بسم الله الرّحمن الرّحيم

In the name of ALLAH

The beneficient, the merciful.

To:

My best friends

SHOHADAY ABADAN

who died while they were defending

their homeland and their country.

Their place in heaven is guaranteed by God
and history will remember them as heroes.

Shapes, Measures and Elliptic Equations

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Submitted in accordance with the requirements for the degree of

Doctor of Philosophy

The University of Leeds

Department of Applied Mathematical Studies

September 1996

The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others.

Abstract

A measurable set - a *shape* - can be considered as a measure; the present thesis treats the inverse problem - to characterize those measures which can be considered as shapes, in a very generalized sense - by solving some optimal shape and optimal shape design problems which are governed by linear or nonlinear elliptic equations. A new method is introduced for solving the usual optimal shape problems, and also a new set of problems which are defined in terms of a pair of elements, a shape (defined by its boundary) and an optimal control associated with it. The problems are considered in polar and cartesian coordinates separately.

The new method to attack these problems, which is applicable to both system of coordinates, consists in using the variational form of the elliptic equations and then applying the process of embedding into some appropriate spaces of measures; thus the problem is replaced by a measure-theoretical one in which one seeks to minimize a linear form over a subset of positive Radon measures defined by linear equalities. The optimal solution is approximated then by a finite combination of atomic measures so that the optimal shape design problem is transformed into a finite linear programming problem. The solution of this problem is used to construct the optimal shape and its associated optimal control. The advantages of this new method with respect to other methods, and the existence of the optimal solution in each case, have been carefully considered.

Acknowledgments

The completion of a work of this nature is not possible without a great deal of assistance from many people. "*Who does not appreciate the people, cannot appreciate the God*" (Prophet Mohammed (S)).

I wish to express my sincere thanks to my supervisor, Dr J. E. Rubio, Reader in the Department of Applied Mathematical Studies, for his supervision, help, encouragement and invaluable ideas and also for his friendship, which I value highly.

Thanks are also due to the Government and the Ministry of Higher Education of Islamic Republic of Iran, and also the Shahid Chamran University of Ahwaz for providing the opportunity and financial support.

Finally, this work would not have been possible without the encouragement of my family. I am grateful to my father, my mother and my father and mother in-law for their supports. I should like to thank my wife Mrs *Farzaneh Mohajery*, who has been endured much additional family responsibility; I appreciate her assistance throughout this research.

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Chapter 1

Introduction

The study of optimal shape design tries to answer the question,"*What is the best shape for a physical system?*" We will discuss several such physical systems, mostly those that can be described by an elliptic partial differential equation; the optimal shape minimizes a certain performance criterion.

Broadly speaking, the term *optimal shape design* (OSD) is used whenever a function is to be minimized with respect to a particular geometric element (or elements). In general, the element is a curve, a domain (an open measurable set), or a point. Traditionally, OSD has been treated as a branch of the calculus of variations and more specifically of optimal control; this subject interfaces with several fields including optimization, optimal control theory, differential equations (or inequalities) and their numerical solutions.

1.1 OSD in Calculus of Variations

The foundations of the calculus of variations were laid by great mathematicians like Bernoulli, Euler, Lagrange and Weirstrass. As a matter of fact, all problems of Minima and Maxima in functional analysis properly belong to the calculus of variations; when this minimization or maximization takes place over a set of geometrical elements, the problem is usually an OSD problem. Many of these problems can be found in text books on the calculus of variations (see for instance [63] and [21]). One of the most famous and oldest of them is the *free-boundary* problem, in which the solution of a differential equation has to satisfy certain conditions on the boundary of a variable domain; in many particular cases the domain has to be determined as a part of the solution (see [27] for more details). In this thesis we shall consider some of these problems.

1.2 OSD and Optimization

The fundamental problem of optimization is to arrive at the best possible decision in any given set of circumstances. The study of optimization techniques is attractive because of its very wide field of application such as operational research, economics, aerospace, pure geometry, physics, control theory, chemical engineering and many other subjects (it has been claimed that everyone is optimizing something all the time!!). Walsh in his book [60] has given many examples of optimization problems in diverse fields. The one factor that has influenced this growth and extension of optimization theory more than any other, has been the parallel development of computer equipment with which optimization theory can be applied to broad classes of problems.

An optimization problem sometimes can be considered as an OSD problem and vice versa. An OSD problem is obviously an optimization problem; depending upon the physical structural of the problems, optimization problems can be classified as optimal and nonoptimal control problems. Again, if the minimization or maximization associated with a control problem takes place over a set of geometric elements, the problem can be considered as an OSD problem.

Optimality of an OSD problem defined by partial differential equations has been studied in many ways, as in [8] and [9] by the dynamic interpretation of optimal shape design, in [19], [44] and [43] by some direct calculation of shape variations, in [64] and [41] via minimax differentiability method, in [40] via the mapping method. The manner of solution has an important role in the numerical computation of optimal shapes. In the present work, for the first time, we consider an optimality of an OSD problem by changing the problem into a measure-theoretical one. In this procedure, we will apply many optimization methods and techniques to reach an optimal solution.

1.3 OSD in Optimal Control Theory

An optimal control problem is a mathematical programming problem involving a number of stages, where each stage evolves from the previous stage in a prescribed manner. In studying an optimal control problem, one usually requires:

- i) A real closed (time) interval $J = [t_0, t_1]$, in which the controlled system will be involved.
- ii) A bounded-closed set $U \subset \mathbb{R}^m$ on which u takes its values.
- iii) A differential equation describing the controlled system, satisfied by the trajectory function $t \in J \rightarrow X(t) \in \mathbb{R}^n$ and the control function $t \in J \rightarrow u(t) \in U$, where u is a measurable function.
- iv) An observation function $f_\circ[t, X(t), u(t)]$ which is given.

A classical optimal control problem is to find an admissible control u which satisfies the differential equation and minimizes the functional $\mathbf{I} : \mathcal{F} \rightarrow \mathbb{R}$ defined by $\mathbf{I}(p) =$

$\psi(f_o(t,p))$, $\forall p \in \mathcal{F}$, where $p = [X(\cdot), u(\cdot)]$, ψ is given and $\mathcal{F}$ is the set of admissible trajectory and control pairs.

The following problems are among the main fields of study and developments of optimal control theory:

- Existence of an optimal control.
- Necessary (and possibly sufficient) conditions for u to be an optimal control.
- Constructive algorithms amenable to numerical computations for the approximation of optimal controls.

Clearly the development of such theory depends on the form of the differential equation describing the controlled system. The theory described in the works of Pontryagin, Boltyanskii, Gamkerlidze and Mischenko in [46], and also Hestenes in [29], is applicable to controlled systems defined by a family of ordinary differential operators. To see the development of optimal control theory, the reader is recommended to have a look at [34], [30], and [15], for instance.

By performing changes of variables which bring the variable domain into a fixed domain, one can convert a problem of OSD into an optimal control one, where the control variable is the coefficient of the partial differential equation defined in the OSD problem. This method allowed Begis and Glowinski in [5] and Morice in [39] to devise a satisfactory method for optimal design problems; it has, however, two important shortcoming points:

- It is difficult to take into account geometrical constraints.
- A completely new study of the (new) state equation is necessary.

Thus, solving an OSD problem in general needs special methods.

One of the attractive and powerful recent methods for solving an optimal control problem is based on an idea of Young in [63], consisting of the replacement of the classical variational problem by problems in measure spaces. An early version of this approach was carried out in 1967 by Ghouila-Houri in [22]. In 1977, this method was employed for the first time by Wilson and Rubio in [62] and [52] on an optimal control problem defined by a diffusion equation; then the method has been theoretically established by Rubio in his book [50] in 1986. The application of the new method was extended and improved gradually; it was completed for systems governed by diffusion and wave equations in [31], [16] and [17]. Moreover it was extended for the elliptic equations in [53]. But no attempt has yet been made to solve an OSD problem via this ideas; we attempt such an application in this thesis.

1.4 Optimal Shape Design Theory

In an OSD process, the objective is to optimize certain criteria involving the solution of a partial differential equation with respect to a particular geometrical element (or elements) appearing in the partial differential equation. In optimal shape design theory, attempts are directed to computerize the design process to create a new shape design or improve an existing design. From a mathematical point of view, an optimal shape design problem is (usually) defined as follows:

Let u_D be the solution of a partial differential equation related to the geometrical element D; let $\mathrm{I}(u_D, D)$ be a real-valued function of u_D and D. We say that we have an OSD problem to solve if we find D^* in a class $\mathcal{F}$ of allowable geometrical elements, to minimize I. Symbolically, one may write an OSD problem as:

$$\text{Minimize}: \qquad \mathrm{I}(u_D, D)$$

$$\text{Subject to}: \qquad \Lambda(u_D, D) = 0,$$

where Λ is an operator that for every $D \in \mathcal{F}$ defines a unique u_D, and the minimization takes place over the set $\mathcal{F}$. Here we deal only with the cases that Λ is an elliptic operator. To take some examples, one can see [7] (Appendix 4) for the design of a nozzle at low speed with a required velocity of air in some prescribed region, and [44] for some examples in the optimization of an electromagnet, a wing and minimum drag problems.

Although OSD is a branch of optimal control theory with geometrical elements as controls, there are two notable differences between classical problems of optimal control and OSD. The first is that the control sets in the latter are classes of gometrical elements which do not have *natural* algebrical or topological structures. The second is that the state u_D here is related to D which is not fixed; this makes it difficult to examine the convergence of sequences of the form $\{u_{D_n}\}$, which is one of the key steps to the establishment of the existence for OSD. However, sometimes one may convert a shape design problem into an optimal control problem with controls appearing in the coefficients of the differential equation (see for example [5], [39] and [44]).

Historical background and references: OSD has been studied in a wide variety of fields; it is difficult to give a complete account of the previous works. The studies of OSD were started a long time ago; Bernoulli's speediest descent problem, for example, can be viewed as such a problem. Hadamard's book [25] in 1910, is considered as the most influential early work; he gave a formula for computing the derivative of Green's function of Laplace operator with respect to the normal variations of domain.

Later on, studies were made only for those problems with an explicit solution for the partial differential equation, as the work of Miele [37] in 1965 on the optimization of wing profiles at supersonic speed. Eventually the method was extended to problems of structural engineering; in particular those possible to convert into optimal control problems. Meanwhile, OSD has been studied in great depth by the French school of Applied Mathematics (at universities of Parise and Nice, specially) and the treatment using the optimal control technique of distributed systems, seems to have begun in 1972 with Lions in [35] and with Cea, Giaan and Michel (1973) in [10], where the first algorithm is found. Optimality conditions were found concurrently by Pironneau [43] and Murat and Simon (1976) [40] for problems with Dirichlet conditions, by Dervieux and Plamerion (1975) in [13] for Neumann problem, by Rousselet (1976) [49] for eigenvalue problem. The existence of solution was then studied by Murat and Simon (1976) [40] and Zolezio (1979) [64]. Numerical methods based on the above results were devised and tested by Begis and Glowinski (1975) [5] and Morice (1974) [39] for the technique of mapping and also many others like Pironneau (1983) [44] and Haslinger and Neittaanmaki (1978) [27] by use of the finite element method.

Optimal shape design is an applications-oriented subject; the use of OSD can be found in many engineering branches, because systems described by partial differential equations have particular shape design applications in industry. We can describe some of them as mechanical engineering (for designing airplanes, wings at supersonic speed and in Fluid dynamic), civil engineering, electrical engineering (for electromagnet and antenna design), marine industry (for design of submerged in naval hydrodynamics) and chemical engineering (for change the anode surface to a given fixed shape) for more detail see [27], [44], [43] and [7]. Thus, OSD problems have been studied extensively by engineers where the state equations are governed by partial differential equations with suitable boundary conditions. The results can be found in textbooks, for example Haug and Arora [28] (1979), Vanderplaats [59] (1984) and Komkov [32] (1988), and in some conference proceedings like that one which edited by Adelman [1] (1986). In the last twenty years, there has been increased mathematical interest in studying the question of the existence of optimal shapes, numerical approximation, convergence and sensitivity analysis, dynamical interpretation of OSD algorithms, topologies and compactness properties, and optimality conditions that are referenced in [27] and [36]. However, enormous work has been done by mathematicians for OSD problems governed by variational inequalities; see [7], [27] and [36].

In general, most methods of solving an OSD problem are related to the numerical solution of (partial) differential equations. The exception is the mapping method which maps the solution spaces of the partial differential equations in an OSD problem on a fixed domain, and then converts the shape design problem into an optimal control problem with controls appearing in the coefficients of the differential equation (see [40] and [44] for example). We can also consider, the recent works in [26] on finite element method, [61] on boundary-element method, [41] on minimax (computing the shape derivative by differentiating a Min Max problem with respect to an appropriate vector field), and on the Least-Squares method in [4].

Up to now, there has been no attempt to solve an OSD problem by applying the measure-theoretical method. Also, all OSD problems considered have been based on not more than one geometrical element (which, indeed, has usually been a domain); thus efforts have been directed to obtain an optimal element as the optimal solution.

In this task, we introduce a new approach to attack an OSD problem by transferring the problem into a new one in which positive Radon measures are involved. The method has been successfully applied on some optimal control problems as explained above, but never applied on an OSD problem before. This method has some important advantages in comparison with others, such as automatic existence theorems, achieving the global minimizer, and applying a linear treatment for nonlinear problems. Moreover, an OSD problem has normally been defined with respect to a particular geometrical element (which usually has been a domain). Here we also consider a new and larger set of OSD problems; those are defined in terms of a pair of geometrical elements (a domain and its boundary). We attack these problems and obtain their solution by use of the new approach.

1.5 Outline of the work

In the present thesis, we are going to solve some OSD problems in which they are defined with respect to a pair of geometrical elements. This pair consists of a measurable set (in $\mathbb{R}^2$) that can be regarded as a domain, and a simple closed curve which is the boundary of the measurable set and contains a given point. Based on the simple property of curves, the related OSD problem depends on the geometry which is used. We shall solve the appropriate OSD problems in polar coordinates (in chapters 2 and 3) where $0 \leq \theta \leq 2\pi$ and $r \geq 0$, and in cartesian coordinates (in chapters 4, 5 and 6) where the boundary curve consists of fixed and variable parts.

In the whole of the thesis we use a common approach, to extend the problem into a measure-theoretical one which is defined on a class of positive Radon measures. Then, the new problem is approximated by a finite linear programming one in which its results approximate the optimal solution of the OSD problem. We remind the reader that all partial differential equations involved are chosen as different cases of elliptic equations.

In Chapter 2: The solution of an OSD problem which is defined in terms of a pair of geometrical elements, a set C and its boundary ∂C, is studied in Chapter 2. By introducing the set of admissible pairs, $\mathcal{F}$, a classical OSD problem is introduced (in (2.4)) as the finding of the minimizer pair in $\mathcal{F}$ for the given functional

$$\mathrm{I}(C, \partial C) = \int_C f_o dA + \int_{\partial C} \frac{1}{\sqrt{r^2 + w^2}} h_o ds,$$

where area of C is fixed and $w = \frac{dr}{d\theta}$ is the control function while ∂C is represented by $r = r(\theta)$.

By some analysis, the necessary conditions for admissibility of a pair $(C, \partial C)$ in the classical formulation, are characterized as integral equalities which are mentioned in (2.6), (2.8) and (2.11) in section 2.2. To be sure that $\mathcal{F}$ is not empty and the problem has a solution in $\mathcal{F}$, we try to somehow enlarge this set; the basis of this metamorphosis is the fact that an admissible pair can be considered as a pair of positive Radon measures, say (μ_c, ν_c), which is proved by means of Proposition 1. Moreover the transformation $(C, \partial C) \longrightarrow (\mu_c, \nu_c)$ is an injection (Proposition 2) and it changes the classical problem OSD into a measure-theoretical one. Then by enlarging the image of the transformation, we change the problem into a new nonclassical one (defined in (2.13)), where the involved pair (μ, ν) belongs to $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ (indeed, the measure satisfying the conditions of 2.13 can be approximated, in weakly* sense, by the actual pair $(C, \partial C)$). The new problem has some important advantages which are listed in section 2.4.

Regarding the simple compactness properties of the weak* topology and the concepts of Proposition 3, the existence of an optimizer for problem (2.13) is proved in Theorem 1. Problem (2.13) is linear in terms of the variables μ and ν; thus it is an infinite-dimensional linear programming problem. Our attempt is to approximate its solution with the solution of a finite dimensional linear one. In the first step of approximation, by introducing countable total sets in an appropriate space and then choosing a finite number of their functions, the solution of the problem can be approximated by one in which the number of constraints is finite (Proposition 6). Moreover it is proved that the optimal measures μ^* and ν^* have the following forms:

$$\mu^* = \sum_{i=1}^{n} \alpha_i^* \delta(Z_i^*), \; \nu^* = \sum_{j=1}^{m} \beta_j^* \delta(z_j^*),$$

where δ is a unitary atomic measure and the coefficients $\alpha_i^* \geq 0$, $\beta_j^* \geq 0$ and points Z_i^*, z_j^* (belong to a dense subset of Ω and ω respectively) are unknowns. So in the second step, by using discretization on appropriate spaces the problem is approximated by one (in (2.17)) with unknowns α_i's and β_j's. By introducing the function ξ (Proposition 8) in section 2.7, we show that the measure μ can be identified in terms of the boundary measure ν; hence the problem is approximated by a finite linear programming problem (in (2.19)) in which its unknowns are only β_j's. But this replacing may cause some limitations which are discussed in section 2.8. In the end of the chapter, based on the continuity of the integrand functions, two examples are given in detail. We can also claim that this chapter is an answer of the interesting question: "*When can a measure* $\mu \in \mathcal{M}^+(\Omega)$ *be approximated by a shape* C *associated with the given measure* η *as* $\int_\Omega f d\mu = \int_C f d\eta$, $\forall f \in C(\Omega)$?"

In Chapter 3: Based on the concepts of the previous chapter, we are going to solve a similar problem in which the solution of the following elliptic equation on C

$$\operatorname{div}(k(\theta, r)\nabla u) - f(\theta, r, u) = 0, \tag{1.1}$$

with the Neumann condition

$$\nabla u.\mathbf{n}_{|\partial C} = v, \tag{1.2}$$

is involved (here $u : \Omega \to U \subset \mathbb{R}$ and $v : [0, 2\pi] \to V \subset \mathbb{R}$). We say the quadruplet $(C, \partial C, u, v)$ is admissible when u and v satisfy (1.1) and (1.2), and the pair $(C, \partial C)$ is defined as Chapter 2. Let the set of all admissible quadruplets be denoted by $\mathcal{F}$, then the aim of Chapter 3 is to find the minimizer of

$$\mathbf{I}(C, \partial C, u, v) = \int_C f_\circ(\theta, r, u, \nabla u)\, dr d\theta + \int_{\partial C} h_\circ(\theta, r, w, v)\, ds,$$

over $\mathcal{F}$ (problem (3.4)); here (r, u) is the trajectory and (w, v) is the control pair. In general, it is difficult to identify a classical solution for the elliptic problem; thus (by proving Proposition 9) we apply the variational form of the elliptic problem as

$$\int_C (k \nabla u \nabla \varphi + f\varphi)\, r dr d\theta - \int_{\partial C} k\varphi v\, ds = 0,\ \forall \varphi \in H^1(C), \tag{1.3}$$

and look for a bounded weak solution u satisfying (1.3) for all φ in $H^1(C)$ (the Sobolev space of order 1 on C). We attack the problem by use of the Radon measures. An admissible $(C, \partial C, u, v)$ defines two positive Radon measures as

$$\lambda_u(F) = \int_C F(\theta, r, u, \nabla u)\, dr d\theta,\ \sigma_v(G) = \int_J G(\theta, r, w, v)\, d\theta;$$

here $F \in C(\Omega')$ and $G \in C(\omega')$ where $\Omega' = \Omega \times U \times U'$ and $\omega' = \omega \times V$ (where $\nabla u \in U'$). Thus there is an injection transformation between $\mathcal{F}$ and a subset of $\mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$, which changes the problem into a measure-theoretical one. Then the problem is extended to a bigger space defined by all pair of measures (λ, σ) satisfying some linear conditions.

Considering the variational equality (1.3), the first set of conditions are already introduced (see (3.13)). Because the restriction of the measures λ and σ over Ω and ω are the measures μ and ν in Chapter 2, measure σ must satisfy the same conditions as for ν (see (3.15)). Stokes's theorem defines another relation between λ and σ (see (3.16)). The last set of conditions is obtained by use of Green's formula (see (3.17)). Thus we replace the problem with new one (at (3.18)) which definitely has a minimizer. Then the minimizer is approximated by a solution of a finite linear programming problem; for this reason we apply the same total sets as Chapter 2 for the related spaces. For the rest, in section 3.4 we show that the set of functions φ_n such that $\varphi_n = r^n \cos n\theta$ or $\varphi_n = r^n \sin n\theta$ for $n = 1, 2, 3, \ldots$, is total in $H^1(C)$ and can be applied as a part of the approximation scheme. Applying discretization on Ω' and ω' gives requisted finite linear program (shown in (3.21)). This chapter ends with a numerical example.

<u>**In Chapter 4:**</u> It is difficult to introduce a linear condition which determines the property of a closed curve being simple, in cartesian coordinates; thus in the following chapters, we consider those measurable sets D whose boundary ∂D consists of a variable part Γ and a fixed part between two given points, so that it is certainly simple. A domain D is called admissible if the elliptic equation

$$\Delta u(X) + f(X, u) = v(X) \tag{1.4}$$

with the boundary condition

$$u_{|\partial D} = 0, \tag{1.5}$$

has a bounded solution on D. Let $\mathcal{D}$ be the set of all admissible domains. We deal in Chapter 4 with solving an optimal shape problem (an OSD problem with a fixed control), which is to find the optimal domain for the functional $\mathrm{I}(D) = \int_D f_\circ(X,u)\,dX$ over $\mathcal{D}_M$ (the set of domains D where Γ is determined by joining M segments); here u is the solution of the elliptic equations over D. The process of solution is achieved in two stages. First for a fixed domain, by using the density property and the idea of approximating a curve by broken lines, Γ (and hence ∂D) can be determined with fixed number of points (M-representation). Then D and any integral on D can be considered as a function of M variables. Based on the elliptic equations, the generalized solution u is determined (in Proposition 13) by the following integral equality

$$\int_D (u\Delta\psi + \psi f)\,dX = \int_D \psi v\,dX\ ;\ \forall\psi \in H_0^1(D). \tag{1.6}$$

Then by introducing measure $\mu_u(F) \equiv \int_\Omega F\,d\mu_u = \int_D F(X,u)\,dX,\ \forall F \in C(\Omega)$, ($\Omega \equiv U \times \overline{D}$ that $u \in U$), we transfer the problem into a measure-theoretical one in which more than the set of equalities induced by (1.6) the measure must be projected on (x,y)-space as the Lebesgue measure (condition (4.11)). Then we enlarge the underlying space to reach an infinite linear system of equations that the unknown is a measure in $\mathcal{M}^+(\Omega)$ (see (4.12)).

In section 4.3 we show that the set of functions

$$\psi_i = xy(y-1)(x - x_1 + y - Y_1)(x - x_2 + y - Y_2)\ldots(x - x_M + y - Y_M)q_i,$$

in which that Y_m's are given and $q_i \in \{1, x, y, x^2, xy, y^2, \ldots\}$, is total in $H_0^1(D)$. By the use of this total set and putting an appropriate discretization on Ω, one can approximate the solution of the problem with the solution of a finite linear system (see (4.24)). Hence the value of $\mathrm{I}(D)$ is calculated as a function of M variables, $x_1, x_2, \ldots, x_M$, for any given domain D.

In the second stage (section 4.4), considering the previous one, a vector function $\mathbf{J}$:

$D \in \mathcal{D}_M \rightarrow \mathrm{I}(D)$ is set up. Using a standard minimization algorithm on $\mathbf{J}$ (like *AMOEBA*), gives the minimizer domain for $\mathbf{J}$; then Theorem 3, proves that the M-representation determined by this minimizer, is the optimal solution for the problem. Many examples for the linear and nonlinear cases of elliptic problem are given in section 4.5.

In Chapter 5: Here we consider an optimal shape problem similar to those in Chapter 4, however the control function v in this case is not fixed; rather, it represents a further means of optimization, so that the performance criterion is:

$$\mathrm{I}(D,v) = \int_D f_1(X,u(X))\,dX + \int_D f_2(X,v(X))\,dX. \tag{1.7}$$

By fixing the domain, we change the problem into an optimal control one which is to find the optimal control v_D^* for the given D. Then the classical control problem is replaced with a measure-theoretical one by introducing measures

$$\mu(F) = \int_D F(X,u(X))dX, \forall F \in C(\Omega),\ \nu(G) = \int_D G(X,v(X))dX, \forall G \in C(\omega)$$

in Proposition 16, in which their projections on (x,y)-plane are the respective Lebesgue measures (here $\omega = D \times V$ that $v(X) \in V$). The existence of the optimal solution for the new formulation is proved in Theorem 4. We limited the number of constraints by use of total sets (see (5.17)); then Theorem 5 shows that the solution of the new problem can introduce two piecewise-constant functions which are close enough to the optimal trajectory and control function of the optimal control problem. Moreover, this solution can be obtained by the solution of a finite linear program by discretization (see (5.39)). Hence the value of $\mathrm{I}(D, v_D^*)$ is determined as a function of M variables.

To obtain the optimal shape in the next approach, we establish the function

$$\mathbf{J} : D \in \mathcal{D}_M \longrightarrow \mathbf{I}(D, v_D^*) \in \mathbb{R},$$

in section 5.4 which is a vector function. By use of a standard minimization algorithm, the minimizer pair for **J** is obtained. Then Proposition 20 shows that this pair estimates nearly optimal domain and control for the mentioned OSD problem. In the numerical examples we use the same data as in examples of Chapter 4; thus the reader can compare the result of the controlled system with those for the uncontrolled one.

In Chapter 6: We consider a different type of elliptic equation by changing the boundary condition into $u_{|\partial D} = v$, that is, v is a boundary control; hence we try to solve an OSD problem with the objective function

$$\mathbf{I}(D, v) = \int_D f_1(X, u(X))\, dX + \int_{\partial D} f_2(s, v(s))\, ds.$$

This change results in the following integral equality as the variational form of the elliptic equation

$$\int_D (u\Delta\psi + \psi f)\, dX - \int_{\partial D} v(\nabla\psi.\mathrm{n})\, ds = \int_D \psi g\, dX,\ \forall\psi \in H_0^1(D).$$

For the fixed domain, the weak solution u and the control function introduce two positive Radon measures as

$$\mu_u(F) = \int_D F(X, u(X))\, dX, \forall F \in C(\Omega),\ \nu_v(G) = \int_{\partial D} G(s, V(s))\, ds, \forall F \in C(\omega),$$

where $\omega = \partial D \times V$. We replace the problem with a measure-theoretical one and follows the concepts of chapter 5 for the rest of the process to reach the solution.

Some proofs of the above given results and many other related materials is described in the following Appendixes:

In Appendix A we introduce and prove the way of calculating the function ξ in Chapter 2, when the discretization is used.

Appendix B, related to the concepts of Chapters 4, 5 and 6, explains why we chose the admissible set $\mathcal{D}_{\mathcal{M}}$ for a fixed M. Also it is discussed there what could happen as M tends to infinity.

The frequent use of the subroutine $AMOEBA$ requires the description of this program in Appendix C; we also mentioned some limitations in using $AMOEBA$ in this Appendix.

Appendix D introduces the way of obtaining the suboptimal control function for the results of the numerical examples in Chapter 5. It is also explained there how one can plot this suboptimal control in 3-dimensions.

Chapter 2

Shapes and Measures

2.1 Introduction

A measurable set - a "*shape*" - can be considered as a measure. Indeed, let $\Omega \subset \mathbb{R}^2$ be a bounded domain, and $C \subset \Omega$ a measurable subset. If $\eta \in \mathcal{M}^+(\Omega)$ (that is, η is a positive Radon measure on the Borel subsets of the given set Ω), then the following function F defined by

$$\mathbf{F} : f \longrightarrow \int_C f d\eta \, , \forall f \in C(\Omega) \tag{2.1}$$

is a continuous linear functional, and therefore it can be identified with a unique measure η_C in $\mathcal{M}^+(\Omega)$, by the Riesz Representation theorem (see for instance [55]).

In this Chapter, by defining an appropriate optimal shape design problem, we attempt to treat a more general version of the inverse problem; that is: "*When can a measure $\mu \in \mathcal{M}^+(\Omega)$ be approximated (weakly*, to be sure) by a shape C associated with* *the given measure η as:*

$$(\mu(f) \equiv) \int_{\Omega} f d\mu = \int_{C} f d\eta, \ \forall f \in C(\Omega)". \tag{2.2}$$

We remind the reader that not only the shape C but also its boundary ∂C will be involved in the optimal shape design problem. By solving this problem, the shape C which satisfies (2.2), and also the curve ∂C will be determined. In the first stage, by introducing necessary conditions, the appropriate classical optimal shape design problem will be setup. Then, by a process of embedding, this problem will be extended to a measure-theoretical one in which one looks for two unknown positive Radon measures. The new formulation has some advantages; especially, it always has a solution, as shown in an existence theorem. Changing the problem into an infinite dimensional linear programming problem helps to approximate the solution with the solution of the appropriate finite linear programming problem. Meanwhile, we will show that one of the measures can be evaluated in terms of the other one; hence the number of unknowns will be decreased. In the final stage the optimal control, the optimal shape and the minimum value of the performance criteria will be illustrated (approximately of course) from the result of the appropriate finite linear programming problem. We state here that one may use some standard minimization algorithms (like $AMOEBA$ in [47]) and penalty method (see for example [60] for details) as well, to consider some logical limitations, more details will be given in section 2.8. At the end of the Chapter some numerical examples will be given.

2.2 The optimal shape design problem

2.2.1 Classical form of the problem

In order to define a classical optimal shape design (control) problem we need to describe its several components, such as the differential equation satisfied by the controlled system, the performance criterion, conditions, etc. The conditions that we shall put on the functions and sets will serve two important purposes. First, they are the kind of reasonable conditions which are usually met when considering classical problems; second, they will allow the modification of these classical problems into other problems which appear to have some advantages.

Let r and u be two real-valued functions, and θ a real variable; then consider:

- (i) The closed interval $J = [0, 2\pi]$ in $\mathbb{R}$; its interior in the real line is $J^\circ = (0, 2\pi)$.
- $(ii) A = [0, 1]$, a bounded, closed set in $\mathbb{R}$. The trajectory of the controlled system is constrained to stay in this set for all $\theta \in J$. In other words: $0 \le r(\theta) \le 1, \forall \theta \in J$.
- (iii) $r = r_a$, an element of A which is to be the initial and final states of the trajectory of the controlled system.
- (iv) The bounded closed subset W of $\mathbb{R}$, the set in which the control function takes values.
- (v) Consider the following differential equation:

$$\dot{r}(\theta) = w(\theta) \equiv g(\theta, r, w), \; \theta \in J^\circ, \tag{2.3}$$

where the trajectory function $\theta \in J \longrightarrow r(\theta) \in A$ is absolutely continuous and the control function $\theta \in J \longrightarrow w(\theta) \in W$ is Lebesgue-measurable; this differential equation describes the controlled system.

- (vi) Let $\Omega = J \times A$ (the unit disk in polar coordinate), $\omega = J \times A \times W$, and $f_\circ : \Omega \longrightarrow \mathbb{R}$ and $h_\circ : \omega \longrightarrow \mathbb{R}$ be two integrable functions in appropriate spaces. These functions make the integrands in the performance criterion for the related optimal shape design problem.

- (vii) Let ∂C be a continuous simple closed curve and C be a (measurable) set which is bounded by ∂C in the polar plane. Here C and ∂C are the geometrical objects of the classical optimal shape design problem; a simple closed curve is a curve with the same initial and final points which does not cut itself. It means that the curve $r = f(\theta)$ defined on $[\theta_1, \theta_2]$ is called simple closed, whenever f satisfies in the following conditions:

$$(\theta_1, f(\theta_1)) = (\theta_2, f(\theta_2));$$
$$(\theta, f(\theta)) = (\theta', f(\theta')) \text{ when } \theta \neq \theta' \text{ and } \theta, \theta' \in (\theta_1, \theta_2] (\text{see [2]}).$$

In a classical optimal shape design problem, the optimization takes place on the set of all admissible (geometrical) elements which are related to the problem. In our case, these elements are defined as follows.

Definition 1 : *An admissible pair $(C, \partial C)$ is a pair consisting of measurable set $C \subset \Omega$ and its continuous simple closed boundary, ∂C (which are mentioned before), so if ∂C is defined by the trajectory function $r(\theta)$ then:*

- *$i)$ the differential equation (2.3) holds.*

- *$ii)$ The boundary condition $r(0) = r(2\pi) = r_a$ is satisfied.*

- *iii) The area of C is a given number.*

The set of all admissible pairs $(C, \partial C)$ is denoted by $\mathcal{F}$.

The appropriate optimal shape design problem related to the mentioned question, is as follows:

$$\begin{aligned} \textbf{Minimize}: \mathbf{I}(C, \partial C) &= \int_C f_\circ dA + \int_{\partial C} \frac{1}{\sqrt{r^2 + w^2}} h_\circ ds. \\ \textbf{Subject to}: \quad & (C, \partial C) \in \mathcal{F}; \\ & \text{the area of } C = \text{given}; \\ & r_a = \text{given}. \end{aligned} \qquad (2.4)$$

We know that in the polar coordinates when $r > 0$ and $0 \leq \theta \leq 2\pi$, the curve $r = f(\theta)$ is simple. Therefore with these constraints that have appeared in the determination of Ω and ω, searching for the mentioned closed and simple curve is completely possible. But in the $xy-$plane (orthognal coordinates), finding the similar necessary conditions, for a curve to be simple and closed, is much more difficult.

In general the set of all admissible pairs, $\mathcal{F}$, may be empty or may not contain the optimal pair (see [44] and [36]). Even if the set $\mathcal{F}$ is nonempty, and a minimizing pair for (2.4) does exist in $\mathcal{F}$, it may be difficult to characterize it; necessary conditions are not always helpful because the information that they give may be impossible to interpret. Also the optimal pair may be very difficult or impossible to estimate numerically; there are no comprehensive computational methods for this purpose. However using some effort could be better directed towards finding an alternative way, perhaps one using other spaces, other sets, different things and so on. We shall apply such an approach in the rest of this Chapter.

We shall effect the transformation of this classical optimal shape design into another, nonclassical, problem which appears to have better properties in some respects. Before this, in the next section, we shall analyse further this classical problem, to gain some idea on how to find the minimizer for the problem (2.4).

2.2.2 Conditions

To identify the optimal pair, it is necessary to point out some characteristics of the admissible pairs $(C, \partial C)$ in $\mathcal{F}$.

We first consider the boundary conditions; let B be an open ball in $\mathbb{R}^2$ containing $J \times A$, and denote by $C'(B)$ the space of real-valued continuously differentiable functions on B such that they and their first derivatives are bounded on B (this space is the same as that of all real-valued functions that are uniformly continuous on B together with their derivatives). Let $\phi \in C'(B)$ and define:

$$\phi^g(\theta, r, w) = \phi_r(\theta, r)w + \phi_\theta(\theta, r) \tag{2.5}$$

for all $(\theta, r, w) \in \omega$. The function ϕ^g is in the space $C(\omega)$ and we have:

$$\begin{aligned}(\int_{\partial C} \frac{1}{\sqrt{(r^2 + w^2)}} \phi^g(\theta, r, w) ds =) \int_J \phi^g(\theta, r(\theta), w(\theta)) d\theta \\ = \int_0^{2\pi} \{\phi_r(\theta, r(\theta))\, \dot{r}(\theta) + \phi_\theta(\theta, r(\theta))\}\, d\theta \\ = \int_0^{2\pi} \dot{\phi}(\theta, r(\theta)) d\theta = \delta_\phi \end{aligned} \tag{2.6}$$

for all $\phi \in C'(B)$ (regard that in polar coordinates $ds^2 = r^2 d\theta^2 + dr^2 = (r^2 + w^2) d\theta^2$).

We now consider a special case of (2.5). Let $\mathcal{D}(J^\circ)$ be the space of infinitely differentiable real-valued functions with compact support in J°. Define:

$$\psi^g(\theta, r, w) = r(\theta)\psi'(\theta) + \dot{w}(\theta)\psi(\theta) \tag{2.7}$$

for all $\psi \in \mathcal{D}(J^\circ)$. Now for an admissible pair $(\partial C, C)$ and $\psi \in \mathcal{D}(J^\circ)$ we have

$$\left(\int_{\partial C} \frac{1}{\sqrt{(r^2 + w^2)}} \psi^g(\theta, r, w) ds = \right) \int_J \psi^g(\theta, r(\theta), w(\theta)) d\theta =$$

$$r(\theta)\psi(\theta) \,|_0^{2\pi} + \int_0^{2\pi} [\dot{r}(\theta) - w(\theta)]\psi(\theta) d\theta = r(2\pi)\psi(2\pi) - r(0)\psi(0). \tag{2.8}$$

Since $(C, \partial C)$ is an admissible pair satisfies (2.3) on J°, and, since the function ψ has compact support in J°, so $supp(\psi) \subset J^\circ = (0, 2\pi)$; then 0 and 2π do not belong to $supp(\psi)$, and therefore

$$\psi(0) = \psi(2\pi) = 0.$$

Hence the right-hand side of (2.8) is zero. We note that the equality (2.8) also can be derived from (2.6) by choosing

$$\phi(\theta, r, w) = r(\theta)\psi(\theta)\,;\ (\theta, r, w) \in \omega. \tag{2.9}$$

It is important to single out this special case of (2.6); because later on, when we want to consider the approximation, we shall be forced to consider problems in which (2.6) is satisfied only for finite number of functions in $C'(B)$; it will be necessary then to include among these some functions of type (2.9). So we wish to make sure that we do not overlook these.

The same situation arises for another special choice of functions in $C'(B)$; put

$$\phi(\theta, r, w) = \Theta(\theta);\ (\theta, r, w) \in \omega, \tag{2.10}$$

that is, a function which depends on the variable θ only; then $\phi^g(\theta, r, w) = \dot{\Theta}(\theta)$, for all $(\theta, r, w) \in \omega$, also is a function of θ only. We are led thus to consider a subset of $C(\omega)$, to be denoted by $C_1(\omega)$, of the functions in this space which depend only on the variable θ; its elements will be denoted as a function of three variables, $(\theta, r, w) \longrightarrow f(\theta, r, w)$, even if their value does not change when r or u, or both change. The equation (2.6) with the choice (2.10) implies that

$$(\int_{\partial C} \frac{1}{\sqrt{(r^2 + w^2)}} f(\theta, r, w) ds =) \int_J f(\theta, r(\theta), w(\theta)) d\theta = a_f \, ; \; f \in C_1(\omega) \tag{2.11}$$

where a_f is the integral of $f(., r, u)$ over $[0, 2\pi]$, independent of r and u; we have put f for Θ in (2.10).

The set of equalities (2.6) and its special cases (2.8) and (2.11), are properties of the admissible pairs in the classical formulation of the optimal shape design problem. By suitably generalizing them we shall transform this into another nonclassical one which appears to have much better properties in some respects.

2.3 Metamorphosis

It appears that the situation mentioned in section (2.2) may become more favorable if the set $\mathcal{F}$ could somehow be made larger; if we could only enlarge this set. Of course, in a given classical problem, the set of admissible pairs is fixed. If we somehow add elements to it, we are changing the problem, and considering a new one, a different formulation nevertheless. This is precisely our intention; the basis of this *metamorphosis* is the fact that an admissible pair $(C, \partial C) \in \mathcal{F}$ can be considered as something else (like a pair of measures), that is, a transformation can be established between the admissible pairs and other mathematical objects. This transformation is an injection (one-to-one mapping), so the optimal pair and its image under the transformation can be identified. It is possible then to augment the set of all images of optimal pairs under this transformation.

Hence we will replace C and ∂C with the measures μ_c and ν_c respectively by the following proposition.

Proposition 1 : *Let $C, \partial C, \Omega$ and ω are defined as before, then there exist two unique positive Radon measures $\mu_c \in \mathcal{M}^+(\Omega)$ and $\nu_c \in \mathcal{M}^+(\omega)$ so that:*

$$\mu_c(g) = \int_C g dA, \ \forall g \in C(\Omega)$$

and

$$\nu_c(h) = \int_J h d\theta (= \int_{\partial C} \frac{1}{\sqrt{(r^2 + w^2)}} h ds), \ \forall h \in C(\omega).$$

Proof: $\mathbb{R}^2$, and therefore by the Heine-Borel theorem its closed subset Ω, are locally compact Hausdorff topological spaces. Let $g \in C(\Omega)$. Since g is continuous, by using again of the Heine-Borel theorem, it has the compact support. So $g \in C_c(\Omega)$ and consequently, $C(\Omega) \subset C_c(\Omega)$. Now for all $g \in C(\Omega)$ we define the functional Λ_C as below:

$$\Lambda_C \ : \ g \in C(\Omega) \longrightarrow \int_C g dA \in \mathbb{R}.$$

Λ_C is a linear and positive functional in $C(\Omega)$ because of:

- $\Lambda_C(a_1 g_1 + a_2 g_2) = \int_C (a_1 g_1 + a_2 g_2) dA = a_1 \int_C g_1 dA + a_2 \int_C g_2 dA$
  $= a_1 \Lambda_C(g_1) + a_2 \Lambda_C(g_2)$; for all $a_1, a_2 \in \mathbb{R}$ and $g_1, g_2 \in C(\Omega)$.
- If $g(\theta, r) \geq 0$, $\forall (\theta, r) \in \Omega$ then abviously $\Lambda_C(g) = \int_C g dA \geq 0$.

Hence, the conditions of the Reisz Representation Theorem in [55] are satisfied and thus there exists a unique positive Radon measure, say, $\mu_c \in \mathcal{M}^+(\Omega)$ so that:

$$\mu_c(g) = \Lambda_C(g) = \int_C g dA, \ \forall g \in C(\Omega).$$

Similarly, by defining:

$$\Lambda_{\partial C} \ : \ h \in C(\omega) \longrightarrow \int_J h d\theta (= \int_{\partial C} \frac{1}{\sqrt{(r^2 + w^2)}} h ds) \in \mathbb{R},$$

$\Lambda_{\partial C}$ is a positive linear functional on $C(\omega)$. Using again Riesz Representation Theorem, we obtain the unique positive Radon measure, say, $\nu_c \in \mathcal{M}^+(\Omega)$ such that:

$$\nu_c(h) = \Lambda_{\partial C}(h) = \int_J h d\theta (= \int_{\partial C} \frac{1}{\sqrt{(r^2 + w^2)}} h ds), \ \forall h \in C(\omega).$$

□

The above Proposition shows that each pair $(C, \partial C)$ in $\mathcal{F}$ can be regarded as a pair of measures (μ_c, ν_c) in the appropriate subset of $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$.The Uniqueness of μ_c and ν_c in the Proposition 1, state that there exist a transformation

$$(C, \partial C) \longrightarrow (\mu_c, \nu_c)$$

between those two sets. The following proposition shows that this transformation is an injection (a.e.).

