# Functor Homology for Augmented Algebras


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# Abstract

The gamma homology theory of Robinson and Whitehouse [Whi94] for commutative algebras was developed to encode information about homotopy commutativity. It has a description in terms of functor homology described by Pirashvili and Richter [PR00].

The symmetric homology theory of Fiedorowicz, [Fie] and [Aul10], for associative algebras is defined in terms of functor homology and, in the case of a group algebra, is related to stable homotopy theory by a theorem of Fiedorowicz, [Aul10, Corollary 40].

Both gamma homology and symmetric homology are constructed by building a symmetric group action into Hochschild homology, albeit in very different ways. We provide a comparison map for the two theories in the case of an augmented, commutative algebra over a commutative ring.

Fiedorowicz's hyperoctahedral homology theory is a homology theory for associative algebras with involution and in the case of a group algebra is related to equivariant stable homotopy theory [Fie, Theorem 1]. It is the homology theory associated to the hyperoctahedral crossed simplicial group, whose associated category is denoted $\Delta H$ [FL91].

We define the *category of involutive, non-commutative sets*, denoted $\mathcal{IF}(as)$, and prove that this category is isomorphic to $\Delta H$. The category $\mathcal{IF}(as)$ has a more combinatorial definition than $\Delta H$ and contains the category of non-commutative sets of Pirashvili and Richter [PR02] as a subcategory.

There is a standard complex, provided by Gabriel and Zisman [GZ67, Appendix 2], that computes the hyperoctahedral homology of an involutive, associative algebra. In the case of an augmented, involutive, associative algebra we provide a smaller complex for computing hyperoctahedral homology which is built from the epimorphisms in the category $\Delta H$ and the augmentation ideal of the algebra.

# Introduction

# Homology Theories For Algebras

This thesis concerns homology theories for algebras over commutative rings. The study of such objects was first introduced by Hochschild [Hoc45] as a method for classifying algebra extensions of associative algebras over a field. Cartan and Eilenberg [CE56] constructed the analogous homology theory for algebras over a commutative ring. This homology theory is called *Hochschild homology*. Since then a great many homology theories have been introduced to encode information about algebras equipped with extra structure.

For example, for a commutative algebra over a commutative ring one can consider its *Harrison homology*, its *gamma homology* and its *André-Quillen homology*. These theories encode information about commutative algebra extensions, homotopy commutativity and smoothness of the commutative algebra respectively.

There are homology theories that encode group actions and involutions on an algebra. Many of these are generalizations of Hochschild homology. Most famously, Connes [Con83] and Tsygan [Tsy83] independently introduced the notion of *cyclic homology* for an associative algebra. One example of particular interest is the cyclic homology of a group algebra which is isomorphic to the $S^1$-equivariant homology of the free loop space on the classifying space of the group.

Fiedorowicz and Loday [FL91] and Krasauskas [Kra87] extended the construction of cyclic homology to other families of groups through the notion of a *crossed simplicial group*. For example, for an associative algebra with involution one can define a *dihedral homology theory* which is intimately related to $O(2)$-equivariant homology theory.

Two examples of crossed simplicial groups, the symmetric groups and the hyperoctahedral groups, do not generalize to algebraic homology theories straightforwardly from Hochschild homology. However, Fiedorowicz [Fie] successfully defines a *symmetric homology theory* for associative algebras and a *hyperoctahedral homology theory* for associative algebras with involution. Furthermore he proves that these homology theories, in the case of a group algebra, are related to the homology of infinite loop spaces.

Each of the homology theories thus far mentioned can be described in terms of *functor homology*. There exists a small category $\mathbf{C}$ such that we can define a homology theory for functors from $\mathbf{C}$ to the category of modules over the ground ring in such a way that, if we choose the right functor, we recover the homology theory for algebras over the ground ring.

The existence of functor homology theories provides new tools for calculation and methods from category theory that were previously unavailable. Functor homology interpretations and methods form a key part of this thesis.

# New Material

The new material in this thesis fits into two sections.

## A Comparison Map for Symmetric Homology and Gamma Homology

The gamma homology theory of Robinson and Whitehouse [Whi94] and the symmetric homology theory of Fiedorowicz, [Fie] and [Aul10], are both constructed by building a symmetric group action into Hochschild homology, albeit in very different ways. We provide a comparison map for the two theories in the case of an augmented, commutative algebra.

There is a standard complex that computes the gamma homology of a commutative algebra called the *Robinson-Whitehouse complex*. In Part III we provide a splitting of the Robinson-Whitehouse complex for an augmented, commutative algebra. One summand of this splitting can be constructed using only the elements of the algebra that lie in the augmentation ideal. Furthermore, we show that when the ground ring contains $\mathbb{Q}$, the homology of this summand is the gamma homology of the algebra.

There is also a standard complex for computing the symmetric homology of an associative algebra, constructed by Gabriel and Zisman [GZ67]. In Part VII we provide a surjective map of chain complexes which leads to a long exact sequence in homology comparing the symmetric homology of an augmented, commutative algebra with a summand of its gamma homology.

When the ground ring contains $\mathbb{Q}$, we introduce a normalized version of Harrison homology which allows us to deduce that, in this case, our long exact sequence compares the symmetric homology of an augmented, commutative algebra with its gamma homology.

## Hyperoctahedral Homology

We prove a series of results for Fiedorowicz's hyperoctahedral homology theory.

To any crossed simplicial group there is an associated category. The category associated to the hyperoctahedral crossed simplicial group is denoted $\Delta H$. In Part IV we provide a presentation for $\Delta H$. In Part V we define the *category of involutive, non-commutative sets*, denoted $\mathcal{IF}(as)$. We prove that this category is isomorphic to $\Delta H$. The category $\mathcal{IF}(as)$ has a more combinatorial definition than $\Delta H$ and contains the category of non-commutative sets of Pirashvili and Richter [PR02] as a subcategory.

In Part VI we generalize an argument of Ault [Aul10] and prove that the hyperoctahedral homology of an augmented, involutive, associative algebra can be calculated using only the epimorphisms in the category $\Delta H$ and the augmentation ideal of the algebra.

# Structure of The Thesis

## Part I

Part I is an extensive background chapter and is designed to be a point of reference. The reader should feel free to skip familiar material.

Chapter 1 collates the definitions of a number of categories that will occur frequently throughout the thesis.

Throughout the thesis we will use simplicial methods freely. Chapter 2 collates the necessary definitions and results from simplicial homotopy theory for reference.

In order to motivate Fiedorowicz's symmetric and hyperoctahedral homology theories we state his results for group algebras and the homology of infinite loop spaces. Chapter 3 collates the necessary background on spectra and infinite loop spaces required to state these results.

As stated, this thesis concerns homology theories for algebras with varying amounts of structure over a commutative ring. In Chapter 4 we define different types of algebra over a commutative ring together with examples.

Finally, in Chapter 5 we recall the constructions of some classical homology theories for algebras, namely Hochschild homology, Harrison homology, André-Quillen homology and cyclic homology.

## Part II

Part II recalls the theory of functor homology, or the homology of small categories. Every homology theory discussed in this thesis has a functor homology interpretation and we will use a number of techniques and tools from functor homology throughout the thesis.

Of particular importance is the tensor product of modules over a small category and its associated Tor functors.

We recall two complexes used for computing Tor over a small category. The first is a construction of Gabriel and Zisman. The second involves computing resolutions in functor categories. These two complexes are isomorphic and each has its benefits when it comes to calculating Tor groups and proving results.

# Part III

Part III concerns gamma homology, which we will frequently write as Γ-homology. Γ-homology for commutative algebras was developed by Robinson and Whitehouse as a homology theory to encode information about homotopy commutativity. The details first appeared in the thesis of Whitehouse [Whi94]. We recall the definition of Γ-homology as functor homology following Pirashvili and Richter [PR00]. We then specialize to the case of commutative algebras as defined by Whitehouse. Finally, we provide some new material. In the case of an augmented, commutative algebra we provide a splitting of the Robinson-Whitehouse complex and analyse the summands.

# Part IV

Part IV outlines the theory of crossed simplicial groups following Fiedorowicz and Loday [FL91].

A crossed simplicial group is a simplicial object built from a family of groups whose face and degeneracy maps are not group homomorphisms but "crossed" or "twisted" group homomorphisms. Important examples include the cyclic groups, the dihedral groups, the symmetric groups and the hyperoctahedral groups. Fiedorowicz and Loday show that there is a standard way to associate a functor homology theory to a crossed simplicial group. In this fashion one can recover Hochschild homology and cyclic homology of an associative algebra and define a dihedral homology theory for associative algebras with an involution. However, they also show that the symmetric and hyperoctahedral functor homology theories defined in this standard way are isomorphic to Hochschild homology.

Fortunately Fiedorowicz was able to define more interesting symmetric and hyperoctahedral homology theories which have connections to the homology of infinite loop spaces.

We recall the equivalent definitions of a crossed simplicial group together with some important examples. We then discuss Hochschild homology, cyclic homology, dihedral homology, symmetric homology and hyperoctahedral homology in the functor homology setting, motivating each with an interesting result.

# Part V

To each crossed simplicial group there is an associated category. The category associated to the symmetric crossed simplicial group is denoted $\Delta S$ and the category associated to the hyperoctahedral crossed simplicial group is denoted $\Delta H$.

Pirashvili and Richter [PR02] introduced the category of non-commutative sets, denoted $\mathcal{F}(as)$, as an isomorphic variant of $\Delta S$.

In Part V we introduce the *category of involutive, non-commutative sets*, denoted $\mathcal{IF}(as)$, as an isomorphic variant of the category $\Delta H$. The category $\mathcal{IF}(as)$ contains $\mathcal{F}(as)$ as a subcategory and there is a commutative diagram of categories

$$\begin{array}{ccc} \Delta H & \xrightarrow{\cong} & \mathcal{IF}(as) \\ \uparrow & & \uparrow \\ \Delta S & \xrightarrow[\cong]{} & \mathcal{F}(as) \end{array}$$

where the vertical arrows are inclusions of subcategories.

# Part VI

In Part VI we present the epimorphism construction for the hyperoctahedral homology of an augmented, involutive, associative algebra. Recall that $\Delta H$ is the category associated to the hyperoctahedral crossed simplicial group. Let $\mathrm{Epi}\Delta H$ be the subcategory of epimorphisms. The epimorphism construction consists of a functor between the undercategories

$$[x]\backslash\Delta H \to [x]\backslash\mathrm{Epi}\Delta H$$

for each object $[x]$ in $\Delta H$.

These functors allow us to define a chain homotopy equivalence between a standard complex computing the hyperoctahedral homology of an augmented, involutive, associative algebra and a smaller complex constructed from the category $\mathrm{Epi}\Delta H$ and the elements of the augmentation ideal.

# Part VII

We begin Part VII by defining an isomorphic variant of the category $\Delta S$. This variant is more convenient when describing the comparison map.

We define a normalized version of Harrison homology for the case where the ground ring contains $\mathbb{Q}$.

Using the splitting of the Robinson-Whitehouse complex in Part III we define a surjective map between a chain complex computing the symmetric homology and a chain complex computing a summand of the $\Gamma$-homology of an augmented, commutative algebra. As a corollary we obtain a long exact sequence in homology.

Finally, we use the normalized Harrison homology to show that when the ground ring contains $\mathbb{Q}$, the long exact sequence in homology compares the symmetric homology and the $\Gamma$-homology of an augmented, commutative algebra.

## Appendices

Appendices A through D contain long technical proofs that would have otherwise interrupted the flow of the material in certain sections. Appendix E contains a table summarizing details of all the homology theories contained within the thesis.

## Conventions

Throughout the thesis we will use the following conventions.

- $k$ is a commutative ring with unit and
- an unadorned tensor product $\otimes$ will denote the tensor product of $k$-modules.

# Part I

# Background Material

# Chapter 1

# Categories

## Introduction

We collect the definitions of some standard categories that we will use throughout the thesis, including the constructions of the under-category and over-category.

## 1.1 Definitions

**Definition 1.1.1.** Let $\mathbf{C}$ be a small category. We denote the set of objects in $\mathbf{C}$ by $\mathrm{Ob}(\mathbf{C})$ and the set of morphisms in $\mathbf{C}$ by $\mathrm{Hom}(\mathbf{C})$.

**Definition 1.1.2.** Let $\mathbf{C}$ be any category and let $\alpha \in \mathrm{Hom}(\mathbf{C})$. We denote the *source* of the morphism $\alpha$ by $src(\alpha)$ and the target of the morphism $\alpha$ by $trg(\alpha)$.

**Definition 1.1.3.** A morphism $\alpha \in \mathrm{Hom}_{\mathbf{C}}(C_1, C_2)$ is an *epimorphism* if for any two morphisms $f$, $g \in \mathrm{Hom}_{\mathbf{C}}(C_2, C_3)$ the equality

$$f \circ \alpha = g \circ \alpha$$

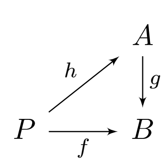
implies that $f = g$; in other words $\alpha$ is *right cancellable*.

**Definition 1.1.4.** A morphism $\alpha \in \mathrm{Hom}_{\mathbf{C}}(C_1, C_2)$ is a *monomorphism* if for any two morphisms $f$, $g \in \mathrm{Hom}_{\mathbf{C}}(C, C_1)$ the equality

$$\alpha \circ f = \alpha \circ g$$

implies that $f = g$; in other words $\alpha$ is *left cancellable*.

**Definition 1.1.5.** An object $P$ in $\mathbf{C}$ is *projective* if for any morphism $f \in \mathrm{Hom}_{\mathbf{C}}(P, B)$ and epimorphism $g \in \mathrm{Hom}_{\mathbf{C}}(A, B)$ there exists $h \in \mathrm{Hom}_{\mathbf{C}}(P, A)$ such that the diagram



commutes.

**Definition 1.1.6.** Let $\mathcal{I}$ be an indexing category. A *set of generators* in $\mathbf{C}$ is a set

$$\{C_i \in \mathrm{Ob}\,(\mathbf{C}) : i \in \mathcal{I}\}$$

such that for any two morphism $f,\ g \in \mathrm{Hom}_{\mathbf{C}}\,(X, Y)$ if $f \neq g$ there exists $i \in \mathcal{I}$ and $h \in \mathrm{Hom}_{\mathbf{C}}\,(C_i, X)$ such that

$$h \circ f \neq h \circ g.$$

**Definition 1.1.7.** A *set of projective generators* is a set of generators such that each object is projective.

**Definition 1.1.8.** A category $\mathbf{C}$ is said to be *abelian* if it satisfies the following properties:

- $\mathbf{C}$ has a zero object,
- for each pair of objects $C_1$ and $C_2$ there is a product $C_1 \times C_2$ in $\mathbf{C}$,
- every morphism in $\mathbf{C}$ has a kernel and a cokernel,
- every monomorphism in $\mathbf{C}$ is the kernel of its cokernel and
- every epimorphism in $\mathbf{C}$ is the cokernel of its kernel.

An arbitrary abelian category will henceforth be denoted by $\mathbf{A}$.

## 1.2 Categories

We will use the following categories without reference.

**Definition 1.2.1.** Let **Fin** denote the category of finite sets and set maps.

**Definition 1.2.2.** Let $\mathbf{Fin}_\star$ denote the category whose objects are based finite sets and whose morphisms are basepoint-preserving maps of sets.

**Definition 1.2.3.** Let **Set** denote the category of sets and set maps.

**Definition 1.2.4.** Let $\mathbf{Set}_\star$ denote the category whose objects are based sets and whose morphisms are basepoint-preserving maps of sets.

**Definition 1.2.5.** Let $k$ be a commutative ring. Let **kMod** denote the category of $k$-modules and $k$-module homomorphisms.

**Definition 1.2.6.** Let $k$ be a commutative ring. Let **ChCpx** denote the category of non-negatively graded chain complexes of $k$-modules and chain maps.

**Definition 1.2.7.** Let $\mathbf{A}$ be an abelian category. Let $\mathbf{ChCpx}(\mathbf{A})$ denote the category of non-negatively graded chain complexes in $\mathbf{A}$ and chain maps.

**Definition 1.2.8.** Let **Top** denote the category of compactly-generated, Hausdorff $CW$-topological spaces and continuous maps.

**Definition 1.2.9.** Let $\mathbf{Top}_\star$ denote the category of based compactly-generated, Hausdorff $CW$-topological spaces and basepoint-preserving continuous maps.

**Definition 1.2.10.** Let $\mathbf{C}$ and $\mathbf{D}$ be categories. We denote the category whose objects are functors $\mathbf{C} \to \mathbf{D}$ and whose morphisms are natural transformations by $\mathrm{Fun}\,(\mathbf{C}, \mathbf{D})$.

**Definition 1.2.11.** Let $\mathbf{Cat}$ denote the category of small categories. The objects are small categories and the morphisms are functors between small categories.

## 1.3 The Under-Category

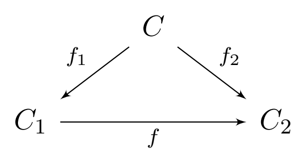
For an object $C \in \mathbf{C}$ we define the category $(C\backslash\mathbf{C})$ of *objects under* $C$.

**Definition 1.3.1.** The objects of the category $(C\backslash\mathbf{C})$ are all morphisms in $\mathbf{C}$ of the form $C \to C_1$.

**Definition 1.3.2.** A morphism

$$\left(C \xrightarrow{f_1} C_1\right) \to \left(C \xrightarrow{f_2} C_2\right)$$

in $(C\backslash\mathbf{C})$ consists of a morphism $f \in \mathrm{Hom}_{\mathbf{C}}\,(C_1, C_2)$ such that the triangle




commutes in $\mathbf{C}$.

We will denote such a morphism by

$$(f, f_1, f_2)\colon \left(C \xrightarrow{f_1} C_1\right) \to \left(C \xrightarrow{f_2} C_2\right).$$

The identity morphism on

$$C \xrightarrow{f_1} C_1$$

is $(id_{C_1}, f_1, f_1)$.

**Definition 1.3.3.** Composition of morphisms

$$(f, f_1, f_2)\colon \left(C \xrightarrow{f_1} C_1\right) \to \left(C \xrightarrow{f_2} C_2\right)$$

and

$$(g, f_2, f_3)\colon \left(C \xrightarrow{f_2} C_2\right) \to \left(C \xrightarrow{f_3} C_3\right)$$

in $(C\backslash\mathbf{C})$ is defined by

$$(g, f_2, f_3) \circ (f, f_1, f_2) = (g \circ f, f_1, f_3).$$

**Definition 1.3.4.** The category $(C\backslash\mathbf{C})$, with objects as in Definition 1.3.1, morphisms as in Definition 1.3.2 and composition as in Definition 1.3.3 is called the *category of objects under $C$*.

*Remark 1.3.5.* There exists a functor

$$\left(-\backslash\mathbf{C}\right) : \mathbf{C}^{op} \to \mathbf{Cat}$$

which on objects assigns the under-category $\left(C\backslash\mathbf{C}\right)$ to an object $C$ in $\mathbf{C}$.

The assignment on morphisms is given as follows. Let $f \in \mathrm{Hom}_{\mathbf{C}}\left(C, C_1\right)$. We define a functor

$$(f\backslash\mathbf{C})\colon (C_1\backslash\mathbf{C}) \to (C\backslash\mathbf{C})$$

on objects by

$$(f\backslash\mathbf{C})\left(C_1 \xrightarrow{f'} C_2\right) = C \xrightarrow{f'\circ f} C_2.$$

On morphisms we define $(f\backslash\mathbf{C})$ to send the morphism $(g, f', f'')$ in $(C_1\backslash\mathbf{C})$ to the morphism $(g, f' \circ f, f'' \circ f)$ in $(C\backslash\mathbf{C})$.

## 1.4 The Over-Category

For an object $C \in \mathbf{C}$ we define the category $(\mathbf{C}/C)$ of *objects over $C$*.

**Definition 1.4.1.** The objects of the category $(\mathbf{C}/C)$ are all morphisms in $\mathbf{C}$ of the form $C_1 \to C$.

**Definition 1.4.2.** A morphism

$$\left(C_1 \xrightarrow{f_1} C\right) \to \left(C_2 \xrightarrow{f_2} C\right)$$

in $(\mathbf{C}/C)$ consists of a morphism $f \in \mathrm{Hom}_{\mathbf{C}}\left(C_1, C_2\right)$ such that the triangle

$$\begin{array}{ccc} C_1 & \xrightarrow{\quad f \quad} & C_2 \\ & {\scriptstyle f_1}\searrow \quad \swarrow{\scriptstyle f_2} & \\ & C & \end{array}$$

commutes in $\mathbf{C}$. We will denote such a morphism by

$$(f_1, f_2, f)\colon \left(C_1 \xrightarrow{f_1} C\right) \to \left(C_2 \xrightarrow{f_2} C\right).$$

The identity morphism on

$$C_1 \xrightarrow{f_1} C$$

is $(f_1, f_1, id_{C_1})$.

**Definition 1.4.3.** The composition of morphisms

$$(f_1, f_2, f)\colon \left(C_1 \xrightarrow{f_1} C\right) \to \left(C_2 \xrightarrow{f_2} C\right)$$

and

$$(f_2, f_3, g)\colon \left(C_2 \xrightarrow{f_2} C\right) \to \left(C_3 \xrightarrow{f_3} C\right)$$

in $(\mathbf{C}/C)$ is defined by

$$(f_2, f_3, g) \circ (f_1, f_2, f) = (f_1, f_3, g \circ f).$$

**Definition 1.4.4.** The category $(\mathbf{C}/C)$ with objects as in Definition 1.4.1, morphisms as in Definition 1.4.2 and composition as in Definition 1.4.3 is called the *category of objects over* $C$.

*Remark 1.4.5.* There exists a functor

$$\big(\mathbf{C}/-\big) : \mathbf{C} \to \mathbf{Cat}$$

which on objects assigns the over-category $\big(\mathbf{C}/C\big)$ to an object $C$ in $\mathbf{C}$.

The assignment on morphisms is given as follows. Let $f \in \mathrm{Hom}_{\mathbf{C}}(C, C_1)$. The functor

$$(\mathbf{C}/f)\colon (\mathbf{C}/C_1) \to (\mathbf{C}/C)$$

is defined on objects by

$$(\mathbf{C}/f)\left(C_2 \xrightarrow{f'} C_1\right) = \left(C_2 \xrightarrow{f \circ f'} C\right).$$

On morphisms we define $(\mathbf{C}/f)$ to send the morphism $(f', f'', g)$ in $(\mathbf{C}/C_1)$ to the morphism $(f \circ f', f \circ f'', g)$ in $(\mathbf{C}/C)$.

# Chapter 2

# Simplicial Homotopy Theory

## Introduction

We recall standard material from simplicial homotopy theory that we will use throughout the thesis. We recall the definitions of a simplicial set, together with the notions of geometric realization and homotopy groups. We also recall the standard chain complex associated to a simplicial set, together with the normalized and degenerate subcomplexes.

## 2.1 Simplicial Sets

**Definition 2.1.1.** A *simplicial set* $X_\star$ is a graded set indexed on the non-negative integers together with maps

$$\partial_i \in \mathrm{Hom}_{\mathbf{Set}}\left(X_n, X_{n-1}\right)$$

for $n \geqslant 1$ and $0 \leqslant i \leqslant n$ and

$$s_i \in \mathrm{Hom}_{\mathbf{Set}}\left(X_n, X_{n+1}\right)$$

for $n \geqslant 0$ and $0 \leqslant i \leqslant n$, satisfying the *simplicial identities* below.

- $\partial_i \circ \partial_j = \partial_{j-1} \circ \partial_i$ if $i < j$,
- $s_i \circ s_j = s_{j+1} \circ s_i$ if $i \leqslant j$,
- $\partial_i \circ s_j = \begin{cases} s_{j-1} \circ \partial_i & i < j \\ id_{X_n} & i = j,\, j+1 \\ s_j \circ \partial_{i-1} & i > j+1. \end{cases}$

**Definition 2.1.2.** The maps $\partial_i$ of Definition 2.1.1 are called *face maps* and the maps $s_i$ are called *degeneracy maps*.

**Definition 2.1.3.** The elements of the set $X_n$ are called $n$-*simplices*.

**Definition 2.1.4.** An $n$-simplex $y \in X_n$ is said to be *degenerate* if it is in the image of a degeneracy map, that is, if there exists $x \in X_{n-1}$ and $0 \leqslant i \leqslant n-1$ such that $y = s_i(x)$.

**Definition 2.1.5.** A morphism of simplicial sets $f\colon X_\star \to Y_\star$ consists of a morphism $f_n \in \mathrm{Hom}_{\mathbf{Set}}(X_n, Y_n)$ for each $n \geqslant 0$ such that diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle \partial_i}\downarrow & & \downarrow{\scriptstyle \partial_i} \\ X_{n-1} & \xrightarrow[f_{n-1}]{} & Y_{n-1} \end{array}$$

commutes for all $n \geqslant 1$ and $0 \leqslant i \leqslant n$ and the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle s_i}\uparrow & & \uparrow{\scriptstyle s_i} \\ X_{n-1} & \xrightarrow[f_{n-1}]{} & Y_{n-1} \end{array}$$

commutes for all $n \geqslant 1$ and $0 \leqslant i \leqslant n-1$.

**Definition 2.1.6.** We denote by **sSet** the category whose objects are the simplicial sets of Definition 2.1.1 and whose morphisms are defined in Definition 2.1.5. Composition is defined degree-wise.

### 2.1.1 The nerve of a category

One example of a simplicial set is the nerve of a small category, [Lod98, Appendix B.12]. Let $\mathbf{C}$ be a small category. The *nerve of* $\mathbf{C}$, denoted $N_\star\mathbf{C}$ , is a simplicial set described as follows. The set $N_n(\mathbf{C})$ consists of all strings of composable morphisms of length $n$ in the category $\mathbf{C}$. That is, for $n \geqslant 1$,

$$N_n\mathbf{C} = \left\{ C_0 \xrightarrow{f_1} C_1 \xrightarrow{f_2} \cdots \xrightarrow{f_n} C_n \;:\; C_i \in \mathrm{Ob}(\mathbf{C}),\; f_i \in \mathrm{Hom}_{\mathbf{C}}(C_{i-1}, C_i) \right\}.$$

$N_0\mathbf{C}$ is defined to be $\mathrm{Ob}(\mathbf{C})$, the set of of objects in $\mathbf{C}$.

Henceforth, to ease notation, an element of $N_n\mathbf{C}$, for $n \geqslant 1$, will be denoted $(f_n, \ldots, f_1)$.

*Remark 2.1.7.* It will be assumed that for an arbitrary element $(f_n, \ldots, f_1)$ of $N_n(\mathbf{C})$, $f_1$ has domain $C_0$ unless otherwise stated.

The face maps $\partial_i\colon N_n\mathbf{C} \to N_{n-1}\mathbf{C}$ are defined to either omit or compose morphisms in the string:

$$\partial_i(f_n, \ldots, f_1) = \begin{cases} (f_n, \ldots, f_2) & i = 0, \\ (f_n, \ldots, f_{i+1} \circ f_i, \ldots, f_1) & 1 \leqslant i \leqslant n-1, \\ (f_{n-1}, \ldots, f_1) & i = n. \end{cases}$$

The degeneracy maps $s_j \colon N_n\mathbf{C} \to N_{n+1}\mathbf{C}$ are defined by inserting identities into the string. That is,

$$s_j(f_n, \ldots, f_1) = \begin{cases} \left(f_n, \ldots, f_1, id_{C_0}\right) & j = 0, \\ \left(f_n, \ldots, id_{C_j}, f_j, \ldots, f_1\right) & 1 \leqslant j \leqslant n. \end{cases}$$

*Remark 2.1.8.* The nerve construction is functorial. Let $F \colon \mathbf{C} \to \mathbf{D}$ be a functor between small categories. There is an induced map of simplicial sets

$$N_\star(\mathbf{C}) \to N_\star(\mathbf{D})$$

defined in degree $n$ by

$$\left(C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} C_n\right) \mapsto \left(F(C_0) \xrightarrow{F(f_1)} \cdots \xrightarrow{F(f_n)} F(C_n)\right).$$

In other words, the nerve construction is a functor

$$N_\star(-) : \mathbf{Cat} \to \mathbf{sSet}.$$

## 2.2 The Category $\Delta$

**Definition 2.2.1.** The *simplicial category* $\Delta$ has as objects the sets $[n] = \{0, 1, \ldots, n\}$ for $n \geqslant 0$. An element $f \in \mathrm{Hom}_\Delta\left([n], [m]\right)$ is a map of sets such that if $i < j$ in $[n]$ then $f(i) \leqslant f(j)$ in $[m]$. Such a morphism is called *order-preserving*.

**Definition 2.2.2.** For each set $[n]$ in $\Delta$ and each $0 \leqslant i \leqslant n+1$ the morphism

$$\delta_i \in \mathrm{Hom}_\Delta\left([n], [n+1]\right)$$

is defined to be the unique order-preserving injection such that $i \notin \mathrm{Im}(\delta_i)$. The morphism $\delta_i$ is called the $i^{th}$ *face map*.

**Definition 2.2.3.** For each set $[n]$ in $\Delta$ and $0 \leqslant j \leqslant n$ the morphism

$$\sigma_j \in \mathrm{Hom}_\Delta\left([n+1], [n]\right)$$

is defined to be the unique order-preserving surjection such that $\sigma_j^{-1}(j) = \{j, j+1\}$. The morphism $\sigma_j$ is called the $j^{th}$ *degeneracy map.*

**Proposition 2.2.4 ([Lod98, Corollary B.3]).** *The category $\Delta$ has as objects the sets $[n]$ for $n \geqslant 0$. Morphisms in $\Delta$ are generated by the face maps*

$$\delta_i \in \mathrm{Hom}_\Delta\left([n], [n+1]\right)$$

*for each $n \geqslant 0$ and $0 \leqslant i \leqslant n+1$ and the degeneracy maps*

$$\sigma_j \in \mathrm{Hom}_\Delta\left([n+1], [n]\right)$$

*for each $n \geqslant 0$ and $0 \leqslant j \leqslant n$ subject to the relations*

- $\delta_j \circ \delta_i = \delta_i \circ \delta_{j-1}$ *for* $i < j$,
- $\sigma_j \circ \sigma_i = \sigma_i \circ \sigma_{j+1}$ *for* $i \leqslant j$ *and*
- $\sigma_j \circ \delta_i = \begin{cases} \delta_i \circ \sigma_{j-1} & i < j \\ id_{[n]} & i = j,\, j+1 \\ \delta_{i-1} \circ \sigma_j & i > j+1. \end{cases}$

*This is a presentation of the category* $\Delta$. □

**Theorem 2.2.5 ([Lod98, Theorem B.2]).** *For* $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$ *there is a unique decomposition*

$$\varphi = \delta_{i_1} \circ \cdots \delta_{i_r} \circ \sigma_{j_1} \circ \cdots \circ \sigma_{j_s}$$

*such that* $i_1 \leqslant \cdots \leqslant i_r$ *and* $j_1 \leqslant \cdots \leqslant j_s$ *with* $m = n - s + r$. *If the set of indices is empty then* $\varphi = id_{[n]}$. □

**Proposition 2.2.6.** *A simplicial set* $X_\star$ *is equivalent to a functor*

$$X_\star \colon \Delta^{op} \to \mathbf{Set}$$

*and a morphism of simplicial sets is equivalent to a natural transformation of functors.* □

## 2.3 Simplicial Objects in a Category

**Definition 2.3.1.** A simplicial object $X_\star$ in a category $\mathbf{C}$ consists of an object $X_n$ in $\mathbf{C}$ for each $n \geqslant 0$ together with morphisms

$$\partial_i \in \mathrm{Hom}_{\mathbf{C}}\left(X_n, X_{n-1}\right)$$

for $n \geqslant 1$ and $0 \leqslant i \leqslant n$ and

$$s_i \in \mathrm{Hom}_{\mathbf{C}}\left(X_n, X_{n+1}\right)$$

for $n \geqslant 0$ and $0 \leqslant i \leqslant n$ satisfying the identities of Definition 2.1.1.

**Definition 2.3.2.** A morphism of simplicial objects $f \colon X_\star \to Y_\star$ in a category $\mathbf{C}$ consists of a morphism $f_n \in \mathrm{Hom}_{\mathbf{C}}\left(X_n, Y_n\right)$ for each $n \geqslant 0$ such that diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle \partial_i}\downarrow & & \downarrow{\scriptstyle \partial_i} \\ X_{n-1} & \xrightarrow[f_{n-1}]{} & Y_{n-1} \end{array}$$

commutes for all $n \geqslant 1$ and $0 \leqslant i \leqslant n$ and the diagram

$$\begin{array}{ccc} X_n & \xrightarrow{f_n} & Y_n \\ {\scriptstyle s_i}\uparrow & & \uparrow{\scriptstyle s_i} \\ X_{n-1} & \xrightarrow[f_{n-1}]{} & Y_{n-1} \end{array}$$

commutes for all $n \geqslant 1$ and $0 \leqslant i \leqslant n-1$.

*Remark 2.3.3.* Equivalently, a simplicial object $X_\star$ in $\mathbf{C}$ is a functor

$$X_\star \colon \Delta^{op} \to \mathbf{C}$$

and a morphism of simplicial objects in $\mathbf{C}$ is a natural transformation of functors.

**Definition 2.3.4.** Let $\mathbf{C}$ be a category. We denote the category of *simplicial objects in* $\mathbf{C}$ and simplicial morphisms by $\mathbf{sC}$.

### 2.3.1 Simplicial objects with a group action

**Definition 2.3.5.** Let $G$ be a group and let $X_\star \in \mathbf{sC}$. A *$G$-action* on $X_\star$ is a level-wise action of $G$ on $X_n$ such that

- $\partial_i(gx) = g(\partial_i(x))$ for $x \in X_n$, $g \in G$, $n \geqslant 1$ and $0 \leqslant i \leqslant n$ and
- $s_j(gx) = g(s_j(x))$ for $x \in X_n$, $g \in G$, $n \geqslant 0$ and $0 \leqslant j \leqslant n$.

## 2.4 Simplicial Homotopy Theory

### 2.4.1 Geometric realization

**Definition 2.4.1.** The *geometric $n$-simplex*, $\Delta^n$ is the subspace

$$\Big\{(t_0, \ldots, t_n) : 0 \leqslant t_i \leqslant 1, \ \sum t_i = 1\Big\} \subseteq \mathbb{R}^{n+1}.$$

**Proposition 2.4.2 ([Wei94, Subsection 8.1.6]).** *Let $n \geqslant 0$. For each face map*

$$\delta_i \in \operatorname{Hom}_\Delta\big([n], [n+1]\big)$$

*there is an induced map*

$$\delta_{i\star} \colon \Delta^n \to \Delta^{n+1}$$

*determined by*

$$(t_0, \ldots, t_n) \mapsto (t_0, \ldots, 0, t_i, \ldots, t_n).$$

*Furthermore, for each degeneracy map*

$$\sigma_j \in \operatorname{Hom}_\Delta\big([n+1], [n]\big)$$

*there is an induced map*

$$\sigma_{j\star} \colon \Delta^{n+1} \to \Delta^n$$

*determined by*

$$(t_0, \ldots, t_{n+1}) \mapsto (t_0, \ldots, t_j + t_{j+1}, \ldots, t_{n+1}).$$

□

**Definition 2.4.3.** Let $X_\star$ be a simplicial set. Consider the topological space

$$\coprod_{n \geqslant 0} X_n \times \Delta^n.$$

We define an equivalence relation on this space as follows. Let $n \geqslant 0$.

- For $\delta_i \in \mathrm{Hom}_\Delta\big([n], [n+1]\big)$ we define

  $$\big(\partial_i(x), s\big) \sim \big(x, \delta_{i\star}(s)\big)$$

  for $x \in X_{n+1}$ and $s \in \Delta^n$.

- For $\sigma_j \in \mathrm{Hom}_\Delta\big([n+1], [n]\big)$ we define

  $$\big(s_j(x), s\big) \sim \big(x, \sigma_{j\star}(s)\big)$$

  for $x \in X_n$ and $s \in \Delta^{n+1}$.

**Definition 2.4.4.** Let $X_\star$ be a simplicial set. We define the *geometric realization* of $X_\star$, denoted $|X_\star|$, to be the quotient of the topological space

$$\coprod_{n \geqslant 0} X_n \times \Delta^n$$

by the equivalence relation of Definition 2.4.3.

**Definition 2.4.5.** The functor

$$|-| : \mathbf{sSet} \to \mathbf{Top}$$

is defined on objects by sending a simplicial set $X_\star$ to its geometric realization. Given $f \in \mathrm{Hom}_{\mathbf{sSet}}(X_\star, Y_\star)$,

$$|f| : \; |X_\star| \to |Y_\star|$$

is defined by

$$\big[(x, s)\big] \mapsto \big[(f(x), s)\big] .$$

### 2.4.2 Simplicial homotopy groups

**Definition 2.4.6.** A simplicial set $X_\star$ is called *fibrant* if it satisfies the following condition.

- For every $n$ and $k$ with $0 \leqslant k \leqslant n+1$, if $x_i$ for $0 \leqslant i \leqslant k-1$ and $k+1 \leqslant i \leqslant n+1$ are $n$-simplices such that

  $$\partial_i(x_j) = \partial_{j-1}(x_i)$$

  for $i < j$ and $i,\, j \neq k$, then there exists $y \in X_{n+1}$ such that $\partial_i(y) = x_i$ for all $i \neq k$.

*Remark 2.4.7.* This condition is sometimes referred to as the *Kan condition.*

**Definition 2.4.8.** Let $X_\star$ be a fibrant simplicial set with a basepoint $\star \in X_0$. We will write $\star$ for the image of the basepoint under any composite of the degeneracy maps $s_0$. For each $n \geqslant 1$ let

$$Z_n \coloneqq \left\{x \in X_n : \partial_i(x) = \star \ \forall 0 \leqslant i \leqslant n\right\}.$$

We define an equivalence relation on $Z_n$ as follows. We say that $x$ and $x'$ in $Z_n$ are *homotopic*, written $x \sim x'$, if there exists $y \in X_{n+1}$ such that

$$\partial_i(y) = \begin{cases} \star & i < n \\ x & i = n \\ x' & i = n+1. \end{cases}$$

**Definition 2.4.9.** Let $X_\star$ be a fibrant simplicial set. We define the $n^{th}$ *homotopy group*, $\pi_n(X_\star)$, to be the quotient of

$$Z_n \coloneqq \left\{x \in X_n : \partial_i(x) = \star \ \forall 0 \leqslant i \leqslant n\right\}$$

by the equivalence relation of Definition 2.4.8.

**Proposition 2.4.10 ([Wei94, Section 8.3]).** *Let $X_\star$ be a fibrant simplicial set. There is an isomorphism of groups*

$$\pi_n(X_\star) \cong \pi_n\left(|X_\star|\right)$$

*for each $n \geqslant 1$.* □

### 2.4.3 Simplicial homotopies

**Definition 2.4.11.** Let $A_\star$ and $B_\star$ be simplicial objects in an abelian category $\mathbf{A}$. Two simplicial maps

$$f_\star, g_\star \colon A_\star \to B_\star$$

are called *pre-simplicially homotopic* if there exists morphisms $h_i \in \mathrm{Hom}_{\mathbf{A}}(A_n, B_{n+1})$ for $0 \leqslant i \leqslant n$ such that

- $\partial_0 \circ h_0 = f_\star$,
- $\partial_{n+1} \circ h_n = g_\star$,
- $\partial_i \circ h_j = \begin{cases} h_{j-1} \circ \partial_i & i < j \\ \partial_i \circ h_{i-1} & i = j > 0 \\ h_j \circ \partial_{i-1} & i > j+1. \end{cases}$

Furthermore, $f_\star$ and $g_\star$ are called *simplicially homotopic* if the morphisms $h_i$ also satisfy

- $s_i \circ h_j = \begin{cases} h_{j+1} \circ s_i & i \leqslant j \\ h_j \circ s_{i-1} & i > j. \end{cases}$

**Definition 2.4.12.** A simplicial object $X_\star$ in a category $\mathbf{C}$ is called *augmented* if it comes equipped with a morphism in $\mathbf{C}$

$$\varepsilon\colon X_0 \to X_{-1}$$

to a fixed object $X_{-1}$ satisfying

$$\varepsilon \circ \partial_0 = \varepsilon \circ \partial_1$$

for the face maps

$$\partial_0,\, \partial_1\colon X_1 \to X_0.$$

**Proposition 2.4.13 ([Wei94, Section 8.4.6]).** *Let $A_\star$ be a a simplicial object in an abelian category $\mathbf{A}$ with augmentation*

$$\varepsilon\colon A_0 \to A_{-1}.$$

*Suppose there exist morphisms*

$$h_n\colon A_n \to A_{n+1}$$

*for $n \geqslant -1$ satisfying either the conditions*

- $\varepsilon \circ h_{-1} = id_{A_{-1}}$,
- $\partial_{n+1} \circ h_n = id_{A_n}$ *for* $n \geqslant 0$,
- $\partial_0 \circ h_0 = h_{-1} \circ \varepsilon$ *and*
- $\partial_i \circ h_n = h_{n-1} \circ \partial_i$ *for* $n \geqslant 1$ *and* $0 \leqslant i \leqslant n$

*or the conditions*

- $\varepsilon \circ h_{-1} = id_{A_{-1}}$,
- $\partial_0 \circ h_n = id_{A_n}$ *for* $n \geqslant 0$,
- $\partial_1 \circ h_0 = h_{-1} \circ \varepsilon$ *and*
- $\partial_i \circ h_n = h_{n-1} \circ \partial_{i-1}$ *for* $n \geqslant 1$ *and* $1 \leqslant i \leqslant n+1$.

*Then*

$$\pi_\star(A) \cong \begin{cases} A_{-1} & \star = 0 \\ 0 & \textit{else.} \end{cases}$$

□

## 2.5 Chain Complexes Associated to Simplicial Sets

### 2.5.1 The associated chain complex

Let $\mathbf{A}$ be an abelian category. There is a standard construction which associates a chain complex to a simplicial object in $\mathbf{A}$.

**Definition 2.5.1.** Let $A_\star$ be a simplicial object in an abelian category $\mathbf{A}$. The *associated* or *unnormalized* chain complex $C_\star(A)$ is defined as follows. In degree $n$ one takes $C_n(A) = A_n$. The boundary map $d\colon A_n \to A_{n-1}$ is given by the alternating sum of the face maps. That is,

$$d = \sum_{i=0}^{n} (-1)^i \partial_i.$$

**Definition 2.5.2.** We define a functor

$$C_\star(-)\colon \mathbf{sA} \to \mathbf{ChCpx}(\mathbf{A})$$

by

$$A_\star \mapsto C_\star(A)$$

on objects. For a morphism $f_\star \in \mathrm{Hom}_{\mathbf{sA}}(A_\star, B_\star)$, $C_\star(f_\star)$ is the map of chain complexes given degree-wise by $f_\star$.

**Definition 2.5.3.** Let $A_\star$ be a simplicial object in an abelian category $\mathbf{A}$. We define the $n^{th}$ *homology group* of $A_\star$, denoted $H_n(A)$, to be

$$H_n\left(C_\star(A)\right).$$

**Proposition 2.5.4 ([Wei94, Theorem 8.3.8]).** *Let $\mathbf{A}$ be an abelian category and let $A_\star \in \mathbf{sA}$. Let $C_\star(A)$ be the associated chain complex. There is an isomorphism of groups*

$$\pi_n(A) \cong H_n(C_\star(A))$$

*for each $n \geqslant 0$.* □

**Proposition 2.5.5 ([Lod98, Lemma 1.0.9]).** *Let $h_\star$ be a pre-simplicial homotopy between simplicial maps*

$$f_\star,\, g_\star\colon A_\star \to B_\star$$

*in the sense of Definition 2.4.11. The induced maps*

$$f_\star,\, g_\star\colon C_\star(A) \to C_\star(B)$$

*on the associated chain complexes are chain homotopic.* □

### 2.5.2 The associated chain complex of a simplicial set

Given a simplicial set, one can form a simplicial $k$-module in a standard way.

**Definition 2.5.6.** Let

$$k[-]\colon \mathbf{Set} \to \mathbf{kMod}$$

be the functor that takes a set $X$ and returns the free $k$-module generated by $X$. For $f \in \mathrm{Hom}_{\mathbf{Set}}(X, Y)$,

$$k[f]\colon k[X] \to k[Y]$$

is the $k$-linear extension of $f$.

By applying this functor level-wise, we obtain a functor from the category of simplicial sets to the category of simplicial $k$-modules. By abuse of notation we will denote this functor

$$k[-]\colon \mathbf{sSet} \to \mathbf{skMod}.$$

**Definition 2.5.7.** Let $X_\star$ be a simplicial set. We define the *chain complex associated to* $X_\star$ to be

$$C_\star\left(k[X_\star]\right).$$

We denote this chain complex by $C_\star(X)$.

**Definition 2.5.8.** We define the $n^{th}$ *homology group* of a simplicial set $X_\star$, denoted $H_n(X)$ to be the group

$$H_n\left(C_\star(X)\right).$$

### 2.5.3 The degenerate subcomplex

The chain complex $C_\star(A)$ of Definition 2.5.1 was defined using only part of the simplicial structure, namely the face maps. We can use the degeneracy maps to define a subcomplex of $C_\star(A)$.

**Definition 2.5.9.** Let $A_\star$ be a simplicial object in an abelian category $\mathbf{A}$. For $n \geqslant 0$, let $D_n(A)$ denote the $k$-submodule of $C_n(A)$ generated by degenerate elements. That is,

$$D_n(A) = \left\{ \sum_{i \text{ finite}} \lambda_i s_{j_i}(x_i) : \lambda_i \in k,\ x_i \in C_{n-1}(A),\ 0 \leqslant j_i \leqslant n-1\ \forall i \right\}.$$

**Definition 2.5.10.** Let $A_\star$ be a simplicial object in an abelian category $\mathbf{A}$. The *degenerate subcomplex* of $C_\star(A)$, denoted $D_\star(A)$, is defined to have the $k$-module $D_n(A)$ of Definition 2.5.9 in degree $n$. The boundary map is the restriction of the boundary map of $C_\star(A)$.

**Proposition 2.5.11 ([Wei94, Theorem 8.3.8]).** *Let $A_\star$ be a simplicial object in an abelian category $\mathbf{A}$. The degenerate subcomplex, $D_\star(A)$, is acyclic.* □

### 2.5.4 The normalized subcomplex

**Definition 2.5.12.** Let $A_\star$ be a simplicial object in an abelian category $\mathbf{A}$. Let $n \geqslant 0$. We define the *normalized subcomplex* $N_\star(A)$ of $C_\star(A)$ in degree $n$ by

$$N_n(A) = \bigcap_{i=0}^{n-1} \operatorname{Ker}\left(\partial_i\colon A_n \to A_{n-1}\right).$$

The boundary map $d_n\colon N_n(A) \to N_{n-1}(A)$ is defined to be $d_n = (-1)^n \partial_n$, where $\partial_n\colon A_n \to A_{n-1}$ is the $n^{th}$ face map.

*Remark 2.5.13.* The normalized chain complex is sometimes referred to as the *Moore complex.*

**Proposition 2.5.14 ([Wei94, Lemma 8.3.7]).** *There is an isomorphism of chain complexes*

$$C_\star(A) \cong N_\star(A) \oplus D_\star(A). \qquad \square$$

**Corollary 2.5.15.** *There is an isomorphism of chain complexes*

$$\frac{C_\star(A)}{D_\star(A)} \cong N_\star(A). \qquad \square$$

**Corollary 2.5.16.** *The canonical projection map*

$$C_\star(A) \to N_\star(A)$$

*is a quasi-isomorphism. In particular, there is an isomorphism*

$$H_n(C_\star(A)) \cong H_n(N_\star(A))$$

*for each* $n \geqslant 0$. □

**Definition 2.5.17.** Let $\mathbf{A}$ be an abelian category. The *normalized chain functor*

$$N_\star \colon \mathbf{sA} \to \mathbf{ChCpx}(\mathbf{A})$$

is defined on objects by

$$A_\star \mapsto N_\star(A).$$

For $f \in \mathrm{Hom}_{\mathbf{sA}}(A, B)$, $N_\star(f)$ is the induced map of chain complexes

$$\frac{C_\star(A)}{D_\star(A)} \to \frac{C_\star(B)}{D_\star(B)}.$$

We conclude this chapter by stating a theorem that ties all of this material together, namely the *Dold-Kan correspondence.*

**Theorem 2.5.18 ([Wei94, Theorem 8.4.1]).** *For any abelian category* $\mathbf{A}$*, the normalized chain functor* $N_\star$ *is an equivalence of categories*

$$s\mathbf{A} \simeq \mathbf{ChCpx}(\mathbf{A})$$

*between simplicial objects in* $\mathbf{A}$ *and non-negatively graded chain complexes in* $\mathbf{A}$*.*

*Furthermore, under this correspondence simplicial homotopy corresponds to homology of chain complexes and simplicially homotopic morphisms correspond to chain homotopic maps.* □

# Chapter 3

# Spectra

## Introduction

We recall the based, free and Moore loop spaces. We recall the suspension functor and the loop-suspension adjunction. We recall the definitions of spectra and infinite loop spaces, in particular the functor $Q$, defined as the composite of the infinite suspension functor and the infinite loop functor.

All the material in this chapter can be found in [Ada78, Chapter 1].

## 3.1 Loop Spaces

### 3.1.1 Path space

**Definition 3.1.1.** Let $I$ be the based unit interval $[0,1]$ with $0$ as a basepoint. Let $X \in \mathbf{Top}_\star$. The *path space of* $X$ is defined to be

$$PX \coloneqq \mathrm{Hom}_{\mathbf{Top}_\star}(I, X) \in \mathbf{Top}_\star.$$

**Proposition 3.1.2 ([Wei94, Application 5.3.4]).** *Let $X \in \mathbf{Top}_\star$. The path space $PX$ is contractible.* □

### 3.1.2 Based loop functor

**Definition 3.1.3.** For $n \geqslant 1$, the $n^{th}$ *loop space functor* is the functor

$$\Omega^n \coloneqq \mathrm{Hom}_{\mathbf{Top}_\star}(S^n, -) : \mathbf{Top}_\star \to \mathbf{Top}_\star.$$

**Proposition 3.1.4 ([Wei94, Application 5.3.4]).** *There is a fibration, called the* path space fibration,

$$\Omega X \to PX \to X. \qquad \square$$

### 3.1.3 Free loop functor

**Definition 3.1.5.** For $n \geqslant 1$, the $n^{th}$ *free loop space functor* is the functor

$$\mathcal{L}^n := \mathrm{Hom}_{\mathbf{Top}}\left(S^n, -\right) : \mathbf{Top} \to \mathbf{Top}.$$

### 3.1.4 Moore loop space

**Definition 3.1.6.** Let $X \in \mathbf{Top}_\star$ with basepoint $\star$. The *Moore loop space*, denoted $\Omega^M(X)$, is the subset of

$$\mathrm{Hom}_{\mathbf{Top}_\star}\big([0,\infty), X\big) \times [0,\infty)$$

consisting of all pairs of the form $(f, r)$ such that $f(t) = \star$ for all $t \geqslant r$.

**Proposition 3.1.7.** *Let $X \in \mathbf{Top}_\star$. There is a homotopy equivalence*

$$\Omega^M(X) \simeq \Omega(X)$$

*between the Moore loop space of $X$ and the based loop space of $X$.* $\square$

## 3.2 Smash Products

**Definition 3.2.1.** Let $X$, $Y \in \mathbf{Top}_\star$. The *smash product* of $X$ and $Y$ is defined to be

$$X \wedge Y := \frac{X \times Y}{X \vee Y}.$$

**Definition 3.2.2.** Let $G$ be a group. Let $X$, $Y \in \mathbf{Top}_\star$ be such that $X$ has a right $G$-action and $Y$ has a left $G$-action. The *$G$-equivariant smash product* of $X$ and $Y$, denoted $X \wedge_G Y$, is defined to be the quotient of $X \wedge Y$ by the subspace generated by all differences

$$\big[(xg, y)\big] - \big[(x, gy)\big].$$

## 3.3 Suspensions

**Definition 3.3.1.** For $n \geqslant 1$, the *$n$-fold suspension functor* is the functor

$$\Sigma^n := S^n \wedge - : \mathbf{Top}_\star \to \mathbf{Top}_\star.$$

## 3.4 The Loop-Suspension Adjunction

Recall that the functor

$$\Sigma\colon \mathbf{Top}_\star \to \mathbf{Top}_\star$$

is left adjoint to the functor

$$\Omega\colon \mathbf{Top}_\star \to \mathbf{Top}_\star.$$

That is, for $X, Y \in \mathbf{Top}_\star$, there is a natural isomorphism of homotopy classes of maps

$$[\Sigma X, Y] \cong [X, \Omega Y].$$

More generally, there is an isomorphism of homotopy classes of maps

$$[\Sigma^n X, Y] \cong [X, \Omega^n Y]$$

for each $n \geqslant 1$.

## 3.5 Spectra and Infinite Loop Spaces

**Definition 3.5.1.** A morphism $f \in \mathrm{Hom}_{\mathbf{Top}_\star}(X, Y)$ is called a *weak equivalence* if the induced map

$$\pi_n(f)\colon \pi_n(X) \to \pi_n(Y)$$

is an isomorphism for all $n \geqslant 0$.

**Definition 3.5.2.** A based topological space $X$ is called an *infinite loop space* if there exist spaces $X_n \in \mathbf{Top}_\star$ for $n \geqslant 0$ such that $X_0 = X$ and there exist weak equivalences

$$X_n \to \Omega X_{n+1}$$

for each $n \geqslant 0$.

**Definition 3.5.3.** A *spectrum*, $E$, consists of based spaces $E_n \in \mathbf{Top}_\star$ for $n \geqslant 0$ together with *structure maps*

$$\gamma_n\colon \Sigma E_n \to E_{n+1}$$

for each $n \geqslant 0$.

**Definition 3.5.4.** Let $E$ and $F$ be spectra. A *morphism of spectra* $f\colon E \to F$ consists of a map $f_n \in \mathrm{Hom}_{\mathbf{Top}_\star}(E_n, F_n)$ for each $n \geqslant 0$ such that the diagram

$$\begin{array}{ccc} \Sigma E_n & \xrightarrow{\gamma_n^E} & E_{n+1} \\ {\scriptstyle \Sigma(f_n)}\downarrow & & \downarrow{\scriptstyle f_{n+1}} \\ \Sigma F_n & \xrightarrow[\gamma_n^F]{} & F_{n+1} \end{array}$$

commutes for each $n \geqslant 0$.

**Definition 3.5.5.** We denote by **Sp** the category whose objects are spectra with morphisms as in Definition 3.5.4. Composition is defined level-wise.

*Example 3.5.6.* Let $X \in \mathbf{Top}_\star$. The *suspension spectrum of* $X$, denoted $\Sigma^\infty X$, has

$$(\Sigma^\infty X)_n = \Sigma^n X.$$

Each structure map

$$\Sigma(\Sigma^n X) \to \Sigma^{n+1} X$$

is the identity.

The suspension spectrum construction is functorial. Given $f \in \mathrm{Hom}_{\mathbf{Top}_\star}(X, Y)$ there is an induced map of suspension spectra

$$\Sigma^\infty X \to \Sigma^\infty Y.$$

**Definition 3.5.7.** We call the functor

$$\Sigma^\infty \colon \mathbf{Top}_\star \to \mathbf{Sp}$$

the *infinite suspension functor*.

**Definition 3.5.8.** Let $E$ be a spectrum with structure maps

$$\Sigma E_n \to E_{n+1}.$$

We say that $E$ is an $\Omega$*-spectrum* if the adjoint maps

$$E_n \to \Omega E_{n+1}$$

are weak equivalences.

**Definition 3.5.9.** Let $E$ be a spectrum. We define the *associated* $\Omega$*-spectrum* to have spaces

$$F_n \coloneqq colim_m \, \Omega^m E_{n+m}$$

where the colimit is taken with respect to iterated applications of $\Omega$ to the adjoints

$$E_i \to \Omega E_{i+1}$$

of the structure maps

$$\Sigma E_i \to E_{i+1}.$$

The structure maps $\Sigma F_n \to F_{n+1}$ are obtained by shifting the indices, using the fact that the functor $\Omega$ commutes with colimits.

