Lie bracket. This property, paired with the fact that $[X, Y]$ is vertical, is enough to conclude the right-invariance of $[X, Y]$. □

We would like to show that, as $\mathcal{C}^\infty(M)$–modules, $\Gamma(AG)$ and $\mathfrak{X}^R(G)$ are equivalent. To do this, we first show that vertical vector fields on G can be identified with sections of another vector bundle. The bundle we are interested in is the pullback bundle of $AG \to M$ by the tangent projection $\beta\colon G \to M$. We have the following pullback diagram,

$$\begin{array}{ccc} \beta^! AG & \xrightarrow{\beta^!} & AG \\ \downarrow & & \downarrow \\ G & \xrightarrow{\beta} & M. \end{array} \tag{1.7}$$

Recall that for $X \in \Gamma(AG)$ we use the notation $X^!$ to denote its corresponding pullback section in $\Gamma(\beta^! AG)$. It is the unique section such that $\beta^! \circ X^! = X \circ \beta$.

Proposition 1.39. *The map* $\Phi\colon \beta^! AG \to T^\alpha G$ *defined on fibres by*

$$\Phi_g = T_{1_{\beta(g)}}(R_g)\colon \{g\} \times T_{1_{\beta(g)}}(G_{\beta(g)}) \to T_g(G_{\alpha(g)}),$$

is an isomorphism of vector bundles over G.

Proof. It should be clear that each fibre Φ_g is a linear isomorphism which preserves the base point $g \in G$. It remains only to show that Φ is smooth. This arises immediately when we view Φ as the composition of the following smooth maps,

$$\begin{array}{ccccc} \beta^! AG & \longrightarrow & TG * TG & \xrightarrow{T(\kappa)} & TG \\ (g, Y_{1_{\beta(g)}}) & \longmapsto & (Y_{1_{\beta(g)}}, 0_g) & \longmapsto & T_{1_{\beta(g)}}(R_g)(Y_{1_{\beta(g)}}). \end{array} \tag{1.8}$$

□

This result leads us to the following consequence.

Corollary 1.40. *Let* G *be a Lie groupoid on* M. *For* $X \in \Gamma(AG)$, *the section* $\overrightarrow{X} := \Phi(X^!)$ *of* $T^\alpha G$ *is a right-invariant vector field of* G.

Proof. Let $X \in \Gamma(AG)$, for any $g \in G^x$, $x \in M$, we have

$$\overrightarrow{X}(g) = \Phi_g(X^!(g)) = \Phi_g(g, X(x)) = T_{1_x}(R_g)(X(x)).$$

On the other hand,

$$\overrightarrow{X}(1_x) = \Phi_{1_x}(X^!(1_x)) = \Phi_{1_x}(1_x, X(x)) = X(x).$$

Thus, we have the relation $\overrightarrow{X}(g) = T_{1_x}(R_g)(\overrightarrow{X}(1_x))$, and so $\overrightarrow{X}$ is a right-invariant vector field of G. □

We now have all the components we require to specify an identification between the sections of AG and right-invariant vector fields of G. The checks are all straightforward and are left to the reader.

Proposition 1.41. *The map* $\Gamma(AG) \to \mathfrak{X}^R(G)$, $X \mapsto \overrightarrow{X}$, *is an isomorphism of* $\mathcal{C}^\infty(M)$*–modules with inverse* $\mathfrak{X}^R(G) \to \Gamma(AG)$, $X \mapsto X \circ 1$. □

We can now use this isomorphism to transfer the Lie algebra structure on $\mathfrak{X}^R(G)$ to $\Gamma(AG)$. We define a Lie bracket by

$$[X, Y] = [\overrightarrow{X}, \overrightarrow{Y}] \circ 1, \tag{1.9}$$

for $X, Y \in \Gamma(AG)$. Because of the isomorphism of modules, this Lie bracket should have the relationship with the module structure on $\Gamma(AG)$ that we desire.

Proposition 1.42. *The vector bundle* $AG \to M$, *with the Lie algebra structure on* $\Gamma(AG)$ *defined by* (1.9), *equipped with the anchor* a_G, *is a Lie algebroid.*

Proof. We only need to check the 'Leibniz rule' (1.4) given in the definition of a Lie algebroid. Let $X, Y \in \Gamma(AG)$, $f \in \mathcal{C}^\infty(M)$, and $x \in M$. We first observe that

$$\overrightarrow{X}(f \circ \beta)(1_x) = T(\beta)(\overrightarrow{X}(1_x))(f) = T(\beta)(X(x))(f) = (a(X)(f))(x).$$

Hence, it follows that

$$\begin{aligned} [X, fY] &= [\overrightarrow{X}, \overrightarrow{fY}] \circ 1 \\ &= [\overrightarrow{X}, (f \circ \beta)\overrightarrow{Y}] \circ 1 \\ &= ((f \circ \beta)[\overrightarrow{X}, \overrightarrow{Y}] + \overrightarrow{X}(f \circ \beta)\overrightarrow{Y}) \circ 1 \\ &= (f \circ \beta \circ 1)([\overrightarrow{X}, \overrightarrow{Y}] \circ 1) + (\overrightarrow{X}(f \circ \beta) \circ 1)(\overrightarrow{Y} \circ 1) \\ &= f[X, Y] + a(X)(f)Y. \end{aligned}$$ □

With this structure, we call $AG \to M$ the *Lie algebroid of* G.

Example 1.43. Let M be a smooth manifold and consider the pair groupoid $M \times M$ on base M, as defined in Example 1.5. Let us construct the Lie algebroid of this Lie groupoid. Observe that, for $x \in M$,

$$A_x(M \times M) = T_{1_x}(M \times M)_x = T_{(x,x)}(M \times \{x\}) \cong T_x M.$$

Moreover, it is clear to see that the induced anchor map is just the identity map $\mathrm{id}_{TM} \colon TM \to TM$. This immediately implies that the Lie bracket is given precisely by the standard bracket of vector fields. Hence, the Lie algebroid $A(M \times M) \to M$ is nothing more than the tangent bundle $TM \to M$. ⊠

1.2.3 Morphisms of Lie algebroids and Lie subalgebroids

We now look at the morphisms in the category of Lie algebroids. Let us first recall some results about vector bundle morphisms and introduce some notation.

Let $q' \colon E' \to M'$ and $q \colon E \to M$ be smooth vector bundles. Given a vector bundle morphism $\varphi \colon E' \to E$ over a smooth map $f \colon M' \to M$, we can form the pullback bundle $f^! E$ over M'. As a $\mathcal{C}^\infty(M')$-module, we can identify $\Gamma(f^!(E))$ with the tensor product $\mathcal{C}^\infty(M') \otimes \Gamma(E)$ via the isomorphism $u' \otimes X \mapsto u' X^!$ (see [24, Section 2.26]). We also have a vector bundle morphism $\varphi^! \colon E' \to f^! E$ over M', given by the mapping $X' \mapsto (q'(X'), \varphi(X'))$. This induces a map on sections $\varphi^! \colon \Gamma(E') \to \Gamma(f^! E)$, which is linear over $\mathcal{C}^\infty(M')$. Hence, for any $X' \in \Gamma(E')$, we have

$$\varphi^!(X') = \sum_i u'_i \otimes X_i, \tag{1.10}$$

for some $u'_i \in \mathcal{C}^\infty(M')$, $X_i \in \Gamma(E)$. Alternatively, this equation can be written in the form

$$\varphi \circ X' = \sum_i u'_i (X_i \circ f). \tag{1.11}$$

We refer to both (1.10) and (1.11) as φ*–decompositions* of X'.

Definition 1.44. Let A' and A be Lie algebroids on bases M' and M with anchor maps a' and a, respectively. A *morphism of Lie algebroids* is a vector bundle morphism $\varphi \colon A' \to A$ over a smooth map $f \colon M' \to M$ satisfying $a \circ \varphi = T(f) \circ a'$, and such that for any $X', Y' \in \Gamma(A')$ with φ–decompositions

$$\varphi^!(X') = \textstyle\sum_i u'_i \otimes X_i\,, \quad \varphi^!(Y') = \sum_j v'_j \otimes Y_j\,,$$

we have

$$\varphi^!([X',Y']) = \sum_{i,j} u'_i v'_j \otimes [X_i, Y_j] + \sum_j a'(X')(v'_j) \otimes Y_j - \sum_i a'(Y')(u'_i) \otimes X_i.$$

We also say that (φ, f) is a morphism of Lie algebroids.

We leave to the reader the checks needed to prove the well-definedness of the definition.

Proposition 1.45. *Given morphisms of Lie algebroids $\psi\colon A'' \to A'$ and $\varphi\colon A' \to A$, the composition $\varphi \circ \psi\colon A'' \to A$ is a morphism of Lie algebroids.* □

Remark 1.46. Given Lie algebroids A' and A on the same base M, and a vector bundle morphism $\varphi\colon A' \to A$ over M, the conditions that φ is a morphism of Lie algebroids reduce to $a \circ \varphi = a'$ and $\varphi([X',Y']) = [\varphi(X'), \varphi(Y')]$ for all sections $X', Y' \in \Gamma(A')$.

Definition 1.47. Let A' and A be Lie algebroids on the same base manifold M. An *isomorphism of Lie algebroids over M* is a morphism of Lie algebroids $\varphi\colon A' \to A$ over M which is also a diffeomorphism.

We now look at our first natural example of a morphism of Lie algebroids.

Proposition 1.48. *Let $F\colon M' \to M$ be a smooth map. Then the differential of F, $T(F)\colon TM' \to TM$, is a morphism of Lie algebroids over F.*

Proof. Since the anchor maps of TM' and TM are just identity maps, the anchor preservation condition for $T(F)$ to be a morphism is trivial. To check the bracket condition, let $X', Y' \in \mathfrak{X}(M')$ and choose $T(F)$–decompositions

$$T(F) \circ X' = \textstyle\sum_i u'_i (X_i \circ F)\,, \quad T(F) \circ Y' = \sum_j v'_j (Y_j \circ F)\,,$$

for some $u'_i, v'_j \in \mathcal{C}^\infty(M')$, $X_i, Y_j \in \mathfrak{X}(M)$. Observe that for any $w \in \mathcal{C}^\infty(M)$ we have

$$\begin{aligned} X'(Y'(w \circ F)) &= X'\left(\textstyle\sum_j v'_j (Y_j(w) \circ F)\right) \\ &= \textstyle\sum_j \left(X'(v'_j)(Y_j(w) \circ F) + v'_j X'(Y_j(w) \circ F)\right) \\ &= \textstyle\sum_j X'(v'_j)(Y_j(w) \circ F) + \sum_{i,j} v'_j u'_i (X_i(Y_j(w)) \circ F). \end{aligned}$$

Similarly, we can show that

$$Y'(X'(w \circ F)) = \textstyle\sum_i Y'(u'_i)(X_i(w) \circ F) + \sum_{i,j} u'_i v'_j (Y_j(X_i(w)) \circ F).$$

Hence,

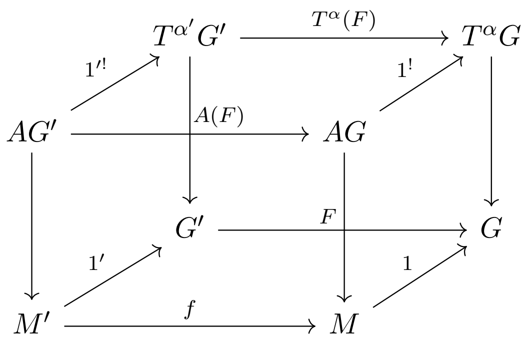
$$\begin{aligned} [X',Y'](w \circ F) = \textstyle\sum_{i,j} & u'_i v'_j ([X_i, Y_j](w) \circ F) \\ & + \textstyle\sum_j X'(v'_j)(Y_j(w) \circ F) - \sum_i Y'(u'_i)(X_i(w) \circ F). \end{aligned}$$

However, this is precisely the condition that

$$T(F)^!([X',Y']) = \sum_{i,j} u'_i v'_j \otimes [X_i, Y_j] + \sum_j X'(v'_j) \otimes Y_j - \sum_i Y'(u'_i) \otimes X_i. \qquad \square$$

In the study of Lie groups, a homomorphism of Lie groups gives rise to a homomorphism of the respective Lie algebras. A similar phenomenon occurs for morphisms in the category of Lie groupoids.

Let $F\colon G' \to G$ be a morphism of Lie groupoids over a smooth map $f\colon M' \to M$. The differential $T(F)$ restricts to a vector bundle morphism $T^{\alpha}(F)\colon T^{\alpha'}G' \to T^{\alpha}G$ because of the property $\alpha \circ F = f \circ \alpha'$. We define $A(F)\colon AG' \to AG$ to be the unique vector bundle morphism over $f\colon M' \to M$ such that the following diagram of morphisms commutes:



(1.12)

Proposition 1.49. *For a morphism of Lie groupoids $F\colon G' \to G$ over $f\colon M' \to M$, the map $A(F)\colon AG' \to AG$ is a morphism of Lie algebroids over f.*

Proof. The anchor preservation condition, $T(f)\circ a_{G'} = a_G \circ A(F)$, is easily verified by utilising the commutativity relations of (1.12) and the property that $\beta \circ F = f \circ B'$.

To establish the bracket condition, we first take any $X', Y' \in \Gamma(AG')$ with $A(F)$–decompositions

$$A(F)\circ X' = \textstyle\sum_i u_i'(X_i \circ f)\,, \quad A(F)\circ Y' = \textstyle\sum_j v_j'(Y_j \circ f)\,,$$

where $u_i', v_j' \in \mathcal{C}^{\infty}(M')$, and $X_i, Y_j \in \Gamma(AG)$. By recalling the vector bundle isomorphisms $\Phi'\colon \beta'^{!}AG \to T^{\alpha'}G'$ and $\Phi\colon \beta^{!}AG \to T^{\alpha}G$ defined in Proposition 1.39, observe that

$$\begin{aligned} T(F)\circ \overrightarrow{X'} &= T(F)\circ \Phi' \circ X' \circ \beta' \\ &= \Phi \circ A(F) \circ X' \circ \beta' \\ &= \Phi \circ \Big(\textstyle\sum_i u_i'(X_i \circ f)\Big) \circ \beta' \\ &= \textstyle\sum_i (u_i' \circ \beta')(\Phi \circ X_i \circ f \circ \beta') \\ &= \textstyle\sum_i (u_i' \circ \beta')(\overrightarrow{X_i} \circ F). \end{aligned}$$

Similarly, we can also show that $T(F)\circ \overrightarrow{Y'} = \sum_j (v_j' \circ \beta')(\overrightarrow{Y_j} \circ F)$. Hence, by Proposition 1.48, we have

$$\begin{aligned} T(F)\circ [\overrightarrow{X'}, \overrightarrow{Y'}] = {} & \textstyle\sum_{i,j} (u_i' \circ \beta')(v_j' \circ \beta')([\overrightarrow{X_i}, \overrightarrow{Y_j}] \circ F) \\ & + \textstyle\sum_j \overrightarrow{X'}(v_j' \circ \beta')(\overrightarrow{Y_j} \circ F) - \sum_i \overrightarrow{Y'}(u_i' \circ \beta')(\overrightarrow{X_i} \circ F). \end{aligned}$$

Note that $\overrightarrow{X'}$ are $\overrightarrow{Y'}$ are β'-related to $a_{G'}(X')$ and $a_{G'}(Y')$, respectively. That is, $\overrightarrow{X'}(u' \circ \beta') = a_{G'}(X')(u') \circ \beta'$, and $\overrightarrow{Y'}(u' \circ \beta') = a_{G'}(Y')(u') \circ \beta'$ for any $u \in \mathcal{C}^{\infty}(M')$.

From this inspection, it follows that

$$\begin{aligned} A(F)\circ[X',Y'] &= T(F)\circ[\overrightarrow{X'},\overrightarrow{Y'}]\circ 1' \\ &= \sum\nolimits_{i,j} u'_i v'_j([X_i,Y_j]\circ f) \\ &\qquad + \sum\nolimits_j a_{G'}(X')(v'_j)(Y_j\circ f) - \sum\nolimits_i a_{G'}(Y')(u'_i)(X_i\circ f). \qquad \square \end{aligned}$$

One can check that the assignment $G \mapsto AG$, $F \mapsto A(F)$ gives a well-defined functor from the category of Lie groupoids to the category of Lie algebroids. We denote this functor by A and it is appropriately named the *Lie functor*.

We now give the statement of a useful criterion for when a vector bundle morphism of Lie algebroids is a morphism of Lie algebroids.

Proposition 1.50 ([27, Proposition 1.5]). *Let $\varphi\colon A' \to A$ be a vector bundle morphism of Lie algebroids over a map $f\colon M' \to M$,*

$$\begin{array}{ccc} A' & \xrightarrow{\ \varphi\ } & A \\ \downarrow & & \downarrow \\ M' & \xrightarrow{\ f\ } & M \end{array} \tag{1.13}$$

and suppose that φ is a fibrewise surjection. If we have $a\circ\varphi = T(f)\circ a'$ and for any $X_1, X_2 \in \Gamma(A')$, $Y_1, Y_2 \in \Gamma(A)$, we have

$$\varphi\circ X_i = Y_i\circ f,\ \forall i=1,2 \implies \varphi\circ[X_1,X_2] = [Y_1,Y_2]\circ f, \tag{1.14}$$

then (φ, f) is a morphism of Lie algebroids. $\square$

Remark 1.51 ([47, page 162]). The previous result still holds if the assumption that φ is a fibrewise surjection is replaced with the condition that φ is of constant rank.

Lemma 1.52. *Let (φ, f) be a vector bundle morphism of Lie algebroids as in* (1.13)*, which satisfies the property* (1.14)*. If for any $X_1, X_2 \in \Gamma(A')$, $Y_1, Y_2 \in \Gamma(A)$, we have*

$$\varphi\circ X_1 \equiv Y_1\circ f, \quad \varphi\circ X_2 \equiv Y_2\circ f,$$

in a neighbourhood of a point $y \in M'$, then

$$\varphi([X_1,X_2](y)) = [Y_1,Y_2](f(y)). \qquad \square$$

We finish this section with a discussion of the sub-objects in the category of Lie algebroids.

Definition 1.53. Let A be a Lie algebroid on base M. A Lie algebroid A' on base M' is a *Lie subalgebroid of* A if there exist injective immersions $\widetilde{i}\colon A' \to A$ and $i\colon M' \to M$ such that $(\widetilde{i}, i)$ is a morphism of Lie algebroids.

If, in addition, the maps $\widetilde{i}$ and i are smooth embeddings, we call A' an *embedded Lie subalgebroid of* A.

Proposition 1.54. *Let A be a Lie algebroid on a base manifold M, and M' a closed embedded submanifold of M. Then a vector subbundle $A' \to M'$ of $A|_{M'} \to M'$ is a Lie subalgebroid of A if and only if the following properties hold:*

(i) the anchor map $a\colon A \to TM$ restricts to a map $A' \to TM'$;

(ii) if $X, Y \in \Gamma(A)$ satisfy $X|_{M'}, Y|_{M'} \in \Gamma(A')$, then $[X,Y]|_{M'} \in \Gamma(A')$;

(iii) if $X, Y \in \Gamma(A)$ satisfy $X|_{M'} = 0$ and $Y|_{M'} \in \Gamma(A')$, then $[X,Y]|_{M'} = 0$. $\square$

1.2.4 Direct product Lie algebroids

Let $A_1 \to M_1$ and $A_2 \to M_2$ be Lie algebroids and consider the Cartesian product vector bundle $A_1 \times A_2 \to M_1 \times M_2$. We would like to put a Lie algebroid structure on $A_1 \times A_2$ such that the two projections $\widetilde{\mathrm{pr}}_1 \colon A_1 \times A_2 \to A_1$ and $\widetilde{\mathrm{pr}}_2 \colon A_1 \times A_2 \to A_2$ are morphisms of Lie algebroids over the projections $\mathrm{pr}_1 \colon M_1 \times M_2 \to M_1$ and $\mathrm{pr}_2 \colon M_1 \times M_2 \to M_2$, respectively. Let us suppose that $A_1 \times A_2$ has such a Lie algebroid structure, then in particular we must have

$$a_1 \circ \widetilde{\mathrm{pr}}_1 = T(\mathrm{pr}_1) \circ a\,, \qquad a_2 \circ \widetilde{\mathrm{pr}}_2 = T(\mathrm{pr}_1) \circ a\,,$$

where a is the anchor map of $A_1 \times A_2$, and a_1 and a_2 are the anchors of A_1 and A_2, respectively. Viewed as a map from $A_1 \times A_2$ to $TM_1 \times TM_2$, it follows that $a = (a_1 \circ \widetilde{\mathrm{pr}}_1, a_2 \circ \widetilde{\mathrm{pr}}_2)$. That is, for a pair $X = (X_1, X_2) \in A_1 \times A_2$, we have $a(X) = (a_1(X_1), a_2(X_2))$.

The bracket condition for $\widetilde{\mathrm{pr}}_1$ tells us that if the sections $X, Y \in \Gamma(A_1 \times A_2)$ have $\widetilde{\mathrm{pr}}_1$–decompositions

$$\widetilde{\mathrm{pr}}_1^!(X) = \textstyle\sum_i u_i^1 \otimes X_i^1\,, \qquad \widetilde{\mathrm{pr}}_1^!(Y) = \textstyle\sum_j v_j^1 \otimes Y_j^1\,,$$

where $X_i^1, Y_j^1 \in \Gamma(A_1)$ and $u_i^1, v_j^1 \in \mathcal{C}^\infty(M_1 \times M_2)$, then

$$\widetilde{\mathrm{pr}}_1^!([X,Y]) = \sum\nolimits_{i,j} u_i^1 v_j^1 \otimes [X_i^1, Y_j^1] + \sum\nolimits_j a(X)(v_j^1)Y_j^1 - \sum\nolimits_i a(Y)(u_i^1)X_i^1. \quad (1.15)$$

Similarly, if X and Y have $\widetilde{\mathrm{pr}}_2$–decompositions given by

$$\widetilde{\mathrm{pr}}_2^!(X) = \textstyle\sum_k u_k^2 \otimes X_k^2\,, \qquad \widetilde{\mathrm{pr}}_2^!(Y) = \textstyle\sum_l v_l^2 \otimes Y_l^2\,,$$

where $X_k^2, Y_l^2 \in \Gamma(A_2)$ and $u_k^2, v_l^2 \in \mathcal{C}^\infty(M_1 \times M_2)$, then the assumption that $\widetilde{\mathrm{pr}}_2$ is a morphism of Lie algebroids implies

$$\widetilde{\mathrm{pr}}_2^!([X,Y]) = \sum\nolimits_{k,l} u_k^2 v_l^2 \otimes [X_k^2, Y_l^2] + \sum\nolimits_l a(X)(v_l^2)Y_l^2 - \sum\nolimits_k a(Y)(u_k^2)X_k^2. \quad (1.16)$$

We have a natural identification between the Cartesian product vector bundle $A_1 \times A_2$ and the Whitney sum $\mathrm{pr}_1^!(A_1) \oplus \mathrm{pr}_2^!(A_2)$. From this viewpoint, we can write

$$X = \textstyle\sum_i \left(u_i^1 \otimes X_i^1\right) \oplus \sum_k \left(u_k^2 \otimes X_k^2\right), \qquad Y = \sum_j \left(v_j^1 \otimes Y_j^1\right) \oplus \sum_l \left(v_l^2 \otimes Y_l^2\right),$$

and then equations (1.15) and (1.16) immediately imply that

$$\begin{aligned}[X,Y] = &\left(\sum\nolimits_{i,j} u_i^1 v_j^1 \otimes [X_i^1, Y_j^1] + \sum\nolimits_j a(X)(v_j^1)Y_j^1 - \sum\nolimits_i a(Y)(u_i^1)X_i^1\right) \qquad (1.17)\\ &\oplus \left(\sum\nolimits_{k,l} u_k^2 v_l^2 \otimes [X_k^2, Y_l^2] + \sum\nolimits_l a(X)(v_l^2)Y_l^2 - \sum\nolimits_k a(Y)(u_k^2)X_k^2\right).\end{aligned}$$

If we also utilise the identification between the vector bundles $T(M_1 \times M_2)$ and $\mathrm{pr}_1^!(TM_1) \oplus \mathrm{pr}_2^!(TM_2)$, the anchor map has the form

$$a(X) = \sum_i \left(u_i^1 \otimes a_1(X_i^1)\right) \oplus \sum_k \left(u_k^2 \otimes a_2(X_k^2)\right). \quad (1.18)$$

Removing all prior assumptions, it is straightforward to check that $A_1 \times A_2$ becomes a Lie algebroid when equipped with a bracket defined by (1.17) and an anchor map defined by (1.18). We call it the *direct product Lie algebroid* of A_1 and A_2. By construction, it is the unique Lie algebroid such that $\widetilde{\mathrm{pr}}_1$ and $\widetilde{\mathrm{pr}}_2$ are morphisms of Lie algebroids. Thus, unsurprisingly, it follows that it is the direct product of A_1 and A_2 in the category of Lie algebroids.

Proposition 1.55. *Let A_1, A_2 and A_3 be Lie algebroids with base manifolds M_1, M_2 and M_3, respectively. Then the direct product Lie algebroid $(A_1 \times A_2) \times A_3$ with base $(M_1 \times M_2) \times M_3$ is isomorphic to the direct product Lie algebroid $A_1 \times (A_2 \times A_3)$ with base $M_1 \times (M_2 \times M_3)$.* □

Proposition 1.56. *Let G_1 and G_2 be Lie groupoids with base manifolds M_1 and M_2, respectively. Then the Lie algebroid $A(G_1 \times G_2)$ of the Cartesian product groupoid $G_1 \times G_2$ with base manifold $M_1 \times M_2$ is isomorphic to the direct product Lie algebroid $AG_1 \times AG_2$.* □

1.2.5 Actions of Lie algebroids

The notion of a Lie algebra action extends in a natural way to a notion of a Lie algebroid action.

Definition 1.57. Let A be a Lie algebroid on M, and let $f\colon M' \to M$ be a smooth map. Then a *Lie algebroid action* of A on f is a homomorphism of Lie algebras $\Gamma(A) \to \mathfrak{X}(M')$, $X \mapsto X^\dagger$, which is also a homomorphism of $\mathcal{C}^\infty(M)$-modules, and satisfies the following condition:

$$T(f) \circ X^\dagger = a(X) \circ f,$$

for all $X \in \Gamma(A)$.

Given an action of a Lie algebroid $A \to M$ on a smooth map $f\colon M' \to M$, for every $m' \in M'$, we have an induced linear map $A_{f(m')} \to T_{m'}M'$, $v \mapsto v^\dagger$, defined by $v^\dagger = \tilde{v}^\dagger(m')$, where $\tilde{v}$ is any smooth section of A satisfying $\tilde{v}(f(m')) = v$. We also acquire an induced Lie algebroid structure on the pullback vector bundle $f^!A \to M'$. The anchor map of this Lie algebroid structure is defined by

$$a^\dagger\colon f^!(A) \to TM, \quad (v, m') \mapsto v^\dagger.$$

Again we recall that, as a $\mathcal{C}^\infty(M')$-module, $\Gamma(f^!(A))$ can be identified with the tensor product $\mathcal{C}^\infty(M') \otimes \Gamma(A)$. Then, the Lie bracket is defined by

$$\left[\sum_i u'_i \otimes X_i, \sum_j v'_j \otimes Y_j\right] = \sum_{i,j} u'_i v'_j \otimes [X_i, Y_j] \,+\, u'_i X_i^\dagger(v'_j) \otimes Y_j \,-\, v'_j Y_j^\dagger(u'_i) \otimes X_i.$$

If we view the anchor as a map of sections, then we have

$$a^\dagger\left(\sum_i u'_i \otimes X_i\right) = \sum_i u'_i X_i^\dagger.$$

It is not too hard to show that this structure determines a well-defined Lie algebroid.[2] We refer to a Lie algebroid of this form as an *action Lie algebroid* and denote it by $A \lhd M'$ or $A \lhd f$.

In a similar manner that a Lie group action gives rise to a Lie algebra action, a Lie groupoid action gives rise to a Lie algebroid action. Indeed, if G is a Lie groupoid on base M and θ is a Lie groupoid action of G on a smooth map $f\colon M' \to M$, then we have a Lie algebroid action of AG on f defined by

$$X^\dagger(m') = T_{1_{f(m')}}(\theta_{m'})(X(f(m'))), \tag{1.19}$$

for $X \in \Gamma(AG)$, $m' \in M'$.

[2] See [47, Proposition 4.1.2] or [27, Theorem 2.4] for details.

Proposition 1.58 ([27, Theorem 2.5]). *Let G be a Lie groupoid with base manifold M and $f\colon M' \to M$ a smooth map. Suppose that we have a Lie groupoid action of G on f. Then the Lie algebroid of the action groupoid $G \lhd f$ is isomorphic to the action Lie algebroid $AG \lhd f$.* □

§ 1.3 Double Lie structures

A key attribute that the notion of a groupoid has over the notion of a group is the ability to produce interesting objects by taking groupoid objects in a category. Note that a group object in the category of groups is nothing more than an abelian group; however, when we consider a groupoid object in the category of groupoids a new and interesting object appears. In this section, we will consider groupoid objects in the categories of Lie groupoids, Lie algebroids and vector bundles.

1.3.1 Double Lie groupoids

The introduction of double groupoids is accredited to Ehresmann [21]. The development of a Lie theory for double Lie groupoids was initiated by Mackenzie [43, 45]. Let us begin with the definition of a double groupoid.

Definition 1.59. A *double groupoid* is a quadruple of sets $(S; H, V; M)$, such that H and V have groupoid structures with base M, and S has two groupoid structures, one with base V and one with base H, such that the structure maps of $S \rightrightarrows V$ are morphisms of groupoids over the corresponding structure maps of $H \rightrightarrows M$.

The groupoid structure on S with base V is called the *horizontal structure*; likewise the groupoid structure on S with base H is called the *vertical structure*. The two groupoid structures with base M are referred to as the *side groupoids*, and M itself is often referred to as the *double base*.

We often illustrate the groupoid structures of a double groupoid diagrammatically:

$$\begin{array}{ccc} S & \overset{\widetilde{\alpha}_H, \widetilde{\beta}_H}{\rightrightarrows} & V \\ {\scriptstyle \widetilde{\alpha}_V, \widetilde{\beta}_V} \downdownarrows & & \downdownarrows {\scriptstyle \alpha_V, \beta_V} \\ H & \underset{}{\overset{\alpha_H, \beta_H}{\rightrightarrows}} & M. \end{array} \tag{1.20}$$

Given a double groupoid $(S; H, V; M)$, let us clarify the notation that we will use throughout. We denote the source and target projections of the groupoid $H \rightrightarrows M$ by α_H and β_H, respectively, and the partial multiplication, identity and inversion maps by κ_H, 1^H and ι_H, respectively. Similarly, we denote the structure maps of $V \rightrightarrows M$ by α_V, β_V, κ_V, 1^V and ι_V.

For the horizontal groupoid structure $S \rightrightarrows V$, we denote the source and target projections by $\widetilde{\alpha}_H$ and $\widetilde{\beta}_H$, respectively, and the partial multiplication, identity and inversion maps by $\widetilde{\kappa}_H$, $\widetilde{1}^H$ and $\widetilde{\iota}_H$, respectively. For $(s_2, s_1) \in S *_V S$, we denote the product $\widetilde{\kappa}_H(s_2, s_1)$ by $s_2 \boxminus s_1$, and the inverse of any $s \in S$ by $s^{-1(H)}$. (Here, we are using H as a subscript and superscript to indicate the horizontal structure.)

Similarly, for the vertical groupoid structure $S \rightrightarrows H$, we will denote the structure maps by $\widetilde{\alpha}_V$, $\widetilde{\beta}_V$, $\widetilde{\kappa}_V$, $\widetilde{1}^V$ and $\widetilde{\iota}_V$. For a pair $(s_2, s_1) \in S *_H S$, we denote the product $\widetilde{\kappa}_V(s_2, s_1)$ by $s_2 \boxminus s_1$, and we use $s^{-1(V)}$ to denote the inverse of an element $s \in S$. For any $m \in M$, we denote the *double identity* $\widetilde{1}^H_{1^V_m} = \widetilde{1}^V_{1^H_m}$ by 1^2_m.

Remark 1.60. The definition of a double groupoid $(S; H, V; M)$ is symmetrical in the following sense. The condition that the structure maps of the horizontal structure are groupoid morphisms over the corresponding structure maps of $H \rightrightarrows M$ is equivalent to the condition that the structure maps of the vertical structure are groupoid morphisms over the corresponding structure maps of $V \rightrightarrows M$.

Definition 1.61. A *double Lie groupoid* is a double groupoid $(S; H, V; M)$ with smooth manifold structures on each set, such that all four groupoid structures are Lie groupoid structures, and the double source map $(\widetilde{\alpha}_V, \widetilde{\alpha}_H)\colon S \to H \times_\alpha V$ is a surjective submersion. Here, $H \times_\alpha V$ is just the pullback manifold given by $(\alpha_H \times \alpha_V)^{-1}(\Delta_M)$.

We call $(S; H, V; M)$ a *weak double Lie groupoid* when the condition that the double source map is surjective is removed.

In some sense, a double groupoid can be seen as a groupoid object in the category of groupoids.[3] In fact, this type of interpretation can be extended. That is, every weak double Lie groupoid is a groupoid object in the category of Lie groupoids. However, this statement alludes to the fact that the domains of the partial multiplications are Lie groupoids themselves, which is not immediately obvious. The following result will provide us with a mechanism to prove this fact.

Proposition 1.62.[4] *Let $F_1\colon G_1 \to G$ and $F_2\colon G_2 \to G$ be partial fibrations over smooth maps $f_1\colon M_1 \to M$ and $f_2\colon M_2 \to M$, respectively. Suppose that f_1 and f_2 are transversal to each other. Then the pullback $G_1 \times_G G_2$ of F_1 and F_2 is an embedded submanifold of the product $G_1 \times G_2$ and has a Lie groupoid structure with base the pullback manifold $M_1 \times_M M_2$. Moreover, $G_1 \times_G G_2$ is an embedded Lie subgroupoid of the Cartesian product groupoid $G_1 \times G_2$.*

Proof. Let $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\kappa}$, $\widetilde{1}$ and $\widetilde{\iota}$ denote the restrictions of the structure maps of the Cartesian product groupoid $G_1 \times G_2$ to the pullbacks $G_1 \times_G G_2$ and $M_1 \times_M M_2$. We first check that restricting the codomains of these maps give structure maps for a groupoid structure on $G_1 \times_G G_2$ with base $M_1 \times_M M_2$. To do this, it suffices just to verify that these maps are well-defined. However, this is a simple check using the fact that F_1 and F_2 are morphisms of groupoids over f_1 and f_2, respectively.

Since f_1 and f_2 are transversal to each other, it follows that $M_1 \times_M M_2$ is an embedded submanifold of $M_1 \times M_2$. Let us now check that $G_1 \times_G G_2$ is an embedded submanifold of the product $G_1 \times G_2$.

Consider the maps defined by

$$\psi_1\colon f_1^! G \to G\,,\ (m_1, g) \mapsto g\,; \quad \psi_2\colon f_2^! G \to G\,,\ (m_2, g) \mapsto g.$$

It is straightforward to verify that these maps are transversal to each other, again using the fact that f_1 and f_2 are transversal to each other. Thus, we get an embedded submanifold $S := (\psi_1 \times \psi_2)^{-1}(\Delta_G)$ of $f_1^! G \times f_2^! G$.

[3]For the notion of a groupoid object in a category, see for example [41, Section XII.1]

[4]This result was given for fibrations by Brown and Mackenzie [7, Proposition 1.2]. Here, we strengthen the result to include partial fibrations.

Next, consider the map $F := F_1^! \times F_2^! \colon G_1 \times G_2 \to f_1^! G \times f_2^! G$. Since F_1 and F_2 are partial fibrations it follows that F is a submersion. Thus, $F^{-1}(S)$ is an embedded submanifold of $G_1 \times G_2$. However, one can see that $F^{-1}(S)$ is precisely $G_1 \times_G G_2$.

It remains to show that $G_1 \times_G G_2$ is a Lie groupoid on $M_1 \times_M M_2$. Let us first show that the source projection $\widetilde{\alpha}$ is a submersion.

Consider the diagonal map $f \colon M_1 \times_M M_2 \to M$, $(m_1, m_2) \mapsto f_1(m_1) = f_2(m_2)$, and the induced pullback manifold given by the diagram

$$\begin{array}{ccc} f^! G & \longrightarrow & G \\ {\scriptstyle p}\big\downarrow & & \big\downarrow{\scriptstyle \alpha} \\ M_1 \times_M M_2 & \xrightarrow{\;\;f\;\;} & M. \end{array}$$

Note, since α is a submersion, it follows that p is a submersion. We also have a diffeomorphism $\psi \colon f^! G \to S$ given by $((m_1, m_2), g) \mapsto ((m_1, g), (m_2, g))$. One can also check that the restriction of F, to domain $G_1 \times_G G_2$ and codomain S, remains a submersion. Let us denote this restriction by $\widetilde{F}$.

We can see that the source projection can be decomposed as $\widetilde{\alpha} = p \circ \psi^{-1} \circ \widetilde{F}$. Hence, $\widetilde{\alpha}$ is a submersion as it is just a composition of submersions.

Since inversion in the Cartesian product groupoid $G_1 \times G_2$ is a diffeomorphism, $\widetilde{\iota}$ is also a diffeomorphism, as it is a restriction of the domain and codomain to embedded submanifolds. Now since $\widetilde{\beta} = \widetilde{\alpha} \circ \widetilde{\iota}$, it follows that the target projection $\widetilde{\beta}$ is also a submersion.

The partial multiplication $\widetilde{\kappa}$ and the identity map $\widetilde{1}$ are smooth since they are just restrictions of smooth maps to embedded submanifolds.[5] Hence $G_1 \times_G G_2$ is a Lie groupoid; moreover, it is an embedded Lie subgroupoid of $G_1 \times G_2$. □

Consider a weak double Lie groupoid $(S; H, V; M)$. Since the double source map $(\widetilde{\alpha}_V, \widetilde{\alpha}_H) \colon S \to H \times_\alpha V$ is a submersion, it follows that both the source projections $\widetilde{\alpha}_H$ and $\widetilde{\alpha}_V$ are partial fibrations over α_H and α_V, respectively. Using the property that the inversion maps are diffeomorphisms, one can also deduce that the target projections $\widetilde{\beta}_H$ and $\widetilde{\beta}_V$ are partial fibrations over β_H and β_V, respectively. Now Proposition 1.62 implies that the domains $S *_V S$ and $S *_H S$ of the partial multiplications in S are embedded Lie subgroupoids of the Cartesian product groupoids $S \times S \rightrightarrows H \times H$ and $S \times S \rightrightarrows V \times V$ with base manifolds $H * H$ and $V * V$, respectively.

The double source map of the weak double Lie groupoid $(S; H, V; M)$ being a submersion also has another important implication. It induces another Lie groupoid structure with base M. Referred to as the *core groupoid*, it was first introduced for double Lie groupoids by Mackenzie and Brown [7, 43].

Let us first consider a general double groupoid $(S; H, V; M)$. The *core* of $(S; H, V; M)$ is defined to be the subset of S given by

$$C := \{c \in S \mid \exists m \in M \text{ s.t. } \widetilde{\alpha}_H(c) = 1_m^V,\ \widetilde{\alpha}_V(c) = 1_m^H\}.$$

Indeed, the core has a natural groupoid structure on M, with source and target projections defined by $\alpha_C := \alpha_V \circ \widetilde{\alpha}_H|_C$, and $\beta_C := \beta_V \circ \widetilde{\beta}_H|_C$, respectively. For any

[5] See for example [34, Theorem 5.27, Corollary 5.30] for proofs of these standard results.

pair $(c_2, c_1) \in C * C$, we define their product by

$$c_2 \boxbslash c_1 := \left(c_2 \boxbar \widetilde{1}^V_{\widetilde{\beta}_V(c_1)}\right) \boxminus c_1 = \left(c_2 \boxminus \widetilde{1}^H_{\widetilde{\beta}_H(c_1)}\right) \boxbar c_1.$$

The identity map is given by $1^C \colon M \to C$; $m \mapsto 1^C_m := 1^2_m$, and for any $c \in C$, we define its inverse by

$$c^{-1(C)} := c^{-1(H)} \boxminus \widetilde{1}^H_{\widetilde{\beta}_H(c)^{-1}} = c^{-1(V)} \boxbar \widetilde{1}^V_{\widetilde{\beta}_V(c)^{-1}}.$$

The checks that these structure maps are well-defined and do define a groupoid over M are straightforward and left to the reader.

Definition 1.63. Let $(S; H, V; M)$ be a double groupoid. The groupoid $C \rightrightarrows M$ described above is called the *core groupoid* of $(S; H, V; M)$.

It was stated in [7] that the core groupoid of a double Lie groupoid is in fact a Lie groupoid. We extend this result to include the core groupoids of weak double Lie groupoids.

Proposition 1.64. *Let $(S; H, V; M)$ be a weak double Lie groupoid. Then the core groupoid $C \rightrightarrows M$ has a Lie groupoid structure.*

Proof. On inspection, it should be clear that we can express the core C as the subspace $(\widetilde{\alpha}_V, \widetilde{\alpha}_H)^{-1}(1^H \times 1^V(\Delta_M))$. Hence, C is an embedded submanifold of S, using the fact that $(\widetilde{\alpha}_V, \widetilde{\alpha}_H)$ is a submersion and the identity maps are smooth embeddings.

It is not hard to verify that all the structure maps of the core groupoid are smooth. The fact that the inversion map $\iota_C \colon C \to C$ is self-inverse also implies that it is a diffeomorphism.

It remains to show that the source and target projections are submersions. Let us start with the source projection $\alpha_C \colon C \to M$. Fix $c \in C$ and let $m = \alpha_C(c)$; we need to show that the tangent map $T_c(\alpha_C) \colon T_c C \to T_m M$ is surjective. Note that the tangent space of the core at c is just given by

$$T_c C = \{X \in T_c S \mid \exists Y \in T_m M \text{ s.t. } T(\widetilde{\alpha}_H)(X) = T(1^V)(Y),\ T(\widetilde{\alpha}_V)(X) = T(1^H)(Y)\}.$$

Take $Y \in T_m M$, and consider $\widetilde{Y} := (T(1^H)(Y), T(1^V)(Y)) \in T_{(1^H_m, 1^V_m)}(H \times_\alpha V)$. Since the double source map $(\widetilde{\alpha}_V, \widetilde{\alpha}_H) \colon S \to H \times_\alpha V$ is a submersion, there exists $X \in T_c S$ such that $T(\widetilde{\alpha}_V, \widetilde{\alpha}_H)(X) = \widetilde{Y}$. That is,

$$(T(\widetilde{\alpha}_V)(X), T(\widetilde{\alpha}_H)(X)) = (T(1^H)(Y), T(1^V)(Y)).$$

Hence, it follows that $X \in T_c C$. Moreover,

$$T(\alpha_C)(X) = T(\alpha_V)(T(\widetilde{\alpha}_H)(X)) = T(\alpha_V)(T(1^V)(Y)) = Y.$$

Thus, α_C is a submersion. Since we can express the target projection as a composition of submersions, $\beta_C = \alpha_C \circ \iota_C$, it is also a submersion. □

Let us consider some examples of double Lie groupoids and their corresponding core groupoids.[6] We will see some examples of weak double Lie groupoids appearing in Chapters 3 and 4.

[6] The following are standard examples given in the literature; compare with the examples given in [63, 7, 43].

Example 1.65. Consider a smooth manifold M. Let $\Theta\colon M^4 \to M^4$ be the diffeomorphism defined by $(m, x, y, z) \mapsto (m, y, x, z)$. We can construct a double Lie groupoid $(M^4; M \times M, M \times M; M)$, whose side groupoids are given by the pair groupoid $M \times M$ on M. The vertical structure is given by the pair groupoid M^4 on $M \times M$, and the horizontal structure is defined so that its image $\Theta(M^4)$ is a pair groupoid on $M \times M$, and so that Θ is an isomorphism of groupoids over $M \times M$. The core groupoid is also isomorphic to the pair groupoid $M \times M$ on M. ⊠

Example 1.66. Let G be a Lie groupoid on a base manifold M. We can form a double Lie groupoid structure on the quadruple $(G \times G; G, M \times M; M)$. The side groupoids for which are the pair groupoid $M \times M$ on M, and the original Lie groupoid G on M. The horizontal structure is given by the Cartesian product $G \times G$ on base $M \times M$, and the vertical structure is given by the pair groupoid $G \times G$ on base G. Diagrammatically, we have

$$\begin{array}{ccc} G \times G & \rightrightarrows & M \times M \\ \downdownarrows & & \downdownarrows \\ G & \rightrightarrows & M. \end{array}$$

It is not hard to see that the core groupoid is in fact isomorphic to the original Lie groupoid $G \rightrightarrows M$. ⊠

Example 1.67. Let $(S_1; H_1, V_1; M_1)$ and $(S_2; H_2, V_2; M_2)$ be double Lie groupoids. We can give $S_1 \times S_2$ Cartesian product groupoid structures on base manifolds $H_1 \times H_2$ and $V_1 \times V_2$. Moreover, $H_1 \times H_2$ and $V_1 \times V_2$ have Cartesian product groupoid structures on the base manifold $M_1 \times M_2$, and it is straightforward to check that $(S_1 \times S_2; H_1 \times H_2, V_1 \times V_2; M_1 \times M_2)$ forms a double Lie groupoid. This is called the *Cartesian product double groupoid*. An identical construction can also be done for weak double Lie groupoids.

Example 1.68. Recall from Example 1.15 that the anchor of a Lie groupoid $G \rightrightarrows M$ is defined as the map $\chi_G = (\beta, \alpha)\colon G \to M \times M$. Let us introduce the notation $\widetilde{\chi}_G$ for the similar map $(\alpha, \beta)\colon G \to M \times M$.

Consider two Lie groupoids H and V on the same base manifold M, with anchor maps χ_H and χ_V which are transversal to each other. We can form the pullback[7] groupoid $\widetilde{\chi}_H^{!!}(V \times V)$ on base H, and also the pullback groupoid $\chi_V^{!!}(H \times H)$ on base V. We have the following pullback diagrams

$$\begin{array}{ccc} \widetilde{\chi}_H^{!!}(V \times V) & \longrightarrow & V \times V \\ \downarrow & & \downarrow {\scriptstyle \chi_{V \times V}} \\ H \times H & \xrightarrow{\widetilde{\chi}_H \times \widetilde{\chi}_H} & M^4 \end{array} \qquad \begin{array}{ccc} \chi_V^{!!}(H \times H) & \longrightarrow & H \times H \\ \downarrow & & \downarrow {\scriptstyle \chi_{H \times H}} \\ V \times V & \xrightarrow{\chi_V \times \chi_V} & M^4. \end{array}$$

Note, via the diffeomorphism $\chi_V^{!!}(H \times H) \to \widetilde{\chi}_H^{!!}(V \times V)$, $(v, g, h, u) \mapsto (g, u, v, h)$, we can give $\widetilde{\chi}_H^{!!}(V \times V)$ a Lie groupoid structure with base V. We denote $\widetilde{\chi}_H^{!!}(V \times V)$ with these two groupoid structures by $\square(H, V)$. It is straightforward to check that

[7]See Section 1.1.4 for the construction of a pullback groupoid.

this forms a double Lie groupoid,

$$\begin{array}{ccc} \square(H,V) & \rightrightarrows & V \\ \downdownarrows & & \downdownarrows \\ H & \rightrightarrows & M. \end{array}$$

As a manifold, the core is diffeomorphic to the pullback of χ_H and χ_V. It is an embedded Lie subgroupoid of the Cartesian product groupoid $H \times V$ via inclusion over the diagonal map. $\boxtimes$

We finish this subsection by giving the notion of a morphism for these double structures.

Definition 1.69. Let $(S; H, V, M)$ and $(S'; H', V'; M')$ be double groupoids with maps $F\colon S' \to S$, $F_H\colon H' \to H$, $F_V\colon V' \to V$ and $f\colon M' \to M$. We say that $(F; F_H, F_V; f)\colon (S'; H', V'; M') \to (S; H, V, M)$ is a *morphism of double groupoids* if the four pairs (F, F_H), (F, F_V), (F_H, f) and (F_V, f) are all morphisms of groupoids.

When, in addition, $(S; H, V, M)$ and $(S'; H', V'; M')$ are weak double Lie groupoids and the maps F, F_H, F_V and f are all smooth, we say that $(F; F_H, F_V; f)$ is a *morphism of double Lie groupoids*. If F is also a diffeomorphism then we call $(F; F_H, F_V; f)$ an *isomorphism of double Lie groupoids.*

1.3.2 $\mathcal{VB}$-groupoids

In this subsection, we consider the groupoid objects in the category of vector bundles.

Definition 1.70. A *$\mathcal{VB}$-groupoid* is a quadruple of smooth manifolds $(\Omega; A, G; M)$, such that Ω and G are Lie groupoids with bases A and M, respectively, and Ω and A are vector bundles over G and M, respectively, such that the structure maps of the Lie groupoid $\Omega \rightrightarrows A$ are vector bundle morphisms over the corresponding structure maps of $G \rightrightarrows M$.

$$\begin{array}{ccc} \Omega & \xrightarrow{\widetilde{q}} & G \\ {\scriptstyle \widetilde{\alpha},\widetilde{\beta}}\downdownarrows & & \downdownarrows{\scriptstyle \alpha,\beta} \\ A & \xrightarrow{q} & M. \end{array} \tag{1.21}$$

The observant reader should spot that we have made no assumption that the double source map $(\widetilde{q}, \widetilde{\alpha})\colon \Omega \to G \times_M A$ is a surjective submersion, as we did for double Lie groupoids. However, the following result of Li-Bland and Ševera shows that this property is still upheld.

Proposition 1.71 ([35, Appendix A., Lemma 2])**.** *Let $(\Omega; A, G; M)$ be a $\mathcal{VB}$-groupoid, then the double source map $(\widetilde{q}, \widetilde{\alpha})\colon \Omega \to G \times_M A$ is a surjective submersion.* $\square$

As an immediate consequence, we see that every $\mathcal{VB}$-groupoid is an example of a double Lie groupoid.

We define the *core* of a $\mathcal{VB}$-groupoid $(\Omega; A, G; M)$ to be the subset of Ω given by

$$K \coloneqq \{k \in \Omega \mid \exists m \in M \text{ s.t. } \widetilde{\alpha}(k) = 0_m,\ \widetilde{q}(k) = 1_m\}.$$

A secondary consequence of the previous proposition is that $K = (\widetilde{q}, \widetilde{\alpha})^{-1}(1 \times 0(\Delta_M))$ is an embedded submanifold of Ω. Moreover, the core K becomes a vector bundle over M by restricting the structure maps of the vector bundle $\Omega \to G$. We call $K \to M$ the *core vector bundle* of the $\mathcal{VB}$-groupoid.

Example 1.72. Let G be a Lie groupoid with base manifold M. Consider the tangent prolongation groupoid of $G \rightrightarrows M$ defined in Example 1.7. Recall that the structure maps of $TG \rightrightarrows TM$ are obtained by applying the tangent functor to each of the structure maps of G, and hence, in particular, they are all vector bundle morphisms.

$$\begin{array}{ccc} TG & \longrightarrow & G \\ {\scriptstyle T(\alpha),T(\beta)}\Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow{\scriptstyle \alpha,\beta} \\ TM & \longrightarrow & M \end{array}$$

Thus, $(TG; TM, G; M)$ is an example of a $\mathcal{VB}$-groupoid. The core vector bundle is given by $AG \to M$. ⊠

Example 1.73. We now generalise the construction of Example 1.9. Let G be a Lie groupoid on base M. We will construct a Lie groupoid structure on T^*G with base A^*G, the source and target projections for which are defined in the following way. Take $\varphi \in T^*_gG$ and define $\widetilde{\alpha}(\varphi) \in A^*_{\alpha(g)}G$ and $\widetilde{\beta}(\varphi) \in A^*_{\beta(g)}G$ by

$$\langle \widetilde{\alpha}(\varphi), X \rangle = \langle \varphi, T(L_g)(X - T(1)(a(X))) \rangle,$$

$$\langle \widetilde{\beta}(\varphi), Y \rangle = \langle \varphi, T(R_g)(Y) \rangle,$$

for $X \in A_{\alpha(g)}G$ and $Y \in A_{\beta(g)}G$. To define the product of $\varphi_2 \in T^*_hG$ and $\varphi \in T^*_gG$ such that $\widetilde{\alpha}(\varphi_2) = \widetilde{\beta}(\varphi_2)$, we first observe that, since the partial multiplication of G is a submersion, for any $Z \in T_{hg}G$, there exists $X \in T_hG$ and $Y \in T_gG$ with $T(\kappa)(X, Y) = Z$. We define

$$\langle \varphi_2\varphi_1, Z \rangle = \langle \varphi_2, X \rangle + \langle \varphi_1, Y \rangle.$$

To define the identity element of $\psi \in A^*_mG$, we first make the observation that any $Z \in T_{1_m}G$ can be written as $Z = T(1)(x) + X$ for some $x \in T_mM$ and $X \in A_mG$. We then define $\widetilde{1}_\psi \in T^*_{1_m}G$ by

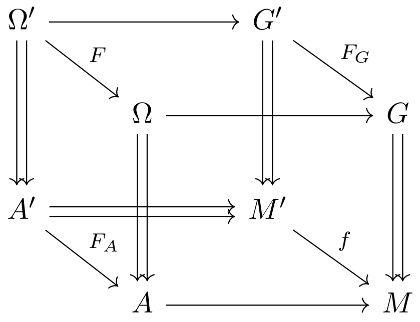
$$\langle \widetilde{1}_\psi, Z \rangle = \langle \psi, X \rangle.$$

One can check that this defines a Lie groupoid structure $T^*G \rightrightarrows A^*G$. Furthermore, this structure actually gives rise to a $\mathcal{VB}$-groupoid $(T^*G; A^*G, G; M)$. We call this the *dual* $\mathcal{VB}$-groupoid[8] of $(TG; TM, G; M)$. The core vector bundle of $(T^*G; A^*G, G; M)$ is given by the tangent bundle $TM \to M$. ⊠

We now briefly define morphisms for $\mathcal{VB}$-groupoids.

[8] There exists an abstract notion of dual $\mathcal{VB}$-groupoid; see [44, §1] for more details.

Definition 1.74. Let $(\Omega; A, G; M)$ and $(\Omega'; A', G'; M')$ be $\mathcal{VB}$-groupoids with maps $F\colon \Omega' \to \Omega$, $F_A\colon A' \to A$, $F_G\colon G' \to G$ and $f\colon M' \to M$. We say that $(F; F_A, F_G; f)$ is a *morphism of $\mathcal{VB}$-groupoids* if (F, F_A) and (F_G, f) are morphisms of Lie groupoids, and (F, F_G) and (F_A, f) are vector bundle morphisms.



1.3.3 $\mathcal{LA}$-groupoids

In this final subsection, we introduce the groupoid objects in the category of Lie algebroids.

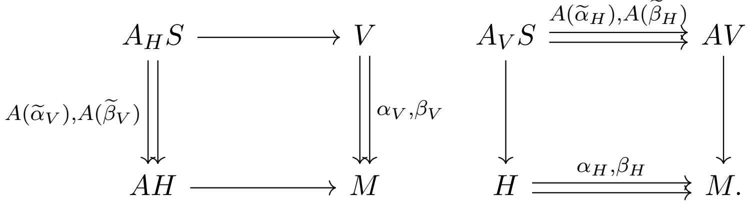
Definition 1.75. An *$\mathcal{LA}$-groupoid* is a $\mathcal{VB}$-groupoid $(\Omega; A, G; M)$ such that both $\Omega \to G$ and $A \to M$ are Lie algebroids, and the structure maps of the Lie groupoid $\Omega \rightrightarrows A$ are morphisms of Lie algebroids over the corresponding structure maps of the Lie groupoid $G \rightrightarrows M$.

Example 1.76. Let G be a Lie groupoid with base manifold M. Consider the $\mathcal{VB}$-groupoid $(TG; TM, G; M)$ of Example 1.72. Recall that the structure maps of the tangent prolongation groupoid $TG \rightrightarrows TM$ are obtained by applying the tangent functor to each of the structure maps of $G \rightrightarrows M$. Hence, by Proposition 1.48, all of these structure maps are morphisms of Lie algebroids over the corresponding structure maps of $G \rightrightarrows M$. Thus, $(TG; TM, G; M)$ is an $\mathcal{LA}$-groupoid. ⊠

Example 1.77. Let $(S; H, V; M)$ be a double Lie groupoid. We can consider the Lie algebroid $A_H S \to V$ of the horizontal structure $S \rightrightarrows V$, and the Lie algebroid $AH \to M$ of the side groupoid $H \rightrightarrows M$. By applying the Lie functor to the structure maps of the vertical structure $S \rightrightarrows H$, we produce structure maps for a Lie groupoid structure on $A_H S$ with base AH. Note, the partial multiplication for which is given by the composite

$$A_H S *_{AH} A_H S \xrightarrow{\cong} A(S *_V S) \xrightarrow{A(\widetilde{\kappa}_V)} A_H S,$$

where here we have used the property that the Lie functor preserves pullbacks. By Proposition 1.49, the structure maps of $A_H S \rightrightarrows AH$ are all morphisms of Lie algebroids over the corresponding structure maps of $V \rightrightarrows M$. Hence, $(A_H S; AH, V; M)$ is an $\mathcal{LA}$-groupoid.



