The Boundary Control of the Wave Equation

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Abstract

This thesis is devoted to the solution of optimal control problems governed by linear and nonlinear wave equations and the estimation of the errors in approximating these solutions.

First, the boundary control of a linear wave equation with an integral performance criterion and fixed final states is considered. This problem is modified into the one consisting of the minimization of a linear functional over a set of positive Radon measures, the optimal measure is then approximated by a finite combination of atomic measures and so the problem is converted to a finite-dimensional linear programming problem. The solution of this problem is used to construct a piecewise-constant control.

In estimating the integral performance criterion and fixed final states from the mentioned finite-dimensional linear program, some errors occur. We have established some general results concerning these errors, and estimate them in term of the number of linear constraints appeared in the finite-dimensional linear program.

Finally, the existence and numerical estimation of the distributed control of a nonlinear wave equation with an integral performance criterion and fixed final states is considered. Again by means of the well-known process of embedding, the problem is replaced by another one in which the minimum of a linear form is sought over a subset of pairs of positive Radon measures defined by linear equalities. The minimization in the new problem is global, and it can be approximated by the solution of a finite-dimensional linear program. However, the final states in this case are only reached asymptotically, that is, as the number of constraints being considered tends to infinity.

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Chapter 1

Introduction

1.1 General background

Optimal control theory has been rapidly developing during the last 30 years, and much of the newly found theory has been used in the solution of an enormous variety of engineering, biological, and social problems. Many of the problems of design in airframe, shipbuilding, electronic, and other engineering fields are, in essence, problems of control (see [5], [25] and [27]).

To define a classical control problem, we require to describe the components of the problem, such as the differential equation satisfied by the controlled system, the space where the problem has a solution, the performance criterion and so on. Then since we will modify the classical problem, we need to put some conditions on the functions and sets which appear in control problem to allow the modifications which seem to have some advantages over the classical formulation. So to start the definitions we need, (i) a real closed time interval $I = [t_0, t_1]$, with $t_0 < t_1$, in which the controlled system will be involved, (ii) a bounded and closed subset $\mathcal{U}$ of $I\!R^m$ (set of admissible controls) that in which the control functions take values, (iii) a differential equation describing the control system, satisfied by the trajectory function $t \in I \longrightarrow Y(t) \in I\!R^n$ and control function $t \in\longrightarrow u(t) \in \mathcal{U}$, where $u(t)$ is a measurable function, and (iv) an observation function $f_0[t, Y(t), u(t)]$ which is assumed to be known. We can put further conditions on this function as necessary.

A classical optimal control problem is that of finding an admissible control $u \in \mathcal{U}$ which satisfies the differential equation describing the controlled system and minimizes the functional $J : W \longrightarrow I\!R$ defined by

$$J(p) = \Psi(f_0(t,p)), \quad p \in W \tag{1.1}$$

where $p = [Y(\cdot), u(\cdot)]$, Ψ is given and W is the set of admissible pairs of trajectories and controls.

The following problems are among the main fields of study and development of optimal control theory:

(i) Existence of optimal control.

(ii) Necessary (or possibly necessary and sufficient) conditions for u to be an optimal control.

(iii) Constructive algorithms amenable to numerical computations for the approximation of an optimal control.

Clearly the development of such theory depends on the form of differential equation describing the controlled system. A substantial literature has been developed in the field of optimal controls. It seems impossible to review here its development or even that of any its branches. Lerner [40], [41], [42], [43] and Fel'dbaum [15], [16], [17], [18] in their pioneering papers, Hestenes [30], Pontryagin et al [51], and recently Rubio [59] considered the case where the system is controlled by ordinary differential equations. Barbu [1] considered the problem for systems governed by variational inequalities. We may consider the system whose state $Y(t) \in I\!R^n$ is given by the solution of a partial differential equation combined with some appropriate boundary and initial conditions. Considerable research has been published in this area, let us mention only Butkovskiy [5], Lions [45], Wang [83], Russell [67]-[69], and Egorov [11]. In this thesis, we deal with the control theory of hyperbolic partial differential equations. These are the equations of propagation (wave) processes and their properties correspond to the properties of these processes. We will consider some control theoretic questions arising in connection with such propagating processes.

1.2 Boundary optimal control of the linear wave equation

Boundary value controls are practically motivated, in fact, in most physical systems, control is typically applied on the boundary of the controller medium. For example, in control systems regulating the temperature of material bodies and in systems containing chemical reactors the control is applied on the boundary. Boundary value control of a system causes changes in the state of the system to be propagated from the boundary into the interior of the state. The linear wave equation with boundary control is the archetype mathematical system for many control processes. Some well-known examples modeled by this kind of equation are the study of structural vibration in a distributed elastic medium, electrical transmission along a lossless line [27], and the propagation of sound in gases.

To give a short explanation in mathematical terms, assume $n \geq 1$ be a positive integer and Ω a bounded, open, connected region in $I\!R^n$ with smooth boundary $\Gamma \equiv \partial\Omega$. Define

$$L = \Delta_n Y - \frac{\partial^2 Y}{\partial t^2},$$

$$\Delta_n = \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}.$$

Consider a control system whose evolution in time is described by function $Y = Y(x,t)$, defined in $\Omega \times (0,T)$, where T is a given positive number, satisfying

$$L[Y] = 0, \quad (x,t) \in \Omega \times (0,T) \tag{1.2}$$

$$Y(x,0) = Y_0(x), \; \frac{\partial Y}{\partial t}(x,0) = Y_1(x), \; x \in \Omega \tag{1.3}$$

$$Y(\xi,t) = u(\xi,t), \; (\xi,t) \in \Gamma \times (0,T), \tag{1.4}$$

reaching a specified state at time T,

$$Y(x,T) = g_1(x),\ \frac{\partial Y}{\partial t}(x,T) = g_2(x),\ x \in \Omega, \tag{1.5}$$

and also minimizing a particular performance criterion, where $(\xi, t) \in \Gamma \times [0,T] \to u(\xi,t)$, is the control function.

This problem is considered in the work of many authors. We name only Butkovskiy [5], Lions [45], Russell [67], [68], [69], Lagnese [38], [39], and Malnowski [46]. In each normally the concern is on the exact controllability of some class of hyperbolic equations. However they differ mainly in (i) the geometry of the body Ω, for example some assumed Ω is a domain with simple geometry such as parallelepiped (see Graham [25]) or sphere (see Graham and Russell [26]), (ii) the value of terminal states $g_1(x)$ and $g_2(x)$, (iii) the determination method of terminal states, (iv) the measure for the deviation of $Y(x,T)$ from $g_1(x)$ and $Y_t(x,T)$ from $g_2(x)$.

The control function u will be termed *admissible* if it is measurable function on $[0,T]$ and

(a) $u(\xi,t) \in \mathcal{U}$, a.e. for $(\xi,t) \in \Gamma \times [0,T]$

(b) $Y(x,T) = g_1(x)$, $Y_t(x,T) = g_2(x)$, a.e. for $x \in \Omega$, $g_1(x)$ and $g_2(x)$ are fixed functions in $L_2(\Omega)$. Let $\mathcal{U}$ be the set of admissible controls. In general this set may be empty, there are many control problems without solution because the desired terminal states not can be reached by means of an admissible control. When $n = 1$, Ω is an interval in $I\!R^1$, in this case Russell [68] showed that for $Y(x,T) \equiv 0$ and $\frac{\partial Y}{\partial t}(x,T) \equiv 0$, $x \in [0,1]$, the system is controllable if $T > 2$. Herget [29] considered the case when the system is nullcontrollable, i.e., the set of reachable states are dense in $L_2(\Omega)$. For the case $n \geq 2$, Graham and Russell [26] considered the problem where Ω is the sphere

$$\Omega = \{x \in I\!R^n |\ \|x\|_e < 1\},$$

and $\Gamma \equiv \partial\Omega$ is the set

$$\Gamma = \{x \in I\!R^n |\ \|x\|_e = 1\},$$

with $\|\cdot\|_e$ denoting the Euclidean norm, and $Y(x,T) \equiv 0$, $\frac{\partial Y}{\partial t}(x,T) \equiv 0$, $x \in \Omega$. They showed that the problem is controllable if $T > \text{diam}\Omega = 2$. Russell [69] and Lagnese [39] showed that for general Ω, when $n \geq 2$, the wave equation is exactly controllable in some finite time T and the set of controllable states includes in $H^2(\Omega) \times H^1(\Omega)$, then T is unspecified if n is even, but if n is odd, exact controllability is possible in any time $T > \text{diam}\Omega$.

We may reduce the control problems (1.2)-(1.5) to moment problems. These moment problems will be studied by employing methods developed by Butkovskiy [5] and Courant & Hilbert [8].

In the following we define an optimal control problem associated with the above problem. Assume that the set of admissible controls $\mathcal{U}$ is non-empty, then the optimal control problem consists of finding a $u \in \mathcal{U}$ which minimizing the functional

$$J(u) = \int_0^T \int_\Gamma f_0[\xi, t, u(t,\xi)]d\xi dt \tag{1.6}$$

where $f_0 \in C(\sum)$, the space of continuous functions on $\sum = \Gamma \times [0,T] \times \mathcal{U}$, with the uniform topology. We modify the problem in which we seek the minimum of a functional defined on a set of positive Radon measures. We show the existence of such a minimizing measure, and show that this measure can be approximated by a piecewise constant control.

The main approach that is used here is based on an idea of Young [86], consisting of the replacement of classical variational problems by problems in measure spaces. The early and very principle version of this approach was carried out by Ghouila-Houri [23]. This method was employed for the first time by Wilson and Rubio [85] on an optimal control problem with a diffusion equation, then by some others, for example, Rubio and Wilson [64], Rubio [58] and Kamyad et al [34].

1.3 Optimal control for a nonlinear wave equation

Let $n \geq 1$ be a positive integer and Ω a bounded, open, connected region in $I\!R^n$, with smooth boundary Γ and T a positive real number, define $Q_T := \Omega \times (0,T)$ a bounded cylinder of height T, Γ_T the lateral surface of the cylinder Q_T, D_0 and D_T respectively the base and the top of it. Define

$$L[Y] = \frac{\partial^2 Y}{\partial^2 t} - \triangle_n Y + |Y|^\rho Y,$$

for $\rho > 0$. Now consider a control system whose evolution in time is described by the real function $Y = Y(x,t)$, defined in Q_T and satisfying

$$L[Y] = v, \quad (x,t) \in Q_T \tag{1.7}$$

where $v \in L_2(Q_T)$, with the boundary condition

$$Y = 0, \quad \text{on } \Gamma_T \tag{1.8}$$

and the initial conditions

$$\begin{aligned} Y(x,0) &= Y_0(x), \quad x \in D_0 \\ Y_t(x,0) &= Y_1(x), \quad x \in D_0 \end{aligned} \tag{1.9}$$

while the continuous functions Y_0 and Y_1 are given, and achieves specified states

$$\begin{aligned} Y(x,T) &= g_1(x), \quad x \in D_T \\ Y_t(x,T) &= g_2(x), \quad x \in D_T \end{aligned} \tag{1.10}$$

where continuous functions $g_1(x)$ and $g_2(x)$ are given.

The function Y in (1.7) represents some field quantity and the inhomogeneous term v represents sources energy in the field. Possible interactions of the field with itself are described by the non-linear term $|Y|^\rho Y$. We call $(x,t) \in Q_T \to v(x,t) \in V \subset I\!R$ the control function.

Lions [44] showed that the problem (1.7)-(1.9) at each t has a solution in the Sobolev spaces $V_1 = H_0^1(\Omega) \cap L^P(\Omega)$, where $P = \rho + 2$. He also established uniqueness for the case $\rho = 2$ and $n = 3$. In the case $\Omega = I\!R^n$ we ignore the boundary condition (1.8) and have the pure initial values, or Cauchy problem. For $n = 3$, Jörgens [32] has obtained a classical solution of the Cauchy problem for certain class of equations. Sather [71] in his pioneering paper considered existence of the weak solution of the problem (1.7)-(1.9) in generalized sense. Segal [74] and Schiff [72] also considered the existence of classical and weak solutions for the problem (1.7)-(1.9) when n and ρ have some fixed values.

A pair (Y, v) of trajectory function Y and control function v is said to be *admissible* if:

(1) The function $(x,t) \to Y(x,t)$ is a solution of (1.7)-(1.9).

(2) The terminal relationships

$$Y(x,T) = g_1(x), \quad x \in D_T$$

$$Y_t(x,T) = g_2(x), \quad x \in D_T$$

are satisfied.

Let $\mathcal{F}$ be the set of all admissible pairs which assumed to be non-empty. The optimal control problem associated with this control system consists of finding an admissible pair (Y, v) which minimizes the functional

$$J = \int_{Q_T} f_0(Y(x,t), x, t)dxdt + \int_{Q_T} f_1(v(x,t), x, t)dxdt \tag{1.11}$$

where f_0, f_1 are continuous, non-negative, real-valued functions on $I\!R^{n+2}$ with specified properties.

We first write the integral relationships satisfied by the solution of the problem (1.7)-(1.10) (see Mikhailov [48]), then transform the problem, instead of minimizing over a set of admissible trajectory-control pairs, we minimize it over a subset of a product of two measure spaces. In this way we change the problem to a linear form and so we benefit from the whole paraphernalia of linear analysis. We show that there is always a minimizer for our measure theoretical problem. This minimizer is *global*, i.e., the value reached is close to the global minimum of the problem. Next we obtain the approximation value of the optimal pair Radon measure that enable us to construct a piecewise constant control $v(x,t)$ corresponding to the desired final states $Y(x,T)$ and $Y_t(x,T)$.

Like the linear case, the final states are reached only asymptotically, that is, as the number of constraints associated with the measure-theoretical problem tends to infinity, $Y(x,T)$ and $Y_t(x,T)$ respectively in $L_2(Q_T)$, tend to $g_1(x)$ and $g_2(x)$.
The first time this idea was employed by Rubio [58] who considered nonlinear optimal control problems in Hilbert spaces, Rubio and Holden [63] considered the idea in the control problem of a nonlinear diffusion equation with a small nonlinearity. Recently, many researches have been carried out for nonlinear optimal control problems. Rubio in [61]-[62] considered the case for nonlinear diffusion equations and in [60] for the nonlinear elliptic equations, where the control function were assumed to be in the boundary. Farahi [12] and then Farahi, Rubio and Wilson [14] considered the optimal control problem for nonlinear wave equations, while they used distributed control for their purposes. Nevertheless, much more works still are needed to do (see Section 4.8). We should mention also that independently of these works there has been much research on dual methods, especially by the Lipzig school, see [35], [36], [65], and [66].

1.4 Outline of the thesis

In this thesis, we will only discuss existence and optimality conditions of optimal controls governed by linear and nonlinear wave equations. For other respects such as necessary and sufficient conditions for controllability, stability of numerical methods and applications of optimal control problems, one can refer to references listed in the previous Sections 1.1-1.3.
In Section 2.1 we consider the existence of an optimal control for the one-dimensional wave equation with the same initial, boundary and final conditions as in Section 1.2. We denote the set of all admissible controls by $\mathcal{U}$. Our optimal control problem consists of finding a $u(\cdot) \in \mathcal{U}$ which minimizes the functional

$$J[u(\cdot)] = \int_0^T f_0[t, u(t)]dt,$$

where $f_0 \in C([0,T] \times \mathcal{U})$. This problem may or may not have a solution in $\mathcal{U}$. We then replace the problem by another one in which the minimum of a linear functional is considered over a set of positive Radon measures on Ω. Then we consider a linear program for determining an approximation to the optimal control. In particular we obtain an approximation to the optimal Radon measure μ^*, which is introduced in this chapter. By changing the problem to a finite-dimensional linear program, we show a practical way to obtain an approximation to the measure μ^*. Also we obtain the approximate value of the optimal control corresponding to the final desired functions. Further in Section 2.2 we extend the purposes of Section 2.1 to n-dimensions, in this section we consider the existence of a control function for the n-dimensional linear wave equations with the same initial, boundary and final conditions respectively as (1.3), (1.4), (1.5). Then we want to find an admissible control u which minimizes a functional such as

$$J[u(\cdot,\cdot)] = \int_0^T \int_{\partial\omega} f_0[\xi, t, u(t,\xi)]d\xi dt,$$

where $f_0 \in C(\sum)$, the space of continuous functions on $\sum = \Gamma \times [0,T] \times \mathcal{U}$. Like Section 2.1, we again modify the control problem to a linear programming problem for determining an approximation to the optimal control, then we construct a numerical algorithm to find the approximate value of the optimal control. In each section some example will be given to specify the procedure. The paper of Farahi, Rubio and Wilson [13] is covered by a part of this chapter.

In Chapter 3, we establish some general results for approximating our optimal control problem by a linear programming problem. We will assume the control set $\mathcal{U}$ is non-empty. By procedure described in Chapter 2, we change the control problem (1.2)-(1.6) to a problem of minimizing the linear functional

$$\mu \longrightarrow \mu(f_0) \tag{1.12}$$

over the set Q of positive Radon measures on Ω which satisfy the equalities

$$\begin{aligned} \mu(\psi_n) &= \alpha_n, \qquad n = 1,2,... \\ \mu(G_r) &= a_{G_r}, \qquad r = 0,1,... \end{aligned} \tag{1.13}$$

where ψ_n's and G_r's are known functions and α_n's and a_{G_r}'s are known scalars. This problem is one of linear programming, all the functions in (1.12) and (1.13) are linear in the variable μ. This linear programming problem consists of minimizing the function (1.12) on the subset Q of positive Radon measures described by the equalities (1.13). Since the underlying space is not finite-dimensional and the number of equalities (1.13) is not finite, this problem is an infinite-dimensional linear programming problem. Proposition 2.1 in Chapter 2 shows that this problem has a minimizing solution, μ^*, say , in Q. We develop an intermediate program, semi-infinite, by considering the minimization of $\mu \longrightarrow \mu(f_0)$ not over the set Q but over a subset of Q defined by requiring that only a finite number of constraints in (1.13) be satisfied, $Q(M_1, M_2)$. Now we approximate the solution of (1.12)-(1.13) by the solution of the following problem:

Minimize

$$\mu \longrightarrow \mu(f_0) \tag{1.14}$$

over the subset $Q(M_1, M_2)$ of positive Radon measures on Ω which only satisfy the equalities

$$\begin{aligned} \mu(\psi_n) &= \alpha_n, \qquad n = 1, 2, ..., M_1, \\ \mu(G_r) &= a_{G_r}, \qquad r = 0, 1, ..., M_2, \end{aligned} \tag{1.15}$$

where M_1 and M_2 are two positive integers, in some cases one of them may be infinity. The solution of this final problem for given functions ψ_n's and G_r's depends on the numbers M_1 and M_2 appearing in the linear program; we have not made this fact explicit in the notation, since we believe that it is a very difficult problem. We shall call the solution of the new semi-infinite linear programming problem $\mu^*_{Q(M_1,M_2)}(f_0)$, the results of Proposition 3.1 in this chapter tell us that if we increase these numbers M_1 and M_2, then $\mu^*_{Q(M_1,M_2)}(f_0)$ approaches the $\mu^*(f_0)$. First in Section 3.2 we discuss about

$$\max |Y_{M_1}(x, T) - g_1(x)|$$

where $Y_{M_1}(x, T)$ indicates the solution of the system (1.14)-(1.15) at the time $t = T$ when only M_1 in (1.15) is finite and $g_1(x)$ is the given final state continuous function. In this chapter we will focus then on the estimation of the maximum value of

$$|\mu^*(f_0) - \mu^*_{Q(M_1,M_2)}(f_0)| \tag{1.16}$$

where M_1 and M_2 are two positive integers. In Section 3.3 we estimate the maximum value of (1.16) when only M_2 is finite. In the numerical examples of Chapter 2, we have established the solutions of some control problems in which we have chosen G_r's as semi-continuous functions defined on $[0, T]$. It has been shown in Rubio [59] that linear combinations of these functions can approximate arbitrarily well any function in $C_1(I \times \mathcal{U})$, the class of all continuous functions depending only on t. By using these pulse functions, as shown in Section 3.4, and from (1.14)-(1.15) we can find the approximate value of $\mu^*(f_0)$ with fewer number of M_2 than the number used in Section 3.3. Finally in Section 3.5, we construct a method to estimate the maximum value of (1.16) when both M_1 and M_2 are finite, in this construction one may establish an upper-bound as well as a lower-bound for $\mu^*(f_0)$, the solution of the linear system (1.12)-(1.13), then the difference of these upper-lower bounds gives the missing error in the computation of $\mu^*(f_0)$. We show that the sequence

$$\left\{\mu^*_{Q(M_1,M_2)}(f_0),\ M_1 = 1, 2, ...,\ M_2 = 1, 2, ...\right\}$$

is a nondecreasing convergent sequence bounded above by the value $\mu^*(f_0)$, so by choosing fixed numbers M_1 and M_2 we will find a lower-bound for $\mu^*(f_0)$. Unfortunately we could not provide any analysis leading to an accurate upper-bound for $\mu^*(f_0)$, but by using the dual of the linear program (1.14)-(1.15) and an iterative method we can estimate an upper-bound for $\mu^*(f_0)$ and so estimate the maximum value of (1.16). Although many useful concepts and results have been proved in this chapter, there are still few unanswered questions related to these concepts and results (see Section 4.8).

In Chapter 4 we study the existence of an optimal distributed control for the n-dimensional nonlinear wave equation (1.7) with boundary condition as (1.8) and initial conditions as (1.9) with exception $Y(x, 0) \equiv 0$, $x \in D_0$, and with terminal conditions as (1.10). Our optimal control problem consists of finding an admissible control v which minimizes a functional like (1.11). This chapter is in fact an extension of Rubio [62] from nonlinear diffusion equations to nonlinear wave equations. Firstly, we define certain spaces which are to be used in construction of a framework for the treatment of nonlinear wave equations and consider the sufficient conditions for existence solution of equation (1.7). Next we write some integral relationships satisfied by the solution of this equation and then proceed to transform the problem to a linear case, in this metamorphosis we replace the problem by a new one in which the minimum of the functional (1.11) is sought over a set of product of two measure spaces. The new formulation has some advantages: there is an automatic existence theory (Proposition 4.2), and the minimization in it is *global*, that is, the value reached from the new problem is close to the global infimum of the problem, but as well as in the case of the linear wave equations considered in Chapter 2, the final states are reached only asymptotically (Theorem 4.2), that is as the number of (linear) constraints in the new problem tends to infinity. Now as in Chapter 2 we face a practical way to obtain an approximation to the optimal pair Radon measures (μ^*, ν^*) which are introduced in this chapter. Finally we compute the approximate optimal control corresponding to the final desired states (1.10). The theory is confirmed by proving several propositions and a theorem and also by computing the nearly optimal control and the desired final state for one example and the nearly optimal control for the another one. A part of research contained in this chapter appeared in Farahi [12] and further more in Farahi, Rubio and Wilson [14].

Chapter 2

The Optimal Control of the Linear Wave Equation

2.1 An optimal control problem for one-dimensional wave equation

2.1.1 Introduction

In this section we consider a control system whose evolution in time is described by the function $(x,t) \longrightarrow Y(x,t)$, defined in $(0,S) \times (0,T)$, where T and S are positive numbers, satisfying the wave-equation

$$Y_{tt}(x,t) = a^2 Y_{xx}(x,t), \tag{2.1}$$

where a is the velocity of wave propagation in the given medium, and $Y(x,t)$ describes the variation of the oscillation at point x and time t. The initial conditions are:

$$\begin{aligned} Y(x,0) &= f(x) \\ Y_t(x,0) &= h(x), \end{aligned} \tag{2.2}$$

the boundary conditions:

$$Y(0,t) = u(t)$$
$$Y(S,t) = u(t), \tag{2.3}$$

where $t \in [0,T] \longrightarrow u(t) \in \mathbb{R}$ is the control function. We define the control u to be *admissible* if it is Lebesgue measurable function on $[0,T]$ and

(a) $u(t) \in [-1,1]$ a.e. for $t \in [0,T]$.

(b) The solution of the partial differential equation (2.1) corresponding to the initial conditions (2.2) and the boundary conditions (2.3) satisfies the terminal conditions

$$Y(x,T) = g_1(x)$$
$$Y_t(x,T) = g_2(x), \tag{2.4}$$

where $g_1(x),\ g_2(x) \in L_2(0,S)$.

Let $\mathcal{U}$ be the set of admissible controls. In general this set may be empty. Problems of this type have been considered by, for example, [67] and [46] (in which the control function was assumed to be differentiable) and by, for example, [5].

We may reduce the above control problem to a moment problem. This moment problem will be studied by employing methods developed by [5], [28], and [79]. We define in the following an optimal control problem associated with the above problem. Let $\mathcal{U}$ be a non-empty set, and let the optimal control problem consists of finding an admissible control u which minimizes the functional

$$I[u(\cdot)] = \int_0^T f_0[t,u(t)]dt, \tag{2.5}$$

where $f_0 \in C(\Omega)$, the space of continuous functions on $\Omega=[0,T] \times [-1,1]$, with the topology of uniform convergence.

This control problem may or may not have a solution in $\mathcal{U}$. In the following we replace the problem by another one in which the minimum of a linear functional is considered over a set of Radon measures on $\Omega=[0,T] \times [-1,1]$.

2.1.2 Modified control problem

We consider the solution of (2.1)-(2.3) in the sense of Butkovskiy [5]

$$Y(x,t) = Y_1(x,t) + Y_2(x,t),$$

where

$$Y_1(x,t) = \sum_{n=1}^{\infty}(A_n \cos\frac{\pi nat}{S} + B_n \sin\frac{\pi nat}{S}) \sin\frac{\pi nx}{S},$$
$$A_n = \frac{2}{S}\int_0^S f(\xi)\sin\left(\frac{\pi n\xi}{S}\right) d\xi,$$
$$B_n = \frac{2}{Sna}\int_0^S h(\xi)\sin\left(\frac{\pi n\xi}{S}\right) d\xi,$$

and

$$Y_2(x,t) = \int_0^t K(x,t-\tau)u(\tau)d\tau,$$

where

$$K(x,t) = \sum_{n=1}^{\infty} c_n \sin\frac{\pi nx}{S} \sin\frac{\pi nat}{S},$$

and $c_n = \frac{2a}{S}[1-(-1)^n]$. When the control function is applied at one side of the boundary, then $c_n = \frac{4a}{S}$.

Since $g_1(x),\ g_2(x) \in L_2(0,S)$, they possess a half-range Fourier series

$$g_1(x) = \sum_{n=1}^{\infty} a_n \sin\frac{\pi nax}{S}$$
$$g_2(x) = \sum_{n=1}^{\infty} b_n \sin\frac{\pi nax}{S}.$$

Hence, by assuming $a = 1$, $S = \pi$ the optimal control problem reduces to finding a measurable control function $u(t) \in [-1, 1]$ for $t \in [0, T]$ which satisfies

$$\int_0^T \psi_n(t, u(t))\, dt = \alpha_n,$$

where

$$\begin{aligned} \psi_n(t, u(t)) &= \frac{2}{\pi}[1 - (-1)^n](\sin(nt) + \cos(nt))\, u(t), \\ \alpha_n &= A_n - B_n + a_n(\sin(nT) - \cos(nT)) \\ &\quad + \frac{b_n}{n}(\sin(nT) + \cos(nT)), \quad n = 1, 2, \ldots \end{aligned} \tag{2.6}$$

and minimizes the functional (2.5). In general, a minimizing solution to this problem may not exist; in the following we replace the problem by another one in which the minimum of a linear functional is sought over a set of Radon measures on $\Omega = [0, T] \times [-1, 1]$. In fact we proceed to enlarge the set $\mathcal{U}$, we proceed as follows:

(1) For a fixed control function $u(\cdot)$ the mapping

$$\Lambda_F : F \longrightarrow \int_0^T F[t, u(t)]dt, \qquad F \in C(\Omega)$$

defines a positive linear functional on $C(\Omega)$.

(2) There exists a unique positive Radon representing measure μ on Ω such that

$$\int_0^T F[t, u(t)]dt = \int_\Omega F d\mu \equiv \mu(F), \qquad F \in C(\Omega). \tag{2.7}$$

In particular the above equality is valid for $F = f_0$. Now we replace the original minimization problem by one in which we will find the minimum of $\mu(f_0)$ over a set Q of positive Radon measures on Ω. These measures are required to have certain properties which are abstracted from the definition of admissible controls. First from (2.7),

$$|\mu(F)| \leq T \sup_\Omega |F(t, u)|;$$

hence

$$\mu(1) \leq T.$$

Next, the measures in Q must satisfy an abstracted version of equation (2.6),

$$\mu(\psi_n) = \alpha_n, \qquad n = 1, 2, \ldots.$$

Note that this is possible since $\psi_n \in C(\Omega)$. Finally, suppose that $G \in C(\Omega)$ does not depend on u, that is,

$$G(t, u_1) = G(t, u_2),$$

for all $t \in [0, T]$, u_1, $u_2 \in [-1, 1]$.Then measures in Q must satisfy

$$\int_\Omega G d\mu = \int_0^T G[t, u] dt = a_G,$$

where u is an arbitrary number in the set $[-1, 1]$, and a_G is the Lebesgue integral of $G(\cdot, u)$ over $[0, T]$, independent of u. This property of Q will be used in the next section, when we use a theorem due to Ghouila-Houri [23]. Let $\mathcal{M}^+(\Omega)$ be the set of all positive Radon measures on Ω. Then the set Q is defined as a subset of $\mathcal{M}^+(\Omega)$

$$Q = S_1 \cap S_2 \cap S_3$$

where

$$S_1 = \left\{\mu \in \mathcal{M}^+(\Omega) : \mu(1) \leq T\right\},$$
$$S_2 = \left\{\mu \in \mathcal{M}^+(\Omega) : \mu(\psi_n) = \alpha_n,\ n = 1, 2, \ldots\right\},$$
$$S_3 = \left\{\mu \in \mathcal{M}^+(\Omega) : \mu(G) = a_G, G \in C(\Omega), \text{independent of } u\right\}.$$

Now if we topologize the space $\mathcal{M}(\Omega)$ by the weak*-topology, it can be seen that S_1 is compact in $\mathcal{M}(\Omega)$ with respect to weak*-topology (see [33] page 26, [59] Chapter 2). The set S_2 can be written as

$$S_2 = \bigcap_{n=1}^{\infty} M_n$$

where $M_n = \{\mu \in \mathcal{M}^+(\Omega) : \ \mu(\psi_n) = \alpha_n\}$ is closed because it is the inverse image of a closed set on the real line (the set $\{\alpha_n\}$), under a continuous map $\mu \longrightarrow \mu(\psi_n) \in I\!R$

(see [59]). Since the infinite intersection of closed sets is closed, so S_2 is closed. By a similar argument S_3 is closed. Therefore Q is a closed subset of the compact set S_1, and hence is compact. The functional $I : Q \longrightarrow I\!R$, defined by

$$I(\mu) = \int_\Omega f_0 d\mu = \mu(f_0) \in I\!R, \qquad \mu \in Q, \tag{2.8}$$

is a linear continuous functional on a compact set Q, therefore attains its minimum at least in one extreme point on Q. We have shown, thus, the purpose of the following proposition:

Proposition 2.1 The measure- theoretical problem, which consists of finding the minimum of the functional (2.8) over the subset Q of $\mathcal{M}^+(\Omega)$, possesses a minimizing solution μ^*, say, in Q.

2.1.3 Approximation of the optimal control by a piecewise-constant control

It can be shown that the action of the optimal measure might be approximated by that of a piecewise constant control. With each piecewise constant admissible control $u(\cdot)$ we may associate a measure μ_u in $\mathcal{M}^+(\Omega) \cap S_1 \cap S_2 \cap S_3$. Let Q_1 be the set of all such measures μ_u; then Theorem 1 of Ghouila-Houri [23] shows that, when $\mathcal{M}(\Omega)$ is given the weak*-topology, Q_1 is dense in $\mathcal{M}^+(\Omega) \cap S_1 \cap S_2 \cap S_3$. A basis of closed neighborhoods in this topology is given by the sets of the form

$$\{\mu : |\mu(F_n)| \leq \epsilon, n = 1, 2, ...k + 1\}, \qquad (\epsilon > 0)$$

where k is a positive integer, $F_n \in C(\Omega)$. It is therefore possible to find a measure μ_u corresponding to a piecewise constant control in any weak*-neighborhood of μ^*. In particular we can put

$$F_1 = f_0, F_2 = \psi_1, ...F_{k+1} = \psi_k;$$

and a piecewise control $u_k(\cdot)$ can be found such that

$$|\int_0^T f_0[t, u_k(t)]dt - \mu^*(f_0)| \leq \epsilon,$$

$$|\int_0^T \psi_n[t, u_k(t)]dt - \alpha_n| \leq \epsilon, \qquad n = 1, 2, ..., k.$$

Therefore, by using the piecewise constant control $u_k(\cdot)$, we can get within ϵ of the minimum value $\mu^*(f_0)$.

The question of whether the final state $g_1(\cdot)$ in (2.4) is (approximately) attained is a difficult problem for this equation, because, we believe, the lack of a damping term. The only general result that can be achieved depends for its proof on a property of compact sets, of which we proceed to remind the reader. We make the following definition (see Royden [56], Choquet [6]):

The set S is said to be compact if it has the *finite intersection property*; that is, let F_α be any collection of closed sets in S such that any finite number of them has a nonempty intersection; then the total intersection $\bigcap_\alpha F_\alpha$ is nonempty.

Lemma. 2.1 If S is compact, then any sequence in S has a convergent subsequence.

Proof. Let $X = \{x_n,\ n = 1, 2, ...\}$ be a sequence in S and $r_1 < r_2 < ... < r_n < ...$ a strictly increasing sequence of natural numbers; then the sequence X' in S given by $X' = \{x_{r_n},\ n = 1, 2, ...\}$ is a subsequence of X. Let F_{r_n} be the set $\left\{x_{r_n}, x_{r_{n+1}}, ...\right\}$, then $\left\{\bar{F_{r_n}}\right\}$ is a collection of closed sets with the finite intersection property, and so there is a point x which belongs to $\bigcap \bar{F_{r_n}}$. The point x is the limit point of the subsequence X', since for any open set O containing x we have $x \in \bar{F_{r_n}}$, and so there must be an N such that $x_{r_n} \in O$ with $n \geq N$. □

Now we can prove the following proposition.

Proposition 2.2 Consider our general problem P:

$$\min \mu(f_0)$$

subject to the conditions

$$\mu(\psi_j) = \alpha_j,\ j = 1, 2, ... \tag{2.9}$$

where $\mu \in Q$.

Let μ_k be the solution in $\mathcal{M}^+(\Omega)$ of the following problem, to be called P_k:

$$\min \mu(f_0)$$

subject to the conditions

$$\mu(\psi_j) = \alpha_j,\ j = 1, 2, ..., k \tag{2.10}$$

where $\mu \in S_1$.

Then the sequence $\{\mu_k\}$ has a convergent subsequence, which converges weakly* to $\mu^* \in \mathcal{M}^+(\Omega)$. Further:

(a)

$$\begin{aligned} &\mu_k(\psi_j) \longrightarrow \mu(\psi_j) = \alpha_j,\ j = 1, 2, ... \\ &\mu_k(f_0) \longrightarrow \mu(f_0). \end{aligned} \tag{2.11}$$

(b)

$$\inf_{P_k} \mu(f_0) = (\mu^*, f_0) \geq \inf_{P} \mu(f_0). \tag{2.12}$$

(c) Each μ_k can be considered to be an admissible control u_k. Then any control in the convergent subsequence will give a final state close to $g_1(\cdot)$ for sufficiently large values of the index k in the index set of the subsequence.

Proof. (a) Since S_1 is compact, we can extract a subsequence so that

$$\mu_k \longrightarrow \mu^* \ \text{weakly}^*$$

and (2.11) are satisfied.

(b) Assume $Q(k)$ be the set of measures in $\mathcal{M}^+(\Omega)$ satisfying (2.10), so $\mathcal{M}^+(\Omega) \supset Q(k) \supset Q$. Then the solution of the problem (2.10) (the proof of the existence of solution of the problem (2.10) is being based on the same considerations on Proposition 2.1), satisfies the following inequality

$$\inf_{P_k} \mu(f_0) = \inf_{Q(k)} \mu(f_0) = (\mu^*, f_0) \geq \inf_{P} \mu(f_0) = \inf_{Q} \mu(f_0). \tag{2.13}$$

(c) We can find a control u_k that approximates well the measure μ_k in P_k. Then this can be modified slightly so that the conditions for P_k are exact. □

In general, this result is as far as we can go searching for an admissible control that gives rise to a final state near $g_1(\cdot)$. But in the many cases studied in the rest of the chapter, we have found that the desired final states can be approximated very well by only small values of k; and of course no attention has been paid whether this index is, or is not, in the index set of convergent subsequence.