**Definition 3.5.10.** Let $E$ be an $\Omega$-spectrum. We call $E_0$ the 0*-space of* $E$ and denote it $\Omega^\infty E$.

*Remark 3.5.11.* The 0-space of an $\Omega$-spectrum is an infinite loop space.

**Definition 3.5.12.** Let

$$\Omega^\infty \colon \mathbf{Sp} \to \mathbf{Top}_\star$$

denote the functor which takes a spectrum and returns the 0-space of the associated Ω-spectrum. We call $\Omega^\infty$ the *infinite loop functor*.

*Remark 3.5.13.* One can consider $\Omega^\infty$ as the composite of two functors, the first sends a spectrum to its associated Ω-spectrum and the second sends the Ω-spectrum to its 0-space.

**Definition 3.5.14.** We define the functor

$$Q \colon \mathbf{Top}_\star \to \mathbf{Top}_\star$$

to be the composite

$$\Omega^\infty \circ \Sigma^\infty.$$

The functor $Q$ is sometimes called the *stabilization functor*, [May77].

*Remark 3.5.15.* The functor $Q$ takes a based topological space, forms the suspension spectrum, replaces it with the associated Ω-spectrum and then takes the 0-space.

# Chapter 4

# Algebras

## Introduction

Throughout the thesis we will be concerned with homology theories for associative algebras over a commutative ring with varying levels of extra structure. This chapter describes the different types of algebras we will consider, together with examples. In particular we recall the definitions of associative, commutative and involutive algebras over a commutative ring. We recall the definition of an augmented algebra and describe the generators of the $k$-module $A^{\otimes n}$ in this case. Finally, we recall the definition of a module over an algebra and the definition of the Kähler differentials.

## 4.1 Associative Algebras

Recall that $k$ is a commutative ring with unit.

**Definition 4.1.1.** An *associative $k$-algebra* is a ring $A$ together with a ring homomorphism

$$\varphi\colon k \to A,$$

called the *structure map*.

*Example 4.1.2.* The ring $k$ together with the identity map on $k$ is an associative $k$-algebra.

*Example 4.1.3.* The ring of $n \times n$ matrices with entries in $k$, denoted $M_n(k)$, together with the diagonal map,

$$\lambda \mapsto \begin{pmatrix} \lambda & 0 & \cdots & 0 \\ 0 & \lambda & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda \end{pmatrix}$$

is an associative $k$-algebra.

*Example 4.1.4.* Let $G$ be a group. The *group algebra of* $G$ over $k$, denoted $k[G]$, is an associative $k$-algebra. It consists of all finite $k$-linear combinations of elements of $G$. That is,

$$k[G] = \left\{ \sum_{g \in G} \lambda_g g \,:\, \lambda_g \in k \right\},$$

where all but finitely many $\lambda_g$ are zero.

Addition is defined by the formula

$$\sum_{g \in G} \lambda_g g + \sum_{g \in G} \mu_g g = \sum_{g \in G} (\lambda_g + \mu_g) g.$$

Scalar multiplication is defined by

$$\lambda \sum_{g \in G} \lambda_g g = \sum_{g \in G} (\lambda \lambda_g) g.$$

Multiplication is defined by

$$\left( \sum_{g \in G} \lambda_g g \right) \left( \sum_{g \in G} \mu_g g \right) = \sum_{g \in G,\, h \in G} (\lambda_g \mu_h)(gh).$$

The ring homomorphism

$$\varphi \colon k \to k[G]$$

is determined by

$$\lambda \mapsto \lambda e$$

where $e$ is the identity element of the group.

*Example 4.1.5.* The polynomial ring $k\left[x_1, \ldots x_n\right]$ together with the ring homomorphism

$$\varphi \colon k \to k\left[x_1, \ldots x_n\right]$$

determined by sending $\lambda \in k$ to the constant polynomial $\lambda$ is a $k$-algebra.

*Example 4.1.6.* Let $M$ be a $k$-module. Let

$$T(M) = k \oplus \bigoplus_{i=1}^{\infty} M^{\otimes i}.$$

We call $T(M)$ the *tensor module of* $M$. This module can be equipped with an algebra structure by defining the multiplication

$$M^{\otimes i} \otimes M^{\otimes j} \to M^{\otimes (i+j)}$$

to be concatenation:

$$\left(m_1 \otimes \cdots \otimes m_i\right) \otimes \left(m_{i+1} \otimes \cdots \otimes m_{i+j}\right) \mapsto \left(m_1 \otimes \cdots \otimes m_{i+j}\right).$$

The unit is $1_k$. We call $T(M)$ with this algebra structure the *tensor algebra of* $M$.

*Example 4.1.7.* Let $M$ be a $k$-module. Recall that the symmetric group $\Sigma_n$ acts on $M^{\otimes n}$ on the left by permuting the factors. Let $\sigma \in \Sigma_n$. The action is defined by

$$\sigma\left(m_1 \otimes \cdots \otimes m_n\right) = \left(m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(n)}\right).$$

Let $S^n(M)$ be the quotient of $M^{\otimes n}$ by this $\Sigma_n$-action. Define

$$S(M) \coloneqq \bigoplus_{n \geqslant 0} S^n(M)$$

with $S^0(M) = k$. When equipped with concatenation as multiplication, $S(M)$ is called the *symmetric algebra on $M$*.

*Example 4.1.8.* We change the $\Sigma_n$ action of Example 4.1.7 to include the sign of the permutation:

$$\sigma\left(m_1 \otimes \cdots \otimes m_n\right) = sgn(\sigma)\left(m_{\sigma^{-1}(1)} \otimes \cdots \otimes m_{\sigma^{-1}(n)}\right).$$

We define $E^n(M)$ to be the quotient of $M^{\otimes n}$ by this $\Sigma_n$-action. Define

$$E(M) \coloneqq \bigoplus_{n \geqslant 0} E^n(M)$$

with $E^0(M) = k$. When equipped with concatenation as multiplication we call $E(M)$ the *exterior algebra on $M$*. An element of $E^n(M)$ is usually denoted

$$m_1 \wedge \ldots \wedge m_n.$$

### 4.1.1 Tensor powers of algebras

**Definition 4.1.9.** Let $A$ be an associative $k$-algebra. For each $n \geqslant 2$, we define $A^{\otimes n}$ to be the $k$-module consisting of all finite $k$-linear combinations of elementary tensors

$$a_1 \otimes \cdots \otimes a_n$$

such that each $a_i \in A$.

## 4.2 Commutative Algebras

**Definition 4.2.1.** An associative $k$-algebra $A$ is called *commutative* if it is a commutative ring.

*Example 4.2.2.* The commutative ring $k$ is a commutative $k$-algebra.

*Example 4.2.3.* If $G$ is an abelian group then the group algebra $k[G]$ is a commutative $k$-algebra.

*Example 4.2.4.* The polynomial algebra $k\left[x_1, \ldots, x_n\right]$ is a commutative $k$-algebra.

*Example 4.2.5.* If $M$ is a finite dimensional, free $k$-module then the symmetric algebra $S(M)$ is a polynomial algebra.

## 4.3 Involutive Algebras

**Definition 4.3.1.** Let $A$ and $B$ be associative $k$-algebras with structure maps

$$\varphi_1 \colon k \to A$$

and

$$\varphi_2 \colon k \to B$$

respectively. An assignment $f \colon A \to B$ is called an *anti-homomorphism* if

- $f(1_A) = 1_B$,
- $f\big(\varphi_1(\lambda)a\big) = \varphi_2(\lambda)f(a)$ for all $\lambda \in k$ and $a \in A$,
- $f(a_1 + a_2) = f(a_1) + f(a_2)$ for all $a_1$, $a_2 \in A$ and
- $f(a_1a_2) = f(a_2)f(a_1)$.

**Definition 4.3.2.** An *involution* on an associative $k$-algebra $A$ is an anti-homomorphism

$$A \to A$$

which will be denoted by

$$a \mapsto \overline{a}.$$

An associative $k$-algebra equipped with an involution is called *involutive*.

*Remark 4.3.3.* Let $A$ be an associative $k$-algebra with structure map

$$\varphi \colon k \to A.$$

By the definition of anti-homomorphism, an involution on $A$ satisfies the following relations for all $a_1$, $a_2 \in A$ and $\lambda \in k$.

- $\overline{a_1 + a_2} = \overline{a_1} + \overline{a_2}$,
- $\overline{a_1a_2} = \overline{a_2}\,\overline{a_1}$,
- $\overline{\overline{a}} = a$ and
- $\overline{\varphi(\lambda)} = \varphi(\lambda)$.

*Remark 4.3.4.* We follow the convention of [Lod98, Section 5.2] in defining an involution. Our definition of involution is sometimes referred to as an anti-involution. See, for example, [ABB+04, Section 2.1.2].

*Example 4.3.5.* Any commutative $k$-algebra has a trivial involution defined by $\overline{a} = a$.

*Example 4.3.6.* Any group algebra $k[G]$ has an involution determined by

$$\overline{g} = g^{-1}.$$

## 4.4 Augmented Algebras

### 4.4.1 Definitions and properties

**Definition 4.4.1.** An associative $k$-algebra $A$ is said to be *augmented* if it is equipped with a $k$-algebra homomorphism $\varepsilon\colon A \to k$. Henceforth we will denote an augmented algebra by $A_\varepsilon$.

**Definition 4.4.2.** Let $A_\varepsilon$ be an augmented, associative $k$-algebra. We define the *augmentation ideal* of $A_\varepsilon$ to be $\mathrm{Ker}(\varepsilon)$ and we denote it by $I$.

We therefore have a short exact sequence of $k$-modules

$$0 \longrightarrow I \longrightarrow A_\varepsilon \xrightarrow{\varepsilon} k \longrightarrow 0.$$

**Proposition 4.4.3 ([LV12, Section 1.1.1]).** *Let $A_\varepsilon$ be an augmented, associative $k$-algebra with augmentation ideal $I$. There is an isomorphism of $k$-modules*

$$A_\varepsilon \cong I \oplus k.$$ □

*Remark 4.4.4.* It follows from Proposition 4.4.3 that every element $a \in A_\varepsilon$ can be written uniquely in the form $y + \lambda$ where $y \in I$ and $\lambda \in k$.

### 4.4.2 Tensor powers of an augmented algebra

Let $A_\varepsilon$ be an augmented, associative $k$-algebra with augmentation ideal $I$ and consider the $k$-module $A_\varepsilon^{\otimes n}$.

Consider an elementary tensor

$$a_1 \otimes \cdots \otimes a_n \in A_\varepsilon^{\otimes n}.$$

Following Remark 4.4.4, each $a_i$ can be written uniquely in the form $y_i + \lambda_i$ where $y_i \in I$ and $\lambda_i \in k$. We can therefore write

$$a_1 \otimes \cdots \otimes a_n = (y_1 + \lambda_1) \otimes \cdots \otimes (y_n + \lambda_n)\,.$$

Since the tensor product distributes over direct sums we can re-write this elementary tensor as a $k$-linear combination of elementary tensors in which each tensor factor is either an element $y_i$ of the augmentation ideal $I$, or an element $\lambda_i$ of the ground ring $k$.

*Example 4.4.5.* Consider the tensor $a_1 \otimes a_2$ in $A_\varepsilon^{\otimes 2}$. Since $A_\varepsilon$ is augmented we can write

$$a_1 \otimes a_2 = (y_1 + \lambda_1) \otimes (y_2 + \lambda_2)$$

with $y_i \in I$ and $\lambda_i \in k$. Distributing tensor products over the direct sums we see that

$$\begin{aligned}(y_1 + \lambda_1) \otimes (y_2 + \lambda_2) &= (y_1 \otimes (y_2 + \lambda_2)) + (\lambda_1 \otimes (y_2 + \lambda_2)) \\ &= (y_1 \otimes y_2) + (y_1 \otimes \lambda_1) \\ &\quad + (\lambda_1 \otimes y_2) + (\lambda_1 \otimes \lambda_2).\end{aligned}$$

**Definition 4.4.6.** Let $A_\varepsilon$ be an augmented, associative $k$-algebra with augmentation ideal $I$. An elementary tensor

$$a_1 \otimes \cdots \otimes a_n$$

in $A_\varepsilon^{\otimes n}$ is called a *split tensor* if $a_i \in I$ or $a_i \in k$ for each $1 \leqslant i \leqslant n$.

*Remark 4.4.7.* It follows that for an augmented, associative $k$-algebra $A_\varepsilon$, the $k$-module $A_\varepsilon^{\otimes n}$ is generated by all finite $k$-linear combinations of split tensors.

In fact, one can go further. Let $a_1 \otimes \cdots \otimes a_n$ be a split tensor in $A_\varepsilon^{\otimes n}$. Let $\mathcal{I} = \{i_1, \ldots, i_k\} \subseteq \underline{n}$ be the subset of indices such that $a_i \in I$ if and only if $i \in \mathcal{I}$. The tensor factors $a_i$ for $i \notin \mathcal{I}$ are elements of the ground ring $k$. Since the tensor product is $k$-linear, $a_1 \otimes \cdots \otimes a_n$ is equal to

$$\left(\prod_{i \notin \mathcal{I}} a_i\right) (a_1' \otimes \cdots \otimes a_n')$$

where

$$a_i' = \begin{cases} a_i & i \in \mathcal{I} \\ 1_k & i \notin \mathcal{I}. \end{cases}$$

If $\mathcal{I}$ is empty then $a_1 \otimes \cdots \otimes a_n$ is equal to

$$\left(\prod_{i \notin \mathcal{I}} a_i\right) \left(\underbrace{1_k \otimes \cdots \otimes 1_k}_{n}\right).$$

*Example 4.4.8.* Let

$$y_1 \otimes \lambda_2 \otimes \lambda_3 \otimes y_4$$

be a $k$-module generator of $A_\varepsilon^{\otimes 4}$, where $y_1$, $y_4 \in I$ and $\lambda_2$, $\lambda_3 \in k$. This is equal in $A_\varepsilon^{\otimes 4}$ to

$$(\lambda_2 \lambda_3) (y_1 \otimes 1_k \otimes 1_k \otimes y_4).$$

Let $\lambda_1 \otimes \lambda_2 \otimes \lambda_3$ be a $k$-module generator of $A_\varepsilon^{\otimes 3}$ where $\lambda_i \in k$ for each $1 \leqslant i \leqslant 3$. This elementary tensor is equal in $A_\varepsilon^{\otimes 3}$ to

$$(\lambda_1 \lambda_2 \lambda_3) (1_k \otimes 1_k \otimes 1_k).$$

We can therefore write any generator of $A_\varepsilon^{\otimes n}$ as a $k$-linear combination of elementary tensors $a_1 \otimes \cdots \otimes a_n$ such that, for each $1 \leqslant i \leqslant n$, either $a_i \in I$ or $a_i = 1_k$.

**Definition 4.4.9.** Let $A_\varepsilon$ be an augmented, associative $k$-algebra with augmentation ideal $I$. A *basic tensor* in $A_\varepsilon^{\otimes n}$ is an elementary tensor

$$a_1 \otimes \cdots \otimes a_n$$

such that either $a_i \in I$ or $a_i = 1_k$ for each $1 \leqslant i \leqslant n$.

*Remark 4.4.10.* It follows that for an augmented, associative $k$-algebra $A_\varepsilon$, the $k$-module $A_\varepsilon^{\otimes n}$ is generated by all finite $k$-linear combinations of basic tensors.

**Definition 4.4.11.** Let $a_1 \otimes \cdots \otimes a_n$ be a basic tensor of $A_\varepsilon^{\otimes n}$. A tensor factor $a_i$ is called *trivial* if $a_i = 1_k$ and is called *non-trivial* if $a_i \in I$.

## 4.5 Modules over an Algebra

**Definition 4.5.1.** A *bimodule* over an associative $k$-algebra $A$ is a $k$-module $M$ with a $k$-linear action of $A$ on the left and right satisfying

$$(am)a' = a(ma')$$

where $a$, $a' \in A$ and $m \in M$.

An $A$-bimodule $M$ is said to be *symmetric* if $am = ma$ for all $a \in A$ and $m \in M$.

*Remark 4.5.2.* In general, the ground ring $k$ is not an $A$-bimodule. However, for an augmented, associative $k$-algebra $A_\varepsilon$, $k$ is an $A_\varepsilon$-bimodule with the structure maps

$$A_\varepsilon \times k \to k$$

and

$$k \times A_\varepsilon \to k$$

obtained from the augmentation map $\varepsilon$ and the multiplication map $\mu\colon k \times k \to k$ in the ground ring. That is, the structure maps are the composites

$$\begin{array}{ccc} A_\varepsilon \times k & \longrightarrow & k \\ {\scriptstyle \varepsilon \times 1_k}\downarrow & \nearrow_{\mu} & \\ k \times k & & \end{array}$$

and

$$\begin{array}{ccc} k \times A_\varepsilon & \longrightarrow & k \\ {\scriptstyle 1_k \times \varepsilon}\downarrow & \nearrow_{\mu} & \\ k \times k & & \end{array}$$

**Definition 4.5.3.** Let $A$ be an involutive $k$-algebra. We define an *involutive* $A$-bimodule to be an $A$-bimodule $M$ together with an endomorphism

$$M \to M$$

denoted by

$$m \mapsto \overline{m}$$

such that

$$\overline{ama'} = \overline{a'}\ \overline{m}\ \overline{a}$$

for all $a$, $a' \in A$ and all $m \in M$.

### 4.5.1 Kähler differentials

**Definition 4.5.4.** Let $A$ be a unital, commutative $k$-algebra with structure map $\varphi\colon k \to A$. Let $a$, $b \in A$ and $\lambda \in k$.

Let $\Omega^1_{A|k}$ denote the $A$-module generated by symbols $da$ for $a \in A$ subject to the relations

- $d(\varphi(\lambda)) = 0$
- $d(\varphi(\lambda)a) = \varphi(\lambda)d(a)$,
- $d(a+b) = d(a) + d(b)$ and
- $d(ab) = a(d(b)) + b(d(a))$.

The $A$-module $\Omega^1_{A|k}$ is called the *module of Kähler differentials*.

*Remark 4.5.5.* There are two alternative constructions of the module of Kähler differentials. Firstly, let $\mu\colon A \otimes A \to A$ be the multiplication map. Let $J \coloneqq \operatorname{Ker}(\mu)$. Then

$$\Omega^1_{A|k} \cong \frac{J}{J^2}.$$

Secondly, consider the $A$-module $A \otimes_k A$. Denote by

$$\langle ab \otimes c - a \otimes bc + ac \otimes b \rangle$$

the $A$-submodule generated by all $k$-linear combinations of the form

$$ab \otimes c - a \otimes bc + ac \otimes b.$$

Then

$$\Omega^1_{A|k} \cong \frac{A \otimes_k A}{\langle ab \otimes c - a \otimes bc + ac \otimes b \rangle}.$$

**Definition 4.5.6.** Let $dA$ denote the $A$-submodule of $\Omega^1_{A|k}$ generated by all differences of the form

$$[1_A \otimes a] - [a \otimes 1_A].$$

### 4.5.2 Higher order Kähler differentials

**Definition 4.5.7.** Let $A$ be a unital commutative $k$-algebra. The $n^{th}$ *module of Kähler differentials*, denoted $\Omega^n_{A|k}$, is defined to be the $A$-module

$$E^n\left(\Omega^1_{A|k}\right)$$

in the notation of Example 4.1.8.

*Remark 4.5.8.* The $A$-module $\Omega^n_{A|k}$ is spanned by elements of the form

$$da_1 \wedge \ldots \wedge da_n.$$

# Chapter 5

# Classical Homology Theories for Algebras

## Introduction

We recall some classical homology theories for algebras over a commutative ring. We define Hochschild homology and provide a splitting of the Hochschild complex for an augmented, associative $k$-algebra.

We recall the definition of the shuffle complex as a subcomplex of the Hochschild complex. We recall the Eulerian idempotents in the case $k \supseteq \mathbb{Q}$ and describe a decomposition of the Hochschild complex into a direct sum following Gerstenhaber and Schack.

We recall Harrison homology and André-Quillen homology for a commutative $k$-algebra, together with their connections to Hochschild homology.

Finally, we recall the definition of cyclic homology for an associative $k$-algebra.

## 5.1 Hochschild Homology of an Associative Algebra

### 5.1.1 A brief history

Hochschild cohomology for associative algebras over a field was introduced by Hochschild [Hoc45] in order to classify the extensions of an associative algebra. Hochschild homology and cohomology for algebras over a commutative ring were introduced by Cartan and Eilenberg [CE56].

### 5.1.2 Hochschild homology

The material in this subsection can be found in Loday [Lod98, Chapter 1].

**Definition 5.1.1.** Let $A$ be an associative $k$-algebra and let $M$ be an $A$-bimodule. Let $C_\star(A, M)$ be the simplicial $k$-module with

$$C_n(A, M) = M \otimes A^{\otimes n},$$

the $k$-module generated $k$-linearly by all elementary tensors $(m \otimes a_1 \otimes \cdots \otimes a_n)$. We define the face maps

$$\partial_i \colon C_n(A, M) \to C_{n-1}(A, M)$$

and the degeneracy maps

$$s_j \colon C_n(A, M) \to C_{n+1}(A, M)$$

on generators by

- $\partial_0(m \otimes a_1 \otimes \cdots \otimes a_n) = (ma_1 \otimes a_2 \otimes \cdots \otimes a_n)$,
- $\partial_i(m \otimes a_1 \otimes \cdots \otimes a_n) = (m \otimes a_1 \otimes \cdots \otimes a_i a_{i+1} \otimes \cdots \otimes a_n)$ for $1 \leqslant i \leqslant n-1$,
- $\partial_n(m \otimes a_1 \otimes \cdots \otimes a_n) = (a_n m \otimes a_1 \otimes \cdots \otimes a_{n-1})$,
- $s_0(m \otimes a_1 \otimes \cdots \otimes a_n) = (m \otimes 1_A \otimes a_1 \otimes \cdots \otimes a_n)$,
- $s_j(m \otimes a_1 \otimes \cdots \otimes a_n) = \big(m \otimes a_1 \otimes \cdots \otimes 1_A \otimes a_{j+1} \otimes \cdots \otimes a_n\big)$ for $1 \leqslant j \leqslant n$.

**Definition 5.1.2.** Let

$$b \coloneqq \sum_{i=0}^{n} (-1)^i \partial_i \colon C_n(A, M) \to C_{n-1}(A, M)$$

We will refer to $b$ as the *Hochschild boundary map.*

**Definition 5.1.3.** The *Hochschild complex*, which by abuse of notation we will denote $C_\star(A, M)$, is the associated chain complex of the simplicial $k$-module of Definition 5.1.1:

$$\cdots \xrightarrow{\ b\ } M \otimes A^{\otimes 2} \xrightarrow{\ b\ } M \otimes A \xrightarrow{\ b\ } M \longrightarrow 0.$$

**Definition 5.1.4.** We define the $n^{th}$ *Hochschild homology group of $A$ with coefficients in $M$*, denoted $HH_n(A, M)$, to be

$$H_n\big(C_\star(A, M)\big).$$

*Remark 5.1.5.* Observe that we can consider $A$ as an $A$-bimodule. When we set $M = A$ we will denote the Hochschild complex by $C_\star(A)$ and the Hochschild homology by $HH_\star(A)$.

**Proposition 5.1.6 ([Lod98, Subsection 1.1.6]).** *The Hochschild homology of an associative $k$-algebra $A$ with coefficients in $M$ in degree zero is the* module of coinvariants *of $M$ by $A$.*

*In particular, if we take $M = A$, then*

$$HH_0(A) = \frac{A}{[A, A]},$$

*where $[A, A]$ is the commutator $k$-submodule of $A$.* □

**Proposition 5.1.7 ([Lod98, Proposition 1.1.10]).** *Let $A$ be a unital and commutative $k$-algebra. Let $M$ be a symmetric $A$-bimodule. There is a canonical isomorphism of $k$-modules*

$$HH_1(A, M) \cong M \otimes_A \Omega^1_{A|k}.$$ □

## 5.2 Normalized Hochschild Homology for Augmented Algebras

### 5.2.1 Hochschild complex for an augmented algebra

Let $A_\varepsilon$ be an augmented, associative $k$-algebra with augmentation ideal $I$. We observe, using Definition 4.4.9, that

$$C_n(A_\varepsilon, M) = M \otimes A_\varepsilon^{\otimes n}$$

is generated $k$-linearly by all basic tensors $m \otimes a_1 \otimes \cdots \otimes a_n$.

### 5.2.2 The degenerate subcomplex $D_\star(A, M)$

For the Hochschild complex $C_\star(A, M)$, the degenerate subcomplex in degree $n$ is defined, as in Subsection 2.5.3, to consist of finite $k$-linear combinations of elements in the image of degeneracy maps. That is,

$$D_n(A, M) = \left\{ \sum_{i \text{ finite}} \lambda_i s_{j_i} \left( m^i \otimes a_1^i \otimes \cdots \otimes a_{n-1}^i \right) : \lambda_i \in k,\ 0 \leqslant j_i \leqslant n-1 \right\}.$$

The $k$-module $D_n(A, M)$ is spanned by tensors $(m \otimes a_1 \otimes \cdots \otimes a_n)$ such that at least one $a_i$ is equal to $1_A \in A$, for $1 \leqslant i \leqslant n$. That is,

$$D_n(A, M) = \left\{ \sum_{i \text{ finite}} \lambda_i \left( m^i \otimes a_1^i \otimes \cdots \otimes a_n^i \right) : \forall i\ \exists\ 1 \leqslant j \leqslant n \text{ s.t. } a_j^i = 1_A \right\}.$$

In the case of an augmented, associative $k$-algebra we can use the notion of basic tensor from Definition 4.4.9 to provide another description of the degenerate subcomplex. Let $A_\varepsilon$ be an augmented, associative $k$-algebra and $M$ an $A_\varepsilon$-bimodule. Then $D_n(A_\varepsilon, M)$ is

$$\left\{ \sum_{i \text{ finite}} \lambda_i \left( m^i \otimes a_1^i \otimes \cdots \otimes a_n^i \right) : \forall i\ \left( m^i \otimes a_1^i \otimes \cdots \otimes a_n^i \right) \text{ is basic and } \exists\ 1 \leqslant j \leqslant n \text{ s.t. } a_j^i = 1_k \right\}.$$

### 5.2.3 The chain complex $C_\star(I, M)$

We will demonstrate that there is a well-defined chain complex $C_\star(I, M)$ where $M$ is a symmetric $A_\varepsilon$-bimodule. This is the Hochschild complex built from the augmentation ideal $I$. For the case where $M$ is flat over $k$, $C_\star(I, M)$ is a subcomplex of $C_\star(A_\varepsilon, M)$ and we obtain a splitting result.

**Proposition 5.2.1.** *Let $A_\varepsilon$ be an augmented, associative $k$-algebra with augmentation ideal $I$. Let $M$ be an $A_\varepsilon$-bimodule. There is a well-defined chain complex $C_\star(I, M)$ with boundary map $b$.*

*Proof.* Since $I$ is a $k$-subalgebra of $A_\varepsilon$ we observe that the Hochschild boundary map, whose actions on elementary tensors either multiply tensor factors together or act on the coefficients by an element of $I$ is well-defined. $\square$

**Lemma 5.2.2.** *Let $A_\varepsilon$ be an augmented, associative $k$-algebra with augmentation ideal $I$. Let $i\colon I \to A$ be the inclusion of $I$ into $A$ as a $k$-submodule. We claim that*

$$i^{\otimes n}\colon I^{\otimes n} \to A^{\otimes n}$$

*is an injective map of $k$-modules.*

*Proof.* Recall from Proposition 4.4.3 that there is an isomorphism of $k$-modules

$$A_\varepsilon \cong I \oplus k.$$

Consider the $k$-module $A_\varepsilon^{\otimes n}$. By distributing tensor products over direct sums we obtain an isomorphism of $k$-modules

$$A_\varepsilon^{\otimes n} \cong I^{\otimes n} \oplus \left(\bigoplus_n I^{\otimes n-1}\right) \oplus \cdots \oplus \left(\bigoplus_n I\right) \oplus k^{\otimes n}.$$

In particular, $I^{\otimes n}$ can be considered as a subobject of $A_\varepsilon^{\otimes n}$ in a canonical way and the map

$$i^{\otimes n}\colon I^{\otimes n} \to A^{\otimes n}$$

is an isomorphism onto the first summand. Hence $i^{\otimes n}$ is injective. $\square$

**Lemma 5.2.3.** *Let $M$ be an $A_\varepsilon$-bimodule which is flat over $k$. The complex $C_\star(I, M)$ is a subcomplex of $C_\star(A_\varepsilon, M)$.*

*Proof.* By Lemma 5.2.2, $i^{\otimes n}\colon I^{\otimes n} \to A_\varepsilon^{\otimes n}$ is an injective map. Since $M$ is flat over $k$ the map

$$id_M \otimes i^{\otimes n}\colon M \otimes I^{\otimes n} \to M \otimes A_\varepsilon^{\otimes n}$$

is an injective map for $n \geqslant 1$.

Since the Hochschild boundary map $b$ is well-defined on $C_\star(I, M)$ we deduce that

$$(id_M \otimes i^{\otimes n-1}) \circ b = b \circ (id_M \otimes i^{\otimes n})$$

and so $C_\star(I, M)$ is a subcomplex of $C_\star(A_\varepsilon, M)$. $\square$

**Proposition 5.2.4.** *Let $A_\varepsilon$ be an augmented, associative $k$-algebra. Let $M$ be an $A_\varepsilon$-bimodule which is flat over $k$. There is an isomorphism of chain complexes*

$$C_\star(A_\varepsilon, M) \cong C_\star(I, M) \oplus D_\star(A_\varepsilon, M).$$

*Proof.* Recall from Subsection 5.2.2 that the degenerate subcomplex $D_\star(A_\varepsilon, M)$ is generated by the elementary tensors with at least one trivial tensor factor.

Consider the quotient of chain complexes

$$\frac{C_\star(A_\varepsilon, M)}{D_\star(A_\varepsilon, M)}.$$

We can choose representatives such that the quotient complex is generated $k$-linearly, in degree $n$, by equivalence classes of the form

$$[m \otimes y_1 \otimes \cdots \otimes y_n]$$

where each $y_i$ is an element of the augmentation ideal, $I$, and the boundary map is induced from the Hochschild boundary map, $b$.

There is an isomorphism of chain complex

$$\frac{C_\star(A_\varepsilon, M)}{D_\star(A_\varepsilon, M)} \cong C_\star(I, M)$$

determined by the map

$$[m \otimes y_1 \otimes \cdots \otimes y_n] \mapsto m \otimes y_1 \otimes \cdots \otimes y_n.$$

The inverse is determined by the map that sends an elementary tensor $m \otimes y_1 \otimes \cdots \otimes y_n$ to its equivalence class in the quotient.

However, Corollary 2.5.15 tells us that the quotient complex is isomorphic to the normalized subcomplex $N_\star(A_\varepsilon, M)$.

We deduce that there is an isomorphism

$$C_\star(I, M) \cong N_\star(A_\varepsilon, M).$$

We deduce from Proposition 2.5.14 that we have a splitting

$$C_\star(A_\varepsilon, M) \cong C_\star(I, M) \oplus D_\star(A_\varepsilon, M)$$

as required. □

**Corollary 5.2.5.** *Under the hypotheses of Proposition 5.2.4, the canonical projection map*

$$C_\star(A_\varepsilon, M) \to C_\star(I, M)$$

*is a quasi-isomorphism. The inclusion of chain complexes*

$$C_\star(I, M) \to C_\star(A_\varepsilon, M)$$

*induces the inverse map on homology. In particular,*

$$HH_\star(A_\varepsilon, M) \cong H_\star(C_\star(I, M)).$$

*Proof.* This follows upon combining Proposition 5.2.4 and Corollary 2.5.16. □

## 5.3 Shuffles and the Hochschild Complex

### 5.3.1 Shuffles and shuffle operators

We include some background material about shuffles and their relationship with the Hochschild complex. The details included here can be found in Loday [Lod98, Chapter 4]. We begin with the definition of a *shuffle*. Let $\Sigma_n$ denote the symmetric group on the set $\{1, \ldots, n\}$.

**Definition 5.3.1.** For $1 \leqslant i \leqslant n-1$, a permutation $\sigma \in \Sigma_n$ is called an *i-shuffle* if

$$\sigma(1) < \sigma(2) < \cdots < \sigma(i)$$

and

$$\sigma(i+1) < \sigma(i+2) < \cdots < \sigma(n).$$

**Definition 5.3.2.** We define an element $sh_{i,n-i}$ in the group algebra $k[\Sigma_n]$ by

$$sh_{i,n-i} = \sum_{\substack{i-\text{shuffles} \\ \sigma \in \Sigma_n}} sgn(\sigma)\sigma.$$

That is, we take the signed sum over all the $i$-shuffles in $\Sigma_n$.

**Definition 5.3.3.** We define the *total shuffle operator*, $sh_n$, in $k[\Sigma_n]$ to be the sum of the elements $sh_{i,n-i}$. That is,

$$sh_n = \sum_{i=1}^{n-1} sh_{i,n-i}.$$

### 5.3.2 Shuffles and the Hochschild complex

For this subsection we will restrict to the case where $A$ is commutative and $M = A$. In particular, $C_n(A) = A \otimes A^{\otimes n}$ and we denote an elementary tensor in this module by $a_0 \otimes a_1 \otimes \cdots \otimes a_n$, where $a_0$ is considered to be an element of the coefficient module. We will explain in Subsection 5.3.3 how to deal with coefficients in an arbitrary symmetric $A$-bimodule $M$.

Recall firstly that the symmetric group $\Sigma_n$ acts on the left of $C_n(A)$ by permuting the tensor factors. Let $\sigma \in \Sigma_n$. The action is determined by

$$\sigma(a_0 \otimes a_1 \cdots \otimes a_n) = a_0 \otimes a_{\sigma^{-1}(1)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$$

and extending $k$-linearly.

With this action we see that the total shuffle operator $sh_n$ of Definition 5.3.3 is an endomorphism

$$sh_n \colon C_n(A) \to C_n(A).$$

**Proposition 5.3.4 ([Lod98, Section 4.2.8]).** *The total shuffle operators of Definition 5.3.3*

$$sh_n \colon C_n(A) \to C_n(A)$$

*form a chain map. That is,*

$$b \circ sh_n = sh_{n-1} \circ b$$

*for all $n \geqslant 1$.* □

**Definition 5.3.5.** Let $a_0 \otimes a_1 \otimes \cdots \otimes a_n$ be an elementary tensor of the $k$-module $C_n(A)$. Let

$$\sum_{i=1}^{n-1} sh_{i,n-i}(a_0 \otimes a_1 \otimes \cdots \otimes a_n)$$

denote the linear combination of tensors obtained by applying the total shuffle operator. Let $Sh_n(A)$ denote the submodule of $C_n(A)$ generated by all such $k$-linear combinations obtained from the $k$-module generators of $C_n(A)$.

**Definition 5.3.6.** The *shuffle complex*, $Sh_\star(A)$ is defined to have the module $Sh_n(A)$ of Definition 5.3.5 in degree $n$ with boundary map induced from $b$, the Hochschild differential.

### 5.3.3 The rational and commutative case

In the case where the ground ring $k$ contains $\mathbb{Q}$ and $A$ is a commutative $k$-algebra, we can decompose the Hochschild complex into a direct sum of subcomplexes.

Following Loday [Lod98, 4.5.4], when $k \supseteq \mathbb{Q}$, the total shuffle operator is the multiplication in the *commutative graded cotensor Hopf algebra of $A$*.

It follows from the theory of such objects that we have a set of elements of $\mathbb{Q}[\Sigma_n]$ called the *Eulerian idempotents*. Let $e_{\Sigma_n}$ denote the identity element in $\mathbb{Q}[\Sigma_n]$.

**Proposition 5.3.7 ([Lod98, Proposition 4.5.3]).** *For $n \geqslant 1$ and $1 \leqslant i \leqslant n$, there exist elements $e_n^{(i)} \in \mathbb{Q}[\Sigma_n]$ satisfying*

1. $e_{\Sigma_n} = e_n^{(1)} + \cdots + e_n^{(n)}$,
2. $e_n^{(i)} e_n^{(j)} = 0$ *if* $i \neq j$,
3. $e_n^{(i)} e_n^{(i)} = e_n^{(i)}$. □

**Definition 5.3.8.** The elements $e_n^{(i)}$ of Proposition 5.3.7 are called the *Eulerian idempotents.*

*Example 5.3.9 ([Lod98, 4.5.7]).* The following are examples of the Eulerian idempotents, where permutations are written in cycle notation.

- $e_1^{(1)} = e_{\Sigma_1}$,
- $e_2^{(1)} = \frac{1}{2}\left(e_{\Sigma_2} + (1\,2)\right)$,

- $e_2^{(2)} = \frac{1}{2}\left(e_{\Sigma_2} - (1\,2)\right)$,
- $e_3^{(1)} = \frac{1}{3}e_{\Sigma_3} - \frac{1}{6}((1\,2\,3) + (1\,3\,2) - (1\,2) - (2\,3)) - \frac{1}{3}(1\,3)$.

As the Eulerian idempotents $e_n^{(i)}$ are elements of $\mathbb{Q}[\Sigma_n]$, they act on $A^{\otimes n}$. This is extended to an action on $C_n(A, M) = M \otimes A^{\otimes n}$ by $1_M \otimes e_n^{(i)}$.

**Proposition 5.3.10 ([Lod98, Proposition 4.5.9]).** *For $n \geqslant 1$ and $1 \leqslant i \leqslant n-1$,*

$$b \circ e_n^{(i)} = e_{n-1}^{(i)} \circ b$$

*and*

$$b \circ e_n^{(n)} = 0. \qquad \square$$

*Remark 5.3.11.* From Proposition 5.3.10 we deduce that for each $i \geqslant 1$ we have a subcomplex of the Hochschild complex, $e_\star^{(i)}C_\star(A, M)$. We can describe this complex as follows. Let $(m \otimes a_1 \otimes \cdots \otimes a_n)$ be a $k$-module generator of $C_n(A, M)$. Let $e_n^{(i)}(m \otimes a_1 \otimes \cdots \otimes a_n)$ denote the $k$-linear combination of tuples obtained by applying the Eulerian idempotent to our generator. This is simply a linear combination of permutations of our generator. The subcomplex $e_\star^{(i)}C_\star(A, M)$ is generated in degree $n$ by all linear combinations of the form $e_n^{(i)}(m \otimes a_1 \otimes \cdots \otimes a_n)$, arising from $k$-module generators of $C_n(A, M)$.

Combining the idempotent properties of Proposition 5.3.7 and the compatibility with the Hochschild boundary map from Proposition 5.3.10 we deduce the following theorem of Gerstenhaber and Schack, [GS87].

**Theorem 5.3.12 ([Lod98, Theorem 4.5.10]).** *Let $k$ be a commutative ring containing $\mathbb{Q}$. Let $A$ be a $k$-algebra and let $M$ be a symmetric $A$-bimodule. The Eulerian idempotents $e_n^{(i)}$ naturally split the Hochschild complex into a direct sum of subcomplexes:*

$$C_\star(A, M) = \bigoplus_{i \geqslant 0} e_\star^{(i)} C_\star(A, M).$$

*Thus, this provides a decomposition of Hochschild homology:*

$$H_n(A, M) = H_n(e_n^{(1)}C_n(A, M)) \oplus \cdots \oplus H_n(e_n^{(n)}C_n(A, M)). \qquad \square$$

*Remark 5.3.13.* This decomposition is known as the *Hodge decomposition* or, sometimes, as the *$\lambda$-decomposition* of Hochschild homology.

## 5.4 Harrison Homology for Commutative Algebras

### 5.4.1 A brief history

*Harrison cohomology* for commutative algebras was introduced by Harrison in 1962 [Har62] to classify extensions of commutative algebras.

In 1968, Barr [Bar68] gave a description of *Harrison homology* and proved that, over a field of characteristic zero, Harrison homology is a direct summand of Hochschild homology.

In papers of 1987 [GS87] and 1991 [GS91], Gerstenhaber and Schack prove that this summand is part of the Hodge decomposition of Hochschild homology described in Theorem 5.3.12.

### 5.4.2 The Harrison complex as a quotient of the Hochschild complex

Recall the shuffle complex, $Sh_\star(A)$, of Definition 5.3.6.

**Definition 5.4.1.** Let $A$ be a commutative $k$-algebra. The *Harrison complex* of $A$ is defined to be the quotient of the Hochschild complex $C_\star(A)$ by the shuffle complex $Sh_\star(A)$. That is,

$$CHarr_\star(A) := \frac{C_\star(A)}{Sh_\star(A)}$$

with boundary map induced from the Hochschild boundary map.

**Definition 5.4.2.** Let $A$ be a commutative $k$-algebra. The $n^{th}$ *Harrison homology* of $A$, denoted $Harr_n(A)$, is defined to be

$$H_n\left(CHarr_\star(A)\right).$$

**Definition 5.4.3.** Let $A$ be a commutative $k$-algebra. Let $M$ be a symmetric $A$-bimodule. We define the *Harrison complex of $A$ with coefficients in $M$* to be the complex

$$M \otimes_A CHarr_\star(A).$$

We denote the Harrison complex of $A$ with coefficients in $M$ by $CHarr_\star(A, M)$.

**Definition 5.4.4.** The $n^{th}$ *Harrison homology of $A$ with coefficients in $M$* is defined to be

$$H_n(CHarr_\star(A, M)).$$

and we denote it by $Harr_n(A, M)$.

### 5.4.3 The Harrison complex as a subcomplex of $C_\star(A)$

In the case where $k$ is a commutative ring containing $\mathbb{Q}$ we can also express the Harrison complex as a subcomplex of the Hochschild complex. Recall that in Theorem 5.3.12 we stated the Hodge decomposition of the Hochschild complex into a direct sum of subcomplexes.

In this case the summand $e_\star^{(1)}C_\star(A, M)$ is naturally isomorphic to the Harrison complex.

**Proposition 5.4.5 ([Bar68, Proposition 2.5]).** *For each $n \geqslant 2$ there is an $e_n \in \mathbb{Q}[\Sigma_n]$ with the following properties:*

- *$e_n$ is a polynomial in $sh_n$ without constant term,*
- *$sgn(e_n) = e_{\Sigma_n}$ where $sgn$ is extended linearly over $\mathbb{Q}[\Sigma_n]$,*
- *$b \circ e_n = e_{n-1} \circ b$,*
- *$e_n^2 = e_n$,*
- *$e_n \circ s_{i,n-i} = s_{i,n-i}$ for $1 \leqslant i \leqslant n-1$.* □

The final property implies that $(e_{\Sigma_n} - e_n)$ vanishes on shuffles for each $n \geqslant 2$.

Barr observes that there are natural isomorphisms of chain complexes

$$e_\star C_\star(A) \cong Sh_\star(A)$$

and

$$(e_{\Sigma_n} - e_\star)\, C_\star(A) \cong CHarr_\star(A).$$

**Theorem 5.4.6 ([GS87, Theorem 1.3 (i)]).** $e_n = e_n^{(2)} + \cdots + e_n^{(n)}$. □

We deduce that $e_{\Sigma_n} - e_n = e_n^{(1)}$ and so $e_n^{(1)}$ vanishes on shuffles and $e_\star^{(1)} C_\star(A, M)$ is naturally isomorphic to the Harrison complex.

**Corollary 5.4.7.** *There is a natural isomorphism of $k$-modules*

$$Harr_n(A, M) \cong H_n\left(e_\star^{(1)} C_\star(A, M)\right)$$

*for each $n \geqslant 0$.* □

## 5.5 André-Quillen Homology for Commutative Algebras

### 5.5.1 A brief history

The basis of André-Quillen homology comes from work of Lichtenbaum and Schlessinger [LS67]. The homology theory itself was introduced independently by Quillen [Qui70] and André [And74]. It is used to analyse the smoothness of an algebra and is closely related to Hochschild homology.

### 5.5.2 André-Quillen homology

**Definition 5.5.1.** Let $A$ be commutative $k$-algebra. A *simplicial resolution* of $A$ is an acyclic, $A$-augmented, simplicial commutative $k$-algebra, $P_\star$. Such a resolution is called *free* if each $P_n$ is a symmetric algebra over a free $k$-module.

As discussed in [Lod98, Section 3.5], given a commutative $k$-algebra, a free simplicial resolution always exists and any two are homotopy equivalent.

**Definition 5.5.2.** Let $A$ be a commutative $k$-algebra. Let $P_\star$ be a free resolution of $A$. We define

$$\mathbb{L}_n(A) := \Omega^1_{P_n|k} \otimes_{P_n} A$$

and

$$\mathbb{L}^q_n(A) := \Omega^q_{P_n|k} \otimes_{P_n} A.$$

**Definition 5.5.3.** Let $A$ be a commutative $k$-algebra and let $M$ be an $A$-module. The $n^{th}$ *André-Quillen homology of $A$ with coefficients in $M$* is defined to be

$$AQ_n(A, M) := H_n\left(\mathbb{L}_\star(A) \otimes_A M\right).$$

The $q^{th}$ *higher André-Quillen homology* is defined to be

$$AQ^q_n(A, M) := H_n\left(\mathbb{L}^q_\star(A) \otimes_A M\right).$$

**Theorem 5.5.4 ([Lod98, Theorem 3.5.8]).** *Let $A$ be a commutative $k$-algebra. Suppose $k$ contains $\mathbb{Q}$. There is an isomorphism of $k$-modules*

$$HH_n(A) \cong \bigoplus_{p+q=n} AQ^q_p(A)$$

*for each $n \geqslant 0$.* □

**Theorem 5.5.5 ([Lod98, Proposition 4.2.10]).** *Suppose that $k$ contains $\mathbb{Q}$. Let $A$ be a flat commutative $k$-algebra. There is an isomorphism of $k$-modules*

$$AQ_{n-1}(A) \cong Harr_n(A)$$

*for $n \geqslant 1$.* □

## 5.6 Cyclic Homology of an Associative Algebra

### 5.6.1 A brief history

Cyclic homology for associative algebras was introduced independently by Connes [Con83], working on non-commutative geometry, and Tsygan [Tsy83], working on matrix Lie algebras.

### 5.6.2 Cyclic homology

**Definition 5.6.1.** For $n \geqslant 1$, we define the *cyclic group of order $n$* to be

$$C_n := \left\langle t_n \mid t_n^n \right\rangle.$$

**Definition 5.6.2.** Let $n \geqslant 1$. We define an action of $C_n$ on $A^{\otimes n}$ to be determined $k$-linearly by

$$t_n(a_1 \otimes \cdots \otimes a_n) = (-1)^{n+1}(a_n \otimes a_1 \otimes \cdots \otimes a_{n-1})$$

on elementary tensors.

**Definition 5.6.3.** Let $n \geqslant 1$. We define the $n^{th}$ *norm operator* to be

$$N_n \coloneqq \sum_{i=0}^{n-1} t_n^i \colon A^{\otimes n} \to A^{\otimes n}.$$

**Definition 5.6.4.** For each $n \geqslant 1$, let

$$b' \colon C_{n+1}(A) \to C_n(A)$$

be defined by

$$b' \coloneqq \sum_{i=0}^{n-1} (-1)^i \partial_i.$$

**Definition 5.6.5.** Let $A$ be an associative $k$-algebra. We define the *cyclic bicomplex of* $A$, denoted $CC_{\star,\star}(A)$, to be the bicomplex

$$\begin{array}{ccccccccc}
\vdots & & \vdots & & \vdots & & \vdots & & \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
A^{\otimes 3} & \xleftarrow{1-t_3} & A^{\otimes 3} & \xleftarrow{N_3} & A^{\otimes 3} & \xleftarrow{1-t_3} & A^{\otimes 3} & \xleftarrow{N_3} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
A^{\otimes 2} & \xleftarrow{1-t_2} & A^{\otimes 2} & \xleftarrow{N_2} & A^{\otimes 2} & \xleftarrow{1-t_2} & A^{\otimes 2} & \xleftarrow{N_2} & \cdots \\
\downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \downarrow{\scriptstyle b} & & \downarrow{\scriptstyle -b'} & & \\
A & \xleftarrow{0} & A & \xleftarrow{1} & A & \xleftarrow{0} & A & \xleftarrow{1} & \cdots
\end{array}$$

**Definition 5.6.6.** Let $A$ be an associative $k$-algebra. We define the $n^{th}$ *cyclic homology of* $A$, denoted $HC_n(A)$, to be

$$H_n(\mathrm{Tot}(CC_{\star,\star}(A))).$$

**Proposition 5.6.7 ([Lod98, Section 2.1.12]).** *For an associative $k$-algebra, $A$, there exists a canonical isomorphism of $k$-modules*

$$HC_0(A) \cong HH_0(A, A) = \frac{A}{[A, A]},$$

*where $[A, A]$ denotes the commutator submodule of $A$.* □

**Proposition 5.6.8 ([Lod98, Proposition 2.1.14]).** *Let $A$ be a unital, commutative $k$-algebra. There is a natural isomorphism of $k$-modules*

$$HC_1(A) \cong \frac{\Omega^1_{A|k}}{dA},$$

*where $\Omega^1_{A|k}$ is the module of Kähler differentials of Definition 4.5.4 and $dA$ is the $A$-submodule of Definition 4.5.6.* □

# Part II

# Homology of Small Categories

# Introduction

Many homology theories for algebras generalize to the setting of functor homology in the sense that there exists a small category **C**, sometimes known as an *indexing category*, and a homology theory for functors of the form $\mathbf{C} \to \mathbf{kMod}$, such that, if we choose the right functor, we recover our homology theory for algebras.

In Chapter 6 we will describe a tensor product for functors of the form $\mathbf{C} \to \mathbf{kMod}$ and its associated Tor groups. These Tor groups have an axiomatic characterization and universal properties. We will see later that many homology theories, including gamma homology in Part III and homology theories arising from crossed simplicial groups in Part IV, can be defined in terms of Tor groups of this sort.

Another benefit of functor homology interpretations is that one obtains more chain complexes with which to calculate homology. In Chapter 8, we will recall a chain complex of Gabriel and Zisman [GZ67], whose homology groups are the Tor groups over a small category. In Chapter 9 we describe another chain complex that can be used to compute Tor groups. This complex is obtained by computing resolutions in functor categories. In Chapter 10, we demonstrate that these two chain complexes are isomorphic. Both chain complexes will be used frequently throughout the rest of the thesis.

The material in these chapters is well-known to experts and is scattered throughout the literature. We gather together the material we require and our treatment is purely expository.

# Chapter 6

# The Tensor Product of Modules over a Small Category

## Introduction

Let $\mathbf{C}$ be a small category. In Section 6.1 we recall the categories of *left* and *right* $\mathbf{C}$-*modules* and the important properties of these categories. In Section 6.2 we define the *tensor product of* $\mathbf{C}$-*modules* as a bifunctor $- \otimes_{\mathbf{C}} -$ into the category of $k$-modules. In Section 6.3 we define the functors $\mathrm{Tor}_{\star}^{\mathbf{C}}(-,-)$ to be the left derived functors of the tensor product of $\mathbf{C}$-modules and describe the axiomatic characterization of these Tor functors. Finally, in Section 6.4 we describe the case of the functor $k^{\star} \otimes_{\mathbf{C}} -$, where $k^{\star}$ is the trivial right $\mathbf{C}$-module. This case will be particularly important in later chapters.

The material in this chapter can be found in [PR02, Section 1.6].

## 6.1 Modules over a Small Category

**Definition 6.1.1.** Let **CMod** denote the functor category $\mathrm{Fun}(\mathbf{C}, \mathbf{kMod})$. The objects are functors $\mathbf{C} \to \mathbf{kMod}$ and the morphisms are natural transformations of functors. We call **CMod** the *category of left* $\mathbf{C}$-*modules*.

**Definition 6.1.2.** Let **ModC** denote the functor category $\mathrm{Fun}(\mathbf{C}^{op}, \mathbf{kMod})$. The objects are functors $\mathbf{C}^{op} \to \mathbf{kMod}$ and the morphisms are natural transformations of functors. We call **ModC** the *category of right* $\mathbf{C}$-*modules*.

**Definition 6.1.3.** We define the *trivial left* $\mathbf{C}$-*module*,

$$k_{\star} \colon \mathbf{C} \to \mathbf{kMod}$$

to be the functor that sends every object of $\mathbf{C}$ to the trivial $k$-module, $k$, and every morphism of $\mathbf{C}$ to the identity map on $k$, $id_k$.

**Definition 6.1.4.** We define the *trivial right* $\mathbf{C}$*-module*,

$$k^\star \colon \mathbf{C}^{op} \to \mathbf{kMod}$$

to be the functor that sends every object of $\mathbf{C}^{op}$ to the trivial $k$-module, $k$, and every morphism of $\mathbf{C}^{op}$ to the identity map on $k$, $id_k$.

**Proposition 6.1.5 ([PR02, Section 1.6]).** *The categories* $\mathbf{CMod}$ *and* $\mathbf{ModC}$ *of left and right* $\mathbf{C}$*-modules are abelian.* □

**Proposition 6.1.6 ([PR02, Section 1.6]).** *The functors*

$$P_C \coloneqq k[\mathrm{Hom}_{\mathbf{C}}(C, -)],$$

*where* $C$ *runs through all objects of* $\mathbf{C}$*, form a set of projective generators for the category* $\mathbf{CMod}$. □

**Proposition 6.1.7 ([PR02, Section 1.6]).** *The functors*

$$P^C \coloneqq k[\mathrm{Hom}_{\mathbf{C}}(-, C)],$$

*where* $C$ *runs through all objects of* $\mathbf{C}$*, form a set of projective generators for the category* $\mathbf{ModC}$. □

## 6.2 The Tensor Product of C-modules

**Definition 6.2.1.** Let $F$ be an object in $\mathbf{CMod}$ and $G$ an object in $\mathbf{ModC}$. Consider the $k$-module

$$\bigoplus_{C \in \mathrm{Ob}(\mathbf{C})} G(C) \otimes_k F(C).$$

We denote by

$$\left\langle G(\alpha)(x) \otimes y - x \otimes F(\alpha)(y) \right\rangle$$

the $k$-submodule generated by the set

$$\left\{ G(\alpha)(x) \otimes y - x \otimes F(\alpha)(y) : \alpha \in \mathrm{Hom}(\mathbf{C}),\ x \in src(G(\alpha)),\ y \in src(F(\alpha)) \right\}.$$

**Definition 6.2.2.** Let $G$ be an object of $\mathbf{ModC}$ and $F$ be an object of $\mathbf{CMod}$. We define the tensor product $G \otimes_{\mathbf{C}} F$ to be the $k$-module

$$\frac{\bigoplus_{C \in \mathrm{Ob}(\mathbf{C})} G(C) \otimes_k F(C)}{\left\langle G(\alpha)(x) \otimes y - x \otimes F(\alpha)(y) \right\rangle}.$$

This quotient module is spanned $k$-linearly by equivalence classes of elementary tensors in $\bigoplus_{C \in \mathrm{Ob}(\mathbf{C})} G(C) \otimes_k F(C)$ which we will denote by

$$[x \otimes y].$$

Definition 6.2.2 defines an assignment

$$- \otimes_{\mathbf{C}} - \colon \mathbf{ModC} \times \mathbf{CMod} \to \mathbf{kMod}$$

on objects.

We extend $- \otimes_{\mathbf{C}} -$ to an assignment on morphisms.