Similarly, we may take the Lie algebroid $A_V S \to H$ of the vertical structure $S \rightrightarrows H$, and the Lie algebroid $AV \to M$ of the side groupoid $V \rightrightarrows M$. Then, applying the Lie functor to the structure maps of the horizontal structure $S \rightrightarrows V$ gives structure maps for a Lie groupoid $A_V S \rightrightarrows AV$. An analogous argument shows that $(A_V S; AV, H; M)$ is an $\mathcal{LA}$-groupoid. ⊠

We finish by introducing the corresponding notion of a morphism.

Definition 1.78. Let $(\Omega; A, G; M)$ and $(\Omega'; A', G'; M')$ be $\mathcal{LA}$-groupoids with maps $F\colon \Omega' \to \Omega$, $F_A\colon A' \to A$, $F_G\colon G' \to G$ and $f\colon M' \to M$. We say that $(F; F_A, F_G; f)$ is a *morphism of* $\mathcal{LA}$*-groupoids* if (F, F_A) and (F_G, f) are morphisms of Lie groupoids, and (F, F_G) and (F_A, f) are morphisms of Lie algebroids.

Chapter 2

Poisson geometry

In this chapter, we give an overview of the key topics in Poisson geometry. After providing a standard treatment of the theory, we go on to outline the prevalence of Lie groupoids and Lie algebroids within the field. We also describe the role that double Lie structures play within Poisson and symplectic geometry.

§ 2.1 Poisson structures

In this section, we will present preliminaries on Poisson manifolds, coisotropic submanifolds, Poisson Lie groups, Lie bialgebras and Manin triples.

2.1.1 Poisson manifolds

Let us begin with a discussion of the primary objects of Poisson geometry.

Definition 2.1. A *Poisson manifold* is a smooth manifold P endowed with a Lie algebra structure on its space of functions $\mathcal{C}^\infty(P)$ such that the bracket satisfies the following property:

$$\{f, gh\} = g\{f, h\} + h\{f, g\}, \tag{2.1}$$

for all $f, g, h \in C^\infty(M)$. We call a Lie bracket satisfying this condition a *Poisson bracket.*

It follows from condition (2.1), that for any function $f \in \mathcal{C}^\infty(P)$ on a Poisson manifold P, the linear operator $\{f, \cdot\}\colon \mathcal{C}^\infty(P) \to \mathcal{C}^\infty(P)$ is a derivation. Thus, this derivation defines a vector field on P which we denote by X_f. Explicitly, this vector field is defined by

$$X_f(g) = \{f, g\},$$

for all $g \in \mathcal{C}^\infty(P)$. We refer to vector fields on a Poisson manifold of this form as *Hamiltonian vector fields.*

Proposition 2.2. *The map $\mathcal{C}^\infty(P) \to \mathfrak{X}(P)$, $f \mapsto X_f$, is a Lie algebra homomorphism.*

Proof. Firstly note that the linearity of the map follows immediately from the bilinearity of the Poisson bracket. Now for any $f, g, h \in \mathcal{C}^\infty(P)$, the Jacobi identity for the Poisson bracket implies that

$$\begin{aligned}[X_f, X_g](h) &= X_f(X_g(h)) - X_g(X_f(h)) \\ &= \{f, \{g, h\}\} - \{g, \{f, h\}\} \\ &= \{\{f, g\}, h\} \\ &= X_{\{f,g\}}(h).\end{aligned}$$

□

Let (x^i) be local coordinates on a given Poisson manifold P. We can construct a (rough) contravariant 2-tensor field $\pi\colon P \to TP \otimes TP$, defined locally by

$$\pi = \pi^{ij} \frac{\partial}{\partial x^i} \otimes \frac{\partial}{\partial x^j},$$

where the component functions are given by $\pi^{ij} = \{x^i, x^j\}$. Since each component function of π is smooth in every smooth coordinate chart, it follows that π is smooth. Since the bracket is anti-symmetric, each of the component functions satisfies the property that $\pi^{ij} = -\pi^{ji}$, and hence π is alternating. Alternating contravariant 2-tensor fields are often referred to simply as *bivector fields*. Due to this, one will often see π referred to as the *Poisson bivector* of P.

One may wonder whether the Jacobi identity for the bracket can be expressed in a form involving only this bivector field. To give a satisfactory answer to this query, we need to introduce some more machinery.

Recall that a *k-multivector field* on a smooth manifold M, is a smooth section of the tensor bundle $\bigwedge^k TM$. We will use the notation $\Omega_k(M) = \Gamma(\bigwedge^k TM)$ to denote this space of sections. Note that a Poisson bivector π, as described above, is a 2-multivector field.

Those with any interest in differential geometry will have undoubtedly come across the Lie derivative – an operator $\mathcal{L}_X\colon \Omega_k(M) \to \Omega_k(M)$ defined for any vector field $X \in \mathfrak{X}(M)$. It provides a natural extension of the usual bracket of vector fields; indeed $\mathcal{L}_X Y = [X, Y]$, for any $X, Y \in \mathfrak{X}(M)$. In fact, it is possible to formulate a further extension to give a bracket of multivector fields.

Theorem 2.3. [1] *Let M be a smooth manifold. There exists a unique local type extension of the Lie derivative to a biderivation, $[\![\cdot\,,\cdot]\!]\colon \Omega_k(M)\times\Omega_l(M) \to \Omega_{k+l-1}(M)$, such that*

$$[\![X_1 \wedge \cdots \wedge X_k, V]\!] = \sum_{i=1}^{k} (-1)^{i+1} X_1 \wedge \cdots \wedge \hat{X}_i \wedge \cdots \wedge X_k \wedge \mathcal{L}_{X_i} V, \tag{2.2}$$

for all $X_i \in \mathfrak{X}(M)$, $V \in \Omega_\bullet(M)$. □

This biderivation is called the *Schouten bracket* (also sometimes referred to as the *Schouten-Nijenhuis bracket*). It has the following properties:

(i) $[\![A, B]\!] = (-1)^{ab}[\![B, A]\!]$,

(ii) $[\![A, B \wedge C]\!] = [\![A, B]\!] \wedge C + (-1)^{(a+1)b} B \wedge [\![A, C]\!]$,

(iii) $(-1)^{ac}[\![[\![A, B]\!], C]\!] + (-1)^{ba}[\![[\![B, C]\!], A]\!] + (-1)^{cb}[\![[\![C, A]\!], B]\!] = 0$,

[1] A proof can be found in [60, Theorem 1.1].

for all $A \in \Omega_a(M)$, $B \in \Omega_b(M)$, $C \in \Omega_c(M)$. One can think of this third property as a graded version of the Jacobi identity. We can also deduce the following property,[2] which gives a relationship for when a differential form is paired with the Schouten bracket of two multivector fields:

(iv) $i_{[\![A,B]\!]}\omega = (-1)^{(a+1)b} i_A d(i_B\omega) + (-1)^a i_B d(i_A\omega) - i_{A\wedge B} d\omega,$

for $A \in \Omega_a(M)$, $B \in \Omega_b(M)$, and $\omega \in \Omega^{a+b-1}(M)$.

Returning to the case of a Poisson manifold P, it should be clear from the construction of the bivector field π that the corresponding Poisson bracket can be expressed in the form $\{f, g\} = \pi(df, dg)$, for $f, g \in \mathcal{C}^\infty(P)$. Moreover, with a little extra work, one can now show that the Jacobi identity for the Poisson bracket implies that

$$[\![\pi, \pi]\!] = 0. \tag{2.3}$$

We omit the computation here;[3] instead we prefer to give a proof of the converse result:

Proposition 2.4. *Let π be a bivector field on a smooth manifold P, satisfying $[\![\pi, \pi]\!] = 0$. Then P is a Poisson manifold with Poisson bracket defined by*

$$\{f, g\} = \pi(df, dg),$$

for any $f, g \in \mathcal{C}^\infty(P)$.

Proof. The anti-symmetry of the bracket and property (2.1) follow immediately from the fact π is a bivector field. It remains to show that the bracket satisfies the Jacobi identity.

Let $\omega = df \wedge dg \wedge dh$, for some functions $f, g, h \in \mathcal{C}^\infty(P)$. By property (iv) above, we have

$$i_{[\![\pi,\pi]\!]}\omega = (-1)^6 i_\pi d(i_\pi\omega) + (-1)^2 i_\pi d(i_\pi\omega) - i_{\pi\wedge\pi} d\omega = 2 i_\pi d(i_\pi\omega).$$

Note that the term $i_{\pi\wedge\pi} d\omega$ vanishes as ω is a closed form. It is a straightforward computation to show that

$$i_\pi\omega = \circlearrowleft \pi(df, dg)dh = \circlearrowleft \{f, g\}dh.$$

Here, $\circlearrowleft$ denotes the sum over the circular permutations of f, g, h. Another standard computation gives

$$i_{[\![\pi,\pi]\!]}\omega = 2 i_\pi d\big(\circlearrowleft \{f, g\}dh\big) = 2 \circlearrowleft \pi(d\{f, g\}, dh) = 2 \circlearrowleft \{\{f, g\}, h\}.$$

The coefficient here will depend on how the wedge product has been defined. Now $[\![\pi, \pi]\!] = 0$ implies that $\circlearrowleft \{\{f, g\}, h\} = 0$, which is precisely the Jacobi identity. □

A bivector field on a smooth manifold that satisfies equation (2.3) is called a *Poisson tensor*. We will use the term *Poisson structure* to refer to either a Poisson bracket or a Poisson tensor. In subsequent sections, we may denote a Poisson manifold as a pair (P, π) to indicate the Poisson structure.

[2]The computation is straightforward, see [4] for further details.

[3]Full details can be found in [60, Proposition 1.4] or [47, Proposition 10.1.3].

Any Poisson tensor π on a manifold P has an associated vector bundle morphism $\pi^{\#}\colon T^*P \to TP$, defined by

$$\langle \psi, \pi^{\#}(\varphi) \rangle = \pi(\varphi, \psi),$$

for $\varphi, \psi \in T^*P$. Here, the bracket $\langle \cdot, \cdot \rangle$ denotes the natural pairing of a vector bundle and its dual – this notation will be adopted throughout. We call $\pi^{\#}$ the *Poisson anchor* of P.

Every smooth manifold can be given the *trivial Poisson structure*, where the bracket maps any two smooth functions to the zero function. Given a Poisson manifold (P, π), we can give P another Poisson structure by assigning the Poisson tensor to be $-\pi$. This is often referred to as the *opposite Poisson structure*, and we will use the short-hand notation $\overline{P}$ to indicate when we would like to consider P as a Poisson manifold with this structure. Let us now take a look at some more interesting examples.

Example 2.5. Let (M, ω) be a symplectic manifold. Nondegeneracy of the symplectic form ω implies that the vector bundle morphism $\omega^\flat\colon TM \to T^*M$, defined by

$$\langle \omega^\flat(X), Y \rangle = \omega(X, Y),$$

for $X, Y \in TM$, is an isomorphism. Consider a bivector field π with anchor defined by $\pi^{\#} = -(\omega^\flat)^{-1}$. That π is alternating follows immediately from the fact that ω is alternating. Note that, in particular, we have $\omega^\flat(X_f) = -df$.

We claim that π defines a Poisson structure on M. All that needs to be checked is that the associated bracket given by

$$\{f, g\} = \pi(df, dg) = \omega(X_f, X_g),$$

for $f, g \in \mathcal{C}^\infty(M)$, satisfies the Jacobi identity. Let $f, g, h \in \mathcal{C}^\infty(M)$, and observe that

$$\begin{aligned} d\omega(X_f, X_g, X_h) &= \circlearrowleft \Big(X_f(\omega(X_g, X_h)) - \omega([X_f, X_g], X_h) \Big) \\ &= \circlearrowleft \Big(\{f, \{g, h\}\} - \big(\{f, \{g, h\}\} - \{g, \{f, h\}\} \big) \Big) \\ &= \circlearrowleft \{g, \{f, h\}\}. \end{aligned}$$

Again, we are using $\circlearrowleft$ here to denote the sum over the circular permutations of f, g, h. However, since ω is a closed form, we have $d\omega = 0$, and thus the Jacobi identity is immediate. ⊠

Example 2.6. Let $\mathfrak{g}$ be a Lie algebra. The dual vector space $\mathfrak{g}^*$ can be given a natural Poisson structure. We first make the observation that for any $f \in \mathcal{C}^\infty(\mathfrak{g}^*)$ and $\xi \in \mathfrak{g}^*$ we have $df(\xi) \in T^*_\xi \mathfrak{g}^* \cong \mathfrak{g}$. We thus can define a bracket by

$$\{f, g\}(\xi) = \langle [df(\xi), dg(\xi)], \xi \rangle,$$

for all $f, g \in \mathcal{C}^\infty(\mathfrak{g}^*)$, $\xi \in \mathfrak{g}^*$. It is routine to check that this defines a Poisson bracket on $\mathfrak{g}^*$. This is called the *Lie-Poisson structure*. ⊠

Example 2.7. Suppose that (P, π_P) and (Q, π_Q) are Poisson manifolds. The product manifold $P \times Q$ inherits a natural Poisson structure given by $\pi_{P\times Q} = \pi_P \oplus \pi_Q$. One can easily verify that the corresponding Poisson bracket is given by

$$\{f, g\}_{P\times Q}(x, y) = \{f(\cdot, y), g(\cdot, y)\}_P(x) + \{f(x, \cdot), g(x, \cdot)\}_Q(y),$$

for $x \in P$, $y \in Q$, and $f, g \in \mathcal{C}^\infty(P \times Q)$. We call this the *product Poisson structure* on $P \times Q$. ⊠

2.1.2 Poisson maps and coisotropic submanifolds

We now consider the morphisms and sub-objects in the category of Poisson manifolds; these are called Poisson maps and Poisson submanifolds, respectively. We also discuss coisotropic submanifolds, which play a role in Poisson geometry similar to that played by Lagrangian submanifolds in symplectic geometry.

Definition 2.8. Let P and Q be Poisson manifolds. A smooth map $F\colon P \to Q$ is a *Poisson map* if it preserves the Poisson brackets. That is,

$$\{f, g\}_Q \circ F = \{f \circ F, g \circ F\}_P, \tag{2.4}$$

for all $f, g \in C^\infty(Q)$.

Alternatively, we can express this condition between the Poisson brackets as a condition between the Poisson tensors.

Proposition 2.9. *Let (P, π_P) and (Q, π_Q) be Poisson manifolds, and $F\colon P \to Q$ a smooth map. Then F is a Poisson map if and only if the Poisson tensors are F-related. That is,*

$$(T(F) \otimes T(F))(\pi_P(x)) = \pi_Q(F(x)), \tag{2.5}$$

for every $x \in P$.

Proof. Suppose that F is a Poisson map. Let $x \in P$, and observe that for any $f, g \in C^\infty(Q)$, we have

$$\begin{aligned}\langle (df \otimes dg)(F(x)), (T(F) \otimes T(F))(\pi_P(x))\rangle &= \langle (d(f \circ F) \otimes d(g \circ F))(x), \pi_P(x)\rangle \\ &= \{f \circ F, g \circ F\}_P(x) \\ &= \{f, g\}_Q(F(x)) \\ &= \langle (df \otimes dg)(F(x)), \pi_Q(F(x))\rangle.\end{aligned}$$

Now consider an arbitrary 2-covector field φ of Q. In a smooth chart containing $F(x)$, we can write φ locally as

$$\varphi = \sum_{i,j} u_{ij} dx_i \otimes dx_j,$$

where $u_{ij} \in \mathcal{C}^\infty(Q)$, and (x_i) are local coordinates on Q. Hence, by the calculation above, it follows that

$$\begin{aligned}\langle \varphi(F(x)), &(T(F) \otimes T(F))(\pi_P(x))\rangle \\ &= \sum_{i,j} u_{ij}(F(x))\langle (dx_i \otimes dx_j)(F(x)), (T(F) \otimes T(F))(\pi_P(x))\rangle \\ &= \sum_{i,j} u_{ij}(F(x))\langle (dx_i \otimes dx_j)(F(x)), \pi_Q(F(x))\rangle \\ &= \langle \varphi(F(x)), \pi_Q(F(x))\rangle.\end{aligned}$$

Thus, equation (2.5) holds. The converse is even simpler to show. □

Remark 2.10. Note that the identity (2.5) can also be expressed in terms of the Poisson anchors. Then the previous result can be restated as F is Poisson if and only if

$$T(F) \circ \pi_P^\#(F^*\varphi) = \pi_Q^\#(\varphi) \circ F,$$

for all $\varphi \in \Omega^1(Q)$.

As a consequence of this result, and from the definition of the product Poisson structure seen in Example 2.7, we arrive at the following:

Proposition 2.11. *Suppose* (P, π_P), (Q, π_Q), *and* (M, π_M) *are Poisson manifolds, and* $F\colon P \times Q \to M$ *is a smooth map. Then, with* $P \times Q$ *equipped with the product Poisson structure,* F *is a Poisson map if and only if*

$$\pi_M(F(x, y)) = T(F_x)(\pi_Q(y)) + T(F_y)(\pi_P(x)),$$

for all $x \in P$, $y \in Q$. *Here, we define* $F_x\colon Q \to M$ *and* $F_y\colon P \to M$ *by the relations* $F_x(y) = F_y(x) = F(x, y)$, *for* $x \in P$, $y \in Q$. □

We also have another criterion for a Poisson map involving the Hamiltonian vector fields.

Proposition 2.12. *Let* P *and* Q *be Poisson manifolds, and* $F\colon P \to Q$ *a smooth map. Then* F *is a Poisson map if and only if*

$$T(F) \circ X_{f \circ F} = X_f \circ F$$

for every $f \in \mathcal{C}^\infty(Q)$.

Proof. Take any $f, g \in \mathcal{C}^\infty(Q)$ and $x \in Q$, and observe that

$$(T(F) \circ X_{f \circ F}(x))(g) = X_{f \circ F}(x)(g \circ F) = \{f \circ F, g \circ F\}(x).$$

On the other hand,

$$(X_f \circ F)(x)(g) = X_f(F(x))(g) = \{f, g\}(F(x)).$$

Hence,

$$T(F) \circ X_{f \circ F} = X_f \circ F,$$

for every $f \in \mathcal{C}^\infty(Q)$, if and only if

$$\{f \circ F, g \circ F\} = \{f, g\} \circ F,$$

for every $f, g \in \mathcal{C}^\infty(Q)$. □

The following class of Poisson maps are useful for lifting trajectories of Hamiltonian vector fields in a Poisson manifold. We only provide a definition here, further details can be found in [8, Section 6.2].

Definition 2.13. A Poisson map $F\colon P \to Q$ is *complete* if whenever the Hamiltonian vector field X_f is complete, for $f \in \mathcal{C}^\infty(Q)$, the Hamiltonian vector field $X_{f \circ F}$ is also complete.

Another useful notion for smooth maps between Poisson manifolds is the following:

Definition 2.14. Let P and Q be Poisson manifolds, and $F\colon P \to Q$ a smooth map. We say that F is an *anti-Poisson map* if $F\colon P \to \overline{Q}$ is a Poisson map.

We now begin a brief discussion of two important classes of submanifolds of Poisson manifolds.

Definition 2.15. Let P be a Poisson manifold. We say that S is a *Poisson submanifold of* P if S is a submanifold of P that is equipped with a Poisson structure such that the inclusion map is a Poisson map.

It turns out that the submanifolds in Poisson geometry that perform the most crucial role are those which we coin *coisotropic*. They have a similar function in Poisson geometry to what Lagrangian submanifolds have in symplectic geometry.

Definition 2.16. Let (P, π) be a Poisson manifold and C an embedded submanifold of P. We call C a *coisotropic submanifold of* P if

$$\pi^{\#}((TC)^{\circ}) \subseteq TC.$$

Here, $(TC)^{\circ}$ is the *conormal bundle of* C *in* P given by

$$(TC)^{\circ} = \{\varphi \in T_x^*P \mid x \in C\,,\ \langle \varphi, T_x C\rangle = 0\}.$$

Example 2.17. Let (P, π) be a Poisson manifold. Consider the embedded submanifold Δ_P of the product manifold $P \times P$. Recall that $T(\Delta_P) = \Delta_{TP}$, and so the conormal bundle of Δ_P in $P \times P$ is given by

$$T(\Delta_P)^{\circ} = \{(\varphi, \psi) \in T_x^*P \times T_x^*P \mid \langle (\varphi, \psi), (X, X)\rangle = 0\,,\ \forall X \in T_x P,\, x \in P\}.$$

Thus, $(\varphi, \psi) \in T_{(x,x)}(\Delta_P)^{\circ}$ if and only if

$$\langle \varphi, X\rangle + \langle \psi, X\rangle = 0,$$

for all $X \in T_x P$. This holds if and only if $\varphi = -\psi$. However, in this scenario

$$\left(\pi^{\#} \oplus (-\pi^{\#})\right)(\varphi, \psi) = (\pi^{\#}(\varphi), \pi^{\#}(\varphi)) \in T(\Delta_P).$$

Hence, we have proved that Δ_P is a coisotropic submanifold of $P \times \overline{P}$. ⊠

Proposition 2.18 ([63, Corollary 2.2.5])**.** *Let* $F\colon P \to Q$ *be a Poisson map and* C *a coisotropic submanifold of* Q*. If* F *is transverse to* C*, then* $F^{-1}(C)$ *is a coisotropic submanifold of* P*.* □

One important application of coisotropic submanifolds is that they give another useful criterion for a map to be Poisson.

Proposition 2.19. *Let* $F\colon P \to Q$ *be a smooth map of Poisson manifolds. Then* F *is a Poisson map if and only if the graph of* F *is a coisotropic submanifold of* $P \times \overline{Q}$*.*

Proof. Let us denote the graph of F by C. The conormal bundle of C in $P \times Q$ is given by

$$(TC)^{\circ} = \{(\varphi, \psi) \in T_x^*P \times T_{F(x)}^*Q \mid \langle (\varphi, \psi), T_{(x,F(x))}C\rangle = 0\,,\ x \in P\}.$$

Recall that $TC = \mathrm{Gr}(T(F))$. Hence, $(\varphi, \psi) \in (T_{(x,F(x))}C)^{\circ}$ if and only if

$$\langle (\varphi, \psi), (X, T(F)(X))\rangle = 0,$$

for all $X \in T_x P$. This is equivalent to the relation $\varphi + T_x^*(F)(\psi) = 0$.

Hence, C is coisotropic in $P \times \overline{Q}$ if and only if

$$\left(\pi_P^{\#}(F^*\psi)(x), -\pi_Q^{\#}(-\psi)(F(x))\right) \in TC,$$

for all $\psi \in \Omega^1(Q)$, $x \in P$. Which holds true if and only if

$$T(F) \circ \pi_P^{\#}(F^*\psi) = \pi_Q^{\#}(\psi) \circ F,$$

for all $\psi \in \Omega^1(Q)$. That is, if and only if F is a Poisson map. □

Proposition 2.20 ([37, Lemma 3.33])**.** *Let $F\colon (P, \pi_P) \to (Q, \pi_Q)$ be a Poisson map, and C a coisotropic submanifold of P. Then the graph of $F|_C : C \to Q$,*

$$\Gamma(F|_C) = \{(x, F(x)) \mid x \in C\},$$

is a coisotropic submanifold of the Poisson manifold $P \times \overline{Q}$ if and only if

$$\pi_P^{\#}((TC)^\circ) \subseteq \ker(T(F)).$$

Proof. By definition, the conormal bundle of $\Gamma(F|_C)$ in $P \times Q$ is given by

$$T(\Gamma(F|_C))^\circ = \left\{(\theta, \psi) \in T_x^*P \times T_{F(x)}^*Q \mid x \in C, \langle \theta \oplus \psi, T_x(\Gamma(F|_C))\rangle = 0\right\}.$$

Note that, given $x \in C$, $(\theta, \psi) \in T_{(x,F(x))}(\Gamma(F|_C))^\circ$, and an arbitrary $X \in T_xC$, we have

$$\langle \theta \oplus \psi, X \oplus T_x(F)(X)\rangle = \langle \theta + T_x^*(F)(\psi), X\rangle.$$

By setting $\varphi = \theta + T_x^*(F)(\psi)$, the above equality implies that the condition

$$\langle \theta \oplus \psi, T_x(\Gamma(F|_C))\rangle = 0$$

is equivalent to the condition that $\varphi \in (T_xC)^\circ$. Hence, the conormal bundle can be described as

$$T(\Gamma(F|_C))^\circ = \left\{(\varphi - T_x^*(F)(\psi), \psi) \mid x \in C, \varphi \in (T_xC)^\circ, \psi \in T_{F(x)}^*Q\right\}.$$

Therefore, $\Gamma(F|_C)$ is coisotropic in $P \times \overline{Q}$ if and only if

$$T_x(F)\left(\pi_P^{\#}(\varphi - T_x^*(F)(\psi))\right) = -\pi_Q^{\#}(\psi), \tag{2.6}$$

for all $x \in C$, $\varphi \in (T_xC)^\circ$, and $\psi \in T_{F(x)}^*Q$. Note that we can write this equation as

$$T_x(F)(\pi_P^{\#}(\varphi)) - T_x(F) \circ \pi_P^{\#} \circ T_x^*(F)(\psi) = -\pi_Q^{\#}(\psi).$$

Since F is Poisson, we have

$$T_x(F) \circ \pi_P^{\#} \circ T_x^*(F) = \pi_Q^{\#},$$

for all $x \in P$. Thus, the condition (2.6) holds if and only if

$$T(F)(\pi_P^{\#}(\varphi)) = 0,$$

for all $\varphi \in (TC)^\circ$. That is, if and only if

$$\pi_P^{\#}((TC)^\circ) \subseteq \ker(T(F)).$$ □

2.1.3 Poisson Lie groups

In this subsection, we will look at Poisson manifolds which also have a Lie group structure. In particular, we will focus on Poisson Lie groups. They are geometric objects of this type, which also come with a compatibility property between their Poisson and group structures.

Before giving a definition of a Poisson Lie group, we first introduce some results about multivector fields on a Lie group. If $F\colon M \to N$ is a smooth map of manifolds, and V a k-multivector field on M, then to simplify notation we will write $T(F)(V(x))$ in place of $(\otimes^k T(F))(V(x))$, for $x \in M$.

Definition 2.21. We call a multivector field V on a Lie group G *multiplicative* if for every $g, h \in G$,

$$V(hg) = T(L_h)(V(g)) + T(R_g)(V(h)). \tag{2.7}$$

Note, an immediate consequence of this is that $V(e) = 0$, where e is the identity element of G.

Remark 2.22. Let G be a Lie group with Lie algebra $\mathfrak{g}$. Given any vector $V_e \in \bigwedge^k \mathfrak{g}$, we can construct a multiplicative k-multivector field V by defining

$$V(g) = T(L_g)(V_e) - T(R_g)(V_e),$$

for every $g \in G$.

Given any k-multivector field V on a Lie group G, we can define the following map,

$$V^R\colon G \to \bigwedge^k \mathfrak{g}\,,\ V^R(g) = T(R_{g^{-1}})(V(g)),$$

where $\mathfrak{g}$ denotes the Lie algebra of G. When V is multiplicative, it is natural to try to express equation (2.7) in terms of V^R. This leads to the following alternative criterion for when a multivector field is multiplicative:

Proposition 2.23. *A k-multivector field V on a Lie group G is multiplicative if and only if*

$$V^R(hg) = V^R(h) + \mathrm{Ad}_h(V^R(g)), \tag{2.8}$$

for all $g, h \in G$. □

Remark 2.24. When V is a bivector field on a Lie group G, equation (2.8) is precisely the condition that V^R is a 1-cocycle[4] on G with respect to the adjoint representation of G on $\mathfrak{g} \wedge \mathfrak{g}$.

The next result gives an insight into the behaviour of multiplicative multivector fields when they have been operated on by the Lie derivative. It gives yet another criterion for the multiplicity of a multivector field.

Proposition 2.25. *Let V be a multivector field on a connected Lie group G. Then V is multiplicative if and only if $V(e) = 0$ and $\mathcal{L}_X V$ is left-invariant for every left-invariant vector field X on G.*

[4] An exposition of the cohomology theories of Lie groups and Lie algebras can be found in the appendix of [36].

Proof. Suppose that V is multiplicative. We have already seen this immediately implies that $V(e) = 0$. Let $\bar{X}$ be a left-invariant vector field on G with flow θ_t, and $X = \bar{X}(e)$. The left-invariance of $\bar{X}$ implies that $\theta_t = R_{\exp tX}$. Observe that, for any $g \in G$,

$$\begin{aligned}
\mathcal{L}_{\bar{X}} V(g) &= \left.\frac{d}{dt}\right|_{t=0} T(\theta_{-t})(V(\theta_t(g))) \\
&= \left.\frac{d}{dt}\right|_{t=0} T(R_{\exp(-tX)})(V(g \exp tX)) \\
&= \left.\frac{d}{dt}\right|_{t=0} T(R_{\exp(-tX)})(T(L_g)(V(\exp tX)) + T(R_{\exp tX})(V(g))) \\
&= \left.\frac{d}{dt}\right|_{t=0} T(R_{\exp(-tX)} \circ L_g)(V(\exp tX)).
\end{aligned}$$

Hence, for any $g, h \in G$, we have

$$\begin{aligned}
\mathcal{L}_{\bar{X}} V(hg) &= \left.\frac{d}{dt}\right|_{t=0} T(R_{\exp(-tX)} \circ L_{hg})(V(\exp tX)) \\
&= T(L_h)\left(\left.\frac{d}{dt}\right|_{t=0} T(R_{\exp(-tX)} \circ L_g)(V(\exp tX)) \right) \\
&= T(L_h)(\mathcal{L}_{\bar{X}} V(g)).
\end{aligned}$$

Hence, $\mathcal{L}_{\bar{X}} V$ is left-invariant. We omit the proof of the converse statement here, but the reader can consult [60, Proposition 10.5] for the details. □

Remark 2.26. A k-multivector field V on a Lie group G also gives rise to a map $V^L : G \to \bigwedge^k \mathfrak{g}$, defined by $V^L(g) = T(L_{g^{-1}})(V(g))$. There is a corresponding version of Proposition 2.23 for V^L. That is, V is multiplicative if and only if

$$V^L(hg) = V^L(g) + \mathrm{Ad}_{g^{-1}}(V^L(h)), \tag{2.9}$$

for all $h, g \in G$.

In Proposition 2.25, we saw that the Lie derivative preserves left-invariance when it operates on a multiplicative multivector field. An analogous result holds for right-invariance. The statement is given by replacing the term 'left-invariant' by 'right-invariant' whenever it occurs in Proposition 2.25.

We now collect some further properties of multivector fields given by the Schouten bracket of two multivector fields.

Proposition 2.27. *Let V and W be multivector fields on a Lie group G. Suppose that V is left-invariant and W is right-invariant, then $[\![V, W]\!] = 0$.*

Proof. Let $\bar{X}$ be a left-invariant vector field on G with $X = \bar{X}(e)$, and W any right-invariant multivector field on G. Then, for any $g \in G$, we have

$$\begin{aligned}
[\![\bar{X}, W]\!](g) &= \mathcal{L}_{\bar{X}} W(g) \\
&= \left.\frac{d}{dt}\right|_{t=0} T(R_{\exp(-tX)})(W(g \exp tX)) \\
&= \left.\frac{d}{dt}\right|_{t=0} T(R_{\exp(-tX)})(T(R_{\exp tX})(W(g))) \\
&= \left.\frac{d}{dt}\right|_{t=0} W(g) \\
&= 0.
\end{aligned}$$

In the general case, when V is any left-invariant multivector field on G we use the computation above, and property (2.2) of the Schouten bracket. □

In fact, in the case of a connected Lie group, we also have a reverse statement.

Proposition 2.28. [5] *Let V be a multivector field on a connected Lie group G. Then V is left-invariant if and only if $\mathcal{L}_Y V = 0$ for every right-invariant $Y \in \mathfrak{X}(G)$.* □

We also have an analogous result with the terms 'left-invariant' and 'right-invariant' interchanged.

Proposition 2.29. *Suppose that V and W are multiplicative multivector fields on a connected Lie group G. Then $[\![V, W]\!]$ is multiplicative.*

Proof. First note that, since V and W are multiplicative, $V(e) = 0$ and $W(e) = 0$. It follows that $[\![V, W]\!](e) = 0$. The simplest way to see this is to write down local coordinate expressions for V and W, and then use the properties of the Schouten bracket to find a local coordinate expression for $[\![V, W]\!]$. The result should then be obvious.

Now take any two vector fields $X, Y \in \mathfrak{X}(G)$. By the graded Jacobi identity for the Schouten bracket, we can deduce

$$\mathcal{L}_X [\![V, W]\!] = [\![V, \mathcal{L}_X W]\!] + [\![\mathcal{L}_X V, W]\!].$$

Iterating this result, we find that

$$\mathcal{L}_Y \mathcal{L}_X [\![V, W]\!] = [\![\mathcal{L}_Y V, \mathcal{L}_X W]\!] + [\![V, \mathcal{L}_Y \mathcal{L}_X W]\!] + [\![\mathcal{L}_Y \mathcal{L}_X V, W]\!] + [\![\mathcal{L}_X V, \mathcal{L}_Y W]\!].$$

Suppose now that X is left-invariant, and Y is right-invariant. Then since V and W are multiplicative, $\mathcal{L}_X V$ and $\mathcal{L}_X W$ are left-invariant by Proposition 2.25, and $\mathcal{L}_X V$ and $\mathcal{L}_X W$ are right-invariant by Remark 2.26. Thus, by Proposition 2.27, $[\![\mathcal{L}_Y V, \mathcal{L}_X W]\!] = 0$ and $[\![\mathcal{L}_X V, \mathcal{L}_Y W]\!] = 0$.

Moreover, since G is connected, $\mathcal{L}_Y \mathcal{L}_X W = 0$ and $\mathcal{L}_Y \mathcal{L}_X V = 0$ by Proposition 2.28. Hence, we can conclude that

$$\mathcal{L}_Y \mathcal{L}_X [\![V, W]\!] = 0.$$

This holds true for every right-invariant vector field Y, and so by Proposition 2.28 again $\mathcal{L}_X [\![V, W]\!]$ must be left-invariant for every left-invariant vector field X. This property, coupled with the fact $[\![V, W]\!](e) = 0$, is enough to conclude the multiplicity of $[\![V, W]\!]$ by Proposition 2.25. □

Let us now turn our attention towards Poisson Lie groups. We will see that they are a particular class of Lie groups which come endowed with a multiplicative multivector field.

Definition 2.30. A *Poisson Lie group* is a Lie group G with a Poisson structure such that the multiplication in the group $\kappa \colon G \times G \to G$ is a Poisson map.

For completeness, let us briefly define the morphisms and sub-objects in the category of Poisson Lie groups.

[5] When V is just a vector field, we are reduced to the standard result in Lie group theory. The proof of this proposition can be produced in a similar manner.

Definition 2.31. A *morphism of Poisson Lie groups* is a Lie group homomorphism $\phi\colon G \to H$ between Poisson Lie groups that is also a Poisson map.

A *Poisson Lie subgroup* is a Lie subgroup S of a Poisson Lie group G, endowed with a Poisson structure that also makes it a Poisson submanifold of G.

In light of Proposition 2.11, for any Poisson Lie group (G, π), the condition that the multiplication is a Poisson map is equivalent to the condition that

$$\pi(hg) = T(L_h)(\pi(g)) + T(R_g)(\pi(h)),$$

for all $g, h \in G$. This gives us the following criterion for a Poisson Lie group.

Proposition 2.32. *Let G be a Lie group with a Poisson structure π. Then G is a Poisson Lie group if and only if π is multiplicative.* □

The simplest example of a Poisson Lie group is any Lie group equipped with the trivial Poisson structure (see page 34). Let us briefly discuss some more interesting examples.

Example 2.33. Let $\mathfrak{g}$ be a Lie algebra. We saw that the dual vector space $\mathfrak{g}^*$ obtains the Lie-Poisson structure in Example 2.6. In fact, with this Poisson structure $\mathfrak{g}^*$ becomes a Poisson Lie group. ⊠

Example 2.34. Consider the abelian Lie group $\mathbb{R}^2$ with global coordinates (x, y). Note that any Poisson structure on $\mathbb{R}^2$ is completely determined by $\{x, y\}$. It can be shown that such a Poisson structure is multiplicative if and only if

$$\{x, y\} = ax + by,$$

for some $a, b \in \mathbb{R}$. ⊠

Example 2.35. The following is an example of a non-abelian Lie group with a non-trivial Poisson structure. Consider the matrix group

$$M = \left\{ \begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \;\middle|\; x, y \in \mathbb{R},\ x > 0 \right\}.$$

Given any fixed $\lambda \neq 0$, we can identify M with $\mathbb{R}^2$ via the diffeomorphism

$$\begin{pmatrix} x & y \\ 0 & 1 \end{pmatrix} \mapsto \left(\frac{\log(x)}{\lambda}, \frac{y}{\sqrt{x}} \right).$$

The group structure on M can be transferred over to $\mathbb{R}^2$ and is given by

$$(x_1, y_1)(x_2, y_2) = \left(x_1 + x_2, y_2 e^{\frac{\lambda x_1}{2}} + y_1 e^{-\frac{\lambda x_2}{2}} \right),$$

for any $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$. We will denote this Lie group by $\mathbb{R}^2_\lambda$. Mikami showed that all multiplicative Poisson structures on $\mathbb{R}^2_\lambda$ are determined by

$$\{x, y\} = a \sinh\left(\frac{\lambda x}{2}\right) + by,$$

for some $a, b \in \mathbb{R}$ [52, Proposition 2.6]. ⊠

Let (G, π) be a Poisson Lie group with Lie algebra $\mathfrak{g}$. Remark 2.24 tells us that π^R can be interpreted as a 1-cocycle of G. One reason for taking this perspective is that π^R actually gives rise to a 1-cocycle of the Lie algebra $\mathfrak{g}$, as we will see in what follows.

Suppose V is a k-multivector field on a smooth manifold M, which has $V(x) = 0$ for some $x \in M$. We define the *intrinsic derivative*[6] of V at x to be the map

$$d_x V\colon T_x M \to \bigwedge^k T_x M\,;\; X \mapsto \mathcal{L}_{\bar{X}} V(x),$$

where $\bar{X} \in \mathfrak{X}(M)$ is any vector field satisfying $\bar{X}(x) = X$. To prove that this map is well-defined is a standard exercise and relies on the fact that $V(x) = 0$. We call the dual map, $(d_x V)^*\colon \bigwedge^k T_x^* M \to T_x^* M$, the *linearisation of V at x*.

In the scenario where V is a multiplicative k-multivector field on a Lie group G, the intrinsic derivative of V at e can be interpreted in another way. Consider the usual derivative of V^R at e – the linear map $T_e V^R\colon \mathfrak{g} \to \bigwedge^k \mathfrak{g}$. Here $\mathfrak{g}$ denotes the Lie algebra of G. Note that given any $X \in \mathfrak{g}$, with corresponding left-invariant vector field denoted by $\bar{X}$, we have

$$\begin{aligned} T_e V^R(X) &= \frac{d}{dt}\bigg|_{t=0} V^R(\exp tX) \\ &= \frac{d}{dt}\bigg|_{t=0} T(R_{(\exp tX)^{-1}})(V(\exp tX)) \\ &= \frac{d}{dt}\bigg|_{t=0} T(R_{\exp -tX})(V(\exp tX)) \\ &= \mathcal{L}_{\bar{X}} V(e). \end{aligned}$$

We see that the usual derivative of V^R at e coincides with the intrinsic derivative of V at e. An analogous argument shows that $T_e V^L$ also coincides with $d_e V$.

Proposition 2.36. *Let V be a multiplicative bivector field on a Lie group G. Then the intrinsic derivative of V at e, $\epsilon = d_e V\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, defines a 1-cocycle of $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$. That is,*

$$\mathrm{ad}_X(\epsilon(Y)) - \mathrm{ad}_Y(\epsilon(X)) - \epsilon([X, Y]) = 0,$$

for all $X, Y \in \mathfrak{g}$.

Remark 2.37. In the equation above, we are using the slightly ambiguous, but shorter, notation ad_X, to denote the representation $\mathrm{ad}_X \otimes 1 + 1 \otimes \mathrm{ad}_X$. In this second expression, ad_X denotes the usual adjoint representation of $\mathfrak{g}$ on $\mathfrak{g}$.

Proof. Let $X, Y \in \mathfrak{g}$, with corresponding left-invariant vector fields denoted by $\bar{X}$ and $\bar{Y}$, respectively. By the definition of intrinsic derivative, it is clear that

$$\epsilon([X, Y]) = d_e V([X, Y]) = \mathcal{L}_{[\bar{X}, \bar{Y}]} V(e).$$

Since V is multiplicative, Proposition 2.25 tells us that $\mathcal{L}_{\bar{X}} V$ and $\mathcal{L}_{\bar{Y}} V$ are also left-

[6]For a deeper intuition behind this definition, the reader should consult [23, p64].

invariant. It follows that

$$\begin{aligned}\mathrm{ad}_X(\epsilon(Y)) &= \mathrm{ad}_X(\mathcal{L}_{\bar{Y}}V(e))\\ &= \frac{d}{dt}\bigg|_{t=0}\mathrm{Ad}_{\exp tX}(\mathcal{L}_{\bar{Y}}V(e))\\ &= \frac{d}{dt}\bigg|_{t=0}T(R_{\exp(-tX)})(T(L_{\exp tX})(\mathcal{L}_{\bar{Y}}V(e)))\\ &= \frac{d}{dt}\bigg|_{t=0}T(R_{\exp(-tX)})(\mathcal{L}_{\bar{Y}}V(\exp tX))\\ &= \mathcal{L}_{\bar{X}}\mathcal{L}_{\bar{Y}}V(e).\end{aligned}$$

A similar result holds for $\mathrm{ad}_Y(\epsilon(X))$. Hence,

$$\mathrm{ad}_X(\epsilon(Y)) - \mathrm{ad}_Y(\epsilon(X)) - \epsilon([X,Y]) = \mathcal{L}_{\bar{X}}\mathcal{L}_{\bar{Y}}V(e) - \mathcal{L}_{\bar{Y}}\mathcal{L}_{\bar{X}}V(e) - \mathcal{L}_{[\bar{X},\bar{Y}]}V(e) = 0.$$

The last step can be realised by utilising the graded Jacobi identity of the Schouten bracket. □

The next result gives us a useful relationship between a multiplicative multivector field and its intrinsic derivative.

Proposition 2.38. *Let V be a multiplicative multivector field on a connected Lie group G. Then $V \equiv 0$ if and only if $d_eV = 0$. Moreover, if W is another multiplicative multivector field on G, then $V \equiv W$ if and only if $d_eV = d_eW$.*

Proof. That $V \equiv 0$ implies $d_eV = 0$ is clear. To prove the converse statement, note that $d_eV = 0$ means that $\mathcal{L}_XV(e) = 0$ for every left-invariant vector field $X \in \mathfrak{X}(G)$. On the other hand, by Proposition 2.25 $\mathcal{L}_XV$ is left-invariant whenever X is left-invariant, because of the multiplicity of V. Hence $\mathcal{L}_XV \equiv 0$, for every left-invariant vector field X. However, then V must be right-invariant by Proposition 2.28. Since $V(e) = 0$, it follows that $V \equiv 0$.

To prove the second assertion, note that the difference $V - W$ is also a multiplicative multivector field on G. Now applying the first result to $V - W$ and employing linearity leads to the desired conclusion. □

Consider a multiplicative bivector field V on a Lie group G. The linearisation of V at the identity element e gives us a map

$$[\cdot\,,\cdot]_V : \mathfrak{g}^* \wedge \mathfrak{g}^* \to \mathfrak{g}^*.$$

For $\xi, \eta \in \mathfrak{g}^*$, we have a convenient way to interpret the vector $[\xi,\eta]_V$. First take any $X \in \mathfrak{g}$ and choose $\bar{\xi}, \bar{\eta} \in \Omega^1(G)$, $\bar{X} \in \mathfrak{X}(G)$ such that $\bar{\xi}(e) = \xi$, $\bar{\eta}(e) = \eta$ and $\bar{X}(e) = X$. Then, using properties of the Lie derivative and the fact $V(e) = 0$, we have

$$\begin{aligned}\langle [\xi,\eta]_V, X\rangle &= \langle \xi\wedge\eta, d_eV(X)\rangle\\ &= \langle \bar{\xi}\wedge\bar{\eta}, \mathcal{L}_{\bar{X}}V\rangle(e)\\ &= \left(\mathcal{L}_{\bar{X}}\langle\bar{\xi}\wedge\bar{\eta}, V\rangle - \langle\mathcal{L}_{\bar{X}}(\bar{\xi}\wedge\bar{\eta}), V\rangle\right)(e)\\ &= \mathcal{L}_{\bar{X}}(V(\bar{\xi},\bar{\eta}))(e) - \langle\mathcal{L}_{\bar{X}}(\bar{\xi}\wedge\bar{\eta})(e), V(e)\rangle\\ &= \langle d_e(V(\bar{\xi},\bar{\eta})), X\rangle.\end{aligned}$$

That is, we have $[\xi,\eta]_V = d_e(V(\bar{\xi},\bar{\eta}))$. The reason why we have used a bracket to denote this map becomes apparent in light of the following theorem.

Theorem 2.39. [7] *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. A multiplicative bivector field π is a Poisson tensor on G if and only if $[\cdot\,,\cdot]_\pi$ defines a Lie bracket on $\mathfrak{g}^*$.*

Proof. Let $\xi, \eta, \zeta \in \mathfrak{g}^*$ and choose $\bar{\xi}, \bar{\eta}, \bar{\zeta} \in \Omega^1(G)$ with $\bar{\xi}(e) = \xi$, $\bar{\eta}(e) = \eta$ and $\bar{\zeta}(e) = \zeta$. As we saw above,

$$[\xi, \eta]_\pi = d_e(\pi(\bar{\xi}, \bar{\eta})) = di_\pi(\bar{\xi} \wedge \bar{\eta})(e).$$

It follows that,

$$[[\xi, \eta]_\pi, \zeta]_\pi = di_\pi(di_\pi(\bar{\xi} \wedge \bar{\eta}) \wedge \bar{\zeta})(e).$$

Now consider the multiplicative multivector field $[\![\pi, \pi]\!]$. We denote $\omega = \bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta}$. By property (iv) of the Schouten bracket we have

$$\begin{aligned} i_{[\![\pi,\pi]\!]}(\bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta}) &= (-1)^6 i_\pi di_\pi \omega + (-1)^2 i_\pi di_\pi \omega - i_{\pi \wedge \pi} d\omega \\ &= 2 i_\pi di_\pi \omega - i_\pi i_\pi d\omega \\ &= 2 i_\pi di_\pi (\bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta}). \end{aligned}$$

Here we have used the property $i_\pi \circ i_\pi = 0$. Observe that,

$$i_\pi(\bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta}) = \circlearrowleft i_\pi(\bar{\xi} \wedge \bar{\eta})\bar{\zeta},$$

where $\circlearrowleft$ denotes the summation over the circular permutations of $\bar{\xi}$, $\bar{\eta}$, $\bar{\zeta}$. Thus, using the standard properties of exterior differentiation,

$$\begin{aligned} i_{[\![\pi,\pi]\!]}(\bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta}) &= 2 i_\pi d \left(\circlearrowleft i_\pi(\bar{\xi} \wedge \bar{\eta})\bar{\zeta} \right) \\ &= 2 \circlearrowleft \left(i_\pi(di_\pi(\bar{\xi} \wedge \bar{\eta}) \wedge \bar{\zeta}) + i_\pi(i_\pi(\bar{\xi} \wedge \bar{\eta}) d\bar{\zeta}) \right). \end{aligned}$$

Let us first focus on the second term of this expression. Note that,

$$\begin{aligned} d(i_\pi(i_\pi(\bar{\xi} \wedge \bar{\eta}) d\bar{\zeta}))(e) &= d(i_\pi(\bar{\xi} \wedge \bar{\eta}) i_\pi d\bar{\zeta})(e) \\ &= i_\pi(\bar{\xi} \wedge \bar{\eta})(e) di_\pi d\bar{\zeta}(e) + i_\pi d\bar{\zeta}(e) di_\pi(\bar{\xi} \wedge \bar{\eta})(e) \\ &= 0, \end{aligned}$$

since $\pi(e) = 0$. Hence,

$$di_{[\![\pi,\pi]\!]}(\bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta})(e) = 2 \circlearrowleft di_\pi(di_\pi(\bar{\xi} \wedge \bar{\eta}) \wedge \bar{\zeta})(e) = 2 \circlearrowleft [[\xi, \eta]_\pi, \zeta]_\pi.$$

On the other hand, for any $X \in \mathfrak{g}$,

$$\begin{aligned} \langle d_e[\![\pi, \pi]\!](X), \xi \wedge \eta \wedge \zeta \rangle &= \langle \mathcal{L}_{\bar{X}}[\![\pi, \pi]\!], \bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta} \rangle(e) \\ &= \mathcal{L}_{\bar{X}} \langle [\![\pi, \pi]\!], \bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta} \rangle(e) - \langle [\![\pi, \pi]\!](e), \mathcal{L}_{\bar{X}}(\bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta})(e) \rangle \\ &= d_e \langle [\![\pi, \pi]\!], \bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta} \rangle(X) \\ &= \langle di_{[\![\pi,\pi]\!]}(\bar{\xi} \wedge \bar{\eta} \wedge \bar{\zeta})(e), X \rangle \\ &= 2 \circlearrowleft \langle [[\xi, \eta]_\pi, \zeta]_\pi, X \rangle. \end{aligned}$$

Finally, we can conclude that $[\cdot\,,\cdot]_\pi$ satisfies the Jacobi identity if and only if we have $d_e[\![\pi, \pi]\!] = 0$. By Proposition 2.38, this occurs if and only if $[\![\pi, \pi]\!] \equiv 0$. $\square$

We finish this section by giving a notion of an action for Poisson Lie groups.

Definition 2.40. Let G be a Poisson Lie group and P a Poisson manifold. A *Poisson action* is a Lie group action $\theta\colon G \times P \to P$ such that θ is a Poisson map when $G \times P$ is given the product Poisson structure.

[7] We follow a similar method of proof to that given in [17, Theorem 2.2].

2.1.4 Lie bialgebras

We now present a brief introduction to Lie bialgebras. They arise as the infinitesimal objects associated to Poisson Lie groups. We will assume that all Lie algebras in this subsection are finite-dimensional.

From our study of Poisson Lie groups, the result of Theorem 2.39 implies that every Poisson Lie group (G, π), with Lie algebra $\mathfrak{g}$, gives rise to a second Lie algebra structure on the dual space $\mathfrak{g}^*$. The Lie bracket for which, $[\cdot\,,\cdot]_\pi$, is given by the linearisation of π at the identity element e. Moreover, from Proposition 2.36 we can deduce that the dual of this Lie bracket, which is given by $d_e\pi\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, is a 1-cocycle of $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$.

These properties that this pair of Lie algebras have can be abstracted to give the following definition.

Definition 2.41. Let $\mathfrak{g}$ be a Lie algebra with dual vector space $\mathfrak{g}^*$. The pair $(\mathfrak{g}, \mathfrak{g}^*)$ is called a *Lie bialgebra* if $\mathfrak{g}^*$ has a Lie algebra structure such that the dual map to its Lie bracket, $\epsilon\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$, is a 1-cocycle of $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$.

Often a Lie bialgebra is alternatively denoted by $(\mathfrak{g}, \epsilon)$, when it is necessary to emphasize the corresponding 1-cocycle.

It should be clear from the above passage that any Poisson Lie group (G, π), with Lie algebra $\mathfrak{g}$, induces a Lie bialgebra structure on the pair $(\mathfrak{g}, \mathfrak{g}^*)$ with the corresponding 1-cocycle given by $\epsilon = d_e\pi$. We call $(\mathfrak{g}, \mathfrak{g}^*)$ the *tangent Lie bialgebra to* (G, π).

Given an arbitrary Lie bialgebra $(\mathfrak{g}, \epsilon)$, the 1-cocycle condition for $\epsilon\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ states that

$$\operatorname{ad}_X(\epsilon(Y)) - \operatorname{ad}_Y(\epsilon(X)) - \epsilon([X, Y]) = 0, \tag{2.10}$$

for every $X, Y \in \mathfrak{g}$. Let us try to unravel this condition further.

We first note that the Lie bracket on $\mathfrak{g}^*$ and the cocycle ϵ are related by the equation

$$\langle \xi \wedge \eta, \epsilon([X, Y])\rangle = \langle [\xi, \eta], [X, Y]\rangle,$$

for $X, Y \in \mathfrak{g}$, and $\xi, \eta \in \mathfrak{g}^*$.

Next, we make the observation that

$$\begin{aligned}
\langle \xi \wedge \eta, \operatorname{ad}_X(\epsilon(Y))\rangle &= \langle \xi \wedge \eta, (\operatorname{ad}_X \otimes 1 + 1 \otimes \operatorname{ad}_X)(\epsilon(Y))\rangle \\
&= \langle (\xi \circ \operatorname{ad}_X) \wedge \eta + \xi \wedge (\eta \circ \operatorname{ad}_X), \epsilon(Y)\rangle \\
&= \langle (-\operatorname{ad}^*_X \xi) \wedge \eta + \xi \wedge (-\operatorname{ad}^*_X \eta), \epsilon(Y)\rangle \\
&= \langle [\eta, \operatorname{ad}^*_X \xi] + [\operatorname{ad}^*_X \eta, \xi], Y\rangle \\
&= \langle \operatorname{ad}_\eta(\operatorname{ad}^*_X \xi) - \operatorname{ad}_\xi(\operatorname{ad}^*_X \eta), Y\rangle \\
&= \langle \operatorname{ad}^*_X \eta, \operatorname{ad}^*_\xi Y\rangle - \langle \operatorname{ad}^*_X \xi, \operatorname{ad}^*_\eta Y\rangle.
\end{aligned}$$

By a symmetrical argument, it follows that

$$\langle \xi \wedge \eta, \operatorname{ad}_Y(\epsilon(X))\rangle = \langle \operatorname{ad}^*_Y \eta, \operatorname{ad}^*_\xi X\rangle - \langle \operatorname{ad}^*_Y \xi, \operatorname{ad}^*_\eta X\rangle.$$

Hence, by nondegeneracy of the pairing of wedge products, condition (2.10) is equivalent to the condition

$$\begin{aligned}
&\langle \operatorname{ad}^*_X \eta, \operatorname{ad}^*_\xi Y\rangle - \langle \operatorname{ad}^*_X \xi, \operatorname{ad}^*_\eta Y\rangle - \langle \operatorname{ad}^*_Y \eta, \operatorname{ad}^*_\xi X\rangle \\
&\quad + \langle \operatorname{ad}^*_Y \xi, \operatorname{ad}^*_\eta X\rangle - \langle [\xi, \eta], [X, Y]\rangle = 0,
\end{aligned} \tag{2.11}$$

for every $X, Y \in \mathfrak{g}$, and $\xi, \eta \in \mathfrak{g}^*$.

This equation indicates the symmetry between the roles played by $\mathfrak{g}$ and $\mathfrak{g}^*$ in the definition of a Lie bialgebra. In fact, if we denote the dual of the Lie algebra bracket on $\mathfrak{g}$ by $\psi\colon \mathfrak{g}^* \to \mathfrak{g}^* \wedge \mathfrak{g}^*$, then condition (2.11) is equivalent to ψ being a 1-cocycle of $\mathfrak{g}^*$ with respect to the adjoint representation of $\mathfrak{g}^*$ on $\mathfrak{g}^* \wedge \mathfrak{g}^*$. We have proved the following result:

Proposition 2.42. *$(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra if and only if $(\mathfrak{g}^*, \mathfrak{g})$ is a Lie bialgebra.*

Remark 2.43. Let $\mathfrak{g}$ be a Lie algebra with Lie bracket $[\cdot\,,\cdot]$. We denote by $\bar{\mathfrak{g}}$ the Lie algebra, which as a vector space is just $\mathfrak{g}$, but whose Lie bracket is given by $-[\cdot\,,\cdot]$. Note that $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra if and only if $(\bar{\mathfrak{g}}, \mathfrak{g}^*)$ is a Lie bialgebra. We call the Lie bialgebra $(\mathfrak{g}^*, \bar{\mathfrak{g}})$ the *flip* of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$.

Definition 2.44. Two Lie bialgebras $(\mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{h}, \mathfrak{h}^*)$ are *isomorphic* if there exists an isomorphism of Lie algebras $\phi\colon \mathfrak{g} \to \mathfrak{h}$ whose dual map $\phi^*\colon \mathfrak{h}^* \to \mathfrak{g}^*$ is an isomorphism of Lie algebras.