Proposition 2 : *The transformation $(C, \partial C) \longrightarrow (\Lambda_C, \Lambda_C)$ of the admissible pair in $\mathcal{F}$ into the pair of linear mapping $(\Lambda_C, \Lambda_{\partial C})$, defined in the proof of Proposition1, is an injection.*

Proof: It must be shown that if $(C_1, \partial C_1)$ and $(C_2, \partial C_2)$ are not equal in $\mathcal{F}$, then we have $(\Lambda_{C_1}, \Lambda_{\partial C_1}) \neq (\Lambda_{C_2}, \Lambda_{\partial C_2})$. Let $(C_1, \partial C_1) \neq (C_2, \partial C_2)$ then because ∂C_1 and ∂C_2 are two simple closed curves which are the boundary of C_1 and C_2, we have $C_1 \neq C_2$ and $\partial C_1 \neq \partial C_2$. If we have $\Lambda_{C_1}(f) = \Lambda_{C_2}(f)$ for all $f \in C(\Omega)$, then

$$\int_{C_1} f dA = \int_{C_2} f dA, \forall f \in C(\Omega).$$

Therefore $\int_{C_1 - C2} f dA = 0$, for all continuous functions f in $C(\Omega)$. Hence $C_1 - C_2$ is an empty set (a.e.), or equally $C_1 = C_2$ (a.e.), which contradicts with $C_1 \neq C_2$. (Also as Rubio did in [50], let J_1 be the subinterval of J so that $r_1(\theta) \neq r_2(\theta)$ for all $\theta \in J_1$; then, one may make $F \in C(\omega)$ independent from w, equal zero on J_1, and such that it is positive on the appropriate portion of the graph $r_1(.)$ and zero on that of $r_2(.)$. Then the related linear functions are not equal.) □

Now each pair $(C, \partial C)$ can be identify with the pair of related linear functionals $(\Lambda_C, \Lambda_{\partial C})$. Some may think that we have not achieved something new, that we are simply writing some integrals in different way. But in reality, we have achieved something deep and useful, when identifying the optimal pairs with the positive Radon measures. Consider the equalities (2.6), (2.8) and (2.11), their left-hand sides are all integrals which are exactly the same type as that appearing in the definition of the positive linear functional $\Lambda_{\partial C}$ in Proposition 1. These equalities can then be written by using the definition of the related Radon measures ν_c as follow:

$$\nu_c(\phi^g) = \delta_\phi, \ \phi \in C'(B);$$

$$\nu_c(\psi^g) = 0, \ \psi \in \mathcal{D}(J^\circ);$$

$$\nu_c(f) = a_f, \ f \in C_1(\omega). \tag{2.12}$$

Besides, by applying the definition of Λ_C and its related Radon measure μ_c, the performance criterion in (2.3) can be written as

$$\mathrm{I}(C, \partial C) = \mu_c(f_\circ) + \nu_c(h_\circ).$$

The image of the set of all admissible pairs in $\mathcal{F}$ under the transformation $(C, \partial C) \to (\mu_c, \nu_c)$ (in the Proposition 2) is in the set of all those pairs of positive linear functionals $(\Lambda_C, \Lambda_{\partial C})$ on $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$, or equally those pairs of Radon measures (μ_c, ν_c) which satisfy the equalities (2.12). We shall now enlarge this image which, we repeat, can be identified with $\mathcal{F}$ itself (remember that the transformation $(C, \partial C) \longrightarrow (\mu_c, \nu_c)$ is injective); and define the new, nonclassical, problem. The classical problem can be rephrased as follows:

Among those pairs of positive Radon measures on $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ of the type (μ_c, ν_c), we seek one for which the number $\mu_c(f_\circ) + \nu_c(h_\circ)$ is minimum.

But in the new nonclassical problem simply do this:

We shall consider all pairs (μ, ν) of positive Radon measures in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ which satisfy (2.12), and seek to minimize the function $(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$ over this new, larger set of positive Radon measures. (we shall discuss later about the reasons for taking this approach.)

We should emphasize that what we are doing is to consider the problem as defined over all measures in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ which satisfy the conditions (2.12), as shown in (2.13). The measures satisfy (2.13) can be approximated (in weakly* sense) by actual pairs $(C, \partial C)$.

Thus by using these concepts we can put our nonclassical problem in its definitive form, which will be used in the rest of the Chapter. As a conclusion, the new nonclassical optimal shape design problem that we treat to find its minimizer, say $(\mu^*, \nu^*) \in \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$, is as follows:

$$\begin{aligned}
\text{Minimize}: \quad & \mathbf{I}(C, \partial C) = \mu(f_\circ) + \nu(h_\circ) \\
\text{subject to}: \quad & \nu(\phi^g) = \delta_\phi, \ \phi \in C'(B); \\
& \nu(\psi^g) = 0, \ \psi \in \mathcal{D}(J^\circ); \\
& \nu(f) = a_f, \ f \in C_1(\omega).
\end{aligned} \tag{2.13}$$

In the next, we shall examine the advantages of this new nonclassical problem with respect to the old one in section 2.2. Also we will indicate how the optimal pair of the measures can be used so that a reasonable modification of the original problem comes to be solved, and the optimal shape can be found.

2.4 The advantages of the new formulation

As mentioned before, in the classical form of the optimal shape design problem, generally the minimization of the performance criterion in (2.5) over the set $\mathcal{F}$ is not possible, the infimum may not attained at any admissible pair; it is not possible, then to write the necessary conditions for the problem. Conditions which guarantee the existence of a minimum take usually the form of some sort of convexity requirements on the sets or functions; these conditions may or may not be artificial when imposed on a particular system. Also if the minimizer pair exists, it may be difficult to characterize it. Moreover the minimizer pair may be very difficult or sometimes impossible to be estimated numerically; there are no comprehensive methods for this purpose.

But in the nonclassical optimal shape design problem, which has been formulated as a measure-theoretical problem in (2.13), there are some characteristics which make this new problem more effective. Let the subset of $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ defined by the equalities (2.12) be denoted by Q; then:

- i) The existence of an optimal pair of measures in the set Q minimizing $(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$ is guaranteed because of the automatical existence theorem. We shall examine the interesting relationships between a particular topology on the set Q and existence properties.

- ii) The function $(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$, as well as the functions appearing in the left-hand side of the equalities (2.12) - those that defined the set $Q \subset \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ - are linear in their arguments, measures ν and μ. This fact forms the basis of our approach; since the functions involved are linear even for those problems normally classed as nonlinear, the whole machinery properties of linear analysis can be used to attack the problem. So the computational methods for getting the solution are much easier.

- iii) Since the set $\mathcal{F}$ of admissible pairs can be considered, by means of the injection function $(C, \partial C) \longrightarrow (\Lambda_C, \Lambda_{\partial C})$, as a subset of Q, therefore

$$\inf_{\mathcal{F}} \mathrm{I}(C, \partial C) \geq \inf_{Q} \mathrm{I}(\mu, \nu).$$

  Thus, here, the minimization is *global*, that is, the global infimum of the problem can be approximated well. So in the nonclassical form, the global minimizer of the problem will be found, or rather, a reasonable approximation to it.
  As explained in [50], the infimum associated with the new formulation can be strictly less than the classical infimum.

In the next section we will explain that why the minimizer pair of measures (μ^*, ν^*) for the problem (2.13) exists.

2.5 Existence

It is the aim of this section to show that the linear function $(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$ always has at least one optimizer (an optimal pair of measures) in the set Q under the conditions on the functions and sets of the problem (2.13). This is based on simple compactness properties of the weak* topology. The following proposition which is proved in [50] Chapter 2, will be used to prove the way to reach to the existence of the optimal pair.

Proposition 3 : *If S is a compact subset of the Hausdorff space X and the function $y : S \longrightarrow \mathbb{R}$ is lower semi continuous (lsc) in S, then:*
(i) $\inf_s y(s) < -\infty$
(ii) There is an element $s_\circ \in S$ such that $y(s_\circ) \leq y(s)$, for all $s \in S$; that is, the infimum of y is attained on S.

We assume that Q is nonempty. Of course that the set Q may be nonempty while $\mathcal{F}$ is empty; one of the advantages of the nonclassical formulation. The space $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ of all pairs of Radon measures will take on the role of the space X in the above Proposition but no topology has been put on it yet. We try to find a Hausdorff topology on this space so that Q is compact and the function $(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$ is lower semicontinuous. Of course, if no optimal measure does exist under our hypothesis we will never find such topology. But, as we shall see below, a Hausdorff topology can be found in which the set Q is compact and the function $(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$ is not only lower semi continuous but actiually continuous.

There are several ways of characterizing the topology we have in mind, known as weak* topology, or vague topology on the space $\mathcal{M}(\Omega) \times \mathcal{M}(\omega)$. We note that this space is a linear space, which will become a locally convex topological vector space when given the weak* topology; this can be defined by the family of semi norms $(\mu, \nu) \longrightarrow$

$|\mu(F)| + |\nu(H)|$, $(F, H) \in C(\Omega) \times C(\omega)$; then, where gives rise to a basis of neighborhoods of zero of $\mathcal{M}(\Omega) \times \mathcal{M}(\omega)$ is:

$$U_\varepsilon = \{(\mu, \nu) \in \mathcal{M}(\Omega) \times \mathcal{M}(\omega) : |\mu(F_j)| + |\nu(H_j)| < \varepsilon; j = 1, 2, \ldots, r\},$$

for every $\varepsilon > 0$ and all finite subset $\{(F_j, H_j) \in C(\Omega) \times C(\omega); j = 1, 2, \ldots, r\}$. Hence $\mathcal{M}(\Omega) \times \mathcal{M}(\omega)$ is a Hausdorff topological vector space (see [58] Chapter 19 and [11] Chapter 3 and 4). The following Proposition shows that Q is dense in $\mathcal{M}(\Omega) \times \mathcal{M}(\omega)$.

Proposition 4 : *The set of measures $Q \subset \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$, is compact in the topology induced by the weak* topology on $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$.*

Proof: Denote $Q_\nu \subset \mathcal{M}^+(\omega)$ the set of those measures $\nu \in \mathcal{M}^+(\omega)$ which satisfies in the equalities (2.13). Then by Proposition$II.2$ in [50], Q_ν is dense in $\mathcal{M}^+(\omega)$ by the induced weak* topology on $\mathcal{M}^+(\omega)$. Moreover, $Q = \mathcal{M}^+(\Omega) \times Q_\nu$ (because no measures of $\mathcal{M}^+(\Omega)$ is involved in the conditions of (2.13)), and each subset of Q is the form of $\mathcal{M}^+(\Omega) \times Q'$ where $Q' \subset Q_\nu$. Thus, by regarding the definition of compactness, Q is a compact subset of $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ under the same topology. □

The proof of the following Proposition is much the same as that one in [50] Chapter 2, so it is omitted.

Proposition 5 : *The function $(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$, mapping Q into the real line, is continuous.*

The last two Propositions state that Q is compact and the function $(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$ is continuous, therefore it is lower semi continuous. Now by applying the Proposition 3 the following Theorem will be obtained.

Theorem 1 : *There exist an optimal pair of measures* (μ^*, ν^*) *in the set* $Q \subset \mathcal{M}^+(\Omega)\times \mathcal{M}^+(\omega)$ *of pair of measures that satisfy the equations (2.12), for which* $\mu^*(f_\circ)+\nu^*(h_\circ) \leq \mu(f_\circ) + \nu(h_\circ)$, *for all* $(\mu, \nu) \in Q$.

As a result of the Theorem 1, one can state that the problem (2.13) has an optimal solution in Q; but still it is difficult to obtain the exact solution, the underlying spaces are not a finite dimensional, the number of equations are not finite, etc. Hence we look for a suboptimal solution. In the next section, we will explain how the solution of (2.13) can be approximated by a solution of a finite linear programming problem.

2.6 Approximation

As noted before, the problem defined in (2.13) is a linear programming problem; all the functions are linear in the terms of the variables μ and ν; moreover these measures are required to be positive. But this linear programming problem is not finite-dimensional, because not only the underlying space, $\mathcal{M}(\Omega) \times \mathcal{M}(\omega)$, is infinite-dimensional but also the number of equalities in (2.13) is not finite. (This kind of problems is called *an infinite-dimensional linear programming problem*; there is a large and growing literature on such problems, for example see [14]). In our case it is possible to approximate the solution of this problem by the solution of a finite-dimensional linear one (which is much more common and easier to deal with) of sufficiently large dimensions. Besides, by increasing the dimensionality of the problem the accuracy of approximation can be increased. In this section we will first consider the minimization of $(\mu, \nu) \longrightarrow \mu(f_\circ)+\nu(h_\circ)$ over a subset of $\mathcal{M}^+(\Omega)\times\mathcal{M}^+(\omega)$ which contains those pairs of measures (μ, ν) satisfying only a finite number of constraints in (2.13). We remind the reader that a *total* set in an appropriate space is a set such that the linear combinations of its elements are uniformly dense - that is, dense in the topology of uniform convergence - in the related space.

For the first set of equations in (2.13), let the set $\{\phi_i \,;\, i = 1, 2, 3, \dots\}$ be a countable total set in $C'(B)$. These functions can be taken as monomials.
In $\mathcal{D}(J^\circ)$ consider the functions defined by:

$$\sin 2\pi j\theta \quad , \quad 1 - \cos 2\pi j\theta,\ j = 1, 2, 3, \dots, \tag{2.14}$$

and then we introduce the sequence of functions $\{\chi_h : h = 1, 2, 3, \dots\}$ as

$$\chi_h = r\psi'(\theta) + w\psi(\theta)$$

when the function ψ are the *sin* and *cos* functions in (2.14). The set of these functions is total in $\mathcal{D}(J^\circ)$.
Also let the set $\{f_s : s = 1, 2, \dots\}$ be total in $C_1(\omega)$; we will talk about these function later. Now we have the important following proposition which its proof is much like as the proof of Proposition $III.1$ in [50].

Proposition 6 : *For positive integer numbers M_1, M_2, M_2 consider the problem of minimizing*

$$(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$$

over the set $Q(M_1, M_2, M_3)$ of measures in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ satisfying

$$\begin{aligned} \nu(\phi_i^g) &= \delta_{\phi_i},\ i = 1, 2, \dots, M_1; \\ \nu(\chi_h) &= 0,\ h = 1, 2, \dots, M_2; \\ \nu(f_s) &= a_s,\ s = 1, 2, \dots, M_3. \end{aligned} \tag{2.15}$$

If M_1, M_2, M_3 tends to infinity, then

$$\inf_{Q(M_1,M_2,M_3)}[\mu(f_\circ)+\nu(h_\circ)] \longrightarrow \inf_{Q}[\mu(f_\circ)+\nu(h_\circ)].$$

Up to now, in the first stage of approximation, we have limited the number of constraints in the original linear program. But the underlying space is still infinite-dimensional. Next we are going to approximate this problem with the finite-dimensional one.

It will be assumed that the first function appearing in the first set of equations in (2.15) is $\phi_1^g(z) = 1$ for $z = (\theta, r, w) \in \omega$; then the first equality will be written as $\nu(1) = 2\pi$.

Here we remember the fact that a unitary atomic measure with support the singleton point set z, to be denoted by $\delta(z) \in \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$, is characterized by $\delta(z)(F) = F(z), F \in C(\omega), z \in \omega$. Then from the Proposition $III.2$ of [50] Chapter 3 (which has been taken from [48]), one can conclude that the measure ν^* in the set $Q(M_1, M_2, M_3)$ at which the function $(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$ attains its minimum has the form:

$$\nu^* = \sum_{j=1}^{m} \beta_j^* \delta(z_j^*),$$

where $m = M_1 + M_2 + M_3$; $z_j^* \in \omega$ and $\beta_j^* \geq 0$ for $j = 1, 2, \ldots, m$. Now for the other measure, let us consider a finite number of arbitrary continuous functions $F_1, F_2, \ldots F_n$ on Ω which is a Hausdorff topological space, so that $F_1(Z) = 1$ for all $Z \in \Omega$. Because μ is a positive Radon measure we suppose $\mu(F_1) = c_1 > 0$; in the other words, without loss of generality we assume μ is nonzero (which is what we are usually looking for).

Therefore, from the Theorem $A.5$ in [50] (Appendix), μ^* has the form:

$$\mu^* = \sum_{i=1}^{n} \alpha_i^* \delta(Z_i^*),$$

with $\alpha_i^* \geq 0, Z_i^* \in \Omega$ for $i = 1, 2, \ldots, n$. Here $\delta(Z)$ is a unitary atomic measure on Ω with support at Z. The following proposition has been proved.

Proposition 7 : *The optimal measure μ^* and ν^* for the function*

$$(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$$

with the constraints in (2.15) have the from

$$\mu^* = \sum_{i=1}^{n} \alpha_i^* \delta(Z_j^*)\,,\ \nu^* = \sum_{j=1}^{m} \beta_j^* \delta(z_j^*).$$

Thus, the measure-theoretical optimization problem is equivalent to a nonlinear optimization problem in which the unknowns are the coefficients α_i^*, β_j^* and supports $\{Z_i^*\}$, $\{z_j^*\}$ for $i = 1, 2, \ldots, n,\ j = 1, 2, \ldots, m$. It would be much more convenient if we could minimize the function $(\mu, \nu) \longrightarrow \mu(f_\circ) + \nu(h_\circ)$ only with respect to the coefficients α_i^* and β_j^*; which would be a linear programming problem.

The answer of that possibility, lies in the next stage of approximation, where we introduce dense sets in Ω and ω. Let D_Ω and D_ω be two countable dense subset of Ω and ω respectively; then as a result of Proposition $III.3$ in [50], measures $\mu_\circ \in \mathcal{M}^+(\Omega)$ and $\nu_\circ \in \mathcal{M}^+(\omega)$ of the form

$$\mu_\circ = \sum_{i=1}^{n} \alpha_i \delta(Z_i)\,,\ \nu_\circ = \sum_{j=1}^{m} \beta_j \delta(z_j) \tag{2.16}$$

exist such that $Z_i \in D_\Omega, z_j \in D_\omega$ and they can approximate μ^* and ν^* (respectively).

This result suggests that the problem (2.15) can be approximated by the following linear programming one which Z_i and z_j for $i = 1, 2, \ldots, N, j = 1, 2, \ldots, M$, belong to dense subsets of Ω and ω respectively.

$$\begin{array}{lll}
\textbf{Minimize:} & \sum_{i=1}^{N} \alpha_i f_o(Z_i) + \sum_{j=1}^{M} \beta_j h_o(z_j) & \\
\textbf{Subject to:} & \alpha_i \geq 0 & i = 1, 2, \ldots, N; \\
& \beta_j \geq 0, & j = 1, 2, \ldots, M; \\
& \sum_{j=1}^{M} \beta_j \phi_k(z_j) = \delta_{\phi_k}, & k = 1, 2, \ldots, M_1; \\
& \sum_{j=1}^{M} \beta_j \chi_h(z_j) = 0, & h = 1, 2, \ldots, M_2; \\
& \sum_{j=1}^{M} \beta_j f_s(z_j) = a_s, & s = 1, 2, \ldots, M_3.
\end{array} \tag{2.17}$$

For the last set of equations in (2.17) we define:

$$f_s(\theta) = \begin{cases} 1 & \text{if } \theta \in J_s \\ 0 & \text{otherwise,} \end{cases}$$

where $J_s = [\frac{2\pi(s-1)}{M_3}, \frac{2\pi s}{M_3}]$ and a_s in (2.17) is written for the integral of f_s over J. Since these functions are not continuous, two remarks need to be made concerning their suitability:

- (i) Each of the functions $f_s, s = 1, 2, \ldots, M_3$, is the limit of an increasing sequence of positive continuous functions, f_{s_k}; then if ν is any positive Radon measure in $\mathcal{M}^+(\omega)$, $\nu(f_s) = \lim_{k\to\infty} \nu(f_{s_k})$.
- (ii) Consider now the set of all such functions, for all positive integers M_3; the linear combinations of these functions can approximate a function in $C_1(\omega)$ arbitrarily well (see [3] Theorem 24.4), in the sense that the essential supremum

(see [18]) of the error function can be made to tend to zero by choosing in an appropriate manner, a sufficient number of terms in the corresponding expansion.

Now, by using the solution $\{\alpha_1^*, \alpha_2^*, \ldots, \alpha_N^*, \beta_1^*, \beta_2^*, \ldots, \beta_M^*\}$ of the problem (2.17), one is able to construct the pair of suboptimal trajectory and control functions. Of course, we only need to construct the control function, $w(\cdot)$, since the trajectory function, $r(\cdot)$, then is the corresponding solution of the differential equation (2.2), with the initial values $r(0) = r_a, r(2\pi) = r_a$. The construction of the control function is based on the methods introduced in [50] ChapterIV. This pair of trajectory (shape) and control functions, turns out to be the solution to the modified shape design problem; we note that the functions $f_\circ, h_\circ$ and $\frac{dr}{d\theta}$ will be required to be Lipschitz rather than merely continuous for these properties to hold (see [50] Chapter IV).

2.7 Relationships between the measures μ and ν

As mentioned before, the measure μ is not involved in the constraints of (2.13) and it appears only in the performance criteria of the optimal shape design problem. We would like to express this measure μ in terms of the boundary measure ν. If this were be possible, the β_j's would be the only unknowns in (2.17). To confirm this possibility, first we prove the following proposition.

Proposition 8 : *Suppose $\mu = \sum_{i=1}^{N} \alpha_i \delta(Z_j)$, then there exists a ν-measurable function ξ so that*

$$\nu(\xi_i) = \begin{cases} \alpha_i & Z_i \in C \\ 0 & \text{otherwise.} \end{cases}$$

(Here the dependence of the function ξ on the point Z_i is shown as $\xi_i \equiv \xi_{Z_i}$).

Proof: To define the function ξ we use the idea of generating an electromagnetic field by an infinite wire. An infinite wire that carries a fixed current, which is perpendicular to the polar plane, at an arbitrary point Z_i, produces an electromagnetic field B at distance ρ from Z_i. This field has two components B_ρ and B_θ (in plane) in the direction of ρ and perpendicular to it (see figure 2.1, that ρ is the line segment between the points Z_i and z). The components B_ρ and B_z are zero, so $B = B_\theta = \frac{k}{2\pi\rho}$ where k is a constant (see [45]). We wish to have a circulation equal to 1 at $z = (r, \theta)$,

$$\int_{\partial C} B.dl = \int_0^{2\pi} \frac{k}{2\pi\rho} \rho d\theta = 1.$$

Hence we should choose $k = 1$; so in our case $B = B_\theta = \frac{1}{2\pi\rho}$. Moreover, in cylindrical

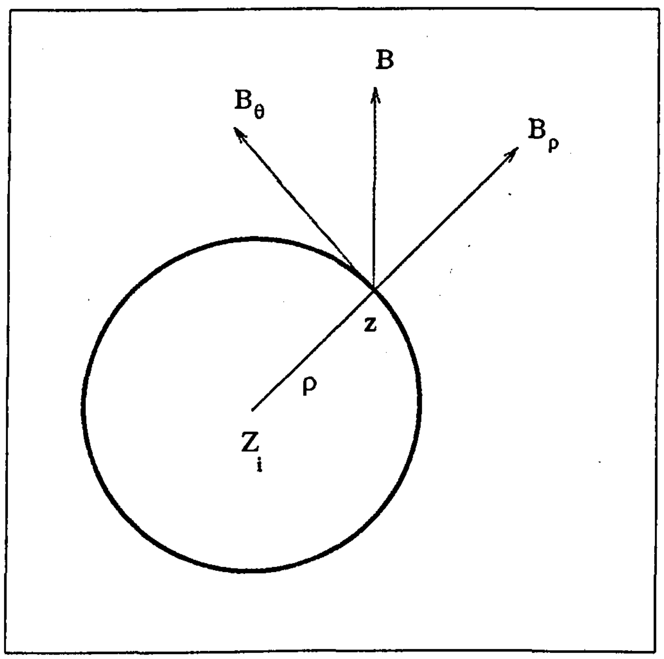

Figure 2.1: Electromagnetic field produced by an infinite wire

coordinates, with the unit vectors u_ρ, u_θ and u_z, we have:

$$\begin{aligned} CurlB &= \frac{1}{\rho}[\frac{\partial B_z}{\partial \theta} - \frac{\partial}{\partial z}(\rho B_\theta)]u_\rho - [\frac{\partial B_z}{\partial \rho} - \frac{\partial B_\rho}{\partial z}]u_\theta + \frac{1}{\rho}[\frac{\partial}{\partial \rho}(\rho B_\theta) - \frac{\partial B_\rho}{\partial \theta}]u_z \\ &= -\frac{1}{\rho}[\frac{\partial}{\partial z}(\frac{1}{2\pi})]u_\rho + \frac{1}{\rho}[\frac{\partial}{\partial \rho}(\frac{1}{2\pi})]u_z. \quad (2.18) \end{aligned}$$

Hence, if $\rho \neq 0$ then (2.18) shows that Curl$B = 0$ therefore by Stokes's Theorem the circulation is zero unless $\rho = 0$. In other words the circulation at $Z_i \in C$ is nonzero. So we define the function $\xi_i^\dagger$ as the integrand function for calculating the circulation at any point $z \in \omega$. So we have

$$\nu(\xi_i^\dagger) = \int_{\partial C} B.dl.$$

Thus $\nu(\xi_i^\dagger)$ is the circulation of B at Z_i, and so

$$\nu(\xi_i^\dagger) = \begin{cases} 1 & Z_i \in C \\ 0 & \text{otherwise.} \end{cases}$$

To complete the proof, it is enough to define:

$$\xi_i(z) = \alpha_i \xi_i^\dagger(z), \ \forall z \in \omega.$$

□

Now by applying Proposition 8, in the related equation (2.16) one could have $\mu = \sum_{i=1}^N \nu(\xi_i)\delta(Z_i)$ and hence

$$\mu(f_\circ) = \sum_{j=1}^M \beta_j \gamma_\circ,$$

where $\gamma_\circ = \sum_{i=1}^N f_\circ \xi_i$ is a function defined in Ω. So the performance criteria in (2.17) changes into

$$\sum_{j=1}^M \beta_j (h_\circ + \gamma_\circ).$$

The main point here is that α_i's, in the definition of ξ, are still unknown and our effort is to find them. For a given partition $\{\Omega_i\}_{i=1}^N$ on Ω, we call each Ω_i a quasi-rectangular subset of Ω; consider an extra condition on μ that for all $(\mu, \nu) \in Q(M_1, M_2, M_3)$ we have:

$$\mu(\Omega_i) = area\ of\ \Omega_i,\ \forall i = 1, 2, \ldots, N.$$

Let select a set of points $\{Z_1, Z_2, \ldots, Z_N\}$ as a subset of a dense set in Ω where $Z_i \in \Omega_i$ for all $i = 1, 2, \ldots, N$. Then from (2.16) we have:

$$\mu(\Omega_i) = \sum_{i=1}^{N} \alpha_i 1_{\Omega_i}(Z_i) = \alpha_i.$$

Thus by regarding the above discretization on Ω, one can consider α_i in the definition of ξ as the area of the quasi-rectangular Ω_i, for each $i = 1, 2, \ldots, N$. (In Appendix A, we have shown how one can compute ξ by putting a discretization on Ω.)

Finally, if we chose the L number nodes in a dense subset of ω by discretization, the optimal shape design problem in (2.3) can be approximated with the result of the finite linear programming problem below in which the unknowns are the β_n's. We put $z_n = (\theta_n, r_n, u_n)$, which is a node in the discretizations.

$$\begin{aligned}
\text{Minimize:} \quad & \sum_{n=1}^{L} \beta_n (h_\circ + \gamma_\circ)(z_n) & \\
\text{subject to:} \quad & \beta_n \geq 0, & n = 1, 2, \ldots, L; \\
& \sum_{n=1}^{L} \beta_n \phi_i(z_n) = \delta_{\phi_i}, & i = 1, 2, \ldots, M_1; \\
& \sum_{n=1}^{L} \beta_n \psi_j(z_n) = 0, & j = 1, 2, \ldots, M_2; \\
& \sum_{n=1}^{L} \beta_n f_s(z_n) = a_s, & s = 1, 2, \ldots, M_3; \\
& \sum_{n=1}^{L} \beta_n (\frac{1}{2} r_n^2) = \text{given area}. &
\end{aligned} \tag{2.19}$$

We note that the last equation in (2.19) is separated from the second set of the equations to emphasize the area of C; the equation shows the area of C because of

$$\int_0^{2\pi} (\frac{1}{2}r^2)d\theta = \text{area of C}.$$

2.8 Limitations

As Rubio in [50] has proved, the resulted trajectory from the solution of (2.19) is close to the real one when the functions in the performance criteria are the Lipschitz functions. We remind the reader that a function f is said to satisfy a Lipschitz condition with a Lipschitz constant k on D, if there is a constant k such that for all $x, y \in D$ we have

$$\mid f(x) - f(y) \mid \leq k \mid x - y \mid .$$

But the function ξ, and therefore $\gamma_\circ$, which appears in the performance criterion, is not Lipschitz. So the optimal solution of (2.19) may not be an accurate approximation for (2.3). However, if the function ξ is considered in the context of our discretization scheme, one can see from Figure 2.2 below that one can replace ξ by a function with Lipschitz constant $\frac{1}{h^2}$.

Thus, since a large Lipschitz constant needs a large number of equalities in (2.19), to achieve a given accuracy, finer discretization may need a large number of such equalities.

If the function $f_\circ$ is not a continuous, as in *Example* 2 below, we can not expect good approximation; however, we have found that the resulting shape provides a good

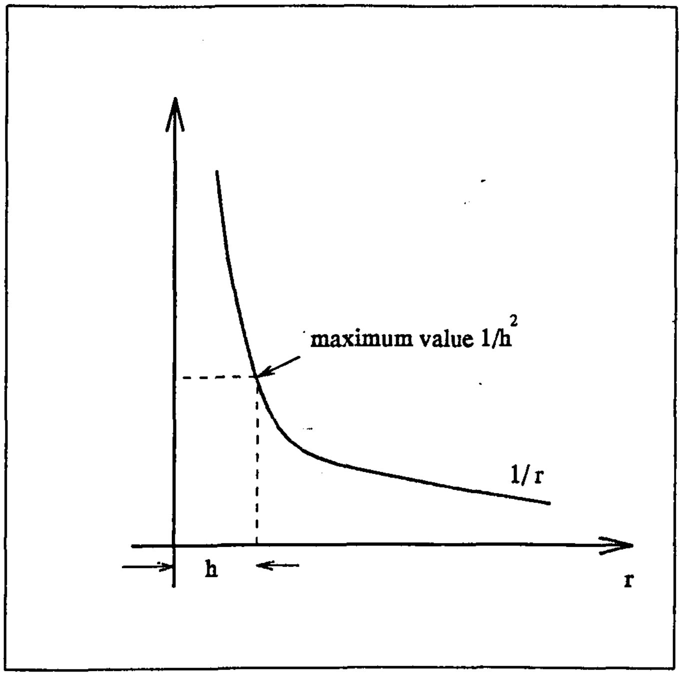

Figure 2.2: A function with Lipschitz constant $\frac{1}{h^2}$

starting point for a standard minimization algorithm, like $AMOEBA$ in [47]. In our case, the result of (2.19) is a very good initial solution, because it is satisfied in all the necessary conditions of the problem and in some sense makes the performance criterion minimum. By using a standard algorithm the initial solution, initial shape, can be improved into another one that is the nearest one to the minimizer of the problem (2.4). Plainly the necessary conditions (like area condition) may be applied by using the penalty method with the minimization algorithm (see [60]).

2.9 Numerical examples

2.9.1 Example 1

In this example, we looked for the optimal shape with the area of 0.6 which is located inside the Lemniscate $r = \sin 2\theta, 0 \leq \theta \leq 2\pi$ as much as possible; this shape is supposed to contain the fixed points $(0, 0.5)$ and $(2\pi, 0.5)$ as its initial and final points. Thus we chose $h_\circ = 0$ and

$$f_\circ(\theta, r) = \begin{cases} 0 & \text{r} \geq \sin 2\theta \\ r - \sin 2\theta & \text{otherwise.} \end{cases}$$

$f_\circ$is a continuous function. We wish to minimize the integral of $f_\circ$ (which is negative inside the Lemniscate) on C; that is, C will be as such is allowed by the constraints, inside the Lemniscate. (Note that the fixed point $(0, 0.5)$ is outside of the Lemniscate; so it must cause that a part of the optimal shape to be located outside of Lemniscate). By trial and error we chose $W = [-0.3\ ,\ 0.3]$; then

$$\omega = [0\ ,\ 2\pi] \times [0\ ,\ 1] \times W, \text{ and } \Omega = [0\ ,\ 2\pi] \times [0\ ,\ 1].$$

Then by selecting the following 10 points in W:

$$w\ :\ -0.3, -\frac{21}{90}, -\frac{15}{90}, -\frac{9}{90}, -\frac{3}{90}, \frac{3}{90}, \frac{9}{90}, \frac{15}{90}, \frac{21}{90}, 0.3,$$

and choosing 10 angles in $[0\ ,\ 2\pi]$:

$$\theta\ :\ \frac{\pi}{10}, \frac{3\pi}{10}, \ldots, \frac{19\pi}{10},$$

and also by 10 values in $A = [0\ ,\ 1]$ as:

$$r\ :\ 0, \frac{1}{9}, \frac{2}{9}, \ldots 1,$$

a discretization on ω was made with $M = 10 \times 10 \times 10 = 1000$ nodes $z = (r, \theta, w)$; each component of nodes is a rational number (we supposed $\pi = 3.141592654$) and hence all nodes belong to the dense subset of ω.

With respect to Ω, we divided it by 10 circles with radius

$$r\ :\ 0.1, 0.2, \ldots, 1,$$

and the following 10 lines

$$\theta\ :\ \frac{2\pi}{10}, \frac{4\pi}{10}, \ldots, 2\pi,$$

into $N = 10 \times 10 = 100$ subdivisions; and the node Z_i in subdivision i ($i = 1, 2, \ldots, N$) was selected as the top left corner of each subdivision. We emphasize that each 10 subdivision of Ω which has the same radius (i.e. same r_i), has the equal area. Hence:
For a fixed k, the area of the inside part of the circle with radius r_k which is located outside the circle with radius r_{k-1} is equal to:

$$\pi(r_k^2 - r_{k-1}^2) = \pi(r_k + r_{k-1})(r_k - r_{k-1}) = \frac{\pi(2k-1)}{100}.$$

Now each ten α_i's corresponded is one-tenth of this area; it means, $\alpha_i = \frac{\pi(2k-1)}{1000}$.
Finally when k takes value from 1 to 10, 100 values for α_i's will be determined. According to the discretization, the calculation of the function ξ is explained in Appendix A.
The results of the appropriate finite linear program (2.19) is presented in the following table.

LNCON	STATE	VALUE	LOWER BOUND	UPPER BOUND	LAGR MULT	RESIDUAL
L 1	EQ	0.1554312E-14	0.	0.	-0.6396E-02	0.1554E-14
L 2	EQ	3.141593	3.141593	3.141593	0.2308E-02	0.1066E-13
L 3	EQ	-0.1062483E-12	0.	0.	-0.4027E-03	-0.1062E-12
L 4	EQ	-0.3106404E-12	0.	0.	0.7529E-03	-0.3106E-12
L 5	EQ	-0.1731948E-12	0.	0.	-0.1027E-03	-0.1732E-12
L 6	EQ	0.9925394E-13	0.	0.	-0.1323E-02	0.9925E-13
L 7	EQ	-0.8298917E-14	0.	0.	-0.1390E-02	-0.8299E-14
L 8	EQ	0.1403322E-12	0.	0.	0.4485E-03	0.1403E-12
L 9	EQ	0.3574918E-13	0.	0.	0.3102E-03	0.3575E-13
L 10	EQ	0.1654232E-12	0.	0.	-0.6917E-03	0.1654E-12
L 11	EQ	0.6283185	0.6283185	0.6283185	-0.2924E-02	-0.2220E-15
L 12	EQ	0.6283185	0.6283185	0.6283185	-0.1207E-01	0.4441E-15
L 13	EQ	0.6283185	0.6283185	0.6283185	-0.9974E-02	0.4441E-15
L 14	EQ	0.6283185	0.6283185	0.6283185	-0.1109E-17	0.
L 15	EQ	0.6283185	0.6283185	0.6283185	-0.2596E-02	-0.2220E-15
L 16	EQ	0.6283185	0.6283185	0.6283185	-0.1633E-01	-0.3331E-15
L 17	EQ	0.6283185	0.6283185	0.6283185	-0.1053E-01	-0.4441E-15
L 18	EQ	0.6283185	0.6283185	0.6283185	-0.6583E-02	-0.2220E-15
L 19	EQ	0.6283185	0.6283185	0.6283185	-0.3740E-03	0.
L 20	EQ	0.6283185	0.6283185	0.6283185	-0.4899E-02	0.5662E-14
L 21	EQ	0.6000000	0.6000000	0.6000000	0.5993E-01	-0.6661E-15

EXIT E04MBF - OPTIMAL LP SOLUTION FOUND.

LP OBJECTIVE FUNCTION = 1.568288D-03

NO. OF ITERATIONS = 65

We remind the reader that the subroutine $EO4MAF$ from NAG-library Routine have been applied for solving the related finite linear program.
From these results, on the base of (2.3), we obtained the suboptimal control (which is plotted in Figure 2.3) and the following points which are located on the boundary of the suboptimal shape:

$$(0.252828138409, 0.4241515554631)\ ,\ (0.628318530718, 0.5367986888222),$$

$$(1.192201841392, 0.3676336888977)\ ,\ (1.256637061435, 0.3483031221167),$$

$$(1.680677884428, 0.2210908701640)\ ,\ (1.884955592153, 0.1598075554112),$$

$$(2.246363104187, 0.0513852974929)\ ,\ (2.320256157762, 0.07355321664839),$$

$$(2.513274122871, 0.0156478248144)\ ,\ (3.012859572856, 0.1655234806540),$$

$$(3.141592653589, 0.2041434102453)\ ,\ (3.275060367330, 0.2441837299362),$$

$$(3.769911184307, 0.3926389956762)\ ,\ (4.300393415010, 0.2334943201414),$$

$$(4.398229715025, 0.2041434289707)\ ,\ (4.846031655481, 0.0698028414956),$$

$$(4.847613316953, 0.0693283430353)\ ,\ (5.026548245743, 0.1230088291382),$$

$$(5.468624725335, 0.2556317914605)\ ,\ (5.654866776461, 0.3115044145691),$$

$$(6.283185307179, 0.4999999999999).$$

Linking these points, creates the optimal shape which is plotted with the Lemniscate in Figure 2.4. Note that the point $(0, 0.5)$ is outside of Lemniscate and that this curve is not simple; in spite of all these, most of the shape is inside it. So we did not apply any standard minimization Algorithm. However one may use it to get a better result.

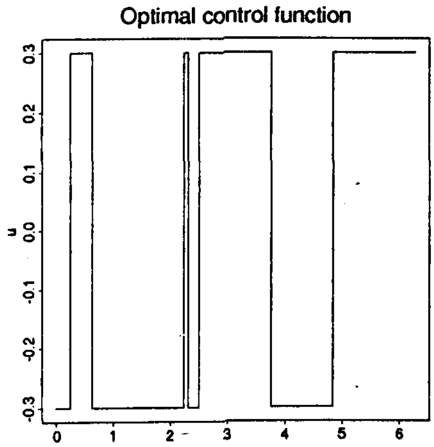

Figure 2.3: The optimal control function

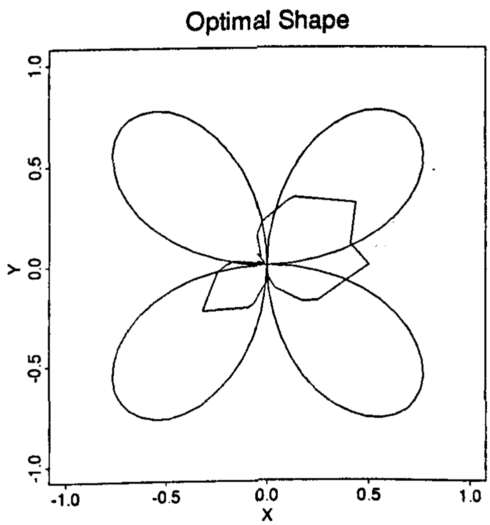

Figure 2.4: The optimal shape and the Lemniscate

2.9.2 Example 2

For the second example, chose one $h_\circ = 0$ and

$$f_\circ(\theta, r) = \begin{cases} 1 & r\sin(\theta) \leq 0.25 \\ 0 & \text{otherwise.} \end{cases}$$

In the other words, the above function $f_\circ$ states that we are looking for a closed shape C satisfying in the conditions of (2.4) and located under the line $r\sin(\theta) = 0.25$. We remind the reader that the function $f_\circ$ is not a continuous function and therefore it is not a Lipschitz function. For this reason (as shown in section 2.8), we will not anticipate that the resulting shape from (2.19) does take place completely under the line $r\sin\theta = 0.25$ and hence we will use a standard minimization algorithm (for example $AMOEBA$ in [47]) as explained.

Finding Optimal Control and Trajectory:

To find the optimal control and shape, first we discretize on Ω and ω. We selected everything the same as in Example 1, except $W = [-0.1, 0.1]$; then we setup the appropriate finite linear program in (2.19) by using 21 equations ($M_1 = 2, M_2 = 8, M_3 = 10$). To solve the problem, we applied the $EO4MAF$ from NAG-library Routine; the summary of the result is shown in the following table.

LNCON	STATE	VALUE	LOWER BOUND	UPPER BOUND	LAGR MULT	RESIDUAL
L 1	EQ	-0.7766357E-14	0.	0.	-0.4164	-0.7766E-14
L 2	EQ	3.141593	3.141593	3.141593	0.1731	-0.1008E-12
L 3	EQ	-0.3487488E-13	0.	0.	-0.1088E-01	-0.3487E-13
L 4	EQ	-0.1365852E-12	0.	0.	0.4233E-02	-0.1366E-12
L 5	EQ	0.9259538E-12	0.	0.	0.1293E-02	0.9260E-12
L 6	EQ	-0.1600386E-12	0.	0.	-0.9263E-02	-0.1600E-12
L 7	EQ	-0.5015779E-13	0.	0.	-0.1069E-01	-0.5016E-13
L 8	EQ	-0.2739475E-13	0.	0.	0.4933E-02	-0.2739E-13
L 9	EQ	-0.4025669E-12	0.	0.	-0.2377E-02	-0.4026E-12
L 10	EQ	0.2997602E-12	0.	0.	-0.3872E-02	0.2998E-12
L 11	EQ	0.6283185	0.6283185	0.6283185	-0.2469E-01	0.6661E-15
L 12	EQ	0.6283185	0.6283185	0.6283185	-0.1521	0.
L 13	EQ	0.6283185	0.6283185	0.6283185	-0.2014	-0.3331E-15
L 14	EQ	0.6283185	0.6283185	0.6283185	-0.1005	-0.2220E-15
L 15	EQ	0.6283185	0.6283185	0.6283185	-0.4468E-01	0.6661E-15
L 16	EQ	0.6283185	0.6283185	0.6283185	-0.1662E-01	-0.4441E-15
L 17	EQ	0.6283185	0.6283185	0.6283185	-0.1556E-01	-0.5773E-14
L 18	EQ	0.6283185	0.6283185	0.6283185	-0.4140E-01	-0.3331E-15
L 19	EQ	0.6283185	0.6283185	0.6283185	-0.5923E-01	-0.3331E-15
L 20	EQ	0.6283185	0.6283185	0.6283185	-0.8159E-01	-0.6606E-13
L 21	EQ	0.6000000	0.6000000	0.6000000	-0.1096	-0.2520E-13

EXIT E04MBF - OPTIMAL LP SOLUTION FOUND.

LP OBJECTIVE FUNCTION = 1.448595D-02

NO. OF ITERATIONS = 78

The resulting optimal control function, given by the solution of the reminded linear program, was modified in the Figure 2.5 by using Rubio's method in [50] Chapter 5.

In the case of equation $w(\theta) = \frac{dr}{d\theta}$, the trajectory function $r(\theta)$ (shape) can be taken by integrating from the above optimal control function over the interval $[0\ ,\ 2\pi]$. Because the control is a piecewise-constant function, the mentioned integration gives us the following 21 points which are located on the boundary of the suboptimal shape:

$$(0.000595979095, 0.4999404020895)\ ,\ (0.628318530717, 0.437168145991),$$

$$(1.256637061435, 0.3743362919838)\ ,\ (1.831261918818, 0.3168738053893),$$

$$(1.884955592153, 0.3115044379758)\ ,\ (2.206005111473, 0.2793994855654),$$

$$(2.513274122871, 0.2486725839677)\ ,\ (3.101783117978, 0.1898216835801),$$

$$(3.141592653589, 0.1858407299596)\ ,\ (3.141592887656, 0.1858407065529),$$

$$(3.509669220769, 0.2226483458974)\ ,\ (3.769911184307, 0.2486725465170),$$

$$(4.074511522101, 0.2791325852891)\ ,\ (4.119983818585, 0.2836798156829),$$

$$(4.398229715025, 0.3115044098877)\ ,\ (4.727550645424, 0.3444365083256),$$

$$(5.026548245743, 0.3743362732585)\ ,\ (5.494982199890, 0.4211796763514),$$

$$(5.654866776461, 0.4371681366292)\ ,\ (6.145061554817, 0.4861876224997),$$

$$(6.283185307179, 0.4999999999999).$$

Linking these points creates the shape which is plotted in the Figure 2.6, with the line $r \sin\theta = 0.25$. (Note that here X and Y in the Figure are the Cartesian axes.)