**Definition 6.2.3.** Let $\Theta \colon G_1 \Rightarrow G_2$ be a morphism in **ModC** and $\Psi \colon F_1 \Rightarrow F_2$ be a morphism in **CMod**. We define a map of $k$-modules

$$\Theta \otimes_{\mathbf{C}} \Psi \colon \frac{\bigoplus_{C \in \mathrm{Ob}(\mathbf{C})} G_1(C) \otimes_k F_1(C)}{\left\langle G_1(\alpha)(x) \otimes y - x \otimes F_1(\alpha)(y) \right\rangle} \to \frac{\bigoplus_{C \in \mathrm{Ob}(\mathbf{C})} G_2(C) \otimes_k F_2(C)}{\left\langle G_2(\alpha)(x) \otimes y - x \otimes F_2(\alpha)(y) \right\rangle}$$

to be determined by

$$[x \otimes y] \mapsto \left[\Theta_C(x) \otimes \Psi_C(y)\right].$$

**Definition 6.2.4.** Let $\mathbf{C}$ be a small category. We call the bifunctor

$$- \otimes_{\mathbf{C}} - \colon \mathbf{ModC} \times \mathbf{CMod} \to \mathbf{kMod},$$

defined on objects in Definition 6.2.2 and on morphisms in Definition 6.2.3, the *tensor product of* $\mathbf{C}$*-modules.*

**Proposition 6.2.5 ([PR02, Section 1.6]).** *The bifunctor*

$$- \otimes_{\mathbf{C}} - \colon \mathbf{ModC} \times \mathbf{CMod} \to \mathbf{kMod}$$

*is right exact with respect to both variables and preserves direct sums.* □

## 6.3 The Functors $\mathrm{Tor}^{\mathbf{C}}_{\star}(-,-)$

**Proposition 6.3.1 ([PR02, Section 1.6]).** *The left derived functors of* $- \otimes_{\mathbf{C}} -$ *with respect to each variable are isomorphic.* □

**Definition 6.3.2.** Let $\mathbf{C}$ be a small category. We denote the left derived functors of the bifunctor

$$- \otimes_{\mathbf{C}} - \colon \mathbf{ModC} \times \mathbf{CMod} \to \mathbf{kMod}$$

by $\mathrm{Tor}^{\mathbf{C}}_{\star}(-,-)$.

**Proposition 6.3.3.** *There exist functors*

$$\mathrm{Tor}^{\mathbf{C}}_{n}(-,-) \colon \mathbf{ModC} \times \mathbf{CMod} \to \mathbf{kMod}$$

*for all* $n \geqslant 0$*, covariant in each variable, satisfying the following axioms.*

- ***T1*** $\mathrm{Tor}^{\mathbf{C}}_{0}(G, F) = G \otimes_{\mathbf{C}} F$.
- ***T2*** *If* $0 \Rightarrow G_1 \Rightarrow G_2 \Rightarrow G_3 \Rightarrow 0$ *is a short exact sequence of right* $\mathbf{C}$*-modules and* $F$ *is any left* $\mathbf{C}$*-module there is a long exact sequence of* $k$*-modules*

$$
\begin{array}{l}
\cdots \longrightarrow \operatorname{Tor}_n^{\mathbf{C}}(G_1, F) \longrightarrow \operatorname{Tor}_n^{\mathbf{C}}(G_2, F) \longrightarrow \operatorname{Tor}_n^{\mathbf{C}}(G_3, F) \longrightarrow \\
\longrightarrow \operatorname{Tor}_{n-1}^{\mathbf{C}}(G_1, F) \longrightarrow \cdots \longrightarrow \operatorname{Tor}_1^{\mathbf{C}}(G_3, F) \longrightarrow \\
\longrightarrow G_1 \otimes_{\mathbf{C}} F \longrightarrow G_2 \otimes_{\mathbf{C}} F \longrightarrow G_3 \otimes_{\mathbf{C}} F \longrightarrow 0
\end{array}
$$

*which is natural in the second variable.*

***T3*** *If $0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow F_3 \Rightarrow 0$ is a short exact sequence of left* $\mathbf{C}$*-modules and $G$ is any right* $\mathbf{C}$*-module there is a long exact sequence of $k$-modules*

$$
\begin{array}{l}
\cdots \longrightarrow \operatorname{Tor}_n^{\mathbf{C}}(G, F_1) \longrightarrow \operatorname{Tor}_n^{\mathbf{C}}(G, F_2) \longrightarrow \operatorname{Tor}_n^{\mathbf{C}}(G, F_3) \longrightarrow \\
\longrightarrow \operatorname{Tor}_{n-1}^{\mathbf{C}}(G, F_1) \longrightarrow \cdots \longrightarrow \operatorname{Tor}_1^{\mathbf{C}}(G, F_3) \longrightarrow \\
\longrightarrow G \otimes_{\mathbf{C}} F_1 \longrightarrow G \otimes_{\mathbf{C}} F_2 \longrightarrow G \otimes_{\mathbf{C}} F_3 \longrightarrow 0
\end{array}
$$

*which is natural in the first variable.*

***T4*** $\operatorname{Tor}_n^{\mathbf{C}}(G, P) = 0$ *if $P$ is a projective left* $\mathbf{C}$*-module and $n > 0$.*

***T5*** $\operatorname{Tor}_n^{\mathbf{C}}(P, F) = 0$ *if $P$ is a projective right* $\mathbf{C}$*-module and $n > 0$.*

*Furthermore, if*

$$F_n(-,-) : \mathbf{ModC} \times \mathbf{CMod} \to \mathbf{kMod}$$

*for $n \geqslant 0$ is another family of functors satisfying the conditions* ***T1***-***T5*** *then there is a natural isomorphism*

$$\operatorname{Tor}_n^{\mathbf{C}}(-,-) \cong F_n(-,-)$$

*for each $n \geqslant 0$.* □

## 6.4 The Functor $k^\star \otimes_{\mathbf{C}} -$

Recall the trivial right **C**-module, $k^\star$, of Definition 6.1.4.

By Proposition 6.2.5, the functor

$$k^\star \otimes_{\mathbf{C}} - : \mathbf{CMod} \to \mathbf{kMod}$$

is right exact and gives rise to left derived functors $\operatorname{Tor}_\star^{\mathbf{C}}(k^\star, -)$, subject to the axiomatic characterization of Proposition 6.3.3.

Let $F$ be an object in **CMod**. In this case

$$k^\star \otimes_{\mathbf{C}} F = \frac{\bigoplus_{C \in \operatorname{Ob}(\mathbf{C})} k \otimes F(C)}{\left\langle k^\star(\alpha)(x) \otimes y - x \otimes F(\alpha)(y) \right\rangle}$$

where

$$\left\langle k^\star(\alpha)(x) \otimes y - x \otimes F(\alpha)(y) \right\rangle$$

is generated by all differences of the form

$$1_k \otimes y - 1_k \otimes F(\alpha)(y)$$

where $\alpha$ runs through all morphisms in $\mathbf{C}$ and $y$ runs through all elements in the domain of $F(\alpha)$.

# Chapter 7

# Homological Algebra in Functor Categories

We can define a number of standard properties from homological algebra for functor categories by doing so object-wise.

Let $\mathbf{C}$ be a small category and let $\mathbf{A}$ be an abelian category.

**Definition 7.0.1.** A functor $F \in \mathrm{Fun}(\mathbf{C}, \mathbf{A})$ is said to be *free*, *projective* or *injective* if $F(C)$ is *free*, *projective* or *injective* respectively for every object $C \in \mathbf{C}$.

**Definition 7.0.2.** A sequence of functors

$$0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow F_3 \Rightarrow 0$$

is said to be a *short exact sequence* in $\mathrm{Fun}(\mathbf{C}, \mathbf{A})$ if

$$0 \to F_1(C) \to F_2(C) \to F_3(C) \to 0$$

is a short exact sequence in $\mathbf{A}$ for each object $C \in \mathbf{C}$.

**Definition 7.0.3.** A sequence of functors

$$\cdots \Rightarrow F_2 \Rightarrow F_1 \Rightarrow F_0 \Rightarrow 0$$

in $\mathrm{Fun}(\mathbf{C}, \mathbf{A})$ is called a *chain complex* if

$$\cdots \to F_2(C) \to F_1(C) \to F_0(C) \to 0$$

is a chain complex in $\mathbf{A}$ for each object $C \in \mathbf{C}$.

**Definition 7.0.4.** A chain complex

$$\cdots \Rightarrow F_2 \Rightarrow F_1 \Rightarrow F_0 \Rightarrow 0$$

in $\mathrm{Fun}(\mathbf{C}, \mathbf{A})$ is said to be a *free resolution* of $F \in \mathrm{Fun}(\mathbf{C}, \mathbf{A})$ if

$$\cdots \to F_2(C) \to F_1(C) \to F_0(C) \to 0$$

is a free resolution for $F(C)$ in $\mathbf{A}$ for each object $C \in \mathbf{C}$.

**Definition 7.0.5.** Let $\mathbf{C}$ be a small category and let $\mathbf{D}$ be any category. A simplicial object $X_\star$ in the functor category $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ is a family of functors

$$X_n \colon \mathbf{C} \to \mathbf{D}$$

indexed by the non-negative integers together with natural transformations $\partial_i \colon X_n \Rightarrow X_{n-1}$ for $0 \leqslant i \leqslant n$ and $s_j \colon X_n \Rightarrow X_{n+1}$ for $0 \leqslant j \leqslant n$ satisfying the simplicial identities of Definition 2.1.1.

**Proposition 7.0.6.** *A simplicial object* $X_\star$ *in* $\mathrm{Fun}(\mathbf{C}, \mathbf{D})$ *is isomorphic to a functor*

$$\mathbf{C} \to \mathbf{sD}. \qquad \square$$

# Chapter 8

# The Construction of Gabriel and Zisman

## Introduction

Let $\mathbf{C}$ be a small category and let $F \in \mathbf{CMod}$. Gabriel and Zisman [GZ67, Appendix 2] construct a standard chain complex for computing the $k$-modules $\mathrm{Tor}_n^{\mathbf{C}}(k^\star, F)$ of Section 6.4.

In Section 8.1 we describe the chain complex $C_\star(\mathbf{C}, F)$ of Gabriel and Zisman, as the chain complex associated to a simplicial $k$-module. The rest of the chapter is dedicated to proving that the homology groups of $C_\star(\mathbf{C}, F)$ are isomorphic to the groups $\mathrm{Tor}_n^{\mathbf{C}}(k^\star, F)$ from Chapter 6. Sections 8.2 and 8.3 prove that $C_\star(\mathbf{C}, -)$ is an exact functor. In Section 8.4 we prove that the functors $C_\star(\mathbf{C}, -)$ are naturally isomorphic to the functors $\mathrm{Tor}_n(k^\star, -)$ using the axiomatic characterization of Proposition 6.3.3.

## 8.1 The Simplicial $k$-module $C_\star(\mathbf{C}, F)$

Recall the nerve of a small category from Subsection 2.1.1 and, in particular, the convention of Remark 2.1.7.

**Definition 8.1.1.** Let $F \in \mathbf{CMod}$.

We define a simplicial $k$-module $C_\star(\mathbf{C}, F)$ as follows. In degree $n$ we define

$$C_n(\mathbf{C}, F) = \bigoplus_{(f_n, \ldots, f_1)} F(C_0)$$

where the sum runs through all elements $(f_n, \ldots, f_1)$ of $N_n\mathbf{C}$.

We write a generator of $C_n(\mathbf{C}, F)$ in the form

$$(f_n, \ldots, f_1, x)$$

where $(f_n, \dots, f_1) \in N_n\mathbf{C}$ indexes the summand and $x \in F(C_0)$. A general element of $C_n(\mathbf{C}, F)$ is then a $k$-linear combination of such generators.

The face maps $\partial_i \colon C_n(\mathbf{C}, F) \to C_{n-1}(\mathbf{C}, F)$ are determined by

$$\partial_i(f_n, \dots, f_1, x) = \begin{cases} (f_n, \dots, f_2, F(f_1)(x)) & i = 0, \\ (f_n, \dots, f_{i+1} \circ f_i, \dots, f_1, x) & 1 \leqslant i \leqslant n-1, \\ (f_{n-1}, \dots, f_1, x) & i = n. \end{cases}$$

The degeneracy maps $s_j \colon C_n(\mathbf{C}, F) \to C_{n+1}(\mathbf{C}, F)$ are determined by

$$s_j(f_n, \dots, f_1, x) = \begin{cases} (f_n, \dots, f_1, id_{C_0}, x) & j = 0, \\ \left(f_n, \dots, id_{C_j}, f_j, \dots, f_1, x\right) & 1 \leqslant j \leqslant n. \end{cases}$$

*Remark 8.1.2.* These are analogous to the face and degeneracy maps in the nerve construction of Subsection 2.1.1 with one exception. For $\partial_0$, when we omit the first morphism in the string, we must act on $x \in F(C_0)$ using $F(f_1)$ in order to obtain an element in the codomain of $\partial_0$.

**Definition 8.1.3.** Let $\mathbf{C}$ be a small category. Let $F \in \mathbf{CMod}$ and let $C_\star(\mathbf{C}, F)$ be the simplicial $k$-module of Definition 8.1.1. We denote the homology of the associated chain complex by $H_\star(\mathbf{C}, F)$. The homology groups $H_\star(\mathbf{C}, F)$ are called the *homology groups of the small category* $\mathbf{C}$ *with coefficients in the functor* $F$.

## 8.2 Functoriality of $C_\star(\mathbf{C}, -)$

Definition 8.1.1 gives us an assignment

$$C_\star(\mathbf{C}, -) : \mathbf{CMod} \to \mathbf{skMod}$$

on objects. We will extend this to an assignment on morphisms.

**Definition 8.2.1.** Let $n \geqslant 0$. Let $\Theta \colon F_1 \Rightarrow F_2$ be a morphism of left $\mathbf{C}$-modules. We define a map of simplicial $k$-modules

$$C_\star(\mathbf{C}, \Theta) : C_\star(\mathbf{C}, F_1) \to C_\star(\mathbf{C}, F_2)$$

to be determined in degree $n$ by

$$(f_n, \dots, f_1, x) \mapsto (f_n, \dots, f_1, \Theta_{C_0}(x)).$$

**Proposition 8.2.2.** *The assignment*

$$C_\star(\mathbf{C}, -) \colon \mathbf{CMod} \to \mathbf{skMod}$$

*defined on objects in Definition 8.1.1 and on morphisms in Definition 8.2.1 is functorial.*□

## 8.3 Exactness of $C_\star(\mathbf{C}, -)$

We claim that $C_\star(\mathbf{C}, -)$ is an exact functor. That is, we claim that the functor $C_\star(\mathbf{C}, -)$ takes short exact sequences of left $\mathbf{C}$-modules to short exact sequences of simplicial $k$-modules.

**Proposition 8.3.1.** *Let*

$$0 \Longrightarrow F_1 \overset{\Theta}{\Longrightarrow} F_2 \overset{\Psi}{\Longrightarrow} F_3 \Longrightarrow 0.$$

*be a short exact sequence of left* $\mathbf{C}$*-modules. There is a short exact sequence of simplicial* $k$*-modules*

$$0 \to C_\star(\mathbf{C}, F_1) \xrightarrow{C_\star(\mathbf{C},\Theta)} C_\star(\mathbf{C}, F_2) \xrightarrow{C_\star(\mathbf{C},\Psi)} C_\star(\mathbf{C}, F_3) \to 0.$$

*Proof.* In order to prove that we have a short exact sequence of simplicial $k$-modules it suffices to show that, for each $n \geqslant 0$,

$$0 \to C_n(\mathbf{C}, F_1) \xrightarrow{C_n(\mathbf{C},\Theta)} C_n(\mathbf{C}, F_2) \xrightarrow{C_n(\mathbf{C},\Psi)} C_n(\mathbf{C}, F_3) \to 0$$

is a short exact sequence of $k$-modules, since we already have compatibility with the simplicial structure by Proposition 8.2.2.

Consider the short exact sequence

$$0 \Longrightarrow F_1 \overset{\Theta}{\Longrightarrow} F_2 \overset{\Psi}{\Longrightarrow} F_3 \Longrightarrow 0.$$

By Definition 7.0.2 we have an exact sequence of $k$-modules

$$0 \to F_1(C) \xrightarrow{\Theta_C} F_2(C) \xrightarrow{\Psi_C} F_3(C) \to 0$$

for each object $C \in \mathbf{C}$.

Take the direct sum of all such short exact sequences indexed by elements $(f_n, \ldots, f_1)$ of $N_n\mathbf{C}$. An arbitrary coproduct of short exact sequences of $k$-modules is itself a short exact sequence of $k$-modules. We therefore have a short exact sequence

$$0 \to \bigoplus_{(f_n,\ldots,f_1)} F_1(C_0) \xrightarrow{\bigoplus \Theta_{C_0}} \bigoplus_{(f_n,\ldots,f_1)} F_2(C_0) \xrightarrow{\bigoplus \Psi_{C_0}} \bigoplus_{(f_n,\ldots,f_1)} F_3(C_0) \to 0.$$

That is, we have a short exact sequence

$$0 \to C_n(\mathbf{C}, F_1) \xrightarrow{C_n(\mathbf{C},\Theta)} C_n(\mathbf{C}, F_2) \xrightarrow{C_n(\mathbf{C},\Psi)} C_n(\mathbf{C}, F_3) \to 0$$

as required.

Hence, $C_\star(\mathbf{C}, -)$ is an exact functor. □

## 8.4 Axiomatic Characterization

In Subsection 8.2 we demonstrated that we have a functor

$$C_\star(\mathbf{C}, -)\colon \mathbf{CMod} \to \mathbf{skMod}.$$

Since taking the associated chain complex of a simplicial $k$-module is functorial and taking homology of a chain complex is functorial, by composition we obtain functors

$$H_n(\mathbf{C}, -)\colon \mathbf{CMod} \to \mathbf{kMod}$$

for each $n \geqslant 0$.

We claim that the functors $H_n(\mathbf{C}, -)$ are naturally isomorphic to the functors $\mathrm{Tor}_n^{\mathbf{C}}(k^\star, -)$ of Subsection 6.4. In order to prove this we will use Proposition 6.3.3.

**Lemma 8.4.1.** *There is an isomorphism of $k$-modules*

$$H_0(\mathbf{C}, F) \cong k^\star \otimes_{\mathbf{C}} F.$$

*Proof.* By definition,

$$H_0(\mathbf{C}, F) = \frac{C_0(\mathbf{C}, F)}{\mathrm{Im}(\partial_0 - \partial_1)}.$$

By definition,

$$C_0(\mathbf{C}, F) = \bigoplus_{C \in \mathrm{Ob}(\mathbf{C})} F(C).$$

Let $(f, x)$ be a generator in $C_1(\mathbf{C}, F)$. That is, $f \in \mathrm{Hom}_{\mathbf{C}}(C_0, C_1)$ and $x \in F(C_0)$. We see that

$$(\partial_0 - \partial_1)(f, x) = (F(f)(x) - x).$$

Since the face maps are $k$-linear we see that $\mathrm{Im}(\partial_0 - \partial_1)$ is the $k$-submodule of $C_0(\mathbf{C}, F)$ generated by all differences $x - F(f)(x)$ where $f$ runs through all morphisms in the category $\mathbf{C}$ and $x$ runs through all elements of the domain of $F(f)$.

By comparison with Subsection 6.4, we see that there is an isomorphism

$$H_0(\mathbf{C}, F) \cong k^\star \otimes_{\mathbf{C}} F$$

induced by the isomorphism $F(C) \cong k \otimes_k F(C)$. □

**Lemma 8.4.2.** *Given a short exact sequence*

$$0 \Rightarrow F_1 \Rightarrow F_2 \Rightarrow F_3 \Rightarrow 0$$

*of left $\mathbf{C}$-modules, we have a long exact sequence*

$$
\cdots \longrightarrow H_n(\mathbf{C}, F_1) \longrightarrow H_n(\mathbf{C}, F_2) \longrightarrow H_n(\mathbf{C}, F_3) \longrightarrow
$$
$$
\longrightarrow H_{n-1}(\mathbf{C}, F_1) \longrightarrow \cdots \longrightarrow H_1(\mathbf{C}, F_3) \longrightarrow
$$
$$
\longrightarrow H_0(\mathbf{C}, F_1) \longrightarrow H_0(\mathbf{C}, F_2) \longrightarrow H_0(\mathbf{C}, F_3) \longrightarrow 0
$$

*of $k$-modules.*

*Proof.* We demonstrated in Proposition 8.3.1 that given such a short exact sequence of left **C**-modules we obtain a short exact sequence of simplicial $k$-modules

$$0 \to C_\star(\mathbf{C}, F_1) \to C_\star(\mathbf{C}, F_2) \to C_\star(\mathbf{C}, F_3) \to 0.$$

It follows that we have a short exact sequence of the associated chain complexes and, by a standard construction, we obtain the necessary long exact sequence in homology. □

**Lemma 8.4.3.** *Let $P$ be a projective object in* **CMod**. *The homology of the chain complex $C_\star(\mathbf{C}, P)$ vanishes in positive degrees.*

*Proof.* It suffices to prove this for the set of projective generators

$$P_C = k[\mathrm{Hom}_{\mathbf{C}}(C, -)]$$

of Proposition 6.1.6.

Consider

$$C_\star(\mathbf{C}, P_C) = \bigoplus_{(f_n, \ldots, f_1)} k[\mathrm{Hom}_{\mathbf{C}}(C, C_0)].$$

A generator is of the form $(f_n, \ldots, f_1, f_0)$ where $f_0 \in \mathrm{Hom}_{\mathbf{C}}(C, C_0)$.

For each $n \geqslant 0$, let

$$h_n \colon C_n(\mathbf{C}, P_C) \to C_{n+1}(\mathbf{C}, P_C)$$

be the map of $k$-modules determined by

$$(f_n, \ldots, f_1, f_0) \mapsto (f_n, \ldots, f_1, f_0, id_C).$$

That is, we send the element $f_0 \in \mathrm{Hom}_{\mathbf{C}}(C, C_0)$ in the summand indexed by $(f_n, \ldots, f_1)$ to the element $id_C \in \mathrm{Hom}_{\mathbf{C}}(C, C)$ in the summand indexed by $(f_n, \ldots, f_0)$.

We claim that the maps $h_n$ for $n \geqslant 0$ form a contracting homotopy for the chain complex $C_\star(\mathbf{C}, P_C)$.

It is straightforward to check that these map satisfy the conditions:

- $\partial_0 \circ h_n$ is the identity map for each $n \geqslant 0$ and
- $\partial_{i+1} \circ h_n = h_{n-1} \circ \partial_i$ for $0 \leqslant i \leqslant n$.

It follows that

$$\left(\sum_{i=0}^{n+1}(-1)^i\partial_i\right)\circ h_n + h_{n-1}\circ\left(\sum_{i=0}^{n}(-1)^i\partial_i\right)$$

is the identity map on $C_n\left(\mathbf{C}, P_C\right)$ as required. □

**Theorem 8.4.4.** *Let* $\mathbf{C}$ *be a small category and let* $F$ *be a left* $\mathbf{C}$*-module. There are natural isomorphisms of* $k$*-modules*

$$\mathrm{Tor}_n^{\mathbf{C}}(k^\star, F) \cong H_n(\mathbf{C}, F)$$

*for all* $n \geqslant 0$*.*

*Proof.* We combine Lemmata 8.4.1, 8.4.2 and 8.4.3 and apply the axiomatic characterization of Proposition 6.3.3. □

# Chapter 9

# Resolutions for Trivial Modules

## Introduction

We can construct more chain complexes for computing Tor over a small category using more familiar methods from homological algebra. In this chapter we construct resolutions for the trivial modules $k_\star \in \mathbf{CMod}$ and $k^\star \in \mathbf{ModC}$ using the nerves of the over-category and under-category respectively. Section 9.1 recalls the necessary details of the over-category and describes the resolution of $k_\star \in \mathbf{CMod}$. Section 9.2 recalls the necessary details of the under-category and describes the resolution of $k^\star \in \mathbf{ModC}$.

## 9.1 A Resolution for the Trivial Left C-module

Recall the trivial left $\mathbf{C}$-module, $k_\star$, from Definition 6.1.3. Recall the over-category of Section 1.4 and the nerve construction of Subsection 2.1.1. We will form a resolution of $k_\star$ in $\mathbf{CMod}$ from the nerve of the over-category.

### 9.1.1 The nerve of the over-category

Combining Remarks 1.4.5 and 2.1.8 we observe that there is a functor

$$N_\star\left(\mathbf{C}/-\right) : \mathbf{C} \to \mathbf{sSet}.$$

For an object $C \in \mathbf{C}$, $N_n(\mathbf{C}/C)$ is the set of all strings of composable morphisms of length $n+1$ in $\mathbf{C}$ whose final codomain is $C$. That is, the set of elements of the form

$$C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} C_n \xrightarrow{f} C.$$

Such an element will be denoted by $(f, f_n, \ldots, f_1)$. This is notation for an element in degree $n$ of the simplicial set $N_\star(\mathbf{C}/C)$ and is not to be confused with the notation for morphisms in the over-category as introduced in Definition 1.4.2.

Given a morphism $g \in \mathrm{Hom}_{\mathbf{C}}\left(C, C'\right)$ in $\mathbf{C}$, $N_\star(\mathbf{C}/g)$ is defined in degree $n$ by

$$N_n(\mathbf{C}/g)(f, f_n, \ldots, f_1) = (g \circ f, f_n, \ldots, f_1).$$

Recall that for each object $C \in \mathbf{C}$, $N_\star(\mathbf{C}/C)$ the face and degeneracy maps are given by

- $\partial_0(f, f_n, \ldots, f_1) = (f, f_n, \ldots, f_2)$,
- $\partial_i(f, f_n, \ldots, f_1) = (f, f_n, \ldots, f_{i+1} \circ f_i, \ldots, f_1)$ for $1 \leqslant i \leqslant n-1$,
- $\partial_n(f, f_n, \ldots, f_1) = (f \circ f_n, f_{n-1}, \ldots, f_1)$,
- $s_0(f, f_n, \ldots, f_1) = \left(f, f_n, \ldots, f_1, id_{C_0}\right)$,
- $s_j(f, f_n, \ldots, f_1) = \left(f, f_n, \ldots, id_{C_j}, f_j, \ldots, f_1\right)$ for $1 \leqslant j \leqslant n$.

By Proposition 7.0.6, $N_\star\left(\mathbf{C}/-\right)$ is a simplicial object in the category $\mathrm{Fun}\left(\mathbf{C}, \mathbf{Set}\right)$. It follows that $k[N_\star(\mathbf{C}/-)]$ is a simplicial object in the category **CMod**.

### 9.1.2 A resolution for $k_\star$

We claim that the chain complex associated to $k[N_\star(\mathbf{C}/-)]$ is a free resolution of the trivial $\mathbf{C}$-module, $k_\star$. Following Definition 7.0.4, we prove this by showing that for each object $C \in \mathbf{C}$, $k[N_\star(\mathbf{C}/C)]$ is a free resolution of the trivial $k$-module, $k$.

**Proposition 9.1.1.** *For each object $C \in \mathbf{C}$ the chain complex $k[N_\star(\mathbf{C}/C)]$ has homology isomorphic to $k$ concentrated in degree zero.*

*Proof.* Consider the morphism

$$\varepsilon\colon k\left[N_0\left(\mathbf{C}/C\right)\right] \to k$$

determined by

$$\left(C_0 \xrightarrow{f_1} C\right) \mapsto 1_k.$$

This is an augmentation in the sense of Definition 2.4.12.

For each $C \in \mathbf{C}$ and $n \geqslant 0$, let

$$h_n\colon k[N_n(\mathbf{C}/C)] \to k[N_{n+1}(\mathbf{C}/C)]$$

be the morphism determined by

$$h_n(f, f_n, \ldots, f_1) = (-1)^{n+1}\left(id_C, f, f_n, \ldots, f_1\right).$$

Furthermore, we define

$$h_{-1}\colon k \to k\left[N_0\left(\mathbf{C}/C\right)\right]$$

to be determined by

$$1_k \mapsto \left( C \xrightarrow{id_C} C \right).$$

It is straightforward to check that the maps $h_n$ for $n \geqslant -1$ satisfy the first set of conditions of Proposition 2.4.13.

We deduce that

$$H_n \left( k[N_\star(\mathbf{C}/C)] \right) \cong \begin{cases} k & n = 0 \\ 0 & \text{else.} \end{cases}$$

□

We deduce the following theorem.

**Theorem 9.1.2.** *The chain complex associated to the simplicial object $k[N_\star(\mathbf{C}/-)]$ in the category* **CMod** *is a free resolution of the trivial* **C***-module, $k_\star$.*

*In particular, for a functor $G \in$* **ModC** *the groups $\mathrm{Tor}_\star^{\mathbf{C}}(G, k_\star)$ are naturally isomorphic to the homology groups of the chain complex of $k$-modules*

$$\cdots \to G \otimes_{\mathbf{C}} k[N_2(\mathbf{C}/-)] \to G \otimes_{\mathbf{C}} k[N_1(\mathbf{C}/-)] \to G \otimes_{\mathbf{C}} k[N_0(\mathbf{C}/-)] \to 0.$$

□

## 9.2 A Resolution for the Trivial Right C-module

Recall the trivial right **C**-module, $k^\star$, of Definition 6.1.4. Recall the under-category of Section 1.3.

We will use the under-category to construct a free resolution of $k^\star$.

### 9.2.1 The nerve of the under-category

Combining Remarks 1.3.5 and 2.1.8 we observe that there is a functor

$$N_\star \left( -\backslash \mathbf{C} \right) : \mathbf{C}^{op} \to \mathbf{sSet}.$$

For an object $C \in \mathbf{C}^{op}$, $N_n(C\backslash\mathbf{C})$ is the set of all strings of composable morphisms of length $(n+1)$ in $\mathbf{C}^{op}$ whose initial domain is $C$. That is, the set of elements of the form

$$C \xrightarrow{f} C_0 \xrightarrow{f_1} \cdots \xrightarrow{f_n} C_n.$$

Such an element will be denoted by $(f_n, \ldots, f_1, f)$. This is notation for an element in degree $n$ of the simplicial set $N_\star(C\backslash\mathbf{C})$ and is not to be confused with the notation for morphisms in the under-category as introduced in Definition 1.3.2.

Given a morphism $g \in \mathrm{Hom}_{\mathbf{C}}\left(C', C\right)$, $N_\star(g\backslash\mathbf{C})$ is defined in degree $n$ by

$$N_n(g\backslash\mathbf{C})\left(f_n, \ldots, f_1, f\right) = \left(f_n, \ldots, f_1, f \circ g\right).$$

Recall that for each object $C \in \mathbf{C}^{op}$, $N_\star(C\backslash\mathbf{C})$ the face and degeneracy maps are given by

- $\partial_0(f_n, \ldots, f_1, f) = (f_n, \ldots, f_2, f_1 \circ f)$,
- $\partial_i(f_n, \ldots, f_1, f) = (f_n, \ldots, f_{i+1} \circ f_i, \ldots, f_1, f)$ for $1 \leqslant i \leqslant n-1$,
- $\partial_n(f_n, \ldots, f_1, f) = (f_{n-1}, \ldots, f_1, f)$,
- $s_0(f_n, \ldots, f_1, f) = \big(f_n, \ldots, f_1, id_{C_0}, f\big)$,
- $s_j(f_n, \ldots, f_1, f) = \Big(f_n, \ldots, id_{C_j}, f_j, \ldots, f_1, f\Big)$ for $1 \leqslant j \leqslant n$.

By Proposition 7.0.6, $N_\star\left(-\backslash\mathbf{C}\right)$ is a simplicial object in $\mathrm{Fun}\left(\mathbf{C}^{op}, \mathbf{Set}\right)$. It follows that $k\left[N_\star\left(-\backslash\mathbf{C}\right)\right]$ is a simplicial object in the category **ModC**.

### 9.2.2 A resolution for $k^\star$

We claim that the chain complex associated to $k[N_\star(-\backslash\mathbf{C})]$ is a free resolution of the trivial $\mathbf{C}^{op}$-module, $k^\star$. Following Definition 7.0.4, we prove this by showing that for each object $C \in \mathbf{C}^{op}$, $k[N_\star(C\backslash\mathbf{C})]$ is a free resolution of the trivial $k$-module, $k$.

**Proposition 9.2.1.** *For each object $C \in \mathbf{C}$ the chain complex $k[N_\star(C\backslash\mathbf{C})]$ has homology isomorphic to $k$ concentrated in degree zero.*

*Proof.* Consider the morphism

$$\varepsilon \colon k\left[N_0\left(C\backslash\mathbf{C}\right)\right] \to k$$

determined by

$$\left(C \xrightarrow{f_1} C_0\right) \mapsto 1_k.$$

This is an augmentation in the sense of Definition 2.4.12.

For each $C \in \mathbf{C}^{op}$ and each $n \geqslant 0$, let

$$h_n \colon k[N_n(C\backslash\mathbf{C})] \to k[N_{n+1}(C\backslash\mathbf{C})]$$

be the morphism determined by

$$h_n(f_n, \ldots, f_1, f) = (f_n, \ldots, f_1, f, id_C)\,.$$

Furthermore, we define

$$h_{-1} \colon k \to k\left[N_0\left(C\backslash\mathbf{C}\right)\right]$$

to be determined by

$$1_k \mapsto \left(C \xrightarrow{id_C} C\right).$$

It is straightforward to check that the maps $h_n$ for $n \geqslant -1$ satisfy the second set of conditions of Proposition 2.4.13.

We deduce that

$$H_n\left(k[N_\star(C\backslash\mathbf{C})]\right) \cong \begin{cases} k & n = 0 \\ 0 & \text{else.} \end{cases}$$

□

We deduce the following theorem.

**Theorem 9.2.2.** *The chain complex associated to the simplicial object $k[N_\star(-\backslash\mathbf{C})]$ in the category* **ModC** *is a free resolution of the trivial $\mathbf{C}^{op}$-module, $k^\star$.*

*In particular, for a functor $F \in$* **CMod** *the groups $\mathrm{Tor}_\star^{\mathbf{C}}(k^\star, F)$ are naturally isomorphic to the homology groups of the chain complex of $k$-modules*

$$\cdots \to k[N_2(-\backslash\mathbf{C})] \otimes_{\mathbf{C}} F \to k[N_1(-\backslash\mathbf{C})] \otimes_{\mathbf{C}} F \to k[N_0(-\backslash\mathbf{C})] \otimes_{\mathbf{C}} F \to 0.$$

□

# Chapter 10

# Connecting $k^\star \otimes_{\mathbf{C}} -$ and $C_\star(\mathbf{C}, -)$

## 10.1 The Isomorphism of Chain Complexes

**Lemma 10.1.1 ([Aul10, Section 1.3]).** *For each $n \geqslant 0$, there is an isomorphism of $k$-modules*

$$\varphi_n \colon k[N_n(-\backslash\mathbf{C})] \otimes_{\mathbf{C}} F \to C_n(\mathbf{C}, F)$$

*determined by*

$$\left[\left(C_n \xleftarrow{f_n} \cdots \xleftarrow{f_1} C_0 \xleftarrow{f_0} C\right) \otimes x\right] \mapsto (f_n, \ldots, f_1, F(f_0)(x)).$$

□

Furthermore, these isomorphisms are compatible with the boundary maps and we obtain the following theorem.

**Theorem 10.1.2 ([Aul10, Section 1.3]).** *Let $\mathbf{C}$ be a small category and $F \in \mathbf{CMod}$. There is an isomorphism of chain complexes*

$$C_\star(\mathbf{C}, F) \cong k[N_\star(-\backslash\mathbf{C})] \otimes_{\mathbf{C}} F.$$

□

**Corollary 10.1.3.** *Let $\mathbf{C}$ be a small category and $F \in \mathbf{CMod}$. There exist isomorphisms $k$-modules*

$$\mathrm{Tor}_n^{\mathbf{C}}(k^\star, F) \cong H_n(C_\star(\mathbf{C}, F)) \cong H_n(k[N_\star(-\backslash\mathbf{C})] \otimes_{\mathbf{C}} F)$$

*for each $n \geqslant 0$.* □

# Part III

# Gamma Homology

# Introduction

Robinson and Whitehouse developed gamma homology, which we will frequently write as Γ-homology, for commutative $k$-algebras to encode information about homotopy commutativity. The details of Γ-homology were first written in the thesis of Whitehouse [Whi94]. Γ-homology is closely related to stable homotopy theory as demonstrated by Pirashvili [Pir00b] and Pirashvili and Richter [PR00]. One way to view the construction of Γ-homology is as building a symmetric group action into the Hochschild complex. We shall see in Chapter 24 that there is another theory, called symmetric homology, that builds a symmetric group action into the Hochschild complex. In Part VII, under some conditions, we will provide a comparison map between symmetric homology and Γ-homology.

As mentioned, Γ-homology for commutative algebras was introduced in the thesis of Whitehouse [Whi94]. This included a definition of a standard complex for computing Γ-homology, which has become known as the *Robinson-Whitehouse complex*. Pirashvili [Pir00b] and Pirashvili and Richter [PR00] generalized Γ-homology to the setting of functor homology in the sense of Part II. In particular, they provide a version of the Robinson-Whitehouse complex in the functor homology setting.

We will take the description of Γ-homology as functor homology to be our definition and will specialize to the results of Whitehouse [Whi94].

In Chapter 11 we recall the categories Γ and Ω and some related categories. In Chapter 12 we define the categories of Γ-modules and Ω-modules, following Section 6.1 and recall Pirashvili's Dold-Kan Type theorem. This theorem proves an equivalence between categories of certain types of Γ-modules and Ω-modules.

In Chapter 13 we define Γ-homology in terms of functor homology. We define a Loday functor for the category Γ and recover the definition of Γ-homology for a commutative algebra given by Whitehouse. In Chapter 14 we define the Robinson-Whitehouse complex for a Γ-module following Pirashvili and Richter and specialize to the complex defined by Whitehouse.

In Chapter 15 we recall the calculations of Γ-homology for some important commutative algebras. Robinson and Whitehouse describe Γ-homology of a commutative algebra in degree 0 in terms of the Kähler differentials. Richter and Robinson calculate Γ-homology for group algebras and polynomial algebras.

An important feature of Γ-homology, which is crucial to defining the comparison map of Part VII, is its relation to Harrison homology. In Chapter 16 we recall some of the properties of Γ-homology, including its relation to tree spaces, following Whitehouse. In Chapter 17 we continue in this vein, describing a variant of the Γ-complex in the case when the ground ring contains $\mathbb{Q}$ and the relationship between Γ-homology and Harrison homology.

In Chapter 18, we present some new material. We examine the Robinson-Whitehouse complex for an augmented, commutative algebra and prove that it splits into a direct sum of chain complexes, one summand of which is built from the augmentation ideal. Furthermore we prove that when the ground ring contains $\mathbb{Q}$, the homology of this summand is isomorphic to the Γ-homology of the augmented, commutative algebra.

# Chapter 11

# The Categories $\Gamma$ and $\Omega$

## 11.1 The Categories $\Gamma$ and $\Omega$

**Definition 11.1.1.** Let $\Gamma$ denote the category whose objects are the finite based sets $[n] = \{\underline{0}, 1, \ldots, n\}$ for $n \geqslant 0$ and whose morphisms are basepoint-preserving maps of sets.

*Remark 11.1.2.* The category $\Gamma$ is a skeleton of $\mathbf{Fin}_\star$.

**Definition 11.1.3.** Let $\Omega$ denote the category whose objects are the finite sets $\underline{n} = \{1, \ldots, n\}$ for $n \geqslant 1$ and whose morphisms are surjections of sets.

### 11.1.1 Related categories

**Definition 11.1.4.** Let $\Omega_\emptyset$ be the category obtained from $\Omega$ by adding the empty set $\emptyset$ as a zero object. That is,

$$\mathrm{Ob}\,(\Omega_\emptyset) = \mathrm{Ob}\,(\Omega) \coprod \{\emptyset\}\,.$$

The morphisms of $\Omega_\emptyset$ are the morphisms of $\Omega$ together with unique maps

$$\underline{n} \to \emptyset$$

and

$$\emptyset \to \underline{n}$$

for each $n \geqslant 1$.

**Definition 11.1.5.** For $n \geqslant 2$, let $\underline{n}/\Omega/\underline{1}$ denote the category whose objects are all composites in $\Omega$ of the form

$$\underline{n} \to \underline{r} \to \underline{1}$$

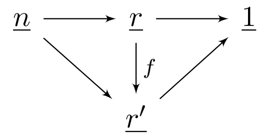
where $1 < r < n$. A morphism from

$$\underline{n} \to \underline{r} \to \underline{1}$$

to

$$\underline{n} \to \underline{r'} \to \underline{1}$$

in $\underline{n}/\Omega/\underline{1}$ is a morphism $f \in \mathrm{Hom}_\Omega\left(\underline{r}, \underline{r'}\right)$ such that the diagram




commutes. We call $\underline{n}/\Omega/\underline{1}$ the *category of objects strictly under* $\underline{n}$ *and over* $\underline{1}$ *in* $\Omega$.

# Chapter 12

# Γ-modules and Pirashvili's Dold-Kan Type Theorem

## Introduction

We define the categories of Γ-modules and Ω-modules following Section 6.1. We recall Pirashvili's Dold-Kan Type theorem. This theorem proves an equivalence of categories between categories of Γ-modules and categories of Ω-modules. This theorem is the key to the fact that we can calculate Γ-homology using only epimorphisms, that is morphisms in Ω, rather than morphisms in Γ.

## 12.1 Γ-modules and Ω-modules

We recall the notions of Γ-modules and Ω-modules following Section 6.1.

**Definition 12.1.1.** Let $\Gamma\mathbf{Mod}$ denote the *category of left* Γ*-modules.* The objects are functors $\Gamma \to \mathbf{kMod}$ and the morphisms are natural transformations of functors.

**Definition 12.1.2.** Let $\Gamma\mathbf{Mod}_0$ denote the subcategory of $\Gamma\mathbf{Mod}$ whose objects are those left Γ-modules $F$ satisfying $F\left([0]\right) = 0$.

**Definition 12.1.3.** Let $\mathbf{Mod}\Gamma$ denote the *category of right* Γ*-modules.* The objects are functors $\Gamma^{op} \to \mathbf{kMod}$ and the morphisms are natural transformations of functors.

**Definition 12.1.4.** Let $\Omega\mathbf{Mod}$ denote the *category of left* Ω*-modules.* The objects are functors $\Omega \to \mathbf{kMod}$ and the morphisms are natural transformations of functors.

## 12.2 Pirashvili's Dold-Kan Type Theorem

**Theorem 12.2.1 ([Pir00a, Theorem 3.1]).** *There exists an equivalence of categories*

$$\Gamma\mathbf{Mod}_0 \simeq \Omega\mathbf{Mod}.$$

□

*Remark 12.2.2.* The key step in proving this equivalence is to construct the *cross-effect functor* [Pir00a, Section 2],

$$cr\colon \Gamma\mathbf{Mod}_0 \to \Omega\mathbf{Mod}.$$

The name *Dold-Kan Type theorem* arises from the fact that the construction of the cross-effect functor $cr$ is inspired by the construction of the functor

$$K\colon \mathbf{ChCpx}\,(\mathbf{A}) \to \mathbf{sA}$$

in the Dold-Kan Correspondence. See [Wei94, Section 8.4] for details of the functor $K$.

The following variant of the theorem is obtained by Pirashvili as a corollary. The key point is that any left $\Gamma$-module $F$ can be written as

$$F = F' \oplus F_0$$

where $F' \in \Gamma\mathbf{Mod}_0$ and $F_0$ is a constant object in $\Gamma\mathbf{Mod}$ with value $F\big([0]\big)$. The fact that the category $\Gamma$ has a zero object, namely $[0]$, is crucial to this result.

**Corollary 12.2.3.** *There is an equivalence of categories*

$$\Gamma\mathbf{Mod} \simeq \Omega_{\emptyset}\mathbf{Mod}.$$ □

# Chapter 13

# $\Gamma$-homology as Functor Homology

## 13.1 The Functor $t$

**Definition 13.1.1.** Let

$$t \coloneqq \mathrm{Hom}_{\mathbf{Set}_\star}(-,k) : \Gamma^{op} \to \mathbf{kMod}$$

where the commutative ring $k$ is considered to be a based set with basepoint 0.

The functor $t$ is sometimes referred to as the *based $k$-cochain functor* [Rob18, Section 3.4].

The functor $t$ arises as the cokernel of a natural transformation of projective generators in $\mathbf{Mod}\Gamma$. We follow Pirashvili [Pir00b, Section 1.4]. Recall from Proposition 6.1.7 that the functors

$$\Gamma^n = k\left[\mathrm{Hom}_\Gamma\left(-,[n]\right)\right]$$

for $n \geqslant 0$ are projective generators for the category $\mathbf{Mod}\Gamma$.

**Definition 13.1.2.** For each $i \in [n]$ we define the *characteristic function*,

$$\chi_i\colon [n] \to k$$

by

$$\chi_i(j) = \begin{cases} 0 & i \neq j \\ 1_k & i = j. \end{cases}$$

*Remark 13.1.3.* For $i \neq 0$, $\chi_i$ is a morphism of based sets.

**Definition 13.1.4.** Let $i \in [n]$ be a non-zero element. We define the morphism

$$p_i\colon [n] \to [1]$$

in $\Gamma$ by

$$p_i(j) = \begin{cases} 0 & i \neq j \\ 1 & i = j. \end{cases}$$

**Definition 13.1.5.** We define the morphism

$$p\colon [2] \to [1]$$

in $\Gamma$ by

$$p(j) = \begin{cases} 0 & j = 0 \\ 1 & j = 1,\, 2. \end{cases}$$

**Definition 13.1.6.** Let $\alpha\colon \Gamma^2 \Rightarrow \Gamma^1$ be the natural transformation of right $\Gamma$-modules determined by

$$\alpha\left([n] \xrightarrow{f} [2]\right) = p_1 \circ f + p_2 \circ f - p \circ f.$$

**Definition 13.1.7.** Let $\beta\colon \Gamma^1 \Rightarrow t$ be the natural transformation of right $\Gamma$-modules determined by

$$\beta\left([n] \xrightarrow{g} [1]\right) = \sum_{g(i)=1} \chi_i.$$

**Proposition 13.1.8 ([Pir00b, Section 1.4]).** *There is an exact sequence*

$$\Gamma^2 \overset{\alpha}{\Rightarrow} \Gamma^1 \overset{\beta}{\Rightarrow} t \Rightarrow 0$$

*of right* $\Gamma$*-modules.* □

## 13.2 Γ-homology as Functor Homology

**Definition 13.2.1.** Let $F$ be a left $\Gamma$-module. We define the $\Gamma$*-homology of* $F$ by

$$H\Gamma_\star(F) \coloneqq \operatorname{Tor}^{\Gamma}_{\star}(t, F).$$

## 13.3 The Loday Functor and Γ-homology of a Commutative Algebra

**Definition 13.3.1.** Let $A$ be a commutative $k$-algebra and let $M$ be a symmetric $A$-bimodule. We define the *Loday functor*

$$\mathscr{L}(A, M)(-)\colon \Gamma \to \mathbf{kMod}$$

on objects by

$$\mathscr{L}(A, M)\left([n]\right) = A^{\otimes n} \otimes M.$$

For an element $f \in \operatorname{Hom}_\Gamma\left([p], [q]\right)$,

$$\mathscr{L}(A, M)(f)\colon A^{\otimes p} \otimes M \to A^{\otimes q} \otimes M$$

is determined by

$$\left(a_1 \otimes \cdots \otimes a_p \otimes m\right) \mapsto \left(\prod_{i \in f^{-1}(1)} a_i\right) \otimes \cdots \otimes \left(\prod_{i \in f^{-1}(q)} a_i\right) \otimes \left(\prod_{i \in f^{-1}(0)} a_i\right) m$$

where an empty product is understood to be $1_A \in A$.

**Definition 13.3.2.** Let $A$ be a commutative $k$-algebra and let $M$ be a symmetric $A$-bimodule. We define the $\Gamma$-*homology of* $A$ *with coefficients in* $M$ by

$$H\Gamma_\star(A, M) \coloneqq H\Gamma_\star\left(\mathscr{L}(A, M)\right).$$

# Chapter 14

# The Robinson-Whitehouse Complex

## Introduction

There is a standard chain complex that computes Γ-homology, called the *Robinson-Whitehouse complex.* This complex was first defined in the thesis of Sarah Whitehouse for commutative algebras over a commutative ground ring [Whi94, Definition II.4.1]. We recall the more general version for all left Γ-modules defined by Pirashvili and Richter [PR00, Section 2].

## 14.1 The Nerve of Ω

**Definition 14.1.1.** Let $N\Omega_n(\underline{x}, \underline{1})$ denote the set of strings of composable morphisms of length $n$ in the category Ω whose initial domain is the set $\underline{x}$ and whose final codomain is the set $\underline{1}$.

An element

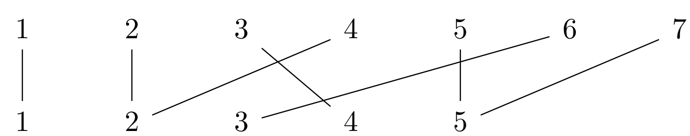
$$\underline{x} \xrightarrow{f_1} \underline{x_1} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}} \xrightarrow{f_n} \underline{1}$$

of $N\Omega_n(\underline{x}, \underline{1})$ will be denoted

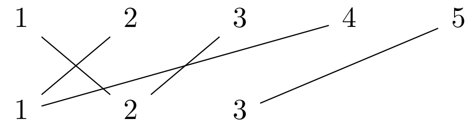
$$\left[f_n \mid \cdots \mid f_1\right].$$

We will find it useful to represent elements of $N\Omega_n(\underline{x}, \underline{1})$ as graphs.

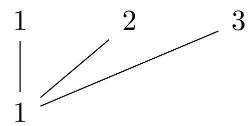
*Example 14.1.2.* Let $f_1 \in \mathrm{Hom}_\Omega(\underline{7}, \underline{5})$ be represented by the graph



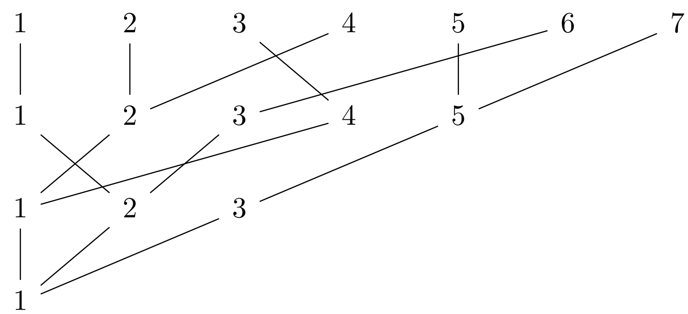
Let $f_2 \in \mathrm{Hom}_\Omega(\underline{5}, \underline{3})$ be represented by the graph




Let $f_3 \in \mathrm{Hom}_\Omega(\underline{3}, \underline{1})$ be represented by the graph



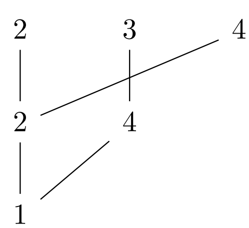

We represent the element $\left[f_3 \mid f_2 \mid f_1\right]$ in $N\Omega_3(\underline{7}, \underline{1})$ with the graph



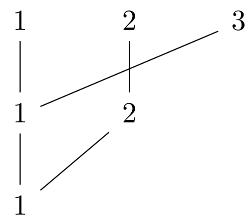

**Definition 14.1.3.** Let $\left[f_n \mid \cdots \mid f_1\right]$ be an element of $N\Omega_n(\underline{x}, \underline{1})$ thought of as a graph. We define the $i^{th}$ *component* of $\left[f_n \mid \cdots \mid f_1\right]$, denoted $\left[f^i_{n-1} \mid \cdots \mid f^i_1\right]$ to be the sub-graph consisting of the preimage of $i \in \underline{x_{n-1}}$, re-indexed such that

- the domain of the $j^{th}$ morphism is the set $\underline{\left|f_j^{-1} \ldots f_{n-1}^{-1}(i)\right|}$ and
- the final codomain is $\underline{1}$.

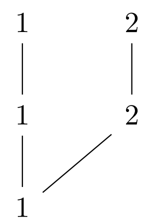
*Example 14.1.4.* Consider $\left[f_3 \mid f_2 \mid f_1\right]$ in $N\Omega_3(\underline{7}, \underline{1})$ from Example 14.1.2. Consider the element $1 \in \underline{3}$. The sub-graph of $\left[f_3 \mid f_2 \mid f_1\right]$ consisting of the preimage of $1 \in \underline{3}$ is



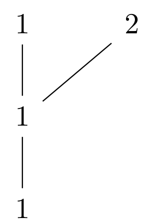

Re-indexing according to Definition 14.1.3 we see that $\left[f^1_2 \mid f^1_1\right]$ in $N\Omega_2(\underline{3}, \underline{1})$ is represented by the graph




Similarly we see that $\left[f_2^2 \mid f_1^2\right]$ in $N\Omega_2\left(\underline{2},\underline{1}\right)$ is represented by the graph




and $\left[f_2^3 \mid f_1^3\right]$ in $N\Omega_2\left(\underline{2},\underline{1}\right)$ is represented by the graph




**Definition 14.1.5.** Let $\left[f_n \mid \cdots \mid f_1\right]$ be an element of $N\Omega_n\left(\underline{x},\underline{1}\right)$. Let

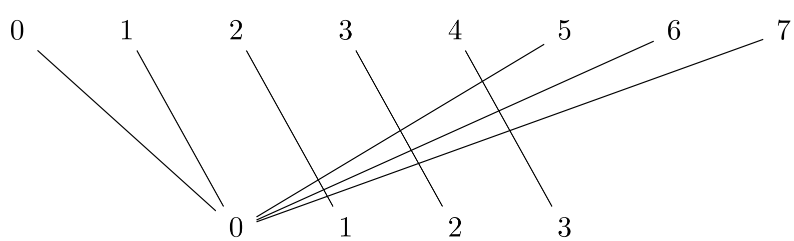
$$r_i = \left|f_1^{-1} \dots f_{n-1}^{-1}(i)\right|.$$

We define $l_i \in \mathrm{Hom}_\Gamma\left([x],[r_i]\right)$ to be non-zero only on elements of $f_1^{-1} \dots f_{n-1}^{-1}(i)$ in which case it is strictly order-preserving.

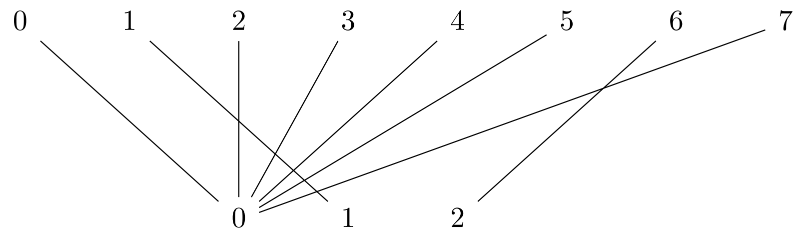
*Example 14.1.6.* Consider $\left[f_3 \mid f_2 \mid f_1\right]$ in $N\Omega_3\left(\underline{7},\underline{1}\right)$ from Example 14.1.2. In this case

- $f_1^{-1}f_2^{-1}(1) = \{2,3,4\}$ and $r_1 = 3$,
- $f_1^{-1}f_2^{-1}(2) = \{1,6\}$ and $r_2 = 2$ and
- $f_1^{-1}f_2^{-1}(3) = \{5,7\}$ and $r_3 = 2$

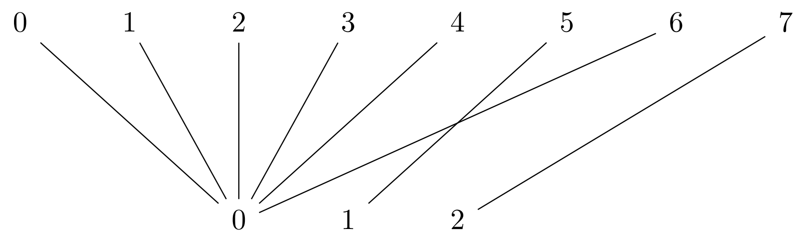
We can represent $l_1 \in \mathrm{Hom}_\Gamma\left([7],[3]\right)$ by the graph




We can represent $l_2 \in \mathrm{Hom}_\Gamma\left([7],[2]\right)$ by the graph




We can represent $l_3 \in \mathrm{Hom}_\Gamma\big([7],[2]\big)$ by the graph




## 14.2 The Robinson-Whitehouse Complex

Let $F$ be a left $\Gamma$-module. We define the simplicial $k$-module $C\Gamma_\star(F)$ as follows. In degree zero we set

$$C\Gamma_0(F) \coloneqq F\big([1]\big).$$

In degree $n$ we set

$$C\Gamma_n(F) \coloneqq \bigoplus_{x\geqslant 1} k\left[N\Omega_n\left(\underline{x},\underline{1}\right)\right] \otimes F\big([x]\big).$$

### 14.2.1 Face maps

Let $n=1$ and let $[f_1]\otimes y$ be a generator of $C\Gamma_1(F)$. That is, $f_1 \in \mathrm{Hom}_\Omega\left(\underline{x},\underline{1}\right)$ and $y \in F\big([x]\big)$.