2.1.4 Approximation to the optimal measure

We now develop a method for the estimation of a nearly-optimal piecewise constant control. In this development we follow Kamyad, Rubio, and Wilson [34]. In the first step, we obtain an approximation to the optimal measure μ^* by constructing a sequence of finite-dimensional approximation to the infinite-dimensional linear programming problem defined by (2.8) and with restrictions defined by S_1, S_2 and S_3. Next, we construct a piecewise-constant control function corresponding to the finite-dimensional problem. The infinite-dimensional linear programming problem can be written in the following form:

Minimize

$$I(\mu) = \mu(f_0)$$

subject to

$$\begin{aligned} &\mu(1) \le T \\ &\mu(\psi_n) = \alpha_n, \qquad n = 1, 2, ... \\ &\mu(G) = a_G, \qquad G \in C(\Omega), \text{independent of } u. \end{aligned} \tag{2.14}$$

The functions ψ_n, α_n, $n = 1, 2, ...$, are defined by (2.6). We define $G_i(\cdot, u) = t^i$ as a monomial in t only, then from Weierstrass Approximation Theorem (see, e.g., [2]) any continuous function on $[0, T]$, can be uniformly approximated by a finite linear combinations of elements of the set

$$\left\{G_i(\cdot, u) = t^i \,, i = 1, 2, ...\right\}.$$

Of course, the measures μ in (2.14) are required to be positive Radon measures on Ω. Now, by using monomials G_i, the minimization problem (2.14) changes to the following one:

Minimize

$$I(\mu) = \mu(f_0)$$

subject to

$$\begin{aligned} &\mu(1) \le T \\ &\mu(\psi_n) = \alpha_n, \qquad n = 1, 2, ... \\ &\mu(G_i) = a_{G_i}, \qquad i = 1, 2, ... \end{aligned} \tag{2.15}$$

where a_{G_i} is the Lebesgue integral of $G_i(\cdot, u)$ over $[0, T]$.

The approximation of this program by a sequence of finite-dimensional programs is based on the following constructions:

(a) Only a finite number M_1 of functions ψ_n will be considered; this number can be as large as required.

(b) Only a finite number M_2 of functions G_i in (2.15) will be considered. We assume

$Q(M_1, M_2)$ be the set of measures in $\mathcal{M}^+(\Omega)$ satisfying

$$\begin{aligned} &\mu(1) \le T \\ &\mu(\psi_n) = \alpha_n, \qquad n = 1, 2, ..., M_1 \\ &\mu(G_i) = a_{G_i}, \qquad i = 1, 2, ..., M_2, \end{aligned} \tag{2.16}$$

we will show in Proposition 3.1, that if $\mu^*_{Q(M_1,M_2)}(f_0)$, is the minimum of $\mu \longrightarrow \mu(f_0)$ over the set $Q(M_1, M_2)$, then

$$\lim_{M_1, M_2 \to \infty} \mu^*_{Q(M_1,M_2)}(f_0) = \inf \mu_Q(f_0) = \mu^*(f_0).$$

(c) The set $\Omega = [0, T] \times [-1, 1]$ will be covered with a grid, by taking $m_1 + 1$ and $m_2 + 1$ points along the t-axis and u-axis respectively, these points will be equidistant, at a distance $\frac{T}{m_1}$ and $\frac{2}{m_2}$ each separately in the order mentioned. Now Ω is divided to $N = m_1 \cdot m_2$ equal volume rectangles Ω_j, $j = 1, 2, ..., N$; we choose Z_j as the center Ω_j. Since for many applications it is convenient to approximate continuous functions by functions of an elementary nature, we shall not take functions $G_i(\cdot, u) = t^i$, $i = 1, 2, ..., M_2$, in (b), but lower-semicontinuous functions, we choose these functions to be the characteristic functions of the individual rectangles formed by dividing the set $\Omega = [0, T] \times [-1, 1]$ to m_1 partition along t-axis. In fact these lower-semicontinuous step functions are defined as:

$$f_i(t, u) = \begin{cases} 1 & t \in J_i \\ 0 & \text{otherwise} \end{cases}$$

where $J_i = [(i-1)\frac{T}{m_1}, i\frac{T}{m_1})$, $i = 1, 2, ..., m_1$, so any continuous function on $[0, T]$ can be uniformly approximated by linear combinations of these functions (see Bartle [2], Theorem 24.4). Of course, there will be then as many functions as rectangles, that is,

$$M_2 = m_1.$$

Now, instead of the infinite-dimensional linear programming problem (2.15), we shall consider a finite-dimensional one in which the measures are positive Radon measures on Ω supported by the grid defined above; each such measure is defined by a set of non-negative real numbers a_j, $j = 1, 2, ..., N$, where $N = m_1 \cdot m_2$ is the number of points in the grid. Since one non-negative extra slack variable a_{N+1} is necessary to handle the first inequality in (2.15), (see Gass [22], chapter 4), thus the linear programming problem consists of minimizing the linear form

$$\sum_{j=1}^{N+1} a_j f_0(Z_j)$$

over the set $a_j \geq 0$, subject to

$$\begin{aligned}
&\sum_{j=1}^{N+1} a_j = T \\
&\sum_{j=1}^{N+1} a_j \psi_n(Z_j) = \alpha_n, \qquad n = 1, 2, ..., M_1 \\
&\sum_{j=1}^{N+1} a_j f_i(Z_j) = a_{f_i}, \qquad i = 1, 2, ..., M_2.
\end{aligned} \tag{2.17}$$

It is well known (see, e. g., [22], Chapter 3) that the linear form $\sum_{j=1}^{N+1} a_j f_0(Z_j)$ attains its minimum at an extreme point of the set in $I\!R^{N+1}$ defined by the constraints (2.17) and the requirement that $a_j \geq 0$, $j = 1, 2, ..., N+1$, and that such an extreme point has at most $M = M_1 + M_2 + 1$ nonzero coefficients; this is fortunate, because we usually choose N much larger than that M. In Chapter 3, we show that, if we take M_1, M_2 sufficiently large, then the solution of the linear programming problem (2.17) gives a good approximation to the solution of (2.14), i.e., the infima of the two problems are as close as desired.

The procedure to construct a pieciewise constant control function from the solution $\{a_j, j = 1, 2, ..., N\}$ of the linear programming problem (2.17) which approximates the action of the optimal measure, is based on the analysis in Rubio [59] (Chapter 5).

Assume that the set $\omega_N = \{Z_l, l = 1, 2, ..., N\}$ is the set constructed by dividing the interval $[0, T]$ and $[-1, 1]$ respectively into m_1 and m_2 equal subintervals. From Theorem III.1 of [59], the measure $\mu^*_{Q(M_1,M_2)}$ in the set $Q(M_1, M_2)$ satisfying (2.16) and at which $I(\mu) = \mu(f_0)$ attains its minimum, for sufficiently large N, has the following form

$$\mu^*_{Q(M_1,M_2)} = \sum_{l=1}^{N} a_l \delta(Z_l) \tag{2.18}$$

where $Z_l \in \omega_N$ and $\delta(Z) \in \mathcal{M}^+(\Omega)$ is the unitary atomic measure with support the singleton set $\{Z\}$. This measure is characterized by

$$\delta(Z)F = F(Z), \quad F \in C(\Omega),\ Z \in \Omega.$$

Of course the equality in (2.18) is not exact and some error will appear that can considered as the error present in numerical computation.

Define $H_{ij} = \int_{[t_{i-1},t_i)\times[u_{j-1},u_j)} d\mu^*_{Q(M_1,M_2)}$, by Ghouila-Houri's Theorem [23]

$$H_{ij} = \int_{\Omega_l} d\mu^*_{Q(M_1,M_2)}(t,u) = \int_{[t_{i-1},t_i)\times[u_{j-1},u_j)} d\mu^*_{Q(M_1,M_2)}(t,u) =$$

$$\mu^*([t_{i-1},t_i) \times [u_{j-1},u_j))$$

for $1 \le i \le m_1$, $1 \le j \le m_2$, and Ω_l; $l = 1,2,...,N$, is the lth subrectangle of the partition Ω, where $l = m_1(j-1)+i$. As we mentioned before, the center of the subrectangle Ω_l has been chosen as Z_l, that is $Z_l = Z_{m_1(j-1)+i}$, thus

$$H_{ij} = \int_{\Omega_l} d\mu^*_{Q(M_1,M_2)}(t,u) = \int_{[t_{i-1},t_i)\times[u_{j-1},u_j)} d\mu^*_{Q(M_1,M_2)}(t,u) =$$

$$\int_{\Omega_l} d[\sum_{k=1}^{N} a_k \delta(Z_k)] = a_l$$

where $l = m_1(j-1)+i$.

We can proceed now to construct the piecewise constant control u; which approximates the action of μ^* on the functions f_0, ψ_n, $n = 1,2,...,M_1$ and f_i, $i = 1,2,...,M_2$. Let

$$u(t) = u_l = u_{m_1(j-1)+i}$$

for $t \in B_{ij}$, where

$$B_{ij} = [t_{i-1} + \sum_{k<j} H_{ik}, t_{i-1} \sum_{k\leq j} H_{ik}) \tag{2.19}$$

where t_i's are the number $\frac{T}{m_1}$, $\frac{2T}{m_1}$, ..., T, and H_{ik} defined as above. Since those intervals B_{ij} for which $H_{ij} = 0$ are reduced to a point, they do not contribute any u, so they can be ignored. We shall consider in the analysis to follow only those H_{ij} which are positive. We may obtain all B_{ij} in the same way. Then, the piecewise constant control u is defined by taking the value u_l in the subinterval corresponding to the value $a_l = a_{m_1(j-1)+i}$.

In fact, to summarize the procedure of constructing the piecewise constant control $u(\cdot)$ from the solution of the linear programming problem (2.17), first, for $l = 1, 2, ..., N$, we identify the indices l such that the components a_l of the extreme point are positive, and the corresponding values t_i associated with them. With each such t_i, we identify one subinterval $J_i = [t_{i-1}, t_i)$ of the m_1 partition interval of $[0, T]$. To each of these subintervals J_i, there will correspond at least one, and usually several, of these values a_l which just selected. We then partition these subintervals J_i into further subintervals B_{ij}; for each index l with these properties, of length equal to a_l, and then, the control u is defined by taking $u(t) = u_l$ on it. These subintervals which partition J_i then can be put together to cover the main interval $[0, T]$.

Since at most $M_1 + M_2 + 1$ of a_l's are non zero, the control therefore will suffer at most $M_1 + M_2 + 1$ jumps.

In the next section we shall give some numerical results obtained by using these techniques.

2.1.5 Numerical results

We will apply the method described in the previous section for the estimation of pieciewise constant controls for the one-dimensional wave equations. In all the examples, the criterion functions is $f_0 = u^2$ and $a = 1, T = \pi, S = \pi, M_1 = 5, M_2 = 10,$ $m_1 = 10, m_2 = 20$. So it is assumed that the set $\Omega = [0, \pi] \times [-1, 1]$ is divided to

$N = 200$ subrectangles with centers at $z_l = (t_l, u_l)$; $l = 1, 2, ..., N$. We define

$$\begin{aligned} u_1 = u_2 &= \ldots = u_{10} = -0.95 \\ u_{11} = u_{12} &= \ldots = u_{20} = -0.85 \\ &\vdots \\ u_{191} = u_{192} &= \ldots = u_{200} = 0.95 \end{aligned}$$

and

$$\begin{aligned} t_1 = t_{11} &= \ldots = t_{191} = \frac{\pi}{20} \\ t_2 = t_{12} &= \ldots = t_{192} = \frac{3\pi}{20} \\ &\vdots \\ t_{10} = t_{20} &= \ldots = t_{200} = \frac{19\pi}{20}. \end{aligned}$$

Then we use a NAG fortran library routine based on the modified simplex method to solve the linear system (2.17).

Example 2.1 Consider the linear wave equation

$$Y_{tt}(x,t) = Y_{xx}(x,t),$$

the initial conditions are:

$$\begin{aligned} Y(x,0) &= f(x) = \sin(x) \\ Y_t(x,0) &= h(x) = 0, \end{aligned}$$

the boundary conditions are:

$$\begin{aligned} Y(0,t) &= u(t) \\ Y(S,t) &= u(t). \end{aligned}$$

We assume that $g_1 = 0$, $g_2 = 0$, $x \in [0, \pi]$. In this example, after 72 iterations the cost function converges to the value 3.95714×10^{-1}, and we find the following computational results:

$a_{101} = 0.31415926$ $a_{110} = 0,31415927$ $a_{112} = 0.00463199$

$a_{119} = 0.00463426$ $a_{122} = 0.30952727$ $a_{123} = 0.02667988$

$a_{128} = 0.02667596$ $a_{129} = 0.30952500$ $a_{133} = 0.28747939$

$a_{138} = 0.28748330$ $a_{144} = 0.31415927$ $a_{145} = 0.17911889$

$a_{146} = 0.17912431$ $a_{147} = 0.31415926$ $a_{155} = 0.13504037$

$a_{156} = 0.13503495$

Now to construct the pieciewise constant control function for this example, we use the method introduced above, we have, e.g.,

$$H_{ij} = a_{101} = a_{m_1(j-1)+i} = 0.31415926$$

for $i = 1$, $j = 11$, since $t_1 = \frac{\pi}{10}$, so by (2.19), $B_{1,11} = [0.0, 0.31415926)$ and if $t \in B_{1,11}$, then $u(t) = u_{101} = 0.05$. We define all the subintervals B_{ij}'s in the same way and then draw the pieciewise constant control function in an appropriate manner in the corresponding subintervals.

The graphs of the piecewise constant control function and the approximation of the final function $g_1(x)$ can be seen in Fig. 2.1 and Fig. 2.2, respectively. We mention that in all the numerical examples $Y(x, T)$ is approximated by $\sum_{n=1}^{16} \beta_n \sin(nx)$, where

$$\beta_n = (-1)^n A_n + \frac{2}{\pi}[1 - (-1)^n](-1)^{n+1} \int_0^\pi \sin(nt) u_k(t) dt.$$

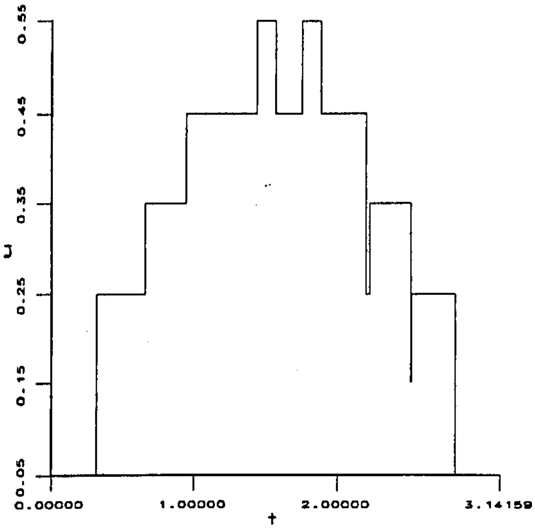

Figure 2.1: Graph of the pieciewise constant control function for Example 2.1

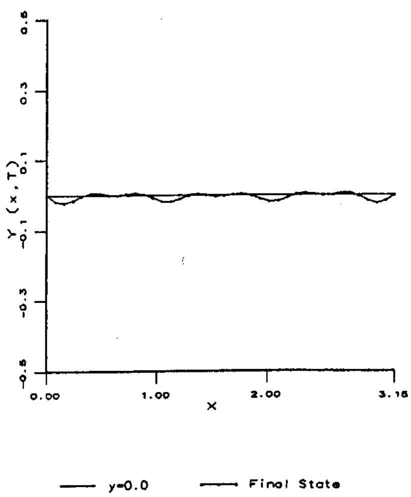

Figure 2.2: Final state for Example 2.1

Example 2.2 Consider the linear wave equation

$$Y_{tt}(x,t) = Y_{xx}(x,t),$$

the initial conditions are:

$$Y(x,0) = f(x) = \cos(x)$$
$$Y_t(x,0) = h(x) = 0,$$

the boundary conditions are:

$$Y(0,t) = u(t)$$
$$Y(S,t) = 0.$$

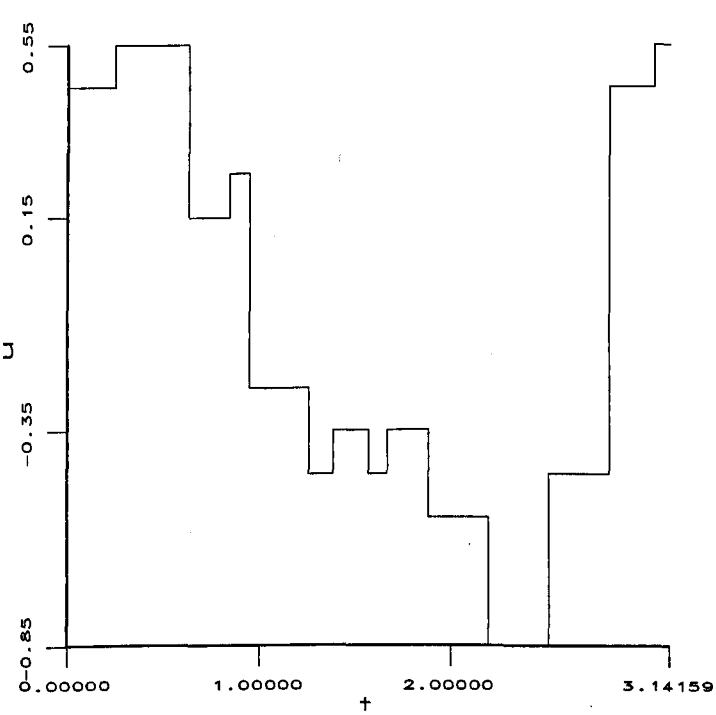

Figure 2.3: Graph of the pieciewise constant control function for Example 2.2

We assume that $g_1 = 0$, $g_2 = 0$, $x \in [0, \pi]$. In this example after 44 iterations the cost function converges to the value 1.086860. The graphs of the piecewise constant control function and the approximation of the final function $g_1(x)$ can be seen in Fig. 2.3 and Fig. 2.4, respectively.

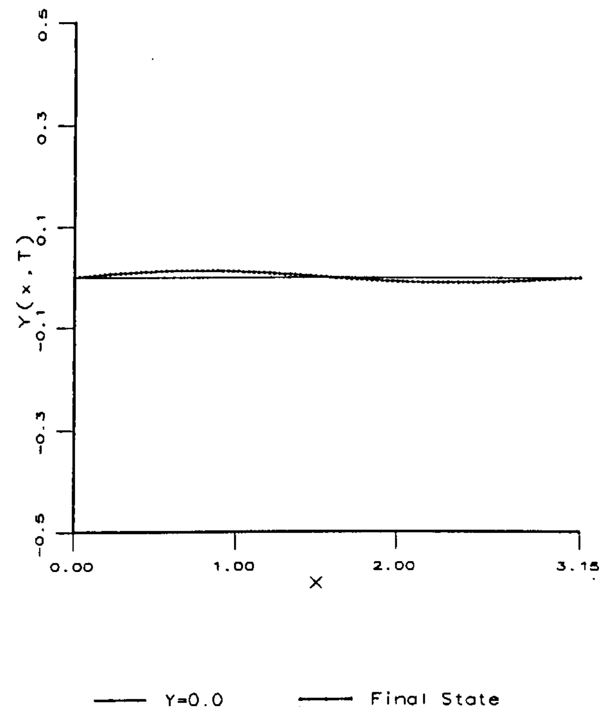

Figure 2.4: Final state for Example 2.2

Example 2.3 Consider the linear wave equation

$$Y_{tt}(x,t) = Y_{xx}(x,t),$$

the initial conditions are:

$$Y(x,0) = f(x) = 0$$
$$Y_t(x,0) = h(x) = 0,$$

the boundary conditions are:

$$Y(0,t) = u(t)$$
$$Y(S,t) = u(t).$$

We assume $g_1 = \sin(x)$, $g_2 = 0$, $x \in [0, \pi]$. For this example, the total number of iterations is 39, and the cost function takes a value of 7.903563×10^{-1}. The graphs of the piecewise constant control function and the approximation of the final function $g_1(x)$ can be seen in Fig. 2.5 and Fig. 2.6, respectively.

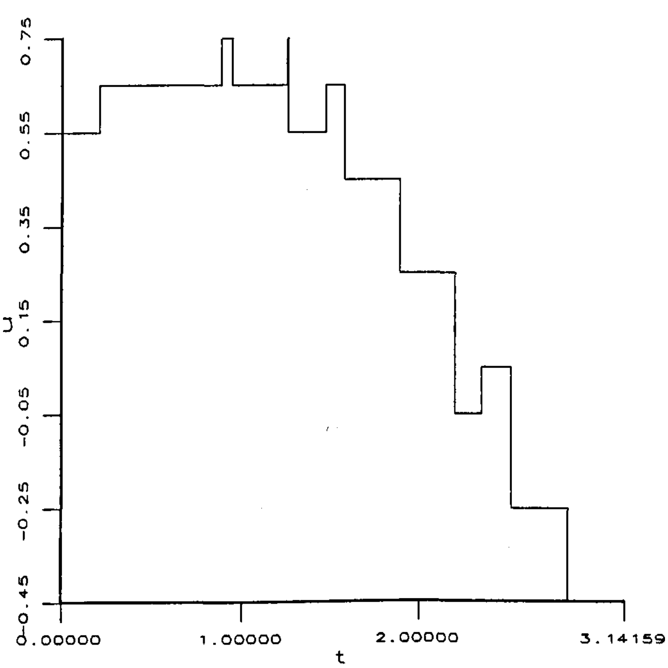

Figure 2.5: Graph of the pieciewise constant control function for Example 2.3

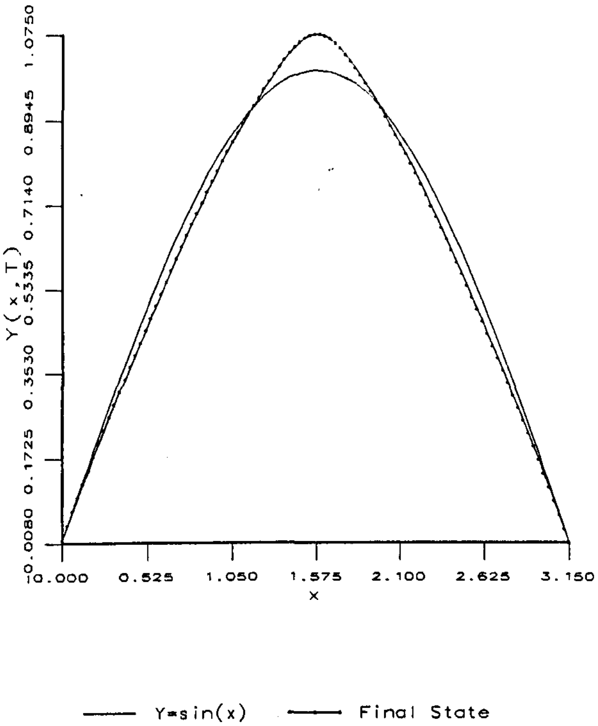

Figure 2.6: Final state for Example 2.3

Example 2.4 Consider the linear wave equation

$$Y_{tt}(x,t) = Y_{xx}(x,t),$$

the initial conditions are:

$$Y(x,0) = f(x) = 0$$
$$Y_t(x,0) = h(x) = \cos(x),$$

the boundary conditions are:

$$Y(0,t) = u(t)$$
$$Y(S,t) = 0.$$

We assume $g_1 = \sin(x)$, $g_2 = 0$, $x \in [0, \pi]$. For this example the total number of iterations is 47, and the cost function takes a value of 7.351063×10^{-1}. The graphs of the piecewise constant control function and the approximation of the final function $g_1(x)$ can be seen in Fig. 2.7 and Fig. 2.8, respectively.

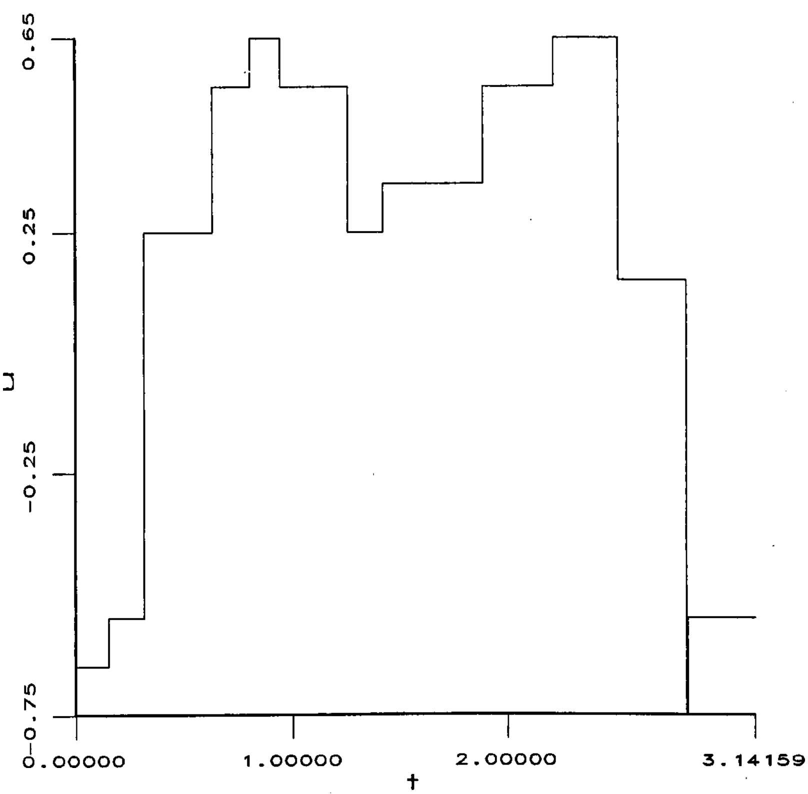

Figure 2.7: Graph of the pieciewise constant control function for Example 2.4

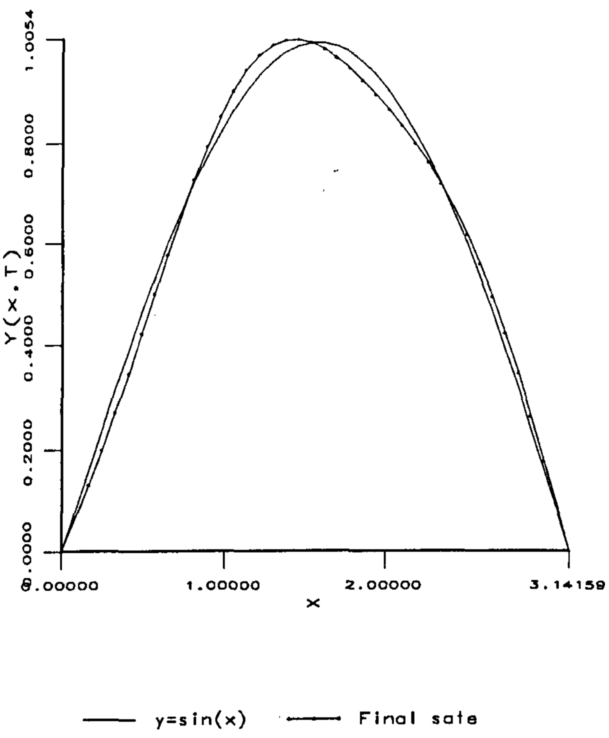

Figure 2.8: Final state for Example 2.4

2.2 Existence of an optimal control for the n-dimensional wave equation

2.2.1 Introduction

Here we consider the existence of an optimal control for the n-dimensional wave equation

$$Y_{tt}(x,t) = a^2 \nabla^2 Y(x,t), \tag{2.20}$$

where $(x,t) \longrightarrow Y(x,t), (x,t) \in \omega \times (0,T)$, and $\nabla^2 Y$ is the *Laplacian* of the function Y in $I\!R^n$.

The boundary condition is:

$$Y(x,t) = u(x,t), \qquad (x,t) \in \partial\omega \times (0,T) \tag{2.21}$$

and the initial conditions are:

$$\begin{aligned} Y(x,0) &= f(x), \qquad x \in \omega \\ Y_t(x,0) &= h(x), \qquad x \in \omega, \end{aligned} \tag{2.22}$$

where ω is a bounded open set in $I\!R^n$, with boundary $\partial\omega \in C^1$, and $u(x,t) \in I\!R$, where $(x,t) \in \partial\omega \times [0,T]$, is the control function.

We say that the control u is *admissible* if it is a measurable function on $\partial\omega \times [0,T]$ and

(a) $u(t) \in [-1,1]$ a.e. for $(x,t) \in \partial\omega \times [0,T]$.
(b) $Y(x,T) = g_1(x)$ a.e. for $x \in \omega$; $g_1 \in L_2(\omega)$ is the desired final state.
(c) $Y_t(x,T) = g_2(x)$ a.e. for $x \in \omega$; $g_2 \in L_2(\omega)$ is a given continuous function.

We define the set of admissible controls as $\mathcal{U}$. Our optimal control problem consists of finding a $u(\cdot,\cdot) \in \mathcal{U}$ which minimizes

$$J(u) = \int_0^T \int_{\partial\omega} f_0[\xi, t, u(t,\xi)]\, d\xi\, dt, \tag{2.23}$$

where $f_0 \in C(\Sigma)$, the space of continuous functions on

$$\Sigma = \partial\omega \times [0,T] \times [-1,1],$$

with the topology of uniform convergence.

2.2.2 Modified control problem

The solution of equation (2.20) with boundary condition (2.21) and initial conditions (2.22) is

$$Y(x,t) = -a^2 \int_{\partial\omega} \int_0^t u(y,\tau) \frac{\partial}{\partial \nu_y}[K(x,y,t-\tau)] d\tau dy$$
$$+ \int_\omega [K(x,y,t)h(x) - K_t(x,y,t)f(x)]dy$$

(see Roach [54], Chapter 9). Here $\frac{\partial}{\partial \nu_y}[K(x,y,t)]$ is the normal derivative with respect to its second variable y, and

$$K(x,y,t) = \sum_{n=1}^{\infty} a_n(x)a_n(y)H(t)\frac{\sin(k_n a t)}{k_n a},$$

where a_n denotes the orthonormal eigenfunction with corresponding eigenvalue λ_n defined by the equation

$$(\frac{\partial^2}{\partial t^2} - \nabla^2)u_n = \lambda_n u_n.$$

Here $k_n^2 = \lambda_n$ and $H(t)$ is the *Heaviside* function and is included to emphasize the fact that the solution is identically zero for $t < 0$. Thus if we assume a=1, we have

$$Y(x,T) = -\int_0^T \int_{\partial\omega} u(y,t)\frac{\partial}{\partial \nu_y}[\sum_{n=1}^{\infty} a_n(x)a_n(y)\frac{\sin(k_n(T-t))}{k_n}]dydt$$

$$+ \int_\omega [\sum_{n=1}^{\infty} a_n(x)a_n(y)\frac{\sin(k_n(T))}{k_n}h(x) - \sum_{n=1}^{\infty} a_n(x)a_n(y)\cos(k_nT)f(x)]dy$$
$$= \sum_{n=1}^{\infty}[-\int_0^T \int_{\partial\omega} u(y,t)\frac{\partial}{\partial \nu_y}a_n(y)\frac{\sin(k_n(T-t))}{k_n}dydt$$
$$+ \int_\omega (a_n(y)\frac{\sin(k_n(T))}{k_n}h(x) - a_n(y)\cos(k_nT)f(x))dy]a_n(x).$$

If we assume $\int_{\partial\omega}[\frac{\partial}{\partial \nu_y}a_n(y)]u(y,t)dy = v_n(t)$, then

$$Y(x,T) = \sum_{n=1}^{\infty}[-\int_0^T v_n(t)\frac{\sin(k_n(T-t))}{k_n}dt$$

$$+ \int_\omega (a_n(y)\frac{\sin(k_n(T))}{k_n}h(x) - a_n(y)\cos(k_nT)f(x))dy]a_n(x).$$

From the above we find $Y_t(x,T)$ as

$$\frac{\partial Y}{\partial t} = \sum_{n=1}^{\infty}[-\int_0^t \int_{\partial\omega}(u(y,\tau)\frac{\partial}{\partial\nu_y}a_n(y)dy)\cos(k_n(t-\tau))d\tau$$

$$+\int_{\omega}(a_n(y)\cos(k_nt)h(x) + a_n(y)k_n\sin(k_nt)f(x))dy]a_n(x),$$

where by assumption

$$\int_{\partial\omega}\frac{\partial}{\partial\nu_y}a_n(y)u(y,\tau)dy = v_n(\tau),$$

we get,

$$\frac{\partial Y}{\partial t}(x,T) = \sum_{n=1}^{\infty}[-\int_0^T v_n(t)\cos(k_n(T-t))dt$$

$$+\int_{\omega}(a_n(y)\cos(k_nT)h(x) + a_n(y)k_n\sin(k_nT)f(x))dy]a_n(x).$$

Thus,

$$Y(x,T) = \sum_{n=1}^{\infty}[-\int_0^T v_n(t)\frac{\sin(k_n(T-t))}{k_n}dt$$

$$+\int_{\omega}(a_n(y)\frac{\sin(k_n(T))}{k_n}h(x) - a_n(y)\cos(k_nT)f(x))dy]a_n(x)$$

and

$$\frac{\partial Y}{\partial t}(x,T) = \sum_{n=1}^{\infty}[-\int_0^T v_n(t)\cos(k_n(T-t))dt$$

$$+\int_{\omega}(a_n(y)\cos(k_nT)h(x) + a_n(y)k_n\sin(k_nT)f(x))dy]a_n(x).$$

Since the desired final states belong to $L_2(\omega)$, we can expand them in terms of the sequence of orthonormal eigenfunctions $a_n(x)$,

$$Y(x,T) = g_1(x) = \sum_{n=1}^{\infty} c_n a_n(x),$$
$$\frac{\partial Y}{\partial t}(x,T) = g_2(x) = \sum_{n=1}^{\infty} d_n a_n(x).$$

Thus,

$$\int_0^T[-v_n(t)\frac{\sin(k_n(T-t))}{k_n}]dt + \int_{\omega}a_n(y)[\frac{\sin(k_n(T))}{k_n}h(x) - \cos(k_nT))f(x)]dy = c_n,$$

and

$$\int_0^T [-v_n(t)\cos(k_n(T-t))]dt + \int_\omega a_n(y)[\cos(k_nT)h(x) + k_n \sin(k_nT)f(x)]dy = d_n.$$

But,

$$\cos k_n(T-t) = \cos k_nT \cdot \cos k_nt + \sin k_nT \cdot \sin k_nt$$
$$\sin k_n(T-t) = \sin k_nT \cdot \cos k_nt - \cos k_nT \cdot \sin k_nt,$$

so

$$\int_0^T [-v_n(t) \cdot \frac{1}{k_n}(\sin k_nT \cdot \cos k_nt - \cos k_nT \cdot \sin k_nt]dt$$
$$+ \int_\omega a_n(y)[\frac{\sin k_nT}{k_n} \cdot h(x) - \cos(k_nT) \cdot f(x)]dy = c_n,$$

and

$$\int_0^T [-v_n(t)(\cos k_nT \cdot \cos k_nt + \sin k_nT \cdot \sin k_nt]dt$$
$$+ \int_\omega a_n(y)[\cos k_nT \cdot h(x) + k_n \cdot \sin(k_nT) \cdot f(x)]dy = d_n.$$

Let

$$\int_0^T [v_n(t) \cdot \sin k_nt]dt = X$$
$$\int_0^T [v_n(t) \cdot \cos k_nt]dt = Y,$$

then by some manipulations and using Cramer's Rule, we find

$$X = c_nk_n \cos k_nT - d_n \sin k_nT + k_n \int_\omega a_n(y)f(x)dy,$$
$$Y = -c_nk_n \sin k_nT - d_n \cos k_nT + \int_\omega a_n(y)h(x)dy,$$

or

$$\int_0^T v_n(t)(\sin k_nt + \cos k_nt)dt =$$

$$c_nk_n(\cos k_nT - \sin k_nT) - d_n(\cos k_nT + \sin k_nT)$$

$$+\int_{\omega}[h(x)+k_nf(x)]a_n(y)dy.$$

Assume

$$c_nk_n(\cos k_nT-\sin k_nT)-d_n(\cos k_nT+\sin k_nT)$$

$$+\int_{\omega}[h(x)+k_nf(x)]a_n(y)dy=\delta_n,\quad n=1,2,..., \tag{2.24}$$

then we have

$$\int_0^T v_n(t)(\sin k_nt+\cos k_nt)dt=\delta_n,\quad n=1,2,...,$$

or

$$\int_0^T\int_{\partial\omega}\frac{\partial}{\partial\nu_y}a_n(y)\cdot u(y,t)(\sin k_nt+\cos k_nt)dydt=\delta_n,\quad n=1,2,....$$

Since we assumed $\partial\omega\in C^1$, we can let the parametric equation of $\partial\omega$ be in the following form,

$$Y=Y(\xi_1(s_1,s_2,...s_{n-1}),\xi_2(s_1,s_2,...s_{n-1}),...,\xi_n(s_1,s_2,...s_{n-1}))$$

where

$$0\le s_i\le 1,\quad i=1,2,...n-1$$

(see Kamyad [33], page 13).