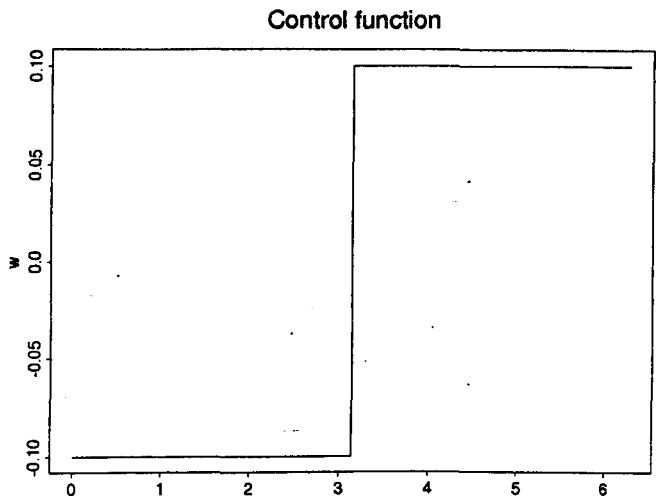

Figure 2.5: The optimal control function

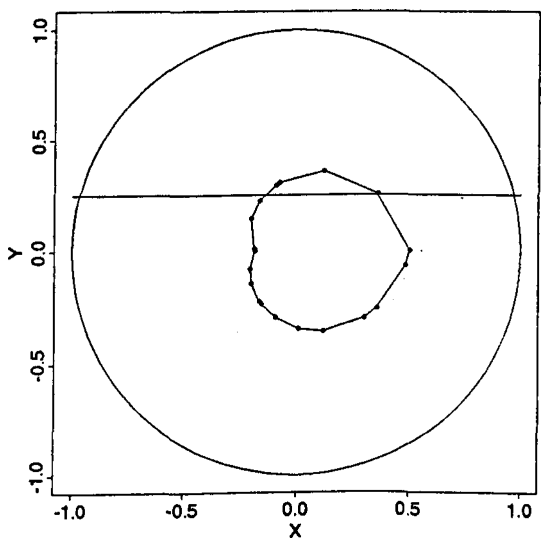

Figure 2.6: The suboptimal trajectory function

The optimal shape

Figure 2.6 shows that a part of the shape C is located over the line $r\sin\theta = 0.25$, which was predicted before in section 2.8, because of the limitations; for this reason, a standard minimization algorithm and the penalty method have been used. As we knew, the optimal shape must go through the initial point $(0, 0.5)$ and the same final point $(2\pi, 0.5)$ when its area is 0.6. Hence we need a constraint to present the area when each point (r, θ) on the boundary of C, is satisfied at the conditions $0 \leq r \leq 1$ and $0 \leq \theta \leq 2\pi$. Also, it is necessary to have another condition on the shape to be located under the line. Precisely it can be done by selecting a suitable performance criterion like the previous one in the finite linear programming case.

Let θ_i, $i = 1, 2, \ldots, 19$, be the fixed resulted angles from (2.19) without the initial and the final value; and suppose $Z_i = (\theta_i, r_i)$, $i = 1, 2, \ldots, 19$, are nineteen points in Ω. Assume $r_i : i = 1, 2, \ldots, 19$, are variables and

$$f_i(Z_j) = \begin{cases} 0 & r_j \sin(\theta_i) \leq 0.25 \\ 1 & \text{otherwise.} \end{cases}$$

We define:

$$F_o(Z_i) = \sum_{i=1}^{19} f_i(Z_i),$$

which is obviously a function of $(r_1, r_2, \ldots, r_{19})$.

Let $\theta_0 = 0$, $r_0 = 0.5$, $\theta_{20} = 2\pi$, $r_{20} = 0.5$, and the area of that part of the shape which is located between the lines $\theta = \theta_{i-1}$ and $\theta = \theta_i$, be estimated by the same part of the circle with radius $\frac{r_i + r_{i-1}}{2}$. Thus the area condition for the optimal shape C is introduced by:

$$P(r_1, r_2, \ldots, r_{19}) \equiv \sum_{i=1}^{20} \frac{(r_i + r_{i-1})^2(\theta_i - \theta_{i-1})}{8} = 0.6,$$

which is also a function of the $(r_1, r_2, \ldots, r_{19})$ as well. Hence for the resulted fixed values of θ_i's, we can look for the answer of the following constrained minimization problem over all $(r_1, r_2, \ldots, r_{19}) \in [0\,,\,1]^{19} \subset \mathbb{R}^{19}$.

$$\begin{array}{ll} \text{Minimize:} & F_o(r_1, r_2, \ldots, r_{19}) \\ \text{subject to:} & P(r_1, r_2, \ldots, r_{19}) = 0.6. \end{array} \tag{2.20}$$

To get the solution of (2.20), it may possible to use one of the related constrained minimization programs; but for the standard algorithm (like $AMOEBA$) it is better to replace the above constraint problem with an unconstrained one. There are several ways to do this (see for example [60]). We chose the penalty method and applied the function $c[P(r_1, r_2, \ldots, r_{19}) - 0.6]$ as a penalty function for a real positive constant c (for more details see [60]). Hence the appropriate unconstrained optimization problem at this stage is as follows:

$$\text{Minimize:} \qquad F_o(r_1, r_2, \ldots, r_{19}) + c[P(r_1, r_2, \ldots, r_{19}) - 0.6]. \tag{2.21}$$

In spite of the fact that F_o is not a continuous function, some standard minimization algorithms for continuous performance criteria like, $E04JAF$ NAG-library Routine, are applicable. These algorithms can be run by a little change in the performance criteria to make it continuous without any changes in the value. One of the advantages of applying this type of algorithm is that they (usually) give the global minimizer for the given function. For this purpose let us to define:

$$f_i'(Z_j) = \begin{cases} 0 & r_j \sin(\theta_i) \leq 0.25 \\ (r_j - a_i)(\frac{\sin\theta_i}{\epsilon_i}) & 0.25 < r_j \sin(\theta_i) < 0.25 + \epsilon_i \\ 1 & 0.25 + \epsilon_i \leq r_j \sin(\theta_i) \end{cases}$$

where $a_i = \frac{0.25}{\sin(\theta_i)}$ and $0 < \epsilon_i < 1$, for $i = 1, 2, \ldots, 19$. Now for each i, the function f_i' is a continuous function and therefore the function

$$F_o'(Z_i) = \sum_{i=1}^{19} f_i'(Z_i),$$

is also continuous. Moreover, because for a sufficiently small value of ϵ_i the value of $(r_j - a_i)(\frac{\sin\theta_i}{\epsilon_i})$ is a large enough positive number then in the minimization algorithm, this value would be disregarded automatically (note that for $0.25 < r_j \sin(\theta_i) < 0.25 + \epsilon_i$ the value of the $(r_j - a_i)$ is positive). As a result, the problem (2.21) can be replaced by the following one for the mentioned algorithms,

$$\text{Minimize}: \qquad F_o'(r_1, r_2, \ldots, r_{19}) + c[P(r_1, r_2, \ldots, r_{19}) - 0.6]. \qquad (2.22)$$

We applied the $E04JAF$ minimization algorithm from NAG Routine library to find the solution of the (2.22) with $\epsilon_i = 0.11, \forall i = 1, 2, \ldots, 19$ and $c = 20$. Also the previous result from the linear programming problem was used as an initial solution which was necessary for the Routine. The optimal value of the performance criteria (2.22) was *zero* which means that all of the points has been located below the line. The resulted r_i's from $E04JAF$ were:

0.35523058385207 , 0.14012364797159 , 0. , 0. , 0.19260268747351

0.38302730144207 , 0.31290102243751 , 0.29205929667316 , 0.30138263551773

0.34559652292155 , 0.39060671726193 , 0.42975811313137 , 0.45079795177859

0.48921228505258 , 0.53850854481107 , 0.59098608536728 , 0.65079219282540

0.69405223541451 , 0.75117213275203.

The optimal shape (∂C) with the line $r\sin(\theta) = 0.25$ are plotted in the Figure 2.7. Note that X and Y in the Figure, are the Cartesian axes.

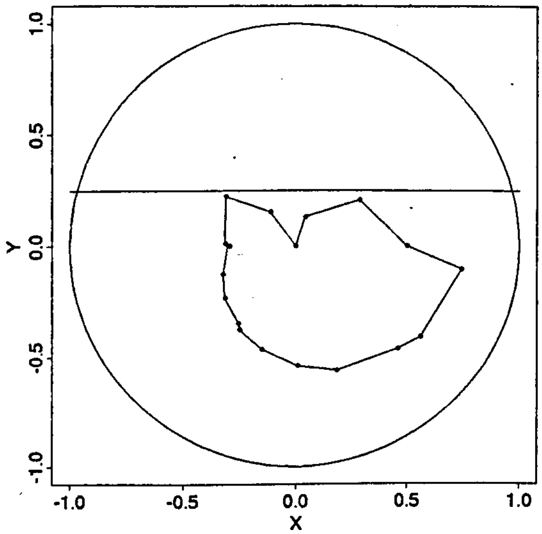

Figure 2.7: The Optimal Shape

Chapter 3

Shapes, Measures and Elliptic Equations in Polar Coordinates

3.1 Introduction

In the present chapter, we consider again $J = [0, 2\pi]$, $J^\circ = (0, 2\pi)$, $A = [0, 1]$, $\Omega = J \times A$, and consider the variables θ and r in polar coordinates, to belong to J and A respectively. We also assume that the curve ∂C is a simple and closed curve in $\mathbb{R}^2$, contains the fixed point (θ_a, r_a); the curve is defined by the equation $r = r(\theta) \in A, \forall \theta \in J$. Moreover we consider $w = \frac{dr}{d\theta}$ as a bounded function on J which takes values in the bounded set $W \subset \mathbb{R}$. Let $C \subset \mathbb{R}$ be a Lebesgue measurable set which is determined by ∂C as its boundary; it is supposed that C has a fixed area. We remind the reader that the pair $(C, \partial C)$ is the same as one in Chapter 2. Let $u : \Omega \longrightarrow \mathbb{R}$, a differentiable and bounded function in $C^2(C)$ in which its first derivatives are bounded in C, be a solution for the elliptic problem

$$\operatorname{div}(k(\theta, r)\nabla u) - f(\theta, r, u) = 0, \tag{3.1}$$

with the Neumann condition

$$\nabla u.\mathrm{n}_{|\partial C} = v. \tag{3.2}$$

Here it is supposed that the function u takes values in the bounded set $U \subset \mathbb{R}$, $k(\theta, r)$ is a positive function in $C^1(C)$, $f : \Omega \times U \longrightarrow \mathbb{R}$ is a bounded function in $C(\Omega \times U)$, n is the outward normal vector on ∂C, and $v : J \longrightarrow \mathbb{R}$ is a bounded Lebesgue measurable function which takes values in the bounded set $V \subset \mathbb{R}$. In this Chapter, the functions $v = v(\theta)$ and $w = w(\theta)$ are considered as the pair of control functions, and the functions $r = r(\theta)$ and $u = u(\theta, r)$ are regarded as the pair of trajectory functions in a classical optimal control (or shape design) problem.

Definition 2 : *The quadruplet $(C, \partial C, u, v)$, defined above, is called admissible if the elliptic equations (3.1) and (3.2) have a bounded solution on C. The set of all admissible quadruplets is denoted by $\mathcal{F}$.*

Based on the mentioned concepts, the aim of this Chapter is to find the minimizer of the following performance criterion, I, over the set $\mathcal{F}$ by applying the similar method as explained in the previous Chapter.

$$\mathbf{I}(C, \partial C, u, v) = \int_C f_\circ(\theta, r, u, \nabla u)\, dr d\theta + \int_{\partial C} h_\circ(\theta, r, w, v)\, ds; \tag{3.3}$$

here $f_\circ$ and $h_\circ$ are two given continuos functions.

Indeed, when one regards the functions v (appeared in (3.2)) and w (corresponding to $\partial C$and C) as the control functions, and the functions u (appeared in (3.1) and (3.2)) and r (corresponding to $\partial C$and C) as the trajectory functions, we are going to solve the following optimal shape design problem over $\mathcal{F}$:

$$\begin{aligned} \textbf{Minimize}: \ \mathbf{I}(C, \partial C, u, v) = \int_C f_\circ dr d\theta + \int_{\partial C} h_\circ ds \\ \textbf{Subject to}: \qquad (C, \partial C, u, v) \in \mathcal{F}; \\ \text{the area of } C = \text{given}; \\ \theta_a \text{ and } r_a = \text{given}; \\ \operatorname{div}(k(\theta, r)\nabla u) - f(\theta, r, u) = 0; \\ \nabla u.\mathbf{n}_{|\partial C} = v. \end{aligned} \tag{3.4}$$

3.2 Weak solution

Definition 3 : *The function u is called a classical solution of the elliptic equations (3.1) and (3.2) whenever $u \in C^1(C) \cap C^2(C)$ and satisfies the equations (3.1) and (3.2).*

It is difficult to identify a classical solution for the general case of the elliptic Neumann problem; thus usually it has been tried to find a *weak* (or *generalized*) solution of the problem, which is more applicable in our work. The main idea in this replacement, is to change the elliptic problem into the variational form; the following Proposition shows that how this can be done. It is necessary to introduce a new space first.

Definition 4 : *The Sobolev space of order 1 on C is denoted by $H^1(C)$ and defined as*

$$H^1(C) = \left\{ h \in L_2(C) : \frac{\partial h}{\partial \theta} \in L_2(C), \frac{\partial h}{\partial r} \in L_2(C) \right\}.$$

We follow Mikhailov in [38] to prove the below Proposition.

Proposition 9 : *Let u be the classical solution of (3.1) and (3.2), then we have the following integral equality*

$$\int_C (k\nabla u \nabla\varphi + f\varphi)\, r dr d\theta - \int_{\partial C} k\varphi v \, ds = 0, \ \forall\varphi \in H^1(C). \tag{3.5}$$

Proof: By multiplying (3.1) with a function $\varphi \in H^1(C)$ and then integrating over C, we obtain

$$\int_C \varphi \operatorname{div}(k\nabla u)\, r dr d\theta - \int_C \varphi f \, r dr d\theta = 0.$$

Because $\operatorname{div}(k\nabla u) = k\Delta u + \nabla u \nabla k$ (see for instance [38]), thus

$$\int_C \varphi k \Delta u \, r dr d\theta + \int_C \varphi k \nabla u \nabla k \, r dr d\theta - \int_C \varphi f \, r dr d\theta = 0. \tag{3.6}$$

Green's formula (see [38]) gives rise to

$$\int_C \varphi k \Delta u \, r dr d\theta = \int_{\partial C} \varphi k \frac{\partial u}{\partial \mathrm{n}} \, ds - \int_C \nabla u \nabla \varphi k \, r dr d\theta. \tag{3.7}$$

But $\nabla(\varphi k) = \varphi \nabla k + k \nabla \varphi$; hence by considering (3.2) and applying (3.7) in (3.6), the equality (3.5) is obtained. □

Definition 5 : *A bounded function* $u \in H^1(C)$ *is called a bounded weak solution of the problem (3.1) and (3.2) if it satisfies the equality (3.5) for all function* $\varphi \in H^1(C)$.

Note that the existence of a classical and a bounded weak solution for a problem like (3.1) and (3.2) has been considered in many references; in Chapter 4 we will explain some of these results very briefly. Considering the above Proposition, instead of looking for the minimizer of (3.4) in $\mathcal{F}$ we seek the optimal solution of the following problem in the same admissible set.

$$\begin{aligned}
\textbf{Minimize}: \quad & \mathbf{I}(C, \partial C, u, v) = \int_C f_\circ \, dr d\theta + \int_{\partial C} h_\circ \, ds \\
\textbf{Subject to}: \quad & (C, \partial C, u, v) \in \mathcal{F}; \\
& \text{the area of } C = \text{given}; \\
& \theta_a \text{ and } r_a = \text{given}; \\
& \int_C (k\nabla u \nabla \varphi + f\varphi)\, r dr d\theta - \int_{\partial C} k\varphi v \, ds = 0, \ \forall \varphi \in H^1(C). \quad (3.8)
\end{aligned}$$

It is usually difficult and sometimes impossible to obtain the solution of the problem like (3.8); some of these difficulties are explained in Chapter 2. In the next section, we will replace the problem with the new one in which positive Radon measures are involved. The following integral equality, based on the Green's formula, will be used later;

$$\int_C (u\Delta\varphi + \nabla u \nabla \varphi)\, r dr d\theta = \int_{\partial C} \varphi v \, ds, \ \forall \varphi \in H^1(C). \tag{3.9}$$

3.3 Metamorphosis

In general, the minimization of (3.8) over $\mathcal{F}$ is not easy. The infimum may not be attained at any admissible quadruplet; it is not possible, for instance to write necessary conditions for this problem. We proceed then to transform it into a measure-theoretical form. Because $u \in H^1(C)$ and bounded, then ∇u is a bounded real-valued function; let ∇u takes values in the bounded set U', then we define

$$\Omega' = \Omega \times U \times U', \ \omega' = \omega \times V \, (\text{that } \omega = J \times A \times W).$$

An admissible quadruplet $(C, \partial C, u, v) \in \mathcal{F}$ introduces two functionals. A bounded weak solution of (3.1) and (3.2) defined on C, determines a linear, bounded and positive functional

$$u_C(\cdot) : F \longrightarrow \int_C F(\theta, r, u, \nabla u)\, dr d\theta \tag{3.10}$$

on the space $C(\Omega')$. Also a control function v, defined on ∂C which satisfis (3.2), introduces a linear, bounded and positive functional

$$v_{\partial C}(\cdot) : G \longrightarrow \int_J G(\theta, r, w, v)\, d\theta (\equiv \int_{\partial C} \frac{1}{\sqrt{r^2 + w^2}} G\, ds) \tag{3.11}$$

on the space $C(\omega')$. On the base of the Riesz Representation Theorem (see [55]), the above functionals represent two positive Radon measures as shown in the following Proposition; the proof is similar to the proof of Proposition 1, so it is omitted.

Proposition 10 : *There exists uniquely a pair of positive Radon measures* $\lambda_u \in \mathcal{M}^+(\Omega')$ *and* $\sigma_v \in \mathcal{M}^+(\omega')$, *so that*

$$\begin{aligned} \lambda_u(F) &= \int_C F(\theta, r, u, \nabla u)\, dr d\theta \equiv u_C(F), & \forall F \in C(\Omega'), \\ \sigma_v(G) &= \int_J G(\theta, r, w, v)\, d\theta \equiv v_{\partial C}(G), & \forall G \in C(\omega'). \end{aligned} \tag{3.12}$$

Proposition 10 shows that each admissible quadruplet $(C, \partial C, u, v) \in \mathcal{F}$ can be considered as a pair of measures (λ_u, σ_v) in the appropriate subset of $\mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$, say $\mathcal{F}$ again; thus there exist the transformation

$$(C, \partial C, u, v) \in \mathcal{F} \longrightarrow (\lambda_u, \sigma_v) \in \mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega').$$

As we showed in Chapter 2 (see Proposition 2), this transformation is injective. Hence someone may think that nothing is changed and the same difficulties as before (existing the optimal pair, achieving to the minimizer, belonging the minimizer to $\mathcal{F}$ and so on) still remain. So, we will extend the underlying space of the problem; instead of seeking in the set of all pairs (λ_u, σ_v), $\mathcal{F}$, we look for the minimizer of the functional I in a subset of $\mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$ defined by some linear equalities which will be explained later. Indeed, we are going to find the minimizer of the functional

$$(\lambda, \sigma) \in \mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega') \longrightarrow \lambda(f_\circ) + \sigma(h_\circ\sqrt{r^2 + w^2}),$$

in which λ and σ are satisfied some linear equalities defined by properties of admissible quadruplets. We remind the reader that the advantages of the new formulation have been studied in Chapter 2.

According to the new formulation, Proposition 9 shows that an admissible pair of measures (λ, σ) must satisfy

$$\lambda(F_\varphi) + \sigma(G_\varphi) = 0, \; \forall \varphi \in H^1(C) \tag{3.13}$$

where

$$F_\varphi \equiv rk\nabla u \nabla \varphi + rf\varphi, \; G_\varphi \equiv -k\varphi v\sqrt{r^2 + w^2}. \tag{3.14}$$

The condition (3.13) does not cover all properties of an admissible pair; it just modifies the weak solution of the elliptic problem. Somehow these kind of properties must come into account.

The admissibility of the curve ∂C (and hence the set C) has been characterized by equalities (2.6), (2.8) and(2.11) in Chapter 2. Moreover the restriction of the measures λ and σ over Ω and ω respectively, are the measures μ and ν defined in Chapter 2. Thus we have

$$\begin{aligned} &\sigma(\phi^g) = \delta_\phi, \quad \forall \phi \in C'(B); \\ &\sigma(\psi^g) = 0, \quad \forall \psi \in \mathcal{D}(J^o); \\ &\sigma(f) = a_f, \quad \forall f \in C_1(\omega). \end{aligned} \tag{3.15}$$

Also there is a relationship between the set C and ∂C that the simple and closed curve ∂C is the boundary of C. This fact introduces a relation between the measures λ and σ. In the previous Chapter, this fact has been considered by computing the measures μ in terms of the measure ν with applying a special function; but here we are going to show this relation by use of the Stokes's (or Green's) Theorem in polar coordinates. Let $\rho, \tau \in C^1(\Omega)$, then from Stokes's Theorem we have:

$$\int_C [\frac{\partial}{\partial r}(r\rho) - \frac{\partial \tau}{\partial \theta}]\, dr d\theta = \int_J [\tau w + \rho r]\, d\theta.$$

So, we have:

$$\lambda(\rho + r\frac{\partial \rho}{\partial r} - \frac{\partial \tau}{\partial \theta}) - \sigma(\tau w + \rho r) = 0, \ \forall \rho, \tau \in C^1(\Omega). \tag{3.16}$$

Moreover, the definition of u_C in (3.10), that represents measure λ_C in Proposition 10, shows that for $(\theta, r, u, t) \in \Omega'$ there is a relation between the variables $u \in U$ and $t \in U'$ (that $t = \nabla u$). Let $(\theta, r, u, t) \in \Omega' \longrightarrow f'(\theta, r, u, t) \in \mathbb{R}$ be a function in $C(\Omega')$; whenever the measure λ_C (or equally the functional u_C) is applied on a function like $f' \in C(\Omega')$, this relation should be considered. In other words, the variables u and t

are not independent from each other and this dependency should be regarded in the determination of the measures λ and σ; it is also very important to regard this fact in the numerical examples when we identify the variables u and t just by some (finite) values in the appropriate bounded sets (see Example). From Green's formula, the equality (3.9) shows this relation for every function $\varphi \in H^1(C)$ as

$$\lambda(ru\Delta\varphi + r\nabla u\nabla\varphi) = \sigma(\varphi v\sqrt{r^2+w^2}),\ \forall\varphi \in H^1(C). \tag{3.17}$$

As a result, to find the minimizer of I over $\mathcal{F}$, one can search for the minimizer of the functional $(\lambda, \sigma) \longrightarrow \lambda(f_\circ) + \sigma(h_\circ\sqrt{r^2+w^2})$ over a subset Q of $\mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$ defined by all pairs (λ, σ) which satisfied the conditions (3.13), (3.15), (3.16), and (3.17). Thus, instead of solving the problem (3.8), we look for the minimizer of the following new problem over Q:

$$\begin{aligned}
&\textbf{Minimize:} && \mathrm{i}(\lambda,\sigma) = \lambda(f_\circ) + \sigma(h_\circ\sqrt{r^2+w^2}) \\
&\textbf{Subject to:} && \sigma(\phi^g) = \delta_\phi, && \forall\phi \in C'(B); \\
& && \sigma(\psi^g) = 0, && \forall\psi \in \mathcal{D}(J^\circ); \\
& && \sigma(f) = a_f, && \forall f \in C_1(\omega); \\
& && \lambda(F_\varphi) + \sigma(G_\varphi) = 0, && \forall\varphi \in H^1(C); \\
& && \lambda(\rho + r\frac{\partial\rho}{\partial r} - \frac{\partial\tau}{\partial\theta}) - \sigma(\tau w + \rho r) = 0, && \forall\rho,\tau \in C^1(\Omega); \\
& && \lambda(ru\Delta\varphi + r\nabla u\nabla\varphi) = \sigma(\varphi v\sqrt{r^2+w^2}), && \forall\varphi \in H^1(C).
\end{aligned} \tag{3.18}$$

The following theorem states that the above problem has a minimizer. To prove the Theorem the reader can follow Rubio in [50] as we did for proof of the Theorem 1 in previous Chapter.

Theorem 2 : *There exists an optimal pair of measures* (λ^*, σ^*) *in the set* $Q \subset \mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$ *such that for which*

$$\mathrm{i}(\lambda^*, \sigma^*) \leq \mathrm{i}(\lambda, \sigma), \ \forall (\lambda, \sigma) \in Q.$$

We remind the reader, since the set $\mathcal{F}$ of admissible quadruplets can be considered, by means of the mentioned injective transformation, as a subset of Q, therefore

$$\inf_{\mathcal{F}} \mathbf{I}(C, \partial C, u, v) \geq \inf_{Q} \mathrm{i}(\lambda, \sigma).$$

Thus, in (3.18) the minimization is global, that is the global infimum of the problem can be obtained. So in the nonclassical form of the optimal shape design problem (problem (3.18)), the global minimizer will be illustrated.

3.4 Approximation

All the equations in the problem (3.18) are linear in their arguments λ and σ. It is an infinite linear program; the number of equations and the dimension of the underlying space are infinite. In this section we are going to approximate the solution of the problem by the solution of an appropriate finite linear programming problem so that not only the number of equations is finite, but the underlying space on which minimization takes place on it, will be a finite dimensional space. This important can be happened by use of a total set in each space $H^1(C), C^1(\Omega), C_1(\omega), \mathcal{D}(J^\circ)$ and $C'(B)$.

In Chapter 2 we introduced the total sets in the spaces $C_1(\omega), \mathcal{D}(J^\circ)$ and $C'(B)$; here, we identify the total sets in the other spaces. Let P be the $\mathbb{C}$-vector space with the basis $\left\{Z^n, \overline{Z}^n : Z \in \Omega\right\}$ (note that indeed $\Omega = \{Z \in \mathbb{C} : \mid Z \mid \leq 1\}$). Under multiplication, P is an algebra and satisfies in the conditions of Stone's-Wierstrass Theorem (see for instance [54]); hence it is dense in $C(\Omega)$. Regarding the polar coordinates, each $Z \in \Omega$ can be rewritten as $Z = r(\cos\theta + i\sin\theta)$, where $r =\mid Z \mid$. Also $Z^n = r^n(\cos n\theta + i\sin n\theta)$ and $\overline{Z}^n = r^n(\cos n\theta - i\sin n\theta)$; thus if one consider Ω as a subset of $\mathbb{R}^2$, the set of functions $r^n \cos n\theta$ and $r^n \sin n\theta$ that $n = 1, 2, 3, \ldots,$ is a base for P (indeed these functions can be regarded as projections of the function $Z \rightarrow (r^n \cos n\theta, r^n \sin n\theta)$ on $\mathbb{R}$). Hence the set of functions φ_n that $\varphi_n = r^n \cos n\theta$ or $\varphi_n = r^n \sin n\theta$ for $n = 1, 2, 3, \ldots,$ is dense in $C^1(\Omega) \subset C(\Omega)$, for all $(\theta, r) \in \Omega$; moreover, by Theorem 3 in Chapter III of [38], $C^1(\Omega)$ is dense in $H^1(C)$. Thus, as a conclution of the above discussion, the set of functions $\varphi_n, n = 1, 2, 3, \ldots,$ is also total in $H^1(C)$.

Now consider the following problem which is resulted from (3.18) just by choosing a finite number of functions in the appropriate total sets;

$$\begin{aligned}
&\text{Minimize}: \quad \mathrm{i}(\lambda, \sigma) = \lambda(f_\circ) + \sigma(h_\circ\sqrt{r^2 + w^2}) && \\
&\text{Subject to}: \quad \sigma(\phi_k^g) = \delta_{\phi_k}, && k = 1, 2, \ldots, M_1; \\
&\sigma(\chi_l) = 0, && l = 1, 2, \ldots, M_2; \\
&\sigma(f_s) = a_s, && s = 1, 2, \ldots, M_3; \\
&\lambda(F_i) + \sigma(G_i) = 0, && i = 1, 2, \ldots, M_4; \\
&\lambda(D_j) + \sigma(E_j) = 0, && j = 1, 2, \ldots, M_5; \\
&\lambda(H_r) + \sigma(I_r) = 0, && r = 1, 2, \ldots, M_6. \qquad (3.19)
\end{aligned}$$

Here

$$D_j \equiv ru\Delta\varphi_j + r\nabla u\nabla\varphi_j, \qquad E_j \equiv -(\varphi_j v\sqrt{r^2+w^2});$$

$$F_i \equiv F_{\varphi_i}, \qquad G_i \equiv G_{\varphi_i};$$

$$H_r = H_{ij} \equiv \varphi_i + r\tfrac{\partial\varphi_i}{\partial r} - \tfrac{\partial\varphi_j}{\partial\theta}, \quad I_r = I_{ij} \equiv -(\varphi_j w + \varphi_i r). \tag{3.20}$$

Now we have the following Proposition which shows that the solution of the problem (3.18) can be approximated by the solution of (3.19); for proof, one can follows Rubio in [50] Chapter III.

Proposition 11 : *For positive integer numbers $M_1, M_2, M_3, M_4, M_5, M_6$, let $Q_{M'}$ be the set of the pairs $(\lambda,\sigma) \in \mathcal{M}^+(\Omega') \times \mathcal{M}^+(\omega')$ which satisfy the constraints of (3.19). If $M_1, M_2, M_3, M_4, M_5, M_6$, tends to infinity then*

$$\inf_{Q_{M'}} \mathrm{i}(\lambda,\sigma) \longrightarrow \inf_{Q} \mathrm{i}(\lambda,\sigma);$$

in other words, the solution of (3.19) tends to the solution of (3.18).

We have already limited the number of constraints of (3.18) in the first stage of approximation; but the underlying space, $Q_{M'}$, is still infinite-dimensional. We shall approximate now the solution of this problem with the solution of a finite linear programming one. Let (λ^*,σ^*) be the optimal solution of (3.19) (the existence of the solution can be obtained from Theorem 2). By applying Theorem $A.5$ of [50], as shown in Chapter 2, one can obtain

$$\lambda^* = \sum_{n=1}^{N} \alpha_n^* \delta(Z_n^*)\,,\ \sigma^* = \sum_{m=1}^{M} \beta_m^* \delta(z_m^*),$$

that for each $n = 1,2,\ldots,N$ and $m = 1,2,\ldots,M$, we have $\alpha_n^* \geq 0, \beta_m^* \geq 0$, and also

Z_n^* and z_m^* belong to the dense subsets of Ω' and ω' respectively; here M and N are two positive integers and $\delta(z)$ is a unitary atomic measure with support the singleton point set $\{z\}$.

Up to here, the measure-theoretical optimization problem is equivalent to a nonlinear optimization one in which the unknowns are the coefficients α_n^*, β_m^* and supports $\{Z_n^*\}, \{z_m^*\}$ for $n = 1, 2, \ldots, N$, and $m = 1, 2, \ldots, M$. It would be much more convenient if we could minimize the functional i only with respect to the coefficients, which would cause the problem to change into a finite linear program.

In the next stage of approximation, let $D_{\Omega'}$ and $D_{\omega'}$ be two countable dense subset of Ω' and ω' respectively. Then, (as a result of Proposition $III.3$ in [50]) measures λ^* and σ^* can be approximated by

$$\lambda = \sum_{n=1}^{N} \alpha_n \delta(Z_n) \, , \, \sigma = \sum_{m=1}^{M} \beta_m \delta(z_m)$$

where $Z_m \in D_{\Omega'}, z_n \in D_{\omega'}$. This result suggests that the problem (3.19) can be approximated by the following linear programming one which the points Z_n and z_m are chosen from a finite subset of a countable dense subsets in the appropriate space by putting discretization on Ω' and ω'. Hence the only unknowns are the coefficients α_n and β_m, for $n = 1, 2, \ldots, N$, and $m = 1, 2, \ldots, M$.

$$
\begin{aligned}
\textbf{Minimize:} \quad & \sum_{n=1}^{N} \alpha_n f_\circ(Z_n) + \sum_{m=1}^{M} \beta_m h_\circ(z_m)\sqrt{r_m^2 + w_m^2} & \\
\textbf{Subject to:} \quad & \alpha_n \geq 0, & n = 1,2,\ldots,N; \\
& \beta_m \geq 0, & m = 1,2,\ldots,M; \\
& \sum_{m=1}^{M} \beta_m \phi_k^g(z_m) = \delta_{\phi_k}, & k = 1,2,\ldots,M_1; \\
& \sum_{m=1}^{M} \beta_m \chi_l(z_m) = 0, & l = 1,2,\ldots,M_2; \\
& \sum_{m=1}^{M} \beta_m f_s(z_m) = a_s, & s = 1,2,\ldots,M_3; \\
& \sum_{n=1}^{N} \alpha_n F_i(Z_n) + \sum_{m=1}^{M} \beta_m G_i(z_m) = 0, & i = 1,2,\ldots,M_4; \\
& \sum_{n=1}^{N} \alpha_n D_j(Z_n) + \sum_{m=1}^{M} \beta_m E_j(z_m) = 0, & j = 1,2,\ldots,M_5; \\
& \sum_{n=1}^{N} \alpha_n H_r(Z_n) + \sum_{m=1}^{M} \beta_m I_r(z_m) = 0, & r = 1,2,\ldots,M_6; \\
& \sum_{m=1}^{M} \beta_m \left(\frac{1}{2} r_m^2\right) = \text{given area}; & (3.21)
\end{aligned}
$$

here is assumed that $Z_n = (\theta_n, r_n, u_n, t_n) \in \Omega'$ and $z_m = (\theta_m, r_m, w_m, v_m) \in \omega'$. The last equation in (3.21) represents the area condition as explained in previous Chapter.

3.5 Numerical example

As an example, we chose one that $f_\circ = 0$, $h_\circ = v^2$, $f = u(u - 0.5)$ and $k(\theta, r) = 1$. We remind the reader that in polar coordinates

$$\nabla\varphi = \frac{\partial\varphi}{\partial r}\mathrm{u_r} + \frac{1}{r}\frac{\partial\varphi}{\partial\theta}\mathrm{u_\theta},\ \forall\varphi \in H^1(C);$$

also it is supposed that $\nabla u = u_1\mathbf{u_r} + u_2\mathbf{u_\theta}$ where $u_1 \in U_1$ and $u_2 \in U_2$. So, for this problem we also chose $W = [-0.3, 0.3]$, $V = [-10, 10]$, $U = [-5, 5]$ and $U' = U_1 \times U_2 = [-15, 15]^2$.

To set up the finite linear program (3.21) for this example, the appropriate discretization was made on Ω' and ω' as follows. By selecting:

- 10 angles on $J = [0, 2\pi]$ for θ as: $\frac{\pi}{10}, \frac{3\pi}{10}, \ldots, \frac{19\pi}{10}$;
- in A, 10 values for r as: $0, \frac{1}{9}, \frac{2}{9}, \ldots, 1$;
- 10 values for w in W as; $-0.3, \frac{-21}{90}, \frac{-15}{90}, \ldots, 0.3$
- in V, 10 values for v as: $-10, \frac{-70}{9}, \frac{-50}{9}, \ldots, 10$;

a discretization with $M = 10^4$ nodes $z = (\theta, r, w, v)$ was put on ω'. With respect to Ω', we also chose :

- 10 values in each sets J and A for θ and r as above;
- in U, 10 values for u as: $-5, \frac{-35}{9}, \frac{-25}{9}, \ldots, 5$;
- 10 values in each sets U_1 and U_2 for u_1 and u_2 as: $-15, \frac{-105}{9}, \frac{-75}{9}, \ldots, 15$;

hence we made a discretization on Ω' with $M = 10^5$ nodes $Z = (\theta, r, u, u_1, u_2)$. Each component of nodes Z and z is a rational number (we supposed that $\pi = 3.141592654$) and hence all nodes belong to the dense subset of Ω' and ω' respectively.

For the first three set of equations in (3.21), the same 20 equations as in examples of Chapter 2 have been applied ($M_1 = 2, M_2 = 8, M_3 = 10$) with the same values for the fixed point and area. We also chose $M_4 = 5, M_5 = 2$ and $M_6 = 2$; then the linear program (3.21) was run with 30 equations and 110000 variables. We applied the

$E04MBF$ NAG-Routine to solve the problem. The optimal value of performance criterion was 274.23683327352. Based on the equation $w(\theta) = \frac{dr}{d\theta}$, as Rubio in [50], the suboptimal control function w and the following points of the boundary of the (approximate) optimal shape were obtained:

$$(0.6283185482025\ ,\ 0.31150437270739),\ (1.1843659597573\ ,\ 0.14469009363621),$$

$$(1.2446240940434\ ,\ 0.12661264732457),\ (1.2566370964050\ ,\ 0.13021654966397),$$

$$(1.7049525491271\ ,\ 0),\ (1.8104140347910\ ,\ 0),$$

$$(1.8849556446075\ ,\ 0),\ (2.3533600786619\ ,\ 0.12696729964365),$$

$$(2.5132741928101\ ,\ 0.78993049407780),\ (2.9936742628018\ ,\ 0.22311313562577),$$

$$(3.1415927410126\ ,\ 0.17873757737069),\ (3.6413286509549\ ,\ 0.32865841819894),$$

$$(3.7699112892151\ ,\ 0.29008361386263),\ (4.2646965540754\ ,\ 0.14164798492600),$$

$$(4.3982298374176\ ,\ 0.10158798657002),\ (4.6766879544327\ ,\ 0.01805052361967),$$

$$(4.8688439868599\ ,\ 0.07569735943546),\ (5.0265483856201\ ,\ 0.12300870047393),$$

$$(5.4861769095531\ ,\ 0.26089732005427),\ (5.6548669338226\ ,\ 0.31150435023696),$$

$$(6.0230599912156\ ,\ 0.42196231744179),\ (6.2831854820251\ ,\ 0.5).$$

Linking these points together, gives rise the optimal shape; the optimal control function (function w) and the optimal shape is plotted in Figures 3.1 and 3.2 respectively.

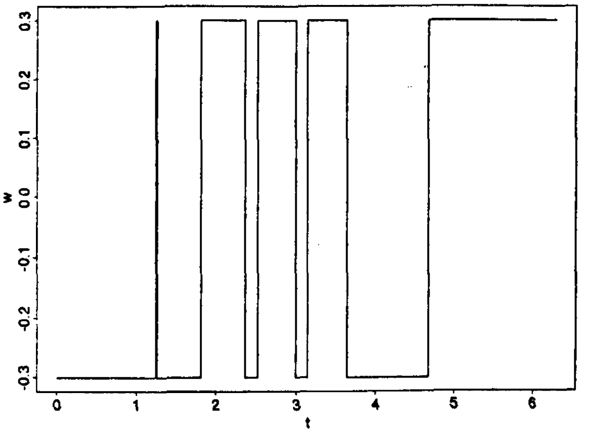

Figure 3.1: The optimal control function w

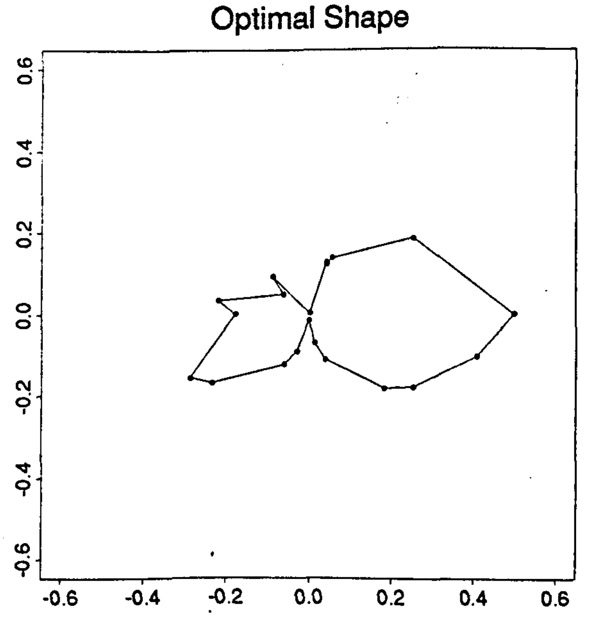

Figure 3.2: The optimal shape

Chapter 4

Shapes, Measures and Elliptic Equations (Fixed Control)

4.1 Introduction

Let $D \subset \mathbb{R}^2$ be a bounded domain with a piecewise-smooth, closed and simple boundary ∂D. We assume that some part of ∂D is fixed and the rest, Γ, with the given initial and final points $A(x_a, y_a)$ and $B(x_b, y_b)$ respectively, is not fixed (see Figure 4.1).

Suppose we choose an appropriate (variable) curve Γ joining A and B, so that D is well-defined. Let $X \in D \longrightarrow u(X) \in \mathbb{R}$, where $X = (x, y) \in \mathbb{R}^2$, is a bounded solution of the following elliptic partial differential equation on the domain D

$$\Delta u(X) + f(X, u) = v(X) \tag{4.1}$$

with the boundary condition

$$u_{|\partial D} = 0, \tag{4.2}$$

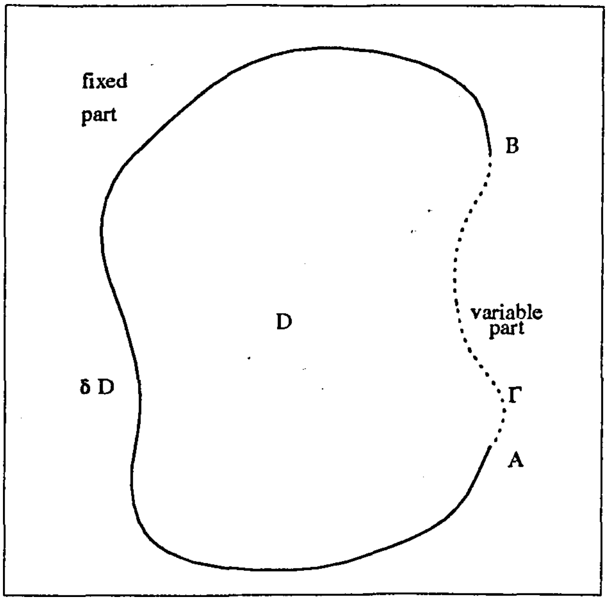

Figure 4.1: A domain D in its general form

where $X \in D \longrightarrow v(X) \in \mathbb{R}$ is (in this chapter) a bounded fixed control function; the function f is assumed to be a bounded and continuous real-valued function in $L_2(D \times \mathbb{R})$. We remind the reader that the equation (4.1) with the boundary condition (4.2), is known as the *Dirichlet problem* (see for instance [38], [33], [20]). A domain D as above, is called an *admissible domain* if the elliptic equation (4.1) and (4.2) has a bounded solution on D; we denote by $\mathcal{D}$ as the set of all such admissible domains.

In this chapter, we consider first the problem of minimizing the following functional on the set $\mathcal{D}$ defined above:

$$\mathrm{I}(D) = \int_D f_\circ(X, u)\, dX, \tag{4.3}$$

where $f_\circ$ is a given continuous, nonnegative, real-valued function on $D \times \mathbb{R}$. Then we will find the minimizer domain D^* in $\mathcal{D}_M$, a subset of $\mathcal{D}$ to be defined later, for the functional (4.3) in the following way:

- (1) In the first step, we will obtain a solution of (4.1) and (4.2) for a fixed admissible domain in a class to be denoted as $\mathcal{D}_M$. By using the density property of an appropriate subset of points on $\mathbb{R}^2$, we establish the fact that the boundary ∂D of a domain $D \in \mathcal{D}$ can be determined by a countable subset of points in $\mathbb{R}^2$ which is dense in ∂D. This countable set will be called *the representative set of* D. Because the simple, closed curve ∂D is the boundary of D, the domain also is determined by this representative set. Moreover, an approximation to this domain is denoted by a finite set, to be called an *M-representation* . For a fixed number M, we shall denote $\mathcal{D}_M$ the set of all such M-representations. Therefore, the variable part Γ of ∂D is defined by a finite set of M real variables. Then any integral like (4.3) - with a fixed control - is simply a function of this finite number of real variables. The problem (4.1) and (4.2) will then be generalized and the variational form of the problem will be obtained. Next, by using the representation set and the generalized form, the problem will be changed into a measure-theoretical one, which has some advantages. The new formulation helps us replace the problem with an infinite dimensional linear system of equations; then we shall approximate this system with a finite one. Hence the solution of (4.1) and (4.2) will be approximated by the solution of the appropriate finite linear system as a function of M variables. So, we will be able to approximate the value of I(D) for any given domain $D \in \mathcal{D}_M$.

We want to emphasize that in this and the following Chapters we will consider M as a fixed positive integer number, and that we will search for an optimal domain in the class $\mathcal{D}_M$. It is not at all obvious that as $M \longrightarrow \infty$ a sequence $\{D_i^*\}$ of optimal domains, $D_i^* \in \mathcal{D}_i, i = 1, 2, 3, \ldots$, tends - in any sense whatsoever - to a domain $D \in \mathcal{D}$. We shall discuss this problem in Appendix B.