We define the face maps

$$\partial_0,\, \partial_1 \colon C\Gamma_1(F) \to C\Gamma_0(F)$$

to be determined by

$$\partial_0\big([f_1]\otimes y\big) = F(f_1)(y)$$

and

$$\partial_1\big([f_1]\otimes y\big) = \sum_{i=1}^{x} F(p_i)(y)$$

where the $p_i$ are the morphisms of Definition 13.1.4.

Let $n \geqslant 2$ and let $\left[f_n \mid \cdots \mid f_1\right]\otimes y$ be a generator of $C\Gamma_n(F)$. That is,

$$\left[f_n \mid \cdots \mid f_1\right] = \underline{x} \xrightarrow{f_1} \underline{x_1} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}} \xrightarrow{f_n} \underline{1}$$

is an element of $N\Omega_n(\underline{x}, \underline{1})$ and $y \in F\big([x]\big)$.

We define the face maps
$$\partial_i \colon C\Gamma_n(F) \to C\Gamma_{n-1}(F)$$
for $0 \leqslant i \leqslant n$ to be determined by
$$\partial_0\Big(\big[f_n \mid \cdots \mid f_1\big] \otimes y\Big) = \big[f_n \mid \cdots \mid f_2\big] \otimes F(f_1)(y),$$
$$\partial_i\Big(\big[f_n \mid \cdots \mid f_1\big] \otimes y\Big) = \big[f_n \mid \cdots \mid f_{i+1} \circ f_i \mid \cdots \mid f_1\big] \otimes y$$
for $1 \leqslant i \leqslant n-1$ and
$$\partial_n\Big(\big[f_n \mid \cdots \mid f_1\big] \otimes y\Big) = \sum_{i \in \underline{x_{n-1}}} \Big[f^i_{n-1} \mid \cdots \mid f^i_1\Big] \otimes F\left(l_i\right)(y)$$
where the $l_i$ are the morphisms of Definition 14.1.5.

### 14.2.2 Degeneracy maps

Let $n = 0$ and let $y \in C\Gamma_0(F)$. We define the degeneracy map
$$s_0 \colon C\Gamma_0(F) \to C\Gamma_1(F)$$
to be determined by
$$s_0(y) = \big[id_{\underline{1}}\big] \otimes y.$$

Let $n \geqslant 1$ and let $\big[f_n \mid \cdots \mid f_1\big] \otimes y$ be a generator of $C\Gamma_n(F)$. That is,
$$\big[f_n \mid \cdots \mid f_1\big] = \underline{x} \xrightarrow{f_1} \underline{x_1} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}} \xrightarrow{f_n} \underline{1}$$
is an element of $N\Omega_n(\underline{x}, \underline{1})$ and $y \in F\big([x]\big)$.

We define the degeneracy maps
$$s_j \colon C\Gamma_n(F) \to C\Gamma_{n+1}(F)$$
for $0 \leqslant j \leqslant n$ to be determined by
$$s_0\Big(\big[f_n \mid \cdots \mid f_1\big] \otimes y\Big) = \big[f_n \mid \cdots \mid f_1 \mid id_{\underline{x}}\big] \otimes y$$
and
$$s_j\Big(\big[f_n \mid \cdots \mid f_1\big] \otimes y\Big) = \Big[f_n \mid \cdots \mid id_{\underline{x_j}} \mid f_j \mid \cdots \mid f_1\Big] \otimes y$$
for $1 \leqslant j \leqslant n$.

### 14.2.3 The Robinson-Whitehouse complex

**Definition 14.2.1.** Let $F \in \mathbf{CMod}$. Let $C\Gamma_\star(F)$ be defined by

$$C\Gamma_0(F) \coloneqq F\left([1]\right)$$

and

$$C\Gamma_n(F) \coloneqq \bigoplus_{x \geqslant 1} k\left[N\Omega_n\left(\underline{x}, \underline{1}\right)\right] \otimes F\left([x]\right)$$

with face maps defined in Subsection 14.2.1 and degeneracy maps defined in Subsection 14.2.2. We call the associated chain complex of $C\Gamma_\star(F)$ the *Robinson-Whitehouse complex of $F$*.

*Remark 14.2.2.* If we take $F = \mathscr{L}(A, M)$, the Loday functor of Definition 13.3.1, we recover the chain complex of [Whi94, Definition II.4.1].

**Theorem 14.2.3 ([PR00, Theorem 1]).** *Let $F$ be a left $\Gamma$-module. There is an isomorphism of $k$-modules*

$$H\Gamma_n(F) \cong H_n\left(C\Gamma_\star(F)\right)$$

*for each $n \geqslant 0$.* □

*Remark 14.2.4.* We observe from the definition of the Robinson-Whitehouse complex and Pirashvili's Dold-Kan Type Theorem that in order to compute Γ-homology it is sufficient to work with the category Ω. That is, we only require epimorphisms. We will see in Part VI that for certain homology theories arising from crossed simplicial groups we also have reduction to epimorphisms in certain cases. However these do not arise from Dold-Kan Type theorems. We will comment more on this in the introduction to Part VI.

### 14.2.4 Normalized Robinson-Whitehouse complex

Since the Robinson-Whitehouse complex is a chain complex associated to a simplicial $k$-module we can form the normalized complex following Subsection 2.5.12.

**Definition 14.2.5.** The *normalized Robinson-Whitehouse complex*, $NC\Gamma_\star(F)$, considered as a quotient of $C\Gamma_\star(F)$, is generated in degree $n$ by equivalence classes of the form

$$\left[\left[f_n \mid \cdots \mid f_1\right] \otimes y\right],$$

where $\left[f_n \mid \cdots \mid f_1\right]$ denotes a string of morphisms

$$\underline{x} \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}} \xrightarrow{f_n} \underline{1}$$

in $N_n\Omega(\underline{x}, \underline{1})$ such that each $f_i$ for $1 \leqslant i \leqslant n$ is not an identity morphism and $y \in F\left([x]\right)$. The differential is induced from the differential in the Robinson-Whitehouse complex.

**Proposition 14.2.6.** *Let $F \in \Gamma\mathbf{Mod}$. There is an isomorphism of $k$-modules*

$$H\Gamma_n(F) \cong H_n\left(NC\Gamma_\star(F)\right)$$

*for each $n \geqslant 0$.* □

# Chapter 15

# Calculations of Γ-homology

## 15.1 Calculations of Γ-homology

**Proposition 15.1.1 ([RW02, Proposition 5.5 (1)]).** *For $A$ a commutative $k$-algebra and $M$ a symmetric $A$-bimodule there is an isomorphism of $k$-modules*

$$H\Gamma_0(A, M) \cong \Omega^1_{A|k} \otimes_A M,$$

*where $\Omega^1_{A|k}$ is the module of Kähler differentials of Definition 4.5.4.* □

**Proposition 15.1.2 ([RR04, Proposition 2.1]).** *The Γ-homology of the integral group ring on an abelian group $G$ is isomorphic to the integral homology of the Eilenberg-Mac Lane spectrum of the group $G$. That is,*

$$H\Gamma_\star\left(\mathbb{Z}[G], \mathbb{Z}\right) \cong H\mathbb{Z}_\star HG.$$ □

*Remark 15.1.3.* The more general case where $\mathbb{Z}$ is replaced by an arbitrary commutative ring $k$ also holds, see [RR04, Proposition 3.1].

**Proposition 15.1.4 ([RR04, Proposition 3.2]).** *If $A$ is an augmented, commutative $k$-algebra and some abelian group algebra $k[G]$ is étale over $A$, then the Γ-homology of $A$ is isomorphic to that of $k[G]$, thus*

$$H\Gamma_\star\left(A, k\right) \cong Hk_\star HG.$$

*In particular, the Γ-homology of a polynomial algebra $k[x]$ is isomorphic to the $k$-homology of $H\mathbb{Z}$ because $k\left[x, x^{-1}\right] \cong k[\mathbb{Z}]$ is étale over $k[x]$.* □

**Theorem 15.1.5 ([RR04, Theorem 3.3]).** *Let $A$ be a smooth, augmented $k$-algebra. The Γ-homology of $A$ consists of the direct sum of as many copies of $Hk_\star H\mathbb{Z}$ as the dimension of the module of Kähler differentials $\Omega^1_{A|k} \otimes_A k$. That is, there is an isomorphism of $k$-modules*

$$H\Gamma_\star\left(A, k\right) \cong \Omega^1_{A|k} \otimes_A Hk_\star H\mathbb{Z}.$$ □

# Chapter 16

# Tree Spaces, the Category $\Gamma$ and Harrison Homology

## Introduction

When the ground ring $k$ contains $\mathbb{Q}$, $\Gamma$-homology coincides with Harrison homology. This fact will be very useful when we come to define a comparison map between symmetric homology, to be discussed in Chapter 24, and $\Gamma$-homology in Part VII. In this chapter we recall the details required to prove that $\Gamma$-homology and Harrison homology coincide in this case following Whitehouse [Whi94].

$\Gamma$-homology can be seen, in one sense, as a way of building an action of the symmetric groups into Hochschild homology. This action occurs via tree spaces. Loosely speaking, one forms a collection of topological spaces, $\{T_n\}$ for $n \geqslant 2$, from labelled trees, each of which has only one non-trivial reduced homology group. The symmetric group $\Sigma_n$ acts on the topological space $T_n$ and the non-trivial reduced homology group gives a representation of $\Sigma_n$. This representation is also related to the category $\Omega$.

## 16.1 The Space of Fully-Grown Trees

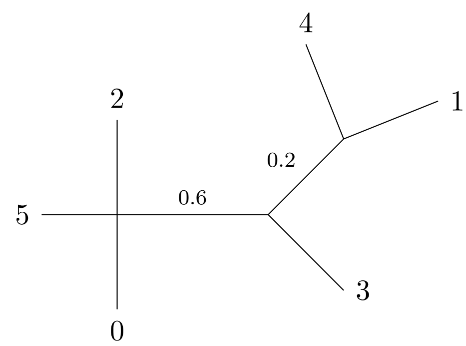
The material in this subsection follows [Whi94, Section II.2].

**Definition 16.1.1.** A *tree* is a compact contractible one-dimensional polyhedron. It is always triangulated so that each vertex is either an *end*, belonging to exactly one edge called a *free edge*, or a *node*, belonging to at least three edges. Edges that are not free are called *internal edges*.

**Definition 16.1.2.** Let $n \geqslant 2$. An *$n$-tree* is a tree such that

- it has exactly $n+1$ ends, labelled by $0, 1, \ldots, n$
- each internal edge $\alpha$ has a *length*, $l(\alpha)$, $0 < l(\alpha) \leqslant 1$.

*Example 16.1.3.* A 5-tree.

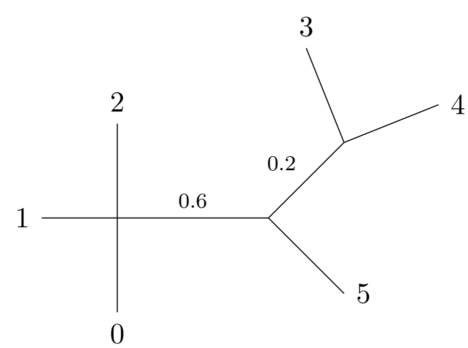



**Definition 16.1.4.** An *isomorphism of $n$-trees* is a homeomorphism which is isometric on edges and which preserves the labelling of the ends.

**Definition 16.1.5.** We denote by $\overline{T_n}$ the space of isomorphism classes of $n$-trees. See [RW96, Section 1] for details of the space of $\overline{T_n}$.

**Definition 16.1.6.** An $n$-tree has a *cyclic labelling* if it can be drawn in such a way that the labels $0, \ldots, n$ occur in order as one goes around the tree.

*Example 16.1.7.* A 5-tree with cyclic labelling.




**Definition 16.1.8.** The topological space $T_n$, called the *space of fully-grown $n$-trees*, consists of isomorphism classes of trees that have at least one internal edge length equal to 1.

**Proposition 16.1.9 ([Whi94, Proposition II.2.4]).** *The space of fully-grown $n$-trees, $T_n$, can be triangulated as a simplicial complex such that every simplex is the face of an $(n-3)$-simplex. Furthermore, every $(n-4)$-simplex of $T_n$ is a face of precisely three $(n-3)$-simplicies.* □

*Remark 16.1.10.* A simplex of $T_n$ corresponds to an equivalence class of fully-grown $n$-trees under label-preserving homeomorphism. In other words, it corresponds to the shape of an $n$-tree. The faces of the simplicial complex correspond to shrinking an internal edge of the tree to zero.

## 16.2 Topological Properties of the Space of Fully-Grown Trees

### 16.2.1 Homotopy type of the space of fully-grown trees

**Theorem 16.2.1 ([Whi94, Theorem II.2.5]).** *There is a homotopy equivalence*

$$T_n \simeq \bigvee_{(n-1)!} S^{n-3}.$$

□

**Corollary 16.2.2.** *The space of fully-grown trees, $T_n$, has one non-trivial reduced homology group with coefficients in $k$,*

$$\overline{H}_{n-3}(T_n, k) = k^{\oplus (n-1)!}.$$

□

### 16.2.2 The module $V_n$

The symmetric group $\Sigma_n$ acts on the space of fully-grown trees $T_n$ by permuting the non-zero labels on $n$-trees. Let $\sigma \in \Sigma_n$. The group $\Sigma_n$ acts on the left of $T_n$ by applying the permutation $\sigma$ to the non-zero labels on $n$-trees. In fact, $\Sigma_n$ also acts on the right of $T_n$ by applying the permutation $\sigma^{-1}$ to the non-zero labels of an $n$-tree. The homology group $\overline{H}_{n-3}(T_n, k)$ gives a representation of $\Sigma_n$ and has the structure of both a left and right $k\Sigma_n$-module. Observe that the actions described here extend to formal $k$-linear combinations of $n$-trees.

**Definition 16.2.3.** We denote the homology group $\overline{H}_{n-3}(T_n, k)$, considered as a right $k\Sigma_n$-module, by $V_n$.

### 16.2.3 The cycle $c_n$

**Proposition 16.2.4 ([Whi94, Section III.1]).** *The $(n-3)$-dimensional simplicies of $T_n$ given by $n$-trees with a cyclic labelling can be oriented so that they form a cycle in the simplicial complex of Proposition 16.1.9.* □

*Remark 16.2.5.* This result is non-trivial to prove. Full combinatorial details are given in [Whi94, Section III.1].

**Definition 16.2.6.** We will denote the cycle corresponding to that of Proposition 16.2.4 in the associated chain complex by $c_n$.

*Remark 16.2.7.* The cycle $c_n$ is a $k$-linear combination over all isomorphism classes of $n$-trees with $(n-2)$ internal edges and a cyclic labelling.

The right action of the symmetric group $\Sigma_n$ on $T_n$ restricts to an action of $\Sigma_{n-1}$, where we only permute the labels $1, \ldots, n-1$.

**Proposition 16.2.8 ([Whi94, Section III.1]).** *Recall the right $k\Sigma_n$-module $V_n$ of Definition 16.2.3. Let $\sigma \in \Sigma_{n-1}$. The k-linear combination $c_n\sigma$, where $\sigma$ runs through all elements of $\Sigma_{n-1}$, form a basis for $V_n$.* □

## 16.3 The Connection with the Category $\Omega$

We have the following relationship between the category $\Omega$ and the space of fully-grown trees. Recall the category $\underline{n}/\Omega/\underline{1}$ from Definition 11.1.5.

**Proposition 16.3.1 ([Whi94, Proposition II.3.2]).** *There is a $\Sigma_n$-equivariant map*

$$\Phi\colon \left|N_\star\left(\underline{n}/\Omega/\underline{1}\right)\right| \to T_n,$$

*from the realization of the nerve of the category $\underline{n}/\Omega/\underline{1}$ to the space of fully-grown trees, which is a homotopy equivalence.*

*In particular,*

$$\overline{H}_{n-3}\left(\left|N_\star\left(\underline{n}/\Omega/\underline{1}\right)\right|, k\right) = \overline{H}_{n-3}\left(T_n, k\right)$$

*is the only non-trivial reduced homology group of $\left|N_\star\left(\underline{n}/\Omega/\underline{1}\right)\right|$.* □

# Chapter 17

# Gamma Homology in Characteristic Zero

## Introduction

We recall a variant of the Γ-complex when the ground ring contains $\mathbb{Q}$ and prove that its homology coincides with Harrison homology, up to a shift in degree.

## 17.1 The Rational Γ-complex

Suppose $k \supseteq \mathbb{Q}$. Let $A$ be a flat commutative $k$-algebra and let $M$ be a symmetric $A$-bimodule.

Recall that $A^{\otimes n} \otimes M$ is a left $k\Sigma_n$-module, where $\Sigma_n$ acts by permuting the tensor factors.

**Definition 17.1.1.** We define

$$V_n\Gamma(A, M) \coloneqq V_n \otimes_{k\Sigma_n} \left(A^{\otimes n} \otimes M\right).$$

We consider a $k$-module generator of $V_n\Gamma(A, M)$ to be an equivalence class of the form

$$\left[c_n\sigma \otimes (a_1 \otimes \cdots \otimes a_n \otimes m)\right]$$

where $c_n\sigma$ is a generator of $V_n$ and $(a_1 \otimes \cdots \otimes a_n \otimes m)$ is a generator of $A^{\otimes n} \otimes M$.

The boundary map

$$d\colon V_n\Gamma(A, M) \to V_{n-1}\Gamma(A, M)$$

is determined by

$$\left[c_n\sigma \otimes (a_1 \otimes \cdots \otimes a_n \otimes m)\right] \mapsto \left[c_{n-1} \otimes b\left(\sigma\left(a_1 \otimes \cdots \otimes a_n \otimes m\right)\right)\right]$$

where $b$ is the Hochschild boundary map.

We call $V_\star\Gamma(A, M)$ the *rational* $\Gamma$*-complex*.

**Proposition 17.1.2.** *Let $k \supseteq \mathbb{Q}$. Let $A$ be a flat commutative $k$-algebra and let $M$ be a symmetric $A$-bimodule. There is an isomorphism of $k$-modules*

$$H\Gamma_{n-1}(A, M) \cong H_n\big(V_\star\Gamma(A, M)\big)$$

*for each $n \geqslant 1$.* □

*Remark 17.1.3.* We obtain the *rational* $\Gamma$*-complex* by considering a filtration of the $\Gamma$-complex $C\Gamma_\star\big(\mathscr{L}(A, M)\big)$. For full details see [Whi94, Sections II.4 and III.4].

## 17.2 Relationship with Harrison Homology

Recall the Eulerian idempotent $e_n^{(1)}$ from Definition 5.3.8.

**Lemma 17.2.1 ([Whi94, Proposition III.3.2]).** *There is an isomorphism*

$$V_n \cong e_n^{(1)} k\Sigma_n$$

*of right $k\Sigma_n$-modules determined by*

$$c_n\sigma \mapsto e_n^{(1)}\sigma$$

*for $\sigma \in \Sigma_{n-1}$.* □

**Proposition 17.2.2 ([Whi94, Theorem III.4.2]).** *Let $k \supseteq \mathbb{Q}$. Let $A$ be a flat commutative $k$-algebra and let $M$ be a symmetric $A$-bimodule. There is an isomorphism of $k$-modules*

$$H\Gamma_{n-1}(A, M) \cong Harr_n(A, M)$$

*for each $n \geqslant 1$.*

*Proof.* There is an isomorphism of chain complexes between the rational $\Gamma$-complex and the Harrison complex, viewed as a subcomplex of the Hochschild complex.

The isomorphism is induced from the maps

$$\Theta_n \colon V_n\Gamma\left(A, M\right) \to e_n^{(1)} C_n\left(A, M\right)$$

determined by

$$[c_n\sigma \otimes (a_1 \otimes \cdots \otimes a_n \otimes m)] \mapsto e_n^{(1)}\sigma(a_1 \otimes \cdots \otimes a_n \otimes m)$$

and

$$\Psi_n \colon e_n^{(1)} C_n\left(A, M\right) \to V_n\Gamma\left(A, M\right)$$

determined by

$$e_n^{(1)}(a_1 \otimes \cdots \otimes a_n \otimes m) \mapsto [c_n \otimes (a_1 \otimes \cdots \otimes a_n \otimes m)].$$ □

*Remark 17.2.3.* It follows from Theorem 5.5.5 that when the ground ring $k$ contains $\mathbb{Q}$, $\Gamma$-homology coincides with André-Quillen homology.

# Chapter 18

# Gamma Homology of an Augmented Algebra

## Introduction

We prove that the Γ-complex for an augmented, commutative $k$-algebra splits as a direct sum, one summand of which is built from the augmentation ideal. The key is to define the *pruning map* of Definition 18.2.4.

## 18.1 The Γ-complex for the Augmentation Ideal

### 18.1.1 Basic tensors

Let $A_\varepsilon$ be an augmented, commutative $k$-algebra with augmentation ideal $I$ and let $M$ be a symmetric $A_\varepsilon$-bimodule. In this case $C\Gamma_n(A,M)$ is generated $k$-linearly by tensors of the form

$$\left[f_n \mid \cdots \mid f_1\right] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m$$

where

$$\left[f_n \mid \cdots \mid f_1\right] = \underline{x} \xrightarrow{f_1} \underline{x_1} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}} \xrightarrow{f_n} \underline{1}$$

is an element of $N\Omega_n\left(\underline{x}, \underline{1}\right)$, $(a_1 \otimes \cdots \otimes a_x)$ is a basic tensor in $A^{\otimes x}$ and $m \in M$. Recall from Definition 4.4.9 that a basic tensor is such that each $a_i$ is either an element of the augmentation ideal $I$ or is equal to $1_k$.

### 18.1.2 The Γ-complex for the augmentation ideal

Let $A_\varepsilon$ be an augmented, commutative $k$-algebra with augmentation ideal $I$ and let $M$ be a symmetric $A_\varepsilon$-bimodule. We can form the Γ-complex for the augmentation ideal $I$

rather that the algebra $A_\varepsilon$. In order to check that this is well-defined we need to show that reducing to the augmentation ideal is compatible with the face maps. Let

$$\left[f_n \mid \cdots \mid f_1\right] \otimes (y_1 \otimes \cdots \otimes y_x) \otimes m$$

be a generator of

$$C\Gamma_n(I, M) = \bigoplus_{x \geqslant 1} kN\Omega_n\left(\underline{x}, \underline{1}\right) \otimes I^{\otimes x} \otimes M.$$

In order to ease notation we let

$$f_\star \coloneqq \mathscr{L}(A, M)(f).$$

The only face maps that interact with the tensor $(y_1 \otimes \cdots \otimes y_x) \in I^{\otimes x}$ are $\partial_0$ and $\partial_n$.

We observe that

$$\partial_0\left(\left[f_n \mid \cdots \mid f_1\right] \otimes (y_1 \otimes \cdots \otimes y_x) \otimes m\right)$$

is equal to

$$\left[f_n \mid \cdots \mid f_2\right] \otimes f_{1\star}\left(y_1 \otimes \cdots \otimes y_x\right) \otimes m.$$

Since $f_1$ is a surjective map of sets, the action of $f_{1\star}$ on $(y_1 \otimes \cdots \otimes y_x)$ permutes tensor factors and multiplies tensor factors together. Since the augmentation ideal $I$ is closed under multiplication we see that

$$\partial_0\left(\left[f_n \mid \cdots \mid f_1\right] \otimes (y_1 \otimes \cdots \otimes y_x) \otimes m\right)$$

is an element of $C\Gamma_{n-1}\left(I, M\right)$ as required.

The final face map $\partial_n$ is compatible with the augmentation ideal. Since $I$ is a $k$-subalgebra of $A_\varepsilon$ we have a compatible action of the augmentation ideal $I$ on the $A_\varepsilon$-bimodule $M$ by restriction.

**Definition 18.1.1.** Let $A_\varepsilon$ be an augmented $k$-algebra with augmentation ideal $I$ and let $M$ be a symmetric $A_\varepsilon$-bimodule. We define the chain complex $C\Gamma_\star(I, M)$ in degree $n$ by

$$C\Gamma_n(I, M) \coloneqq \bigoplus_{x \geqslant 1} k\left[N\Omega_n\left(\underline{x}, \underline{1}\right)\right] \otimes I^{\otimes x} \otimes M$$

with the boundary map given by the alternating sum of the face maps.

**Proposition 18.1.2.** *Let $A_\varepsilon$ be an augmented, commutative $k$-algebra with augmentation ideal $I$ and let $M$ be a symmetric $A_\varepsilon$-bimodule which is flat over $k$. The chain complex $C\Gamma_\star(I, M)$ is a subcomplex of $C\Gamma_\star\left(A_\varepsilon, M\right)$.*

*Proof.* The proof is similar to that of Lemma 5.2.3. Note also that the proof of Lemma 5.2.2 implies that we do not need any flatness condition on $I$ for this result. □

### 18.1.3 Normalization

We can view $C\Gamma_\star(I,M)$ as a chain complex associated to a simplicial $k$-module by considering the degeneracy maps of Subsection 14.2.2, which insert identities into the string of morphisms. We can therefore form the normalized complex following Subsection 2.5.12.

**Definition 18.1.3.** The normalized complex, $NC\Gamma_\star(I,M)$, considered as a quotient of $C\Gamma_\star(I,M)$, is generated in degree $n$ by equivalence classes of the form

$$\left[\left[f_n \mid \cdots \mid f_1\right] \otimes (y_1 \otimes \cdots \otimes y_x) \otimes m\right],$$

where $\left[f_n \mid \cdots \mid f_1\right]$ denotes a string of morphisms

$$\underline{x} \xrightarrow{f_1} \underline{x_1} \xrightarrow{f_2} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}} \xrightarrow{f_n} \underline{1}$$

in $N_n\Omega(\underline{x},\underline{1})$ such that no $f_i$ is an identity map and $(y_1 \otimes \cdots \otimes y_x) \otimes m \in I^{\otimes x} \otimes M$. The differential is induced from the differential of $C\Gamma_\star(I,M)$.

We deduce the following proposition from Corollary 2.5.16.

**Proposition 18.1.4.** *There is an isomorphism of $k$-modules*

$$H_n\left(C\Gamma_\star(I,M)\right) \cong H_n\left(NC\Gamma_\star(I,M)\right)$$

*for each $n \geqslant 0$.* □

## 18.2 The Pruning Map

Let $A_\varepsilon$ be an augmented, commutative $k$-algebra with augmentation ideal $I$ and let $M$ be a symmetric $A_\varepsilon$-bimodule which is flat over $k$. We will demonstrate that the chain complex $C\Gamma_\star(I,M)$ is a direct summand of the chain complex $C\Gamma_\star(A_\varepsilon,M)$. We do so by providing a splitting map called the *pruning map*.

**Definition 18.2.1.** Let

$$\left[f_n \mid \cdots \mid f_1\right] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m$$

be a generator of $C\Gamma_n(A_\varepsilon,M)$ such that $(a_1 \otimes \cdots \otimes a_x) \otimes m$ is a basic tensor in the sense of Definition 4.4.9.

Let $L = \{l_1, \ldots, l_h\}$ be the set such that $a_i \in I$ if and only if $i \in L$. Let

$$m_i := \operatorname{Im}\left(\left(f_i \circ \cdots \circ f_1\right)\big|_L\right).$$

Let

$$\widetilde{f_1} \colon \underline{|L|} \to \underline{|m_1|}$$

denote the map obtained from $f_1$ by restricting the domain to the set $L$, restricting the codomain to $m_1$ and re-indexing both domain and codomain in the canonical way.

For $2 \leqslant i \leqslant n$ let

$$\widetilde{f}_i \colon \underline{|m_{i-1}|} \to \underline{|m_i|}$$

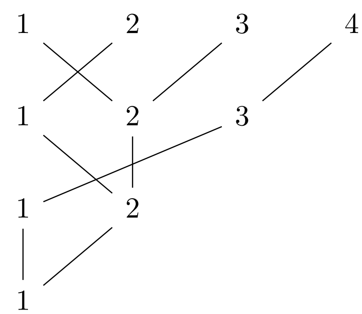
denote the map obtained from $f_i$ by restricting the domain to the set $m_{i-1}$, restricting the codomain to the set $m_i$ and re-indexing both domain and codomain in the canonical way.

*Remark 18.2.2.* Observe that this ensures that the maps $\widetilde{f}_i$ are surjective.

*Example 18.2.3.* Let

$$\left[f_3 \mid f_2 \mid f_1\right] \otimes (y_1 \otimes 1_k \otimes 1_k \otimes y_4) \otimes m$$

be a generator of $C\Gamma_3\,(A_\varepsilon, M)$ where $y_1,\ y_4 \in I$ and $\left[f_3 \mid f_2 \mid f_1\right]$ is represented by the graph



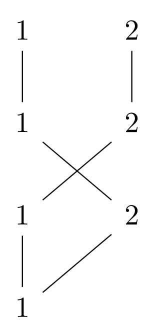

In this case,

- $L = \{1, 4\}$,
- $m_1 = \{2, 3\}$
- $m_2 = \{1, 2\}$ and
- $m_3 = \{1\}$.

We observe that

$$\left[\widetilde{f}_3 \mid \widetilde{f}_2 \mid \widetilde{f}_1\right]$$

is represented by the graph



**Definition 18.2.4.** Let $A_\varepsilon$ be an augmented, commutative $k$-algebra with augmentation ideal $I$. Let $M$ be a symmetric $A_\varepsilon$-bimodule which is flat over $k$. We define the *pruning map*

$$P_\star \colon C\Gamma_\star(A_\varepsilon, M) \to C\Gamma_\star(I, M)$$

to be the $k$-linear map of chain complexes determined in degree $n$ by

$$\begin{array}{c}\left[f_n \mid \cdots \mid f_1\right] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m \\ \big\downarrow \\ \left[\widetilde{f_n} \mid \cdots \mid \widetilde{f_1}\right] \otimes \left(a_{l_1} \otimes \cdots \otimes a_{l_h}\right) \otimes m\end{array}$$

*Remark 18.2.5.* Intuitively, the pruning map removes the trivial tensor factors from the basic tensor $(a_1 \otimes \cdots \otimes a_x) \in A_\varepsilon^{\otimes x}$ and prunes the graph in order to preserve the permutations and multiplications of the non-trivial tensor factors.

*Example 18.2.6.* Let

$$\left[f_3 \mid f_2 \mid f_1\right] \otimes (y_1 \otimes 1_k \otimes 1_k \otimes y_4) \otimes m$$

be the generator of $C\Gamma_3(A_\varepsilon, M)$ from Example 18.2.3. We observe that

$$P_3\left(\left[f_3 \mid f_2 \mid f_1\right] \otimes (y_1 \otimes 1_k \otimes 1_k \otimes y_4) \otimes m\right)$$

is equal to

$$\left[\widetilde{f_3} \mid \widetilde{f_2} \mid \widetilde{f_1}\right] \otimes (y_1 \otimes y_4) \otimes m.$$

The proof that the pruning map is a well-defined map of chain complexes can be found in Appendix A.

## 18.3 Properties of the Pruning Map

**Theorem 18.3.1.** *Let $A_\varepsilon$ be an augmented, commutative $k$-algebra with augmentation ideal $I$. Let $M$ be a symmetric $A_\varepsilon$-bimodule which is flat over $k$. Let*

$$i \colon C\Gamma_\star(I, M) \to C\Gamma_\star(A_\varepsilon, M)$$

*denote the inclusion of the subcomplex. The composite*

$$P_\star \circ i \colon C\Gamma_\star(I, M) \to C\Gamma_\star(I, M)$$

*is the identity map.*

*Proof.* An element in the image of $i$ is a $k$-linear combination of generators of the form

$$\left[f_n \mid \cdots \mid f_1\right] \otimes (y_1 \otimes \cdots \otimes y_x) \otimes m$$

such that $(y_1 \otimes \cdots \otimes y_x) \in I^{\otimes x}$. In particular, $(y_1 \otimes \cdots \otimes y_x)$ contains no trivial factors. By construction, the pruning map $P_\star$ is the identity on such elements. ☐

**Corollary 18.3.2.** *Under the conditions of Theorem 18.3.1, there is an isomorphism of chain complexes*

$$C\Gamma_\star\left(A_\varepsilon, M\right) \cong C\Gamma_\star(I, M) \oplus \operatorname{Ker}\left(P_\star\right). \qquad \square$$

**Corollary 18.3.3.** *Under the conditions of Theorem 18.3.1, there is an isomorphism of $k$-modules*

$$H\Gamma_n\left(A_\varepsilon, M\right) \cong H\Gamma_n(I, M) \oplus H_n\left(\operatorname{Ker}(P_\star)\right)$$

*for each $n \geqslant 0$.* $\square$

*Remark 18.3.4.* In particular, under the conditions of Theorem 18.3.1, the chain complex $C\Gamma_\star\left(I, M\right)$ is a direct summand of the $\Gamma$-complex $C\Gamma_\star\left(A_\varepsilon, M\right)$ and the homology of $C\Gamma_\star\left(I, M\right)$ is a direct summand of the $\Gamma$-homology of $A_\varepsilon$ with coefficients in $M$.

# Part IV

# Crossed Simplicial Groups

# Introduction

Hochschild [Hoc45] introduced a cohomology theory in order to classify algebra extensions for associative algebras over a field. Cartan and Eilenberg introduced the corresponding homology theory for algebras over a commutative ring, as discussed in Section 5.1.

Connes [Con83] and Tsygan [Tsy83] independently introduced the notion of cyclic homology. Cyclic homology can be viewed as being constructed from Hochschild homology by incorporating an action of the cyclic groups, as discussed in Section 5.6.

This raises a natural question: can one build actions of other families of groups into Hochschild homology and obtain an interesting homology theory? Fiedorowicz and Loday [FL91] and Krasauskas [Kra87] independently answered this question in the affirmative with the theory of crossed simplicial groups.

In Chapter 19 we recall the equivalent definitions of crossed simplicial groups together with examples and the necessary theory that we will require for the rest of the thesis. Of particular importance is the fact that given a crossed simplicial group there is a standard way to associate a functor homology theory.

In Chapters 20 and 21 we describe the functor homology generalizations of Hochschild homology and cyclic homology and recover the algebraic theories of Sections 5.1 and 5.6.

In Chapter 22 we introduce the reflexive crossed simplicial group and its associated category.

In Chapter 23 we describe the dihedral crossed simplicial group and its associated homology theory. We recall its relation to $O(2)$-equivariant homology.

A result of Fiedorowicz and Loday [FL91, Theorem 6.16] implies that the homology theories associated to the symmetric crossed simplicial group and the hyperoctahedral crossed simplicial group in the standard way are isomorphic to Hochschild homology. Fortunately, Fiedorowicz [Fie] introduced interesting variants of symmetric homology and hyperoctahedral homology.

In Chapter 24 we describe the symmetric crossed simplicial group and discuss the problems with the homology theory constructed in the standard way. We recall Fiedorowicz's construction of symmetric homology and state a result connecting the symmetric homology of a group algebra to the homology of infinite loop spaces.

In Chapter 25 we describe the hyperoctahedral crossed simplicial group. We provide a presentation of the associated category. We recall Fiedorowicz's construction of hyperoctahedral homology and state a result connecting the hyperoctahedral homology of a group algebra to the $\mathbb{Z}/2\mathbb{Z}$-equivariant homology of an infinite loop space.

# Chapter 19

# Crossed Simplicial Groups

## Introduction

In this section we recall the general theory of *crossed simplicial groups* following Fiedorowicz and Loday [FL91].

In Section 19.1 we recall several equivalent definitions of crossed simplicial groups.

In Section 19.2 we provide some important examples of crossed simplicial groups, known as the *fundamental crossed simplicial groups*. These include the cyclic, dihedral, symmetric and hyperoctahedral groups.

To each crossed simplicial group there is an associated category. In Section 19.3 we describe the subcategories of epimorphisms and monomorphisms for categories associated to crossed simplicial groups. The subcategory of epimorphisms will play an important role in Part VI.

Finally in Sections 19.4 and 19.5 we describe how to associate a functor homology theory to a crossed simplicial group and the connection to equivariant homology theories.

## 19.1 Definitions

**Definition 19.1.1.** A sequence of groups $\{G_n\}$, $n \geqslant 0$, is a *crossed simplicial group* if it is equipped with the following structure. There is a small category $\Delta G$, which is part of the structure, such that,

1. the objects of $\Delta G$ are the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$,
2. $\Delta G$ contains $\Delta$ as a subcategory, where $\varphi \in \operatorname{Hom}_{\Delta}\left([n], [m]\right)$ is written as $\left(\varphi, id_{[n]}\right) \in \operatorname{Hom}_{\Delta G}\left([n], [m]\right)$,
3. there is an isomorphism of groups $\operatorname{Aut}_{\Delta G}([n]) \cong G_n$, where $g \in G_n$ is written as $\left(id_{[n]}, g\right) \in \operatorname{Hom}_{\Delta G}\left([n], [n]\right)$,

4. an element of $\mathrm{Hom}_{\Delta G}\big([n],[m]\big)$ can be written uniquely as a composite

$$\Big(\varphi, id_{[n]}\Big) \circ \Big(id_{[n]}, g\Big)$$

where $g \in G_n$ and $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$. Such a composite will be written $(\varphi, g)$.

*Remark 19.1.2.* The definition of Fiedorowicz and Loday [FL91, Definition 1.1] states that

$$\mathrm{Aut}_{\Delta G}([n]) = G_n^{op}.$$

Our definition is equivalent since there is a natural isomorphism

$$G \xrightarrow{\cong} G^{op}$$

given by

$$g \mapsto g^{-1}.$$

**Notation 19.1.3.** Condition 3 of Definition 19.1.1 tells us that

$$\mathrm{Aut}_{\Delta G}([n]) = G_n.$$

We will abuse notation and let the identity element of $G_n$ for *any* crossed simplicial group be denoted by $id_{[n]}$, making it clear which crossed simplicial group we are considering at any given time.

*Remark 19.1.4.* Condition 4 of Definition 19.1.1 implies that a composite of the form

$$\Big(id_{[m]}, g\Big) \circ \Big(\varphi, id_{[n]}\Big)$$

in $\Delta G$, where $\Big(id_{[m]}, g\Big) \in \mathrm{Hom}_{\Delta G}\big([m],[m]\big)$ and $\Big(\varphi, id_{[n]}\Big) \in \mathrm{Hom}_{\Delta G}\big([n],[m]\big)$ has a unique expression

$$\big(g_\star(\varphi), \varphi^\star(g)\big) \in \mathrm{Hom}_{\Delta G}\big([n],[m]\big)\,,$$

where $g_\star(\varphi) \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$ and $\varphi^\star(g) \in G_n$.

We deduce the following two lemmata from the uniqueness of the expression

$$\big(g_\star(\varphi), \varphi^\star(g)\big)$$

in Remark 19.1.4.

**Lemma 19.1.5.** *Given $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$ there is a well-defined map of sets*

$$\varphi^\star \colon G_m \to G_n$$

*defined by*

$$g \mapsto \varphi^\star(g). \qquad \square$$

**Lemma 19.1.6.** *Given $g \in G_n$ there is a well defined map of sets*

$$g_\star\colon \mathrm{Hom}_\Delta\big([n],[m]\big) \to \mathrm{Hom}_\Delta\big([n],[m]\big)$$

*defined by*

$$\varphi \mapsto g_\star(\varphi).$$ □

**Proposition 19.1.7 ([FL91, Proposition 1.6]).** *The functions of Lemmata 19.1.5 and 19.1.6 induce the following relations.*

*(I)* *Let $\varphi \in \mathrm{Hom}_\Delta\big([n],[m]\big)$ and let $g,\, g' \in G_m$. Then*

$$\Big(\big(g \circ g'\big)_\star(\varphi), \varphi^\star\left(g \circ g'\right)\Big) = \Big(g_\star\left(g'_\star(\varphi)\right), \left(g'_\star(\varphi)\right)^\star(g) \circ \varphi^\star\left(g'\right)\Big)$$

*in* $\mathrm{Hom}_{\Delta G}\big([n],[m]\big)$.

*(II)* *Let $\varphi \in \mathrm{Hom}_\Delta\big([n],[m]\big)$, $\psi \in \mathrm{Hom}_\Delta\big([m],[l]\big)$ and $g \in G_l$. Then*

$$\big(g_\star\left(\psi \circ \varphi\right), \left(\psi \circ \varphi\right)^\star(g)\big) = \Big(g_\star(\psi) \circ \psi^\star(g)_\star(\varphi), \varphi^\star\left(\psi^\star(g)\right)\Big)$$

*in* $\mathrm{Hom}_{\Delta G}\big([n],[l]\big)$.

*(III)* *Let $g \in G_n$. Then $id^\star_{[n]}(g) = g$ and $g_\star\Big(id_{[n]}\Big) = id_{[n]}$.*

*(IV)* *Let $\varphi \in \mathrm{Hom}_\Delta\big([n],[m]\big)$. Then $\Big(id_{[m]}\Big)_\star(\varphi) = \varphi$ and $\varphi^\star\Big(id_{[m]}\Big) = id_{[n]}$.* □

Taken together, Lemmata 19.1.5 and 19.1.6 and Proposition 19.1.7 tell us that given a crossed simplicial group we have the functions $\varphi^\star$ and $g_\star$ and that these functions satisfy the given relations. In fact, the converse is also true.

**Proposition 19.1.8 ([FL91, Proposition 1.6]).** *A collection of groups $\{G_n\}$ for $n \geqslant 0$, together with maps of sets $\varphi^\star$ and $g_\star$ as in Lemmata 19.1.5 and 19.1.6 which satisfy the relations of Proposition 19.1.7 determine a crossed simplicial group.*

*Proof.* We must show that we can define a category $\Delta G$ with the given data. We take as objects the sets $[n] = \{0, \dots, n\}$ for $n \geqslant 0$. We define $\mathrm{Hom}_{\Delta G}\big([n],[m]\big)$ to consist of all pairs $(\varphi, g)$ such that $g \in G_n$ and $\varphi \in \mathrm{Hom}_\Delta\big([n],[m]\big)$.

We define composition of morphisms as follows. Let $(\varphi, g) \in \mathrm{Hom}_{\Delta G}\big([n],[m]\big)$ and $(\psi, h) \in \mathrm{Hom}_{\Delta G}\big([m],[l]\big)$. We define

$$(\psi, h) \circ (\varphi, g) = \big(\psi \circ h_\star(\varphi), \varphi^\star(h) \circ g\big) \in \mathrm{Hom}_{\Delta G}\big([n],[l]\big).$$

One shows that composition is associative using Proposition 19.1.7.

It is clear that $\Delta$ is a subcategory of $\Delta G$ by restricting to morphisms of the form $(\varphi, id)$. Furthermore, since the only automorphism of the set $[n]$ in the category $\Delta$ is the identity map, we observe that $\mathrm{Aut}_{\Delta G}\big([n]\big)$ consists of all morphisms of the form $\Big(id_{[n]}, g\Big)$ for $g \in G_n$. Therefore, $\mathrm{Aut}_{\Delta G}\big([n]\big) \cong G_n$. □

**Lemma 19.1.9 ([FL91, Lemma 1.3]).** *A crossed simplicial group $\{G_n\}$ has the structure of a simplicial set.* □

**Proposition 19.1.10 ([FL91, Proposition 1.7]).** *A crossed simplicial group is a simplicial set $G_\star$ such that $G_n$ is a group for each $n$, together with a group homomorphism $\rho_n \colon G_n \to \Sigma_{n+1} = \mathrm{Aut}_{\mathbf{Set}}\big([n]\big)$ for each $n$, such that*

1. $\partial_i(gg') = \partial_i(g)\partial_{g(i)}(g')$,
2. $s_i(gg') = s_i(g)s_{g(i)}(g')$ *and*
3. *the diagrams*

$$\begin{array}{ccc} [n-1] & \xrightarrow{\delta_i} & [n] \\ {\scriptstyle \rho_{n-1}\left(\partial_i(g)\right)}\big\downarrow & & \big\downarrow{\scriptstyle \rho_n(g)} \\ [n-1] & \xrightarrow[\delta_{g(i)}]{} & [n] \end{array}$$

*and*

$$\begin{array}{ccc} [n+1] & \xrightarrow{\sigma_i} & [n] \\ {\scriptstyle \rho_{n-1}\left(s_i(g)\right)}\big\downarrow & & \big\downarrow{\scriptstyle \rho_n(g)} \\ [n+1] & \xrightarrow[\sigma_{g(i)}]{} & [n] \end{array}$$

*commute as maps of sets.* □

## 19.2 Examples

We collate some examples of crossed simplicial groups.

*Example 19.2.1.* By [FL91, Proposition 1.4], every simplicial group is a crossed simplicial group.

*Example 19.2.2.* The following are the *fundamental crossed simplicial groups*:

- the trivial crossed simplicial group $\{id_{[n]}\}$ for $n \geqslant 0$,
- the cyclic crossed simplicial group, $\{C_{n+1}\}$ for $n \geqslant 0$,
- the reflexive crossed simplicial group, $\{\mathbb{Z}/2\mathbb{Z}_n\}$ for $n \geqslant 0$,
- the dihedral crossed simplicial group, $\{D_{n+1}\}$ for $n \geqslant 0$,
- the symmetric crossed simplicial group, $\{\Sigma_{n+1}\}$ for $n \geqslant 0$,
- the family of product groups, $\{\mathbb{Z}/2\mathbb{Z} \times \Sigma_{n+1}\}$ for $n \geqslant 0$,
- the hyperoctahedral crossed simplicial group, $\{H_{n+1}\}$ for $n \geqslant 0$.

*Remark 19.2.3.* We note that the reflexive crossed simplicial group is not the constant simplicial group at $\mathbb{Z}/2\mathbb{Z}$. We will discuss this further in Section 22.

The term *fundamental* is motivated by the *classification of crossed simplicial groups.*

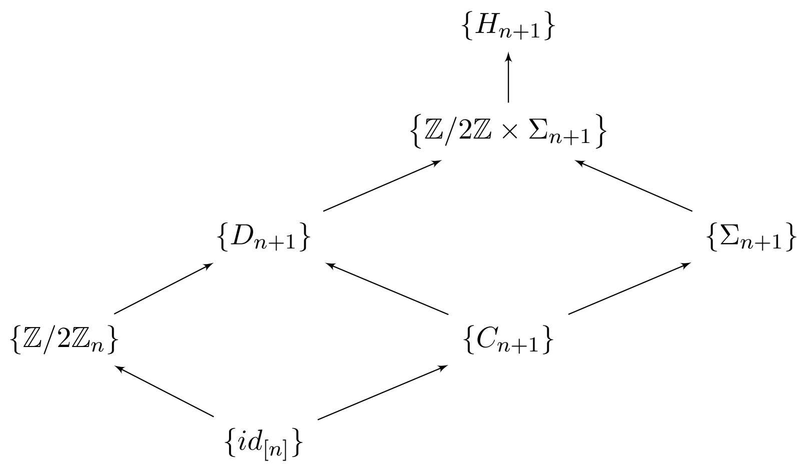
**Theorem 19.2.4 ([FL91, Theorem 3.6]).** *For any crossed simplicial group $G_\star$ there exists an exact sequence, unique up to isomorphism, of crossed simplicial groups*

$$id_\star \to G'_\star \to G_\star \to G''_\star \to id_\star$$

*such that $G'_\star$ is a simplicial group and $G''_\star$ is a fundamental crossed simplicial group.* □

*Remark 19.2.5.* In other words, every crossed simplicial group occurs as an extension of a fundamental crossed simplicial group by a simplicial group.

The fundamental crossed simplicial groups fit into the following diagram of Krasauskas [Kra87], where all arrows denote inclusions of crossed simplicial groups.




*Example 19.2.6.* A further example of a crossed simplicial group, which occurs as an extension of the symmetric crossed simplicial group is the family of braid groups, $\{B_{n+1}\}$ for $n \geqslant 0$.

## 19.3 Epimorphisms and Monomorphisms

We will demonstrate that a morphism $(\varphi, g) \in \mathrm{Hom}_{\Delta G}\big([n],[m]\big)$ is an epimorphism if and only if $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$ is an epimorphism. Recall that the epimorphisms in the category $\Delta$ are the surjective order-preserving maps of sets.

We have an inclusion functor

$$i\colon \Delta \to \Delta G.$$

This functor is the identity on objects and sends a morphism $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$ to $\Big(\varphi, id_{[n]}\Big) \in \mathrm{Hom}_{\Delta G}\big([n],[m]\big)$. The functor $i$ is faithful.

**Lemma 19.3.1.** *If the morphism $(\varphi, g) \in \mathrm{Hom}_{\Delta G}\big([n],[m]\big)$ is an epimorphism then the morphism $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$ is an epimorphism.*

*Proof.* The morphism $(\varphi, g)$ can be written as a composite

$$\left(\varphi, id_{[n]}\right) \circ \left(id_{[n]}, g\right)$$

in $\mathrm{Hom}_{\Delta G}\left([n],[m]\right)$. Since the composite is an epimorphism we know that $\left(\varphi, id_{[n]}\right) \in \mathrm{Hom}_{\Delta G}\left([n],[m]\right)$ is an epimorphism.

The functor $i\colon \Delta \to \Delta G$ is faithful. Since $\left(\varphi, id_{[n]}\right) \in \mathrm{Hom}_{\Delta G}\left([n],[m]\right)$ is an epimorphism, $\varphi \in \mathrm{Hom}_{\Delta}\left([n],[m]\right)$ is an epimorphism since a faithful functor reflects epimorphisms. □

For the opposite implication, recall that an epimorphism in $\Delta$ can be written uniquely as a composite of degeneracy maps $\sigma_j$ by Theorem 2.2.5. The simplicial identity $\sigma_j \circ \delta_j = id_{[n]}$ from Proposition 2.2.4 implies that each $\sigma_j$ has a right inverse, namely the face map $\delta_j$. Therefore an epimorphism in $\Delta$ has a right inverse given by a composite of face maps.

**Lemma 19.3.2.** *If the morphism $\varphi \in \mathrm{Hom}_{\Delta}\left([n],[m]\right)$ is an epimorphism then the morphism $(\varphi, g) \in \mathrm{Hom}_{\Delta G}\left([n],[m]\right)$ is an epimorphism for all $g \in G_n$.*

*Proof.* Suppose, for some object $\underline{l}$ in $\Delta G$, that

$$(\psi, h) \circ (\varphi, g) = \left(\psi', h'\right) \circ (\varphi, g)$$

for some $(\psi, h)$, $\left(\psi', h'\right) \in \mathrm{Hom}_{\Delta G}\left([m],[l]\right)$. This expression can be rewritten as

$$(\psi, h) \circ \left(\varphi, id_{[n]}\right) \circ \left(id_{[n]}, g\right) = (\psi', h') \circ \left(\varphi, id_{[n]}\right) \circ \left(id_{[n]}, g\right).$$

We can cancel on the right using the morphism $\left(id_{[n]}, g^{-1}\right)$ to obtain

$$(\psi, h) \circ \left(\varphi, id_{[n]}\right) = (\psi', h') \circ \left(\varphi, id_{[n]}\right).$$

Since $\varphi$ is an epimorphism in $\Delta$ it has a right inverse, say $\mu \in \mathrm{Hom}_{\Delta}\left([m],[n]\right)$. We can therefore cancel on the right using the morphism $\left(\mu, id_{[m]}\right)$ to obtain

$$(\psi, h) = \left(\psi', h'\right)$$

as required. □

**Corollary 19.3.3.** *A morphism $(\varphi, g) \in \mathrm{Hom}_{\Delta G}\left([n],[m]\right)$ is an epimorphism if and only if $\varphi \in \mathrm{Hom}_{\Delta}\left([n],[m]\right)$ is an epimorphism.* □

Recall that the monomorphisms in the category $\Delta$ are the injective order-preserving maps of sets. We observe that a faithful functor also reflects monomorphisms. The simplicial identity $\sigma_j \circ \delta_j = id_{[n]}$ from Proposition 2.2.4 implies that a monomorphism in $\Delta$ has a left inverse given by a composite of degeneracy maps. Similar arguments to Lemmata 19.3.1 and 19.3.2 prove the following result.

**Corollary 19.3.4.** *A morphism $(\varphi, g) \in \mathrm{Hom}_{\Delta G}\big([n],[m]\big)$ is a monomorphism if and only if $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$ is a monomorphism.* □

**Definition 19.3.5.** We denote by Epi$\Delta G$ the subcategory of $\Delta G$ whose objects are the sets $[n] = \{0,\dots,n\}$ for $n \geqslant 0$ and whose morphisms are the epimorphisms in the category $\Delta G$.

**Definition 19.3.6.** We denote by Mono$\Delta G$ the subcategory of $\Delta G$ whose objects are the sets $[n] = \{0,\dots,n\}$ for $n \geqslant 0$ and whose morphisms are the monomorphisms in the category $\Delta G$.

## 19.4 Crossed Simplicial Groups and Functor Homology

One can construct functor homology theories from crossed simplicial groups. Following Definition 6.1.4 we recall the trivial right $\Delta G^{op}$-module

$$k^{\star}\colon \Delta G^{op} \to \mathbf{kMod}.$$

**Definition 19.4.1.** Let $\{G_\star\}$ be a crossed simplicial group with associated category $\Delta G$. Let $F\colon \Delta G^{op} \to \mathbf{kMod}$ be a functor. We define the homology theory associated to $\{G_\star\}$ for the functor $F$ by

$$HG_n(F) \coloneqq \mathrm{Tor}_n^{\Delta G^{op}}(k^\star, F)$$

for $n \geqslant 0$.

## 19.5 Realizations of Crossed Simplicial Groups

**Theorem 19.5.1 ([FL91, Theorem 5.3]).** *If $\{G_\star\}$ is a crossed simplicial group then the geometric realization $|G_\star|$ is a topological group.* □

**Proposition 19.5.2 ([Lod98, Section 6.3.6]).** *There are isomorphisms of topological groups*

$$|C_\star| \cong S^1$$

*and*

$$|D_\star| \cong O(2)$$

*where $O(2)$ is the orthogonal group. That is, the group of $2 \times 2$ matrices, $M$, with entries in $k$ satisfying*

$$M^T M = M M^T = I$$

*where $I$ is the identity matrix.* □

**Proposition 19.5.3.** *Let $\{G_\star\}$ be a crossed simplicial group with inclusion maps*

$$i_n\colon G_n \to G_{n+1},$$

*which are group homomorphisms, for $n \geqslant 1$. The geometric realization $|G_\star|$ is contractible.*

*Proof.* Analogously to [FL91, Example 6], the inclusion maps satisfy $\partial_i \circ i_n - i_{n-1} \circ \partial_i$ for $0 \leqslant i \leqslant n$ and $\partial_{n+1} \circ i_n = id_{G_n}$ and therefore they form a contracting homotopy. □

**Corollary 19.5.4.** *The simplicial sets $\{\Sigma_{n+1}\}$, $\{H_{n+1}\}$ and $\{B_{n+1}\}$ of the symmetric, hyperoctahedral and braid groups are contractible.* □

Recall the based loop functor $\Omega$ from Definition 3.1.3.

**Proposition 19.5.5 ([FL91, Proposition 5.8]).** *For a crossed simplicial group, $\{G_\star\}$, there is a homotopy equivalence*

$$|G_\star| \simeq \Omega B\Delta G,$$

*where $B\Delta G$ is the classifying space of the category $\Delta G$.* □

### 19.5.1 Relationship to equivariant homology theory

The functor homology theories associated to crossed simplicial groups are related to equivariant homology via the Borel construction.

**Definition 19.5.6.** Let $G$ be a topological group. Let $X$ be a $G$-space and $EG$ be a contractible space with free $G$-action. We define the space

$$EG \times_G X$$

to be the quotient of the product space $EG \times X$ by the equivalence relation

$$(y, gx) \sim (yg, x)$$

for all $y \in EG$, $x \in X$ and $g \in G$.