Thus

$$\delta_n=\int_0^T\int_A\frac{\partial}{\partial\nu_y}a_n(\xi_1(s),\xi_2(s),...,\xi_n(s))\cdot u(\xi_1(s),\xi_2(s),...,\xi_n(s),t)$$

$$\times[\sin k_nt+\cos k_nt]B(s)dsdt,$$

where $s=(s_1,s_2,...,s_{n-1})$ and $B(s)$ is the Jacobian of this transformation from x-coordinates to s-coordinates and $ds=ds_1.ds_2....ds_{n-1}$, and $A=[0,1]\times[0,1],...,\times[0,1]$, (n-1 times). If we assume

$$\psi_n(s,t,u)=\frac{\partial}{\partial\nu_y}a_n(\xi_1(s),\xi_2(s),...,\xi_n(s))\cdot u(\xi_1(s),\xi_2(s),...,\xi_n(s),t)$$

$$\times [\sin k_n t + \cos k_n t] B(s), \tag{2.25}$$

then our problem reduces to finding a measurable control $u(y,t) \in [-1,1]$ where $(y,t) \in \partial\omega \times [0,T]$ or $(y(s),t) \in [0,1] \times [0,1] \times, ..., \times [0,1] \times [0,T]$ (n times) which satisfies

$$\int_0^T \int_A \psi_n(s,t,u(s,t))dsdt = \delta_n, \quad n = 1,2,..., \tag{2.26}$$

and minimizes

$$J = \int_0^T \int_A f_0^\wedge(s,t,u(s,t))dsdt, \tag{2.27}$$

where $f_0^\wedge(s,t,u(s,t)) = f_0(s,t,u(s,t)) \cdot B(s)$.

In general, this problem may have no solution, so we proceed to enlarge the set $\mathcal{U}$, the set of admissible controls, and replace this problem by another one in which the minimum of a linear functional is considered over $\Omega = A \times [0,T] \times [-1,1]$. We proceed as follows:

(1) For a fixed u, the mapping

$$\Lambda_F : F \longrightarrow \int_0^T \int_A F[s,t,u(s,t)]dsdt, \qquad F \in C(\Omega),$$

defines a positive linear functional over $C(\Omega)$.

(2) There exists a unique positive Radon representing measure μ on Ω such that

$$J = \int_0^T \int_A F(s,t,u(s,t))dsdt = \int_\Omega F d\mu \equiv \mu(F), \quad F \in C(\Omega) \tag{2.28}$$

(Riesz representation theorem).

Now, we replace the original minimization problem by one in which we are to find the minimum of $\mu(f_0)$ over a set Q of positive Radon measures. We proceed as before. Firstly from (2.28) we have

$$|\mu(F)| \leq T \sup_\Omega |F(t,u)|,$$

hence,

$$|\mu(1)| \leq T.$$

Next, the measures in Q must satisfy

$$\mu(\psi_n) = \delta_n, \qquad n = 1, 2, \ldots.$$

Note that this is possible because $\psi_n \in C(\Omega)$, $n = 1, 2, \ldots$.
Finally, assuming $G \in C(\Omega)$ and independent of u, that is, if $(s,t) \in A \times [0,T]$ and $u_1, u_2 \in [-1,1]$, then $G(s,t,u_1) = G(s,t,u_2)$; then the measures in Q must satisfy

$$\int_\Omega G d\mu = \int_0^T \int_A G(s,t,u(s,t)) ds dt = a_G,$$

where a_G is the Lebesgue integral of G. As before, if $\mathcal{M}^+(\Omega)$ be the set of all positive Radon measures on Ω, we have

$$Q = S_1 \cap S_2 \cap S_3,$$

where

$$\begin{aligned} S_1 &= \left\{\mu \in \mathcal{M}^+(\Omega) : \mu(1) \leq T\right\}, \\ S_2 &= \left\{\mu \in \mathcal{M}^+(\Omega) : \mu(\psi_n) = \delta_n\right\}, \\ S_3 &= \left\{\mu \in \mathcal{M}^+(\Omega) : \mu(G) = a_G, G \in C(\Omega), \text{independent of } u\right\}. \end{aligned}$$

If we topologize the space $\mathcal{M}(\Omega)$ by the weak*-topology, it can be seen that (Section 2.1) S_1 is compact, S_2 and S_3 are closed, therefore Q is a closed subset of the compact set S_1, and then is compact, (see also [34]). Therefore the functional $I : Q \longrightarrow I\!R$ defined by

$$I(\mu) = \int_\Omega f_0 d\mu = \mu(f_0) \in I\!R, \qquad \mu \in Q \tag{2.29}$$

attains its minimum on Q. Thus, the measure-theoretical problem, which consists of finding the minimum of the functional (2.29) over the subset Q of $\mathcal{M}^+(\Omega)$, possesses a minimizing solution μ^*, say, in Q.

We now show that the action of the optimal measure could be approximated by that of a piecewise constant control. With each piecewise constant control $u(\cdot)$ we associate a measure μ_u in $\mathcal{M}^+(\Omega) \cap S_1 \cap S_2 \cap S_3$. Let Q_1 be the set of all such measures μ_u, then an extension of a theorem of Ghouila-Houri (see [33]) shows that, when $M(\Omega)$ is given the weak*-topology, Q_1 is dense in $\mathcal{M}^+(\Omega) \cap S_1 \cap S_2 \cap S_3$. A basis of closed neighborhoods in this topology is given by the sets of the form

$$\{\mu : |\mu(F_n)| \leq \epsilon, n = 1, 2, ...k+1\}, \qquad (\epsilon > 0)$$

where k is an integer, $F_n \in C(\Omega)$. In any weak*-neighborhood of μ^*, it is therefore possible to find a measure μ_u corresponding to a piecewise constant control. In particular we can put

$$F_1 = f_0, F_2 = \psi_1, ...F_{k+1} = \psi_k;$$

a piecewise constant control $u_k(\cdot)$ can be found such that

$$|\int_0^T F_n[t, u_k(t)]dt - \mu^*(F_n)| \leq \epsilon, \qquad n = 1, 2, ..., k+1.$$

Thus, by using the piecewise constant control $u_k(\cdot)$, we can get within ϵ of the minimum value $\mu^*(f_0)$.

2.2.3 Approximation to the optimal measure

Now, as before, we intend to develop a method for the estimation of piecewise constant control. We proceed just as in Section 2.1. Our infinite dimensional problem can be written in the following form:

Minimize

$$I(\mu) = \mu(f_0)$$

such that

$$\begin{aligned} &\mu(1) \leq T \\ &\mu(\psi_n) = \delta_n, \qquad n = 1, 2, ... \\ &\mu(G) = a_G, \qquad G \in C(\Omega), \text{independent of } u. \end{aligned} \tag{2.30}$$

The functions δ_n and ψ_n, $n = 1, 2, ...,$ are respectively as (2.24) and (2.25). Of course, the measures μ are required to be positive Radon measures on Ω. The approximation of the program (2.30) by a sequence of finite-dimensional program is based on the constructions that we described it precisely in Section 2.1. In fact

(a) We consider only a finite number M_1 of the functions ψ_n; this number can be as large as required.

(b) We cover the set $\Omega = A \times [0, T] \times [-1, 1] = [0, 1]^{n-1} \times [0, T] \times [-1, 1]$ with a grid, by taking $m_1 + 1$, $m_2 + 1$, and $m_3 + 1$ points along the s_i-axis, $i = 1, 2, ..., n-1$, t-axis, and u-axis respectively. These points will be equidistant, by $\frac{1}{m_1}$, $\frac{T}{m_2}$, and $\frac{2}{m_3}$ respectively. Then the set Ω is divided into $N = m_1^{n-1} m_2 m_3$ equal volume cuboids Ω_j, $j = 1, 2, ..., N$; we choose $Z \equiv (s_i;\ i = 1, 2, ..., n-1,\ t,\ u)$ as the center of each cuboid. We will number then the points Z in Ω sequentially, from 1 to N.

(c) We consider only M_2 functions of G in (2.30). These functions are not continuous functions as above, but pulselike, lower-semicontinuous functions. We choose these functions G to be the characteristic functions of the individual rectangular cuboids that formed by dividing the set Ω into m_1^{n-1} partition along s_i-axis; $i = 1, 2, ..., n-1$, and m_2 partition along t- axis. We shall denote these functions as G_l, $l = 1, 2, ..., M_2$. To specify these G_l's precisely, we find the numbers

$$s_i^0 = 0 < s_i^1 < ... < s_i^{m_1} = 1, \quad \text{for } i = 1, 2, ..., n-1$$

on s_i-axis and the numbers

$$t_0 = 0 < t_1 < ... < t_{m_2} = T$$

on t-axis. Now define $G_l(\cdot,\cdot,u)$ on $[0,1]^{n-1} \times [0,T]$ as follows:

$$G_l(s,t,u) = \begin{cases} 1 & \text{if } t \in [t_{i-1}, t_i),\ s = (s_1, s_2, ..., s_{n-1}) \in \prod_{j=1}^{n-1}[s_j^{i_j}, s_j^{i_j-1}) \\ 0 & \text{otherwise} \end{cases}$$

where

$$\begin{aligned} i_1 &= 1,2,...,m_1 \\ i_2 &= 1,2,...,m_1 \\ &\vdots \\ i_{n-1} &= 1,2,...,m_1 \\ i &= 1,2,...,m_2 \end{aligned}$$

and

$$l = [m_1^{n-2}(i_{n-1}-1) + m_1^{n-3}(i_{n-2}-1) + ... + m_1(i_2-1) + i_1] + m_1^{n-1}(i-1).$$

So each G_l will be unity in one of the cuboids defined above and zero elsewhere. There will be then as many functions as cuboids, that is

$$M_2 = m_1^{n-1} m_2.$$

Remark: The linear combinations of the functions G_l, can approximate any function arbitrarily well in $C_2(\Omega)$ where $\Omega = A \times [0,T] \times [-1,1]$ [$C_2(\Omega)$, is used for a subspace of $C(\Omega)$ which depends only on variables $s = (s_1, s_2, ..., s_{n-1})$ and t]. This means that for any $G \in C_2(\Omega)$ there exists a sequence $\{G^l\}$ of functions in the subspace spanned by the functions $G_l(\cdot,\cdot,u)$ defined as above, such that

$$G^l(s,t,u) \longrightarrow G(s,t,u)$$

uniformly in Ω. (see [31]). The functions G_l are not continuous, however, each of the G_l's is the limit of an increasing sequence of positive continuous functions, $\{G_{lk}\}$; then, if μ is any positive Radon measure on Ω,

$$\mu(G_l) = \lim_{k\to\infty} \mu(G_{lk}).$$

Now instead of the infinite-dimensional linear programming problem (2.30), we consider one in which the measures are positive Radon measures on Ω supported by the grid defined above; each such measure is defined by a set of non-negative real numbers a_j, $j = 1, 2, ..., N$, where $N = m_1^{n-1} m_2 m_3$ is the number of points Z in the grid. Since one non-negative dummy variable a_{N+1} is necessary to handle the first inequality in (2.30) (see [22]), thus the infinite-dimensional linear programming problem (2.30) would be approximated by the following finite-dimensional linear programming problem:

Minimize

$$\sum_{j=1}^{N+1} a_j f_0(Z_j)$$

over the set $a_j \geq 0$, subject to

$$\sum_{j=1}^{N+1} a_j = T$$
$$\sum_{j=1}^{N+1} a_j \psi_n(Z_j) = \delta_n, \qquad n = 1, 2, ..., M_1,$$
$$\sum_{j=1}^{N+1} a_j G_i(Z_j) = a_{G_i}, \qquad i = 1, 2, ..., M_2. \tag{2.31}$$

The solution of the linear programming problem (2.31) gives rise to a Radon measure which is a linear combination of $N + 1$ atomic measures

$$\mu = \sum_{j=1}^{N+1} a_j \delta(Z_j)$$

where the a_j's are the solution of (2.31) and the Z_j's are the corresponding values of Z which define the support of each atomic measure. It is clear that

$$Z_j \equiv (s_{ji}; i = 1, 2, ..., n-1, t_j, u_j).$$

We shall use a_j's and Z_j's to define a pieciewise constant control u which is admissible and, for sufficiently large values of M_1, M_2, and N, gives a value to the performance criteria $\sum_{j=1}^{N+1} a_j f_0(Z_j)$ which is near to the infimum of $\mu(f_0)$ over the set Q.
The procedure of the construction of this pieciewise constant control function which approximates the action of the optimal measure is based on the analysis described in Section 2.1.

2.2.4 Numerical results

Example 2.5 Consider the wave equation

$$Y_{tt} = \nabla^2 Y_{xx}$$

where $Y(x,y,t) :(x,y,t) \in \omega \times [0,T]$, and ω is the rectangle $[0,\pi] \times [0,\pi]$ in the xy-plane. Since the boundary $\partial\omega$ is divided to four individual segments which are defined by variable s, $0 \leq s \leq 1$ (see Appendix 2.3), so the control function is symmetric in the four edges of the boundary $\partial\omega$. The initial conditions are:

$$Y(x,y,0) = f(x,y) = 0, \qquad (x,y) \in \omega$$
$$Y_t(x,y,0) = h(x,y) = 0, \qquad (x,y) \in \omega$$

and the boundary condition is:

$$Y(x,y,t) = u(x,y,t), \qquad (x,y,t) \in \partial\omega \times (0,T).$$

We are looking the control function $u(x,y,t)$ such that

$$Y(x,y,T) = g_1(x,y) = (0.1) \cdot \frac{2}{\pi} \cdot \sin x \cdot \sin y,$$
$$Y_t(x,y,T) = g_2(x,y) = 0,$$

at a specified time T.

We choose $T = 1.5\pi$. Then $(s,t,u) \in \Gamma$ where $s \in [0,1]$, $t \in [0,T]$, and $u \in [-1,1]$. We divided the set $[0,1]$ on the s-axis into 8 equal subintervals and the set

$[0, T]$=$[0, 1.5\pi]$ on the t-axis into 20 subintervals and the set $[-1, 1]$ on the u-axis into 15 equal subintervals, so that $\Gamma = [0, 1] \times [0, T] \times [-1, 1]$ is divided into 2400 equal subsets. We assume $Z_k = Z(s_k, t_k, u_k)$, $k = 1, 2, ...2400$, where

$$\begin{aligned}
u_1 = u_2 &= \ldots = u_{160} = -\frac{14}{15} \\
u_{161} = u_{162} &= \ldots = u_{320} = -\frac{12}{15} \\
&\vdots \\
u_{2241} = u_{2242} &= \ldots = u_{2400} = \frac{14}{15},
\end{aligned}$$

so in general $u_{160k+1} = u_{160k+2} = ... = u_{160(k+1)} = \frac{(2k-14)}{15}$; $k = 0, 1, ...14$.

Also we choose s_k as follows:

$$\begin{aligned}
s_1 = s_2 &= \ldots = s_{20} = s_{161} = \ldots = s_{180} = \ldots = s_{2260} = \frac{1}{16} \\
s_{21} = s_{22} &= \ldots = s_{40} = s_{181} = \ldots = s_{200} = \ldots = s_{2280} = \frac{3}{16} \\
&\vdots \\
s_{141} = s_{142} &= \ldots = s_{160} = s_{301} = \ldots = s_{320} = \ldots = s_{2400} = \frac{15}{16}
\end{aligned}$$

and in general $s_{160k+20i+j} = \frac{(2i+1)}{16}$; $i = 0, 1, ...7$, $j = 1, 2, ...20$, $k = 0, 1, ..., 14$, and we choose,

$$\begin{aligned}
t_1 = t_{21} &= \ldots = t_{141} = \frac{1.5\pi}{40} \\
t_{161} = t_{181} &= \ldots = t_{301} = \frac{1.5\pi}{40} \\
&\vdots \\
t_{2041} = t_{2061} &= \ldots = t_{2381} = \frac{1.5\pi}{40},
\end{aligned}$$

and,

$$\begin{aligned} t_2 = t_{22} &= \ldots = t_{142} = \frac{4.5\pi}{40} \\ t_{162} = t_{182} &= \ldots = t_{302} = \frac{4.5\pi}{40} \\ &\vdots \\ t_{2042} = t_{2062} &= \ldots = t_{2382} = \frac{4.5\pi}{40}, \end{aligned}$$

and,

$$\begin{aligned} t_{20} = t_{40} &= \ldots = t_{160} = \frac{58.5\pi}{40} \\ t_{180} = t_{220} &= \ldots = t_{320} = \frac{58.5\pi}{40} \\ &\vdots \\ t_{2060} = t_{2100} &= \ldots = t_{2400} = \frac{58.5\pi}{40}, \end{aligned}$$

and in general $t_{160k+20i+j} = \frac{(2j-1)\times 1.5\pi}{40}; i = 0, 1, \ldots 7, j = 1, 2, \ldots 20, k = 0, 1, \ldots, 14.$

Our linear programming problem consists of minimizing the following real function

$$\sum_{j=1}^{2401} a_j f_0(Z_j),$$

over the set of coefficients $a_j \geq 0, j = 1, 2, \ldots 2401$, such that

$$\begin{aligned} &\sum_{j=1}^{2401} a_j = T \\ &\sum_{j=1}^{2401} a_j \psi_i(Z_j) = \delta_i, \qquad i = 1, 2, \ldots M_1, \\ &\sum_{j=1}^{2401} a_j f_{kl}(Z_j) = c^{kl}, \qquad i = 1, 2, \ldots M_2. \end{aligned}$$

In this linear programming problem one extra dummy variable, a_{2401}, is necessary to handle the first inequality in (2.30).

Here we choose the criterion function to be $f_0(s, t, u) = u^2$, and we define the functions f_{kl}'s as

$$f_{kl}(s,t,u) = 1, \quad (s,t) \in j_{kl}$$

$$f_{kl}(s,t,u) = 0, \quad \text{otherwise},$$

where $j_{kl} = [(k-1)d, kd) \times [(l-1)d', ld')$, d=$\frac{1}{8}$, $d' = \frac{1.5\pi}{20}$, and $k = 1,2,...8$, and $l = 1,2,...20$.

We know that c^{kl} is the integral of function f_{kl} over j_{kl}, that is,

$$c^{kl} = \int_0^1 \int_0^{1.5\pi} f_{kl}(s,t)dtds = \int_{(k-1)d}^{kd} \int_{(l-1)d'}^{ld'} f_{kl}(s,t)dtds$$

$$= \int_{(k-1)d}^{kd} \int_{(l-1)d'}^{ld'} dtds = \frac{1}{8} \times \frac{1.5\pi}{20} = \frac{1.5\pi}{160}, k = 1,2,...8, l = 1,2,...20.$$

By definition of the functions f_{kl} and due to the choice of the s_i's and t_j's, for $k = 1,2,...8$, and $l = 1,2,...20$, the following correspondence is established:

$$(s_k, t_l) \Longleftrightarrow (s_{160n+20(k-1)+l}, t_{160n+20(k-1)+l}),$$

$n = 0,1,...14$, $l = 1,2,...8$, and $k = 1,2,...20$.

Thus,

$$f_{kl}(s_k, t_l) = f_{kl}(s_{160n+20(k-1)+l}, t_{160n+20(k-1)+l}) = 1$$

$$f_{kl}(s,t) = 0, \quad \text{otherwise.}$$

From the equation

$$\sum_{j=1}^{2400} a_j f_{kl}(Z_j) = c^{kl},$$

$$k = 1,2,...8,\ l = 1,2,...20, \text{ and } Z_j = (s_j, t_j),$$

we have

$$\begin{aligned}
a_1 f_{11}(s_1,t_1) + a_2 f_{11}(s_2,t_2) + \ldots + a_{2400} f_{11}(s_{2400},t_{2400}) &= \frac{1.5\pi}{160} \\
a_1 f_{21}(s_1,t_1) + a_2 f_{21}(s_2,t_2) + \ldots + a_{2400} f_{21}(s_{2400},t_{2400}) &= \frac{1.5\pi}{160} \\
&\vdots \\
a_1 f_{208}(s_1,t_1) + a_2 f_{208}(s_2,t_2) + \ldots + a_{2400} f_{208}(s_{2400},t_{2400}) &= \frac{1.5\pi}{160},
\end{aligned}$$

or by definition f_{kl} we find that:

$$\begin{aligned}
a_1 + a_{161} + \ldots + a_{2241} &= \frac{1.5\pi}{160} \\
a_2 + a_{162} + \ldots + a_{2242} &= \frac{1.5\pi}{160} \\
&\vdots \\
a_{160} + a_{320} + \ldots + a_{2400} &= \frac{1.5\pi}{160}.
\end{aligned}$$

Thus $\sum_{j=1}^{2400} a_j = 1.5\pi = T$, but since $\sum_{j=1}^{2401} a_j = T$, we find that the slack variable a_{2401}=0. Also we have

$$\sum_{j=1}^{2400} a_j \psi_i(Z_j) = \delta_i, \qquad i = 1, 2, ..., M_1,$$

so for $M_1 = 4$

$$\begin{aligned}
a_1\psi_1(Z_1) + a_2\psi_1(Z_2) + \ldots + a_{2400}\psi_1(Z_{2400}) &= \delta_1 \\
a_1\psi_2(Z_1) + a_2\psi_2(Z_2) + \ldots + a_{2400}\psi_2(Z_{2400}) &= \delta_2 \\
a_1\psi_3(Z_1) + a_2\psi_3(Z_2) + \ldots + a_{2400}\psi_3(Z_{2400}) &= \delta_3 \\
a_1\psi_4(Z_1) + a_2\psi_4(Z_2) + \ldots + a_{2400}\psi_4(Z_{2400}) &= \delta_4,
\end{aligned}$$

where $Z_j = (s_j, t_j, u_j)$; $j = 1, 2, ...2400$.

We have shown in Appendix 2.3 that:

$$\psi_1 = -8\sin(s\pi)\cdot u(s,t)\cdot(\sin t + \cos t)$$
$$\psi_2 = -12\sin(s\pi)\cdot u(s,t)\cdot(\sin\sqrt{5}t + \cos\sqrt{5}t)$$
$$\psi_3 = -24\sin(3s\pi)\cdot u(s,t)\cdot(\sin 3t + \cos 3t)$$
$$\psi_4 = -8[2\sin(2s\pi) - \sin(4s\pi)]\cdot u(s,t)\cdot(\sin\sqrt{10}t + \cos\sqrt{10}t)$$

and

$$\delta_1 = 0.1$$
$$\delta_i = 0, \qquad i \geq 2.$$

So our control problem is as follows:

Minimize the functional

$$\sum_{j=1}^{2400} a_j u_j^2,$$

subject to,

$$\sum_{j=1}^{2400} a_j[-8\sin(s_j\pi)\cdot u_j\cdot(\sin t_j + \cos t_j)] = 0.1$$
$$\sum_{j=1}^{2400} a_j[-12\sin(s_j\pi)\cdot u_j\cdot(\sin\sqrt{5}t_j + \cos\sqrt{5}t_j)] = 0$$
$$\sum_{j=1}^{2400} a_j[-24\sin(3s_j\pi)\cdot u_j\cdot(\sin 3t_j + \cos 3t_j)] = 0$$
$$\sum_{j=1}^{2400} a_j[-16\sin(2s_j\pi) + 8\sin(4s_j\pi)]\cdot u_j\cdot(\sin\sqrt{10}t_j + \cos\sqrt{10}t_j)] = 0$$

and,

$$
\begin{aligned}
a_1 + a_{161} + \ldots + a_{2241} &= \frac{1.5\pi}{160} \\
a_2 + a_{162} + \ldots + a_{2242} &= \frac{1.5\pi}{160} \\
&\vdots \\
a_{160} + a_{320} + \ldots + a_{2400} &= \frac{1.5\pi}{160}.
\end{aligned}
$$

In this example after 631 iterations the cost function converges to the value 1.214811×10^{-3} while CPU time is 629 seconds. The graph of the piecewise constant control function can be seen in Fig. 2.9.

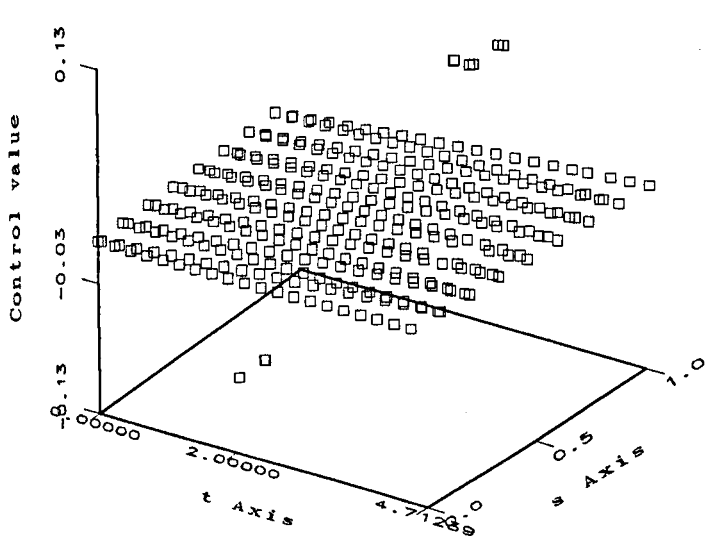

Figure 2.9: Graph of the pieciewise constant control for Exampe 2.5

2.3 Appendix

In Example 2.5 we have assumed the region ω to be rectangle $[0,\pi]\times[0,\pi]$. Here we find the set of orthonormal eigenfunctions $a_n(x,y)$, $n = 1,2,...$, and corresponding eigenvalues λ_n, associated with the equation

$$(\frac{\partial^2}{\partial t^2}-\nabla^2)u_n = \lambda_n u_n.$$

The orthonormal eigenfunctions $a_n(x,y)$, (x,y)$\in\omega$, and corresponding eigenvalues λ_n's, and k_n's are as follows (see [28], Chapter 6)

$$\begin{aligned}
a_1(x,y) &= \frac{2}{\pi}\sin x\sin y, & \lambda_1 &= 1, & k_1 &= 1,\\
a_2(x,y) &= \frac{2}{\pi}\sin x\sin 2y, & \lambda_2 &= \frac{5}{2}, & k_2 &= \sqrt{\frac{5}{2}},\\
a_3(x,y) &= \frac{2}{\pi}\sin 2x\sin 2y, & \lambda_3 &= 4, & k_3 &= 2,\\
a_4(x,y) &= \frac{2}{\pi}\sin x\sin 3y, & \lambda_4 &= 5, & k_4 &= \sqrt{5},\\
a_5(x,y) &= \frac{2}{\pi}\sin 2x\sin 3y, & \lambda_5 &= \frac{13}{2}, & k_5 &= \sqrt{\frac{13}{2}},\\
a_6(x,y) &= \frac{2}{\pi}\sin 3x\sin 3y, & \lambda_6 &= 9, & k_6 &= 3,\\
a_7(x,y) &= \frac{2}{\pi}\sin 2x\sin 4y, & \lambda_7 &= 10, & k_7 &= \sqrt{10},\\
&\vdots
\end{aligned}$$

Now we consider,

$$\psi_n(s,t,u(s,t)) = \frac{\partial}{\partial\nu_y}a_n(\xi_1(s),\xi_2(s))\cdot u(\xi_1(s),\xi_2(s),t)\times[\sin k_n t+\cos k_n t]B(s),$$

where $\frac{\partial}{\partial\nu_y}$ denotes the outward normal derivative to $\partial\omega$. We know that $\partial\omega = \partial\omega_1\cup\partial\omega_2\cup\partial\omega_3\cup\partial\omega_4$, where

$$\partial\omega_1 = \begin{cases}\xi_1(s) = \pi s\\ \xi_2(s) = 0,\end{cases}$$

$$\partial\omega_2 = \begin{cases} \xi_1(s) = \pi \\ \xi_2(s) = \pi s, \end{cases}$$

$$\partial\omega_3 = \begin{cases} \xi_1(s) = \pi(1-s) \\ \xi_2(s) = \pi, \end{cases}$$

$$\partial\omega_4 = \begin{cases} \xi_1(s) = 0 \\ \xi_2(s) = \pi(1-s), \end{cases}$$

and $s \in [0,1]$. Thus for example

$$\frac{\partial a_1}{\partial \nu_y} = \sum_{i=1}^{4} \frac{\partial a_1}{\partial \nu_i},$$

where,

$$\begin{aligned}\frac{\partial a_1}{\partial \nu_1} = (\text{grad}a_1)_{\partial\omega_1}(0,-1) &= (\frac{2}{\pi}\cos x \sin y, \frac{2}{\pi}\sin x \cos y)_{\partial\omega_1}(0,-1)\\ &= -\frac{2}{\pi}\sin(s\pi)\cos(0) = -\frac{2}{\pi}\sin(s\pi),\end{aligned}$$

$$\begin{aligned}\frac{\partial a_1}{\partial \nu_2} = (\text{grad}a_1)_{\partial\omega_2}(1,0) &= (\frac{2}{\pi}\cos x \sin y, \frac{2}{\pi}\sin x \cos y)_{\partial\omega_2}(1,0)\\ &= \frac{2}{\pi}\cos(\pi)\sin(s\pi) = -\frac{2}{\pi}\sin(s\pi),\end{aligned}$$

$$\begin{aligned}\frac{\partial a_1}{\partial \nu_3} = (\text{grad}a_1)_{\partial\omega_3}(0,1) &= (\frac{2}{\pi}\cos x \sin y, \frac{2}{\pi}\sin x \cos y)_{\partial\omega_3}(0,1)\\ &= \frac{2}{\pi}\sin(\pi - s\pi)\cos(\pi) = -\frac{2}{\pi}\sin(s\pi),\end{aligned}$$

$$\begin{aligned}\frac{\partial a_1}{\partial \nu_4} = (\text{grad}a_1)_{\partial\omega_4}(-1,0) &= (\frac{2}{\pi}\cos x \sin y, \frac{2}{\pi}\sin x \cos y)_{\partial\omega_4}(-1,0)\\ &= -\frac{2}{\pi}\cos(0)\sin(\pi - s\pi) = -\frac{2}{\pi}\sin(\pi s).\end{aligned}$$

So $\frac{\partial a_1}{\partial \nu} = -\frac{8}{\pi}\sin(\pi s)$, and $B(s) = \sqrt{(\xi_1'^2 + \xi_2'^2)} = \pi$. Thus we have

$$\psi_1 = -8\sin(\pi s)\cdot u(\xi_1(s), \xi_2(s), t)[\sin t + \cos t], (s,t) \in [0,1]\times[0,T].$$

Similarly, we find that

$$\psi_2 = \psi_3 = \psi_5 = 0,$$

and

$$\psi_4 = [-12\sin(\pi s)]u(\xi_1(s), \xi_2(s), t)[\sin\sqrt{5}t + \cos\sqrt{5}t],$$
$$\psi_6 = [-24\sin(3\pi s)]u(\xi_1(s), \xi_2(s), t)[\sin 3t + \cos 3t],$$
$$\psi_7 = [-16\sin(2\pi s) + 8\sin(4\pi s)]u(\xi_1(s), \xi_2(s), t)[\sin\sqrt{10}t + \cos\sqrt{10}t],$$

and so on. In Example (2.5), we have chosen

$$\psi_1 = \psi_1,$$

$$\psi_2 = \psi_4,$$

$$\psi_3 = \psi_6,$$

$$\psi_4 = \psi_7.$$

From (2.24) we have:

$$\delta_n = c_n k_n(\cos k_n T - \sin k_n T) - d_n(\cos k_n T + \sin k_n T)$$
$$+ \int_\omega [h(x) + k_n f(x)]a_n(y)dy \quad , n = 1, 2, \ldots.$$

In this example

$$g_1(x,y) = (0.1)\frac{2}{\pi}\sin x \sin y$$

and

$$g_2(x,y) = 0,$$

so $c_1 = 0.1$, $c_2 = c_3 = ... = 0$, and $d_1 = d_2 = d_3 = ... = 0$, also

$$f(x,y) \equiv 0, \qquad (x,y) \in \omega$$
$$h(x,y) \equiv 0, \qquad (x,y) \in \omega,$$

so

$$\delta_1 = c_1 k_1(\cos k_1 T - \sin k_1 T).$$

But $k_1 = 1$ and $T = \frac{3\pi}{2}$, so

$$\delta_1 = (0.1)(-\sin\frac{3\pi}{2}) = 0.1.$$

Similarly we find that

$$\delta_i = 0, i \geq 2.$$

Chapter 3

Approximation

3.1 Introduction

In this chapter, we consider the problem of approximating an optimal control problem by a class of linear programming problems. The primal problem to be investigated is the following

Minimize

$$\mu : f_0 \longrightarrow \mu(f_0)$$

subject to

$$\begin{aligned} \mu(\psi_n) &= \alpha_n, \qquad & n &= 1, 2, ..., \\ \mu(G_r) &= a_{G_r}, \qquad & r &= 0, 1, ..., \end{aligned} \tag{3.1}$$

where $\mu \in \mathcal{M}^+(\Omega)$, the set of positive Radon measures on the given Ω, and we want to approximate (3.1) by a problem of the following form

Minimize

$$\mu : f_0 \longrightarrow \mu(f_0)$$

subject to

$$\begin{aligned} \mu(\psi_n) &= \alpha_n, \qquad & n &= 1, 2, ..., M_1, \\ \mu(G_r) &= a_{G_r}, \qquad & r &= 0, 1, ..., M_2, \end{aligned} \tag{3.2}$$

which M_1 and M_2 are two given positive integers, in some cases one of them may be infinity. In this approximation we focus our attention on the maximum value of

$$|\mu^*(f_0) - \mu^*_{Q(M_1,M_2)}(f_0)|$$

where $\mu^*(f_0)$ is the optimal value of the primal problem (3.1) and $\mu^*_{Q(M_1,M_2)}(f_0)$ is the minimum of the problem (3.2); $Q(M_1, M_2)$ is the set of measures in $\mathcal{M}^+(\Omega)$ satisfying (3.2).

We consider the approximations, however, from a more general case in which only one of the two positive integers M_1 or M_2 are finite, to a particular case, in which both of M_1 and M_2 are finite.

In Section 3.2, we discuss about

$$\max |Y_{M_1}(x, T) - g_1(x)| \tag{3.3}$$

where $Y_{M_1}(x, T)$ is the terminal value of the solution of the optimal control problem which is determined by using only M_1 constraints of the equations (3.1), and $g_1(x)$ is the final given value.

In Section 3.3, we will consider the value of

$$\max |\mu^*(f_0) - \mu^*_{Q(M_1,M_2)}(f_0)| \tag{3.4}$$

where M_2 is finite and $M_1 = \infty$.

In Section 3.4, we consider the case when instead of using the monomials $G_r = t^r$, $r = 0, 1, ..., M_2$ in the second equations of (3.2), we use a sequence of step functions $\{f_s\}$ whose linear combinations can approximate a function in $C_1(\Omega)$ arbitrarily well, when $\Omega = [0, T] \times [-1, 1]$ and $C_1(\Omega)$ is the subspace of $C(\Omega)$ which depends only on the variable t. This means that for any $f \in C_1(\Omega)$, there exists a sequence $\{f^s\}$ of simple functions in the subspace spanned by $\{f_s\}$ such that

$$f^s \longrightarrow f$$

uniformly when s tends to infinity (see Hewitt and Stromberg [31], Theorem 11.35 and Bartel [2], Theorem 24.4).

Use of these simple functions $\{f_s\}$ brings a remarkable advantage which is used in this section.

In the last section, Section 3.5, finally, we shall consider the value of (3.4) when M_1 and M_2 both are finite. In this case, in fact, we consider a sequence of finite-dimensional linear programming problems whose solutions approximate the solution of the optimal control problem (3.1).

Several authors have proposed and studied this kind of approximation, that is, approximating an infinite-dimensional linear programming problem by a finite-dimensional one. It appears that the first attempt was done by Vershik and Temel't [81], Temel't [78] and Vershik [80], and then followed by many others. Here we employ the method that used by Dahleh and Pearson [9], Mendlovitz [47] and then developed in the work of Staffans [76]-[77].

In each section we will give some numerical results to show how accurate each procedure is.

3.2 The first type of approximation

In Chapter 2 we considered the following one-dimensional wave equation

$$Y_{tt}(x,t) = a^2 Y_{xx}(x,t)$$

with the initial conditions:

$$Y(x,0) = f(x)$$
$$Y_t(x,0) = h(x),$$

and with the boundary conditions:

$$Y(0,t) = u(t)$$
$$Y(S,t) = u(t),$$

where $t \in [0,T] \longrightarrow u(t) \in I\!R$ was called the control function. We defined the control u is *admissible* if it is Lebesgue measurable function on $[0,T]$ and

(a) $u(t) \in [-1,1]$ a.e. for $t \in [0,T]$.