We say that $(\mathfrak{g}, \mathfrak{g}^*)$ and $(\mathfrak{h}, \mathfrak{h}^*)$ are *dual* Lie bialgebras if $(\mathfrak{g}, \mathfrak{g}^*)$ is isomorphic to the flip $(\mathfrak{h}^*, \bar{\mathfrak{h}})$.

We have seen that a Poisson Lie group (G, π), with Lie algebra $\mathfrak{g}$, gives rise to a Lie bialgebra structure on $(\mathfrak{g}, \mathfrak{g}^*)$. As it happens, there is actually a reverse construction. This process relies on the following result, which can be seen as a converse to Proposition 2.36.

Proposition 2.45. [8] *Let G be a connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. Suppose that $\epsilon\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle of $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$. Then there exists a unique multiplicative bivector field V on G such that $d_e V = \epsilon$.* □

This proposition effectively gives us an integration result for Lie bialgebras. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra, and G be the connected, simply-connected Lie group with Lie algebra $\mathfrak{g}$. Then the dual of the Lie bracket on $\mathfrak{g}^*$ is a 1-cocycle $\epsilon\colon \mathfrak{g} \to \mathfrak{g} \wedge \mathfrak{g}$ of $\mathfrak{g}$ with respect to the adjoint representation of $\mathfrak{g}$ on $\mathfrak{g} \wedge \mathfrak{g}$. Proposition 2.45 now provides a unique multiplicative bivector field π on G satisfying $d_e\pi = \epsilon$. Moreover, the induced bracket $[\cdot\,,\cdot]_\pi$ is precisely the Lie bracket on $\mathfrak{g}^*$ that is already given. Hence, by Theorem 2.39 we can conclude that π is also Poisson, and so (G, π) is a Poisson Lie group. We have proved Drinfel'd's theorem:

Theorem 2.46 ([18, Theorem 3])**.** *Let G be a connected, simply-connected Lie group, and suppose that its Lie algebra $\mathfrak{g}$ has a given Lie bialgebra structure $(\mathfrak{g}, \mathfrak{g}^*)$. Then there exists a unique multiplicative Poisson structure π on G such that $(\mathfrak{g}, \mathfrak{g}^*)$ is the tangent Lie bialgebra to (G, π).* □

We now focus on another construction that arises from a Lie bialgebra. Let $(\mathfrak{g}, \mathfrak{g}^*)$ be an arbitrary Lie bialgebra, and consider the vector space $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$. We claim that $\mathfrak{d}$ has a natural Lie algebra structure, with bracket[9] given by

$$[X \oplus \xi, Y \oplus \eta] = \big([X, Y] + \mathrm{ad}^*_\xi Y - \mathrm{ad}^*_\eta X\big) \oplus \left([\xi, \eta] + \mathrm{ad}^*_X \eta - \mathrm{ad}^*_Y \xi\right), \tag{2.12}$$

[8] A rigorous proof can be found in [60, Theorem 10.9]. The method used particularises the results of [32, §3].

[9] In the next subsection on Manin triples, the intuition as to why this bracket is *natural* will become apparent.

for $X \oplus \xi, Y \oplus \eta \in \mathfrak{d}$. This bracket is clearly bilinear and anti-symmetric, and so to show it defines a Lie bracket on $\mathfrak{d}$, it remains only to show that the Jacobi identity is satisfied. By bilinearity, it suffices to check the Jacobi identity holds just in the cases where elements have only a $\mathfrak{g}$ or $\mathfrak{g}^*$ component.

Let us denote the natural inclusions of $\mathfrak{g}$ and $\mathfrak{g}^*$ into $\mathfrak{d}$ by

$$\iota_1\colon \mathfrak{g} \to \mathfrak{d}\,,\ X \mapsto \bar{X}; \qquad \iota_2\colon \mathfrak{g}^* \to \mathfrak{d}\,,\ \xi \mapsto \bar{\xi}. \tag{2.13}$$

For triples $(\bar{X}, \bar{Y}, \bar{Z})$ of elements with only a $\mathfrak{g}$ component, and for triples $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$ of elements with only a $\mathfrak{g}^*$ component, the Jacobi identity just follows from the fact that the Lie brackets on $\mathfrak{g}$ and $\mathfrak{g}^*$ both satisfy the Jacobi identity.

Let us next verify the Jacobi identity for triples of the form $(\bar{X}, \bar{Y}, \bar{\xi})$ where $X, Y \in \mathfrak{g}$, and $\xi \in \mathfrak{g}^*$. Observe,

$$\begin{aligned}
&[\bar{X}, [\bar{Y}, \bar{\xi}]] + [\bar{Y}, [\bar{\xi}, \bar{X}]] + [\bar{\xi}, [\bar{X}, \bar{Y}]] \qquad (2.14)\\
&\quad = [\bar{X}, -\operatorname{ad}^*_\xi Y \oplus \operatorname{ad}^*_Y \xi] + [\bar{Y}, \operatorname{ad}^*_\xi X \oplus -\operatorname{ad}^*_X \xi] + \Big(\operatorname{ad}^*_\xi([X,Y]) \oplus -\operatorname{ad}^*_{[X,Y]} \xi\Big)\\
&\quad = \Big(\Big(-[X, \operatorname{ad}^*_\xi Y] - \operatorname{ad}^*_{\operatorname{ad}^*_Y \xi} X\Big) \oplus \operatorname{ad}^*_X(\operatorname{ad}^*_Y \xi)\Big)\\
&\qquad + \Big(\Big([Y, \operatorname{ad}^*_\xi X] + \operatorname{ad}^*_{\operatorname{ad}^*_X \xi} Y\Big) \oplus -\operatorname{ad}^*_Y(\operatorname{ad}^*_X \xi)\Big)\\
&\qquad + \Big(\operatorname{ad}^*_\xi([X,Y]) \oplus \Big(-\operatorname{ad}^*_X(\operatorname{ad}^*_Y \xi) + \operatorname{ad}^*_Y(\operatorname{ad}^*_X \xi)\Big)\Big)\\
&\quad = \Big(-[X, \operatorname{ad}^*_\xi Y] - \operatorname{ad}^*_{\operatorname{ad}^*_Y \xi} X + [Y, \operatorname{ad}^*_\xi X] + \operatorname{ad}^*_{\operatorname{ad}^*_X \xi} Y + \operatorname{ad}^*_\xi([X,Y])\Big) \oplus 0.
\end{aligned}$$

Since this expression has no $\mathfrak{g}^*$ component, we can view it as an element of $\mathfrak{g}$. Let us denote it by

$$\Lambda = -[X, \operatorname{ad}^*_\xi Y] - \operatorname{ad}^*_{\operatorname{ad}^*_Y \xi} X + [Y, \operatorname{ad}^*_\xi X] + \operatorname{ad}^*_{\operatorname{ad}^*_X \xi} Y + \operatorname{ad}^*_\xi([X,Y]).$$

Then, for any $\eta \in \mathfrak{g}^*$, we have

$$\begin{aligned}
\langle \eta, \Lambda \rangle &= -\langle \eta, [X, \operatorname{ad}^*_\xi Y]\rangle - \langle \eta, \operatorname{ad}^*_{\operatorname{ad}^*_Y \xi} X\rangle + \langle \eta, [Y, \operatorname{ad}^*_\xi X]\rangle \\
&\quad + \langle \eta, \operatorname{ad}^*_{\operatorname{ad}^*_X \xi} Y\rangle + \langle \eta, \operatorname{ad}^*_\xi([X,Y])\rangle \\
&= \langle \operatorname{ad}^*_X \eta, \operatorname{ad}^*_\xi Y\rangle + \langle \operatorname{ad}^*_Y \xi, \operatorname{ad}^*_\eta X\rangle - \langle \operatorname{ad}^*_Y \eta, \operatorname{ad}^*_\xi X\rangle \\
&\quad - \langle \operatorname{ad}^*_X \xi, \operatorname{ad}^*_\eta Y\rangle - \langle [\xi,\eta], [X,Y]\rangle.
\end{aligned} \tag{2.15}$$

Now since $(\mathfrak{g}, \mathfrak{g}^*)$ is a Lie bialgebra, we notice that this expression vanishes because of the identity (2.11). By nondegeneracy of the pairing $\langle \cdot\,, \cdot\rangle$ of $\mathfrak{g}$ with its dual space $\mathfrak{g}^*$, it follows that the Jacobi identity is satisfied.

The remaining case of triples of the form $(\bar{X}, \bar{\xi}, \bar{\eta})$ where $X \in \mathfrak{g}$, and $\xi, \eta \in \mathfrak{g}^*$, has a completely symmetric argument to the above. Therefore, we can conclude that the bracket defined by (2.12) satisfies the Jacobi identity, and hence provides $\mathfrak{d}$ with a Lie algebra structure.

Definition 2.47. For a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, we write $\mathfrak{g} \bowtie \mathfrak{g}^*$ to denote the Lie algebra $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$ defined by the Lie bracket (2.12). We call $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ the *Drinfel'd double Lie algebra* of the Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$.

2.1.5 Manin triples

Any study of Lie bialgebras would be incomplete without touching on the notion of a Manin triple. Manin triples were first introduced by Drinfel'd (see [19, p803]). They are useful objects in our study primarily because there is a natural correspondence between them and Lie bialgebras. Again, we will assume that all Lie algebras in this subsection are finite-dimensional.

Definition 2.48. A *Manin triple* $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ consists of a Lie algebra $\mathfrak{p}$ endowed with a symmetric, nondegenerate, ad–invariant bilinear form $\langle \cdot \mid \cdot \rangle$ on $\mathfrak{p}$, and isotropic Lie subalgebras $\mathfrak{p}_1$, $\mathfrak{p}_2$ of $\mathfrak{p}$, such that $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ as a vector space.

Let us see how a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$ gives rise to a Manin triple. Consider the Drinfel'd double Lie algebra $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$. We can define a symmetric bilinear form $\langle \cdot \mid \cdot \rangle$ on $\mathfrak{d}$ as follows:

$$\langle X \oplus \xi \mid Y \oplus \eta \rangle = \langle \xi, Y \rangle + \langle \eta, X \rangle. \tag{2.16}$$

Nondegeneracy follows immediately from the nondegeneracy of the pairing $\langle \cdot\,, \cdot \rangle$. By this construction of $\langle \cdot \mid \cdot \rangle$, it should be clear that $\mathfrak{g}$ and $\mathfrak{g}^*$ form isotropic Lie subalgebras of $\mathfrak{d}$. It remains to check the ad-invariance of $\langle \cdot \mid \cdot \rangle$. Making use of bilinearity, we only need to check the cases for which the elements of $\mathfrak{d}$ have only a $\mathfrak{g}$ or $\mathfrak{g}^*$ component.

Let us consider the case of triples of the form $(\bar{X}, \bar{Y}, \bar{\xi})$, with $X, Y \in \mathfrak{g}$, and $\xi \in \mathfrak{g}^*$. We have,

$$\begin{aligned} \langle [\bar{X}, \bar{Y}] \mid \bar{\xi} \rangle + \langle \bar{Y} \mid [\bar{X}, \bar{\xi}] \rangle &= \langle \xi, [X, Y] \rangle + \langle \bar{Y} \mid -\operatorname{ad}^*_\xi X \oplus \operatorname{ad}^*_X \xi \rangle \\ &= \langle \xi, [X, Y] \rangle + \langle \operatorname{ad}^*_X \xi, Y \rangle \\ &= \langle \xi, \operatorname{ad}_X Y \rangle - \langle \xi, \operatorname{ad}_X Y \rangle \\ &= 0. \end{aligned}$$

We have a symmetric argument for triples of the form $(\bar{X}, \bar{\xi}, \bar{\eta})$, with $X \in \mathfrak{g}$, and $\xi, \eta \in \mathfrak{g}^*$. For triples $(\bar{X}, \bar{Y}, \bar{Z})$ or triples $(\bar{\xi}, \bar{\eta}, \bar{\zeta})$, where $X, Y, Z \in \mathfrak{g}$, and $\xi, \eta, \zeta \in \mathfrak{g}^*$, the proof of ad–invariance is even simpler, as all terms on the right-hand side of (2.16) vanish.

Hence, we can conclude that $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ is a Manin triple with respect to the bilinear form $\langle \cdot \mid \cdot \rangle$ defined by (2.16).

It turns out that we also have a construction that goes in the opposite direction – from Manin triples to Lie bialgebras. We show this construction in the proof of the following theorem.

Theorem 2.49. *There is a one-to-one correspondence between the isomorphism classes of Lie bialgebras and the isomorphism classes of Manin triples.*

Proof. We have seen that given a Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$, we can construct a Manin triple $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$, where $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$ is the Drinfel'd double Lie algebra. Conversely, suppose $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$ is a Manin triple with respect to a symmetric, nondegenerate, ad-invariant, bilinear form on $\mathfrak{p}$, which we will denote by $\langle \cdot \mid \cdot \rangle$. We aim to construct a Lie bialgebra, and then show that these two constructions are mutually inverse.

For the inclusions of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ into $\mathfrak{p}$, we will use the notation

$$\mathfrak{p}_1 \hookrightarrow \mathfrak{p}\,,\ X \mapsto \bar{X}; \qquad \mathfrak{p}_2 \hookrightarrow \mathfrak{p}\,,\ Y \mapsto \bar{Y}. \tag{2.17}$$

We first define a linear map $\phi : \mathfrak{p}_2 \to \mathfrak{p}_1^*$ by

$$\langle \phi(Y), X \rangle = \langle \bar{Y} \mid \bar{X} \rangle,$$

for $Y \in \mathfrak{p}_2$, and $X \in \mathfrak{p}_1$. We claim that this map gives an isomorphism of vector spaces. To show injectivity, suppose that $\phi(Y) = 0$ for some $Y \in \mathfrak{p}_2$. This implies that $\langle \bar{Y} \mid \bar{X} \rangle = 0$ for every $X \in \mathfrak{p}_1$. On the other hand, $\langle \bar{Y} \mid \bar{Z} \rangle = 0$ for every $Z \in \mathfrak{p}_2$, since $\mathfrak{p}_2$ is an isotropic subspace of $\mathfrak{p}$. Because $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$, it follows that $\langle \bar{Y} | P \rangle = 0$, for every $P \in \mathfrak{p}$. Thus, by nondegeneracy, we deduce that $Y = 0$, and therefore ϕ is injective.

To show surjectivity, suppose that $\xi \in \mathfrak{p}_1^*$. By nondegeneracy, the map $\psi : \mathfrak{p} \to \mathfrak{p}^*$, given by $P \mapsto \langle P \mid \cdot \rangle$, is an isomorphism. Hence, there exists $X \in \mathfrak{p}_1$, and $Y \in \mathfrak{p}_2$, such that $\xi \oplus 0 = \psi(X \oplus Y)$. Then, for every $Z \in \mathfrak{p}_1$, we have

$$\langle \xi, Z \rangle = \langle X \oplus Y \mid \bar{Z} \rangle = \langle \bar{X} \mid \bar{Z} \rangle + \langle \bar{Y} \mid \bar{Z} \rangle = \langle \bar{Y} \mid \bar{Z} \rangle.$$

The last equality uses the fact $\mathfrak{p}_1$ is an isotropic subspace of $\mathfrak{p}$. Hence $\xi = \phi(Y)$, and so the map is surjective.

We can now transfer the Lie algebra structure on $\mathfrak{p} = \mathfrak{p}_1 \oplus \mathfrak{p}_2$ to the vector space $\mathfrak{q} := \mathfrak{p}_1 \oplus \mathfrak{p}_1^*$ via the isomorphism $\mathrm{id} \oplus \phi$. More precisely, for $X_1 \oplus \xi_1, X_2 \oplus \xi_2 \in \mathfrak{q}$, take the unique elements $Y_1, Y_2 \in \mathfrak{p}_2$ such that $\phi(Y_i) = \xi_i$, for $i = 1, 2$. Next, take the unique elements $X_3 \in \mathfrak{p}_1$, $Y_3 \in \mathfrak{p}_2$ satisfying

$$[X_1 \oplus Y_1, X_2 \oplus Y_2]_{\mathfrak{p}} = X_3 \oplus Y_3.$$

Then we define the Lie bracket on $\mathfrak{q}$ by

$$[X_1 \oplus \xi_1, X_2 \oplus \xi_2]_{\mathfrak{q}} = X_3 \oplus \phi(Y_3).$$

Now, since $\mathfrak{p}_2$ is a Lie subalgebra of $\mathfrak{p}$, this Lie bracket on $\mathfrak{q}$ restricts to a Lie bracket on $\mathfrak{p}_1^*$, which we denote by $[\cdot\,,\cdot]_{\mathfrak{p}_1^*}$.

Note, we can also transfer the bilinear form $\langle \cdot \,|\, \cdot \rangle$ of $\mathfrak{p}$ onto $\mathfrak{q}$ via the isomorphism $\mathrm{id} \oplus \phi$. To be precise, we set

$$\langle X_1 \oplus \xi_1 | X_2 \oplus \xi_2 \rangle_{\mathfrak{q}} = \langle X_1 \oplus Y_1 | X_2 \oplus Y_2 \rangle.$$

We claim that $(\mathfrak{p}_1, \mathfrak{p}_1^*)$ forms a Lie bialgebra, where the Lie bracket on $\mathfrak{p}_1^*$ is the bracket $[\cdot\,,\cdot]_{\mathfrak{p}_1^*}$ constructed above. To show this we need to check that the cocycle condition (2.10) is satisfied, or equivalently show that the identity (2.11) is satisfied.

We further claim that the Lie bracket on $\mathfrak{q}$ is exactly the Lie bracket defined by (2.12), and the bilinear form on $\mathfrak{q}$ is exactly (2.16), so that the Manin triple that one constructs from the Lie bialgebra $(\mathfrak{p}_1, \mathfrak{p}_1^*)$ is in fact isomorphic to the original Manin triple $(\mathfrak{p}, \mathfrak{p}_1, \mathfrak{p}_2)$. Furthermore, these two constructions are mutually inverse to one another.

Let us first show that the Lie bracket on $\mathfrak{q}$ satisfies (2.12). For the inclusions of $\mathfrak{p}_1$ and $\mathfrak{p}_2$ into $\mathfrak{p}$, we will also use the notation

$$\mathfrak{p}_1 \hookrightarrow \mathfrak{q}\,,\ X \mapsto \bar{X}; \qquad \mathfrak{p}_1^* \hookrightarrow \mathfrak{q}\,,\ \xi \mapsto \bar{\xi}. \tag{2.18}$$

Take elements $X_1 \oplus \xi_1, X_2 \oplus \xi_2 \in \mathfrak{q}$. By bilinearity, we can write

$$[X_1 \oplus \xi_1, X_2 \oplus \xi_2]_{\mathfrak{q}} = [\bar{X}_1, \bar{X}_2]_{\mathfrak{q}} + [\bar{X}_1, \bar{\xi}_2]_{\mathfrak{q}} + [\bar{\xi}_1, \bar{X}_2]_{\mathfrak{q}} + [\bar{\xi}_1, \bar{\xi}_2]_{\mathfrak{q}}. \tag{2.19}$$

It should be clear from the construction that we have the following two equalities:

$$[\bar{X}_1, \bar{X}_2]_{\mathfrak{q}} = [X_1, X_2]_{\mathfrak{p}_1}, \qquad [\bar{\xi}_1, \bar{\xi}_2]_{\mathfrak{q}} = [\xi_1, \xi_2]_{\mathfrak{p}_1^*}.$$

Let us compute the remaining two terms from the right-hand side of equation (2.19).

Since ϕ is an isomorphism, there exists a unique element $Y_1 \in \mathfrak{p}_2$, such that $\phi(Y_1) = \xi_2$. We also have unique elements $X_3 \in \mathfrak{p}_1$, $Y_2 \in \mathfrak{p}_2$, such that $[\bar{X}_1, \bar{Y}_1]_{\mathfrak{p}} = X_3 \oplus Y_2$. Thus,

$$[\bar{X}_1, \bar{\xi}_2]_{\mathfrak{q}} = X_3 \oplus \phi(Y_2).$$

We compute X_3 by observing that for any $Y_3 \in \mathfrak{p}_2$ we have the following

$$\begin{aligned}
\langle \phi(Y_3), X_3 \rangle &= \langle \bar{Y}_3 \mid \bar{X}_3 \rangle \\
&= \langle \bar{Y}_3 \mid X_3 \oplus Y_2 \rangle \\
&= \langle \bar{Y}_3 \mid [\bar{X}_1, \bar{Y}_1]_{\mathfrak{p}} \rangle \\
&= \langle [\bar{Y}_1, \bar{Y}_3]_{\mathfrak{p}} \mid \bar{X}_1 \rangle \\
&= \langle \phi([Y_1, Y_3]_{\mathfrak{p}_2}), X_1 \rangle \\
&= \langle [\phi(Y_1), \phi(Y_3)]_{\mathfrak{p}_1^*}, X_1 \rangle \\
&= \langle \phi(Y_3), -\operatorname{ad}^*_{\phi(Y_1)} X_1 \rangle.
\end{aligned}$$

By nondegeneracy of the pairing $\langle \cdot\,, \cdot \rangle$, and from the fact that ϕ is an isomorphism, we deduce

$$X_3 = -\operatorname{ad}^*_{\phi(Y_1)} X_1 = -\operatorname{ad}^*_{\xi_2} X_1. \tag{2.20}$$

On the other hand, for any $X_4 \in \mathfrak{p}_1$ we have

$$\begin{aligned}
\langle \phi(Y_2), X_4 \rangle &= \langle \bar{Y}_2 \mid \bar{X}_4 \rangle \\
&= \langle X_3 \oplus Y_2 \mid \bar{X}_4 \rangle \\
&= \langle [\bar{X}_1, \bar{Y}_1]_{\mathfrak{p}} \mid \bar{X}_4 \rangle \\
&= -\langle \bar{Y}_1 \mid [\bar{X}_1, \bar{X}_4]_{\mathfrak{p}_1} \rangle \\
&= -\langle \phi(Y_1), [X_1, X_4]_{\mathfrak{p}_1} \rangle \\
&= \langle \operatorname{ad}^*_{X_1} \phi(Y_1), X_4 \rangle.
\end{aligned}$$

Again, by utilising the nondegeneracy of the pairing $\langle \cdot\,, \cdot \rangle$, we find

$$\phi(Y_2) = \operatorname{ad}^*_{X_1} \phi(Y_1) = \operatorname{ad}^*_{X_1} \xi_2. \tag{2.21}$$

Hence, combining the results of (2.20) and (2.21) gives

$$[\bar{X}_1, \bar{\xi}_2]_{\mathfrak{q}} = -\operatorname{ad}^*_{\xi_2} X_1 \oplus \operatorname{ad}^*_{X_1} \xi_2.$$

Now, since $[\cdot\,, \cdot]_{\mathfrak{q}}$ is a well-defined Lie bracket, by anti-symmetry we can infer that

$$[\bar{\xi}_1, \bar{X}_2]_{\mathfrak{q}} = \operatorname{ad}^*_{\xi_1} X_2 \oplus -\operatorname{ad}^*_{X_2} \xi_1.$$

Putting all this together, we find that $[X_1 \oplus \xi_1, X_2 \oplus \xi_2]_{\mathfrak{q}}$ is given by the expression

$$\Big([X_1, X_2]_{\mathfrak{p}_1} + \operatorname{ad}^*_{\xi_1} X_2 - \operatorname{ad}^*_{\xi_2} X_1\Big) \oplus \Big([\xi_1, \xi_2]_{\mathfrak{p}_1^*} + \operatorname{ad}^*_{X_1} \xi_2 - \operatorname{ad}^*_{X_2} \xi_1\Big).$$

Up to relabelling, this is exactly how the Lie bracket given by equation (2.12) is defined.

Also observe that, we have

$$\begin{aligned}\langle X_1 \oplus \xi_1 | X_2 \oplus \xi_2 \rangle_{\mathfrak{q}} &= \langle X_1 \oplus \phi^{-1}(\xi_1) | X_2 \oplus \phi^{-1}(\xi_2) \rangle_{\mathfrak{p}} \\ &= \langle X_1 | \phi^{-1}(\xi_2) \rangle_{\mathfrak{p}} + \langle \phi^{-1}(\xi_1) | X_2 \rangle_{\mathfrak{p}} \\ &= \langle \phi(\phi^{-1}(\xi_2)), X_1 \rangle + \langle \phi(\phi^{-1}(\xi_1)), X_2 \rangle \\ &= \langle \xi_2, X_1 \rangle + \langle \xi_1, X_2 \rangle.\end{aligned}$$

Therefore, we see that the bilinear form $\langle \cdot \,|\, \cdot \rangle_{\mathfrak{q}}$ is indeed given by equation (2.16).

It remains to show that the Lie algebra structures we now have on $\mathfrak{p}_1$ and $\mathfrak{p}_1^*$ make $(\mathfrak{p}_1, \mathfrak{p}_1^*)$ into a Lie bialgebra. That is, we need to check that the Lie brackets satisfy equation (2.11). However, we note that the computations (2.14) and (2.15) combined show that relation (2.11) is in fact equivalent to the condition that the Lie bracket on $\mathfrak{q}$ satisfies the Jacobi identity. Moreover, since $\mathfrak{q}$ is a well-defined Lie algebra, the Jacobi identity certainly holds, and so we are done. □

§ 2.2 Groupoids in Poisson geometry

The second half of this chapter is dedicated to illustrating the importance of Lie groupoids, Lie algebroids, and double Lie structures in Poisson and symplectic geometry. We begin with the observation that every Poisson manifold gives rise to a Lie algebroid structure on its cotangent bundle.

2.2.1 The cotangent Lie algebroid

Let (P, π) be a Poisson manifold. The Poisson anchor $\pi^{\#}\colon T^*P \to TP$ is an anchor map for a Lie algebroid structure on the cotangent bundle $T^*P \to P$, with Lie bracket defined by

$$[\varphi, \psi] = \mathcal{L}_{\pi^{\#}(\varphi)}(\psi) - \mathcal{L}_{\pi^{\#}(\psi)}(\varphi) - d(\pi(\varphi, \psi)), \tag{2.22}$$

for $\varphi, \psi \in \Omega^1(P)$. It is the unique Lie algebroid structure on T^*P with anchor map $\pi^{\#}$ satisfying $[df, dg] = d\{f, g\}$ for $f, g \in \mathcal{C}^\infty(P)$. We call this the *cotangent Lie algebroid of P*.

Proposition 2.50 ([47, Theorem 10.4.2])**.** *Let C be a coisotropic closed embedded submanifold of a Poisson manifold P. Then the conormal bundle $(TC)^\circ \to C$ is a Lie subalgebroid of the cotangent Lie algebroid T^*P.* □

Given a coisotropic closed embedded submanifold C of a Poisson manifold P, we call the induced Lie algebroid on $(TC)^\circ \to C$ the *conormal Lie algebroid of C in P*.

Example 2.51. Let (P, π) be a Poisson manifold. Suppose that $\pi(x) = 0$ for some $x \in P$, then $\{x\}$ is a coisotropic closed embedded submanifold P. The corresponding conormal Lie algebroid is a Lie algebra structure on the cotangent space T_x^*P.

Consider a Poisson Lie group (G, π) with Lie algebra $\mathfrak{g}^*$. Then $\pi(e) = 0$, where e is the identity element. Thus, we get an induced Lie algebra structure on $T_e^*G \cong \mathfrak{g}^*$. This Lie algebra structure coincides with the one given by the linearisation of π at e, as in Theorem 2.39. ⊠

2.2.2 Symplectic groupoids

Symplectic groupoids first arose as a means to construct symplectic realisations of Poisson manifolds.

Definition 2.52. Let P be a Poisson manifold. A *symplectic realisation of* P is a surjective submersion $F\colon \Sigma \to P$, from a symplectic manifold Σ, which is also a Poisson map.

Weinstein [12] and Karasev [29] proved the existence of a symplectic realisation for every Poisson manifold.[10] Briefly, they constructed a symplectic realisation by patching together the target projections of *local* symplectic groupoids. Note that the key property utilised is that the target projection of a symplectic groupoid is a symplectic realisation of the base manifold. In this section, we will introduce symplectic groupoids and their basic properties.

Definition 2.53. Let Σ be a symplectic manifold and a Lie groupoid on base P. We say that $\Sigma \rightrightarrows P$ is a *symplectic groupoid* if the graph of the partial multiplication

$$\Gamma(\kappa) = \{(h, g, \kappa(h, g)) \mid (h, g) \in \Sigma * \Sigma\}\,,$$

is a Lagrangian submanifold of the product manifold $\Sigma \times \Sigma \times \overline{\Sigma}$.

Theorem 2.54 ([53, Proposition 1.5, Theorem 1.6(iii)])**.** *Let $\Sigma \rightrightarrows P$ be a symplectic groupoid. Then:*

(i) 1_P *is a Lagrangian submanifold of* Σ*;*

(ii) The inversion map $\iota\colon \Sigma \to \Sigma$ *is an anti-symplectomorphism;*

(iii) For any $x \in P$*, the tangent spaces of* G_x *and* G^x *are symplectically orthogonal;*

(iv) There exists a unique Poisson structure on P *such that the target projection* $\beta\colon \Sigma \to P$ *is a Poisson map and the source projection* $\alpha\colon \Sigma \to P$ *is an anti-Poisson map.* □

The notion of a Lie groupoid action extends to give a sensible concept of action for symplectic groupoids.

Definition 2.55. Let M' be a symplectic manifold and Σ a symplectic groupoid on base M. A *symplectic groupoid action* is a Lie groupoid action of Σ on a smooth map $J\colon M' \to M$ such that the graph of the action map

$$\Gamma = \{(g, m', g \cdot m') \in \Sigma \times M' \times M' \mid \alpha(g) = J(m')\}$$

is a Lagrangian submanifold of $\Sigma \times M' \times \overline{M'}$. J is called the *moment map* of the action.

Proposition 2.56 ([53, Theorem 3.8])**.** *Let M' be a symplectic manifold and Σ a symplectic groupoid on base M. Suppose that $\theta\colon \Sigma * M' \to M'$ is a symplectic groupoid action of Σ on a moment map $J\colon M' \to M$. Then J is a Poisson map.*

[10] However, not every Poisson manifold admits a complete symplectic realisation. Obstructions to their existence were introduced by Crainic and Fernandes [14] (see also [15]).

Proof. Let us denote the graph of the action θ by Γ and the symplectic structures on Σ and M' by ω_Σ and $\omega_{M'}$, respectively. The symplectic structure on $G\times M'\times\overline{M'}$ is thus given by $\widetilde{\omega} = \omega_\Sigma \oplus \omega_{M'} \oplus (-\omega_{M'})$. Take any arbitrary $f, g \in \mathcal{C}^\infty(M)$ and $m' \in M'$. Since the source projection α is surjective, there exists $g \in \Sigma$ with $\alpha(g) = J(m')$, and so the triple $\gamma = (g, m', g\cdot m')$ is an element of the graph Γ. Consider the tangent vector $(-X_{f\circ\alpha}(g), X_{f\circ J}(m'), 0) \in T_g\Sigma \oplus T_{m'}M' \oplus T_{g\cdot m'}M'$, and take any $(X, Y, Z) \in T_\gamma\Gamma$. In particular, we have $T(\alpha)(X) = T(J)(Y)$. Now observe that

$$\begin{aligned}
\widetilde{\omega}((-X_{f\circ\alpha}(g), X_{f\circ J}(m'), 0), (X, Y, Z)) &= \omega_\Sigma(-X_{f\circ\alpha}(g), X) + \omega_{M'}(X_{f\circ J}(m'), Y)\\
&= \langle -\omega_\Sigma^\flat(X_{f\circ\alpha}(g)), X\rangle + \langle \omega_{M'}^\flat(X_{f\circ J}(m')), Y\rangle\\
&= \langle d(f\circ\alpha)(g), X\rangle - \langle d(f\circ J)(m'), Y\rangle\\
&= \langle (\alpha^* df)(g), X\rangle - \langle (J^* df)(m'), Y\rangle\\
&= \langle df(\alpha(g)), T(\alpha)(X)\rangle - \langle df(J(m')), T(J)(Y)\rangle\\
&= \langle df(J(m')), T(\alpha)(X) - T(J)(Y)\rangle\\
&= 0
\end{aligned}$$

Thus, $(-X_{f\circ\alpha}(g), X_{f\circ J}(m'), 0) \in (T_\gamma\Gamma)^\perp = T_\gamma\Gamma$. It follows that

$$T(J)(X_{f\circ J}(m')) = T(\alpha)(-X_{f\circ\alpha}(g)).$$

However, since the source projection α is an anti-Poisson map, by Proposition 2.12 we have $T(\alpha)\circ X_{f\circ\alpha} = -X_f\circ\alpha$. Hence,

$$T(J)(X_{f\circ J}(m')) = X_f(\alpha(g)) = X_f(J(m')).$$

Since this identity holds for every $m' \in M'$, Proposition 2.12 implies that J is a Poisson map. □

The following result of Xu gives us a useful method of constructing symplectic groupoid actions.

Proposition 2.57 ([65, Theorem 3.1])**.** *Let M' be a symplectic manifold and Σ an α–simply connected symplectic groupoid on base M. Then a smooth map $J\colon M' \to M$ is a complete symplectic realisation if and only if there exists a symplectic groupoid action of Σ on J.* □

Remark 2.58. Note that, if we replace the condition that $J\colon M' \to M$ be a Poisson map, in the proposition, with the condition that J be an anti-Poisson map, one instead gets a *right* symplectic groupoid action of Σ on J.

2.2.3 Poisson groupoids

In this section, we unify the notions of Poisson Lie groups and symplectic groupoids. Recall that a Poisson Lie group was defined to be a Lie group G equipped with a Poisson structure for which the group multiplication is a Poisson map. In light of Proposition 2.19, this is equivalent to the condition that the graph of the multiplication is a coisotropic submanifold of $G\times G\times\overline{G}$. On the other hand, a symplectic groupoid was defined to be a Lie groupoid $\Sigma \rightrightarrows P$ equipped with a symplectic structure for which the graph of the partial multiplication is a Lagrangian submanifold of $\Sigma\times\Sigma\times\overline{\Sigma}$. The symplectic structure on Σ gives rise to a Poisson structure on Σ, and the condition above implies that the graph of the partial multiplication is a coisotropic submanifold of $\Sigma \times \Sigma \times \overline{\Sigma}$ with respect to this Poisson structure. It should now be immediately clear how one can define a general object which incorporates both of these notions.

Definition 2.59. Let G be a Poisson manifold and a Lie groupoid on base P. We say that $G \rightrightarrows P$ is a *Poisson groupoid* if the graph of the partial multiplication

$$\Gamma(\kappa) = \{(h, g, \kappa(h, g)) \mid (h, g) \in G * G\}\,,$$

is a coisotropic submanifold of the product manifold $G \times G \times \overline{G}$.

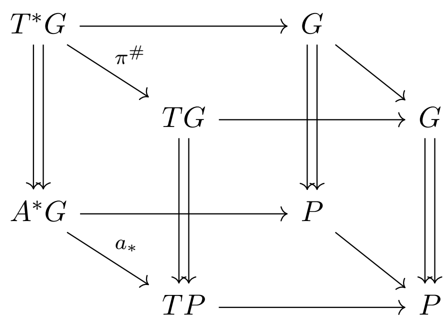
The following result offers another interpretation of the relationship between the Poisson structure and the groupoid operations of a Poisson groupoid.

Proposition 2.60 ([49, Proposition 8.1], [2])**.** *Let (G, π) be a Poisson manifold and a Lie groupoid on base P. Then $G \rightrightarrows P$ is a Poisson groupoid if and only if the Poisson anchor*

$$\begin{array}{ccc} T^*G & \xrightarrow{\;\pi^\#\;} & TG \\ \downdownarrows & & \downdownarrows \\ A^*G & \xrightarrow{\;a_*\;} & TP. \end{array} \tag{2.23}$$

is a morphism of Lie groupoids over a map $a_*\colon A^*G \to TP$. □

Here, the Lie groupoid structure on T^*G with base A^*G is defined as in Example 1.73. Note that for a Poisson groupoid G with base manifold P, we actually have a morphism of $\mathcal{VB}$-groupoids:



We now state an analogue of Theorem 2.54 for Poisson groupoids. The properties in this theorem can be derived by standard arguments involving coisotropic submanifolds (see for example the original formulation [63]), or alternatively by making use of the previous proposition (as in [47, §11]).

Theorem 2.61 ([63, Theorem 4.2.3])**.** *Let $G \rightrightarrows P$ be a Poisson groupoid. Then:*

(i) *1_P is a coisotropic submanifold of G;*

(ii) *The inversion map $\iota\colon G \to G$ is an anti-Poisson map;*

(iv) *There exists a unique Poisson structure on P such that the target projection $\beta\colon G \to P$ is a Poisson map and the source projection $\alpha\colon G \to P$ is an anti-Poisson map.* □

Given a Poisson groupoid G on a base manifold P, the conormal bundle of 1_P is given by the dual bundle $A^*G \to P$. Moreover, the previous result immediately implies that this dual bundle has the structure of a conormal Lie algebroid. We call $A^*G \to P$ the *dual Lie algebroid*.

Proposition 2.62. *Let $G \rightrightarrows P$ be a Poisson groupoid, with the associated Poisson structure on G denoted by π. Then the maps $a_*\colon A^*G \to TP$ and $-a_*^*\colon T^*P \to AG$ are morphisms of Lie algebroids over P.* □

The following criterion gives us a useful tool for verifying whether one has a symplectic groupoid. We will make use of this result in a later section.

Proposition 2.63. *Let Σ be a symplectic manifold and a Lie groupoid on base P. Denote the Poisson structure associated to Σ by π. Then $\Sigma \rightrightarrows P$ is a symplectic groupoid if and only if $\Sigma \rightrightarrows P$ is a Poisson groupoid with respect to π.* □

We also have an analogue of Proposition 2.62 for symplectic groupoids:

Proposition 2.64. *Let $\Sigma \rightrightarrows P$ be a symplectic groupoid, with the associated Poisson structure on G denoted by π. Then the maps $a_*\colon A^*\Sigma \to TP$ and $-a_*^*\colon T^*P \to A\Sigma$ are isomorphisms of Lie algebroids over P.* □

Let us now consider a few of the core examples of Poisson groupoids.

Example 2.65. Let (P, π_P) be a Poisson manifold. We claim that the product Poisson manifold $P \times \overline{P}$, with the pair groupoid structure on base P, is a Poisson groupoid.

Let Γ denote the graph of the partial multiplication of the pair groupoid on P, and let π denote the Poisson structure $\pi_P \oplus (-\pi_P)$. Take an arbitrary element $\gamma = ((x, y), (y, z), (x, z)) \in \Gamma$, and note that

$$T_\gamma\Gamma = \{(X \oplus Y, Y \oplus Z, X \oplus Z) \mid X \in T_xP,\ Y \in T_yP,\ Z \in T_zP\}.$$

Thus, $(\varphi_1 \oplus \psi_1, \varphi_2 \oplus \psi_2, \varphi_3 \oplus \psi_3) \in (T_\gamma\Gamma)^\circ$ if and only if

$$\langle \varphi_1 \oplus \psi_1, X \oplus Y\rangle + \langle \varphi_2 \oplus \psi_2, Y \oplus Z\rangle + \langle \varphi_3 \oplus \psi_3, X \oplus Z\rangle = 0$$

for every $X \in T_xP$, $Y \in T_yP$, and $Z \in T_zP$. This occurs, if and only if we have $\langle \varphi_1 + \varphi_3, X\rangle = 0$ for every $X \in T_xP$, $\langle \psi_1 + \varphi_2, Y\rangle = 0$ for every $Y \in T_yP$, and $\langle \psi_2 + \psi_3, Z\rangle = 0$ for every $Z \in T_zP$. Furthermore, this is the case if and only if $\varphi_3 = -\varphi_1$, $\varphi_2 = -\psi_1$ and $\psi_3 = -\psi_2$. Hence, elements of $(T_\gamma\Gamma)^\circ$ have the form

$$\big(\varphi \oplus \psi, (-\psi) \oplus \theta, (-\varphi) \oplus (-\theta)\big),$$

where $\varphi \in T_x^*P$, $\psi \in T_y^*P$, and $\theta \in T_z^*P$. Given such an element, observe that

$$\Big(\pi^\# \oplus \pi^\# \oplus (-\pi^\#)\Big)\big(\varphi \oplus \psi, (-\psi) \oplus \theta, (-\varphi) \oplus (-\theta)\big)$$
$$= \Big(\pi_P^\#(\varphi) \oplus (-\pi_P^\#(\psi)), (-\pi_P^\#(\psi)) \oplus (-\pi_P^\#(\theta)), \pi_P^\#(\varphi) \oplus (-\pi_P^\#(\theta))\Big) \in T_\gamma\Gamma.$$

Therefore, $\big(\pi^\# \oplus \pi^\# \oplus (-\pi^\#)\big)\big((T_\gamma\Gamma)^\circ\big) \subseteq T_\gamma\Gamma$, and so the graph of the partial multiplication of the pair groupoid is a coisotropic submanifold of the product manifold $(P \times \overline{P}) \times (P \times \overline{P}) \times (\overline{P} \times P)$. Hence, $P \times \overline{P} \rightrightarrows P$ is a Poisson groupoid. ⊠

Proposition 2.66. *Let $G \rightrightarrows P$ and $H \rightrightarrows Q$ be Poisson groupoids. Then the Cartesian product groupoid $G \times H$ on base $P \times Q$ is a Poisson groupoid when $G \times H$ is equipped with the product Poisson structure.*

This result is really a corollary of the following simple lemma.

Lemma 2.67. *Let W be a subspace of a direct sum vector space $V_1 \oplus V_2$. Suppose that $W = W_1 \oplus W_2$ for some subspaces W_1 and W_2 of V_1 and V_2, respectively. Then $W^\circ = W_1^\circ \oplus W_2^\circ$.*

Proof. First observe that

$$W^\circ = \{\varphi \oplus \psi \in V_1^* \oplus V_2^* \mid \langle \varphi \oplus \psi, X \oplus Y \rangle = 0\,,\ \forall X \oplus Y \in W_1 \oplus W_2\}.$$

Thus, $\varphi \oplus \psi \in W^\circ$, if and only if

$$\langle \varphi, X \rangle + \langle \psi, Y \rangle = 0,$$

for every $X \in W_1$, $Y \in W_2$. This is the case, if and only if $\langle \varphi, X \rangle = 0$ for every $X \in W_1$, and $\langle \psi, Y \rangle = 0$ for every $Y \in W_2$. Moreover, this occurs if and only if $\varphi \oplus \psi \in W_1^\circ \oplus W_2^\circ$. □

Proof of Proposition 2.66. Let π_G and π_H denote the Poisson tensors of G and H, respectively, and let π denote the product Poisson structure on $G \times H$. Also, let Γ, Γ_G and Γ_H denote the graphs of the partial multiplications of the groupoids $G \times H$, G and H, respectively. Fix an element $\gamma = ((g_2, h_2), (g_1, h_1), (g_2 g_1, h_2 h_1)) \in \Gamma$. It follows that the elements $\gamma_G := (g_2, g_1, g_2 g_1)$ and $\gamma_H := (h_2, h_1, h_2 h_1)$ lie in the graphs Γ_G and Γ_H, respectively. It should be clear that Γ is diffeomorphic to the product manifold $\Gamma_G \times \Gamma_H$. Hence, the tangent space $T_\gamma \Gamma$ is isomorphic to the direct sum $T_{\gamma_G} \Gamma_G \oplus T_{\gamma_H} \Gamma_H$. Moreover, by Lemma 2.67, $(T_\gamma \Gamma)^\circ$ is isomorphic to $(T_{\gamma_G} \Gamma_G)^\circ \oplus (T_{\gamma_H} \Gamma_H)^\circ$. Now since G and H are Poisson groupoids, we have

$$\left(\pi_G^\# \oplus \pi_G^\# \oplus (-\pi_G^\#)\right) \left((T_{\gamma_G} \Gamma_G)^\circ\right) \subseteq T_{\gamma_G} \Gamma_G,$$

and

$$\left(\pi_H^\# \oplus \pi_H^\# \oplus (-\pi_H^\#)\right) \left((T_{\gamma_H} \Gamma_H)^\circ\right) \subseteq T_{\gamma_H} \Gamma_H.$$

It follows that

$$\left(\pi^\# \oplus \pi^\# \oplus (-\pi^\#)\right) \left((T_\gamma \Gamma)^\circ\right) \subseteq T_\gamma \Gamma.$$

Thus the graph Γ of the partial multiplication of $G \times H$ is a coisotropic submanifold of $(G \times H) \times (G \times H) \times (\overline{G \times H})$, and so $G \times H \rightrightarrows P \times Q$ is a Poisson groupoid. □

Next, we briefly define the corresponding notion of morphism for Poisson groupoids.

Definition 2.68. Let G and G' be Poisson groupoids with base manifolds P and P', respectively. Then a *morphism of Poisson groupoids* is a morphism $F\colon G \to G'$ of Lie groupoids over a smooth map $f\colon P \to P'$ such that F is also a Poisson map.

If, in addition, F is also a diffeomorphism, we say that F is an *isomorphism of Poisson groupoids* over f.

We now turn our attention to the concept of *duality* for Poisson groupoids. There are varying ways in which one can give a sensible definition of duality. We will take the following approach:

Definition 2.69. A *Lie algebroid pair* (A, A^*) is a pair of Lie algebroids A and A^* with the same base manifold P, which are dual as vector bundles.

If the Lie algebroid A has anchor map denoted by a and Lie bracket denoted by $[\cdot\,,\cdot]$, the Lie algebroid $\overline{A}$ denotes the vector bundle $A \to P$, equipped with the anchor map $-a$ and the Lie bracket $-[\cdot\,,\cdot]$. We call the Lie algebroid pair $(A^*, \overline{A})$ the *flip* of (A, A^*).

We say that two Lie algebroid pairs (A, A^*) and (B, B^*) are *isomorphic* if there exists an isomorphism of Lie algebroids $\phi\colon A \to B$ such that the dual map $\phi^*\colon B^* \to A^*$ is an isomorphism of Lie algebroids.

The Lie algebroid pairs that we will be most interested in are those that arise from Poisson groupoids. Given a Poisson groupoid $G \rightrightarrows P$, we saw that $A^*G \to P$ has a Lie algebroid structure. We call the Lie algebroid pair (AG, A^*G) the *tangent Lie bialgebroid of* G.[11]

Definition 2.70. Let G and H be Poisson groupoids on the same base manifold P, with tangent Lie bialgebroids (AG, A^*G) and (AH, A^*H), respectively. We say that G and H are *dual* Poisson groupoids if (AG, A^*G) and $(A^*H, \overline{AH})$ are isomorphic.[12]

Example 2.71. Let G be a Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. By Theorem 2.46, there exists a unique simply-connected Poisson Lie group G^* with tangent Lie bialgebra $(\mathfrak{g}^*, \bar{\mathfrak{g}})$. Thus, G and G^* are dual Poisson Lie groups.[13] ⊠

Example 2.72. Let M be a symplectic manifold with associated Poisson structure π. By Example 2.65, we know that the pair groupoid $M \times \overline{M}$ on base M is a Poisson groupoid. In fact, Proposition 2.63 implies further that $M \times \overline{M} \rightrightarrows M$ is a symplectic groupoid. We have seen in Example 1.43 that the Lie algebroid of $M \times \overline{M}$ is the tangent bundle TM, it follows that the dual Lie algebroid is the cotangent Lie algebroid T^*M. Hence, (TM, T^*M) is the tangent Lie bialgebroid of $M \times \overline{M} \rightrightarrows M$. Now the Poisson anchor $\pi^{\#}\colon T^*M \to TM$ is an isomorphism of Lie algebroids, and so it follows that $-\pi^{\#}\colon T^*M \to \overline{TM}$ is an isomorphism of Lie algebroids. However, by anti-symmetry we know that $(\pi^{\#})^* = -\pi^{\#}$. Thus, (TM, T^*M) and $(T^*M, \overline{TM})$ are isomorphic Lie algebroid pairs, and so the symplectic groupoid $M \times \overline{M}$ is dual to itself. ⊠

Example 2.73. Let $\Sigma \rightrightarrows P$ be a symplectic groupoid. In Proposition 2.64 we saw that we have isomorphisms of Lie algebroids $a_*\colon A^*\Sigma \to TP$ and $-a_*^*\colon T^*P \to A\Sigma$. It follows that $a_*^*\colon T^*P \to \overline{A\Sigma}$ is an isomorphism of Lie algebroids, and hence $(A^*\Sigma, \overline{A\Sigma})$ and (TP, T^*P) are isomorphic Lie algebroid pairs. Thus, $\Sigma \rightrightarrows P$ is dual to the Poisson groupoid $P \times \overline{P} \rightrightarrows P$. Note that the previous example is just a special case of this example. ⊠

We finish this section by unifying the notions of a Poisson action and a symplectic groupoid action under a single notion of an action for Poisson groupoids.

Definition 2.74. Let P be a Poisson manifold and G a Poisson groupoid on base M. A *Poisson groupoid action* is a Lie groupoid action of G on a smooth map $J\colon P \to M$

[11] More generally, a *Lie bialgebroid* is a Lie algebroid pair (A, A^*) with a compatibility condition that generalises that of a Lie bialgebra. See [49] for the general theory.

[12] Compare with [63, Definition 4.4.1].

[13] This notion of duality for Poisson Lie groups differs slightly from the original definition of Drinfel'd [18]. The Poisson tensor of the dual Poisson Lie group G^* will differ by a minus sign.

such that the graph of the action map

$$\Gamma = \{(g, p, g \cdot p) \in G \times P \times P \mid \alpha(g) = J(p)\}$$

is a coisotropic submanifold of $G \times P \times \overline{P}$. As with symplectic groupoid actions, we also call J the *moment map* of the action.

Theorem 2.75 ([63, Theorem 4.3.1, Remark 4.3.2])**.** *Let P be a Poisson manifold and G a Poisson groupoid on base M. Suppose that there is a Poisson groupoid action of G on a moment map $J\colon P \to M$ such that the orbit space P/G has a smooth manifold structure for which the natural projection $p\colon P \to P/G$ is a submersion. Then, P/G has a unique Poisson structure that makes p a Poisson map.* □

When the hypotheses of the above theorem are met, the orbit space P/G equipped with this Poisson structure is called a *Poisson reduced space.*

Remark 2.76. Similar reduction results have been given for Poisson actions and symplectic groupoid actions. Reduction for Poisson actions was first studied by Semenov-Tian-Shansky [59] and Lu [38]. The Poisson reduced spaces for symplectic groupoid actions were introduced by Weinstein and Mikami [53] and also studied by Xu [66].

2.2.4 Double Lie structures in Poisson and symplectic geometry

In this final subsection, we describe some of the roles that double Lie structures play in Poisson and symplectic geometry.

Definition 2.77. Let $(S; H, V; P)$ be a weak double Lie groupoid and π a Poisson structure on S. Then $(S; H, V; P)$ is a *Poisson double groupoid* if the horizontal and vertical structures on S are Poisson groupoids with respect to π.

Definition 2.78. Let $(S; H, V; P)$ be a weak double Lie groupoid and S a symplectic manifold with associated Poisson structure π. Then $(S; H, V; P)$ is a *symplectic double groupoid* if it is a Poisson double groupoid with respect to π.

Example 2.79. Let M be a symplectic manifold. We saw in Example 2.72 that the pair groupoid $M \times \overline{M}$ on base M is a symplectic groupoid. By the same reasoning, the pair groupoid $M \times \overline{M} \times \overline{M} \times M$ on base $M \times \overline{M}$ is also a symplectic groupoid. Hence, the double Lie groupoid $(M \times \overline{M} \times \overline{M} \times M; M \times \overline{M}, M \times \overline{M}; M)$ of Example 1.65 is a symplectic double groupoid. ⊠

Example 2.80. Let $G \rightrightarrows P$ be a Poisson groupoid. We saw in Example 2.65 that the pair groupoid $G \times \overline{G}$ on base G is a Poisson groupoid. On the other hand, Proposition 2.66 shows that the Cartesian product groupoid $G \times \overline{G}$ on base $P \times \overline{P}$ is a Poisson groupoid. It follows that the double Lie groupoid $(G \times \overline{G}; G, P \times \overline{P}, P)$ of Example 1.66 is a Poisson double groupoid. When $G \rightrightarrows P$ is taken to be a symplectic groupoid, the above construction leads to a symplectic double groupoid. ⊠

An observant reader may have noted that the side groupoids for these examples of symplectic double groupoids are in fact dual Poisson groupoids. This is not a coincidence, as we will see in Chapter 4, the side groupoids of any symplectic double groupoid are Poisson groupoids in duality.

We now consider some examples of $\mathcal{LA}$-groupoids that naturally arise in Poisson geometry.

Example 2.81. Let (G,π) be a Poisson Lie group with Lie algebra $\mathfrak{g}$. We saw in Theorem 2.39 that the linearisation of π at the identity e defines a Lie algebra structure on the dual vector space $\mathfrak{g}^*$. We also saw in Example 1.9 that T^*G has a Lie groupoid structure with base $\mathfrak{g}^*$. On the other hand, since G is a Poisson manifold, T^*G possesses the cotangent Lie algebroid structure on G.

$$\begin{array}{ccc} T^*G & \longrightarrow & G \\ \downdownarrows & & \downdownarrows \\ \mathfrak{g}^* & \longrightarrow & \{\cdot\} \end{array}$$

Moreover, the structures are compatible in the sense that $(T^*G;\mathfrak{g}^*,G;\{\cdot\})$ forms an $\mathcal{LA}$-groupoid. ⊠

Example 2.82. The previous example can be extended to a construction for Poisson groupoids. Consider a Poisson groupoid G on base P. We have the dual Lie algebroid $A^*G \to P$, and the cotangent Lie algebroid $T^*G \to G$. Furthermore, we saw in Example 1.73 that T^*G has a Lie groupoid structure on base A^*G.

$$\begin{array}{ccc} T^*G & \longrightarrow & G \\ \downdownarrows & & \downdownarrows \\ A^*G & \longrightarrow & P \end{array}$$

It turns out that these Lie structures form an $\mathcal{LA}$-groupoid $(T^*G;A^*G,G;P)$. ⊠

Chapter 3

Symplectic double groupoids of Poisson Lie groups

In this chapter, we will analyse the construction of a symplectic double groupoid for every pair of dual Poisson Lie groups.

Groupoids were first introduced into symplectic geometry by Karasev [29], Weinstein [62, 12] and Zakrzewski [68, 69]. As mentioned in the introduction, symplectic groupoids arose in relation to the existence of symplectic realisations. The construction of local symplectic groupoids gave a positive answer to the question of whether every Poisson manifold admits a symplectic realisation. Given an arbitrary Poisson manifold P, Weinstein outlined the construction of a local symplectic groupoid Σ with base manifold P [12, Chapter III][61].

The result is stronger in the case of Poisson Lie groups. In [39], Lu and Weinstein showed that every Poisson Lie group G gives rise to a (global) symplectic groupoid Σ with base manifold G. In fact, it turns out that Σ has another groupoid structure with base manifold the dual Poisson Lie group G^*. Moreover, the compatibility between these groupoid structures satisfies that of a symplectic double groupoid,

$$\begin{array}{ccc} \Sigma & \rightrightarrows & G^* \\ \downdownarrows & & \downdownarrows \\ G & \rightrightarrows & \{\cdot\}. \end{array}$$

We will give a full exposition of Lu and Weinstein's results. In Chapter 4, we will see that there is a reverse procedure; that is, for a symplectic double groupoid with a singleton base manifold, the side groupoids obtain Poisson structures that make them dual Poisson Lie groups.

§ 3.1 The Drinfel'd double Lie group

In the previous chapter, we saw that a Poisson Lie group gives rise to a Lie bialgebra, from which one can form the Drinfel'd double Lie algebra. This can be integrated to give another Lie group. In this section, we look at some of the possible Poisson structures on this Lie group and examine how they correspond to the Poisson structures of the original Poisson Lie group and its dual.

3.1.1 The adjoint and coadjoint representations of the Drinfel'd double

Let (G, π_G) be a simply-connected Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}, \mathfrak{g}^*)$. We denote the unique simply-connected Poisson Lie group with tangent Lie bialgebra $(\mathfrak{g}^*, \bar{\mathfrak{g}})$ by (G^*, π_{G^*}). Thus, G and G^* are dual Poison Lie groups. Recall that the Lie brackets on $\mathfrak{g}$ and $\bar{\mathfrak{g}}$ only differ by a minus sign (see Remark 2.43).

In Chapter 2, we saw that we can construct another Lie algebra structure on the direct sum $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$, by defining a Lie bracket in the following way:

$$[X \oplus \xi, Y \oplus \eta] = \big([X, Y] + \mathrm{ad}^*_\xi Y - \mathrm{ad}^*_\eta X\big) \oplus ([\xi, \eta] + \mathrm{ad}^*_X \eta - \mathrm{ad}^*_Y \xi)\,, \tag{3.1}$$

for $X, Y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$. We called this the Drinfel'd double Lie algebra (see Definition 2.47) and denoted it by $\mathfrak{d} = \mathfrak{g} \bowtie \mathfrak{g}^*$. The simply-connected Lie group D with corresponding Lie algebra $\mathfrak{d}$ we call the *Drinfel'd double Lie group.*

The Drinfel'd double Lie algebra $\mathfrak{d}$ is also endowed with a natural symmetric, nondegenerate, ad–invariant bilinear form, given by

$$\langle X \oplus \xi \mid Y \oplus \eta \rangle = \langle \xi, Y \rangle + \langle \eta, X \rangle, \tag{3.2}$$

where $X, Y \in \mathfrak{g}$, and $\xi, \eta \in \mathfrak{g}^*$. Equipped with this bilinear form, $\mathfrak{g}$ and $\mathfrak{g}^*$ form isotropic Lie subalgebras of $\mathfrak{d}$, and $(\mathfrak{d}, \mathfrak{g}, \mathfrak{g}^*)$ forms a Manin triple.