- (2) For a fixed positive integer number M, we are going to solve the optimal shape problem, which is to find the minimizer domain for the functional (4.3) over $\mathcal{D}_M$. The previous step states that how one can determine a solution for the elliptic equations (4.1) and (4.2) for any arbitrary M-representation $D \in \mathcal{D}_M$; this solution is a function of M variables. The solution defines a unique value for $\mathbf{I}(D)$ in terms of the finite number of variables as explained. Therefore, one can define a function

$$\mathbf{J} \,:\, D \in \mathcal{D}_M \longrightarrow \mathbf{I}(D) \in \mathbb{R}; \tag{4.4}$$

here $\mathbf{J}$ is a function of a finite number M of variables; in fact, it is a vector function. To find the minimizer of the optimal shape problem, it is now enough to identify the minimizer of $\mathbf{J}$. The application of a standard minimization algorithm (like Nelder and Mead [42]), gives us that minimizer. The minimizer is a set of points (an M-representation) which introduces the optimal domain for the functional $\mathbf{I}$ in $\mathcal{D}_M$; indeed it presents the optimal shape (domain) and also determines the minimal value of the performance criterion for the mentioned optimal shape problem.

In spite of the fact that there are some other methods for solving the problem, for instance the methods involving finite elements and finite differences (see [44] for example), our method has some advantages. It is applicable to solve the related optimal shape design and control problems (see the following chapters), it can determine the optimal shape (domain) and the optimal distributed or boundary control functions at the same time. Moreover the computation is much easier than the others because of the linearity properties of the replaced system. We shall also give some numerical examples to see how this method can be applied.

4.2 Solution in the fixed domain $D \in \mathcal{D}_M$

In the present section we are going to obtain the solution of (4.1) and (4.2) on a given domain, D, so as to calculate $\mathrm{I}(D)$. A domain in $\mathcal{D}$ is identified by the variable part (Γ) of its boundary; replacing Γ with the representative set and applying the variational form of the equations (4.1) and (4.2), change the problem into a measure-theoretical one in which its result will approximate the generalized solution of (4.1) and (4.2) in the given domain D. Therefore we shall be able to compute the value of $\mathrm{I}(D)$ and set up an appropriate function in the next section.

4.2.1 Representative sets

Let $D \in \mathcal{D}$ be a fixed, open and bounded subset of $\mathbb{R}^2$ which is an admissible domain for the elliptic partial differential equations (4.1) and (4.2). Let ∂D be the piecewise-smooth, simple and closed curve in $\mathbb{R}^2$ that is the boundary of the given domain D; thus ∂D and also its subset Γ are fixed. In general the curve ∂D, and hence Γ, can be regarded as an infinite set of points. More specifically, by applying the density property, one can regard Γ as a known countable set as follows.

The space $\mathbb{R}^2$ contains many countable dense subsets; for example if we denote $\mathbb{Q}$ as the set of rational numbers, then $\mathbb{Q}\times\mathbb{Q}$ is a countable dense subset of $\mathbb{R}^2$ (see [54] for example). Let $D_\circ$ be a given countable dense subset of $\mathbb{R}^2$, then the set $D_\circ \bigcap \Gamma$ is a countable dense subset of Γ; thus by the density property, the known set $D_\circ \bigcap \Gamma$ determines a sequence of points in $\mathbb{R}^2$ like $\{(x_k, y_k)\}_{k\in\mathrm{N}}$ so that $(x_k, y_k) \in \Gamma$ for all $k \in \mathbb{N}$. Whenever $D_\circ$ is fixed, this sequence determines Γ and hence the domain D uniquely; so, the set

$D_\circ \bigcap \Gamma$ and the domain D are equivalent and one can characterize the domain D just by this sequence of points uniquely. Therefore, we have proved the following Proposition.

Proposition 12 : *For a fixed dense subset $D_\circ \subset \mathbb{R}^2$, any domain $D \in \mathcal{D}$ is determined uniquely by $D_\circ \bigcap \Gamma$, as a sequence of its boundary points.*

Definition 6 : *For a given countable dense subset $D_\circ$ in $\mathbb{R}^2$ and for the given domain D, the countable dense subset of Γ*

$$D_\circ \bigcap \Gamma \equiv \{(x_k, y_k)\}_{k \in \mathbb{N}},$$

which determines the domain D, is called "the representative set of D".

Since a domain D is characterized by its representative set, as above, one can consider its representative set, for a known countable dense set $D_\circ$, instead of the domain D . Moreover, because each curve in $\mathbb{R}^2$ can be approximated by a finite set of broken lines, the curve Γ will be approximated by the finite set of broken lines in which their corners belong to the representative set of D. Consequently, to identify a representative set approximately, one can determine these finite number of corners (see Figure 4.2). In the section 4.3, without loosing generality, we shall show that these corners may have a fixed y-direction (i.e. each has a fixed y-component, see Figure 4.2).

Thus an arbitrary domain could be shown approximately with a finite set of its boundary points (corners). By replacing $\mathcal{D}$ with $\mathcal{D}_M$ for a fixed number M of points (see Appendix (B)), it will be shown how the value of $\mathrm{I}(D)$ can be computed from these points. In section 4.4 we shall introduce a way to identify those finite points in which $\mathrm{I}(D)$ has the minimum value. Hence the missing part of the boundary, Γ, and therefore the optimal domain will be obtained.

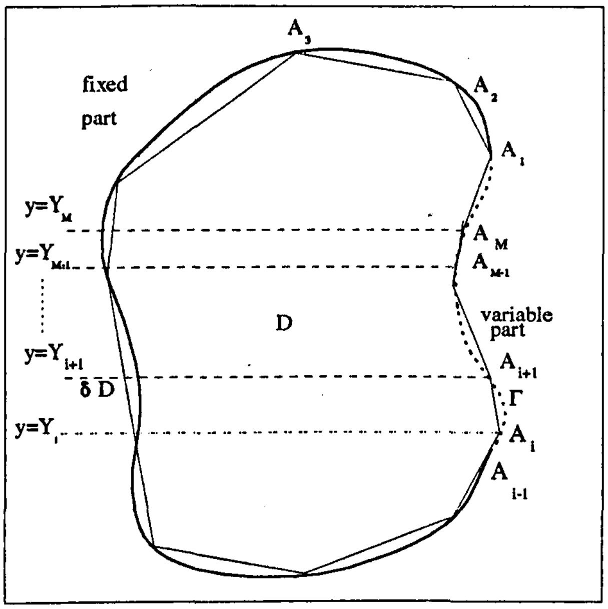

Figure 4.2: Approximating curve Γ with broken lines

To calculate the value of $\mathbf{I}(D)$ for a given domain D, it is necessary, first, to identify the solution of the partial differential equations (4.1) and (4.2). For this reason, in the following, the variational form of the problem (4.1) and (4.2) will be considered.

4.2.2 Generalized solution

Definition 7 : *The function $u(X)$: $D \longrightarrow \mathbb{R}$ is called a classical solution of (4.1) and (4.2) whenever $u(X) \in C^2(D) \bigcap C^1(D)$ and also satisfies (4.1) and (4.2).*

In general, it is difficult and sometimes impossible to identify a classical solution for the problem like (4.1) and (4.2); thus usually one tries to find a generalized or *weak* solution of them. Also the generalized solution is more applicable than the classical one in some branches like calculus of variations. In our method, especially whenever one wants to change the problem into a measure-theoretical form, this kind of solution is more appropriate.

For these reasons, it is necessary to introduce the new spaces, the new functions and also the new variational form of the problem (4.1) and (4.2) as follows.

Definition 8 : *The space $H_0^1(D)$ is defined as follows:*

$$H_0^1(D) = \left\{\psi \in H^1(D) \ : \ \psi_{|\partial D} = 0\right\};$$

$H^1(D)$ is the Sobolev space of order 1 which is defined as

$$H^1(D) = \left\{h \in L_2(D) \ : \ \frac{\partial h}{\partial x} \in L_2(D), \frac{\partial h}{\partial y} \in L_2(D)\right\}.$$

Proposition 13 : *Let u be the classical solution of (4.1) and (4.2), then we have the following integral equality*

$$\int_D (u\Delta\psi + \psi f)\, dX = \int_D \psi v\, dX\ ;\ \forall\psi \in H_0^1(D). \tag{4.5}$$

Proof: Multiplying (4.1) by the function $\psi \in H_0^1(D)$ and then integrating over D,

$$\int_D (\psi\Delta u + \psi f)\, dX = \int_D \psi v\, dX. \tag{4.6}$$

Green's formula (see for instance [38]) gives:

$$\int_D (\psi\Delta u - u\Delta\psi)\, dX = \int_{\partial D} (\psi\frac{\partial u}{\partial \mathrm{n}} - u\frac{\partial \psi}{\partial \mathrm{n}})\, dS,$$

where n is the unit vector normal to the boundary ∂D and directed outward with respect to D. Because $\psi_{|\partial D} = 0$ and $u_{|\partial D} = 0$, then

$$\int_{\partial D} (\psi\frac{\partial u}{\partial \mathrm{n}} - u\frac{\partial \psi}{\partial \mathrm{n}})\, dS = 0 \tag{4.7}$$

Now the equality (4.5) simply follows by applying (4.7) and (4.6). □

Definition 9 : *A function $u \in H^1(D)$ is called a generalized solution of the problem (4.1) and (4.2) when it satisfies in the equality (4.5) for all functions $\psi \in H_0^1(D)$.*

Indeed the equality (4.5), which introduces the generalized solution, is just an integral representation of the original elliptic problem (4.1) and (4.2). Now we are going to find this generalized solution for the given domain D. Conditions for the existence of the classical and of the generalized solution of the problem (4.1) and (4.2), and also other properties of them such as boundedness and uniqueness, have been considered in many references, like [38], [33] and [20]. For instance, in the linear case when the function $f(X, u)$ in (4.2) is assumed to be a linear function of u of the form

$$f(X, u) = a(X)u,$$

if $v(X) \in L_2(D)$ and moreover the function $a(X)$ is a nonnegative function on D ($a(X) \geq 0, \ \forall X \in D$), then there exist a unique generalized solution $u \in H_0^1(D)$ for the problem (4.1) and (4.2). This solution is bounded because $v(X)$ is supposed to be a bounded function on the domain D (for details see [38] chapter IV, especially Theorem 1 and for the more general case Theorem 7). Also one can similarly find the sufficient conditions for the bounded generalized solution for the nonlinear case of the elliptic equations, in the literature [33] (for example Theorem 7.1 in Chapter 4).

The bounded generalized solution can be represented by a positive Radon measure and then one can replace the problem with a measure-theoretical one. Hence instead of looking for the generalized solution on the given domain D, one prefers to seek for its related measure, defined on the appropriate space. In the Metamorphosis, this matter will be discussed.

4.2.3 Metamorphosis

The following Proposition, which is the base of our metamorphosis, shows that the generalized solution can be regarded as a positive Radon measure. Moreover, it also indicates that the representing measure is unique. We remind the reader that for the rest of the Chapter, $\Omega \equiv U \times \overline{D}$, where $U \subset \mathbb{R}$ is the smallest bounded set in which the bounded generalized solution $u(\cdot)$ takes values.

Proposition 14 : *Let $u(X)$ be a bounded generalized solution of (4.1) and (4.2). There exist a unique positive Radon measure, say μ_u, in $\mathcal{M}^+(\Omega)$ so that:*

$$\mu_u(F) \equiv \int_\Omega F \, d\mu_u = \int_D F(X,u) \, dX \; ; \; \forall F \in C(\Omega). \tag{4.8}$$

Proof: By applying the Riesz Representation Theorem ([55]), similar to the Proposition (1), one can obtain the equality (4.8) easily; the detail is omitted. □

By the above Proposition, the equality (4.5) changes into the following:

$$\mu_u(F_\psi) = \gamma_\psi \qquad ; \qquad \forall \psi \in H_0^1(D) \tag{4.9}$$

where

$$F_\psi = u\Delta\psi + f\psi \qquad ; \qquad \gamma_\psi = \int_D \psi v \, dX. \tag{4.10}$$

Also, $\mathrm{I}(D)$ in (4.3) is changed to $\mathrm{I}(D) = \mu_u(f_\circ)$.

It is clear that the measure μ_u projects on the (x, y)-space as the respective Lebesgue measure; hence we should have

$$\mu_u(\xi) = a_\xi,$$

where $\xi : \Omega \to \mathbb{R}$ depends only on variable X (i.e. $\xi \in C_1(\Omega)$), and a_ξ is the Lebesgue integral of ξ over D, i.e. $a_\xi = \int_D \xi \, dX$.

Therefore the problem can be described as follows:

Find a measure $\mu_u \in \mathcal{M}^+(\Omega)$ so that it satisfies the following constraints:

$$\begin{aligned} \mu_u(F_\psi) &= \gamma_\psi, \qquad \forall \psi \in H_0^1(D); \\ \mu_u(\xi) &= a_\xi, \qquad \forall \xi \in C_1(\Omega). \end{aligned} \tag{4.11}$$

Let us now consider a more general version of the problem. We extend the underlying space; instead of finding a measure $\mu_u \in \mathcal{M}^+(\Omega)$, defined by Proposition 14, satisfying equalities (4.11), we seek a measure $\mu \in \mathcal{M}^+(\Omega)$ which satisfies just the conditions

$$\begin{aligned} \mu(F_\psi) &= \gamma_\psi, \qquad \forall \psi \in H_0^1(D); \\ \mu(\xi) &= a_\xi, \qquad \forall \xi \in C_1(\Omega). \end{aligned} \tag{4.12}$$

Hence we have $\mathrm{I}(D) = \mu(f_\circ)$. The system (4.12) is linear because all the functions in the right-hand-side of equations are linear functions in their argument μ. But the number of equations is not finite and also the underlying space is not finite-dimensional. In the next, we are going to approximate a solution of (4.12) by another one in which the number of equations and also the underlying space are finite.

4.3 Approximation

The linear system (4.12) is not finite-dimensional; indeed the number of equations is not finite. We shall develop the system by requiring that only a finite number of the constraints are satisfied. This will be achieved by choosing countable sets of functions whose linear combinations are dense in the appropriate spaces, and then selecting a finite number of these.

__First set of functions:__ Consider the first set of equalities in (4.12); we are going to introduce the set

$$\left\{\psi_i \in H_0^1(D) : i = 1, 2, \ldots\right\}$$

so that the linear combinations of the functions $\psi_i \in H_0^1(D)$ are uniformly dense - that is, dense in the topology of the uniform convergence - in the space $H_0^1(D)$. For instance, these functions can be taken to be a special subset of polynomials in the components of x and y, as follows.

We know that the vector space of polynomials with the variable x and y, $P(x, y)$, is dense in $C^\infty(\overline{D})$; therefore the set $P_0(x, y)$:

$$P_0(x, y) = \{p(x, y) \in P(x, y) \mid p(x, y) = 0, \forall (x, y) \in \partial D\},$$

is dense (uniformly of course) in the space $\left\{h \in C^\infty(\overline{D}) : h_{|\partial D} = 0\right\} \equiv C_0^\infty(\overline{D})$. By the way, the set

$$Q(x, y) = \left\{1, x, y, x^2, xy, y^2, x^3, x^2y, xy^2, y^3, \ldots\right\},$$

is a countable base for the vector space $P(x, y)$ and hence each elements of $P(x, y)$ and also $P_0(x, y)$, is a linear combination of the elements in $Q(x, y)$. In the other hand, by theorem 3 of Mikhailov [38] page 131, the space $C^\infty(\overline{D})$ is dense in $H^1(D)$; thus the space $C_0^\infty(\overline{D})$ will be dense in $H_0^1(D)$ (see the definition of $H_0^1(D)$). Consequently, the space $P_0(x, y)$ is uniformly dense in $H_0^1(D)$. As before, let $\{(x_k, y_k)\}_{k \in \mathbb{N}}$ be the representation set for the fixed D; we define the function ψ_i for each $i \in \mathbb{N}$ as follows:

$$\psi_i(x,y) = \prod_{k \in \mathbb{N}} (x - x_k + y - y_k)\tau(x,y)q_i(x,y) \tag{4.13}$$

where $\tau(x, y)$ is a function which is zero on the fixed part of ∂D, and q_i is an element of the countable set $Q(x, y)$. Then the set

$$\{\psi_i(x,y) : i = 1, 2, \dots\},$$

is total (uniformly dense in the topology of the uniform convergence) in the space $H_0^1(D)$. We remind the reader that the term

$$\prod_{k \in \mathbb{N}} (x - x_k + y - y_k) = (x - x_1 + y - y_1)(x - x_2 + y - y_2)\dots \tag{4.14}$$

in (4.13) implies that $\psi_{i|_\Gamma} = 0$.

Despite the fact that ψ_i is zero on the boundary of D for each i, there is no guarantee that the value of (4.14) is convergent at every point $(x, y) \in D$; hence we have the same difficulty for ψ_i in (4.13). Besides, if the function $\prod_{k \in \mathbb{N}}(x - x_k + y - y_k)$, or equally $\psi_i(x, y)$, is uniformly convergent on D, it may be too difficult or impossible to characterize them; there is no comprehensive method to calculate F_{ψ_i} and γ_{ψ_i} in (4.12) for every $(x, y) \in D$. In fact this difficulty is caused by the number of points in the representative set, which is infinite. We will approximate the boundary ∂D by a finite number of points in the representative set.

Approximating ∂D with broken lines: The general idea of selecting a finite set of points instead of the curve ∂D, comes from the approximation of a curve by broken lines. For the given D and hence for the given Γ, let $A_m = (x_m, y_m), m = 1, 2, \ldots, M$, be a finite points of ∂D in the representative set of D (we suppose $A_1 = B$). We link together each pair of consecutive points A_m and A_{m+1} for $m = 1, 2, \ldots, M-1$. The set of segments $A_m A_{m+1}, m = 1, 2, \ldots, M-1$, defines a curve. We close this curve by joining the points A_1 and A_M together. Now the resulted shape, which is denoted by ∂D_M, is an approximation for ∂D; we also call D_M to the domain which introduced by its boundary ∂D_M. The domain D_M is called a *M-approximated domain of D* (domains D, D_M and their boundaries are shown in Figure 4.3). We remind the reader that this method, approximation by broken lines, is more convenient when the fixed part of ∂D is too complicated to be denoted by a formula.

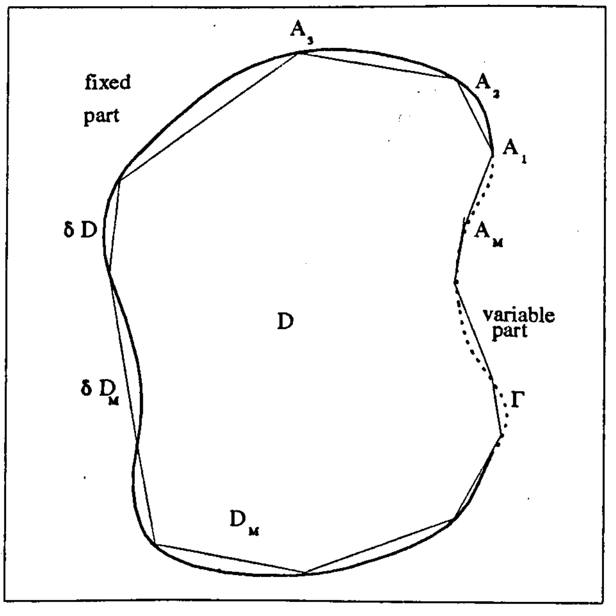

Figure 4.3: Approximation of ∂D by finite number of segments

It is possible that by increasing the number of points, M, the curve ∂D_M will become closer and closer (in the Euclidean metric) to the curve ∂D, and hence one may conclude that the minimizer of I over $\mathcal{D}_M$, if one exists, tends to the minimizer of I over $\mathcal{D}$, if one exists. In the Appendix B, we have explained some of the difficulties that arise and have discussed these matters. Here, we will fix the number of points (M) and look for the minimizer of (4.3) amongst all admissible D_M's.

As a result, the equality (4.13), for each i, changes into

$$\psi_i = (x - x_1 + y - y_1)(x - x_2 + y - y_2)\dots(x - x_M + y - y_M)\tau q_i. \tag{4.15}$$

It is necessary to mention that whenever the fixed part of ∂D is defined explicitly, for instance an expression of the form $y = h(x)$, we have $\tau = y - h(x)$. Hence ψ_i will have the following form:

$$\psi_i = (x - x_1 + y - y_1)(x - x_2 + y - y_2)\dots(x - x_M + y - y_M)(y - h(x))q_i. \tag{4.16}$$

Moreover one may define $\psi_i(x, y)$ so that it will be zero on each segment $A_m A_{m+1}$ where the points $A_m, m = 1, 2, \dots, M$ are belong to the both parts of ∂D, as follows:

$$\psi_i = (y - \frac{y_2 - y_1}{x_2 - x_1}(x - x_1) - y_1)\dots(y - \frac{y_K - y_{K-1}}{x_K - x_{K-1}}(x - x_{K-1}) - y_{K-1})$$
$$(x - x_{K+1} + y - y_{K+1})\dots(x - x_M + y - y_M)q_i; \tag{4.17}$$

here it is supposed that the points $A_1, A_2, \dots, A_K$, belong to the fixed part of ∂D and $A_{K+1}, A_{K+2}, \dots, A_M$ are in Γ (this expression of ψ_i is more convenient when the fixed part of ∂D is too complicated to be given by an explicit function).

For the rest of this chapter and also for the following Chapter, we suppose that the fixed part of ∂D is the union of the following three segments:

1) The part of the line $y = 0$ between the points $(1, 0)$ and$(0, 0)$

2) The part of the line $x = 0$ between the points $(0, 0)$ and$(0, 1)$

3) The part of the line $y = 1$ between the points $(0, 1)$ and$(1, 1)$;

hence $A = (1, 0)$ and $B = (1, 1)$ (see Figure 4.4). Also we denote $\mathcal{D}_M$, for a fixed number M, as the class of all M-approximated domains D_M.

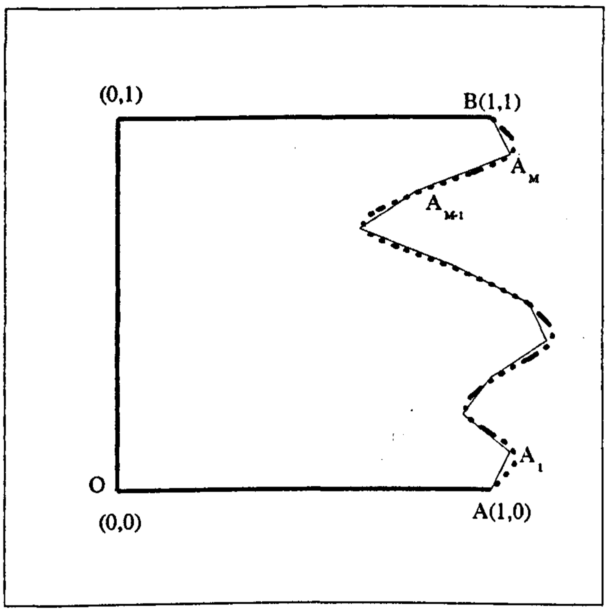

Figure 4.4: ∂D in our assumption

By the assumption made above, the function ψ_i in (4.16) will be chosen as

$$\psi_i(x, y) = xy(y-1)(x - x_1 + y - y_1)(x - x_2 + y - y_2)\dots$$
$$(x - x_M + y - y_M)q_i(x, y), \qquad (4.18)$$

where each $A_M = (x_m, y_m), m = 1, 2, \dots, M$, is an unknown point in Γ; $\psi_i(x, y)$ is zero at the points $A(1, 0)$ and $B(1, 1)$, hence there is no need to consider the terms

$(x + y - 1)$ and $(x + y - 2)$ in (4.18). Here we have actually $2M$ unknowns to determine, $x_1, x_2, \ldots, x_M, y_1, y_2, \ldots, y_M$. It would be more convenient if one, somehow, could reduce the number of unknowns, without losing the generality.

Let the value of the components $y_1, y_2, \ldots, y_M$, be fixed, for a given positive integer M. In other words suppose that for each $m = 1, 2, \ldots, M$, the point A_m is located somewhere on the line $y = Y_m, x \geq 0$. Because x_m is a free term, the point A_m could be anywhere on the line for every m (see Figure 4.5). Therefore points A_m and A_{m+1} can be chosen so that they belong to Γ and hence the part of Γ between the lines $y = Y_m$ and $y = Y_{m+1}$ can be approximated by the segment $A_m A_{m+1}$, especially whenever the number M is large. It means, we do not lose generality by fixing $y_1, y_2, \ldots, y_M$. Thus, from now on, we suppose that in (4.18) the components $y_1, y_2, \ldots, y_M$ are fixed with the values $Y_1, Y_2, \ldots, Y_M$, respectively; so

$$\psi_i(x, y) = xy(y-1)(x - x_1 + y - Y_1)(x - x_2 + y - Y_2)\ldots \\ (x - x_M + y - Y_M) \quad q_i(x, y). \quad (4.19)$$

Definition 10 : *For a fixed number M, the set of finite points in Γ, i.e. the set*

$$\{A_M = (x_m, Y_m), m = 1, 2, \ldots, M\},$$

with the fixed components $Y_1, Y_2, \ldots, Y_M$, and unknowns $x_1, x_2, \ldots, x_M$, is called the M-representation of D. *Indeed, this set introduces the M-approximated domain D_M.*

Second set of functions: For the second set of equations in (4.12), let L be a given positive integer number and divide D into L (not necessary equal) parts $D_1, D_2, \ldots, D_L$, so that by increasing L the area of each $D_s, s = 1, 2, \ldots, L$, will be decreased. Then,

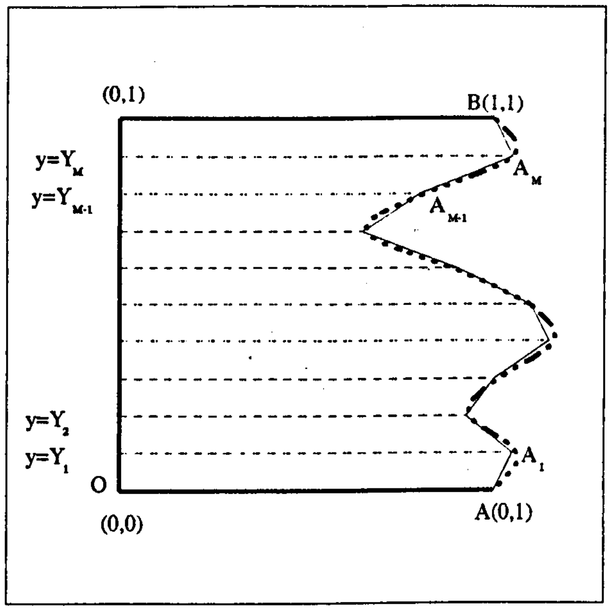

Figure 4.5: ∂D with the M lines

for each $s = 1, 2, \ldots, L$, we define:

$$\xi_s(x, y, u) = \begin{cases} 1 & \text{if } (x, y) \in D_s \\ 0 & \text{otherwise.} \end{cases}$$

These functions are not continuous, but each of them is the limit of an increasing sequence of positive continuous functions,$\{\xi_{s_k}\}$; then if μ is any positive Radon measure on Ω, $\mu(\xi_s) = \lim_{k\to\infty} \mu(\xi_{s_k})$. Now consider the set $\{\xi_j : j = 1, 2, \ldots\}$ of all such functions, for all positive integer L. The linear combination of these functions can approximate a function in $C_1(\Omega)$ arbitrary well, in the sense that the essential supremum of the error function can be made to tend to zero by choosing in an appropriate manner, a sufficient number of terms in the corresponding expansion (see [50] chapter 5).

As a result, the problem (4.12) can be replaced by another one in which we are looking for the measure $\mu \in \mathcal{M}^+(\Omega)$, so that it satisfies the following constraints:

$$\begin{aligned} \mu(F_i) &= \gamma_i, \qquad i = 1, 2, \ldots; \\ \mu(\xi_j) &= a_j, \qquad j = 1, 2, \ldots. \end{aligned} \tag{4.20}$$

where the functions ψ_i's and ξ_j's belong to the above mentioned total sets, and

$$F_i \equiv F_{\psi_i} \; , \; \gamma_i \equiv \gamma_{\psi_i} \; , \; a_j \equiv a_{\xi_j}.$$

To approximate the system of equations in (4.20) with a finite system of equations, we choose a finite number of equations and thus set up the following finite linear system of equations:

$$\begin{aligned} \mu_{M_1,M_2}(F_i) &= \gamma_i, \qquad i = 1, 2, \ldots, M_1; \\ \mu_{M_1,M_2}(\xi_j) &= a_j, \qquad j = 1, 2, \ldots, M_2, \end{aligned} \tag{4.21}$$

where M_1 and M_2 are two positive integers. If we denote by $Q(M_1, M_2)$ the set of positive Radon measures in $\mathcal{M}^+(\Omega)$ which satisfy equalities (4.21), and also denote by Q the set of positive Radon measures in $\mathcal{M}^+(\Omega)$ which satisfy equalities (4.12), by regarding the property of the total sets one can easily prove the following Proposition by considering the proof of Proposition $III.1$ in [50].

Proposition 15 : *If $M_1, M_2 \longrightarrow \infty$; then $Q(M_1, M_2) \longrightarrow Q$, hence for the large enough numbers M_1 and M_2 the set Q can be identified by $Q(M_1, M_2)$.*

Therefore, instead of seeking a measure $\mu \in Q$ we prefer to seek the measure $\mu_{M_1,M_2} \in Q(M_1, M_2)$; but even if the number of equations in (4.21) is finite, the underlying space $Q(M_1, M_2)$ is still not finite-dimensional. It is possible to define finite linear systems whose solutions can be used to approximate that for (4.21). A measure μ_{M_1,M_2} in the set $Q(M_1, M_2)$ can be characterized by a result of Rosenbloom [48], which was proved in Theorem $A.5$ Appendix in [50], that μ_{M_1,M_2} in (4.21) has the form

$$\mu_{M_1,M_2} = \sum_{n=1}^{M_1+M_2} \alpha_n \delta(Z_n), \tag{4.22}$$

with triples $Z_n \in \Omega$ and the coefficients $\alpha_n \geq 0$ for $n = 1, 2, \ldots, M_1 + M_2$, where $\delta(z) \in \mathcal{M}^+(\Omega)$ is supposed to be a unitary atomic measure with support the singleton set $\{z\}$.

This structural result points the way toward a further approximation scheme; the measure problem is equivalent to a nonlinear one in which the unknowns are the coefficients α_n and supports $\{Z_n\}, n = 1, 2, \ldots, M_1 + M_2$. It would be more convenient if one could find the solution only with respect to the coefficients α_n in (4.22); this would be a linear system of equations (a type of linear programming problem). The answer lies in approximating this support, by introducing a set dense in Ω. Proposition $III.3$ of [50] Chapter 3, states that the measure μ_{M_1,M_2} in (4.22) has the following form

$$\mu_{M_1,M_2} = \sum_{n=1}^{N} \alpha_n \delta(Z_n), \tag{4.23}$$

where $Z_n, n = 1, 2, \ldots, N$, belongs to a dense subset of Ω. Note that the elements $Z_n, n = 1, 2, \ldots, N$, are fixed; the only unknowns are the numbers $\alpha_n, n = 1, 2, \ldots, N$.

Now let put a discretization on Ω, with the nodes $Z_n = (x_n, y_n, u_n), n = 1, 2, \ldots, N$, in a dense subset of Ω; then we can set up the following linear system in which the unknowns are the coefficients $\alpha_n, n = 1, 2, \ldots, N$:

$$\begin{aligned} \alpha_n \geq 0, \qquad & n = 1, 2, \ldots, N; \\ \sum_{n=1}^{N} \alpha_n F_i(Z_n) = \gamma_i, \qquad & i = 1, 2, \ldots, M_1; \\ \sum_{n=1}^{N} \alpha_n \xi_j(Z_n) = a_j, \qquad & j = 1, 2, \ldots, M_2. \end{aligned} \tag{4.24}$$

We remind the reader that we should not be surprised if we find more than one solution for the problem, (even if the problem (4.1) and (4.2) satisfies the necessary conditions for having a unique bounded generalized solution). It is true that, in this case, μ_u in (4.18) is also unique by Proposition 4.2; but remember that the generalized solution must satisfy the equality (4.5) for all $\psi \in H_0^1(D)$; there we have chosen just a finite M_1 number of them for (4.21) and also for (4.24), to obtain the measure μ_{M_1,M_2}. Thus μ_{M_1,M_2} may not be unique because of this reduction. Each solution introduces a measure μ_{M_1,M_2} via the equality (4.23) which has the same properties (approximately) as the measure μ_u, the representative measure for the generalized solution $u(X)$. Indeed we achieve an approximate solution for the elliptic problem in the given domain D.

We have shown in this section how to find the representative set of a domain $D \in \mathcal{D}$ and then approximate it, and hence D, by a finite set of its boundary points (the M-representation of D). As a result of this, one can obtain a solution (approximately) for the problem (4.1) and (4.2) for any given domain $D \in \mathcal{D}_M$, via the related linear system. Therefore we are able to calculate the value of $\mathrm{I}(D)$ for each given domain D. In the next section, we shall explain how one can find the optimal domain for the functional (4.3) in $\mathcal{D}_M$ by applying the results of this section.

4.4 The optimal solution

The main aim of the present section is to find an optimal domain $D^* \in \mathcal{D}_M$ so that the value of $\mathrm{I}(D^*)$ in (4.3) will be the minimum on the set $\mathcal{D}_M$. In the other words, we are going to identify the lowest value of $\mathrm{I}(D)$ for every admissible domain D to determine its related minimizer domain D^*. The process of finding an approximation to D^*, is as follows:

Each $D \in \mathcal{D}_M$ is an M-approximated domain with the mentioned M-representation set, like $\{A_1, A_2, \ldots, A_M,\}$ as explained above. By applying the result of the previous section, a solution of (4.1) and (4.2) can be found as a function of the finite number of unknowns (the finite unknown components of an unknown M-representation). Thus we will be able to calculate $\mathrm{I}(D)$ for every $D \in \mathcal{D}_M$; and hence, we can define the following function that it is a function of finite number M of variables,

$$\mathbf{J} : D \in \mathcal{D}_M \longrightarrow \mathrm{I}(D).$$

By applying a standard minimization algorithm on $\mathbf{J}$, the optimal value of the variables (optimal M-representation) will be obtained. These values identify the optimal domain D^* for (4.3). Indeed, instead of identifying the optimal domain D^*, we are going to determine its M-representation

$$\{A_m = (x_m^*, Y_m), m = 1, 2, \ldots, M\},$$

or other words, the components, $x_1^*, x_2^*, \ldots, x_M^*$.

To calculate $\mathrm{I}(D)$ for an arbitrary $D \in \mathcal{D}_M$, it is necessary to obtain a solution for (4.1) and (4.2) in D. This solution is approximated by a solution of the linear system

(4.24) according to the variables, $x_m, m = 1, 2, \ldots, M$. As mentioned in section 4.3, the solution of (4.24) is not necessary unique. Let us to specify one of them for each D; there are some possibilities, for example, by solving the following linear programming problem, one may chose that one in which the value of $\int_D f_\circ(X, u)dX$ (for a given D) is minimum according to the variables $\alpha_n, n = 1, 2, \ldots, N$:

$$
\begin{aligned}
&\text{Minimize:} && \sum_{n=1}^{N} \alpha_n f_\circ(Z_n) && \\
&\text{Subject to:} && \alpha_n \geq 0, && n = 1, 2, \ldots, N; \\
& && \sum_{n=1}^{N} \alpha_n F_i(Z_n) = \gamma_i, && i = 1, 2, \ldots, M_1; \\
& && \sum_{n=1}^{N} \alpha_n \xi_j(Z_n) = a_j, && j = 1, 2, \ldots, M_2. \qquad (4.25)
\end{aligned}
$$

For the given examples in the present Chapter, the solution will be specified by applying a certain subroutine for solving the system.

As a result, for each D, the value of $\mathbf{I}(D)$ below:

$$\mathbf{I}(D) = \int_D f_\circ(X, u)\, dX \equiv \mu(f_\circ) \simeq \mu_{M_1, M_2}(f_\circ),$$

is defined uniquely in terms of the variables $x_m, m = 1, 2, \ldots, M$.

So, for an arbitrary domain $D \in \mathcal{D}_M$, we approximate $\mathbf{I}(D) \cong \sum_{n=1}^{N} \alpha_n f_\circ(Z_n)$ in the mentioned manner uniquely. In other words, we set up a function, $\mathbf{J}$, on $\mathcal{D}_M$ in which for each $D \in \mathcal{D}_M$ shows a value for $\mathbf{I}(D)$:

$$\mathbf{J} : D \in \mathcal{D}_M \longrightarrow \mathbf{I}(D) \cong \mu_{M_1, M_2}(f_\circ) \in \mathbb{R}; \qquad (4.26)$$

here, in the sense of (4.23), $\mu_{M_1,M_2}(f_\circ) = \sum_{n=1}^{N} \alpha_n f_\circ(Z_n)$. Clearly **J** is a function of the variables $x_1, x_2, \ldots, x_M$, and hence can be regarded as a vector function:

$$\mathbf{J} : (x_1, x_2, \ldots, x_M) \in \mathbb{R}^M \longrightarrow \mu_{M_1,M_2}(f_\circ) \in \mathbb{R}. \tag{4.27}$$

It is not possible in general to ascertain continuity properties of this function (see for instance [44]); we can say, however, that, since this is a real-valued function which is bounded below, and is defined on a compact set (since constraints are to be put in the variables) it is possible to find a sequence of points P_i say so that the value of the function along the sequence tends to the (finite) infimum of the function. The coordinate values corresponding to the points in the sequence are of course finite. The same properties can be ascertained for similar functions to be found in the next two Chapters.

Now, suppose that $(x_1^*, x_2^*, \ldots, x_M^*)$ is the minimizer of the vector function **J**; it can be identified by using one of the related minimization methods (for instance the method introduced by Nelder and Mead, see [60] and [42]). For this, one can apply standard Algorithms and Routines (like $AMOEBA$ [47] or $EO4JAF$-NAG Library Routine). Let $D^* \in \mathcal{D}_M$ be the domain which is introduced by the minimizer $(x_1^*, x_2^*, \ldots, x_M^*)$; indeed, it is defined according to its M-representation, the set

$$\{A_m^* = (x_m^*, Y_m), m = 1, 2, \ldots, M\}.$$

We assume in the following theoretical result that the minimization algorithm used (such as $AMOEBA$) is perfect; that is, that it comes out with the *global minimum* of J in its (compact) domain. (The same, rather optimistic, assumption, is made in deriving some related results in the following Chapters). Thus,

Theorem 3 : *Let M, M_1 and M_2 be the given positive integer numbers which were defined in section 4.3, and D^* be the minimizer of (4.27) as mentioned above. Then D^* is the minimizer domain of the functional (4.3) over $\mathcal{D}_M$ and the value of $\mathbf{I}(D^*)$ can be approximated by $\mathbf{J}(D^*)$; moreover $\mathbf{J}(D^*) \longrightarrow \mathbf{I}(D^*)$ as M_1 and M_2 tend to infinity.*

Proof: Suppose D^* is not the minimizer of $\mathbf{I}$ in (4.3); hence at least there exists a domain, call D', in $\mathcal{D}_M$ so that

$$\mathbf{I}(D') < \mathbf{I}(D^*). \tag{4.28}$$

Proposition 14 shows that there is a unique measure, call μ', in $\mathcal{M}^+(\Omega)$ so that $\mathbf{I}(D') = \mu'(f_\circ)$. In the other hand, Proposition 15 states that for sufficiently large numbers M_1 and M_2, $\mu'(f_\circ)$ can be approximated by $\mu'_{M_1,M_2}(f_\circ)$ in $Q(M_1, M_2)$. Thus, by the definition of the function $\mathbf{J}$, we have

$$\mathbf{I}(D') \cong \mu'_{M_1,M_2}(f_\circ) = \mathbf{J}(D').$$

In the same way, one can show that $\mathbf{J}(D^*)$ approximates $\mathbf{I}(D^*)$; so

$$\mathbf{I}(D^*) \cong \mu^*_{M_1,M_2}(f_\circ) = \mathbf{J}(D^*).$$

Therefore from (4.28) we have

$$\mathbf{J}(D') < \mathbf{J}(D^*),$$

which is in contrary with the fact that D^* is the minimizer of $\mathbf{J}$. Consequently, D^* is the minimizer of $\mathbf{I}$ over $\mathcal{D}_M$. Moreover, from Proposition 15 it follows that $\mathbf{J}(D^*)$ tends to $\mathbf{I}(D^*)$ as $M_1, M_2 \longrightarrow \infty$. $\square$

4.5 Numerical Examples

For the next two sets of examples, we consider the elliptic equations (4.1) and (4.2) for which for each admissible domain $D \in \mathcal{D}_M$ the function $v(x, y)$ (the fixed control function) is defined as:

$$v(x,y) = \begin{cases} 1 & \text{if } (x,y) \in D \cap C \\ 0 & \text{otherwise,} \end{cases}$$

where C is the square $[\frac{1}{4}, \frac{3}{4}] \times [\frac{1}{4}, \frac{3}{4}]$ (see Figure 4.6); then the right-hand-side of the first set of equations in (4.24), γ_i, is

$$\gamma_i = \int_{D \cap C} \psi_i \, dX \; ; \; i = 1, 2, \ldots, M_1. \tag{4.29}$$

As explained in section 4.3, an admissible domain like $D \in \mathcal{D}_M$ is bounded by its boundary, ∂D, which includes a union of three segments of lines and a simple curve between the points $A(0, 1)$ and $B(1, 1)$ (see Figure 4.4). For a fixed number M, this curve and therefore the domain was defined by the set of M points (the M-representation set) $\{A = (x_m, Y_m), m = 1, 2, \ldots, M\}$ with the known components $Y_1, Y_2, \ldots, Y_M$. In the following examples, we take $M = 8$ and also

$$Y_1 = 0.15 \; , \; Y_2 = 0.25 \; , \; Y_3 = 0.35 \; , \; Y_4 = 0.45 \; ,$$

$$Y_5 = 0.55 \; , \; Y_6 = 0.65 \; , \; Y_7 = 0.75 \; , \; Y_8 = 0.85$$

(see Figure 4.6); hence each $x_1, x_2, \ldots, x_8$, defines a domain . By an extra constraint on $x_2, x_3, \ldots, x_7$,

$$x_m \geq \frac{3}{4} \, ; \, m = 2, 3, \ldots, 7,$$

the calculation of (4.29) will be simple and the valve of γ_i for any $D \in \mathcal{D}_M$ is defined as

$$\gamma_i = \int_{\frac{1}{4}}^{\frac{3}{4}} \int_{\frac{1}{4}}^{\frac{3}{4}} \psi_i(x, y) \, dx dy \, ; \, i = 1, 2, \ldots, M_1.$$

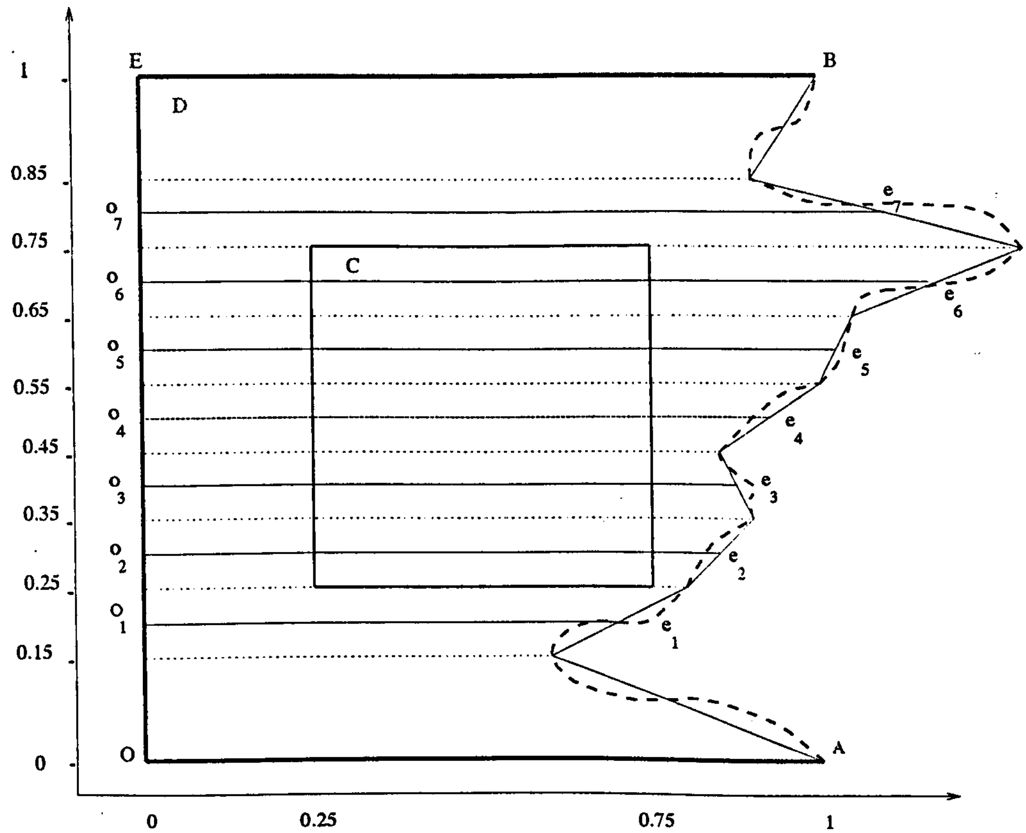

Figure 4.6: An admissible domain D under the assumptions of the numerical work

We also assume that the function $u(\cdot)$ takes value in the bounded set $U = [-1, 1]$; therefore $\Omega = \overline{D} \times U$, for each given domain $D \in \mathcal{D}_M$. One may obtain the set U by trial and error so as to be sure that the appropriate finite linear system in 4.24 has a solution.