(b) The solution of the wave equation corresponding to the given initial and boundary conditions satisfies the following terminal relations

$$Y(x,T) = g_1(x)$$
$$Y_t(x,T) = g_2(x),$$

where $g_1(x) \in L_2(0,S)$ and $g_2(x) \in L_2(0,S)$. We assumed the control u be admissible, then the optimal control problem consisted of finding an admissible control u which minimizes the functional

$$I[u(\cdot)] = \int_0^T f_0[t,u(t)]dt, \tag{3.5}$$

where $f_0 \in C(\Omega)$, the space of continuous functions on $\Omega=[0,T] \times [-1,1]$, with the topology of uniform convergence.

We showed in Chapter 2 that

$$Y(x,t) = Y_1(x,t) + Y_2(x,t),$$

where

$$Y_1(x,t) = \sum_{n=1}^{\infty}(A_n \cos\frac{\pi nat}{S} + B_n \sin\frac{\pi nat}{S}) \sin\frac{\pi nx}{S},$$
$$A_n = \frac{2}{S}\int_0^S f(\xi)\sin\left(\frac{\pi n\xi}{S}\right)d\xi,$$
$$B_n = \frac{2}{Sna}\int_0^S h(\xi)\sin\left(\frac{\pi n\xi}{S}\right)d\xi,$$

and

$$Y_2(x,t) = \int_0^t K(x,t-\tau)u(\tau)d\tau,$$

where

$$K(x,t) = \sum_{n=1}^{\infty} c_n \sin \frac{\pi n x}{S} \sin \frac{\pi n a t}{S}.$$

This control problem is then reduced to finding a measurable control function $u(t) \in [-1,1]$ for $t \in [0,T]$ which satisfies

$$\int_0^T \psi_n(t, u(t))\, dt = \alpha_n, n = 1, 2, ..., \tag{3.6}$$

and minimizes the functional (3.5), where,

$$\psi_n(t, u(t)) = \frac{2}{\pi}[1 - (-1)^n](\sin(nt) + \cos(nt))\, u(t)$$

$$\alpha_n = A_n - B_n + a_n(\sin(nT) - \cos(nT)) + \frac{b_n}{n}(\sin(nT) + \cos(nT)),$$

$$n = 1, 2, ..., \tag{3.7}$$

a_n and b_n are the Fourier coefficients of $g_1(x)$ and $g_2(x)$ respectively in the expansion of these functions over the interval $[0, S]$ into the function $\sin(nx)$, $n = 1, 2, ...$ Then, this problem is reduced again to the following one,

Minimize

$$\mu : f_0 \longrightarrow \mu(f_0) \tag{3.8}$$

over the set Q of positive Radon measures as a subset of $\mathcal{M}^+(\Omega)$ (the set of all positive Radon measures on Ω) with the weak*-topology, such that

$$\begin{aligned} \mu(\psi_n) &= \alpha_n, \qquad n = 1, 2, ..., \\ \mu(G) &= a_G, \qquad G \in C(\Omega), \text{independent of } u. \end{aligned} \tag{3.9}$$

By Proposition 2.1, there exists an optimal measure μ^* in the set Q which satisfies the equalities (3.9) and for which $\mu^*(f_0) \leq \mu(f_0)$, for all $\mu \in Q$.

We assume $Q(M_1, M_2)$ be the set of measures in $\mathcal{M}^+(\Omega)$ satisfying

$$\mu(\psi_n) = \alpha_n, \qquad n = 1, 2, ..., M_1$$

$$\mu(G_r) = a_{G_r}, \qquad r = 0, 1, 2, ...M_2, G_r = t^r,$$

then we can prove the following proposition.

Proposition 3.1 As M_1 and M_2 tend to infinity

$$\eta(M_1, M_2) = \inf_{Q(M_1, M_2)} \mu(f_0)$$

tends to

$$\eta = \inf_Q \mu(f_0).$$

Proof. We prove this proposition in several steps,
(a) The sequence $\{\eta(M_1, M_2), M_1 = 1, 2, ..., M_2 = 1, 2, ...\}$ converges as M_1, M_2 tend to infinity, (see [59] Chapter 3). Thus, the double sequence $\eta(M_1, M_2)$ converges as M_1 and M_2 tend to infinity, to a number, say, ξ.
(b) We prove that this limit ξ equals η, $\inf_Q \mu(f_0)$. First we show that the double limit ξ can be computed sequentially. It is known that

$$\xi = \lim_{M_1 \to \infty} [\lim_{M_2 \to \infty} \eta(M_1, M_2)]$$

provided that the $\lim_{M_2 \to \infty} \eta(M_1, M_2)$ exists. We show that when M_1 is fixed and M_2 varies, $\lim_{M_2 \to \infty} \eta(M_1, M_2)$ exists; we have

$$Q(M_1, 1) \supset Q(M_1, 2) \supset ... \supset Q(M_1, M_2) \supset ...,$$

$$\eta(M_1, 1) \leq \eta(M_1, 2) \leq ... \leq \eta(M_1, M_2) \leq ...,$$

for $M_1 = 1, 2, ...$ The sequence $\{\eta(M_1, M_2), M_2 = 1, 2, ...\}$ is nondecreasing and bounded above; it converges to a number, say, $\zeta(M_1)$, thus

$$\lim_{M_2 \to \infty} \eta(M_1, M_2) = \zeta(M_1).$$

(c) The double limit ξ can be computed sequentially. Define

$$Q(M_1) = \bigcap_{M_2=1}^{\infty} Q(M_1, M_2);$$

then

$$\zeta(M_1) = \lim_{M_2\to\infty} \eta(M_1, M_2) = \inf_{Q(M_1)} \mu(f_0).$$

We have

$$Q(1) \supset Q(2) \supset ... \supset Q(M_1) \supset ... \supset Q,$$

thus,

$$\zeta(1) \leq \zeta(2) \leq ... \leq \zeta(M_1) \leq ... \leq \eta,$$

as expected, the sequence $\zeta(M_1)$ converges, necessarily, to the same number ξ introduced in (a), and so

$$\xi \leq \eta \tag{3.10}$$

(d) Let

$$P = \bigcap_{M_1=1}^{\infty} Q(M_1),$$

then from

$$Q(1) \supset Q(2) \supset ... \supset Q(M_1) \supset ... \supset Q,$$

we have $Q \subset P$, and $\xi = \lim_{M_1\to\infty} \zeta(M_1) = \inf_P \mu(f_0)$.

We can show that $P \subset Q$. If $\mu \in P$, then $\mu(\psi_n) = \alpha_n, n = 1, 2, ...$ Also if $\mu \in P$, then $\mu(G) = a_G$ for all continuous functions G independent of u in the subspace spanned by the set $\{G_r = t^r,\ r = 0, 1, ...\}$, by linearity. This implies that we prove the equality $\mu(G) = a_G = \int_0^T G(t)dt$ holds for all $G \in C[0, T]$ and independent of u, since the set $\{G_r = t^r,\ r = 0, 1, ...\}$ is dense in the space of all continuous functions G on the compact interval $[0, T]$ and with the values in R, so for such functions G there exists a sequence $\{G^j\}$ of functions in this subspace for which

$$\sup_{[0,T]} |G(t) - G^j(t)|$$

tends to zero as j tends to infinity. Then,

$$\begin{aligned}|\mu(G) - a_G| &= |\mu(G) - a_G - \mu(G^j) + a_{G^j}| \\ &= |\int_\Omega [G(t) - G^j(t)]d\mu + a_G - a_{G^j}| \\ &\leq T \cdot \epsilon(j) + |a_G - a_{G^j}|.\end{aligned}$$

The two last terms in this expression tend to zero as j tends to infinity, (see [2], page 316), while the first term is independent of j, thus $\mu(G) = a_G$, and then $P \subset Q$, so

$$\eta = \inf_Q \mu(f_0) \leq \xi = \inf_P \mu(f_0). \tag{3.11}$$

Now from (3.10) and (3.11), we find that $\eta = \xi$, and the contention of the proposition follows. □

Let $Y_{M_1}(x, T)$, $x \in [0, S]$ be the final state attained by assuming $\mu \in Q(M_1)$. The question of whether the distance between $Y_{M_1}(x, T)$ and $g_1(x)$, the given terminal condition, in L_p $(1 \leq p \leq \infty)$, is small enough is a difficult problem for this equation, because, we believe, the lack of a damping term.

Assume $\mu^*_{Q(M_1)}(f_0)$ be the infimum of the optimal control problem (3.1) which is determined by using only M_1 constraints in the first equations of (3.1), by [85], for any $\epsilon > 0$ and an integer M_1, we can find a piecewise constant control u_{M_1}, such that by using this control one can get within ϵ of the minimum value $\mu^*(f_0)$ in L_1, that is

$$|\int_0^T f_0[t, u_{M_1}(t)]dt - \mu^*(f_0)| \leq \epsilon, \tag{3.12}$$

where $\mu^*(f_0) = \inf_Q \mu(f_0)$.

We can put together this result and those from Proposition 2.2 and indicate that given $\epsilon > 0$, we may chose M_1 so that u_{M_1} tends to satisfy the conditions of the problem. However, since Proposition 2.2 is based on the extraction of a convergent subsequence, it may be difficult in general to identify this control. In fact this result is as far as we can go in the search of an admissible control $u_{M_1}(\cdot)$ that gives rise to a final state $Y_{M_1}(x, T)$ near $g_1(x)$. In the especial cases as examples carried out in Chapter 2, we found that the desired final states $g_1(x)$'s were approximated very well in L_∞ with only small values of

M_1. In the following example, we will show these matters by indicating the maximum difference between $Y_{M_1}(x,T)$ and $g_1(x)$, that is

$$\|Y_{M_1}(x,T) - g_1(x)\|_{L_\infty}$$

for some examples in Chapter 2.

Example 3.1 Let

$$Y_{tt}(x,t) = Y_{xx}(x,t),$$

the initial conditions are:

$$Y(x,0) = f(x)$$
$$Y_t(x,0) = h(x),$$

the boundary conditions are:

$$Y(0,t) = u(t)$$
$$Y(\pi,t) = u(t).$$

We assume that the control function u is admissible, so the solution of the above optimal problem satisfies the terminal conditions (2.4) for some given functions $g_1(x)$, $g_2(x)$ and minimizes the functional $I[u(\cdot)]$ in (3.5). We showed in Chapter 2 that this optimal control problem can be reduced to the following one:

Minimize

$$\mu : f_0 \longrightarrow \mu(f_0)$$

subject to

$$\mu(\psi_n) = \alpha_n, \qquad n = 1,2,...,$$
$$\mu(G_r) = a_{G_r}, \qquad r = 0,1,..., \tag{3.13}$$

where from (3.7)

$$\psi_n(t,u(t)) = c_n\,(\sin(nt)+\cos(nt))\,u(t),$$
$$\alpha_n = A_n - B_n + (-1)^{n+1}a_n + (-1)^{n+1}\frac{b_n}{n},$$

and A_n, B_n, a_n, b_n, c_n are as in Section 2.1.

In chapter 2 we employed a discretization method to solve such a linear programming problem as (3.13) for various initial and terminal conditions defined in Examples 2.1-2.4, and then we used the associated pieciewise constant controls to draw the final states $Y(x,T)$. In the following table, we will show the maximum difference between $Y_{M_1}(x,T)$; the obtained terminal value, and $g_1(x)$; the given terminal value, for each of these examples. In all cases we have chosen $f_0 = u^2$, $M_1 = 15$, while in Example 2.1, $f(x) = \sin(x)$, $h(x) \equiv 0$, $g_1(x) \equiv 0$, $g_2(x) \equiv 0$, and in Example 2.2, $f(x) = \cos(x)$, $h(x) \equiv 0$, $g_1(x) \equiv 0$, $g_2(x) \equiv 0$, and in Example 2.3, $f(x) \equiv 0$, $h(x) \equiv 0$, $g_1(x) = \sin(x)$, $g_2(x) \equiv 0$, and in Example 2.4, $f(x) \equiv 0$, $h(x) = \cos(x)$, $g_1(x) = \sin(x)$, and $g_2(x) \equiv 0$, when $x \in [0,\pi]$.

M_1	Example	$\max\lvert Y_{M_1}(x,T) - g_1(x)\rvert$
15	Example 2.1	1.07×10^{-2}
15	Example 2.2	1.43×10^{-2}
15	Example 2.3	7.41×10^{-2}
15	Example 2.4	5.20×10^{-2}

3.3 The second type of approximation

In this section we consider the variation of M_2 in the following problem:

Minimize

$$\mu : f_0 \longrightarrow \mu(f_0)$$

over the set $Q(M_2) = \bigcap_{M_1=1}^{\infty} Q(M_1, M_2)$ of positive Radon measures as a subset of $\mathcal{M}^+(\Omega)$, satisfying

$$\mu(\psi_n) = \alpha_n, \qquad n = 1, 2, ...$$
$$\mu(G_r) = a_{G_r}, \qquad r = 0, 1, ...M_2, \; G_r = t^r,$$

where $\psi_n(t, u)$, and α_n, $n = 1, 2, ...$, are known functions. Let $\mu^*_{Q(M_2)}$ be the optimal measure over $Q(M_2)$, that is, $\mu^*_{Q(M_2)}(f_0) = \inf_{Q(M_2)} \mu(f_0)$, which satisfies

$$\mu^*_{Q(M_2)}(\psi_n) = \alpha_n, \qquad n = 1, 2, ...$$
$$\mu^*_{Q(M_2)}(G_r) = a_{G_r}, \qquad r = 0, 1, ...M_2. \tag{3.14}$$

Here we want to find M_2 such that for a given $\epsilon > 0$, (3.14) is satisfied and

$$|\mu^*_{Q(M_2)}(f_0) - \inf_Q \mu(f_0)| \leq \epsilon \tag{3.15}$$

where Q is the set of positive Radon measures μ satisfying (3.9), and $\mu^*(f_0) = \inf_Q \mu(f_0)$. Consider the following Lemma from [33] (Appendix B.2).

Lemma. 3.1 Let f_0 be a continuous function defined on a compact set $\Omega \subseteq I\!R^P$, and let $\epsilon > 0$ be given. We can divide Ω into a finite number of subsets, say, $\Omega_j, j = 1, 2, ..., m$ of equal volume or measure such that for every $x, x' \in \Omega_j, j = 1, 2, ..., m$

$$|f_0(x) - f_0(x')| \leq \epsilon.$$

Now suppose $\epsilon_1 > 0$ is given, then by Lemma 3.1 we can find numbers

$$0 = t_0 < t_1 < t_2 < ... < t_i < ... < t_R = \pi \tag{3.16}$$

and Borel sets $V_1, V_2, ..., V_j, ..., V_S$, forming a partition on $U = [-1, 1]$, such that for any $i = 1, 2, ..., R$, $j = 1, 2, ..., S$,

$$(t, t') \in [t_{i-1}, t_i), \; (u, u') \in V_j, \qquad |f_0(t, u) - f_0(t', u')| < \epsilon_1. \tag{3.17}$$

Let

$$K_{ij} = \mu^*_{Q(M_2)}([t_{i-1}, t_i) \times V_j), \tag{3.18}$$

then

$$\sum_{j=1}^{S} K_{ij} = \sum_{j=1}^{S} \mu^*_{Q(M_2)}([t_{i-1}, t_i) \times V_j).$$

Now, we define

$$\begin{aligned} F_i(t,u) &= 1 \qquad \text{if } (t,u) \in [t_{i-1}, t_i) \times U \\ F_i(t,u) &= 0 \qquad \text{otherwise,} \end{aligned}$$

then

$$\begin{aligned} \mu^*_{Q(M_2)}(F_i) &= \int_\Omega F_i d\mu^*_{Q(M_2)} = \int_{[t_0,t_1)\times U} F_i d\mu^*_{Q(M_2)} + \int_{[t_1,t_2)\times U} F_i d\mu^*_{Q(M_2)} \\ &+ \dots + \int_{[t_{i-1},t_i)\times U} F_i d\mu^*_{Q(M_2)} + \dots + \int_{[t_{R-1},t_R]\times U} F_i d\mu^*_{Q(M_2)} \\ &= \int_{[t_{i-1},t_i)\times U} d\mu^*_{Q(M_2)} = \mu^*_{Q(M_2)}([t_{i-1}, t_i) \times U); \end{aligned}$$

and so,

$$\sum_{j=1}^{S} K_{ij} = \mu^*_{Q(M_2)}(F_i). \tag{3.19}$$

We note that the functions F_i's, $i = 1, 2..., R$, are dependent on the time only. For $i = 1$, we extend F_i to the part of the t-axis for which $t \leq t_0$ and call this extension F_1^1, defined as

$$F_1^1(t_0 - h) = F_1^1(t_0 + h) = F_1(t_0 + h),\ 0 < h \leq \Delta_1 = t_1 - t_0.$$

For $i = R$ we extend F_R in a similar manner beyond $t = t_R$, and call this extension F_R^1. Let $\hat{F}_i$ be the function F_i, $i = 2, ..., R-1$ and F_i^1 for $i = 1, R$; then the $\hat{F}_i$'s have the following shape:

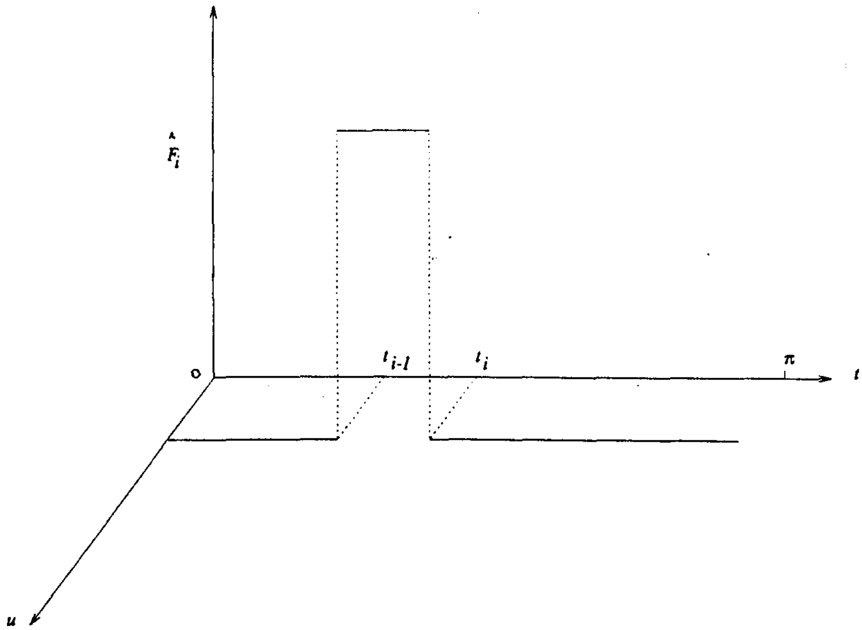

Figure 3.1: The graph of $\hat{F}_i$

Let $P_i^{M_2}$ be the function of time consisting of the first M_2+1 terms of the Chebyshev approximation of $\hat{F}_i$, $i = 1, 2, ...R$. Since the set$\{1, t, ..., t^{M_2}\}$, for every $t \in [t_0, t_R]$, satisfies the Haar condition (see [37] Chapter 6), thus it is a basis for $P_i^{M_2}$, and we can write

$$P_i^{M_2} = \sum_{r=0}^{M_2} \beta_{ir} t^r,\ i = 1, 2, ..., R.$$

Since $\mu^*_{Q(M_2)}$ satisfies (3.14),

$$\mu^*_{Q(M_2)}(G_r) = a_r,\ r = 0, 1, ..., M_2,\ G_r = t^r,$$

thus,

$$\begin{aligned}\mu^*_{Q(M_2)}(P_i^{M_2}) &= \mu^*_{Q(M_2)}(\sum_{r=0}^{M_2} \beta_{ir} G_r) = \sum_{r=0}^{M_2} \beta_{ir} a_r \\ &= \int_J [P_i^{M_2}(t) - \hat{F}_i(t)]dt + \int_J \hat{F}_i(t)dt \\ &= \Delta_i + \int_J [P_i^{M_2}(t) - \hat{F}_i(t)]dt\end{aligned}$$

where $\Delta_i = t_i - t_{i-1}$ and $J = [t_0, t_R] = [0, \pi]$.

We have

$$\mu^*_{Q(M_2)}(\hat{F}_i) = \mu^*_{Q(M_2)}(\hat{F}_i - P_i^{M_2}) + \mu^*_{Q(M_2)}(P_i^{M_2})$$
$$= \Delta_i + \int_J [P_i^{M_2}(t) - \hat{F}_i(t)]dt + \mu^*_{Q(M_2)}(\hat{F}_i - P_i^{M_2}) = \Delta_i + \delta_i^{M_2},$$

where,

$$\delta_i^{M_2} = \mu^*_{Q(M_2)}(\hat{F}_i - P_i^{M_2}) + \int_J [P_i^{M_2}(t) - \hat{F}_i(t)]dt.$$

We define

$$H_{ij} = K_{ij}(1 + \rho_i^{M_2}),$$

where,

$$\rho_i^{M_2} = -\frac{\delta_i^{M_2}}{\Delta_i + \delta_i^{M_2}}.$$

Then by (3.19) we have

$$\sum_{j=1}^{S} H_{ij} = \sum_{j=1}^{S} K_{ij} + \sum_{j=1}^{S} K_{ij}\rho_i^{M_2} = \mu^*_{Q(M_2)}(\hat{F}_i) + \rho_i^{M_2}\mu^*_{Q(M_2)}(\hat{F}_i)$$
$$= \mu^*_{Q(M_2)}(\hat{F}_i)(1 + \rho_i^{M_2}) = (\Delta_i + \delta_i^{M_2})(1 - \frac{\delta_i^{M_2}}{\Delta_i + \delta_i^{M_2}}) \qquad (3.20)$$
$$= \Delta_i.$$

Also we have

$$K_{ij} = \frac{H_{ij}}{1 + \rho_i^{M_2}} = \xi_i^{M_2} H_{ij}, \qquad (3.21)$$

where

$$\xi_i^{M_2} = \frac{1}{1 + \rho_i^{M_2}} = \frac{1}{1 - \frac{\delta_i^{M_2}}{\Delta_i + \delta_i^{M_2}}} = 1 + \frac{\delta_i^{M_2}}{\Delta_i}.$$

Now we proceed to construct a piecewise constant control which approximates the action of $\mu^*_{Q(M_2)}$ on the function f_0. Let u_j be an element of V_j, for $j = 1, 2, ...S$. Define

$u(t) = u_j$ if $t \in B_{ij}$, where

$$B_{ij} = [t_{i-1} + \sum_{k<j} H_{ik}, t_{i-1} + \sum_{k\leq j} H_{ik}).$$

Since those intervals B_{ij} for which $H_{ij} = 0$ are reduced to a point, they do not contribute anything to integrals such as those in the definition of the numbers l_{ij} below, and can be ignored. Thus without loss of generality, we assume $H_{ij} > 0$, $i = 1, 2, ..., R$, $j = 1, 2, ...S$. With each piecewise constant admissible control $u(\cdot)$ we may associate a measure μ_u; if μ_q is the measure associated with the piecewise constant control constructed above, then

$$l_{ij} = \int_{B_{ij}\times V_j} f_0(t,u)d\mu_q = \int_{B_{ij}} f_0(t,u_j)dt$$

for $i = 1, 2, ..., R$, $j = 1, 2, ..., S$. Then

$$H_{ij}I_{ij} \leq l_{ij} \leq H_{ij}S_{ij} \tag{3.22}$$

where

$$I_{ij} = \inf\{f_0(t,u) : (t,u) \in [t_{i-1}, t_i) \times V_j\}$$

$$S_{ij} = \sup\{f_0(t,u) : (t,u) \in [t_{i-1}, t_i) \times V_j\}.$$

If

$$T_{ij} = \int_{[t_{i-1},t_i)\times V_j} f_0 d\mu^*_{Q(M_2)},$$

then from (3.18),

$$K_{ij}I_{ij} \leq T_{ij} \leq K_{ij}S_{ij},$$

thus

$$\begin{aligned} H_{ij}I_{ij} \leq l_{ij} &\leq H_{ij}S_{ij} \\ -K_{ij}S_{ij} \leq -T_{ij} &\leq -K_{ij}I_{ij}, \end{aligned}$$

and so,

$$H_{ij}I_{ij} - K_{ij}S_{ij} \leq l_{ij} - T_{ij} \leq H_{ij}S_{ij} - K_{ij}I_{ij},$$

but from (3.21) $K_{ij} = \xi_i^{M_2} H_{ij}$, so,

$$H_{ij}(I_{ij} - \xi_i^{M_2} S_{ij}) \leq l_{ij} - T_{ij} \leq H_{ij}(S_{ij} - \xi_i^{M_2} I_{ij}),$$

or,

$$|l_{ij} - T_{ij}| \leq H_{ij} \max\left\{|I_{ij} - \xi_i^{M_2} S_{ij}|, |S_{ij} - \xi_i^{M_2} I_{ij}|\right\},$$

but,

$$\max\left\{|I_{ij} - \xi_i^{M_2} S_{ij}|, |S_{ij} - \xi_i^{M_2} I_{ij}|\right\}$$

$$= \max\left\{|I_{ij} - S_{ij} + S_{ij} - \xi_i^{M_2} S_{ij}|, |S_{ij} - I_{ij} + I_{ij} - \xi_i^{M_2} I_{ij}|\right\}$$
$$\leq \max\left\{|\epsilon_1 + (1 - \xi_i^{M_2}) S_{ij}|, |\epsilon_1 + (1 - \xi_i^{M_2}) I_{ij}|\right\},$$

so

$$|l_{ij} - T_{ij}| \leq H_{ij} \left\{\epsilon_1 + |1 - \xi_i^{M_2}| \max(|I_{ij}|, |S_{ij}|)\right\} \tag{3.23}$$

Adding all the inequalities in (3.23) with respect to i, j, we obtain

$$\left|\int_\Omega f_0 d\mu^*_{Q(M_2)} - \int_J f_0[t, u(t)]dt\right| = \left|\mu^*_{Q(M_2)}(f_0) - \int_J f_0[t, u(t)]dt\right|$$

$$\leq \epsilon_1 \sum_{i=1}^{R} \sum_{j=1}^{S} H_{ij} + \sum_{i=1}^{R} \sum_{j=1}^{S} H_{ij} |1 - \xi_i^{M_2}| \max(|I_{ij}|, |S_{ij}|)$$

$$\leq \epsilon_1 (t_R - t_0) + \sum_{i=1}^{R} \sum_{j=1}^{S} H_{ij} |1 - \xi_i^{M_2}| \max(|I_{ij}|, |S_{ij}|).$$

We know that $|1 - \xi_i^{M_2}| = \frac{|\delta_i^{M_2}|}{\Delta_i}$, thus

$$|1 - \xi_i^{M_2}| \max(|I_{ij}|, |S_{ij}|) = \frac{|\delta_i^{M_2}|}{\Delta_i} \max(|I_{ij}|, |S_{ij}|).$$

If we assume

$$\max(|I_{ij}|, |S_{ij}|) = s_{ij}$$

then if $s = \max_{ij} s_{ij}$, we have

$$|1 - \xi_i^{M_2}| \max(|I_{ij}|, |S_{ij}|) \leq \frac{|\delta_i^{M_2}|}{\Delta_i} s.$$

If we choose M_2 large enough so that

$$|\delta_i^{M_2}| \leq \frac{\epsilon_1 \cdot \Delta_i}{s},$$

then

$$|\mu^*_{Q(M_2)}(f_0) - \int_J f_0[t, u(t)]dt| \leq \epsilon_1(t_R - t_0) + \epsilon_1 \sum_{i=1}^{R} \sum_{j=1}^{S} H_{ij} = 2\epsilon_1(t_R - t_0).$$

This value of M_2 will be referred to as M_2^1. Thus, we have proved that by choosing M_2^1 large enough so that

$$|\delta_i^{M_2^1}| = |\mu^*_{Q(M_2)}(\hat{F}_i - P_i^{M_2^1}) + \int_J (P_i^{M_2^1} - \hat{F}_i)dt| \leq \frac{\epsilon_1 \cdot \Delta_i}{s},$$

there exists a piecewise constant control u such that

$$|\mu^*_{Q(M_2)}(f_0) - \int_J f_0[t, u(t)]dt| \leq 2\epsilon_1(t_R - t_0). \tag{3.24}$$

By a similar way if one assume

$$\hat{K}_{ij} = \mu^*([t_{i-1}, t_i) \times V_j)$$

then

$$\sum_{j=1}^{S} \hat{K}_{ij} = \mu^*(\hat{F}_i) = \Delta_i.$$

Define $\hat{u}(t) = u_j$ if $t \in \hat{B}_{ij}$, where

$$\hat{B}_{ij} = [t_{i-1} + \sum_{k<j} \hat{K}_{ik}, t_{i-1} + \sum_{k\leq j} \hat{K}_{ik}),$$

with each piecewise constant admissible control $u(\cdot)$ we may associate a measure μ_p; then

$$L_{ij} = \int_{\hat{B}_{ij}\times V_j} f_0(t,\hat{u})d\mu_p = \int_{\hat{B}_{ij}} f_0(t,u_j)dt$$

for $i = 1, 2, ..., R$, $j = 1, 2, ..., S$. Now

$$\hat{K}_{ij} I_{ij} \leq L_{ij} \leq \hat{K}_{ij} S_{ij},$$

assume

$$\hat{T}_{ij} = \int_{[t_{i-1},t_i)\times V_j} f_0 d\mu^*,$$

so

$$\hat{K}_{ij} I_{ij} \leq \hat{T}_{ij} \leq \hat{K}_{ij} S_{ij},$$

and

$$|L_{ij} - \hat{T}_{ij}| \leq \hat{K}_{ij}\epsilon_1.$$

Adding all the above inequalities with respect to i and j, we find that

$$|\mu^*(f_0) - \int_J f_0[t,\hat{u}(t)]dt| \leq \epsilon_1(t_R - t_0). \tag{3.25}$$

We assume $M_2 = M_2^1$. Let

$$L_{ij} = \int_{\hat{B}_{ij}} f_0(t,u_j)dt,\ i = 1, 2, ..., R,\ j = 1, 2, ..., S,$$

where

$$\hat{B}_{ij} = [t_{i-1} + \sum_{k<j} \hat{K}_{ik}, t_{i-1} + \sum_{k\leq j} \hat{K}_{ik}),$$

and

$$l_{ij} = \int_{B_{ij}} f_0(t,u_j)dt,\ i = 1, 2, ..., R,\ j = 1, 2, ..., S,$$

where

$$B_{ij} = [t_{i-1} + \sum_{k<j} H_{ik}, t_{i-1} + \sum_{k\le j} H_{ik}),$$

and $\hat{K}_{ij}$ and H_{ij} respectively are defined with respect to μ^* and $\mu^*_{Q(M_2)}$. We have

$$I_{ij}H_{ij} \le l_{ij} \le H_{ij}S_{ij}$$

$$I_{ij}\hat{K}_{ij} \le L_{ij} \le \hat{K}_{ij}S_{ij},$$

where

$$I_{ij} = \inf\{f_0(t,u):\ (t,u) \in [t_{i-1}, t_i) \times V_j\}$$
$$S_{ij} = \sup\{f_0(t,u):\ (t,u) \in [t_{i-1}, t_i) \times V_j\}.$$

Since,

$$I_{ij} \le \frac{l_{ij}}{H_{ij}} \le S_{ij}$$

$$I_{ij} \le \frac{L_{ij}}{\hat{K}_{ij}} \le S_{ij},$$

adding all the above inequalities with respect to j,

$$I_{ij} \le \frac{\sum_{j=1}^{S} l_{ij}}{\sum_{j=1}^{S} H_{ij}} \le S_{ij},$$

and

$$I_{ij} \le \frac{\sum_{j=1}^{S} L_{ij}}{\sum_{j=1}^{S} \hat{K}_{ij}} \le S_{ij},$$

or,

$$I_{ij} \le \frac{\int_{\Delta_i} f_0[t,u(t)]dt}{\Delta_i} \le S_{ij}$$
$$I_{ij} \le \frac{\int_{\Delta_i} f_0[t,\hat{u}(t)]dt}{\Delta_i} \le S_{ij}. \qquad (3.26)$$

But we have

$$(t,t') \in [t_{i-1}, t_i),\ (u,u') \in V_j, \qquad |f_0(t,u) - f_0(t',u')| < \epsilon_1$$

so, $|S_{ij} - I_{ij}| < \epsilon_1$, and from (3.26) we have

$$-\epsilon_1 < \frac{\int_{\Delta_i} f_0[t, \hat{u}(t)]dt}{\Delta_i} - \frac{\int_{\Delta_i} f_0[t, u(t)]dt}{\Delta_i} < \epsilon_1$$

or

$$-\epsilon_1 \Delta_i < \int_{\Delta_i} f_0[t, \hat{u}(t)]dt - \int_{\Delta_i} f_0[t, u(t)]dt < \epsilon_1 \Delta_i. \tag{3.27}$$

Adding all the inequalities in (3.27) with respect to $i = 1, 2, ..., R$ we obtain

$$-\epsilon_1(t_R - t_0) < \int_J f_0[t, \hat{u}(t)]dt - \int_J f_0[t, u(t)]dt < \epsilon_1(t_R - t_0),$$

or

$$|\int_J f_0[t, \hat{u}(t)]dt - \int_J f_0[t, u(t)]dt| < \epsilon_1(t_R - t_0). \tag{3.28}$$

Now by the inequality

$$|A - B| \leq |A - C| + |C - D| + |D - B|,$$

and from (3.24), (3.25) and (3.28) we have

$$|\mu^*(f_0) - \mu^*_{Q(M_2)}(f_0)| \leq |\mu^*(f_0) - \int_J f_0[t, \hat{u}(t)]dt|$$

$$+|\mu^*_{Q(M_2)}(f_0) - \int_J f_0[t, u(t)]dt| + |\int_J f_0[t, \hat{u}(t)]dt - \int_J f_0[t, u(t)]dt|$$

$$\leq \epsilon_1(t_R - t_0) + 2\epsilon_1(t_R - t_0) + \epsilon_1(t_R - t_0) = 4\epsilon_1(t_R - t_0) = 4\pi\epsilon_1.$$

Now if we want (3.15) to be satisfied, i.e., for a given $\epsilon > 0$

$$|\mu^*(f_0) - \mu^*_{Q(M_2)}(f_0)| \leq \epsilon,$$

we must have

$$4\epsilon_1(t_R - t_0) = 4\pi\epsilon_1 \leq \epsilon \text{ or } \epsilon_1 \leq \frac{\epsilon}{4\pi},$$

so (3.15) is satisfied if M_2 is large enough so that,

$$|\delta_i^{M_2}| = |\mu^*_{Q(M_2)}(\hat{F}_i - P_i^{M_2}) + \int_J (P_i^{M_2} - \hat{F}_i)dt| \leq \frac{\Delta_i \cdot \epsilon}{4\pi s}, \tag{3.29}$$

for $i = 1, 2, ..., R$. To estimate such positive integer M_2, we consider the following problem.

Assume $R+1$ distinct points $0 = t_0 < t_1 < ... < t_i < ... < t_R = \pi$ and the functions $\hat{F}_i$, $i = 1, 2, ..., R$; which defined before, be given. If the function $P_i^{M_2}$ consists of the first $M_2 + 1$ terms of the Chebyshev approximation of $\hat{F}_i$, $i = 1, 2, ...R$, then the error of approximation is

$$E_{M_2}(\hat{F}_i) \doteq \|\hat{F}_i - P_i^{M_2}\|$$

where$\|\cdot\|$ is L^1 norm in $[0, \pi]$, i.e.,

$$\|\hat{F}_i - P_i^{M_2}\| = \int_0^\pi |\hat{F}_i - P_i^{M_2}|dt.$$

Since the set $\{1, t, ..., t^{M_2}\}$ forms a basis for $P_i^{M_2}$ in any interval such as $[0, \pi]$, thus

$$P_i^{M_2} = \sum_{r=0}^{M_2} \beta_{ir} t^r,$$

and so

$$E_{M_2}(\hat{F}_i) \doteq \|\hat{F}_i - \sum_{r=0}^{M_2} \beta_{ir} t^r\|, \quad 0 \leq t \leq \pi.$$

We define $H_i(t)$, $i = 2, ..., R-1$ as follows:

$$H_i(t) = \begin{cases} 0 & 0 \leq t \leq t_{i-1} - e \\ \frac{1}{2e}(t - t_{i-1}) + \frac{1}{2} & t_{i-1} - e \leq t \leq t_{i-1} + e \\ 1 & t_{i-1} + e \leq t \leq t_i - e \\ \frac{-1}{2e}(t - t_i) + \frac{1}{2} & t_i - e \leq t \leq t_i + e \\ 0 & t_i + e \leq t \leq \pi \end{cases}$$

and define $H_1(t)$ as

$$H_1(t) = \begin{cases} 0 & t \leq -e \\ \frac{1}{2e}(t) + \frac{1}{2} & -e \leq t \leq e \\ 1 & e \leq t \leq t_1 - e \\ \frac{-1}{2e}(t - t_1) + \frac{1}{2} & t_1 - e \leq t \leq t_1 + e \\ 0 & t_1 + e \leq t \leq \pi \end{cases}$$

and $H_R(t)$ as

$$H_R(t) = \begin{cases} 0 & 0 \leq t \leq t_{R-1} - e \\ \frac{1}{2e}(t - t_{R-1}) + \frac{1}{2} & t_{R-1} - e \leq t \leq t_{R-1} + e \\ 1 & t_{R-1} + e \leq t \leq \pi - e \\ \frac{-1}{2e}(t - \pi) + \frac{1}{2} & \pi - e \leq t \leq \pi + e \\ 0 & t \geq \pi + e \end{cases}$$

when $e < \frac{t_i - t_{i-1}}{2}$ is a positive number.

