STOCHASTIC PROCESS MODELS FOR DYNAMIC TRAFFIC ASSIGNMENT

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The candidate confirms that the work submitted is his own and that appropriate credit has been given where reference has been made to the work of others

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Abstract

This research explored the idea of unifying the deterministic and stochastic process approaches together, and developing a generalised framework of dynamic traffic assignment models to include day-to-day and within day variations in traffic flow. The framework of models is also aimed at capturing individual drivers' adaptation of route choices based on the route costs experienced through suitable driver learning models. In this thesis, the route flows within a day in a given departure period are modelled as random variables, and their evolution over a period of time (a number of days) is modelled as stochastic processes based on the laws of probability. The interactions among the route flows from various departure periods over the network links in space and time, are modelled through dynamic link loading procedures. Stochastic processes under certain mild conditions admit a unique stationary probability distribution which can be modelled by using simulation techniques. Alternatively, the moments (e.g., mean and variance) of the equilibrium (stationary) probability distribution can also be estimated. This research has advanced the idea of estimating the properties of equilibrium probability distribution by making a particular contribution in formulating the methodology for computing the Jacobians of route travel times with respect to the route inflows in a doubly dynamic assignment context using an analytical procedure, which are necessary for estimating the variance-covariance matrices of stationary route flows.

In this modelling framework, there are three modules – the first one is a day-to-day route choice model defined as a discrete time stochastic process, the second is a continuous time dynamic network loading of the route flows considering the complete spatio-temporal effects of the traffic flows that use the road links at the same time, and the third is the drivers' learning and adjusting model based on a linear filter. The main idea of estimating the properties of stationary probability distribution in this research builds on two earlier results: firstly, when the demand is sufficiently large, the equilibrium probability distribution converges approximately to a Multivariate Normal distribution and its mean coincides with the SUE flows; secondly, the variance can be estimated by an approximation procedure.

The equilibrium probability distribution can also be worked out using the commonly followed Monte Carlo simulation technique, which involves simulating the route choice process as a multinomial probability distribution over a number of days, and then summarising the properties e.g., the mean and the variance of the stationary probability distribution. This procedure though simple, is time consuming and the main difficulty lies in detecting the stationarity of the process. Based on the necessary conditions, simple and practically useable tests for identifying the stationarity of a stochastic process have been introduced. These tests involve analysing autocorrelations and computing large lag standard errors in autocorrelations.

The stationary variance-covariance of route flows obtained by the variance approximation model, was compared with that computed by the simulation procedure. Overall, the variance approximation model performs satisfactorily. Variance-covariance of route flows has been found sensitive primarily to the input logit choice parameter, which defines the boundaries of the validity of the variance approximation model. Variance-covariance is also affected by the memory length with the shorter memory systems essentially producing highly variant systems. Similarly, the variance-covariance of route flows is also sensitive to the memory weight, and the lower memory weight ($0 <$ memory weight $<< 1$) produces the same effect as that of shorter memory systems.

The Jacobians of the travel times worked out in this thesis have much wider applicability, and a few possible situations have been listed here among many others. Firstly, the optimisation based user equilibrium programs can be speeded up by defining the descent direction with the help of the Jacobians. Secondly, the Jacobians may be found very useful in defining the dynamic road user pricing problems. Finally, the sensitivity analysis of user equilibrium problems requires the computation of the Jacobians.

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Chapter 1
Introduction

1.1 Motivation

In transport network modelling exercises, it is a common experience to the modellers that the observed traffic flows and the modelled traffic flows are usually at a distance from each other and their convergence may depend on several factors such as the routing of the vehicles, fineness in the calibration of the model, stability of the flows and delays across the iterations within the model etc., for any given set of inputs including the trip matrix and traffic counts. Usually, it is assumed that the trip matrix and counts are accurate enough and the difference between the modelled and observed flows is reduced by altering one or more of the calibration parameters before repeating the model run and checking the validity of the model again. The whole modelling framework follows the idea that some self consistent state of flows (costs) exists, which can reproduce itself for any given set of costs (flows). Such a state is commonly referred as a state of equilibrium and the corresponding flows and costs are interpreted as long term averages of any transient state of the system. The validation stage of the modelling exercise usually relies on counts averaged over a shorter period and any difference between the modelled and observed flows is not sufficiently explained. This approach completely ignores the variable nature of the traffic flows, which is inherent to any given transport network system.

In fact, the traffic flows on roads vary both during the day and across the days. Consider the traffic count data collected over a period of a month during 6:00 – 12:00 everyday and summarised in Figure 1.1. This figure clearly illustrates the building up of traffic flow during the morning peak period with a peak flow rate of over 3000 vehicles, which then stabilises at approximately half the peak during the rest of the morning period. This is usually referred to as within day variation of traffic flow. Similarly, there is a considerable variation of traffic flows across the days with clearly marked week day and week end patterns. This type of variation in traffic flow across the days is referred as day-to-day variation.

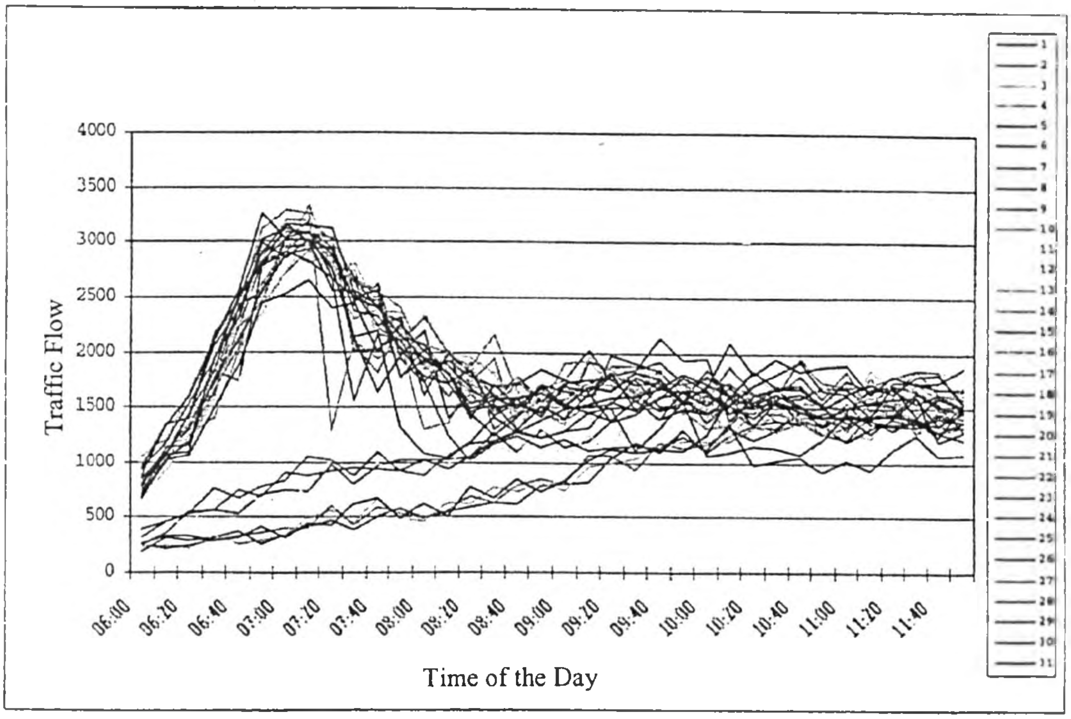

Figure 1.1 Day to Day and Within Day Variability of Traffic Flows
(Source: External Vehicle Speed Control, Deliverable D 11.3 (1999), ITS Leeds and MIRA)

In the past, there were several attempts made to address this question, especially in the case of variation of traffic flow within a day, e.g., Friesz et al (2001), Li et al (2003), Han (2003), etc. These models, commonly known as dynamic traffic assignment models, assume the traffic flow to be a deterministic variable which can take a single value at a point of time and solve for user equilibrium or system optimal type solutions which are dynamic extensions of Wardrop (1952) principles. Some researchers introduced the concept of variation in the perceived costs of drivers by adding an error term to the average route costs into the modelling framework, and formulated fixed point conditions for the user equilibrium (Daganzo and Sheffi, 1977), which is commonly referred to as Stochastic User Equilibrium. In spite of its name, this model assumes the flow as a deterministic variable, and one can work out exactly the number of drivers that will choose any particular route, given the cost difference between the alternatives.

From the above, it is clear that the question of variability in traffic flows cannot be addressed satisfactorily within the framework of conventional deterministic models as it essentially predicts a single outcome given a set of circumstances. By contrast, a wider range of modelling options are now becoming available with stochastic models which assume the traffic flow as a random variable, and such models predict a set of outcomes weighted by their likelihoods or probabilities, just as a truly stochastic model would be expected to perform.

In the literature there are several stochastic process models developed, for example, Cascetta (1989), Cascetta and Cantarella (1991), Cantarella and Cascetta (1995) and Watling (1996), which also assume the traffic flow to be a random variable and solve for equilibrium in probability distribution – stochastic analogue of deterministic equilibrium. In fact, a state of deterministic equilibrium may be considered as a particular instance of an equilibrium probability distribution. Davis and Nihan (1993) in their asymptotic results showed that the stochastic processes converge to multivariate Normal distributions, approximately, with the mean equivalent to the Stochastic User Equilibrium (SUE) flows. This important result forms the basis for all the attempts in unifying the deterministic and stochastic approaches in traffic modelling. Cantarella and Cascetta (1995) noted that, provided the O-D demand and capacity values are large enough, the deterministic equilibrium link and path flows become closer to the mean values obtained through a stochastic process model.