We call $EG \times_G X$ the *Borel construction.*

**Theorem 19.5.7 ([FL91, Corollary 6.13]).** *Let $\{G_\star\}$ be a crossed simplicial group. Let*

$$X\colon \Delta G^{op} \to \mathbf{Set}$$

*be a functor and let*

$$k[X]\colon \Delta G^{op} \to \mathbf{kMod}$$

*be the functor obtained by composing with the free $k$-module functor. There is a canonical isomorphism of $k$-modules*

$$HG_n\left(k[X]\right) \cong H_n\left(E|G_\star| \times_{|G_\star|} |X_\star|, k\right)$$

*for each $n \geqslant 0$.* □

# Chapter 20

# Hochschild Homology and the Trivial Crossed Simplicial Group

## Introduction

We describe the trivial crossed simplicial group and its associated homology functor homology theory, Hochschild homology. In particular, we recover the algebraic Hochschild homology theory of Section 5.1.

## 20.1 The Trivial Crossed Simplicial Group

Following Definition 19.1.1 we will describe the category associated to the trivial crossed simplicial group $\left\{id_{[n]}\right\}$. The objects are the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$. A morphism $[n] \to [m]$ is a pair $\left(\varphi, id_{[n]}\right)$ where $\varphi \in \operatorname{Hom}_{\Delta}\left([n], [m]\right)$. The relations of Proposition 19.1.7 tell us that composition of morphisms corresponds to composition in $\Delta$. Therefore the category associated to the crossed simplicial group $\left\{id_{[n]}\right\}$ is $\Delta$.

## 20.2 Hochschild Homology as Functor Homology

Following Definition 19.4.1 and Subsection 20.1 we make the following definition.

**Definition 20.2.1.** Let $F \colon \Delta^{op} \to \mathbf{kMod}$ be a functor. We define the $n^{th}$ *Hochschild homology of* $F$ by

$$HH_n(F) \coloneqq \operatorname{Tor}_n^{\Delta^{op}}\left(k^{\star}, F\right)$$

for $n \geqslant 0$.

Since $F$ is a functor of the form $\Delta^{op} \to \mathbf{kMod}$ it is a simplicial $k$-module by Remark 2.3.3.

**Definition 20.2.2.** Let $F\colon \Delta^{op} \to \mathbf{kMod}$ be a functor. We denote the associated chain complex by $C_\star(F)$.

**Theorem 20.2.3 ([Lod98, Theorem 6.22]).** *Let $F\colon \Delta^{op} \to \mathbf{kMod}$ be a functor. There is a canonical isomorphism of $k$-modules*

$$\operatorname{Tor}_n^{\Delta^{op}}\left(k^\star, F\right) \cong H_n\left(C_\star(F)\right),$$

*for each $n \geqslant 0$.* □

**Definition 20.2.4.** Recall the simplicial $k$-module $C_\star(A, M)$ of Definition 5.1.1. We denote the corresponding functor by

$$\mathscr{L}(A, M)\colon \Delta^{op} \to \mathbf{kMod}.$$

We call $\mathscr{L}(A, M)$ the *Loday functor*. When we set $M = A$ we denote the Loday functor by $\mathscr{L}(A)$.

**Corollary 20.2.5 ([Lod98, Corollary 6.2.3]).** *Let $\mathscr{L}(A, M)$ be the Loday functor of Definition 20.2.4. There is a canonical isomorphism of $k$-modules*

$$HH_n(\mathscr{L}(A, M)) \cong HH_n(A, M),$$

*for each $n \geqslant 0$.* □

### 20.2.1 Hochschild homology and loop spaces

**Theorem 20.2.6 ([Lod98, Corollary 7.3.13]).** *There is a canonical isomorphism of $k$-modules*

$$HH_n(k[G]) \cong H_n(\mathcal{L}BG, k)$$

*for each $n \geqslant 0$, where $\mathcal{L}$ is the free loop functor and $BG$ is the classifying space of the group $G$.* □

# Chapter 21

# Cyclic Homology

## Introduction

We describe the cyclic crossed simplicial group and its associated functor homology theory, cyclic homology. In particular, we recover the algebraic cyclic homology theory of Section 5.6. We provide an example of the relation between cyclic homology and $S^1$-equivariant homology theory.

## 21.1 The Category $\Delta C$

**Definition 21.1.1.** For $n \geqslant 0$, we define an action of $C_{n+1}$ on the set $[n]$ to be determined by

$$t_{n+1}(i) = \begin{cases} i-1 & i > 0 \\ n & i = 0. \end{cases}$$

We define the category associated to the cyclic crossed simplicial group $\{C_\star\}$ following Definition 19.1.1.

**Definition 21.1.2.** The *cyclic category*, denoted $\Delta C$, has the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$ as objects. An element of $\mathrm{Hom}_{\Delta C}\left([n],[m]\right)$ is a pair $(\varphi, g)$ where $g \in C_{n+1}$ and $\varphi \in \mathrm{Hom}_{\Delta}\left([n],[m]\right)$. Composition is defined as in Proposition 19.1.8. Rather than derive formulae for the maps $\varphi^\star$ and $g_\star$ we will work with the presentation of Proposition 21.1.4.

*Remark 21.1.3.* The category $\Delta C$ is isomorphic to Connes' category $\Lambda$, [Con83, Section 2].

**Proposition 21.1.4 ([Lod98, 6.1.1]).** *The category $\Delta C$ has as objects the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$. Morphisms in $\Delta C$ are generated by*

- $\left(\delta_i, id_{[n]}\right)$*, for each $n \geqslant 0$ and $0 \leqslant i \leqslant n+1$ where the $\delta_i$ are the face maps of Definition 2.2.2,*

- $\left(\sigma_j, id_{[n+1]}\right)$ *for each* $n \geqslant 0$ *and* $0 \leqslant j \leqslant n$ *where the* $\sigma_j$ *are the degeneracy maps of Definition 2.2.3 and*
- $\left(id_{[n]}, t_{n+1}\right)$ *for* $n \geqslant 0$

*subject to*

- *the relations of the category* $\Delta$,
- $\left(id_{[n]}, t_{n+1}\right)^{n+1} = \left(id_{[n]}, id_{[n]}\right)$ *for* $n \geqslant 0$,
- $\left(id_{[n+1]}, t_{n+2}\right) \circ \left(\delta_i, id_{[n]}\right) = \begin{cases} \left(\delta_{n+1}, id_{[n+1]}\right) & i = 0, \\ (\delta_{i-1}, t_{n+1}) & 1 \leqslant i \leqslant n+1, \end{cases}$
- $\left(id_{[n]}, t_{n+1}\right) \circ \left(\sigma_j, id_{[n+1]}\right) = \begin{cases} \left(\sigma_n, t_{n+2}^2\right) & j = 0, \\ \left(\sigma_{j-1}, t_{n+2}\right) & 1 \leqslant j \leqslant n. \end{cases}$

*This is a presentation of the category* $\Delta C$. □

## 21.2 Cyclic Homology as Functor Homology

**Definition 21.2.1.** A *cyclic module* $F$ is a functor

$$F\colon \Delta C^{op} \to \mathbf{kMod}.$$

**Definition 21.2.2.** Let $F\colon \Delta C^{op} \to \mathbf{kMod}$ be a cyclic module. We define the $n^{th}$ *cyclic homology of* $F$ by

$$HC_n(F) \coloneqq \operatorname{Tor}_n^{\Delta C^{op}}\left(k^\star, F\right)$$

for $n \geqslant 0$.

**Definition 21.2.3.** Let $A$ be an associative $k$-algebra. We extend the functor $\mathscr{L}(A)$ of Definition 20.2.4 to a functor

$$\mathscr{L}(A)\colon \Delta C^{op} \to \mathbf{kMod}$$

by defining

$$\mathscr{L}(A)(t_{n+1})(a_0 \otimes \cdots \otimes a_n) = (a_n \otimes a_0 \otimes \cdots \otimes a_{n-1})$$

for each $n \geqslant 0$.

**Theorem 21.2.4 ([Lod98, Theorem 6.2.8]).** *Let* $A$ *be a unital, associative* $k$*-algebra. There exists a canonical isomorphism of* $k$*-modules*

$$HC_n(\mathscr{L}(A)) \cong HC_n(A)$$

*for each* $n \geqslant 0$. □

### 21.2.1 Cyclic homology and $S^1$-equivariant homology

Recall from Proposition 19.5.2 that there is an isomorphism of topological groups $|C_\star| \cong S^1$. Recall the free loop functor from Definition 3.1.5.

**Theorem 21.2.5 ([Lod98, Corollary 7.3.13]).** *Let $G$ be a discrete group with classifying space $BG$ and group algebra $k[G]$. There are canonical isomorphisms of $k$-modules*

$$HC_n(k[G]) \cong H_n\left(ES^1 \times_{S^1} \mathcal{L}BG, k\right)$$

*for each $n \geqslant 0$.* □

# Chapter 22

# The Reflexive Crossed Simplicial Group

The reflexive crossed simplicial group, $\{\mathbb{Z}/2\mathbb{Z}_n\}$ for $n \geqslant 0$, encodes information about involutions.

**Definition 22.0.1.** For each $n \geqslant 0$ we define

$$\mathbb{Z}/2\mathbb{Z}_n := \left\langle r_n \mid r_n^2 = id_{[n]} \right\rangle .$$

**Definition 22.0.2.** We define an action of $\mathbb{Z}/2\mathbb{Z}_n$ on $[n]$ by

$$r_n(i) = n - i.$$

**Definition 22.0.3.** The *reflexive category*, denoted $\Delta R$ has as objects the sets $[n] = \{0, \dots, n\}$ for $n \geqslant 0$. An element of $\mathrm{Hom}_{\Delta R}\left([n], [m]\right)$ is a pair $(\varphi, g)$ where $g \in \mathbb{Z}/2\mathbb{Z}_n$, $\varphi \in \mathrm{Hom}_{\Delta}\left([n], [m]\right)$ and composition is defined as in Proposition 19.1.8. Rather than derive formulae for the maps $\varphi^{\star}$ and $g_{\star}$ we will work with the presentation of Proposition 22.0.4.

**Proposition 22.0.4.** *The category $\Delta R$ has as objects the sets $[n] = \{0, \dots, n\}$ for $n \geqslant 0$. Morphisms in $\Delta R$ are generated by*

- $\left(\delta_i, id_{[n]}\right)$, *for each $n \geqslant 0$ and $0 \leqslant i \leqslant n+1$ where the $\delta_i$ are the face maps of Definition 2.2.2,*
- $\left(\sigma_j, id_{[n+1]}\right)$ *for each $n \geqslant 0$ and $0 \leqslant j \leqslant n$ where the $\sigma_j$ are the degeneracy maps of Definition 2.2.3 and*
- $\left(id_{[n]}, r_n\right)$ *for $n \geqslant 0$*

*subject to*

- *the relations of the category $\Delta$,*

- $\left(id_{[n]}, r_n\right)^2 = \left(id_{[n]}, id_{[n]}\right)$ *for* $n \geqslant 0$,
- $\left(id_{[n]}, r_n\right) \circ \left(\delta_i, id_{[n-1]}\right) = (\delta_{n-i}, r_{n-1})$ *for* $n \geqslant 1$, $0 \leqslant i \leqslant n$ *and*
- $\left(id_{[n]}, r_n\right) \circ \left(\sigma_j, id_{[n+1]}\right) = \left(\sigma_{n-j}, r_{n+1}\right)$ *for* $n \geqslant 0$, $0 \leqslant j \leqslant n$.

*This is a presentation of the category* $\Delta R$. □

*Remark 22.0.5.* The reflexive crossed simplicial group has recently been used in the study of *real topological Hochschild homology* [DMPP17].

# Chapter 23

# Dihedral Homology

## Introduction

The homology theory associated to the crossed simplicial group $\{D_{n+1}\}$ for $n \geqslant 0$ is called *dihedral homology.* Dihedral homology was first introduced independently by Loday [Lod87] and Krasauskas, Lapin and Solov'ev [KLS87] in 1987. Further results were published by Dunn [Dun89] and Lodder [Lod90].

In Section 23.2 we describe the dihedral crossed simplicial group and define the associated functor homology theory. We recover the algebraic dihedral homology theory for involutive, associative algebras defined by Lodder [Lod90].

## 23.1 The Dihedral Groups

**Definition 23.1.1.** For each $n \geqslant 0$ we define the dihedral group to be

$$D_{n+1} \coloneqq \left\langle r_n,\, t_{n+1} \mid r_n^2 = t_{n+1}^{n+1} = id_{[n]},\ r_n t_{n+1} r_n = t_{n+1}^n \right\rangle .$$

## 23.2 The Category $\Delta D$

We define the category associated the dihedral crossed simplicial group $\{D_{n+1}\}$ following Definition 19.1.1.

**Definition 23.2.1.** The *dihedral category*, denoted $\Delta D$, has as objects the sets $[n] = \{0, \dots, n\}$ for $n \geqslant 0$. An element of $\mathrm{Hom}_{\Delta D}\big([n],[m]\big)$ is a pair $(\varphi, g)$ such that $g \in D_{n+1}$ and $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$. Composition is defined as Proposition 19.1.8. Rather than derive formulae for the maps $\varphi^{\star}$ and $g_{\star}$ we will work with the presentation of Proposition 23.2.2.

**Proposition 23.2.2 ([Lod90, 1.1.1]).** *The category $\Delta D$ has as objects the sets $[n] = \{0, \dots, n\}$ for $n \geqslant 0$. Morphisms in $\Delta D$ are generated by*

- *the generators of Proposition 21.1.4 and*
- $\left(id_{[n]}, r_n\right)$ *for* $n \geqslant 0$

*subject to*

- *the relations of Proposition 21.1.4,*
- *the relations of Definition 23.1.1 for* $\left(id_{[n]}, r_n\right)$ *and* $\left(id_{[n]}, t_{n+1}\right)$*,*
- $\left(id_{[n]}, r_n\right) \circ \left(\delta_i, id_{[n-1]}\right) = (\delta_{n-i}, r_{n-1})$ *for* $n \geqslant 1$*,* $0 \leqslant i \leqslant n$ *and*
- $\left(id_{[n]}, r_n\right) \circ \left(\sigma_j, id_{[n+1]}\right) = \left(\sigma_{n-j}, r_{n+1}\right)$ *for* $n \geqslant 0$*,* $0 \leqslant j \leqslant n$*.*

*This a presentation of the category* $\Delta D$*.* □

**Definition 23.2.3.** Let $F\colon \Delta D^{op} \to \mathbf{kMod}$ be a functor. We define the $n^{th}$ *dihedral homology of* $F$ by

$$HD_n(F) \coloneqq \operatorname{Tor}_n^{\Delta D^{op}}\left(k^\star, F\right)$$

for $n \geqslant 0$.

Recall from Subsection 21.2 that we extended the Loday functor of Definition 20.2.4 to a functor

$$\mathscr{L}(A)\colon \Delta C^{op} \to \mathbf{kMod}.$$

**Definition 23.2.4.** Let $A$ be an involutive, associative $k$-algebra with involution

$$a \mapsto \overline{a}.$$

We extend the functor $\mathscr{L}(A)$ of Definition 21.2.3 to a functor

$$\mathscr{L}(A)\colon \Delta D^{op} \to \mathbf{kMod}$$

by defining

$$\mathscr{L}(A)(r_n)(a_0 \otimes \cdots \otimes a_n) = \left(\overline{a_0} \otimes \overline{a_n} \otimes \cdots \otimes \overline{a_1}\right).$$

**Definition 23.2.5.** Let $A$ be an involutive, associative $k$-algebra with involution

$$a \mapsto \overline{a}.$$

We define the $n^{th}$ *dihedral homology of* $A$ to be

$$HD_n(A) \coloneqq HD_n\left(\mathscr{L}(A)\right)$$

for $n \geqslant 0$.

*Remark 23.2.6.* There is a theorem of Lodder [Lod90, Theorem 3.3.3] connecting the dihedral homology of certain Moore loop spaces with the $O(2)$-equivariant homology of certain free loop spaces.

# Chapter 24

# Symmetric Homology

## Introduction

In Section 24.2 we describe the symmetric crossed simplicial group. We also introduce the category of non-commutative sets, following Pirashvili and Richter [PR02].

In Section 24.3 we describe the problems with describing an interesting symmetric homology theory, both in the algebraic setting and functor homology setting.

In Sections 24.4 and 24.5 we recall Fiedorowicz's symmetric bar construction and define symmetric homology. Finally, we recall a result of Fiedorowicz connecting the homology of a group algebra to the homology of infinite loop spaces.

## 24.1 The Symmetric Groups

**Proposition 24.1.1.** *The symmetric group $\Sigma_{n+1}$ on $[n] = \{0, \dots, n\}$ for $n \geqslant 1$ is generated by the transpositions $\theta_i = (i\ i+1)$ for $0 \leqslant i \leqslant n-1$ subject to the relations*

- $\theta_i^2 = id_{[n]}$ *for each* $0 \leqslant i \leqslant n-1$,
- $\theta_i \circ \theta_j = \theta_j \circ \theta_i$ *for* $j \neq i+1, i-1$ *and*
- $(\theta_i \circ \theta_{i+1})^3 = id_{[n]}$ *for* $0 \leqslant i \leqslant n-2$. $\square$

## 24.2 The Category $\Delta S$

We define the category associated to the symmetric crossed simplicial group $\{\Sigma_{n+1}\}$ following Definition 19.1.1.

**Definition 24.2.1.** The *symmetric category*, denoted $\Delta S$, has as objects the sets $[n] = \{0, \dots, n\}$ for $n \geqslant 0$. An element of $\mathrm{Hom}_{\Delta S}\big([n], [m]\big)$ is pair $(\varphi, \sigma)$ such that $\sigma \in \Sigma_{n+1}$

and $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$. Composition is as defined in Proposition 19.1.8. Rather than derive formulae for the maps $\varphi^\star$ and $\sigma_\star$ we will work with the presentation of Proposition 24.2.2.

**Proposition 24.2.2.** *The category $\Delta S$ has as objects the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$. Morphisms are generated by*

- $\big(\delta_i, id_{[n]}\big)$, *for each $n \geqslant 0$ and $0 \leqslant i \leqslant n+1$ where the $\delta_i$ are the face maps of Definition 2.2.2,*
- $\big(\sigma_j, id_{[n+1]}\big)$ *for each $n \geqslant 0$ and $0 \leqslant j \leqslant n$ where the $\sigma_j$ are the degeneracy maps of Definition 2.2.3 and*
- $\big(id_{[n]}, \theta_i\big)$ *for $n \geqslant 1$ and $0 \leqslant i \leqslant n-1$*

*subject to*

- *the relations of the category $\Delta$,*
- *the relations of Proposition 24.1.1 for the $\big(id_{[n]}, \theta_i\big)$,*
- $$\big(id_{[n]}, \theta_k\big) \circ \big(\delta_i, id_{[n-1]}\big) = \begin{cases} (\delta_i, \theta_k) & k < i-1, \\ \big(\delta_{i-1}, id_{[n-1]}\big) & k = i-1, \\ \big(\delta_{i+1}, id_{[n-1]}\big) & k = i, \\ (\delta_i, \theta_{k-1}) & k > i, \end{cases}$$
- $$\big(id_{[n]}, \theta_k\big) \circ \big(\sigma_j, id_{[n+1]}\big) = \begin{cases} \big(\sigma_j, \theta_k\big) & k < j-1, \\ \big(\sigma_{j-1}, \theta_j\theta_{j-1}\big) & k = j-1, \\ \big(\sigma_{j+1}, \theta_j\theta_{j+1}\big) & k = j, \\ \big(\sigma_j, \theta_{k+1}\big) & k > j. \end{cases}$$

*This is a presentation of the category $\Delta S$.* □

### 24.2.1 The category of non-commutative sets

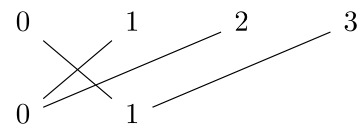
Pirashvili and Richter [PR02, Section 1.2] introduce the category of non-commutative sets. This category is isomorphic to the category $\Delta S$ but has the advantage that one does not need to work with the functions $\varphi^\star$ and $g_\star$ of Proposition 19.1.7.

**Definition 24.2.3.** The *category of non-commutative sets*, denoted $\mathcal{F}(as)$, has as objects the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$. An element $f \in \mathrm{Hom}_{\mathcal{F}(as)}\big([n],[m]\big)$ is a map of sets $f\colon [n] \to [m]$ such that for each singleton $i \in [m]$, the preimage $f^{-1}(i)$ is a totally ordered set.

For $f \in \mathrm{Hom}_{\mathcal{F}(as)}\big([n],[m]\big)$ and $g \in \mathrm{Hom}_{\mathcal{F}(as)}\big([m],[l]\big)$, the composite is the map of sets $g \circ f$ with the totally ordered set $(g \circ f)^{-1}(i)$, for each singleton $i \in [l]$, given by the ordered union of ordered sets:

$$(g \circ f)^{-1}(i) := \coprod_{j \in g^{-1}(i)} f^{-1}(j).$$

*Example 24.2.4.* Let $f\colon [3] \to [1]$ be the map of sets represented by the graph




with $f^{-1}(0) = \{1 < 2\}$ and $f^{-1}(1) = \{3 < 0\}$. Then $f \in \mathrm{Hom}_{\mathcal{F}(as)}\big([3],[1]\big)$.

**Proposition 24.2.5 ([PR02, Lemma 1.1]).** *There is an isomorphism of categories*

$$\mathcal{F}(as) \cong \Delta S. \qquad \square$$

*Remark 24.2.6.* Pirashvili and Richter [PR02, Theorem 1.3] use the category of non-commutative sets to obtain an alternative functor homology interpretation of cyclic homology.

## 24.3 Complications for Symmetric Homology

### 24.3.1 Problems with the Loday functor

Let $A$ be an associative $k$-algebra and recall the Loday functor of Definition 20.2.4. This functor does not extend to a functor

$$\mathscr{L}(A)\colon \Delta S^{op} \to \mathbf{kMod}$$

by defining the action

$$\mathscr{L}(A)(\sigma)\left(a_0 \otimes \cdots \otimes a_n\right) = a_{\sigma^{-1}(0)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$$

for $\sigma \in \Sigma_{n+1}$.

*Example 24.3.1.* Consider the commutative diagram

$$\begin{array}{ccc}
[1] & \xrightarrow{\left(\delta_2, id_{[1]}\right)} & [2] \\
{\scriptstyle \left(id_{[1]}, (0\ 1)\right)} \downarrow & & \downarrow {\scriptstyle \left(id_{[2]}, (0\ 1)\right)} \\
[1] & \xrightarrow[\left(\delta_2, id_{[1]}\right)]{} & [2]
\end{array}$$

in $\Delta S$. We observe that the diagram

$$\begin{array}{ccc}
A^{\otimes 3} & \xrightarrow{\mathscr{L}\left(\delta_2, id_{[1]}\right)} & A^{\otimes 2} \\
{\scriptstyle \mathscr{L}\left(id_{[2]}, (0\ 1)\right)}\downarrow & & \downarrow{\scriptstyle \mathscr{L}\left(id_{[1]}, (0\ 1)\right)} \\
A^{\otimes 3} & \xrightarrow[\mathscr{L}\left(\delta_2, id_{[1]}\right)]{} & A^{\otimes 2}
\end{array}$$

obtained by applying the "functor" $\mathscr{L}(A)$ does not commute in **kMod** since the upper composite takes the form

$$\begin{array}{ccc}
a_0 \otimes a_1 \otimes a_2 & \longmapsto & a_2 a_0 \otimes a_1 \\
& & \downarrow \\
& & a_1 \otimes a_2 a_0
\end{array}$$

whereas the lower composite takes the form

$$\begin{array}{ccc}
a_0 \otimes a_1 \otimes a_2 & & \\
\downarrow & & \\
a_1 \otimes a_0 \otimes a_2 & \longmapsto & a_2 a_1 \otimes a_0.
\end{array}$$

### 24.3.2 Problems with the symmetric homology for a well-defined functor

Suppose that $F\colon \Delta S^{op} \to \mathbf{kMod}$ is a well-defined functor.

Following Definition 19.4.1 we wish to define a homology theory

$$HS_n(F) \coloneqq \mathrm{Tor}_n^{\Delta S^{op}}\left(k^\star, F\right).$$

However, the following theorem tells us that the homology theory so defined does not give any new information about our functor $F$.

**Theorem 24.3.2 ([FL91, Theorem 6.16]).** *For a functor* $F\colon \Delta S^{op} \to \mathbf{kMod}$ *there is an isomorphism of* $k$*-modules*

$$HH_n(F) \cong \mathrm{Tor}_n^{\Delta S^{op}}\left(k^\star, F\right)$$

*for each* $n \geqslant 0$. □

## 24.4 The Symmetric Bar Construction

Let $A$ be a unital, associative $k$-algebra. We define an assignment

$$B_A^{sym}(-)\colon \Delta S \to \mathbf{kMod}$$

on objects by

$$B_A^{sym}\big([n]\big) = A^{\otimes n+1}.$$

We extend this to include an assignment on morphisms. Recall that an element $(\varphi, \sigma) \in \mathrm{Hom}_{\Delta S}\big([n],[m]\big)$ is defined by

$$(\varphi, \sigma) = \Big(\varphi, id_{[n]}\Big) \circ \Big(id_{[n]}, \sigma\Big).$$

We will define the assignment $B_A^{sym}$ on these two morphisms separately.

**Definition 24.4.1.** Let $\varphi \in \mathrm{Hom}_{\Delta S}\big([n],[m]\big)$. We define

$$B_A^{sym}\Big(\varphi, id_{[n]}\Big)(a_0 \otimes \cdots \otimes a_n) = \left(\prod_{i\in\varphi^{-1}(0)}^{<} a_i\right) \otimes \cdots \otimes \left(\prod_{i\in\varphi^{-1}(m)}^{<} a_i\right)$$

on elementary tensors and extend $k$-linearly. Here $\prod^{<}$ denotes the ordered product of elements $a_i$ by increasing index. An empty product is defined to be the multiplicative unit $1_A$.

**Definition 24.4.2.** Let $\sigma \in \Sigma_{n+1}$. We define

$$B_A^{sym}\Big(id_{[n]}, \sigma\Big)(a_0 \otimes \cdots \otimes a_n) = a_{\sigma^{-1}(0)} \otimes \cdots \otimes a_{\sigma^{-1}(n)}$$

on elementary tensors and extend $k$-linearly.

**Definition 24.4.3.** We define a functor

$$B_A^{sym}\colon \Delta S \to \mathbf{kMod}$$

on objects by $B_A^{sym}\big([n]\big) = A^{\otimes n+1}$. For $(\varphi, \sigma) \in \mathrm{Hom}_{\Delta S}\big([n],[m]\big)$ we define

$$B_A^{sym}(\varphi, \sigma) = B_A^{sym}\Big(\varphi, id_{[n]}\Big) \circ B_A^{sym}\Big(id_{[n]}, \sigma\Big)$$

which were defined in Definitions 24.4.1 and 24.4.2 respectively.

## 24.5 Symmetric Homology

**Definition 24.5.1.** Let $F\colon \Delta S \to \mathbf{kMod}$ be a functor. We define the $n^{th}$ *symmetric homology of* $F$ to be

$$HS_n(F) := \mathrm{Tor}_n^{\Delta S}\big(k^\star, F\big)$$

for $n \geqslant 0$.

**Definition 24.5.2.** Let $A$ be an associative $k$-algebra. We define the $n^{th}$ *symmetric homology of* $A$ to be

$$HS_n(A) := \mathrm{Tor}_n^{\Delta S}\big(k^\star, B_A^{sym}\big)$$

for $n \geqslant 0$.

### 24.5.1 Fiedorowicz's theorem for symmetric homology

We will recall a theorem of Fiedorowicz which demonstrates that symmetric homology of an algebra defined via the symmetric bar construction is of great interest. This theorem tells us that the symmetric homology of a group algebra is the homology of the based loop space of the infinite loop space associated to the classifying space of the group.

Recall the based loop functor $\Omega$ of Definition 3.1.3 and the functor $Q$ of Definition 3.5.14.

**Theorem 24.5.3 ([Aul10, Corollary 40]).** *Let $G$ be a group with classifying space $BG$ and group algebra $k[G]$. There is an isomorphism of $k$-modules*

$$HS_n\left(k[G]\right) \cong H_n\left(\Omega Q\left(BG\right), k\right)$$

*for each $n \geqslant 0$.* □

# Chapter 25

# The Hyperoctahedral Crossed Simplicial Group

## Introduction

In Section 25.1 we recall the hyperoctahedral groups and provide a presentation.

In Section 25.2 we describe the hyperoctahedral crossed simplicial group and provide a presentation of the associated category.

In Sections 25.3 and 25.4 we recall Fiedorowicz's hyperoctahedral bar construction and hyperoctahedral homology. We state a result of Fiedorowicz connecting the hyperoctahedral homology of a group algebra to the equivariant homology of infinite loop spaces.

## 25.1 Hyperoctahedral Groups

**Definition 25.1.1.** The *hyperoctahedral group*, $H_{n+1}$ for $n \geqslant 0$, is defined to be the semi-direct product

$$H_{n+1} \coloneqq \left(\mathbb{Z}/2\mathbb{Z}\right)^{n+1} \rtimes \Sigma_{n+1}$$

where $\Sigma_{n+1}$ acts on $\left(\mathbb{Z}/2\mathbb{Z}\right)^{n+1}$ by permuting the factors.

An element of the hyperoctahedral group $H_n$ is written as a tuple

$$(z_0, \ldots, z_n; \sigma)$$

where each $z_i \in \mathbb{Z}/2\mathbb{Z}$ and $\sigma \in \Sigma_{n+1}$.

The identity element of $H_{n+1}$ is

$$id_{[n]} \coloneqq \left(1, \ldots, 1; id_{[n]}\right),$$

where we abuse notation following Notation 19.1.3.

Composition is defined by

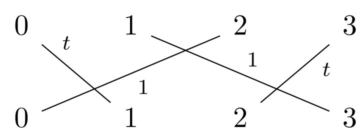
$$(z'_0, \ldots, z'_n; \sigma_2) \circ (z_0, \ldots, z_n; \sigma_1) = \left( z'_{\sigma_1(0)} z_0, \ldots, z'_{\sigma_1(n)} z_n; \sigma_2 \circ \sigma_1 \right).$$

*Remark 25.1.2.* The group $H_{n+1}$ is sometimes denoted as a wreath product

$$\Sigma_{n+1} \int \mathbb{Z}/2\mathbb{Z}.$$

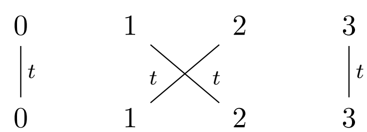
It will be helpful to think of an element of $H_{n+1}$ as a permutation, $\sigma$, of the set $[n] = \{0, \ldots, n\}$, drawn as a graph with an edge joining $i$ and $\sigma(i)$ for each $0 \leqslant i \leqslant n$, with the edge joining $i$ and $\sigma(i)$ labelled by $z_{\sigma^{-1}(i)} \in \mathbb{Z}/2\mathbb{Z}$.

*Example 25.1.3.* Consider $g_1 = (t, 1, 1, t; (0\ 1\ 3\ 2)) \in H_4$. We can represent $g_1$ as follows.

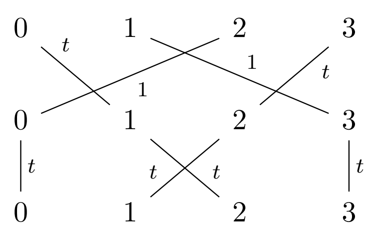



Composition of group elements corresponds to composition of permutations and multiplication of labels.

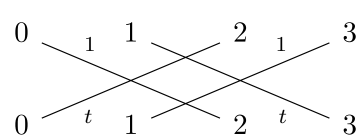
*Example 25.1.4.* Let $g_2 = (t, t, t, t; (1\ 2)) \in H_4$. We represent $g_2$ with the following labelled permutation.



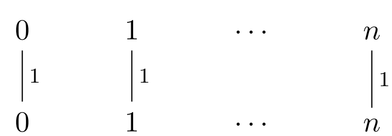

The element $g_2 g_1 \in H_4$ is represented by the composite




which is




The identity element of the group $H_{n+1}$ is the identity permutation of $[n]$, with each edge labelled with $1 \in \mathbb{Z}/2\mathbb{Z}$,



### 25.1.1 Generators and relations

It will be useful to have a presentation of the hyperoctahedral groups in terms of generators and relations. Methods for deriving a presentation of a group extension are well-known and can be found in [Joh90, Chapter 10.2] for example.

**Definition 25.1.5.** Let $t_i$ denote the element $\Big(1, \ldots, 1, t, 1, \ldots, 1; id_{[n]}\Big)$ in $H_{n+1}$, where $t$ occurs in the $i^{th}$ position.

Equivalently, as a labelled permutation, we represent $t_i$ by

$$\begin{array}{cccccccc} 0 & 1 & \cdots & i-1 & i & i+1 & \cdots & n \\ \big|{\scriptstyle 1} & \big|{\scriptstyle 1} & & \big|{\scriptstyle 1} & \big|{\scriptstyle t} & \big|{\scriptstyle 1} & & \big|{\scriptstyle 1} \\ 0 & 1 & \cdots & i-1 & i & i+1 & \cdots & n \end{array}$$

**Definition 25.1.6.** Let $\theta_i^H$ denote the element

$$(1, \ldots, 1; \theta_i)$$

in $H_{n+1}$, where $\theta_i = (i\ i+1) \in \Sigma_{n+1}$.

**Proposition 25.1.7.** *The hyperoctahedral group $H_{n+1}$ is generated by the elements $t_i$ for $0 \leqslant i \leqslant n$ and $\theta_j^H$ for $0 \leqslant j \leqslant n-1$ subject to*

- *the relations of Proposition 24.1.1 for the $\theta_i^H$,*
- *$t_i \circ t_j = t_j \circ t_i$ for all $i$ and $j$,*
- *$t_i^2 = id_{[n]}$ for each $0 \leqslant i \leqslant n$,*
- *$\theta_j^H \circ t_i = t_i \circ \theta_j^H$ for $i < j$ and $i > j+1$,*
- *$\theta_i^H \circ t_{i+1} = t_i \circ \theta_i^H$ and*
- *$\theta_i^H \circ t_i = t_{i+1} \circ \theta_i^H$.*

*This is a presentation of the group $H_{n+1}$.*

*Proof.* We begin by demonstrating that the given elements generate $H_{n+1}$. Consider a general element

$$g = (z_0, \ldots, z_n; \sigma)$$

where each $z_i \in \mathbb{Z}/2\mathbb{Z}$ and $\sigma \in \Sigma_{n+1}$.

By Proposition 24.1.1, $\sigma \in \Sigma_{n+1}$ can be written as a composite of transpositions, say

$$\sigma = \theta_{i_k} \circ \cdots \circ \theta_{i_1}.$$

Let $J = \{j_1, \ldots, j_l\} \subseteq [n]$ be the set of indices such that $z_i = t$ in the element $g$ if and only if $i \in J$.

We observe that

$$g = \theta_{i_k}^H \circ \cdots \circ \theta_{i_1}^H \circ t_{j_l} \circ \cdots \circ t_{j_1}.$$

It follows from Definition 25.1.5 that each $t_i$ squares to the identity and that $t_i \circ t_j = t_j \circ t_i$ for all $i$ and $j$. We deduce that there are $2^{n+1}$ possible composites of elements of the form $t_i$. The elements $\theta_j^H$ must satisfy the same relations as stated in Proposition 24.1.1, since composition of such elements corresponds to composition of transpositions. There are therefore $(n+1)!$ possible composites of the elements $\theta_j^H$. Therefore the group generated by the elements $t_i$ for $0 \leqslant i \leqslant n$ and $\theta_j^H$ for $0 \leqslant j \leqslant n-1$ has at least $2^{n+1}(n+1)! = |H_{n+1}|$ distinct elements.

We check the final three relations.

Suppose $i < j$ or $i > j+1$. We observe that

$$\begin{aligned}
\theta_j^H \circ t_i &= (1, \dots, 1; \theta_j) \circ \left(1, \dots, 1, t, 1, \dots, 1; id_{[n]}\right) \\
&= (1, \dots 1, t, 1 \dots, 1; \theta_j) \\
&= \left(1, \dots, 1, t, 1, \dots, 1; id_{[n]}\right) \circ (1, \dots, 1; \theta_j)
\end{aligned}$$

where $t$ occurs in the $i^{th}$ position throughout. This holds since, in the composites, the transposition $\theta_j = (j\ j+1)$ acts on the entries in position $j$ and position $j+1$, both of which are 1. We conclude that $\theta_j^H \circ t_i = t_i \circ \theta_j^H$ for $i < j$ and $i > j+1$.

Finally,

$$\begin{aligned}
\theta_i^H \circ t_i &= (1, \dots, 1; \theta_i) \circ \left(1, \dots, 1, t, 1, \dots, 1; id_{[n]}\right) \\
&= (1, \dots, 1, t, 1, \dots, 1; \theta_i)
\end{aligned}$$

where $t$ is in the $i^{th}$ position. It follows that

$$\begin{aligned}
\theta_i^H \circ t_i \circ \theta_i^H &= (1, \dots, 1, t, 1, \dots, 1; \theta_i) \circ (1, \dots, 1; \theta_i) \\
&= \left(1, \dots, 1, t, 1, \dots, 1; \theta_i^2\right)
\end{aligned}$$

where $t$ now occurs in position $i+1$ by the action of $\theta_i = (i\ i+1)$ in the composite. Since $\theta_i^2 = id_{[n]}$ we conclude that $\theta_i^H \circ t_i = t_{i+1} \circ \theta_i^H$ and $\theta_i^H \circ t_{i+1} = t_i \circ \theta_i^H$ as required.

The final three relations tell us that any composite of generators not written in the form

$$\theta_{i_k}^H \circ \cdots \circ \theta_{i_1}^H \circ t_{j_l} \circ \cdots \circ t_{j_1}$$

can be rewritten in this form. Therefore the group generated by the elements $t_i$ for $0 \leqslant i \leqslant n$ and $\theta_j^H$ for $0 \leqslant j \leqslant n-1$ has at most $2^{n+1}(n+1)! = |H_{n+1}|$ distinct elements.□

## 25.2 The Category $\Delta H$

The family of hyperoctahedral groups, $\{H_{n+1}\}_{n \geqslant 0}$, forms a crossed simplicial group, [FL91, Theorem 3.3]. We provide a description of the associated category $\Delta H$.

Following [FL91, Definition 1.1], the objects of $\Delta H$ will be the sets $[n] = \{0, \dots, n\}$. An element of $\mathrm{Hom}_{\Delta H}([n], [m])$ will be a pair $(\varphi, g)$ where $g \in H_{n+1}$ and $\varphi \in \mathrm{Hom}_{\Delta}\left([n], [m]\right)$.

### 25.2.1 Composition in $\Delta H$

We will describe the composition of morphisms in the category $\Delta H$.

Recall from Remark 19.1.4 that a composite of the form

$$\Big(id_{[m]}, g\Big) \circ \Big(\varphi, id_{[n]}\Big),$$

where $\Big(\varphi, id_{[n]}\Big) \in \mathrm{Hom}_{\Delta H}([n],[m])$ and $\Big(id_{[m]}, g\Big) \in \mathrm{Hom}_{\Delta H}([m],[m])$, has a unique decomposition of the form

$$\big(g_\star(\varphi), \varphi^\star(g)\big)$$

where $g_\star(\varphi) \in \mathrm{Hom}_\Delta\big([n],[m]\big)$ and $\varphi^\star(g) \in H_{n+1}$.

We begin by defining the order-preserving map $g_\star(\varphi)$. Recall that an order-preserving map is completely determined by the cardinality of the preimage of each element of the codomain.

Since $g \in H_{m+1}$, it is of the form $(z_0, \ldots, z_m; \sigma)$ where each $z_i \in \mathbb{Z}/2\mathbb{Z}$ and $\sigma \in \Sigma_{m+1}$. In particular, $\sigma$ is a permutation of the set $\underline{m}$.

**Definition 25.2.1.** Let $g = (z_0, \ldots, z_m; \sigma) \in H_{m+1}$ and $\varphi \in \mathrm{Hom}_\Delta\big([n],[m]\big)$. We define $g_\star(\varphi) \in \mathrm{Hom}_{\Delta H}\big([n],[m]\big)$ to be the order-preserving map determined by

$$\Big|g_\star(\varphi)^{-1}(i)\Big| = \Big|\varphi^{-1}(\sigma^{-1}(i))\Big|$$

for each $i \in [m]$.

We now describe the element $\varphi^\star(g) \in H_{n+1}$.

**Definition 25.2.2.** For $j \in [m]$, let

$$\varphi^{-1}\Big(\sigma^{-1}(j)\Big) = \big\{l_1, \ldots l_p\big\} \subseteq [n],$$

and

$$g_\star(\varphi)^{-1}(j) = \big\{m_1, \ldots, m_p\big\} \subseteq [n].$$

Let $g = (z_0, \ldots, z_m; \sigma) \in H_{m+1}$ and $\varphi \in \mathrm{Hom}_\Delta\big([n],[m]\big)$. We define

$$\varphi^\star(g) = (w_0, \ldots, w_n; \rho) \in H_{n+1}$$

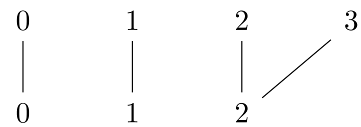
as follows.

Let $j \in [m]$. The permutation $\rho$ sends the elements of $\varphi^{-1}\big(\sigma^{-1}(j)\big)$ to the elements of $g_\star(\varphi)^{-1}(j)$.

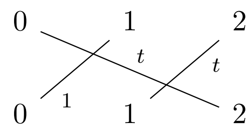
If $z_{\sigma^{-1}(j)} = 1$, then $\rho(l_i) = m_i$ and $w_{\rho^{-1}(m_i)} = 1$.

If $z_{\sigma^{-1}(j)} = t$, then $\rho(l_i) = m_{p-i}$ and $w_{\rho^{-1}(m_i)} = t$.

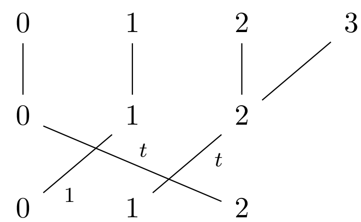
*Example 25.2.3.* Let $\varphi \in \mathrm{Hom}_\Delta\big([3],[2]\big)$ be represented by




and let $g = \big(t, 1, t; (0\ 2\ 1)\big) \in H_3$ be represented by the labelled permutation




The diagram

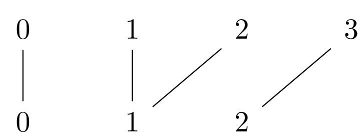



represents the composite $\Big(id_{[2]}, g\Big) \circ \Big(\varphi, id_{[3]}\Big) \in \mathrm{Hom}_{\Delta H}\big([3],[2]\big)$. To ease notation, let $\sigma = (0\ 2\ 1)$.

We define the order-preserving map $g_\star(\varphi) \in \mathrm{Hom}_{\Delta}\big([3],[2]\big)$ by

- $\big|g_\star(\varphi)^{-1}(0)\big| = \big|\varphi^{-1}(\sigma^{-1}(0))\big| = 1$,
- $\big|g_\star(\varphi)^{-1}(1)\big| = \big|\varphi^{-1}(\sigma^{-1}(1))\big| = 2$ and
- $\big|g_\star(\varphi)^{-1}(2)\big| = \big|\varphi^{-1}(\sigma^{-1}(2))\big| = 1$.

Pictorially, $g_\star(\varphi)$ is the order-preserving map



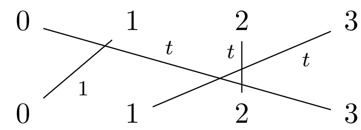

We define $\varphi^\star(g) = (w_0, \ldots, w_3; \rho) \in H_4$ as follows.

The permutation $\rho$ sends the set $\varphi^{-1}\left(\sigma^{-1}(0)\right) = \{1\}$ to the set $g_\star(\varphi)^{-1}(0) = \{0\}$. Since $z_{\sigma^{-1}(0)} = 1$, $w_{\rho^{-1}(0)} = w_1 = 1$.

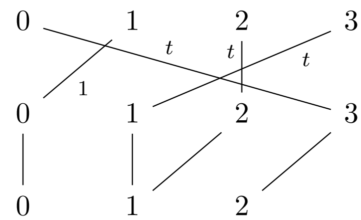
The permutation $\rho$ sends the set $\varphi^{-1}\left(\sigma^{-1}(1)\right) = \{2, 3\}$ to the set $g_\star(\varphi)^{-1}(1) = \{1, 2\}$. We note that $z_{\sigma^{-1}(1)} = t$. Therefore, $\rho(2) = 2$ and $\rho(3) = 1$. Furthermore, $w_{\rho^{-1}(2)} = w_2 = t$ and $w_{\rho^{-1}(1)} = w_3 = t$.

Finally, the permutation $\rho$ sends the set $\varphi^{-1}\left(\sigma^{-1}(2)\right) = \{0\}$ to the set $g_\star(\varphi)^{-1}(2) = \{3\}$. Since $z_{\sigma^{-1}(2)} = t$, $w_{\rho^{-1}(3)} = w_0 = t$.

Pictorially, $\varphi^\star(g) = \big(t, 1, t, t; (0\,3\,1)\big) \in H_4$ is



Finally, we observe that $\left(g_\star(\varphi), \varphi^\star(g)\right)$ is the pair represented by



It is shown in [FL91, Section 3] that the functions $\varphi^\star$ and $g_\star$ satisfy the relations of Proposition 19.1.7.

We use the constructions of $g_\star(\varphi)$ and $\varphi^\star(g)$ to define composition of morphisms in the category $\Delta H$.

**Definition 25.2.4.** Let $(\varphi, g) \in \mathrm{Hom}_{\Delta H}([l],[m])$ and let $(\psi, h) \in \mathrm{Hom}_{\Delta H}([m],[n])$. We define the composite in $\Delta H$ by

$$(\psi, h) \circ (\varphi, g) \coloneqq \left(\psi \circ h_\star(\varphi), \varphi^\star(h) \circ g\right).$$

*Remark 25.2.5.* Recall that Proposition 19.1.8 ensures that this composition is associative.

### 25.2.2 The category $\Delta H$ and a presentation

**Definition 25.2.6.** The *hyperoctahedral category*, denoted $\Delta H$, has as objects the sets $[n] = \{0, \ldots, n\}$, for $n \geqslant 0$. An element of $\mathrm{Hom}_{\Delta H}([n],[m])$ is a pair $(\varphi, g)$ where $\varphi \in \mathrm{Hom}_{\Delta}\left([n],[m]\right)$ is an order-preserving map and $g \in H_{n+1}$. Composition is defined in Definition 25.2.4.

**Proposition 25.2.7.** *The category $\Delta H$ has as objects the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$. Morphisms in $\Delta H$ are generated by*

- $\left(\delta_i, id_{[n]}\right)$*, for each $n \geqslant 0$ and $0 \leqslant i \leqslant n+1$ where the $\delta_i$ are the face maps of Definition 2.2.2,*
- $\left(\sigma_j, id_{[n+1]}\right)$ *for each $n \geqslant 0$ and $0 \leqslant j \leqslant n$ where the $\sigma_j$ are the degeneracy maps of Definition 2.2.3,*
- $\left(id_{[n]}, t_i\right)$ *for each $n \geqslant 0$ and $0 \leqslant i \leqslant n$, where the $t_i$ are the group elements of Definition 25.1.5 and*
- $\left(id_{[n]}, \theta_i^H\right)$ *for each $n \geqslant 1$ and $0 \leqslant i \leqslant n-1$, where the $\theta_i^H$ are the transpositions of Definition 25.1.6.*

*The generators aforementioned are subject to*

- *the relations of Proposition 24.2.2,*
- *the relations of Proposition 25.1.7,*
- $\left(id_{[n+1]}, t_j\right) \circ \left(\delta_i, id_{[n]}\right) = \left(\delta_i, t_{j-1}\right)$ *if* $i < j$*,*
- $\left(id_{[n+1]}, t_i\right) \circ \left(\delta_i, id_{[n]}\right) = \left(\delta_i, id_{[n]}\right)$*,*
- $\left(id_{[n+1]}, t_j\right) \circ \left(\delta_i, id_{[n]}\right) = \left(\delta_i, t_j\right)$ *if* $i > j$*,*
- $\left(id_{[n]}, t_i\right) \circ \left(\sigma_j, id_{[n+1]}\right) = \left(\sigma_j, t_i\right)$ *if* $i < j$*,*
- $\left(id_{[n]}, t_i\right) \circ \left(\sigma_i, id_{[n+1]}\right) = \left(\sigma_i, \theta_i^H t_{i+1} t_i\right)$ *and*
- $\left(id_{[n]}, t_i\right) \circ \left(\sigma_j, id_{[n+1]}\right) = \left(\sigma_j, t_{i+1}\right)$ *if* $i > j$

*for each* $n \geqslant 0$*. This is a presentation of the category* $\Delta H$*.*

*Proof.* Recall that an element of $\mathrm{Hom}_{\Delta H}([n],[m])$ is a pair $(\varphi, g)$ where $g \in H_{n+1}$ and $\varphi \in \mathrm{Hom}_{\Delta}\left([n],[m]\right)$.

By Proposition 2.2.4 we know that the face maps, $\delta_i$, and the degeneracy maps, $\sigma_j$, generate all order-preserving maps and we have the unique decomposition of Theorem 2.2.5. Furthermore, by Proposition 25.1.7, we know that the elements $\theta_i^H$ and $t_i$ generate all elements of the group $H_n$.

Suppose, therefore, that

$$\varphi = \delta_{i_1} \circ \cdots \circ \delta_{i_r} \circ \sigma_{j_1} \circ \cdots \sigma_{j_s}$$

and

$$g = \theta_{k_1}^H \circ \cdots \circ \theta_{k_m}^H \circ t_{l_1} \circ \cdots \circ t_{l_n}.$$

We observe that

$$(\varphi, g) = \left(\delta_{i_1}, id_{[m-1]}\right) \circ \cdots \circ \left(\sigma_{j_s}, id_{[n]}\right) \circ \left(id_{[n]}, \theta_{k_1}^H\right) \circ \cdots \circ \left(id_{[n]}, t_{l_n}\right).$$

It follows that the given morphisms generate all morphisms in $\Delta H$ and the category generated by the given morphisms contains all the morphisms of $\Delta H$.

The relations follow from computing the appropriate versions of $g_\star(\varphi)$ and $\varphi^\star(g)$ using Subsection 25.2.

The relations tell us that any composite of generators can be re-written in the form

$$\left(\delta_{i_1}, id\right) \circ \cdots \circ \left(\sigma_{j_s}, id\right) \circ \left(id, \theta_{k_1}^H\right) \circ \cdots \circ \left(id, t_{l_n}\right)$$

and so any morphism in the category generated by the given morphisms is a morphism in $\Delta H$ as required. □

## 25.3 The Hyperoctahedral Bar Construction

We define the *hyperoctahedral bar construction* following Fiedorowicz [Fie, Section 2].

Let $A$ be an involutive, associative $k$-algebra with the involution denoted by

$$a \mapsto \overline{a}.$$

We define an assignment

$$B_A^{oct} \colon \Delta H \to \mathbf{kMod}$$

on objects by

$$[n] \mapsto A^{\otimes n+1}.$$

**Definition 25.3.1.** For $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$ we define

$$B_A^{oct}\left(\varphi, id_{[n]}\right)(a_0 \otimes \cdots \otimes a_n) = \left(\prod_{i \in \varphi^{-1}(0)}^{<} a_i\right) \otimes \cdots \otimes \left(\prod_{i \in \varphi^{-1}(m)}^{<} a_i\right)$$

on elementary tensors and extend $k$-linearly, where $\prod^{<}$ denotes the ordered product of elements $a_i$ by increasing index. An empty product is defined to be the multiplicative unit $1_A \in A$.

**Definition 25.3.2.** Let $g = (z_0, \dots, z_n; \sigma) \in H_{n+1}$, so $z_i \in \mathbb{Z}/2\mathbb{Z}$ and $\sigma \in \Sigma_{n+1}$. We define

$$B_A^{oct}\left(id_{[n]}, g\right)(a_0 \otimes \cdots \otimes a_n) = a_{\sigma^{-1}(0)}^{z_{\sigma^{-1}(0)}} \otimes \cdots \otimes a_{\sigma^{-1}(n)}^{z_{\sigma^{-1}(n)}}$$

on elementary tensors and extend $k$-linearly, where

$$a^{z_i} = \begin{cases} a & z_i = 1 \\ \overline{a} & z_i = t. \end{cases}$$

*Remark 25.3.3.* The assignment $B_A^{oct}$ acts on a morphism $(\varphi, g) \in \mathrm{Hom}_{\Delta H}\big([n],[m]\big)$ by sending it to the morphism of $k$-modules that permutes the factors of the tensor according to the underlying permutation of $g$, applies the involution according the labels of $g$ and multiplies tensor factors, or inserts identities, according to the order-preserving map $\varphi$.

**Definition 25.3.4.** Let $A$ be an involutive, associative $k$-algebra. We define a functor

$$B_A^{oct} \colon \Delta H \to \mathbf{kMod}$$

on objects by

$$[n] \mapsto A^{\otimes n+1}.$$

For a morphism $(\varphi, g) \in \mathrm{Hom}_{\Delta H}\big([n],[m]\big)$ we define

$$B_A^{oct}(\varphi, g) = B_A^{oct}\left(\varphi, id_{[n]}\right) \circ B_A^{oct}\left(id_{[n]}, g\right),$$

where $B_A^{oct}\left(\varphi, id_{[n]}\right)$ and $B_A^{oct}\left(id_{[n]}, g\right)$ are defined in Definitions 25.3.1 and 25.3.2 respectively.

We call $B_A^{oct}$ the *hyperoctahedral bar construction.*

The proof that the hyperoctahedral bar construction is a well-defined functor can be found in Appendix B.

## 25.4 Hyperoctahedral Homology

**Definition 25.4.1.** Let $A$ be an involutive, associative algebra. We define the $n^{th}$ *hyperoctahedral homology of $A$* to be

$$HO_n(A) \coloneqq \mathrm{Tor}_n^{\Delta H}\left(k^\star, B_A^{oct}\right)$$

for $n \geqslant 0$.

**Definition 25.4.2.** Following Definition 8.1.3 we define the chain complex

$$C_\star\left(\Delta H, B_A^{oct}\right),$$

whose homology groups are the hyperoctahedral homology groups of $A$.

**Definition 25.4.3.** Following Chapter 9 we define the chain complex

$$k\left[N_\star\left(-\backslash\Delta H\right)\right] \otimes_{\Delta H} B_A^{oct}(-)$$

whose homology groups are the hyperoctahedral homology groups of $A$.

**Proposition 25.4.4.** *There exist isomorphisms of graded $k$-modules*

$$HO_\star(A) \cong H_\star\left(\Delta H, B_A^{oct}\right) \cong H_\star\left(k[N_\star(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)\right).$$

□

### 25.4.1 Hyperoctahedral homology of the ground ring

**Proposition 25.4.5.** *The hyperoctahedral homology groups of the ground ring $k$ are*

$$HO_n(k) \cong \begin{cases} k & n = 0, \\ 0 & \textit{else.} \end{cases}$$

*Proof.* The key is to prove that the nerve of the category $\Delta H$ is contractible.

Recall from Lemma 19.1.9 that the crossed simplicial group $\{H_\star\}$ has the structure of a simplicial set. Furthermore, recall from Corollary 19.5.4 that this simplicial set is contractible. Proposition 19.5.5 implies that we have a homotopy equivalence

$$\Omega B\Delta H \simeq |H_\star|$$

where $\Omega$ is the based loop functor of Definition 3.1.3 and $B\Delta H$ is the classifying space of the category $\Delta H$.

We deduce that $\Omega B\Delta H$ is a contractible space. Recall the path space fibration

$$\Omega B\Delta H \to PB\Delta H \to B\Delta H$$

of Proposition 3.1.4. By Proposition 3.1.2, the path space $PB\Delta H$ is contractible. By examining the long exact sequence of homotopy groups associated to a fibration we deduce that $B\Delta H$ is a contractible space.

By definition,

$$B\Delta H \coloneqq |N_\star(\Delta H)|$$

and so $N_\star(\Delta H)$ is a contractible simplicial set. In particular, the homology of the simplicial set $N_\star(\Delta H)$ is isomorphic to $k$ concentrated in degree zero.

To conclude the proof we demonstrate that the chain complex $C_\star\left(\Delta H, B_k^{oct}\right)$, whose homology groups are the hyperoctahedral homology of the ground ring $k$, is isomorphic to the chain complex associated to the nerve of $\Delta H$.