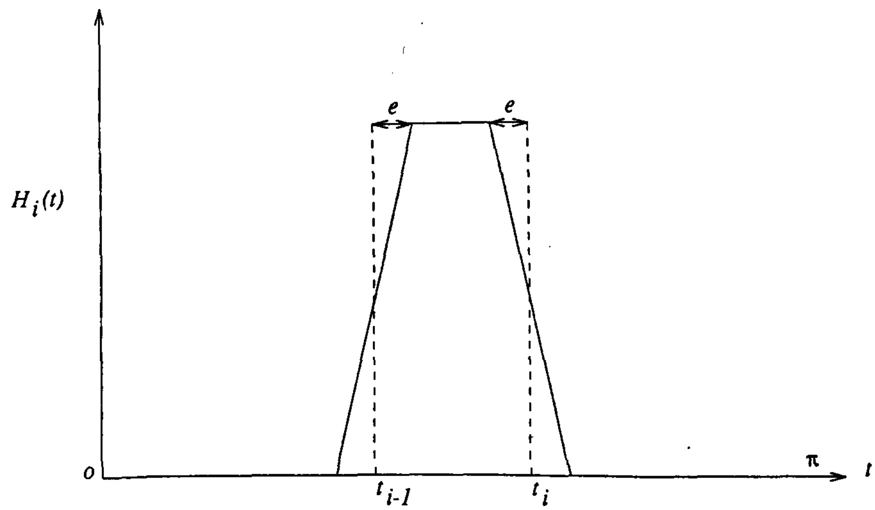

Figure 3.2: The graph of $H_i(t)$

Now

$$\|\hat{F}_i - P_i^{M_2}\| \leq \|\hat{F}_i - H_i\| + \|H_i - P_i^{M_2}\|,$$

where

$$\begin{aligned}\|\hat{F}_i - H_i\| &= \int_0^{\pi} |\hat{F}_i - H_i|dt = \int_0^{t_{i-1}-e} |\hat{F}_i - H_i|dt + \int_{t_{i-1}-e}^{t_{i-1}+e} |\hat{F}_i - H_i|dt \\ &+ \int_{t_{i-1}+e}^{t_i-e} |\hat{F}_i - H_i|dt + \int_{t_i-e}^{t_i+e} |\hat{F}_i - H_i|dt + \int_{t_i+e}^{\pi} |\hat{F}_i - H_i|dt \\ &= 0 + \int_{t_{i-1}-e}^{t_{i-1}+e} |\hat{F}_i - H_i|dt + 0 + \int_{t_i-e}^{t_i+e} |\hat{F}_i - H_i|dt + 0 \\ &= \int_{t_{i-1}-e}^{t_{i-1}} |\hat{F}_i - H_i|dt + \int_{t_{i-1}}^{t_{i-1}+e} |\hat{F}_i - H_i|dt \\ &+ \int_{t_i-e}^{t_i} |\hat{F}_i - H_i|dt + \int_{t_i}^{t_i+e} |\hat{F}_i - H_i|dt \\ &= \frac{1}{4}e + \frac{1}{4}e + \frac{1}{4}e + \frac{1}{4}e = e.\end{aligned}$$

Thus it is clear that as $e \to 0$, $\|\hat{F}_i - H_i\| \to 0$. We have

$$E_{M_2}(\hat{F}_i) \doteq \|\hat{F}_i - P_i^{M_2}\| \le e + \|H_i - P_i^{M_2}\|. \tag{3.30}$$

From Propositions 3.4-3.5 in Appendix 3.6 we have

$$\|H_i - P_i^{M_2}\| = \int_0^{\pi} |H_i - P_i^{M_2}|dt \le \pi e_n(H_i) \le 6\pi \times \frac{1}{2e}(\frac{\pi}{2M_2}) = \frac{3\pi^2}{2eM_2}.$$

Thus (3.30) reduces to the following inequality

$$E_{M_2}(\hat{F}_i) \le e + \frac{3\pi^2}{2eM_2}. \tag{3.31}$$

We now return to the main problem, from (3.29),

$$|\delta_i^{M_2}| = |\mu^*_{Q(M_2)}(\hat{F}_i - P_i^{M_2}) + \int_J (P_i^{M_2} - \hat{F}_i)dt|$$

$$\le |\mu^*_{Q(M_2)}(\hat{F}_i - P_i^{M_2})| + \int_J |P_i^{M_2} - \hat{F}_i|dt = |\mu^*_{Q(M_2)}(\hat{F}_i - P_i^{M_2})| + E_{M_2}(\hat{F}_i),$$

but

$$
\begin{aligned}
|\mu^*_{Q(M_2)}(\hat{F}_i - P_i^{M_2})| &= |\int_\Omega (P_i^{M_2} - \hat{F}_i) d\mu^*_{Q(M_2)}| \\
&\leq \int_\Omega |P_i^{M_2} - \hat{F}_i| d\mu^*_{Q(M_2)} \\
&= E_{M_2}(\hat{F}_i),
\end{aligned}
$$

so

$$|\delta_i^{M_2}| \leq E_{M_2}(\hat{F}_i) + E_{M_2}(\hat{F}_i) = 2E_{M_2}(\hat{F}_i),$$

and by (3.31),

$$|\delta_i^{M_2}| \leq 2e + \frac{3\pi^2}{eM_2}.$$

We assume $e = M_2^{-\frac{1}{2}}$, we may choose M_2 so large that $e \leq \frac{\Delta_i}{2}$, then

$$|\delta_i^{M_2}| \leq \frac{2}{M_2^{\frac{1}{2}}} + \frac{3\pi^2}{M_2^{\frac{1}{2}}}.$$

Thus in order (3.15) to be satisfied, we need to estimate M_2 such that for any $i = 1, 2, ..., R$,

$$\frac{2}{M_2^{\frac{1}{2}}} + \frac{3\pi^2}{M_2^{\frac{1}{2}}} \leq \frac{\epsilon \cdot \Delta_i}{4\pi s},$$

where $\Delta_i = t_i - t_{i-1}$, and $\epsilon > 0$ is given.

3.4 Consideration of pulse functions in the second type of approximation

In the numerical examples of Chapter 2, instead of using monomials $G_r = t^r$, we used piecewise constant continuous functions of the time only, called *pulse* functions, and in this way we found some accurate results. These signal time functions, defined over the time interval $(0, T)$, are like Rademacher and Walsh functions (see, for example, [82], [19], [3]) and can approximate any continuous function depending only on time arbitrarily well. Here we consider employment of such functions in the second type of approximation.

The problem which considered in the previous section is

Minimize

$$\mu : f_0 \longrightarrow \mu(f_0)$$

over the set $Q(M_2)$ of positive Radon measures satisfying

$$\mu(\psi_n) = \alpha_n, \qquad n = 1, 2, ...$$
$$\mu(G_r) = a_{G_r}, \qquad r = 0, 1, ...M_2, G_r = t^r.$$

We looked for number M_2 such that for a given positive $\epsilon > 0$,

$$|\mu^*_{Q(M_2)}(f_0) - \inf_Q \mu(f_0)| \le \epsilon.$$

Now instead of using monomials G_r in the second equalities above, we use a number M_2 of functions of the time only, to replace the functions $G_r, r = 0, 1, 2, ..., M_2$, where these new functions will be denoted by f_s, $s = 1, 2, ..., M_2$ and defined as

$$f_s(t) = 1 \qquad \text{if } t \in j_s$$
$$f_s(t) = 0 \qquad \text{otherwise,} \tag{3.32}$$

with $j_s = [t_0 + (s-1)d, t_0 + sd)$, and $d = \frac{\Delta t}{M_2}$, where $\Delta t = t_R - t_0 = \pi - 0 = \pi$, and M_2 is in fact the number of subintervals in partition of interval the $[0, \pi]$.

These functions are not continuous, and two remarks need to be made concerning their suitability:

(i) Each of functions $f_s, s = 1, 2, ..., M_2$ is the limit of an increasing sequence of positive continuous functions $\{f_{sk}\}$, so, if μ is any positive Radon measure on Ω then $\mu(f_s) = \lim_{k\to\infty} \mu(f_{sk})$.

(ii) Consider now the set of all such functions, for all positive integers M_2. The linear combinations of these functions can approximate any function in $C_1(\Omega)$ [here $C_1(\Omega)$, is the class of all continuous functions depending only on t] arbitrarily well, in the sense

that the essential supremum (see Friedman [20], page 57) of the error function can be made to tend to zero by choosing in an appropriate manner a sufficient number of terms in the corresponding expansion (see Rubio [59]).
By this replacement, the above problem changes to the following
Minimize

$$\mu : f_0 \longrightarrow \mu(f_0)$$

over the set $Q(M_2)$ of positive Radon measures satisfying

$$\begin{aligned} \mu(\psi_n) &= \alpha_n, \qquad n = 1, 2, ... \\ \mu(f_s) &= a_s, \qquad s = 1, 2, ..., M_2, \end{aligned} \tag{3.33}$$

where f_s is defined in (3.32) and a_s is the integral of f_s on $[0, T]$, i.e., $a_s = \int_0^T f_s dt = \frac{T}{M_2}$. Now the problem is to find M_2 such that for a given $\epsilon > 0$, (3.33) is satisfied and

$$|\mu^*_{Q(M_2)}(f_0) - \inf_Q \mu(f_0)| \leq \epsilon. \tag{3.34}$$

where in (3.34) $\mu^*_{Q(M_2)}(f_0) = \inf_{Q(M_2)} \mu(f_0)$ over the new set $Q(M_2)$.

The analysis leading to the second type of approximation can be carried out in the same way as before. We assume for any $\epsilon_1 > 0$ the partition in (3.16), i.e.,

$$0 = t_0 < t_1 < t_2 < ... < t_i < ... < t_R = \pi,$$

is such that $t_i - t_{i-1} = h > 0$ (h is independent of i) where $h = \frac{\Delta t}{R} = \frac{t_R - t_0}{R} = \frac{\pi}{R}$. By Lemma 3.1 there are Borel sets $V_1, V_2, ..., V_j, ..., V_S$, forming a partition on $U = [-1, 1]$, such that for any $i = 1, 2, ..., R$, $j = 1, 2, ...S$, we have, as (3.17),

$$(t, t') \in [t_{i-1}, t_i), \ (u, u') \in V_j, \qquad |f_0(t, u) - f_0(t', u')| < \epsilon_1.$$

Assume M_2 is chosen such that $h = n \cdot \frac{\Delta t}{M_2}$, for some positive integer n, then $M_2 = nR$.

Now, as before, we define

$$F_i(t,u) = 1 \qquad \text{if } (t,u) \in [t_{i-1}, t_i) \times U$$

$$F_i(t,u) = 0 \qquad \text{otherwise}, \quad i = 1, 2, ..., R,$$

and define $\hat{F}_i$ be the projection of F_i, $i = 1, 2, ..., R$, along the u-axis. Let $P_i^{M_2}$ be the function of time consisting of a number M_2 of f_s, $s = 1, 2, ..., M_2$, in the corresponding expansion of $\hat{F}_i$, $i = 1, 2, ..., R$, described below:
For $n = 1$,

$$P_i^{M_2} = \sum_{s=1}^{M_2} \alpha_s f_s, \text{ where } \alpha_i = 1, \ \alpha_s = 0, \text{ if } s \neq i,$$

so in this case, $P_i^{M_2} = f_i = \hat{F}_i$.
In Figure 3.3, we compare the graphs of $\hat{F}_i$ and $P_i^{M_2}$ in this case:

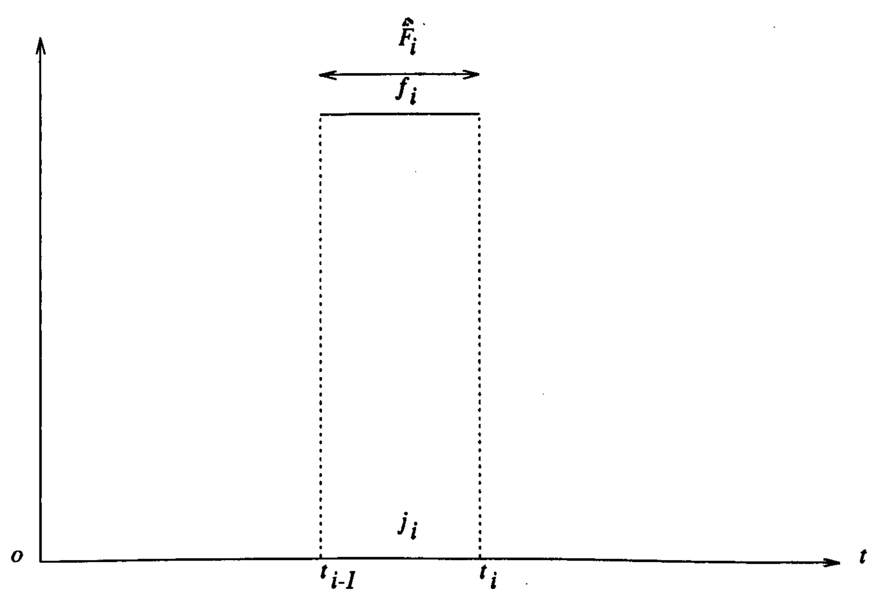

Figure 3.3: The graphs of $\hat{F}_i$ and $P_i^{M_2}$

For $n = 2$,

$$P_i^{M_2} = \sum_{s=1}^{M_2=2R} \alpha_s f_s, \text{ where } \alpha_{2i-1} = \alpha_{2i} = 1,\ \alpha_s = 0, \text{ if } s \neq 2i,\ s \neq 2i-1,$$

so in this case, $P_i^{M_2} = f_{2i-1} + f_{2i} = \hat{F}_i$.

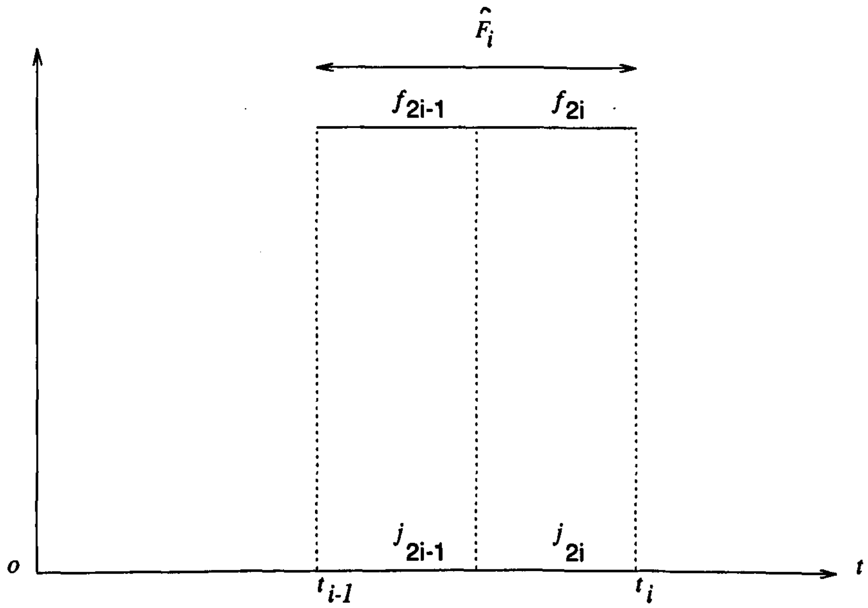

Figure 3.4: The graphs of $\hat{F}_i$ and $P_i^{M_2}$

For $n = 3$,

$$P_i^{M_2} = \sum_{s=1}^{M_2=3R} \alpha_s f_s, \text{ where } \alpha_{3i-2} = \alpha_{3i-1} = \alpha_{3i} = 1,\ \ \alpha_s = 0,\ \text{ otherwise}$$

so in this case, $P_i^{M_2} = f_{3i-2} + f_{3i-1} + f_{3i} = \hat{F}_i$.

The situation is the same for $n = 4, 5, \ldots$. Thus,

$$\mu^*_{Q(M_2)}(P_i^{M_2}) = \mu^*_{Q(M_2)}(\hat{F}_i) = \int_J \hat{F}_i dt = \Delta_i = h,$$

and as in the analysis of previous section

$$\delta_i^{M_2} = \mu^*_{Q(M_2)}(\hat{F}_i - P_i^{M_2}) + \int_J [P_i^{M_2}(t) - \hat{F}_i(t)]dt = 0.$$

Now, by using (3.21), (3.22), (3.23), (3.24) we can find a piecewise constant control u

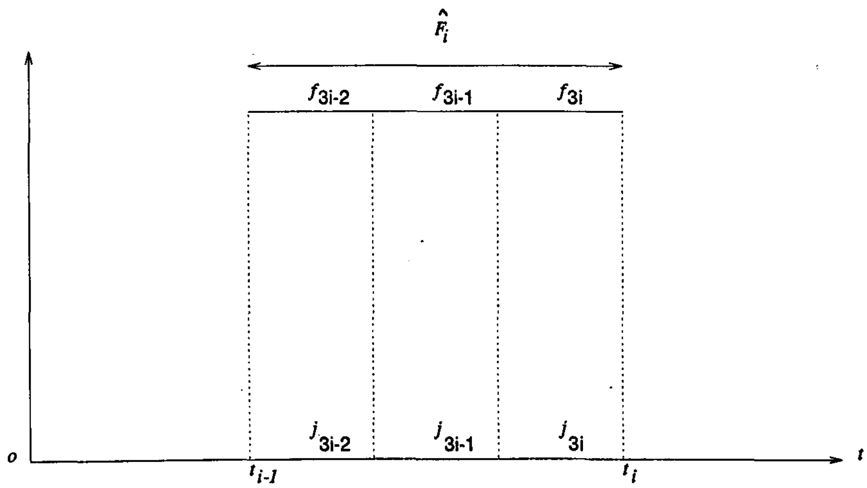

Figure 3.5: The graphs of $\hat{F}_i$ and $P_i^{M_2}$

such that,

$$|\mu^*_{Q(M_2)}(f_0) - \int_J f_0[t, u(t)]dt| \le \epsilon_1(t_R - t_0). \tag{3.35}$$

As we proved earlier in this chapter, we can find a piecewise constant control $\hat{u}$ such that

$$|\mu^*_Q(f_0) - \int_J f_0[t, \hat{u}(t)]dt| \le \epsilon_1(t_R - t_0). \tag{3.36}$$

Now from (3.17), by assuming $t \in [t_{i-1}, t_i)$ and $u \in V_j$, and using (3.26), (3.27), and (3.28) we find that

$$|\mu^*_{Q(M_2)}(f_0) - \mu^*_Q(f_0)| \le 3\pi\epsilon_1.$$

If we want (3.34) to be satisfied, that is for a given $\epsilon > 0$,

$$|\mu^*_{Q(M_2)}(f_0) - \mu^*_Q(f_0)| \le \epsilon,$$

we must have $\epsilon_1 \le \frac{\epsilon}{3\pi}$. So for a given $\epsilon > 0$, (3.34) will be satisfied if we can find the positive integers R and S, where R defining the following equidistant partition on $[0, T]$,

$$0 = t_0 < t_1 < t_2 < \ldots < t_i < \ldots < t_R = \pi$$

$h = t_i - t_{i-1}$, $i = 1, 2, \ldots, R$, and S denoting the Borel sets $V_1, V_2, \ldots, V_j, \ldots, V_S$, a partition on $U = [-1, 1]$, such that for any $i = 1, 2, ..., R$, $j = 1, 2, ..., S$, we have

$$(t, t') \in [t_{i-1}, t_i),\ (u, u') \in V_j, \qquad |f_0(t, u) - f_0(t', u')| \leq \frac{\epsilon}{3\pi}. \tag{3.37}$$

Now we can choose

$$M_2 = nR, \tag{3.38}$$

where n is a positive integer. It is clear that the best choice of M_2 is $M_2 = R$. Consider the following examples.

Example 3.2 In Example 2.3, we assumed $f_0 = u^2$, where $u \in [-1, 1]$ and $t \in [0, \pi]$.

Suppose $\epsilon > 0$ is given and we want to find M_2 such that

$$|\mu^*_{Q(M_2)}(f_0) - \inf_Q \mu(f_0)| \leq \epsilon.$$

We divide $U = [-1, 1]$ into S equidistant partitions V_j, $j = 1, 2, ..., S$ of length k, i.e., $d(V_j) = k$ and $[0, \pi]$ into R equidistant partitions $[t_{i-1}, t_i)$, $i = 1, 2, ..., R$ where $k = \Delta u = \frac{1-(-1)}{S} = \frac{2}{S}$, and $h = \Delta t = \frac{\pi - 0}{R} = \frac{\pi}{R}$, such that by (3.37)

$$(u, u') \in V_j, (t, t') \in [t_{i-1}, t_i) \quad |f_0(u) - f_0(u')| = |u^2 - u'^2| \leq \frac{\epsilon}{3\pi}.$$

Since $f_0 = u^2$ is independent of t, for a given $\epsilon > 0$, we may choose $R = 1$, and the length k small enough such that

$$(u, u') \in V_j, (t, t') \in [t_0, t_R) \quad |f_0(u) - f_0(u')| = |u^2 - u'^2| \leq \frac{\epsilon}{3\pi}.$$

Now by (3.38), we have $M_2 = nR = n$ where n can be any positive integer.

Example 3.3 In the above example we assume $f_0 = ut$, where $u \in [-1, 1]$, and $t \in [0, \pi]$.

Again we divide $[-1, 1]$ into S equidistant partitions V_j of length k and $[0, \pi]$ into R

equidistant partitions $[t_i, t_{i-1})$ of length h where $k = \frac{2}{S}$ and $h = \frac{\pi}{R}$, such that by (3.37)

$$(u, u') \in V_j, (t, t') \in [t_{i-1}, t_i), \quad |f_0(t, u) - f_0(t', u')| \leq \epsilon_1 = \frac{\epsilon}{3\pi}.$$

To find k, h, we have,

$$|f_0(t, u) - f_0(t', u')| = |tu - t'u'| = |t(u - u') + u'(t - t')|$$

$$\leq |t||u - u'| + |u'||t - t'| \leq \pi|u - u'| + |t - t'| \leq \pi k + h,$$

now by assuming, for example, $k \leq \frac{\epsilon_1}{2\pi}$, $h \leq \frac{\epsilon_1}{2}$,

$$|t - t'| \leq h, |u - u'| \leq k, \quad |f_0(t, u) - f_0(t', u'| \leq \epsilon_1 = \frac{\epsilon}{3\pi},$$

but $h = \frac{\pi}{R} \leq \frac{\epsilon_1}{2}$, and $k = \frac{2}{S} \leq \frac{\epsilon_1}{2\pi}$, so $R \geq \frac{2\pi}{\epsilon_1}$ and $S \geq \frac{4\pi}{\epsilon_1}$. If, for example, we are going to find M_2 such that by approximating $\mu_Q^*(f_0)$ with the solution of linear programming (3.33), $\mu_{Q(M_2)}^*(f_0)$, the absolute value of error be less than 0.05, i.e.,

$$|\mu_{Q(M_2)}^*(f_0) - \mu_Q^*(f_0)| \leq \epsilon = \frac{1}{2} \times 10^{-1},$$

we need to choose $\epsilon_1 = \frac{\epsilon}{3\pi} = \frac{1}{60\pi}$, so $R \geq \frac{2\pi}{\epsilon_1} = 120\pi^2 \approx 1185$. Now by (3.38) $M_2 = nR$, thus we can choose $M_2 = 1185$.

3.5 The third type of approximation

In this section we are going to construct a method to estimate the value of $\mu_Q^*(f_0)$ when M_1 and M_2 both are finite positive integers . This method does not give the exact value of the optimal solution but enables us to find an approximate value for it. First of all we change the original problem (3.1),

Minimize

$$\mu : f_0 \longrightarrow \mu(f_0)$$

subject to

$$\mu(\psi_n) = \alpha_n, \qquad n = 1, 2, ...,$$
$$\mu(G_r) = a_{G_r}, \qquad r = 0, 1, ...,$$

to a district linear programming problem, this metamorphosis can be done by using several propositions and a theorem. The new linear problem will have infinitely many constraints and variables and of course its solution is $\mu_Q^*(f_0)$. Now if we limit the number of nonzero variables in this metamorphosed problem and solve a semi-infinite linear programming problem, with finite number of variables but infinitely many equations, then the solutions that we get are feasible for the primal one, hence, they are superoptimal solutions, i.e., the value $\mu_Q^*(f_0)$ lies below of these solutions.
If instead we drop all but only M_1 and $M_2 + 1$ equations in (3.1), then we get the following semi-infinite linear program
Minimize

$$\mu : f_0 \longrightarrow \mu(f_0)$$

subject to

$$\mu(\psi_n) = \alpha_n, \qquad n = 1, 2, ..., M_1$$
$$\mu(G_r) = a_{G_r}, \qquad r = 0, 1, ..., M_2.$$

We call the solutions of this new reduced problem as $\mu^*_{Q(M_1,M_2)}(f_0)$. The sequence

$$\left\{\mu^*_{Q(M_1,M_2)}(f_0), M_1 = 1, 2, ..., M_2 = 1, 2, ...\right\},$$

is a nondecreasing convergent sequence bounded above by the primal optimal value $\mu^*(f_0)$ (see Proposition 3.1), so any term of this sequence is a lower bound for $\mu^*(f_0)$. By using the both solutions at the same time, one can get an upper bound and a lower bound for the optimal value $\mu_Q^*(f_0)$. The difference of these upper-lower bounds gives the missing error in the calculation of $\mu_Q^*(f_0)$. To specify the method we consider the variation of M_1 and M_2 in the following problem

Minimize

$$\mu : f_0 \longrightarrow \mu(f_0)$$

over the set of positive Radon measures $Q(M_1, M_2)$ as a subset of $\mathcal{M}^+(\Omega)$, such that

$$\begin{aligned} \mu(\psi_n) &= \alpha_n, \qquad n = 1, 2, ..., M_1 \\ \mu(G_r) &= a_{G_r}, \qquad r = 0, 1, ..., M_2, G_r = t^r \end{aligned} \tag{3.39}$$

where $\psi_n(t, u)$, and $\alpha_n, n = 1, 2, ..., M_1$, are known functions and $\Omega = [0, \pi] \times [-1, 1]$. Let $\mu^*_{Q(M_1,M_2)}$ be the optimal measure over $Q(M_1, M_2)$, that is, for which $\mu^*_{Q(M_1,M_2)}(f_0) = \inf_{Q(M_1,M_2)} \mu(f_0)$ and which satisfies (3.39). As we proved in Section 3.2

$$\lim_{M_1, M_2 \to \infty} \mu^*_{Q(M_1,M_2)}(f_0) = \mu^*_Q(f_0),$$

where Q is the set of positive Radon measures satisfying

$$\begin{aligned} \mu(\psi_n) &= \alpha_n, \qquad n = 1, 2, ... \\ \mu(G_r) &= a_{G_r}, \qquad r = 0, 1, ... \end{aligned}$$

For given positive integers M_1, M_2, we want to find $\epsilon > 0$, such that $\mu^*_{Q(M_1,M_2)}$ satisfies (3.39) and

$$|\mu^*_{Q(M_1,M_2)}(f_0) - \mu^*_Q(f_0)| \leq \epsilon. \tag{3.40}$$

The value ϵ in (3.40) is the estimated error in approximating the primal value $\mu^*_Q(f_0)$ by $\mu^*_{Q(M_1,M_2)}(f_0)$.

It is assumed that the first function appearing in the set of second equalities is $G_0 = 1$ and as we defined before $a_{G_0} = \int_0^T dt = T = \pi$, so we define Q as the set of positive

Radon measures satisfying

$$\mu(1) = T = \pi$$
$$\mu(\psi_n) = \alpha_n, \quad n = 1, 2, ...$$
$$\mu(G_r) = a_{G_r}, \quad r = 1, 2, ... \tag{3.41}$$

Now we construct a method to find an approximate solution for (3.41). This method is related to some methods appearing in [80], [76], [77] and [9]. In this construction we follow two basic steps:

(i) **First step**

Here we will describe a method to find a lower bound for $\mu_Q^*(f_0)$. Assume that $F_1 = 1, F_2 = \psi_1, ..., F_{M_1+1} = \psi_{M_1}, F_{M_1+2} = G_1, ..., F_{M_1+M_2+1} = G_{M_2}$, thus $Q(M_1, M_2)$ is the set of positive Radon measures on Ω satisfying

$$\mu(F_i) = \alpha_i, \quad \alpha_1 = \pi, i = 1, 2, ..., M_1 + M_2 + 1.$$

Proposition 3.2 Let the functions $F_1, F_2, ..., F_{M_1+M_2+1}$ be continuous on a compact Hausdorff topological space Ω, with $F_1(Z) = 1, Z \in \Omega$, and let $Q(M_1, M_2)$ be the set of positive Radon measures on Ω, $Q(M_1, M_2) = \{\mu \in \mathcal{M}^+(\Omega) : \mu(F_i) = \alpha_i, i = 1, 2, ..., M_1 + M_2 + 1\}$ with $\alpha_1 > 0$. Then if $Q(M_1, M_2)$ is nonempty, it is a compact convex subset of $\mathcal{M}^+(\Omega)$, and if $\mu^*_{Q(M_1,M_2)}$ is an extremal point of $Q(M_1, M_2)$, then it is of the form

$$\mu^*_{Q(M_1,M_2)} = \sum_{i=1}^{M_1+M_2+1} a_i \delta(Z_i), \quad Z_i \in \Omega, a_i \geq 0$$

where $\delta(Z_i)$ is the atomic measure with support the singleton set $\{Z_i\}$.

Proof. See [55] and [59]. □

Using Proposition 3.2, the minimization problem (3.39) changes to the following one:

Minimize

$$\mu_{Q(M_1,M_2)}(f_0) = \sum_{i=1}^{M_1+M_2+1} a_i f_0(Z_i), \quad Z_i \in \Omega, a_i \geq 0$$

subject to

$$\sum_{i=1}^{M_1+M_2+1} a_i = \pi = \alpha_1$$
$$\sum_{i=1}^{M_1+M_2+1} a_i\psi_j(Z_i) = \alpha_{j+1}, \qquad j = 1,2,...,M_1$$
$$\sum_{i=1}^{M_1+M_2+1} a_i G_r(Z_i) = \alpha_{G_r}, \qquad r = 1,2,...,M_2. \tag{3.42}$$

Since we have not used discretization, this problem sets up a semi-infinite linear programming problem; the Z_i's are in the set Ω but there are only a finite number of constraints. Since (3.42) contains less restrictions compared to the original problem (3.41) and since the number of these restrictions increases by increasing the number of variables, it is clear that the corresponding optimal values of the truncated problem (3.42) form a nondecreasing sequence, bounded from above by the infimum of the original problem posed on Q (Proposition 3.1). Therefore the solution of (3.42) gives us $\mu^*_{Q(M_1,M_2)}(f_0)$, which it is a suboptimal solution, i.e., it is a lower bound for the primal solution $\mu^*_Q(f_0)$.

(ii) **Second step**

To find an upper bound for $\mu^*_Q(f_0)$, we need to prove the following proposition.

Proposition 3.3 Let ω be a countable dense subset of Ω. Given $\epsilon' > 0$, a measure $\nu \in \mathcal{M}^+(\omega)$ can be found such that

$$|\mu^*_{Q(M_1,M_2)}(f_0) - \nu(f_0)| < \epsilon'$$
$$|\mu^*_{Q(M_1,M_2)}(F_i) - \nu(F_i)| < \epsilon',$$
$$i = 1,2,...,M_1 + M_2 + 1$$

and

$$\nu = \sum_{k=1}^{M_1+M_2+1} a_k\delta(Z_k), \quad Z_k \in \omega$$

Proof. The proof is same as that of Proposition III.3 in [59]. □

Let $P(M_1, M_2, \epsilon') \subseteq \mathbb{R}^N$ be the set of N ordered tuples $(a_1, a_2, ..., a_N)$ satisfying

$$
\begin{aligned}
a_i &\geq 0 \\
|\sum_{i=1}^{N} a_i - \pi| &\leq \epsilon' \\
|\sum_{i=1}^{N} a_i \psi_j(Z_i) - \alpha_{j+1}| &\leq \epsilon', \qquad j = 1, 2, ..., M_1 \\
|\sum_{i=1}^{N} a_i G_r(Z_i) - a_{G_r}| &\leq \epsilon', \qquad r = 1, 2, ..., M_2, \qquad (3.43)
\end{aligned}
$$

where $Z_i \in \omega$, $i = 1, 2, ..., N$ and ω is a dense subset of Ω.

Theorem. 3.1 For every $\epsilon' > 0$, the problem of minimizing the functional

$$\sum_{i=1}^{N} a_i f_0(Z_i),$$

on the set $P(M_1, M_2, \epsilon')$ has a solution for $N = N(\epsilon')$ sufficiently large, the solution satisfies

$$\mu^*_{Q(M_1,M_2)}(f_0) + \rho(\epsilon') \leq \sum_{i=1}^{N} a_i f_0(Z_i) \leq \mu^*_{Q(M_1,M_2)}(f_0) + \epsilon'$$

where the non-positive $\rho(\epsilon')$ tends to zero as ϵ' tends to zero.

Proof. The proof is same as that of Theorem III.1 of [59]. □

The parameter ϵ' appearing in the Theorem 3.1 can be considered as the error present in numerical computation of the expressions involved in the definition of the set $P(M_1, M_2, \epsilon')$. When setting up the linear programming problem akin to the theorem, it was assumed that ϵ' is equal to zero.

By this theorem our linear programming problem consists of minimizing the linear form

$$\mu_{Q(M_1,M_2)}(f_0) = \sum_{i=1}^{N} a_i f_0(Z_i)$$

over the set of coefficients $a_i \geq 0$, subject to

$$\begin{aligned}
&\sum_{i=1}^{N} a_i = \pi = \alpha_1 \\
&\sum_{i=1}^{N} a_i \psi_j(Z_i) = \alpha_{j+1}, \qquad j = 1, 2, ..., M_1 \\
&\sum_{i=1}^{N} a_i G_r(Z_i) = a_{G_r}, \qquad r = 1, 2, ..., M_2, \qquad (3.44)
\end{aligned}$$

where $Z_i \in \omega$; a dense subset of Ω. Define $\omega_N = \{Z_k, k = 1, 2, ..., N\}$, then it is obvious that $\omega_N \subseteq \omega$.

Assume

$$S_{HK} = \left\{ \left(\frac{\pi}{2H} + \frac{(i-1)\pi}{H}, -1 + \frac{1}{K} + \frac{2(j-1)}{K}\right) |, \; i = 1, 2, ..., H, \; j = 1, 2, ..., K \right\}$$

where H and K are positive integers, then $\omega = \cup_{H\&K \geq 1} S_{HK}$. Now let λ be the following transformation

$$\lambda : N \times N \longrightarrow N$$

defined by $\lambda(i, j) = H(j-1) + i, i = 1, 2, ..., H, \; j = 1, 2, ..., K$, and let the set $\omega_M = \{Z_l, l = 1, 2, ..., M = HK\} \subset \omega$ be the set of points that are defined as

$$Z_l = Z_{\lambda(i,j)} = Z(t_i, u_j) = \left(\frac{\pi}{2H} + \frac{(i-1)\pi}{H}, -1 + \frac{1}{K} + \frac{2(j-1)}{K}\right) \in S_{HK}$$

$$i = 1, 2, ..., H, \; j = 1, 2, ..., K$$

where $l = H(j-1) + i$; for sufficiently large H and K, the set ω_M will contain the set ω_N.

Now we assume that M_1 and M_2 tend to infinity, thus in the linear programming problem (3.44), N tends to infinity, and since ω_N is contained in ω_M, so $M = HK$ tends to infinity. Once M_1, M_2, N tend to infinity, the problem (3.44) changes to the following one

Minimize

$$\sum_{i=1}^{\infty} a_i f_0(Z_i), \quad Z_i \in \omega, \; a_i \geq 0$$

subject to

$$\sum_{i=1}^{\infty} a_i = \pi = \alpha_1$$
$$\sum_{i=1}^{\infty} a_i \psi_j(Z_i) = \alpha_{j+1}, j = 1, 2, \ldots$$
$$\sum_{i=1}^{\infty} a_i G_r(Z_i) = a_{G_r}, \; r = 1, 2, \ldots, \tag{3.45}$$

when $\omega = \cup_{H\&K \geq 1} S_{HK}$.