Usually, stochastic process models are characterised with the help of Monte Carlo techniques to generate a pseudo-random observation of the whole process over a given period of time, referred to as a realisation. But such methods are not free from drawbacks, such as the difficulty in identifying the stationarity of the process and the presence of highly autocorrelated flows. In addition, they require a lot of computer time even with fast modern computers for networks and demand levels of reasonable practical significance. Hence, there is a need for developing fast and efficient methods of working out the equilibrium probability distribution.

Returning back to the Davis and Nihan's asymptotic result, in order to describe a Normal distribution completely, one would require its mean and variance. Since the mean, which is equal to the SUE flows, can be estimated by some method such as the Method of Successive Averages, we only need to estimate the variance. Following this lead, Hazelton and Watling (2004) made an estimate of the variance in the case of day-to-day dynamic, but static within day, conditions assuming standard BPR style cost-flow functions. In order to include the within day variability, there is a need to extend their modelling framework to incorporate dynamic cost flow functions. However, in such a case, working out some of the parameters required to proceed with the variance approximation, (for example the Jacobians of the travel time functions with respect to the path inflows) needs a detailed specification for their computation, and poses a major challenge to the modellers. Based on this line of investigation the following objectives have been formulated for this research work.

1.2 Objectives

The main aim of this research project is to develop an integrated mathematical model addressing the day-to-day and within day variation of traffic flows which unifies the stochastic and deterministic approaches. Within this framework, the following list states the specific objectives of the research work undertaken:

- to specify a combined stochastic process model representing day-to-day and within day dynamics of traffic flows, and to approximate the mean and variance of the stationary probability distribution of the stochastic process;
- to develop an analytic modelling framework for working out the Jacobians of the dynamic travel time functions with respect to the path inflows;
- to identify and implement a suitable dynamic network loading model representing the within day dynamics of traffic flows, which would facilitate the above objectives;
- to develop an implementation framework for the unified model using suitable computer programming;
- to develop and implement a doubly dynamic simulation model and study its properties; and
- to compare the properties of equilibrium probability distribution obtained by the method of approximation and the method of simulation.

1.3 Structure of the Thesis

This thesis is divided into eight chapters including this introductory one, and three appendices as listed below:

Chapter 2 reviews the dynamic traffic assignment models;

Chapter 3 discusses dynamic network loading methods in the context of within day assignment process;

Chapter 4 proposes a theoretical framework for the combined day-to-day and within day stochastic process model;

Chapter 5 formulates an analytic model for deriving the travel time derivatives with respect to the path inflows;

Chapter 6 analyses the behaviour of the drivers across the days in a doubly dynamic simulation experiment;

Chapter 7 discusses the overall framework of computer programs for approximating the equilibrium probability distribution and illustrates the principles using suitable numerical examples; and

Chapter 8 summarises and concludes the research carried out.

Appendix A lists the papers published and conferences at which presentations were made on this research;

Appendix B shows the approximated and simulated variance-covariance matrices for a five-link network; and

Appendix C includes the conditional covariance, Jacobian matrices of route choice probability functions and travel time functions, and variance-covariance matrices by the approximation method and simulation method, in the case of a grid network.

Chapter 2

Review of Dynamic Traffic Assignment Models

Dynamic traffic assignment in the literature commonly refers to the modelling of traffic flows on street networks considering the variations in the demand within a day (within a peak hour, for example) and capturing the spatio-temporal congestion effects through suitable dynamic link travel time functions. Usually such models are aimed at solving for either dynamic system optimal (Li et al 2003) or dynamic user equilibrium (e.g., Heydecker and Addison 2005, Friesz and Mookherjee 2006) or dynamic stochastic user equilibrium (Han 2003), and as they consider the traffic flow as a deterministic variable, the solutions arising out of such approaches naturally tend to be deterministic, representing a single outcome at each moment within a day. Or in other words, the deterministic approaches ignore the fluctuations in the demand/ supply and hence cannot explain the random variations in the traffic flow on the street network (Cascetta, 1989). In addition, the deterministic approaches, primarily driven by the conventional optimisation algorithms, aim to solve the equilibrium state and so, are unable to represent the transient states in the evolution towards equilibrium (Cantarella and Cascetta, 1995).

On the other hand, there are a few other dynamic assignment models which concentrate mainly on the modelling of traffic flows over a sequence of days and consider the evolution of traffic flows, either as a stochastic process (Cascetta 1989, Cantarella and Cascetta 1995, Watling 1996 and Watling and Hazelton 2003) or as a deterministic process (Friesz et al 1994, Smith and Wisten 1995). However, these models do not deal with the spatio-temporal congestion effects since they adopt static link cost functions, and hence are called day-to-day dynamic assignment models. Usually, the day-to-day models represent the state of the traffic flow system on any given day by linking the system states in the past over a period of time by incorporating suitable driver learning models. Driver learning models deal with driver psychology (e.g., memory length, habit), information acquisition process (e.g., route guidance systems, external sources of information such as the weather reports), and represent the process of consolidating the experiences helping further in developing new perceptions for the following day. Various types of learning models, usually called the filters, are described in Horowitz (1984), Cantarella and Cascetta (1995) and Jotisankasa and Polak (2005) among many others.

In fact, the purview of dynamic traffic assignment is much wider and includes day-to-day variations in the demand besides the within day variations and even capturing the delays due to the congestion arising from traffic flow propagation in time and space. Such models are called *doubly dynamic traffic assignment* models, e.g., Cascetta and Cantarella (1991), represent such doubly dynamic approaches. Doubly dynamic traffic assignment is the main subject of this thesis, which is aimed at developing an alternative approach to the commonly followed simulation methods.

This chapter reviews the day-to-day dynamic assignment models which are based on deterministic as well as stochastic process modelling approaches. It then introduces the concept of equilibrium for stochastic process models, and also reviews driver learning models, an essential element of the day-to-day models. Following the review of day-to-day models, principles involved in carrying out the within day dynamic assignment are outlined in this chapter. This chapter also touches upon doubly dynamic models. Finally, simple numerical examples are included to demonstrate the principles of stochastic equilibrium.

2.1 Review of Day-to-day Dynamic Assignment Models

Day-to-day dynamic assignment models deal with the evolution of path /link flows during the peak period/ hour over a number of days and consider *drivers' learning and adjusting* of their departure time/ route choices based on their past experience, potentially combining pre-trip information obtained through exogenous sources such as radio/TV, and solve for *equilibrium in the deterministic or stochastic sense.* A few models that deal with the random day-to-day fluctuations in the demand may culminate in an equilibrium state of a stochastic process described by an equilibrium probability distribution, while most other models assume deterministic demand variables and hence solve for deterministic equilibrium. A day-to-day model needs to incorporate a built-in within day model, which may be dynamic or static in nature. A day-to-day dynamic model with an in-built static within day model is referred to as a 'pure' day-to-day dynamic model, which is the main focus of this section.

2.1.1 Stochastic Process Modelling Approaches

Stochastic processes are mathematical abstractions of empirical processes whose development is based on probabilistic laws (Doob 1952). In the context of traffic network models, these methods consider the demand as a discrete random variable. Under such assumptions, the system is visualised as occupying various possible states whose realisation is modelled using methods such as Markov models. Stochastic models are very flexible in that they can incorporate any driver behaviour at an appropriate level of aggregation.

Cascetta (1989) introduced the stochastic process models for dynamic traffic assignment in a day-to-day context, but assumed a static within day model. That is to say that the model considers the evolution of system states over a sequence of discrete epochs of time based on a constant within day demand profile. In a day-to-day context, the route choices (i.e., the current system state) are assumed to be a function of the previous experiences (i.e., past system states) such as flows/costs along the used routes up to a finite memory length of the drivers – essentially a Markov chain property. A stochastic process model can then be used to characterise the probability distribution of traffic flows on the network. When the probabilities of choosing a route are stable over successive days, the system is said to have reached an equilibrium state and the corresponding probability distribution of flows is called *equilibrium probability distribution.*

Consider a network of directed links serving O-D demand represented by $\mathbf{Q} = \{\ldots, q_k, \ldots\}$ where q_k is the O-D demand for a particular *commodity* k, each commodity defining a combination of origin, destination and (discrete) departure period within a day. Each commodity k is served by a set of routes R_k with $|R_k|$

elements; the full route set across all commodities thus has dimension, $\rho = \sum_{k=1}^{K} |R_k|$. Let **f** be the ρ- vector of commodity route flows and **c(f)** be the vector of commodity route costs.

It is assumed that all the trip makers of commodity k are rational in their behaviour when choosing their route, in an attempt to minimise their perceived cost of travel. This means that the population of drivers in commodity k will have an aggregate memory which is identical to the aggregate memory case of Cantarella and Cascetta (1995). For each commodity k and route $r \in R_k$, the perceived travel cost $\hat{C}_r^{(n)k}$ at the start of day n is given by

$$\hat{C}_r^{(n)k} = C_r^{(n-1)k} + \eta_r^{(n)k} \qquad (2.1)$$

where, $C_r^{(n-1)k}$ is the population-mean perceived cost for commodity k and route r at the end of day n-1, and $\eta_r^{(n)k}$ is a random variable describing unobserved attributes contributing to the population-dispersion of the perceived attractiveness of route r by commodity k. The ρ-vector $\mathbf{C}^{(n-1)}$ represents the collection of population-mean perceived costs across all routes and commodities. The probability of choosing route r on day n is then given by:

$$\mathrm{p}_r^k(\mathbf{C}^{(n-1)}) = \mathrm{Prob}\left(C_r^{(n-1)k} + \eta_r^{(n)k} < C_i^{(n-1)k} + \eta_i^{(n)k}\right) \qquad \forall i \neq r \qquad (2.2)$$

$\mathbf{p}^k(.)$ then represents the vector (of dimension $|R_k|$) of route choice probabilities for the commodity k, and **p**(.) denotes the collection of the choice probability vectors over all the commodities (i.e. **p**(.) is a vector of dimension ρ). The functional form of the path choice probabilities depends on the joint probability density function assumed for the residuals $\{\eta_r^{(n)k} : r \in R_k\}$ for each commodity k, resulting (for example) in a logit model, if independent Gumbel distributions are assumed, and a probit model for a multivariate Normal distribution.