Our way to find an optimal domain for functional (4.3) is an iterative method. For a given domain D, in other words for the given set of variables $x_1 = X_1, x_2 = X_2, \ldots, x_8 = X_8$, in the M-representation form, we will set up the linear system (4.24) to find the solution of the related elliptic equations (4.1) and (4.2), which is necessary to calculate the value of $\mathrm{I}(D)$ according to the X_m's. Then the standard minimization algorithm changes the value of $X_1, X_2, \ldots, X_8$, to new ones for which the value of $\mathrm{I}(D)$ is supposed to be less than previous; by these new values introduce a new domain. Again, in the next iteration, an appropriate linear system (4.24) for the new domain will be solved to calculate the value of $\mathrm{I}(D)$ and see whether $\mathrm{I}(D)$ is smaller than the previous on in the former iteration or not. In the next iteration, if the value is not smaller, the Algorithm changes the domain with the suitable one; if it has been smaller, the Algorithm seeks again for the other domain like $D' \in \mathcal{D}_M$ with the smaller value of $\mathrm{I}(D')$ than $\mathrm{I}(D)$. The iteration will be stopped whenever the optimal domain is obtained; note that we assume in this discussion that the standard minimization Algorithm ($AMOEBA$) is qualified to obtain the global minimizer without any restriction (see Appendix C).

Now for a given domain D with the given values $x_1 = X_1, x_2 = X_2, \ldots, x_8 = X_8$, we must consider an appropriate discretization on Ω for solving the linear system (4.24); because our method is iterative, the discretizations depends on the values $X_1, X_2, \ldots, X_8$ at each iteration.

4.5.1 Discretization

To establish the linear system (4.24) for a given domain D with the M-representation $\{A = (x_m, Y_m), m = 1, 2, \ldots, M\}$, we need to put a discretization on Ω. For this reason, we select $N = 740$ nodes $Z_n = (x_n, y_n, u_n)$ in Ω, so that each component is a rational number; hence these nodes belong to a dense subset of Ω and therefore the statement (4.23) and consequently, the linear system (4.24) can be determined. Since $u_{|\partial D} = 0$ (the Dirichlet condition (4.2)), for each $(x_n, y_n) \in \partial D$, we should have $Z_n = (x_n, y_n, 0)$. This fact has been taken into account in the discretization by choosing the following 36 nodes as follows:

The below 10 points of $\mathbb{R}^3$ in which their projection on the (x, y)-plane, belong to the line $y = 0$,

$$Z_1 = (0,0,0)\,,\ Z_2 = (0.15,0,0)\,,\ Z_3 = (0.25,0,0)\,,\ldots,\ Z_9 = (0.85,0,0)\,,$$

$$Z_{10} = (1,0,0);$$

the points $Z_{11}, Z_{12}, \ldots, Z_{19}$, so that their projection on the (x, y)-plane locate on the line $x = 0$,

$$Z_{11} = (0,0.15,0), Z_{12} = (0,0.25,0), \ldots, Z_{18} = (0,0.85,0), Z_{19} = (0,1,0);$$

the following nodes that their projection on the (x, y)-plane is on the line $y = 1$,

$$Z_{20} = (0.15,1,0), Z_{21} = (0.25,1,0), \ldots, Z_{27} = (0.85,1,0), Z_{28} = (0,1,0);$$

and finally 8 points corresponded to the M-representation set as

$$Z_{29} = (X_1,0.15,0), Z_{30} = (X_2,0.25,0), \ldots, Z_{36} = (X_8,0.85,0).$$

The rest of the nodes are related to the interior points of D. We consider $Z_n = (x_n, y_n, u_n)$ for $n = 37, 38, \ldots, 740$ as

$$Z_n = Z_{36+88(i-1)+11(j-1)+k}$$

where

$$x_{36+88(i-1)+11(j-1)+k} = \frac{(i+0.5)X_j}{10} \ , \ y_{36+88(i-1)+11(j-1)+k} = Y_j,$$

$$u_{36+88(i-1)+11(j-1)+k} = \frac{2(k-1)}{10} - 1$$

for $i = 1, 2, \ldots, 8, j = 1, 2, \ldots, 8$, and $k = 1, 2, \ldots, 11$. Indeed the value of x_n's are one of the following values:

$$0.15X_j \ , \ 0.25X_j \ , \ldots , \ 0.85X_j;$$

and the component u_n takes one of the below numbers:

$$-1, -0.8, \ldots, -0.2, 0, 0.2, \ldots, 0.8, 1.$$

Now, the set Ω is covered by a grid, defined by taking all points in Ω with coordinates $Z_n = (x_n, y_n, u_n), n = 1, 2, \ldots, 740$, which have been already expressed. To solve the corresponded linear system (4.24) it is necessary to identify its equations first.

4.5.2 Equations in the linear system

To find an approximated solution for (4.1) and (4.2) in the domain$D$, we consider the mentioned linear system in (4.24) for $M_1 = 10$ and $M_2 = 8$. Thus, for the first set of equations, the function ψ_i has been defined by (4.19) as

$$\psi_i(x, y) = xy(y-1)(x - X_1 + y - Y_1)(x - X_2 + y - Y_2) \ldots$$
$$(x - X_8 + y - Y_8) \quad q_i(x, y).$$

where the polynomial $q_i(x,y)$ for each $i = 1, 2, \ldots, 10$, is selected as follows:

$$q_1(x,y) = 1, q_2(x,y) = x, q_3(x,y) = y, q_4(x,y) = x^2, q_5(x,y) = xy,$$

$$q_6(x,y) = y^2, q_7(x,y) = x^3, q_8(x,y) = x^2y, q_9(x,y) = xy^2, q_{10}(x,y) = y^3.$$

For the second set of equations in (4.24), we divide the domain D into 8 parts, say $D_1, D_2, \ldots, D_8$, as follows:

D_1 is the region of D between the lines $y = 0$ and $y = 0.2$ (OAe_1o_2 in Figure 4.6),
D_2 is the region of D between the lines $y = 0.2$ and $y = 0.3$ ($o_1e_1e_2o_2$ in Figure 4.6),
D_3 is the region of D between the lines $y = 0.3$ and $y = 0.4$ ($o_2e_2e_3o_3$ in Figure 4.6),
D_4 is the region of D between the lines $y = 0.4$ and $y = 0.5$ ($o_3e_3e_4o_4$ in Figure 4.6),
D_5 is the region of D between the lines $y = 0.5$ and $y = 0.6$ ($o_4e_4e_5o_5$ in Figure 4.6),
D_6 is the region of D between the lines $y = 0.6$ and $y = 0.7$ ($o_5e_5e_6o_6$ in Figure 4.6),
D_7 is the region of D between the lines $y = 0.7$ and $y = 0.8$ ($o_6e_6e_7o_7$ in Figure 4.6),
D_8 is the region of D between the lines $y = 0.8$ and $y = 1$ (o_7e_7BE in Figure 4.6),
where the $x-$component of the points $e_l, l = 1, 2, \ldots, 7$, in the $(x, y)-$plane is

$$x_{e_l} = \frac{1}{2}(X_{l+1} - X_l) + X_l \, ; \; l = 1, 2, \ldots, 7.$$

Therefore the value a_j, the right-hand-side of the second set of equations in (4.24), is

$$a_j = \int_D \xi_j(x,y)\, dX = \text{area of } Dj \, ; \; \forall j = 1, 2, \ldots, 8;$$

thus by some calculation one can get:

$$a_1 = 0.15 + 0.075(X_1 - 1) + 0.05X_1 + 0.025(x_{e_1} - X_1),$$

$$a_2 = 0.05x_{e_1} + 0.025(X_2 - x_{e_1}) + 0.05X_2 + 0.025(x_{e_2} - X_2),$$

$$a_3 = 0.05x_{e_2} + 0.025(X_3 - x_{e_2}) + 0.05X_3 + 0.025(x_{e_3} - X_3),$$

$$a_4 = 0.05x_{e_3} + 0.025(X_4 - x_{e_3}) + 0.05X_4 + 0.025(x_{e_4} - X_4),$$

$$a_5 = 0.05x_{e_4} + 0.025(X_5 - x_{e_4}) + 0.05X_5 + 0.025(x_{e_5} - X_5),$$

$$a_6 = 0.05x_{e_5} + 0.025(X_6 - x_{e_5}) + 0.05X_6 + 0.025(x_{e_6} - X_6),$$

$$a_7 = 0.05x_{e_6} + 0.025(X_7 - x_{e_6}) + 0.05X_7 + 0.025(x_{e_7} - X_7),$$

$$a_8 = 0.05x_{e_7} + 0.025(X_8 - x_{e_7}) + 0.15X_8 + 0.075(1 - X_8).$$

Hence in our case, the linear system (4.24) is

$$\begin{aligned} &\alpha_n \geq 0, && n = 1,2,\ldots,740; \\ &\sum_{n=1}^{740} \alpha_n F_i(Z_n) = \gamma_i, && i = 1,2,\ldots,10; \\ &\sum_{n=1}^{740} \alpha_n \xi_j(Z_n) = a_j, && j = 1,2,\ldots,8. \end{aligned} \tag{4.30}$$

To find the nonnegative unknowns α_n's we apply the $E04MBF - NAG$ Library Routine Document. Although this Routine is usually used for finding the minimizer of a linear programming problem, it is also suitable for finding an admissible solution of a linear system (like (4.30)) by selecting $LINOBJ = .FALSE.$, when the objective function, $CVES$, is not referenced. The result shows a nonnegative value for each $\alpha_n, n = 1,2,\ldots,740$, that satisfy the linear system. By applying these values in (4.23), one can calculate the value of I(D) for a given function $f_\circ$. As mentioned in Section 4.4, this value, I(D), is a function of the variables $X_1, X_2, \ldots, X_8$; thus we have set up the function **J** in (4.27). By applying a standard minimization algorithm ($AMOEBA$) we are going to obtain the optimal domain in $\mathcal{D}_M$ for (4.3). We remind the reader that the functions F_i and also the values of γ_i, $i = 1, 2, \ldots, 10$, has been calculated by the package "*Maple V.3*".

4.5.3 Minimization and penalty functions

Up to now, the function **J** in (4.27) has been established as a function of the variables $X_1, X_2, \ldots, X_8$. We apply the Downhill Simplex Method in Multidimension by using the Subroutine $AMOEBA$ (see [47]) with the conditions $X_1 \geq 0, X_8 \geq 0$ and $X_m \geq 0.75, m = 2, 3, \ldots, 7$; besides these conditions, we also consider an upper bound for variables, for example suppose they are not higher than 2. These conditions are applied by means of a penalty method to change the constraint minimization problem into an unconstrainted one (for instance see [60]). There are several possibilities for applying this method; one may define the same penalty function as Walsh in [60] (like the example of the previous Chapter), or may apply the transformation function (see [6] and [15]). We apply the following penalty function; let

$$T_1 = \begin{cases} \max(0.000001 - X_m, 0) & \text{if } m = 1 \text{ or } m = 2 \\ \max(0.750001 - X_m, 0) & \text{if } m = 2, 3, \ldots, 7, \end{cases}$$

and

$$T_2 = \max(X_m - 1.99999, 0);$$

then we consider

$$P_m(X_m) = \sqrt{T_1} + \sqrt{T_2}\,;\ \forall m = 1, 2, \ldots, 8.$$

The penalty function $\mathbf{P}(X_1, X_2, \ldots, X_8)$ is defined as:

$$\mathbf{P}(X_1, X_2, \ldots, X_8) = \sum_{m=1}^{8} P_m(X_m).$$

Then we change the form of the objective function into the new one for $AMOEBA$ that is shown by $\mathbf{J}'$ as follows;

$$\mathbf{J}'(D) = \begin{cases} 10^7 & \text{if } \mathbf{P}(X_1, X_2, \ldots, X_8) \neq 0 \\ \mathbf{J(D)} & \text{if } \mathbf{P}(X_1, X_2, \ldots, X_8) = 0. \end{cases}$$

If one of the constraints is violated, the value of $\mathbf{J}'(D)$ will be 10^7 which is too big; to suppress, the minimization algorithm will ignore this value by finding new values for $X_1, X_2, \ldots, X_8$, that satisfy in all constraints to achieve the value of $\mathbf{J}(D)$ which is much less than 10^7.

To start, $AMOEBA$ needs an initial value for variables X_m, when $m = 1, 2, \ldots, 8$, (a given domain). Each time that $AMOEBA$ needs to calculate a value for the objective function , $\mathbf{J}'$, the linear system (4.30) with the conditions $\alpha_n \geq 0$ for $n = 1, 2, \ldots, 740$, must be solved. At any iteration the new domain is illustrated and the new value for $\mathbf{J}'$ is calculated; comparing this value with the previous one leads the algorithm to find a domain with a smaller value of the objective function in the new iteration. This procedure is repeating till the optimal domain is characterized. In the next, two examples are given; one for the linear case and the other for the nonlinear case of the elliptic equation in (4.1) and (4.2). We chose the function $f_\circ$ as:

$$f_\circ = (u - 0.1)^2,$$

this function, indeed, can be considered as a distribution of heat in the surface for the system governed by an elliptic equations.

4.5.4 Example 1

In the linear case defined by the partial differential equations (4.1), (4.2) and $f(x, y, u) = 0$, the function F_i in (4.30) is

$$F_i = u\Delta\psi_i\,;\; i = 1, 2, \ldots, 10.$$

We used the initial values $X_m = 1.0, m = 1, 2, \ldots, 8$, as a given domain for starting the algorithm; also the stopping tolerance for the program (variable $ftol$ in the Subroutine $AMOEBA$) has been chosen as 10^{-7}. Here are the results:

- The optimal value of **I** $= 0.70469099432415$;
- The number of iterations $= 827$;
- The value of the variables in the final step:

  $X_1 = 1.033028, X_2 = 1.390598, X_3 = 1.422364, X_4 = 0.97706,$

  $X_5 = 1.017410, X_6 = 0.958974, X_7 = 1.018387, X_8 = 0.951333.$

These values represent the optimal domain. The initial and the final domain has been shown in the Figure 4.7, and also the alteration of the objective function, according to the number of iterations, has been plotted in the Figure 4.8.

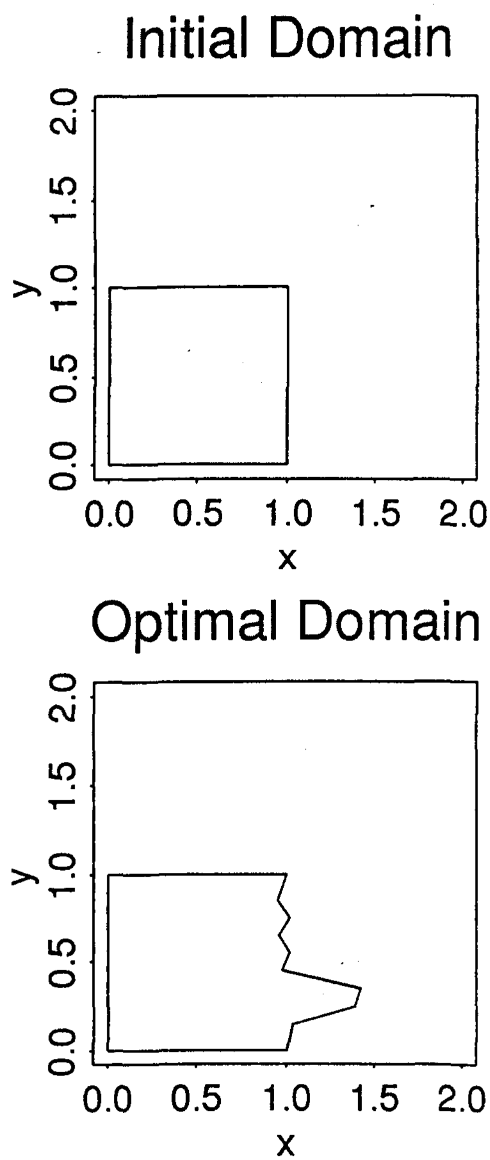

Figure 4.7: The initial and the optimal domain for the starting initial values $X_m = 1, m = 1, 2, \ldots, 8$, in the linear case.

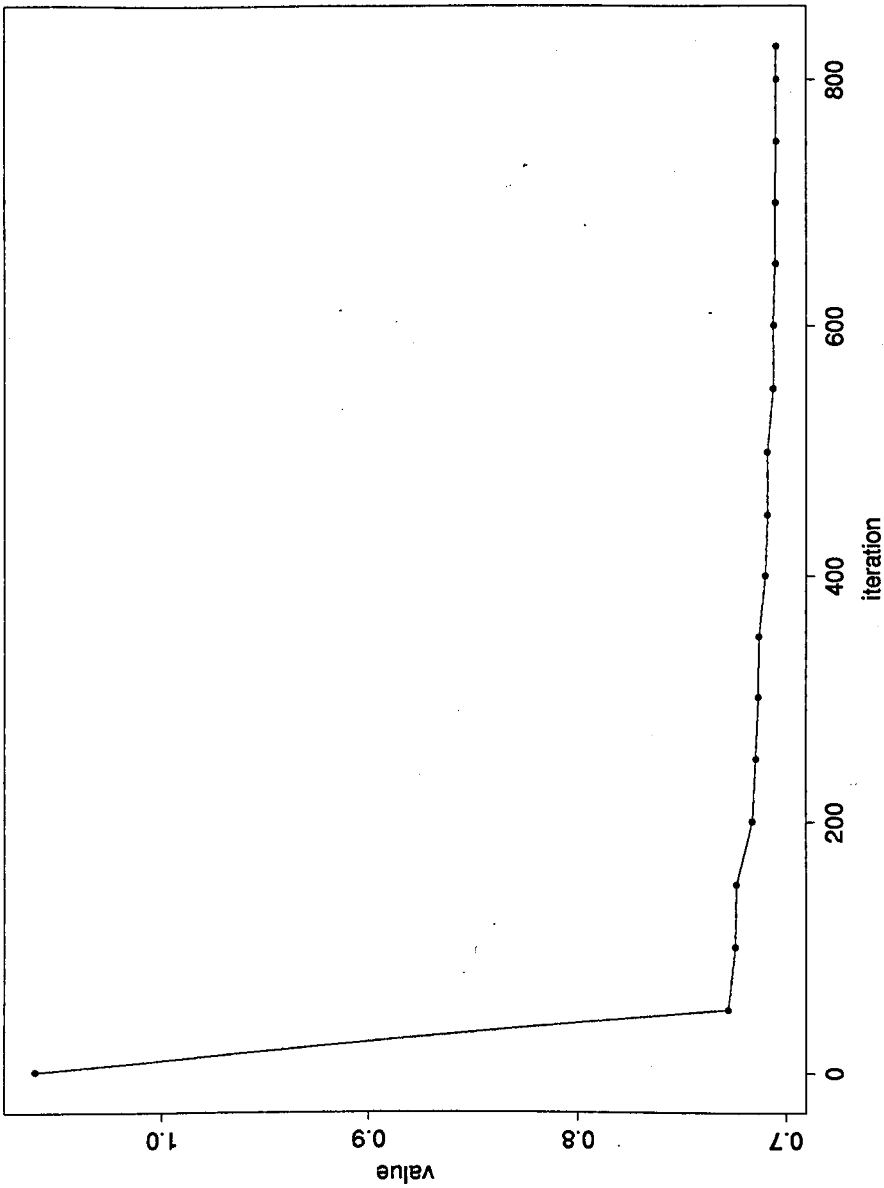

Figure 4.8: Changes of the objective function according to iterations in the linear case with the starting valves $X_m = 1, m = 1, 2, \ldots, 8$.

Also we applied the initial values $X_m = 1.1, m = 1, 2, \ldots, 8$, and ran the same program with this; the obtained results are:

- The optimal value of $\mathbf{I} = 0.85045367617752$;
- The number of iterations $= 389$;
- The value of the variables in the final step:

  $X_1 = 1.150384, X_2 = 1.081058, X_3 = 1.076638, X_4 = 1.078285,$

  $X_5 = 1.714226, X_6 = 1.050096, X_7 = 1.107141, X_8 = 1.014125.$

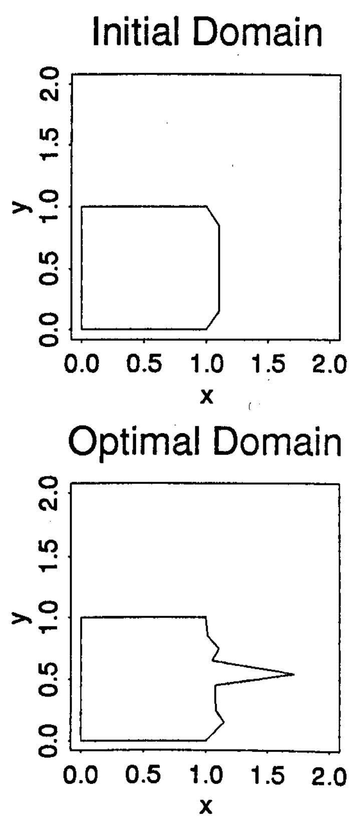

Figure 4.9: The initial and the optimal domain for the starting initial values $X_m = 1.1, m = 1, 2, \ldots, 8$, in the linear case.

These values represent the optimal domain which has been shown in the Figure 4.9 with the initial domain. The changes of the objective function according to the number of iterations was also plotted in the Figure 4.10.

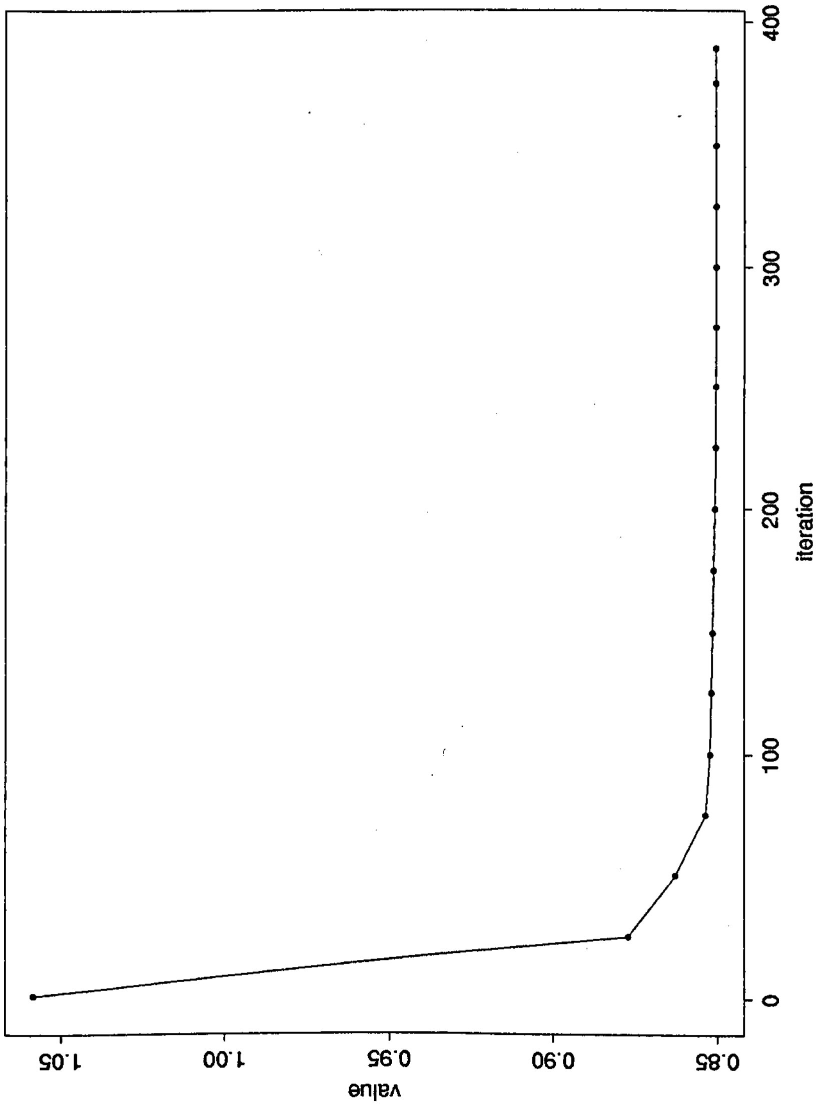

Figure 4.10: Changes of the objective function according to iterations in the linear case with the starting valves $X_m = 1.1, m = 1, 2, \ldots, 8$.

We remind the reader that the difference between the two values of the objective function and variables associated with the use of two different initial values (in this Example and also in Example 2), is caused by some limitation in the Subroutine $AMOEBA$ and it is not in relation with the method used. This fact will be discussed in Appendix C.

4.5.5 Example 2

For the nonlinear case of the partial differential equations (4.1) and (4.2), we have taken $f(x, y, u) = 0.25u^2$. As in *Example* 1 we used the initial values $X_m = 1.0, m = 1, 2, \ldots, 8$, as a given domain for starting the algorithm with the same value for stopping tolerance. The obtained results are:

- The optimal value of **I** $= 0.45467920356379$;
- The number of iterations $= 502$;
- The value of the variables in the final step:

  $X_1 = 1.050197, X_2 = 1.085212, X_3 = 0.750001, X_4 = 0.768701,$

  $X_5 = 1.129861, X_6 = 1.137751, X_7 = 0.977838, X_8 = 1.615668,$

which represent the optimal domain. The initial and the final domain has been shown in the Figure 4.11, and also the change of the objective function, according to the number of iterations, has been plotted in the Figure 4.12.

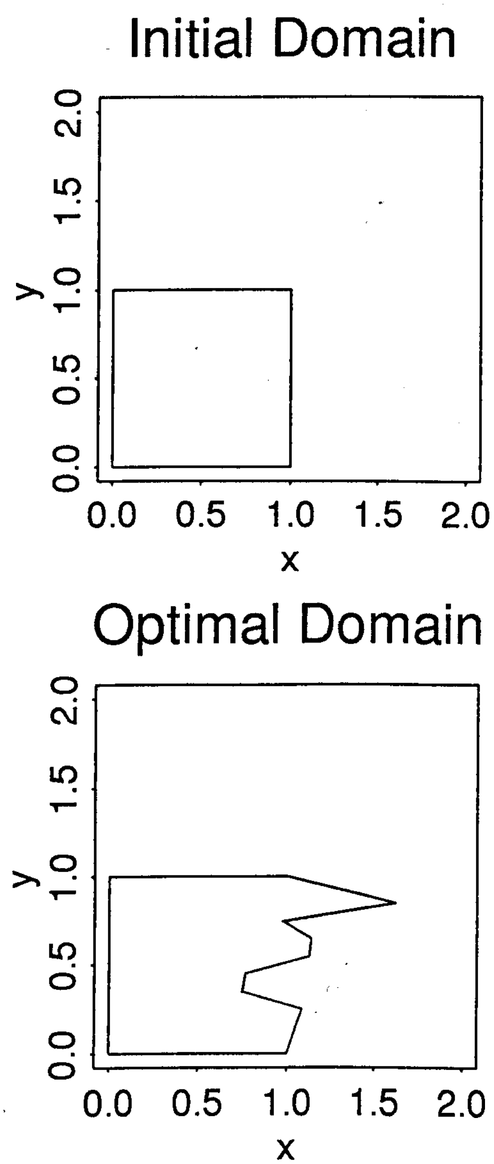

Figure 4.11: The initial and the optimal domain for the starting initial values $X_m = 1, m = 1, 2, \ldots, 8$, in the nonlinear case.

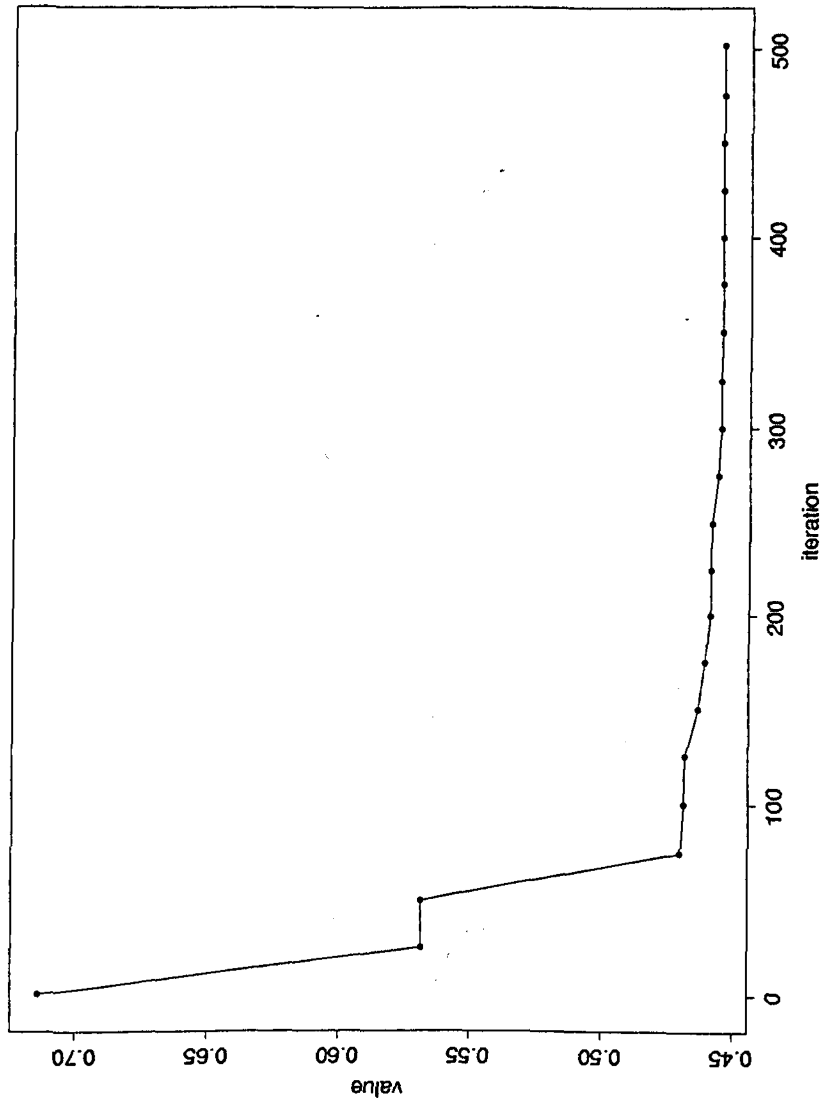

Figure 4.12: Changes of the objective function according to iterations in the nonlinear case with the starting valves $X_m = 1, m = 1, 2, \ldots, 8$.

As in *Example* 1, we chose the initial values $X_m = 1.1, m = 1, 2, \ldots, 8$, and ran the program again with this initial; here are the results:

- The optimal value of $\mathbf{I} = 0.40243494655212$;

- The number of iterations $=$ *586*;

- The value of the variables in the final step:

  $X_1 = 0.825538, X_2 = 0.952122, X_3 = 0.923957, X_4 = 0.96417,$

  $X_5 = 1.358162, X_6 = 1.088290, X_7 = 1.250303, X_8 = 1.884825,$

these values represent the optimal domain which has been shown in the Figure 4.13 with the initial domain. The changes of the objective function according to the number of iterations was also plotted in the Figure 4.14.

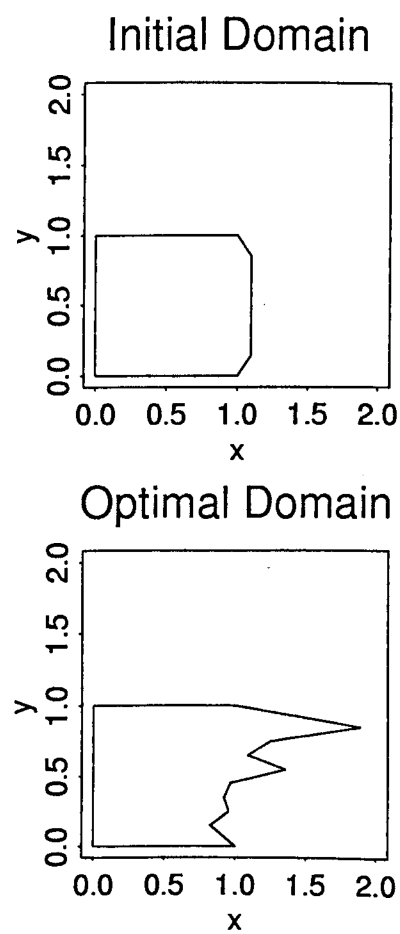

Figure 4.13: The initial and the optimal domain for the starting initial values $X_m = 1.1, m = 1, 2, \dots, 8$, in the nonlinear case.

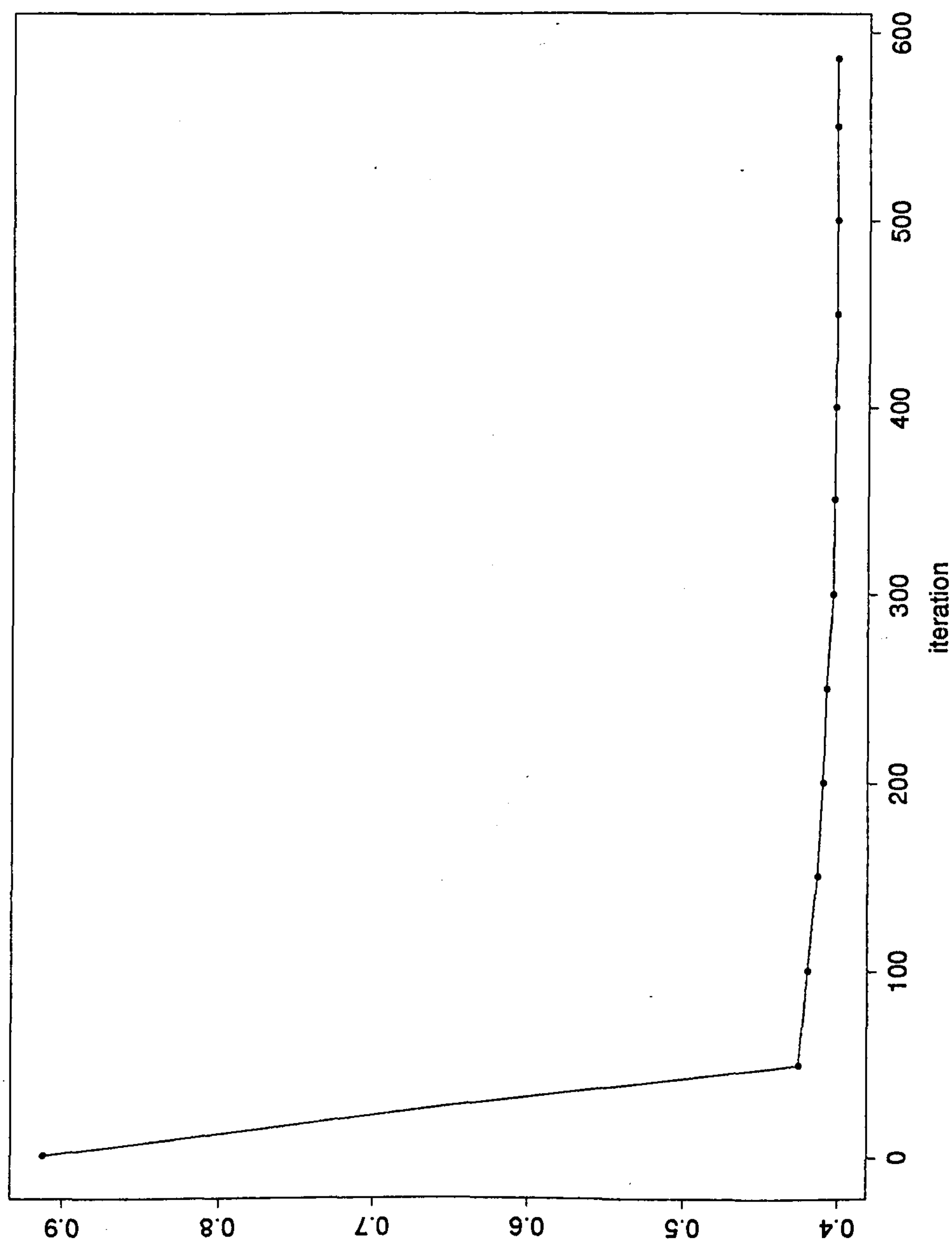

Figure 4.14: Changes of the objective function according to iterations in the nonlinear case with the starting valves $X_m = 1.1, m = 1, 2, \ldots, 8$.

Chapter 5

Shapes, Measures and Elliptic Equations (Variable Control)

5.1 Introduction

In this chapter, as in the previous one, we assume $D \subset \mathbb{R}^2$ to be a bounded domain with a piecewise-smooth, closed and simple boundary ∂D which consists of a fixed and a variable part. The fixed part is a union of three segments, part of the line $y = 0$ between the points $(1, 0)$ and $(0, 0)$, part of the line $x = 0$ between the points $(0, 0)$ and $(0, 1)$, and part of the line $y = 1$ between the points $(0, 1)$ and $(1, 1)$. The variable part is a curve Γ with the initial and the final points $A = (1, 0)$ and $B = (1, 1)$ respectively; Γ is a simple curve but not a closed one; it does not cut itself between the points A and B (see Figure 4.4).

A domain D as defined above, and the pair (D, v) are called admissible if the elliptic equation

$$\Delta u(X) + f(X,u) = v(X) \tag{5.1}$$

with the boundary condition

$$u_{|\partial D} = 0, \tag{5.2}$$

has a (unique) bounded solution on the domain D. We remind the reader that the functions $u(\cdot)$ and $f(\cdot,\cdot)$ have the same properties as in the previous chapter; however, $v : D \longrightarrow \mathbb{R}$ is a Lebesgue measurable function which is defined as a bounded *Distributed control* function. This function is assumed to take values on the bounded set V.

The set of all admissible domains is denoted by $\mathcal{D}$; indeed it contains all mentioned admissible domains like D for all possible curves like Γ. It was explained in the previous chapter how an admissible domain D (or in other words Γ), can be defined by a countable dense subset of its points, called the *representative set of* D. Then, by means of the procedure of approximating a curve with broken lines, these countable points, and hence Γ, is approximated with a number M of its points; this was called the *M-representation of* D. For a fixed number M, without losing generality, the points in the M-representation set can have the y-components fixed, like $y_m = Y_m, m = 1, 2, \ldots, M$ (see chapter 4, section 4.3). Thus an admissible domain $D \in \mathcal{D}$ can be identified by its M-representation set such as:

$$\{A_m = (x_m, Y_m), m = 1, 2, \ldots, M\}.$$

In this way, for a given fixed M, we replace $\mathcal{D}$ with $\mathcal{D}_M$, the set of all admissible M-representations; we also call $\mathbf{F}$ the set of all admissible pairs (D, v) such that $D \in \mathcal{D}_M$. Let $f_1, f_2 : D \times \mathbb{R} \longrightarrow \mathbb{R}$ be two functions in $C(D \times \mathbb{R})$, further, we assume that there is a constant $K > 0$ so that the function f_1 satisfies

$$| f_1(X, u(X)) | \leq K | u |, \tag{5.3}$$

for all pairs $(X, u(X))$ where $X \in D$. The present chapter is going to identify the minimizer pair of domain and control, $(D^*, v^*_{D^*})$ for the functional

$$\mathbf{I}(D, v) = \int_D f_1(X, u(X))\, dX + \int_D f_2(X, v(X))\, dX, \tag{5.4}$$

over the set $\mathbf{F}$. This optimal pair will be characterized in two stages:

- (i) In the first stage, we are going to determine the optimal control function for each given domain. For fixed domain $D \in \mathcal{D}_M$, or in other words for a fixed values of $x_1 = X_1, x_2 = X_2, \ldots, x_M = X_M$, we will use the generalized form of the equations (5.1) and (5.2) to introduce the classical form of the optimal control problem. Then the problem will be changed into a measure-theoretical one. The new problem has a solution because of existence theorems. We also replace the problem with an infinite dimensional linear programming one, and then approximate it by a finite one. Hence the optimal control and a solution of (5.1) and (5.2) will be characterized (approximately) from the solution of the appropriate finite linear programming. Thus, at the end of this stage, we will be able to determine the nearly optimal control function, v^*_D, for the given domain D; also we can calculate the minimum value of the performance criterion for any given domain like D, $\mathbf{I}(D, v^*_D)$, in terms of the finite number of variables $X_1, X_2, \ldots, X_M$.

- (ii) We have shown (in the Chapter 4) that each domain $D \in \mathcal{D}_M$ and hence each control function $v : D \longrightarrow \mathbb{R}$ defined on D, is a function of the variables $X_1, X_2, \ldots, X_M$; also in the stage one, we calculate the value of $\mathbf{I}(D, v_D^*)$ in terms of these variables. To introduce the optimal pair $(D^*, v_{D^*}^*)$ for the functional $\mathbf{I}$ in (5.4), in the second stage, we will define a function, say $\mathbf{J}$,

$$\mathbf{J} : D \in \mathcal{D}_M \longrightarrow \mathbf{I}(D, v_D^*) \in \mathbb{R},$$

which is a vector function with the variables $X_1, X_2, \ldots, X_M$. Then, by applying an iterative standard minimization algorithm, like the Algorithm introduced by Nelder and Mead [42], we will obtain $x_1^*, x_2^*, \ldots, x_M^*$, the global minimizer of the function $\mathbf{J}$. This minimizer which, indeed, is an M-representation, shown by the values $x_1^*, x_2^*, \ldots, x_M^*$, introduces the (nearly) optimal shape (domain), call it D^*. Then, in the manner which has been explained in the first stage, the associated suboptimal control function to the domain D^*, say $v_{D^*}^*$, will be determined, The pair $(D^*, v_{D^*}^*)$ will be the minimizer of the of the functional $\mathbf{I}$ over the set $\mathbf{F}$.

The new method has some advantages:

- An automatic existence theorem: there always is a minimizer for the measure-theoretical problem.
- The problem is changed into a linear one even if the performance criterion is non-linear: then one can use the whole paraphanalia of linear analysis for dealing with such problem; thus the computation is much easier.
- Our minimization is global: the value reached, say, numerically is close to what one could reasonably call the global infimum of the problem (here it is supposed that the standard minimization algorithm gives us the global minimizer).
- The optimal shape (domain) and the optimal control function can be determined at the same time.

In the last section of this chapter, some numerical examples for the linear and nonlinear cases of the elliptic equations, will be given. These examples will show how the method is applied.

5.2 The optimal shape design problem

In order to define an optimal shape design problem, it is necessary to describe its several components, such as the (partial) differential equation satisfied by the controlled system, a function to be minimized with respect to a particular geometrical element (performance criterion) and the admissible space in which the minimization takes place. We have already defined in the Introduction section all the necessary components for the optimal shape design problem which we are going to solve.