If the set Q is non-empty, then the solution of (3.45) is $\mu_Q^*(f_0)$. The new linear programming problem (3.45) is an infinite-dimensional linear programming. To find an upper bound for the solution of (3.45), we restrict the variables in (3.45) while the constraints are still infinite, i.e., we consider the following problem

Minimize

$$\sum_{i=1}^{N'} a_i f_0(Z_i), \quad Z_i \in \omega, \; a_i \geq 0$$

subject to

$$\sum_{i=1}^{N'} a_i = \pi = \alpha_1$$
$$\sum_{i=1}^{N'} a_i \psi_j(Z_i) = \alpha_{j+1}, j = 1, 2, \ldots$$
$$\sum_{i=1}^{N'} a_i G_r(Z_i) = a_{G_r}, \; r = 1, 2, \ldots. \tag{3.46}$$

In fact, in (3.46), we restrict the number of nonzero variables to N' and we want to solve again a semi-infinite linear programming problem with infinitely many equations. The solutions that we get are feasible for the primal problem (3.45), hence, they are superoptimal, i.e., the values of the objective function of (3.46) lie above the primal value, so we can get an upper bound for the primal value $\mu_Q^*(f_0)$.

Although it is clear in principle what we should do, but still two major obstacles remain: How do we solve the truncated problems (3.42) and (3.46)?

Let us first discuss the solution of the problem (3.42). As mentioned before, this problem is a semi-infinite linear programming problem. We have chosen here a method due to Glashoff and Gustafson (see [24]) to solve such a problem. This is an iterative method and it consists of six steps. The most difficult problem encountered in using this method was that of finding an initial solution from which one can start the method and find successive values towards the minimum. This was done here by means of finite-dimensional linear program, obtained by discretization. In step (E_3) of the method, we used the AMOEBA routine (see [52]) based on Nelder-Mead [50] to find the minimum of a function of several scalar variables. Of course, since we had a constrained domain $\Omega = [0,\pi] \times [-1,1]$ in this minimization, we used the following transformation from [4],

$$\tau = \pi - \pi \sin^2 t$$

$$\vartheta = 1 - 2\sin^2 u.$$

The situation is more complicated for the problem (3.46). In fact we did not find in literature any method to solve such kind of problems, except in special cases, where the semi-finite problem (3.46) changes to a finite-dimensional one. (e.g., see Mendlovitz [47], Staffans [76]-[77]). We attempt here a method to find an approximate value of the solution (3.46), (see [9]).

The dual of the problem (3.46) is as follows

Maximize

$$\sum_{i=1}^{\infty} \beta_i \alpha_i$$

subject to

$$\sum_{i=1}^{\infty} \beta_i F_i(Z_1) \le f_0(Z_1)$$

$$\sum_{i=1}^{\infty} \beta_i F_i(Z_2) \le f_0(Z_2)$$

$$\vdots$$

$$\sum_{i=1}^{\infty} \beta_i F_i(Z_{N'}) \le f_0(Z_{N'}), \tag{3.47}$$

where the F_i's are the known functions ψ_i's and G_r's.

We restrict once more the variables, this time in (3.47) that is, we assume

$$\beta_i = 0, \quad i > M$$

for some positive integer M. So this new problem is as follows

Maximize

$$\sum_{i=1}^{M} \beta_i \alpha_i$$

subject to

$$\beta_1 F_1(Z_1) + ... + \beta_M F_M(Z_1) \leq f_0(Z_1)$$
$$\beta_1 F_1(Z_2) + ... + \beta_M F_M(Z_2) \leq f_0(Z_2)$$
$$\vdots$$
$$\beta_1 F_1(Z_{N'}) + ... + \beta_M F_M(Z_{N'}) \leq f_0(Z_{N'}). \tag{3.48}$$

Thus in (3.48) we have restricted again the number of nonzero variables. This new problem is a finite-dimensional linear programming problem with finite variables and constraints. The solutions of this final problem are a lower bound for the solution of the semi-infinite problem (3.47) (problem (3.47) is a maximization problem). Now let the number of variables M in (3.48) increases, for larger value of M the value of objective function in (3.48) increases; if we assume that the value of the objective function of (3.48) in this case is $I_{M,N'}(f_0)$, then by increasing M we give rise to successive values of $I_{M,N'}(f_0)$. Since the sequence $\{I_{M,N'}(f_0), M = 1, 2, ...\}$ converges to superoptimal solution of (3.47), one can find, by increasing M, an approximate value of the superoptimal to any desired accuracy.

By using the both solutions of (3.42) and (3.47) at the same time, we have access to an upper bound as well as a lower bound of the optimal value $\mu_Q^*(f_0)$, and the difference of these upper-lower bounds gives the missing error bound that permits one to truncate the problem and gets within ϵ of the optimal solution.

To illustrate the method, we consider the following example. Note that N' can be any positive integer for which we can find a feasible solution in (3.48). In the next example, we have chosen it as $N' = 800$, which is in fact the number of the points in discretization of Ω.

Example 3.4 Consider the following one-dimensional linear wave equation (Example 2.3)

$$Y_{tt}(x,t) = Y_{xx}(x,t),$$

the initial conditions are:

$$Y(x,0) = f(x)$$

$$Y_t(x,0) = h(x),$$

and the boundary conditions are:

$$Y(0,t) = u(t)$$

$$Y(S,t) = u(t).$$

The problem is to find the admissible control $u(t)$ such that at $t = T = \pi$,

$$Y(x,T) = \sin(x)$$
$$Y_t(x,T) = 0,$$

and minimizes the functional $I = \int_0^\pi u(t)^2 dt$. This control problem is reduced to the following one

Minimize

$$\mu : f_0 \longrightarrow \mu(f_0)$$

over the set Q of positive Radon measures such that,

$$\mu(\psi_n) = \alpha_n, \qquad n = 1,2,...,$$
$$\mu(f_s(t)) = b_s, \qquad s = 1,2,...$$

$f_0 = u^2$, $\psi_n(t,u(t)) = \frac{4}{\pi}(\sin(2n-1)t + \cos(2n-1)t)\,u(t)$, $\alpha_n = -a_n \cos((2n-1)\pi)$, where a_n is the Fourier coefficient of $\sin(x)$, $n = 1,2,...$, $f_s(t)$ is defined as (3.32) and $b_s = \int_0^\pi f_s(t)dt$, $s = 1, 2, \ldots$. In $Q(M_1, M_2)$; a subset of positive Radon measures of $\mathcal{M}^+(\Omega)$ satisfying in (3.39), we assume $M_1 = 6$, $M_2 = 10$, so as Example 2.3

$$\psi_1 = (\sin t + \cos t)u$$
$$\psi_2 = (\sin 3t + \cos 3t)u$$
$$\psi_3 = (\sin 5t + \cos 5t)u$$
$$\psi_4 = (\sin 7t + \cos 7t)u$$
$$\psi_5 = (\sin 9t + \cos 9t)u$$
$$\psi_6 = (\sin 11t + \cos 11t)u$$

and

$$f_s(t) = \begin{cases} 1 & \frac{(s-1)\pi}{10} \leq t < \frac{s\pi}{10} \\ 0 & \text{otherwise} \end{cases}$$

Thus the linear programming problem akin to the (3.42) is as the following

Minimize

$$\sum_{i=1}^{16} a_i f_0(Z_i)$$

subject to

$$\sum_{i=1}^{16} a_i \psi_j(Z_i) = \alpha_j, \qquad j = 1, 2, \ldots, 6$$
$$\sum_{i=1}^{16} a_i f_s(Z_i) = b_s, \qquad s = 1, 2, \ldots, 10,$$

$a_i \geq 0$, $Z_i \in \Omega = [0, \pi] \times [-1, 1]$. Of course we do not need $\sum_{i=1}^{16} a_i = \pi$, since the last equalities automatically give this result.

This is a semi-infinite linear programming problem with a suboptimal solution. For treating this problem numerically, we choose here the simplex algorithm of Glashoff and Gustafson. To start the algorithm one need a basic set from which can be iterated toward the minimum. Here we find a basic set by means of a finite-dimensional linear program obtained by discretization. We cover the set $\Omega = [0, \pi] \times [-1, 1]$ with a grid by taking $m_1 + 1$ and $m_2 + 1$ points along the t-axis, and u-axis each separately in the ordered mentioned. These points are equidistant of $\frac{\pi}{m_1}$ and $\frac{2}{m_2}$ respectively. Now Ω is divided into $N = m_1 \cdot m_2$ number of equal volume rectangles Ω_j, $j = 1, 2, ..., N$, and $Z_j = (t_j, u_j)$ is chosen as the center of Ω_j. We assume $m_1 = 10$, $m_2 = 10$, so $N = 100$. Then a finite-dimensional simplex program is run, with 100 variables and, of course, 16 constraints. The minimum obtained is 0.807839 with only 16 variables

$$a_{30},\ a_{39},\ a_{40},\ a_{48},\ a_{49},\ a_{57},\ a_{58},\ a_{66},\ a_{67},\ a_{71},\ a_{75},\ a_{81},\ a_{82},\ a_{83},\ a_{84},\ a_{85}$$

which are nonzero. Now for the basic set we choose

$$\begin{aligned}
s_1 &= (t_{30}, u_{30}) = (-0.5, \frac{19\pi}{20}) \\
s_2 &= (t_{39}, u_{39}) = (-0.3, \frac{17\pi}{20}) \\
s_3 &= (t_{40}, u_{40}) = (-0.3, \frac{19\pi}{20}) \\
s_4 &= (t_{48}, u_{48}) = (-0.1, \frac{15\pi}{20}) \\
s_5 &= (t_{49}, u_{49}) = (-0.1, \frac{17\pi}{20}) \\
s_6 &= (t_{57}, u_{57}) = (0.1, \frac{13\pi}{20}) \\
s_7 &= (t_{58}, u_{58}) = (0.1, \frac{15\pi}{20}) \\
s_8 &= (t_{66}, u_{66}) = (0.3, \frac{11\pi}{20}) \\
s_9 &= (t_{67}, u_{67}) = (0.3, \frac{13\pi}{20}) \\
s_{10} &= (t_{71}, u_{71}) = (0.5, \frac{\pi}{20}) \\
s_{11} &= (t_{75}, u_{75}) = (0.5, \frac{9\pi}{20}) \\
s_{12} &= (t_{81}, u_{81}) = (0.7, \frac{\pi}{20}) \\
s_{13} &= (t_{82}, u_{82}) = (0.7, \frac{3\pi}{20})
\end{aligned}$$

$$s_{14} = (t_{83}, u_{83}) = (0.7, \frac{5\pi}{20})$$
$$s_{15} = (t_{84}, u_{84}) = (0.7, \frac{7\pi}{20})$$
$$s_{16} = (t_{85}, u_{85}) = (0.7, \frac{9\pi}{20}).$$

Then the simplex algorithm of Glashoff and Gustafson is applied by using this basic set, and after several iteration the value of 0.7853981 is resulted as the best improvement for the value of objective function. So

$$\mu^*_{(6,10)}(f_0) = 0.7853981,$$

and this value is a suboptimal value, i.e., it is a lower bound for $\mu^*_Q(f_0)$.

We apply then the method introduced in this section for the estimation of a superoptimal solution for the problem (3.47). We start to produce successive values of $I_{M,N'}(f_0)$ by increasing M and using the solutions of the linear programming problem (3.48) in an appropriate manner by choosing right functions ψ_i's and f_s's. Note that for a successful termination of this scheme we require that the following error test for all values of M and M' to be greater than a positive integer N, be satisfied,

$$|I_{M,N'}(f_0) - I_{M',N'}(f_0)| \leq \frac{1}{2} \times 10^{-n},$$

where n is a positive integer and indicates the correct significant digits in the values of $I_{M,N'}(f_0)$ (see [7]). Of course the scheme also terminates if a feasible solution is not found.

To get an overview of this method, we show the results of the solutions of the linear program (3.48) for various numbers of M, in the following table.

M	$I_{M,N'}(f_0) = \sum_{j=1}^{M} a_j f_0(Z_j)$	M	$I_{M,N'}(f_0) = \sum_{j=1}^{M} a_j f_0(Z_j)$
16	0.7855724	160	0.8076010
20	0.7856818	170	0.8110139
30	0.7860472	180	0.8115679
40	0.7865888	190	0.8126507
50	0.7874906	192	0.8132239
60	0.7884763	193	0.8167052
70	0.7900899	194	0.8167381
80	0.7908173	195	0.8174612
90	0.8003187	196	0.8175006
100	0.8014668	197	0.8191731
110	0.8016821	198	0.8191749
120	0.8019248	199	0.8202239
130	0.8021950	200	0.8202265
140	0.8026118	201	0.8202269
150	0.8040056	202	0.8202272

We did not find feasible solution for this problem when we increased M. In Figure 3.6, we show the graph of the function $I_{M,N'}(f_0)$ in term of some values of M, as a piece-wise constant function. Since from the number $N = 202$, by choosing $M > N$, we did not find feasible solution, so the final point of the sequence $\{I_{M,N'}(f_0)\}$ is the number 0.8202272, i.e., the sequence $\{I_{M,N'}(f_0)\}$ converges to 0.8202272, which this number is a superoptimal value.

Therefore, we have found a lower bound as well as an upper bound for the primal solution $\mu_Q^*(f_0)$, that is

$$0.785398 < \mu_Q^*(f_0) < 0.8202272$$

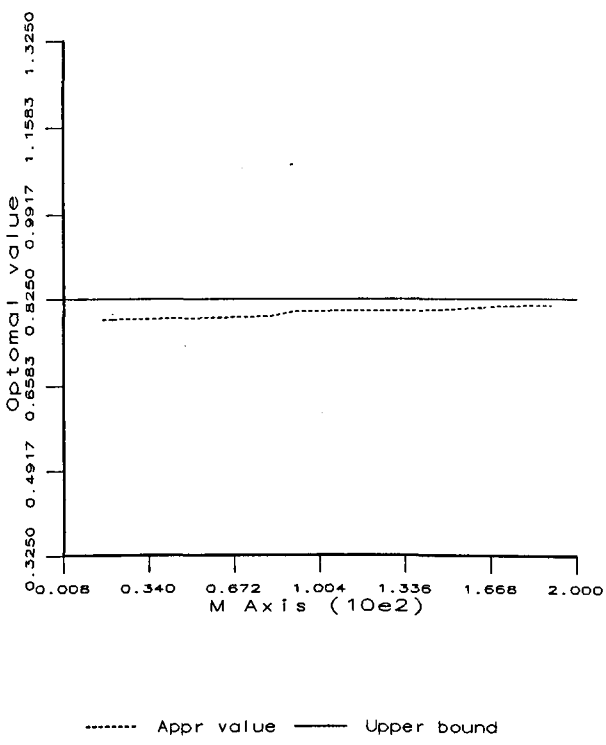

Figure 3.6: District graph of $I_{M,N'}(f_0)$

In Example 2.3, with $M_1 = 6$, $M_2 = 10$ we found $\mu_Q^*(f_0) \approx 7.903563 \times 10^{-1}$, that the most absolute error in this computation is

$$d = |7.903563 \times 10^{-1} - 0.8202272| = 0.0298709.$$

3.6 Appendix

Remark. If the function $f(x)$ satisfies

$$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|^{\alpha}$$

for $x_1,\ x_2 \in [a, b]$ and $\alpha > 0$, then $f(x)$ is said to satisfy the Lipschitz condition of order α with constant K on $[a, b]$.

Proposition 3.4 Let the numbers

$$0 = t_0 < t_1 < ... < t_i < ... < t_R = \pi$$

define a partition in the interval $[0, \pi]$. Then the functions $H_i(t)$ defined as

$$H_i(t) = \begin{cases} 0 & 0 \leq t \leq t_{i-1} - e \\ \frac{1}{2e}(t - t_{i-1}) + \frac{1}{2} & t_{i-1} - e \leq t \leq t_{i-1} + e \\ 1 & t_{i-1} + e \leq t \leq t_i - e \\ \frac{-1}{2e}(t - t_i) + \frac{1}{2} & t_i - e \leq t \leq t_i + e \\ 0 & t_i + e \leq t \leq \pi \end{cases}$$

for $i = 2, 3, ..., R - 1$, and $H_1(t)$ defined as

$$H_1(t) = \begin{cases} 0 & t \leq -e \\ \frac{1}{2e}(t) + \frac{1}{2} & -e \leq t \leq e \\ 1 & e \leq t \leq t_1 - e \\ \frac{-1}{2e}(t - t_1) + \frac{1}{2} & t_1 - e \leq t \leq t_1 + e \\ 0 & t_1 + e \leq t \leq \pi \end{cases}$$

and $H_R(t)$ as

$$H_R(t) = \begin{cases} 0 & 0 \leq t \leq t_{R-1} - e \\ \frac{1}{2e}(t - t_{R-1}) + \frac{1}{2} & t_{R-1} - e \leq t \leq t_{R-1} + e \\ 1 & t_{R-1} + e \leq t \leq \pi - e \\ \frac{-1}{2e}(t - \pi) + \frac{1}{2} & \pi - e \leq t \leq \pi + e \\ 0 & t \geq \pi + e. \end{cases}$$

are satisfying the Lipschitz conditions for $t \in [0, \pi]$, when $e < \frac{t_i - t_{i-1}}{2}$ is a positive number.

Proof. We choose $\alpha = 1$ and $K = \frac{1}{2e}$, thus $H_i(t)$, $i = 1, 2, ..., R$ will satisfies the Lipschitz condition if

$$|H_i(\tau_1) - H_i(\tau_2)| \leq \frac{1}{2e}|\tau_1 - \tau_2| \tag{3.49}$$

for every $\tau_1 \leq \tau_2$ and $\tau_1,\ \tau_2 \in [0, \pi]$.

1. It is clear that if $\tau_1, \tau_2 \in [0, t_{i-1} - e] \cup [t_i + e, \pi]$ then (3.49) is satisfied.

2. If $\tau_1 \in [0, t_{i-1} - e] \cup [t_i + e, \pi]$ and $\tau_2 \in [t_{i-1} + e, t_i - e]$ or $\tau_2 \in [0, t_{i-1} - e] \cup [t_i + e, \pi]$ and $\tau_1 \in [t_{i-1} + e, t_i - e]$ then $|H_i(\tau_1) - H_i(\tau_2)| = 1$ and $|\tau_1 - \tau_2| \geq 2e$, or $\frac{|\tau_1 - \tau_2|}{2e} \geq 1$, thus

$$|H_i(\tau_1) - H_i(\tau_2)| \leq \frac{1}{2e}|\tau_1 - \tau_2|.$$

3. If $\tau_1 \in (t_{i-1} - e, t_{i-1} + e)$ and $\tau_2 = t_{i-1} + e$, then from the following shape $\frac{H_i(\tau_2)}{H_i(\tau_1)} = \frac{\tau_2 - \alpha}{\tau_1 - \alpha}$, or, $\frac{1}{H_i(\tau_2) - H_i(\tau_1)} = \frac{\tau_2 - \alpha}{\tau_2 - \tau_1}$, (since in this case $H_i(\tau_2) = 1$), thus

$$|H_i(\tau_2) - H_i(\tau_1)| = |\tau_2 - \tau_1| \cdot \frac{1}{|\tau_2 - \alpha|} = \frac{1}{2e}|\tau_2 - \tau_1|.$$

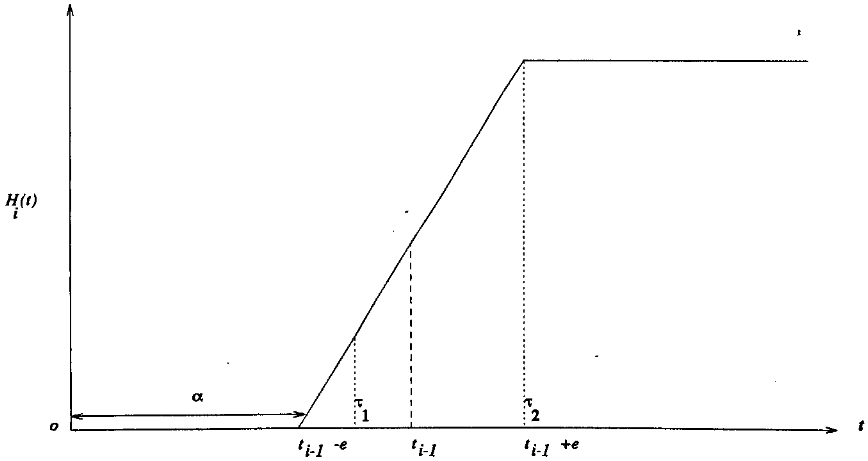

Figure 3.7: The graph of $H_i(t)$

4. From above it is clear that if $\tau_1 \in [t_{i-1} - e, t_{i-1} + e]$ and $\tau_2 \in [t_{i-1} + e, t_i - e]$ then (3.49) is satisfied.

5. If $\tau_1 \in (t_{i-1} - e, t_{i-1} + e]$ and $\tau_2 \in (t_{i-1} - e, t_{i-1} + e]$, then from the following shape

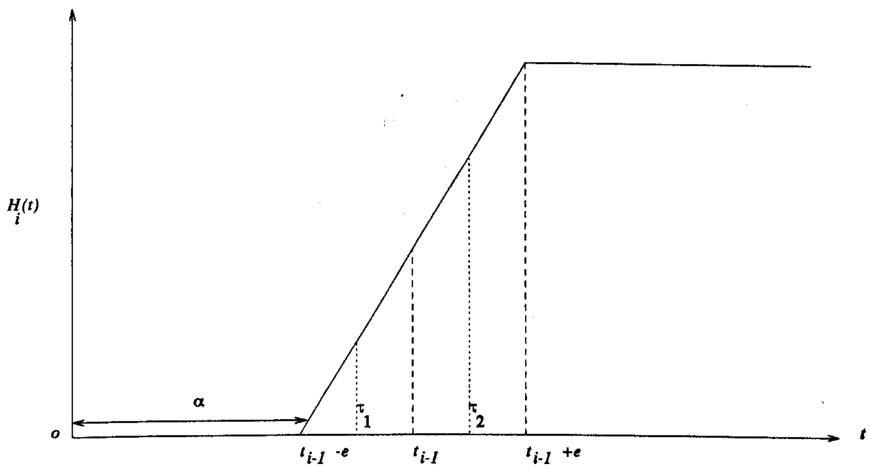

Figure 3.8: The graph of $H_i(t)$

$\frac{H_i(\tau_2)}{H_i(\tau_1)} = \frac{\tau_2 - \alpha}{\tau_1 - \alpha}$, or, $\frac{H_i(\tau_2) - H_i(\tau_1)}{H_i(\tau_1)} = \frac{\tau_2 - \tau_1}{\tau_1 - \alpha}$, or $|H_i(\tau_2) - H_i(\tau_1)| = |\tau_2 - \tau_1| \cdot \frac{|H_i(\tau_1)|}{|\tau_1 - \alpha|}$, but $\frac{|H_i(\tau_1)|}{|\tau_1 - \alpha|} = \frac{1}{2e}$, so

$$|H_i(\tau_2) - H_i(\tau_1)| = \frac{|\tau_2 - \tau_1|}{2e}.$$

6. If $\tau_1 \le t_{i-1} - e$ and $\tau_2 \in (t_{i-1} - e, t_{i-1} + e]$, then from the following shape

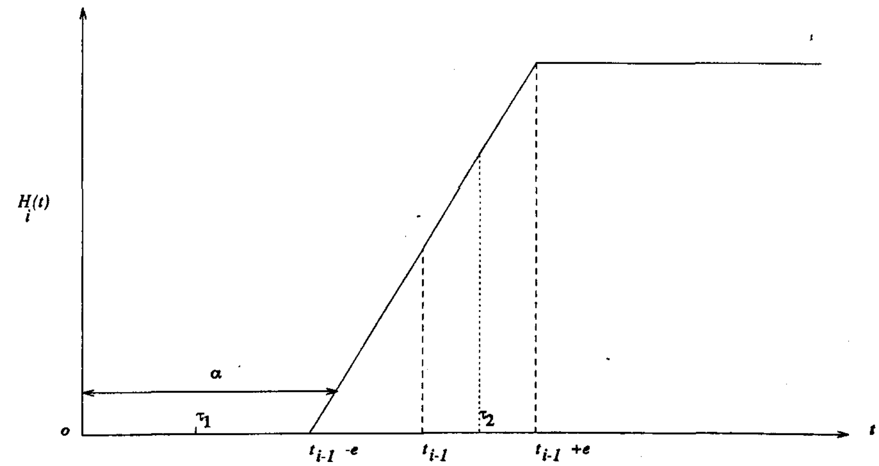

Figure 3.9: The graph of $H_i(t)$

$|H_i(\tau_2) - H_i(\tau_1)| = |H_i(\tau_2)|$, but, $\frac{|H_i(\tau_2)|}{|\tau_2-\alpha|} = \frac{1}{2e}$, so

$$|H_i(\tau_2)| = |H_i(\tau_2) - H_i(\tau_1)|\frac{|\tau_2 - \alpha|}{2e} \le \frac{|\tau_2 - \tau_1|}{2e}.$$

7. Proof for other choices of τ_1 and τ_2 turns back to the proof for one of the above mentioned cases. □

Proposition 3.5 Let $e_n(f)$ now denotes the minimax error in approximation of the function $f \in [a, b]$ by an algebraic polynomial of degree $\le n$. If f satisfies the Lipschitz condition with constant K and order α, then

$$e_n(f) \le 6K(\frac{b-a}{2n})^\alpha$$

Proof. See [53]. □

We recall that in Proposition 3.5, $e_n(f) \doteq \min \max_{a\le x\le b} |f - P_n(x)|$.

Chapter 4

The Global Control of Nonlinear Wave Equations

4.1 Introduction

In Chapter 2 we introduced an approach for the treatment of boundary control problems associated with linear wave equations, based on the replacement of the classical optimization problem by one in measure spaces. However, in this chapter we apply these ideas to a control problem associated with a nonlinear wave equation and distributed control.

In Section 4.2 we define the equation and the functional spaces in which we are going to find its solution. In Sections 4.3-4.5, we transform the problem; instead of minimizing the integral performance criterion over a set of admissible pairs of trajectory and control, we minimize it not over one measure space, as in Chapter 2, but over a subset of a product of two measure spaces so as to change the problem to a linear form and to benefit from the whole paraphernalia of linear analysis. In Section 4.6 we obtain an approximation to the optimal pair Radon measures (μ^*, ν^*), by using a linear programming scheme, and then we obtain a piecewise constant control function $v(x,t)$ corresponding to the desired final states $u(x,T)$ and $u_t(x,T)$. And finally in Section 4.7 we bring two examples and obtain the corresponding controls and show the graphs of these pieciewise constant controls .

Problems of this type have been considered in (e.g. Rubio and Holden [63], and Rubio [60], [61], [62]) in which in the nonlinear diffusion equations the control function were assumed to be in the boundary, whilst we use distributed control for our purpose.

4.2 The equation- functional spaces

Let D be a bounded open region in the n-dimensional space $I\!R^n$ with smooth boundary ∂D, and T a positive real number, define:

$$Q_T := D \times (0,T), \text{ a bounded cylinder of height } T > 0,$$
$$\Gamma_T := \partial D \times (0,T), \text{ the lateral surface of the cylinder } Q_T,$$
$$D_0 := D \times \{0\}, \text{ the base of the cylinder } Q_T,$$
$$D_T := D \times \{T\}, \text{ the top of the cylinder } Q_T.$$

We will consider the following wave equation which appears in Relativistic Quantum Mechanics. (See [72], [73], [44]).

$$u_{tt} - \Delta u + |u|^\rho u = v, \tag{4.1}$$

for the real function $u = u(x,t)$, $(x,t) \in Q_T$, where

$$\Delta u = \sum_{i=1}^{n} \frac{\partial^2 u}{\partial x_i^2},$$

and $\rho > 0$, with the boundary condition

$$u = 0 \text{ on } \Gamma_T, \tag{4.2}$$

and the initial conditions

$$u(x,0) = u_0(x) = 0, \qquad x \in D, \tag{4.3}$$

$$u_t(x,0) = u_1(x), \qquad x \in D, \tag{4.4}$$

where the continuous function u_1 is given.

The problem is *nonlinear* due to the presence of the term $|u|^\rho u$.

In (4.1), the function $(x,t) \in Q_T \longrightarrow v(x,t) \in V \subset I\!R$ is the *Distributed Control* function which takes the values in a *bounded* set V.

To have the tools to resolve the problem, we need to introduce certain spaces of functionals which is to be used through of this chapter.

(a) We shall constantly use the usual real spaces $L^P(D)$, $1 \le P \le \infty$,

$$L^P(D) = \left\{u : \text{real} - \text{valued, Lebesgue measurable functions in } D, \text{ with } \|u\|_{P,D} < \infty\right\}$$

where the norm is,

$$\|u\|_{P,D} = \left(\int_D |u(x)|^P dx\right)^{\frac{1}{P}}, \text{ if } 1 \le P < \infty. \tag{4.5}$$

Similarly, $L^\infty(D)$ is the class of all essentially bounded real-valued measurable functions u on D, with the norm

$$\|u\|_\infty = \text{ess sup}\,|u(x)|, \qquad x \in D \tag{4.6}$$

when ess sup indicates the essential supremum, (see [10], [20]). $L^P(D)$ is a Banach spaces where given the norm $\|u\|_{P,D}$ (see [20]).

(b) We designate by (f,g) the scalar product in $L^2(D)$, i.e.,

$$(f,g) = \int_D f(x)g(x)dx. \tag{4.7}$$

(c) We shall repeatedly make use of Sobolev spaces, we put:

$$H^0(D) = \left\{u : u \in L^2(D)\right\}$$

so $H^0(D)$ consists of all real-valued functions which are square summable on D in the sense of Lebesgue, and

$$H^1(D) = \left\{ u : u \in L^2(D), \frac{\partial u}{\partial x_i} \in L^2(D),\ i = 1, 2, ..., n \right\} \tag{4.8}$$

which is equipped with the norm

$$\|u\|_{H^1(D)} = (|u|^2 + \sum_{i=1}^{n} (\frac{\partial u}{\partial x_i})^2)^{\frac{1}{2}}. \tag{4.9}$$

We also use $H_0^1(D)$, defined as follows:

$$\begin{aligned} H_0^1(D) &= \text{closure of } C_0^\infty(D) \text{ in } H^1(D) \\ &= \text{subspace of } H^1(D) \text{ of functions null on } \Gamma_T, \end{aligned} \tag{4.10}$$

where C_0^∞ is the space of C^∞ functions on D with compact support in D. The space $H^1(D)$, and hence $H_0^1(D)$, are Hilbert spaces (see [21],[44]). To study the problem (4.1)-(4.4) we introduce the space

$$V_1 = H_0^1(D) \bigcap L^P(D) \tag{4.11}$$

where

$$P = \rho + 2. \tag{4.12}$$

The space V_1 is equipped with the norm

$$v_1 \in V_1 \longrightarrow \|v_1\|_{H_0^1(D)} + \|v_1\|_{L^P(D)} \tag{4.13}$$

which makes it a Banach space. Indeed, by Sobolev's imbedding theorem (see [44]),

$$H_0^1(D) \subset L^P(D), \qquad \text{if } P \leq \frac{2n}{n-2}, \qquad n \geq 3$$

from which we get

$$V_1 = H_0^1(D), \text{ if } \rho \leq \frac{4}{n-2} \qquad (\rho < \infty, \text{ when } n = 2). \tag{4.14}$$

(d) If X is a Banach space, we designate by $L^P(0,T;X)$ the space of functions $t \longrightarrow f(t)$, from $(0,T) \longrightarrow X$ that are measurable, with values in X, and such that

$$\left(\int_0^T \|f(t)\|_X^P dt\right)^{\frac{1}{P}} = \|f\|_{L^P(0,T;X)} < \infty, \text{ if } P < \infty;$$

if $P = \infty$, we replace the norm by

$$\text{ess sup}\, \|f(t)\|_X = \|f\|_{L^\infty(0,T;X)}, \qquad t \in (0,T).$$

Naturally we have:

$$L^P(0,T;L^P(D)) = L^P(Q_T).$$

4.3 A formulation of the initial-boundary value problem

In this section we shall consider weak solutions of equation (4.1) which satisfy the boundary condition (4.2) and the initial conditions (4.3) and (4.4) in generalized senses (see Mihkailov [48], Rubio [62], Sather [71], Wilcox [84]). The concept of weak solution is defined relative to a certain class of *test functions* that we use later:

$$\Psi = \{\psi : \psi \in L^1(I; H_0^1(D)), \text{ where its normal derivative } \psi' \in L^1(I; H^0(D)),$$
$$\text{and } \psi(t) \equiv 0 \text{ in a neighborhood of } t = T\},$$

when $I = \{t, 0 \leq t < T < \infty\} = [0,T)$.

The following condition will be imposed on the inhomogeneous term v:

$$v \in L^2(Q_T). \tag{4.15}$$

Definition 4.1 Let the hypothesis (4.15) holds along with conditions $u_0(x) \in V_1$, (V_1 as (4.11)), $u_1 \in L^2(D)$ on the initial data. A function u is a *weak solution* of the initial-boundary value problem (4.1)-(4.4) if

$$u \in L^\infty(0,T;V_1) \tag{4.16}$$

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(D)) \tag{4.17}$$

$$\begin{aligned}\int_0^T \{(-u'(t),\psi'(t)) - (\triangle u(t),\psi(t)) + (u(t)^\rho u(t),\psi(t))\}\,dt \\ = (u_1,\psi(0)) + \int_0^T (v(t),\psi(t))dt, \qquad \psi \in \Psi\end{aligned} \tag{4.18}$$

and,

$$u(0) = u_0(x). \tag{4.19}$$

Definition 4.1 is equivalent to the classical formulation (4.1)-(4.4) in case the solution and data are smooth functions of (x,t) and x in Q_T and in D respectiviely (see, for instance ,[71]).

4.4 Existence and transformation

Now we demonstrate the existence of the solution of the problem (4.1)-(4.4) and then transform it to a new problem. Here we have not established uniqueness of the solution. Lions [44] showed that if $0 < \rho \leq \frac{2}{n-2}$, then the solution of the initial-boundary value problem (4.1)-(4.4) is unique. He chose $n = 3$, $\rho = 2$, (see also Schiff [72] where he took $n = 3$, $\rho = 2$). Segal [74] proved that the problem has unique global solution in one and two space dimensions, and in three dimensions when $\rho = 2$. We should mention that the class of initial data and inhomogeneous term v in the following existence theorem is the same as the class employed in the definition of the weak solution.
The proof of the next theorem can be found in [44] and in [70].

Theorem. 4.1 Suppose that D is a bounded open set. We are given v, u_0, u_1, such that:

$$v \in L^2(Q_T) \tag{4.20}$$

$$u_0 \in V_1 \ (V_1 \text{ as } (4.11)\ ) \tag{4.21}$$

and,

$$u_1 \in L^2(D), \tag{4.22}$$

then there exists a function u where

$$u \in L^\infty(0,T;V_1) \tag{4.23}$$

$$\frac{\partial u}{\partial t} \in L^\infty(0,T;L^2(D)) \tag{4.24}$$

and satisfying the equation (4.1)

$$\frac{\partial^2 u}{\partial t^2} - \triangle u + |u|^\rho u = v \text{ in } \mathrm{Q_T} \tag{4.25}$$

and the initial conditions (4.3)-(4.4),

$$u(0) = u_0 \tag{4.26}$$

$$\frac{\partial u}{\partial t}(0) = u_1. \tag{4.27}$$

From (4.10) and (4.11), $u = 0$ on Γ_T; the condition (4.2) is thus incorporated in (4.23). From (4.23)-(4.25) we get in particular that u and $\frac{\partial u}{\partial t}$ are continuous functions on $[0,T]$ in such a way that (4.26) and (4.27) make sense (for details see [44]).

A pair (u, v) of trajectory function u and control function v is said to be *admissible* if:

(i) The function $(x,t) \longrightarrow u(x,t)$ is a solution of (4.1), that is in $L^\infty(0,T;H_0^1(D) \cap L^P(D))$ and satisfies (4.2), (4.3) and (4.4).

(ii) The control function is continuous in Q_T.

(iii) The terminal relationships

$$u(x,T) = g_1(x), \qquad x \in D$$
$$u_t(x,T) = g_2(x), \qquad x \in D \tag{4.28}$$

are satisfied; $g_1(x)$ and $g_2(x)$ are given continuous functions on D_T.

The set of admissible pairs will be denoted by $\mathcal{F}$, and assumed to be nonempty. We make a further point that, since the control set V is bounded, that is, there is a constant M_V so that

$$|v(x,t)| \leq M_V, \qquad (x,t) \in \overline{Q}_T,$$

thus $v \in L^2(Q_T)$, (4.20), and so by Theorem 4.1 and (4.23), $u \in L^\infty(0,T;V_1)$, so

$$\text{ess sup } \|u\|_{V_1} < \infty,$$

by (4.13), $\|u\|_{V_1} = \|u\|_{H_0^1(D)} + \|u\|_{L^p(D)}$, and since by (4.9)

$$\|u\|_{H_0^1(D)} = (|u|^2 + \sum_{i=1}^{n}(\frac{\partial u}{\partial x_i})^2)^{\frac{1}{2}}$$

we conclude that $u(x,t)$ is bounded. Thus there is a set $A \subset I\!R$, so that

$$u(x,t) \in A, \qquad \forall (x,t) \in \overline{Q}_T. \tag{4.29}$$

The set A is the smallest such sets, i.e., the intersection of all such sets for all possible admissible controls. Thus every point in our set A will be a state that can be reached by an admissible control inside the time interval $[0,T]$. We will use this property in the proof of Theorem 4.2.