While the behavioural choice-side of the model is quite conventional, a simple linear learning filter is used to replicate drivers building up their experience of travel costs on a day-by-day basis following the completion of each day's trip. Please refer to section 2.1.3 for a review of drivers' learning models. Cascetta (1989) assumed a simple weighted average approach similar to the other simulation experiments, for example, Horowitz (1984). Thus following the completion of trips on any day (n-1), the population-mean perceived costs are updated based on a weighted average of costs actually incurred in a finite number of previous days m, using the form:

$$\mathbf{C}^{(n)} = s(\lambda)^{-1}\left\{ \mathbf{c}(\mathbf{F}^{n-1}) + \lambda\mathbf{c}(\mathbf{F}^{n-2}) + \ldots + \lambda^{m-1}\mathbf{c}(\mathbf{F}^{n-m}) \right\} \qquad \forall\, 0 < \lambda < 1 \qquad (2.3)$$

$$s(\lambda) = \sum_{j=1}^{m} \lambda^{j-1} = (1-\lambda^{m})/(1-\lambda) \qquad (2.4)$$

where, s(λ) is simply a scaling factor to make the weights sum to unity and **c**(.) is the commodity route cost-flow function as defined above, and $\mathbf{F}^n$ is a vector random variable of dimension ρ denoting the network path flows by commodity on day n. Assuming that for any day n and for each commodity k, all q^k drivers wishing to travel make their travel choices independently, conditional on their experiences in past days, then the number of drivers taking each possible route on day n by each commodity k, conditional on the costs (2.3) experienced in the past, is obtained as:

$$\mathbf{F}^{(n)k} \,\big|\, \mathbf{C}^{(n-1)} \sim \textit{Multinomial}\ (q^k, \mathbf{p}^k(\mathbf{C}^{(n-1)})) \text{ independently, for k} = 1,2,,\ldots\text{K} \qquad (2.5)$$

where, $\mathbf{F}^{(n)k}$ is the vector of route flows on day n by the commodity k. Typically, the working to implement (2.5) would involve calculating the multinomial probability of choosing a route conditionally, given the costs up to the previous day (assuming that the route choice is a function of the costs only) in the space of discrete demand and feasible route choices. These probabilities are called transition probabilities and the matrix of such cells is referred to as a transition matrix, denoted by M. Then the equilibrium probability distribution denoted by π, is calculated such that,

$$\pi^T = \pi^T M \tag{2.6}$$

where, T denotes the transposing operator and M is the transition probability matrix. The elements of π represent the probability that the system occupies a particular state as defined by the random variable commodity flow $F_r^{(n)k}$, over a given route on any given day. Equation (2.6) means that the probability distribution is invariant, with itself repeating over time. For real sized problems, the dimensions of the vector π are extremely large and so directly solving this condition is not a viable option. As such, it is common to use simulation methods to approximate the equilibrium probability distribution, which involves repeating the route choice process over a number of days using simulation techniques, and then working out the summaries such as means, variances at the end of the simulation. For simple networks catering to a very few trips, an equilibrium probability distribution can be directly solved. One such example is included in section 2.4 of this chapter. However, for larger networks this method is infeasible and requires either a time consuming simulation process or even more sophisticated alternatives as described in Chapter 4.

2.1.2 Equilibrium Probability Distribution of a Stochastic Process

An equilibrium state of a stochastic process is defined in terms of an invariant probability distribution (e.g., probability distribution of route flows) which is conceptually different from that of a deterministic process which is defined in terms of deterministic variables (e.g., route flows). This difference is arising due to the underlying assumption that the stochastic process is defined in terms of random variables as opposed to deterministic variables. Equilibrium probability distribution is of interest to the researchers, as then one can describe the stochastic process using summaries such as means and variances, and similarly draw comparisons with the equilibrium state of a deterministic process. A stochastic process is called *stationary*, if at least one probability distribution is time invariant, it is *ergodic*, if stationary and exactly one stationary probability distribution exists. Finally, a stochastic process is called *regular*, if ergodic and converges to the same probability distribution irrespective of its starting state (Olofsson 2005). Cascetta (1989) uses the following sufficient conditions for the stochastic process of the traffic flow in path choice space to have a unique (ergodic), steady-state probability vector:

- Path choice probabilities are time homogeneous. They are invariant under temporal translation, given the previous states;
- Path choice probabilities depend on a finite number of the previous states; and
- Path choice probabilities are positive for all feasible paths. All the feasible paths have non-zero probability of being chosen each time.

Consider the stochastic process described in the previous paragraphs occupying system states defined in path flows F^n. Then the probability of the drivers of commodity k choosing route r on day n is given based on the previous system states, as below:

$$p_r^{k(n)} = p_r^{k(n)}\left[F^{n-1}, F^{n-2}, \ldots\right] \tag{2.7}$$

Under the assumption that the path choice probabilities are time invariant, they can be moved over time scale without affecting the probabilities of choosing the routes. Thus, moving back the route choice probabilities over the duration of χ, then we have,

$$p_r^{k(n)}\left[F^{n-1} = F_h, F^{n-2} = F_i, \ldots\right] = p_r^{k(n-\chi)}\left[F^{n-1-\chi} = F_h, F^{n-2-\chi} = F_i, \ldots\right] \tag{2.8}$$

Assuming that the drivers remember their experiences of route costs over a finite number of days m, representing the memory length, then

$$p_r^{k(n)}\left[F^{n-1},F^{n-2},...\right]=p_r^{k(n)}\left[F^{n-1},F^{n-2},...F^{n-m}\right] \tag{2.9}$$

Equation (2.9) implies that the route flow system is an m-dependent Markov chain which can be transformed to a 1-dependent Markov chain using an appropriate transformation of state space. Let the transformed state be F^{*n} corresponding to the day n, which includes the states on m-1 previous days plus the current state on day n, which can be written as below:

$$F^{*n}=\left(F^{n-m+1},F^{n-m+2},......,F^{n}\right) \tag{2.10}$$

Then under the third assumption all the system states can communicate with each other, making the system irreducible. Applying the results from the Markov chain theory (Cox and Miller 1972, Bath 1984), an irreducible a-periodic finite Markov chain admits a unique, steady-state probability distribution π, given by the equation (2.6). The one-step transition probabilities (clearly, the cell values of the matrix, M) between the transformed states can thus be computed as the probability given by the multinomial distribution by equation (2.4). Further, it was also proved that, if the process is stationary and ergodic in path choice space, the same properties hold for the processes in path and link flow spaces (Cascetta 1989).

2.1.3 Driver Learning Models

Many drivers in general repeat their trips, for example for the purpose of work, but on any given day the (generalised) cost of travel is not known beforehand. Instead, the drivers develop some kind of perception of the cost of travelling by various routes based on their past experiences. The perceived cost is also complemented by external sources of information, for example, weather report, route guidance systems. This section reviews various approaches followed in modelling the drivers' learning process, which is one of the important components of the day-to-day modelling.

There are various approaches in the literature on day-to-day traveller learning, see Jotisankasa and Polak (2005) for a comprehensive review. While it is noted that the modelling of driver learning is complex and a wide researching subject by itself, the three most common approaches followed are summarised here.

2.1.3.1 Weighted Average Approach

It is assumed that the drivers tend to remember their experiences over a finite number of days and develop their perception based on the most recent set of experienced costs. Thus, at the end of the day n-1 (equivalently at the beginning of the day n), the drivers update their perceived cost for the routes given by a linear combination of the experienced costs weighted by an appropriate weighting system. Commonly drivers tend to remember the most recent experience very well compared to the oldest. Thus, usually the most recent experience will be weighed higher relative to the oldest experience. Within the notational framework used so far, this can be expressed as,

$$C^{(n)} = s(\lambda)^{-1} \sum_{j=1}^{m} \lambda^{j-1}\, c(F^{(n-j)}) \qquad (2.11)$$

where, λ is the memory weight and can take values between (0,1) and s(λ) is the sum to m terms of the memory weight as given by equation (2.4). This approach has been widely used in the research, for example Horowitz (1984), Cascetta (1989), Nakayama et al (1999) and Watling and Hazelton (2003), but little practical investigation of the issues involved in calibrating the model has been reported.

2.1.3.2 Adaptive Expectation Approach

While the weighted average approach was criticised for its inability in accounting for the difference between the actual and perceived costs (Iida et al 1992), an adaptive approach which combines the perceived and actual costs from the previous day has been developed. This model in its simplest form can be expressed as,

$$C^{(n)} = \lambda\, c(F^{(n-1)}) + (1-\lambda)\, C^{(n-1)} \qquad 0 \le \lambda \le 1 \qquad (2.12)$$

This model has been widely used in the research, for example, Cascetta and Cantarella (1991), Ben Akiva (1991), Iida et al (1992). In this approach, the perceived cost on day n is bounded by the perceived cost on the previous day and the actual cost on day n-1, and so the travellers ignore all their other experiences.