We observe that the chain complex $C_\star\left(\Delta H, B_k^{oct}\right)$ is generated $k$-linearly in degree $p$ by elements of the form

$$\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0]\right) \otimes (1_k \otimes \cdots \otimes 1_k)$$

where

$$\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0]\right)$$

is an element of $N_p(\Delta H)$. We note that since all tensor products are taken over $k$ there is an isomorphism of chain complexes

$$C_\star\left(\Delta H, B_k^{oct}\right) \to k\left[N_\star(\Delta H)\right]$$

determined in degree $p$ by

$$\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0]\right) \otimes (1_k \otimes \cdots \otimes 1_k) \mapsto \left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0]\right).$$

We conclude that

$$HO_\star(k) \cong \begin{cases} k & \star = 0 \\ 0 & \text{else.} \end{cases}$$

as required. □

*Remark 25.4.6.* By Corollary 19.5.4, the same proof works for the symmetric homology and braid homology of the ground ring $k$. In the case of symmetric homology Ault [Aul10, Lemma 18 and Corollary 19] provides an alternative proof of this fact.

### 25.4.2 Fiedorowicz's theorem for hyperoctahedral homology

Let $G$ be a group with an involution denoted

$$g \mapsto \overline{g}.$$

*Remark 25.4.7.* Every group satisfies this criterion with the map

$$g \mapsto g^{-1}.$$

This result also holds for groups with a second involution.

Let $BG$ be the classifying space of $G$. Recall the based loops functor of Definition 3.1.3 and the functor $Q$ of Definition 3.5.14 and consider the space $\Omega Q(BG)$.

We form the suspension spectrum of $\Omega Q(BG)$ following Example 3.5.6. In level $n$ we have the space $\Sigma^n \Omega Q(BG)$ with the structure maps

$$id_{\Sigma^n} \colon \Sigma\left(\Sigma^n \Omega Q(BG)\right) \to \Sigma^{n+1}\Omega Q(BG).$$

Each space $\Sigma^n \Omega Q(BG)$ has a $\mathbb{Z}/2\mathbb{Z}$-action induced from the map

$$g \mapsto \overline{g}^{-1}$$

on $G$.

Let $E\mathbb{Z}/2\mathbb{Z}_\star$ denote the total space of $\mathbb{Z}/2\mathbb{Z}$ with a disjoint basepoint. Recall the equivariant smash product of Definition 3.2.2. We can take the equivariant smash product level-wise to obtain a spectrum with the $n^{th}$ space

$$E\mathbb{Z}/2\mathbb{Z}_\star \wedge_{\mathbb{Z}/2\mathbb{Z}} \Sigma^n \Omega Q(BG).$$

The structure map

$$\gamma_n \colon E\mathbb{Z}/2\mathbb{Z}_\star \wedge_{\mathbb{Z}/2\mathbb{Z}} \Sigma\left(\Sigma^n \Omega Q(BG)\right) \to E\mathbb{Z}/2\mathbb{Z}_\star \wedge_{\mathbb{Z}/2\mathbb{Z}} \Sigma^{n+1}\Omega Q(BG)$$

given by

$$id_{\mathbb{Z}/2\mathbb{Z}} \wedge id_{\Sigma^n}.$$

*Remark 25.4.8.* This is well-defined since the smash product is associative.

We may now state Fiedorowicz's theorem for hyperoctahedral homology. This theorem tells us that the hyperoctahedral homology of a group algebra is isomorphic to the homology of the infinite loop space associated to the spectrum

$$E\mathbb{Z}/2\mathbb{Z}_\star \wedge_{\mathbb{Z}/2\mathbb{Z}} \Sigma^n \Omega Q(BG).$$

**Theorem 25.4.9 ([Fie, Theorem 1]).** *Let $G$ be a discrete group with involution. Let $BG$ be the classifying space of the group and let $k[G]$ be its group algebra. There is an isomorphism of $k$-modules*

$$HO_n\left(k[G]\right) \cong H_n\left(\Omega^\infty\left(E\mathbb{Z}/2\mathbb{Z}_\star \wedge_{\mathbb{Z}/2\mathbb{Z}} \Omega Q(BG)\right), k\right)$$

*for each $n \geqslant 0$.* □

# Part V

# The Category of Involutive, Non-Commutative Sets

# Chapter 26

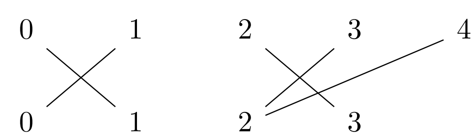
# The Category of Involutive, Non-Commutative Sets

We define the category of *involutive, non-commutative sets* and demonstrate that it is isomorphic to the category $\Delta H$ of Definition 25.2.6. One advantage of the category of involutive non-commutative sets is that one does not need to calculate the unique decompositions in $\Delta H$ as discussed in Subsection 25.2. The category $\mathcal{IF}(as)$ contains the category $\mathcal{F}(as)$ of Subsection 24.2.1 as a subcategory.

We will denote the category of involutive, non-commutative sets by $\mathcal{IF}(as)$. It will have as objects the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$. An element $f \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([n], [m]\big)$ will be a map of sets such that the preimage of each singleton $i \in [m]$ is a totally ordered set such that each element comes adorned with a superscript label from the group $\mathbb{Z}/2\mathbb{Z}$.

*Remark 26.0.1.* Henceforth we will say that a morphism in $\mathcal{IF}(as)$ is a map of sets together with a *labelled, ordered set* for each preimage. In particular, note that we will use *preimage* to mean preimage of a singleton.

*Example 26.0.2.* Let $f_1 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([4], [3]\big)$ have underlying map of sets




with the following labelled, ordered sets as preimages:

- $f_1^{-1}(0) = \big\{1^1\big\}$,
- $f_1^{-1}(1) = \big\{0^t\big\}$,
- $f_1^{-1}(2) = \big\{3^t < 4^1\big\}$ and
- $f_1^{-1}(3) = \big\{2^t\big\}$.

### 26.0.1 Composition

We must define composition of morphisms in $\mathcal{IF}(as)$. We will denote composition in $\mathcal{IF}(as)$ by $\bullet$ in order to distinguish from the composition of maps of sets. In particular, we use $\circ$ for two morphisms in $\mathcal{IF}(as)$ if we are referring to the composite of the underlying maps of sets. In order to ease notation we have chosen not to introduce notation for the forgetful functor $\mathcal{IF}(as) \to \mathbf{Set}$.

Let $f_1 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[m]\big)$ and $f_2 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([m],[l]\big)$.

In order to define the composite $f_2 \bullet f_1 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[l]\big)$ we must provide a map of sets and describe the labelled total orderings on each of the preimages.

As a map of sets, $f_2 \bullet f_1$ is the composite of the underlying map of sets $f_2 \circ f_1$.

In order to specify a labelled, ordered set for the preimage of each singleton in $\underline{l}$ under the composite we first make a definition.

**Definition 26.0.3.** We define an action of $\mathbb{Z}/2\mathbb{Z}$, which will be denoted by a superscript, on finite, ordered sets with $\mathbb{Z}/2\mathbb{Z}$-labels by

$$\left\{j_1^{\delta_{j_1}} < \cdots < j_r^{\delta_{j_r}}\right\}^t = \left\{j_r^{t\delta_{j_r}} < \cdots < j_1^{t\delta_{j_1}}\right\}.$$

That is, we invert the ordering and multiply each label by $t \in \mathbb{Z}/2\mathbb{Z}$.

*Example 26.0.4.*

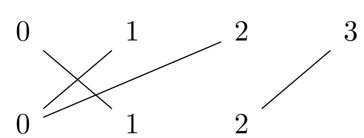
$$\left\{0^t < 2^1 < 1^t\right\}^t = \left\{1^1 < 2^t < 0^1\right\}$$

**Definition 26.0.5.** Let $f_1 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[m]\big)$ and $f_2 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([m],[l]\big)$. We define $f_2 \bullet f_1 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[l]\big)$ to have underlying map of sets $f_2 \circ f_1$. We define the labelled totally ordered set $(f_2 \bullet f_1)^{-1}(i)$ to be the ordered coproduct of labelled, ordered sets

$$\coprod_{j^{\delta_j} \in f_2^{-1}(i)} f_1^{-1}(j)^{\delta_j}.$$

*Example 26.0.6.* Let $f_1 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([4],[3]\big)$ be the morphism defined in Example 26.0.2.

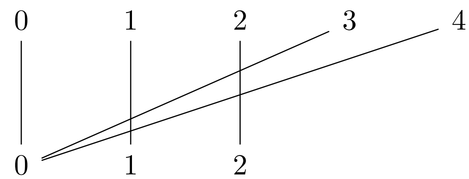
Let $f_2 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([3],[2]\big)$ have underlying map of sets




with the following labelled, ordered sets as preimages:

- $f_2^{-1}(0) = \{1^t < 2^t\}$,
- $f_2^{-1}(1) = \{0^1\}$ and
- $f_2^{-1}(2) = \{3^t\}$.

The underlying map of sets $f_2 \circ f_1 \colon [4] \to [2]$ is




We describe the labelled total orderings on each preimage.

For simplicity, we start with $(f_2 \bullet f_1)^{-1}(1)$. Firstly, $f_2^{-1}(1) = \{0^1\}$. Since this a singleton with label 1, the preimage

$$(f_2 \bullet f_1)^{-1}(1) = f_1^{-1}(0)^1 = \left\{1^1\right\}.$$

We see that $f_2^{-1}(2) = \{3^t\}$. The set is a singleton and the label is $t$ so the preimage $(f_2 \bullet f_1)^{-1}(2)$ is the labelled ordered set $f_1^{-1}(3)$ with each label multiplied by $t$ and the order inverted. That is,

$$(f_2 \bullet f_1)^{-1}(2) = f_1^{-1}(3)^t = \left\{2^1\right\}.$$

Finally we consider $(f_2 \bullet f_1)^{-1}(0)$. We have $f_2^{-1}(0) = \{1^t < 2^t\}$ and so the labelled ordered set

$$(f_2 \bullet f_1)^{-1}(0) = f_1^{-1}(1)^t \coprod f_1^{-1}(2)^t.$$

Firstly,

$$f_1^{-1}(1)^t = \left\{0^t\right\}^t = \left\{0^1\right\}.$$

Furthermore,

$$f_1^{-1}(2)^t = \left\{3^t < 4^t\right\}^t = \left\{4^t < 3^1\right\}.$$

We conclude that

$$(f_2 \bullet f_1)^{-1}(0) = f_1^{-1}(1)^t \coprod f_1^{-1}(2)^t = \left\{0^1 < 4^t < 3^1\right\}.$$

In summary, the preimage data of $f_2 \bullet f_1 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([4],[2]\big)$ is

- $(f_2 \bullet f_1)^{-1}(0) = \{0^1 < 4^t < 3^1\}$,
- $(f_2 \bullet f_1)^{-1}(1) = \{1^1\}$ and
- $(f_2 \bullet f_1)^{-1}(2) = \{2^1\}$.

The proof that composition is associative can be found in Appendix C.1

**Definition 26.0.7.** The *category of involutive, non-commutative sets*, $\mathcal{IF}(as)$, has as objects the sets $[n] = \{0, \ldots, n\}$ for $n \geqslant 0$. An element of $\mathrm{Hom}_{\mathcal{IF}(as)}([n],[m])$ is a map of sets with a total ordering on each preimage such that each element of the domain comes adorned with a superscript label from the group $\mathbb{Z}/2\mathbb{Z}$. Composition of morphisms is as defined in Definition 26.0.5.

# Chapter 27

# The Isomorphism

We define two mutually inverse functors between the categories $\Delta H$ and $\mathcal{IF}(as)$.

## 27.1 The Functor $F\colon \Delta H \to \mathcal{IF}(as)$

We define an assignment

$$F\colon \Delta H \to \mathcal{IF}(as)$$

to be the identity on objects.

We will extend this to include an assignment on morphisms.

We note that a morphism $(\varphi, g) \in \mathrm{Hom}_{\Delta H}\big([n],[m]\big)$ is defined as a composite

$$\Big(\varphi, id_{[n]}\Big) \circ \Big(id_{[n]}, g\Big).$$

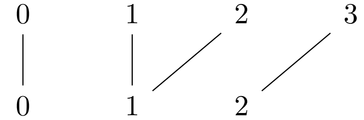
We will define $F$ on these two morphisms separately.

**Definition 27.1.1.** Let $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$. We define

$$F\Big(\varphi, id_{[n]}\Big) \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[m]\big)$$

to be the underlying map of sets, $\varphi$, with the standard total ordering on each preimage with 1 for every label.

*Example 27.1.2.* Let $\varphi \in \mathrm{Hom}_{\Delta}\big([3],[2]\big)$ be represented by




The map $F\Big(\varphi, id_{[3]}\Big) \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([3],[2]\big)$ is the underlying map of sets given by $\varphi$ with

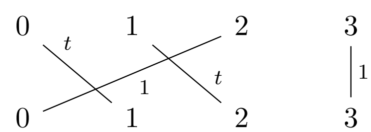
- $\left(F\left(\varphi, id_{[3]}\right)\right)^{-1}(0) = \{0^1\}$,
- $\left(F\left(\varphi, id_{[3]}\right)\right)^{-1}(1) = \{1^1 < 2^1\}$ and
- $\left(F\left(\varphi, id_{[3]}\right)\right)^{-1}(2) = \{3^1\}$.

**Definition 27.1.3.** Let $g = (z_0, \ldots, z_n; \sigma) \in H_{n+1}$, so each $z_i \in \mathbb{Z}/2\mathbb{Z}$ and $\sigma \in \Sigma_{n+1}$. We define $F\left(id_{[n]}, g\right) \in \mathrm{Hom}_{\mathcal{IF}(as)}\left([n], [n]\right)$ to be the underlying map of sets defined by the permutation, $\sigma$. Each preimage is a singleton and we define the label on

$$\left(F\left(id_{[n]}, g\right)\right)^{-1}(i)$$

to be $z_{\sigma^{-1}(i)}$.

*Example 27.1.4.* Let $g = \left(t, t, 1, 1; (0\,1\,2)\right) \in H_4$ be represented by




The morphism $F\left(id_{[3]}, g\right)$ in $\mathrm{Hom}_{\mathcal{IF}(as)}\left([3], [3]\right)$ is given by the underlying map of sets with

- $\left(F\left(id_{[3]}, g\right)\right)^{-1}(0) = \{2^1\}$,
- $\left(F\left(id_{[3]}, g\right)\right)^{-1}(1) = \{0^t\}$,
- $\left(F\left(id_{[3]}, g\right)\right)^{-1}(2) = \{1^t\}$ and
- $\left(F\left(id_{[3]}, g\right)\right)^{-1}(3) = \{3^1\}$.

**Definition 27.1.5.** Let $(\varphi, g) \in \mathrm{Hom}_{\Delta H}\left([n], [m]\right)$. We define

$$F(\varphi, g) = F\left(\varphi, id_{[n]}\right) \bullet F\left(id_{[n]}, g\right),$$

where $F\left(\varphi, id_{[n]}\right)$ and $F\left(id_{[n]}, g\right)$ are defined in Definitions 27.1.1 and 27.1.3 respectively.

The proof that the assignment $F$ is functorial can be found in Appendix C.2.

**Definition 27.1.6.** We define a functor

$$F\colon \Delta H \to \mathcal{IF}(as)$$

to be the identity on objects. For a morphism $(\varphi, g) \in \mathrm{Hom}_{\Delta H}\left([n],[m]\right)$ we define

$$F(\varphi, g) = F\left(\varphi, id_{[n]}\right) \bullet F\left(id_{[n]}, g\right)$$

as in Definition 26.0.5.

## 27.2 The Functor $G\colon \mathcal{IF}(as) \to \Delta H$

We define $G$ to be the identity on objects.

In order to define $G$ on morphisms we need the following construction.

A morphism $f \in \mathrm{Hom}_{\mathcal{IF}(as)}\left([n],[m]\right)$ determines a morphism $(\varphi_f, g_f) \in \mathrm{Hom}_{\Delta H}\left([n],[m]\right)$ as follows.

**Definition 27.2.1.** Let $f \in \mathrm{Hom}_{\mathcal{IF}(as)}\left([n],[m]\right)$. We define $\varphi_f \in \mathrm{Hom}_{\Delta}\left([n],[m]\right)$ to be the order-preserving map determined by

$$\left|\varphi_f^{-1}(i)\right| = \left|f^{-1}(i)\right|.$$

**Definition 27.2.2.** Let $f \in \mathrm{Hom}_{\mathcal{IF}(as)}\left([n],[m]\right)$. Let $g_f = \left(z_0, \dots, z_n; \sigma_f\right) \in H_{n+1}$ be defined as follows.

For $i \in [m]$, let

$$f^{-1}(i) := \left\{i_1^{\delta_1} < \cdots < i_k^{\delta_k}\right\}$$

and let

$$\varphi_f^{-1}(i) = \{j_1 < \cdots < j_k\}.$$

For each $i \in [m]$ let $\sigma_f$ send the elements of the underlying ordered set of $f^{-1}(i)$,

$$\{i_1 < \ldots < i_k\}$$

to the elements of the naturally ordered set of $\varphi_f^{-1}(i)$,

$$\{j_1 < \ldots < j_k\}$$

by $\sigma_f(i_l) = j_l$. Let $z_{\sigma^{-1}(j_l)} = \delta_l$ for $1 \leqslant l \leqslant k$.

*Remark 27.2.3.* Observe that we define the element $g_f \in H_{n+1}$ by encoding the labels of the preimages $f^{-1}(i)$ with the labels $z_i$ and encoding the total ordering data of $f^{-1}(i)$ in the permutation $\sigma$.

**Lemma 27.2.4.** *Let $f \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[m]\big)$. Let $\varphi_f$ and $g_f$ be as in Definitions 27.2.1 and 27.2.2. Let $F$ be the functor of Definition 27.1.6. Then*

$$F\left(\varphi_f, g_f\right) = f$$

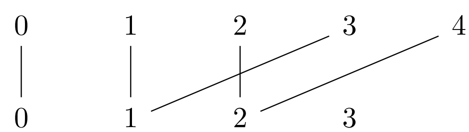
*in* $\mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[m]\big)$.

*Proof.* Observe that, by construction, the composition $\varphi_f \circ \sigma_f$ is the underlying map of sets for $f \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[m]\big)$. Furthermore, using Definitions 26.0.5, 27.1.1 and 27.1.3, we observe that for each $i \in [m]$,

$$\begin{aligned}
\Big(F\left(\varphi_f, g_f\right)\Big)^{-1}(i) &= \Big(F\left(\varphi_f, id_{[n]}\right) \bullet F\left(id_{[n]}, g_f\right)\Big)^{-1}(i) \\
&= \coprod_{j_l \in \left(F\left(\varphi_f, id_{[n]}\right)\right)^{-1}(i)} \Big(F\left(id_{[n]}, g_f\right)\Big)^{-1}(j_l) \\
&= \coprod_{j_l \in \varphi_f^{-1}(i)} \sigma_f^{-1}\left(j_l\right)^{\delta_l} \\
&= \left\{ i_1^{\delta_1} < \cdots < i_k^{\delta_k} \right\} \\
&= f^{-1}(i)
\end{aligned}$$

as labelled, ordered sets. □

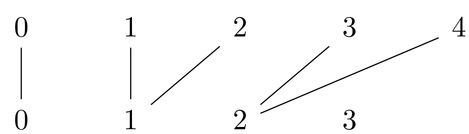
*Example 27.2.5.* Let $f \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([4],[3]\big)$ have underlying map of sets




with

- $f^{-1}(0) = \left\{0^t\right\}$,
- $f^{-1}(1) = \left\{1^1 < 3^t\right\}$ and
- $f^{-1}(2) = \left\{4^t < 2^1\right\}$.

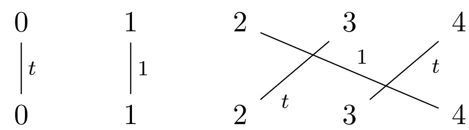
We note that $\left|f^{-1}(0)\right| = 1$, $\left|f^{-1}(1)\right| = 2$, $\left|f^{-1}(2)\right| = 2$ and $\left|f^{-1}(3)\right| = 0$. Therefore, $\varphi_f \in \mathrm{Hom}_{\Delta}\big([4],[3]\big)$ is represented by




We note that

- $f^{-1}(0) = \{0^t\}$ so $\sigma_f(0) = 0$ and $z_{\sigma^{-1}(0)} = t$,
- $f^{-1}(1) = \{1^1 < 3^t\}$ so $\sigma_f(1) = 1$, $\sigma_f(3) = 1$, $z_{\sigma^{-1}(1)} = 1$ and $z_{\sigma^{-1}(2)} = t$ and
- $f^{-1}(2) = \{4^t < 2^1\}$ so $\sigma_f(4) = 3$, $\sigma_f(2) = 4$, $z_{\sigma^{-1}(3)} = t$ and $z_{\sigma^{-1}(4)} = 1$.

That is, $g_f = \big(t, 1, 1, t, t; (2\,4\,3)\big) \in H_5$ and is represented by




We define an assignment

$$G\colon \mathrm{Hom}(\mathcal{IF}(as)) \to \mathrm{Hom}(\Delta H)$$

given by

$$f \mapsto (\varphi_f, g_f).$$

**Lemma 27.2.6.** *As a map of sets*

$$F \circ G\colon \mathrm{Hom}\left(\mathcal{IF}(as)\right) \to \mathrm{Hom}\left(\mathcal{IF}(as)\right)$$

*is the identity.*

*Proof.* Let $f \in \mathrm{Hom}_{\mathcal{IF}(as)}\left([n], [m]\right)$. We note that

$$\begin{aligned}(F \circ G)(f) &= F\left(\varphi_f, g_f\right) \\ &= f\end{aligned}$$

by Lemma 27.2.4. □

**Lemma 27.2.7.** *As a map of sets*

$$G \circ F\colon \mathrm{Hom}(\Delta H) \to \mathrm{Hom}(\Delta H)$$

*is the identity.*

*Proof.* Let $(\varphi, g) \in \mathrm{Hom}_{\Delta H}\left([n], [m]\right)$. Since $g \in H_{n+1}$ it is of the form $(z_0, \ldots, z_n; \sigma)$ where each $z_i \in \mathbb{Z}/2\mathbb{Z}$ and $\sigma \in \Sigma_{n+1}$. $F(\varphi, g)$ is the map of sets $\varphi \circ \sigma$ with the preimage data

$$\left(F\left(\varphi, g\right)\right)^{-1}(i) = \coprod_{j \in \varphi^{-1}(i)} \sigma^{-1}(j)^{\delta_j}$$

where $\delta_j = z_{\sigma^{-1}(j)}$.

Consider $(G \circ F)(\varphi, g)$. This is a morphism in $\Delta H$, constructed by the method provided at the beginning of this subsection, which we will denote by $(\varphi', g')$. We note that, by construction, $\varphi'$ is an order-preserving map with

$$\begin{aligned}\left|\left(\varphi'\right)^{-1}(i)\right| &= \left|(\varphi \circ \sigma)^{-1}(i)\right| \\ &= \left|\sigma^{-1}(\varphi^{-1}(i))\right| \\ &= \left|\varphi^{-1}(i)\right|\end{aligned}$$

where the last equality follows from the fact that $\sigma$ is a bijection of sets. We conclude that $\varphi' = \varphi$.

Recall that
$$\left(F\left(\varphi, g\right)\right)^{-1}(i) := \coprod_{j \in \varphi^{-1}(i)} \sigma^{-1}(j)^{\delta_j}.$$

Suppose that
$$\left(F\left(\varphi, g\right)\right)^{-1}(i) = \left\{ i_1^{\delta_1} < \cdots < i_k^{\delta_k} \right\}.$$

By construction, the underlying permutation of $g'$ sends the elements $i_1, \ldots, i_k$ to the elements of the ordered set
$$\left\{ \left|(\varphi')^{-1}(1)\right| + \cdots + \left|(\varphi')^{-1}(i-1)\right| + 1 < \cdots < \left|(\varphi')^{-1}(1)\right| + \cdots + \left|(\varphi')^{-1}(i)\right| \right\},$$
preserving the order, with the labels $\delta_i$ for $1 \leqslant i \leqslant k$. Since we know that $\varphi' = \varphi$ this is equal to the ordered set

$$\left\{ \left|\varphi^{-1}(1)\right| + \cdots + \left|\varphi^{-1}(i-1)\right| + 1 < \cdots < \left|\varphi^{-1}(1)\right| + \cdots + \left|\varphi^{-1}(i)\right| \right\}.$$

That is, the underlying permutation of $g'$ sends the elements of $\left(F\left(\varphi, g\right)\right)^{-1}(i)$ to the elements of $\varphi^{-1}(i)$ preserving the order. Furthermore the labels of $g$ and $g'$ match. We conclude that $g' = g$.

Hence $(G \circ F)(\varphi, g) = (\varphi, g)$ as required. □

**Corollary 27.2.8.** *The assignment $G \colon \mathcal{IF}(as) \to \Delta H$ is functorial.*

*Proof.* It is clear that $G$ respects identities. We note that
$$\begin{aligned} G(g \circ f) &= G((F \circ G)(g) \circ (F \circ G)(f)) && (F \circ G = id) \\ &= G(F(G(g) \circ G(f))) && (F \text{ functorial}) \\ &= (G \circ F)(G(g) \circ G(f)) \\ &= G(g) \circ G(f) && (G \circ F = id). \end{aligned}$$

Hence $G$ is functorial as required. □

**Definition 27.2.9.** We define a functor
$$G \colon \mathcal{IF}(as) \to \Delta H$$
to be the identity on objects. For a morphism $f \in \mathrm{Hom}_{\mathcal{IF}(as)}\left([n], [m]\right)$ we define
$$G(f) = \left(\varphi_f, g_f\right)$$
where $\left(\varphi_f, g_f\right)$ is defined using Definitions 27.2.1 and 27.2.2.

**Theorem 27.2.10.** *There is an isomorphism of categories*
$$F \colon \Delta H \to \mathcal{IF}(as)$$
*with inverse given by $G \colon \mathcal{IF}(as) \to \Delta H$.*

*Proof.* Both functors are the identity on objects. Lemma 27.2.6 demonstrates that $F \circ G$ is the identity on morphisms in $\mathcal{IF}(as)$ and Lemma 27.2.7 demonstrates that $G \circ F$ is the identity on morphisms in $\Delta H$. □

## 27.3 A Commutative Diagram of Categories

**Definition 27.3.1.** We have inclusions of the symmetric group $\Sigma_{n+1}$ into the hyperoctahedral group $H_{n+1}$,

$$i_{n+1} \colon \Sigma_{n+1} \to H_{n+1}$$

defined by

$$\sigma \mapsto (1, \ldots, 1; \sigma)$$

for each $n \geqslant 0$.

**Definition 27.3.2.** Let $I \colon \Delta S \to \Delta H$ be the inclusion of the subcategory $\Delta S$ into $\Delta H$. $I$ is the identity on objects. For $(\varphi, \sigma) \in \mathrm{Hom}_{\Delta S}\big([n], [m]\big)$,

$$I(\varphi, \sigma) = \big(\varphi, i_{n+1}(\sigma)\big)$$

where $i_n$ is the inclusion map of Definition 27.3.1.

**Definition 27.3.3.** Let $J$ denote the inclusion of the subcategory $\mathcal{F}(as)$ into $\mathcal{IF}(as)$. $J$ is the identity on objects. Let $f \in \mathrm{Hom}_{\mathcal{F}(as)}\big([n], [m]\big)$. Recall that $f$ is a map of sets such that the preimage of each singleton $i \in [m]$ is a totally ordered set. We define $J(f) \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([n], [m]\big)$ by labelling each element of each preimage with $1 \in \mathbb{Z}/2\mathbb{Z}$.

As a corollary to Proposition 24.2.5 and Theorem 27.2.10 we obtain the following result.

**Proposition 27.3.4.** *There is a commutative diagram of categories*

$$\begin{array}{ccc} \Delta H & \xrightarrow{\cong} & \mathcal{IF}(as) \\ {\scriptstyle I}\uparrow & & \uparrow{\scriptstyle J} \\ \Delta S & \xrightarrow[\cong]{} & \mathcal{F}(as) \end{array}$$

*where $I$ and $J$ are the functors of Definitions 27.3.2 and 27.3.3 respectively.* □

# Part VI

# Epimorphism Constructions

## 27.4 Ault's Results

Ault [Aul10, Section 6] proves that for an augmented associative $k$-algebra one can calculate the symmetric homology using the epimorphisms in the category $\Delta S$ and the augmentation ideal. He does this by constructing a symmetric bar construction built from the augmentation ideal of the algebra. This is a functor from the category of epimorphisms in $\Delta S$ to the category of $k$-modules. Having constructed this functor he forms a chain complex following the material in Chapter 9 and proves that it is chain homotopic to the complex formed by symmetric bar construction. We summarize Ault's results here.

Recall the material of Section 19.3.

**Definition 27.4.1.** Let $\text{Epi}\Delta S$ be the subcategory of epimorphisms in $\Delta S$.

**Definition 27.4.2.** Let $A_\varepsilon$ be an augmented associative $k$-algebra with augmentation ideal $I$. We define the functor

$$B_I^{sym} \colon \text{Epi}\Delta S \to \mathbf{kMod}$$

to be given on objects by $[n] \mapsto I^{\otimes n+1}$. The action of $B_I^{sym}$ on a morphism of $\text{Epi}\Delta S$ is the restriction of the symmetric bar construction $B_A^{sym}$ of Definition 24.4.3.

**Definition 27.4.3.** We define the chain complex

$$C_\star\left(\text{Epi}\Delta S, B_I^{sym}\right)$$

following Definition 8.1.3.

Henceforth, in order to ease notation we will denote this chain complex by $CS_\star(I)$.

Ault proves that there is an isomorphism of graded $k$-modules

$$HS_\star(A_\varepsilon) \cong H_\star\left(\text{Epi}\Delta S, B_I^{sym}\right) \oplus k_0$$

where $k_0$ is the graded $k$-module consisting of $k$ in degree 0 and zero in all other degrees.

Furthermore, Ault makes the following definition.

**Definition 27.4.4.** We define the *reduced symmetric homology* of $A_\varepsilon$ to be

$$\widetilde{HS}_\star(A_\varepsilon) = H_\star\left(\text{Epi}\Delta S, B_I^{sym}\right).$$

### 27.4.1 A note on Dold-Kan Type theorems

Recall from Chapter 12 and Remark 14.2.4, that in the case of Γ-homology we had a reduction to epimorphisms that relied on Pirashvili's Dold-Kan Type theorem. It was originally hoped that we could construct a Dold-Kan Type theorem in the cases of symmetric homology and hyperoctahedral homology. Despite our best efforts this has not been possible. As remarked in Section 12.2, Pirashvili's Dold-Kan Type theorem relies on the fact that the category Γ has a zero object which, alas, $\Delta S$ and $\Delta H$ do not.

## 27.5 Summary of Part VI

We can generalize Ault's epimorphism construction to the hyperoctahedral homology of an augmented, involutive associative $k$-algebra. The argument we provide is a modification of that given by Ault. The main difference is that by analysing the generators of the chain complex

$$k\left[N_\star\left(-\backslash\Delta H\right)\right]\otimes_{\Delta H} B_A^{oct}(-)$$

of Definition 25.4.3, we can split the complex into a direct sum and prove results by considering one of the summands.

It is straightforward but tedious to check that if we let $A$ be an augmented, associative algebra and replace $\Delta H$ and $B_A^{oct}$ by $\Delta S$ and $B_A^{sym}$ throughout Sections 27.6 to 27.10 we recover the definitions and results of [Aul10, Section 6].

Elsewhere in the thesis we have denoted an augmented $k$-algebra by $A_\varepsilon$ in order to distinguish it from an arbitrary $k$-algebra. Throughout Part VI we will let $A$ denote an augmented, involutive, associative $k$-algebra, in order to ease notation.

With this in mind, let $A$ be an augmented, involutive, associative $k$-algebra with augmentation ideal $I$.

In Section 27.6 we provide a splitting of the chain complex

$$k[N_\star(-\backslash\Delta H)]\otimes_{\Delta H} B_A^{oct}(-)$$

into a direct sum

$$C_\star\left(\Delta H, I\right)\oplus C_\star\left(\Delta H, k\right).$$

We show that the homology of the summand $C_\star\left(\Delta H, k\right)$ is isomorphic to $k$ concentrated in degree zero. We define the reduced hyperoctahedral homology to be the homology of the summand $C_\star\left(\Delta H, I\right)$.

In Section 27.7 we construct a hyperoctahedral bar construction for the augmentation ideal. This is a functor

$$B_I^{oct}\colon \mathrm{Epi}\Delta H\to \mathbf{kMod}.$$

We form the chain complex

$$k\left[N_\star\left(-\backslash\mathrm{Epi}\Delta H\right)\right]\otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-)$$

following Chapter 9.

In Sections 27.8 and 27.9 we form the *epimorphism construction* as a functor

$$[x]\backslash\Delta H\to [x]\backslash\mathrm{Epi}\Delta H,$$

for each object $[x]$ in $\Delta H$ and use this to define a chain map

$$C_\star\left(\Delta H, I\right)\to k\left[N_\star\left(-\backslash\mathrm{Epi}\Delta H\right)\right]\otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-).$$

Finally, in Section 27.10 we prove that this chain map is a homotopy equivalence, demonstrating that the chain complex

$$k\left[N_\star\left(-\backslash \text{Epi}\Delta H\right)\right] \otimes_{\text{Epi}\Delta H} B_I^{oct}(-)$$

built from the epimorphisms in $\Delta H$ and the augmentation ideal is sufficient to calculate the hyperoctahedral homology of an augmented, involutive, associative $k$-algebra.

## 27.6 Reduced Hyperoctahedral Homology

We will prove that there is a splitting of the chain complex

$$k[N_\star(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$$

when $A$ is an augmented, involutive, associative $k$-algebra.

In order to ease notation, for the remainder of this section we will define

$$f \coloneqq (\varphi, g) \in \text{Hom}_{\Delta H}\left([x], [x_0]\right)$$

and

$$f_i \coloneqq (\varphi_i, g_i) \in \text{Hom}_{\Delta H}\left([x_{i-1}], [x_i]\right)$$

for $1 \leqslant i \leqslant p$ to be composable morphisms.

### 27.6.1 Choosing representatives

**Definition 27.6.1.** Let $x \geqslant 0$. Let $Y_x$ denote a basic tensor in $B_A^{oct}\left([x]\right) = A^{\otimes x+1}$. Let $S \subseteq [x]$ be the subset of indices whose tensor factors in $Y_x$ are non-trivial. Let $s = |S|$.

Let $Y_{x,S}$ denote the basic tensor in $A^{\otimes s}$ obtained by omitting the trivial factors.

If $S$ is empty, that is, if all the factors of $Y_x$ are trivial we define $Y_{x,S} = 1_k$.

**Definition 27.6.2.** Suppose $s \geqslant 1$. Let

$$i_S^x \in \text{Hom}_\Delta\left([s-1], [x]\right)$$

be the unique order-preserving injection such that $\text{Im}\left(i_S^x\right) = S$.

We define

$$i_{x,S} \coloneqq \left(i_S^x, id_{[s-1]}\right) \in \text{Hom}_{\Delta H}\left([s-1], [x]\right).$$

*Remark 27.6.3.* Observe that

$$B_A^{oct}\left(i_{x,S}\right)\left(Y_{x,S}\right) = Y_x.$$

**Definition 27.6.4.** Let $Y_x$ denote a basic tensor in $B_A^{oct}\left([x]\right) = A^{\otimes x+1}$ such that every tensor factor is trivial. That is, $Y_x = 1_k^{\otimes x+1}$. Let

$$\delta_0^{0,x} := \underbrace{\delta_0 \circ \cdots \circ \delta_0}_{x} \in \mathrm{Hom}_{\Delta}\left([0],[x]\right).$$

We define

$$\delta^{0,x} := \left(\delta_0^{0,x}, id_{[0]}\right) \in \mathrm{Hom}_{\Delta H}\left([0],[x]\right).$$

*Remark 27.6.5.* Observe that

$$B_A^{oct}\left(\delta^{0,x}\right)(1_k) = 1_k^{\otimes x+1}.$$

Recall the notation of Definition 6.2.2.

**Proposition 27.6.6.** *Let*

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

*be a generator in the quotient module*

$$k[N_p(-\backslash \Delta H)] \otimes_{\Delta H} B_A^{oct}(-) = \frac{\bigoplus_{x\geqslant 0} k[N_p([x]\backslash \Delta H)] \otimes B_A^{oct}([x])}{\left\langle k[N_p(\alpha\backslash \Delta H)](x) \otimes y - x \otimes B_A^{oct}(\alpha)(y)\right\rangle}$$

*such that $Y_x$ has at least one non-trivial factor.*

*This is equal to a generator of the form*

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ i_{x,S}} [s-1]\right) \otimes Y_{x,S}\right]$$

*where $Y_{x,S}$ contains no trivial factors.*

*Proof.* Let

$$\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x$$

be an elementary tensor in

$$\bigoplus_{x\geqslant 0} k[N_p([x]\backslash \Delta H)] \otimes B_A^{oct}([x]).$$

Let $S \subseteq [x]$ be the non-empty subset of indices whose factors are elements of the augmentation ideal $I$. Following Definition 27.6.2 and Remark 27.6.3 we write

$$B_A^{oct}\left(i_{x,S}\right)(Y_{x,S}) = Y_x.$$

It follows that we have an equality of elementary tensors

$$\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x = \left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes B_A^{oct}\left(i_{x,S}\right)(Y_{x,S}).$$

We note that the difference

$$\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f \circ i_{x,S}} [s-1]\right) \otimes Y_{x,S} - \left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes B_A^{oct}\left(i_{x,S}\right)(Y_{x,S})$$

is an element of the $k$-module

$$\left\langle k[N_p(\alpha \backslash \Delta H)](x) \otimes y - x \otimes B_A^{oct}(\alpha)(y)\right\rangle,$$

and so

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f \circ i_{x,S}} [s-1]\right) \otimes Y_{x,S}\right] = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes B_A^{oct}(i_{x,S})(Y_{x,S})\right]$$

in $k[N_p(-\backslash \Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$.

By definition, $Y_{x,S}$ contains no non-trivial factors. □

**Proposition 27.6.7.** *Let*

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

*be a generator in the quotient module*

$$k[N_p(-\backslash \Delta H)] \otimes_{\Delta H} B_A^{oct}(-) = \frac{\bigoplus_{x \geqslant 0} k[N_p([x]\backslash \Delta H)] \otimes B_A^{oct}([x])}{\left\langle k[N_p(\alpha \backslash \Delta H)](x) \otimes y - x \otimes B_A^{oct}(\alpha)(y)\right\rangle}$$

*such that* $Y_x = 1_k^{\otimes x+1}$.

*This is equal to a generator of the form*

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f \circ \delta^{0,x}} [0]\right) \otimes 1_k\right].$$

*Proof.* Consider the elementary tensor

$$\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes 1_k^{\otimes x+1}.$$

By Definition 27.6.4 and Remark 27.6.5 we write

$$B_A^{oct}\left(\delta^{0,x}\right)(1_k) = 1_k^{\otimes x+1}$$

and therefore we have an equality of elementary tensors

$$\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes 1_k^{\otimes x+1} = \left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes B_A^{oct}\left(\delta^{0,x}\right)(1_k).$$

We observe that the difference

$$\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ\delta^{0,x}} [0]\right) \otimes 1_k - \left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes B_A^{oct}\left(\delta^{0,x}\right)(1_k)$$

is an element of the $k$-module

$$\left\langle k[N_p(\alpha\backslash\Delta H)](x) \otimes y - x \otimes B_A^{oct}(\alpha)(y)\right\rangle,$$

and so

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ\delta^{0,x}} [0]\right) \otimes 1_k\right] = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes B_A^{oct}\left(\delta^{0,x}\right)(1_k)\right]$$

in $k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$ as required. □

### 27.6.2 Choosing generators

**Definition 27.6.8.** Let $C_p\left(\Delta H, I\right)$ denote the $k$-submodule of

$$k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$$

generated $k$-linearly by the equivalence classes

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

such that $Y_x$ contains at least one non-trivial factor.

**Proposition 27.6.9.** *The $k$-submodule $C_p\left(\Delta H, I\right)$ of*

$$k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$$

*is equal to the $k$-submodule generated $k$-linearly by equivalence classes*

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

*such that $Y_x$ contains no trivial factors.*

*Proof.* Proposition 27.6.6 tells us that any generator in $C_p\left(\Delta H, I\right)$ has a representative such that $Y_x$ contains no trivial factors. Conversely, if

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

is an equivalence class such that $Y_x$ contains no trivial factors then it is a generator of $C_p\left(\Delta H, I\right)$, □

**Definition 27.6.10.** Let $C_p(\Delta H, k)$ denote the $k$-submodule of

$$k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$$

generated $k$-linearly by equivalence classes

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

such that $Y_x$ contains only trivial factors.

**Proposition 27.6.11.** *The $k$-submodule $C_p(\Delta H, k)$ of*

$$k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$$

*is equal to the $k$-submodule generated $k$-linearly by equivalence classes*

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

*such that $x = 0$ and $Y_x = 1_k$.*

*Proof.* Proposition 27.6.7 tells us that any generator in $C_p(\Delta H, k)$ has a representative such that $Y_x = 1_k$. Conversely, if

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

is an equivalence class such that $Y_x = 1_k$ then it is a generator of $C_p(\Delta H, k)$. □

### 27.6.3 Splittings

**Proposition 27.6.12.** *There is an isomorphism of $k$-modules*

$$k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-) \cong C_p(\Delta H, I) \oplus C_p(\Delta H, k).$$

*Proof.* By Propositions 27.6.6 and 27.6.7 we can choose generators of

$$k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$$

to be the equivalence classes

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

such that either $Y_x \in I^{\otimes x+1}$ or $Y_x = 1_k$ and $x = 0$.

Clearly there exists an inclusion map

$$i_1 \colon C_p(\Delta H, k) \to k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-).$$

Define a map of $k$-modules

$$\pi_1 \colon k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-) \to C_p\left(\Delta H, k\right)$$

on generators by

$$\pi_1\left(\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]\right) = \begin{cases} \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right] & Y_x = 1_k \\ 0 & \text{else.} \end{cases}$$

We observe that $\pi_1 \circ i_1$ is the identity map on $C_p\left(\Delta H, k\right)$ and the kernel of $\pi_1$ is equal to $C_p\left(\Delta H, I\right)$.

This provides the required splitting. The isomorphism

$$\pi \colon k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-) \to C_p\left(\Delta H, I\right) \oplus C_p\left(\Delta H, k\right)$$

is determined by the map that sends a generator

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

of $k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$ to

$$\begin{cases} \left(\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right], 0\right) & Y_x \in I^{\otimes x+1} \\ \left(0, \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]\right) & Y_x = 1_k. \end{cases}$$

□

Consider the chain complex $k[N_\star(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$ and denote the boundary map $d$.

**Lemma 27.6.13.** *The boundary map*

$$d \colon k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-) \to k[N_{p-1}(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$$

*restricts to a boundary map*

$$d \colon C_p\left(\Delta H, I\right) \to C_{p-1}\left(\Delta H, I\right).$$

*Proof.* It suffices to show that

$$d\left(\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]\right)$$

lies in $C_{p-1}(\Delta H, I)$ for a generator

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

of $C_p(\Delta H, I)$.

Since the boundary map $d$ is the alternating sum of face maps it suffices to show this for each face map.

It is clear for the face maps $\partial_i$ for $1 \leqslant i \leqslant p$ since these face maps compose adjacent morphisms in the string or, in the case of $\partial_p$, omit the final morphism. None of these morphisms interact with the elementary tensor $Y_x$.

For the remaining face map, $\partial_0$, we note that

$$\partial_0 \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right] = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0]\right) \otimes B_A^{oct}(f)(Y_x)\right].$$

The elementary tensor $B_A^{oct}(f)(Y_x)$ may contain trivial factors but it must contain at least one non-trivial factor since $Y_x$ had non-trivial tensor factors. Proposition 27.6.6 tells us that

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0]\right) \otimes B_A^{oct}(f)(Y_x)\right]$$

is equal to a generator of the form

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \leftarrow [x']\right) \otimes Y_{x_0}\right]$$

such that $Y_{x_0}$ contains no trivial factors. Therefore

$$\partial_0 \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

lies in $C_{p-1}(\Delta H, I)$ as required. □

**Lemma 27.6.14.** *The boundary map*

$$d\colon k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-) \to k[N_{p-1}(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$$

*restricts to a boundary map*

$$d\colon C_p(\Delta H, k) \to C_{p-1}(\Delta H, k).$$

*Proof.* It is clear that this is true following the argument of the previous lemma. We note that in this case $B_A^{oct}(f)(Y_x)$ is an elementary tensor consisting of only trivial factors and the lemma follows from Proposition 27.6.7. □

**Corollary 27.6.15.** *The chain complex* $k[N_\star(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-)$ *splits as the direct sum of chain complexes*

$$C_\star(\Delta H, I) \oplus C_\star(\Delta H, k).$$

*Proof.* Combining the isomorphism $\pi$ from the proof of Proposition 27.6.12 with Lemmata 27.6.13 and 27.6.14 we deduce that we have the following commutative diagram for each $p \geqslant 1$.

$$\begin{array}{ccc} k[N_p(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-) & \xrightarrow{\pi} & C_p(\Delta H, I) \oplus C_p(\Delta H, k) \\ \downarrow{\scriptstyle d} & & \downarrow{\scriptstyle (d,d)} \\ k[N_{p-1}(-\backslash\Delta H)] \otimes_{\Delta H} B_A^{oct}(-) & \xrightarrow[\pi]{} & C_{p-1}(\Delta H, I) \oplus C_{p-1}(\Delta H, k) \end{array}$$

This gives the required splitting. □

### 27.6.4 Reduced hyperoctahedral homology

**Proposition 27.6.16.** *There is an isomorphism of* $k$*-modules*

$$H_n\left(C_\star(\Delta H, k)\right) \cong \begin{cases} k & n = 0 \\ 0 & \textit{else}. \end{cases}$$

*Proof.* We define an augmentation

$$\varepsilon\colon C_0(\Delta H, k) \to k,$$

in the sense of Definition 2.4.12, to be determined by

$$[f \otimes 1_k] \mapsto 1_k.$$

It is straightforward to check that the morphisms

$$h_p\colon C_p(\Delta H, k) \to C_{p+1}(\Delta H, k)$$

determined by

$$\left[\left(f_p, \ldots, f_1, f\right) \otimes 1_k\right] \mapsto \left[\left(f_p, \ldots, f_1, f, id_{[0]}\right) \otimes 1_k\right]$$

for $p \geqslant 0$ and

$$h_{-1}\colon k \to C_0(\Delta H, k)$$

determined by

$$1_k \mapsto \left[id_{[0]} \otimes 1_k\right]$$

satisfy the second set of conditions of Proposition 2.4.13. □

**Definition 27.6.17.** Let $A$ be an augmented, involutive, associative $k$-algebra. We define the $n^{th}$ *reduced hyperoctahedral homology of* $A$ to be

$$\widetilde{HO}_n(A) \coloneqq H_n\left(C_\star(\Delta H, I)\right)$$

for $n \geqslant 0$.

As a corollary to Corollary 27.6.15 and Proposition 27.6.16 we obtain the following proposition.

**Proposition 27.6.18.** *There exist isomorphisms of $k$-modules*

$$HO_n(A) \cong \widetilde{HO}_n(A)$$

*for $n \geqslant 1$ and*

$$HO_0(A) \cong \widetilde{HO}_0(A) \oplus k. \qquad \square$$

## 27.7 The Functor $B_I^{oct}$

Let $A$ be an augmented, associative $k$-algebra. Let $I$ be the augmentation ideal. Recall the category Epi$\Delta H$ from Definition 19.3.5. Consider an assignment

$$B_I^{oct} \colon \mathrm{Epi}\Delta H \to \mathbf{kMod}$$

given on objects by

$$[n] \mapsto I^{\otimes n+1}.$$

Let $y_0 \otimes \cdots \otimes y_n$ be an elementary tensor in $I^{\otimes n+1}$. For a morphism

$$(\varphi, g) \in \mathrm{Hom}_{\mathrm{Epi}\Delta H}\left([n],[m]\right)$$

we define

$$B_I^{oct}(\varphi, g)(y_0 \otimes \cdots \otimes y_n) = B_A^{oct}(\varphi, g)(y_0 \otimes \cdots \otimes y_n).$$

We observe that the assignment is well-defined on morphisms since we are only considering epimorphisms in $\Delta H$. The morphism $B_A^{oct}(\varphi, g)$ acts on an elementary tensor by first permuting the factors and applying the involution according to $g \in H_{n+1}$. It then multiplies tensor factors according to the order-preserving map $\varphi$. The augmentation ideal $I$ is closed under both of these operations.

The assignment is functorial since it is the restriction of the functor $B_A^{oct}$ to the subcategory Epi$\Delta H$.

**Definition 27.7.1.** Let $A$ be an augmented, associative $k$-algebra with augmentation ideal $I$. We define a functor

$$B_I^{oct} \colon \mathrm{Epi}\Delta H \to \mathbf{kMod}$$

on objects by

$$[n] \to I^{\otimes n+1}.$$

On morphisms we define $B_I^{oct}$ to be the restriction of $B_A^{oct}$ to the subcategory Epi$\Delta H$.

**Definition 27.7.2.** Following Definition 8.1.3 we define the chain complex

$$C_\star\left(\mathrm{Epi}\Delta H, B_I^{oct}\right).$$

**Definition 27.7.3.** Following Subsection 9.2.2 we define the chain complex

$$k\left[N_\star\left(-\backslash \mathrm{Epi}\Delta H\right)\right]\otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-).$$

Recall that Theorem 10.1.2 provides an isomorphism of chain complexes

$$C_\star\left(\mathrm{Epi}\Delta H, B_I^{oct}\right)\cong k\left[N_\star\left(-\backslash \mathrm{Epi}\Delta H\right)\right]\otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-).$$

Recall from Proposition 27.6.9 that the chain complex $C_\star(\Delta H, I)$ is generated by equivalence classes of the form

$$\left[\left([x_p]\xleftarrow{f_p}\cdots\xleftarrow{f_1}[x_0]\xleftarrow{f}[x]\right)\otimes Y_x\right]$$

such that $Y_x$ contains no trivial factors.

**Definition 27.7.4.** We denote by

$$i\colon k\left[N_\star\left(-\backslash \mathrm{Epi}\Delta H\right)\right]\otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-)\to C_\star(\Delta H, I)$$

the inclusion of chain complexes.

We will show that the inclusion map $i$ is a chain homotopy equivalence.

## 27.8 The Epimorphism Construction

In this section we will describe the *epimorphism construction for hyperoctahedral homology.* This construction is the key to defining a chain map

$$\chi\colon C_\star(\Delta H, I)\to k\left[N_\star\left(-\backslash \mathrm{Epi}\Delta H\right)\right]\otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-)$$

such that $\chi\circ i$ is equal to the identity and $i\circ\chi$ is homotopic to the identity, where $i$ is the inclusion of chain complexes from Definition 27.7.4.

The following proposition uses the unique decomposition of morphisms in the category $\Delta$ to decompose any morphism in $\Delta H$ into a composite of an epimorphism and a monomorphism.

**Proposition 27.8.1.** *Let $(\varphi, g)\in \mathrm{Hom}_{\Delta H}\left([x],[z]\right)$. Let $r=\left|\mathrm{Im}(\varphi)\right|$. The morphism $(\varphi, g)$ can be written uniquely in the form*

$$[x]\xrightarrow{(\pi_\varphi, g)}[r-1]\xrightarrow{\left(i_\varphi, id_{[r-1]}\right)}[z]$$

*where $(\pi_\varphi, g)$ is an epimorphism in $\Delta H$ and $i_\varphi$ is a monomorphism in $\Delta$.*

*Proof.* By Theorem 2.2.5 the morphism $\varphi \in \mathrm{Hom}_\Delta\big([x],[z]\big)$ has a unique decomposition in $\Delta$ of the form $i_\varphi \circ \pi_\varphi$ where $\pi_\varphi \in \mathrm{Hom}_\Delta\big([x],[r-1]\big)$ is an epimorphism and $i_\varphi \in \mathrm{Hom}_\Delta\big([r-1],[z]\big)$ is a monomorphism. We note that

$$\begin{aligned}(\varphi, g) &= \big(i_\varphi \circ \pi_\varphi, g\big) \\ &= \Big(i_\varphi, id_{[r-1]}\Big) \circ \big(\pi_\varphi, g\big)\,.\end{aligned}$$

Suppose $\Big(i', id_{[y]}\Big) \circ \big(\pi', h\big)$ is another such decomposition. That is,

$$(\varphi, g) = \Big(i', id_{[y]}\Big) \circ \big(\pi', h\big)\,.$$

By the composition rule in $\Delta H$ we see that

$$\Big(i', id_{[y]}\Big) \circ \big(\pi', h\big) = \big(i' \circ \pi', h\big)\,.$$

It follows that $g = h$ and, by Theorem 2.2.5, we deduce that $i' = i_\varphi$ and $\pi' = \pi_\varphi$. □

**Notation 27.8.2.** In order to ease notation we adopt the following convention. Whenever we use the decomposition of Proposition 27.8.1 we define

$$i'_\varphi \coloneqq \Big(i_\varphi, id_{[r-1]}\Big) \in \mathrm{Hom}_{\Delta H}\big([r-1],[z]\big)\,.$$

**Notation 27.8.3.** For $(\varphi, g) \in \mathrm{Hom}_{\Delta H}\big([x],[z]\big)$ we will let $r = \big|\mathrm{Im}\,(\varphi)\big|$. For $(\varphi_i, g_i) \in \mathrm{Hom}_{\Delta H}\big([x],[z_i]\big)$ we will let $r_i = \big|\mathrm{Im}\,(\varphi_i)\big|$.

For each $x \geqslant 0$, we have an assignment on objects, which we will denote by

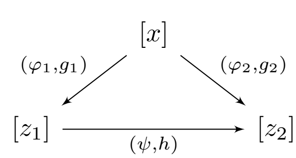
$$E_x \colon [x]\backslash\Delta H \to [x]\backslash\mathrm{Epi}\Delta H$$

given by

$$\left([x] \xrightarrow{(\varphi,g)} [z]\right) \mapsto \left([x] \xrightarrow{(\pi_\varphi,g)} [r-1]\right)\,.$$

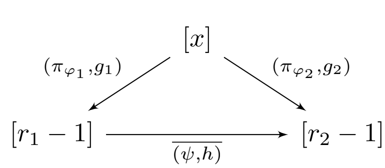
We will extend this to also give an assignment on morphisms and demonstrate that the assignment is functorial.

**Lemma 27.8.4.** *Let*



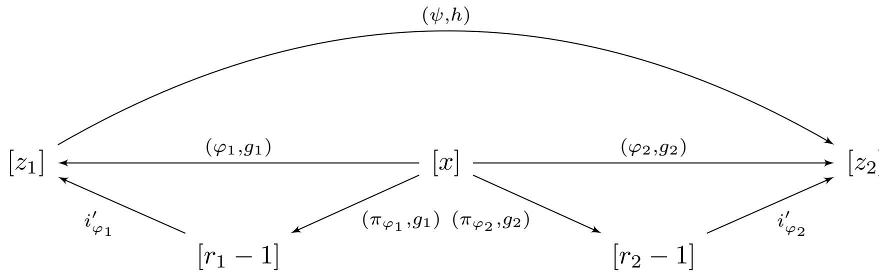

*be a morphism in* $[x]\backslash\Delta H$.

*We claim that* $(\psi, h)$ *induces a morphism*



*in* $[x]\backslash\text{Epi}\Delta H$.

*Proof.* Using Notation 27.8.2, we have a commutative diagram




in $\Delta H$.

The upper triangle is the morphism in $[x]\backslash\Delta H$. The two lower triangles are the unique decompositions of Proposition 27.8.1.

By construction $\text{Im}(i'_{\varphi_1}) = \text{Im}((\varphi_1, g_1))$ and so,

$$\begin{aligned}\text{Im}\left((\psi,h)\circ i'_{\varphi_1}\right) &= \text{Im}((\psi,h)\circ(\varphi_1,g_1))\\ &= \text{Im}((\varphi_2,g_2))\\ &= \text{Im}(i'_{\varphi_2}).\end{aligned}$$

In particular, $(\psi, h)\circ i'_{\varphi_1}$ has image equal to $\text{Im}(\varphi_2)$.