The optimization problem associated with this equation is as follows:

Let f_0, f_1 be continuous, non-negative, real-valued functions on $I\!R^{n+2}$, and assume that there is a constant $h > 0$ so that

$$f_0(u(x,t),x,t) \leq h|u|, \qquad (u,x,t) \in A \times \overline{Q}_T.$$

Then we wish to find a minimizer pair (u,v) in $\mathcal{F}$ for the functional

$$J = \int_{Q_T} f_0(u(x,t),x,t)dxdt + \int_{Q_T} f_1(v(x,t),x,t)dxdt. \tag{4.30}$$

Now we transform the problem with a view at generalization. Suppose that the function u is an admissible trajectory function. Take $0 < \epsilon < T$, find the scalar product of (4.1) by a function $\psi(x,t) \in H_0^1(Q_T)$, since by (4.8), $\psi(x,t) \in L^2(Q_T)$, we can use (4.7):

$$(v \cdot \psi) = ((u_{tt} - \text{div}(\nabla u) + |u|^\rho u) \cdot \psi), \qquad \forall \psi(x,t) \in H_0^1(Q_T)$$

or by (4.7)

$$\int_{Q_{T-\epsilon}} v\psi dxdt = \int_{Q_{T-\epsilon}} (u_{tt} - \text{div}(\nabla u) + |u|^\rho u)\psi dxdt,$$

or,

$$\int_{Q_{T-\epsilon}} v\psi dxdt = \int_{Q_{T-\epsilon}} (u_{tt} - \text{div}(\nabla u))\psi dxdt + \int_{Q_{T-\epsilon}} (|u|^\rho u)\psi dxdt.$$

Since

$$u_{tt}\psi = (u_t\psi)_t - u_t\psi_t,$$

and

$$\psi \text{div}(\nabla u) = \text{div}(\psi \nabla u) - \nabla u \nabla \psi,$$

so

$$\int_{Q_{T-\epsilon}} v\psi dxdt = \int_{Q_{T-\epsilon}} (u_t\psi)_t dxdt - \int_{Q_{T-\epsilon}} \text{div}(\psi \nabla u)dxdt$$
$$+ \int_{Q_{T-\epsilon}} (\nabla u \nabla \psi - u_t\psi_t)dxdt + \int_{Q_{T-\epsilon}} (|u|^\rho u)\psi dxdt.$$

Now by Ostrogradskii's formula

$$\int_{Q_{T-\epsilon}} \operatorname{div}(A(x))dxdt = \int_{\Gamma_{T-\epsilon}} A(x)\cdot n ds dt,$$

and

$$\int_{Q_{T-\epsilon}} (u_t\psi)_t dt dx = \int_{D_{T-\epsilon}} (u_t\psi)dx - \int_{D_0} (u_t\psi)dx,$$

which yield

$$\begin{aligned}\int_{Q_{T-\epsilon}} v\psi dxdt = & \int_{D_{T-\epsilon}} (u_t\psi)dx - \int_{D_0} (u_t\psi)dx - \int_{\Gamma_{T-\epsilon}} \psi(\nabla u\cdot n)dsdt \\ & + \int_{Q_{T-\epsilon}} (\nabla u \nabla \psi - u_t\psi_t)dxdt + \int_{Q_{T-\epsilon}} (|u|^\rho u)\psi dxdt.\end{aligned}$$

By initial condition (4.4) and taking into account the final condition of u_t and the property of ψ on the boundary, and the continuity of the $u(x,t)$ in $\overline{Q}_T$, as $\epsilon \longrightarrow 0$, we find that

$$\begin{aligned}&\int_{Q_T} (u_t\psi_t - \nabla u \nabla \psi - (|u|^\rho u)\psi)dxdt + \int_{Q_T} (v\psi)dxdt \\ &= \int_{D_T} g_2\psi dx - \int_{D_0} u_1\psi dx\end{aligned}$$

for every $\psi \in H_0^1(\overline{Q}_T)$.

Since $u_t\psi_t = (u\psi_t)_t - u\psi_{tt}$,

$$\int_{Q_T} (u_t\psi_t)dxdt = \int_{Q_T} (u\psi_t)_t dxdt - \int_{Q_T} (u\psi_{tt})dxdt,$$

but by (4.3) $u(x,0) = u_0(x) = 0$, and by (4.28) $u(x,T) = g_1(x)$, so

$$\int_{Q_T} (u\psi_t)_t dxdt = \int_{D_T} u(x,T)\psi_t(x,T)dx - \int_{D_0} u(x,0)\psi_t(x,0)dx$$

$$= \int_{D_T} g_1(x)\psi_t(x,T)dx.$$

Thus

$$\int_{Q_T} (u_t\psi_t)dxdt = \int_{D_T} g_1(x)\psi_t(x,T)dx - \int_{Q_T} (u\psi_{tt})dxdt,$$

and also since

$$u \triangle \psi = \operatorname{div}(u \bigtriangledown \psi) - \bigtriangledown u \bigtriangledown \psi$$

by using again the Ostrogradskii's formula we have

$$\begin{aligned}\int_{Q_T} (\bigtriangledown u \bigtriangledown \psi) dxdt &= \int_{Q_T} \operatorname{div}(u \bigtriangledown \psi) dxdt - \int_{Q_T} u \triangle \psi dxdt \\ &= \int_{\Gamma_T} u \frac{\partial \psi}{\partial n} dsdt - \int_{Q_T} u \triangle \psi dxdt,\end{aligned}$$

and from the boundary condition (4.2), $u(x,t)|_{\Gamma_T} = 0$, so,

$$\int_{Q_T} (\bigtriangledown u \bigtriangledown \psi) dxdt = -\int_{Q_T} u \triangle \psi dxdt.$$

Thus the above integral form changes to:

$$\begin{aligned}&\int_{D_T} g_1(x)\psi_t(x,T)dx - \int_{Q_T} (u\psi_{tt})dxdt - \int_{Q_T} (|u|^\rho u\psi)dxdt \\ &+ \int_{Q_T} u \triangle \psi dxdt + \int_{Q_T} (v\psi)dxdt = \int_{D_T} g_2\psi dx - \int_{D_0} u_1\psi dx,\end{aligned}$$

or

$$\begin{aligned}\int_{Q_T} (u\psi_{tt} - u \triangle \psi + |u|^\rho u\psi)dxdt - \int_{Q_T} (v\psi)dxdt \\ = \int_{D_T} (g_1\psi_t - g_2\psi)dx + \int_{D_0} u_1\psi dx,\end{aligned} \tag{4.31}$$

for every $\psi \in H_0^1(\overline{Q}_T)$.

We proceed to transform the problem in the next section and we will follow Chapter 2 in this transformation.

4.5 Metamorphosis

In general, the minimization of the functional (4.30) over $\mathcal{F}$ may be not possible, the infimum is not attained at any admissible pair; it is not even possible, for instance, to write necessary conditions for this problem. We then proceed to transform the problem, so in the following, we replace the problem by the one in which the minimum of the functional (4.30) calculated over a set $\mathcal{S}$ of positive Radon-measures on $\Omega \times \omega$, that will be defined. Those measures in $\mathcal{S}$ should have some properties which can be deduced from the definition of admissible control.

We notice that any solution of (4.1)-(4.4) defines a linear-bounded positive functional

$$u(\cdot,\cdot) : F \longrightarrow \int_{Q_T} F(u, x, t)dxdt$$

in the space $C(\Omega)$ of continuous real-valued functions F, where $\Omega := A \times Q_T$. Also, a control v defines a linear-bounded-positive functional

$$v(\cdot,\cdot) : G \longrightarrow \int_{Q_T} G(v(t, x), x, t)dxdt$$

in the space $C(\omega)$ of continuous functions G, where $\omega := V \times Q_T$.

By the Riesz representation Theorem, an admissible pair (u, v) defines a pair of unique Radon measure (μ, ν), on $\Omega \times \omega$, such that

$$\int_{Q_T} F(u, x, t)dxdt = \int_{\Omega} F d\mu,$$

for all $F \in C(\Omega)$, and

$$\int_{Q_T} G(v, x, t)dxdt = \int_{\omega} G d\nu,$$

for all $G \in C(\omega)$. Thus (4.31) changes to:

$$\int_{\Omega} F_{\psi} d_{\mu} + \int_{\omega} G_{\psi} d\nu = \alpha_{\psi}, \ \forall \psi \in H_0^1(\overline{Q}_T), \tag{4.32}$$

where

$$\begin{aligned} F_{\psi}(u(x,t), x, t) &:= u\psi_{tt} - u \bigtriangleup \psi + |u|^{\rho} u\psi \\ G_{\psi}(v(x,t), x, t) &:= -v\psi \\ \alpha_{\psi} &:= \int_{D_T} (g_1\psi_t - g_2\psi)dx + \int_{D_0} u_1 \psi dx. \end{aligned} \tag{4.33}$$

Thus, the minimization of the functional (4.30) over $\mathcal{F}$ is equivalent to the minimization of

$$I(\mu,\nu) = \mu(f_0) + \nu(f_1) \tag{4.34}$$

where

$$\mu(f_0) := \int_\Omega f_0 d\mu$$

$$\nu(f_1) := \int_\omega f_1 d\nu$$

over the set of measures (μ,ν) corresponding to admissible pair (u,v) subject to:

$$\mu(F_\psi) + \nu(G_\psi) = \alpha_\psi, \qquad \forall\psi \in H_0^1(\overline{Q}_T). \tag{4.35}$$

So far we have just changed only the appearance of the problem. Now we consider the extension of our problem, and the minimization (4.34) over the set Q of all pairs of measures $(\mu,\nu) \in \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ satisfying (4.35). Measures in Q, where

$$Q = \Big\{(\mu,\nu) : \ (\mu,\nu) \in \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega) \text{ and satisfy } (4.35)\Big\}$$

should have some extra properties which are derived from the definition of admissible pairs (u,v). In fact if

$$\xi : \Omega \longrightarrow I\!R$$

depends only on (x,t), then

$$\mu(\xi) = \int_{Q_T} \xi(x,t)dxdt = a_\xi \tag{4.36a}$$

the Lebesgue integral of ξ over Q_T. Also if a function

$$\zeta : \omega \longrightarrow I\!R$$

depends only on (x, t), then

$$\nu(\zeta) = \int_{Q_T} \zeta(x,t)dxdt = b_\zeta \tag{4.36b}$$

the Lebesgue integral of ζ over Q_T.

Now we have defined a new optimization problem. In the next section we analyse this new problem and show that it always has at least one optimizer, then we start the process of approximation, that is, of building a framework so as to construct admissible trajectory-control pairs which are nearly minimizers of the functional (4.30) and satisfies the terminal conditions (4.28) as nearly as desired.

4.6 Existence and approximation

Now back to our original problem, we determine under what conditions there is a pair of optimal measures (μ, ν) for the function

$$(\mu, \nu) \longrightarrow \mu(f_0) + \nu(f_1)$$

in the set $Q \subset \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ defined by

$$\begin{aligned} \mu(F_\psi) + \nu(G_\psi) &= \alpha_\psi, \qquad \forall \psi \in H_0^1(\overline{Q}_T) \\ \mu(\xi) &= a_\xi \\ \nu(\zeta) &= b_\zeta, \text{ for all } \xi,\ \zeta \end{aligned} \tag{4.37}$$

where $\xi : \Omega \longrightarrow I\!R$ and $\zeta : \Omega \longrightarrow I\!R$ depend only on (x, t). We assume that the set of measures Q is nonempty, it may be because the system is controllable, even though, of course, the set Q may be nonempty while the set of trajectory function u, and control function v, i.e., the set $\mathcal{F}$ is empty, one of the advantages of the present formulation. We try to find a topology on the space $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ in which Q is compact and the function $(\mu, \nu) \longrightarrow \mu(f_0) + \nu(f_1)$ is continuous. This topology can be the *weak* – topology* on

$$\mathcal{S} = \left\{(\mu, \nu) : (\mu, \nu) \in \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)\right\}.$$

Now we can prove:

Proposition 4.1 The set Q of measures (μ, ν) defined as those measures in $\mathcal{S}$ satisfying (4.37) is compact in the topology induced by the weak*-topology on $\mathcal{S}$.

Proof. Note that (4.36) imply that; writing 1_Ω and 1_ω for the characteristic functions on Ω and ω respectively and L for the Lebesgue measure of D,

$$\mu(1_\Omega) = TL, \quad \nu(1_\omega) = TL,$$

then the set

$$\{(\mu,\nu) : (\mu,\nu) \in \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega),\ \mu(1_\Omega) + \nu(1_\omega) = 2TL > 0\},$$

is compact in the weak*-topology (see Alaoglu Theorem, in Royden [56], page 202). Thus Q that satisfies (4.37) is a subset of the above compact set. Also, we can write

$$Q = \bigcap_{\psi \in H_0^1(\overline{Q}_T)} \{(\mu,\nu) : \mu(f_\psi) + \nu(G_\psi) = \alpha_\psi\},$$

remember that the last two sets of equalities in (4.37) are implied by the first set. The linear functional $(\mu,\nu) \in Q \longrightarrow \mu(F) + \nu(G) \in \mathbb{R}$ is continuous (see Chapter 2, for the continuity of $(\mu,\nu) \in Q \longrightarrow \mu(F) + \nu(G) \in \mathbb{R}$), but the set $\{(\mu,\nu) : \mu(f_\psi) + \nu(G_\psi) = \alpha_\psi\}$ that it is the inverse image of the closed singleton set $\{\alpha_\psi\} \subset \mathbb{R}$ is a closed set, so the set Q is closed and since it is a subset of a compact set, is compact.□

Proposition 4.2 There exists an optimal measure (μ^*, ν^*) in the set Q that minimizes $I(\mu,\nu) = \mu(f_0) + \nu(f_1)$

Proof. By Proposition II.I of Rubio [59], the continuous function $(\mu,\nu) \longrightarrow \mu(f_0) + \nu(f_1)$ that is a mapping from the compact set Q into the real line attains its minimum on the set Q, i.e., exists a pair of measure $(\mu^*, \nu^*) \in Q$, such that

$$(\mu^*, \nu^*) \leq \mu(f_0) + \nu(f_1)$$

for all $(\mu,\nu) \in Q$. □

The proof of the following proposition is much like that of Theorem 7.1 in [34] and Proposition 2 in [57].

Proposition 4.3 The set $Q_1 \subset Q$ of measures (u, v), which are piecewise-constant functions on Ω and ω respectively and satisfy (4.35) and (4.36), is weakly*-dense in Q.

We later use this proposition to prove our important theorem of approximation. Up to here we have developed an infinite-dimensional program by considering minimization of

$$I(\mu, \nu) = \mu(f_0) + \nu(f_1)$$

over the set Q. Now we are going to consider the minimization of $I(\mu, \nu)$, not over the set Q but over a subset of Q defined by requiring that only a finite number of constraints (4.37) is satisfied. This will be achieved by choosing a countable set of functions that is *total* in $H_0^1(\overline{Q}_T)$. In the following we define precisely what we mean here by this concept. (see Kreyszig [37] page 168).

Definition 4.2 The set $\left\{\psi_i,\ \psi_i \in H_0^1(\overline{Q}_T), i = 1, 2, ...\right\}$ is a total set (fundamental set) in $H_0^1(\overline{Q}_T)$, if for any given $\psi \in H_0^1(\overline{Q}_T)$, and any $\epsilon > 0$, there exists an integer $N > 0$ and scalars a_i, $i = 1, 2, ..., N$, so that

$$\begin{aligned}
&\max_{\overline{Q}_T} |\psi - \sum_{i=1}^{N} a_i \psi_i| < \epsilon \\
&\max_{\overline{Q}_T} |\psi_t - \sum_{i=1}^{N} a_i \psi_{it}| < \epsilon \\
&\max_{\overline{Q}_T} |\psi_{tt} - \sum_{i=1}^{N} a_i \psi_{itt}| < \epsilon \\
&\max_{\overline{Q}_T} | \triangle \psi - \sum_{i=1}^{N} a_i \triangle \psi_i| < \epsilon.
\end{aligned}$$

So the span of this total set is dense in $H_0^1(\overline{Q}_T)$.
We also take sets of functions $\{\xi_j, j = 1, 2, ...\}$ and $\{\zeta_k, k = 1, 2, ...\}$ which are total in the respective subspaces of $C(\Omega)$ and $C(\omega)$. We shall write

$$F_i := F_{\psi_i},\ G_i := G_{\psi_i},\ \alpha_i = \alpha_{\psi_i},\ i = 1, 2, ...$$

$$a_j := a_{\xi_j},\ j = 1, 2, ..., \ b_k = b_{\zeta_k},\ k = 1, 2, ...$$

Then we have the important following proposition for our first result of approximation; its proof is much like the proof of Proposition 3.1.

Proposition 4.4 Let M_1, M_2, M_3 be positive integers. Consider the problem of minimizing

$$(\mu, \nu) \longrightarrow \mu(f_0) + \nu(f_1)$$

over the set $Q(M_1, M_2, M_3)$ of measures $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ satisfying

$$\begin{aligned} \mu(F_i) + \nu(G_i) &= \alpha_i,\ i = 1, 2, ..., M_1 \\ \mu(\xi_j) &= a_j,\ j = 1, 2, ..., M_2 \\ \nu(\zeta_k) &= b_k,\ k = 1, 2, ..., M_3 \end{aligned}$$

then as $M_1, M_2, M_3 \longrightarrow \infty$

$$\eta(M_1, M_2, M_3) = \inf_{Q(M_1, M_2, M_3)} [\mu(f_0) + \nu(f_1)]$$

tends to

$$\eta = \inf_{Q} [\mu(f_0) + \nu(f_1)].$$

It is clear that $\eta \leq \inf_{\mathcal{F}} J$, where J defined as (4.30).

Note that we have limited the number of constraints in the original problem, but underlying space still is infinite-dimensional. In fact the problem is:

Minimize

$$I(\mu, \nu) = \mu(f_0) + \nu(f_1) \tag{4.38a}$$

over the set $Q(M_1, M_2, M_3)$ subject to:

$$\mu(F_i) + \nu(G_i) = \alpha_i,\ i = 1, 2, ..., M_1 \tag{4.38b}$$

$$\mu(\xi_j) = a_j,\ j = 1, 2, ..., M_2 \tag{4.38c}$$

$$\nu(\zeta_k) = b_k,\ k = 1, 2, ..., M_3. \tag{4.38d}$$

This problem is one of the linear programming, all functions in (4.38) are linear in μ and ν, and the pairs (μ, ν) are positive, in fact it is a semi-infinite linear programming, since the unknowns μ and ν are in $Q(M_1, M_2, M_3)$.

Now we consider the construction of the suboptimal pairs of trajectories and controls for (4.30). We first obtain optimal measures (μ^*, ν^*) from problem like (4.38). There are several methods to find the numerical solution to such problems, for example, the method described in Glashoff and Gustafson [24] and the discretization method that we used in Chapter 2. The existence of such a minimizer shown in Appendix 4.9, then we obtain a weak* approximation to (μ^*, ν^*) by set of two piecewise-constant functions (u, v) and by means of results given in Proposition 4.3.

Since the control function v obtained in this way is a piecewise-constant function and D is a bounded domain, thus for each $(x,t) \in D \times (0,T)$, $v(x,t) \in L^2(Q_T)$, so v can serves as control function (4.15) for a weak solution of the problem (4.1)-(4.4). We denote this solution by u_v, where $u_v \in L^\infty(0, T, V_1)$, and V_1 is as (4.11). (See [44]). This pair (u_v, v) of trajectory and control function is not exactly admissible, i.e., while M_1, M_2, M_3 are not large enough, (u_v, v) may not belong to $\mathcal{F}$, however it is *asymptotically admissible*. This concept, borrowed from Rubio [62], is defined as follows:

Definition 4.3 The pair (u_v, v) of trajectory and control function that has been found as above procedure is called asymptotically admissible if:

(a) The trajectory u_v is the weak solution of (4.1)-(4.4) corresponding to the admissible control $v \in L^2(Q_T)$.

(b) For every $\psi \in H_0^1(\overline{Q}_T)$, the trajectory u_v satisfies the constraints (4.31).

(c) The final values of $u_v(0,T)$ and $u_{vt}(0,T)$ tend respectively in $L_2(D_T)$ to the indicated functions $g_1(x)$ and $g_2(x)$ in (4.28) as $M_1, M_2, M_3 \longrightarrow \infty$.

We now prove in the following theorem, that (u_v, v) is asymptotically admissible and if M_1, M_2, M_3 tend to ∞ then the value of the functional J defined by (4.30) at (u_v, v) tends to $\eta = \inf_Q I(\mu, \nu)$.

Theorem. 4.2 Let (u_v, v) be the pair constructed as explained above. Then under the appropriate conditions on the approximations involved:

(i) The trajectory u_v as the weak solution of (4.1)-(4.4) satisfies the constraints (4.18) for every test function and the relations (4.31) for every $\psi \in H_0^1(\overline{Q}_T)$.
(ii) As $M_1, M_2, M_3 \longrightarrow \infty$

$$\|u_v(\cdot, T) - g_1\|_{L_2} \longrightarrow 0$$

$$\|u_{vt}(\cdot, T) - g_2\|_{L_2} \longrightarrow 0$$

where g_1 and g_2 are described as (4.28),
(iii)

$$J(u_v, v) \longrightarrow \inf_Q I(\mu, \nu) = \eta,$$

when M_1, M_2, M_3 tend to ∞. $J(u_v, v)$ is the value of J at the pair (u_v, v).
Proof. Since the proof is slightly subtle and long, we break it into 3 manageable parts. We assume in (4.4) $u_1(x, 0) = 0, x \in D$. Assume $M_1 > 0$ is fixed. Then write $\epsilon := \frac{1}{M_1}$ and fix the values of M_2 and M_3. Let (μ^*, ν^*) be the minimizer for the functional (4.38a) over the set of measures $Q(M_1, M_2, M_3)$, Appendix 4.9 shows how such a minimizer exists. Because of the density of Q_1 in Q, Proposition 4.3, we can find a pair of piecewise constant trajectory-control functions (u, v) on Q_1, so that

$$|\{\mu_u(f_0) + \nu_v(f_1)\} - \{\mu^*(f_0) + \nu^*(f_1)\}| < \epsilon \tag{4.39a}$$

$$|\mu_u(F_i) + \nu_v(G_i) - \alpha_i| < \epsilon, \; i = 1, 2, ..., M_1 \tag{4.39b}$$

$$|\mu(\xi_j) - a_j| < \epsilon, \; j = 1, 2, ..., M_2 \tag{4.39c}$$

$$|\nu(\zeta_k) - b_k| < \epsilon, \; k = 1, 2, ..., M_3. \tag{4.39d}$$

In the above inequalities, we have used the pair (μ_u, ν_v) to denote measures in $\mathcal{M}^+(\Omega)\times \mathcal{M}^+(\omega)$ generated by the pair trajectory-control (u, v).

Now we prove that the pair (u_v, v) obtained as described above is asymptotically admissible, i.e., (i) and (ii) of the theorem are satisfied.

(i) As explained above we apply the piecewise constant control function $v \in L^2(\overline{Q}_T)$ to find the weak solution of the problem (4.1)-(4.4), called u_v. So by Definition 4.1, the trajectory function u_v satisfies (4.18), i.e.,

$$\int_0^T \{(-u_v'(t), \psi'(t)) - (\triangle u_v(t), \psi(t)) + (|u_v(t)|^\rho u_v(t), \psi(t))\}\, dt$$

$$= (u_1, \psi(0)) + \int_0^T (v(t), \psi(t))dt, \;\; \psi \in \Psi$$

and by the procedure ended to (4.31), we can show that

$$\int_{Q_T} (u_v\psi_{tt} - u_v \triangle \psi + |u_v|^\rho u_v \psi)dxdt - \int_{Q_T} (v\psi)dxdt$$

$$= \int_{D_T} (g_1\psi_t - g_2\psi)dx + \int_{D_0} u_1\psi dx,$$

for every $\psi \in H_0^1(\overline{Q}_T)$.

Thus the trajectory function u_v satisfies the constraints (4.18) for every $\psi \in \Psi$ and constraints (4.31) for every $\psi \in H_0^1(\overline{Q}_T)$ and so (i) is satisfied.

(ii) To prove the second part of the theorem, choose the functions $\psi_i \in H_0^1(\overline{Q}_T)$, $i = 1, 2, ..., M_1$ as the following linear combination,

$$\psi_i = \theta_i + \eta_i \tag{4.40}$$

such that θ_i's and η_i's have the following specific characteristics:

(a) The function $\theta_i \in H_0^1(\overline{Q}_T)$ is chosen so that the Lebesgue measure of the support of θ_i in $\overline{Q}_T$ does not exceed $\epsilon_1 > 0$ to be indicated below, and so that the maximum of the numbers

$$|\theta_i(x,t)|, \quad |\theta_{itt}(x,t)|, \quad |\triangle\theta_i(x,t)|, \qquad (x,t) \in \overline{Q}_T \tag{4.41}$$

are not greater than $\epsilon_2 > 0$ which will be defined later. For example, if we assume that the total set $\{\psi_i\}$ in $H_0^1(\overline{Q}_T)$ is such that

$$\psi_i = x(1-x)\sin^2(\pi i t), \quad 0 \le x \le 1, \;\; T = 1,$$

then $\theta_i \in H_0^1(\overline{Q}_T)$ can be chosen as $\theta_i = 0$, for every $(x,t) \in \overline{Q}_T$, and of course the maximum value of the numbers in (4.41) are zero.

Now from (4.33) we have:

$$F_{\theta_i} := u\theta_{itt} - u\triangle\theta_i + |u|^\rho u\theta_i,$$

so

$$\begin{aligned}|F_{\theta_i}(x,t)| &= |u(x,t)\theta_{itt} - u(x,t)\triangle\theta_i + |u(x,t)|^\rho u(x,t)\theta_i| \\ &\le |u(x,t)|\,|\theta_{itt}| + |u(x,t)|\,|\triangle\theta_i| + |u(x,t)|^{\rho+1}\,|\theta_i| \\ &\le \epsilon_2[2l(A) + (l(A))^{\rho+1}],\end{aligned}$$

by the properties $|\theta_i(x,t)|$, $|\theta_{itt}(x,t)|$ and $|\triangle\theta_i|$ on $\overline{Q}_T$ assumed above, and from (4.29) where we assumed $u(x,t) \in A \subset I\!R, \forall (x,t) \in \overline{Q}_T$, here $l(A)$ is defined as the length of the bounded set A. Since the set A is the intersection of all the sets that contain $u(x,t)$ for every $(x,t) \in \overline{Q}_T$, then $l(A)$ is finite and is as small as possible. Now we choose $\epsilon_2 > 0$ such that $\epsilon_2[2l(A) + l(A)^{\rho+1}] \le 1$, which gives the result that,

$$|F_{\theta_i}(x,t)| \le 1,$$

on $\Omega := A \times Q_T$. The Lebesgue measure of the support of $F_{\theta_i}(\cdot,\cdot)$ in Ω is in fact equal the product of the Lebesgue measure of the support of θ_i in Q_T and the Lebesgue measure of the support of u in A which as we defined the first one is less than ϵ_1, thus the

Lebesgue measure of the support of $F_{\theta_i}(\cdot,\cdot)$ is less than ϵ_1.meas$A$. Now we choose ϵ_1 such that ϵ_1.meas$A < \epsilon$. So by choosing ϵ_1 accordingly, the Lebesgue measure of the support of $F_{\theta_i}(\cdot,\cdot)$ will be less than ϵ.

(b) The second part of the linear combination (4.40), i.e., the function $\eta_i(x,t) \in H_0^1(\overline{Q}_T)$ is chosen so that $\eta_i(x,T) = 0$ and $\eta_{it}(x,T) = 0$ on D. Obviously it is easy to find such functions, e.g., in the case of choosing the total set $\{\psi_i\}$ in $H_0^1(\overline{Q}_T)$ as $\psi_i = x(1-x)\sin^2(\pi i t)$ in (a), then $\eta_i(x,t) = x(1-x)\sin^2(\pi i t)$. So

$$\int_{D_T} (g_1\eta_{it}(\cdot,T) - g_2\eta_i(\cdot,T))dx = 0, \quad i = 1,2,...,M_1. \tag{4.42}$$

Since the set $\{\psi_i\}$ is total in $H_0^1(\overline{Q}_T)$, and $\eta_i(x,t) \in H_0^1(\overline{Q}_T)$, then by the definition, for any given $\epsilon_3 > 0$ there exists an integer N_i^1 and scalars a_{ij} such that

$$\max_{\overline{Q}_T} |\eta_i - \sum_{j=1}^{N_i^1} a_{ij}\psi_j| < \epsilon_3$$

$$\max_{\overline{Q}_T} |\eta_{it} - \sum_{j=1}^{N_i^1} a_{ij}\psi_{jt}| < \epsilon_3$$

$$\max_{\overline{Q}_T} |\eta_{itt} - \sum_{j=1}^{N_i^1} a_{ij}\psi_{jtt}| < \epsilon_3$$

$$\max_{\overline{Q}_T} | \triangle \eta_i - \sum_{j=1}^{N_i^1} a_{ij} \triangle \psi_j| < \epsilon_3.$$

From (4.33)

$$F_{\eta_i} := u\eta_{itt} - u \triangle \eta_i + |u|^\rho u\eta_i,$$

so

$$-\epsilon_3|2u + |u|^\rho u| + \sum_{j=1}^{N_i^1} a_{ij}F_{\psi_j} < F_{\eta_i} < \epsilon_3|2u + |u|^\rho u| + \sum_{j=1}^{N_i^1} a_{ij}F_{\psi_j}$$

or

$$|F_{\eta_i} - \sum_{j=1}^{N_i^1} a_{ij}F_{\psi_j}| < \epsilon_3|2u + |u|^\rho u|. \tag{4.43}$$

Similarly, since from (4.33)

$$G_{\eta_i} = -v\eta_i,$$

we find that

$$|G_{\eta_i} - \sum_{j=1}^{N_i^1} a_{ij} G_{\psi_j}| < \epsilon_3 |v|. \tag{4.44}$$

From (4.38),

$$\mu^*(F_{\psi_i}) + \nu^*(G_{\psi_i}) = \alpha_i, \quad i = 1, 2, ..., M_1$$

where

$$\alpha_i = \int_{D_T} (g_1 \psi_{it} - g_2 \psi_i) dx.$$

Now if we choose $M_1 \geq N_i^1$, then

$$\mu^*(\sum_{j=1}^{N_i^1} a_{ij} F_{\psi_j}) + \nu^*(\sum_{j=1}^{N_i^1} a_{ij} G_{\psi_j}) = \sum_{j=1}^{N_i^1} a_{ij} \alpha_j$$

$$i = 1, 2, ..., M_1$$

where

$$\sum_{j=1}^{N_i^1} a_{ij} \alpha_j = \sum_{j=1}^{N_i^1} a_{ij} \int_{D_T} (g_1 \psi_{it} - g_2 \psi_i) dx$$

$$= \int_{D_T} [g_1 \sum_{j=1}^{N_i^1} a_{ij} \psi_{jt} - g_2 \sum_{j=1}^{N_i^1} a_{ij} \psi_j] dx.$$

But the set $\{\psi_j\}$ is total, so for given $\epsilon_3 > 0$ we can find N_i^2 such that

$$|\int_{D_T} (g_1 \sum_{j=1}^{N_i^2} a_{ij} \psi_{jt}) dx - \int_{D_T} g_1 \eta_{it}(x, T) dx| < \epsilon_3 \int_{D_T} |g_1| dx$$

and,

$$|\int_{D_T} (g_2 \sum_{j=1}^{N_i^2} a_{ij} \psi_j) dx - \int_{D_T} g_2 \eta_i(x, T) dx| < \epsilon_3 \int_{D_T} |g_2| dx.$$

We assume that $N_i = \max\{N_i^1, N_i^2\}$, since $\eta_{it}(x,T) = 0$, $\eta_i(x,T) = 0$, thus

$$|\int_{D_T} \sum_{j=1}^{N_i} a_{ij}(g_1\psi_{jt} - g_2\psi_j)dx| = |\mu^*(\sum_{j=1}^{N_i} a_{ij}F_{\psi_j}) + \nu^*(\sum_{j=1}^{N_i} a_{ij}G_{\psi_j})|$$
$$< \epsilon_3 \int_{D_T} (|g_1| + |g_2|)dx. \tag{4.45}$$

Now from (4.43)-(4.45), we may choose N_i so large that

$$|\mu^*(F_{\eta_i}) + \nu^*(G_{\eta_i})| < \frac{\epsilon}{2}. \tag{4.46}$$

By Proposition 4.3 there exists a pair trajectory-control (u, v), and so the pair $(\mu_u, \nu_v) \in Q_1$ of measures generated by this pair (u, v), such that (4.39b) is satisfied,

$$|\mu_u(F_{\eta_i}) + \nu_v(G_{\eta_i}) - \mu^*(F_{\eta_i}) - \nu^*(G_{\eta_i})| < \frac{\epsilon}{2},$$

so by (4.46) and above inequality

$$|\mu_u(F_{\eta_i}) + \nu_v(G_{\eta_i})| < \epsilon, \quad i = 1, 2, ..., M_1. \tag{4.47}$$

Thus we have

$$F_i = F_{\eta_i} + F_{\theta_i},$$

with $\eta_i(x,t) \in H_0^1(\overline{Q}_T)$, $\theta_i(x,t) \in H_0^1(\overline{Q}_T)$, $\eta_i(x,T) = 0$, $\eta_{it}(x,T) = 0$ on D, and $|F_{\theta_i}| < 1$ on Ω, and the Lebesgue measure of $\text{supp}F_{\theta_i} < \epsilon$, $i = 1, 2, ..., M_1$, and (4.47) is satisfied.

Now we obtain the weak trajectory u_v corresponding to the control v as explained before. Then by writing μ_{u_v} for the measure corresponding to this trajectory function u_v and by the manipulations leading to (4.31), we find that

$$\mu_{u_v}(F_i) + \nu_v(G_i) = \int_{D_T} (u_v(\cdot, T)\psi_{it} - u_{vt}(\cdot, T)\psi_i)dx, \tag{4.48}$$
$$i = 1, 2, ..., M_1.$$

But the pair (μ^*, ν^*) satisfies (4.35), so

$$\mu^*(F_i) + \nu^*(G_i) = \int_{D_T} (g_1\psi_{it} - g_2\psi_i)dx = \alpha_i, \quad i = 1, 2, ..., M_1. \tag{4.49}$$

By subtraction of two relation (4.48) and (4.49) we find that

$$|\int_{D_T} [(u_v(\cdot, T) - g_1)\psi_{it} + (g_2 - u_{vt}(\cdot, T))\psi_i]dx| = |\mu^*(F_i) + \nu^*(G_i) - \mu_{u_v}(F_i) - \nu_v(G_i)|$$

$$= |\mu^*(F_i) + \nu^*(G_i) - \mu_u(F_i) + \mu_u(F_i) - \mu_{u_v}(F_i) - \nu_v(G_i)|$$

$$\leq |\mu_u(F_i) + \nu_v(G_i) - \mu^*(F_i) - \nu^*(G_i)| + |\mu_u(F_i) - \mu_{u_v}(F_i)|$$

where by (4.49) and (4.39b)

$$|\mu_u(F_i) + \nu_v(G_i) - \mu^*(F_i) - \nu^*(G_i)| = |\mu_u(F_i) + \nu_v(G_i) - \alpha_i| < \epsilon, \quad i = 1, 2, ..., M_1,$$

thus

$$|\int_{D_T} [(u_v(\cdot, T) - g_1)\psi_{it} + (g_2 - u_{vt}(\cdot, T))\psi_i]dx| \leq \epsilon + |\mu_u(F_i) - \mu_{u_v}(F_i)|,$$

but,

$$|\mu_u(F_i) - \mu_{u_v}(F_i)| = |(\mu_u - \mu_{u_v})(F_i)| = |(\mu_u - \mu_{u_v})(F_{\eta_i} + F_{\theta_i})|$$
$$\leq |(\mu_u - \mu_{u_v})(F_{\eta_i})| + |(\mu_u - \mu_{u_v})(F_{\theta_i})|,$$

where we have,

$$|(\mu_u - \mu_{u_v})(F_{\eta_i})| = |\mu_u(F_{\eta_i}) + \nu_v(G_{\eta_i}) - \nu_v(G_{\eta_i}) - \mu_{u_v}(F_{\eta_i})|$$
$$\leq |\mu_u(F_{\eta_i}) + \nu_v(G_{\eta_i})| + |\mu_{u_v}(F_{\eta_i}) + \nu_v(G_{\eta_i})|.$$

From (4.47) we infer the first part of the right-hand side is less than ϵ, and

$$|\mu_{u_v}(F_{\eta_i}) + \nu_v(G_{\eta_i})| = |\int_{D_T} [u_v(0, T)\eta_{it}(\cdot, T) - u_{vt}(0, T)\eta_i(\cdot, T)]dx| = 0,$$

since $\eta_i(\cdot,T)=0$ and $\eta_{it}(\cdot,T)=0$, for all $x\in D_T$, also

$$|(\mu_u-\mu_{u_v})(F_{\theta_i})|\le \max|F_{\theta_i}||\text{supp}F_{\theta_i}|\le 1\cdot\epsilon=\epsilon,$$

so

$$|(\mu_u-\mu_{u_v})(F_i)|\le 2\epsilon. \tag{4.50}$$

Thus,

$$|\int_{D_T}[(u_v(\cdot,T)-g_1)\psi_{it}+(g_2-u_{vt}(\cdot,T))\psi_i]dx|\le 3\epsilon=\frac{3}{M_1},$$
$$i=1,2,...,M_1. \tag{4.51}$$

It follows from (4.51) that as $M_1\longrightarrow\infty$

$$\|u_v(0,T)-g_1\|_{L_2}\longrightarrow 0$$

$$\|u_{vt}(0,T)-g_2\|_{L_2}\longrightarrow 0$$

(See Appendix 4.10).
Note that we need also $M_2, M_3\longrightarrow\infty$, a requirement of the proof of Proposition 4.3 (see [34] and [57]). So part (ii) of the theorem is satisfied.