2.1.3.3 Bayesian Approach

The third approach is based on Bayesian updating of the perceived travel time. Jha et al (1998) and Chen and Mahmassani (2004) adopted a two stage updating process – pre-trip (prior) and post-trip (posterior), in which the perceived travel time and travel time information provided from external sources (e.g., Advanced Traveller Information System, ATIS) are modelled as random variables with an assumed probability distribution. Pre-trip updating incorporates the information received into the historic perceptions, and the post-trip updating combines the experienced travel time with the pre-trip perception to become the historic perception for the following day. The drivers choose routes and departure periods based on the mean of the probability distribution of the perceived travel time. This model assumes that the underlying mean travel time is steady, and hence can not deal with significant changes in the network flow patterns.

2.1.4 Deterministic Process Modelling Approaches

The previous sections 2.1.1 – 2.1.3, dealt with the stochastic processes and the associated driver learning models. Now, the aim of this section is to introduce the deterministic process models, which consider the evolution of deterministic variables over time rather than random variables. Or in other words, deterministic process models ignore the random fluctuations of demand/ supply. Therefore, the path flow vector is equal to the average path flow vector as given below:

$$F^{(n)k} \equiv E\left[F^{(n)k} \middle| F^{(n-1)k}\right] \qquad \forall k \tag{2.13}$$

$$F^{(n)} \equiv E\left[F^{(n)} \middle| F^{(n-1)}\right] \tag{2.14}$$

Friesz et al (1994) formulated a continuous time deterministic process model, similar to an auctioneer increasing or decreasing the selling price of a good based on the demand, and applied the principles to compute the day-to-day user equilibrium. This process involves computing the excess travel demand or excess travel cost over the equilibrium solution after each time period, thus depicting the day-to-day adjustment of the process from one realisable disequilibrium state to another.

The rate of change of path flow in continuous time represented by the variable f(t), and the rate of change of perceived O-D cost represented by C(t), are described as an adjustment process over time which is considered as a continuous variable, with the dynamics specified as follows:

$$\frac{df(t)}{dt} = \varsigma(ETC) \tag{2.15}$$

$$\frac{dC(t)}{dt} = \xi(ETD) \tag{2.16}$$

$$f(0) = f^0 \tag{2.17}$$

$$C(0) = C^0 \tag{2.18}$$

where, ς, ξ are matrices of constants of appropriate dimensions, ETC is the vector of excess travel costs and ETD is the vector of excess travel demands. Equation (2.15) represents the path flows varying in proportion to excess travel costs, i.e., the path flow decreases if the excess travel cost increases, or vice versa. Excess travel cost is the difference between the actual and perceived travel costs. Similarly, equation (2.16) represents the perceived travel costs varying over time in response to the excess travel demand. Excess travel demand is the difference between the desired and actual travel demand for a given O-D pair. Equations (2.17) and (2.18) represent the initial conditions.

Friesz et al (1994) assumed BPR style cost flow functions for the link costs. They have shown that the process converges to a Wardrop's solution asymptotically from any given initial condition. The impact of the provision of information on the rate of convergence was investigated. Provision of complete information accelerated the process of settling down towards the equilibrium state and their results supported the findings of Mahmassani's (1990) experimental results that investigated exactly the same issue.

Smith and Wisten (1995) also formulated a day-to-day dynamic assignment model assuming deterministic flow variables in continuous time, and their route swapping rules are identical to the equations (2.15 and 2.16). However, the within day loading model divides the time of the day into a number of discrete departure periods and allows for the modelling of vehicle interactions at the bottlenecks through a deterministic queuing model. But, their model deals with only a single O-D pair serving two links which are paths as well, and hence cannot deal with the general case of multiple O-D pairs and links being shared by multiple paths.

Continuous time deterministic process approaches assume that the route choices are revised in continuous time, whereas due to the activity constraints certain types of trips (e.g., work, school trips) are repeatable over days (Watling and Hazelton 2003). Therefore, it would be natural to model such trips using discrete time approaches as done by Horowitz (1984), Cantarella and Cascetta (1995), Watling and Hazelton (2003). A simple discrete time deterministic process aimed at solving for *Stochastic User Equilibrium* (SUE) is defined by the following equations:

$$\boldsymbol{C}^{(n)} = \beta\, \boldsymbol{c}(\boldsymbol{f}^{(n-1)}) + (1-\beta)\, \boldsymbol{C}^{(n-1)} \qquad \beta > 0 \tag{2.19}$$

$$f_r^{(n)k} = q_k\, p_r^k(\boldsymbol{C}^{(n)}) \qquad \forall r, k \tag{2.20}$$

It is important to note that despite the name, SUE refers to deterministic equilibrium because for a given set of costs, one can exactly work out the route flow state occupied by the system defined by the proportions **p**(.), and hence it is different from stochastic equilibrium which is defined by a steady state probability distribution as defined in section 2.1.2.

2.2 Within Day Dynamic Assignment Approaches

The aim of this section is to briefly review the principles involved in carrying out within day dynamic traffic assignment. A within day dynamic assignment model considers a non-uniform demand profile during the day and allows for the modelling of time varying path and link flows through drivers' choice of departure time and/or route. A within day dynamic model thus models the spatio-temporal interaction between the time varying demand and supply through a *dynamic network loading method,* seeking the path/link flows consistent with *dynamic user equilibrium* or *dynamic system optimal* conditions. Peeta and Ziliaskopoulos (2001) provides a comprehensive review of the within day DTA models. More recently, Szeto and Lo (2005) reviews the analytical formulation of DTA models, and also touches upon the issue of differentiability of route cost functions in addition to the solution properties of DTA.

Dynamic User Equilibrium (DUE) is reached when all the drivers entering the network at each instant of time are assigned to the routes with equal and minimal costs, and all unused routes have equal or greater costs.

Now for dynamic user equilibrium, we need

$$f_r^k(t) > 0 \;\Rightarrow\; C_r^k(t) = C_r^{k*}(t) \qquad \forall\, r \in R_k \text{ and } \forall\, k,t \tag{2.21}$$

$$f_r^k(t) = 0 \;\Rightarrow\; C_r^k(t) \geq C_r^{k*}(t) \qquad \forall\, r \in R_k \text{ and } \forall\, k,t \tag{2.22}$$

where, $C_r^{k*}(t)$ is the least travel cost for route r on O-D movement k at continuous departure time t. Janson (1991) formulated a convex non-linear programming model aimed at solving for the dynamic user equilibrium defined by (2.21 and 2.22) as an extension to the static user equilibrium principles. Thus, the objective function of the DUE is very similar to the static user equilibrium in continuous time, and is indicated below:

$$\text{Minimise } z = \int_t \sum_a \int_0^{x_a(t)} \tau_a^t(\omega)\, d\omega\, dt \tag{2.23}$$

Subject to $$\sum_{r \in R_k} f_r^k(t) = q_k(t) \tag{2.24}$$

$$f_r^k(t) \geq 0 \quad \forall\, k,t \tag{2.25}$$

$$x_a(t) = \int_{\omega} \sum_{k} \sum_{r \in R_k} f_r^k(\omega)\, \Gamma_{a,t}^r(\omega)\, d\omega \qquad \forall a,t \tag{2.26}$$

where, $x_a(t)$ is the number of vehicles on the link a at time t, and

$$\Gamma_{a,t}^r(\omega) = \begin{cases} 1 & \text{traffic on path r departing at } \omega \text{ is present on link a at time t} \\ 0 & \text{otherwise} \end{cases}.$$

This approach was used by many others including Jayakrishnan et al (1995), Ran and Boyce (1996). Lin and Lo (2000) demonstrated that minimising the objective function (2.23) need not necessarily result in an assignment that satisfies the complementarity in (2.21 and 2.22). Han and Heydecker (2006) modified the objective function to solve for DUE in the case of separable cost functions. Typically, the objective function is calculated at each incremental interval of time, using the outflow that is assigned throughout that increment together with the costs at the end of it. However, their approach still needs to be tested over larger networks.

The Dynamic System Optimal (DSO) formulation aims at minimising the overall travel time on the network at each instant. Thus, the objective function is to

Minimise $$\sum_{k} \sum_{r \in R_k} \int_{\omega} C_r^k(\omega)\, f_r^k(\omega)\, d\omega \tag{2.27}$$

Subject to $$\frac{dx_a(t)}{dt} = \int_{\omega} [u_a(\omega) - v_a(\omega)]\, d\omega \tag{2.28}$$

$$x_a(t) \geq 0 \tag{2.29}$$

$$v_a(t) \geq 0 \tag{2.30}$$

where, $u_a(t)$ and $v_a(t)$ are the inflow and out flow rates for the link a. Merchant and Nemhauser (1978a) formulated DSO problem and followed by many others including Friesz et al (1989), Ziliaskopoulos (2000), Li et al (2003), Chow (2006).

The DUE paradigm assumes that all drivers have perfect knowledge of the route costs, and this limitation is relaxed by assuming that the drivers perceive route costs, which includes an error in perception over the average route cost as given by (2.1). When the errors in perception η_r^k are assumed independent Gumbel variables, then the route choice probabilities can be modelled based on logit principle (Dial 1971) as below:

$$p_r^k(t) = \frac{e^{-\theta C_r^k(t)}}{\sum_{s \in R_k} e^{-\theta C_s^k(t)}} \qquad \forall r,k \tag{2.31}$$

where θ is the dispersion parameter and $C_r^k(t)$ is the mean perceived route cost for route r, by commodity k, at time t.

When all the drivers entering the network at each instant are allocated to their least perceived cost routes, then the assignment is said to be in *Dynamic Stochastic User Equilibrium* (DSUE).