Recall that ad–invariance means that for any $X, Y, Z \in \mathfrak{g}$, and $\xi, \eta, \zeta \in \mathfrak{g}^*$, the following holds

$$\langle [X \oplus \xi, Y \oplus \eta] \mid Z \oplus \zeta \rangle + \langle Y \oplus \eta \mid [X \oplus \xi, Z \oplus \zeta] \rangle = 0.$$

Note that this is equivalent to Ad–invariance of the bilinear form, which means that for every $d \in D$, $X, Y \in \mathfrak{g}$, and $\xi, \eta \in \mathfrak{g}^*$, we have

$$\langle \mathrm{Ad}_d(X \oplus \xi) \mid \mathrm{Ad}_d(Y \oplus \eta) \rangle = \langle X \oplus \xi \mid Y \oplus \eta \rangle.$$

Another observation is that the natural pairing between $\mathfrak{d}$ and its dual space $\mathfrak{d}^*$ (which we identify with $\mathfrak{g}^* \oplus \mathfrak{g}$) is given by

$$\langle \xi \oplus X, Y \oplus \eta \rangle = \langle \xi, Y \rangle + \langle \eta, X \rangle = \langle X \oplus \xi \mid Y \oplus \eta \rangle,$$

where $X, Y \in \mathfrak{g}$, and $\xi, \eta \in \mathfrak{g}^*$.

We denote the natural projections by $p_1 \colon \mathfrak{d} \to \mathfrak{g}$ and $p_2 \colon \mathfrak{d} \to \mathfrak{g}^*$, and the inclusions by $i_1 \colon \mathfrak{g} \to \mathfrak{d}$, $X \mapsto \bar{X}$, and $i_2 \colon \mathfrak{g}^* \to \mathfrak{d}$, $\xi \mapsto \bar{\xi}$. These inclusions integrate to give unique Lie group homomorphisms,

$$\phi_1 \colon G \to D\,,\ g \mapsto \bar{g}; \quad \phi_2 \colon G^* \to D\,,\ u \mapsto \bar{u},$$

satisfying $(\phi_1)_* = i_1$, $(\phi_2)_* = i_2$.

Let us also fix some notation for the natural maps involving the dual space $\mathfrak{d}^* = \mathfrak{g}^* \oplus \mathfrak{g}$. We denote the projections by $q_1 \colon \mathfrak{d}^* \to \mathfrak{g}^*$ and $q_2 \colon \mathfrak{d}^* \to \mathfrak{g}$, and the inclusions by $j_1 \colon \mathfrak{g}^* \to \mathfrak{d}^*$ and $j_2 \colon \mathfrak{g} \to \mathfrak{d}^*$. Note that these projections and inclusions are related by $i_1^* = q_1$, $i_2^* = q_2$, $j_1^* = p_1$ and $j_2^* = p_2$. We also let $\Theta \colon \mathfrak{d} \to \mathfrak{d}^*$ denote the canonical linear isomorphism given by the mapping $X \oplus \xi \mapsto \xi \oplus X$, for $X \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$. This involution relates the projections in $\mathfrak{d}$ and $\mathfrak{d}^*$ by $p_1 = q_2 \circ \Theta$ and $p_2 = q_1 \circ \Theta$.

Let us now examine the adjoint and coadjoint representations of $\mathfrak{d}$ and D in more detail. We first observe how the involution $\Theta \colon \mathfrak{d} \to \mathfrak{d}^*$ defined above can allow us to describe the coadjoint representations in terms of the adjoint representations.

Lemma 3.1. *The coadjoint representation of* $\mathfrak{d}$ *on* $\mathfrak{d}^*$ *is given by*

$$\mathrm{ad}^*_{X\oplus\xi}(\eta\oplus Y) = \Theta(\mathrm{ad}_{X\oplus\xi}(Y\oplus\eta)), \tag{3.3}$$

for $X, Y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$. *Similarly, the coadjoint representation of* D *on* $\mathfrak{d}^*$ *is given by*

$$\mathrm{Ad}^*_d(\xi\oplus X) = \Theta(\mathrm{Ad}_d(X\oplus\xi)), \tag{3.4}$$

for $d \in D$, $X \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$.

Proof. The results can be proved directly from the definition of the Lie bracket on $\mathfrak{d}$ given by (3.1). However, it is simpler to utilise the ad–invariance of the bilinear form defined by (3.2). Let $X, Y, Z \in \mathfrak{g}$, and $\xi, \eta, \zeta \in \mathfrak{g}^*$, and observe that

$$\begin{aligned}\langle \mathrm{ad}^*_{X\oplus\xi}(\eta\oplus Y), Z\oplus\zeta\rangle &= -\langle \eta\oplus Y, \mathrm{ad}_{X\oplus\xi}(Z\oplus\zeta)\rangle \\ &= -\langle Y\oplus\eta \mid \mathrm{ad}_{X\oplus\xi}(Z\oplus\zeta)\rangle \\ &= \langle \mathrm{ad}_{X\oplus\xi}(Y\oplus\eta) \mid Z\oplus\zeta\rangle \\ &= \langle \Theta(\mathrm{ad}_{X\oplus\xi}(Y\oplus\eta)), Z\oplus\zeta\rangle.\end{aligned}$$

Hence, by nondegeneracy of the pairing, the first result follows. For the second formula, take any $d \in D$, $X, Y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$, and observe

$$\begin{aligned}\langle \mathrm{Ad}^*_d(\xi\oplus X), Y\oplus\eta\rangle &= \langle \xi\oplus X, \mathrm{Ad}_{d^{-1}}(Y\oplus\eta)\rangle \\ &= \langle X\oplus\xi \mid \mathrm{Ad}_{d^{-1}}(Y\oplus\eta)\rangle \\ &= \langle \mathrm{Ad}_d(X\oplus\xi) \mid Y\oplus\eta\rangle \\ &= \langle \Theta(\mathrm{Ad}_d(X\oplus\xi)), Y\oplus\eta\rangle.\end{aligned}$$

Again, the result then follows from the nondegeneracy of the pairing. □

Remark 3.2. The second result (3.4) of Lemma 3.1 gives us the following identities

$$\begin{aligned}q_1(\mathrm{Ad}^*_d(\xi\oplus X)) &= p_2(\mathrm{Ad}_d(X\oplus\xi)),\\ q_2(\mathrm{Ad}^*_d(\xi\oplus X)) &= p_1(\mathrm{Ad}_d(X\oplus\xi)),\end{aligned} \tag{3.5}$$

for all $d \in D$, $X_1 \in \mathfrak{g}$, $\xi_1 \in \mathfrak{g}^*$.

The previous lemma implies that the coadjoint representations can be understood in terms of the adjoint representations. From (3.1) we can see that the adjoint representation of $\mathfrak{d}$ can be formulated in terms of the adjoint and coadjoint representations of $\mathfrak{g}$ and $\mathfrak{g}^*$. We now try to relate the adjoint representation of D to the adjoint and coadjoint representations of G and G^*.

Proposition 3.3. *The adjoint representation of* D *on* $\mathfrak{d}$ *has the following properties:*

(i) $\mathrm{Ad}_{\bar g}(\bar X) = \overline{\mathrm{Ad}_g(X)}$, *for all* $g \in G$, $X \in \mathfrak{g}$.

(ii) $\mathrm{Ad}_{\bar u}(\bar\xi) = \overline{Ad_u(\xi)}$, *for all* $u \in G^*$, $\xi \in \mathfrak{g}^*$.

(iii) $p_2(\mathrm{Ad}_{\bar g}(X\oplus\xi)) = \mathrm{Ad}^*_g(\xi)$, *for all* $g \in G$, $X \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$.

(iv) $p_1(\mathrm{Ad}_{\bar u}(X\oplus\xi)) = \mathrm{Ad}^*_u(X)$, *for all* $u \in G^*$, $X \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$.

Proof. Denote by $\widetilde{L}_d$ and $\widetilde{R}_d$, respectively, the left and right translations in D by an element $d \in D$, and L_g and R_g, the left and right translations in G by an element $g \in G$. Let $g, h \in G$, and observe that since ϕ_1 is a group homomorphism, we have

$$\begin{aligned}\widetilde{L}_{\bar{g}} \circ \widetilde{R}_{\bar{g}^{-1}} \circ \phi_1(h) &= \phi_1(g)\phi_1(h)\phi_1(g)^{-1} \\ &= \phi_1(ghg^{-1}) \\ &= \phi_1 \circ L_g \circ R_{g^{-1}}(h).\end{aligned}$$

As h was arbitrarily chosen, it follows that $\widetilde{L}_{\bar{g}} \circ \widetilde{R}_{\bar{g}^{-1}} \circ \phi_1 = \phi_1 \circ L_g \circ R_{g^{-1}}$, and applying the tangent functor at the identity gives

$$\mathrm{Ad}_{\bar{g}} \circ i_1 = i_1 \circ \mathrm{Ad}_g\,. \tag{3.6}$$

Evaluating this equation at an element $X \in \mathfrak{g}$ proves part *(i)*. The statement of *(ii)* is proved in an analogous way.

To show statement *(iii)*, replace g with g^{-1} in (3.6) and take the dual of both sides of the equation to give

$$i_1^* \circ \mathrm{Ad}_{\bar{g}}^* = \mathrm{Ad}_g^* \circ i_1^*\,.$$

Evaluating this equation at an arbitrary element $\xi \oplus X \in \mathfrak{d}^*$, and recalling that $i_1^* = q_1$, gives

$$q_1(\mathrm{Ad}_{\bar{g}}^*(\xi \oplus X)) = \mathrm{Ad}_g^*(q_1(\xi \oplus X)).$$

It follows by equation (3.5) in Remark 3.2 that

$$p_2(\mathrm{Ad}_{\bar{g}}(X \oplus \xi)) = \mathrm{Ad}_g^*(\xi).$$ □

We can describe the Lie brackets of $\mathfrak{g}$ and $\mathfrak{g}^*$ in terms of the Lie bracket of the Drinfel'd double Lie algebra by

$$[X, Y] = p_1([\bar{X}, \bar{Y}]), \quad [\xi, \eta] = p_2([\bar{\xi}, \bar{\eta}]),$$

for $X, Y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$. On the other hand, these brackets can be obtained by linearising the Poisson tensors $-\pi_{G^*}$ and π_G, respectively, at the identity elements. We can combine these two facts to find alternative expressions for the Poisson tensors, which involve the adjoint representation of the Drinfel'd double Lie group.

Proposition 3.4 ([39],[37, Theorem 2.31])**.** *The Poisson tensors on G and G^* are given by the following formulas:*

$$(\pi_G)^L(g)(\xi, \eta) = -\langle p_2(\mathrm{Ad}_{\bar{g}}(\bar{\xi})), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))\rangle, \tag{3.7}$$

for $g \in G$, and $\xi, \eta \in \mathfrak{g}^*$;

$$(\pi_{G^*})^R(u)(X, Y) = \langle p_2(\mathrm{Ad}_{\bar{u}^{-1}}(\bar{X})), p_1(\mathrm{Ad}_{\bar{u}^{-1}}(\bar{Y}))\rangle, \tag{3.8}$$

for $u \in G^*$, and $X, Y \in \mathfrak{g}$.

Proof. To show the first statement, let us first define a bivector field π by setting $\pi^L(g)(\xi, \eta)$ equal to the right-hand side of (3.7), for $g \in G$, and $\xi, \eta \in \mathfrak{g}^*$. We aim to show that π is identical to π_G. The fact that π is anti-symmetric is apparent from the following computation. For $g \in G$, $\xi, \eta \in \mathfrak{g}^*$, we have

$$\begin{aligned}
\pi^L(g)(\xi,\eta) + \pi^L(g)(\eta,\xi) &= -\langle p_2(\mathrm{Ad}_{\bar{g}}(\bar{\xi})), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))\rangle - \langle p_2(\mathrm{Ad}_{\bar{g}}(\bar{\eta})), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\xi}))\rangle \\
&= -\langle p_1(\mathrm{Ad}_{\bar{g}}(\bar{\xi})) \oplus p_2(\mathrm{Ad}_{\bar{g}}(\bar{\xi})) \mid p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta})) \oplus p_2(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))\rangle \\
&= -\langle \mathrm{Ad}_{\bar{g}}(\bar{\xi}) \mid \mathrm{Ad}_{\bar{g}}(\bar{\eta})\rangle \\
&= -\langle \bar{\xi} \mid \bar{\eta}\rangle \\
&= 0.
\end{aligned}$$

Next, we would like to show that π is multiplicative. By Remark 2.26, it is equivalent to show that

$$\pi^L(hg) = \pi^L(g) + \mathrm{Ad}_{g^{-1}}(\pi^L(h)), \tag{3.9}$$

for all $h, g \in G$. To deduce this equation, let us first fix $h, g \in G$, $\xi, \eta \in \mathfrak{g}^*$, and consider

$$\pi^L(hg)(\xi,\eta) = -\langle p_2(\mathrm{Ad}_{\bar{h}\bar{g}}(\bar{\xi})), p_1(\mathrm{Ad}_{\bar{h}\bar{g}}(\bar{\eta}))\rangle = -\langle \mathrm{Ad}^*_{hg}(\xi), p_1(\mathrm{Ad}_{\bar{h}\bar{g}}(\bar{\eta}))\rangle. \tag{3.10}$$

Applying the results of Proposition 3.3, we find

$$\begin{aligned}
p_1(\mathrm{Ad}_{\bar{h}\bar{g}}(\bar{\eta})) &= p_1(\mathrm{Ad}_{\bar{h}}(p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta})) \oplus p_2(\mathrm{Ad}_{\bar{g}}(\bar{\eta})))) \\
&= p_1(\mathrm{Ad}_{\bar{h}}(\overline{p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))})) + p_1(\mathrm{Ad}_{\bar{h}}(\overline{p_2(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))})) \\
&= \mathrm{Ad}_h(p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))) + p_1(\mathrm{Ad}_{\bar{h}}(\overline{\mathrm{Ad}^*_g(\eta)})).
\end{aligned}$$

Hence, equation (3.10) becomes

$$\pi^L(hg)(\xi,\eta) = -\langle \mathrm{Ad}^*_{hg}(\xi), \mathrm{Ad}_h(p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta})))\rangle - \langle \mathrm{Ad}^*_{hg}(\xi), p_1(\mathrm{Ad}_{\bar{h}}(\overline{\mathrm{Ad}^*_g(\eta)}))\rangle. \tag{3.11}$$

However, we can compute

$$\begin{aligned}
\langle \mathrm{Ad}^*_{hg}(\xi), \mathrm{Ad}_h(p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta})))\rangle &= \langle \mathrm{Ad}^*_{h^{-1}}(\mathrm{Ad}^*_{hg}(\xi)), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))\rangle \\
&= \langle \mathrm{Ad}^*_g(\xi), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))\rangle \\
&= \langle p_2(\mathrm{Ad}_{\bar{g}}(\bar{\xi})), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))\rangle \\
&= -\pi^L(g)(\xi,\eta),
\end{aligned}$$

and

$$\begin{aligned}
\langle \mathrm{Ad}^*_{hg}(\xi), p_1(\mathrm{Ad}_{\bar{h}}(\overline{\mathrm{Ad}^*_g(\eta)}))\rangle &= \langle \mathrm{Ad}^*_h(\mathrm{Ad}^*_g(\xi)), p_1(\mathrm{Ad}_{\bar{h}}(\overline{\mathrm{Ad}^*_g(\eta)}))\rangle \\
&= \langle p_2(\mathrm{Ad}_{\bar{h}}(\overline{\mathrm{Ad}^*_g(\xi)})), p_1(\mathrm{Ad}_{\bar{h}}(\overline{\mathrm{Ad}^*_g(\eta)}))\rangle \\
&= -\pi^L(h)(\mathrm{Ad}^*_g(\xi), \mathrm{Ad}^*_g(\eta)) \\
&= -\mathrm{Ad}_{g^{-1}}(\pi^L(h))(\xi,\eta)
\end{aligned}$$

Thus, since equation (3.11) holds for all $\xi, \eta \in \mathfrak{g}^*$, we arrive at equation (3.9).

Now let us consider the linearisation of π at the identity e of G. For any $X \in \mathfrak{g}$, and

$\xi, \eta \in \mathfrak{g}^*$, we have

$$\begin{aligned}
\langle [\xi, \eta]_\pi, X \rangle &= T_e \pi^L(X)(\xi, \eta) \\
&= \frac{d}{dt}\bigg|_{t=0} T(L_{\exp(-tX)})\pi(\exp tX)(\xi, \eta) \\
&= -\frac{d}{dt}\bigg|_{t=0} \langle p_2(\mathrm{Ad}_{\overline{\exp tX}}(\bar{\xi})), p_1(\mathrm{Ad}_{\overline{\exp tX}}(\bar{\eta})) \rangle \\
&= -\langle \frac{d}{dt}\bigg|_{t=0} \left(p_2(\mathrm{Ad}_{\exp t\bar{X}}(\bar{\xi}))\right), 0\rangle - \langle \xi, \frac{d}{dt}\bigg|_{t=0} \left(p_1(\mathrm{Ad}_{\exp t\bar{X}}(\bar{\eta}))\right)\rangle \\
&= -\langle \xi, \frac{d}{dt}\bigg|_{t=0} \left(p_1(\mathrm{Ad}_{\exp t\bar{X}}(\bar{\eta}))\right)\rangle \\
&= -\langle \xi, p_1(\mathrm{ad}_{\bar{X}}(\bar{\eta}))\rangle \\
&= -\langle \xi, -\mathrm{ad}^*_\eta(X)\rangle \\
&= \langle [\xi, \eta], X\rangle.
\end{aligned}$$

Hence, the linearisation of π at the identity of G defines precisely the Lie bracket on $\mathfrak{g}^*$ given by linearising π_G at the identity. By Theorem 2.39, we deduce that π is Poisson, and thus (G, π) forms a Poisson Lie group. Moreover, (G, π) has the same tangent Lie bialgebra as (G, π_G) and so by the uniqueness statement of Theorem 2.46 we have $\pi_G = \pi$. The second formula can be shown using a similar method. □

Remark 3.5. The Poisson structures on G and G^* are also given by the following formulas, which can be verified using the result of Proposition 3.4, or proved directly.

$$(\pi_G)^R(g)(\xi, \eta) = \langle p_2(\mathrm{Ad}_{\bar{g}^{-1}}(\bar{\xi})), p_1(\mathrm{Ad}_{\bar{g}^{-1}}(\bar{\eta}))\rangle, \tag{3.12}$$

for $g \in G$, and $\xi, \eta \in \mathfrak{g}^*$;

$$(\pi_{G^*})^L(u)(X, Y) = -\langle p_2(\mathrm{Ad}_{\bar{u}}(\bar{X})), p_1(\mathrm{Ad}_{\bar{u}}(\bar{Y}))\rangle, \tag{3.13}$$

for $u \in G^*$, and $X, Y \in \mathfrak{g}$.

3.1.2 Poisson structures on the Drinfel'd double

We will now try to construct two different Poisson structures on D. Firstly, consider the element $\pi_0 \in \mathfrak{d} \wedge \mathfrak{d}$ defined by

$$\pi_0(\xi \oplus X, \eta \oplus Y) = \langle \eta, X\rangle - \langle \xi, Y\rangle, \tag{3.14}$$

for $X, Y \in \mathfrak{g}$, $\xi, \eta \in \mathfrak{g}^*$. Note that π_0 is just the standard symplectic form on the vector space $\mathfrak{d} = \mathfrak{g} \oplus \mathfrak{g}^*$. By Remark 2.22, there is a multiplicative bivector field π_- on D, given by

$$\pi_-(d) = \frac{1}{2}(T(R_d)(\pi_0) - T(L_d)(\pi_0)), \tag{3.15}$$

for $d \in D$. In a similar way, we can also define another bivector field π_+ on D by

$$\pi_+(d) = \frac{1}{2}(T(R_d)(\pi_0) + T(L_d)(\pi_0)), \tag{3.16}$$

where $d \in D$.

Proposition 3.6 ([37, Proposition 2.34(1)]). *The two bivector fields π_- and π_+ are Poisson structures on D.*

Proof. We first check whether the linearisation of π_- at the identity element e of D defines a Lie algebra structure on $\mathfrak{d}^*$. If this is the case, then Theorem 2.39 implies that π_- is Poisson. For $\xi \oplus X, \eta \oplus Y \in \mathfrak{d}^*$, and $Z \oplus \zeta \in \mathfrak{d}$, observe that

$$\begin{aligned}
\langle [\xi \oplus X, \eta \oplus Y], Z \oplus \zeta \rangle &= T_e \pi_-^R (Z \oplus \zeta)(\xi \oplus X, \eta \oplus Y) \\
&= \frac{d}{dt}\bigg|_{t=0} T(R_{\exp -t(Z\oplus\zeta)})\pi_-(\exp t(Z \oplus \zeta))(\xi \oplus X, \eta \oplus Y) \\
&= \frac{1}{2}\frac{d}{dt}\bigg|_{t=0} \left(\pi_0 - \mathrm{Ad}_{\exp t(Z\oplus\zeta)}(\pi_0)\right)(\xi \oplus X, \eta \oplus Y) \\
&= -\frac{1}{2}\frac{d}{dt}\bigg|_{t=0} \left(\mathrm{Ad}_{\exp t(Z\oplus\zeta)}(\pi_0)\right)(\xi \oplus X, \eta \oplus Y) \\
&= -\frac{1}{2}\mathrm{ad}_{Z\oplus\zeta}(\pi_0)(\xi \oplus X, \eta \oplus Y) \\
&= \frac{1}{2}\pi_0(\mathrm{ad}^*_{Z\oplus\zeta}(\xi \oplus X), \eta \oplus Y) + \frac{1}{2}\pi_0(\xi \oplus X, \mathrm{ad}^*_{Z\oplus\zeta}(\eta \oplus Y)).
\end{aligned}$$

Making use of Lemma 3.1, we find that

$$\begin{aligned}
\pi_0(\mathrm{ad}^*_{Z\oplus\zeta}(\xi \oplus X), \eta \oplus Y) &= \pi_0(\Theta(\mathrm{ad}_{Z\oplus\zeta}(X \oplus \xi)), \eta \oplus Y) \\
&= \langle \eta, p_1(\mathrm{ad}_{Z\oplus\zeta}(X \oplus \xi))\rangle - \langle p_2(\mathrm{ad}_{Z\oplus\zeta}(X \oplus \xi)), Y\rangle \\
&= \langle \eta, [Z, X] + \mathrm{ad}^*_\zeta X - \mathrm{ad}^*_\xi Z\rangle - \langle [\zeta, \xi] + \mathrm{ad}^*_Z \xi - \mathrm{ad}^*_X \zeta, Y\rangle \\
&= \langle \eta, [Z, X]\rangle - \langle [\zeta, \eta], X\rangle + \langle [\xi, \eta], Z\rangle \\
&\quad - \langle [\zeta, \xi], Y\rangle + \langle \xi, [Z, Y]\rangle - \langle \zeta, [X, Y]\rangle.
\end{aligned}$$

A similar computation shows that

$$\begin{aligned}
\pi_0(\xi \oplus X, \mathrm{ad}^*_{Z\oplus\zeta}(\eta \oplus Y)) &= \langle [\zeta, \eta], X\rangle - \langle \eta, [Z, X]\rangle + \langle \zeta, [Y, X]\rangle \\
&\quad - \langle \xi, [Z, Y]\rangle + \langle [\zeta, \xi], Y\rangle - \langle [\eta, \xi], Z\rangle.
\end{aligned}$$

By combining these results, we can deduce that

$$\langle [\xi \oplus X, \eta \oplus Y], Z \oplus \zeta \rangle = \frac{1}{2}\left(2\langle [\xi, \eta], Z\rangle - 2\langle \zeta, [X, Y]\rangle\right) = \langle [\xi, \eta] \oplus -[X, Y], Z \oplus \zeta\rangle.$$

Hence, we have shown that the linearisation of π_- at the identity element of D defines a well defined Lie bracket on $\mathfrak{d}^*$, given by

$$[\xi \oplus X, \eta \oplus Y] = [\xi, \eta] \oplus -[X, Y], \tag{3.17}$$

where $\xi \oplus X, \eta \oplus Y \in \mathfrak{d}^*$. Thus, by Theorem 2.39, π_- is Poisson.

To show π_+ is also Poisson, first let us define the following bivector fields on D by

$$\pi_0^R(d) := T(R_d)(\pi_0), \quad \pi_0^L(d) := T(L_d)(\pi_0),$$

for $d \in D$. This allows us to write our bivector fields π_- and π_+ on D as

$$\pi_\pm = \frac{1}{2}(\pi_0^R \pm \pi_0^L).$$

Note that π_0^R is right-invariant and π_0^L is left-invariant. Recall that the Schouten bracket of a left-invariant multivector field and a right-invariant multivector field on a Lie group vanishes (see Proposition 2.27). Thus, we have the following results,

$$\begin{aligned}
[\![\pi_+,\pi_+]\!] &= \frac{1}{4}[\![\pi_0^R+\pi_0^L,\pi_0^R+\pi_0^L]\!] \\
&= \frac{1}{4}\left([\![\pi_0^R,\pi_0^R]\!]+[\![\pi_0^L,\pi_0^L]\!]+[\![\pi_0^L,\pi_0^R]\!]+[\![\pi_0^R,\pi_0^L]\!]\right) \\
&= \frac{1}{4}\left([\![\pi_0^R,\pi_0^R]\!]+[\![\pi_0^L,\pi_0^L]\!]\right),
\end{aligned}$$

and

$$\begin{aligned}
[\![\pi_-,\pi_-]\!] &= \frac{1}{4}[\![\pi_0^R-\pi_0^L,\pi_0^R-\pi_0^L]\!] \\
&= \frac{1}{4}\left([\![\pi_0^R,\pi_0^R]\!]+[\![\pi_0^L,\pi_0^L]\!]-[\![\pi_0^L,\pi_0^R]\!]-[\![\pi_0^R,\pi_0^L]\!]\right) \\
&= \frac{1}{4}\left([\![\pi_0^R,\pi_0^R]\!]+[\![\pi_0^L,\pi_0^L]\!]\right).
\end{aligned}$$

Whence, $[\![\pi_+,\pi_+]\!]=[\![\pi_-,\pi_-]\!]=0$, and so π_+ is also Poisson. □

We have shown that π_- is a multiplicative Poisson structure, and so (D,π_-) is a Poisson Lie group. It has tangent Lie bialgebra $(\mathfrak{d},\mathfrak{d}^*)$, where the Lie algebra structure on $\mathfrak{d}^*$ is given by (3.17).

On the other hand, π_+ is not multiplicative as $\pi_+(e)\neq 0$, and therefore does not generate a Poisson Lie group structure on D.

Proposition 3.7 ([37, Proposition 2.34(3)])**.** *Suppose that $d\in D$ can be written as $d=\bar{g}\bar{u}$, for some $g\in G$, $u\in G^*$. Then the following relation holds for π_+:*

$$(T(L_{\bar{g}^{-1}}\circ R_{\bar{u}^{-1}})\pi_+(d))(\xi\oplus X,\eta\oplus Y)=\pi_0(\xi\oplus X,\eta\oplus Y)+\pi_G^L(g)(\xi,\eta)-\pi_{G^*}^R(u)(X,Y),$$

for $X,Y\in\mathfrak{g}$, $\xi,\eta\in\mathfrak{g}^*$. We also have the following relation for π_- given by

$$(T(L_{\bar{g}^{-1}}\circ R_{\bar{u}^{-1}})\pi_-(d))(\xi\oplus X,\eta\oplus Y)=\pi_G^L(g)(\xi,\eta)+\pi_{G^*}^R(u)(X,Y),$$

where $X,Y\in\mathfrak{g}$, $\xi,\eta\in\mathfrak{g}^*$.

Proof. To prove the first formula, take $d\in D$ that satisfies $d=\bar{g}\bar{u}$ for some $g\in G$, $u\in G^*$, and notice that

$$T(L_{\bar{g}^{-1}}\circ R_{\bar{u}^{-1}})\pi_+(d)=\frac{1}{2}\left(\mathrm{Ad}_{\bar{g}^{-1}}(\pi_0)+\mathrm{Ad}_{\bar{u}}(\pi_0)\right).$$

To evaluate this further, let us try to compute the terms $\mathrm{Ad}_{\bar{g}^{-1}}(\pi_0)$ and $\mathrm{Ad}_{\bar{u}}(\pi_0)$. Let us first fix elements $\xi\oplus X,\eta\oplus Y\in\mathfrak{d}^*$.

Note that by Lemma 3.1, we have the following

$$\begin{aligned}
\mathrm{Ad}^*_{\bar{g}}(\xi\oplus X) &= \Theta(\mathrm{Ad}_{\bar{g}}(X\oplus\xi)) \\
&= p_2(\mathrm{Ad}_{\bar{g}}(X\oplus\xi))\oplus p_1(\mathrm{Ad}_{\bar{g}}(X\oplus\xi)) \\
&= \mathrm{Ad}^*_g(\xi)\oplus\left(\mathrm{Ad}_g(X)+p_1(\mathrm{Ad}_{\bar{g}}(\bar{\xi}))\right),
\end{aligned}$$

where in the last line we have made use of the results of Proposition 3.3.

Implementing this calculation, and recalling the definition of π_0, we find

$$\begin{aligned}\mathrm{Ad}_{\bar{g}^{-1}}(\pi_0)(\xi\oplus X,\eta\oplus Y) &= \pi_0\left(\mathrm{Ad}^*_{\bar{g}}(\xi\oplus X),\mathrm{Ad}^*_{\bar{g}}(\eta\oplus Y)\right)\\ &= \langle \mathrm{Ad}^*_g(\eta),\mathrm{Ad}_g(X)+p_1(\mathrm{Ad}_{\bar{g}}(\bar{\xi}))\rangle\\ &\quad -\langle \mathrm{Ad}^*_g(\xi),\mathrm{Ad}_g(Y)+p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))\rangle\\ &= \langle \eta, X\rangle + \langle \mathrm{Ad}^*_g(\eta), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\xi}))\rangle\\ &\quad -\langle \xi, Y\rangle - \langle \mathrm{Ad}^*_g(\xi), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))\rangle\\ &= \langle \eta, X\rangle + \langle p_2(\mathrm{Ad}_{\bar{g}}(\bar{\eta})), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\xi}))\rangle\\ &\quad -\langle \xi, Y\rangle - \langle p_2(\mathrm{Ad}_{\bar{g}}(\bar{\xi})), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\eta}))\rangle\\ &= \pi_0(\xi\oplus X,\eta\oplus Y) + 2(\pi^L_G(g)(\xi,\eta)).\end{aligned}$$

A similar computation shows that,

$$\mathrm{Ad}_{\bar{u}}(\pi_0)(\xi\oplus X,\eta\oplus Y) = \pi_0(\xi\oplus X,\eta\oplus Y) - 2(\pi^R_{G^*}(u)(X,Y)).$$

Combining these results, we have

$$\begin{aligned}(T(L_{\bar{g}^{-1}}\circ R_{\bar{u}^{-1}})\pi_+(d))(\xi\oplus X,\eta\oplus Y) &= \frac{1}{2}\left(\mathrm{Ad}_{\bar{g}^{-1}}(\pi_0)+\mathrm{Ad}_{\bar{u}}(\pi_0)\right)(\xi\oplus X,\eta\oplus Y)\\ &= \pi_0(\xi\oplus X,\eta\oplus Y)+\pi^L_G(g)(\xi,\eta)-\pi^R_{G^*}(u)(X,Y).\end{aligned}$$

The second formula now follows straightforwardly;

$$\begin{aligned}(T(L_{\bar{g}^{-1}}\circ R_{\bar{u}^{-1}})\pi_-(d))(\xi\oplus X,\eta\oplus Y) &= \frac{1}{2}\left(\mathrm{Ad}_{\bar{g}^{-1}}(\pi_0)-\mathrm{Ad}_{\bar{u}}(\pi_0)\right)(\xi\oplus X,\eta\oplus Y)\\ &= \pi^L_G(g)(\xi,\eta)+\pi^R_{G^*}(u)(X,Y).\end{aligned}$$ □

In a similar fashion, we can also prove the following comparable descriptions for π_+ and π_-.

Proposition 3.8. *Suppose that $d\in D$ can be written as $d=\bar{v}\bar{h}$, for some $v\in G^*$, $h\in G$. Then the following relation holds for π_+:*

$$(T(L_{\bar{v}^{-1}}\circ R_{\bar{h}^{-1}})\pi_+(d))(\xi\oplus X,\eta\oplus Y) = \pi_0(\xi\oplus X,\eta\oplus Y) - \pi^R_G(h)(\xi,\eta)+\pi^L_{G^*}(v)(X,Y),$$

for $X,Y\in\mathfrak{g}$, $\xi,\eta\in\mathfrak{g}^*$. We also have the following relation for π_- given by

$$(T(L_{\bar{v}^{-1}}\circ R_{\bar{h}^{-1}})\pi_-(d))(\xi\oplus X,\eta\oplus Y) = \pi^R_G(h)(\xi,\eta)+\pi^L_{G^*}(v)(X,Y),$$

where $X,Y\in\mathfrak{g}$, $\xi,\eta\in\mathfrak{g}^*$. □

It is straightforward to show that $\phi_1\colon G\to D$ and $\phi_2\colon G^*\to D$ are immersions. Let us consider the case when ϕ_1 and ϕ_2 are also injective. In this situation, we can identify G and G^* with Lie subgroups of D. We have seen that D is a Poisson Lie group when it is equipped with the Poisson structure π_-. The following result implies that, in this scenario, (G,π_G) and (G^*,π_{G^*}) are Poisson Lie subgroups of (D,π_-).

Proposition 3.9 ([37, Proposition 2.36(1)]). *The following maps are Poisson:*

$$\phi_1\colon (G,\pi_G)\to(D,\pi_-),\quad g\mapsto\bar{g};$$

$$\phi_2\colon (G^*,\pi_{G^*})\to(D,\pi_-),\quad u\mapsto\bar{u}.$$

Proof. Let us fix $g \in G$. We need to show that

$$T(\phi_1)(\pi_G(g)) = \pi_-(\bar{g}), \tag{3.18}$$

to prove that ϕ_1 is a Poisson map.

By the formula given in Proposition 3.7, for any $\xi \oplus X, \eta \oplus Y \in \mathfrak{d}^*$, we find

$$(T(L_{\bar{g}^{-1}})\pi_-(\bar{g}))(\xi \oplus X, \eta \oplus Y) = \pi_G^L(g)(\xi, \eta) + \pi_{G^*}^R(e)(X, Y) = \pi_G^L(g)(\xi, \eta).$$

On the other hand,

$$\begin{aligned}(T(L_{\bar{g}^{-1}} \circ \phi_1)\pi_G(g))(\xi \oplus X, \eta \oplus Y) &= (T(\phi_1 \circ L_{g^{-1}})\pi_G(g))(\xi \oplus X, \eta \oplus Y)\\ &= i_1(\pi_G^L(g))(\xi \oplus X, \eta \oplus Y)\\ &= \pi_G^L(g)(i_1^*(\xi \oplus X), i_1^*(\eta \oplus Y))\\ &= \pi_G^L(g)(q_1(\xi \oplus X), q_1(\eta \oplus Y))\\ &= \pi_G^L(g)(\xi, \eta).\end{aligned}$$

This shows that

$$T(L_{\bar{g}^{-1}}) \circ T(\phi_1)(\pi_G(g)) = T(L_{\bar{g}^{-1}})(\pi_-(\bar{g})),$$

and then we are reduced to (3.18) by applying the linear isomorphism $T_e(L_{\bar{g}})$ to both sides of this equation. We can show that ϕ_2 is a Poisson map using an analogous method. □

We have seen some of the importance of the Poisson structure π_-. Let us now focus on our second Poisson structure π_+ and what properties it possesses.

Proposition 3.10.[1] *Suppose that $d \in D$ can be decomposed as $d = \bar{g}\bar{u} = \bar{v}\bar{h}$ for some $g, h \in G$, $u, v \in G^*$. Then π_+ is nondegenerate at d.*

Proof. Let $d \in D$ be an element that has the factorisations $d = \bar{g}\bar{u} = \bar{v}\bar{h}$, for some $g, h \in G$, $u, v \in G^*$. We first make the simple observation that $\pi_+(d)$ is nondegenerate if and only if $(T(L_{\bar{g}^{-1}}) \circ T(R_{\bar{u}^{-1}})\pi_+(d))$ is nondegenerate.

Suppose that $\xi \oplus X \in \mathfrak{d}^*$ satisfies

$$(T(L_{\bar{g}^{-1}} \circ R_{\bar{u}^{-1}})\pi_+(d))(\xi \oplus X, \eta \oplus Y) = 0,$$

for every $\eta \oplus Y \in \mathfrak{d}^*$. We need to show that this implies that $\xi \oplus X = 0$.

By Proposition 3.7, we can show that for every $\eta \in \mathfrak{g}^*$,

$$\begin{aligned}(T(L_{\bar{g}^{-1}} \circ R_{\bar{u}^{-1}})\pi_+(d))(\xi \oplus X, \bar{\eta}) &= \pi_0(\xi \oplus X, \bar{\eta}) - \pi_G^L(\eta, \xi)\\ &= \langle \eta, X\rangle + \langle p_2(\mathrm{Ad}_{\bar{g}}(\bar{\eta})), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\xi}))\rangle\\ &= \langle \mathrm{Ad}_g^*(\eta), \mathrm{Ad}_g(X)\rangle + \langle \mathrm{Ad}_g^*(\eta), p_1(\mathrm{Ad}_{\bar{g}}(\bar{\xi}))\rangle\\ &= \langle \mathrm{Ad}_g^*(\eta), \mathrm{Ad}_g(X) + p_1(\mathrm{Ad}_{\bar{g}}(\bar{\xi}))\rangle\\ &= \langle \mathrm{Ad}_g^*(\eta), p_1(\mathrm{Ad}_{\bar{g}}(\bar{X})) + p_1(\mathrm{Ad}_{\bar{g}}(\bar{\xi}))\rangle\\ &= \langle \mathrm{Ad}_g^*(\eta), p_1(\mathrm{Ad}_{\bar{g}}(X \oplus \xi))\rangle\end{aligned}$$

[1]A statement of this result was given by Semenov-Tian-Shansky [59, Proposition 6]. We follow the proof outlined by Lu [37, Proposition 2.35].

Similarly, it can be shown that

$$(T(L_{\bar{g}^{-1}} \circ R_{\bar{u}^{-1}})\pi_+(d))(\xi \oplus X, \bar{Y}) = -\langle p_2(\mathrm{Ad}_{\bar{u}^{-1}}(X \oplus \xi)), \mathrm{Ad}^*_{u^{-1}}(Y)\rangle$$

for all $Y \in \mathfrak{g}$.

By our initial assumption, these two equations must vanish for every $\eta \in \mathfrak{g}^*$ and $Y \in \mathfrak{g}$, and so we deduce that $p_1(\mathrm{Ad}_{\bar{g}}(X \oplus \xi)) = 0$, and $p_2(\mathrm{Ad}_{\bar{u}^{-1}}(X \oplus \xi)) = 0$. Thus there exists $Z \in \mathfrak{g}$ and $\zeta \in \mathfrak{g}^*$, such that $\mathrm{Ad}_{\bar{u}^{-1}}(X \oplus \xi) = \bar{Z}$ and $\mathrm{Ad}_{\bar{g}}(X \oplus \xi) = \bar{\zeta}$.

Rearranging these two equations leads us to the identity $\mathrm{Ad}_{\bar{u}}(\bar{Z}) = \mathrm{Ad}_{\bar{g}^{-1}}(\bar{\zeta})$. However, then $\mathrm{Ad}_{\bar{h}}(\bar{Z}) = \mathrm{Ad}_{\bar{v}^{-1}}(\bar{\zeta})$, utilising the property that $\bar{g}\bar{u} = \bar{v}\bar{h}$. Now applying the results of Proposition 3.3 we arrive at the equation

$$\mathrm{Ad}_h(Z) \oplus 0 = 0 \oplus \mathrm{Ad}_{v^{-1}}(\zeta).$$

Thus, $\mathrm{Ad}_h(Z) = 0 \in \mathfrak{g}$, and $\mathrm{Ad}_{v^{-1}}(\zeta) = 0 \in \mathfrak{g}^*$. Finally, we conclude that both $Z = 0$ and $\zeta = 0$; then it follows that $X \oplus \xi = 0$. □

The next result shows that when the Drinfel'd double D is equipped with the Poisson structure π_+, we have natural Poisson actions of G and G^* on D.

Proposition 3.11 ([39, Proposition 2]). *The following maps are Poisson and anti-Poisson, respectively:*

$$\sigma_1 \colon (G, \pi_G) \times (D, \pi_+) \to (D, \pi_+), \quad (g, d) \mapsto \bar{g}d;$$

$$\sigma_2 \colon (D, \pi_+) \times (G^*, \pi_{G^*}) \to (D, \pi_+), \quad (d, u) \mapsto d\bar{u}.$$

Proof. We will prove that the first map is Poisson; the second map can be shown to be anti-Poisson using a similar method. Firstly, to simplify notation let us write $\sigma = \sigma_1$ and $\sigma(g, d) = \bar{g}d = \sigma_g(d) = \sigma_d(g)$, for $g \in G$, $d \in D$. Note that $\sigma_g = L_{\bar{g}}$, and $\sigma_d = R_d \circ \phi_1$. By Proposition 2.11, we only need to show that

$$\pi_+(\bar{g}d) = T(\sigma_g)(\pi_+(d)) + T(\sigma_d)(\pi_G(g)),$$

for all $g \in G$, $d \in D$. Observe that

$$\begin{aligned}
\pi_+(\bar{g}d) - T(\sigma_g)(\pi_+(d)) &= \frac{1}{2}(T(R_{\bar{g}d})(\pi_0) + T(L_{\bar{g}d})(\pi_0)) \\
&\quad - T(L_{\bar{g}}) \left(\frac{1}{2}(T(R_d)(\pi_0) + T(L_d)(\pi_0)) \right) \\
&= \frac{1}{2} \Big(T(R_d) \circ T(R_{\bar{g}})(\pi_0) - T(L_{\bar{g}}) \circ T(R_d)(\pi_0) \Big) \\
&= T(R_d)(\pi_-(\bar{g})),
\end{aligned}$$

for $g \in G$, $d \in D$. Hence, it remains to show that

$$T(R_d)(\pi_-(\bar{g})) = T(\sigma_d)(\pi_G(g)) = T(R_d) \circ T(\phi_1)(\pi_G(g)),$$

for all $g \in G$, $d \in D$. Equivalently, we just need to show that

$$T(\phi_1)(\pi_G(g)) = \pi_-(\bar{g}),$$

for all $g \in G$, to prove that σ_1 is a Poisson map. However, this is just the statement that ϕ_1 is a Poisson map, which was proved in Proposition 3.9. □

We now look to induce Poisson structures on the product manifolds $G\times G^*$ and $G^*\times G$, using the Poisson structure π_+ defined on D. We do this by first constructing local diffeomorphisms from $G \times G^*$ and $G^* \times G$ into D.

Proposition 3.12 ([26, Lemma 5.2]). *The maps given by*

$$\Phi_1 \colon G \times G^* \to D, \quad (g, u) \mapsto \bar{g}\bar{u};$$

$$\Phi_2 \colon G^* \times G \to D, \quad (v, h) \mapsto \bar{v}\bar{h};$$

are local diffeomorphisms.

Proof. To prove the statement about Φ_1, it is equivalent to show that Φ_1 is étale at every point of $G \times G^*$. That is, for every $(g, u) \in G \times G^*$, we are required to show that the linear map,

$$T_{(g,u)}\Phi_1 \colon T_{(g,u)}(G \times G^*) \to T_{\bar{g}\bar{u}}D,$$

is an isomorphism.

Let us fix $g \in G$, $u \in G^*$. We first make the canonical identification between the vector spaces $T_{(g,u)}(G \times G^*)$ and $T_gG \oplus T_uG^*$, and note that

$$\dim(T_gG \oplus T_uG^*) = \dim(\mathfrak{g} \oplus \mathfrak{g}^*) = \dim(\mathfrak{d}) = \dim(T_{\bar{g}\bar{u}}D).$$

It thus suffices to show that $T_{(g,u)}\Phi_1$ is injective.

Let $\tilde{X} \in T_gG$, $\tilde{\xi} \in T_uG^*$, and suppose

$$T\Phi_1(\tilde{X}, \tilde{\xi}) = 0. \tag{3.19}$$

It remains to show that both $\tilde{X} = 0$, and $\tilde{\xi} = 0$.

Let $X \in \mathfrak{g}$, $\xi \in \mathfrak{g}^*$ denote the unique vectors such that $\tilde{X} = T(L_g)(X)$, $\tilde{\xi} = T(L_u)(\xi)$. Observe that the curve $t \mapsto g \exp tX$ in G has tangent vector $T(L_g)(X)$ at g, and the curve $t \mapsto u \exp t\xi$ in G^* has tangent vector $T(L_u)(\xi)$ at u. Hence,

$$\begin{aligned} T\Phi_1(\tilde{X}, \tilde{\xi}) &= T\Phi_1(T(L_g)(X), 0) + T\Phi_1(0, T(L_u)(\xi)) \\ &= \left.\frac{d}{dt}\right|_{t=0} \Big(\Phi_1(g \exp tX, u)\Big) + \left.\frac{d}{dt}\right|_{t=0} \Big(\Phi_1(g, u \exp t\xi)\Big). \end{aligned} \tag{3.20}$$

Note that

$$\begin{aligned} \Phi_1(g \exp tX, u) &= \phi_1(g \exp tX)\phi_2(u) \\ &= \phi_1(g)\phi_1(\exp tX)\phi_2(u) \\ &= \phi_1(g) \exp(t(\phi_1)_*X)\phi_2(u) \\ &= \bar{g} \exp(t\bar{X})\bar{u}. \end{aligned}$$

Also, notice that

$$\begin{aligned} \bar{u}^{-1} \exp(t\bar{X})\bar{u} &= (L_{\bar{u}^{-1}} \circ R_{\bar{u}})(\exp t\bar{X}) \\ &= \exp(t(L_{\bar{u}^{-1}} \circ R_{\bar{u}})_*\bar{X}) \\ &= \exp(t \operatorname{Ad}_{\bar{u}^{-1}}(\bar{X})). \end{aligned}$$

Thus, $\exp(t\bar{X})\bar{u} = \bar{u} \exp(t \operatorname{Ad}_{\bar{u}^{-1}}(\bar{X}))$, and so

$$\Phi_1(g \exp tX, u) = \bar{g}\bar{u} \exp(t \operatorname{Ad}_{\bar{u}^{-1}}(\bar{X})).$$

Similarly, we have

$$\Phi_1(g, u \exp t\xi) = \bar{g}\bar{u} \exp t\bar{\xi}.$$

Hence, equation (3.20) becomes

$$T\Phi_1(\tilde{X}, \tilde{\xi}) = \frac{d}{dt}\bigg|_{t=0} \Big(\bar{g}\bar{u} \exp(t(\mathrm{Ad}_{\bar{u}^{-1}}(\bar{X}) + \bar{\xi}))\Big) = T(L_{\bar{g}\bar{u}})(\mathrm{Ad}_{\bar{u}^{-1}}(\bar{X}) + \bar{\xi}).$$

By the assumption (3.19), this implies that $\mathrm{Ad}_{\bar{u}^{-1}}(\bar{X}) + \bar{\xi} = 0$. Rearranging and making use of Proposition 3.3 gives $X \oplus \mathrm{Ad}_u(\xi) = 0$. It follows that $X = 0$, and $\xi = 0$, and thus $\tilde{X} = 0$, and $\tilde{\xi} = 0$. □

These two local diffeomorphisms Φ_1 and Φ_2 give us a way of defining Poisson structures on $G \times G^*$ and $G^* \times G$. Denote by π_1 the unique Poisson structure such that $\Phi_1 \colon (G \times G^*, \pi_1) \to (D, \pi_+)$ is a Poisson map, and similarly denote by π_2 the unique Poisson structure such that $\Phi_2 \colon (G^* \times G, \pi_2) \to (D, \pi_+)$ is a Poisson map.

Explicitly, π_1 is given by

$$\pi_1(g, u) = T_{(g,u)}(\Phi_1)^{-1}(\pi_+(\bar{g}\bar{u})),$$

for $(g, u) \in G \times G^*$, and π_2 is given by

$$\pi_2(v, h) = T_{(v,h)}(\Phi_2)^{-1}(\pi_+(\bar{v}\bar{h})),$$

for $(v, h) \in G^* \times G$. In terms of the Poisson anchors we have the following relations:

$$\pi_1^{\#}(g, u) = T_{(g,u)}(\Phi_1)^{-1} \circ \pi_+^{\#}(\bar{g}\bar{u}) \circ T^*_{(g,u)}(\Phi_1)^{-1};$$

$$\pi_2^{\#}(v, h) = T_{(v,h)}(\Phi_2)^{-1} \circ \pi_+^{\#}(\bar{v}\bar{h}) \circ T^*_{(v,h)}(\Phi_2)^{-1}.$$

Next we will see that the formulas for π_+ given in Propositions 3.7 and 3.8 induce similar formulas for π_1 and π_2.

Proposition 3.13. *For $(g, u) \in G \times G^*$, and $\xi \oplus X, \eta \oplus Y \in \mathfrak{d}^*$, we have the following formula for π_1:*

$$\begin{aligned}(T(L_{(g^{-1},e)} \circ R_{(e,u^{-1})})\pi_1(g, u))(\xi \oplus X, \eta \oplus Y) &= \pi_0(\xi \oplus X, \eta \oplus Y)\\ &\quad + \pi_G^L(g)(\xi, \eta) - \pi_{G^*}^R(u)(X, Y);\end{aligned}$$

and for $(v, h) \in G^* \times G$, we have the following formula for π_2:

$$\begin{aligned}(T(L_{(v^{-1},e)} \circ R_{(e,h^{-1})})\pi_2(v, h))(X \oplus \xi, Y \oplus \eta) &= \pi_0(\xi \oplus X, \eta \oplus Y)\\ &\quad - \pi_G^R(h)(\xi, \eta) + \pi_{G^*}^L(v)(X, Y).\end{aligned}$$

Proof. Fix $(g, u) \in G \times G^*$, and $\xi \oplus X, \eta \oplus Y \in \mathfrak{d}^*$. Firstly, we note that the following relations are easily verified:

$$\Phi_1 \circ L_{(g^{-1},e)} = L_{\bar{g}^{-1}} \circ \Phi_1;$$

$$\Phi_1 \circ R_{(e,u^{-1})} = R_{\bar{u}^{-1}} \circ \Phi_1.$$

By applying the tangent functor to these equations and rearranging, it follows that

$$T_{(g,u)}(L_{(g^{-1},e)} \circ R_{(e,u^{-1})}) = T_{(e,e)}(\Phi_1)^{-1} \circ T_{\bar{g}\bar{u}}(L_{\bar{g}^{-1}} \circ R_{\bar{u}^{-1}}) \circ T_{(g,u)}(\Phi_1).$$

Hence, we have

$$\begin{aligned}
&(T(L_{(g^{-1},e)} \circ R_{(e,u^{-1})})\pi_1(g,u))(\xi \oplus X, \eta \oplus Y) \\
&\quad = ((T_{(e,e)}(\Phi_1)^{-1} \circ T_{\bar{g}\bar{u}}(L_{\bar{g}^{-1}} \circ R_{\bar{u}^{-1}}) \circ T_{(g,u)}(\Phi_1))\pi_1(g,u))(\xi \oplus X, \eta \oplus Y) \\
&\quad = (T(L_{\bar{g}^{-1}} \circ R_{\bar{u}^{-1}})\pi_+(\bar{g}\bar{u}))(T^*_{(e,e)}(\Phi_1)^{-1}(\xi \oplus X), T^*_{(e,e)}(\Phi_1)^{-1}(\eta \oplus Y)) \\
&\quad = (T(L_{\bar{g}^{-1}} \circ R_{\bar{u}^{-1}})\pi_+(\bar{g}\bar{u}))(\xi \oplus X, \eta \oplus Y) \\
&\quad = \pi_0(\xi \oplus X, \eta \oplus Y) + \pi^L_G(g)(\xi, \eta) - \pi^R_{G^*}(u)(X, Y).
\end{aligned}$$

The formula for π_2 can be proved in a similar manner. □

§ 3.2 The Lu & Weinstein symplectic double groupoid

In this section we finally describe Lu and Weinstein's construction of a symplectic double groupoid for every pair of dual Poisson Lie groups G and G^*. The results were announced in [39] and further details were given in [37].

3.2.1 A symplectic double groupoid of G and G^*

Let G and G^* be dual Poisson Lie groups and D the corresponding Drinfel'd double Lie group. For now, we assume that G and G^* are both simply-connected. We begin the construction by considering a specific submanifold of the product manifold $G \times G^* \times G^* \times G$. In the previous section, we saw that the maps $\Phi_1 \colon G \times G^* \to D$ and $\Phi_2 \colon G \times G^* \to D$ are local diffeomorphisms. In particular, it follows that Φ_1 and Φ_2 are transversal to each other. Thus, the pullback $\Sigma = (\Phi_1 \times \Phi_2)^{-1}(\Delta_D)$ is an embedded submanifold of $G \times G^* \times G^* \times G$ of codimension equal to $\dim D = 2 \dim G$. Explicitly, we can express

$$\Sigma = \{(g, u, v, h) \in G \times G^* \times G^* \times G \mid \bar{g}\bar{u} = \bar{v}\bar{h}\}.$$

Note that, by definition of codimension, we have

$$\operatorname{codim} \Sigma = \dim(G \times G^* \times G^* \times G) - \dim \Sigma.$$

This becomes,

$$2 \dim G = 4 \dim G - \dim \Sigma,$$

and thus, $\dim \Sigma = 2 \dim G$.

We now aim to give Σ the structure of a double groupoid with side groupoids given by G and G^*. We make the initial observation that Σ is contained in the manifold $\square(G, G^*)$ of Example 1.68. Moreover, we claim that the structure maps of $\square(G, G^*)$ restrict to Σ to define a double groupoid $(\Sigma; G, G^*; \{\cdot\})$.

$$\begin{array}{ccc}
\Sigma & \overset{\alpha_*,\, \beta_*}{\rightrightarrows} & G^* \\
{\scriptstyle \alpha,\beta}\downdownarrows & & \downdownarrows \\
G & \rightrightarrows & \{\cdot\}.
\end{array}$$

To verify this that this is the case, it is sufficient to check that the restrictions of the structure maps are well-defined. The checks are routine and left to the reader.

Let us explicitly write down the structure maps. The vertical structure $\Sigma \rightrightarrows G$, has source and target projections given by

$$\alpha\colon \Sigma \to G\,, \quad (g,u,v,h) \mapsto h;$$

$$\beta\colon \Sigma \to G\,, \quad (g,u,v,h) \mapsto g.$$

The partial multiplication $\kappa\colon \Sigma *_G \Sigma \to \Sigma$ is defined by

$$(g_2,u_2,v_2,h_2) \boxminus (g_1,u_1,v_1,h_1) = (g_2,u_2u_1,v_2v_1,h_1).$$

Note that the space of compatible pairs is determined by

$$\Sigma *_G \Sigma = \{((g_2,u_2,v_2,h_2),(g_1,u_1,v_1,h_1)) \in \Sigma \times \Sigma \mid h_2 = g_1\}.$$

The identity map is given by

$$1^G\colon G \to \Sigma\,, \quad g \mapsto (g,e,e,g);$$

and the inversion map is given by

$$\iota\colon \Sigma \to \Sigma\,, \quad (g,u,v,h) \mapsto (h,u^{-1},v^{-1},g).$$

On the other hand, the horizontal structure $\Sigma \rightrightarrows G^*$ has source and target projections given by

$$\alpha_*\colon \Sigma \to G^*\,, \quad (g,u,v,h) \mapsto u;$$

$$\beta_*\colon \Sigma \to G^*\,, \quad (g,u,v,h) \mapsto v.$$

The partial multiplication $\kappa_*\colon \Sigma *_{G^*} \Sigma \to \Sigma$ is defined by

$$(g_2,u_2,v_2,h_2) \boxbar (g_1,u_1,v_1,h_1) = (g_2g_1,u_1,v_2,h_2h_1);$$

where the space of compatible elements is governed by

$$\Sigma *_{G^*} \Sigma = \{((g_2,u_2,v_2,h_2),(g_1,u_1,v_1,h_1)) \in \Sigma \times \Sigma \mid u_2 = v_1\}.$$

Lastly, the identity map is given by

$$1^{G^*}\colon G^* \to \Sigma\,, \quad u \mapsto (e,u,u,e);$$

and the inversion map is given by

$$\iota_*\colon \Sigma \to \Sigma\,, \quad (g,u,v,h) \mapsto (g^{-1},v,u,h^{-1}).$$

Proposition 3.14. *With the structure maps defined as above,* $(\Sigma; G, G^*; \{\cdot\})$ *is a weak double Lie groupoid.*

Proof. Since the structure maps of the double Lie groupoid $\square(G,G^*)$ are smooth, the structure maps of Σ must also be smooth because they are simply restrictions of the domains and codomains of these maps to embedded submanifolds. In particular, since the inversion maps are self-inverse they must also be diffeomorphisms. It remains only to show that the double source map and the source and target projections of the horizontal and vertical structures are all submersions.