In the present Chapter, we seek in the admissible set F, for the minimizer pair of domain and control function, $(D^*, v^*_{D^*})$, for the functional (5.4), so that the elliptic equation (5.1) with the Dirichlet condition (5.2), is satisfied. Indeed we are going to find the solution of the following (classical) shape design problem over the set of admissible pairs, F.

$$\begin{aligned}
\text{Minimize:} \quad & \mathrm{I}(D,v) = \int_D f_1(X,u(X))\,dX + \int_D f_2(X,v(X))\,dX \\
\text{Subject to:} \quad & \Delta u(X) + f(X,u) = v(X); \\
& u_{|\partial D} = 0. \qquad (5.5)
\end{aligned}$$

As we mentioned in Chapter 4, in general, it is difficult to characterize a classical solution for the elliptic equations (5.1) and (5.2). By applying the variational form of the elliptic problem (see the Proposition 4.1), we prefer to obtain a bounded weak solution (generalized solution) of the problem; so the functions $u(\cdot)$ and $v(\cdot)$ in (5.1) and(5.2) must satisfy the general equalities mentioned in (4.5). Consequently, the optimal shape design problem (5.5) changes into the following one:

$$\begin{aligned} \textbf{Minimize:} \quad & \mathbf{I}(D,v) = \int_D f_1(X,u(X))\,dX + \int_D f_2(X,v(X))\,dX \\ \textbf{Subject to:} \quad & \int_D (u\Delta\psi + \psi f)\,dX = \int_D \psi v\,dX\,;\ \forall \psi \in H_0^1(D). \end{aligned} \tag{5.6}$$

To solve the above optimal shape design problem, in the first step we will find out how one can calculate the minimum value of $\mathbf{I}(D,v)$ for a given domain $D \in \mathcal{D}_M$, subject to the mentioned conditions. In other words, for a given domain $D \in \mathcal{D}_M$ the optimal shape design problem becomes an optimal control problem; hence one should find an optimal pair of trajectory and control functions which satisfy the conditions of (5.6). Then the minimum value for $\mathbf{I}(D,v)$ can be calculated. Afterwards, in the next step, it is possible to look for an admissible domain D^* which gives the minimal value $\mathbf{I}(D^*, v^*_{D^*})$ between the domains in $\mathcal{D}_M$. In the following section we will characterize the optimal pair of trajectory and control functions for a given domain $D \in \mathcal{D}_M$, according to the mentioned conditions in (5.6).

5.3 The control problem for a fixed domain

In this section we suppose that $D \in \mathcal{D}_M$ is a given admissible domain. For this fixed domain, the optimal shape design problem (5.6) changes into a classical optimal control problem which is to find a pair of trajectory function , u, and the control function, v, so that they satisfy the following conditions:

$$\int_D (u\Delta\psi + \psi f)\,dX = \int_D \psi v\,dX\,;\ \forall \psi \in H_0^1(D), \tag{5.7}$$

and minimizes the function

$$(\mathrm{I}(D,v) \equiv)\mathrm{i}(u,v) = \int_D f_1(X,u(X))\,dX + \int_D f_2(X,v(X))\,dX. \qquad (5.8)$$

In the present section we are going to find the minimizer pair of functions,by solving the classical control problem. We will change the problem into the measure-theoretical one and identify its related space; this new formulation has some advantages. By applying Rosenbloom theorem [48] and discretization method, the problem will be approximated by a finite linear programming one in which its result identifies the trajectory and the optimal control function for the given domain approximately.

5.3.1 The classical optimal control problem

In the sense of the classical form of a control problem, we assume that the function $u : D \longrightarrow \mathbb{R}$ is the trajectory and the function $v : D \longrightarrow \mathbb{R}$ is the control function. An admissible pair of trajectory and control function is defined as follows.

Definition 11 : *A pair of the functions* (u,v) *is called* admissible *if:*

- *i) The trajectory function* $u \in H^1(D)$ *is bounded and takes values in the bounded set* U*; moreover here we assume that* U *is the intersection of all such bounded sets.*
- *ii) The trajectory function is zero on the boundary of* D *(i.e. on* ∂D *).*
- *iii) The function* v *is the bounded control function which takes values on the bounded set* V*. This function also is supposed to be Lebesgue-measurable on* D*.*

- *iv) The functions u and v satisfy the condition (5.7) for every $\psi \in H_0^1(D)$.*

The set of all admissible pairs is denoted by $\mathcal{F}$.

We suppose that the set $\mathcal{F}$ is nonempty. In fact, we assume that the elliptic equation (5.1) and (5.2) has a bounded weak solution on D. So, for a fixed domain the optimal shape design problem (5.6) changes into the following optimal control problem over $\mathcal{F}$.

$$\begin{aligned} &\text{Minimize:} && \mathrm{i}(u,v) = \int_D f_1(X,u(X))\,dX + \int_D f_2(X,v(X))\,dX \\ &\text{Subject to:} && \int_D (u\Delta\psi + \psi f)\,dX = \int_D \psi v\,dX\ ;\ \forall\psi \in H_0^1(D). \end{aligned} \tag{5.9}$$

Problems may arise in the quest for the finding the optimal pair; it is difficult to determine the solution of the elliptic equations although we know it exists, there is no comprehensive method to identify an admissible pair (u, v). There may be many methods which estimate numerically the generalized solution (trajectory function) for a fixed distributed control function (see for instance [12] and [24]). But it is difficult to find a general applicable approximation method to estimate numerically the optimal control and its related generalized solution at the same time for a problem like (5.9). Also it is difficult to prove that these methods can find the global minimum for the problem.
We therefore change the problem and consider a new one with different formulation. The basis of this metamorphosis is the fact that an admissible pair (u, v) can be considered as *something else*, that is, a transformation can be established between the admissible pairs and other mathematical entities; this transformation is an injection. It is possible then to set up an applicable method for calculating the image of an admissible pair under the transformation.

5.3.2 Metamorphosis

In general, the minimization of the functional i in (5.9) over $\mathcal{F}$ may not be possible since an optimal control may not exist; even it exists, there is no comprehensive way to characterize the optimal pair either numerically. In the following, by replacing the problem (5.9) with another one, the minimizer of the functional i will be calculated over a set of pairs of positive Radon-measures; these pairs should have some properties which can be deduced from the definition of an admissible pair of control and trajectory functions.

The basis of this metamorphosis consists of replacing the pair (u, v) of an admissible trajectory and control functions with a pair of positive Radon measures. Any weak solution of (5.1) and (5.2) defines a positive and linear functional like

$$u(\cdot) : F \longrightarrow \int_D F(X, u(X))\, dX$$

on $C(\Omega)$, that $\Omega = D \times U$; also a control function v defines a positive and linear functional like

$$v(\cdot) : G \longrightarrow \int_D G(X, v(X))\, dX$$

on $C(\omega)$, that $\omega = D \times V$.

We remind the reader that the transformation between admissible pairs (u, v) in $\mathcal{F}$ and the pairs of linear functionals $(u(\cdot), v(\cdot))$ defined above, is an injection; one can show it easily by using the same method as Rubio did in [50]. Now by applying the Riesz Representation Theorem (see for instance [55]) for the functionals $u(\cdot)$ and $v(\cdot)$, one can deduce the following Proposition.

Proposition 16 : *For each admissible pair $(u, v) \in \mathcal{F}$ there is a pair of positive Radon* *measures* (μ, ν), $\mu \in \mathcal{M}^+(\Omega)$, $\nu \in \mathcal{M}^+(\omega)$, *so that*

$$\begin{aligned} \mu(F) &= \int_D F(X, u(X))\, dX; \qquad \forall F \in C(\Omega), \\ \nu(G) &= \int_D G(X, v(X))\, dX; \qquad \forall G \in C(\omega). \end{aligned} \tag{5.10}$$

Proof: The proof is similar to the Proposition 1, thus it is omitted. □

Definition 12 :*The pair of measures* (μ, ν) *defined in the Proposition 16 is called a* representing pair of measures.

By applying the mentioned transformation between the set of admissible pairs, $\mathcal{F}$, and the set of all representing pairs of measures, the new form of the problem (5.9) is as follows:

$$\begin{aligned} &\text{Minimize :} \qquad \mathrm{i}(\mu, \nu) = \mu(f_1) + \nu(f_2) \\ &\text{Subject to :} \qquad \mu(F_\psi) + \nu(G_\psi) = 0\,;\ \forall \psi \in H_0^1(D). \end{aligned} \tag{5.11}$$

where the functions $F_\psi \in C(\Omega)$ and $G_\psi \in C(\omega)$ are defined as

$$F_\psi = u\Delta\psi + \psi f\,,\ G_\psi = -\psi v. \tag{5.12}$$

So far, we have not achieved anything new, and just changed only the appearance of the problem; nothing else. We will extend the problem and shall consider the minimization of (5.11) over the set of all pairs of measures in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ satisfying the mentioned conditions in (5.11) for all $\psi \in H_0^1(D)$, plus the extra properties which are deduced from the definition of admissible pairs (u, v). These properties indicate that the measures μ and ν project on the (x, y)-plane as the respective Lebesgue measures.

In fact, if a function

$$\xi : \Omega \longrightarrow \mathbb{R}$$

in $C(\Omega)$ depends only on variable $X = (x, y)$ (i.e. $\xi \in C_1(\Omega)$), then

$$\mu(\xi) = \int_D \xi(x,y)\, dX = a_\xi, \tag{5.13}$$

the Lebesgue integral of ξ over D. Also, if a function

$$\zeta : \omega \longrightarrow \mathbb{R}$$

in $C(\omega)$ depends only on variable X (i.e. $\zeta \in C_1(\omega)$), then

$$\nu(\zeta) = \int_D \zeta(x,y)\, dX = b_\zeta, \tag{5.14}$$

the Lebesgue integral of ζ over D. Therefore, instead of solving the optimal control problem (5.9), we are going to solve the following measure-theoretical one in the space $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$:

$$\begin{aligned}
&\text{Minimize :} && \mathrm{i}(\mu,\nu) = \mu(f_1) + \nu(f_2) && \\
&\text{Subject to :} && \mu(F_\psi) + \nu(G_\psi) = 0, && \forall \psi \in H_0^1(D); \\
& && \mu(\xi) = a_\xi, && \forall \xi \in C_1(\Omega); \\
& && \nu(\zeta) = b_\zeta, && \forall \zeta \in C_1(\omega).
\end{aligned} \tag{5.15}$$

As mentioned before, in the classical form of the optimal shape design problem the minimization of the performance criterion in (5.8) over the set $\mathcal{F}$ may not be possible, the infimum may not attained at any admissible pair; also, it is difficult to write the necessary conditions for the problem. If the minimizer pair does exist, it may difficult to characterize it or estimate it numerically. But in the nonclassical optimal control problem, which has been formulated as a measure-theoretical problem in (5.15), there are *three* major characteristics which make the new formulation more effective. Let the subset of $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ which satisfy the conditions in (5.15), be denoted by Q, then,

- (1) The existence of an optimal pair of measures in Q, minimizing the functional $(\mu, \nu) \longrightarrow \mu(f_1) + \nu(f_2)$ is guaranteed because of the automatic existence theorem. We shall deal with this fact below.

- (2) The functional $(\mu, \nu) \longrightarrow \mu(f_1) + \nu(f_2)$ and other functions appearing in (5.15) are linear in their arguments, the measures μ and ν, even for those problems normally classed as nonlinear. So the computational methods for getting the solution are simpler.

- (3) Since the set $\mathcal{F}$ can be considered, by means of the transformation, as a subset of Q, therefore,

$$\inf_Q \mathrm{i}(\mu, \nu) \leq \inf_{\mathcal{F}} \mathrm{i}(u, v);$$

  thus in (5.15) the minimization is *global*, that is, the global infimum of the problem can be approximated.

5.3.3 Existence

We intend to prove the existence of an optimal pair of measures, say (μ^*, ν^*), in the set Q for the function $(\mu, \nu) \longrightarrow \mu(f_1) + \nu(f_2)$ under the conditions on the functions and the sets of the problem given in (5.15). Let assume that the set of measures Q is not empty. In other words, the elliptic equation (5.1) and (5.2) has a bounded generalized solution for a given bounded control function (as mentioned in Chapter (4)) and hence

$\mathcal{F} \neq \phi$, as supposed before. Even though of course the set Q may nonempty while the set of the pairs of trajectory u and control v (i.e. the set $\mathcal{F}$) is empty (one of the advantages of the theoretical-measure formulation). Existence theorems have a very heavy topological content; we note that no topology has yet been put on the set Q, or the (linear) space $\mathcal{M}(\Omega) \times \mathcal{M}(\omega)$ of all Radon measures, positive or otherwise, or on the spaces Ω and ω. One must try to find a topology on the space $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ so that Q is compact in this topology and the function $(\mu, \nu) \longrightarrow \mu(f_1) + \nu(f_2)$ is continuous. This topology can be the *weak*-topology* on

$$S = \left\{(\mu, \nu) \ : \ (\mu, \nu) \in \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)\right\};$$

for more information, the reader is advised to see Chapter 2.

Proposition 17 : *The set Q of measures (μ, ν), defined as those measures in S satisfy the conditions of (5.15), is compact.*

Proof : Let the space $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ be topologized by the weak*-topology and define

$$S_0 = \{(\mu, \nu) \ : \ \mu(1_\Omega) - \nu(1_\omega) \leq W\}$$

where W is a positive fixed number. The set S_0 is a compact set (see for instance [11]). Proposition 16 shows that

$$\mu(1_\Omega) = L = \nu(1_\omega)$$

where L is defined as the Lebesgue measure of D; therefore Q is a subset of the compact set S_0. Moreover the set Q is closed, since one can write

$$Q = \bigcap_{\psi \in H_0^1(D)} \{(\mu, \nu) \ : \ \mu(F_\psi) + \nu(G_\psi) = 0\},$$

where the set $\{(\mu, \nu) \ : \ \mu(F_\psi) + \nu(G_\psi) = 0\}$ is the inverse image of a closed set on the real line, the set $\{0\}$, under the continuous map $(\mu, \nu) \in Q \longrightarrow \mu(F_\psi) + \nu(G_\psi) \in \mathbb{R}$. Since Q is a closed subset of a compact set, it is compact. □

Theorem 4 : *There exists an optimal pair of measures (μ^*, ν^*) in the set Q so that it is the minimizer of* $\mathrm{i}(\mu, \nu)$ *in (5.15).*

Proof : The function $(\mu, \nu) \in Q \longrightarrow \mu(f_1) + \nu(f_2) \in \mathbb{R}$, mapping the compact set Q on the real line, is a continuous function (one may show it easily as the same way as Rubio in [50]). Thus it attains its minimum on the compact set Q by Proposition $II.1$ of [50]; thus there exists a pair of measures $(\mu^*, \nu^*) \in Q$, such that:

$$\mathrm{i}(\mu^*, \nu^*) \leq \mathrm{i}(\mu, \nu)$$

for all $(\mu, \nu) \in Q$. □

Up to now, it has been shown that the problem (5.15) has an optimal solution. In the next we shall explain how this optimal solution could be characterized approximately of course. In the end of this subsection, we present the following Proposition which will be used later to prove the important theorem of approximation.

Proposition 18 : *The set $Q_1 \subset Q$ of measures associated with (u, v) which are piecewise-constant trajectory and control function on Ω and ω respectively and satisfy the mentioned conditions in (5.15), is weak*-dense in Q.*

Proof : See the proof of Theorem 7.1 of Kamyad, Rubio and Wilson in [31]. □

5.3.4 Approximation

Up to here we have developed an infinite-dimensional program by considering minimization of $\mathrm{i}(\mu,\nu)$, over the set Q in (5.15). Now we intend to consider the minimization of $\mathrm{i}(\mu,\nu)$ not over the set Q, but over a subset of $\mathcal{M}^+(\Omega)\times\mathcal{M}^+(\omega)$ defined by requiring that only a finite number of constraints in (5.15) be satisfied; then the solution of the problem (5.15) will be achieved by choosing a countable sets of functions that are *total* in the spaces $H_0^1(D)$, $C_1(\Omega)$ and $C_1(\omega)$, that is, so that the linear combinations of these functions are uniformly dense (dense in the topology of uniform convergence) in the appropriate spaces. The total sets in the spaces $H_0^1(D)$ and $C_1(\Omega)$ are the sets

$$\{\psi_i, i=1,2,3,\dots\}, \{\xi_j, j=1,2,3,\dots\},$$

which have been already defined in Chapter 4. Because $C_1(\Omega)\equiv C_1(\omega)$, we define $\zeta_l:\omega\longrightarrow\mathbb{R}$ so that $\zeta_{l|_D}=\xi_{l|_D}$ for all $l=1,2,3,\dots$; hence the set

$$\{\zeta_l, l=1,2,3,\dots\}$$

is total in $C_1(\omega)$. Thus the problem (5.15) can be replaced by the following one;

$$\begin{array}{llll}
\text{Minimize:} & \mathrm{i}(\mu,\nu)=\mu(f_1)+\nu(f_2) & & \\
\text{Subject to:} & \mu(F_i)+\nu(G_i)=0, & i=1,2,3,\dots; & (5.16)\\
 & \mu(\xi_j)=a_j, & j=1,2,3,\dots; & \\
 & \nu(\zeta_l)=b_l, & l=1,2,3,\dots, &
\end{array}$$

where

$$F_i:=F_{\psi_i}, G_i:=G_{\psi_i} \text{ for } i=1,2,3,\dots,$$

$$a_j := a_{\xi_j} \text{ for } j = 1,2,3,\ldots,$$

$$b_l := b_{\zeta_l} \text{ for } l = 1,2,3,\ldots.$$

Then the important following Proposition can be deduced; its proof is similat to as the proof of the Proposition $III.1$ in [50].

Proposition 19 : *Let M_1, M_2 and M_3 be the positive integers. Consider the problem of minimizing the function*

$$(\mu,\nu) \longrightarrow \mathrm{i}(\mu,\nu)$$

over the set $Q(M_1, M_2, M_3)$ of measures in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ satisfying

$$\begin{aligned} &\mu(F_i) + \nu(G_i) = 0, && i = 1,2,\ldots,M_1;\\ &\mu(\xi_j) = a_j, && j = 1,2,\ldots,M_2;\\ &\nu(\zeta_l) = b_l, && l = 1,2,\ldots,M_3, \end{aligned}$$

then

$$\eta(M_1,M_2,M_3) = \inf_{Q(M_1,M_2,M_3)} [\mathrm{i}(\mu,\nu)]$$

tends to

$$\eta = \inf_Q [\mathrm{i}(\mu,\nu)]$$

whenever $M_1, M_2, M_3 \longrightarrow \infty$.

We remind the reader that, because the set Q in the calculation of η is an extension of the set $\mathcal{F}$, our minimization is *global*; that is, for the given D, $\eta \leq \inf_{\mathcal{F}} \mathrm{i}$; indeed, this is another advantage of the new formulation.

Note that we have limited the number of constraints in the original problem; but the underlying space is still infinite-dimensional. In fact the problem is:

$$\begin{aligned} &\text{Minimize}: && \mathrm{i}(\mu,\nu) = \mu(f_1) + \nu(f_2) && \\ &\text{Subject to}: && \mu(F_i) + \nu(G_i) = 0, && i = 1,2,\ldots,M_1; \\ & && \mu(\xi_j) = a_j, && j = 1,2,\ldots,M_2; \\ & && \nu(\zeta_l) = b_l, && l = 1,2,\ldots,M_3, \end{aligned} \tag{5.17}$$

where the minimization takes place over the set $Q(M_1, M_2, M_3)$. This problem is one of linear programming, in which all functions in (5.17) are linear in their arguments μ and ν. Indeed it is a semi-infinite linear programming problem, since the unknowns, μ and ν, are in Q. Let (μ^*, ν^*) be the minimizer pair, then for the mentioned fixed D, the optimal control, v_D^* can be characterized from the measure ν^* (see below); hence the value of $\mathrm{I}(D, v_D^*)$ is $\mathrm{i}(\mu^*, \nu^*) \equiv \eta(M_1, M_2, M_3)$.

We have reached the main point of this section; how do we construct suboptimal pairs of trajectories and controls for the functional i in (5.9)? We shall proceed in several steps:

- (i) First we shall obtain the optimal pair of measures (μ^*, ν^*) for a problem such as (5.17). The existence of such a minimizer follows from the simple considerations as the existence theorem given in Theorem 4.

- (ii) We obtain a (weak*) approximation to (μ^*, ν^*) by a set of two piecewise-constant functions (u, v) by means of the results given in Proposition 18 (see Appendix D).

- (iii) The control function v obtained above, is in $L_2(D)$, since it is piecewise-constant and D is bounded. It can give rise to a *weak solution* of the system (5.1) and (5.2) to be denoted by u_v. This solution will be in $H_0^1(D)$. Conditions for the existence of such weak solution are given in [38] and [33] for instance.

- (iv) We shall prove below that if the numbers M_1, M_2 and M_3 are sufficiently large and the weak*-approximation in step (ii) above, is sufficiently good, then the value of the functional i at the (u_v, v) defined by (5.9), $\mathrm{i}(u_v, v)$, is close to η and thus is a good suboptimal pair. Note that no use is made of the trajectory u, obtained in step (ii) together with the control v.

To prove the next theorem, we follow Farahi in [15], Rubio in [51] and especially in [53].

Theorem 5 : *Let (u_v, v) be the pair constructed as explained above. Then, under the appropriate conditions on the approximations involved, as M_1, M_2, M_3 tend to ∞, we have*

$$\mathrm{i}(u_v, v) \longrightarrow \inf_Q [\mathrm{i}(\mu, \nu)].$$

Proof: First we are going to show that for a given positive number $\epsilon > 0$, one can choose the positive integers M_1, M_2 and M_3 so that

$$|\ \mathrm{i}(u_v, v) - \inf_{Q(M_1,M_2,M_3)} \mathrm{i}(\mu, \nu)\ | < 2\epsilon. \tag{5.18}$$

Let (μ^*, ν^*) be the minimizer of the problem (5.17) over the set $Q(M_1, M_2, M_3)$ for fixed integers $M_1 > 0$, $M_2 > 0$ and $M_3 > 0$; its existence can be proved by the same arguments as the proof of the Theorem 4. We choose $\epsilon := \frac{1}{M_1}$; then by Proposition 18, because the set Q_1 is weakly* dense in Q, there exists a pair trajectory-control (u, v) so that the pair of measures $(\mu_u, \nu_v) \in Q_1$ generated by the pair (u, v) satisfies

$$| [\mu_u(f_1) + \nu_v(f_2)] - [\mu^*(f_1) + \nu^*(f_2)] | < \epsilon \tag{5.19}$$

$$\mu_u(F_i) + \nu_v(G_i) = 0 \; ; \; i = 1, 2, \ldots, M_1; \tag{5.20}$$

we note, further, that these measures satisfy the rest of the conditions in (5.17) for functions in $C_1(\Omega)$ and $C_1(\omega)$.

Now, by the manner explained in step (iii), one can obtain the weak solution, the trajectory u_v, corresponded to the control v. Let μ_{u_v} be the corresponding measure to the trajectory u_v; hence the trajectory-control pair (u_v, v) introduces the pair of measures (μ_{u_v}, ν_v) in Q. At this stage, we intend to prove that

$$| (\mu_{u_v} - \mu_u)(f_1) | < \epsilon. \tag{5.21}$$

We remind the reader that the function f_1 satisfies (5.3); thus we have

$$| (\mu_{u_v} - \mu_u)(f_1) | \leq K \, | (\mu_{u_v} - \mu_u)(\vartheta) |, \tag{5.22}$$

where $\vartheta : \Omega \longrightarrow U$ is a function defined by $\vartheta(X, u) = u$ for each $(X, u) \in \Omega$. Further, by considering the statements (5.17) and (5.20), the following equalities are satisfied for each $i = 1, 2, \ldots, M_1$:

$$\begin{aligned}
(\mu_{u_v} - \mu_u)(F_i) &= (\mu_{u_v} - \mu_u)(F_i) + [\nu_v(G_i) - \nu_v(G_i)] \\
&= [\mu_{u_v}(F_i) + \nu_v(G_i)] - [\mu_u(F_i) + \nu_v(G_i)] \\
&= 0;
\end{aligned}$$

then by considering the definition of F_i in (5.12), for each $i = 1, 2, \ldots, M_1$, we have

$$(\mu_{u_v} - \mu_u)(F_i) = (\mu_{u_v} - \mu_u)(\vartheta\Delta\psi_i + f\psi_i) = 0. \tag{5.23}$$

As explained before, the set of functions $\psi_i, i = 1, 2, 3, \ldots$, was chosen so that this set is total in the space $H_0^1(D)$; this means that the set of the linear combinations of the functions $\psi_i, i = 1, 2, 3, \ldots$, is uniformly dense in the space $H_0^1(D)$. Thus each function $\psi \in H_0^1(D)$, can be approximated by one of these linear combinations; hence if we consider $\{\psi_i : i = 1, 2, 3, \ldots\}$ as a base for the Hilbert space $H_0^1(D)$, there exist coefficients $c_i \in \mathbb{R}, i = 1, 2, 3, \ldots$, so that, if define $P = \sum_{i=1}^{\infty} c_i\psi_i$, we have

$$\sup_D \mid \psi - P \mid < \epsilon. \tag{5.24}$$

From (5.23), it is calculated that

$$\begin{aligned} \mid (\mu_{u_v} - \mu_u)(\vartheta\Delta P + fP) \mid &= \mid \sum_{i=1}^{\infty} c_i(\mu_{u_v} - \mu_u)(\vartheta\Delta\psi_i + f\psi_i) \mid \\ &= \mid \sum_{i=M_1+1}^{\infty} c_i(\mu_{u_v} - \mu_u)(\vartheta\Delta\psi_i + f\psi_i) \mid \\ &\leq \sum_{i=M_1+1}^{\infty} \mid c_i \mid\mid (\mu_{u_v} - \mu_u)(\vartheta\Delta\psi_i + f\psi_i) \mid \\ &\equiv \mathcal{O} \end{aligned} \tag{5.25}$$

by considering again the equality (5.23), the above statements shows that whenever the number M_1 is increased, the value of $\mathcal{O}$ will decrease; therefore $\frac{\mathcal{O}}{M_1} \longrightarrow 0$ when $M_1 \longrightarrow \infty$ (indeed, $\mathcal{O} \longrightarrow 0$ where $M_1 \longrightarrow \infty$). Thus from (5.24) and (5.25) one can conclude that

$$\mid (\mu_{u_v} - \mu_u)(\vartheta\Delta\psi + f\psi) \mid = o(M_1). \tag{5.26}$$

Now, choose $\psi \in H_0^1(D)$ so that on the given domain D we have

$$| f\psi | \leq \epsilon', \; | \Delta\psi - 1 | \leq \epsilon' \tag{5.27}$$

with

$$\epsilon' := \frac{\epsilon}{2K} \frac{1}{(T+1)L(\Omega)}$$

that $L(\Omega)$ is the Lebesgue measure of Ω and the positive number T will be defined later on. So, by applying (5.26), we have

$$\begin{aligned} | (\mu_{u_v} - \mu_u)(\vartheta) | &= | (\mu_{u_v} - \mu_u)[(\vartheta\Delta\psi + f\psi) - (\vartheta(\Delta\psi - 1) + f\psi)] | \qquad (5.28) \\ &\leq | (\mu_{u_v} - \mu_u)(\vartheta\Delta\psi + f\psi) | + | (\mu_{u_v} - \mu_u)(\vartheta(\Delta\psi - 1) + f\psi)] | \\ &\leq o(M_1) + | (\mu_{u_v} - \mu_u)(\vartheta(\Delta\psi - 1) + f\psi) |; \end{aligned}$$

also note that

$$| (\mu_{u_v} - \mu_u)(\vartheta(\Delta\psi - 1) + f\psi) | \leq \Psi \tag{5.29}$$

where

$$\Psi = \max[\, | \mu_{u_v}(\vartheta(\Delta\psi - 1) + f\psi) |, | \mu_u(\vartheta(\Delta\psi - 1) + f\psi) | \,].$$

Without loss of generality, here we suppose that the right-hand-side of (5.29) is equal with $| \mu_u(\vartheta(\Delta\psi - 1) + f\psi) |$. Moreover, the functions ϑ and f are assumed to be bounded on Ω; by the boundedness of Ω itself, the function $\psi \in H_0^1(D)$ is also bounded (since $C_\infty^0(D)$ is dense in $H_0^1(D)$, see Chapter 4 section 4.3). Thus the function $\vartheta(\Delta\psi-$

$1) + f\psi$ is a bounded function on Ω. Therefore

$$| \mu_u(\vartheta(\Delta\psi - 1) + f\psi) | \leq \max | \vartheta(\Delta\psi - 1) + f\psi | \mu_u(1_\Omega);$$

but $\mu_u(1_\Omega) = L(\Omega)$. As a result,

$$| \mu_u(\vartheta(\Delta\psi - 1) + f\psi) | \leq \max | \vartheta(\Delta\psi - 1) + f\psi | L(\Omega). \tag{5.30}$$

Also, the inequality (5.30) is satisfied if μ_u is changed with μ_{u_v}.

By regarding (5.29) and (5.30), inequality (5.28) implies

$$| (\mu_{u_v} - \mu_u)(\vartheta) | \leq o(M_1) + \max | \vartheta(\Delta\psi - 1) + f\psi | L(\Omega). \tag{5.31}$$

From (5.27) it is deduced that

$$\begin{aligned} | \vartheta(\Delta\psi - 1) + f\psi | &\leq | \vartheta || \Delta\psi - 1 | + | f\psi | \\ &\leq \epsilon'(| \vartheta | +1). \end{aligned} \tag{5.32}$$

The function $| \vartheta |$ is bounded on Ω, because the generalized solution is bounded; let T be the least upper bound of $| \vartheta |$ over Ω (in other words $T \equiv \max_U | \vartheta |$), then, by considering (5.32) we have

$$\max | \vartheta(\Delta\psi - 1) + f\psi | \leq \epsilon'(T + 1). \tag{5.33}$$

Hence, it is concluded from (5.31) and (5.33) that

$$| (\mu_{u_v} - \mu_u)(\vartheta) | \leq o(M_1) + \epsilon'(T + 1)L(\Omega) = \frac{\epsilon}{K}, \tag{5.34}$$

whenever M_1 is taken sufficiently high to make $o(M_1) = \frac{\epsilon}{2K}$. Now the inequality (5.21) can be deduced from (5.22) and (5.34).

Further,

$$\begin{aligned}
\mathrm{i}(u_v, v) - \inf_{Q(M_1,M_2,M_3)} \mathrm{i}(\mu,\nu) &= [\mu_{u_u}(f_1) + \nu_v(f_2)] - [\mu^*(f_1) + \nu^*(f_2)] \\
&= [\mu_{u_v}(f_1) + \nu_v(f_2)] - [\mu^*(f_1) + \nu^*(f_2)] + [\mu_u(f_1) - \mu_u(f_1)] \\
&= [\mu_u(f_1) + \nu_v(f_2)] - [\mu^*(f_1) + \nu^*(f_2)] + [\mu_{u_v}(f_1) - \mu_u(f_1)];
\end{aligned}$$

hence

$$\begin{aligned}
\mid \mathrm{i}(u_v, v) - \inf_{Q(M_1,M_2,M_3)} \mathrm{i}(\mu,\nu) \mid \leq \mid [\mu_u(f_1) + \nu_v(f_2)] - [\mu^*(f_1) + \nu^*(f_2)] \mid \\
+ \mid (\mu_{u_v} - \mu_u)(f_1) \mid
\end{aligned} \tag{5.35}$$

Applying (5.19) and (5.21) in (5.35) shows the following relation:

$$\mid \mathrm{i}(u_v, v) - \inf_{Q(M_1,M_2,M_3)} \mathrm{i}(\mu,\nu) \mid \leq 2\epsilon,$$

which is the inequality (5.18) that we were looking for. Now if $M_1, M_2, M_3 \longrightarrow \infty$, by applying the Theorem 4,

$$\lim_{M_1,M_2,M_3 \longrightarrow \infty} \mathrm{i}(u_v, v) = \inf_Q \mathrm{i}(\mu,\nu).$$

□

In this section we have explained how one can obtain the pair of trajectory and control functions for the problem (5.9). By means of the Proposition 16, the problem has been replaced by a measure-theoretical one. We have identified the corresponding pair

of optimal measures (μ^*, ν^*) as the solution of the linear programming problem in (5.17) defined by just a finite number of constraints. The measure ν^* introduces a piecewise-constant suboptimal control function, say v^*, for the problem, by means of the Proposition 18; in the Appendix D, we explain how one can obtain v^*. This control gives rise to u_{v^*}, a weak solution of the system (5.1) and (5.2). Then, Theorem 5 proved that the value of $\mathbf{i}(u_{v^*}, v^*)$ is a very close approximation for the optimal value of the performance criterion in the general case. Therefore the pair of trajectory and control functions (u_{v^*}, v^*), is the nearly optimal solution for the problem (5.9).
Indeed, for a given domain, we can characterize the suboptimal control function (defined on D of course) for the problem (5.6), say v_D^*, by applying the above procedure. Since, for a fixed domain, the optimal shape design problem (5.6) changes into the optimal control problem (5.9), and the optimal value of the functional I (i.e. $\mathbf{I}(D, v_D^*)$ can be calculated as

$$\mathbf{I}(D, v_D^*) \equiv \mathbf{i}(u_{v_D^*}, v_D^*).$$

Calculating the weak solution $u = u_{v_D^*}$ for the problem (5.1) and (5.2), defined by $v = v_D^*$, is not always easy. For the rest of the work, we need only the optimal control v_D^* (which already have) and the value of $\mathbf{I}(D, v_D^*)$ which is exactly the value of η, defined in the Proposition 19; there is no need to use the function $u_{v_D^*}$ anymore. Proposition 19 shows that we can calculate the optimal value of the functional I as

$$\mathbf{I}(D, v_D^*) \equiv \mathbf{i}(\mu^*, \nu^*) \cong \eta(M_1, M_2, M_3) \tag{5.36}$$

(To see an example about finding the weak solution for the similar case, the reference [15] is recommended for instance.)

5.3.5 Approximation by finite linear programming

The problem (5.17) is a semi-infinite linear programming problem; the number of equations is finite but the underlying space $Q(M_1, M_2, M_3)$ is not a finite-dimensional space. It is possible, therefore, to estimate the solution by a process of discretization. We remind the reader that there are several methods for treating numerically such problem which do not involve discretization, for instance Rudolph method (see for example [56] and [57]) and the Glashoff and Gustafson method (see [23]). We mention also that one may try to solve these kinds of problems in the space $\mathcal{M}^+(\Omega \cup \omega)$ or in $\mathcal{M}^+(\Omega \times \omega)$ in the appropriate manner (see for example [51]).

A pair of measures (μ, ν) in the set $Q(M_1, M_2, M_3)$ can be characterized by a result of Rosenbloom [48], which was proved in Theorem $A.5$ of an Appendix in [50], that μ and ν in (5.17) have the form

$$\mu = \sum_{n=1}^{M_1+M_2} \alpha_n \delta(Z_n) \, , \, \nu = \sum_{k=1}^{M_1+M_3} \beta_k \delta(z_k) \tag{5.37}$$

with triples $Z_n \in \Omega$, $z_k \in \omega$ and the coefficients $\alpha_n \geq 0, \beta_k \geq 0$, for $n = 1, 2, \ldots, M_1+M_2$, $k = 1, 2, \ldots, M_1+M_3$, where $\delta(t)$ is supposed to be a unitary atomic measure with support the singleton set $\{t\}$.

This structural result points the way toward a further approximation scheme; the measure problem is equivalent to a nonlinear one in which the unknowns are the coefficients α_n, β_k, and supports $\{Z_n\}$, $\{z_k\}$ for $n = 1, 2, \ldots, M_1 + M_2$, $k = 1, 2, \ldots, M_1 + M_3$. It would be more convenient if one could find the solution only with respect to the coefficients α_n and β_k in (5.37); this would be a finite linear programming problem. The answer lies in approximating this support, by introducing a set dense in Ω and ω. Proposition $III.3$ of [50] Chapter 3, states that the measures μ and ν in (5.37) have the following form

$$\mu = \sum_{n=1}^{N} \alpha_n \delta(Z_n)\ ,\ \nu = \sum_{k=1}^{K} \beta_k \delta(z_k) \tag{5.38}$$

where $Z_n, n = 1, 2, \ldots, N$, and $z_k, k = 1, 2, \ldots, K$, belongs to dense subset of Ω and ω respectively. Note that the elements $Z_n, n = 1, 2, \ldots, N$, and $z_k, k = 1, 2, \ldots, K$, are fixed; the only unknowns are the numbers $\alpha_n, n = 1, 2, \ldots, N$, and $\beta_k, k = 1, 2, \ldots, K$.

Now let us put a discretization on Ω and ω with the nodes $Z_n = (x_n, y_n, u_n)$ for $n = 1, 2, \ldots, N$, and $z_k, k = 1, 2, \ldots, K$, belong to dense subset of them; then we can set up the following finite linear programming problem in which the unknowns are the coefficients $\alpha_n, n = 1, 2, \ldots, N$, and $\beta_k, k = 1, 2, \ldots, K$:

$$\begin{aligned}
&\text{Minimize}: && \sum_{n=1}^{N} \alpha_n f_1(Z_n) + \sum_{k=1}^{K} \beta_k f_2(z_k) && \\
&\text{Subject to}: && \alpha_n \geq 0, && n = 1, 2, \ldots, N; \\
& && \beta_k \geq 0, && k = 1, 2, \ldots, K; \\
& && \sum_{n=1}^{N} \alpha_n F_i(Z_n) + \sum_{k=1}^{K} \beta_k G_i(z_k) = 0, && i = 1, 2, \ldots, M_1; \\
& && \sum_{n=1}^{N} \alpha_n \xi_j(Z_n) = a_j, && j = 1, 2, \ldots, M_2; \\
& && \sum_{k=1}^{K} \beta_k \zeta_l(z_k) = b_l, && l = 1, 2, \ldots, M_3.
\end{aligned} \tag{5.39}$$

We have shown in this section how one can obtain a solution (approximately) for the optimal control problem given in (5.9) via the finite linear programming problem mentioned in (5.39) above. Therefore we are able to characterize the optimal control v_D^* and moreover, calculate the value of $\mathbf{I}(D, v_D^*)$ for each given domain D. In the next section, we shall show how one can find out the optimal domain and its associated optimal control of course, for the optimal shape design problem shown in (5.5) by applying the result of this section.

5.4 The Optimal Shape

In the previous section we have pointed out how one can calculate the minimum value of $\mathbf{I}(D, v)$ for a given domain D. In the other words, we obtained the approximate pair of trajectory and control functions (u, v) for the optimal control problem (5.9) in which they have characterized the optimal value for the functional $\mathbf{I}(D, v)$ over $\mathcal{F}$. Considering this fact, in the present section we shall develop a procedure for finding (an approximation to) the optimal value of the same functional, over the set of all admissible domains $\mathcal{D}_M$; also, we intend now to solve the optimal shape design problem mentioned in (5.5) by determining the optimal shape and its related optimal control function, to obtain the minimum value of the performance criterion $\mathbf{I}(D, v)$ on $\mathbf{F}$. From the results of the previous section, a function of a finite number of variables, $\mathbf{J}$, will be defined, in which its minimizer will be the (weak*-approximated) optimal domain for (5.5). This minimizer, further, will be characterized by applying a standard minimization algorithm (like Nelder and Mead [42]) over the function $\mathbf{J}$, assumed to be *perfect* in the theoretical work below.

Each domain $D \in \mathcal{D}_M$, as explained in Chapter 4, is defined by a M-representation like $\{A_m = (x_m, Y_m), m = 1, 2, \ldots, M\}$ where the numbers $Y_m, m = 1, 2, \ldots, M$, are fixed; therefore each admissible domain is determined by a set of finite points. Thus, we choose a domain $D \in \mathcal{D}_M$ just by fixing the set of M numbers, the x-components, $\{x_1, x_2, \ldots, x_M\}$. By solving the appropriate finite linear programming problem in (5.19), the nearly optimal value for $\mathbf{I}(D, v)$ (i.e. $\mathbf{I}(D, v_D^*) \equiv \mathbf{i}(\mu^*, \nu^*)$) is calculated as a function of the variables $x_1, x_2, \ldots, x_M$. Consequently, one can define the function

$$\mathbf{J} : D \in \mathcal{D}_M \longrightarrow \mathbf{I}(D, v_D^*) \in \mathbb{R};$$

indeed $\mathbf{J}$ is a function of the variables $x_1, x_2, \ldots, x_M$, and hence it is a vector function. So, let us to redefine this function as follows:

$$\mathbf{J} : (x_1, x_2, \ldots, x_M) \in \mathbb{R}^M \longrightarrow \mathbf{I}(D, v_D^*) \in \mathbb{R}. \tag{5.40}$$

The global minimizer of the vector function $\mathbf{J}$, say $(x_1^*, x_2^*, \ldots, x_M^*)$, can be identified by using one of the appropriate minimization methods; one can apply the method introduced by Nelder and Mead, [60] and [42], for instance from Subroutine $AMOEBA$ in [47] or $EO4JAF$-NAG Library Routine. These routines usually need an initial set of components (initial domain) to start the process of minimization; we also suppose that they give the global minimizer.

Each time that the Algorithm wants to calculate a value for $\mathbf{J}$, a finite linear programming problem like (5.39) should be solved; thus for a given domain D, the optimal control (v_D^*) is characterized. Whenever it reaches to the minimum value for $\mathbf{J}$, the minimizer $(x_1^*, x_2^*, \ldots, x_M^*)$ (the optimal domain D^*), and therefore its associated optimal control have been obtained. So, the optimal domain and the optimal control are determined at the same time; this is the main advantage of the new method. The following Proposition shows that the value of $\mathbf{I}(D^*, v_{D^*}^*)$, obtained by the above process, is the optimal value for the functional (5.4) and hence$(D^*, v_{D^*}^*)$ is the optimal solution for the given optimal shape design problem defined in (5.5).

Proposition 20 : *Let the minimization Algorithm (for finding the minimizer of* $\mathbf{J}$ *in (5.40)), give the global minimizer* $(x_1^*, x_2^*, \ldots, x_M^*)$*. If the domain found by the minimizer is denoted by* D^**, then* $\mathbf{I}(D^*, v_{D^*}^*)$ *is the optimal value of the functional* $\mathbf{I}$ *in (5.5) and hence the pair of domain and control* $(D^*, v_{D^*}^*)$ *is optimal.*

Proof: Assume that the value of $\mathbf{I}(D^*, v_{D^*}^*)$ is not optimal, thus there exists a pair of domain and control $(D', v) \in \mathbf{F}$, where $D' \in \mathcal{D}_M$, so that:

$$\mathbf{I}(D', v) < \mathbf{I}(D^*, v_{D^*}^*).$$

Let v' be the optimal control for the problem (5.9) defined with respect to the given domain D'; then

$$\mathbf{i}(\mu^*, \nu^*) \equiv \mathbf{I}(D', v') \leq \mathbf{I}(D', v) \equiv \mathbf{i}(\mu, \nu).$$

Because the process of embedding defined above, is injective and v' is optimal, we should have $v' = v_{D'}^*$; see Proposition 19 and also Theorem 5. Therefore,

$$\mathbf{I}(D', v_{D'}^*) < \mathbf{I}(D^*, v_{D^*}^*).$$

Now, by considering the definition of the function $\mathbf{J}$, this inequality implies that $\mathbf{J}(D') < \mathbf{J}(D^*)$. Let $(x_1', x_2', \ldots, x_M')$ be the representation of the domain $D' \in \mathcal{D}_M$; by (5.40) we have

$$\mathbf{J}(x_1', x_2', \ldots, x_M') < \mathbf{J}(x_1^*, x_2^*, \ldots, x_M^*).$$

The above inequality states that $(x_1^*, x_2^*, \ldots, x_M^*)$ is not the global minimizer of $\mathbf{J}$, which is a contradiction; since, as explained above, it is supposed that the minimization Algorithm is perfect; it gives rise to the global minimizer. As a result, $\mathbf{I}(D^*, v_{D^*}^*)$ is the optimal value for functional $\mathbf{I}$ over $\mathbf{F}$, so the pair of $(D^*, v_{D^*}^*)$ is optimal. □

5.5 Numerical Examples

We will apply the method introduced in this chapter for estimating nearly optimal domains and controls in the following examples. We will make the same assumptions as in the examples in Chapter 4, same performance criterion, same discretization and so on; thus the reader can compare the results for the controlled system with those for the uncontrolled one.

It is assumed $M = 8$ and that each domain in $\mathcal{D}_M$ is characterized by the set of 8 points, $\{A_m = (x_m, Y_m), m = 1, 2, \ldots, M\}$, with the same constant Y_m's as Chapter 4 (see Figure 4.6) so that $x_m \geq 0$ for each $m = 1, 2, \ldots, M$. We assume that the functions $u(\cdot)$ and $v(\cdot)$ take value in the set $U = V = [-1, 1]$ and hence for each given domain $D \in \mathcal{D}_M$ we have $\Omega = D \times [-1, 1]$ and $\omega = D \times [-1, 1]$; one may obtain sets U and V by trial and error so as to be sure that the appropriate finite linear programming problem (5.39) has a feasible solution in each iteration. We have chosen them as above by solving the finite linear program without using the standard minimization Algorithm, and have also checked the results of the finite linear program in each iteration when used in the minimization Algorithm.