Many authors followed this approach in the literature, including Ran and Boyce (1996), Liu et al (2001) and Han (2003) among many others. The basic model can be expressed mathematically as below:

$$p_r^k(t) = \frac{f_r^k(t)}{q_r^k(t)} \qquad \forall r,k \tag{2.32}$$

$$\sum_{k \in R_k} f_r^k(t) = q_k(t) \tag{2.33}$$

$$f_r^k(t) \geq 0 \quad \forall k,t \tag{2.34}$$

where, $p_r^k(t)$ is given by (2.2). At DSUE, the route choice given by (2.32 – 2.34) should be the same as that given by (2.2), i.e., when the perceived cost does not vary following the dynamic network loading process.

The solution to the DSUE problem can be obtained by analytical methods based on optimisation or by so called simulation methods. One of the analytical procedures is based on solving the variational inequality problem that can be expressed as below (Han 2003):

$$\sum_k \sum_r \Lambda_r^k(t)\left[f_r^k(t) - f_r^{k*}(t)\right] \geq 0 \tag{2.35}$$

where, $\Lambda_r^k(t) = \left[f_r^k(t) - p_r^k(t)q_r^k(t)\right]\dfrac{\partial C_r^k(t)}{\partial f_r^k(t)}$ (2.36)

For the solution to the variational inequality (2.35) to exist, the mapping function (2.36) must be a continuous function of the cost (Nagurney 1993), and for the solution to be unique, the derivative in (2.36) must be strictly monotonic, i.e., the route costs increase with the increase in route flows. Rosa and Maher (2006) have been investigating to extend the DSUE by defining the perception errors at the link levels rather than route levels, thus leading to the formulation of probit-based DSUE.

While the analytical DTA models are usually limited to very small hypothetical test networks primarily aimed at studying the properties of the models, there are other classes of models developed mainly aimed at reflecting the traffic flow phenomena on larger real sized networks though not guaranteeing optimality or even convergence (Ziliaskopoulos et al 2004). This class of models are usually referred to as simulation based DTA models (Peeta and Ziliaskopoulos 2001) although they may not involve the simulation in the sense of realisation of the random variables by random drawings. The so-called simulation based DTA models are usually grouped into microscopic (e.g., Barcelo and Casas 2004, Liu et al 2006), mesoscopic (e.g., Mahmassani 2001, Taylor 2003) and macroscopic (e.g., Bliemer et al 2004) models, depending on the manner in which the traffic flow is treated within the modelling framework.

The basic framework (See Figure 2.1) of a simulation based within day dynamic traffic assignment model (e.g., Bliemer et al 2004) consists of a route choice model and a dynamic network loading model interacting with each other until a convergence between them is achieved. The route choice module allocates the O-D demand to feasible paths usually based on logit principles and requires the input information on travel times on paths/links. Similarly, the network loading model moves the allocated route flows along the links of the network in space and time, resulting in new path travel times based on the experience. A detailed review of the dynamic network loading models is set out in Chapter 3. The convergence is obtained using the Method of Successive Averages (MSA) following the principles set out in Sheffi (1985) which computes the average path flow using the following equation:

$$\overline{F}_r^{k,i} = [(1-(1/i))\,\overline{F}_r^{k,i-1} + (1/i)F_r^{k,i}] \quad (2.37)$$

where i is the current number of iteration, $\overline{F}_r^{k,i-1}$ is the average path flow over route r for commodity k up to (and including) the previous iteration i.e., (i-1), and $F_r^{k,i}$ is the current auxiliary path flows. It is well known that the MSA method converges far too slowly (Ortuzar and Willumsen 1999), and hence usually a large number of iterations (over which the flow averaging is done) is resorted to, so that the difference in flows/costs between successive iterations, in the end, may be considered negligible.

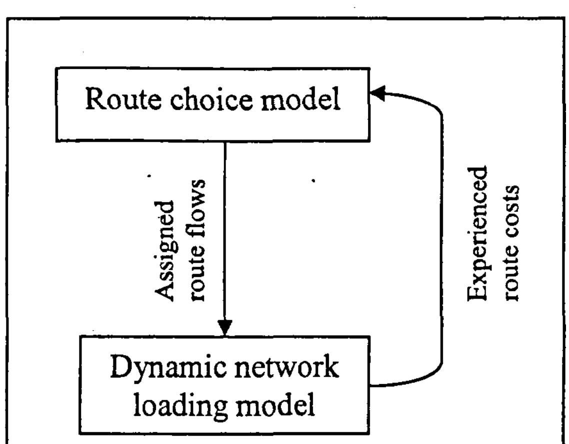

Figure 2.1 Outline of a DTA Model

2.3 Doubly Dynamic Assignment Models

In general the traffic flows on road networks vary within a day as well as across the days (Figure 1.1). Therefore, any traffic assignment model addressing only one of these variations is inadequate to understand the situation. Hence, there is a need to develop a comprehensive framework which addresses both day-to-day and within day variations in traffic flow. Such models are called *doubly dynamic traffic assignment* models, which are the main subject of the present thesis.

Although it is crucial to understand the doubly dynamic nature of the traffic flows for proper planning and management of the road networks, it is surprising to note that only very little research has been published on this topic. Cascetta and Cantarella (1991) developed such a doubly dynamic simulation model in which they defined the route flows on any day as a stochastic process, and used a simulation procedure based on which the summaries, such as means and variances were computed. They included a queuing model to capture the delays on the links. The mathematical specification of the stochastic process is very similar to the one set out in Section 2.1.1 earlier in this chapter. Cascetta and Cantarella (1991) used the adaptive expectation approach (Section 2.1.3.2) to update the perceived route cost at the beginning of any day n as set out by the equation (2.12). The dynamic network loading of the vehicles is handled through a queuing model, the principles of which are described in Chapter 3 – section 3.3.2. The main limitations of this approach are that the queuing models assume that the vehicles travel at average (or free flow) travel time until they reach the state of queuing (i.e., when the outflow rate is more than the exit capacity) and hence under-estimate the delays. Besides, the link travel time relationships are non-differentiable due to the discontinuity in travel time with the increase in flow on the link.

2.4 Numerical Exercises

This section is aimed at demonstrating the steady state (equilibrium) probability distribution and its sensitivity to the input modelling parameter i.e., the driver behaviour represented by θ in the logit choice, and also illustrates the implementation of the stochastic process modelling method over a simple network and discusses the results.

Consider the two link/route network (Figure 2.2) serving a single origin-destination pair with demand, q = 5 drivers with a memory length m = 1 day. The disutility of travel on route r is computed as the cost of travel defined as a function of the route flows. Note that in this example, the cost-flow functions assumed are static, and cannot capture the dynamics of time varying flow. Hence, this example corresponds to the case of day-to-day dynamic, but within day static assignment model.

Assume the cost flow functions for routes 1 and 2 as below:

$$c_1 = 10 + 5f_1 \tag{2.38}$$

$$c_2 = 5 + 10f_2 \tag{2.39}$$

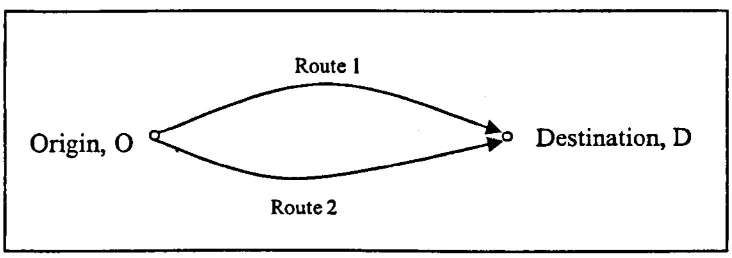

Figure 2.2 Two Link Network

It is assumed that the route choice probability follows the logit principle, and is given as,

$$P_r(c_r) = \frac{e^{-\theta c_r}}{e^{-\theta c_1} + e^{-\theta c_2}} \qquad \text{for} \quad r = 1, 2 \tag{2.40}$$

where, θ is the logit choice parameter which represents a measure of spread in the perceived route costs of the drivers.

The probability of choosing a route on any day is a function of the route costs on the day before, because the memory length parameter 'm' is taken as unity. Hence, the conditional probability distribution of a given number of drivers choosing route 1 on any day 'n' is binomial, and is given by:

$$P\left(f_1^n = j \middle| f_1^{(n-1)} = i\right) = \frac{q!}{(q-j)!\, j!}(\alpha_i)^j (1-\alpha_i)^{(q-j)} \qquad \forall i,j \in \{0,1,2,3,4,5\} \quad (2.41)$$

where, α_i = probability of i drivers choosing the route on the previous day, and ! indicating the factorial operator. Since the number of drivers and the number of routes available are very small, we can work out the entire probability distribution by hand calculations as we can completely enumerate all possible system states. However, it should be noted that such a method is not feasible for real size networks as the demand and the available paths are significantly large, thus increasing the size of the transition probability matrix by manifold. In such cases, simulation methods or even approximation methods would be the alternatives to use. Table 2.1 illustrates the steps in computing the probability of 'i' drivers choosing a route 1 (say), which would help in calculating the elements of the transition probability matrix given later. In this case, θ was set equal to 0.1.