We first focus on the source projection of the vertical structure. Consider an arbitrary element $s = (g,u,v,h) \in \Sigma$; we would like to show that the tangent map $T_s(\alpha)\colon T_s\Sigma \to T_hG$ is a surjection. Fix $Y \in T_hG$, and choose any $\eta \in T_vG^*$. Then, we have $T(\Phi_2)(\eta, Y) \in T_{\bar{v}\bar{h}}D = T_{\bar{g}\bar{u}}D$, and so, since Φ_1 is a local diffeomorphism, there exist vectors $X \in T_gG$ and $\xi \in T_uG^*$ such that $T(\Phi_1)(X, \xi) = T(\Phi_2)(\eta, Y)$. Thus, $(X, \xi, \eta, Y) \in T_s\Sigma$ and, moreover, $T(\alpha)(X, \xi, \eta, Y) = Y$. Hence, the source projection α is a submersion.

Since the target projection of the vertical structure $\beta = \alpha \circ \iota$ is a composite of submersions, it is also a submersion. We can show that the source and target projections of the horizontal structure are submersions using analogous arguments.

Finally, we show that the double source map $\alpha_2 \coloneqq (\alpha, \alpha_*) \colon \Sigma \to G \times G^*$ is a submersion. Consider an arbitrary element $s = (g, u, v, h) \in \Sigma$ and the corresponding tangent map $T_s(\alpha_2) \colon T_s\Sigma \to T_hG \oplus T_uG^*$. Let us denote the inversion map of the Lie group G by $\iota_G \colon G \to G$. Fix any $Y \in T_hG$ and $\xi \in T_uG^*$, and then define $Y' = T(\iota_G)(Y) \in T_{h^{-1}}G$. It follows that $T(\Phi_2)(\xi, Y') \in T_{\bar{u}\bar{h}^{-1}}D = T_{\bar{g}^{-1}\bar{v}}D$. Now, since Φ_1 is a local diffeomorphism, there exists $X' \in T_{g^{-1}}G$ and $\eta \in T_vG^*$ with $T(\Phi_1)(X', \eta) = T(\Phi_2)(\xi, Y')$. Thus, we have $(X', \eta, \xi, Y') \in T_{\iota_*(s)}\Sigma$. We now further define $X = T(\iota_G)(X') \in T_gG$, and make the observation that

$$T(\iota_*)(X', \eta, \xi, Y') = (T(\iota_G)(X'), \xi, \eta, T(\iota_G)(Y')) = (X, \xi, \eta, Y).$$

Therefore, $(X, \xi, \eta, Y) \in T_s\Sigma$ and, moreover, $T(\alpha_2)(X, \xi, \eta, Y) = (Y, \xi)$. This shows that the tangent map $T_s(\alpha_2)$ is surjective. Hence, the double source map is a submersion. □

Our remaining objective is to provide Σ with a symplectic structure that makes $(\Sigma; G, G^*; \{\cdot\})$ a symplectic double groupoid. In the previous section, we built Poisson structures on the Drinfel'd double Lie group D and the product manifolds $G \times G^*$ and $G^* \times G$. In a bid to pull back these Poisson structures onto Σ, we first construct local diffeomorphisms from Σ to D, $G \times G^*$, and $G^* \times G$.

Proposition 3.15. *The maps given by*

$$\Psi_1 \colon \Sigma \to G \times G^*, \quad (g, u, v, h) \mapsto (g, u);$$

$$\Psi_2 \colon \Sigma \to G^* \times G, \quad (g, u, v, h) \mapsto (v, h);$$

are local diffeomorphisms.

Proof. To show that Ψ_1 is a local diffeomorphism we will prove that, for an arbitrary $s = (g, u, v, h) \in \Sigma$, the tangent map

$$T_s(\Psi_1) \colon T_s\Sigma \to T_gG \oplus T_uG^*$$

is an isomorphism. Since $\dim \Sigma = 2 \dim G = \dim(G \times G^*)$, we need only show that $T_s(\Psi_1)$ is injective.

Let $(X, \xi, \eta, Y) \in T_s\Sigma$, and suppose that $T_s(\Psi_1)(X, \xi, \eta, Y) = (0, 0)$. Now, since Ψ_1 is just a projection, we have $T_s(\Psi_1)(X, \xi, \eta, Y) = (X, \xi)$, and so $X = 0$ and $\xi = 0$. We also have the property that

$$T_{(g,u)}(\Phi_1)(X, \xi) = T_{(v,h)}(\Phi_2)(\eta, Y),$$

and since $T_{(v,h)}(\Phi_2)$ is an isomorphism by Proposition 3.12, we also deduce that $\eta = 0$ and $Y = 0$. Hence, $T_s(\Psi_1)$ is an injection and thus Ψ_1 is a local diffeomorphism. A similar proof shows that Ψ_2 is also a local diffeomorphism. □

Since both Ψ_1 and Φ_1 are local diffeomorphisms, their composition

$$\Phi\colon \Sigma \to D; \quad (g,u,v,h) \mapsto \bar{g}\bar{u},$$

is also a local diffeomorphism. This allows us to define a Poisson structure π on Σ – the unique Poisson structure such that $\Phi\colon (\Sigma,\pi) \to (D,\pi_+)$ is a Poisson map. More precisely, π is defined by

$$\pi(s) = T_s(\Phi)^{-1}(\pi_+(d)),$$

for $s = (g,u,v,h) \in \Sigma$, where $d = \bar{g}\bar{u} \in D$. In terms of the Poisson anchor we can formulate this relation as

$$\pi^{\#}(s) = T_s(\Phi)^{-1} \circ \pi_+^{\#}(d) \circ T_s^*(\Phi)^{-1}.$$

We saw in Proposition 3.10 that π_+ is nondegenerate at every $d \in \mathrm{Im}(\Phi)$, that is, $\pi_+^{\#}(d)$ is an isomorphism. It follows by the above relationship that π is nondegenerate at every point of Σ. Hence, π gives rise to a symplectic structure on Σ.

Note that Φ is also equal to the composition of Ψ_2 and Φ_2, and so we have the following commutative diagram of Poisson local diffeomorphisms:

$$\begin{array}{ccc} (\Sigma,\pi) & \xrightarrow{\Psi_2} & (G^* \times G, \pi_2) \\ \Big\downarrow{\scriptstyle \Psi_1} & & \Big\downarrow{\scriptstyle \Phi_2} \\ (G \times G^*, \pi_1) & \xrightarrow{\Phi_1} & (D,\pi_+). \end{array}$$

We now deduce some useful properties of this Poisson structure on Σ.

Proposition 3.16. *With Σ endowed with the Poisson structure π, the target projections $\beta\colon \Sigma \to G$ and $\beta_*\colon \Sigma \to G^*$ are Poisson maps, and the source projections $\alpha\colon \Sigma \to G$ and $\alpha_*\colon \Sigma \to G^*$ are anti-Poisson maps.*

Proof. Let us start by considering the target projection $\beta\colon \Sigma \to G$. We define the following map of Poisson manifolds:

$$\widetilde{\beta}\colon (G \times G^*, \pi_1) \to (G, \pi_G), \quad (g,u) \mapsto g.$$

Note that we have $\beta = \widetilde{\beta} \circ \Psi_1$. Since $\Psi_1\colon (\Sigma,\pi) \to (G \times G^*, \pi_1)$ is a Poisson map, it is sufficient to prove that $\widetilde{\beta}$ is a Poisson map to conclude that β is also a Poisson map.

Take any $g \in G$, $u \in G^*$, and $\varphi, \psi \in T_g^*G$, and observe that

$$\begin{aligned} &T_{(g,u)}(\widetilde{\beta})(\pi_1(g,u))(\varphi,\psi) \\ &\quad = \pi_1(g,u)\left(T^*_{(g,u)}(\widetilde{\beta})(\varphi), T^*_{(g,u)}(\widetilde{\beta})(\psi)\right) \\ &\quad = \pi_1(g,u)(\varphi \oplus 0, \psi \oplus 0) \\ &\quad = (T(L_{(g^{-1},e)} \circ R_{(e,u^{-1})})\pi_1(g,u))\,(T_e^*(L_g)(\varphi) \oplus 0, T_e^*(L_g)(\psi) \oplus 0) \\ &\quad = \langle 0, T_e^*(L_g)(\psi)\rangle - \langle 0, T_e^*(L_g)(\varphi)\rangle \\ &\qquad + \pi_G^L(g)(T_e^*(L_g)(\varphi), T_e^*(L_g)(\psi)) - \pi_{G^*}^R(u)(0,0) \\ &\quad = \pi_G(g)(\varphi,\psi). \end{aligned}$$

This shows that $\widetilde{\beta}$ is a Poisson map, and hence so is the target projection β. Note that in the second to last line of the computation above, we are using the formula for π_1 given in Proposition 3.8.

To prove that the source projection $\alpha\colon \Sigma \to G$ is a Poisson map, we first define the map of Poisson manifolds

$$\widetilde{\alpha}\colon (G^* \times G, \pi_2) \to (G, \pi_G), \quad (v,h) \mapsto h.$$

Since $\alpha = \widetilde{\alpha} \circ \Psi_2$, and $\Psi_2\colon (\Sigma, \pi) \to (G^* \times G, \pi_2)$ is a Poisson map, it is sufficient to show that $\widetilde{\alpha}$ is an anti-Poisson map to prove that the source projection α is also an anti-Poisson map.

Observe that for any $h \in G$, $v \in G^*$, and $\varphi, \psi \in T_h^*G$, we have

$$\begin{aligned}
&T_{(v,h)}(\widetilde{\alpha})(\pi_2(v,h))(\varphi,\psi)\\
&\quad= \pi_2(v,h)\left(T^*_{(v,h)}(\widetilde{\alpha})(\varphi), T^*_{(v,h)}(\widetilde{\alpha})(\psi)\right)\\
&\quad= \pi_2(v,h)(0\oplus\varphi, 0\oplus\psi)\\
&\quad= \left(T(L_{(v^{-1},e)} \circ R_{(e,h^{-1})})\pi_2(v,h)\right)\left(0\oplus T_e^*(R_h)(\varphi), 0\oplus T_e^*(R_h)(\psi)\right)\\
&\quad= \langle 0, T_e^*(R_h)(\psi)\rangle - \langle 0, T_e^*(R_h)(\varphi)\rangle\\
&\qquad - \pi_G^R(h)(T_e^*(R_h)(\varphi), T_e^*(R_h)(\psi)) + \pi_{G^*}^L(v)(0,0)\\
&\quad= -\pi_G(h)(\varphi,\psi).
\end{aligned}$$

Thus, $\widetilde{\alpha}$ is an anti-Poisson map, and hence so is the source projection α. Here, in the second to last line of the computation above, we are using the formula for π_2 given in Proposition 3.8.

Similar arguments show that the source and target projections of the horizontal structure $\Sigma \rightrightarrows G^*$ are also Poisson and anti-Poisson maps, respectively. □

Proposition 3.17. *The domains $\Sigma *_G \Sigma$ and $\Sigma *_{G^*} \Sigma$ of the partial multiplication maps of Σ are coisotropic submanifolds of $\Sigma \times \Sigma$.*

Proof. As a consequence of Proposition 3.16, the map $\alpha \times \beta\colon \Sigma \times \Sigma \to \overline{G} \times G$ is a Poisson map. In Example 2.17, we showed that the diagonal Δ_G is a coisotropic submanifold of $\overline{G} \times G$. Thus, by Proposition 2.18 the domain $\Sigma *_G \Sigma = (\alpha \times \beta)^{-1}(\Delta_G)$ of the partial multiplication in the vertical structure $\Sigma \rightrightarrows G$ is a coisotropic submanifold of $\Sigma \times \Sigma$. Arguing in a similar fashion shows that the domain $\Sigma *_{G^*} \Sigma$ of the partial multiplication in the horizontal structure $\Sigma \rightrightarrows G^*$ is also a coisotropic submanifold of $\Sigma \times \Sigma$. □

We now state Lu and Weinstein's theorem, which is the main result of this section.

Theorem 3.18 ([39, Theorem 3])**.** *With Σ endowed with the Poisson structure π, $(\Sigma; G, G^*; \{\cdot\})$ is a symplectic double groupoid.*

Before giving a complete proof of this theorem, we first prove some preliminaries results.

Let us introduce some new notation to make more compact some of the formulas that will follow. Given vectors $X \in T_gG$, $\xi \in T_uG^*$, $\eta \in T_h^*G$, $Y \in T_v^*G^*$, we denote by $X_L, X_R, Y_L, Y_R \in \mathfrak{g}$ and $\xi_L, \xi_R, \eta_L, \eta_R \in \mathfrak{g}^*$, the unique vectors satisfying

$$X = T(L_g)(X_L) = T(R_g)(X_R), \quad \xi = T(L_u)(\xi_L) = T(R_u)(\xi_R),$$
$$\eta = T_h^*(L_{h^{-1}})(\eta_L) = T_h^*(R_{h^{-1}})(\eta_R), \quad Y = T_v^*(L_{v^{-1}})(Y_L) = T_v^*(R_{v^{-1}})(Y_R).$$

Lemma 3.19. *Let $(X,\xi,\eta,Y)\in T_s\Sigma$, with $s=(g,u,v,h)\in\Sigma$. Then we have the relations*

$$X_L\oplus\xi_R=\mathrm{Ad}_{\bar{g}^{-1}}(\overline{\eta_R})+\mathrm{Ad}_{\bar{u}}(\overline{Y_L}),\tag{3.21}$$

and

$$Y_R\oplus\eta_L=\mathrm{Ad}_{\bar{v}^{-1}}(\overline{X_R})+\mathrm{Ad}_{\bar{h}}(\overline{\xi_L}).\tag{3.22}$$

Proof. Since Σ is the pullback manifold $(\Phi_1\times\Phi_2)^{-1}(\Delta_D)$, we have the equation

$$T(\Phi_1)(X,\xi)=T(\Phi_2)(\eta,Y).$$

The relations follow after computing the tangent maps of Φ_1 and Φ_2. □

Corollary 3.20. *Let $(X,\xi,\eta,Y)\in T_s\Sigma$, with $s=(g,u,v,h)\in\Sigma$. Then we have the relations*

(i) $X_L=p_1(\mathrm{Ad}_{\bar{g}^{-1}}(\overline{\eta_R}))+\mathrm{Ad}^*_u(Y_L)$;

(ii) $\xi_R=\mathrm{Ad}^*_{g^{-1}}(\eta_R)+p_2(\mathrm{Ad}_{\bar{u}}(\overline{Y_L}))$;

(iii) $\eta_L=p_2(\mathrm{Ad}_{\bar{v}^{-1}}(\overline{X_R}))+\mathrm{Ad}^*_h(\xi_L)$;

(iv) $Y_R=\mathrm{Ad}^*_{v^{-1}}(X_R)+p_1(\mathrm{Ad}_{\bar{h}}(\overline{\xi_L}))$. □

Proposition 3.21. *Let $(s_1,s_2)\in\Sigma*_G\Sigma$, with $s_i=(g_i,u_i,v_i,h_i)$, for $i=1,2$. Then the space $(\pi^\#\oplus\pi^\#)(T_{(s_1,s_2)}(\Sigma*_G\Sigma)^\circ)$ consists of elements of the form*

$$\big((0,T(L_{u_1})(\xi),T(L_{v_1})(\eta),T(R_{h_1})(Y)),(T(R_{h_1})(Y),-T(R_{u_2})(\xi),-T(R_{v_2})(\eta),0)\big),$$

where $\xi,\eta\in\mathfrak{g}^*$, $Y\in\mathfrak{g}$ are vectors satisfying $Y\oplus\eta=\mathrm{Ad}_{\bar{h}_1}(\bar{\xi})$.

Proof. Let us fix an arbitrary pair $(s_1,s_2)\in\Sigma*_G\Sigma$, where $s_i=(g_i,u_i,v_i,h_i)$, for $i=1,2$. In particular, we have $h_1=g_2$. We claim that,

$$T_{(s_1,s_2)}(\Sigma*_G\Sigma)^\circ=\{(T^*_{s_1}(\alpha)(\varphi),-T^*_{s_2}(\beta)(\varphi))\in T^*_{s_1}\Sigma\times T^*_{s_2}\Sigma\mid\varphi\in T^*_{h_1}G\}.$$

The inclusion of the right-hand side is easily seen from that fact that $\Sigma*_G\Sigma$ is given by the pullback manifold $(\alpha\times\beta)^{-1}(\Delta_G)$, and the reverse inclusion follows from a dimension count. Therefore, $(\pi^\#\oplus\pi^\#)(T_{(s_1,s_2)}(\Sigma*_G\Sigma)^\circ)$ consists of elements of the form

$$\big(\pi^\#(T^*_{s_1}(\alpha)(\varphi)),-\pi^\#(T^*_{s_2}(\beta)(\varphi))\big),$$

where $\varphi\in T^*_{h_1}G$. Let us define $(X_1,\xi_1,\eta_1,Y_1)=\pi^\#(T^*_{s_1}(\alpha)(\varphi))$, and $(X_2,\xi_2,\eta_2,Y_2)=\pi^\#(T^*_{s_2}(\beta)(\varphi))$, for a fixed $\varphi\in T^*_{h_1}G$. Our next objective is to find more explicit relations between φ and X_i, ξ_i, η_i, Y_i, for $i=1,2$.

Recall, from the proof of Proposition 3.16, the projections $\widetilde{\alpha}\colon G^*\times G\to G$, and $\widetilde{\beta}\colon G\times G^*\to G$, which satisfy $\alpha=\tilde{\alpha}\circ\Psi_2$ and $\beta=\widetilde{\beta}\circ\Psi_1$. Observe that we have

$$\begin{aligned}\pi^\#_{s_1}\circ T^*_{s_1}(\alpha)&=(T_{s_1}(\Psi_2)^{-1}\circ\pi_2^\#(v_1,h_1)\circ T^*_{s_1}(\Psi_2)^{-1})\circ(T^*_{s_1}(\Psi_2)\circ T^*_{(v_1,h_1)}(\widetilde{\alpha}))\\&=T_{s_1}(\Psi_2)^{-1}\circ\pi_2^\#(v_1,h_1)\circ T^*_{(v_1,h_1)}(\widetilde{\alpha}),\end{aligned}$$

and

$$\begin{aligned}\pi^\#_{s_2}\circ T^*_{s_2}(\beta)&=(T_{s_2}(\Psi_1)^{-1}\circ\pi^\#1(g_2,u_2)\circ T^*_{s_2}(\Psi_1)^{-1})\circ(T^*_{s_2}(\Psi_1)\circ T^*_{(g_2,u_2)}(\widetilde{\beta}))\\&=T_{s_2}(\Psi_1)^{-1}\circ\pi_1^\#(g_2,u_2)\circ T^*_{(g_2,u_2)}(\widetilde{\beta}).\end{aligned}$$

Using these relations, we conclude that

$$(\eta_1, Y_1) = T_{s_1}(\Psi_2)(X_1, \xi_1, \eta_1, Y_1) = \pi_2^{\#}(v_1, h_1)(T^*_{(v_1,h_1)}(\widetilde{\alpha})(\varphi)),$$

and

$$(X_2, \xi_2) = T_{s_2}(\Psi_1)(X_2, \xi_2, \eta_2, Y_2) = \pi_1^{\#}(g_2, u_2)(T^*_{(g_2,u_2)}(\widetilde{\beta})(\varphi)).$$

Moreover, since the maps $\widetilde{\alpha}$ and $\widetilde{\beta}$ are projections we also have $T^*_{(v_1,h_1)}(\widetilde{\alpha})(\varphi) = 0 \oplus \varphi$ and $T^*_{(g_2,u_2)}(\widetilde{\beta})(\varphi) = \varphi \oplus 0$.

Now, fix arbitrary vectors $\tilde{X} \in T^*_{v_1}G^*$, $\tilde{\xi} \in T^*_{h_1}G$, and observe that

$$\begin{aligned}
&\langle \eta_1 \oplus Y_1, \tilde{X} \oplus \tilde{\xi}\rangle = \langle \pi_2^{\#}(v_1, h_1)(0 \oplus \varphi), \tilde{X} \oplus \tilde{\xi}\rangle = \pi_2(v_1, h_1)(0 \oplus \varphi, \tilde{X} \oplus \tilde{\xi})\\
&= (T(L_{(v_1^{-1},e)} \circ R_{(e,h_1^{-1})})\pi_2(v_1, h_1))(0 \oplus T^*_e(R_{h_1})(\varphi), T^*_e(L_{v_1})(\tilde{X}) \oplus T^*_e(R_{h_1})(\tilde{\xi}))\\
&= -\langle T^*_e(L_{v_1})(\tilde{X}), T^*_e(R_{h_1})(\varphi)\rangle - \pi^R_G(h_1)(T^*_e(R_{h_1})(\varphi), T^*_e(R_{h_1})(\tilde{\xi})).
\end{aligned}$$

Using the more compact notation discussed on page 78, we can write this as

$$\begin{aligned}
\langle \eta_{1,L} \oplus Y_{1,R}, \tilde{X}_L \oplus \tilde{\xi}_R\rangle &= -\langle \tilde{X}_L, \varphi_R\rangle - \pi^R_G(h_1)(\varphi_R, \tilde{\xi}_R)\\
&= \langle \tilde{X}_L, -\varphi_R\rangle + \langle p_1(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R})), p_2(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\tilde{\xi}_R}))\rangle\\
&= \langle \tilde{X}_L, -\varphi_R\rangle + \langle p_1(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R})), \mathrm{Ad}^*_{h_1^{-1}}(\tilde{\xi}_R)\rangle\\
&= \langle \tilde{X}_L, -\varphi_R\rangle + \langle \mathrm{Ad}_{h_1}(p_1(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}))), \tilde{\xi}_R\rangle\\
&= \langle (-\varphi_R) \oplus \mathrm{Ad}_{h_1}(p_1(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}))), \tilde{X}_L \oplus \tilde{\xi}_R\rangle.
\end{aligned}$$

Here, we have made use of the properties given in Proposition 3.3. The nondegeneracy of the pairing implies that

$$\eta_{1,L} = -\varphi_R\,, \quad Y_{1,R} = \mathrm{Ad}_{h_1}(p_1(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}))).$$

Alternatively, we can write these relations as

$$\eta_{1,R} = -\mathrm{Ad}_{v_1}(\varphi_R)\,, \quad Y_{1,L} = p_1(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R})).$$

Arguing similarly, fix $\tilde{\eta} \oplus \tilde{Y} \in T^*_{g_2}G \oplus T^*_{u_2}G^*$, and observe that

$$\begin{aligned}
&\langle X_2 \oplus \xi_2, \tilde{\eta} \oplus \tilde{Y}\rangle = \langle \pi_1^{\#}(g_2, u_2)(\varphi \oplus 0), \tilde{\eta} \oplus \tilde{Y}\rangle = \pi_1(g_2, u_2)(\varphi \oplus 0, \tilde{\eta} \oplus \tilde{Y})\\
&= (T(L_{(g_2^{-1},e)} \circ R_{(e,u_2^{-1})})\pi_1(g_2, u_2))(T^*_e(L_{g_2})(\varphi) \oplus 0, T^*_e(L_{g_2})(\tilde{\eta}) \oplus T^*_e(R_{u_2})(\tilde{Y}))\\
&= -\langle T^*_e(R_{u_2})(\tilde{Y}), T^*_e(L_{g_2})(\varphi)\rangle + \pi^L_G(g_2)(T^*_e(L_{g_2})(\varphi), T^*_e(L_{g_2})(\tilde{\eta})).
\end{aligned}$$

Switching to our compact notation, this equation becomes

$$\begin{aligned}
\langle X_{2,L} \oplus \xi_{2,R}, \tilde{\eta}_L \oplus \tilde{Y}_R\rangle &= -\langle \tilde{Y}_R, \varphi_L\rangle + \pi^L_G(g_2)(\varphi_L, \tilde{\eta}_L)\\
&= \langle \tilde{Y}_R, -\varphi_L\rangle + \langle p_1(\mathrm{Ad}_{\bar{g}_2}(\overline{\varphi_L})), p_2(\mathrm{Ad}_{\bar{g}_2}(\overline{\tilde{\eta}_L}))\rangle\\
&= \langle \tilde{Y}_R, -\varphi_L\rangle + \langle p_1(\mathrm{Ad}_{\bar{g}_2}(\overline{\varphi_L})), \mathrm{Ad}^*_{g_2}(\tilde{\eta}_L)\rangle\\
&= \langle \tilde{Y}_R, -\varphi_L\rangle + \langle \mathrm{Ad}_{g_2^{-1}}(p_1(\mathrm{Ad}_{\bar{g}_2}(\overline{\varphi_L}))), \tilde{\eta}_L\rangle\\
&= \langle \mathrm{Ad}_{g_2^{-1}}(p_1(\mathrm{Ad}_{\bar{g}_2}(\overline{\varphi_L}))) \oplus (-\varphi_L), \tilde{\eta}_L \oplus \tilde{Y}_R\rangle.
\end{aligned}$$

Again, the nondegeneracy of the pairing implies that

$$X_{2,L} = \mathrm{Ad}_{g_2^{-1}}(p_1(\mathrm{Ad}_{\bar{g}_2}(\overline{\varphi_L})))\,, \quad \xi_{2,R} = -\varphi_L.$$

We can also write these relations as

$$X_{2,R} = p_1(\mathrm{Ad}_{\bar{g}_2}(\overline{\varphi_L}))\,, \quad \xi_{2,L} = -\mathrm{Ad}_{u_2^{-1}}(\varphi_L).$$

Now, since $(X_1, \xi_1, \eta_1, Y_1) \in T_{s_1}\Sigma$, the relation (3.22) of Lemma 3.19 gives

$$Y_{1,R} \oplus \eta_{1,L} = \mathrm{Ad}_{\bar{v}_1^{-1}}(\overline{X_{1,R}}) + \mathrm{Ad}_{\bar{h}_1}(\overline{\xi_{1,L}}). \tag{3.23}$$

Observe that, by property (i) of Corollary 3.20, we have

$$\begin{aligned} X_{1,L} &= p_1(\mathrm{Ad}_{\bar{g}_1^{-1}}(\overline{\eta_{1,R}})) + \mathrm{Ad}^*_{u_1}(Y_{1,L}) \\ &= p_1(\mathrm{Ad}_{\bar{g}_1^{-1}}(\overline{-\mathrm{Ad}_{v_1}(\varphi_R)})) + \mathrm{Ad}^*_{u_1}(p_1(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}))) \\ &= -\,p_1(\mathrm{Ad}_{\bar{g}_1^{-1}}(\mathrm{Ad}_{\bar{v}_1}(\overline{\varphi_R}))) + p_1(\mathrm{Ad}_{\bar{u}_1}(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}))) \\ &= 0, \end{aligned}$$

since $\bar{g}_1^{-1}\bar{v}_1 = \bar{u}_1\bar{h}_1^{-1}$. Let us define vectors $\xi, \eta \in \mathfrak{g}^*$, $Y \in \mathfrak{g}$, such that $\xi = \xi_{1,L}$, $\eta = \eta_{1,L}$, and $Y = Y_{1,R}$. Then equation (3.23) becomes

$$Y \oplus \eta = \mathrm{Ad}_{\bar{h}_1}(\overline{\xi}). \tag{3.24}$$

Notice that by this relation, Y and η are uniquely determined by ξ. We also have

$$(X_1, \xi_1, \eta_1, Y_1) = (0, T(L_{u_1})(\xi), T(L_{v_1})(\eta), T(R_{h_1})(Y)).$$

Finally, let us try to compute $(X_2, \xi_2, \eta_2, Y_2)$ in terms of ξ, η, and Y. We will prove that the following relations hold:

(i) $X_{2,R} = -Y$;

(ii) $\xi_{2,R} = \xi$;

(iii) $\eta_{2,R} = \eta$;

(iv) $Y_{2,R} = 0$.

To prove the first statement (i), we need to show $Y_{1,R} + X_{2,R} = 0$. Recall that we have $Y_{1,R} = \mathrm{Ad}_{h_1}(p_1(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R})))$ and $X_{2,R} = p_1(\mathrm{Ad}_{\bar{g}_2}(\overline{\varphi_L}))$. Also, note that we have $h_1 = g_2$. Pairing $Y_{1,R} + X_{2,R}$ with an arbitrary element $\zeta \in \mathfrak{g}^*$ gives

$$\begin{aligned} \langle Y_{1,R} + X_{2,R}, \zeta\rangle &= \langle \mathrm{Ad}_{h_1}(p_1(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}))), \zeta\rangle + \langle p_1(\mathrm{Ad}_{\bar{h}_1}(\overline{\varphi_L})), \zeta\rangle \\ &= \langle \mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}) \mid \overline{\mathrm{Ad}^*_{h_1^{-1}}(\zeta)}\rangle + \langle \mathrm{Ad}_{\bar{h}_1}(\overline{\mathrm{Ad}^*_{h_1^{-1}}(\varphi_R)}) \mid \overline{\zeta}\rangle \\ &= \langle \mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}) \mid \overline{p_2(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\zeta}))}\rangle + \langle \overline{\mathrm{Ad}^*_{h_1^{-1}}(\varphi_R)} \mid \mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\zeta})\rangle \\ &= \langle \mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}) \mid \overline{p_2(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\zeta}))}\rangle + \langle \overline{p_2(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}))} \mid \mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\zeta})\rangle \\ &= \langle \mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}) \mid \overline{p_2(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\zeta}))}\rangle + \langle \mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}) \mid \overline{p_1(\mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\zeta}))}\rangle \\ &= \langle \mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\varphi_R}) \mid \mathrm{Ad}_{\bar{h}_1^{-1}}(\overline{\zeta})\rangle \\ &= \langle \overline{\varphi_R} \mid \overline{\zeta}\rangle \\ &= 0. \end{aligned}$$

From the nondegeneracy of the pairing, we deduce $Y_{1,R} + X_{2,R} = 0$.

Next, we prove (iv). By the fourth relation of Corollary 3.20, we have

$$\begin{aligned} Y_{2,R} &= \operatorname{Ad}^*_{v_2^{-1}}(X_{2,R}) + p_1(\operatorname{Ad}_{\bar h_2}(\overline{\xi_{2,R}})) \\ &= \operatorname{Ad}^*_{v_2^{-1}}(p_1(\operatorname{Ad}_{\bar g_2}(\overline{\varphi_L}))) + p_1(\operatorname{Ad}_{\bar h_2}(\overline{-\operatorname{Ad}_{u_2^{-1}}(\varphi_L)})) \\ &= p_1(\operatorname{Ad}_{\bar v_2^{-1}}(\operatorname{Ad}_{\bar g_2}(\overline{\varphi_L}))) - p_1(\operatorname{Ad}_{\bar h_2}(\operatorname{Ad}_{\bar u_2^{-1}}(\overline{\varphi_L}))) \\ &= 0. \end{aligned}$$

The last line follows because $\bar v_2^{-1}\bar g_2 = \bar h_2 \bar u_2^{-1}$. This further implies that $Y_2 = 0$.

For the second statement (ii), recall that we have $\xi_{2,R} = -\varphi_L$, and so we need to show that $\xi = -\varphi_L$. By the equation (3.24), this is equivalent to showing that $Y \oplus \eta = -\operatorname{Ad}_{\bar h_1}(\overline{\varphi_L})$. Observe that

$$\eta = \eta_{1,L} = -\varphi_R = -\operatorname{Ad}^*_{h_1}(\varphi_L) = -\,p_2(\operatorname{Ad}_{\bar h_1}(\overline{\varphi_L})).$$

Therefore, it remains only to show that $Y = -\,p_1(\operatorname{Ad}_{\bar h_1}(\overline{\varphi_L}))$. Recall that we have $Y = Y_{1,R} = \operatorname{Ad}_{h_1}(p_1(\operatorname{Ad}_{\bar h_1^{-1}}(\overline{\varphi_R})))$. By pairing Y with an arbitrary element $\zeta \in \mathfrak{g}^*$, we find

$$\begin{aligned} \langle Y, \zeta\rangle &= \langle \operatorname{Ad}_{h_1}(p_1(\operatorname{Ad}_{\bar h_1^{-1}}(\overline{\varphi_R}))), \zeta\rangle \\ &= \langle p_1(\operatorname{Ad}_{\bar h_1^{-1}}(\overline{\varphi_R})), \operatorname{Ad}^*_{h_1^{-1}}(\zeta)\rangle \\ &= \langle \operatorname{Ad}_{\bar h_1^{-1}}(\overline{\varphi_R}) \mid \overline{\operatorname{Ad}^*_{h_1^{-1}}(\zeta)}\rangle \\ &= \langle \overline{\varphi_R} \mid \operatorname{Ad}_{\bar h_1}(\overline{\operatorname{Ad}^*_{h_1^{-1}}(\zeta)})\rangle \\ &= \langle \overline{\operatorname{Ad}^*_{h_1}(\varphi_L)} \mid \operatorname{Ad}_{\bar h_1}(\overline{\operatorname{Ad}^*_{h_1^{-1}}(\zeta)})\rangle \\ &= \langle \overline{p_2(\operatorname{Ad}_{\bar h_1}(\overline{\varphi_L}))} \mid \operatorname{Ad}_{\bar h_1}(\overline{\operatorname{Ad}^*_{h_1^{-1}}(\zeta)})\rangle \\ &= \langle \operatorname{Ad}_{\bar h_1}(\overline{\varphi_L}) \mid \overline{p_1(\operatorname{Ad}_{\bar h_1}(\overline{\operatorname{Ad}^*_{h_1^{-1}}(\zeta)}))}\rangle \\ &= \langle \operatorname{Ad}_{\bar h_1}(\overline{\varphi_L}) \mid \operatorname{Ad}_{\bar h_1}(\overline{\operatorname{Ad}^*_{h_1^{-1}}(\zeta)})\rangle \\ &\quad - \langle \operatorname{Ad}_{\bar h_1}(\overline{\varphi_L}) \mid \overline{p_2(\operatorname{Ad}_{\bar h_1}(\overline{\operatorname{Ad}^*_{h_1^{-1}}(\zeta)}))}\rangle \\ &= \langle \overline{\varphi_L} \mid \overline{\operatorname{Ad}^*_{h_1^{-1}}(\zeta)}\rangle - \langle \operatorname{Ad}_{\bar h_1}(\overline{\varphi_L}) \mid \overline{\operatorname{Ad}^*_{h_1}(\operatorname{Ad}^*_{h_1^{-1}}(\zeta))}\rangle \\ &= -\langle \operatorname{Ad}_{\bar h_1}(\overline{\varphi_L}) \mid \overline{\zeta}\rangle \\ &= \langle -\,p_1(\operatorname{Ad}_{\bar h_1}(\overline{\varphi_L})), \zeta\rangle. \end{aligned}$$

By nondegeneracy of the pairing, the required result follows.

To prove the third statement (iii), first recall that $\eta = \eta_{1,L} = -\varphi_R$. Thus, we need to show that $\eta_{2,R} = -\varphi_R$. By Corollary 3.20 and statement (iv), this is equivalent to showing that

$$X_{2,L} \oplus \xi_{2,R} = -\operatorname{Ad}_{\bar g_2^{-1}}(\overline{\varphi_R}).$$

However, we notice that

$$\xi_{2,R} = -\varphi_L = -\operatorname{Ad}^*_{g_2^{-1}}(\varphi_R) = -\,p_2(\operatorname{Ad}_{\bar g_2^{-1}}(\overline{\varphi_R})).$$

Hence, it remains to show that $X_{2,L} = -\,p_1(\operatorname{Ad}_{\bar g_2^{-1}}(\overline{\varphi_R}))$. Pairing $X_{2,L}$ with an arbitrary element $\zeta \in \mathfrak{g}^*$ gives

$$\begin{aligned}
\langle X_{2,L}, \zeta\rangle &= \langle \mathrm{Ad}_{g_2^{-1}}(p_1(\mathrm{Ad}_{\bar{g}_2}(\overline{\varphi_L}))), \zeta\rangle \\
&= \langle p_1(\mathrm{Ad}_{\bar{g}_2}(\overline{\varphi_L})), \mathrm{Ad}^*_{g_2}(\zeta)\rangle \\
&= \langle \mathrm{Ad}_{\bar{g}_2}(\overline{\varphi_L}) \mid \overline{\mathrm{Ad}^*_{g_2}(\zeta)}\rangle \\
&= \langle \overline{\varphi_L} \mid \mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\mathrm{Ad}^*_{g_2}(\zeta)})\rangle \\
&= \langle \overline{\mathrm{Ad}^*_{g_2^{-1}}(\varphi_R)} \mid \mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\mathrm{Ad}^*_{g_2}(\zeta)})\rangle \\
&= \langle \overline{p_2(\mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\varphi_R}))} \mid \mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\mathrm{Ad}^*_{g_2}(\zeta)})\rangle \\
&= \langle \mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\varphi_R}) \mid \overline{p_1(\mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\mathrm{Ad}^*_{g_2}(\zeta)}))}\rangle \\
&= \langle \mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\varphi_R}) \mid \mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\mathrm{Ad}^*_{g_2}(\zeta)})\rangle \\
&\quad - \langle \mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\varphi_R}) \mid \overline{p_2(\mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\mathrm{Ad}^*_{g_2}(\zeta)}))}\rangle \\
&= \langle \overline{\varphi_R} \mid \overline{\mathrm{Ad}^*_{g_2}(\zeta)}\rangle - \langle \mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\varphi_R}) \mid \overline{\mathrm{Ad}^*_{g_2^{-1}}(\mathrm{Ad}^*_{g_2}(\zeta))}\rangle \\
&= -\langle \mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\varphi_R}) \mid \bar{\zeta}\rangle \\
&= \langle -\, p_1(\mathrm{Ad}_{\bar{g}_2^{-1}}(\overline{\varphi_R})), \zeta\rangle.
\end{aligned}$$

The nondegeneracy of the pairing, gives the required result.

Properties (i)–(iv) lead us to the conclusion that

$$(X_2, \xi_2, \eta_2, Y_2) = (-T(R_{g_2})(Y), T(R_{u_2})(\xi), T(R_{v_2})(\eta), 0).$$

Hence, we have shown that the space $(\pi^\# \oplus \pi^\#)(T_{(s_1,s_2)}(\Sigma *_G \Sigma)^\circ)$ contains only pairs of the form

$$\big((0, T(L_{u_1})(\xi), T(L_{v_1})(\eta), T(R_{h_1})(Y)), (T(R_{h_1})(Y), -T(R_{u_2})(\xi), -T(R_{v_2})(\eta), 0)\big),$$

where $\xi, \eta \in \mathfrak{g}^*$, $Y \in \mathfrak{g}$ are vectors satisfying $Y \oplus \eta = \mathrm{Ad}_{\bar{h}_1}(\bar{\xi})$. □

We are now able to give a proof of Lu and Weinstein's theorem.

Proof of Theorem 3.18. To prove that $(\Sigma; G, G^*; \{\cdot\})$ is a symplectic groupoid, we need only to show that the vertical structure $\Sigma \rightrightarrows G$ and the horizontal structure $\Sigma \rightrightarrows G^*$ are Poisson groupoids with respect to π. We will prove only that the vertical structure is a Poisson groupoid; a similar argument can be used for the horizontal structure.

By definition of a Poisson groupoid, we need to show that the graph of the partial multiplication,

$$\begin{aligned}
\Gamma(\kappa) &= \{(s_2, s_1, \kappa(s_1, s_2)) \mid (s_1, s_2) \in \Sigma *_G \Sigma\} \\
&= \{((g_2, u_2, v_2, h_2), (g_1, u_1, v_1, h_1), (g_2, u_2u_1, v_2v_1, h_1)) \in (\Sigma *_G \Sigma) \times \Sigma\},
\end{aligned}$$

is a coisotropic submanifold of $\Sigma \times \Sigma \times \overline{\Sigma}$.

We consider the graph $\Gamma(\mu)$ of the map

$$\mu\colon \Sigma *_G \Sigma \to D\,, \quad ((g_2, u_2, v_2, h_2), (g_1, u_1, v_1, h_1)) \mapsto \bar{g}_2\bar{u}_2\bar{u}_1.$$

Observe that $\Gamma(\kappa)$ is diffeomorphic to $\Gamma(\mu)$ via the Poisson local diffeomorphism $\mathrm{id}_{\Sigma *_G \Sigma} \times \Phi$. Hence, $\Gamma(\kappa)$ is a coisotropic submanifold of $\Sigma \times \Sigma \times \overline{\Sigma}$, if and only if, $\Gamma(\mu)$ is a coisotropic submanifold of $\Sigma \times \Sigma \times \overline{D}$.

Note that μ is the restriction of the map

$$\tilde{\mu} \colon \Sigma \times \Sigma \to D\,, ((g_2, u_2, v_2, h_2), (g_1, u_1, v_1, h_1)) \mapsto \bar{g}_2 \bar{u}_2 \bar{u}_1,$$

to $\Sigma *_G \Sigma$, and we can write $\tilde{\mu} = \sigma_2 \circ (\Phi \times \alpha_*)$. Recall $\sigma_2 \colon D \times G^* \to D$; $(d, u) \mapsto d\bar{u}$, is an anti-Poisson map by Proposition 3.11, the source projection $\alpha_* \colon \Sigma \to G^*$ is an anti-Poisson map by Proposition 3.16, and $\Phi \colon \Sigma \to D$ is a Poisson map by construction of π. Therefore, $\tilde{\mu}$ is a Poisson map.

By Proposition 2.20, we deduce that $\Gamma(\mu)$ is a coisotropic submanifold of $\Sigma \times \Sigma \times \overline{D}$, if and only if,

$$(\pi^{\#} \oplus \pi^{\#})(T(\Sigma *_G \Sigma)^{\circ}) \subseteq \ker(T(\tilde{\mu})).$$

A simple calculation shows that, for any $(s_1, s_2) \in \Sigma *_G \Sigma$, with $s_i = (g_i, u_i, v_i, h_i)$, for $i = 1, 2$, and any pair $((X_1, \xi_1, \eta_1, Y_1), (X_2, \xi_2, \eta_2, Y_2)) \in T_{(s_1, s_2)}(\Sigma *_G \Sigma)$, we have

$$\begin{aligned} &T(\tilde{\mu})((X_1, \xi_1, \eta_1, Y_1), (X_2, \xi_2, \eta_2, Y_2)) \\ &\quad = T(R_{\bar{u}_1 \bar{u}_2} \circ \phi_1)(X_1) + T(L_{\bar{g}_1} \circ R_{\bar{u}_2} \circ \phi_2)(\xi_1) + T(L_{\bar{g}_1} \circ L_{\bar{u}_1} \circ \phi_2)(\xi_2). \end{aligned}$$

In Proposition 3.21, we saw that the space $(\pi^{\#} \oplus \pi^{\#})(T_{(s_1, s_2)}(\Sigma *_G \Sigma)^{\circ})$ contains only pairs of the form

$$((0, T(L_{u_1})(\xi), T(L_{v_1})(\eta), T(R_{h_1})(Y)), (T(R_{h_1})(Y), -T(R_{u_2})(\xi), -T(R_{v_2})(\eta), 0)),$$

where $\xi, \eta \in \mathfrak{g}^*$, $Y \in \mathfrak{g}$ are vectors satisfying $Y \oplus \eta = \mathrm{Ad}_{\bar{h}_1}(\overline{\xi})$. However, $T(\tilde{\mu})$ sends elements of this form to

$$\begin{aligned} &T(R_{\bar{u}_1 \bar{u}_2} \circ \phi_1)(0) + T(L_{\bar{g}_1} \circ R_{\bar{u}_2} \circ \phi_2)(T(L_{u_1})(\xi)) + T(L_{\bar{g}_1} \circ L_{\bar{u}_1} \circ \phi_2)(-T(R_{u_2})(\xi)) \\ &\quad = T(L_{\bar{g}_1} \circ R_{\bar{u}_2} \circ L_{\bar{u}_1} \circ \phi_2)(\xi) - T(L_{\bar{g}_1} \circ L_{\bar{u}_1} \circ R_{\bar{u}_2} \circ \phi_2)(\xi) \\ &\quad = 0. \end{aligned}$$

This completes the proof. □

Remark 3.22. In the construction of $(\Sigma; G, G^*; \{\cdot\})$ we have assumed that G and G^* are simply-connected. When this condition is removed, one can consider the universal covering groups $\widetilde{G}$ and $\widetilde{G^*}$. By constructing the symplectic double groupoid of $\widetilde{G}$ and $\widetilde{G^*}$ in the manner described above, a symplectic double groupoid of G and G^* can be obtained as a quotient. See [37, Remark 4.6] for the details.

Chapter 4

Symplectic double groupoids of Poisson groupoids

In the previous chapter we saw that any pair of dual Poisson Lie groups, G and G^*, give rise to a symplectic double groupoid whose side groupoids are given by G and G^*. There is of course a reverse procedure: given any symplectic double groupoid with base manifold given by a singleton set, the side groupoids obtain Poisson structures for which they are dual Poisson Lie groups. In fact, Mackenzie proved a more general result:

Theorem 4.1 ([44, Theorem 2.9])**.** *Let $(S; H, V; P)$ be a symplectic double groupoid. Then the side groupoids $H \rightrightarrows P$ and $V \rightrightarrows P$ are Poisson groupoids in duality, and the core groupoid $C \rightrightarrows P$ is a symplectic groupoid.* □

The original proof was given for symplectic double groupoids for which the double source map is also a surjection. However, the proof extends without change to include the symplectic double groupoids defined in this thesis.

§ 4.1 Generalisations of the Lu-Weinstein double groupoid

We have just seen that a symplectic double groupoid induces Poisson structures on its side groupoids which make them dual Poisson groupoids. This section will be devoted to considering scenarios where this process can be reversed. More precisely, given a pair of dual Poisson groupoids, we will assess in which situations there exists a symplectic double groupoid whose side groupoids are given by this pair. In particular, we would like to be able to describe a construction principle for such a symplectic double groupoid, that generalises that of Lu and Weinstein [39]. A symplectic double groupoid will not exist for every pair of dual Poisson groupoids, although one should exist locally (see [63, §4.5]). We are only interested in the cases where a construction can be done globally.

4.1.1 Constructions of weak double Lie groupoids

Before considering any Poisson or symplectic structures, we first try to construct weak double Lie groupoids that give generalisations of the Lu-Weinstein double groupoid of Chapter 3.

We saw that for a pair of dual Poisson Lie groups G and G^*, the structure maps of the Lu-Weinstein double groupoid Σ were inherited from the double Lie groupoid $\Box(G, G^*)$. Moreover, the construction of Σ depended on the relationship of the triple (D, G, G^*), where D was the Drinfel'd double Lie group. We begin by introducing the following notion, which generalises the relationship of this triple.

Definition 4.2. Let M be a smooth manifold. A *Lie groupoid triple on M* is a triple of Lie groupoids (D, H, V) each with base manifold M, equipped with morphisms of Lie groupoids $\phi_1 \colon H \to D$, $h \mapsto \bar{h}$ and $\phi_2 \colon V \to D$, $v \mapsto \bar{v}$ over M, such that the map $\Phi_1 \colon H * V \to D$, $(h, v) \mapsto \bar{h}\bar{v}$ is a submersion. Here, $H * V$ is the pullback manifold defined by $(\alpha_H \times \beta_V)^{-1}(\Delta_M)$.

Remark 4.3. This generalises the notion of a *double Lie group* given by Lu and Weinstein [40, §3]. In particular, a Poisson Lie group G and its dual Poisson Lie group G^*, together with the corresponding Drinfel'd double Lie group D, form a Lie groupoid triple (D, G, G^*) on base a singleton set.

Example 4.4. Let M be a smooth manifold. Consider the pair groupoid $M \times M$ on base M. We have a Lie groupoid triple $(M \times M, M \times M, M \times M)$ on M, where ϕ_1 and ϕ_2 are given by the identity map $\mathrm{id}_{M\times M}$. The map Φ_1 is identical to the partial multiplication of the pair groupoid on M, and so is a diffeomorphism.

Example 4.5. Let G be a Lie groupoid with base manifold M. The anchor map $\chi = (\beta, \alpha) \colon G \to M \times M$ is a morphism of Lie groupoids over M, where $M \times M$ is the pair groupoid on M. We claim that $(M \times M, G, M \times M)$ is a Lie groupoid triple, with $\phi_1 = \chi$ and $\phi_2 = \mathrm{id}_{M\times M}$. We need to check that the map

$$\Phi_1 \colon G * (M \times M) \to M \times M, \quad (g, m_1, m_2) \mapsto (\beta(g), m_2),$$

is a submersion. Note that

$$G * (M \times M) = \{(g, m_1, m_2) \in G \times M \times M \mid \alpha(g) = m_1\}.$$

Hence, we have a diffeomorphism $F \colon G \times M \to G * (M \times M)$, $(g, m) \mapsto (g, \alpha(g), m)$. Moreover, we can write Φ_1 as the composite $F^{-1} \circ (\beta \times \mathrm{id}_M)$. Since the target projection β is a submersion, it follows that Φ_1 is also a submersion. ⊠

Example 4.6. Let H and V be Lie groupoids with the same base manifold M. Suppose that H is locally trivial. Then $(M \times M, H, V)$ is a Lie groupoid triple on M, with ϕ_1 and ϕ_2 equal to the anchor maps χ_H and χ_V, respectively. Here $M \times M$ is just the pair groupoid on M. Let us check that the map

$$\Phi_1 \colon H * V \to M \times M, \quad (h, v) \mapsto (\beta_H(h), \alpha_V(v)),$$

is a submersion. Given $(h, v) \in H * V$, we need to show that the tangent map $T_{(h,v)}(\Phi_1) \colon T_{(h,v)}(H * V) \to T_{\beta_H(h)}M \oplus T_{\alpha_V(v)}M$ is a surjection. Take an arbitrary pair $(Y, Z) \in T_{\beta_H(h)}M \oplus T_{\alpha_V(v)}M$. Now since the source projection $\alpha_V \colon V \to M$ is a submersion, there exists $\xi \in T_vV$ such that $T(\alpha_V)(\xi) = Z$. The condition that H is locally trivial means that the anchor map $\chi_H = (\beta_H, \alpha_H) \colon H \to M \times M$ is a submersion. Since $\alpha_H(h) = \beta_V(v)$, we have $(Y, T(\beta_V)(\xi)) \in T_{\chi_H(h)}(M \times M)$, and so there exists $X \in T_hH$ with

$$T(\chi_H)(X) = (T(\beta_H)(X), T(\alpha_H)(X)) = (Y, T(\beta_V)(\xi)).$$

Thus, $(X,\xi) \in T_{(h,v)}(H * V)$. Moreover,

$$T(\Phi_1)(X,\xi) = (T(\beta_H)(X), T(\alpha_V)(\xi)) = (Y,Z).$$

Hence, the tangent map $T_{(h,v)}(\Phi_1)$ is surjective, and Φ_1 is a submersion. ⊠

For our first construction, let us suppose we are given a Lie groupoid triple (D,H,V) on base manifold M. Let us construct the double groupoid $\square(H,V)$ of Example 1.68. Explicitly, the structure maps for the vertical structure are given as follows. The source and target projections are

$$\widetilde{\alpha}_V\colon \square(H,V) \to H\,, \quad (g,u,v,h) \mapsto h;$$

$$\widetilde{\beta}_V\colon \square(H,V) \to H\,, \quad (g,u,v,h) \mapsto g.$$

The partial multiplication $\widetilde{\kappa}_V\colon \square(H,V) *_H \square(H,V) \to \square(H,V)$ is defined by

$$(g_2,u_2,v_2,h_2) \boxminus (g_1,u_1,v_1,h_1) = (g_2, u_2u_1, v_2v_1, h_1).$$

The identity map is given by

$$\widetilde{1}^V\colon H \to \square(H,V)\,, \quad g \mapsto (g, 1^V_{\alpha_H(g)}, 1^V_{\beta_H(g)}, g);$$

and the inversion map given by

$$\widetilde{\iota}_V\colon \square(H,V) \to \square(H,V)\,, \quad (g,u,v,h) \mapsto (h, u^{-1}, v^{-1}, g).$$

On the other hand, the horizontal structure has source and target projections given by

$$\widetilde{\alpha}_H\colon \square(H,V) \to V\,, \quad (g,u,v,h) \mapsto u;$$
$$\widetilde{\beta}_H\colon \square(H,V) \to V\,, \quad (g,u,v,h) \mapsto v.$$

The partial multiplication $\widetilde{\kappa}_H\colon \square(H,V) *_V \square(H,V) \to \square(H,V)$ is defined by

$$(g_2,u_2,v_2,h_2) \boxbar (g_1,u_1,v_1,h_1) = (g_2g_1, u_1, v_2, h_2h_1).$$

The identity map is given by

$$\widetilde{1}^H\colon V \to \square(H,V)\,, \quad u \mapsto (1^H_{\beta_V(u)}, u, u, 1^H_{\alpha_V(u)});$$

and the inversion map given by

$$\widetilde{\iota}_H\colon \square(H,V) \to \square(H,V)\,, \quad (g,u,v,h) \mapsto (g^{-1}, v, u, h^{-1}).$$

The map $\Phi_1\colon H * V \to D$, $(g,u) \mapsto \bar{g}\bar{u}$ is a submersion by assumption. Let $V * H$ denote the pullback manifold defined by $(\alpha_V \times \beta_H)^{-1}(\Delta_M)$. Then it follows that the map defined by $\Phi_2\colon V * H \to D$, $(v,h) \mapsto \bar{v}\bar{h}$ is also a submersion.

Now consider the embedded submanifold $\widetilde{\Sigma}$ of $\square(H,V)$ defined by $(\Phi_1 \times \Phi_2)^{-1}(\Delta_D)$. We claim that the structure maps of the double groupoid $\square(H,V)$ restrict to $\widetilde{\Sigma}$ to give a well-defined double groupoid $(\widetilde{\Sigma}; H, V; M)$. To show that this is the case, one only needs to check that the restrictions of the structure maps are themselves well-defined, which is straightforward. In fact, we have a stronger result.

Theorem 4.7. *Let (D,H,V) be a Lie groupoid triple on base manifold M. Then $(\widetilde{\Sigma}; H, V; M)$ is a weak double Lie groupoid.*

Proof. We first note that all of the structure maps of the double groupoid $\square(H, V)$ described above are smooth. Since the structure maps of $\widetilde{\Sigma}$ are just restrictions of the domains and codomains of these maps to embedded submanifolds, it follows that they are also smooth. In particular, this shows that the inversion maps must be diffeomorphisms because they are self-inverse.

It remains to show that the source and target projections of the horizontal and vertical structures are submersions, and also that the double source map is a submersion.

Let us first consider the source projection of the vertical structure. Given an element $s = (g, u, v, h) \in \widetilde{\Sigma}$, we need to show that the tangent map $T_s(\widetilde{\alpha}_V)\colon T_s\widetilde{\Sigma} \to T_h H$ is surjective. Take $Y \in T_h H$, then $T(\beta_H)(Y) \in T_{\beta_H(h)}M = T_{\alpha_V(v)}M$. Now since α_V is a submersion, there exists $\eta \in T_v V$ such that $T(\alpha_V)(\eta) = T(\beta_H)(Y)$. Moreover, $T(\Phi_2)(\eta, Y) \in T_{\bar{v}\bar{h}}D = T_{\bar{g}\bar{u}}D$, and so, since Φ_1 is a submersion, there exists a pair $(X, \xi) \in T_{(g,u)}(H * V)$ such that $T(\Phi_1)(X, \xi) = T(\Phi_2)(\eta, Y)$. Thus, $(X, \xi, \eta, Y) \in T_s\widetilde{\Sigma}$ and $T(\widetilde{\alpha}_V)(X, \xi, \eta, Y) = Y$. Hence, the source projection $\widetilde{\alpha}_V$ is a submersion.

The target projection of the vertical structure can also be seen to be a submersion by writing it as a composition of submersions, $\widetilde{\beta}_V = \widetilde{\alpha}_V \circ \widetilde{\iota}_V$. We can show that the source and target projections of the horizontal structure are submersions using a similar style of argument.