Our way to find an optimal domain for functional (5.4) is an iterative method. For a given set of variables $x_1 = X_1, x_2 = X_2, \ldots, x_8 = X_8$, in other words for the given domain D, the finite linear program (5.39) will be solved to find the optimal control and its trajectory pair for the elliptic equations (5.1) and (5.2); thus the value of $\mathrm{I}(D, v_D^*)$ is calculated. Then, the standard minimization Algorithm advises us to change the value of $X_1, X_2, \ldots, X_8$, with the new one so that the functional I, should have less value than before. These new values define a new domain for the next iteration. The process will continue till the Algorithm finds the global minimizer; note that the applied minimization Algorithm, $AMOEBA$, is supposed to cover this fact.

5.5.1 Discretization

To set up the linear programming problem (5.39) for given values $X_1, X_2, \ldots, X_8$, it is necessary first to put an appropriate discretization on the spaces Ω and ω. We select $N = 740$ nodes $Z_n = (x_n, y_n, u_n) \in \Omega$ as same as examples in Chapter 4. Also the nodes $z_k = (x_k, y_k, v_k) \in \omega$ for $k = 1, 2, \ldots, K = 1100$, are chosen as follows:

$$z_k = z_{10(i-1)+11(j-1)+l},$$

for $i = 1, 2, \ldots, 10, j = 1, 2, \ldots, 10, l = 1, 2, \ldots, 11$, where

$$\begin{aligned}
x_{10(i-1)+11(j-1)+l} &= \frac{(i-1)+0.5}{10} X_{i-1}; \\
y_{10(i-1)+11(j-1)+l} &= Y_{j-1}; \\
v_{10(i-1)+11(j-1)+l} &= \frac{2(l-1)}{10} - 1,
\end{aligned}$$

with

$$\begin{aligned}
X_0 &= \frac{1}{3}(X_1 - 1) + 1, & Y_0 &= 0.05; \\
X_9 &= \frac{1}{3}(1 - X_8) + 1, & Y_9 &= 0.95.
\end{aligned}$$

Indeed the value of x_k is one of the following values:

$$0.05X_{i-1}, 0.15X_{i-1}, \ldots, 0.95X_{i-1},$$

and the component v_k takes one of the below numbers:

$$-1, -0.8, \ldots, -0.2, 0.0, 0.2, \ldots, 0.8, 1.$$

Now, the set $\omega = D \times [-1, 1]$ is covered by a grid, defined by taking all points $z_k = (x_k, y_k, u_k)$, $k = 1, 2, \ldots, 1100$, as above. For the linear programming problem mentioned in (5.39), we also select $M_1 = 10$, number of functions ψ_i's and $M_2 = 8$, number of functions ξ_i's in which they have been defined for the examples in Chapter 4. We select $M_3 = 8$, the number of functions in $C_1(\omega)$; since $C_1(\Omega) \equiv C_1(\omega)$, we chose $\zeta_l := \xi_l$ for $l = 1, 2, \ldots, 8$. Although these functions seem to be the same, but we consider that they are applied for the different spaces, and hence they affect on the different set of points; thus they indeed are different. Therefore

$$b_l = \int_D \zeta_l(x, y)\, dX = \text{the area of } D_l(:= a_l),\ \forall l = 1, 2, \ldots, 8;$$

where a_l and D_l are defined in Chapter 4 section 4.5.

For the next examples the integrand functions f_1 and f_2 in the performance criterion in (5.4), are selected as

$$f_1(X, u) = (u - 0.1)^2 \quad , \quad f_2(X, v) = 0;$$

so, the performance criterion is precisely the same as Chapter 4 and hence it is possible for the reader to use these values for any comparison between the controlled and the uncontrolled problems. In one example, the control function is plotted as the way as described in Appendix D.

Consequently, for the following two sets of examples, the finite linear programming in (5.39) is the below one:

$$
\begin{aligned}
&\textbf{Minimize:} && \sum_{n=1}^{740} \alpha_n (u-0.1)^2 \\
&\textbf{Subject to:} && \alpha_n \geq 0, \qquad n = 1,2,\ldots,740; \\
& && \beta_k \geq 0, \qquad k = 1,2,\ldots,1100; \\
& && \sum_{n=1}^{740} \alpha_n F_i(Z_n) + \sum_{k=1}^{1100} \beta_k G_i(z_k), \qquad i = 1,2,\ldots,10; \\
& && \sum_{n=1}^{740} \alpha_n \xi_j(Z_n) = a_j, \qquad j = 1,2,\ldots,8; \\
& && \sum_{k=1}^{1100} \beta_k \zeta_l(z_k) = b_l, \qquad l = 1,2,\ldots,8.
\end{aligned}
\tag{5.41}
$$

To find the optimal value for unknowns α_n and β_k, the $E04MBF - NAG$ Library Routine Document was applied; from the resulted values, one can obtain the optimal control function. Also the result shows the optimal value of $\mathbf{I}(D, v_D^*)$ for the given D. As mentioned before, this value is a function of the variables $X_1, X_2, \ldots, X_8$. Thus the function $\mathbf{J}$ (see (5.40)) has been established. Then, by applying a standard minimization Algorithm ($AMOEBA$) we obtained the optimal domain in $\mathcal{D}_M$ and also characterized its associated suboptimal control. We remind the reader that the functions F_i and $G_i, i = 1,2,\ldots,10$, have been calculated by means of the package "*Maple V.3*".

5.5.2 Minimization and penalty functions

Up to now, the function **J** as a function of variables $X_1, X_2, \ldots, X_8$, has been established. The variables are supposed to satisfy in conditions:

$$0 \leq X_m \leq 2, m = 1, 2, \ldots, 8.$$

These conditions are applied by means of a penalty method to change the constraint minimization problem into an unconstraint one (for instance see [60]). For this reason, the following penalty function is defined; let

$$T_1 = \max(0.000001 - X_m, 0),\ T_2 = \max(X_m - 1.99999, 0);$$

then we consider

$$P_m(X_m) = \sqrt{T_1} + \sqrt{T_2};\ \forall m = 1, 2, \ldots, 8.$$

Hence the penalty function $\mathbf{P}(X_1, X_2, \ldots, X_8)$ is defined as:

$$\mathbf{P}(X_1, X_2, \ldots, X_8) = \sum_{m=1}^{8} P_m(X_m).$$

Thus the objective function for minimization, the function $\mathbf{J}'$, is shown as:

$$\mathbf{J}'(D) = \begin{cases} 10^7 & \text{if } \mathbf{P}(X_1, X_2, \ldots, X_8) \neq 0 \\ \mathbf{J}(\mathbf{D}) & \text{if } \mathbf{P}(X_1, X_2, \ldots, X_8) = 0. \end{cases}$$

Now by applying the Downhill Simplex Method in Multidimension, the Subroutine

$AMOEBA$ (see [47]), one is able to find the minimizer of the problem (5.5). To know how the penalty function and $AMOEBA$ work, the reader is recommended to read the related part of section 4.5 in Chapter 4. In the next, two sets of examples will be given, one for a linear and the other for a nonlinear case of the elliptic equation (5.1) and (5.2).

5.5.3 Example 1

As in $Example$ 1 in the previous Chapter, in the linear case we consider $f(x,y,u) = 0$, the functions F_i and G_i in (5.12) are

$$F_i = u\Delta\psi_i,\ G_i = -\psi_i v,\ i = 1,2,\dots,10.$$

We used the initial values $X_m = 1, m = 1,2,\dots,8$, as a given domain for starting the Algorithm and applied $ftol = 10^{-7}$. Here are the results:

- The optimal value of $\mathrm{I}(D,v) \;=\; 4.4447937006414 \times 10^{-3}$;
- The number of iterations $=$ 395;
- The value of the variables in the final step:
  $X_1 = 0.043932, X_2 = 0.085128, X_3 = 1.178854, X_4 = 0.003257,$
  $X_5 = 0.000717, X_6 = 0.002100, X_7 = 0.004760, X_8 = 1.292132.$

These define the optimal domain; this, together with the initial domain have been plotted in the Figure 5.1, and the obtained optimal control is plotted in the Figure 5.2. We also plotted the change of the objective function, according to the number of iterations, in the Figure 5.3.

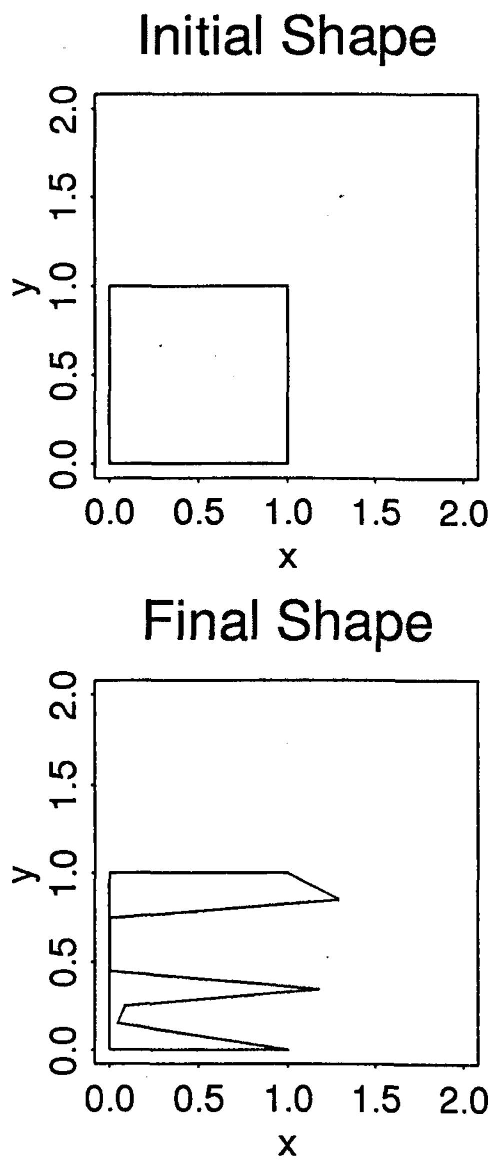

Figure 5.1: The initial and the optimal domain (for the distributed controlled system) with the starting initial values $X_m = 1.0, m = 1, 2, \ldots, 8$, in the linear case.

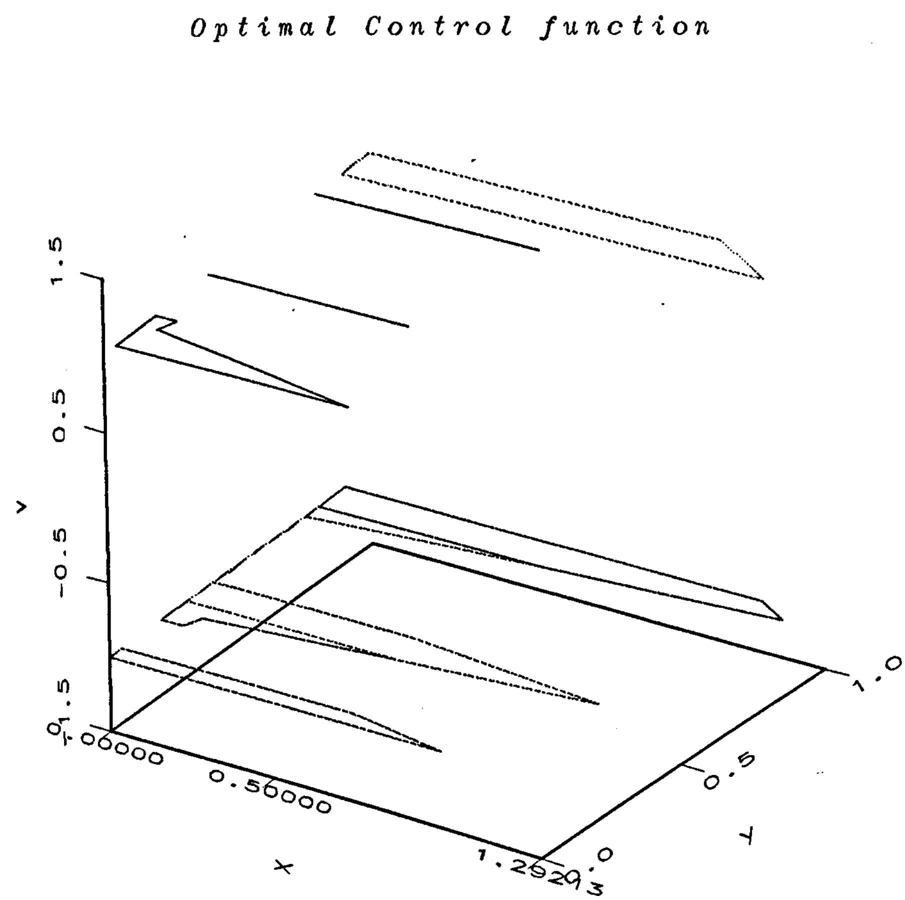

Figure 5.2: The optimal (distributed) control function for the starting initial values $X_m = 1.0, m = 1, 2, \ldots, 8$, in the linear case.

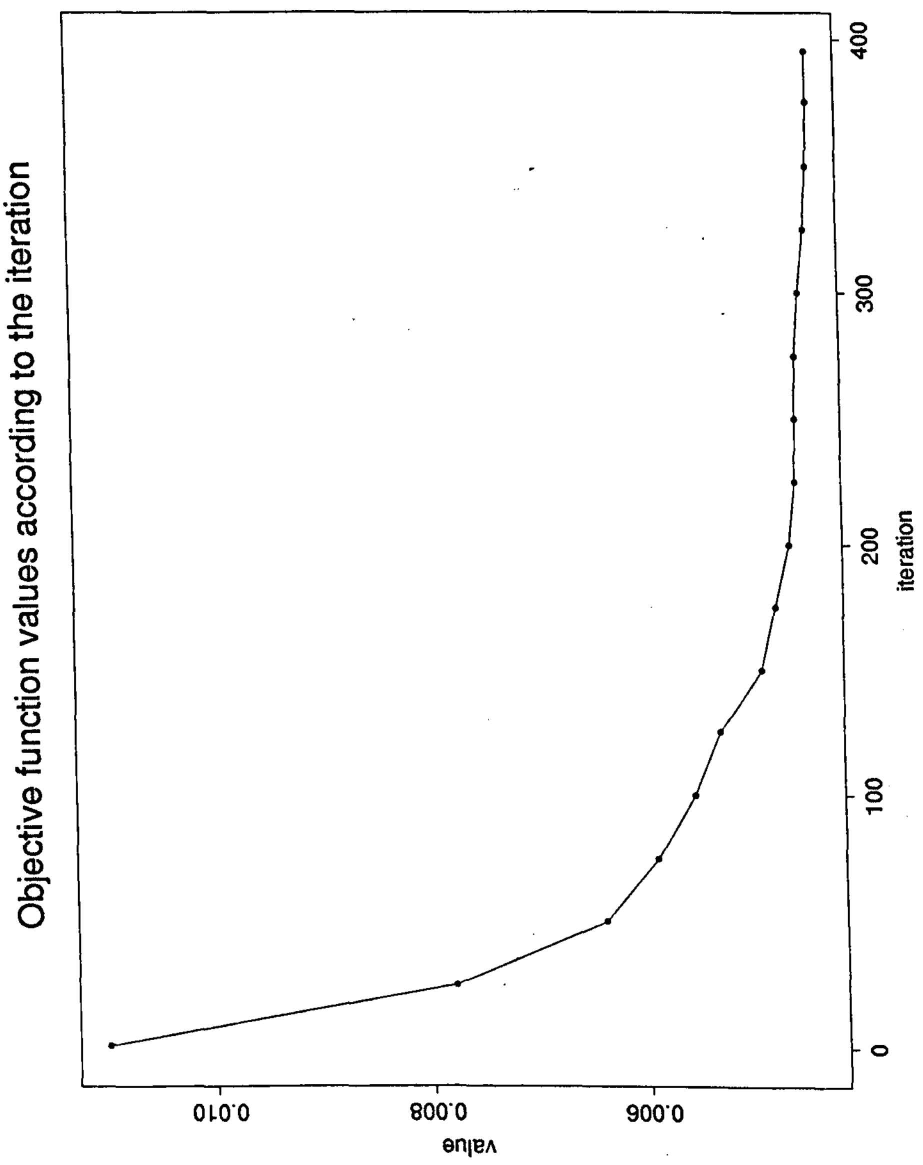

Figure 5.3: Change of the objective function according to iterations (for the distributed controlled system) in the linear case with the starting valves $X_m = 1.0, m = 1, 2, \ldots, 8$.

Further, we applied the initial values $X_m = 1.1, m = 1, 2, \ldots, 8$, and ran the same program with this; the results obtained are:

- The optimal value of $\mathrm{I}(D, v) = 5.9912470738808 \times 10^{-3}$;
- The number of iterations $= 373$;
- The value of the variables in the final step:

  $X_1 = 0.002314, X_2 = 0.018096, X_3 = 1.136087, X_4 = 0.004777, X_5 = 0.320013, X_6 = 1.981772, X_7 = 0.284138, X_8 = 0.594778.$

The initial and final shape and the change of the objective function have been plotted in Figures 5.4 and 5.5.

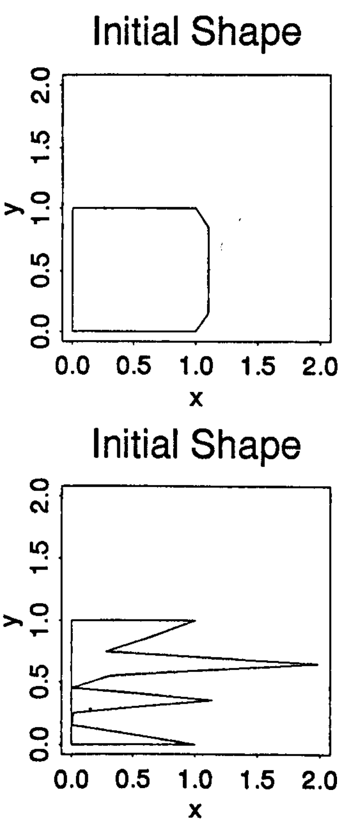

Figure 5.4: The initial and the optimal domain (for the distributed controlled system) with the starting initial values $X_m = 1.1, m = 1, 2, \ldots, 8$, in the linear case.

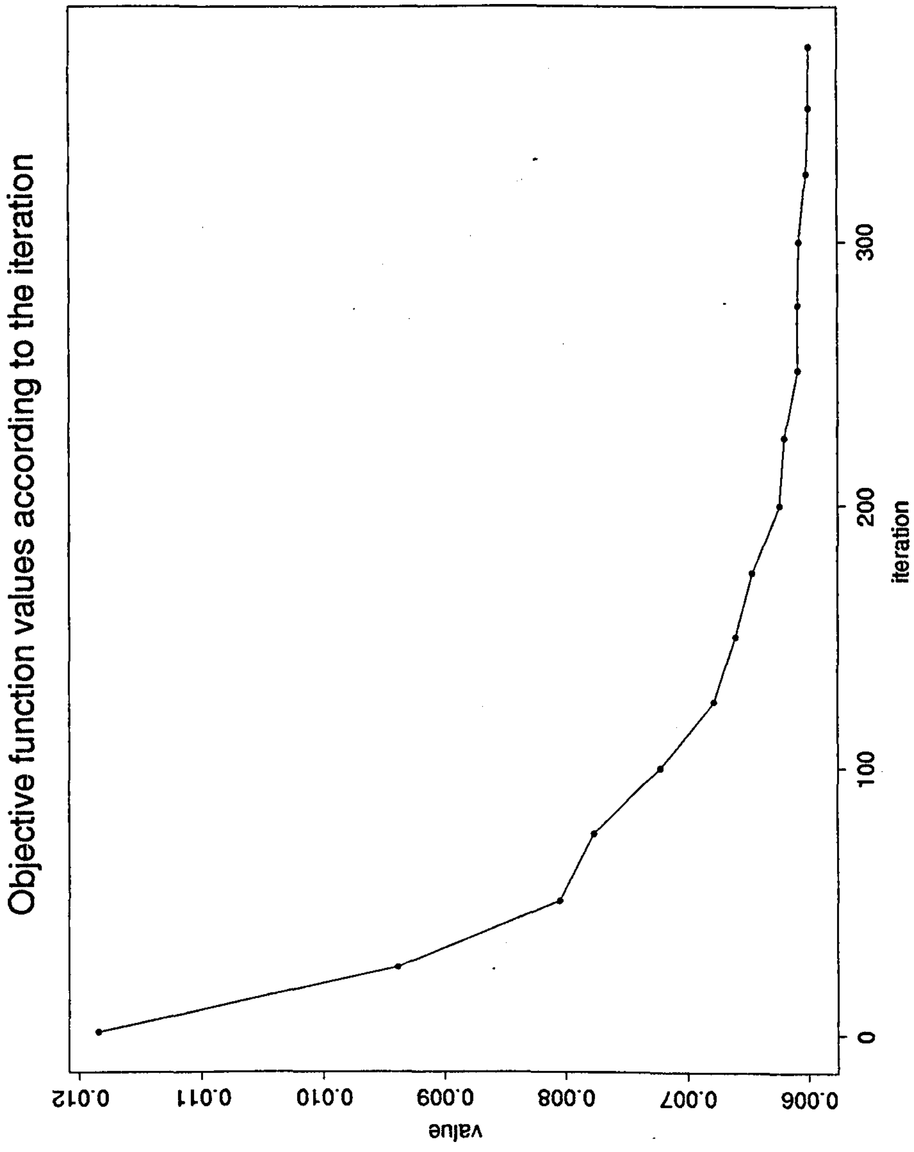

Figure 5.5: Change of the objective function according to iterations (for the distributed controlled system) in the linear case with the starting valves $X_m = 1.1, m = 1, 2, \ldots, 8$.

5.5.4 Example 2

In the nonlinear case, again as in *Example* 2 in Chapter 4, we have taken $f(x, y, u) = 0.25u^2$ and also the same value for stopping tolerance (the variable *ftol* in the subroutine *AMOEBA*). Thus the functions F_i and G_i in (5.12) have been defined as

$$F_i = u\Delta\psi_i + 0.25u^2\psi_i, \; G_i = -\psi_i v, \; i = 1, 2, \ldots, 10.$$

By applying the initial value $X_m = 1, m = 1, 2, \ldots, 8$, as a given domain for starting the standard minimization algorithm (*AMOEBA*), the results are the following:

- The optimal value of $\mathrm{I}(D, v) = 5.9905811520515 \times 10^{-3}$;
- The number of iterations $= 373$;
- The value of the variables in the final step:
  $X_1 = 0.003829, X_2 = 1.982183, X_3 = 0.321985, X_4 = 0.018270,$
  $X_5 = 0.001920, X_6 = 1.134801, X_7 = 0.283892, X_8 = 0.594196.$

According to the above results, the initial and the optimal domain are shown in the Figure 5.6; moreover, the change of the objective function, according to the number of iterations is also plotted in the Figure 5.7 below:

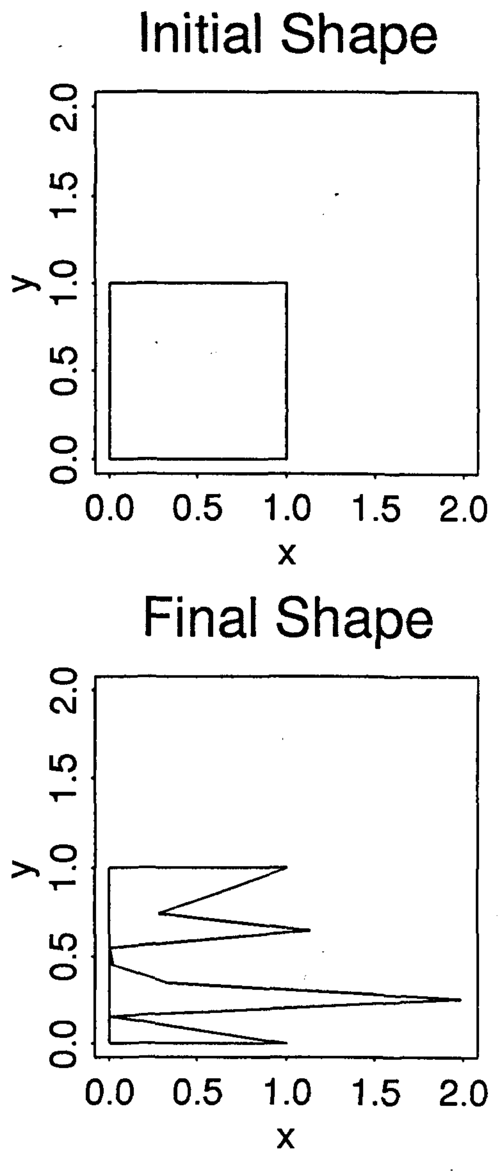

Figure 5.6: The initial and the optimal domain (for the distributed controlled system) with the starting initial values $X_m = 1.0, m = 1, 2, \ldots, 8$, in the nonlinear case.

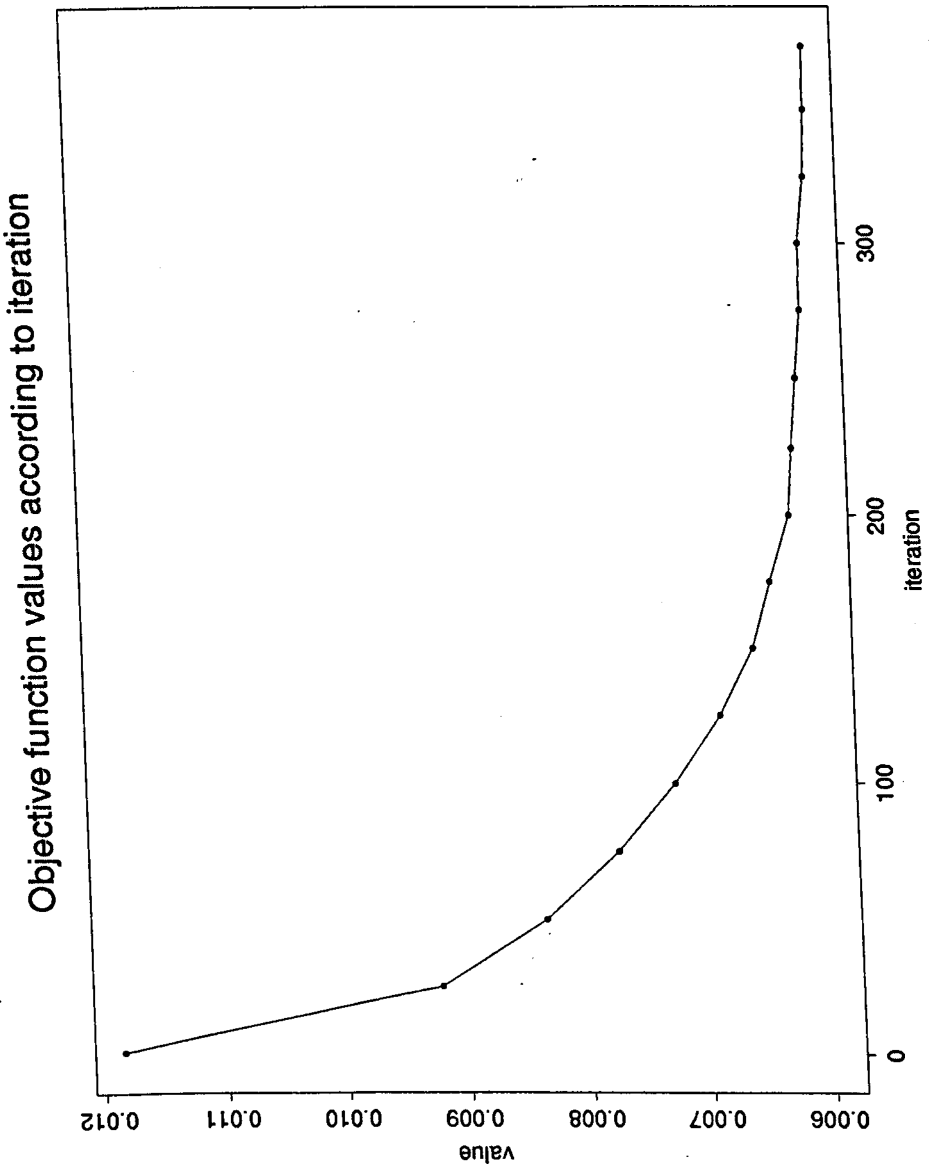

Figure 5.7: Change of the objective function according to iterations (for the distributed controlled system) in the nonlinear case with the starting valves $X_m = 1.0, m = 1, 2, \ldots, 8$.

As in *Example* 1, we also chose initial values $X_m = 1.1, m = 1, 2, \ldots, 8$, and ran the program; here are the results obtained:

- The optimal value of $\mathrm{I}(D, v) = 4.4439439539026 \times 10^{-3}$;
- The number of iterations $= 395$;
- The value of the variables in the final step:
  $X_1 = 0.044476, X_2 = 0.084477, X_3 = 0.004738, X_4 = 0.001490,$
  $X_5 = 0.003245, X_6 = 1.180229, X_7 = 0.000127, X_8 = 1.291236.$

For this case, the initial and the final shape and also the change of the objective function, have been plotted in the Figures 5.8 and 5.9 respectively.

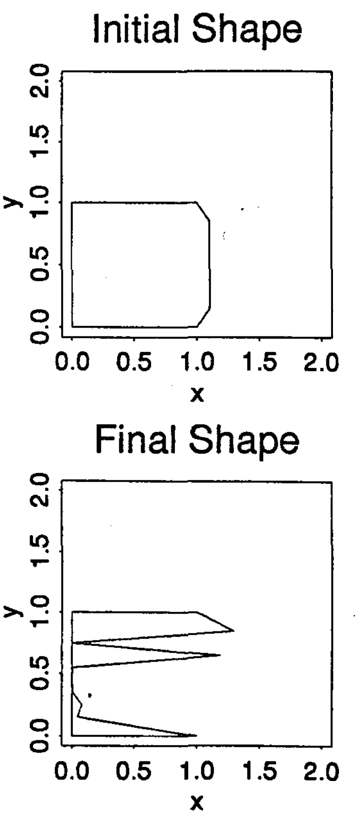

Figure 5.8: The initial and the optimal domain (for the distributed controlled system) with the starting initial values $X_m = 1.1, m = 1, 2, \ldots, 8$, in the nonlinear case.

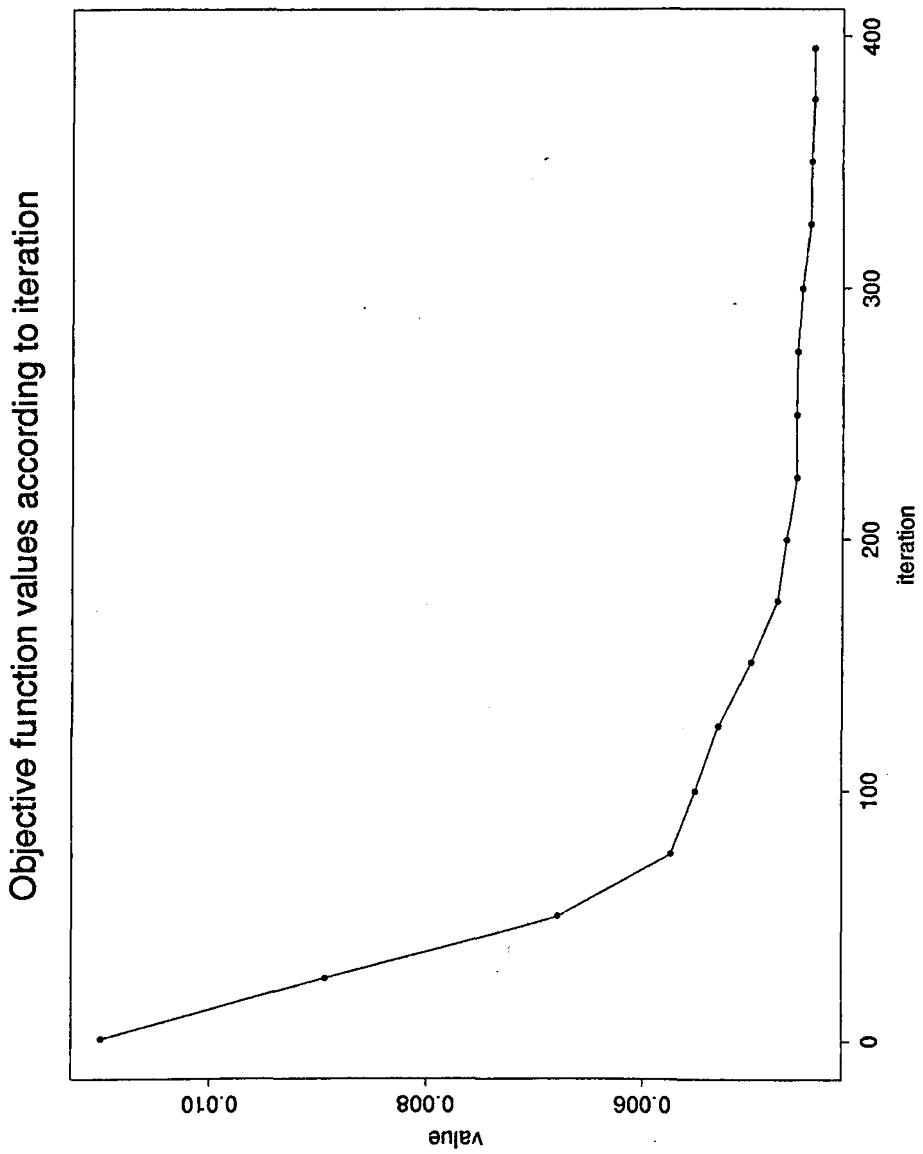

Figure 5.9: Change of the objective function according to iterations (for the distributed controlled system) in the nonlinear case with the starting valves $X_m = 1.1, m = 1, 2, \ldots, 8$.

Chapter 6

Shapes, Measures and Elliptic Equations (Boundary Control)

6.1 Introduction

In the present chapter, we consider again $D \subset \mathbb{R}^2$ as a bounded domain with a piecewise-smooth, closed and simple boundary ∂D which consists of a fixed and a variable part; these parts have been introduced in Chapters 4 and 5 in detail (see Figure 4.4).

Let $f : D \times \mathbb{R} \longrightarrow \mathbb{R}, f \in C(D \times \mathbb{R}); g : D \longrightarrow \mathbb{R}, g \in C(D)$, be two given functions. A domain D is called *admissible* if the elliptic equation

$$\Delta u(X) + f(X, u) = g(X), \tag{6.1}$$

with the boundary condition

$$u_{|\partial D} = v, \tag{6.2}$$

has a bounded solution on the domain D; here are also supposed that $X = (x, y) \in D$, $u : D \longrightarrow \mathbb{R}$ is a bounded function which takes values in the bounded set U, and $v : \partial D \longrightarrow \mathbb{R}$ is a bounded boundary control function, taking values in abounded set V.

As explained in Chapter 4, the variable part of ∂D can be approximated with M number of corners. For a fixed positive integer M, the set of all admissible domains is denoted by $\mathcal{D}_M$.

The aim of this chapter is to identify the optimal domain in $\mathcal{D}_M$, D^*, and its associated optimal control function, $v^*_{D^*}$, for a given optimal shape design problem with a functional performance criterion, $\mathrm{I}(D, v)(D \in \mathcal{D}_M)$, governed by the elliptic equations (6.1) and (6.2). Again, as explained in previous chapters, the optimal pair will be characterized in two stages:

- (i) For a given domain $D \in \mathcal{D}_M$, by applying the generalized form of (6.1) and (6.2) (weak solution), and using the process of embedding, the problem will be replaced by a measure-theoretical one which definitely has a solution because of existence theorems. Then its optimal solution can be approximated sufficiently close by a solution of a finite linear program. Hence the optimal control v^*_D, associated with the fixed domain D, will be characterized. In this manner, for any given domain D, one can calculate the value $\mathrm{I}(D, v^*_D)$; thus in the end of this stage, the following function J can be identified,

$$\mathrm{J} : D \in \mathcal{D}_M \longrightarrow \mathrm{I}(D, v^*_D) \in \mathbb{R}.$$

- (ii) In the next stage, a standard minimization algorithm will be applied on the function J above, to find its minimizer. The result determines the optimal pair of domain and control which indeed is (an approximation of) the optimal solution for the given optimal shape design problem.

As explained in Chapter 4, this new method has some advantages in comparison with similar methods; some of these advantages are listed in previous Chapters.

6.2 Classical optimal Shape and Control problem

For a given admissible domain $D \in \mathcal{D}_M$, let $f_1 : D \times U \longrightarrow \mathbb{R}$ and , $f_2 : \partial D \times V \longrightarrow \mathbb{R}$ be two continuous, non- negative, real-valued function; further we assume that there is a constant $K > 0$ so that the function f_1 satisfies

$$| f_1(X, u(X)) | \leq K | u |, \tag{6.3}$$

for all pairs $(X, u(X))$ where $X \in D$. We define the functional I as the performance criteria for a classical optimal shape design problem as

$$\mathrm{I}(D, v) = \int_D f_1(X, u(X))\, dX + \int_{\partial D} f_2(s, v(s))\, ds, \tag{6.4}$$

where u is the bounded solution of the boundary elliptic equations (6.1) and (6.2); moreover the function v is supposed to be a Lebesgue measurable function which appears in (6.2). We define also

$$\mathbf{F} = \{(D, v) \mid D \in \mathcal{D}_M \text{ is admissible}, v : \partial D \longrightarrow \mathbb{R} \text{ satisfies } (6.2)\}$$

In this Chapter we are going to solve the following optimal shape design problem on **F**.

$$\begin{aligned} \textbf{Minimize:} \quad & \mathbf{I}(D,v) = \int_D f_1(X,u(X))\,dX + \int_{\partial D} f_2(s,v(s))\,ds \\ \textbf{Subject to:} \quad & \Delta u(X) + f(X,u) = g(X) \\ & u_{|\partial D} = v. \end{aligned} \tag{6.5}$$

In general it is difficult to characterize a classical bounded solution for the elliptic equation (6.1) and (6.2); therefore it is too difficult to find the solution of (6.5). By applying the variational form of the elliptic equations, defined by the following Proposition, we will change the problem into the other in which a bounded weak solution of (6.1) and (6.2) is involved.

Proposition 21 : *Let u be the classical solution of (6.1) and (6.2), then we have the following integral equality:*

$$\int_D (u\Delta\psi + \psi f)\,dX - \int_{\partial D} v(\nabla\psi.\mathrm{n})\,ds = \int_D \psi g\,dX,\ \forall \psi \in H_0^1(D). \tag{6.6}$$

that here n *is the outward unit vector on ∂D*

Proof: By applying Green's formula in the same way as we did in the Proposition 13, one can prove this proposition; so the detail is omitted. □

Proposition 6.1 states that the equations (6.1) and (6.2) can be written in a new formulation in (6.6). A function $u \in H^1(D)$ is called *weak (generalized) solution* of the problem (6.1) and (6.2) if it satisfies in the equality (6.6). By applying this fact, we are going now to calculate the value of the functional **I** for a given domain D and its associated control v; for a fixed domain D, the optimal shape problem is changed into an optimal control one as follows:

$$\text{Minimize}: \quad \mathbf{I}(D,v) = \int_D f_1(X,u(X))\,dX + \int_{\partial D} f_2(s,v(s))\,ds;$$

$$\text{Subject to}: \quad \int_D (u\Delta\psi + \psi f)\,dX - \int_{\partial D} v(\nabla\psi.\mathrm{n})\,ds = \int_D \psi g\,dX \;\forall\psi \in H_0^1(D), \tag{6.7}$$

where the minimization takes place on the set of admissible pair of trajectory and control functions (u,v), say $\mathcal{F}$, which was defined in section 5.3 in Chapter 5. We suppose that $\mathcal{F}$ is nonempty; in other words, because D is admissible, the elliptic equations (6.1) and (6.2) have a bounded weak solution.

6.3 Metamorphosis and Approximation

Let D be a fixed domain, and define $\Omega = D \times U$ and $\omega = \partial D \times V$. Then, a bounded weak solution defines a positive and linear functional like

$$u(\cdot) : F \longrightarrow \int_D F(X,u(X))\,dX$$

on $C(\Omega)$; moreover, a control function v, defines the following positive, linear functional

$$v(\cdot) : G \longrightarrow \int_{\partial D} G(s,V(s))\,ds$$

on $C(\omega)$. Therefore there exists an injective transformation between the set $\mathcal{F}$ and the set of all pairs of linear functionals $(u(\cdot),v(\cdot))$ (see Proposition 2). The Riesz Representation Theorem [55] shows that there are measures μ_u and ν_v so that,

$$\mu_u(F) = u(F)\,\forall F \in C\Omega),\; \nu_v(G) = v(G)\,\forall F \in C(\omega),$$

(see for proof Proposition 16).

Now, by applying the mentioned transformation, the new formulation of the problem (6.7) is:

$$\begin{aligned} &\text{Minimize}: && \mathrm{i}(\mu_u,\nu_v) = \mu_u(f_1) + \nu_v(f_2) \\ &\text{Subject to}: && \mu_u(F_\psi) + \nu_v(G_\psi) = c_\psi\,;\; \forall\psi \in H_0^1(D), \end{aligned} \tag{6.8}$$

where

$$F_\psi = u\Delta\psi + \psi f\;,\; G_\psi = -v(\nabla\psi.\mathrm{n}\,|_{\partial D})\;,\; c_\psi = \int_D \psi g. \tag{6.9}$$

So far, we have just changed the appearance of the problem; now we consider the minimization of the problem (6.8) over the set of all pairs of measures (μ,ν) in $\mathcal{M}^+(\Omega)\times\mathcal{M}^+(\omega)$ satisfying the mentioned conditions plus the extra properties:

$$\begin{aligned} \mu(\xi) &= \int_D \xi(x,y)\,dX = a_\xi, \\ \nu(\tau) &= \int_D \tau(s)\,ds = b_\tau, \end{aligned} \tag{6.10}$$

which are deduced from the definition of admissible pair of trajectory and control functions, (u,v). These properties indicate that the measures μ and ν project on the (x,y)-plan and real line respectively, as Lebesgue measures. We remind the reader that here it is supposed $\xi:\Omega\longrightarrow\mathbb{R}$ in $C(\Omega)$ depends only on variable $X=(x,y)$, and $\tau:\omega\longrightarrow\mathbb{R}$ in $C(\omega)$ depends only on variable s. Therefore, we are going to solve the following problem:

$$\begin{aligned}
\textbf{Minimize:} \quad & \mathrm{i}(\mu,\nu) = \mu(f_1) + \nu(f_2) \\
\textbf{Subject to:} \quad & \mu(F_\psi) + \nu(G_\psi) = c_\psi, \qquad \forall \psi \in H_0^1(D); \\
& \mu(\xi) = a_\xi, \qquad \forall \xi \in C_1(\Omega); \\
& \nu(\tau) = b_\tau, \qquad \forall \tau \in C_1(\omega).
\end{aligned} \tag{6.11}$$

The new formulation has some advantages that were explained precisely in Chapter 5; for instance, if we denote $Q \subset \mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ as the set of all pairs of measures (μ, ν) that satisfy the conditions mentioned in (6.11), then Q is compact (see Chapter 5, Proposition 17) and moreover the function $(\mu,\nu) \in Q \longrightarrow \mu(f_1) + \nu(f_2) \in \mathbb{R}$ is continuous. Thus by Proposition $II.1$ of [50], the problem (6.11) definitely has a minimizer in Q.

The problem (6.11) is an infinite-dimensional linear program; but its solution can be achieved by choosing the countable sets of functions that are uniformly dense (total), in the appropriate spaces. Let

$$\{\psi_i, i = 1,2,3,\dots\}\ ,\ \{\xi_j, j = 1,2,3,\dots\}\ ,\ \{\tau_l, l = 1,2,3,\dots\},$$

be total sets in the spaces $H_0^1(D)$, $C_1(\Omega)$ and $C_1(\omega)$ respectively, thus the problem (6.11) can be replaced by the following one;

$$\begin{aligned}
\text{Minimize}: \quad & \mathrm{i}(\mu,\nu) = \mu(f_1) + \nu(f_2) \\
\text{Subject to}: \quad & \mu(F_i) + \nu(G_i) = c_i, && i = 1,2,3,\ldots; \\
& \mu(\xi_j) = a_j, && j = 1,2,3,\ldots; \\
& \nu(\tau_l) = b_l, && l = 1,2,3,\ldots, \qquad (6.12)
\end{aligned}$$

where

$$F_i := F_{\psi_i}, G_i := G_{\psi_i}, c_i := c_{\psi_i} \text{ for } i = 1,2,3,\ldots,$$

$$a_j := a_{\xi_j} \text{ for } j = 1,2,3,\ldots,$$

$$b_l := b_{\tau_l} \text{ for } l = 1,2,3,\ldots.$$

Let us now choose just a finite number of constraints in (6.12) and consider the following minimization problem:

$$\begin{aligned}
\text{Minimize}: \quad & \mathrm{i}(\mu,\nu) = \mu(f_1) + \nu(f_2) \\
\text{Subject to}: \quad & \mu(F_i) + \nu(G_i) = c_i, && i = 1,2,\ldots,M_1; \\
& \mu(\xi_j) = a_j, && j = 1,2,\ldots,M_2; \\
& \nu(\tau_l) = b_l, && l = 1,2,\ldots,M_3. \qquad (6.13)
\end{aligned}$$

Proposition 19 in Chapter 5 shows that the solution of (6.13) tends to the solution of (6.12) whenever $M_1, M_2, M_3 \longrightarrow \infty$; hence the solution of (6.11) can be approximated by a solution of (6.13) when the positive integers M_1, M_2 and M_3 are chosen large enough. Now one can construct suboptimal pair of trajectory and control functions for the functional i in (6.7) via the optimal solution, (μ^*, ν^*), of (6.13); in Chapter 5, this procedure has been explained in detail. Let (u_v, v) be the pair of trajectory and control functions constructed as explained there; then we have the following theorem which guarantees that the pair is a good approximation for the solution of (6.11). The reader can find its proof similar to others in [53] and Chapter 5.