Table 2.1 Route Choice Probabilities for Driver Combinations

	Number of drivers on route 1, i					
	0	1	2	3	4	5
Cost of travel on route 1, c_1	10	15	20	25	30	35
Cost of travel on route 2, c_2	55	45	35	25	15	5
Cost difference, c_1-c_2	-45	-30	-15	0	15	30
Probability of 'i' drivers choosing route 1, $\alpha_i = p_1(c_1)$	0.989	0.953	0.818	0.5	0.182	0.047

The transition probabilities are computed by substituting appropriate probability values from Table 2.1, in equation (2.41). To illustrate, the probability of no driver choosing route 1 on day n, if there were no drivers on the same route the previous day (n-1), is given as,

$$P\left(f_1^n = 0 \middle| f_1^{(n-1)} = 0\right) = \frac{5!}{(5-0)!\, 0!}(0.989)^0 (1-0.989)^{(5-0)} \qquad \text{for i = 0 and j = 0.}$$

Similarly, the second cell on the first row indicates the probability of no drivers (j = 0) choosing route 1, given that there was one driver (i = 1) on that route the previous day, and so on. Figure 2.3 shows the matrix of transition probabilities for all combinations of drivers choosing route 1.

$$M = \begin{bmatrix} 0.0000 & 0.0000 & 0.0000 & 0.0012 & 0.0526 & 0.9463 \\ 0.0000 & 0.0000 & 0.0010 & 0.0194 & 0.1952 & 0.7843 \\ 0.0002 & 0.0045 & 0.0406 & 0.1819 & 0.4075 & 0.3653 \\ 0.0313 & 0.1563 & 0.3125 & 0.3125 & 0.1563 & 0.0313 \\ 0.3653 & 0.4075 & 0.1819 & 0.0406 & 0.0045 & 0.0002 \\ 0.7843 & 0.1952 & 0.0194 & 0.0010 & 0.0000 & 0.0000 \end{bmatrix}$$

Figure 2.3 Transition Probability Matrix for Route 1

It may be observed that the elements of the leading diagonal of the transition matrix are all very small and that the probability mass is concentrated on the second diagonal. This indicates that if there were no drivers on route 1 on the previous day, it is highly likely that all the drivers would choose that route on the following day and vice versa, thus suggesting a flip-flopping behaviour of the drivers. By solving the system of linear equations given by $\pi^T = \pi^T M$, we can obtain the equilibrium probability distribution, which is given by,

$$\pi^T = [0.3633 \quad 0.1091 \quad 0.0233 \quad 0.0136 \quad 0.0523 \quad 0.4383].$$

Figure 2.4 shows the steady state probability distribution shown above and the mean and standard deviation work out to 2.56 and 1.87 respectively.

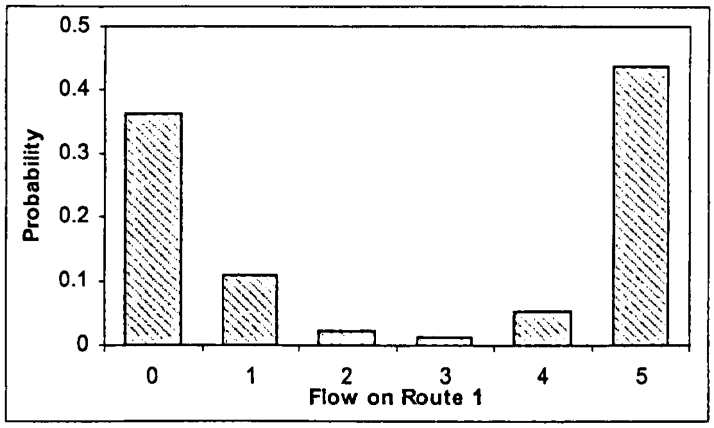

Figure 2.4 Steady State Probability Distribution (θ=0.1)

As indicated by the transition probability matrix earlier, the steady state probability distribution shows that nearly all the drivers are going to change their route every day, representing a flip-flopping behaviour. In a logit route choice modal, the driver behaviour is controlled by the θ parameter, which represents the inverse of the dispersion in the behaviour of the drivers. Clearly, higher values of θ indicate very little dispersion in the driver behaviour, tending all the people to think alike (as in a deterministic case) as represented by the high probability bars at the two extremes in Figure 2.4. Even higher values of θ, such as 1, would just share the probability between the two extremes as shown in Figure 2.5 resulting in a bimodal distribution with probability mass concentrated towards the ends. The mean and standard deviation in this case were 2.5 and 1.87 respectively. On the contrary, reduced values of θ (=0.01) result in a probability distribution (Figure 2.6) which is similar to a Normal distribution, with a mean of 2.58 and a standard deviation of 1.87.

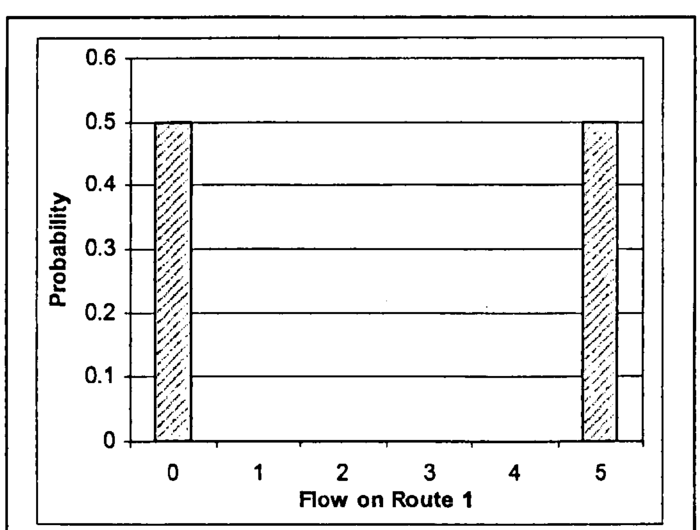

Figure 2.5 Steady State Probability Distribution (θ = 1)

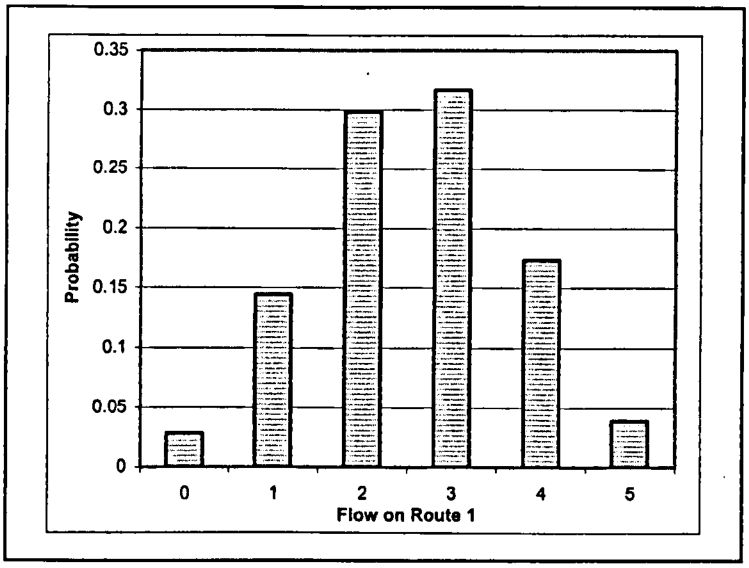

Figure 2.6 Steady State Probability Distribution (θ = 0.01)

Davis and Nihan (1993) shows that the mean of the equilibrium probability distribution coincides with the SUE solution, provided the demand is sufficiently large. Hence, it would be interesting to compare the above results with the SUE solution. For the given network demand as in the numerical example above for the case of $\theta = 0.01$, SUE solution is $f_1 = 2.825$ and $f_2 = 2.175$ with corresponding costs $c_1 = 24.125$ and $c_2 = 26.750$ units. In this case, the mean of the steady state probability distribution as approximated by SUE flows seems to be an over estimate of the true mean, but given that there are only five drivers on the network, the estimate of the mean may be considered satisfactory. However, higher values of the demand are likely to take the approximation much closer to the true mean.

2.5 Summary

This chapter reviewed the principles involved in carrying out the day-to-day dynamic assignment including stochastic process and deterministic process models. It defined the equilibrium probability distribution of a stochastic process and introduced the conditions sufficient for a stochastic process to admit a unique steady state probability distribution. This chapter also introduced the principles of carrying out a within day dynamic assignment by analytical and simulation methods. It also introduced the doubly dynamic assignment available in the literature. The concept of equilibrium for stochastic processes has been demonstrated with a simple numerical day-to-day dynamic, but within day static example. The ensuing chapter introduces the dynamic network loading models essential for carrying out a dynamic traffic assignment.

Chapter 3

Dynamic Network Loading Models

This chapter reviews the dynamic network loading models and describes the basic principles involved. It also discusses in detail the properties of whole link models and illustrates a model with suitable numerical examples. .

3.1 Need for a Dynamic Network Loading Model

A dynamic loading model determines the number of vehicles on the link at each instant of time, computes the travel time, exit flow from each link and even models the interaction of vehicles satisfying various flow propagation constraints such as flow conservation, First-In, First-Out (FIFO), causality, given the route choices. In a static assignment, usually Bureau of Public Roads (BPR) (1964) style cost flow functions are assumed. The general model form of such a cost-flow function is as below:

$$c = c_0 \left\{ 1 + \alpha \left(\frac{f}{Q_{cap}} \right)^{\beta} \right\} \tag{3.1}$$

where, c is the link travel time (cost), c_0 is the free-flow time, f is the link flow, Q_{cap} is the capacity of the link and α, β are calibration constants. In equation (3.1), f represents the flow on the link which is equal to the sum of all the O-D flow rate contributions using that link. In contrast, the dynamic traffic assignment deals with varying demand profiles and needs to move the vehicles in space and time, thus resulting in varying inflow and outflow profiles on each link. Mere temporal extensions of BPR functions cannot capture such link dynamics, as then one would need to specify the link flow rate as either the inflow or outflow or some combination of them. Link travel time function in dynamic assignment should reflect this requirement i.e., it should be capable of capturing the link flow dynamics in space and time. However, day-to-day dynamic models do not explicitly deal with the kind of link flow dynamics in space or time, and hence the BPR style cost flow functions become popular choice (e.g., Cascetta 1989, Friesz et al 1994, Hazelton and Watling 2004).