Finally, let us show that the double source map $\alpha_2 \coloneqq (\widetilde{\alpha}_V, \widetilde{\alpha}_H)\colon \widetilde{\Sigma} \to H \times_\alpha V$ is a submersion. Let $s = (g, u, v, h) \in \widetilde{\Sigma}$, and consider the corresponding tangent map $T_s(\alpha_2)\colon T_s\widetilde{\Sigma} \to T_{(h,u)}(H \times_\alpha V)$. Given an arbitrary pair $(Y, \xi) \in T_{(h,u)}(H \times_\alpha V)$, define $Y' = T(\iota_H)(Y) \in T_{h^{-1}}H$. Observe that

$$T(\beta_H)(Y') = T(\beta_H \circ \iota_H)(Y) = T(\alpha_H)(Y) = T(\alpha_V)(\xi),$$

and so $(\xi, Y') \in T_{(u,h^{-1})}(V * H)$. Furthermore, $T(\Phi_2)(\xi, Y') \in T_{\bar{u}\bar{h}^{-1}}D = T_{\bar{g}^{-1}\bar{v}}D$. Now, since Φ_1 is a submersion there exists a pair $(X', \eta) \in T_{(g^{-1},v)}(H * V)$ with $T(\Phi_1)(X', \eta) = T(\Phi_2)(\xi, Y')$. Thus, we have $(X', \eta, \xi, Y') \in T_{s^{-1(H)}}\widetilde{\Sigma}$. Next, we define $X = T(\iota_H)(X') \in T_g H$, and observe that

$$T(\widetilde{\iota}_H)(X', \eta, \xi, Y') = (T(\iota_H)(X'), \xi, \eta, T(\iota_H)(Y')) = (X, \xi, \eta, Y).$$

Hence, $(X, \xi, \eta, Y) \in T_s\widetilde{\Sigma}$ and moreover $T(\alpha_2)(X, \xi, \eta, Y) = (Y, \xi)$. This shows that the tangent map $T_s(\alpha_2)$ is surjective. Therefore, the double source map is a submersion. $\square$

The core of $(\widetilde{\Sigma}; H, V; M)$ can be identified with $(\phi_1 \times \phi_2)^{-1}(\Delta_D)$. It is a subgroupoid of the Cartesian product groupoid $H \times V$ over the diagonal map.

When we consider the Lie groupoid triple (D, G, G^*), with G and G^* dual Poisson Lie groups and D the Drinfel'd double Lie group, $(\widetilde{\Sigma}; G, G^*; \{\cdot\})$ is precisely the double groupoid constructed by Lu and Weinstein. Note that in this scenario, the map $\Phi_1\colon G \times G^* \to D$ is a local diffeomorphism.

In the general case, if (D, H, V) is a Lie groupoid triple over M such that the map $\Phi_1\colon H * V \to D$ is a local diffeomorphism, we can show that

$$\dim(\widetilde{\Sigma}) = \dim(H) + \dim(V) - \dim(M).$$

However, if H and V are taken as dual Poisson groupoids with M non-trivial, then $\dim(\widetilde{\Sigma}) \neq 2\dim(H) = 2\dim(V)$. This means that $\widetilde{\Sigma}$ has no symplectic structure that makes $(\widetilde{\Sigma}; H, V; M)$ a symplectic double groupoid. On the other hand, when Φ_1

is not a local diffeomorphism, there exist cases where $(\widetilde{\Sigma}; H, V; M)$ can be made a symplectic double groupoid. We will see examples of this in Section 4.1.2.

Example 4.8. Let M be a smooth manifold, and consider the Lie groupoid triple $(M\times M, M\times M, M\times M)$ on M of Example 4.4. Here, the maps ϕ_1 and ϕ_2 are given by the identity map $\mathrm{id}_{M\times M}$. For the (weak) double Lie groupoid $(\widetilde{\Sigma}; M\times M, M\times M; M)$, the manifold $\widetilde{\Sigma}$ has elements of the form

$$s = ((z, y), (y, x), (z, m), (m, x)) \in M^8.$$

Thus, we have a diffeomorphism $F\colon \widetilde{\Sigma} \to M^4$ given by $s \mapsto (z, y, m, x)$. It is straightforward to show that F is an isomorphism of Lie groupoids between the vertical structure of $\widetilde{\Sigma}$ and the pair groupoid M^4 on $M \times M$.

Recall, from Example 1.65, the involutive diffeomorphism $\Theta\colon M^4 \to M^4$ given by $(z, y, m, x) \mapsto (z, m, y, x)$. We can also show that $\Theta \circ F$ is an isomorphism of Lie groupoids between the horizontal structure of $\widetilde{\Sigma}$ and the pair groupoid M^4 on $M\times M$. Hence, $(F; \mathrm{id}_{M\times M}, \mathrm{id}_{M\times M}; \mathrm{id}_M)$ provides an isomorphism of double Lie groupoids between $(\widetilde{\Sigma}; M \times M, M \times M; M)$ and $(M^4; M \times M, M \times M; M)$ of Example 1.65.

Example 4.9. Let G be a Lie groupoid with base manifold M. Consider the Lie groupoid triple $(M\times M, G, M\times M)$ on M of Example 4.5 with $\phi_1 = \chi$, the anchor of G, and $\phi_2 = \mathrm{id}_{M\times M}$. Let us construct the (weak) double Lie groupoid $(\widetilde{\Sigma}; G, M\times M; M)$. As a manifold, $\widetilde{\Sigma}$ consists of elements

$$(g_2, (m_2, m_1), (m_4, m_3), g_1) \in G \times M^4 \times G,$$

such that $\chi(g_2) = (m_4, m_2)$ and $\chi(g_1) = (m_3, m_1)$. It is clear that we have a diffeomorphism $F\colon G \times G \to \widetilde{\Sigma}$ given by

$$(g_2, g_1) \mapsto (g_2, (\alpha(g_2), \alpha(g_1)), (\beta(g_2), \beta(g_1)), g_1).$$

Moreover, if we equip $G\times G$ with the pair groupoid structure on base G, it is straightforward to show that F is a morphism of Lie groupoids over G. On the other hand, if we give $G\times G$ the Cartesian product groupoid structure on base $M\times M$, we can also show that F is a morphism of Lie groupoids over $M \times M$. Hence, $(\widetilde{\Sigma}; G, M \times M; M)$ is isomorphic to the double Lie groupoid $(G \times G; G, M \times M; M)$ of Example 1.66. ⊠

Example 4.10. Let H and V be Lie groupoids with the same base manifold M, and suppose that H is locally trivial. Consider the Lie groupoid triple $(M \times M, H, V)$ of Example 4.6, where ϕ_1 and ϕ_2 are equal to the anchor maps of H and V, respectively. Let us construct the (weak) double Lie groupoid $(\widetilde{\Sigma}; H, V; M)$. As a manifold, we can check that $\widetilde{\Sigma}$ consists of elements

$$(g, u, v, h) \in H \times V \times V \times H,$$

satisfying $\alpha_H(g) = \beta_V(u)$, $\alpha_V(v) = \beta_H(h)$, $\beta_H(g) = \beta_V(v)$, and $\alpha_V(u) = \alpha_H(h)$. However, these are precisely the elements of $\Box(H, V)$. Since the structure maps of $\widetilde{\Sigma}$ are just restrictions of the structure maps of $\Box(H, V)$, it follows that $(\widetilde{\Sigma}; H, V; M)$ is equal to the double Lie groupoid $(\Box(H, V); H, V; M)$ of Example 1.68. ⊠

We will now give another construction of a weak double Lie groupoid starting from a Lie groupoid triple. When the corresponding Φ_1 map is a local diffeomorphism, there are cases where the dimensions are suitable for it to be a symplectic double groupoid.

We start by again considering a Lie groupoid triple (D, H, V) on base manifold M. Suppose there exists a common embedded normal Lie subgroupoid N of H and V such that ϕ_1 and ϕ_2 agree on N and are also injective when restricted to N. We denote the source projection of N by $q\colon N \to M$; it is also equal to the target projection of N. The normality condition gives us actions of the Lie groupoids $H \rightrightarrows M$ and $V \rightrightarrows M$ on $q\colon N \to M$ by conjugation,

$$H * N \to N\,, \quad (h, n) \mapsto h \cdot n := hnh^{-1};$$

$$V * N \to N\,, \quad (v, n) \mapsto v \cdot n := vnv^{-1}.$$

We also have an action of the Lie groupoid $D \rightrightarrows M$ on the inner subgroupoid $ID \to M$ by conjugation,

$$D * ID \to ID\,, \quad (d, n) \mapsto d \cdot n := dnd^{-1}.$$

Thus, we can form the semi-direct product groupoids $H \ltimes N$, $V \ltimes N$ and $D \ltimes ID$ all on base M. (The construction can be recalled from Example 1.28.)

Consider the smooth map $\widetilde{\Phi}_2\colon V * H * N \to D$ defined by $(v, h, n) \mapsto \bar{v}\bar{h}\bar{n}$, where

$$V * H * N = \{(v, h, n) \in V \times H \times N \mid \chi_H(h) = (\alpha_V(v), q(n))\}.$$

We aim to put a double groupoid structure on the manifold $\widetilde{S} = (\Phi_1 \times \widetilde{\Phi}_2)^{-1}(\Delta_D)$. Explicitly, this pullback is given by

$$\widetilde{S} = \{(g, u, v, h; n) \in (H * V) \times (V * H * N) \mid \bar{g}\bar{u} = \bar{v}\bar{h}\bar{n}\}$$

We now construct two groupoid structures on $\widetilde{S}$, one with base H and the other with base V. As the first step in this endeavour, we again consider the two groupoid structures on $\square(H, V)$.

Proposition 4.11. *The map*

$$\delta_V\colon \square(H, V) \to D \ltimes ID\,, \quad (g, u, v, h) \mapsto (\bar{h}, \bar{h}^{-1}\bar{v}^{-1}\bar{g}\bar{u}),$$

is a morphism of groupoids over $\alpha_V\colon V \to M$*. Similarly, the map*

$$\delta_H\colon \square(H, V) \to D \mathbin{\bar{\ltimes}} ID\,, \quad (g, u, v, h) \mapsto (\bar{u}, \bar{h}^{-1}\bar{v}^{-1}\bar{g}\bar{u}),$$

is a morphism of groupoids over $\alpha_H\colon H \to M$*. Here,* $D \mathbin{\bar{\ltimes}} ID$ *denotes the opposite semi-direct product groupoid.*

Proof. We will show that (δ_V, α_V) is a morphism of groupoids; showing that (δ_H, α_H) is also a morphism follows by a similar argument. The notation for the source and target projections of the two groupoid structures has been illustrated in the following diagram:

$$\begin{array}{ccc} \square(H, V) & \xrightarrow{\ \delta_V\ } & D \ltimes ID \\ {\scriptstyle \widetilde{\alpha}_H, \widetilde{\beta}_H}\Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow{\scriptstyle \alpha, \beta} \\ V & \xrightarrow{\ \alpha_V\ } & M. \end{array}$$

Observe that for any element $(g, u, v, h) \in \square(H, V)$, we have

$$\alpha \circ \delta_V(g, u, v, h) = \alpha(\bar{h}, \bar{h}^{-1}\bar{v}^{-1}\bar{g}\bar{u}) = \alpha_D(\bar{h}) = \alpha_H(h) = \alpha_V(u) = \alpha_V \circ \widetilde{\alpha}_H(g, u, v, h),$$

and

$$\beta \circ \delta_V(g,u,v,h) = \beta(\bar{h}, \bar{h}^{-1}\bar{v}^{-1}\bar{g}\bar{u}) = \beta_D(\bar{h}) = \beta_H(h) = \alpha_V(v) = \alpha_V \circ \widetilde{\beta}_H(g,u,v,h).$$

Now, for any pair $((g_2,u_2,v_2,h_2),(g_1,u_1,v_1,h_1)) \in \square(H,V) *_V \square(H,V)$, we have

$$\begin{aligned}\delta_V((g_2,u_2,v_2,h_2) \boxbar (g_1,u_1,v_1,h_1)) &= \delta_V(g_2g_1,u_1,v_2,h_2h_1)\\ &= (\bar{h}_2\bar{h}_1, (\bar{h}_2\bar{h}_1)^{-1}\bar{v}_2^{-1}\bar{g}_2\bar{g}_1\bar{u}_1).\end{aligned}$$

On the other hand, we have

$$\begin{aligned}\delta_V(g_2,u_2,v_2,h_2)\delta_V(g_1,u_1,v_1,h_1) &= (\bar{h}_2, \bar{h}_2^{-1}\bar{v}_2^{-1}\bar{g}_2\bar{u}_2)(\bar{h}_1, \bar{h}_1^{-1}\bar{v}_1^{-1}\bar{g}_1\bar{u}_1)\\ &= (\bar{h}_2\bar{h}_1, (\bar{h}_1^{-1} \cdot (\bar{h}_2^{-1}\bar{v}_2^{-1}\bar{g}_2\bar{u}_2))\bar{h}_1^{-1}\bar{v}_1^{-1}\bar{g}_1\bar{u}_1)\\ &= (\bar{h}_2\bar{h}_1, \bar{h}_1^{-1}\bar{h}_2^{-1}\bar{v}_2^{-1}\bar{g}_2\bar{u}_2\bar{v}_1^{-1}\bar{g}_1\bar{u}_1)\\ &= (\bar{h}_2\bar{h}_1, (\bar{h}_2\bar{h}_1)^{-1}\bar{v}_2^{-1}\bar{g}_2\bar{g}_1\bar{u}_1).\end{aligned}$$

Note that in the third line we have used the fact that $u_2 = v_1$. Hence, we have

$$\delta_V((g_2,u_2,v_2,h_2) \boxbar (g_1,u_1,v_1,h_1)) = \delta_V(g_2,u_2,v_2,h_2)\delta_V(g_1,u_1,v_1,h_1),$$

which shows that δ_V is indeed a morphism of groupoids over α_V. $\square$

In light of this result, we can form the pullback in the category of groupoids of the groupoids $\square(H,V) \rightrightarrows V$ and $H \ltimes N \rightrightarrows M$. This can be identified with the manifold $\widetilde{S}$, with the base manifold identified with V. Similarly, we can form the pullback of the groupoids $\square(H,V) \rightrightarrows H$ and $V \bar{\ltimes} N \rightrightarrows M$. Again this can be identified with the manifold $\widetilde{S}$, and the base can be identified with H. We have the following pullback diagrams,

$$\begin{array}{ccc} & & H \ltimes N \\ & & \downarrow{\scriptstyle \phi_1\times\phi_1} \\ \square(H,V) & \xrightarrow{\delta_V} & D \ltimes ID \end{array} \qquad \begin{array}{ccc} & & V \bar{\ltimes} N \\ & & \downarrow{\scriptstyle \phi_2\times\phi_2} \\ \square(H,V) & \xrightarrow{\delta_H} & D \bar{\ltimes} ID. \end{array}$$

Let us explicitly write down the structure maps of these two groupoids. The groupoid $\widetilde{S} \rightrightarrows H$ has source and target projections given by

$$\widetilde{\alpha}_V \colon \widetilde{S} \to H\,, \quad (g,u,v,h;n) \mapsto h;$$

$$\widetilde{\beta}_V \colon \widetilde{S} \to H\,, \quad (g,u,v,h;n) \mapsto g.$$

The partial multiplication $\widetilde{\kappa}_V \colon \widetilde{S} *_H \widetilde{S} \to \widetilde{S}$ is defined by

$$(g_2,u_2,v_2,h_2;n_2) \boxminus (g_1,u_1,v_1,h_1;n_1) = (g_2, u_2u_1, v_2v_1, h_1; n_1(u_1^{-1} \cdot n_2)).$$

The identity map is given by

$$\widetilde{1}^V \colon H \to \widetilde{S}\,, \quad g \mapsto (g, 1^V_{\alpha_H(g)}, 1^V_{\beta_H(g)}, g; 1^N_{\alpha_H(g)});$$

and the inversion map given by

$$\widetilde{\iota}_V \colon \widetilde{S} \to \widetilde{S}\,, \quad (g,u,v,h;n) \mapsto (h, u^{-1}, v^{-1}, g; u \cdot (n^{-1})).$$

For the groupoid $\widetilde{S} \rightrightarrows V$, the source and target projections are given by

$$\widetilde{\alpha}_H\colon \widetilde{S} \to V\,, \quad (g,u,v,h;n) \mapsto u;$$

$$\widetilde{\beta}_H\colon \widetilde{S} \to V\,, \quad (g,u,v,h;n) \mapsto v.$$

The partial multiplication $\widetilde{\kappa}_H\colon \widetilde{S} *_V \widetilde{S} \to \widetilde{S}$ is defined by

$$(g_2,u_2,v_2,h_2;n_2) \boxbar (g_1,u_1,v_1,h_1;n_1) = (g_2g_1,u_1,v_2,h_2h_1;(h_1^{-1}\cdot n_2)n_1).$$

The identity map is given by

$$\widetilde{1}^H\colon V \to \widetilde{S}\,, \quad u \mapsto (1^H_{\beta_V(u)},u,u,1^H_{\alpha_V(u)};1^N_{\alpha_V(u)});$$

and the inversion map given by

$$\widetilde{\iota}_H\colon \widetilde{S} \to \widetilde{S}\,, \quad (g,u,v,h;n) \mapsto (g^{-1},v,u,h^{-1};h\cdot(n^{-1})).$$

Let us now show that these groupoid structures form a weak double Lie groupoid.

Proposition 4.12. *Let (D,H,V) be a Lie groupoid triple on base manifold M, and let N be a common embedded normal Lie subgroupoid of H and V such that ϕ_1 and ϕ_2 agree on N and are also injective when restricted to N. Then $(\widetilde{S};H,V;M)$ is a double groupoid.*

Proof. We will show that the structure maps of $\widetilde{S} \rightrightarrows V$ are morphisms of groupoids over the structure maps of $H \rightrightarrows M$. We will denote arbitrary elements of $\widetilde{S}$ by $s = (g,u,v,h;n)$.

To see that the source projection $\widetilde{\alpha}_H$ is a morphism of groupoids over α_H, first observe that for any $s \in \widetilde{S}$,

$$\alpha_V \circ \widetilde{\alpha}_H(s) = \alpha_V(u) = \alpha_H(h) = \alpha_H \circ \widetilde{\alpha}_V(s),$$

and also

$$\beta_V \circ \widetilde{\alpha}_H(s) = \beta_V(u) = \alpha_H(g) = \alpha_H \circ \widetilde{\beta}_V(s).$$

Furthermore, for any pair $(s_2,s_1) \in \widetilde{S} *_H \widetilde{S}$,

$$\widetilde{\alpha}_H(s_2 \boxminus s_1) = u_2u_1 = \widetilde{\alpha}_H(s_2)\widetilde{\alpha}_H(s_1).$$

Similarly, to see that the target projection $\widetilde{\beta}_H$ is a morphism of groupoids over β_H, observe that for any $s \in \widetilde{S}$,

$$\alpha_V \circ \widetilde{\beta}_H(s) = \alpha_V(v) = \beta_H(h) = \beta_H \circ \widetilde{\alpha}_V(s),$$

and also

$$\beta_V \circ \widetilde{\beta}_H(s) = \beta_V(v) = \beta_H(g) = \beta_H \circ \widetilde{\beta}_V(s).$$

In addition, for any pair $(s_2,s_1) \in \widetilde{S} *_H \widetilde{S}$,

$$\widetilde{\beta}_H(s_2 \boxminus s_1) = v_2v_1 = \widetilde{\beta}_H(s_2)\widetilde{\beta}_H(s_1).$$

We next show that the partial multiplication of the horizontal structure is a morphism of groupoids over the partial multiplication of H. Observe that for an arbitrary pair $(s_2,s_1) \in \widetilde{S} *_V \widetilde{S}$,

$$\widetilde{\alpha}_V(s_2 \boxbar s_1) = h_2h_1 = \widetilde{\alpha}_V(s_2)\widetilde{\alpha}_V(s_1),$$

and also

$$\widetilde{\beta}_V(s_2 \boxbar s_1) = g_2 g_1 = \widetilde{\beta}_V(s_2)\widetilde{\beta}_V(s_1).$$

It remains to show the following interchange law

$$(s_4 \boxminus s_2) \boxbar (s_3 \boxminus s_1) = (s_4 \boxbar s_3) \boxminus (s_2 \boxbar s_1),$$

for compatible elements. A simple calculation shows that the left-hand side is given by

$$\left(g_4 g_3, u_3 u_1, v_4 v_2, h_2 h_1; (h_1^{-1} \cdot n_2)(h_1^{-1} \cdot (u_2^{-1} \cdot n_4)) n_1 (u_1^{-1} \cdot n_3)\right),$$

and the right-hand side is given by

$$\left(g_4 g_3, u_3 u_1, v_4 v_2, h_2 h_1; (h_1^{-1} \cdot n_2) n_1 (u_1^{-1} \cdot (h_3^{-1} \cdot n_4))(u_1^{-1} \cdot n_3)\right).$$

Thus, we are reduced to showing the following relation,

$$(h_1^{-1} \cdot (u_2^{-1} \cdot n_4)) n_1 = n_1 (u_1^{-1} \cdot (h_3^{-1} \cdot n_4)). \tag{4.1}$$

Observe that

$$\begin{aligned}
\phi_1((h_1^{-1} \cdot (u_2^{-1} \cdot n_4)) n_1) &= \phi_1(h_1^{-1}(u_2^{-1} n_4 u_2) h_1 n_1) \\
&= \bar{h}_1^{-1} \bar{u}_2^{-1} \bar{n}_4 \bar{u}_2 \bar{h}_1 \bar{n}_1 \\
&= (\bar{n}_1 \bar{u}_1^{-1} \bar{g}_1^{-1} \bar{v}_1) \bar{u}_2^{-1} \bar{n}_4 \bar{u}_2 \bar{h}_1 \bar{n}_1 \\
&= \bar{n}_1 \bar{u}_1^{-1} \bar{h}_3^{-1} \bar{u}_2 \bar{u}_2^{-1} \bar{n}_4 (\bar{v}_1 \bar{h}_1 \bar{n}_1) \\
&= \bar{n}_1 \bar{u}_1^{-1} \bar{h}_3^{-1} \bar{n}_4 (\bar{g}_1 \bar{u}_1) \\
&= \bar{n}_1 \bar{u}_1^{-1} \bar{h}_3^{-1} \bar{n}_4 \bar{h}_3 \bar{u}_1 \\
&= \phi_1(n_1 u_1^{-1}(h_3^{-1} n_4 h_3) u_1) \\
&= \phi_1(n_1(u_1^{-1} \cdot (h_3^{-1} \cdot n_4))).
\end{aligned}$$

Here we have used the properties $\phi_1|_N = \phi_2|_N$, $\bar{g}_1 \bar{u}_1 = \bar{v}_1 \bar{h}_1 \bar{n}_1$, $g_1 = h_3$, and $v_1 = u_2$. The injectivity of $\phi_1|_N$ now proves that equation (4.1) holds, as required.

Now let us show that the identity map $\widetilde{1}^H$ is a morphism of groupoids over 1^H. It is clear that, for any $v \in V$, we have the two identities

$$\widetilde{\alpha}_V(\widetilde{1}^H_v) = 1^H_{\alpha_V(v)}, \quad \widetilde{\beta}_V(\widetilde{1}^H_v) = 1^H_{\beta_V(v)}.$$

Moreover, for all pairs $(v_2, v_1) \in V * V$, we have

$$\begin{aligned}
\widetilde{1}^H_{v_2} \boxminus \widetilde{1}^H_{v_1} &= \left(1^H_{\beta_V(v_2)}, v_2 v_1, v_2 v_1, 1^H_{\alpha_V(v_1)}; 1^N_{\alpha_V(v_1)}\left(v_1^{-1} \cdot 1^N_{\alpha_V(v_2)}\right)\right) \\
&= \left(1^H_{\beta_V(v_2 v_1)}, v_2 v_1, v_2 v_1, 1^H_{\alpha_V(v_2 v_1)}; 1^N_{\alpha_V(v_2 v_1)}\right) \\
&= \widetilde{1}^H_{v_2 v_1}.
\end{aligned}$$

Finally, we show that the inversion map $\widetilde{\iota}_H$ is a morphism of groupoids over ι_H. Observe that for any $s \in \widetilde{S}$, we have

$$\widetilde{\alpha}_V \circ \widetilde{\iota}_H(s) = h^{-1} = \iota_H \circ \widetilde{\alpha}_V(s),$$

and also

$$\widetilde{\beta}_V \circ \widetilde{\iota}_H(s) = g^{-1} = \iota_H \circ \widetilde{\beta}_V(s).$$

We also need to show that $\widetilde{\iota}_H(s_2 \boxminus s_1) = \widetilde{\iota}_H(s_2) \boxminus \widetilde{\iota}_H(s_1)$, for any pair $(s_2, s_1) \in \widetilde{S} *_H \widetilde{S}$. We can calculate the left-hand side as

$$\left(g_2^{-1}, v_2 v_1, u_2 u_1, h_1^{-1}; h_1 \cdot ((n_1(u_1^{-1} \cdot n_2))^{-1})\right),$$

and the right-hand side as

$$\left(g_2^{-1}, v_2 v_1, u_2 u_1, h_1^{-1}; (h_1 \cdot (n_1^{-1}))(v_1^{-1} \cdot (h_2 \cdot (n_2^{-1})))\right).$$

It remains to show that the final entries of these two expressions are equal. That is,

$$h_1 \cdot ((n_1(u_1^{-1} \cdot n_2))^{-1}) = (h_1 \cdot (n_1^{-1}))(v_1^{-1} \cdot (h_2 \cdot (n_2^{-1}))) \tag{4.2}$$

To deduce this equality, observe that

$$\begin{aligned}
\phi_1\left(h_1 \cdot ((n_1(u_1^{-1} \cdot n_2))^{-1})\right) &= \phi_1\left(h_1(u_1^{-1} n_2^{-1} u_1) n_1^{-1} h_1^{-1}\right) \\
&= \bar{h}_1 \bar{u}_1^{-1} \bar{n}_2^{-1} \bar{u}_1 \bar{n}_1^{-1} \bar{h}_1^{-1} \\
&= \bar{h}_1 (\bar{n}_1^{-1} \bar{h}_1^{-1} \bar{v}_1^{-1} \bar{g}_1) \bar{n}_2^{-1} \bar{u}_1 \bar{n}_1^{-1} \bar{h}_1^{-1} \\
&= \bar{h}_1 \bar{n}_1^{-1} \bar{h}_1^{-1} \bar{v}_1^{-1} \bar{h}_2 \bar{n}_2^{-1} (\bar{u}_1 \bar{n}_1^{-1} \bar{h}_1^{-1}) \\
&= \bar{h}_1 \bar{n}_1^{-1} \bar{h}_1^{-1} \bar{v}_1^{-1} \bar{h}_2 \bar{n}_2^{-1} (\bar{g}_1^{-1} \bar{v}_1) \\
&= \bar{h}_1 \bar{n}_1^{-1} \bar{h}_1^{-1} \bar{v}_1^{-1} \bar{h}_2 \bar{n}_2^{-1} \bar{h}_2^{-1} \bar{v}_1 \\
&= \phi_1\left((h_1 n_1^{-1} h_1^{-1}) v_1^{-1} (h_2 n_2^{-1} h_2^{-1}) v_1\right) \\
&= \phi_1\left((h_1 \cdot (n_1^{-1}))(v_1^{-1} \cdot (h_2 \cdot (n_2^{-1})))\right)
\end{aligned}$$

Here we have used the properties $\phi_1|_N = \phi_2|_N$, $\bar{g}_1 \bar{u}_1 = \bar{v}_1 \bar{h}_1 \bar{n}_1$, and $g_1 = h_2$. The injectivity of $\phi_1|_N$ now proves that equation (4.2) holds. □

Theorem 4.13. *Let (D, H, V) be a Lie groupoid triple on base manifold M, and let N be a common embedded normal Lie subgroupoid of H and V such that ϕ_1 and ϕ_2 agree on N and are also injective when restricted to N. Then $(\widetilde{S}; H, V; M)$ is a weak double Lie groupoid.*

Proof. To verify that all the structure maps of $(\widetilde{S}; H, V; M)$ described above are smooth is straightforward. This shows, in particular, that the inversion maps are diffeomorphisms because they are self-inverse. Thus, it only remains to check that the source and target projections of the horizontal and vertical structures are submersions, and that the double source map is a submersion.

We first consider the source projection of the vertical structure. We need to show that, for any $s = (g, u, v, h; n) \in \widetilde{S}$, the tangent map $T_s(\widetilde{\alpha}_V) \colon T_s\widetilde{S} \to T_h H$ is a surjection. Take $Y \in T_h H$, and note that $(v, h, n) \in V * H * N$. Now, since $\alpha_V \colon V \to M$ and $q \colon N \to M$ are submersions, there exists $\eta \in T_v V$ and $\nu \in T_n N$, such that

$$T(\alpha_V)(\eta) = T(\beta_H)(Y), \quad T(\alpha_H)(Y) = T(q)(\nu).$$

Which means that $(\eta, Y, \nu) \in T_{(v,h,n)}(V * H * N)$. Since Φ_1 is a submersion, there also exists a pair $(X, \xi) \in T_{(g,u)}(H * V)$ with

$$T(\Phi_1)(X, \xi) = T(\widetilde{\Phi}_2)(\eta, Y, \nu).$$

Hence, $(X, \xi, \eta, Y; \nu) \in T_s\widetilde{S}$, and moreover $T(\widetilde{\alpha}_V)(X, \xi, \eta, Y; \nu) = Y$. This proves the surjectivity of this tangent map, and verifies that the source projection $\widetilde{\alpha}_V$ is a submersion.

Since the inversion map $\widetilde{\iota}_V$ is a diffeomorphism, it follows that the target projection $\widetilde{\beta}_V = \widetilde{\alpha}_V \circ \widetilde{\iota}_V$ is also a submersion. A similar argument shows that the source and target projections of the horizontal structure are submersions too.

Lastly, let us verify that the double source map $\alpha_2 = (\widetilde{\alpha}_V, \widetilde{\alpha}_H) \colon S \to H \times_\alpha V$ is a submersion. Given $s = (g, u, v, h; n) \in \widetilde{S}$, we will show that the tangent map $T_s(\alpha_2) \colon T_s S \to T_{(h,u)}(H \times_\alpha V)$ is surjective. To do this, we first consider an arbitrary pair $(Y, \xi) \in T_{(h,u)}(H \times_\alpha V)$, and define $Y' := T(\iota_H)(Y) \in T_{h^{-1}}H$. Note that,

$$T(\alpha_V)(\xi) = T(\alpha_H)(Y) = T(\beta_H)(Y').$$

Also, since $q \colon N \to M$ is a submersion there exists $\nu' \in T_{h\cdot(n^{-1})}N$, such that $T(\alpha_H)(Y') = T(q)(\nu')$. Hence, $(\xi, Y', \nu') \in T_{(u,h^{-1},h\cdot(n^{-1}))}(V * H * N)$. Now since Φ_1 is a submersion, there exists a pair $(X', \eta) \in T_{(g^{-1},v)}(H * V)$ such that

$$T(\Phi_1)(X', \eta) = T(\widetilde{\Phi}_2)(\xi, Y', \nu'),$$

and so $(X', \eta, \xi, Y'; \nu') \in T_{s^{-1}(H)}\widetilde{S}$. Next, we consider the map $\theta \colon H * N \to N$, $(h, n) \mapsto h\cdot(n^{-1})$, where $H*N = (\alpha_H * q)^{-1}(\Delta_M)$. We define $\nu = T(\theta)(Y', \nu') \in T_n N$, and $X = T(\iota_H)(X') \in T_g H$. Observe that,

$$T(\widetilde{\iota}_H)(X', \eta, \xi, Y'; \nu') = (T(\iota_H)(X'), \xi, \eta, T(\iota_H)(Y'); T(\theta)(Y', \nu')) = (X, \xi, \eta, Y; \nu).$$

Thus, $(X, \xi, \eta, Y; \nu) \in T_s(\widetilde{S})$, and furthermore $T(\alpha_2)(X, \xi, \eta, Y; \nu) = (Y, \xi)$. This shows that the double source map α_2 is a submersion, and hence completes the proof. □

The core of $(\widetilde{S}; H, V; M)$ consists of all the elements $(g, 1^V_{\alpha_H(g)}, v, 1^H_{\alpha_V(v)}; n) \in \widetilde{S}$ satisfying $\bar{v}^{-1}\bar{g} = \bar{n}$. We can identify this space with the pullback of the following diagram,

$$\begin{array}{ccc} & & N \\ & & \big\downarrow{\scriptstyle \phi_1|_N} \\ V * H & \xrightarrow{\ \Phi_2\ } & D, \end{array}$$

where $\Phi_2 \colon V * H \to D$ is the map defined by $(v, h) \mapsto \bar{v}\bar{h}$. Here, $V * H$ is the pullback manifold $(\alpha_V \times \beta_H)^{-1}(\Delta_M)$. Note that since the map Φ_1 is a submersion by assumption, it follows that Φ_2 is a submersion. The dimension of the core is thus given by

$$\dim(C) = (\dim(H) + \dim(V) - \dim(M)) + \dim(N) - \dim(D). \tag{4.3}$$

Also, since $\widetilde{S} = (\Phi_1 \times \widetilde{\Phi}_2)^{-1}(\Delta_D)$, we have

$$\dim(\widetilde{S}) = 2\dim(H) + 2\dim(V) + \dim(N) - 3\dim(M) - \dim(D). \tag{4.4}$$

We now attempt to search for a case where $\widetilde{S}$ and C have the correct dimensions for $(\widetilde{S}; H, V; M)$ to be a symplectic double groupoid. Note that when $\Phi_1 \colon H * V \to D$ is a local diffeomorphism,

$$\dim(H) + \dim(V) - \dim(M) = \dim(D).$$

It follows from equation (4.3) that $\dim(C) = \dim(N)$. If $(\widetilde{S}; H, V; M)$ was a symplectic double groupoid, then the core groupoid $C \rightrightarrows M$ would be a symplectic groupoid, and thus a necessary condition is $\dim(N) = 2\dim(M)$. However, in this scenario, it follows from equation (4.4) that $\dim(\widetilde{S}) = \dim(H) + \dim(V)$. Hence, if additionally H and V are dual Poisson groupoids, $\widetilde{S}$ has the correct dimension for $(\widetilde{S}; H, V; M)$ to be a symplectic double groupoid.

Finally, we remark that when we consider the Lie groupoid triple (D, G, G^*), where G and G^* are dual Poisson Lie groups and D the Drinfel'd double Lie group, and we take N to be the trivial subgroup of G and G^*, this construction leads to precisely the Lu-Weinstein double groupoid.

4.1.2 A generalised Lu-Weinstein symplectic double groupoid

In the previous subsection we saw some constructions of weak double Lie groupoids that generalise the Lu-Weinstein double groupoid. We now consider some cases where the first construction gives rise to a symplectic double groupoid.

Example 4.14. Let M be a symplectic manifold. In Example 4.4, we saw that this gives a Lie groupoid triple $(M \times M, M \times M, M \times M)$, with maps ϕ_1 and ϕ_2 given by the identity map $\mathrm{id}_{M\times M}$. Then, in Example 4.8, we saw that the double groupoid $(\widetilde{\Sigma}; M \times M, M \times M; M)$ was isomorphic to $(M^4; M \times M, M \times M; M)$. Moreover, this was already shown to be a symplectic double groupoid in Example 2.79.

Example 4.15. Let G be a symplectic groupoid on base P. We saw in Example 4.5 that $(P\times P, G, P\times P)$ was a Lie groupoid triple with $\phi_1 = \chi$, the anchor of G, and $\phi_2 = \mathrm{id}_{P\times P}$. The resulting double groupoid $(\widetilde{\Sigma}; G, P \times P; P)$ was shown in Example 4.9 to be isomorphic to the double Lie groupoid $(G \times G; G, P \times P; P)$. However, in Example 2.80 we saw that $(G \times \overline{G}; G, P \times \overline{P}; P)$ was in fact a symplectic double groupoid. Thus, via the aforementioned isomorphism, we can make $(\widetilde{\Sigma}; G, P \times \overline{P}; P)$ a symplectic double groupoid. ⊠

We now take a look at a more interesting example. Consider a pair of dual Poisson Lie groups G and G^*, and a symplectic manifold M. We can form the trivial Lie groupoids $M \times G \times M$ and $M \times G^* \times M$ with base manifold M, which were defined in Section 1.1.4. For the Lie groupoid $M \times G \times M$, recall that the source and target projections are given by

$$\alpha\colon M \times G \times M \to M, \quad (y, g, x) \mapsto x;$$

$$\beta\colon M \times G \times M \to M, \quad (y, g, x) \mapsto y;$$

For compatible elements, the partial multiplication is given by

$$(z, h, y)(y, g, x) = (z, hg, x).$$

For $x \in M$ the corresponding identity element is $(x, 1_G, x)$, and the inverse of an element $(y, g, x) \in M \times G \times M$ is given by (x, g^{-1}, y). The structure maps for the Lie groupoid $M \times G^* \times M$ are defined analogously. We will now show that both of these Lie groupoids can be made into Poisson groupoids in a natural way.

Proposition 4.16. *Let G and G^* be dual Poisson Lie groups, and let M be a symplectic manifold. Then the trivial Lie groupoids $M \times G \times \overline{M}$ and $M \times G^* \times \overline{M}$ on base M are dual Poisson groupoids.*

Proof. Note that we have the obvious diffeomorphism $F\colon M \times G \times M \to M \times M \times G$ given by $(y, g, x) \mapsto (y, x, g)$. Consider the pair groupoid $M \times \overline{M}$ on base M. We saw in Example 2.72 that $M \times \overline{M}$ is a symplectic groupoid. We can also regard G as a Poisson groupoid on a singleton set. Now we can form the Cartesian product groupoid $M \times \overline{M} \times G$ on base M, which is a Poisson groupoid by Proposition 2.66. Moreover, F is an isomorphism of Lie groupoids over M, and as a map $F\colon M \times G \times \overline{M} \to M \times \overline{M} \times G$ is Poisson. Hence, $M \times G \times \overline{M}$ is also a Poisson groupoid.

By a parallel argument, we can show that $M \times G^* \times \overline{M}$ is a Poisson groupoid which is isomorphic to the Cartesian product groupoid $M \times \overline{M} \times G^*$.

We will show that $M \times \overline{M} \times G$ and $M \times \overline{M} \times G^*$ are dual Poisson groupoids. It will then follow that $M \times G \times \overline{M}$ and $M \times G^* \times \overline{M}$ are also dual. Let $\mathfrak{g}$ denote the Lie algebra of G, and let π_M denote the Poisson structure associated to the symplectic manifold M. In light of Proposition 1.56, the tangent Lie bialgebroids of $M \times \overline{M} \times G$ and $M \times \overline{M} \times G^*$ are given by $(TM \times \mathfrak{g}, T^*M \times \mathfrak{g}^*)$ and $(TM \times \mathfrak{g}^*, T^*M \times \overline{\mathfrak{g}})$, respectively. Observe that the map

$$\pi_M^{\#} \times \mathrm{id}_{\mathfrak{g}^*}\colon T^*M \times \mathfrak{g}^* \to TM \times \mathfrak{g}^*$$

is an isomorphism of Lie algebroids (because M is symplectic), and has dual map given by

$$(-\pi_M^{\#}) \times \mathrm{id}_{\mathfrak{g}}\colon T^*M \times \overline{\mathfrak{g}} \to \overline{TM} \times \overline{\mathfrak{g}}.$$

By the same reasoning, this dual map is also an isomorphism of Lie algebroids. Thus, $(T^*M \times \mathfrak{g}^*, \overline{TM} \times \overline{\mathfrak{g}})$ and $(TM \times \mathfrak{g}^*, T^*M \times \overline{\mathfrak{g}})$ are isomorphic Lie algebroid pairs. Hence, $M \times \overline{M} \times G$ and $M \times \overline{M} \times G^*$ are dual Poisson groupoids. □

It should be clear that the underlining principle behind this proof can be extended to give the following result.

Proposition 4.17. *Let G_1 and H_1 be dual Poisson groupoids with base manifold P_1, and let G_2 and H_2 be dual Poisson groupoids with base manifold P_2. Then the Cartesian product groupoids $G_1 \times G_2$ and $H_1 \times H_2$ are dual Poisson groupoids with base $P_1 \times P_2$.* □

Consider the Drinfel'd double Lie group D associated to the Poisson Lie group G. We have already seen that (D, G, G^*) forms a Lie groupoid triple with the usual Lie group homomorphisms

$$\phi_1\colon G \to D,\ g \mapsto \bar{g}; \quad \phi_2\colon G^* \to D,\ u \mapsto \bar{u}. \tag{4.5}$$

Note that we can also form the trivial Lie groupoid $M \times D \times M$ on base M.

Proposition 4.18. *Let M be a smooth manifold, and let G and G^* be dual Poisson Lie groups with associated Drinfel'd double Lie group D. Then we have a Lie groupoid triple on base M given by $(M \times D \times M, M \times G \times M, M \times G^* \times M)$.*

Proof. The following maps

$$\widetilde{\phi}_1\colon M \times G \times M \to M \times D \times M, \quad (y, g, x) \mapsto (y, \bar{g}, x);$$

$$\widetilde{\phi}_2\colon M \times G^* \times M \to M \times D \times M, \quad (y, u, x) \mapsto (y, \bar{u}, x);$$

are morphisms of Lie groupoids over M. Note that we have $\widetilde{\phi}_1 = \mathrm{id}_M \times \phi_1 \times \mathrm{id}_M$ and $\widetilde{\phi}_2 = \mathrm{id}_M \times \phi_2 \times \mathrm{id}_M$, where ϕ_1 and ϕ_2 are the Lie group homomorphisms given in Equation (4.5).

It remains to check that the smooth map

$$\widetilde{\Phi}_1 \colon (M \times G \times M) * (M \times G^* \times M) \to M \times D \times M;$$
$$((z, g, y), (y, u, x)) \mapsto (z, \bar{g}\bar{u}, x),$$

is a submersion. We first observe that we have a natural diffeomorphism given by

$$\widetilde{F} \colon (M \times G \times M) * (M \times G^* \times M) \to M^3 \times G \times G^*;$$
$$((z, g, y), (y, u, x)) \mapsto (z, y, x, g, u).$$

In Proposition 3.12, we saw that the map

$$\Phi_1 \colon G \times G^* \to D, \quad (g, u) \mapsto \bar{g}\bar{u},$$

was a local diffeomorphism. Additionally, the projection map $p \colon M^3 \to M^2$ defined by $(z, y, x) \mapsto (z, x)$ is clearly a submersion. Lastly, note that the map defined by

$$F \colon M \times M \times D \to M \times D \times M, \quad (y, x, d) \mapsto (y, d, x),$$

is also a diffeomorphism. Now we can express $\widetilde{\Phi}_1$ as the composite $F \circ (p \times \Phi_1) \circ \widetilde{F}$, and thus $\widetilde{\Phi}_1$ is a submersion. □

From the Lie groupoid triple $(M \times D \times M, M \times G \times M, M \times G^* \times M)$ on M, we can now form the weak double Lie groupoid $(\widetilde{\Sigma}; M \times G \times M, M \times G^* \times M; M)$. Let us show that this becomes a symplectic double groupoid in a natural way, by first introducing a useful lemma.

Lemma 4.19. *Let $(S_1; H_1, V_1; M_1)$ and $(S_2; H_2, V_2; M_2)$ be symplectic double groupoids. Then the Cartesian product double groupoid $(S_1 \times S_2; H_1 \times H_2, V_1 \times V_2; M_1 \times M_2)$ is a symplectic double groupoid.*

Proof. It is routine to verify that $(S_1 \times S_2; H_1 \times H_2, V_1 \times V_2; M_1 \times M_2)$ is a double Lie groupoid. Let us denote the symplectic structures of S_1 and S_2 by ω_1 and ω_2, respectively. Similarly, we let the associated Poisson structures be denoted by π_1 and π_2, respectively. The product manifold $S_1 \times S_2$ becomes a symplectic manifold with symplectic structure $\omega_1 \oplus \omega_2$ and associated Poisson structure $\pi_1 \oplus \pi_2$. It follows by Proposition 2.66 that the horizontal and vertical structures of the double Lie groupoid $(S_1 \times S_2; H_1 \times H_2, V_1 \times V_2; M_1 \times M_2)$ are Poisson groupoids with respect to $\pi_1 \oplus \pi_2$ and hence the result follows. □

Proposition 4.20. *Let G and G^* be dual Poisson Lie groups, and let M be a symplectic manifold. Then there exists a symplectic structure on $\widetilde{\Sigma}$ such that the weak double Lie groupoid $(\widetilde{\Sigma}; M \times G \times \overline{M}, M \times G^* \times \overline{M}; M)$ is a symplectic double groupoid.*

Proof. We will show that the double groupoid $(\widetilde{\Sigma}; M \times G \times \overline{M}, M \times G^* \times \overline{M}; M)$ is intricately related to the Lu-Weinstein double groupoid $(\Sigma; G, G^*; \{\cdot\})$ and the double groupoid $(M^4; M \times M, M \times M; M)$ of Example 1.65. We first note that elements of $\widetilde{\Sigma}$ take the form

$$s = ((z, g, y), (y, u, x), (z, v, m), (m, h, x)), \tag{4.6}$$

where $z, y, m, x \in M$, $g, h \in G$, $u, v \in G^*$, and such that $\bar{g}\bar{u} = \bar{v}\bar{h}$. Hence, we have the obvious diffeomorphism given by

$$F\colon \widetilde{\Sigma} \to M^4 \times \Sigma, \quad s \mapsto ((z, y, m, x), (g, u, v, h)).$$

Note that M^4 has a pair groupoid structure on base $M \times M$, and Σ has Lie groupoid structures with base manifolds G and G^*. Thus, we can form two Cartesian product groupoid structures on $M^4 \times \Sigma$; one with base $M \times M \times G$, and the other with base $M \times M \times G^*$. We now show that F is actually an isomorphism of Lie groupoids over the diffeomorphism defined by

$$f\colon M \times G \times M \to M \times M \times G, \quad (y, g, z) \mapsto (y, z, g).$$

Let the source and target projections of $\widetilde{\Sigma}$ and Σ be denoted as in the following diagram;

$$\begin{array}{ccc} \widetilde{\Sigma} & \overset{\widetilde{\alpha}_H, \widetilde{\beta}_H}{\rightrightarrows} & M \times G^* \times M \\ {\scriptstyle \widetilde{\alpha}_V, \widetilde{\beta}_V}\downdownarrows & & \downdownarrows \\ M \times G \times M & \rightrightarrows & M \end{array} \qquad \begin{array}{ccc} \Sigma & \overset{\alpha_*, \beta_*}{\rightrightarrows} & G^* \\ {\scriptstyle \alpha, \beta}\downdownarrows & & \downdownarrows \\ G & \rightrightarrows & \{\cdot\}. \end{array}$$

Additionally, let us denote the source and target projections of the pair groupoid M^4 on base $M \times M$ by α_M and β_M, respectively. Take $s \in \widetilde{\Sigma}$ of the form given in Equation (4.6). Then,

$$f \circ \widetilde{\alpha}_V(s) = f(m, h, x) = (m, x, h).$$

On the other hand,

$$(\alpha_M \times \alpha) \circ F(s) = (\alpha_M \times \alpha)((z, y, m, x), (g, u, v, h)) = (m, x, h).$$

Thus, $f \circ \widetilde{\alpha}_V = (\alpha_M \times \alpha) \circ F$. Arguing similarly, we observe that

$$f \circ \widetilde{\beta}_V(s) = f(z, g, y) = (z, y, g),$$

and

$$(\beta_M \times \beta) \circ F(s) = (\beta_M \times \beta)((z, y, m, x), (g, u, v, h)) = (z, y, g).$$

Hence, we also have $f \circ \widetilde{\beta}_V = (\beta_M \times \beta) \circ F$.

Now suppose we are given $s_2, s_1 \in \widetilde{\Sigma}$ satisfying $\widetilde{\alpha}_V(s_2) = \widetilde{\beta}_V(s_1)$. As in Equation (4.6), these elements take the form

$$\begin{aligned} s_2 &= ((z_2, g_2, y_2), (y_2, u_2, x_2), (z_2, v_2, m_2), (m_2, h_2, x_2)), \\ s_1 &= ((z_1, g_1, y_1), (y_1, u_1, x_1), (z_1, v_1, m_1), (m_1, h_1, x_1)), \end{aligned}$$

and satisfy $\bar{g}_2\bar{u}_2 = \bar{v}_2\bar{h}_2$, $\bar{g}_1\bar{u}_1 = \bar{v}_1\bar{h}_1$, and $(m_2, h_2, x_2) = (z_1, g_1, y_1)$. We have

$$\begin{aligned} F(s_2 \boxminus s_1) &= F((z_2, g_2, y_2), (y_2, u_2u_1, x_1), (z_2, v_2v_1, m_1), (m_1, h_1, x_1)) \\ &= ((z_2, y_2, m_1, x_1), (g_2, u_2u_1, v_2v_1, h_1)). \end{aligned}$$

However, note that

$$(g_2, u_2u_1, v_2v_1, h_1) = (g_2, u_2, v_2, h_2) *_G (g_1, u_1, v_1, h_1),$$

and

$$(z_2, y_2, m_1, x_1) = (z_2, y_2, m_2, x_2)(z_1, y_1, m_1, x_1).$$

Hence, it is clear that $F(s_2 \boxminus s_1) = F(s_2)F(s_1)$. This shows that F is an isomorphism of Lie groupoids over f.

Next, we recall the diffeomorphism $\Theta \colon M^4 \to M^4$ of Example 1.65, defined by $(m, x, y, z) \mapsto (m, y, x, z)$. This induces another diffeomorphism $G := (\Theta \times \mathrm{id}_\Sigma) \circ F$ from $\widetilde{\Sigma}$ to $M^4 \times \Sigma$. A similar argument to the above shows that G is also an isomorphism of Lie groupoids, this time over the diffeomorphism defined by

$$g \colon M \times G^* \times M \to M \times M \times G^*, \quad (y, u, x) \mapsto (y, x, u).$$

Moreover, the maps f and g are isomorphisms of Lie groupoids over M. It follows that $(F; f, g; \mathrm{id}_M)$ is an isomorphism of double Lie groupoids from the weak double Lie groupoid $(\widetilde{\Sigma}; M \times G \times M, M \times G^* \times M; M)$ to the Cartesian product double groupoid of $(M^4; M \times M, M \times M; M)$ and the Lu-Weinstein double groupoid $(\Sigma; G, G^*; \{\cdot\})$.

We have seen that both $(M^4; M \times M, M \times M; M)$ and $(\Sigma; G, G^*; \{\cdot\})$ are symplectic double groupoids and thus the corresponding Cartesian product double groupoid $(M^4 \times \Sigma; M \times M \times G, M \times M \times G^*; M)$ is also a symplectic double groupoid by Lemma 4.19. We can give $\widetilde{\Sigma}$ the unique symplectic structure that makes the diffeomorphism F a symplectomorphism. Finally, since $(F; f, g; \mathrm{id}_M)$ is an isomorphism of double Lie groupoids, this symplectic structure makes $(\widetilde{\Sigma}; M \times G \times \overline{M}, M \times G^* \times \overline{M}; M)$ a symplectic double groupoid. □

It should be clear from the proof of Proposition 4.20, that the symplectic double groupoid $(\widetilde{\Sigma}; M \times G \times \overline{M}, M \times G^* \times \overline{M}; M)$ generalises the Lu-Weinstein symplectic double groupoid. Indeed, when M is just a singleton set the construction gives precisely the Lu-Weinstein symplectic double groupoid.

Chapter 5

Actions of double structures and Poisson reduction

In this final chapter, we study some of the possible extensions of the notions of Lie groupoid and Lie algebroid actions to double structures. We first review the actions of double Lie groupoids introduced by Brown and Mackenzie [7]. The definition we give naturally leads to sensible notions of action for $\mathcal{LA}$-groupoids, and we give an in-depth study of these objects.

The second half of this chapter centres on an application of actions of double structures to the study of Poisson reduced spaces. In [67], Xu gave a construction of a symplectic groupoid whose base manifold is the Poisson reduced space of a free and proper Poisson groupoid action of a symplectic groupoid. We present an alternate approach to this construction utilising the actions of double Lie groupoids. We also show how a similar method can be used to construct the cotangent Lie algebroid of the Poisson reduced space of any free and proper Poisson groupoid action.

§ 5.1 Actions of double structures

The purpose of this section is to consider abstract notions of actions of double Lie groupoids and $\mathcal{LA}$-groupoids. After examining actions of double Lie groupoids in some detail, we focus on a particular example of an action of the double Lie groupoid $(G \times G; G, M \times M; M)$ of Example 1.66, which is motivated by the work of Xu [67]. We show that such a double action can be obtained from a pair of Lie groupoid actions of G satisfying certain properties. Moreover, we show that there is also a reverse construction.

We then turn our attention towards the actions of $\mathcal{LA}$-groupoids. It turns out that there are two reasonable notions for an action of an $\mathcal{LA}$-groupoid: an action of an $\mathcal{LA}$-groupoid on a morphism of Lie groupoids, and an action of an $\mathcal{LA}$-groupoid on a morphism of Lie algebroids. Just as a Lie groupoid action gives rise to a Lie algebroid action, we show that an action of a double Lie groupoid gives rise to an action of an $\mathcal{LA}$-groupoid of both types.

5.1.1 Actions of double Lie groupoids

In Section 1.1.3, we defined an action of a Lie groupoid on a smooth map of manifolds. We would like to generalise this notion of an action from the category of smooth manifolds to the category of Lie groupoids. To be more precise, we are seeking a notion of an action of a double Lie groupoid on a morphism of Lie groupoids. This leads us to the following definition proposed by Brown and Mackenzie [7].

Definition 5.1. Let $(S; H, V; M)$ be a double Lie groupoid, $G \rightrightarrows P$ a Lie groupoid, and $F\colon G \to V$ a morphism of Lie groupoids over a smooth map $f\colon P \to M$.

$$\begin{array}{ccccc} & & & & G \\ & & & \overset{F}{\swarrow} & \\ S & \overset{\widetilde{\alpha}_H, \widetilde{\beta}_H}{\rightrightarrows} & V & & \Big\downarrow\!\Big\downarrow \alpha_G, \beta_G \\ \widetilde{\alpha}_V, \widetilde{\beta}_V \Big\downarrow\!\Big\downarrow & & \Big\downarrow\!\Big\downarrow \alpha_V, \beta_V & & P \\ & & & \overset{f}{\swarrow} & \\ H & \overset{\alpha_H, \beta_H}{\rightrightarrows} & M & & \end{array} \tag{5.1}$$

An *action of* $(S; H, V; M)$ *on* (F, f) (or on $G \rightrightarrows P$) consists of a Lie groupoid action $\widetilde{\theta}$ of the horizontal structure $S \rightrightarrows V$ on F, and a Lie groupoid action θ of the Lie groupoid $H \rightrightarrows M$ on f, such that $(\widetilde{\theta}, \theta)$ is a morphism of Lie groupoids.[1]

Note that the action $(\widetilde{\theta}, \theta)$ of a double Lie groupoid $(S; H, V; M)$ on a morphism of Lie groupoids can be expressed diagrammatically as follows,

$$\begin{array}{ccc} S \lhd G & \xrightarrow{\ \widetilde{\theta}\ } & G \\ \Big\downarrow\!\Big\downarrow & & \Big\downarrow\!\Big\downarrow \\ H \lhd P & \xrightarrow{\ \theta\ } & P, \end{array} \tag{5.2}$$

where here the Lie groupoid structure on $S \lhd G$ with base $H \lhd P$ is the unique groupoid structure that makes it an embedded Lie subgroupoid of the Cartesian product groupoid $S \times G$ via inclusion.

Remark 5.2. In the above definition, the condition that $(\widetilde{\theta}, \theta)$ is a morphism of Lie groupoids implies the following properties:

- $\alpha_G(s \cdot g) = \widetilde{\alpha}_V(s) \cdot \alpha_G(g)$ for all $(s, g) \in S \lhd G$;
- $\beta_G(s \cdot g) = \widetilde{\beta}_V(s) \cdot \beta_G(g)$ for all $(s, g) \in S \lhd G$;
- $(s_2 \cdot g_2)(s_1 \cdot g_1) = (s_2 \boxminus s_1) \cdot (g_1 g_2)$ for all $(s_2, g_2), (s_1, g_1) \in S \lhd G$, such that $(s_2, s_1) \in S *_H S$ and $(g_2, g_1) \in G * G$;
- $\widetilde{1}^V_h \cdot 1^G_p = 1^G_{h \cdot p}$ for all $(h, p) \in H \lhd P$;
- $(s \cdot g)^{-1} = s^{-1(V)} \cdot g^{-1}$ for all $(s, g) \in S \lhd G$.

[1] Note that in the original formulation [7, Definition 1.5], conditions (i) and (ii) together are equivalent to $(\widetilde{\theta}, \theta)$ being a morphism of groupoids. Condition (iii) is a consequence of the previous two conditions, and so is redundant.

The first three conditions are immediate from the definition of a morphism of Lie groupoids (Definition 1.11). The fourth and fifth conditions are just consequences of the previous three (Proposition 1.12).