By Proposition 27.8.1 there exists a unique morphism

$$\overline{(\psi,h)}\colon [r_1-1]\to[r_2-1]$$

in $\text{Epi}\Delta H$ such that

$$(\psi,h)\circ i'_{\varphi_1} = i'_{\varphi_2}\circ\overline{(\psi,h)}.$$

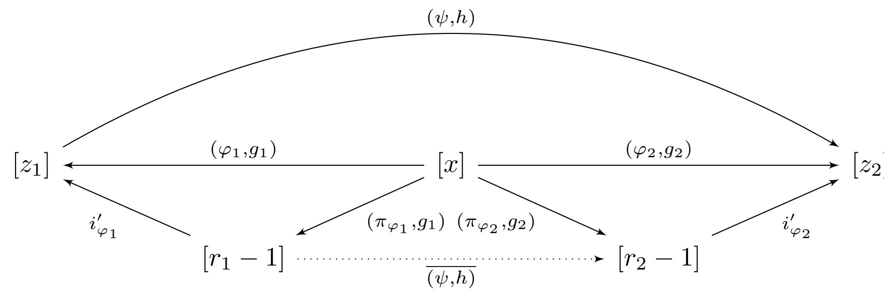
We observe that

$$\begin{aligned}i'_{\varphi_2}\circ\overline{(\psi,h)}\circ(\pi_{\varphi_1},g_1) &= (\psi,h)\circ i'_{\varphi_1}\circ(\pi_{\varphi_1},g_1)\\ &= (\psi,h)\circ(\varphi_1,g_1)\\ &= (\varphi_2,g_2)\\ &= i'_{\varphi_2}\circ(\pi_{\varphi_2},g_2).\end{aligned}$$

Since $i'_{\varphi_2}$ is a monomorphism in $\Delta H$ we can cancel on the left and deduce that

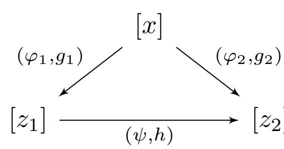
$$\overline{(\psi,h)}\circ(\pi_{\varphi_1},g_1) = (\pi_{\varphi_2},g_2).$$

We therefore have a commutative diagram




as required. ☐

Recall the notation of Definition 1.3.2. We denote the morphism




in $\Delta H$ by

$$\left((\psi, h), (\varphi_1, g_1), (\varphi_2, g_2)\right).$$

We define

$$E_x\colon [x]\backslash\Delta H \to [x]\backslash\mathrm{Epi}\Delta H$$

on morphisms by

$$\left((\psi, h), (\varphi_1, g_1), (\varphi_2, g_2)\right) \mapsto \left(\overline{(\psi, h)}, \left(\pi_{\varphi_1}, g_1\right), \left(\pi_{\varphi_2}, g_2\right)\right).$$

We claim that the assignment $E_x$ is functorial.

**Proposition 27.8.5.** *Let $x \geqslant 0$. Let*

$$E_x\colon [x]\backslash\Delta H \to [x]\backslash\mathrm{Epi}\Delta H$$

*be defined on objects by*

$$\left([x] \xrightarrow{(\varphi,g)} [z]\right) \mapsto \left([x] \xrightarrow{(\pi_\varphi,g)} [r-1]\right)$$

*and on morphisms by*

$$\left((\psi, h), (\varphi_1, g_1), (\varphi_2, g_2)\right) \mapsto \left(\overline{(\psi, h)}, \left(\pi_{\varphi_1}, g_1\right), \left(\pi_{\varphi_2}, g_2\right)\right).$$

*We claim $E_x$ is a covariant functor.*

*Proof.* We observe that $E_x$ respects identities since

$$E_x\left(id_{[z]}, (\varphi, g), (\varphi, g)\right) = \left(id_{[r-1]}, \left(\pi_\varphi, g\right), \left(\pi_\varphi, g\right)\right).$$

We must demonstrate that $E_x$ respects composition.

Let

$$\begin{array}{ccc} & [x] & \\ {}^{(\varphi_1,g_1)}\swarrow & & \searrow^{(\varphi_2,g_2)} \\ [z_1] & \xrightarrow[(\psi,h)]{} & [z_2] \end{array}$$

and

$$\begin{array}{ccc} & [x] & \\ {}^{(\varphi_2,g_2)}\swarrow & & \searrow^{(\varphi_3,g_3)} \\ [z_2] & \xrightarrow[(\psi',h')]{} & [z_3] \end{array}$$

be two morphisms in $[x]\backslash\Delta H$. Write

$$\begin{array}{ccc} & [x] & \\ {}^{(\varphi_1,g_1)}\swarrow & & \searrow^{(\varphi_3,g_3)} \\ [z_1] & \xrightarrow[(\psi',h')\circ(\psi,h)]{} & [z_3] \end{array}$$

for the composite.

By Lemma 27.8.4 we have commutative diagrams

$$\begin{array}{ccc} & [x] & \\ {}^{(\pi_{\varphi_1},g_1)}\swarrow & & \searrow^{(\pi_{\varphi_2},g_2)} \\ [r_1-1] & \xrightarrow[\overline{(\psi,h)}]{} & [r_2-1] \end{array},$$

$$\begin{array}{ccc} & [x] & \\ {}^{(\pi_{\varphi_2},g_2)}\swarrow & & \searrow^{(\pi_{\varphi_3},g_3)} \\ [r_2-1] & \xrightarrow[\overline{(\psi',h')}]{} & [r_3-1] \end{array}$$

and

$$\begin{array}{ccc} & [x] & \\ {}^{(\pi_{\varphi_1},g_1)}\swarrow & & \searrow^{(\pi_{\varphi_3},g_3)} \\ [r_1-1] & \xrightarrow[\overline{(\psi',h')\circ(\psi,h)}]{} & [r_3-1] \end{array}$$

in $\mathrm{Epi}\Delta H$.

We must show that $E_x((\psi',h')\circ(\psi,h)) = E_x((\psi',h'))\circ E_x((\psi,h))$. That is, we must show that

$$\overline{(\psi',h')\circ(\psi,h)} = \overline{(\psi',h')}\circ\overline{(\psi,h)}.$$

Observe that, using the commutative diagrams above, we have

$$\begin{aligned} \overline{(\psi',h')\circ(\psi,h)}\circ(\pi_{\varphi_1},g_1) &= (\pi_{\varphi_3},g_3) \\ &= \overline{(\psi',h')}\circ(\pi_{\varphi_2},g_2) \\ &= \left(\overline{(\psi',h')}\circ\overline{(\psi,h)}\right)\circ(\pi_{\varphi_1},g_1). \end{aligned}$$

Since $(\pi_{\varphi_1}, g_1)$ is an epimorphism we can cancel on the right and deduce that

$$\overline{(\psi', h') \circ (\psi, h)} = \overline{(\psi', h')} \circ \overline{(\psi, h)}$$

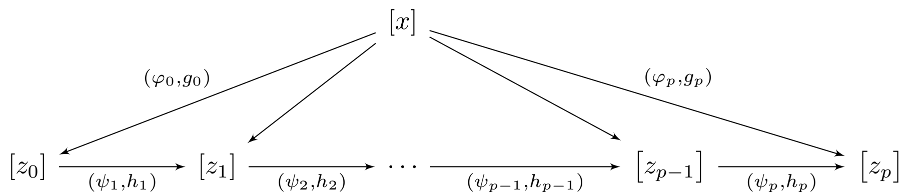
as required. $\square$

Recall Remark 2.1.8.

**Proposition 27.8.6.** *The functor $E_x \colon [x]\backslash\Delta H \to [x]\backslash\mathrm{Epi}\Delta H$ induces a map of simplicial sets*

$$N_\star E_x \colon N_\star([x]\backslash\Delta H) \to N_\star([x]\backslash\mathrm{Epi}\Delta H).$$ $\square$

Let




be an element of $N_p\left([x]\backslash\Delta H\right)$. Recalling the notation of Subsection 9.2.1 we will denote this

$$\Big(\left(\psi_p, h_p\right), \ldots, \left(\psi_1, h_1\right), \left(\varphi_0, g_0\right)\Big).$$

We note that

$$E_x\Big(\left(\psi_p, h_p\right), \ldots, \left(\psi_1, h_1\right), \left(\varphi_0, g_0\right)\Big) = \Big(\overline{(\psi_p, h_p)}, \ldots, \overline{(\psi_1, h_1)}, \left(\pi_{\varphi_0}, g_0\right)\Big),$$

where the morphisms $\overline{(\psi_i, h_i)}$ are obtained as in Lemma 27.8.4.

## 27.9 The Chain Map

We will demonstrate that the chain complex $C_\star\left(\Delta H, I\right)$ is chain homotopic to the chain complex $k\left[N_\star(-\backslash\mathrm{Epi}\Delta H)\right] \otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-)$.

Recall the description of the chain complex $C_\star\left(\Delta H, I\right)$ from Proposition 27.6.9. A generator in degree $p$ is an equivalence class of the form

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

such that $Y_x$ contains no trivial tensor factors.

**Proposition 27.9.1.** *There exists a well-defined map of $k$-modules*

$$\chi_p \colon C_p\left(\Delta H, I\right) \to k\left[N_p(-\backslash\mathrm{Epi}\Delta H)\right] \otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-)$$

*determined by*

$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right] \mapsto \left[k\left[N_p E_x\right]\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$

*for each* $p \geqslant 0$*.*

*Proof.* We must demonstrate that if $[X] = \left[X'\right]$ in $C_p(\Delta H, I)$ then $\chi_p\left([X]\right) = \chi_p\left(\left[X'\right]\right)$ in $k\left[N_p(-\backslash\mathrm{Epi}\Delta H)\right] \otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-)$.

To this end, let

$$f \coloneqq (\varphi, g) \in \mathrm{Hom}_{\Delta H}\left([x'], [x_0]\right)$$

and

$$f_i \coloneqq (\varphi_i, g_i) \in \mathrm{Hom}_{\Delta H}\left([x_{i-1}], [x_i]\right)$$

for $1 \leqslant i \leqslant p$ be composable morphisms in $\Delta H$. Let

$$r = \left|\mathrm{Im}\,(\varphi)\right|$$

and

$$r_i = \left|\mathrm{Im}\,(\varphi_i)\right|.$$

Let

$$\rho \coloneqq (\psi, h) \in \mathrm{Hom}_{\Delta H}\left([x], [x']\right).$$

Let

$$X = \left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ\rho} [x]\right) \otimes Y_x$$

be an elementary tensor in

$$\bigoplus_{x \geqslant 0} k[N_p([x]\backslash\Delta H)] \otimes B_A^{oct}([x])$$

such that $Y_x$ contains at least one non-trivial factor. Let

$$X' = \left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x']\right) \otimes B_A^{oct}(\rho)(Y_x).$$

It follows that $[X] = \left[X'\right]$ in $k\left[N_p\left(-\backslash\Delta H\right)\right] \otimes_{\Delta H} B_A^{oct}(-)$ and since $Y_x$ contains at least one non-trivial factor, $[X] = \left[X'\right]$ in $C_p(\Delta H, I)$, by Proposition 27.6.6.

We must show that $\chi_p\left([X]\right) = \chi_p\left(\left[X'\right]\right)$.

We do this in a number of steps.

In Step 1 we choose representatives

$$[X] = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ\rho\circ i_{x,S}} [s-1]\right) \otimes Y_{x,S}\right]$$

and

$$\left[X'\right] = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ i_{x',S'}} [s'-1]\right) \otimes Y^{\rho}_{x',S'}\right]$$

such that $Y_{x,S}$ and $Y^{\rho}_{x',S'}$ are basic tensors in the sense of Definition 4.4.9, using a similar argument to Proposition 27.6.6.

In Step 2 we use properties of morphisms in $\Delta H$ to show that there is an epimorphism $(\pi_\mu, j)$ in $\Delta H$ such that

$$B^{oct}_A(\pi_\mu, j)(Y_{x,S}) = Y^{\rho}_{x',S'}.$$

In Step 3 we apply the map $\chi_p$ to the equivalence classes $[X]$ and $\left[X'\right]$.

In Step 4 we use the equivalence relation of the $k$-module $k\left[N_p(-\backslash\mathrm{Epi}\Delta H)\right] \otimes_{\mathrm{Epi}\Delta H} B^{oct}_I(-)$ to demonstrate that $\chi_p\left([X]\right) = \chi_p\left(\left[X'\right]\right)$.

**Step 1 (Representing classes by basic tensors).** Consider

$$X = \left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ\rho} [x]\right) \otimes Y_x.$$

Following Definitions 27.6.1 and 27.6.2 and Remark 27.6.3 we let $S$ be the set of indices of the elementary tensor $Y_x$ whose tensor factors are elements of $I$. We let

$$i_{x,S} \coloneqq \left(i^x_S, id_{[s-1]}\right) \in \mathrm{Hom}_{\Delta H}\left([s-1],[x]\right).$$

By Proposition 27.6.6,

$$[X] = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ\rho\circ i_{x,S}} [s-1]\right) \otimes Y_{x,S}\right].$$

Let $S'$ be the set of indices in the elementary tensor $B^{oct}_A(\rho)(Y_x)$ whose tensor factors are non-trivial. Let $s' = \left|S'\right|$. We let

$$i_{x',S'} \coloneqq \left(i^{x'}_{S'}, id_{[s'-1]}\right) \in \mathrm{Hom}_{\Delta H}\left([s'-1],[x']\right).$$

By Proposition 27.6.6

$$\left[X'\right] = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ i_{x',S'}} [s'-1]\right) \otimes B^{oct}_A(\rho)(Y_x)_{x',S'}\right]$$

where $B^{oct}_A(\rho)(Y_x)_{x',S'}$ is the elementary tensor obtained from $B^{oct}_A(\rho)(Y_x)$ by omitting the trivial tensor factors.

To ease notation, let us denote

$$Y^{\rho}_x \coloneqq B^{oct}_A(\rho)(Y_x)$$

and

$$Y^{\rho}_{x',S'} \coloneqq B^{oct}_A(\rho)(Y_x)_{x',S'}.$$

Hence

$$[X'] = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ i_{x',S'}} [s'-1]\right) \otimes Y^{\rho}_{x',S'}\right]$$

**Step 2 (Finding an epimorphism).** We show that there is an epimorphism $(\pi_\mu, j)$ in $\Delta H$ such that

$$B^{oct}_A(\pi_\mu, j)(Y_{x,S}) = Y^{\rho}_{x',S'}.$$

Consider

$$\rho \circ i_{x,S} \in \mathrm{Hom}_{\Delta H}\left([s-1],[x']\right).$$

As a morphism in $\Delta H$ this has a unique expression of the form $(\mu, j)$ where $j \in H_s$ and $\mu \in \mathrm{Hom}_\Delta\left([s-1],[x']\right)$.

As an order-preserving map, $\mu$ has a unique decomposition say $\mu = i_\mu \circ \pi_\mu$. Let $q = \left|\mathrm{Im}\,(\mu)\right|$, so

$$\pi_\mu \colon [s-1] \to [q-1]$$

is an epimorphism and

$$i_\mu \colon [q-1] \to [x']$$

is a monomorphism.

Observe that we have equalities of elementary tensors

$$\begin{aligned} Y^{\rho}_x &= B^{oct}_A\left(\rho \circ i_{x,S}\right)\left(Y_{x,S}\right) \\ &= B^{oct}_A\left(i_\mu \circ \pi_\mu, j\right)\left(Y_{x,S}\right) \\ &= B^{oct}_A\left(i_\mu \circ \pi_\mu, id_{[s-1]}\right)\left(B^{oct}_A\left(id_{[s-1]}, j\right)\left(Y_{x,S}\right)\right). \end{aligned}$$

To ease notation let

$$Y^{j}_{x,S} := B^{oct}_A\left(id_{[s-1]}, j\right)(Y_{x,S}).$$

Observe that $Y^{j}_{x,S}$ is an elementary tensor consisting of $s$ non-trivial factors since it is obtained from $Y_{x,S}$ by permuting and applying the involution to the tensor factors according to $j \in H_s$.

By the functoriality of $B^{oct}_A$ and using Notation 27.8.2 we see that,

$$Y^{\rho}_x = B^{oct}_A\left(i'_\mu\right)\left(B^{oct}_A\left(\pi_\mu, id_{[s-1]}\right)\right)\left(Y^{j}_{x,S}\right).$$

Since the action of $B^{oct}_A\left(i'_\mu\right)$ inserts trivial factors into an elementary tensor we deduce that

$$B^{oct}_A\left(\pi_\mu, id_{[s-1]}\right)\left(Y^{j}_{x,S}\right)$$

is an elementary tensor with the same number of non-trivial factors as $Y^{\rho}_x$.

Furthermore, since $Y^j_{x,S}$ contains no trivial factors and $\pi_\mu$ is surjective we deduce that

$$B^{oct}_A\left(\pi_\mu, id_{[s-1]}\right)\left(Y^j_{x,S}\right)$$

has precisely $s' = \left|S'\right|$ factors, each of which is non-trivial. In other words, we can deduce that $\left|\mathrm{Im}(\mu)\right| = \left|S'\right|$ and

$$\left(\pi_\mu, id_{[s-1]}\right) \in \mathrm{Hom}_{\Delta H}\left([s-1],[s'-1]\right).$$

Since we have an equality of elementary tensors

$$Y^\rho_x = B^{oct}_A\left(i'_\mu\right)\left(B^{oct}_A\left(\pi_\mu, id_{[s-1]}\right)\left(Y^j_{x,S}\right)\right),$$

we deduce that

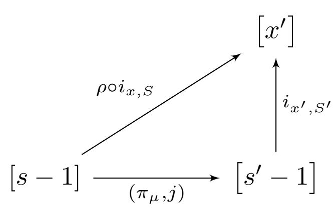
$$i'_\mu = i_{x',S'} \in \mathrm{Hom}_{\Delta H}\left([s'-1],[x']\right).$$

It follows from Proposition 27.8.1 that

$$i_{x',S'} \circ \left(\pi_\mu, j\right)$$

is the unique decomposition of $\rho \circ i_{x,S}$.

We therefore have a commutative diagram




in $\Delta H$.

Recall from Step 1 that $Y^\rho_x$ is an elementary tensor with $s'$ non-trivial factors and $Y^\rho_{x',S'}$ is the elementary tensor obtained by omitting the trivial factors.

We have equalities of elementary tensors

$$\begin{aligned} Y^\rho_x &= B^{oct}_A\left(i_{x',S'}\right)\left(Y^\rho_{x',S'}\right) \\ &= B^{oct}_A\left(i_{x',S'}\right)\left(B^{oct}_A\left(\pi_\mu, j\right)\left(Y_{x,S}\right)\right) \end{aligned}$$

from which we deduce that

$$B^{oct}_A(\pi_\mu, j)(Y_{x,S}) = Y^\rho_{x',S'}.$$

**Step 3 (Applying $\chi_p$).** We demonstrated in Step 2 that

$$\rho \circ i_{x,S} = i_{x',S'} \circ \left(\pi_\mu, j\right) \in \mathrm{Hom}_{\Delta H}\left([s-1],[x']\right).$$

We observe that

$$[X] = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ\rho\circ i_{x,S}} [s-1]\right) \otimes Y_{x,S}\right]$$
$$= \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ i_{x',S'}\circ\left(\pi_\mu, j\right)} [s-1]\right) \otimes Y_{x,S}\right].$$

Since $f \circ i_{x',S'} \in \mathrm{Hom}_{\Delta H}\left([s'-1], [x_0]\right)$ it has a unique expression of the form $(\nu, l)$ where $l \in H_{s'}$ and $\nu \in \mathrm{Hom}_{\Delta}\left([s'-1], [x_0]\right)$. Let $n = \left|\mathrm{Im}\,(\nu)\right|$.

Hence

$$[X] = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{(\nu,l)\circ\left(\pi_\mu, j\right)} [s-1]\right) \otimes Y_{x,S}\right].$$

Therefore

$$\chi_p\,[X] = \left[\left([r_p-1] \xleftarrow{\overline{f_p}} \cdots \xleftarrow{\overline{f_1}} [n-1] \xleftarrow{(\pi_\nu,l)\circ\left(\pi_\mu, j\right)} [s-1]\right) \otimes Y_{x,S}\right],$$

since $\left(\pi_\mu, j\right)$ is an epimorphism in $\Delta H$.

Observe that

$$\left[X'\right] = \left[\left([r_p-1] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f\circ i_{x',S'}} [s'-1]\right) \otimes Y^{\rho}_{x',S'}\right]$$
$$= \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{(\nu,l)} [s'-1]\right) \otimes Y^{\rho}_{x',S'}\right].$$

Therefore

$$\chi_p\left[X'\right] = \left[\left([r_p-1] \xleftarrow{\overline{f_p}} \cdots \xleftarrow{\overline{f_1}} [n-1] \xleftarrow{(\pi_\nu,l)} [s-1]\right) \otimes Y^{\rho}_{x',S'}\right].$$

**Step 4 (Conclusion).** We observe that the difference of

$$\left([r_p-1] \xleftarrow{\overline{f_p}} \cdots \xleftarrow{\overline{f_1}} [n-1] \xleftarrow{(\pi_\nu,l)} [s-1]\right) \otimes Y^{\rho}_{x',S'}$$

and

$$\left([r_p-1] \xleftarrow{\overline{f_p}} \cdots \xleftarrow{\overline{f_1}} [n-1] \xleftarrow{(\pi_\nu,l)\circ\left(\pi_\mu, j\right)} [s-1]\right) \otimes Y_{x,S}$$

is an element of the $k$-module

$$\left\langle k\left[N_p(\alpha \backslash \mathrm{Epi}\Delta H)\right](x) \otimes y - x \otimes B^{oct}_I(\alpha)(y)\right\rangle,$$

defined following Definition 6.2.1, since $(\pi_\mu, j)$ is an epimorphism in $\Delta H$ and

$$B_A^{oct}(\pi_\mu, j)(Y_{x,S}) = Y^\rho_{x',S'}$$

as demonstrated in Step 2.

It follows that $\chi_p\left([X]\right) = \chi_p\left(\left[X'\right]\right)$ in

$$k\left[N_p(-\backslash\mathrm{Epi}\Delta H)\right] \otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-)$$

as required. □

We deduce the following corollary.

**Corollary 27.9.2.** *The maps $\chi_p$ assemble into a map of chain complexes*

$$\chi\colon C_\star\left(\Delta H, I\right) \to k\left[N_\star(-\backslash\mathrm{Epi}\Delta H)\right] \otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-).$$

*Proof.* Proposition 27.8.6 tells us that the $k$-module morphisms $\chi_p$, determined by the epimorphism construction, are compatible with the simplicial structure of the nerve and, therefore, with the boundary maps. □

## 27.10 The Chain Homotopies

Recall the inclusion of chain complexes

$$i\colon k\left[N_\star\left(-\backslash\mathrm{Epi}\Delta H\right)\right] \otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-) \to C_\star\left(\Delta H, I\right)$$

from Subsection 27.7.

**Proposition 27.10.1.** *The chain map*

$$\chi \circ i\colon k\left[N_\star\left(-\backslash\mathrm{Epi}\Delta H\right)\right] \otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-) \to k\left[N_\star\left(-\backslash\mathrm{Epi}\Delta H\right)\right] \otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-)$$

*is equal to the identity.*

*Proof.* We observe that in each degree, the morphism $\chi$ applied to an element in the image of $i$ is the identity since all the morphisms in the representative are epimorphisms. □

**Proposition 27.10.2.** *The chain map*

$$i \circ \chi\colon C_\star\left(\Delta H, I\right) \to C_\star\left(\Delta H, I\right)$$

*is homotopic to the identity map.*

*Proof.* It suffices to provide a presimplicial homotopy between $i \circ \chi$ and the identity map by Proposition 2.5.5. We will make use of Notation 27.8.2.

Let
$$h_j^p \colon C_p\left(\Delta H, I\right) \to C_{p+1}\left(\Delta H, I\right)$$
be determined by mapping an equivalence class
$$\left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right]$$
to
$$\left[\left([x_p] \xleftarrow{f_p} \cdots \leftarrow [x_j] \xleftarrow{i'_{\varphi_j}} [r_j - 1] \xleftarrow{\left(\pi_{\varphi_j}, g_j\right)} \cdots \leftarrow [r-1] \leftarrow [x]\right) \otimes Y_x\right].$$

That is, $h_j^p$ is the map determined by mapping an equivalence class in $C_p\left(\Delta H, I\right)$ to the equivalence class in $C_{p+1}\left(\Delta H, I\right)$ obtained by applying the epimorphism construction to the first $j$ morphisms in the representative, then inserting an inclusion and leaving the remaining morphisms unchanged.

We demonstrate that these morphisms satisfy the conditions of a presimplicial homotopy between $i \circ \chi$ and the identity map.

Let
$$X = \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right].$$

**Step 1.** We show that $\partial_0 \circ h_0^p$ is the identity map.

$$\begin{aligned}
\left(\partial_0 \circ h_0^p\right)(X) &= \partial_0 \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{i'_{\varphi}} [r-1] \xleftarrow{\left(\pi_{\varphi}, g\right)} [x]\right) \otimes Y_x\right] \\
&= \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{i'_{\varphi} \circ \left(\pi_{\varphi}, g\right)} [x]\right) \otimes Y_x\right] \\
&= \left[\left([x_p] \xleftarrow{f_p} \cdots \xleftarrow{f_1} [x_0] \xleftarrow{f} [x]\right) \otimes Y_x\right] \\
&= X.
\end{aligned}$$

Therefore $\left(\partial_0 \circ h_0^p\right)$ is the identity map.

**Step 2.** We show that $\partial_{p+1} \circ h_p^p = i \circ \chi$.

$$\begin{aligned}
\left(\partial_{p+1} \circ h_p^p\right)(X) &= \partial_{p+1} \left[\left([x_p] \xleftarrow{i'_{\varphi_p}} [r_p - 1] \xleftarrow{\overline{f_p}} \cdots \xleftarrow{\overline{f_1}} [r-1] \xleftarrow{\left(\pi_{\varphi}, g\right)} [x]\right) \otimes Y_x\right] \\
&= \left[\left([r_p - 1] \xleftarrow{\overline{f_p}} \cdots \xleftarrow{\overline{f_1}} [r-1] \xleftarrow{\left(\pi_{\varphi}, g\right)} [x]\right) \otimes Y_x\right] \\
&= (i \circ \chi)(X).
\end{aligned}$$

**Step 3.** We show that $\partial_i \circ h_j^p = h_{j-1}^{p-1} \circ \partial_i$ for $0 \leqslant i < j \leqslant p$. On the one hand

$$\left(\partial_i \circ h_j^p\right)(X) = \partial_i \left[\left([x_p] \xleftarrow{f_p} \cdots \leftarrow [x_j] \xleftarrow{i'_{\varphi_j}} [r_j - 1] \leftarrow \cdots \leftarrow [x]\right) \otimes Y_x\right]$$
$$= \left[\left([x_p] \xleftarrow{f_p} \cdots \leftarrow [x_j] \xleftarrow{i'_{\varphi_j}} [r_j - 1] \leftarrow \cdots \xleftarrow{\overline{f_{i+1}} \circ \overline{f_i}} \cdots \leftarrow [x]\right) \otimes Y_x\right].$$

On the other hand

$$\left(h_{j-1}^{p-1} \circ \partial_i\right)(X) = h_{j-1}^{p-1} \left[\left([x_p] \xleftarrow{f_p} \cdots \leftarrow [x_{i+1}] \xleftarrow{f_{i+1} \circ f_i} [x_{i-1}] \leftarrow \cdots \leftarrow [x]\right) \otimes Y_x\right]$$
$$= \left[\left([x_p] \xleftarrow{f_p} \cdots \leftarrow [x_j] \xleftarrow{i'_{\varphi_j}} [r_j - 1] \leftarrow \cdots \xleftarrow{\overline{f_{i+1} \circ f_i}} \cdots \leftarrow [x]\right) \otimes Y_x\right].$$

Equality follows from Proposition 27.8.5.

**Step 4.** We show that $\partial_i \circ h_i^p = \partial_i \circ h_{i-1}^p$ for $0 < i \leqslant p$. On the one hand

$$\left(\partial_i \circ h_i^p\right)(X) = \partial_i \left[\left([x_p] \leftarrow \cdots \leftarrow [x_i] \xleftarrow{i'_{\varphi_i}} [r_i - 1] \xleftarrow{\overline{f_i}} \cdots \leftarrow [r-1] \leftarrow [x]\right) \otimes Y_x\right]$$
$$= \left[\left([x_p] \leftarrow \cdots \leftarrow [x_i] \xleftarrow{i'_{\varphi_i} \circ \overline{f_i}} [r_{i-1} - 1] \leftarrow \cdots \leftarrow [r-1] \leftarrow [x]\right) \otimes Y_x\right].$$

On the other hand

$$\left(\partial_i \circ h_{i-1}^p\right)(X) = \partial_i \left[\left([x_p] \leftarrow \cdots \xleftarrow{f_i} [x_{i-1}] \xleftarrow{i'_{\varphi_{i-1}}} [r_{i-1} - 1] \xleftarrow{\overline{f_i}} \cdots \leftarrow [x]\right) \otimes Y_x\right]$$
$$= \left[\left([x_p] \leftarrow \cdots \leftarrow [x_i] \xleftarrow{f_i \circ i'_{\varphi_{i-1}}} [r_{i-1} - 1] \leftarrow \cdots \leftarrow [x]\right) \otimes Y_x\right].$$

Equality follows from Lemma 27.8.4.

**Step 5.** We show that $\partial_i \circ h_j^p = h_j^p \circ \partial_{i-1}$ for $i > j+1$, $j \geqslant 0$ and $i \leqslant p$. On the one hand

$$\left(\partial_i \circ h_j^p\right)(X) = \partial_i \left[([x_p] \leftarrow \cdots \leftarrow [x_j] \leftarrow [r_j - 1] \leftarrow \cdots \leftarrow [x]) \otimes Y_x\right]$$
$$= \left[\left([x_p] \leftarrow \cdots \xleftarrow{f_i \circ f_{i-1}} \cdots \leftarrow [x_j] \leftarrow [r_j - 1] \leftarrow \cdots \leftarrow [x]\right) \otimes Y_x\right].$$

On the other hand

$$\left(h_j^p \circ \partial_{i-1}\right)(X) = h_j^p \left[\left([x_p] \leftarrow \cdots \leftarrow [x_i] \xleftarrow{f_i \circ f_{i-1}} [x_{i-2}] \leftarrow \cdots \leftarrow [x]\right) \otimes Y_x\right]$$
$$= \left[\left([x_p] \leftarrow \cdots \xleftarrow{f_i \circ f_{i-1}} \cdots \leftarrow [x_j] \leftarrow [r_j - 1] \leftarrow \cdots \leftarrow [x]\right) \otimes Y_x\right].$$

Hence the maps $h_j^p$ form a presimplicial homotopy as required. □

Combining Propositions 27.10.1 and 27.10.2 we deduce the following corollary.

**Corollary 27.10.3.** *The inclusion of chain complexes*

$$i\colon k\left[N_\star\left(-\backslash \mathrm{Epi}\Delta H\right)\right] \otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-) \to C_\star\left(\Delta H, I\right)$$

*is a chain homotopy equivalence.* □

**Theorem 27.10.4.** *Let $A$ be an augmented, involutive, associative $k$-algebra with augmentation ideal $I$. There exist isomorphisms of $k$-modules*

$$\begin{aligned}\widetilde{HO}_n(A) &\cong H_n\left(k\left[N_\star(-\backslash \mathrm{Epi}\Delta H)\right] \otimes_{\mathrm{Epi}\Delta H} B_I^{oct}(-)\right)\\ &\cong H_n\left(C_\star\left(\mathrm{Epi}\Delta H, B_I^{oct}\right)\right)\end{aligned}$$

*for each $n \geqslant 0$.* □

# Part VII

# A Comparison Map for Symmetric Homology and $\Gamma$-Homology

# Introduction

Both symmetric homology and Γ-homology are formed by building a symmetric group action into the Hochschild complex. We prove that in the case of an augmented, commutative algebra there is a comparison map between the symmetric homology and a summand of the Γ-homology. Furthermore, when the ground ring contains $\mathbb{Q}$ we show that our map compares the symmetric homology with the entire Γ-homology.

In Chapter 28 we introduce an isomorphic variant of the category $\Delta S$ of Definition 24.2.1. This variant has as objects the sets $\underline{n}$ for $n \geqslant 1$ and this eases the notation for the comparison map. Throughout Part VII we will work with this version of $\Delta S$.

In Chapter 29 we form a normalized version of Harrison homology. This new description of Harrison homology is required for the properties of the comparison map when $k \supseteq \mathbb{Q}$.

In Chapter 30 we define a normalized symmetric chain complex for an augmented algebra using the standard method of normalization for a chain complex associated to a simplicial $k$-module.

In Chapters 31 and 32 we define a surjective map of chain complexes

$$NCS_\star(I) \to NC\Gamma_\star(I, k)$$

between the normalized symmetric chain complex and the normalized Γ-complex for the augmentation ideal $I$ with coefficients in $k$. We therefore obtain a long exact sequence in homology connecting the reduced symmetric homology with a summand of the Γ-homology. Recalling a theorem from Part III we obtain a long exact sequence in homology connecting the reduced symmetric homology with the entire Γ-homology when $k \supseteq \mathbb{Q}$.

# Chapter 28

# A Variant of $\Delta S$

**Definition 28.0.1.** Let $\Delta'$ be the category whose objects are the sets $\underline{n} = \{1, \ldots, n\}$ for $n \geqslant 1$ and whose morphisms are order-preserving maps.

**Proposition 28.0.2.** *There is an isomorphism of categories* $\Delta \cong \Delta'$. $\square$

**Definition 28.0.3.** Let $\Delta S'$ be the category whose objects are the sets $\underline{n} = \{1, \ldots, n\}$ for $n \geqslant 1$. An element of $\mathrm{Hom}_{\Delta S'}(\underline{n}, \underline{m})$ is a pair $(\varphi, g)$ with $g \in \Sigma_n$ and $\varphi \in \mathrm{Hom}_{\Delta'}(\underline{n}, \underline{m})$. Composition is defined to be compatible with the isomorphism of sets

$$\mathrm{Hom}_{\Delta S'}(\underline{n}, \underline{m}) \cong \mathrm{Hom}_{\Delta S}\big([n-1], [m-1]\big)$$

where the composition on the right hand side was defined in Chapter 24.

**Proposition 28.0.4.** *There is an isomorphism of categories* $\Delta S \cong \Delta S'$ $\square$

Throughout Part VII we will work with $\Delta S'$ but we will denote it $\Delta S$ in order to ease notation.

# Chapter 29

# Normalized Harrison Homology

Suppose $k$ contains $\mathbb{Q}$ and let $A_\varepsilon$ be an augmented commutative $k$-algebra with augmentation ideal $I$. Let $M$ be a symmetric $A_\varepsilon$-bimodule which is flat over $k$. Recall that under these conditions we have a description of the Harrison complex as a subcomplex of the Hochschild complex from Subsections 5.3.3 and 5.4.3. Furthermore, we have the splitting of the Hochschild complex

$$C_\star(A_\varepsilon, M) \cong C_\star(I, M) \oplus D_\star(A_\varepsilon, M),$$

described in Proposition 5.2.4.

In this chapter we combine these methods to provide a normalization of the Harrison complex.

Forming a normalized Harrison homology is not straightforward in general. Whilst we can form a description of the Harrison complex as a subcomplex of the Hochschild complex when the ground ring $k$ contains $\mathbb{Q}$, this is because the Hochschild boundary map is compatible with the Eulerian idempotents. In fact, the Eulerian idempotents are not compatible with individual face and degeneracy maps, so the Harrison subcomplex does not arise as a chain complex associated to a simplicial object.

### 29.0.1 A decomposition of the degenerate subcomplex

Let $A_\varepsilon$ be an augmented, commutative $k$-algebra and let $M$ be a symmetric $A_\varepsilon$-bimodule. Recall the degenerate subcomplex $D_\star(A_\varepsilon, M)$ of the Hochschild complex from Subsection 5.2.2.

**Proposition 29.0.1.** *There is a natural isomorphism of chain complexes*

$$D_\star(A_\varepsilon, M) \cong \bigoplus_{i=1}^{\infty} e_\star^{(i)} D_\star(A_\varepsilon, M).$$

*Proof.* We observe that $\Sigma_n$ acts on the left of $D_n(A_\varepsilon, M)$ in precisely the same way as it acts on $C_n(A_\varepsilon, M)$, by permuting the tensor factors. Therefore $D_n(A_\varepsilon, M)$ can be split naturally using the Eulerian idempotents. □

**Corollary 29.0.2.** *The chain complex* $e_\star^{(i)} D_\star (A_\varepsilon, M)$ *is acyclic.*

*Proof.* Since the degenerate subcomplex $D_\star(A_\varepsilon, M)$ is an acyclic complex, each summand of the decomposition is itself an acyclic complex. □

### 29.0.2 A short exact sequence

By inclusion, $e_\star^{(i)} D_\star(A_\varepsilon, M)$ is a subcomplex of $e_\star^{(i)} C_\star(A_\varepsilon, M)$. In particular, taking $i = 1$, $e_\star^{(1)} D_\star(A_\varepsilon, M)$ is a subcomplex of the Harrison complex.

**Definition 29.0.3.** Denote the quotient map by

$$Q_i \colon e_\star^{(i)} C_\star(A_\varepsilon, M) \to \frac{e_\star^{(i)} C_\star(A_\varepsilon, M)}{e_\star^{(i)} D_\star(A_\varepsilon, M)}.$$

For each $i \geqslant 1$, there is a short exact sequence of chain complexes

$$0 \to e_\star^{(i)} D_\star(A_\varepsilon, M) \to e_\star^{(i)} C_\star(A_\varepsilon, M) \xrightarrow{Q_i} \frac{e_\star^{(i)} C_\star(A_\varepsilon, M)}{e_\star^{(i)} D_\star(A_\varepsilon, M)} \to 0.$$

By a standard construction this gives rise to a long exact sequence in homology. Since the complex $e_\star^{(i)} D_\star(A, M)$ is acyclic by Corollary 29.0.2 we can deduce that the quotient map $Q_i$ is a quasi-isomorphism for each $i \geqslant 1$.

**Proposition 29.0.4.** *There is an isomorphism of* $k$*-modules*

$$H_n\left(e_\star^{(i)} C_\star(A_\varepsilon, M)\right) \cong H_n\left(\frac{e_\star^{(i)} C_\star(A_\varepsilon, M)}{e_\star^{(i)} D_\star(A_\varepsilon, M)}\right)$$

*for each* $n \geqslant 0$ *and* $i \geqslant 1$. □

### 29.0.3 The quotient complex

Let $A_\varepsilon$ be an augmented, commutative $k$-algebra with augmentation ideal $I$. Let $M$ be a symmetric $A_\varepsilon$-bimodule. Recall the chain complex $C_\star(I, M)$ of Proposition 5.2.1.

**Proposition 29.0.5.** *There is a natural isomorphism of chain complexes*

$$C_\star(I, M) \cong \bigoplus_{i=1}^{\infty} e_\star^{(i)} C_\star(I, M).$$

*Proof.* We observe that $\Sigma_n$ acts on the left of $C_n(I, M)$ in precisely the same way as it acts on $C_n(A_\varepsilon, M)$, by permuting the tensor factors. Therefore $C_n(A_\varepsilon, M)$ can be split naturally using the Eulerian idempotents. □

**Lemma 29.0.6.** *For each $i \geqslant 1$ there is an isomorphism of chain complexes*

$$f_i\colon \frac{e_\star^{(i)} C_\star(A_\varepsilon, M)}{e_\star^{(i)} D_\star(A_\varepsilon, M)} \to e_\star^{(i)} C_\star(I, M).$$

*Proof.* We begin by considering the quotient complex

$$\frac{e_\star^{(i)} C_\star(A_\varepsilon, M)}{e_\star^{(i)} D_\star(A_\varepsilon, M)}.$$

The subcomplex $e_\star^{(i)} C_\star(A_\varepsilon, M)$ of the Hochschild complex $C_\star(A_\varepsilon, M)$ is generated by the $k$-linear combinations of the form $e_\star^{(i)}(m \otimes a_1 \otimes \cdots \otimes a_n)$, where $(m \otimes a_1 \otimes \cdots \otimes a_n)$ is a basic tensor in the sense of Definition 4.4.9.

The subcomplex $e_\star^{(i)} D_\star(A_\varepsilon, M)$ of the Hochschild complex is generated by the $k$-linear combinations of the form $e_\star^{(i)}(m \otimes a_1 \otimes \cdots \otimes a_n)$, where we run through all basic tensors with at least one trivial tensor factor.

We can choose representatives such that the quotient complex, in degree $n$, is generated by equivalence classes

$$\left[e_\star^{(i)}(m \otimes y_1 \otimes \cdots \otimes y_n)\right]$$

where each $y_i$ is an element of the augmentation ideal $I$, and the boundary map is induced from the Hochschild boundary map, $b$.

With this choice of representatives, we have a well-defined map of chain complexes

$$f_i\colon \frac{e_\star^{(i)} C_\star(A_\varepsilon, M)}{e_\star^{(i)} D_\star(A_\varepsilon, M)} \to e_\star^{(i)} C_\star(I, M)$$

determined by

$$\left[e_\star^{(i)}(m \otimes y_1 \otimes \cdots \otimes y_n)\right] \mapsto e_\star^{(i)}(m \otimes y_1 \otimes \cdots \otimes y_n)$$

on generators as above in degree $n$.

The inverse is given by the map determined by sending a generator

$$e_\star^{(i)}(m \otimes y_1 \otimes \cdots \otimes y_n)$$

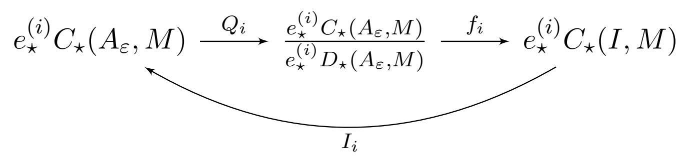
of $e_\star^{(i)} C_\star(I, M)$ to its equivalence class in the quotient. □

**Definition 29.0.7.** Denote by

$$I_i\colon e_\star^{(i)} C_\star(I, M) \to e_\star^{(i)} C_\star(A_\varepsilon, M)$$

the inclusion of chain complexes.

We have a diagram of the form



**Lemma 29.0.8.** *The composite* $(f_i \circ Q_i) \circ I_i$ *is the identity map on the chain complex* $e^{(i)}_\star C_\star(I, M)$.

*Proof.* All three maps are morphisms of chain complexes and it suffices to check the claim on generators in an arbitrary degree. Let $m \otimes y_1 \otimes \cdots \otimes y_n$ be a generator of $e^{(i)}_n C_n(I, M)$. We observe that

$$\begin{aligned} f_i \circ Q_i \circ I_i(m \otimes y_1 \otimes \cdots \otimes y_n) &= f_i \circ Q_i(m \otimes y_1 \otimes \cdots \otimes y_n) \\ &= f_i\left([m \otimes y_1 \otimes \cdots \otimes y_n]\right) \\ &= m \otimes y_1 \otimes \cdots \otimes y_n \end{aligned}$$

as required. □

**Theorem 29.0.9.** *Let* $k \supseteq \mathbb{Q}$ *and let* $A_\varepsilon$ *be a augmented, commutative* $k$*-algebra with augmentation ideal* $I$*. Let* $M$ *be a symmetric* $A_\varepsilon$*-bimodule which is flat over* $k$*. For each* $i \geqslant 1$ *there is an isomorphism of chain complexes*

$$e^{(i)}_\star C_\star(A_\varepsilon, M) \cong e^{(i)}_\star C_\star(I, M) \oplus e^{(i)}_\star D_\star(A_\varepsilon, M).$$

*Proof.* This follows from Lemma 29.0.8 upon observing that

$$\mathrm{Ker}(f_i \circ Q_i) \cong e^{(i)}_\star D_\star(A_\varepsilon, M).$$

□

**Corollary 29.0.10.** *The composite of the the canonical projection map* $Q_i$ *with the isomorphism* $f_i$,

$$f_i \circ Q_i \colon e^{(i)}_\star C_\star(A_\varepsilon, M) \to e^{(i)}_\star C_\star(I, M)$$

*is a quasi-isomorphism. The inclusion of chain complexes*

$$I_i \colon e^{(i)}_\star C_\star(I, M) \to e^{(i)}_\star C_\star(A_\varepsilon, M)$$

*induces the inverse map on homology.*

*In particular, taking* $i = 1$ *there is a quasi-isomorphism*

$$f_1 \circ Q_1 \colon CHarr_\star(A_\varepsilon, M) \to e^{(1)}_\star C_\star(I, M)$$

*and the inclusion of chain complexes*

$$I_1 \colon e^{(1)}_\star C_\star(I, M) \to CHarr_\star(A_\varepsilon, M)$$

*induces the inverse map on homology.*

*That is, there is an isomorphism of* $k$*-modules*

$$H_n\left(e^{(1)}_\star C_\star(I, M)\right) \cong Harr_n\left(A_\varepsilon, M\right)$$

*for each* $n \geqslant 0$*.* □

# Chapter 30

# Normalized Symmetric Homology

Recall the chain complex $CS_\star(I)$ of Definition 27.4.3.

Consider the simplicial set $N_\star(\mathrm{Epi}\Delta S)$. An element in degree $n$ takes the form

$$\underline{x} \xrightarrow{f_1} \underline{x_1} \xrightarrow{f_2} \cdots \xrightarrow{f_n} \underline{x_n}$$

where each $f_i$, for $1 \leqslant i \leqslant n$, is a morphism in $\mathrm{Epi}\Delta S$. That is, each $f_i$ is a surjection of sets with a total ordering on each preimage. As previously we denote such an element by $(f_n, \ldots, f_1)$.

Consider

$$CS_n(I) = \bigoplus_{(f_n,\ldots,f_1)} B_I^{sym}(\underline{x}).$$

A generator takes the form of a pair

$$((f_n, \ldots, f_1), (y_1 \otimes \cdots \otimes y_x)),$$

where $(f_n, \ldots, f_1) \in N_n(\mathrm{Epi}\Delta S)$ indexes the summand and $(y_1 \otimes \cdots \otimes y_x) \in B_I^{sym}(\underline{x})$. The boundary map is given by the alternating sum of the face maps of the simplicial $k$-module $C_\star(\mathrm{Epi}\Delta S, B_I^{sym})$.

Since $CS_\star(I)$ is the chain complex associated to a simplicial set we can form the normalized chain complex following Subsection 2.5.4.

**Definition 30.0.1.** We denote the *normalized symmetric chain complex* by $NCS_\star(I)$.

A generator of $NCS_n(I)$, thought of as a quotient of $CS_n(I)$, is an equivalence class of the form

$$\big[(f_n, \ldots, f_1), (y_1 \otimes \cdots \otimes y_x)\big]$$

where each $f_i$ is a morphism in $\mathrm{Epi}\Delta S$ and is not an identity map. The boundary map is induced from the boundary map of $CS_\star(I)$.

We deduce the following proposition from Corollary 2.5.16.

**Proposition 30.0.2.** *There exist isomorphisms of $k$-modules*

$$\widetilde{HS}_n(A_\varepsilon) \cong H_n(CS_\star(I)) \cong H_n(NCS_\star(I))$$

*for $n \geqslant 0$.* □

# Chapter 31

# A Quotient of the Symmetric Chain Complex

Recall the chain complex $NCS_\star(I)$ of Definition 30.0.1. In degree $n$ we have the $k$-module generated by equivalence classes of the form

$$\left[(f_n, \dots, f_1), (y_1 \otimes \cdots \otimes y_x)\right],$$

where

$$\underline{x} \xrightarrow{f_1} \cdots \xrightarrow{f_n} \underline{x_n}$$

is an element of $N_n(\mathrm{Epi}\Delta S)$, none of the maps $f_i$ are identity maps and $y_1 \otimes \cdots \otimes y_x \in B_I^{sym}(\underline{x})$.

**Definition 31.0.1.** Denote by $NCS_n^1(I)$ the $k$-submodule generated by the equivalence classes for which $\underline{x_n} \neq \underline{1}$.

**Lemma 31.0.2.** *There is a well-defined subcomplex $NCS_\star^1(I)$ of $NCS_\star(I)$.*

*Proof.* In order to show this we must show the differential of $NCS_\star(I)$ induces a well-defined differential on $NCS_\star^1(I)$. It suffices to prove that the image of a generator of $NCS_n^1(I)$ under each face map is an element of $NCS_{n-1}^1(I)$.

The only face map that affects the final codomain of the string is $\partial_n$. We see that

$$\partial_n \left[(f_n, \dots, f_1), (y_1 \otimes \cdots \otimes y_x)\right] = \left[(f_{n-1}, \dots, f_1), (y_1 \otimes \cdots \otimes y_x)\right]$$

where $(f_{n-1}, \dots, f_1)$ denotes

$$\underline{x} \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}}.$$

Since $f_n \in \mathrm{Hom}_{\mathrm{Epi}\Delta S}\left(\underline{x_{n-1}}, \underline{x_n}\right)$ is a surjection as a map of sets and $x_n \neq 1$ we can deduce that $x_{n-1} \neq 1$ and so the differential is well-defined on $NCS_\star^1(I)$. $\square$

**Definition 31.0.3.** We denote the quotient complex by

$$\overline{NCS_\star(I)} = \frac{NCS_\star(I)}{NCS^1_\star(I)}.$$

A $k$-module generator of $\overline{NCS_n(I)}$ is an equivalence class of the form

$$\left[(f_n, \dots, f_1), (y_1 \otimes \cdots \otimes y_x)\right]$$

where $(f_n, \dots, f_1)$ denotes an element of $N_n(\text{Epi}\Delta S)$ of the form

$$\underline{x} \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}} \xrightarrow{f_n} \underline{1}$$

and $(y_0 \otimes \cdots \otimes y_x) \in B_I^{sym}(\underline{x})$.

We will denote the quotient map

$$q \colon NCS_\star(I) \to \overline{NCS_\star(I)}.$$

The differential of $\overline{NCS_\star(I)}$ is induced from the differential of $NCS_\star(I)$. That is, it is induced from the alternating sum of the face maps.

**Lemma 31.0.4.** *For $n \geqslant 1$, the map induced from the final face map,*

$$\partial_n \colon \overline{NCS_n(I)} \to \overline{NCS_{n-1}(I)}$$

*is the zero map.*

*Proof.* A non-trivial element of $\overline{NCS_\star(I)}$ is indexed by a string of composable non-identity morphisms of the form

$$\underline{x} \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}} \xrightarrow{f_n} \underline{1}.$$

The map induced by $\partial_n$ omits the last morphism in the string, so we obtain

$$\underline{x} \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}}.$$

By construction, $f_n$ was a surjection and was not an identity map and so $x_{n-1} > 1$. Therefore, the image of a generator of $\overline{NCS_n(I)}$ under $\partial_n$ is zero in $\overline{NCS_{n-1}(I)}$. □

# Chapter 32

# Mapping to the Γ-complex

Recall the chain complex $NC\Gamma_\star(I,k)$ from Definition 18.1.3. Having formed the quotient map $q$ of Definition 31.0.3, we will now map from the quotient complex to the complex $NC\Gamma_\star(I,k)$. We will construct a map of chain complexes

$$\Phi\colon \overline{NCS_\star(I)} \to NC\Gamma_\star(I,k).$$

Recall that a generator of $NC\Gamma_n(I,k)$ is of the form

$$\big[f_n \mid f_{n-1} \mid \cdots \mid f_1\big] \otimes (y_1 \otimes \cdots \otimes y_x) \otimes 1_k,$$

where $\big[f_n \mid f_{n-1} \mid \cdots \mid f_1\big]$ denotes a string of non-identity morphisms

$$\underline{x} \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}} \xrightarrow{f_n} \underline{1}$$

in $N_n\Omega$ and $(y_1 \otimes \cdots \otimes y_x) \in I^{\otimes x}$.

Consider a string of non-identity morphisms in $N_n(\text{Epi}\Delta S)$ of the form

$$\underline{x} \xrightarrow{f_1} \cdots \xrightarrow{f_{n-1}} \underline{x_{n-1}} \xrightarrow{f_n} \underline{1}.$$

Using the isomorphism $\Delta S \cong \mathcal{F}(as)$ from Proposition 24.2.5, each $f_i$ is a surjection of sets with a total ordering specified on each preimage. If we forget the data on preimages then we simply have a string of morphisms in $N_n\Omega$.

**Definition 32.0.1.** Let $U\colon \text{Epi}\Delta S \to \Omega$ denote the forgetful functor that is the identity on objects and sends a morphism of $\text{Epi}\Delta S$ to the underlying surjection of sets.

**Definition 32.0.2.** Let

$$\Phi_n\colon \overline{NCS_n(I)} \to NC\Gamma_n(I,k)$$

be the map of $k$-modules determined by

$$\big[(f_n,\ldots,f_1),(y_1\otimes\cdots\otimes y_x)\big] \mapsto \big[U(f_n) \mid \cdots \mid U(f_1)\big] \otimes (y_1 \otimes \cdots \otimes y_x) \otimes 1_k.$$

The proof that the maps $\Phi_n$ assemble into a map of chain complexes can be found in Appendix D.

## 32.1 The Comparison Map

**Theorem 32.1.1.** *Let $A_\varepsilon$ be an augmented, commutative $k$-algebra with augmentation ideal $I$. There is a surjective map of chain complexes*

$$\Phi \circ q \colon NCS_\star(I) \to NC\Gamma_\star(I,k).$$

*Proof.* Recall form Definition 31.0.3 that

$$q\colon NCS_\star(I) \to \overline{NCS_\star(I)}$$

is a quotient map and is therefore surjective.

A generator

$$\left[f_n \mid f_{n-1} \mid \cdots \mid f_1\right] \otimes (y_1 \otimes \cdots \otimes y_x) \otimes 1_k$$

of $NC\Gamma_n(I,k)$ is the image of

$$\left[(f_n, \ldots, f_1), (y_1 \otimes \cdots \otimes y_x)\right]$$

in $\overline{NCS_n(I)}$ where we take the total orderings on the preimages of each $f_i$ to be the canonical ones. Hence $\Phi$ is also a surjective map. □

Recall from Section 18.3 that the homology of the chain complex $C\Gamma_\star(I,k)$ is a direct summand of $H\Gamma_\star(A_\varepsilon, k)$.

**Corollary 32.1.2.** *There is a short exact sequence of chain complexes*

$$0 \to \mathrm{Ker}(\Phi\circ q) \to NCS_\star(I) \xrightarrow{\Phi\circ q} NC\Gamma_\star(I,k) \to 0,$$

*which gives rise to the long exact sequence*

$$\cdots \longrightarrow H_n(\mathrm{Ker}(\Phi\circ q) \longrightarrow \widetilde{HS}_n(A_\varepsilon) \longrightarrow H\Gamma_n(I,k) \longrightarrow$$
$$\longrightarrow H_{n-1}(\mathrm{Ker}(\Phi\circ q)) \longrightarrow \cdots \longrightarrow H\Gamma_1(I,k) \longrightarrow$$
$$\longrightarrow H_0(\mathrm{Ker}(\Phi\circ q)) \longrightarrow \widetilde{HS}_0(A_\varepsilon) \longrightarrow H\Gamma_0(I,k) \longrightarrow 0.$$

*connecting the reduced symmetric homology of $A_\varepsilon$ with a direct summand of the $\Gamma$-homology of $A_\varepsilon$.* □

## 32.2 The Comparison Map when $k$ contains $\mathbb{Q}$

Let $A_\varepsilon$ be an augmented $k$-algebra with augmentation ideal $I$. Let $M$ be a symmetric $A_\varepsilon$-bimodule which is flat over $k$. Recall from Section 18.3 that under these conditions there is a splitting of the $\Gamma$-complex

$$C\Gamma_\star(A_\varepsilon, M) \cong C\Gamma_\star(I,M) \oplus \mathrm{Ker}(P_\star)$$

where $P_\star$ is the pruning map of Definition 18.2.4.