3.2 Properties of Dynamic Link Models

As explained previously, dynamic link models are an essential element of a dynamic traffic assignment framework and the performance of the link models can significantly affect the solution to the assignment problem. Carey et al (2003) and Carey (2004) analyse the desirable properties of link flow models in dynamic traffic assignment. Nie and Zhang (2005) compare the performance of dynamic link flow models and in particular, study the property of First-In-First-Out (FIFO) besides identifying the issues in the computation of outflows from links. Dynamic link models are operated upon in each iteration of the assignment, and hence take up the majority of the computing effort required in carrying out the dynamic traffic assignment. Therefore, dynamic link models need to be efficient, and the resulting outflows and travel time profiles should be plausible. For this reason, dynamic link models, in the context of dynamic traffic assignment, should possess the desirable properties identified in Carey et al (2003) which are described in the following paragraphs.

3.2.1 Flow conservation

In a small element of length the quantity changes at a rate equal to the difference between the inflow and outflow, or in other words, traffic flow on a link can neither be created nor disappear, if there are no sources or sinks. Consider a small section of a road link. Let x(t) be the number of vehicles at instant t located in that section of the road and let u(t) and v(t) be the inflow and outflow rates to and from that section (See Figure 3.1).

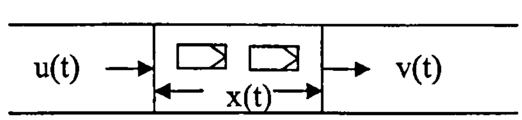

Figure 3.1 Illustration of Flow Conservation

From the previous discussion, it follows that

$$\frac{dx(t)}{dt} = u(t) - v(t) \tag{3.2}$$

Integrating (3.2) with respect to t yields,

$$x(t) = \int_0^t [u(s) - v(s)]\, ds \tag{3.3}$$

where, s is a dummy variable of integration.

3.2.2 First-In, First-Out (FIFO)

In the context of traffic assignment, FIFO implies that the vehicles from a road link depart in the same order as they enter. In a multilane road link case FIFO may not seem a desirable property as the fast moving vehicles can overtake the slow moving ones. But in the case of single lane road links, it is very important to preserve FIFO mainly to ensure that the process of representing the congestion i.e., building up and dissipation of queues in space and time and hence the resulting solutions are plausible (Carey et al 2003).

Let us consider the traffic flow on a link at two successive instances say t and t + Δt. If the travel time for vehicles entering the link at t, is represented by $\tau(t)$, then the corresponding exit time would be t $+\tau(t)$. Similarly, the exit time for vehicles entering the link at t +Δt, would be t + Δt + $\tau(t+\Delta t)$. Then, for FIFO we need,

$$t + \tau(t) \quad < \quad t + \Delta t + \tau(t + \Delta t)$$

$$\Rightarrow 0 \quad < \quad 1 + \frac{\tau(t+\Delta t) - \tau(t)}{\Delta t} \qquad \text{(after transforming and dividing by } \Delta t\text{)}$$

Taking limit as Δt→ 0 and rearranging gives,

$$\frac{d\tau(t)}{dt} > -1$$

From the discussion on flow conservation and FIFO, we can write,

$$\int_0^t u(s)\,ds = \int_0^{t+\tau(t)} v(s)\,ds \tag{3.4}$$

Differentiating equation (3.4) with respect to t and rearranging will yield

$$v(t+\tau(t)) = \frac{u(t)}{1+\dot{\tau}(t)} \tag{3.5}$$

where, $\dot{\tau}(t)$ represents the derivative of travel time. Astarita (1996) provides the proof of equation (3.5). From equation (3.5), it is clear that $\dot{\tau}(t)$ must be greater than -1 for the outflows to be meaningful. Therefore, in order to ensure FIFO, we need to monitor the derivative of travel time throughout the simulation period. Practically, this can also be achieved by monitoring the travel times at each time step through the simulation process, and the computer program can be designed to give an error message as soon as the FIFO is violated.

3.2.3 Causality

It means that delays to a set of vehicles entering the road link during any time period should only be caused by vehicles which have entered the link during that period or earlier but not later (Carey et al 2003). FIFO and causality are two independent conditions, which means for example, causality may be violated even if FIFO is satisfied and vice versa. To illustrate this, consider the inflows to a link at discrete time t then, the corresponding outflows can spread over time period until the earliest vehicle from the following time step t+1 reaches the exit. Although this condition satisfies FIFO, causality is violated as the travel time for vehicles from time period t is affected by vehicles from the following time period t+1. Similarly, satisfying causality need not imply FIFO. For example, if the travel time for vehicles entering at time t is a function of the inflow rate at t, then it satisfies the causality but can still violate FIFO, if the inflow decreases rapidly over time. In this research, in order to ensure the causality condition, a continuous time scale has been adopted which will be introduced later in section 3.4.

The need for a suitable dynamic network loading model was recognised long ago. Merchant and Nemhauser (1978a) were perhaps among the first few who formulated a dynamic traffic assignment model, and since then there have been several formulations proposed by various researchers with each including some form of dynamic network loading model. Cell transmission models, queuing models and whole link models are the dynamic network loading models which have been used by various researchers and the aim of the rest of this chapter is to describe each of these models in detail discussing its properties. Finally, this chapter discusses the implementation of whole link models – one of the methods of moving the vehicles on the street network, and illustrates the model with suitable numerical examples.

3.3 Review of Dynamic Network Loading Models

This section introduces the dynamic link loading methods, and discusses their properties.

3.3.1 Cell Transmission Model

Daganzo (1994) formulated a method called cell transmission model which moves the vehicles on a link in discrete space and time. The cell transmission model divides the link into a number of cells of uniform length such that each cell can be traversed in one time step at free flow speed and applies the following dynamics:

$$x_i^{t+1} = x_i^t + u_i^t - v_{i+1}^t \tag{3.6}$$

$$u_i^t = \min of \begin{cases} x_{i-1}^t \\ Q_i^t \\ \alpha\,[X_i^t - x_i^t] \end{cases} \tag{3.7}$$

where, x_i^t, u_i^t *and* v_i^t represent the number of vehicles in the cell i at time t, the inflow to and outflow from cell i respectively. Q_i^t is the inflow capacity of the cell i, X_i^t is the maximum number of vehicles that can be accommodated in the cell i and α is a parameter. Equation (3.6) conserves the flow and (3.7) constrains the inflowing number of vehicles into the cell i. The cell transmission model calculates the exit flow in the next discrete time interval from any given link (i.e. a series of cells). The concept of delay in this model is implicit in equation (3.7) as the vehicles in the upstream cell are delayed by lack of space in the downstream cell and the flow propagation is actually modelled based on a piecewise linear approximation of the fundamental flow-density diagram.

Daganzo (1995) shows that the cell transmission model is a discrete equivalent of the hydrodynamic model which considers the traffic flow analogous to compressed fluid flow. Lo (1999) and Ziliaskopoulos (2000) embedded the cell transmission model into dynamic traffic assignment problems. The main limitation of this model is that it requires large computer memory, and even the model running times may be very high. Besides, due to the nature of the model, the link travel time is discontinuous in the number of vehicles present on the link, and hence they are non-differentiable.

3.3.2 Deterministic Queuing Model

A deterministic queuing model, also sometimes referred to as a bottleneck model, assumes that the traffic flow arrivals and departures are predictable which is fair as evident in rush hours with high arrival rates and long queues at some junctions every day. It is also assumed that the vehicles move freely from one end of the link to the other end and then possibly incur delay in a vertical queue before leaving the link. The basic model can be specified as below:

$$\frac{dL(t+\alpha)}{dt} = u(t) - v(t+\alpha) \tag{3.8}$$

$$v(t+\alpha) = \begin{cases} Q_e & \text{if } L(t+\alpha) > 0 \\ \min(u(t),\ Q_e) & \text{if } L(t+\alpha) = 0 \end{cases} \tag{3.9}$$

$$\beta(t) = \frac{L(t+\alpha)}{Q_e} \tag{3.10}$$

$$g(t) = t + \alpha + \beta(t). \tag{3.11}$$

where, L(.) is the length of the queue, u(t) and v(t) are the inflow and outflow rates to and from the link respectively at time t. Q_e is the exit capacity of the link. $\beta(t)$ is the delay incurred due to queuing and g(t) is the exit time for vehicles entering the link at time t and α is the free flow travel time.

The rate of change of queue length is defined as the difference between the entry flow rate lagged by free flow travel time and the outflow rate. If there is a queue, the current outflow rate is equal to the exit capacity, otherwise it is equal to the minimum of lagged entry flow rate or the exit capacity under the deterministic assumption. $\beta(t)$ defines the delay incurred in the queue and g(t) defines the exit time for an entry time t. Simply, the travel time on the link is equal to the sum of free flow time and the delay incurred in queuing. There are many examples of using queuing models in dynamic assignment including, Cascetta and Cantarella (1991), Kuwahara and Akamatsu (1997) among many others. The drawback with this model is that it does not predict any delay as long as the inflow rate is less than the exit capacity (because of no queuing) which is clearly an under estimate of realistic delays.

Some assignment models e.g., Ben Akiva et al (2001) and Taylor (2003) made some adjustments to the deterministic queuing model approach by adopting a BPR type flow-delay function/ speed-flow function only for the moving part of link, but also assuming that the traffic flow on the link is uniformly distributed over the length of the link. As a result of this assumption, the model tends to overestimate the delays on lightly congested links and makes it suitable for highly congested ones.