Given an action of a double Lie groupoid $(S; H, V; M)$ on a Lie groupoid $G \rightrightarrows P$, we get action groupoids $S \lessdot G$ and $H \lessdot P$ on base manifolds G and P, respectively. In addition, $S \lessdot G$ also has a Lie groupoid structure on base $H \lessdot P$. We can present these groupoid structures in the following diagram:

$$\begin{array}{ccc} S \lessdot G & \overset{\widetilde{\alpha}_{\lessdot}, \widetilde{\beta}_{\lessdot}}{\rightrightarrows} & G \\ {\scriptstyle \widetilde{\alpha}_V * \alpha_G, \widetilde{\beta}_V * \beta_G} \downdownarrows & & \downdownarrows {\scriptstyle \alpha_G, \beta_G} \\ H \lessdot P & \overset{\alpha_{\lessdot}, \beta_{\lessdot}}{\rightrightarrows} & P. \end{array} \tag{5.3}$$

Proposition 5.3. *Let $(\widetilde{\theta}, \theta)$ be an action of a double Lie groupoid $(S; H, V; M)$ on a morphism of Lie groupoids $F \colon G \to V$ over a smooth map $f \colon P \to M$. Then $(S \lessdot G; H \lessdot P, G; P)$ is a double Lie groupoid.*

Proof. We will use the standard notation[2] for the structure maps of the double Lie groupoid $(S; H, V; M)$. The structure maps of $(S \lessdot G; H \lessdot P, G; P)$ will take the notation indicated in the above diagram (Equation (5.3)). In the Lie groupoid $S \lessdot G \rightrightarrows H \lessdot P$, the product of two compatible elements will just be denoted by usual concatenation.

We need to verify that the structure maps of the action groupoid $S \lessdot G \rightrightarrows G$ are morphisms of Lie groupoids over the corresponding structure maps of $H \lessdot P \rightrightarrows P$.

We first show that the source projection $\widetilde{\alpha}_{\lessdot} \colon S \lessdot G \to G$ is a morphism of Lie groupoids over the source projection $\alpha_{\lessdot} \colon H \lessdot P \to P$. Observe that, for any $(s, g) \in S \lessdot G$, we have

$$\begin{aligned} \alpha_{\lessdot}(\widetilde{\alpha}_V * \alpha_G(s, g)) &= \alpha_{\lessdot}(\widetilde{\alpha}_V(s), \alpha_G(g)) \\ &= \alpha_G(g) \\ &= \alpha_G(\widetilde{\alpha}_{\lessdot}(s, g)), \end{aligned}$$

and

$$\begin{aligned} \alpha_{\lessdot}(\widetilde{\beta}_V * \beta_G(s, g)) &= \alpha_{\lessdot}(\widetilde{\beta}_V(s), \beta_G(g)) \\ &= \beta_G(g) \\ &= \beta_G(\widetilde{\alpha}_{\lessdot}(s, g)). \end{aligned}$$

Moreover, for any $((s_2, g_2), (s_1, g_1)) \in (S \lessdot G) *_{H \lessdot P} (S \lessdot G)$, we have

$$\begin{aligned} \widetilde{\alpha}_{\lessdot}((s_2, g_2)(s_1, g_1)) &= \widetilde{\alpha}_{\lessdot}(s_2 \boxminus s_1, g_2 g_1) \\ &= g_2 g_1 \\ &= \widetilde{\alpha}_{\lessdot}(s_2, g_2)\widetilde{\alpha}_{\lessdot}(s_1, g_1). \end{aligned}$$

We next show that the target projection $\widetilde{\beta}_{\lessdot} \colon S \lessdot G \to G$ is a morphism of Lie groupoids over target projection $\beta_{\lessdot} \colon H \lessdot P \to P$. Observe that, for any $(s, g) \in S \lessdot G$, we

[2]See page 22 for details of the standard notation we use for a double groupoid.

have

$$\begin{aligned}\beta_\lessdot(\widetilde{\alpha}_V * \alpha_G(s,g)) &= \beta_\lessdot(\widetilde{\alpha}_V(s), \alpha_G(g))\\ &= \widetilde{\alpha}_V(s)\cdot\alpha_G(g)\\ &= \alpha_G(s\cdot g)\\ &= \alpha_G(\widetilde{\beta}_\lessdot(s,g)),\end{aligned}$$

and

$$\begin{aligned}\beta_\lessdot(\widetilde{\beta}_V * \beta_G(s,g)) &= \beta_\lessdot(\widetilde{\beta}_V(s), \beta_G(g))\\ &= \widetilde{\beta}_V(s)\cdot\beta_G(g)\\ &= \beta_G(s\cdot g)\\ &= \beta_G(\widetilde{\beta}_\lessdot(s,g)).\end{aligned}$$

Furthermore, for any $((s_2,g_2),(s_1,g_1)) \in (S \lessdot G) *_{H\lessdot P} (S \lessdot G)$, we have

$$\begin{aligned}\widetilde{\beta}_\lessdot((s_2,g_2)(s_1,g_1)) &= \widetilde{\beta}_\lessdot(s_2 \boxminus s_1, g_2g_1)\\ &= (s_2 \boxminus s_1)\cdot(g_2g_1)\\ &= (s_2\cdot g_2)(s_1\cdot g_1)\\ &= \widetilde{\beta}_\lessdot(s_2,g_2)\widetilde{\beta}_\lessdot(s_1,g_1).\end{aligned}$$

We will now show that the partial multiplication $\widetilde{\kappa}_\lessdot$ of the action groupoid $S \lessdot G$ is a morphism of Lie groupoids over the partial multiplication $\kappa_\lessdot$ of $H \lessdot P$. For any $((s_2,g_2),(s_1,g_1)) \in (S \lessdot G) *_G (S \lessdot G)$, we have

$$\begin{aligned}\kappa_\lessdot(\widetilde{\alpha}_V * \alpha_G(s_2,g_2), \widetilde{\alpha}_V * \alpha_G(s_1,g_1)) &= \kappa_\lessdot((\widetilde{\alpha}_V(s_2), \alpha_G(g_2)), (\widetilde{\alpha}_V(s_1), \alpha_G(g_1)))\\ &= (\widetilde{\alpha}_V(s_2)\widetilde{\alpha}_V(s_1), \alpha_G(g_1))\\ &= (\widetilde{\alpha}_V(s_2 \boxbar s_1), \alpha_G(g_1))\\ &= \widetilde{\alpha}_V * \alpha_G(s_2 \boxbar s_1, g_1)\\ &= \widetilde{\alpha}_V * \alpha_G(\widetilde{\kappa}_\lessdot((s_2,g_2),(s_1,g_1))),\end{aligned}$$

and

$$\begin{aligned}\kappa_\lessdot(\widetilde{\beta}_V * \beta_G(s_2,g_2), \widetilde{\beta}_V * \beta_G(s_1,g_1)) &= \kappa_\lessdot((\widetilde{\beta}_V(s_2), \beta_G(g_2)), (\widetilde{\beta}_V(s_1), \beta_G(g_1)))\\ &= (\widetilde{\beta}_V(s_2)\widetilde{\beta}_V(s_1), \beta_G(g_1))\\ &= (\widetilde{\beta}_V(s_2 \boxbar s_1), \beta_G(g_1))\\ &= \widetilde{\beta}_V * \beta_G(s_2 \boxbar s_1, g_1)\\ &= \widetilde{\beta}_V * \beta_G(\widetilde{\kappa}_\lessdot((s_2,g_2),(s_1,g_1))).\end{aligned}$$

Moreover, for any $((s_4,g_4),(s_2,g_2)),((s_3,g_3),(s_1,g_1)) \in (S \lessdot G) *_G (S \lessdot G)$, such that $((s_4,g_4),(s_3,g_3)),((s_2,g_2),(s_1,g_1)) \in (S \lessdot G) *_{H\lessdot P} (S \lessdot G)$, we find that

$$\begin{aligned}\widetilde{\kappa}_\lessdot((s_4,g_4)(s_3,g_3),(s_2,g_2)(s_1,g_1)) &= \widetilde{\kappa}_\lessdot((s_4 \boxminus s_3, g_4g_3),(s_2 \boxminus s_1, g_2g_1))\\ &= ((s_4 \boxminus s_3) \boxbar (s_2 \boxminus s_1), g_2g_1)\\ &= ((s_4 \boxbar s_2) \boxminus (s_3 \boxbar s_1), g_2g_1)\\ &= (s_4 \boxbar s_2, g_2)(s_3 \boxbar s_1, g_1)\\ &= \widetilde{\kappa}_\lessdot((s_4,g_4),(s_2,g_2))\widetilde{\kappa}_\lessdot((s_3,g_3),(s_1,g_1)).\end{aligned}$$

Next, we will show that the identity map $\widetilde{1}^{\lhd}\colon G \to S \lhd G$ is a morphism of Lie groupoids over $1^{\lhd}\colon P \to H \lhd P$. Observe that, for any $g \in G$, we have

$$\begin{aligned}
1^{\lhd}_{\alpha_G(g)} &= \left(1^H_{f(\alpha_G(g))}, \alpha_G(g)\right)\\
&= \left(1^H_{\alpha_V(F(g))}, \alpha_G(g)\right)\\
&= \left(\widetilde{\alpha}_V(\widetilde{1}^H_{F(g)}), \alpha_G(g)\right)\\
&= \widetilde{\alpha}_V * \alpha_G(\widetilde{1}^H_{F(g)}, g)\\
&= \widetilde{\alpha}_V * \alpha_G(\widetilde{1}^{\lhd}_g),
\end{aligned}$$

and

$$\begin{aligned}
1^{\lhd}_{\beta_G(g)} &= \left(1^H_{f(\beta_G(g))}, \beta_G(g)\right)\\
&= \left(1^H_{\beta_V(F(g))}, \beta_G(g)\right)\\
&= \left(\widetilde{\beta}_V(\widetilde{1}^H_{F(g)}), \beta_G(g)\right)\\
&= \widetilde{\beta}_V * \beta_G(\widetilde{1}^H_{F(g)}, g)\\
&= \widetilde{\beta}_V * \beta_G(\widetilde{1}^{\lhd}_g).
\end{aligned}$$

We also see that, for any $(g_2, g_1) \in G * G$,

$$\begin{aligned}
\widetilde{1}^{\lhd}_{g_2 g_1} &= \left(\widetilde{1}^H_{F(g_2 g_1)}, g_2 g_1\right)\\
&= \left(\widetilde{1}^H_{F(g_2)F(g_1)}, g_2 g_1\right)\\
&= \left(\widetilde{1}^H_{F(g_2)} \boxminus \widetilde{1}^H_{F(g_1)}, g_2 g_1\right)\\
&= \left(\widetilde{1}^H_{F(g_2)}, g_2\right)\left(\widetilde{1}^H_{F(g_1)}, g_1\right)\\
&= \widetilde{1}^{\lhd}_{g_2}\widetilde{1}^{\lhd}_{g_1}.
\end{aligned}$$

Finally, we show that inversion map $\widetilde{\iota}_{\lhd}\colon S \lhd G \to S \lhd G$ is a morphism of Lie groupoids over the inversion $\iota_{\lhd}\colon H \lhd P \to H \lhd P$. For any $(s, g) \in S \lhd G$, we have

$$\begin{aligned}
\iota_{\lhd}(\widetilde{\alpha}_V * \alpha_G(s, g)) &= \iota_{\lhd}(\widetilde{\alpha}_V(s), \alpha_G(g))\\
&= (\widetilde{\alpha}_V(s)^{-1}, \widetilde{\alpha}_V(s) \cdot \alpha_G(g))\\
&= (\widetilde{\alpha}_V(s^{-1(H)}), \alpha_G(s \cdot g))\\
&= \widetilde{\alpha}_V * \alpha_G(s^{-1(H)}, s \cdot g)\\
&= \widetilde{\alpha}_V * \alpha_G(\widetilde{\iota}_{\lhd}(s, g)),
\end{aligned}$$

and

$$\begin{aligned}
\iota_{\lhd}(\widetilde{\beta}_V * \beta_G(s, g)) &= \iota_{\lhd}(\widetilde{\beta}_V(s), \beta_G(g))\\
&= (\widetilde{\beta}_V(s)^{-1}, \widetilde{\beta}_V(s) \cdot \beta_G(g))\\
&= (\widetilde{\beta}_V(s^{-1(H)}), \beta_G(s \cdot g))\\
&= \widetilde{\beta}_V * \beta_G(s^{-1(H)}, s \cdot g)\\
&= \widetilde{\beta}_V * \beta_G(\widetilde{\iota}_{\lhd}(s, g)).
\end{aligned}$$

Lastly, observe that, for any $((s_2, g_2), (s_1, g_1)) \in (S \lhd G) *_{H \lhd P} (S \lhd G)$, we have

$$\begin{aligned}
\widetilde{\iota}_{\lhd}((s_2, g_2)(s_1, g_1)) &= \widetilde{\iota}_{\lhd}(s_2 \boxminus s_1, g_2 g_1) \\
&= \Big((s_2 \boxminus s_1)^{-1(H)}, (s_2 \boxminus s_1) \cdot (g_2 g_1)\Big) \\
&= \Big(s_2^{-1(H)} \boxminus s_1^{-1(H)}, (s_2 \cdot g_2)(s_1 \cdot g_1)\Big) \\
&= (s_2^{-1(H)}, s_2 \cdot g_2)(s_1^{-1(H)}, s_1 \cdot g_1) \\
&= \widetilde{\iota}_{\lhd}(s_2, g_2)\widetilde{\iota}_{\lhd}(s_1, g_1).
\end{aligned}$$

To complete the proof, it remains to show that the double source map of the double groupoid $(S \lhd G; H \lhd P, G; P)$ is a surjective submersion. This map is given by

$$\alpha_2^{\lhd} \colon S \lhd G \to (H \lhd P) \times_P G, \quad (s, g) \mapsto (\widetilde{\alpha}_V(s), \alpha_G(g), g),$$

where $(H \lhd P) \times_P G = \{(h, p, g) \in H \times P \times G \mid \alpha_H(h) = f(p),\ p = \alpha_G(g)\}$. We will use the property that the double source map of the original double groupoid $(S; H, V; M)$ is a surjective submersion. This is the map given by

$$\alpha_2 \colon S \to H \times_M V, \quad s \mapsto (\widetilde{\alpha}_V(s), \widetilde{\alpha}_H(s)),$$

where $H \times_M V = \{(h, v) \in H \times V \mid \alpha_H(h) = \alpha_V(v)\}$. To first show surjectivity, consider any triple $(h, p, g) \in (H \lhd P) \times_P G$. We have the relations $\alpha_H(h) = f(p)$ and $p = \alpha_G(g)$. Thus, we deduce

$$\alpha_V(F(g)) = f(\alpha_G(g)) = f(p) = \alpha_H(h),$$

and so $(h, F(g)) \in H \times_M V$. The surjectivity of α_2 immediately implies that there exists an $s \in S$, such that $\widetilde{\alpha}_V(s) = h$ and $\widetilde{\alpha}_H(s) = F(g)$. The latter relation tells us that $(s, g) \in S \lhd G$, and furthermore we find that

$$\alpha_2^{\lhd}(s, g) = (\widetilde{\alpha}_V(s), \alpha_G(g), g) = (h, p, g).$$

Hence, the map $\alpha_2^{\lhd}$ is surjective. To show that this map is also a submersion, we must check that, for any given $(s, g) \in S \lhd G$, the linear map

$$T_{(s,g)}(\alpha_2^{\lhd}) \colon T_{(s,g)}(S \lhd G) \to T_{\alpha_2^{\lhd}(s,g)}((H \lhd P) \times_P G)$$

is a surjection. Take any triple $(Z, Y, X) \in T_{\alpha_2^{\lhd}(s,g)}((H \lhd P) \times_P G)$. In more detail, we have $Z \in T_{\widetilde{\alpha}_V(s)} H$, $Y \in T_{\alpha_G(g)} P$ and $X \in T_g G$, satisfying the relations $T_{\widetilde{\alpha}_V(s)}(\alpha_H)(Z) = T_{\alpha_G(g)}(f)(Y)$ and $Y = T_g(\alpha_G)(X)$. Now observe that

$$\begin{aligned}
T_{F(g)}(\alpha_V)(T_g(F)(X)) &= T_g(\alpha_V \circ F)(X) \\
&= T_g(f \circ \alpha_G)(X) \\
&= T_{\alpha_G(g)}(f)(T_g(\alpha_G)(X)) \\
&= T_{\alpha_G(g)}(f)(Y) \\
&= T_{\widetilde{\alpha}_V(s)}(\alpha_H)(Z).
\end{aligned}$$

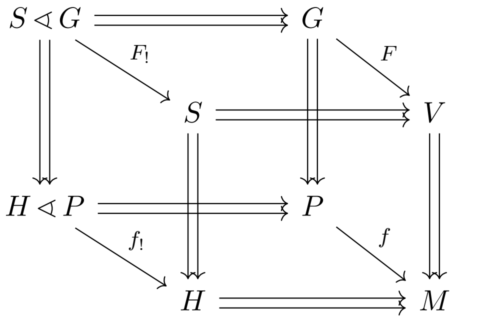
Hence, $(Z, T_g(F)(X)) \in T_{\alpha_2(s)}(H \times_M V)$. Since α_2 is a submersion, it follows that there exists an $\xi \in T_s S$ such that $T_s(\widetilde{\alpha}_V)(\xi) = Z$, and $T_s(\widetilde{\alpha}_H)(\xi) = T_g(F)(X)$. The latter relation implies that $(\xi, X) \in T_{(s,g)}(S \lhd G)$, and moreover

$$T_{(s,g)}(\alpha_2^{\lhd})(\xi, X) = (T_s(\widetilde{\alpha}_V)(\xi), T_g(\alpha_G)(X), X) = (Z, Y, X).$$

Therefore, $T_{(s,g)}(\alpha_2^{\lhd})$ is a surjection, and thus $\alpha_2^{\lhd}$ is a submersion. □

Given an action of a double Lie groupoid $(S; H, V; M)$ on a morphism of Lie groupoids $F\colon G \to V$ over $f\colon P \to M$, we call the corresponding double Lie groupoid $(S \lessdot G; H \lessdot P, G; P)$ an *action double groupoid*.

Note that the action groupoids $S \lessdot G$ and $H \lessdot P$ give rise to the action morphisms $F_!\colon S \lessdot G \to S$ and $f_!\colon H \lessdot P \to H$ over F and f, respectively (see page 7). Moreover, it is routine to show that $F_!$ is also a morphism of Lie groupoids over $f_!$. Hence, we get a morphism of double Lie groupoids $(F_!; f_!, F; f)$ from the action double groupoid $(S \lessdot G; H \lessdot P, G; P)$ to the original double Lie groupoid $(S; H, V; M)$.



Remark 5.4. We can extend this notion of action to include weak double Lie groupoids, and a similar proof to the above shows that the corresponding action double groupoid is also a weak double Lie groupoid.

5.1.2 Actions of $G \times G$

We now consider a special class of actions of double Lie groupoids, which often appear naturally in the literature. Let G be a Lie groupoid on a base manifold M, and consider the double Lie groupoid $(G \times G; G, M \times M; M)$ of Example 1.66.

$$\begin{array}{ccc} G \times G & \overset{\widetilde{\alpha}_G, \widetilde{\beta}_G}{\rightrightarrows} & M \times M \\ {\scriptstyle \alpha_{G\times G}, \beta_{G \times G}}\downdownarrows & & \downdownarrows{\scriptstyle \alpha_{M\times M}, \beta_{M\times M}} \\ G & \overset{\alpha_G, \beta_G}{\rightrightarrows} & M. \end{array} \tag{5.4}$$

Recall that the horizontal structure of $G \times G$ is the Cartesian product groupoid on base $M \times M$, and the vertical structure is the pair groupoid on base G. Here, $M \times M$ also has the pair groupoid structure on base M.

Suppose we have another Lie groupoid Π on a base manifold P, and a smooth map $f\colon P \to M$. By Examples 1.14 and 1.15, the map $f \times f\colon P \times P \to M \times M$ and the anchor $\chi = (\beta, \alpha)\colon \Pi \to P \times P$ are morphisms of Lie groupoids over f and P, respectively. Hence the composite $(f \times f) \circ \chi\colon \Pi \to M \times M$ is a morphism of Lie groupoids over f.

$$\begin{array}{ccc} \Pi & \xrightarrow{(f\times f)\circ\chi} & M \times M \\ {\scriptstyle \alpha, \beta}\downdownarrows & & \downdownarrows{\scriptstyle \alpha_{M\times M}, \beta_{M\times M}} \\ P & \xrightarrow{f} & M. \end{array} \tag{5.5}$$

The aim of this subsection is to detail a one-to-one correspondence between actions of the double Lie groupoid $(G \times G; G, M \times M; M)$ on the morphism $(f \times f) \circ \chi$, with a special class of triples of Lie groupoid actions. A triple of this type appears in Xu's work on symplectic groupoids of Poisson reduced spaces [67]. We will investigate this example in Section 5.2.1.

To show this correspondence, let us first suppose that $(G \times G; G, M \times M; M)$ acts on the morphism of Lie groupoids $((f \times f) \circ \chi, f)$. Thus, we have two Lie groupoid actions

$$\widetilde{\theta}\colon (G \times G) * \Pi \to \Pi\,, \quad ((h,g),\xi) \mapsto (h,g)\cdot\xi, \tag{5.6}$$

$$\theta\colon G * P \to P\,, \quad (g,p) \mapsto g \cdot p, \tag{5.7}$$

acting on the maps $(f \times f) \circ \chi$ and f respectively, such that $(\widetilde{\theta}, \theta)$ is a morphism of Lie groupoids.

In what follows, we will denote the source and target projections of the groupoid structures in the same way that they have been denoted in the above diagrams (Equations (5.4) and (5.5)). We will use the standard notation for the partial multiplication in the horizontal and vertical structures of the double Lie groupoid $(G \times G; G, M \times M; M)$, and we will continue to use concatenation to denote partial multiplication in the Lie groupoid $\Pi \rightrightarrows P$.

We first start by defining the following smooth maps:

$$\theta_L\colon G * \Pi \to \Pi\,, \quad (g,\xi) \mapsto g\cdot\xi := (g, 1_{f\circ\alpha(\xi)})\cdot\xi, \tag{5.8}$$

$$\theta_R\colon \Pi * G \to \Pi\,, \quad (\xi,g) \mapsto \xi\cdot g := (1_{f\circ\beta(\xi)}, g^{-1})\cdot\xi, \tag{5.9}$$

where $G*\Pi$ and $\Pi*G$ are the pullback manifolds defined by $(\alpha_G \times (f\circ\beta))^{-1}(\Delta_M)$ and $((f\circ\alpha)\times\beta_G)^{-1}(\Delta_M)$, respectively. We claim that these maps define left and right Lie groupoid actions of G on the maps $f\circ\beta\colon \Pi \to M$ and $f\circ\alpha\colon \Pi \to M$, respectively. Before verifying this claim, we first deduce some properties of these maps.

Proposition 5.5. *Let the maps θ_L and θ_R be defined as above. For any $\xi \in \Pi$ and $g, h \in G$, which satisfy $\alpha_G(g) = f\circ\beta(\xi)$ and $\beta_G(h) = f\circ\alpha(\xi)$, we have the following properties:*

(1) $\alpha(g\cdot\xi) = \alpha(\xi)$;

(2) $\beta(g\cdot\xi) = g\cdot\beta(\xi)$;

(3) $\alpha(\xi\cdot h) = h^{-1}\cdot\alpha(\xi)$;

(4) $\beta(\xi\cdot h) = \beta(\xi)$.

Proof. First suppose that $g \in G$ and $\xi \in \Pi$ satisfy $\alpha_G(g) = f\circ\beta(\xi)$. We can compute the source and target of the element $g\cdot\xi$ as follows,

$$\alpha(g\cdot\xi) = \alpha\left((g, 1_{f\circ\alpha(\xi)})\cdot\xi\right) = \alpha_{G\times G}(g, 1_{f\circ\alpha(\xi)})\cdot\alpha(\xi) = 1_{f\circ\alpha(\xi)}\cdot\alpha(\xi) = \alpha(\xi),$$

$$\beta(g\cdot\xi) = \beta((g, 1_{f\circ\alpha(\xi)})\cdot\xi) = \beta_{G\times G}(g, 1_{f\circ\alpha(\xi)})\cdot\beta(\xi) = g\cdot\beta(\xi).$$

Now suppose that $h \in G$ and $\xi \in \Pi$ satisfy $\beta_G(h) = f\circ\alpha(\xi)$. Similar computations to the above show that the source and target of the element $\xi\cdot h$ are given as

$$\alpha(\xi\cdot h) = \alpha\left((1_{f\circ\beta(\xi)}, h^{-1})\cdot\xi\right) = \alpha_{G\times G}\left(1_{f\circ\beta(\xi)}, h^{-1}\right)\cdot\alpha(\xi) = h^{-1}\cdot\alpha(\xi),$$

$$\beta(\xi\cdot h) = \beta\left((1_{f\circ\beta(\xi)}, h^{-1})\cdot\xi\right) = \beta_{G\times G}\left(1_{f\circ\beta(\xi)}, h^{-1}\right)\cdot\beta(\xi) = 1_{f\circ\beta(\xi)}\cdot\beta(\xi) = \beta(\xi). \quad \square$$

Proposition 5.6. *The maps θ_L and θ_R define left and right Lie groupoid actions of G on the maps $f \circ \beta \colon \Pi \to M$ and $f \circ \alpha \colon \Pi \to M$, respectively.*

Proof. Let us first show that θ_L is a left Lie groupoid action of G on $f \circ \beta \colon \Pi \to M$. Suppose that $(g, \xi) \in G * \Pi$, so that $\alpha_G(g) = f(\beta(\xi))$. Then, using property *(2)*,

$$f \circ \beta(g \cdot \xi) = f\left(g \cdot \beta(\xi)\right) = \beta_G(g).$$

Next, suppose that $(h, g) \in G * G$ and $\xi \in \Pi$, such that $(g, \xi) \in G * \Pi$. Thus, we have $\alpha_G(h) = \beta_G(g)$, and $\alpha_G(g) = f \circ \beta(\xi)$. It follows by property *(1)* that

$$\begin{aligned} h \cdot (g \cdot \xi) &= (h, 1_{f\circ\alpha(g\cdot\xi)}) \cdot \big((g, 1_{f\circ\alpha(\xi)}) \cdot \xi\big) \\ &= (h, 1_{f\circ\alpha(\xi)}) \cdot \big((g, 1_{f\circ\alpha(\xi)}) \cdot \xi\big) \\ &= \big((h, 1_{f\circ\alpha(\xi)}) \boxminus (g, 1_{f\circ\alpha(\xi)})\big) \cdot \xi \\ &= (hg, 1_{f\circ\alpha(\xi)}) \cdot \xi \\ &= (hg) \cdot \xi. \end{aligned}$$

Lastly, observe that for any $\xi \in \Pi$,

$$1_{f\circ\beta(\xi)} \cdot \xi = (1_{f\circ\beta(\xi)}, 1_{f\circ\alpha(\xi)}) \cdot \xi = \widetilde{1}_{(f\times f)\circ\chi(\xi)} \cdot \xi = \xi.$$

Hence, θ_L defines a left Lie groupoid action of G on $f \circ \beta \colon \Pi \to M$.

Next, let us verify that θ_R defines a right Lie groupoid action of G on the map $f \circ \alpha \colon \Pi \to M$. Suppose that $(\xi, g) \in \Pi * G$, so that $\beta_G(g) = f(\alpha(\xi))$. Property *(3)* implies that

$$f \circ \alpha(\xi \cdot g) = f(g^{-1} \cdot \alpha(\xi)) = \beta_G(g^{-1}) = \alpha_G(g).$$

Moreover, suppose that $(h, g) \in G * G$, and $\xi \in \Pi$ such that $(\xi, h) \in \Pi * G$. That is, $\alpha_G(h) = \beta_G(g)$ and $\beta_G(h) = f \circ \alpha(\xi)$. Then, using property *(4)*, we have

$$\begin{aligned} (\xi \cdot h) \cdot g &= (1_{f\circ\beta(\xi\cdot h)}, g^{-1}) \cdot \big((1_{f\circ\beta(\xi)}, h^{-1}) \cdot \xi\big) \\ &= (1_{f\circ\beta(\xi)}, g^{-1}) \cdot \big((1_{f\circ\beta(\xi)}, h^{-1}) \cdot \xi\big) \\ &= \big((1_{f\circ\beta(\xi)}, g^{-1}) \boxminus (1_{f\circ\beta(\xi)}, h^{-1})\big) \cdot \xi \\ &= \left(1_{f\circ\beta(\xi)}, g^{-1}h^{-1}\right) \cdot \xi \\ &= \left(1_{f\circ\beta(\xi)}, (hg)^{-1}\right) \cdot \xi \\ &= \xi \cdot (hg). \end{aligned}$$

Finally, we observe that for any $\xi \in \Pi$,

$$\xi \cdot 1_{f\circ\alpha(\xi)} = \left(1_{f\circ\beta(\xi)}, 1^{-1}_{f\circ\alpha(\xi)}\right) \cdot \xi = \left(1_{f\circ\beta(\xi)}, 1_{f\circ\alpha(\xi)}\right) \cdot \xi = \widetilde{1}_{(f\times f)\circ\chi(\xi)} \cdot \xi = \xi.$$

Thus, the map θ_R defines a right Lie groupoid action of G on $f \circ \alpha \colon \Pi \to M$. □

The following results give us a relationship between the left and the right Lie groupoid actions of G, and a relationship between these actions and the partial multiplication of the Lie groupoid $\Pi \rightrightarrows P$.

Proposition 5.7. *With the actions defined as above, we have the following property:*

(5) For any $\xi \in \Pi$, $g, h \in G$, such that $\alpha_G(g) = f \circ \beta(\xi)$, and $\beta_G(h) = f \circ \alpha(\xi)$,

$$(g \cdot \xi) \cdot h = g \cdot (\xi \cdot h).$$

Proof. We first note that both sides of the equation above are well-defined by virtue of properties *(1)* and *(4)*. The left-hand side is given by

$$\begin{aligned}(g \cdot \xi) \cdot h &= (1_{f\circ\beta(g\cdot\xi)}, h^{-1}) \cdot \big((g, 1_{f\circ\alpha(\xi)}) \cdot \xi\big) \\ &= \big((1_{f\circ\beta(g\cdot\xi)}, h^{-1}) \boxminus\!\!\!\!\square\, (g, 1_{f\circ\alpha(\xi)})\big) \cdot \xi \\ &= \big(1_{f\circ\beta(g\cdot\xi)} g, h^{-1} 1_{f\circ\alpha(\xi)}\big) \cdot \xi \\ &= \big(1_{\beta_G(g)} g, h^{-1} 1_{\beta_G(h)}\big) \cdot \xi \\ &= (g, h^{-1}) \cdot \xi.\end{aligned}$$

However, we see that the right-hand side is given by

$$\begin{aligned}g \cdot (\xi \cdot h) &= (g, 1_{f\circ\alpha(\xi\cdot h)}) \cdot \big((1_{f\circ\beta(\xi)}, h^{-1}) \cdot \xi\big) \\ &= \big((g, 1_{f\circ\alpha(\xi\cdot h)}) \,\square\!\!\!\!\!\shortmid\;\, (1_{f\circ\beta(\xi)}, h^{-1})\big) \cdot \xi \\ &= \big(g 1_{f\circ\beta(\xi)}, 1_{f\circ\alpha(\xi\cdot h)} h^{-1}\big) \cdot \xi \\ &= \big(g 1_{\alpha_G(g)}, 1_{\alpha_G(h)} h^{-1}\big) \cdot \xi \\ &= (g, h^{-1}) \cdot \xi.\end{aligned}$$

Hence, we deduce the postulated identity. □

Proposition 5.8. *With the actions defined as above, we have the following properties:*

(6) For $(\xi, \eta) \in \Pi * \Pi$ *and* $g \in G$*, such that* $(g, \xi) \in G * \Pi$*, we have*
$$(g \cdot \xi)\eta = g \cdot (\xi\eta);$$

(7) For $(\xi, \eta) \in \Pi * \Pi$ *and* $g \in G$*, such that* $(\eta, g) \in \Pi * G$*, we have*
$$\xi(\eta \cdot g) = (\xi\eta) \cdot g;$$

(8) For $(\xi, g) \in \Pi * G$ *and* $\eta \in \Pi$*, such that* $(\xi \cdot g, \eta) \in \Pi * \Pi$*, we have*
$$(\xi \cdot g)\eta = \xi(g \cdot \eta).$$

Proof. To prove *(6)*, we first note that both sides of the equation are well-defined because of the property *(1)* and the fact that $\beta(\xi\eta) = \beta(\xi)$. Now, observe that

$$\begin{aligned}(g \cdot \xi)\eta &= (g \cdot \xi)(1_{f\circ\beta(\eta)} \cdot \eta) \\ &= \big((g, 1_{f\circ\alpha(\xi)}) \cdot \xi\big) \big((1_{f\circ\beta(\eta)}, 1_{f\circ\alpha(\eta)}) \cdot \eta\big) \\ &= \big((g, 1_{f\circ\alpha(\xi)}) \boxminus (1_{f\circ\beta(\eta)}, 1_{f\circ\alpha(\eta)})\big) \cdot (\xi\eta) \\ &= (g, 1_{f\circ\alpha(\eta)}) \cdot (\xi\eta) \\ &= (g, 1_{f\circ\alpha(\xi\eta)}) \cdot (\xi\eta) \\ &= g \cdot (\xi\eta).\end{aligned}$$

To prove *(7)*, we can first verify that both sides of the equation are well-defined using the property *(4)* and the fact that $\alpha(\xi\eta) = \alpha(\eta)$. Then, we observe that

$$\begin{aligned}\xi(\eta \cdot g) &= (\xi \cdot 1_{f\circ\alpha(\xi)})(\eta \cdot g) \\ &= \Big((1_{f\circ\beta(\xi)}, 1^{-1}_{f\circ\alpha(\xi)}) \cdot \xi\Big) \big((1_{f\circ\beta(\eta)}, g^{-1}) \cdot \eta\big) \\ &= \big((1_{f\circ\beta(\xi)}, 1_{f\circ\alpha(\xi)}) \boxminus (1_{f\circ\beta(\eta)}, g^{-1})\big) \cdot (\xi\eta) \\ &= (1_{f\circ\beta(\xi)}, g^{-1}) \cdot (\xi\eta) \\ &= (1_{f\circ\beta(\xi\eta)}, g^{-1}) \cdot (\xi\eta) \\ &= (\xi\eta) \cdot g.\end{aligned}$$

To prove *(8)*, let us first verify that the right-hand side of the equation is well-defined. Using properties *(2)* and *(3)*, we have

$$\begin{aligned}\beta(g\cdot\eta) &= g\cdot\beta(\eta)\\ &= g\cdot\alpha(\xi\cdot g)\\ &= g\cdot(g^{-1}\cdot\alpha(\xi))\\ &= 1_{\beta_G(g)}\cdot\alpha(\xi)\\ &= \alpha(\xi),\end{aligned}$$

and

$$f\circ\beta(\eta) = f\circ\alpha(\xi\cdot g) = \alpha_G(g).$$

Thus, $(\xi, g\cdot\eta)\in\Pi*\Pi$ and $(g,\eta)\in G*\Pi$, as required. Finally, we observe that

$$\begin{aligned}(\xi\cdot g)\eta &= (\xi\cdot g)(g^{-1}\cdot(g\cdot\eta))\\ &= \big((1_{f\circ\beta(\xi)}, g^{-1})\cdot\xi\big)\big((g^{-1}, 1_{f\circ\alpha(g\cdot\eta)})\cdot(g\cdot\eta)\big)\\ &= \big((1_{f\circ\beta(\xi)}, g^{-1})\boxminus(g^{-1}, 1_{f\circ\alpha(g\cdot\eta)})\big)\cdot(\xi(g\cdot\eta))\\ &= (1_{f\circ\beta(\xi)}, 1_{f\circ\alpha(g\cdot\eta)})\cdot(\xi(g\cdot\eta))\\ &= \widetilde{1}_{f\circ\chi(\xi(g\cdot\eta))}\cdot(\xi(g\cdot\eta))\\ &= \xi(g\cdot\eta).\end{aligned}$$

□

We have seen that an action of the double Lie groupoid $(G\times G; G, M\times M; M)$ on the morphism $((f\times f)\circ\chi, f)$ gives rise to a left and a right action of $G\rightrightarrows M$ on Π, satisfying the properties *(1)–(8)*. Let us now look at the reverse process.

Proposition 5.9. *Let G and Π be Lie groupoids with base manifolds M and P, respectively, and let $\theta\colon G*P\to P$ be a Lie groupoid action of G on a smooth map $f\colon P\to M$. Suppose that we have a left Lie groupoid action of G on $f\circ\beta\colon\Pi\to M$ and a right Lie groupoid action of G on $f\circ\alpha\colon\Pi\to M$, such that properties (1)–(8) are satisfied. Then the smooth map $\widetilde{\theta}\colon(G\times G)*\Pi\to\Pi$ defined by,*

$$((h,g),\xi)\mapsto(h,g)\cdot\xi := (h\cdot\xi)\cdot g^{-1}, \tag{5.10}$$

is a Lie groupoid action of the Lie groupoid $G\times G\rightrightarrows M\times M$ on the smooth map $(f\times f)\circ\chi\colon\Pi\to M\times M$.

Moreover, $(\widetilde{\theta},\theta)$ is an action of the double Lie groupoid $(G\times G; G, M\times M; M)$ on the morphism of Lie groupoids $((f\times f)\circ\chi, f)$.

Proof. To first see that the map $\widetilde{\theta}\colon(G\times G)*\Pi\to\Pi$ is well-defined, we note that if $((h,g),\xi)\in(G\times G)*\Pi$, then $\widetilde{\alpha}_G(h,g) = (f\times f)\circ\chi(\xi)$. This is equivalent to the statement that $f\circ\beta(\xi) = \alpha_G(h)$ and $f\circ\alpha(\xi) = \beta_G(g^{-1})$, and then property *(5)* immediately implies that the map is well-defined. Furthermore, we see that the parentheses used in the above formula for $\widetilde{\theta}$ are unnecessary.

Let us now show that $\widetilde{\theta}$ defines a Lie groupoid action of the Cartesian product groupoid $G\times G\rightrightarrows M\times M$ on the smooth map $(f\times f)\circ\chi\colon\Pi\to M\times M$. Suppose that

$((h,g),\xi) \in (G \times G) * \Pi$ and observe

$$\begin{aligned}(f \times f) \circ \chi((h,g) \cdot \xi) &= (f \times f) \circ \chi((h \cdot \xi) \cdot g^{-1}) \\ &= \big(f \circ \beta(h \cdot (\xi \cdot g^{-1})), f \circ \alpha((h \cdot \xi) \cdot g^{-1})\big) \\ &= \big(\beta_G(h), \alpha_G(g^{-1})\big) \\ &= (\beta_G(h), \beta_G(g)) \\ &= \widetilde{\beta}_G(h,g).\end{aligned}$$

Next, suppose that $((h_2,h_1),(g_2,g_1)) \in (G \times G) * (G \times G)$, and $\xi \in \Pi$ satisfies $((g_2,g_1),\xi) \in (G \times G) * \Pi$. Then,

$$\begin{aligned}(h_2,h_1) \cdot ((g_2,g_1) \cdot \xi) &= (h_2 \cdot ((g_2 \cdot \xi) \cdot g_1^{-1})) \cdot h_1^{-1} \\ &= ((h_2 \cdot (g_2 \cdot \xi)) \cdot g_1^{-1}) \cdot h_1^{-1} \\ &= ((h_2 g_2) \cdot \xi) \cdot (g_1^{-1} h_1^{-1}) \\ &= (h_2 g_2, h_1 g_1) \cdot \xi \\ &= ((h_2,h_1) \boxbar (g_2,g_1)) \cdot \xi.\end{aligned}$$

Lastly, we observe that for any $\xi \in \Pi$,

$$1_{(f \times f) \circ \chi(\xi)} \cdot \xi = (1_{f \circ \beta(\xi)}, 1_{f \circ \alpha(\xi)}) \cdot \xi = (1_{f \circ \beta(\xi)} \cdot \xi) \cdot 1_{f \circ \alpha(\xi)} = \xi.$$

Hence, $\widetilde{\theta}$ indeed defines a Lie groupoid action of $G \times G \rightrightarrows M \times M$ on $(f \times f) \circ \chi$.

Finally, it remains to show that $(\widetilde{\theta}, \theta)$ gives an action of the double Lie groupoid $(G \times G; G, M \times M; M)$ on the morphism of Lie groupoids $((f \times f) \circ \chi, f)$. That is, we need to show that $(\widetilde{\theta}, \theta)$ is a morphism of Lie groupoids. We see that, for $((h,g),\xi) \in (G \times G) * \Pi$, we have

$$\begin{aligned}\alpha((h,g) \cdot \xi) &= \alpha((h \cdot \xi) \cdot g^{-1}) \\ &= \alpha(\xi \cdot g^{-1}) \\ &= g \cdot \alpha(\xi) \\ &= \alpha_{G \times G}(h,g) \cdot \alpha(\xi),\end{aligned}$$

and

$$\begin{aligned}\beta((h,g) \cdot \xi) &= \beta((h \cdot \xi) \cdot g^{-1}) \\ &= \beta(h \cdot \xi) \\ &= h \cdot \beta(\xi) \\ &= \beta_{G \times G}(h,g) \cdot \beta(\xi).\end{aligned}$$

Furthermore, for $(((k,h),\xi),((h,g),\eta)) \in ((G \times G) \lhd \Pi) * ((G \times G) \lhd \Pi)$, we have

$$\begin{aligned}((k,h) \cdot \xi)((h,g) \cdot \eta) &= \big((k \cdot \xi) \cdot h^{-1}\big)\big((h \cdot \eta) \cdot g^{-1}\big) \\ &= \big(((k \cdot \xi) \cdot h^{-1}) \cdot h\big)(\eta \cdot g^{-1}) \\ &= (k \cdot \xi)(\eta \cdot g^{-1}) \\ &= ((k \cdot \xi)\eta) \cdot g^{-1} \\ &= (k \cdot (\xi\eta)) \cdot g^{-1} \\ &= (k,g) \cdot (\xi\eta) \\ &= ((k,h) \boxminus (h,g)) \cdot (\xi\eta).\end{aligned}$$

Hence, we can conclude that $(\widetilde{\theta}, \theta)$ is a morphism of Lie groupoids, and thus gives rise to an action of the double Lie groupoid $(G \times G; G, M \times M; M)$ on the morphism of Lie groupoids $(f \times f) \circ \chi$. Note that in the computations above, we have used all the properties *(1)–(8)*. □

It should be immediately clear that these two constructions that have been described are mutually inverse. To conclude, we combine all of the results of this section to arrive at the following theorem:

Theorem 5.10. *Let* $\Pi \rightrightarrows P$ *be a Lie groupoid with source and target projections denoted by* α *and* β*, respectively, and suppose that there exists a Lie groupoid action of a Lie groupoid* $G \rightrightarrows M$ *on a smooth map* $f \colon P \to M$*. Then there is a one-to-one correspondence between actions of the double Lie groupoid* $(G \times G; G, M \times M; M)$ *on the morphism* $((f \times f) \circ \chi, f)$*, and pairs of left and right Lie groupoid actions of* $G \rightrightarrows M$ *on the maps* $f \circ \beta$ *and* $f \circ \alpha$*, respectively, satisfying the properties (1)–(8).* □

5.1.3 $\mathcal{LA}$-actions of $\mathcal{LA}$-groupoids

We saw in Section 5.1.1 that we can generalise Lie groupoid actions to actions of double Lie groupoids. We now consider an analogous approach to generalise Lie algebroid actions to actions of $\mathcal{LA}$-groupoids. In some sense, we are extending the notion of a Lie algebroid action from the category of smooth manifolds to the category of Lie groupoids.

Definition 5.11. Let $(\Omega; A, V; M)$ be an $\mathcal{LA}$-groupoid, $G \rightrightarrows P$ a Lie groupoid, and $F \colon G \to V$ a morphism of Lie groupoids over a smooth map $f \colon P \to M$.

$$\begin{array}{ccccc} & & & & G \\ & & & \overset{F}{\swarrow} & \\ \Omega & \xrightarrow{\widetilde{q}} & V & & \Big\downarrow\Big\downarrow {\scriptstyle \alpha_G,\beta_G} \\ {\scriptstyle \widetilde{\alpha},\widetilde{\beta}}\Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow{\scriptstyle \alpha,\beta} & & P \\ & & & \overset{f}{\swarrow} & \\ A & \xrightarrow{q} & M & & \end{array} \tag{5.11}$$

An $\mathcal{LA}$-*action* of $(\Omega; A, V; M)$ on (F, f) (or on $G \rightrightarrows P$) consists of a Lie algebroid action of $\Omega \to V$ on F, and a Lie algebroid action of $A \to M$ on f, such that the anchor map $\widetilde{a}^\dagger \colon \Omega \lhd G \to TG$ is a morphism of Lie groupoids over the anchor map $a^\dagger \colon A \lhd P \to TP$.

The morphism $(\widetilde{a}^\dagger, a^\dagger)$ that corresponds to an $\mathcal{LA}$-action of $(\Omega; A, V; M)$ on a Lie groupoid $G \rightrightarrows P$ can be expressed diagrammatically,

$$\begin{array}{ccc} \Omega \lhd G & \xrightarrow{\widetilde{a}^\dagger} & TG \\ \Big\downarrow\Big\downarrow & & \Big\downarrow\Big\downarrow \\ A \lhd P & \xrightarrow{a^\dagger} & TP. \end{array} \tag{5.12}$$

Here, the Lie groupoid structure on $\Omega \lhd G$ with base $A \lhd P$ is the unique groupoid structure that makes $\Omega \lhd G$ an embedded Lie subgroupoid of the Cartesian product groupoid $\Omega \times G \rightrightarrows A \times P$ via inclusion.

Remark 5.12. In the above definition, the condition that $(\widetilde{a}^\dagger, a^\dagger)$ be a morphism of Lie groupoids immediately implies the following properties:

(i) $T(\alpha_G)(\xi^\dagger(g)) = \widetilde{\alpha}(\xi(F(g)))^\dagger$, for all $\xi \in \Gamma(\Omega)$, $g \in G$;

(ii) $T(\beta_G)(\xi^\dagger(g)) = \widetilde{\beta}(\xi(F(g)))^\dagger$, for all $\xi \in \Gamma(\Omega)$, $g \in G$;

(iii) $T(\kappa_G)(\xi^\dagger(h), \eta^\dagger(g)) = \widetilde{\kappa}(\xi(F(h)), \eta(F(g)))^\dagger$, for all $\xi, \eta \in \Gamma(\Omega)$, $(h, g) \in G * G$, such that $(\xi(F(h)), \eta(F(g))) \in \Omega * \Omega$;

(iv) $T(1^G)(X^\dagger(p)) = (\widetilde{1}_{X(f(p))})^\dagger$, for all $X \in \Gamma(A)$, $p \in P$;

(v) $T(\iota_G)(\xi^\dagger(g)) = \widetilde{\iota}(\xi(F(g)))^\dagger$, for all $\xi \in \Gamma(\Omega)$, $g \in G$.

Given an $\mathcal{LA}$-action of an $\mathcal{LA}$-groupoid $(\Omega; A, V; M)$ on a Lie groupoid $G \rightrightarrows P$, we get action Lie algebroids $\Omega \lhd G$ and $A \lhd P$ on base manifolds G and P, respectively. In addition, $\Omega \lhd G$ also has a Lie groupoid structure on base $A \lhd P$. We can present these Lie groupoid and Lie algebroid structures in the following diagram:

$$\begin{array}{ccc} \Omega \lhd G & \xrightarrow{\widetilde{q}_\lhd} & G \\ {\scriptstyle \widetilde{\alpha}*\alpha_G,\, \widetilde{\beta}*\beta_G}\Big\downarrow\!\Big\downarrow & & \Big\downarrow\!\Big\downarrow{\scriptstyle \alpha_G,\, \beta_G} \\ A \lhd P & \xrightarrow{q_\lhd} & P. \end{array} \tag{5.13}$$

Theorem 5.13. *Let $(\Omega; A, V; M)$ be an $\mathcal{LA}$-groupoid, G a Lie groupoid on base P, and $F\colon G \to V$ a morphism of Lie groupoids over a smooth map $f\colon P \to M$. Suppose that we have an $\mathcal{LA}$-action of $(\Omega; A, V; M)$ on (F, f). Then $(\Omega \lhd G; A \lhd P, G; P)$ is an $\mathcal{LA}$-groupoid.*

To give a rigorous proof of this theorem we will first need to prove some preliminary results.

Lemma 5.14. *Let $q\colon E \to M$ and $q'\colon E' \to M'$ be smooth vector bundles of rank k and r, respectively, and let $F\colon E \to E'$ be a vector bundle morphism over $f\colon M \to M'$.*

$$\begin{array}{ccc} E & \xrightarrow{F} & E' \\ \Big\downarrow & & \Big\downarrow \\ M & \xrightarrow{f} & M', \end{array} \tag{5.14}$$

Given any $x \in M$, there exists a local frame $(\sigma_i)_{i=1}^k$ of E defined over a neighbourhood U of x, and a local frame $(\sigma'_i)_{i=1}^r$ of E defined over an open set $V \supseteq f(U)$, such that

(i) when F is a submersion,

$$F \circ \sigma_i(y) = \begin{cases} \sigma'_i \circ f(y) & \forall\, 1 \le i \le r, \\ 0_{f(y)} & \forall\, r < i \le k, \end{cases}$$

for every $y \in U$;

(ii) and when F is an immersion,

$$F \circ \sigma_i(y) = \sigma_i' \circ f(y), \quad \forall\, 1 \leq i \leq k,$$

for every $y \in U$.

Proof. This result is essentially a corollary of the Rank Theorem.[3] Let us begin with the first statement (i). Since F is a submersion there exists neighbourhoods U and V of x and $f(x)$, respectively, such that $F(q^{-1}(U)) \subseteq (q')^{-1}(V)$, and local trivialisations $\Phi\colon q^{-1}(U) \to U \times \mathbb{R}^k$ and $\Phi'\colon (q')^{-1}(V) \to V \times \mathbb{R}^r$, such that

$$\Phi' \circ F \circ \Phi^{-1}\colon U \times \mathbb{R}^k \to V \times \mathbb{R}^r$$

is given by

$$(y, (v_1, \ldots, v_r, v_{r+1}, \ldots, v_k)) \mapsto (f(y), (v_1, \ldots, v_r)).$$

Let $\underline{e}_1, \ldots, \underline{e}_k$ be the standard coordinate bases of $\mathbb{R}^k$, and let $\underline{e}_1', \ldots, \underline{e}_r'$ be the standard coordinate bases of $\mathbb{R}^r$. We can define local sections on U by

$$\sigma_i\colon U \to E, \quad y \mapsto \Phi^{-1}(y, \underline{e}_i),$$

for $i = 1, \ldots, k$. Note that $\Phi \circ \sigma_i$ is smooth by construction, and since Φ is a diffeomorphism, it follows that σ_i is smooth. Clearly $(\sigma_i)_{i=1}^k$ form a local frame for E over U. We can also define smooth local sections on V by

$$\sigma_i'\colon V \to E', \quad z \mapsto (\Phi')^{-1}(z, \underline{e}_i'),$$

for $i = 1, \ldots, r$, and $(\sigma_i')_{i=1}^r$ forms a local frame for E' over V.

Lastly, observe that for any $y \in U$, if $1 \leq i \leq r$,

$$\begin{aligned} F(\sigma_i(y)) &= F(\Phi^{-1}(y, \underline{e}_i)) \\ &= (\Phi')^{-1}(\Phi' \circ F \circ \Phi^{-1}(y, \underline{e}_i)) \\ &= (\Phi')^{-1}(f(y), \underline{e}_i') \\ &= \sigma_i'(f(y)). \end{aligned}$$

On the other hand, if $r < i \leq k$,

$$\begin{aligned} F(\sigma_i(y)) &= F(\Phi^{-1}(y, \underline{e}_i)) \\ &= (\Phi')^{-1}(\Phi' \circ F \circ \Phi^{-1}(y, \underline{e}_i)) \\ &= (\Phi')^{-1}(f(y), \underline{0}) \\ &= 0_{f(y)}. \end{aligned}$$

The second statement can be proved in a similar fashion. □

Lemma 5.15. *Let $q\colon E \to M$ be a smooth vector bundle, and $f\colon M' \to M$ a smooth map. Suppose that (σ_i) is a smooth local frame for E over some neighbourhood U, then $(\sigma_i^!)$ is a smooth local frame for $f^!E$ over $f^{-1}(U)$.* □

The following result is the key step to proving Theorem 5.13.

[3]See [34, Theorem 4.12] for a statement of the Rank Theorem.

Proposition 5.16. *Let Ω and A be Lie algebroids on base manifolds V and M, respectively, and $\widetilde{\varphi}\colon \Omega \to A$ a morphism of Lie algebroids over a map $\varphi\colon V \to M$. Suppose we have a Lie algebroid action of Ω on a smooth map $F\colon G \to V$, a Lie algebroid action of A on a smooth map $f\colon P \to M$, and a smooth map $\varphi_G\colon G \to P$ satisfying $\varphi \circ F = f \circ \varphi_G$. Additionally, we further suppose that $\widetilde{\varphi}$, φ_G and φ are all surjective submersions.*

$$\begin{array}{ccccc} & & & & G \\ & & & \overset{F}{\swarrow} & \downarrow{\scriptstyle \varphi_G} \\ \Omega & \xrightarrow{\widetilde{q}} & V & & \\ {\scriptstyle \widetilde{\varphi}}\downarrow & & \downarrow{\scriptstyle \varphi} & & P \\ & & & \overset{f}{\swarrow} & \\ A & \xrightarrow{q} & M & & \end{array} \tag{5.15}$$

Let $\widetilde{\varphi} * \varphi_G$ denote the restriction of the product map $\widetilde{\varphi} \times \varphi_G\colon \Omega \times G \to A \times P$ to a smooth map $\Omega \lhd G \to A \lhd P$. If $a^\dagger \circ (\widetilde{\varphi} * \varphi_G) = T(\varphi_G) \circ \widetilde{a}^\dagger$, where $\widetilde{a}^\dagger$ and $a^\dagger$ are the anchor maps of the action Lie algebroids $\Omega \lhd G$ and $A \lhd P$, respectively, then $\widetilde{\varphi} * \varphi_G$ is a morphism of Lie algebroids over φ_G.

Proof. We first check that $\widetilde{\varphi} * \varphi_G$ is a vector bundle morphism over φ_G. Observe that for any $(\xi, g) \in \Omega \lhd G$, we have

$$q_\lhd \circ (\widetilde{\varphi} * \varphi_G)(\xi, g) = q_\lhd(\widetilde{\varphi}(\xi), \varphi_G(g)) = \varphi_G(g) = \varphi_G \circ \widetilde{q}_\lhd(\xi, g).$$

Hence, $\widetilde{\varphi} * \varphi_G$ is a fibrewise map. Moreover, for any $\lambda \in \mathbb{R}$, $g \in G$, and any pairs $(\xi, g), (\eta, g) \in (\Omega \lhd G)_g$, we have

$$\begin{aligned} \lambda(\widetilde{\varphi} * \varphi_G)(\xi, g) + (\widetilde{\varphi} * \varphi_G)(\eta, g) &= (\lambda\widetilde{\varphi}(\xi), \varphi_G(g)) + (\widetilde{\varphi}(\eta), \varphi_G(g)) \\ &= (\lambda\widetilde{\varphi}(\xi) + \widetilde{\varphi}(\eta), \varphi_G(g)) \\ &= (\widetilde{\varphi}(\lambda\xi + \eta), \varphi_G(g)) \\ &= \widetilde{\varphi} * \varphi_G(\lambda\xi + \eta, g) \\ &= \widetilde{\varphi} * \varphi_G(\lambda(\xi, g) + (\eta, g)). \end{aligned}$$

Thus $\widetilde{\varphi} * \varphi_G$ is linear on fibres, and is therefore a vector bundle morphism.