Theorem 6 : *Under the appropriate conditions on the approximations involved, if the values M_1, M_2 and M_3 tend to ∞ then*

$$\mathrm{i}(u_v, v) \longrightarrow \inf_Q [\mathrm{i}(\mu, \nu)].$$

The problem (6.13) is a semi-infinite linear programming problem; the number of equations is finite but the underlying space is not a finite-dimensional space. It is possible then to estimate its solution by a process of discretization. a pair of measures (μ, ν) can be characterized by a result of [48] which is proved in [50], that μ and ν in (6.13) have the form like (5.37). By introducing appropriate dense subsets in Ω and ω, and applying the Proposition $III.3$ of [50], one can conclude that μ and ν have the following form

$$\mu = \sum_{n=1}^{N} \alpha_n \delta(Z_n)\,,\ \nu = \sum_{k=1}^{K} \beta_k \delta(z_k)$$

where $Z_n, n = 1, 2, \ldots, N$, and $z_k, k = 1, 2, \ldots, K$, belong to dense subset of Ω and ω respectively, and $\delta(t)$ is the unitary atomic measure with support the singleton set $\{t\}$. Hence, by defining a discretization on Ω and ω with the nodes $Z_n = (x_n, y_n, u_n)$ for $n = 1, 2, \ldots, N$, and $z_k, k = 1, 2, \ldots, K$, the solution of (6.13) can be obtained by solving the following linear programming problem in which its unknowns are the coefficients $\alpha_n, n = 1, 2, \dots, N$, and $\beta_k, k = 1, 2, \dots, K$.

$$
\begin{aligned}
\textbf{Minimize:} \quad & \sum_{n=1}^{N} \alpha_n f_1(Z_n) + \sum_{k=1}^{K} \beta_k f_2(z_k) \\
\textbf{Subject to:} \quad & \alpha_n \geq 0, \qquad n = 1, 2, \dots, N; \\
& \beta_k \geq 0, \qquad k = 1, 2, \dots, K; \\
& \sum_{n=1}^{N} \alpha_n F_i(Z_n) + \sum_{k=1}^{K} \beta_k G_i(z_k) = c_i, \qquad i = 1, 2, \dots, M_1; \\
& \sum_{n=1}^{N} \alpha_n \xi_j(Z_n) = a_j, \qquad j = 1, 2, \dots, M_2; \\
& \sum_{k=1}^{K} \beta_k \tau_l(z_k) = b_l, \qquad l = 1, 2, \dots, M_3.
\end{aligned}
\tag{6.14}
$$

The result of this problem introduces a pair of measures, call (μ^*, ν^*), that for this pair, the value of the functional i, i.e. $\mathrm{i}(\mu^*, \nu^*)$, will be minimum on the set $Q(M_1, M_2, M_3)$, defined by the pairs of measures in $\mathcal{M}^+(\Omega) \times \mathcal{M}^+(\omega)$ satisfying in conditions explained in (6.14). This pair of measures, as explained in Chapter 5, serves the suboptimal pair of trajectory and control functions $(u_{v_D^*}, v_D^*)$. Thus for the fixed domain D, the minimum value of the functional I in the problem (6.5) is approximated as

$$
\mathrm{I}(D, v_D^*) \equiv \mathrm{i}(\mu^*, \nu^*).
$$

6.4 The optimal shape

For a fixed domain, we have explained in the former section how one can find the optimal control v_D^* for the problem (6.5), so that the value of $\mathrm{I}(D, v_D^*)$ is minimum. Hence we have defined the function

$$\mathbf{J} : D \in \mathcal{D}_M \longrightarrow \mathrm{I}(D, v_D^*) \in \mathbb{R}. \tag{6.15}$$

To find the optimal pair of domain and control function in F, say $(D^*, v_{D^*}^*)$, which solves the optimal shape design problem (6.5), it is enough to find the minimizer of J in the same way as pointed out in Chapter 5; details of doing this fact, has been explained completely in the former chapters, thus there is no need to bring them here again. So, we only present some examples. We remind the reader that Proposition 20 in Chapter 5 guarantees that the pair $(D^*, v_{D^*}^*)$ is optimal.

6.5 Numerical work

In this section, we apply the method introduced in the previous sections to solve the appropriate optimal shape design problem in (6.5), defined by functions $g(X) = 0$ (thus $c_{\psi_i} = 0$ in (6.14)), $f_2(s, v) = 0$ and

$$f_1(x,u) = \begin{cases} 400 & \text{-0.05} \leq u \leq 0.05 \\ \frac{1}{u^2} & \text{otherwise.} \end{cases}$$

We will present two examples for the linear and nonlinear cases of the elliptic equations in (6.1) and(6.2); in each example we take $M = 8$. Hence each domain in $\mathcal{D}_M$ is characterized by the set of 8 points, $\{A_m = (x_m, Y_m), m = 1, 2, \ldots, 8\}$, with the same constants Y_m's as Chapter 4 (see Figure 4.6) so that $x_m \geq 0$ for each $m = 1, 2, \ldots, M$. We assume that the function $u(\cdot)$ takes values in the bounded set $U = [-1.0, 1.0]$. The control function is supposed to be zero on ∂D expcet the segment of line $y = 1$; along this segment, it is assumed that $v(s)$ takes values in the bounded set $V = [-1.0, 1.0]$, when $s \in [0,1]$. Thus, in (6.14) we have

$$G_i = -(\frac{\partial \psi_i(s,y)}{\partial y})_{|_{y=1}}.$$

So, for any given domain D, the spaces Ω and ω have been consider as $\Omega = D \times [-1.0, 1.0]$ and $\omega = \partial D \times [-1.0, 1.0]$.

6.5.1 Functions and Discretization

The functions ψ_i's have been chosen the same as those defined in (4.20); it was shown that the set $\{\psi_i : i \in \mathbb{N}\}$ is total in $H_0^1(D)$. For the second set of equations in (6.12), the function ξ_j's are chosen as the same as in the chapter 4 and 5. Also the functions τ_l's in the third set of equations, are selected as the test functions f_s's in Chapter 2 on the interval $[0,1]$.

To set up the finite linear programming (6.14) for the next examples, we choose $M_1 = 3$, M_2 and $M_3 = 10$; thus a_j = area of $D_j, j = 1,2,\ldots,8$ (see Chapter 5), and $b_l = 0.1, l = 1,2,\ldots,10$.

To apply the condition $x_m \geq 0, m = 1,2,\ldots,8$, we have used the penalty method with the same penalty function as defined for the numerical examples in Chapter 5. Moreover we put a discretization on Ω with $N = 1100$ nodes with the points $Z_n = (x_n, y_n, u_n)$, $n = 1,2,\ldots,N$, in the same way as explained in section 5.5. Because the control function is zero on ∂D except the segment of the line $y = 1$, we have put a discretization on ω with $K = 110$ nodes like $z_k = (s_k, v_k), k = 1,2,\ldots,K$; these nodes have been chosen as follows:

$$z_k = z_{11(i-1)+j}$$

for $i = 1, 2, \ldots, 10$ and $j = 1, 2, \ldots, 11$, where

$$s_{11(i-1)+j} = \frac{(i-1)+0.5}{10}$$

$$v_{11(i-1)+j} = \frac{2(j-1)}{10} - 1.0.$$

Hence the total number of variables in the finite linear programming problem (6.14) is $1100 + 110 = 1210$.

In the case of the above concepts, we solved the following examples for the linear and nonlinear case of the elliptic equations; in each case we chose the mentioned subroutine $AMOEBA$ as the standard minimization Algorithm with the initial valves $X_m = 1.0$, for $m = 1, 2, \ldots, 8$. Also, we applied the $E04MBF$ NAG-Library Routine for solving the appropriate finite linear program in each iteration.

6.5.2 Example 1

For the linear case of elliptic equations (6.1) and (6.2), we chose $f(X, u) = 0$, therefore $F_i = u\Delta\psi_i$ in (6.14); we achieved to the following results:

- The optimal value of **I** $= 0.44432256772971$;
- The number of iterations $= 497$;

- The value of the variables in the final step:

  $X_1 = 0.044671, X_2 = 0.000003, X_3 = 0.000018, X_4 = 0.083868,$

  $X_5 = 0.004590, X_6 = 1.181268, X_7 = 0.003360, X_8 = 1.291424,$

According to the results obtained, the suboptimal control function, the initial and the final domain, and the changes diagram of the objective function according to the number of iterations, have been plotted in the Figures 6.1, 6.2 and 6.3.

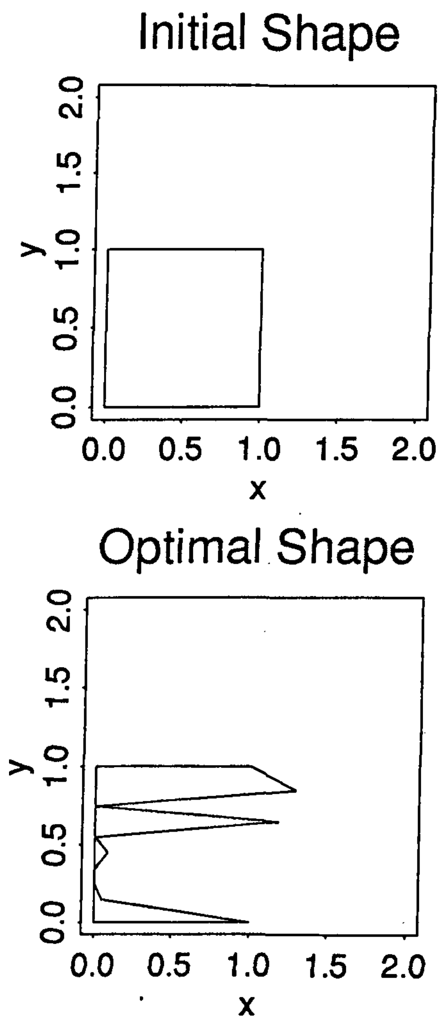

Figure 6.1: The initial and the optimal domain (for the boundary controlled system) with the starting initial values $X_m = 1.0, m = 1, 2, \ldots, 8$, in the linear case.

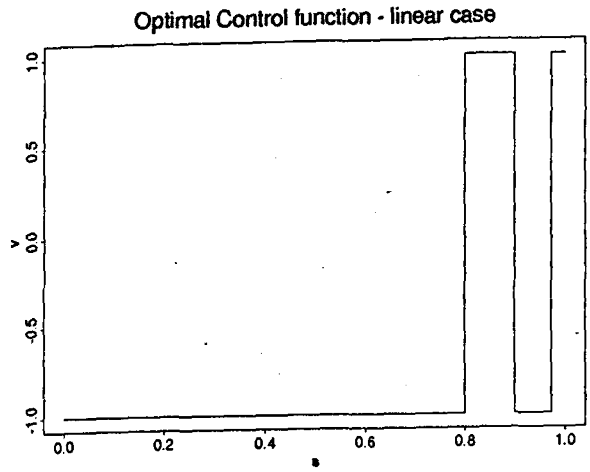

Figure 6.2: The optimal (boundary) control function for the linear case.

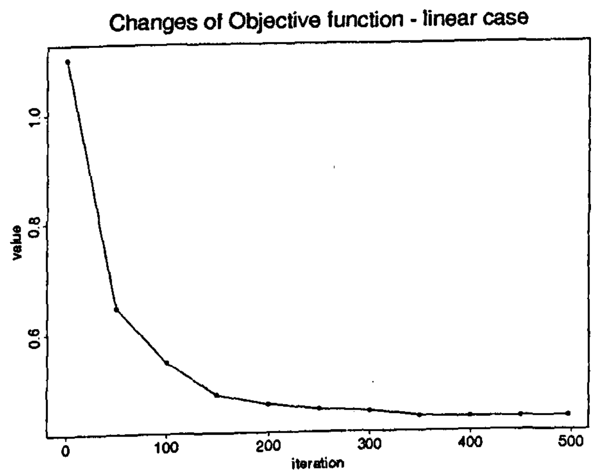

Figure 6.3: Change of the objective function according to iterations (for the boundary controlled system) in the linear case

6.5.3 Example 2

By choosing $f(X,u) = 5u^2$, an example for the nonlinear case of the elliptic equations was given; the result of this example was as follows:

- The optimal value of $\mathbf{I}$ = 0.44432182922939;
- The number of iterations = 492;

- The value of the variables in the final step:

  $X_1 = 0.044691, X_2 = 0.083889, X_3 = 0.004568, X_4 = 0.003356,$

  $X_5 = 0.000026, X_6 = 0.000001, X_7 = 1.181291, X_8 = 1.291379,$

These results have introduced the suboptimal control function, the initial and the final domain, and the changes diagram of the objective function according to the number of iterations; they have been plotted in the Figures 6.4, 6.5 and 6.6.

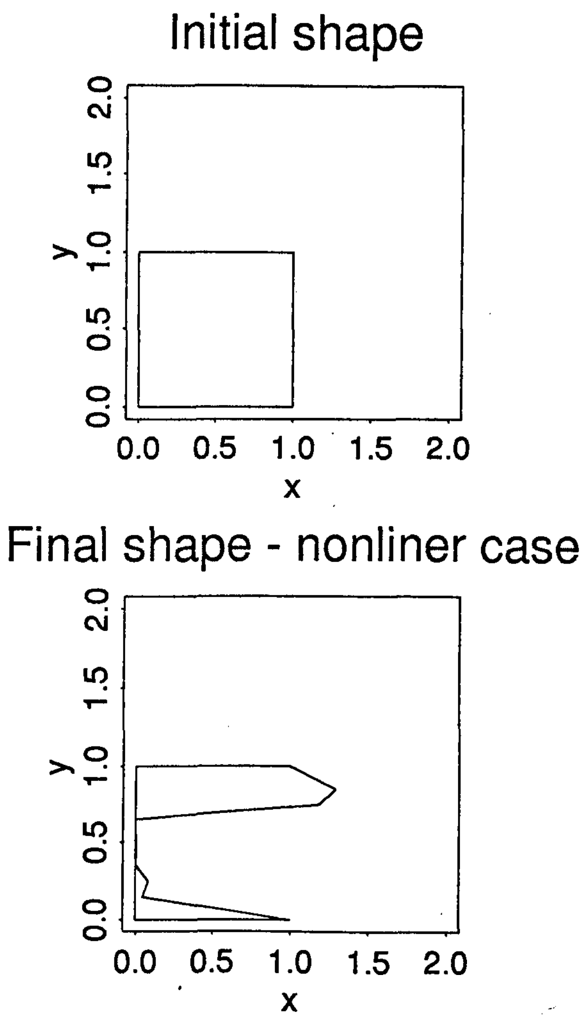

Figure 6.4: The initial and the optimal domain (for the boundary controlled system) with the starting initial values $X_m = 1.0, m = 1, 2, \ldots, 8$, in the nonlinear case.

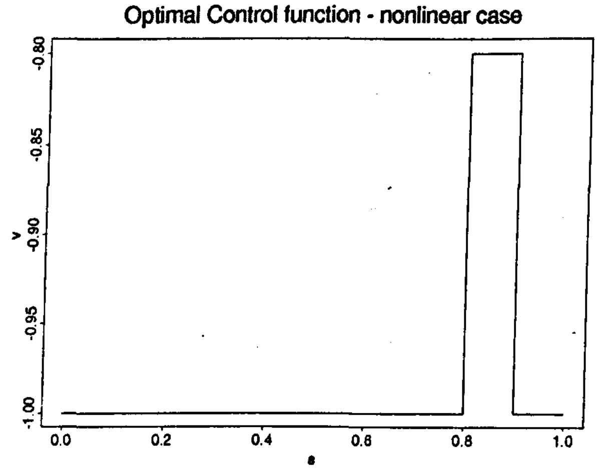

Figure 6.5: The optimal (boundary) control function for the nonlinear case.

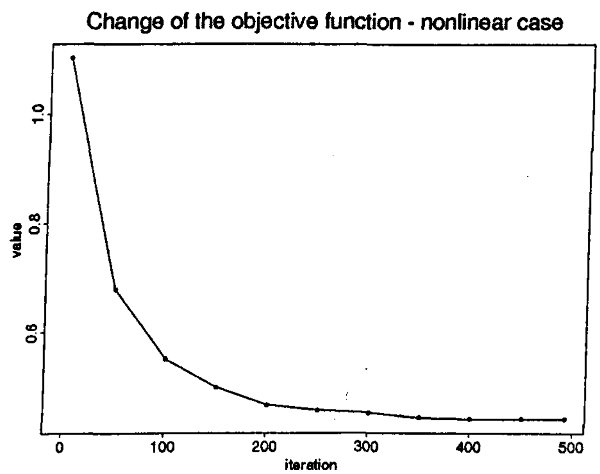

Figure 6.6: Change of the objective function according to iterations (for the boundary controlled system) in the nonlinear case

6.6 Conclusion and recommendation for further research

The solution of optimal shape and optimal shape design problems which are governed by different types of elliptic equations, and defined in terms of a pair of geometrical elements, have been discussed in this work by the use of a new method. The main idea of the solution is based on the replacement of the classical problem by a problem defined on a subset of positive Radon measures, to find a pair of measures (or one measure, sometimes), subject to some related linear conditions.

The new measure-theoretical problem, can then be approximated by a finite linear programming problem by the use of total sets and discretization. The existence of the optimal solution has been immediately proved by the use of compactness properties of the weak* topology via existence theorems. In both systems of coordinates, polar and cartesian, we are able to find the optimal shape and its associated optimal control function together; this makes the method very effective. The new approach enables us to solve also the related optimal control problem (in cartesian coordinates).

There are still some more problems to be solved related to the concepts presented in this thesis; we recommend here some of them for further research works:

- Except in Chapter 2, all the OSD problems defined have been governed by partial differential elliptic equations. We have indeed considered different types of elliptic problems, but no other type of partial differential equations has been studied. Applying the method introduced here to solve those OSD problems which are associated with the solution of hyperbolic or parabolic equations would be of great interest. The works in [31], [51], [16] and [17] on solving a control problem governed by diffusion and wave equations via measures are good guides to apply the method for OSD problems governed by diffusion equations.

- The OSD problems for systems defined by elliptic inequalities are important in many industrial and engineering fields; the solutions of these problems have been considered in many references (see for instance [7]). However, there has been no attempt to solve them by the use of measures. Extending the new method to this field and attacking the OSD problems governed by elliptic inequalities by this approach, is a further research work.

- The solutions of measure-theoretical problems have been approximated by those of a finite linear programming problems in this thesis. The estimation of the error is an open problem even for the simpler related optimal control problems. It has just been tried to find a bound for the error in a particular case in [15]; the same solution may possible in our case.

- Suppose that the measurable set, the geometrical element C (or D) in the definition of an OSD problem, has some fixed holes. Then one can define the similar OSD problems as in this work, over the set of all admissible pairs $(C, \partial C)$ for the mentioned C; there are many examples of this type of problems in industry. Solving these kind of problems by use of measures can be considered as a new work.

Appendix A

Calculating the function ξ in a discretization

To use the linear program (2.19), it is necessary to calculate the function ξ in terms of the components of the appropriate points which are chosen from the discretization on Ω and ω. First of all the function $\xi_i^\dagger$ must be calculated with respect to the components of the points $Z_i = (\theta_i, r_i) \in \Omega$ and $z_j = (\theta_j, r_j, w_j) \in \omega$ which will appear in the discretization on Ω and ω. Afterwards, by the equality

$$\xi_i(z_j) = \alpha_i \xi_i^\dagger(z_j),$$

the function ξ will be determined easily. The coefficient α_i is known.

As proved in the section 2.7, the value of the electromagnetic field in the point $z_j = (\theta_j, r_j, w_j) \in \omega$ from an infinite wire source at $Z_i = (\theta_i, r_i)$, which it carries a fixed current, is $|B| = \frac{1}{2\pi\rho}$, where ρ is the distance between Z_i and z_j. If u_θ and u_r are shown as the unit vectors in the directions of θ and r, the vector field B can be represented in two dimension coordinates by

$$B = B_r u_r + B_\theta u_\theta$$

where B_r and B_θ are the components of B in the directions of r and θ respectively. But $B_r = 0$, then $B = B_\theta u_\theta$; here u_θ is the normal vector to the circle with center $Z_i \in \Omega$, not the u_θ in the Figure A.1, which is anyhow printed in bold face. Therefore $|B_\theta| = |B| \cos b$, that b is the angle between B and u_θ (see figure A.1). Also it was assumed that the circulation is equal by 1; so we have the following line integral equation

$$\int_{\partial C} B.dl = 1. \tag{A.1}$$

From (A.1) one can conclude that

$$\nu(\frac{1}{2\pi\rho}(\cos b)\sqrt{r_j^2 + w_j^2}) = \int_0^{2\pi} \frac{1}{2\pi\rho}(\cos b)(\sqrt{r_j^2 + w_j^2})d\theta = 1.$$

Hence

$$\xi^\dagger(z_j) = \frac{1}{2\pi\rho}(\cos b)(\sqrt{r_j^2 + w_j^2}), \tag{A.2}$$

that by the cosine law in the triangle OZ_iz_j one can get:

$$\rho^2 = r_j^2 + r_i^2 - 2r_jr_i\cos(\theta_j - \theta_i). \tag{A.3}$$

Considering Figure A.1, since θ_j is an exterior angle for Z_iz_jR, $a = \theta_j - n$,

$$\sin a = \sin\theta_j \cos n - \cos\theta_j \sin n;$$

$$\cos a = \cos\theta_j \cos n + \sin\theta_j \sin n;$$

$$\cos n = \frac{OA + OD}{\rho} = \frac{r_j \cos\theta_j - r_i \cos\theta_i}{\rho};$$

$$\sin n = \frac{z_j A - Z_i D}{\rho} = \frac{r_j \sin\theta_j - r_i \sin\theta_i}{\rho}.$$

Since:

$$\rho \sin a = \sin\theta_j (r_j \cos\theta_j - r_i \cos\theta_i) - \cos\theta_j (r_j \sin\theta_j - r_i \sin\theta_i)$$

$$= -r_i \sin(\theta_j - \theta_i),$$

then

$$\sin a = -\frac{r_i}{\rho} \sin(\theta_j - \theta_i); \qquad \text{(A.4)}$$

similarly:

$$\cos a = \frac{r_j}{\rho} - \frac{r_i}{\rho} \cos(\theta_j - \theta_i). \qquad \text{(A.5)}$$

Because $b = \chi + a - \frac{\pi}{2}$; then

$$\cos b = \sin\chi \cos a + \sin a \cos\chi. \qquad \text{(A.6)}$$

Since χ is the angle between the tangent line and the ray at z_j,

$$\tan\chi = \frac{r_j}{(\frac{dr}{d\theta})} = \frac{r_j}{w_j}.$$

Hence:

$$\sin\chi = \frac{r_j}{\sqrt{r_j^2 + w_j^2}}\ ;\ \cos\chi = \frac{w_j}{\sqrt{r_j^2 + w_j^2}}. \tag{A.7}$$

Applying (A.4), (A.5) and (A.7) in (A.6) gives:

$$(\sqrt{r_j^2 + w_j^2})\cos b = \frac{r_j^2}{\rho} - \frac{r_j r_i}{\rho}\cos(\theta_j - \theta_i) - \frac{w_j r_i}{\rho}\sin(\theta_j - \theta_i).$$

Now from (A.2) we have:

$$\xi^\dagger(z_j) = \frac{1}{2\pi\rho^2}(r_j^2 - r_j r_i\cos(\theta_j - \theta_i) - w_j r_i \sin(\theta_j - \theta_i)).$$

Finally by applying (A.3) the function $\xi^\dagger$ can be calculated as:

$$\xi^\dagger(z_j) = \frac{(r_j^2 - r_j r_i\cos(\theta_j - \theta_i) - w_j r_i \sin(\theta_j - \theta_i))}{r_j^2 + r_i^2 - 2 r_j r_i \cos(\theta_j - \theta_i)}. \tag{A.8}$$

So the function ξ can be evaluated from (A.8) by putting the appropriate discretization on ω and Ω. Therefore we are able now to solve the linear programming problem in (2.19) via this discretization.

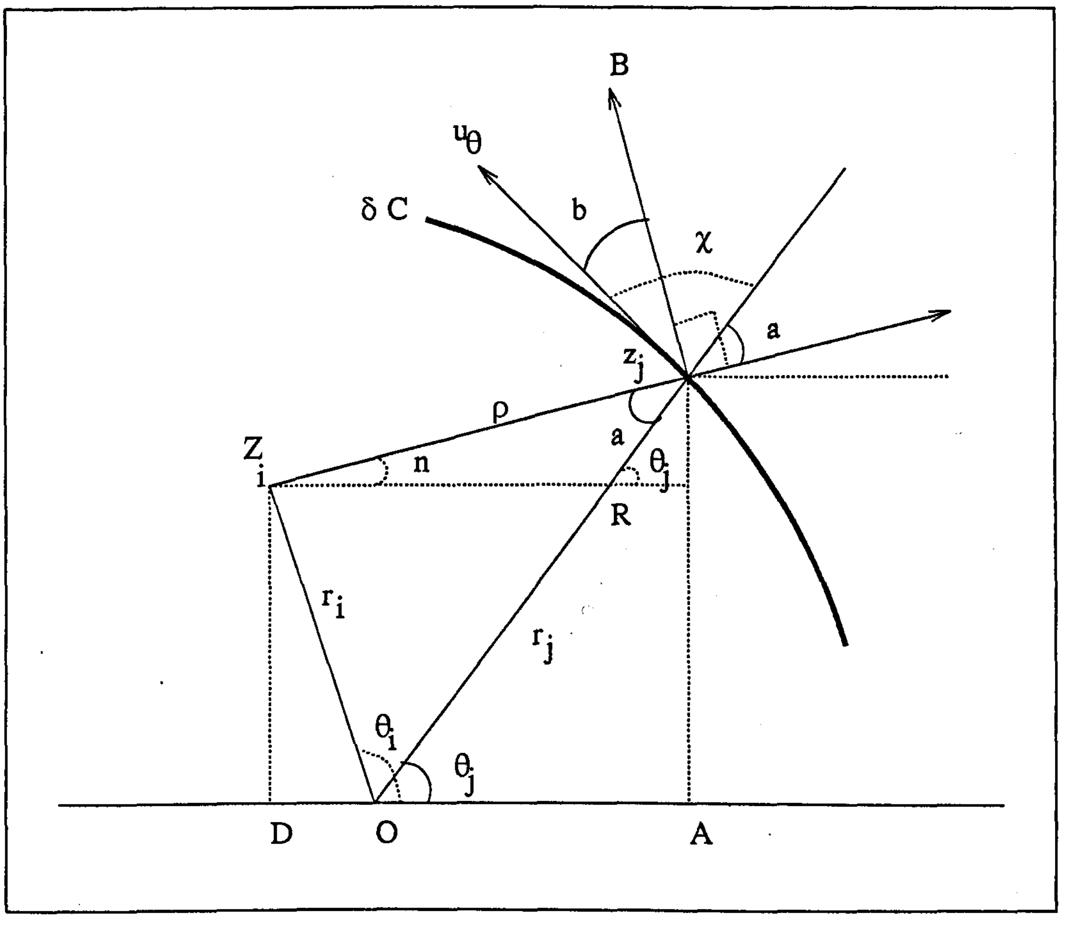

Figure A.1: Angles in calculating ξ

Appendix B

Why $\mathcal{D}_M$ instead of $\mathcal{D}$?

Based on the approximation of a closed and simple curve in $\mathbb{R}^2$ by a set of broken lines, we decided to consider $\mathcal{D}_M$ as the underlying space in which the minimization takes place. Indeed we approximated the variable part of any domain $D \in \mathcal{D}_M$, Γ, by M number of segments (in other words by $M+1$ corners); then we decided to look for the solution of the appropriate optimal shape design problems in Chapters 4,5 and 6 in $\mathcal{D}_M$ instead of $\mathcal{D}$.

As $M \longrightarrow \infty$, if an appropriate optimal shape design problem in $\mathcal{D}_M$ has a minimizer, then this may tend in some topology to the minimizer over $\mathcal{D}$ if such exists. However things can go wrong; for instance:

- There may be no minimizer over $\mathcal{D}_M$.
- There may be no minimizer over $\mathcal{D}$ (or both $\mathcal{D}$ and $\mathcal{D}_M$.)
- The sequence of minimizer over $\mathcal{D}_M$, may not be convergent or may tend in some sense towards a curve that does not define a shape.

On the other hand, let $D^*_M \in \mathcal{D}_M$ be the optimal solution of the appropriate problem over $\mathcal{D}_M$, and $\eta^*_M \in \mathcal{M}^+(\omega)$ be the optimal measure which represents the boundary of D^*_M (∂D^*_M); then because $\mathcal{M}^+(\omega)$ is compact, the sequence $\{\eta^*_M\}_{M=1}^{\infty}$ and hence $\{\partial D^*_M\}_{M=1}^{\infty}$, have a convergent subsequence even they are not convergent. Young in [63] has shown that their related subsequences of broken lines, tends to an infinitesimal zigzag (generalized curve). This is not (necessarily) an admissible curve (see [63] Chapter *VI*). So the solution over $\mathcal{D}_M$ does not tend to the solution over $\mathcal{D}$, even in the weakly*-sense. Also, there is the important point that too oscillatory boundaries (like the infinitesimal zigzag) sometimes cause problem; Pironneau in [44] shows some of these problems.

So, we prefer to fix the number of M in Chapters 4, 5 and 6, and search for the optimal solution of the appropriate optimal shape design problems over $\mathcal{D}_M$; and perhaps do this for several, interesting, values of M.

Appendix C

Some limitations on *AMOEBA*

The *downhill simplex method* (for finding the minimizer of a function with more than one variables), requires only function evaluations, not derivatives. It is not very efficient in terms of the number of function evaluations that it requires. However it may frequently be the best method to use ([47]).

The method is appropriate for the minimizing of a function of N variables; it depends on the comparison of function values at the $N+1$ vertices of a general simplex (a geometric figure consisting, in N dimensions, of $N+1$ points or vertices and all their interconnecting line segments, polygonal faces, etc; for instance in two dimensions, a simplex is a triangle), followed by the replacement of the vertex with the highest value by another point. The simplex adapts itself to the local landscape, and contracts to the final minimum. The method takes a series of steps; most steps just moving the points of the simplex where the function is largest through the opposite face of the simplex to a lower point. The routine name $AMOEBA$ is intended to be descriptive of this kind of behaviors (see [47]).

In one-dimensional minimization, it is possible to bracket a minimum, so that the success of a subsequent isolation is guaranteed. But there is no analogous procedure in multidimensional space. For multidimensional minimization, the best we can do is give the algorithm a starting guess, that is, an N-vector of independent variables as the first point (initial value) to try. The algorithm is then supposed to make its own way till it obtains an (at least local) minimum. Therefore, it is frequently a good idea to restart $AMOEBA$ at a point where it claims to have found a minimum. But in the examples of Chapters 4, 5 and 6 we do not have any idea about the minimizer. Hence we obtained different values for the different initial values; indeed, we obtained the different local minimizers. Also there is no claim that $AMOEBA$ is able to determine the global minimizer (see for instance [42] and [47]). Beside this, it is also advised not to use the method for minimizing a function with more than 6 variables; although, in Chapters 4, 5 and 6, we applied $AMOEBA$ to minimize functions with 8 variables.

Appendix D

Introducing the suboptimal Control

Let the pair (μ^*, ν^*) be the optimal solution of the finite linear programming problem in (5.39), and ν^* be defined as:

$$\nu^* = \sum_{k=1}^{K} \beta_k^* \delta(z_k)$$

where the coefficients $\beta_k^* \geq 0$, and z_k belongs to a dense subset of ω. We are going to introduce the nearly optimal control function, v_D^* for the given optimal measure ν^*, defined on the given domain D. We know that the set of measures associated with the piecewise-constant functions on D is dense in $\mathcal{M}^+(\omega)$; hence we will approximate the optimal measure ν^* by a piecewise-constant function on D, which is called the suboptimal control function.

In the same way as Rubio in [50], for a given $\epsilon_1 > 0$, it is possible to find numbers

$$0 = y_0 < y_1 < \cdots < y_i < \cdots < y_R = 1,$$

and Borel sets $V_1, V_2, \dots, V_j, \dots, V_S$, forming a partition on $\chi \times V$ such that for any $i = 1, 2, \dots, R$, $j = 1, 2, \dots, S$, $k = 1, 2, \dots, M_1$, $y, y' \in [y_{i-1}, y_i)$, (x, v) and (x', v') in V_j, we have

$$| G_k(x, y, v) - G_k(x', y', v') | < \epsilon_1;$$

here it is supposed that $\chi = \{x \mid \exists y : (x, y) \in D\}$.

Let $D_{y_{i-1} y_i} = \{(x, y) \in D : y_{i-1} \le y \le y_i\}$, $K_{ij} \equiv \nu^*(([y_{i-1}, y_i) \times V_j) \cap \omega)$; we define

$$g_i(x, y, v) = \begin{cases} 1 & \text{if } (x, y, v) \in (\chi \times [y_{i-1}, y_i) \times V) \bigcap \omega \\ 0 & \text{otherwise;} \end{cases}$$

then $\sum_{j=1}^{S} K_{ij} = \nu^*((\chi \times [y_{i-1}, y_i) \times V) \cap \omega) = \nu^*(g_i)$. We note that the function g_i depends only on variable y. We also have $\bigcup_{i=1}^{R}(D_{y_{i-1} y_i}) = D$, and the Lebesgue measure of $D_{y_{i-1} y_i}$ is the area of $D_{y_{i-1} y_i}$ (the region in D introduced by lines $y = y_{i-1}, y = y_i$ and the boundary of D); this area is denoted by Δ_i (i.e. the area of $D_{y_{i-1} y_1} = \Delta_i$, see Figure D.1).

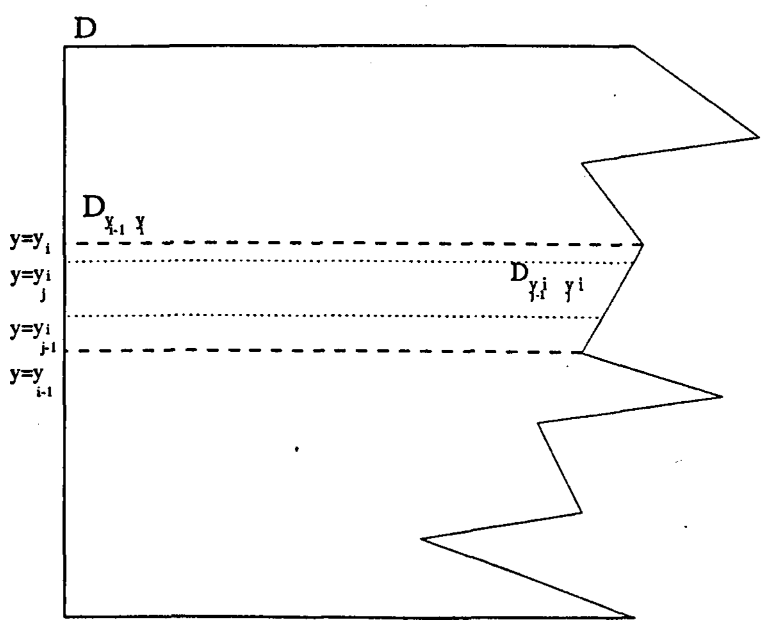

Figure D.1: Domain D and the region $D_{y_{i-1} y_i}$

Now, we follow the same way as Rubio did in Chapter 4 of [50] and apply the Chebyshev approximation of G_i (one may also use the method explained in [31] by some changes); if one defines $H_{ij} = K_{ij}(1 + \rho_i^{M_3})$ where $\rho_i^{M_3} = \frac{-\delta_i^{M_3}}{\Delta_i+\delta_i^{M_3}}$ that $\delta_i^{M_3}$ is introduced in [50], we have $\nu^*(g_i) = \Delta_i + \delta_i^{M_3}$ and $\sum_{j=1}^{S} H_{ij} = \Delta_i$.

We can now proceed to construction of the suboptimal control function which approximates the action of ν^* on the functions $G_i, i = 1, 2, \ldots, M_1$, and ζ_k, for $k = 1, 2, \ldots, M_3$, (note that ζ_k and g_i are similar functions). Let the lines $y = y_{j-1}^i$ and $y = y_j^i$, that

$$y_{i-1} \leq y_{j-1}^i \leq y_j^i \leq y_i,$$

be such that the area of $D_{y_{j-1}^i y_j^i}$ is equal to H_{ij}, and (x_j, v_j) be an element of V_j for $j = 1, 2, \ldots, S$. Define now the control function as follows

$$v(x, y) = v_j, \ \forall (x, y) \in D_{y_{j-1}^i y_j^i}.$$

It is shown in [50] (and similarly in [31]) that this piecewise-constant function approximates the suboptimal control function. By applying the above information on the result of the finite linear programming problem defined in (5.39), we will be able to find the suboptimal control function in the following.

Let $\beta_1, \beta_2, \ldots, \beta_{M_1+M_3}$, be the nonzero coefficients in the definition of ν^*, which resulted from the finite linear programming problem (5.39), and $z_k = (x'_k, y'_k, v_k), k = 1, 2, \ldots, M_1 + M_3$, be its corresponded points in the discretization on ω (in which they were ordered as a decreasing sequence with respect to the components y'_k's and x'_k's). For a number s so that $1 \leq s \leq M$, assume the points

$$(x'_{s1}, y'_{s1}), (x'_{s2}, y'_{s2}), \ldots, (x'_{sp}, y'_{sp}), 1 \leq sj \leq M_1 + M3$$

(for all $j = 1, 2, \ldots, p$), are in D_s and $\beta_{s1}, \beta_{s2}, \ldots, \beta_{sp}$, in $\{\beta_1, \beta_2, \ldots, \beta_{M_1+M_3}\}$ are their corresponding coefficients. Because ν^* is projected on D as Lebesgue measure, $\sum_{i=1}^{p} \beta_{si} =$ area of $D_s \equiv a_s$; indeed for each $i, \beta_{si} = H_{si}$, defined above. For the given value β_{si}, we look for the line $y = y_{si}$ so that the area of the region of D, between the lines $y = y_{si}$ and $y = y_{s(i-1)}$ (shape EFHG or EFOHG in the Figure D.2) is equal to β_{si}; here we assume $y_{s0} = Y^s_{s-1}$ and $Y^s_0 = 1 = y_0$, a is the area of $ABOP$, and we also suppose that the line $y = y_{sj}$ and hence the point (x_{sj}, y_{sj}) (the intersection point of line and the curve Γ), is calculated for each $j = 1, 2, \ldots, i-1$,. Then we have the following three possible cases. In each cases we try to determine the line $y = y_{si}$ and the point (x_{si}, y_{si}) by solving the system of two equations; the first one represents an area condition, and the other a line formulation.

- (I) If $\sum_{j \le i} \beta_{sj} < a$, then the line $y = y_{si}$ is located under the line $y = Y_s$; thus the line and the point (x_{si}, y_{si}) can be obtained from the system of equations below,

$$\begin{cases} 0.5(y_{si} - Y^s_{s-1})(x_{si} - X^s_{s-1}) = \sum_{j \le i} \beta_{sj} (= \text{area of } ABHG); \\ y_{si} = \frac{Y_s - Y^s_{s-1}}{X_s - X^s_{s-1}}(x_{si} - X_s) + Y_s. \end{cases}$$

- (II) If $\sum_{j < i} \beta_{sj} < a$, and If $\sum_{j \le i} \beta_{sj} > a$, then the line $y = y_{si}$ is located above the line $y = Y_s$; hence the solution of the following system of equations gives the line and the point;

$$\begin{cases} 0.5(y_{si} - Y_s)(x_{si} - X_s) = \sum_{j \le i} \beta_{sj} - a (= \text{area of } POHG); \\ y_{si} = \frac{Y^s_s - Y_s}{X^s_s - X_s}(x_{si} - X_s) + Y_s. \end{cases}$$

- **(III)** If $\sum_{j<i} \beta_{sj} > a$, then at least the lines $y = y_{s(i-1)}$ and $y = y_{si}$ are located above the line $y = Y_s$. Hence β_{sj} is the area of the region of D_s which locates between the lines; thus the linear system of the equations below, determines the line and the point;

$$\begin{cases} 0.5(y_{si} - y_{s(i-1)})(x_{si} - x_{s(i-1)}) = \beta_{sj}; \\ y_{si} = \frac{Y_s^* - Y_s}{X_s^* - X_s}(x_{si} - X_s) + Y_s. \end{cases}$$

Therefore, the suboptimal control function $v(\cdot)$ is determined on the region of D located between the lines $y = y_{s(i-1)}$ and $y = y_{si}$, by $v(x, y) = v_{si}$.

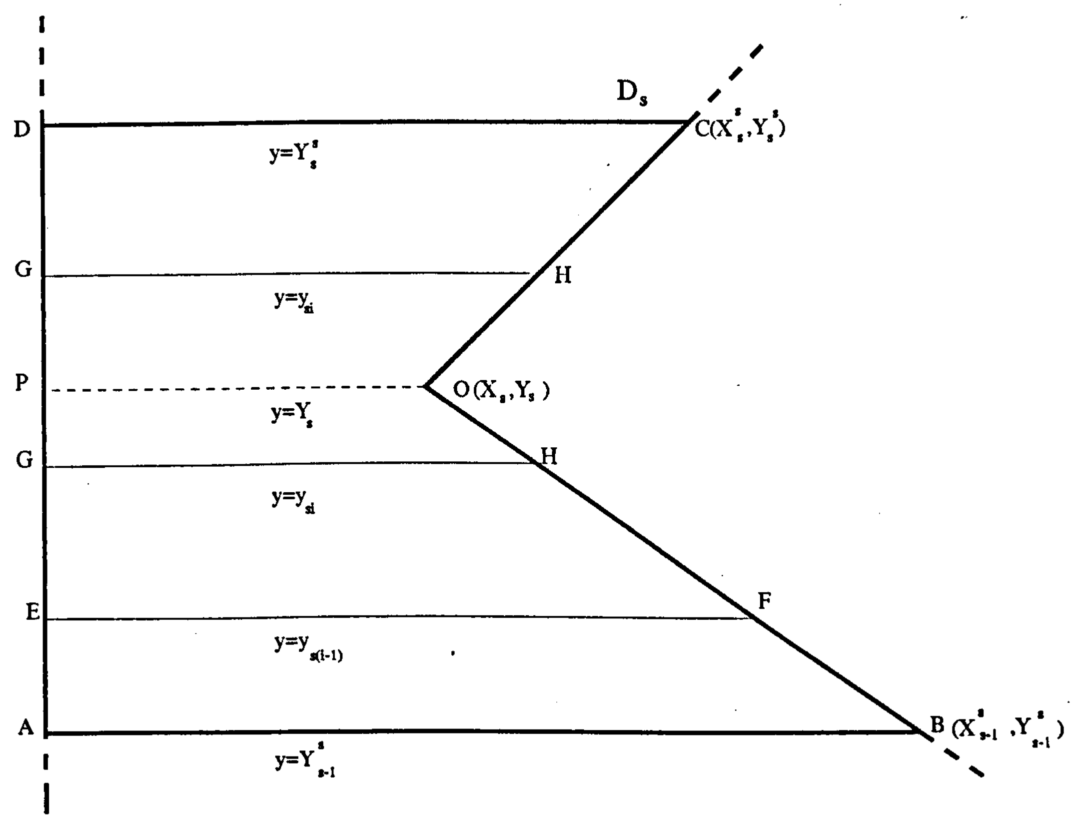

Figure D.2: The possible position of the line $y = y_{si}$ in D_s

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