(iii) To prove the third part of the theorem, we have:

$$|J(u_v,v)-\inf_Q I(\mu,\nu)|=|\mu_{u_v}(f_0)+\nu_v(f_1)-(\mu^*(f_0)+\nu^*(f_1))|=$$
$$|\mu_{u_v}(f_0)+\nu_v(f_1)-(\mu^*(f_0)+\nu^*(f_1))+(\mu_u(f_0)+\nu_v(f_1))-(\mu_u(f_0)+\nu_v(f_1))|$$
$$\le|\mu_u(f_0)+\nu_v(f_1)-(\mu^*(f_0)+\nu^*(f_1))|+|(\mu_{u_v}-\mu_u)(f_0)|$$

where by (4.39a) the first part of the right-hand side of the above inequality is less than ϵ, so

$$|J(u_v,v)-\inf_Q I(\mu,\nu)|\le\epsilon+|(\mu_{u_v}-\mu_u)(f_0)|. \tag{4.52}$$

We recall that the function f_0 satisfies the condition

$$f_0(u(x,t),x,t) \le h|u|,$$

on Ω, so

$$|(\mu_{u_v} - \mu_u)(f_0)| \le h|(\mu_{u_v} - \mu_u)\varpi| \tag{4.53}$$

where $\varpi(u(x,t),x,t) := u$, $(u,x,t) \in \Omega$. We are assuming, without loss of generality, that $u \ge 0$ on $\Omega = A \times Q_T$. Since we have chosen the intersection of those sets A, every $(u,x,t) \in \Omega$ can be reached by an admissible control $v \in V$ inside the interval $[0,T]$, thus F_ψ in the following can be defined.

Further, choose $\psi \in H_0^1(\overline{Q}_T)$ of the form:

$$\psi = \sum_{i=1}^{K} \beta_i \psi_i, \quad \sum_{i=1}^{K} |\beta_i| \le \lambda,$$

with K a fixed integer not greater than M_1, and

$$\lambda \le \frac{1}{4h},$$

then by the linear property

$$\begin{aligned} F_\psi &= u\psi_{tt} - u\Delta\psi + |u|^\rho u\psi = u\sum_{i=1}^{K}\beta_i\psi_{itt} - u\sum_{i=1}^{K}\beta_i\Delta\psi + |u|^\rho u\sum_{i=1}^{K}\beta_i\psi_i \\ &= \sum_{i=1}^{K}\beta_i(u\psi_{itt} - u\Delta\psi_i + |u|^\rho u\psi_i) = \sum_{i=1}^{K}\beta_i F_{\psi_i} \\ &= \sum_{i=1}^{K}\beta_i F_i, \end{aligned}$$

and so

$$\begin{aligned} |(\mu_{u_v} - \mu_u)(F_\psi)| &= |(\mu_{u_v} - \mu_u)(\sum_{i=1}^{K}\beta_i F_i)| \\ &\le |\beta_1||(\mu_{u_v} - \mu_u)(F_1)| + \dots + |\beta_K||(\mu_{u_v} - \mu_u)(F_K)|, \end{aligned}$$

but from (4.50), $|(\mu_{u_v} - \mu_u)F_i| \leq 2\epsilon$, for $i = 1, 2, ..., K$, so

$$|(\mu_{u_v} - \mu_u)(F_\psi)| \leq 2\epsilon \sum_{i=1}^{K} |\beta_i| \leq 2\lambda\epsilon.$$

Since $\psi = \sum_{i=1}^{K} \beta_i \psi_i$ we may choose the coefficients $\beta_i,\ i = 1, 2, ..., K$ so that:

$$\varpi|\psi_{tt}| \leq \frac{\epsilon'}{3}$$
$$\varpi|\triangle\psi - 1| \leq \frac{\epsilon'}{3}$$
$$\varpi^{\rho+1}|\psi| \leq \frac{\epsilon'}{3}$$

on Ω, which imply that

$$|\varpi\psi_{tt} - \varpi(\triangle\psi - 1) + \varpi^{\rho+1}\psi| \leq \epsilon',$$

on Ω; the number ϵ' will be determined below. Then

$$\begin{aligned}|(\mu_{u_v} - \mu_u)\varpi| &= |(\mu_{u_v} - \mu_u)(\varpi\psi_{tt} - \varpi(\triangle\psi - 1) + \varpi^{\rho+1}\psi) \\ &\quad - (\mu_{u_v} - \mu_u)(\varpi\psi_{tt} - \varpi(\triangle\psi) + \varpi^{\rho+1}\psi)| \\ &\leq \epsilon'\text{meas}(Q_T) + |(\mu_{u_v} - \mu_u)F_\psi| \leq \epsilon'\text{meas}(Q_T) + 2\lambda\epsilon.\end{aligned}$$

If we take $\epsilon' \leq \frac{\epsilon}{2\text{meas}(Q_T)h}$, then

$$|(\mu_{u_v} - \mu_u)\varpi| \leq \frac{\epsilon}{2h} + 2\frac{\epsilon}{4h} = \frac{\epsilon}{h},$$

so by (4.53)

$$|(\mu_{u_v} - \mu_u)f_0| \leq \epsilon,$$

and by (4.52)

$$|J(u_v, v) - \inf_{Q} I(\mu, \nu)| \leq 2\epsilon, \tag{4.54}$$

the third contention of the theorem follows. □

4.7 Numerical examples

We will apply the method introduced in this chapter for the estimation of nearly optimal controls. Now we show the details in the following examples.

Example 4.1 Consider the nonlinear wave equation

$$u_{tt} - \Delta u + u^3 = v, \tag{4.55}$$

for the real function $u = u(x,t)$, $(x,t) \in Q_T$, where

$$\begin{aligned} u &= 0, && \text{on } \Gamma_T \\ u(x,0) &= 0, && x \in D \\ u_t(x,0) &= 0, && x \in D \end{aligned}$$

and
$D := (0,1)$, $x \in [0,1]$, $\partial D := \{0,1\}$, $T = 1$, $t \in [0,1]$, $Q_T := (0,1)^2$, $\Gamma_T := \partial D \times (0,1)$, $D_0 := (0,1) \times \{0\}$, $D_T := (0,1) \times \{1\}$.
We assume the terminal relations as follows:

$$u(x,T) = g_1(x) = \sin(\pi x),\ u_t(x,T) = g_2(x) = 0,\ x \in D,$$

and we choose $V = [-4,4]$, $A = [0,4]$, $f_0 = u^2$, $f_1 = 0$, so

$$\Omega := A \times Q_T = [0,4] \times (0,1)^2,\ \omega := V \times Q_T = [-4,4] \times (0,1)^2.$$

The functions ψ in (4.33) are selected as $\psi(x,t) = t^p \sin(l\pi x)$ and six number of them are chosen with values of $l = 1, 2, 3$ and $p = 1, 2$. Thus

$$\begin{aligned}
\psi_1 &= t \sin(\pi x) \\
\psi_2 &= t \sin(2\pi x) \\
\psi_3 &= t \sin(3\pi x) \\
\psi_4 &= t^2 \sin(\pi x) \\
\psi_5 &= t^2 \sin(2\pi x) \\
\psi_6 &= t^2 \sin(3\pi x).
\end{aligned}$$

By (4.33),

$$F_i = F_{\psi_i} = u\psi_{itt} - u \bigtriangleup \psi_i + u^3\psi_i,$$

and

$$G_i = G_{\psi_i} = -v\psi_i$$

and

$$\begin{aligned}
a_i = a_{\psi_i} &= \int_{D_T} (g_1\psi_{it} - g_2\psi_i)dx + \int_{D_0} u_1\psi_i dx \\
&= \int_{D_T} (g_1\psi_{it})dx, \; i = 1, 2, ..., 6.
\end{aligned}$$

Therefore by applying ψ_i's we have:

$$\begin{aligned}
F_1 &= u(t\pi^2 \sin(\pi x)) + u^3(t \sin(\pi x)) \\
F_2 &= u(4t\pi^2 \sin(2\pi x)) + u^3(t \sin(2\pi x)) \\
F_3 &= u(9t\pi^2 \sin(3\pi x)) + u^3(t \sin(3\pi x)) \\
F_4 &= u(2 \sin(\pi x)) + u(t^2\pi^2 \sin(\pi x)) + u^3(t^2 \sin(\pi x)) \\
F_5 &= u(2 \sin(2\pi x)) + u(4t^2\pi^2 \sin(2\pi x)) + u^3(t^2 \sin(2\pi x)) \\
F_6 &= u(2 \sin(3\pi x)) + u(9t^2\pi^2 \sin(3\pi x)) + u^3(t^2 \sin(3\pi x)),
\end{aligned}$$

and

$$\begin{aligned}
G_1 &= -vt\sin(\pi x)\\
G_2 &= -vt\sin(2\pi x)\\
G_3 &= -vt\sin(3\pi x)\\
G_4 &= -vt^2\sin(\pi x)\\
G_5 &= -vt^2\sin(2\pi x)\\
G_6 &= -vt^2\sin(3\pi x),
\end{aligned}$$

and

$$\begin{aligned}
a_1 &= \int_0^1 \sin(\pi x)\sin(\pi x)dx = \frac{1}{2}\\
a_2 &= \int_0^1 \sin(\pi x)\sin(2\pi x)dx = 0\\
a_3 &= \int_0^1 \sin(\pi x)\sin(3\pi x)dx = 0\\
a_4 &= \int_0^1 2\sin(\pi x)\sin(\pi x)dx = 1\\
a_5 &= \int_0^1 2\sin(\pi x)\sin(2\pi x)dx = 0\\
a_6 &= \int_0^1 2\sin(\pi x)\sin(3\pi x)dx = 0.
\end{aligned}$$

Now we will choose 64 functions ξ, we divide the square $[0,1]^2$ into 64 equal squares and choose the 64 functions ξ as being the characteristic functions of the individual squares. In other word the functions ξ_j are defined as the following:

$$\xi_j(x,t) = \begin{cases} 1, & \frac{h-1}{8} \le x < \frac{h}{8}, \frac{l-1}{8} \le t < \frac{l}{8} \\ 0, & \text{otherwise} \end{cases}$$

where $j = 8(l-1)+h$, $h = 1,2,...,8$, $l = 1,2,...,8$. Now by (4.36a)

$$a_j = a_{\xi_j} = \int_{Q_T} \xi_j(x,t)dxdt = \frac{1}{64},\ j = 1,2,...,64.$$

Likewise, the 64 functions ζ are defined and by (4.36b)

$$b_k = b_{\zeta_k} = \int_{Q_T} \zeta_k(x,t)dxdt = \frac{1}{64},\ k = 1,2,...,64.$$

Having defined ψ_i's, ξ_j's and ζ_k's, by (4.38) the problem is transformed to the following: Minimize

$$I(\mu,\nu) = \mu(f_0) + \nu(f_1)$$

subject to:

$$\begin{aligned} \mu(F_i) + \nu(G_i) &= a_i, \qquad & i &= 1,2,...,6 \\ \mu(\xi_j) &= a_j, \qquad & j &= 1,2,...,64 \\ \nu(\zeta_k) &= b_k, \qquad & k &= 1,2,...,64. \end{aligned} \tag{4.56}$$

This system is a semi-infinite linear programming problem, since the unknown is in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ but with only a finite number of constraints. We use a discretization method to deal with the above linear programming problem. We divide the interval $[0,1]$ on the x-axis into 8 equal subintervals, and choose the x_i's as follows:

$$x_{64(k-1)+8(j-1)+i} = \frac{2j-1}{16},$$

$$i = 1,2,...,8,\ j = 1,2,...,8,\ k = 1,2,...,8,$$

except for $x_1,\ x_2,\ ...,\ x_8$ and $x_{57},\ x_{58},\ ...,\ x_{64}$. We divide also the interval $[0,1]$ on t-axis into 8 equal subintervals, and choose the t_i's as:

$$t_{64(k-1)+8(j-1)+i} = \frac{2i-1}{16},$$

$$i = 1,2,...,8,\ j = 1,2,...,8,\ k = 1,2,...,8.$$

We select u_i's as:

$$u_{64(j-1)+i} = \frac{(2j-1)}{4}$$

$$i = 1,2,...,64,\ j = 1,2,...,8,$$

except for $u_1,\ u_2,\ ...,u_8$, and $u_{57},\ u_{58},\ ...u_{64}$. In order to take the boundary condition (4.2) into account in this process, we assume

$$x_1 = x_2 = ...x_8 = 0,\ x_{57} = x_{58} = ... = x_{64} = 1,$$

and

$$u_1 = u_2 = ...u_8 = 0,\ u_{57} = u_{58} = ... = u_{64} = 0.$$

Finally we choose,

$$v_{64(j-1)+i} = -4 + \frac{(2j-1)}{2},$$

$$i = 1, 2, ..., 64,\ j = 1, 2, ..., 8.$$

Now, the set $\Omega = [0,4] \times (0,1)^2$ will be covered with a grid, defined by taking all points in Ω with coordinates $Z_i = (x_i, t_i, u_i)$, $i = 1, 2, ..., 512$ as above. Similarly, the set $\omega = [-4,4] \times (0,1)^2$ will be covered with a grid defined by taking all points in ω with coordinates $z_i = (x_i, t_i, v_i)$, $i = 1, 2, ..., 512$. Then, instead of the semi-infinite linear programming problem (4.56) we shall consider one in which the measures μ and ν are those positive Radon measures on Ω and ω supported by the grids defined by Z_i's and z_i's respectively, each such measure μ and ν is defined by a set of non negative real numbers α_j and β_j, respectively where $j = 1, 2, ..., 512$. Then the linear programming problem consists of minimizing the linear form

$$\sum_{i=1}^{512} \alpha_i u_i^2$$

over the set of coefficients $\alpha_i \geq 0, \beta_j \geq 0$, such that

$$\sum_{i=1}^{512} \alpha_j F_i(Z_j) + \sum_{j=1}^{512} \beta_j G_i(z_j) = a_i, \qquad i = 1, 2, ..., 6$$

$$\sum_{i=1}^{512} \alpha_j \xi_i(Z_j) = \frac{1}{64}, \qquad i = 1, 2, ..., 64$$

$$\sum_{j=1}^{512} \beta_j \zeta_i(z_j) = \frac{1}{64}, \qquad j = 1, 2, ..., 64.$$

This finite-dimensional linear program was solved by a modified simplex method, taking a total of 269 iterations, and the value 1.419222×10^{-1} was resulted for the objective function. Now we will construct the nearly optimal control v by using the method applied in Chapter 2. The graph of the resulting control v is shown in Fig. 4.1.

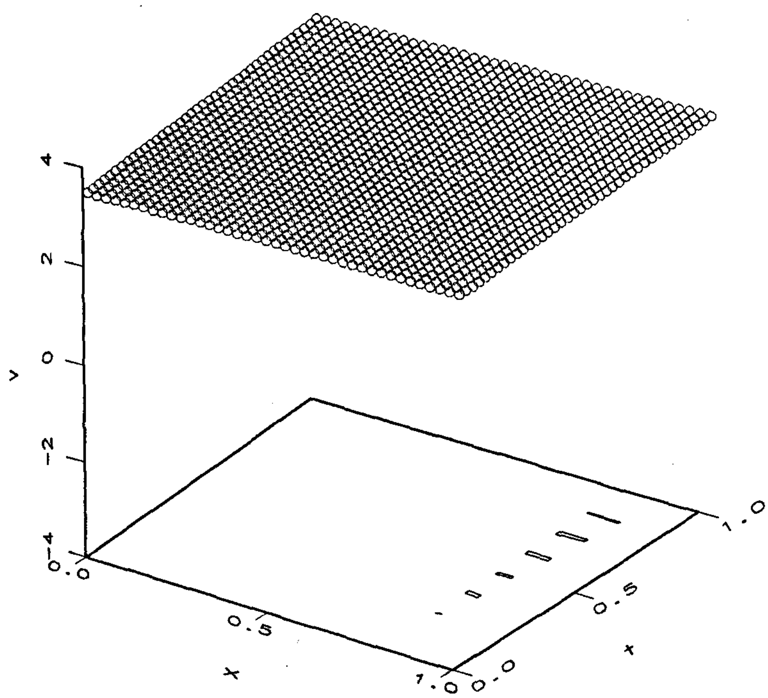

Figure 4.1: The nearly optimal control $v(x,t)$

Then we use the control function v obtained above to find $u(x,T)$, the final value of the solution of the system (4.1)-(4.4) by applying an implicit difference method. In order to obtain an implicit difference method a replacement for the equation (4.55), the region Q_T is covered by a rectilinear grid with sides parallel to the x-axis and t-axis, with h and k being the grid spacing in the x and t directions respectively. So (ih, jk) is a grid point, where i and j are non negative integers which $i = j = 0$ indicate the origin. The functions satisfying (4.55) and the difference equation at the grid point $x = ih$, $t = jk$ are denoted by $u(ih, jk)$ and $u_{i,j}$ respectively. The number of grid points is

$N \cdot M$ and $h = \frac{1-0}{N}$ and $k = \frac{1-0}{M}$. Now we return to the equation (4.55) in the form

$$u_{tt} - \triangle u + u^3 = v,$$

where v is obtained as described above. If we put $x = ih$, $t = jk$ and consider the equation (4.55) at (x, t), then

$$(u_{tt})_{i,j} = (u_{xx})_{i,j} + (v - u^3)_{i,j}.$$

We use the following central difference formulae for the second derivatives (see, for instance, [7]),

$$(u_{tt})_{i,j} = \frac{u_{i,j+1} - 2u_{i,j} + u_{i,j-1}}{k^2}$$

$$(u_{xx})_{i,j} = \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{h^2},$$

then the difference replacement for the equation (4.55) on rectangular grid is

$$u_{i,j+1} - 2u_{i,j} + u_{i,j-1} = m^2(u_{i+1,j} - 2u_{i,j} + u_{i-1,j}) + k^2(v_{i,j} - u_{i,j}^3),$$

where $m = \frac{k}{h}$. Assume $j = 0$, then

$$u_{i,1} - 2u_{i,0} + u_{i,-1} = m^2(u_{i+1,0} - 2u_{i,0} + u_{i-1,0}) + k^2(v_{i,0} - u_{i,0}^3).$$

From initial condition (4.3), we know that $u_{i,0} = 0$, so

$$u_{i,1} + u_{i,-1} = k^2(v_{i,0}).$$

From initial condition (4.4)

$$\frac{\partial u}{\partial t}|_{t=0} = u_t(x, 0) = 0, \ x \in D,$$

so if in the central difference formula for the first derivative, i.e.,

$$(u_t)_{i,j} = \frac{u_{i,j+1} - u_{i,j-1}}{2k},$$

we choose $j = 0$, then $u_{i,-1} = u_{i,1}$, and therefore

$$u_{i,1} = \frac{1}{2}k^2(v_{i,0}),\ 1 \le i \le N-1 \tag{4.57}$$

while from the boundary condition (4.2),

$$u_{0,j} = u_{N,j} = 0,\ 0 \le j \le M. \tag{4.58}$$

Now consider the following implicit difference replacement for the equation (4.55),

$$\frac{1}{k^2}\delta_t^2 u_{i,j} = \frac{1}{h^2}(\frac{1}{4}\delta_x^2 u_{i,j+1} + \frac{1}{2}\delta_x^2 u_{i,j} + \frac{1}{4}\delta_x^2 u_{i,j-1}) + (v_{i,j} - u_{i,j}^3),$$

where

$$\delta_t^2 u_{i,j} = u_{i,j+1} - 2u_{i,j} + u_{i,j-1}$$

$$\delta_x^2 u_{i,j} = u_{i+1,j} - 2u_{i,j} + u_{i-1,j}$$

(see, for instance, Mitchell [49], Chapter 5, Smith [75], Chapter 4). Then an appropriate implicit difference formula for equation (4.55) at the grid point (ih, jk) is :

$$\begin{aligned}
&(\frac{m^2}{4})u_{i-1,j+1} + (-1 - \frac{m^2}{2})u_{i,j+1} + (\frac{m^2}{4})u_{i+1,j+1} \\
&= (-\frac{m^2}{2})u_{i-1,j} + (-2 + k^2u_{i,j}^2 + m^2)u_{i,j} + (-\frac{m^2}{2})u_{i+1,j} \\
&+ (-\frac{m^2}{4})u_{i-1,j-1} + (1 + \frac{m^2}{2})u_{i,j-1} + (-\frac{m^2}{4})u_{i+1,j-1} \\
&- k^2 v_{i,j}.
\end{aligned}$$

By assuming $i = 1, 2, ..., N-1,\ j = 1$, and taking the boundary condition (4.58) into account, we find the following system of linear equations,

$$AU_2 = BU_1 + CU_0 + V_1, \tag{4.59}$$

where A and C are two symmetric tridiagonal matrices as follows:

$$A = \begin{vmatrix} -1-\frac{m^2}{2} & \frac{m^2}{4} & \dots & 0 \\ \frac{m^2}{4} & -1-\frac{m^2}{2} & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -1-\frac{m^2}{2} \end{vmatrix}$$

$$C = -A$$

and B is the following symmetric tridiagonal matrix,

$$B = \begin{vmatrix} -2+k^2u_{1,1}^2+m^2 & -\frac{m^2}{2} & \dots & 0 \\ -\frac{m^2}{2} & -2+k^2u_{2,1}^2+m^2 & \dots & 0 \\ \vdots & \vdots & \dots & \vdots \\ 0 & 0 & \dots & -2+k^2u_{N-1,1}^2+m^2 \end{vmatrix}$$

and,

$$U_2 = \begin{vmatrix} u_{1,2} \\ u_{2,2} \\ \vdots \\ u_{N-1,2} \end{vmatrix}$$

$$U_1 = \begin{vmatrix} u_{1,1} \\ u_{2,1} \\ \vdots \\ u_{N-1,1} \end{vmatrix}$$

$$U_0 = \begin{vmatrix} u_{1,0} \\ u_{2,0} \\ \vdots \\ u_{N-1,0} \end{vmatrix}$$

$$V_1 = \begin{vmatrix} -k^2 v_{1,1} \\ -k^2 v_{2,1} \\ \vdots \\ -k^2 v_{N-1,1} \end{vmatrix}$$

From (4.57) U_1 and from initial condition (4.3) U_0 are known, then from (4.59) we can find U_2. This formula is stable for all $m > 0$ and if it is expanded about the grid point (ih, jk), the truncation error is found to be:

$$h^2k^2[(\frac{-2m^2-1}{12})\frac{\partial^4 u}{\partial x^4} + \frac{1}{360}h^2(\frac{-17-13m^4}{2})\frac{\partial^6 u}{\partial x^6} + ...].$$

By using the iteration relation

$$AU_{j+1} = BU_j + CU_{j-1} + V_j,\ j = 1, 2, ..., M-1$$

we can find U_{j+1} at each step, where

$$U_{j+1} = \begin{vmatrix} u_{1,j+1} \\ u_{2,j+1} \\ \vdots \\ u_{N-1,j+1} \end{vmatrix}$$

We used this implicit difference method to obtain a numerical solution of the above problem. We have chosen $h = \frac{1}{50}$, $k = \frac{1}{50}$, which lead to $M = N = 50$ and $m = 1$. In all calculation, the values of v at internal grid points were taken from $v(x,t)$ obtained above. The graph of the $u(x,T)$, the final value of the solution, produced by this scheme is shown in Fig. 4.2.

Example 4.2 As a second example, we took again the equation (4.55) with the same boundary and initial conditions, but assume

$$f_0 = (u - \sin(\pi x))^2$$

$$f_1 = v^2$$

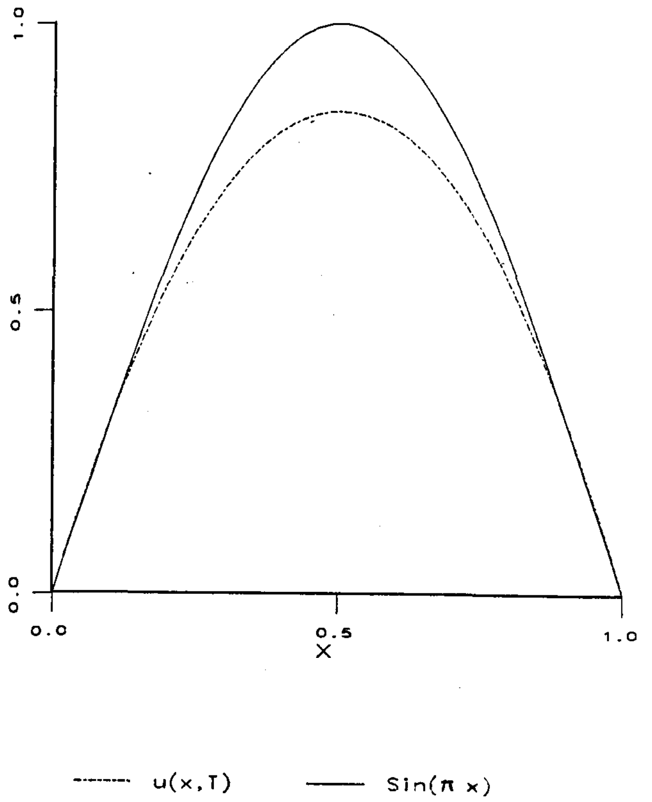

Figure 4.2: Functions $u(x, T)$ and $\sin(\pi x)$

while domains and terminal conditions were the same as in Example 4.1. In this example the total number of iterations was 316, and the cost function took the value 4.137847. Graph of the resulting control function $v(x, t)$ is shown in Fig. 4.3.

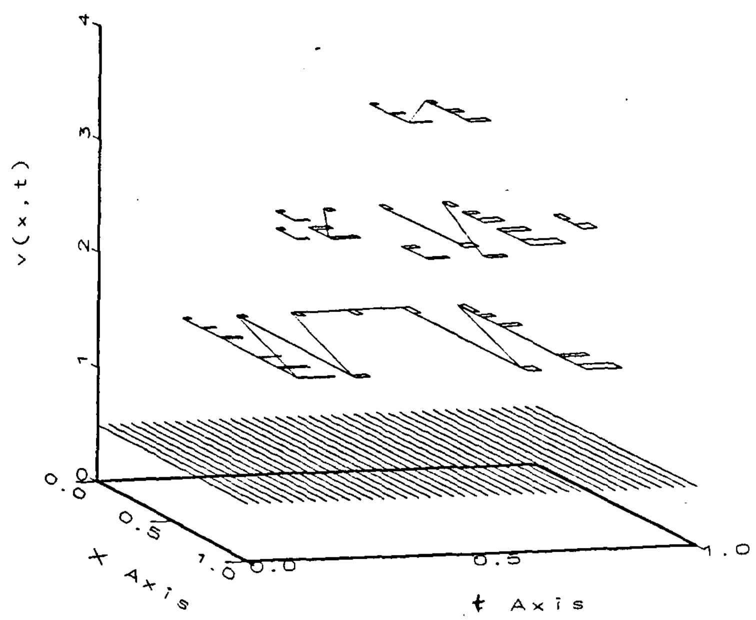

Figure 4.3: The nearly optimal control $v(x,t)$

4.8 Conclusion and recommendations for further research work

In this thesis the solution of optimal control problems governed by linear and nonlinear wave equations and the estimation of the errors in computing these solutions have been considered. The main approach used here is based on the idea of replacement of a classical control problem with a problem in measure space in which one seeks to minimize a linear form over a subset of this measure space which is described by linear equalities. The new measure-theoretical optimization problem was treated as an infinite-dimensional linear programming problem which enabled us to develop a computational method to find the approximate solution of the classical optimal control problem. This computational scheme seems to be adequate for variety of optimal control problems controlled by hyperbolic differential equations. Some basic ideas were established in Chapter 3 to estimate the errors in the approximate solutions of optimal control problems, these ideas are useful in estimating the errors which occur when an infinite linear program is approximated by a finite one. In fact, the concepts and methods introduced in this thesis provide a framework for new treatment for optimal control problems guided by wave equations. There are still some more problems to be solved related to this thesis that are recommended for further research work.

(1) In Chapters 2 and 4, we studied optimal control problems for the linear and non linear wave equations with prescribed initial and boundary conditions defined on a bounded, open, connected domain $\omega \in I\!R^n$ with smooth boundary $\partial\omega$. If the boundary $\partial\omega$ has some jump or if the given data are discontinuous (as in the case, e.g., for wave motion initiated by an impulse) then the solutions also are discontinuous at the curve $\partial\omega$, **shocks**. In the further study of existence and approximation of optimal control, it is interesting and important to investigate the optimal control problems controlled by shock waves. This field is completely new and subtle and needs more explorations.

(2) In Chapter 4, to prove the existence of a classical solution for the nonlinear wave equation

$$u_{tt} - \Delta u + |u|^\rho u = v,$$

we assumed the boundary condition as

$$u = 0 \text{ on the boundary } \Gamma_T,$$

while the control function was the inhomogeneous term $v = v(x, t)$ and the prescribed initial conditions (4.3)-(4.4) were satisfied. It is important to develop the theory of optimal control problems guided by nonlinear wave equations where, as the linear case, the control is applied on the boundary Γ_T. For development of this theory first we need to prove the existence of the solution with the new boundary condition.

(3) In Chapter 4, we considered the optimal control problem with the initial condition (4.3), i.e.,

$$u(x, 0) = u_0(x) = 0, \qquad x \in D.$$

The third problem is to choose $x \longrightarrow u_0(x)$ any real continuous function, where $x \in D$. Although it seems for this case the problem could be solved by the analysis used in Chapter 4, but it needs to be tackled wisely.

(4) In Chapter 3, to find the approximation of the optimal value $\mu^*(f_0)$, we attempted for the lower and upper bounds of this value. However, we were unsuccessful for an accurate upper bound. A further research is to find an appropriate upper bound for $\mu^*(f_0)$ by choosing M_1 and M_2, some fixed positive integers. Although there are some related works (see Vershik and Temel't [81] and Vershik [80]), but we believe that to solve this problem, one need to introduce some new concepts, spaces and topology.

(5) In Chapter 2 we considered the n-dimensional linear wave equation. The fifth problem consists in considering the following linear hyperbolic equation with variable coefficients:

$$Y_{tt}(x,t) - \text{div}(a(x) \bigtriangledown Y(x,t)) + b(x)Y(x,t) = f(x,t)$$

where $(x,t) \in \omega \times [0,T]$, and $f(x,t)$ is a continuous function on $\omega \times [0,T]$ and a, b are continuous functions on ω. We can assume the same boundary condition as in Chapter 2 and define the initial conditions and the set of admissible controls and objective functions as in Chapter 2.

4.9 Appendix

In this appendix we show that the problem

Minimize

$$I(\mu,\nu) = \mu(f_0) + \nu(f_1)$$

over the set $Q(M_1, M_2, M_3)$ subject to:

$$\mu(F_i) + \nu(G_i) = \alpha_i, \qquad i = 1,2,...,M_1$$

$$\mu(\xi_j) = a_j, \qquad j = 1,2,...,M_2$$

$$\nu(\zeta_k) = b_k, \qquad k = 1,2,...,M_3.$$

has a solution.

In fact by the same considerations as Proposition 4.1 and Proposition 4.2, the set of positive measures $Q(M_1, M_2, M_3)$ is compact on the set $\mathcal{S}$ and the function

$$(\mu, \nu) \longrightarrow \mu(f_0) + \nu(f_1) \in \mathbb{R}$$

that maps this compact set on the real line is continuous, so this function attains its minimum on the set $Q(M_1, M_2, M_3)$, i.e., there exists a pair of measures $(\mu_1^*, \nu_1^*) \in Q(M_1, M_2, M_3)$, such that

$$(\mu_1^*, \nu_1^*) = \inf_{Q(M_1, M_2, M_3)} [\mu(f_0) + \nu(f_1)].$$

4.10 Appendix

Since the set $\{\psi_i\}$ is a basis for the space function $H_0^1(\overline{Q}_T)$, we choose the set $\{\psi_i\}$ orthonormal with respect to inner product defined on $\overline{Q}_T$ and such that $\psi_{it}(x, T) = 0$, $\forall x \in D$. Thus for this kind of base, (4.51) changes to

$$|\int_{D_T} [(u_{v_t}(\cdot, T) - g_2)\psi_i]dx| \leq 3\epsilon, \quad i = 1, 2, ..., M_1.$$

Assume

$$w_2(\cdot, T) = u_{v_t}(\cdot, T) - g_2$$

and write $w_2(\cdot, T)$ in term of this orthonormal base,

$$w_2(\cdot, T) = \sum_{j=1}^{\infty} a_j \psi_j(\cdot, T)$$

where

$$\psi_i(\cdot, T)\psi_j(\cdot, T) = \int_{D_T} \psi_i(\cdot, T)\psi_j(\cdot, T)dx = \begin{cases} 1 & i = j \\ 0 & i \neq j. \end{cases}$$

So

$$|\int_{D_T} w_2(\cdot, T)\psi_i(\cdot, T)dx| = |a_i| \leq 3\epsilon, \quad i = 1, 2, ..., M_1. \tag{4.60}$$

Thus the first M_1 Fourier coefficients of $w_2(\cdot,T)$ are bounded tightly, also the tail

$$h(x) = \sum_{j=M_1+1}^{\infty} a_j\psi_j(\cdot,T)$$

of the Fourier series of $w_2(\cdot,T)$ tends to zero as $M_1 \longrightarrow \infty$. Now, we can choose M_1 so large that

$$\|h(x)\|_{L_2} \le \epsilon, \quad x \in D_T,$$

then

$$\|u_{v_t}(\cdot,T) - g_2\|_{L_2} = \|w_2(\cdot,T)\|_{L_2} = \|\sum_{j=1}^{M_1} a_j\psi_j + \sum_{j=M_1+1}^{\infty} a_j\psi_j\|_{L_2}$$

$$\le \|\sum_{j=1}^{M_1} a_j\psi_j\|_{L_2} + \|h\|_{L_2}.$$

But by the orthonormality properties of ψ_j's and from (4.60) we find that

$$\|\sum_{j=1}^{M_1} a_j\psi_j\|_{L_2}^2 = \sum_{j=1}^{M_1} a_j^2 \le 9\epsilon^2 \cdot M_1,$$

so

$$\|\sum_{j=1}^{M_1} a_j\psi_j\|_{L_2} \le 3\epsilon\sqrt{M_1}.$$

Thus

$$\|u_{v_t}(\cdot,T) - g_2\|_{L_2} \le 3\epsilon\sqrt{M_1} + \epsilon = \frac{3}{\sqrt{M_1}} + \frac{1}{M_1},$$

if $M_1 \longrightarrow \infty$, then

$$\|u_{v_t}(\cdot,T) - g_2\|_{L_2} \longrightarrow 0.$$

Similarly, we choose the set $\{\psi_i\}$ orthonormal with respect to an inner product defined on $\overline{Q}_T$ but such that $\psi_i(x,T) = 0$, for every $x \in D$. For this kind of base, (4.51) changes to

$$\left|\int_{D_T} [(u_v(\cdot,T) - g_1)\psi_{it}]dx\right| \le 3\epsilon, \quad i = 1,2,...,M_1.$$

Assume

$$w_1(\cdot,T) = u_v(\cdot,T) - g_1,$$

again in a similar way, we can show that

$$\lim_{M_1 \longrightarrow \infty} \|u_v(\cdot, T) - g_1\|_{L_2} = \lim_{M_1 \longrightarrow \infty} \|w_1(\cdot, T)\| \longrightarrow 0.$$

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