To prove that $\widetilde{\varphi} * \varphi_G$ is a morphism of Lie algebroids, we make use of Proposition 1.50. Since $\widetilde{\varphi} * \varphi_G$ is a fibrewise surjection, and $a^\dagger \circ (\widetilde{\varphi} * \varphi_G) = T(\varphi) \circ \widetilde{a}^\dagger$ by assumption, we only need to check that property (1.14) holds. In this endeavour, we take any sections $\widetilde{\xi}, \widetilde{\eta} \in \Gamma(\Omega \lhd G)$, and $\widetilde{X}, \widetilde{Y} \in \Gamma(A \lhd P)$, satisfying

$$(\widetilde{\varphi} * \varphi_G) \circ \widetilde{\xi} = \widetilde{X} \circ \varphi_G\,, \quad (\widetilde{\varphi} * \varphi_G) \circ \widetilde{\eta} = \widetilde{Y} \circ \varphi_G.$$

We need to show that

$$(\widetilde{\varphi} * \varphi_G) \circ [\widetilde{\xi}, \widetilde{\eta}] = [\widetilde{X}, \widetilde{Y}] \circ \varphi_G.$$

Fix an arbitrary $g \in G$. Let us assume that as vector bundles $\Omega \to V$ and $A \to M$ have ranks k and r, respectively. Since $(\widetilde{\varphi}, \varphi)$ is a vector bundle morphism, and $\widetilde{\varphi}$ is a surjective submersion, by Lemma 5.14 there exists a smooth local frame $(\xi_i)_{i=1}^k$ of Ω defined in a neighbourhood of $F(g)$, and a smooth local frame $(X_i)_{i=1}^r$ of A defined in a neighbourhood of $\varphi(F(g))$, such that

$$\widetilde{\varphi} \circ \xi_i \equiv \begin{cases} X_i \circ \varphi & \forall 1 \le i \le r, \\ 0 \circ \varphi & \forall r < i \le k, \end{cases} \tag{5.16}$$

in a neighbourhood of $F(g)$. By Lemma 5.15 there exist $u_i, v_i \in \mathcal{C}^\infty(G)$, such that

$$\widetilde{\xi} \equiv \sum_{i=1}^{k} u_i \otimes \xi_i, \quad \widetilde{\eta} \equiv \sum_{i=1}^{k} v_i \otimes \xi_i,$$

in some neighbourhood of g. Similarly, there exist $w_j, y_j \in \mathcal{C}^\infty(P)$, such that

$$\widetilde{X} \equiv \sum_{j=1}^{r} w_j \otimes X_j, \quad \widetilde{Y} \equiv \sum_{j=1}^{r} y_j \otimes X_j,$$

in some neighbourhood of $\varphi_G(g)$. Because $(\widetilde{\varphi} * \varphi_G) \circ \widetilde{\xi} = \widetilde{X} \circ \varphi_G$, it follows that

$$\sum_{i=1}^{k} u_i(\widetilde{\varphi} \circ \xi_i \circ F) \equiv \sum_{j=1}^{r} (w_j \circ \varphi_G)(X_j \circ \varphi \circ F), \tag{5.17}$$

in a neighbourhood of g. On the other hand, by the properties given in equation (5.16), we have

$$\sum_{i=1}^{k} u_i(\widetilde{\varphi} \circ \xi_i \circ F) \equiv \sum_{i=1}^{r} u_i(X_i \circ \varphi \circ F), \tag{5.18}$$

in a neighbourhood of g. Equating (5.17) and (5.18), implies that $u_i \equiv w_i \circ \varphi_G$ in a neighbourhood of g, for all $i = 1, \ldots, r$. Using a similar argument, we can also show that $v_i \equiv y_i \circ \varphi_G$ in a neighbourhood of g, for all $i = 1, \ldots, r$.

By recalling the definition of the Lie bracket[4] for the action Lie algebroid $\Omega \lessdot G$, we find that

$$(\widetilde{\varphi} * \varphi_G)([\widetilde{\xi}, \widetilde{\eta}](g)) = (\Lambda_1 + \Lambda_2 + \Lambda_3, \varphi_G(g)),$$

where Λ_1, Λ_2 and Λ_3 are the summands given by

$$\Lambda_1 = \sum_{i,j=1}^{k} u_i(g) v_j(g) (\widetilde{\varphi} \circ [\xi_i, \xi_j])(F(g)),$$

$$\Lambda_2 = \sum_{i,j=1}^{k} u_i(g) \xi_i^\dagger(g)(v_j)(\widetilde{\varphi} \circ \xi_j)(F(g)), \quad \Lambda_3 = -\sum_{i,j=1}^{k} v_j(g) \xi_j^\dagger(g)(u_i)(\widetilde{\varphi} \circ \xi_i)(F(g)).$$

Let us now consider these three summands separately. To compute Λ_1, note that Lemma 1.52 immediately implies that

$$(\widetilde{\varphi} \circ [\xi_i, \xi_j])(F(g)) = ([X_i, X_j] \circ \varphi)(F(g)),$$

whenever $1 \leq i, j \leq r$. Furthermore, we also see that

$$(\widetilde{\varphi} \circ [\xi_i, \xi_j])(F(g)) = 0_{F(g)},$$

whenever $i > r$, or $j > r$. Hence, the first summand is given by

$$\begin{aligned} \Lambda_1 &= \sum_{i,j=1}^{k} u_i(g) v_j(g) (\widetilde{\varphi} \circ [\xi_i, \xi_j])(F(g)) \\ &= \sum_{i,j=1}^{r} u_i(g) v_j(g) ([X_i, X_j] \circ \varphi)(F(g)) \\ &= \sum_{i,j=1}^{r} (w_i \circ \varphi_G)(g)(y_j \circ \varphi_G)(g)[X_i, X_j](f(\varphi_G(g))) \end{aligned}$$

[4]See page 21 for the definition of the Lie bracket for an action Lie algebroid.

For the second summand, note that whenever $1 \leq i \leq r$, we have

$$\xi_i^\dagger(g)(v_j) = \xi_i^\dagger(g)(y_j \circ \varphi_G) = T(\varphi_G)(\xi_i^\dagger(g))(y_j) = \widetilde{\varphi}(\xi_i(F(g)))^\dagger(y_j) = X_i^\dagger(\varphi_G(g))(y_j).$$

Hence, it follows that this second summand is given by

$$\begin{aligned}\Lambda_2 &= \sum_{i,j=1}^{k} u_i(g)\xi_i^\dagger(g)(v_j)(\widetilde{\varphi} \circ \xi_j)(F(g)) \\ &= \sum_{i,j=1}^{r} u_i(g)X_i^\dagger(\varphi_G(g))(y_j)(X_j \circ \varphi)(F(g)) \\ &= \sum_{i,j=1}^{r} (w_i \circ \varphi_G)(g)X_i^\dagger(\varphi_G(g))(y_j)X_j(f(\varphi_G(g))).\end{aligned}$$

We can apply a similar argument for the third summand, and then in total we have obtained

$$\begin{aligned}(\widetilde{\varphi} * \varphi_G)([\widetilde{\xi}, \widetilde{\eta}](g)) &= \Bigg(\sum_{i,j=1}^{r} (w_i \circ \varphi_G)(g)(y_j \circ \varphi_g)(g)[X_i, X_j](f(\varphi_G(g))) \\ &\qquad + \sum_{i,j=1}^{r} (w_i \circ \varphi_G)(g)X_i^\dagger(\varphi_G(g))(y_j)X_j(f(\varphi_G(g))) \\ &\qquad - \sum_{i,j=1}^{r} (y_j \circ \varphi_G)(g)X_j^\dagger(\varphi_G(g))(w_i)X_i(f(\varphi_G(g))),\ \varphi_G(g) \Bigg) \\ &= \left[\sum_{i=1}^{r} w_i \otimes X_i, \sum_{j=1}^{r} y_j \otimes X_j \right] (\varphi_G(g)) \\ &= [\widetilde{X}, \widetilde{Y}](\varphi_G(g)).\end{aligned}$$

Since $g \in G$ was arbitrarily chosen, we conclude that $(\widetilde{\varphi} * \varphi_G) \circ [\widetilde{\xi}, \widetilde{\eta}] = [\widetilde{X}, \widetilde{Y}] \circ \varphi_G$. Hence, $\widetilde{\varphi} * \varphi_G$ is a morphism of Lie algebroids over φ_G. □

We now have the tools to prove the theorem.

Proof of Theorem 5.13. To prove this result, we need to show that the structure maps of the Lie groupoid $\Omega \lessdot G \rightrightarrows A \lessdot P$ are Lie algebroid morphisms over the corresponding structure maps of $G \rightrightarrows P$. By assumption, the structure maps $\widetilde{\alpha}$, $\widetilde{\beta}$, $\widetilde{\kappa}$, $\widetilde{1}$, $\widetilde{\iota}$ of $\Omega \rightrightarrows A$ are all morphisms of Lie algebroids over the structure maps α, β, κ, 1, ι of $V \rightrightarrows M$, respectively. Moreover, since F is a morphism of Lie groupoids over f, we have the relations

$$\alpha \circ F = f \circ \alpha_G, \quad \beta \circ F = f \circ \beta_G, \quad \kappa \circ (F \times F)|_{G*G} = F \circ \kappa_G,$$
$$1 \circ f = F \circ 1^G, \quad \iota \circ F = F \circ \iota_G.$$

Note that the source and target projections of the Lie groupoid $\Omega \lessdot G \rightrightarrows A \lessdot P$ are given by the maps $\widetilde{\alpha} * \alpha_G$ and $\widetilde{\beta} * \beta_G$, respectively (as defined in Proposition 5.16). Similarly, the identity and inversion maps are given by $\widetilde{1} * 1^G$ and $\widetilde{\iota} * \iota_G$. The partial multiplication is given by the composite

$$\begin{array}{ccccc} (\Omega \lhd G) * (\Omega \lhd G) & \xrightarrow{\cong} & (\Omega * \Omega) \lhd (G * G) & \xrightarrow{\widetilde{\kappa} * \kappa_G} & \Omega \lhd G \\ \big\downarrow & & \big\downarrow & & \big\downarrow \\ G * G & \xrightarrow{\mathrm{id}_{G*G}} & G * G & \xrightarrow{\kappa_G} & G. \end{array} \tag{5.19}$$

Let us denote the anchor map of the action Lie algebroid $(\Omega * \Omega) \lhd (G * G) \to G * G$ by $a^\dagger_{\Omega*\Omega}$. Then the condition that $(\widetilde{a}^\dagger, a^\dagger)$ is a morphism of Lie groupoids also gives us the relations

$$T(\alpha_G) \circ \widetilde{a}^\dagger = a^\dagger \circ (\widetilde{\alpha} * \alpha_G), \quad T(\beta_G) \circ \widetilde{a}^\dagger = a^\dagger \circ (\widetilde{\beta} * \beta_G),$$
$$T(\kappa_G) \circ a^\dagger_{\Omega*\Omega} = \widetilde{a}^\dagger \circ (\widetilde{\kappa} * \kappa_G),$$
$$T(1^G) \circ a^\dagger = \widetilde{a}^\dagger \circ (\widetilde{1} * 1^G), \quad T(\iota_G) \circ \widetilde{a}^\dagger = \widetilde{a}^\dagger \circ (\widetilde{\iota} * \iota_G).$$

Since the source and target projections, $\widetilde{\alpha}$ and $\widetilde{\beta}$, the partial multiplication $\widetilde{\kappa}$, and the inversion map $\widetilde{\iota}$ are all surjective submersions, we can apply Proposition 5.16 to deduce that $\widetilde{\alpha} * \alpha_G$, $\widetilde{\beta} * \beta_G$, $\widetilde{\kappa} * \kappa_G$ and $\widetilde{\iota} * \iota_G$ are all morphisms of Lie algebroids.

It remains to show that $\widetilde{1} * 1^G$ is a morphism of Lie algebroids. Note that $\widetilde{1} * 1^G$ is an injective immersion, and hence of constant rank. Since we have already shown that $T(1^G) \circ a^\dagger = \widetilde{a}^\dagger \circ (\widetilde{1} * 1^G)$, by Remark 1.51 and Proposition 1.50, we only need to check that we have the property (1.14). Let us take sections $\widetilde{X}, \widetilde{Y} \in \Gamma(A \lhd P)$, and $\widetilde{\xi}, \widetilde{\eta} \in \Gamma(\Omega \lhd G)$, satisfying

$$(\widetilde{1} * 1^G) \circ \widetilde{X} = \widetilde{\xi} \circ 1^G, \quad (\widetilde{1} * 1^G) \circ \widetilde{Y} = \widetilde{\eta} \circ 1^G.$$

We need to show that

$$(\widetilde{1} * 1^G) \circ [\widetilde{X}, \widetilde{Y}] = [\widetilde{\xi}, \widetilde{\eta}] \circ 1^G.$$

Fix an arbitrary $p \in P$. We assume that as vector bundles $\Omega \to V$ and $A \to M$ have ranks k and r, respectively. Now since $(\widetilde{1}, 1)$ is a vector bundle morphism, and $\widetilde{1}$ is an injective immersion, by part *(ii)* of Lemma 5.14, there exists a smooth local frame $(X_i)_{i=1}^r$ of A defined in a neighbourhood of $f(p)$, and a smooth local frame $(\xi_i)_{i=1}^k$ of Ω defined in a neighbourhood of $1_{f(p)}$, such that

$$\widetilde{1} \circ X_i \equiv \xi_i \circ 1, \quad \forall 1 \le i \le r \tag{5.20}$$

in a neighbourhood of $f(p)$. By Lemma 5.15, there exist $w_i, y_i \in \mathcal{C}^\infty(P)$ such that

$$\widetilde{X} \equiv \sum_{i=1}^r w_i \otimes X_i, \quad \widetilde{Y} \equiv \sum_{i=1}^r y_i \otimes X_i,$$

in some neighbourhood of p. There also exist $u_j, v_j \in \mathcal{C}^\infty(G)$ such that

$$\widetilde{\xi} \equiv \sum_{j=1}^k u_j \otimes \xi_j, \quad \widetilde{\eta} \equiv \sum_{j=1}^k v_j \otimes \xi_j,$$

in some neighbourhood of 1^G_p. Since $(\widetilde{1} * 1^G) \circ \widetilde{X} = \widetilde{\xi} \circ 1^G$, it follows that

$$\sum_{i=1}^r w_i(\widetilde{1} \circ X_i \circ f) \equiv \sum_{j=1}^k (u_j \circ 1^G)(\xi_j \circ 1 \circ f), \tag{5.21}$$

in a neighbourhood of p. On the other hand, by the condition (5.20), we have

$$\sum_{i=1}^{r} w_i(\widetilde{1} \circ X_i \circ f) \equiv \sum_{i=1}^{r} w_i(\xi_i \circ 1 \circ f), \tag{5.22}$$

in a neighbourhood of p. Equating (5.21) and (5.22), implies that $w_i \equiv u_i \circ 1^G$ in a neighbourhood of p, for all $1 \leq i \leq r$, and $u_i \circ 1^G \equiv 0$ in a neighbourhood of p, for all $i > r$. Using a similar argument, we can also show that $y_i \equiv v_i \circ 1^G$ in a neighbourhood of p, for all $1 \leq i \leq r$, and $v_i \circ 1^G \equiv 0$ in a neighbourhood of p, for all $i > r$.

Using the definition of the Lie bracket for the action Lie algebroid $A \lhd P$, we find

$$(\widetilde{1} * 1^G)([\widetilde{X}, \widetilde{Y}](p)) = (\Lambda_1 + \Lambda_2 + \Lambda_3, 1_p^G),$$

where Λ_1, Λ_2 and Λ_3 are the summands given by

$$\Lambda_1 = \sum_{i,j=1}^{r} w_i(p) y_j(p) (\widetilde{1} \circ [X_i, X_j])(f(p)),$$

$$\Lambda_2 = \sum_{i,j=1}^{r} w_i(p) X_i^{\dagger}(p)(y_j)(\widetilde{1} \circ X_j)(f(p)), \quad \Lambda_3 = -\sum_{i,j=1}^{r} y_j(g) X_j^{\dagger}(p)(w_i)(\widetilde{1} \circ X_i)(f(p)).$$

Let us now consider these three summands separately. To compute Λ_1, note that Lemma 1.52 immediately implies that

$$(\widetilde{1} \circ [X_i, X_j])(f(p)) = ([\xi_i, \xi_j] \circ 1)(f(p)),$$

whenever $1 \leq i, j \leq r$. We also have that $(u_i \circ 1^G)(p) = (v_i \circ 1^G)(p) = 0$, for all $i > r$. Hence, the first summand is given by

$$\begin{aligned}
\Lambda_1 &= \sum_{i,j=1}^{r} w_i(p) y_j(p) ([\xi_i, \xi_j] \circ 1)(f(p)) \\
&= \sum_{i,j=1}^{k} (u_i \circ 1^G)(p)(v_j \circ 1^G)(p)[\xi_i, \xi_j](F(1_p^G)).
\end{aligned}$$

For the second summand, we first observe that for $1 \leq i \leq r$,

$$X_i^{\dagger}(p)(v_j \circ 1^G) = T(1^G)(X_i^{\dagger}(p))(v_j) = \widetilde{1}(X_i(f(p)))^{\dagger}(v_j) = \xi_i(1_{f(p)})^{\dagger}(v_j) = \xi_i^{\dagger}(1_p^G)(v_j).$$

Thus, $X_i^{\dagger}(p)(y_j) = \xi_i^{\dagger}(1_p^G)(v_j)$ for $1 \leq i, j \leq r$. Moreover, $\xi_i^{\dagger}(1_p^G)(v_j) = 0$ for $j > r$, and since $u_i \circ 1^G(p) = 0$ for $i > r$, we have

$$\begin{aligned}
\Lambda_2 &= \sum_{i,j=1}^{r} w_i(p) X_i^{\dagger}(p)(y_j)(\widetilde{1} \circ X_j)(f(p)) \\
&= \sum_{i,j=1}^{r} w_i(p) \xi_i^{\dagger}(1_p^G)(v_j)(\xi_j \circ 1)(f(p)) \\
&= \sum_{i,j=1}^{k} (u_i \circ 1^G)(p) \xi_i^{\dagger}(1_p^G)(v_j) \xi_j(F(1_p^G)).
\end{aligned}$$

We can apply a similar argument for the third summand, and then in total we have

$$
\begin{aligned}
(\widetilde{1} * 1^G)([\widetilde{X}, \widetilde{Y}](p)) &= \Bigg(\sum_{i,j=1}^{k} (u_i \circ 1^G)(p)(v_j \circ 1^G)(p)[\xi_i, \xi_j](F(1^G_p)) \\
&\qquad + \sum_{i,j=1}^{k} (u_i \circ 1^G)(p)\xi_i^{\dagger}(1^G_p)(v_j)\xi_j(F(1^G_p)) \\
&\qquad - \sum_{i,j=1}^{k} (v_j \circ 1^G)(p)\xi_j^{\dagger}(1^G_p)(u_i)\xi_i(F(1^G_p)),\ 1^G_p \Bigg) \\
&= \left[\sum_{i=1}^{k} u_i \otimes \xi_i, \sum_{j=1}^{k} v_j \otimes \xi_j \right] (1^G_p) \\
&= [\widetilde{\xi}, \widetilde{\eta}](1^G_p).
\end{aligned}
$$

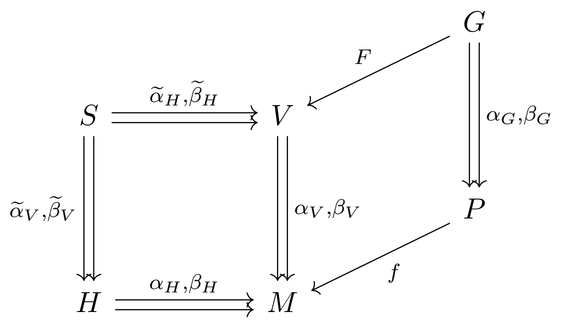
Since $p \in P$ was arbitrarily chosen, we conclude that $(\widetilde{1} * 1^G) \circ [\widetilde{X}, \widetilde{Y}] = [\widetilde{\xi}, \widetilde{\eta}] \circ 1^G$. Hence, the identity map $\widetilde{1} * 1^G$ is a morphism of Lie algebroids over 1^G. This completes the proof of the theorem. □

Given an $\mathcal{LA}$-action of an $\mathcal{LA}$-groupoid $(\Omega; A, V; M)$ on a morphism of Lie groupoids $F\colon G \to V$ over a smooth map $f\colon P \to M$, we call the corresponding $\mathcal{LA}$-groupoid $(\Omega \triangleleft G; A \triangleleft P, G; P)$ an *$\mathcal{LA}$-action $\mathcal{LA}$-groupoid.*

We finish this section by showing how an action of a double Lie groupoid gives rise to an $\mathcal{LA}$-action of an $\mathcal{LA}$-groupoid.

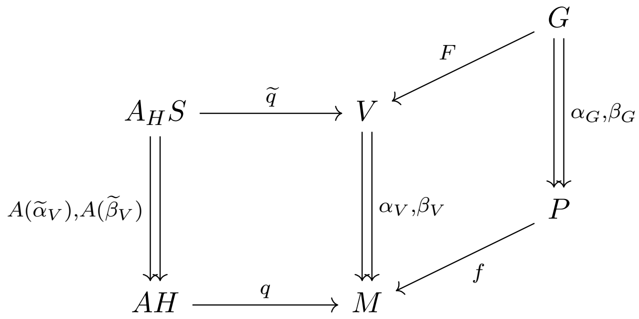
Proposition 5.17. *Let $(\widetilde{\theta}, \theta)$ be an action of a double Lie groupoid $(S; H, V; M)$ on a morphism of Lie groupoids $F\colon G \to V$ over $f\colon P \to M$. Then there exists an $\mathcal{LA}$-action of the $\mathcal{LA}$-groupoid $(A_H S; AH, V; M)$ on the morphism of Lie groupoids (F, f).*

Proof. We denote the structure maps of the groupoids as indicated in the following diagram.



(5.23)

Now consider the $\mathcal{LA}$-groupoid $(A_H S; AH, V; M)$ of Example 1.77. The Lie groupoid action $\widetilde{\theta}$ of $S \rightrightarrows V$ on F gives rise to a Lie algebroid action of $A_H S \to V$ on F by the formula given in (1.19). Similarly, the Lie groupoid action θ of $H \rightrightarrows M$ on f gives rise to a Lie algebroid action of $AH \to M$ on f. We claim that that these two Lie algebroid actions form an $\mathcal{LA}$-action of $(A_HS; AH, V; M)$ on (F, f).



(5.24)

Let the anchor maps of the action Lie algebroids $A_HS \lessdot G$ and $AH \lessdot P$ be denoted by $\widetilde{a}^\dagger$ and $a^\dagger$, respectively. We need to show that $(\widetilde{a}^\dagger, a^\dagger)$ is a morphism of Lie groupoids.

First, we make the observation that, for any $(s, g) \in S \lessdot G$, we have

$$\alpha_G \circ \widetilde{\theta}_g(s) = \alpha_G(s \cdot g) = \widetilde{\alpha}_V(s) \cdot \alpha_G(g) = \theta_{\alpha_G(g)} \circ \widetilde{\alpha}_V(s),$$

and

$$\beta_G \circ \widetilde{\theta}_g(s) = \beta_G(s \cdot g) = \widetilde{\beta}_V(s) \cdot \beta_G(g) = \theta_{\beta_G(g)} \circ \widetilde{\beta}_V(s).$$

Hence, for any $(X, g) \in A_HS \lessdot G$, we have

$$\begin{aligned}
T(\alpha_G)(\widetilde{a}^\dagger(X, g)) &= T(\alpha_G)(T_{\widetilde{1}_{F(g)}}(\widetilde{\theta}_g)(X)) \\
&= T_{1_{\alpha_V(F(g))}}(\theta_{\alpha_G(g)})(T_{\widetilde{1}_{F(g)}}(\widetilde{\alpha}_V)(X)) \\
&= T_{1_{f(\alpha_G(g))}}(\theta_{\alpha_G(g)})(A(\widetilde{\alpha}_V)(X)) \\
&= a^\dagger(A(\widetilde{\alpha}_V)(X), \alpha_G(g)),
\end{aligned}$$

and also

$$\begin{aligned}
T(\beta_G)(\widetilde{a}^\dagger(X, g)) &= T(\beta_G)(T_{\widetilde{1}_{F(g)}}(\widetilde{\theta}_g)(X)) \\
&= T_{1_{\beta_V(F(g))}}(\theta_{\beta_G(g)})(T_{\widetilde{1}_{F(g)}}(\widetilde{\beta}_V)(X)) \\
&= T_{1_{f(\beta_G(g))}}(\theta_{\beta_G(g)})(A(\widetilde{\beta}_V)(X)) \\
&= a^\dagger(A(\widetilde{\beta}_V)(X), \beta_G(g)).
\end{aligned}$$

We also note that, for any $(s_2, h), (s_1, g) \in S \lessdot G$ such that $(s_2, s_1) \in S *_H S$ and $(h, g) \in G * G$, we have

$$\kappa_G \circ (\widetilde{\theta}_h \times \widetilde{\theta}_g)(s_2, s_1) = (s_2 \cdot h)(s_1 \cdot g) = (s_2 \boxminus s_1) \cdot (hg) = \widetilde{\theta}_{hg} \circ \widetilde{\kappa}_V(s_2, s_1).$$

Thus, for any $((X, h), (Y, g)) \in (A_HS \lessdot G) * (A_HS \lessdot G)$, we deduce that

$$\begin{aligned}
T(\kappa_G)(\widetilde{a}^\dagger(X, h), \widetilde{a}^\dagger(Y, g)) &= T(\kappa_G)(T_{\widetilde{1}_{F(h)}}(\widetilde{\theta}_h)(X), T_{\widetilde{1}_{F(g)}}(\widetilde{\theta}_g)(Y)) \\
&= T_{\widetilde{1}_{F(hg)}}(\widetilde{\theta}_{hg})(T_{(\widetilde{1}_{F(h)}, \widetilde{1}_{F(g)})}(\widetilde{\kappa}_V)(X, Y)) \\
&= T_{\widetilde{1}_{F(hg)}}(\widetilde{\theta}_{hg})(A(\widetilde{\kappa}_V)(X, Y)) \\
&= \widetilde{a}^\dagger(A(\widetilde{\kappa}_V)(X, Y), hg).
\end{aligned}$$

We have shown that the axioms of Definition 1.11 have been met, and so the anchor map $\widetilde{a}^\dagger$ is a morphism of Lie groupoids over the anchor map $a^\dagger$. □

5.1.4 $\mathcal{LG}$-actions of $\mathcal{LA}$-groupoids

We now define another type of action for $\mathcal{LA}$-groupoids. It can be interpreted as an extension of the notion of a Lie groupoid action in the category of smooth manifolds to the category of Lie algebroids.

Definition 5.18. Let $(\Omega; E, H; M)$ be an $\mathcal{LA}$-groupoid, $A \to P$ a Lie algebroid, and $F\colon A \to E$ a morphism of Lie algebroids over a smooth map $f\colon P \to M$.

$$\begin{array}{ccccc} & & & & A \\ & & & \overset{F}{\swarrow} & \\ \Omega & \overset{\widetilde{\alpha},\widetilde{\beta}}{\rightrightarrows} & E & & \downarrow{\scriptstyle q_A} \\ \downarrow{\scriptstyle \widetilde{q}} & & \downarrow{\scriptstyle q} & & P \\ & & & \overset{f}{\swarrow} & \\ H & \overset{\alpha,\beta}{\rightrightarrows} & M & & \end{array} \tag{5.25}$$

An *$\mathcal{LG}$-action* of $(\Omega; E, H; M)$ on (F, f) (or on $A \to P$) consists of a Lie groupoid action $\widetilde{\theta}$ of $\Omega \rightrightarrows E$ on F, and a Lie groupoid action θ of $H \rightrightarrows M$ on f, such that $(\widetilde{\theta}, \theta)$ is a morphism of Lie algebroids.

The morphism $(\widetilde{\theta}, \theta)$ that corresponds to an $\mathcal{LG}$-action of $(\Omega; A, V; M)$ on a Lie algebroid $A \to P$ can be expressed diagrammatically,

$$\begin{array}{ccc} \Omega \lhd A & \xrightarrow{\widetilde{\theta}} & A \\ \downarrow & & \downarrow \\ H \lhd P & \xrightarrow{\theta} & P. \end{array} \tag{5.26}$$

Here, the Lie algebroid structure on $\Omega \lhd A$ with base $H \lhd P$ is the unique Lie algebroid structure that makes it an embedded Lie subalgebroid of the direct product Lie algebroid $\Omega \times A$ via inclusion. We also have the action Lie groupoids $\Omega \lhd A$ and $H \lhd P$ on base manifolds A and P, respectively. These Lie groupoid and Lie algebroid structures can be displayed in the following diagram:

$$\begin{array}{ccc} \Omega \lhd A & \overset{\widetilde{\alpha}_\lhd, \widetilde{\beta}_\lhd}{\rightrightarrows} & A \\ \downarrow{\scriptstyle \widetilde{q} * q_A} & & \downarrow{\scriptstyle q_A} \\ H \lhd P & \overset{\alpha_\lhd, \beta_\lhd}{\rightrightarrows} & P. \end{array} \tag{5.27}$$

Theorem 5.19. *Let $(\Omega; E, H; M)$ be an $\mathcal{LA}$-groupoid, A a Lie algebroid on base P, and $F\colon A \to E$ a morphism of Lie algebroids over a smooth map $f\colon P \to M$. Suppose that we have an $\mathcal{LG}$-action of $(\Omega; E, H; M)$ on (F, f). Then $(\Omega \lhd A; A, H \lhd P; P)$ is an $\mathcal{LA}$-groupoid.*

Proof. We need to show that the structure maps of the action groupoid $\Omega \lhd A \rightrightarrows A$ are all morphisms of Lie algebroids over the corresponding structure maps of the action groupoid $H \lhd P \rightrightarrows P$.

We first observe that the source projection $\widetilde{\alpha}_{\lhd}$ can be realised as a composite of the following Lie algebroid morphisms:

$$\begin{array}{ccccc}
\Omega \lhd A & \hookrightarrow & \Omega \times A & \xrightarrow{\widetilde{\mathrm{pr}}_2} & A \\
\downarrow & & \downarrow & & \downarrow \\
H \lhd P & \hookrightarrow & H \times P & \xrightarrow{\mathrm{pr}_2} & P.
\end{array}$$

When we defined direct product Lie algebroids, we saw that the projection maps were morphisms of Lie algebroids (see page 20), and thus $\widetilde{\alpha}_{\lhd}$ is a morphism of Lie algebroids over $\alpha_{\lhd}$.

The target projections $\widetilde{\beta}_{\lhd}$ and $\beta_{\lhd}$ are given by the action maps $\widetilde{\theta}$ and θ, respectively, and so $\widetilde{\beta}_{\lhd}$ is a morphism of Lie algebroids over $\beta_{\lhd}$ by assumption.

We can view the partial multiplication $\widetilde{\kappa}_{\lhd}$ as a composite of the following morphisms of Lie algebroids over $\kappa_{\lhd}$ with restricted codomain:

$$\begin{array}{ccccc}
(\Omega \lhd A) * (\Omega \lhd A) & \hookrightarrow & (\Omega * \Omega) \times A \times A & \xrightarrow{\widetilde{\kappa} \times \widetilde{\mathrm{pr}}_2} & \Omega \times A \\
\downarrow & & \downarrow & & \downarrow \\
(H \lhd P) * (H \lhd P) & \hookrightarrow & (H * H) \times P \times P & \xrightarrow{\kappa \times \mathrm{pr}_2} & H \times P.
\end{array}$$

Hence, $\widetilde{\kappa}_{\lhd}$ is also a morphism of Lie algebroids over $\kappa_{\lhd}$. Note that here we are implicitly making use of Proposition 1.55.

In a similar fashion, the identity map $\widetilde{1}_{\lhd}$ can be viewed as the composite of the following morphisms of Lie algebroids with restricted codomain:

$$\begin{array}{ccccccc}
A & \xrightarrow{\cong} & \Delta_A & \hookrightarrow & A \times A & \xrightarrow{(\widetilde{1} \circ F) \times \mathrm{id}_A} & \Omega \times A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P & \xrightarrow{\cong} & \Delta_P & \hookrightarrow & P \times P & \xrightarrow{(1 \circ f) \times \mathrm{id}_P} & H \times P,
\end{array}$$

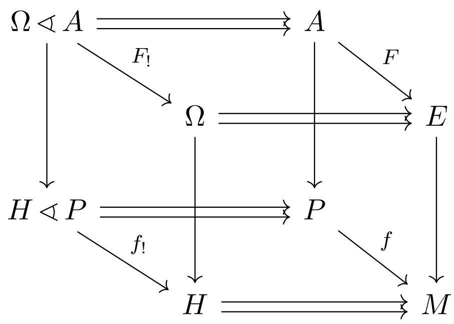
and the inversion map $\widetilde{\iota}_{\lhd}$ can be viewed as a composite of the following morphisms of Lie algebroids with restricted codomain:

$$\begin{array}{ccccccc}
\Omega \lhd A & \xrightarrow{\cong} & \Delta_{\Omega} \lhd A & \hookrightarrow & \Omega \times (\Omega \lhd A) & \xrightarrow{\widetilde{\iota} \times \widetilde{\theta}} & \Omega \times A \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
H \lhd P & \xrightarrow{\cong} & \Delta_H \lhd P & \hookrightarrow & H \times (H \lhd P) & \xrightarrow{\iota \times \theta} & H \times P.
\end{array}$$

Hence, the identity map $\widetilde{1}_{\lhd}$ is a morphism of Lie algebroids over $1_{\lhd}$, and the inversion map $\widetilde{\iota}_{\lhd}$ is a morphism of Lie algebroids over $\iota_{\lhd}$. □

Given an $\mathcal{LG}$-action of an $\mathcal{LA}$-groupoid $(\Omega; E, H; M)$ on a morphism of Lie algebroids $F\colon A \to E$ over a smooth map $f\colon P \to M$, we call the corresponding $\mathcal{LA}$-groupoid $(\Omega \lhd A; A, H \lhd P; P)$ an *$\mathcal{LG}$-action $\mathcal{LA}$-groupoid.*

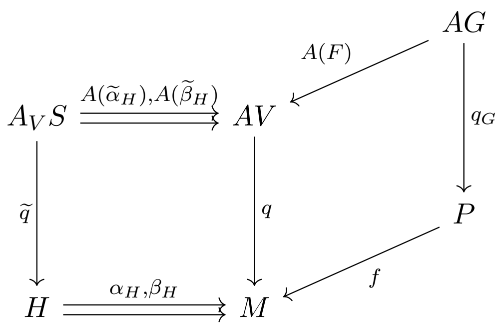
Recall that the action groupoids $\Omega \lessdot A$ and $H \lessdot P$ give rise to the action morphisms $F_!\colon \Omega \lessdot A \to \Omega$ and $f_!\colon H \lessdot P \to H$ over F and f, respectively (see page 7). Furthermore, we claim that $F_!$ is also a morphism of Lie algebroids over $f_!$. Thus, we get a morphism of $\mathcal{LA}$-groupoids $(F_!; F, f_!; f)$ from the $\mathcal{LG}$-action $\mathcal{LA}$-groupoid $(\Omega \lessdot A; A, H \lessdot P; P)$ to the original $\mathcal{LA}$-groupoid $(\Omega; E, H; M)$.



In the previous section we saw how an action of a double Lie groupoid gave rise to an $\mathcal{LA}$-action of an $\mathcal{LA}$-groupoid. It is also true that an action of a double Lie groupoid gives rise to an $\mathcal{LG}$-action of an $\mathcal{LA}$-groupoid.

Proposition 5.20. *Let $(\widetilde{\theta}, \theta)$ be an action of a double Lie groupoid $(S; H, V; M)$ on a morphism of Lie groupoids $F\colon G \to V$ over $f\colon P \to M$. Then there exists an $\mathcal{LG}$-action of the $\mathcal{LA}$-groupoid $(A_V S; AV, H; M)$ on the morphism of Lie algebroids $A(F)$ over f.*

Proof. We consider the $\mathcal{LA}$-groupoid $(A_V S; AV, H; M)$ of Example 1.77. Applying the Lie functor to the morphism of Lie groupoids F gives a morphism of Lie algebroids $A(F)$ by Proposition 1.49. Moreover, applying the Lie functor to the Lie groupoid action $\widetilde{\theta}$ defines a Lie groupoid action $A(\widetilde{\theta})$ of $A_V S \rightrightarrows AV$ on $A(F)$. The proof of this statement is routine and follows in a similar fashion to Example 1.26.



(5.28)

Now, since $A(\widetilde{\theta})$ is a morphism of Lie algebroids over θ we get an $\mathcal{LG}$-action of $(A_V S; AV, H; M)$ on $(A(F), f)$. □

We finish this section with an example of an $\mathcal{LG}$-action on an $\mathcal{LA}$-groupoid arising in Poisson geometry. This example provides a basis for the final section of this chapter.

Example 5.21. Let (G, π_G) be a Poisson Lie group with Lie algebra $\mathfrak{g}$, and (P, π_P) a Poisson manifold. Consider a Poisson action $\theta\colon G \times P \to P$. This gives rise to a Lie algebra action $\mathfrak{g} \to \mathfrak{X}(P)$, $X \mapsto X^\dagger$ which dualizes to give a map $\mathfrak{p}\colon T^*P \to \mathfrak{g}^*$ given by

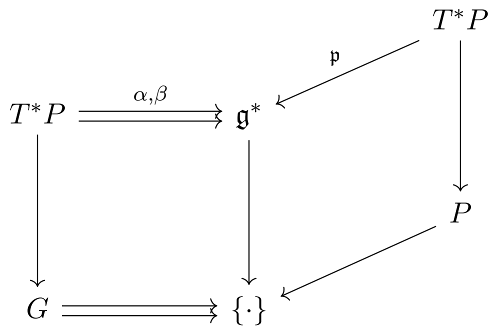
$$\langle \mathfrak{p}(\varphi), X \rangle = \langle \varphi, X^\dagger(p) \rangle,$$

for $\varphi \in T_p^*P$, $X \in \mathfrak{g}$. In Example 1.27, we called $\mathfrak{p}$ the *pith* of the action, and showed that θ lifts to a Lie groupoid action $\widetilde{\theta}\colon T^*G * T^*P \to T^*P$ of $T^*G \rightrightarrows \mathfrak{g}^*$ on $\mathfrak{p}$. Explicitly, this action was defined by

$$\psi \cdot \varphi = \varphi \circ T(\theta_{g^{-1}}),$$

for $\psi \in T_g^*G$ and $\varphi \in T^*P$ with $\alpha(\psi) = \mathfrak{p}(\varphi)$.

Note that, as in Example 2.81, the Poisson Lie group G gives rise to an $\mathcal{LA}$-groupoid $(T^*G; \mathfrak{g}^*, G; \{\cdot\})$. Since P is a Poisson manifold, $T^*P \to P$ has the cotangent Lie algebroid structure. It is also known that the pith $\mathfrak{p}\colon T^*P \to \mathfrak{g}^*$ is a morphism of Lie algebroids [25, Theorem 3.3].



Moreover, Mackenzie proved that the action $\widetilde{\theta}\colon T^*G * T^*P \to T^*P$ is a morphism of Lie algebroids over $\theta\colon G \times P \to P$ [46, Theorem 3.1]. Hence, we get an $\mathcal{LG}$-action of $(T^*G; \mathfrak{g}^*, G; \{\cdot\})$ on the pith.

Let us show that the anchors $\pi_G^{\#} * \pi_P^{\#}$ and $\pi_P^{\#}$ of $T^*G \lhd T^*P \to G \times P$ and $T * P \to P$, respectively, satisfy the property

$$T(\theta) \circ (\pi_G^{\#} * \pi_P^{\#}) = \pi_P^{\#} \circ \widetilde{\theta}.$$

We first make the observation that since the action $\theta\colon G \times P \to P$ is a Poisson map, by Proposition 2.19, the graph of the action $\Gamma(\theta)$ is a coisotropic submanifold of $G \times P \times \overline{P}$. This means that

$$\pi_G^{\#} \oplus \pi_P^{\#} \oplus (-\pi_P^{\#})((T\Gamma(\theta))^{\circ}) \subseteq T\Gamma(\theta).$$

Now fix $g \in G$, $p \in P$ and let $\gamma \coloneqq (g, p, g \cdot p) \in \Gamma(\theta)$. Take arbitrary $\psi \in T_g^*G$ and $\varphi \in T_p^*P$ such that $\alpha(\psi) = \mathfrak{p}(\varphi)$, and arbitrary $X \in T_gG$ and $Y \in T_pP$. Note that the tangent action can be expressed as

$$T(\theta)(X, Y) = T(\theta_p)(X) + T(\theta_g)(Y).$$

Now observe that

$$\begin{aligned}
\langle \psi \cdot \varphi, T(\theta)(X, Y) \rangle &= \langle \varphi \circ T(\theta_{g^{-1}}), T(\theta_p)(X) \rangle + \langle \varphi \circ T(\theta_{g^{-1}}), T(\theta_g)(Y) \rangle \\
&= \langle \varphi, T(\theta_{g^{-1}} \circ \theta_p)(X) \rangle + \langle \varphi, Y \rangle \\
&= \langle \varphi, T(\theta_p \circ L_{g^{-1}})(X) \rangle + \langle \varphi, Y \rangle \\
&= \langle \varphi, (T(L_{g^{-1}})(X))^{\dagger}(p) \rangle + \langle \varphi, Y \rangle \\
&= \langle \mathfrak{p}(\varphi), T(L_{g^{-1}})(X) \rangle + \langle \varphi, Y \rangle \\
&= \langle \alpha(\psi), T(L_{g^{-1}})(X) \rangle + \langle \varphi, Y \rangle \\
&= \langle \psi \circ T(L_{g^{-1}}), T(L_{g^{-1}})(X) \rangle + \langle \varphi, Y \rangle \\
&= \langle \psi, X \rangle + \langle \varphi, Y \rangle.
\end{aligned}$$

Hence, $(\psi, \varphi, -\psi \cdot \varphi) \in (T_\gamma \Gamma(\theta))^\circ$, and moreover $(\pi_G^\#(\psi), \pi_P^\#(\varphi), \pi_P^\#(\psi \cdot \varphi)) \in T_\gamma \Gamma(\theta)$. However, this then implies that

$$T(\theta)(\pi_G^\#(\psi), \pi_P^\#(\varphi)) = \pi_P^\#(\psi \cdot \varphi). \qquad \boxtimes$$

§ 5.2 An application to Poisson reduction

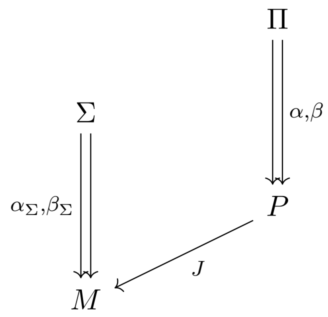
In this final section, we consider an example of the appearance of actions of double Lie structures in Poisson geometry. The majority of the results in this section are stated without proof, but references have been given. We will see that actions of double Lie structures arise in the study of Poisson reduced spaces of Poisson groupoid actions.

5.2.1 Poisson reduced spaces for actions of symplectic groupoids

In [66, Theorem 3.2], Xu constructed a symplectic groupoid for the Poisson reduced space of every free and proper symplectic groupoid action. This was later extended to a construction of a symplectic groupoid for the Poisson reduced space of every free and proper Poisson groupoid action of a symplectic groupoid in [67]. We now discuss this construction and show how there exists an underlying action of a double Lie groupoid.

Let P be a Poisson manifold and Σ an α–simply connected Lie groupoid on base M. Suppose that we have a free and proper Poisson groupoid action $\theta\colon \Sigma * P \to P$, $(g, p) \mapsto g \cdot p$ of Σ on a complete Poisson map $J\colon P \to M$. Since the action is free and proper the orbit space P/Σ has a smooth manifold structure for which the natural projection $p\colon P \to P/\Sigma$ is a submersion. Hence, by Theorem 2.75, P/Σ has a unique Poisson structure for which p is a Poisson map. Recall that we called P/Σ the Poisson reduced space.

Now suppose we have a symplectic groupoid Π with base manifold P. By Theorem 2.54, the source and target projections of Π, denoted as in the following diagram, are anti-Poisson and Poisson maps, respectively.



Since the source and target projections of any symplectic groupoid are complete, it follows that the composite $J \circ \alpha\colon \Pi \to M$ is a complete anti-Poisson map and the composite $J \circ \beta\colon \Pi \to M$ is a complete Poisson map. Hence, by Proposition 2.57 there is an induced right symplectic groupoid action $\theta_R\colon \Pi * \Sigma \to \Pi$, $(\xi, g) \mapsto \xi \cdot g$ of Σ on $J \circ \alpha$, and an induced left symplectic groupoid action $\theta_L\colon \Sigma * \Pi \to \Pi$, $(g, \xi) \mapsto g \cdot \xi$ of Σ on $J \circ \beta$.

In [67], Xu showed that, as a result of the properties of the Poisson and symplectic structures, these actions satisfy the following relations:

Proposition 5.22 ([67, Proposition 2.1]). *Let the actions θ_L and θ_R be defined as above. For any $\xi \in \Pi$ and $g, h \in \Sigma$, which satisfy $\alpha_\Sigma(g) = f \circ \beta(\xi)$ and $\beta_\Sigma(h) = f \circ \alpha(\xi)$, we have the following properties:*

(1) $\alpha(g \cdot \xi) = \alpha(\xi)$;

(2) $\beta(g \cdot \xi) = g \cdot \beta(\xi)$;

(3) $\alpha(\xi \cdot h) = h^{-1} \cdot \alpha(\xi)$;

(4) $\beta(\xi \cdot h) = \beta(\xi)$;

(5) $(g \cdot \xi) \cdot h = g \cdot (\xi \cdot h)$. □

Moreover, it was shown that these actions interact with the partial multiplication of Π in the following way:

Proposition 5.23 ([67, Proposition 2.2]). *With the actions defined as above, we have the following properties:*

(6) For $(\xi, \eta) \in \Pi * \Pi$ and $g \in \Sigma$, such that $(g, \xi) \in \Sigma * \Pi$, we have

$$(g \cdot \xi)\eta = g \cdot (\xi\eta);$$

(7) For $(\xi, \eta) \in \Pi * \Pi$ and $g \in \Sigma$, such that $(\eta, g) \in \Pi * \Sigma$, we have

$$\xi(\eta \cdot g) = (\xi\eta) \cdot g;$$

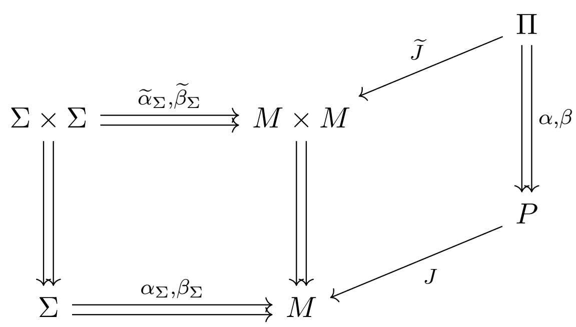
(8) For $(\xi, g) \in \Pi * \Sigma$ and $\eta \in \Pi$, such that $(\xi \cdot g, \eta) \in \Pi * \Pi$, we have

$$(\xi \cdot g)\eta = \xi(g \cdot \eta).$$ □

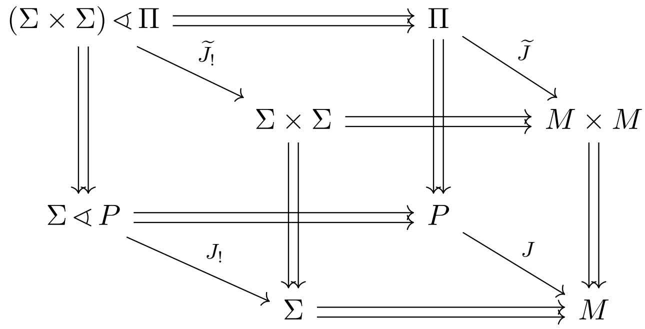
Note that these two propositions show that θ_L and θ_R satisfy the properties *(1)–(8)* discussed in Section 5.1.2. Thus, by Proposition 5.9, we have an induced Lie groupoid action $\widetilde{\theta}\colon (\Sigma \times \Sigma) * \Pi \to \Pi$, defined by

$$((h, g), \xi) \mapsto (h, g) \cdot \xi := (h \cdot \xi) \cdot g^{-1},$$

of the Cartesian product groupoid $\Sigma \times \Sigma \rightrightarrows M \times M$ on the smooth composite map $\widetilde{J} := (J \times J) \circ \chi\colon \Pi \to M \times M$. Furthermore, $(\widetilde{\theta}, \theta)$ defines an action of the double Lie groupoid $(\Sigma \times \Sigma; \Sigma, M \times M; M)$ on the morphism of Lie groupoids $(\widetilde{J}, J)$.



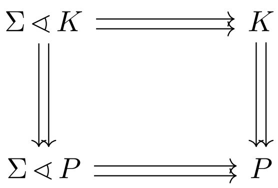
This action gives rise to an action double groupoid $((\Sigma \times \Sigma) \lessdot \Pi; \Sigma \lessdot P, \Pi; P)$, and to a morphism of double Lie groupoids $(\widetilde{J}_!; J_!, \widetilde{J}; J)$ from the action double groupoid to the original double Lie groupoid $(\Sigma \times \Sigma; \Sigma, M \times M; M)$ (see the discussion on page 107).



We now consider the kernels of the morphisms of Lie groupoids $(\widetilde{J}_!, J_!)$ and $(\widetilde{J}, J)$. We use the notation $\widetilde{K} = \ker(\widetilde{J}_!, J_!)$ and $K = \ker(\widetilde{J}, J)$. It is straightforward to check that the structure maps of the action double groupoid restrict to form a double Lie groupoid $(\widetilde{K}; \Sigma \lhd P, K; P)$. Moreover, the horizontal structure $\widetilde{K} \rightrightarrows K$ can be shown to be isomorphic to the action groupoid $\Sigma \lhd K \rightrightarrows K$, where the Lie groupoid action of Σ on $(J \circ \alpha)|_K = (J \circ \beta)|_K$ is given by

$$\Sigma * K \to K, \quad (g, k) \mapsto (g, g) \cdot k.$$

It follows immediately from the relations of Proposition 5.22 that this action is free. Let us make the assumption that the action is also proper, so that the orbit space K/Σ has a smooth manifold structure for which the projection $K \to K/\Sigma$ is a submersion.



We now state a useful result due to Mackenzie:

Proposition 5.24 ([43, Proposition 3.1])**.** *Let $(S; H, V; M)$ be a double Lie groupoid such that the transitivity orbit spaces $\tau^H(S)$ and $\tau(H)$ of the Lie groupoids $S \rightrightarrows V$ and $H \rightrightarrows M$ have smooth manifold structures for which the projections $V \to \tau^H(S)$ and $M \to \tau(H)$ are submersions. Further, suppose that the anchor $\chi_H \colon H \to \mathrm{Im}(\chi_H)$ is a surjective submersion. Then, there is a unique Lie groupoid structure on $\tau^H(S)$ with base manifold $\tau(H)$ for which the projection $V \to \tau^H(S)$ is a morphism of Lie groupoids over the projection $M \to \tau(H)$.* □

Since the transitivity orbit spaces of the two action groupoids $\Sigma \lhd K \rightrightarrows K$ and $\Sigma \lhd P \rightrightarrows P$ are given by K/Σ and P/Σ, respectively, Proposition 5.24 provides us with a Lie groupoid structure on K/Σ with base manifold P/Σ. In fact, $K/\Sigma \rightrightarrows P/\Sigma$ is precisely the Lie groupoid Xu constructs in [67]. It was further shown that K/Σ has a symplectic structure for which $K/\Sigma \rightrightarrows P/\Sigma$ is a symplectic groupoid.

5.2.2 Poisson reduced spaces for actions of Poisson groupoids

We now try a similar approach to investigate the Poisson reduced space of an arbitrary free and proper Poisson groupoid action.

Let P be a Poisson manifold and G a Poisson groupoid on base M. Suppose that we have a free and proper Poisson groupoid action $\theta\colon G * P \to P$, $(g,p) \mapsto g \cdot p$ of G on a moment map $J\colon P \to M$. The assumption that the action is free and proper implies that the orbit space P/G has a smooth manifold structure for which the projection $p\colon P \to P/G$ is a submersion. By Theorem 2.75, there exists a unique Poisson structure on P/G for which p is a Poisson map.

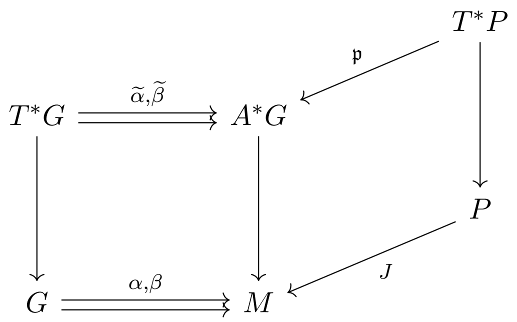
Note that the Lie groupoid action θ gives rise to a Lie algebroid action $\Gamma(AG) \to \mathfrak{X}(P)$, $X \mapsto X^{\dagger}$. Dualizing this infinitesimal action gives a map $\mathfrak{p}\colon T^*P \to A^*G$, which we again call the *pith* of the action.

We now lift θ to a Lie groupoid action $\widetilde{\theta}\colon T^*G * T^*P \to T^*P$ of $T^*G \rightrightarrows A^*G$ on the pith $\mathfrak{p}$. Take any $\psi \in T^*_g G$ and $\varphi \in T^*_p P$ with $\widetilde{\alpha}(\psi) = \mathfrak{p}(\varphi)$. Since the action θ is a submersion, for any $Z \in T_{g\cdot p}P$, there exists $X \in T_g G$ and $Y \in T_p P$ for which $T(\theta)(X,Y) = Z$. We define $\psi \cdot \varphi \in T^*_{g\cdot p}P$ by

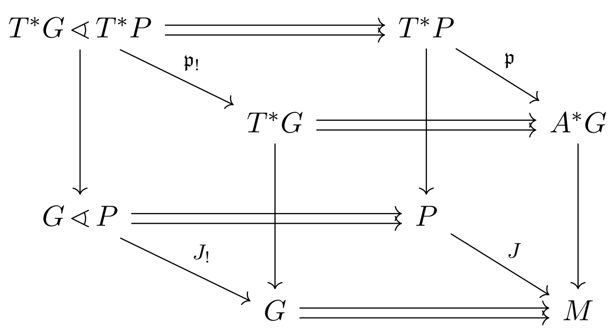
$$\langle \psi \cdot \varphi, Z\rangle = \langle \psi, X\rangle + \langle \varphi, Y\rangle.$$

The condition that $\widetilde{\alpha}(\psi) = \mathfrak{p}(\varphi)$ is sufficient to prove that this definition is well-defined. The check that this defines a Lie groupoid action follows similarly to Example 1.27.

Since $G \rightrightarrows P$ is a Poisson groupoid we have a $\mathcal{LA}$-groupoid $(T^*G; A^*G, G; P)$ as in Example 2.82. It has also been proven that the pith $\mathfrak{p}\colon T^*P \to A^*G$ is a morphism of Lie algebroids over the moment map $J\colon P \to M$ [25, Theorem 3.3]. It was announced in [48] that this action $\widetilde{\theta}\colon T^*G \lessdot T^*P \to T^*P$ gives a morphism of Lie algebroids over $\theta\colon G \lessdot P \to P$. Hence, $(\widetilde{\theta}, \theta)$ defines an $\mathcal{LG}$-action of $(T^*G; A^*G, G; P)$ on the morphism of Lie algebroids $(\mathfrak{p}, J)$.



This action gives rise to an $\mathcal{LG}$-action $\mathcal{LA}$-groupoid $(T^*G \lessdot T^*P; T^*P, G \lessdot P; P)$, and to a morphism of $\mathcal{LA}$-groupoids $(\mathfrak{p}_!; \mathfrak{p}, J_!; J)$ from the $\mathcal{LG}$-action $\mathcal{LA}$-groupoid to the original $\mathcal{LA}$-groupoid $(T^*G; A^*G, G; P)$ (see page 125).



We now consider the kernels of the morphisms of Lie algebroids $(\mathfrak{p}_!, J_!)$ and $(\mathfrak{p}, J)$. We denote these kernels by $\widetilde{K} = \ker(\mathfrak{p}_!, J_!)$ and $K = \ker(\mathfrak{p}, J)$. We assume that the pith is a constant rank map, so that $\widetilde{K}$ and K become Lie subalgebroids of $T^*G \lessdot T^*P$ and T^*P, respectively. Moreover, the structure maps of the $\mathcal{LG}$-action $\mathcal{LA}$-groupoid restrict to form an $\mathcal{LA}$-groupoid $(\widetilde{K}; K, G \lessdot P; P)$. In a similar fashion to the previous section we see that the Lie groupoid $\widetilde{K} \rightrightarrows K$ is isomorphic to an action groupoid $G \lessdot K \rightrightarrows K$. We make the assumption that this action of G on $K \to P$ is free and proper, so that the orbit space K/G has a smooth manifold structure for which the projection $K \to K/G$ is a submersion.

$$\begin{array}{ccc} G \lessdot K & \rightrightarrows & K \\ \downarrow & & \downarrow \\ G \lessdot P & \rightrightarrows & P \end{array}$$

We have the following result of Mackenzie, which gives an analogue of Proposition 5.24 for $\mathcal{LA}$-groupoids.

Proposition 5.25 ([46, Proposition 4.1])**.** *Let $(\Omega; A, H; M)$ be an $\mathcal{LA}$-groupoid such that the transitivity orbit spaces $\tau(\Omega)$ and $\tau(H)$ of the Lie groupoids $\Omega \rightrightarrows A$ and $H \rightrightarrows M$ have smooth manifold structures for which the projections $A \to \tau(\Omega)$ and $M \to \tau(H)$ are submersions. Then, there is a unique Lie algebroid structure on $\tau(\Omega)$ with base manifold $\tau(H)$ for which the projection $A \to \tau(\Omega)$ is a morphism of Lie algebroids over the projection $M \to \tau(H)$.* □

Since the transitivity orbit spaces of the two action groupoids $G \lessdot K \rightrightarrows K$ and $G \lessdot P \rightrightarrows P$ are given by K/G and P/G, respectively, Proposition 5.25 provides us with a Lie algebroid structure on K/G with base manifold P/G. In the case where G is a Poisson Lie group, it was announced that this Lie algebroid is isomorphic to the cotangent Lie algebroid of P/G [46, Theorem 4.2]. We believe the same result holds in the more general case of a Poisson groupoid.

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