**Theorem 32.2.1.** *Let $k \supseteq \mathbb{Q}$. Let $A_\varepsilon$ be an augmented, commutative $k$-algebra with augmentation ideal $I$. There is an isomorphism of $k$-modules*

$$H\Gamma_n\left(I, k\right) \cong H\Gamma_n\left(A_\varepsilon, k\right)$$

*for each $n \geqslant 0$.*

*Proof.* Consider the $\Gamma$-complex for the augmentation ideal $I$, $C\Gamma_\star(I, M)$ from Subsection 18.1.2.

Corollary 2.5.16 tells us that normalization gives a quasi-isomorphism

$$\pi_I \colon C\Gamma_\star(I, k) \to NC\Gamma_\star(I, k).$$

When $k \supseteq \mathbb{Q}$, Proposition 17.2.2 gives an isomorphism

$$\Theta \colon C\Gamma_\star(I, k) \to CHarr_{\star+1}(I, k).$$

From Corollary 29.0.10 there is a quasi-isomorphism

$$I_1 \colon e_\star^{(1)} C_\star(I, k) \to CHarr_\star\left(A_\varepsilon, k\right).$$

Proposition 17.2.2 then provides us with an isomorphism

$$\Psi \colon CHarr_{\star+1}\left(A_\varepsilon, k\right) \to C\Gamma_\star\left(A_\varepsilon, k\right).$$

These maps fit into the following diagram

$$\begin{array}{ccccc} NC\Gamma_\star(I,k) & \xleftarrow{\pi_I} & C\Gamma_\star(I,k) & \xrightarrow{\Theta} & e_\star^{(1)} C_{\star+1}(I,k) \\ & & \vdots & & \downarrow I_1 \\ & & C\Gamma_\star\left(A_\varepsilon, k\right) & \xleftarrow[\Psi]{} & CHarr_{\star+1}\left(A_\varepsilon, k\right), \end{array}$$

where the dotted arrow denotes the composite $\Psi \circ I_1 \circ \Theta$.

Every map in this diagram is at least a quasi-isomorphism so, upon taking homology, we deduce that

$$H\Gamma_n(I, k) \cong H_n(NC\Gamma_\star(I, k)) \cong H_n\left(C\Gamma_\star\left(A_\varepsilon, k\right)\right) = H\Gamma_n\left(A_\varepsilon, k\right)$$

for each $n \geqslant 0$. □

Combining Corollary 32.1.2 and Theorem 32.2.1 we deduce the following theorem.

**Theorem 32.2.2.** *Let $k \supseteq \mathbb{Q}$. Let $A_\varepsilon$ be an augmented, commutative $k$-algebra. There is a long exact sequence*

$$\cdots \longrightarrow H_n(\mathrm{Ker}(\Phi \circ q) \longrightarrow \widetilde{HS}_n(A_\varepsilon) \longrightarrow H\Gamma_n(A_\varepsilon, k) \longrightarrow$$

$$\longrightarrow H_{n-1}(\mathrm{Ker}(\Phi \circ q)) \longrightarrow \cdots \longrightarrow H\Gamma_1(A_\varepsilon, k) \longrightarrow$$

$$\longrightarrow H_0(\mathrm{Ker}(\Phi \circ q)) \longrightarrow \widetilde{HS}_0(A_\varepsilon) \longrightarrow H\Gamma_0(A_\varepsilon, k) \longrightarrow 0.$$

*connecting the reduced symmetric homology of* $A_\varepsilon$ *with the* $\Gamma$*-homology of* $A_\varepsilon$ *with coefficients in* $k$. □

# Appendices

# Appendix A

# The Pruning Map is a Chain Map

Recall from Definition 18.2.4 that the pruning map was defined to be the $k$-linear map of chain complexes determined in degree $n$ by

$$\begin{array}{c}\left[f_n \mid \cdots \mid f_1\right] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m \\ \big\downarrow \\ \left[\widetilde{f_n} \mid \cdots \mid \widetilde{f_1}\right] \otimes \left(a_{l_1} \otimes \cdots \otimes a_{l_h}\right) \otimes m\end{array}$$

In order to check that the pruning map is a well-defined map of chain complexes we must demonstrate that the diagram

$$\begin{array}{ccc} C\Gamma_n\left(A_\varepsilon, M\right) & \xrightarrow{P_n} & C\Gamma_n\left(I, M\right) \\ \Big\downarrow{\scriptstyle \partial_i} & & \Big\downarrow{\scriptstyle \partial_i} \\ C\Gamma_{n-1}\left(A_\varepsilon, M\right) & \xrightarrow[P_{n-1}]{} & C\Gamma_{n-1}\left(I, M\right) \end{array}$$

commutes for all $n \geqslant 1$ and $0 \leqslant i \leqslant n$.

It suffices to check commutativity for a generator

$$\left[f_n \mid \cdots \mid f_1\right] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m$$

of $C\Gamma_n\left(A_\varepsilon, M\right)$, where $(a_1 \otimes \cdots \otimes a_x) \otimes m$ is a basic tensor in the sense of Definition 4.4.9. We do this case by case for each face map.

Let $L = \{l_1, \ldots, l_h\}$ be the set such that $a_p \in I$ if and only if $p \in L$.

### The case $i = 0$

When $i = 0$ the upper composite is given on such a generator by

$$\begin{gathered}\left[f_n \mid \cdots \mid f_1\right] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m \\ \Big\downarrow P_n \\ \left[\widetilde{f_n} \mid \cdots \mid \widetilde{f}_1\right] \otimes \left(a_{l_1} \otimes \cdots \otimes a_{l_h}\right) \otimes m \\ \Big\downarrow \partial_0 \\ \left[\widetilde{f_n} \mid \cdots \mid \widetilde{f}_2\right] \otimes \widetilde{f}_{1\star}\left(a_{l_1} \otimes \cdots \otimes a_{l_h}\right) \otimes m\end{gathered}$$

The first map in the lower composite acts on a generator as follows.

$$\begin{gathered}\left[f_n \mid \cdots \mid f_1\right] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m \\ \Big\downarrow \partial_0 \\ \left[f_n \mid \cdots \mid f_2\right] \otimes f_{1\star}(a_1 \otimes \cdots \otimes a_x) \otimes m\end{gathered}$$

Consider $f_{1\star}(a_1 \otimes \cdots \otimes a_x)$. This is the elementary tensor obtained by permuting and multiplying the factors of $(a_1 \otimes \cdots \otimes a_x)$ according to $f_1$. Since $\widetilde{f_1}$ is the restriction of $f_1$ to the subset $L \subseteq \underline{x}$ we observe that $f_{1\star}(a_1 \otimes \cdots \otimes a_x)$ can only differ from $\widetilde{f}_{1\star}\left(a_{l_1} \otimes \cdots \otimes a_{l_h}\right)$ by containing some additional trivial tensor factors. The pruning map then removes these trivial tensor factors and prunes the graph of $\left[f_n \mid \cdots \mid f_2\right]$ to the graph of $\left[\widetilde{f_n} \mid \cdots \mid \widetilde{f}_2\right]$. We therefore conclude that

$$P_{n-1}\left(\left[f_n \mid \cdots \mid f_2\right] \otimes f_{1\star}(a_1 \otimes \cdots \otimes a_x) \otimes m\right)$$

is equal to

$$\left[\widetilde{f_n} \mid \cdots \mid \widetilde{f}_2\right] \otimes \widetilde{f}_{1\star}\left(a_{l_1} \otimes \cdots \otimes a_{l_h}\right) \otimes m$$

as required.

### The case $1 \leqslant i \leqslant n-1$

For $1 \leqslant i \leqslant n-1$, the upper composite is given on such a generator by

$$\begin{gathered}\left[f_n \mid \cdots \mid f_1\right] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m \\ \Big\downarrow P_n \\ \left[\widetilde{f_n} \mid \cdots \mid \widetilde{f}_1\right] \otimes \left(a_{l_1} \otimes \cdots \otimes a_{l_h}\right) \otimes m \\ \Big\downarrow \partial_i \\ \left[\widetilde{f_n} \mid \cdots \mid \widetilde{f_{i+1}} \circ \widetilde{f}_i \mid \cdots \mid \widetilde{f}_1\right] \otimes \left(a_{l_1} \otimes \cdots \otimes a_{l_h}\right) \otimes m\end{gathered}$$

The lower composite is given on such a generator by

$$\begin{array}{c}
\big[f_n \mid \cdots \mid f_1\big] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m \\
\Big\downarrow \partial_i \\
\big[f_n \mid \cdots \mid f_{i+1} \circ f_i \mid \cdots \mid f_1\big] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m \\
\Big\downarrow P_{n-1} \\
\Big[\widetilde{f_n} \mid \cdots \mid \widetilde{f_{i+1} \circ f_i} \mid \cdots \mid \widetilde{f_1}\Big] \otimes \big(a_{l_1} \otimes \cdots \otimes a_{l_h}\big) \otimes m
\end{array}$$

We note that

$$\widetilde{f_{i+1} \circ f_i} = \widetilde{f_{i+1}} \circ \widetilde{f_i}$$

since

$$(f_{i+1} \circ f_i)\big|_{m_{i-1}} = f_{i+1}\big|_{f_i(m_{i-1})} \circ f_i\big|_{m_{i-1}}$$

as a map of sets.

### The case $i = n$

Let $J = \{j_1, \ldots, j_g\}$ denote the set $f_1^{-1} \ldots f_{n-1}^{-1}(j)$ for each $j \in \underline{x_{n-1}}$. Let $L \cap J = \{k_1, \ldots, k_r\}$.

The lower composite is given on such a generator by

$$\begin{array}{c}
\big[f_n \mid \cdots \mid f_1\big] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m \\
\Big\downarrow \partial_n \\
\sum_{j \in \underline{x_{n-1}}} \Big[f_{n-1}^j \mid \cdots \mid f_1^j\Big] \otimes \big(a_{j_1} \otimes \cdots \otimes a_{j_g}\big) \otimes \left(\prod_{i \notin J} a_i\right) m \\
\Big\downarrow P_{n-1} \\
\sum_{j \in \underline{|m_{n-1}|}} \Big[\widetilde{f_{n-1}^j} \mid \cdots \mid \widetilde{f_1^j}\Big] \otimes \big(a_{k_1} \otimes \cdots \otimes a_{k_r}\big) \otimes \left(\prod_{i \notin J} a_i\right) m
\end{array}$$

The upper composite is given on such a generator by

$$\begin{array}{c}
\big[f_n \mid \cdots \mid f_1\big] \otimes (a_1 \otimes \cdots \otimes a_x) \otimes m \\
\Big\downarrow P_n \\
\Big[\widetilde{f_n} \mid \cdots \mid \widetilde{f_1}\Big] \otimes \big(a_{l_1} \otimes \cdots \otimes a_{l_h}\big) \otimes m \\
\Big\downarrow \partial_n \\
\sum_{j \in \underline{|m_{n-1}|}} \Big[\widetilde{f_{n-1}}^j \mid \cdots \mid \widetilde{f_1}^j\Big] \otimes \big(a_{k_1} \otimes \cdots \otimes a_{k_r}\big) \otimes \left(\prod_{i \in L,\, i \notin J} a_i\right) m
\end{array}$$

We observe that

$$\left(\prod_{i \notin J} a_i\right) = \left(\prod_{i \in L,\, i \notin J} a_i\right)$$

since the extra terms in the left hand product are all equal to $1_k$.

We observe that both $\left[\widetilde{f^j_{n-1}} \mid \cdots \mid \widetilde{f^j_1}\right]$ and $\left[\widetilde{f_{n-1}}^j \mid \cdots \mid \widetilde{f_1}^j\right]$ are equal to the sub-graph of $\left[f_{n-1} \mid \cdots \mid f_1\right]$ corresponding to the elements of $L \cap J$ in $\underline{x}$, re-indexed in the canonical way. We therefore have commutativity as required.

# Appendix B

# The Hyperoctahedral Bar Construction

It is clear that the assignment $B_A^{oct}$ preserves identity morphisms.

In order to show that $B_A^{oct}$ is functorial we must show that it respects composition.

We begin with two lemmata.

**Lemma B.0.1.** *Let $\varphi \in \mathrm{Hom}_\Delta\left([n],[m]\right)$ and $\psi \in \mathrm{Hom}_\Delta\left([m],[l]\right)$. There is an equality*

$$B_A^{oct}\left(\psi \circ \varphi, id_{[n]}\right) = B_A^{oct}\left(\psi, id_{[m]}\right) \circ B_A^{oct}\left(\varphi, id_{[n]}\right).$$

*Proof.* This is the restriction of the hyperoctahedral bar construction to the subcategory $\Delta$. This is the usual bar construction for $\Delta$ and is therefore functorial. □

**Lemma B.0.2.** *Let $g$, $h \in H_{n+1}$. There is an equality*

$$B_A^{oct}\left(id_{[n]}, gh\right) = B_A^{oct}\left(id_{[n]}, g\right) \circ B_A^{oct}\left(id_{[n]}, h\right).$$

*Proof.* Let $g = (z_0, \dots, z_n; \sigma_2)$ and $h = (w_0, \dots, w_n; \sigma_1)$. We observe that both

$$B_A^{oct}\left(id_{[n]}, gh\right)(a_0 \otimes \cdots \otimes a_n)$$

and

$$B_A^{oct}\left(id_{[n]}, g\right) \circ B_A^{oct}\left(id_{[n]}, h\right)(a_0 \otimes \cdots \otimes a_n)$$

are equal to

$$a_{\sigma_1^{-1}\sigma_2^{-1}(0)} \otimes \cdots \otimes a_{\sigma_1^{-1}\sigma_2^{-1}(n)}$$

with the label $z_{\sigma_2^{-1}(i)} w_{\sigma_1^{-1}\sigma_2^{-1}(i)}$ on $a_{\sigma_1^{-1}\sigma_2^{-1}(i)}$. □

We can now begin to demonstrate that $B_A^{oct}$ preserves composition. Let

$$(\varphi, g) \in \mathrm{Hom}_{\Delta H}\left([n],[m]\right)$$

and

$$(\psi, h) \in \mathrm{Hom}_{\Delta H}\left([m], [l]\right).$$

On the one hand, using composition in $\Delta H$ and Lemmata B.0.1 and B.0.2, we see that

$$\begin{aligned} B_A^{oct}\left((\psi, h) \circ (\varphi, g)\right) &= B_A^{oct}\left(\psi \circ h_\star(\varphi), \varphi^\star(h) \circ g\right) \\ &= B_A^{oct}\left(\psi \circ h_\star(\varphi), id_{[n]}\right) \circ B_A^{oct}\left(id_{[n]}, \varphi^\star(h) \circ g\right) \\ &= B_A^{oct}\left(\psi, id_{[m]}\right) \circ B_A^{oct}\left(h_\star(\varphi), id_{[n]}\right) \circ B_A^{oct}\left(id_{[n]}, \varphi^\star(h)\right) \circ B_A^{oct}\left(id_{[n]}, g\right). \end{aligned}$$

On the other hand, we see that

$$B_A^{oct}(\psi, h) \circ B_A^{oct}(\varphi, g) = B_A^{oct}\left(\psi, id_{[m]}\right) \circ B_A^{oct}\left(id_{[m]}, h\right) \circ B_A^{oct}\left(\varphi, id_{[n]}\right) \circ B_A^{oct}\left(id_{[n]}, g\right).$$

Comparing the two expressions we note that checking that $B_A^{oct}$ respects composition is equivalent to checking that the diagram

$$\begin{array}{ccc} A^{\otimes n+1} & \xrightarrow{B_A^{oct}\left(\varphi, id_{[n]}\right)} & A^{\otimes m+1} \\ {\scriptstyle B_A^{oct}\left(id_{[n]}, \varphi^\star(h)\right)}\downarrow & & \downarrow{\scriptstyle B_A^{oct}\left(id_{[m]}, h\right)} \\ A^{\otimes n+1} & \xrightarrow[B_A^{oct}\left(h_\star(\varphi), id_{[n]}\right)]{} & A^{\otimes m+1} \end{array}$$

commutes in **kMod** for all $h \in H_{m+1}$ and $\varphi \in \mathrm{Hom}_\Delta\left([n], [m]\right)$.

By the relations of Proposition 19.1.7 and the unique decomposition of morphisms in $\Delta$, Theorem 2.2.5 it suffices to check the diagrams

$$\begin{array}{ccc} B_A^{oct}\left([n-1]\right) & \xrightarrow{B_A^{oct}\left(\delta_i, id_{[n-1]}\right)} & B_A^{oct}\left([n]\right) \\ {\scriptstyle B_A^{oct}\left(id_{[n-1]}, \delta_i^\star(g)\right)}\downarrow & & \downarrow{\scriptstyle B_A^{oct}\left(id_{[n]}, g\right)} \\ B_A^{oct}\left([n-1]\right) & \xrightarrow[B_A^{oct}\left(g_\star(\delta_i), id_{[n-1]}\right)]{} & B_A^{oct}\left([n]\right) \end{array}$$

and

$$\begin{array}{ccc} B_A^{oct}\left([n+1]\right) & \xrightarrow{B_A^{oct}\left(\sigma_i, id_{[n+1]}\right)} & B_A^{oct}\left([n]\right) \\ {\scriptstyle B_A^{oct}\left(id_{[n+1]}, \sigma_i^\star(g)\right)}\downarrow & & \downarrow{\scriptstyle B_A^{oct}\left(id_{[n]}, g\right)} \\ B_A^{oct}\left([n+1]\right) & \xrightarrow[B_A^{oct}\left(g_\star(\sigma), id_{[n+1]}\right)]{} & B_A^{oct}\left([n]\right) \end{array}$$

commute in **kMod** for all $g \in H_{n+1}$ and $0 \leqslant i \leqslant n$.

**Lemma B.0.3.** *The diagram*

$$\begin{array}{ccc} B_A^{oct}\left([n-1]\right) & \xrightarrow{B_A^{oct}\left(\delta_i, id_{[n-1]}\right)} & B_A^{oct}\left([n]\right) \\ {\scriptstyle B_A^{oct}\left(id_{[n-1]}, \delta_i^\star(g)\right)}\downarrow & & \downarrow{\scriptstyle B_A^{oct}\left(id_{[n]}, g\right)} \\ B_A^{oct}\left([n-1]\right) & \xrightarrow[B_A^{oct}\left(g_\star(\delta_i), id_{[n-1]}\right)]{} & B_A^{oct}\left([n]\right) \end{array}$$

*commutes in* **kMod** *for all* $g \in H_{n+1}$ *and* $0 \leqslant i \leqslant n$.

*Proof.* It suffices to check for elementary tensors. Let $a_0 \otimes \cdots \otimes a_{n-1}$ be an elementary tensor in $A^{\otimes n}$. Let $g = (z_0, \ldots, z_n; \sigma)$ and let $\nu$ be the underlying permutation of $\delta_i^\star(g)$.

We observe that

$$B_A^{oct}\left(id_{[n]}, g\right) \circ B_A^{oct}\left(\delta_i, id_{[n-1]}\right)(a_0 \otimes \cdots \otimes a_{n-1})$$

is the elementary tensor in $A^{\otimes n}$ whose $j^{th}$ factor, for $j \neq \sigma(i)$, is $a_{\delta_i^{-1}\sigma^{-1}(j)}$ with label $z_{\sigma^{-1}(j)}$.

The factor in position $\sigma(i)$ is $1_A^{z_i}$. By Remark 4.3.3 $\overline{1_A} = 1_A$ and so this factor is equal to $1_A$ regardless of the label $z_i$.

On the other hand

$$B_A^{oct}\left(g_\star(\delta_i), id_{[n-1]}\right) \circ B_A^{oct}\left(id_{[n-1]}, \delta_i^\star(g)\right)(a_0 \otimes \cdots \otimes a_{n-1})$$

is the elementary tensor in $A^{\otimes n}$ whose $j^{th}$ factor, for $j \neq \sigma(i)$, is $a_{\nu^{-1}\delta_{\sigma(i)}^{-1}(j)}$ with label $z_{\sigma^{-1}(j)}$, by the construction of $\delta_i^\star(g)$.

We note that the factor in position $\sigma(i)$ is equal to $1_A$. Furthermore, since the diagram

$$\begin{array}{ccc} [n-1] & \xrightarrow{\left(\delta_i, id_{[n-1]}\right)} & [n] \\ {\scriptstyle \left(id_{[n-1]}, \delta_i^\star(g)\right)}\downarrow & & \downarrow{\scriptstyle \left(id_{[n]}, g\right)} \\ [n-1] & \xrightarrow[\left(g_\star(\delta_i), id_{[n-1]}\right)]{} & [n] \end{array}$$

is commutative in $\Delta H$ we observe that

$$\nu^{-1}\delta_{\sigma(i)}^{-1}(j) = \delta_i^{-1}\sigma^{-1}(j)$$

for $j \neq \sigma(i)$ and so all other tensor factors coincide for both composites. $\square$

**Lemma B.0.4.** *The diagram*

$$
\begin{array}{ccc}
B_A^{oct}\left([n+1]\right) & \xrightarrow{B_A^{oct}\left(\sigma_i, id_{[n+1]}\right)} & B_A^{oct}\left([n]\right) \\
\Big\downarrow {\scriptstyle B_A^{oct}\left(id_{[n+1]}, \sigma_i^\star(g)\right)} & & \Big\downarrow {\scriptstyle B_A^{oct}\left(id_{[n]}, g\right)} \\
B_A^{oct}\left([n+1]\right) & \xrightarrow[B_A^{oct}\left(g_\star(\sigma), id_{[n+1]}\right)]{} & B_A^{oct}\left([n]\right)
\end{array}
$$

*commutes in* **kMod** *for all* $g \in H_{n+1}$ *and* $0 \leqslant i \leqslant n$.

*Proof.* Once again, it suffices to check for elementary tensors. Let $a_0 \otimes \cdots \otimes a_{n+1}$ be an elementary tensor in $A^{\otimes n+2}$. Let $g = (z_0, \ldots, z_n; \rho)$.

The only case for this diagram that differs from the previous lemma is the diagram chase for the tensor factors $a_i$ and $a_{i+1}$. Following the upper composite we obtain $(a_i a_{i+1})^{z_i}$ as the $\rho(i)^{th}$ tensor factor.

Suppose $z_i = 1$. By construction, $B_A^{oct}\left(id_{[n+1]}, \sigma_i^\star(g)\right)$ sends $a_i$ to the $\rho(i)^{th}$ tensor factor and $a_{i+1}$ to the $(\rho(i)+1)^{th}$ tensor factor, both with label 1. Applying $B_A^{oct}\left(\sigma_{\rho^{-1}(i)}, id_{[n+1]}\right)$ gives the required commutativity.

On the other hand, suppose $z_i = t$. By construction, $B_A^{oct}\left(id_{[n+1]}, \sigma_i^\star(g)\right)$ sends $a_i$ to the $\left(\rho(i)+1\right)^{th}$ tensor factor and $a_{i+1}$ to the $\rho(i)^{th}$ tensor factor, both with label $t$. Applying $B_A^{oct}\left(\sigma_{\rho^{-1}(i)}, id_{[n+1]}\right)$ we obtain $\overline{a_{i+1}}\ \overline{a_i}$ as the $\rho(i)^{th}$ tensor factor of the lower composite. Commutativity follows upon observing that the upper composite gave

$$(a_i a_{i+1})^{z_i} = \overline{a_i a_{i+1}} = \overline{a_{i+1}}\ \overline{a_i}$$

by Remark 4.3.3. □

Lemmata B.0.3 and B.0.4 demonstrate that the assignment $B_A^{oct}$ preserves composition.

# Appendix C

# The Category of Involutive, Non-Commutative Sets

## C.1 Identity morphisms and associativity

The identity morphism in $\mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[n]\big)$ is the identity map of sets such that each preimage is labelled with $1 \in \mathbb{Z}/2\mathbb{Z}$.

Let $f_1 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[m]\big)$, $f_2 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([m],[p]\big)$ and $f_3 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([p],[q]\big)$.

Clearly, the composition of the underlying maps of sets is associative. By Definition 26.0.5, we observe that an element in the domain of a composite of two morphisms is labelled with the product of the labels of the two maps. This is associative since it is multiplication in the group $\mathbb{Z}/2\mathbb{Z}$.

In order to prove that the composition defined is associative it therefore suffices to check that the total ordering data on the preimages of singletons under

$$\big((f_3 \bullet f_2) \bullet f_1\big)$$

and

$$\big(f_3 \bullet (f_2 \bullet f_1)\big)$$

coincide.

We begin by analysing the total ordering data for the preimage of a composite of two morphisms. Suppose $l_1$ and $l_2$ are distinct elements of $(f_2 \bullet f_1)^{-1}(j)$ for $j \in [p]$. There are two possibilities:

(i) $f_1(l_1) = k_1$, $f_1(l_2) = k_2$, with $k_1 \neq k_2$, and $f_2(k_1) = f_2(k_2) = j$,

(ii) $f_1(l_1) = f_1(l_2) = k$ and $f_2(k) = j$.

Pictorially, we can consider Figures C.1 and C.2.

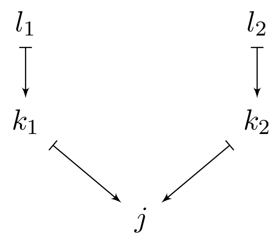



Figure C.1: Case (i)

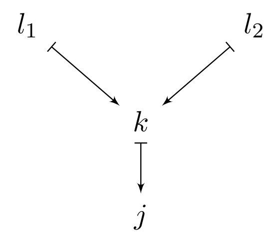



Figure C.2: Case (ii)

Suppose we are in Case (i). We observe that

$$l_1 < l_2 \text{ in } \left(f_2 \bullet f_1\right)^{-1}(j) \Leftrightarrow k_1 < k_2 \text{ in } f_2^{-1}(j).$$

Suppose we are in Case (ii). Let $\delta_k \in \mathbb{Z}/2\mathbb{Z}$ denote the label on $k$ in the preimage $f_2^{-1}(j)$. We observe that

$$l_1 < l_2 \in \left(f_2 \bullet f_1\right)^{-1}(j) \Leftrightarrow \begin{cases} l_1 < l_2 \text{ in } f_1^{-1}(k) \text{ and } \delta_k = 1 \text{ or} \\ l_2 < l_1 \text{ in } f_1^{-1}(k) \text{ and } \delta_k = t. \end{cases}$$

We can now check associativity. Let $i \in [q]$. Let $l_1$ and $l_2$ be distinct elements of

$$\left(f_3 \circ f_2 \circ f_1\right)^{-1}(i).$$

There are three possibilities:

(I) $f_1(l_1) = k_1,\ f_1(l_2) = k_2,\ f_2(k_1) = j_1,\ f_2(k_2) = j_2$ and $f_3(j_1) = f_3(j_2) = i$, where $k_1 \neq k_2$ and $j_1 \neq j_2$,

(II) $f_1(l_1) = k_1,\ f_1(l_2) = k_2,\ f_2(k_1) = f_2(k_2) = j$ and $f_3(j) = i$, where $k_1 \neq k_2$,

(III) $f_1(l_1) = f_1(l_2) = k,\ f_2(k) = j$ and $f_3(j) = i$.

Consider Case (I). Pictorially we can consider Figure C.3.

Consider $\big((f_3 \bullet f_2) \bullet f_1\big)^{-1}(i)$. We observe that

$$\begin{aligned} l_1 < l_2 \text{ in } \big((f_3 \bullet f_2) \bullet f_1\big)^{-1}(i) &\Leftrightarrow k_1 < k_2 \text{ in } \left(f_3 \bullet f_2\right)^{-1}(i) \\ &\Leftrightarrow j_1 < j_2 \text{ in } f_3^{-1}(i). \end{aligned}$$

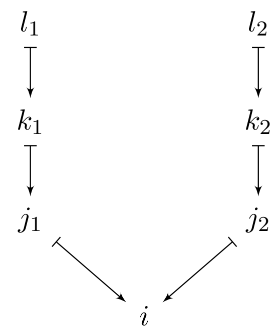



Figure C.3: Case (I)

On the other hand, we observe that

$$l_1 < l_2 \text{ in } \left(f_3 \bullet (f_2 \bullet f_1)\right)^{-1}(i) \Leftrightarrow j_1 < j_2 \text{ in } f_3^{-1}(i).$$

Hence we have associativity of composition in Case (I).

Consider Case (II). Pictorially we can consider Figure C.4.

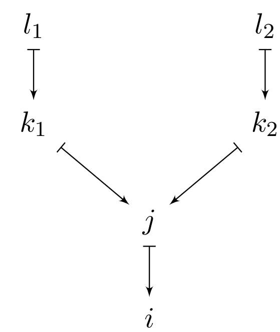



Figure C.4: Case (II)

Consider $\left((f_3 \bullet f_2) \bullet f_1\right)^{-1}(i)$. Let $\delta_j \in \mathbb{Z}/2\mathbb{Z}$ be the label on $j$ in $f_3^{-1}(i)$. We observe that

$$\begin{aligned} l_1 < l_2 \text{ in } \left((f_3 \bullet f_2) \bullet f_1\right)^{-1}(i) &\Leftrightarrow k_1 < k_2 \text{ in } (f_3 \bullet f_2)^{-1}(i) \\ &\Leftrightarrow \begin{cases} k_1 < k_2 \text{ in } f_2^{-1}(j) \text{ and } \delta_j = 1 \text{ or} \\ k_2 < k_1 \text{ in } f_2^{-1}(j) \text{ and } \delta_j = t. \end{cases} \end{aligned}$$

On the other hand,

$$\begin{aligned} l_1 < l_2 \text{ in } \left(f_3 \bullet (f_2 \bullet f_1)\right)^{-1}(i) &\Leftrightarrow \begin{cases} l_1 < l_2 \text{ in } (f_2 \bullet f_1)^{-1}(j) \text{ and } \delta_j = 1 \text{ or} \\ l_2 < l_1 \text{ in } (f_2 \bullet f_1)^{-1}(j) \text{ and } \delta_j = t \end{cases} \\ &\Leftrightarrow \begin{cases} k_1 < k_2 \text{ in } f_2^{-1}(j) \text{ and } \delta_j = 1 \text{ or} \\ k_2 < k_1 \text{ in } f_2^{-1}(j) \text{ and } \delta_j = t. \end{cases} \end{aligned}$$

Hence we have associativity of composition in Case (II).

Finally, consider Case (III). Pictorially we can consider Figure C.5.

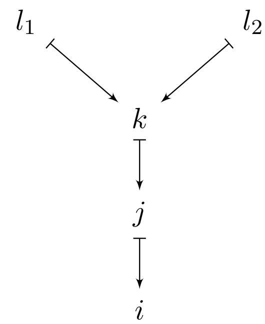



Figure C.5: Case (III)

Let $\delta_j \in \mathbb{Z}/2\mathbb{Z}$ be the label on $j$ in $f_3^{-1}(i)$ and let $\varepsilon_k \in \mathbb{Z}/2\mathbb{Z}$ be the label on $k$ in $f_2^{-1}(j)$. We observe that

$$l_1 < l_2 \text{ in } \big((f_3 \bullet f_2) \bullet f_1\big)^{-1}(i) \Leftrightarrow \begin{cases} l_1 < l_2 \text{ in } f_1^{-1}(k) \text{ and } \delta_j = \varepsilon_k \text{ or} \\ l_2 < l_1 \text{ in } f_1^{-1}(k) \text{ and } \delta_j \neq \varepsilon_k. \end{cases}$$

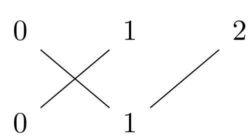
On the other hand,

$$\begin{aligned} l_1 < l_2 \text{ in } \big(f_3 \bullet (f_2 \bullet f_1)\big)^{-1}(i) &\Leftrightarrow \begin{cases} l_1 < l_2 \text{ in } (f_2 \bullet f_1)^{-1}(j) \text{ and } \delta_j = 1 \text{ or} \\ l_2 < l_1 \text{ in } (f_2 \bullet f_1)^{-1}(j) \text{ and } \delta_j = t \end{cases} \\ &\Leftrightarrow \begin{cases} l_1 < l_2 \text{ in } f_1^{-1}(k) \text{ and } \delta_j = \varepsilon_k = 1 \text{ or} \\ l_2 < l_1 \text{ in } f_1^{-1}(k) \text{ and } \delta_j = 1,\, \varepsilon_k = t \text{ or} \\ l_1 < l_2 \text{ in } f_1^{-1}(k) \text{ and } \delta_j = \varepsilon_k = t \text{ or} \\ l_2 < l_1 \text{ in } f_1^{-1}(k) \text{ and } \delta_j = t,\, \varepsilon_k = 1 \end{cases} \\ &\Leftrightarrow \begin{cases} l_1 < l_2 \text{ in } f_1^{-1}(k) \text{ and } \delta_j = \varepsilon_k \text{ or} \\ l_2 < l_1 \text{ in } f_1^{-1}(k) \text{ and } \delta_j \neq \varepsilon_k. \end{cases} \end{aligned}$$

Hence we have associativity of composition in Case (III).

Let's look at an example to illustrate associativity.

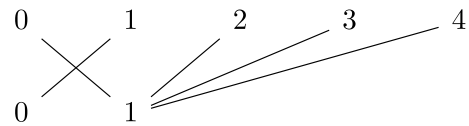
*Example C.1.1.* Let $f_1$ be the morphism defined in Example 26.0.2 and $f_2$ be the morphism defined in Example 26.0.6. Let $f_3 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([2],[1]\big)$ have underlying map of sets




with the following labelled, ordered sets as preimages:

- $f_3^{-1}(0) = \{1^t\}$ and
- $f_3^{-1}(1) = \{2^1 < 0^t\}$.

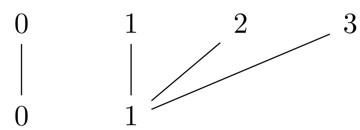
We see that the underlying map of sets $f_3 \circ f_2 \circ f_1$ is




Using the preimage data for $f_2 \bullet f_1$ calculated in Example 26.0.6 we see that

- $(f_3 \bullet (f_2 \bullet f_1))^{-1}(0) = \{1^t\}$ and
- $(f_3 \bullet (f_2 \bullet f_1))^{-1}(1) = \{2^1 < 3^t < 4^1 < 0^t\}$.

On the other hand, $f_3 \bullet f_2 \in \mathrm{Hom}_{\mathcal{IF}(as)}\big([3],[1]\big)$ has underlying map of sets




with

- $(f_3 \bullet f_2)^{-1}(0) = \{0^t\}$ and
- $(f_3 \bullet f_2)^{-1}(1) = \{3^t < 2^1 < 1^1\}$.

We then calculate that

- $((f_3 \bullet f_2) \bullet f_1)^{-1}(0) = \{1^t\}$ and
- $((f_3 \bullet f_2) \bullet f_1)^{-1}(1) = \{2^1 < 3^t < 4^1 < 0^t\}$

as required.

## C.2 Functoriality of $F$

It is clear that the assignment $F$ preserves identity morphisms.

We must demonstrate that the assignment $F$ respects composition. In order to do so we must first prove a couple of lemmata.

**Lemma C.2.1.** *Let $\varphi \in \mathrm{Hom}_\Delta\big([n],[m]\big)$ and $\psi \in \mathrm{Hom}_\Delta\big([m],[p]\big)$. There is an equality*

$$F\left(\psi \circ \varphi, id_{[n]}\right) = F\left(\psi, id_{[m]}\right) \bullet F\left(\varphi, id_{[n]}\right)$$

*in $\mathrm{Hom}_{\mathcal{IF}(as)}\big([n],[p]\big)$.*

*Proof.* On the one hand, $F\left(\psi \circ \varphi, id_{[n]}\right)$ is the underlying map of sets $\psi \circ \varphi$ and

$$\left(F\left(\psi \circ \varphi, id_{[n]}\right)\right)^{-1}(i) = (\psi \circ \varphi)^{-1}(i)$$

with the standard total ordering on elements with every label being 1.

On the other hand, $F\left(\psi, id_{[m]}\right) \bullet F\left(\varphi, id_{[n]}\right)$ has $\psi \circ \varphi$ as the underlying map of sets. Therefore, there is an equality of sets

$$\left(F\left(\psi, id_{[m]}\right) \bullet F\left(\varphi, id_{[n]}\right)\right)^{-1}(i) = (\psi \circ \varphi)^{-1}(i).$$

Furthermore, since $\varphi$ , $\psi$ and $\psi \circ \varphi$ are order-preserving maps, Definitions 26.0.5 and 27.1.1 tell us that the elements of $F\left(\psi, id_{[m]}\right) \bullet F\left(\varphi, id_{[n]}\right)$ have the standard total ordering and that every label is 1 as required. □

**Lemma C.2.2.** *Let $g$, $h \in H_{n+1}$. There is an equality*

$$F\left(id_{[n]}, gh\right) = F\left(id_{[n]}, g\right) \bullet F\left(id_{[n]}, h\right)$$

*in* $\mathrm{Hom}_{\mathcal{IF}(as)}\left([n],[n]\right)$.

*Proof.* Let $g = (z_0, \ldots, z_n; \sigma_2)$ and $h = (w_0, \ldots, w_n; \sigma_1)$, where each $w_i$ and $z_i$ is an element of $\mathbb{Z}/2\mathbb{Z}$ and $\sigma_1$, $\sigma_2 \in \Sigma_{n+1}$.

By definition, $F\left(id_{[n]}, gh\right)$ has underlying map of sets defined by

$$i \mapsto \sigma_2\sigma_1(i).$$

Furthermore, $\left(F\left(id_{[n]}, gh\right)\right)^{-1}(i) = \{\sigma_1^{-1}\sigma_2^{-1}(i)\}$ with label $z_{\sigma_2^{-1}(i)} w_{\sigma_1^{-1}\sigma_2^{-1}(i)}$.

On the other hand, $F\left(id_{[n]}, g\right) \bullet F\left(id_{[n]}, h\right)$ also has underlying map of sets defined by

$$i \mapsto \sigma_2\sigma_1(i).$$

Therefore, as sets,

$$\left(F\left(id_{[n]}, g\right) \bullet F\left(id_{[n]}, h\right)\right)^{-1}(i) = \{\sigma_1^{-1}\sigma_2^{-1}(i)\}.$$

Definition 26.0.5 tells us that the label is given by the product of the labels for

$$\left(F\left(id_{[n]}, g\right)\right)^{-1}(i)$$

and

$$\left(F\left(id_{[n]}, h\right)\right)^{-1}\left(F\left(id_{[n]}, g\right)^{-1}(i)\right).$$

These labels are $z_{\sigma_2^{-1}(i)}$ and $w_{\sigma_1^{-1}\sigma_2^{-1}(i)}$ respectively, as required. □

In order to check that $F$ respects composition, let $(\varphi, g) \in \mathrm{Hom}_{\Delta H}\big([n],[m]\big)$ and $(\psi, h) \in \mathrm{Hom}_{\Delta H}\big([m],[p]\big)$. Using the composition rule in $\Delta H$, Lemma C.2.1 and Lemma C.2.2, we observe that

$$\begin{aligned} F\big((\psi,h)\circ(\varphi,g)\big) &= F\left(\psi\circ h_\star(\varphi), \varphi^\star(h)\circ g\right) \\ &= F\Big(\psi\circ h_\star(\varphi), id_{[n]}\Big) \bullet F\Big(id_{[n]}, \varphi^\star(h)\circ g\Big) \\ &= F\Big(\psi, id_{[m]}\Big) \bullet F\Big(h_\star(\varphi), id_{[n]}\Big) \bullet F\Big(id_{[n]}, \varphi^\star(h)\Big) \bullet F\Big(id_{[n]}, g\Big). \end{aligned}$$

On the other hand, we note that

$$F(\psi,h)\bullet F(\varphi,g) = F\Big(\psi, id_{[m]}\Big) \bullet F\Big(id_{[m]}, h\Big) \bullet F\Big(\varphi, id_{[n]}\Big) \bullet F\Big(id_{[n]}, g\Big).$$

Comparing these two expressions, we observe that showing that $F$ respects composition is equivalent to showing that the diagram

$$\begin{array}{ccc} [n] & \xrightarrow{F\left(\varphi, id_{[n]}\right)} & [m] \\ {\scriptstyle F\left(id_{[n]}, \varphi^\star(h)\right)}\downarrow & & \downarrow{\scriptstyle F\left(id_{[m]}, h\right)} \\ [n] & \xrightarrow[F\left(h_\star(\varphi), id_{[n]}\right)]{} & [m] \end{array}$$

commutes in $\mathcal{IF}(as)$ for all $\varphi \in \mathrm{Hom}_{\Delta}\big([n],[m]\big)$ and all $h \in H_{m+1}$.

By the relations of Proposition 19.1.7 and Theorem 2.2.5 it suffices to check the diagrams

$$\begin{array}{ccc} [n-1] & \xrightarrow{F\left(\delta_i, id_{[n-1]}\right)} & [n] \\ {\scriptstyle F\left(id_{[n-1]}, \delta_i^\star(g)\right)}\downarrow & & \downarrow{\scriptstyle F\left(id_{[n]}, g\right)} \\ [n-1] & \xrightarrow[F\left(g_\star(\delta_i), id_{[n-1]}\right)]{} & [n] \end{array}$$

and

$$\begin{array}{ccc} [n+1] & \xrightarrow{F\left(\sigma_i, id_{[n+1]}\right)} & [n] \\ {\scriptstyle F\left(id_{[n+1]}, \sigma_i^\star(g)\right)}\downarrow & & \downarrow{\scriptstyle F\left(id_{[n]}, g\right)} \\ [n+1] & \xrightarrow[F\left(g_\star(\sigma), id_{[n+1]}\right)]{} & [n] \end{array}$$

commute in $\mathcal{IF}(as)$ for all $g \in H_{n+1}$ and $0 \leqslant i \leqslant n$.

We prove that these diagrams commute in the following two lemmata.

**Lemma C.2.3.** *The diagram*

$$\begin{array}{ccc} [n-1] & \xrightarrow{F\left(\delta_i, id_{[n-1]}\right)} & [n] \\ {\scriptstyle F\left(id_{[n-1]}, \delta_i^{\star}(g)\right)}\Big\downarrow & & \Big\downarrow{\scriptstyle F\left(id_{[n]}, g\right)} \\ [n-1] & \xrightarrow[F\left(\delta_{g(i)}, id_{[n-1]}\right)]{} & [n] \end{array}$$

*commutes in $\mathcal{IF}(as)$ for all $g \in H_{n+1}$ and $0 \leqslant i \leqslant n$.*

*Proof.* Firstly, the diagram commutes as maps of sets since the diagram

$$\begin{array}{ccc} [n-1] & \xrightarrow{\left(\delta_i, id_{[n-1]}\right)} & [n] \\ {\scriptstyle \left(id_{[n-1]}, \delta_i^{\star}(g)\right)}\Big\downarrow & & \Big\downarrow{\scriptstyle \left(id_{[n]}, g\right)} \\ [n-1] & \xrightarrow[\left(\delta_{g(i)}, id_{[n-1]}\right)]{} & [n] \end{array}$$

commutes in $\Delta H$. It therefore suffices to check that the total ordering and labelling data in $\mathcal{IF}(as)$ match for both composites.

Since each morphism in the diagram is injective, all non-empty preimages are singletons and therefore we need only check the labelling data. Since $g$ is an element of $H_{n+1}$ it is of the form $(z_0, \ldots, z_n; \sigma)$ where each $z_i \in \mathbb{Z}/2\mathbb{Z}$ and $\sigma \in \Sigma_{n+1}$.

Firstly,

$$F\left(\left(id_{[n]}, g\right) \circ \left(\delta_i, id_{[n-1]}\right)\right)^{-1}(j) = \begin{cases} \left\{\delta_i^{-1}(\sigma^{-1}(j))\right\} & \sigma^{-1}(j) \neq i \\ \emptyset & \sigma^{-1}(j) = i \end{cases}$$

and the label on $\delta_i^{-1}(\sigma^{-1}(j))$ is $z_{\sigma^{-1}(j)}$.

Since the diagram commutes as a map of sets we know that

$$F\left(\left(id_{[n]}, g\right) \circ \left(\delta_i, id_{[n-1]}\right)\right)^{-1}(j) = F\left(\left(\delta_{g(i)}, id_{[n-1]}\right) \circ \left(id_{[n-1]}, \delta_i^{\star}(g)\right)\right)^{-1}(j)$$

as a set. Furthermore, by the construction of $\delta_i^{\star}(g)$ in Definition 25.2.2 we know that the label on the preimage $F\left(\left(\delta_{g(i)}, id_{[n-1]}\right) \circ \left(id_{[n-1]}, \delta_i^{\star}(g)\right)\right)^{-1}(j)$ is $z_{\sigma^{-1}(j)}$ as required.

Hence the diagram commutes. □

**Lemma C.2.4.** *The diagram*

$$\begin{array}{ccc} [n+1] & \xrightarrow{F\left(\sigma_i, id_{[n+1]}\right)} & [n] \\ {\scriptstyle F\left(id_{[n+1]}, \sigma_i^\star(g)\right)}\downarrow & & \downarrow{\scriptstyle F\left(id_{[n]}, g\right)} \\ [n+1] & \xrightarrow[F\left(\sigma_{g(i)}, id_{[n+1]}\right)]{} & [n] \end{array}$$

*commutes in $\mathcal{IF}(as)$ for all $g \in H_{n+1}$ and $0 \leqslant i \leqslant n$.*

*Proof.* As for the previous lemma, the diagram commutes as maps of sets. In order to check that the diagram commutes in $\mathcal{IF}(as)$ it suffices to check that the total ordering and labelling data match for each composite. Since $g \in H_{n+1}$ it is of the form $(z_0, \ldots, z_n; \rho)$ where each $z_i \in \mathbb{Z}/2\mathbb{Z}$ and $\rho \in \Sigma_{n+1}$.

Observe that

$$F\left(\left(id_{[n]}, g\right) \circ \left(\sigma_i, id_{[n+1]}\right)\right)^{-1}(j) = \begin{cases} \left\{\sigma_i^{-1}(\rho^{-1}(j))\right\} & \rho^{-1}(j) \neq i \\ \{i, i+1\} & \rho^{-1}(j) = i \end{cases}$$

as sets.

If $\rho^{-1}(j) \neq i$, the preimage is a singleton and the labelling data matches for both composites by the same argument as the previous lemma.

Suppose $\rho^{-1}(j) = i$.

If $z_{\rho^{-1}(j)} = 1$ then

$$F\left(\left(id_{[n]}, g\right) \circ \left(\sigma_i, id_{[n+1]}\right)\right)^{-1}(j) = \left\{i^1 < (i+1)^1\right\}.$$

On the other hand, if $z_{\rho^{-1}(j)} = t$ then

$$F(\left(\left(id_{[n]}, g\right) \circ \left(\sigma_i, id_{[n+1]}\right)\right)^{-1}(j) = \left\{(i+1)^t < i^t\right\}.$$

By the construction of $\sigma_i^\star(g)$ in Definition 25.2.2 the same is true for

$$F\left(\left(\sigma_{g(i)}, id_{[n+1]}\right) \circ \left(id_{[n+1]}, \sigma_i^\star(g)\right)\right)^{-1}(j)$$

and the diagram commutes. ∎

Lemmata C.2.3 and C.2.4 demonstrate that the assignment $F$ preserves composition.

# Appendix D

# Comparison Map

We claim that the maps $\Phi_n$ assemble into a map of chain complexes. In order to prove this we must check that the maps $\Phi_n$ are compatible with the differentials. Since both chain complexes are formed from simplicial $k$-modules it suffices to check the compatibility with each of the face maps.

We check that the square

$$\begin{array}{ccc} \overline{NCS_n(I)} & \xrightarrow{\partial_i} & \overline{NCS_{n-1}(I)} \\ \Phi_n \downarrow & & \downarrow \Phi_{n-1} \\ NC\Gamma_n(A,k) & \xrightarrow[\partial_i]{} & NC\Gamma_{n-1}(A,k) \end{array}$$

commutes for each $n \geqslant 1$ and $0 \leqslant i \leqslant n$.

Let $\big[(f_n, \ldots, f_1), (y_1 \otimes \cdots \otimes y_x)\big]$ be a generator of $\overline{NCS_n(I)}$.

### The case $i = 0$

We see that the upper composite takes the form:

$$\begin{array}{c} \big[(f_n, \ldots, f_1), (y_1 \otimes \cdots \otimes y_x)\big] \\ \big\downarrow \partial_0 \\ \big[(f_n, \ldots, f_2), f_{1\star}(y_1 \otimes \cdots \otimes y_x)\big] \\ \big\downarrow \Phi_{n-1} \\ \big[U(f_n) \mid \cdots \mid U(f_2)\big] \otimes f_{1\star}(y_1 \otimes \cdots \otimes y_x) \otimes 1_k. \end{array}$$

The lower composite takes the form:

$$
\begin{array}{c}
\big[(f_n,\dots,f_1),(y_1\otimes\cdots\otimes y_x)\big]\\
\Big\downarrow \Phi_n\\
\big[U(f_n)\mid\cdots\mid U(f_1)\big]\otimes(y_1\otimes\cdots\otimes y_x)\otimes 1_k\\
\Big\downarrow \partial_0\\
\big[U(f_n)\mid\cdots\mid U(f_2)\big]\otimes U(f_1)_\star(y_1\otimes\cdots\otimes y_x)\otimes 1_k.
\end{array}
$$

In order for the diagram to commute we must show that $f_{1\star}(y_1\otimes\cdots\otimes y_x)$ is equal to $U(f_1)_\star(y_1\otimes\cdots\otimes y_x)$. The only difference between $f_1$ and $U(f_1)$ is the total ordering data on preimages. This preimage data is designed to index the order of multiplication for elements of an associative $k$-algebra. Since we are working in the commutative case, this data has no effect. We see this as follows.

Firstly, recall that $f_1=(\varphi,g)$ where $g\in\Sigma_x$ and $\varphi\in\mathrm{Hom}_{\mathrm{Epi}\Delta}\left(\underline{x},\underline{x_1}\right)$ is an order-preserving surjection. By Definition 24.4.3,

$$
f_{1\star}(y_1\otimes\cdots\otimes y_x)=\left(\prod^{<}_{g^{-1}(i)\in\varphi^{-1}(1)} y_i\right)\otimes\cdots\otimes\left(\prod^{<}_{g^{-1}(i)\in\varphi^{-1}(x_1)} y_i\right)
$$

where $\prod^{<}$ denotes the ordered product indexed by increasing values of $g^{-1}(i)$.

Since $I$ is commutative, the order of multiplication in the product is irrelevant. Therefore,

$$
\begin{aligned}
f_{1\star}(y_1\otimes\cdots\otimes y_x)&=\left(\prod_{g^{-1}(i)\in\varphi^{-1}(1)} y_i\right)\otimes\cdots\otimes\left(\prod_{g^{-1}(i)\in\varphi^{-1}(x_1)} y_i\right)\\
&=\left(\prod_{i\in f^{-1}(1)} y_i\right)\otimes\cdots\otimes\left(\prod_{i\in f^{-1}(x_1)} y_i\right)\\
&=U(f_1)_\star(y_1\otimes\cdots\otimes y_x).
\end{aligned}
$$

Hence the square commutes as required.

### The case $1\leqslant i\leqslant n-1$

The upper composite:

$$\begin{gathered}\big[(f_n,\dots,f_1),(y_1\otimes\cdots\otimes y_x)\big]\\ \Big\downarrow \partial_i\\ \big[(f_n,\dots,f_{i+1}\circ f_i,\dots,f_1),(y_1\otimes\cdots\otimes y_x)\big]\\ \Big\downarrow \Phi_{n-1}\\ \big[U(f_n)\mid\cdots\mid U(f_{i+1}\circ f_i)\mid\cdots\mid U(f_1)\big]\otimes(y_1\otimes\cdots\otimes y_x)\otimes 1_k.\end{gathered}$$

The lower composite:

$$\begin{gathered}\big[(f_n,\dots,f_1),(y_1\otimes\cdots\otimes y_x)\big]\\ \Big\downarrow \Phi_n\\ \big[U(f_n)\mid\cdots\mid U(f_1)\big]\otimes(y_1\otimes\cdots\otimes y_x)\otimes 1_k\\ \Big\downarrow \partial_i\\ \big[U(f_n)\mid\cdots\mid U(f_{i+1})\circ U(f_i)\mid\cdots\mid U(f_1)\big]\otimes(y_1\otimes\cdots\otimes y_x)\otimes 1_k.\end{gathered}$$

It is clear that the square commutes in this case since $U$ is a functor and so respects composition of morphisms. That is, $U(f_{i+1})\circ U(f_i)=U(f_{i+1}\circ f_i)$.

### The case $i = n$

Finally, we check $i=n$. As discussed in Section 31, the map

$$\partial_n\colon \overline{NCS_n(I)}\to\overline{NCS_{n-1}(I)}$$

is the zero map. Therefore the upper composite is zero.

The lower composite maps as follows

$$\big[(f_n,\dots,f_1),(y_1\otimes\cdots\otimes y_x)\big]\mapsto \partial_n\Big(\big[U(f_n)\mid\cdots\mid U(f_1)\big]\otimes(y_1\otimes\cdots\otimes y_x)\otimes 1_k\Big).$$

Recall the final face map from the Γ-complex. In order to ease notation, for this subsection only, we write $U(f_i)=g_i$. We see that

$$\partial_n\Big(\big[g_n\mid\cdots\mid g_1\big]\otimes(y_1\otimes\cdots\otimes y_x)\otimes 1_k\Big)=$$

$$\sum_{i\in\underline{x_{n-1}}}\left(\big[g^i_{n-1}\mid\cdots\mid g^i_1\big]\otimes(y_{i_1}\otimes\cdots\otimes y_{i_k})\otimes\left(\varepsilon\left(\prod_{j\notin g_1^{-1}\cdots g_{n-1}^{-1}(i)} y_j\right)\right)\right).$$

We claim that this sum is zero. It suffices to show that for each term in the sum the product

$$\prod_{j\notin g_1^{-1}\cdots g_{n-1}^{-1}(i)} y_j$$

in the final tensor factor is non-empty, since $\varepsilon$ is the augmentation map and each element $y_j$ lies in the kernel.

**Lemma D.0.1.** *For each $i \in \underline{x_{n-1}}$ the product*

$$\prod_{j \notin g_1^{-1} \cdots g_{n-1}^{-1}(i)} y_j$$

*is non-empty.*

*Proof.* Since each map $g_i$ is a surjection we know that $g_1^{-1} \cdots g_{n-1}^{-1}(i)$ is a non-empty subset of $\underline{x}$. Moreover, $g_1^{-1} \cdots g_{n-1}^{-1}(i)$ is equal to $\underline{x_0}$ if and only if $\underline{x_{n-1}} = \underline{1}$. However, since $g_n$ was a non-identity surjection we know that $x_{n-1} \geqslant 2$. Hence $g_1^{-1} \cdots g_{n-1}^{-1}(i)$ is a non-empty strict subset of $\underline{x}$ for each $i \in \underline{x_{n-1}}$.

It follows that the complement of each $g_1^{-1} \cdots g_{n-1}^{-1}(i)$ is a non-empty strict subset of $\underline{x}$ and so the product

$$\prod_{j \notin g_1^{-1} \cdots g_{n-1}^{-1}(i)} y_j$$

is non-empty for each $i \in \underline{x_{n-1}}$ as required. □

We deduce that the lower composite, $\partial_n \circ \Phi_n$ is zero as required.

# Appendix E

# Table of Homology Theories

Here is a table summarizing the homology theories contained in the thesis.

| Name | Notation | Algebra | Applications |
|---|---|---|---|
| Hochschild | $HH_\star$ | Associative | $A_\infty$ obstruction theory, classifying extensions of associative algebras |
| Cyclic | $HC_\star$ | Associative | $S^1$-equivariant homotopy theory, Algebraic $K$-theory |
| Dihedral | $HD_\star$ | Associative with involution | $O(2)$-equivariant homology |
| Symmetric | $HS_\star$ | Associative | Homology of infinite loop spaces |
| Hyperoctahedral | $HO_\star$ | Associative with involution | Equivariant homology of infinite loop spaces |
| Gamma | $H\Gamma_\star$ | Commutative | $E_\infty$ obstruction theory, stable homotopy theory |
| Harrison | $Harr_\star$ | Commutative | Classifying extensions of commutative algebras |
| André-Quillen | $AQ_\star$ | Commutative | Smoothness of an algebra |

# Index of Notation

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**Homology Theories**



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