3.3.3 Whole Link Propagation Model

This section describes the sub-models used to estimate the outflow/travel time on a road link in the analytical network models for dynamic traffic assignment. They derive their name from the fact that the outflow from/travel time on the link is expressed as a function of the whole link attributes such as the number of vehicles currently on the link, inflow rate, outflow rate etc, at any given instant (Heydecker and Addison 1998). Within this class of models, there are two different sub classes viz., exit flow functions and whole link travel time models – deriving their names based on the outputs from such models. Following paragraphs describe different types of whole link models.

3.3.3.1 Exit Flow Functions

One of the earliest attempts in formulating the dynamic traffic assignment model (Merchant and Nemhauser 1978a, b) included a combination of link exit flow function to propagate the traffic on the link and a static link performance function to represent the travel cost as a function of the link volume. The model in general may be specified as below:

$$\frac{dx(t)}{dt} = u(t) - v(t) \qquad (3.12)$$

$$v(t) = g(x(t)) \qquad (3.13)$$

$$\tau(t) = h(x(t)) \qquad (3.14)$$

As seen earlier, equation (3.12) conserves the flow and the exit flow function g(.) in (3.13), representing the process of congestion, is assumed to be a non-decreasing, continuous, concave function. Function h(.) represents the disutility of congestion and is assumed to be a non-negative, non-decreasing, continuous convex functions.

Exit functions were also used by Friesz (1989) and Wie et al (1995). The exit flow function implicitly assumes that the traffic on the link is uniformly distributed over the entire length of the link, and hence any changes in the density propagate instantaneously across the link. While this assumption is suitable for a congested link, for uncongested links it is not. Moreover, Carey (1986, 1987) demonstrated that the exit flow functions violate the FIFO requirement. In addition, Heydecker and Addison (1998) demonstrated that the exit flow functions violate causality. Exit flow functions were also found practically difficult to specify and measure. Due to these reasons, exit functions were not used further by the researchers, who preferred the travel time function formulations which are described in the next section.

3.3.3.2 Travel Time Function Formulations

Travel time function calculates the travel time for vehicles entering the link at each instant of time and is usually expressed as a function of the number of vehicles on the link at that instant. Based on the travel time profile, the outflow profile can be easily computed. General form of the model is as follows:

$$\frac{dx(t)}{dt} = u(t) - v(t) \tag{3.15}$$

$$\tau(t) = h(x(t)) \tag{3.16}$$

$$v(t + \tau(t)) = \frac{u(t)}{1 + \dot{\tau}(t)} \tag{3.17}$$

$\tau(.)$ in equation (3.16) is assumed to be a continuous and non-decreasing function. In order to run the dynamic link loading process, equation (3.16) should be operated in tandem with flow conservation equation (3.15) and flow propagation equation (3.17). Although Friesz et al (1993) and Wu et al (1998) presented similar forms of the model (3.15-17), Astarita (1996) provides the proof of (3.17).

There are various plausible forms for the travel time function in (3.16). For example, the travel time could be a linear function of the number of vehicles on the link (Friesz 1993) or could be even a non-linear power function (e.g., Ran et al 1993 and Ran and Boyce 1996). However, only the linear form of (3.16) is known to satisfy FIFO for various forms of inflows u(.) (Carey et al 2003, Nie and Zhang 2005). Besides, the link travel time and hence the path travel time is continuous in inflows, which is essential for the travel time functions to be differentiable. As noted from Chapter 1, one of the main aims of this research is to compute, analytically, the Jacobians of travel time functions, therefore the differentiability of travel time functions with respect to the inflows, is an essential property. As the whole link travel time functions satisfy these requirements, further part of this research focuses only on such models. The next section of this chapter specifies and then implements a whole link travel time model. It also discusses numerical results, besides discussing the issues in implementing the model.

3.4 Implementation of a Whole Link Model

This section describes the dynamic loading model that is used in this research and illustrates the principles described using suitable numerical examples. Although the theory in sections 3.1 and 3.2 was described in continuous time, in this research the whole link travel time model is implemented in discrete time, but mapped back to a continuous time scale. The details of the model are discussed both for the case of a single link and a combination of links in a general network case. Suitable numerical examples follow.

3.4.1 Preliminaries and Model Specification

Consider the case of a single link which serves one O-D pair with the only link being also the path. It is assumed that the travel time for vehicles entering the link at any continuous instant t, i.e. interval $(t-\delta, t]$ where δ is the step length (also called minor time step), is a linear function of the free flow travel time and congestion related time as shown in equation (3.18) (Friesz et al 1993). It is also assumed that the total period of analysis (say peak hour) is divided into L major periods representing as many departure periods $(w_{i-1}, w_i]$ (for i = 1,2,,..L) such that $(w_0, w_1] \cup (w_1, w_2] \cup .... \cup (w_{L-1}, w_L] = (0, N\delta]$ where N is the total number of minor time steps. Uniform O-D demand flow rates are assumed to have been specified over the major time periods. Now, the model can be formally written as below:

$$\tau(t) = \alpha + \beta\, x(t) \qquad (\alpha > 0, \beta > 0) \qquad (3.18)$$

$$x(t) = \sum_{0}^{t} u(t) - \sum_{0}^{t} v(t)\,. \qquad (3.19)$$

where, α = free flow time, and $1/\beta$ = exit capacity of the link.

In this model δ - the minor time step length is assumed much smaller compared to the free flow travel time α, so that the vehicles entering the link in any minor step cannot leave in the same step. The inflow was transformed into outflow based on the travel time profile obtained by using equations (3.18 and 3.19). It may be noted that the travel time computed for each discrete entry step need not necessarily be an integer again. However, we need to compute the outflow from the link at each discrete exit time step. For this reason, it was assumed that the outflow will spread uniformly over the exit period. Then by the method of linear interpolation, the outflow at each discrete exit step can be obtained. The details of the modelling process are described in the following section.

3.4.2 Model Development

Dynamic network loading model is aimed at moving the vehicles in space and time. In addition to the link specification parameters such as the free flow travel time α, capacity $1/\beta$, the main input to this model includes the departure-time-dependent route flows which are usually obtained from the route choice model described in the previous chapter. The main output from this model is the departure-time-dependent route travel time and is usually passed back to the route choice model. While the route choice model deals with a limited number of departure time periods during the peak hour, in order to capture the interactions amongst the vehicles on the road, the dynamic link loading model slices up each departure period into a number of simulation steps of length δ (also referred as minor time steps), assuming that the demand is uniformly spread over during each departure period. The simulation is carried over the total number of simulation steps N, extending over the interval (0, Nδ] in all departure periods put together.

At each simulation step, the program computes the travel time for vehicles entering the link during that step as a function of the number of vehicles on the link, using equation (3.18). The number of vehicles on the link is computed as the difference between the cumulative inflow and the cumulative outflow, thus explicitly satisfying the flow conservation equation (3.19). A critical requirement in this module is to estimate the outflow v(.) from each link ahead of time. At each simulation step we know the travel time required to reach the end of the link from which we can compute the outflow ahead of the current time step, and stepping through the simulation process recursively works out smoothly. Thus obtained outflow is posted to an integer exit step using the linear interpolation method as described in the following paragraphs.

3.4.3 Computing Outflow by Linear Interpolation

Computing the outflow at each discrete exit step is critical for the overall success of the model. This is a complicated step because the possible scenarios that could arise during the simulation process are very many. For example, the interpolation may need to be carried out over a number of steps depending on the expected earliest and latest exit time of the set of vehicles from any given discrete inflow step. The other extreme could be the need to bypass the interpolation module completely when the earliest and latest exit times fall within the same discrete exit step.

Consider the time-space diagram as shown in Figure 3.2, where two time axes, – the first representing the time when the vehicles enter the link and the other indicating the time when the vehicles leave the link, are represented. Consider the inflow step $(y-\delta, y]$ at which the vehicles enter the link. Then the corresponding earliest and latest exit time g(.) are as indicated below:

$$g(y-\delta) = (y-\delta) + \tau(y-\delta) \tag{3.20}$$

$$g(y) = y + \tau(y) \tag{3.21}$$

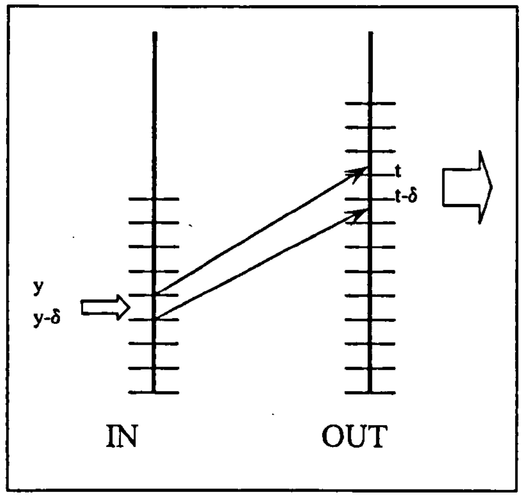

Figure 3.2 Time-space Diagram

Note that the exit times calculated from (3.20) and (3.21) need not be restricted to integers and the vehicles starting their journey at integer entry times may actually reach the exit at real exit times (See Figure 3.3). However, the simulation is based on discrete steps, hence the need for interpolation. We need to extend the notation to proceed with further treatment. Let V(t) be the cumulative outflow by the end of time step t, i.e., the step (t- δ, t] and let $V_y(t)$ be the number of vehicles exiting by the end of time step t that had entered the link in time step (y- δ, y]. Assuming that the outflow due to the inflow from the time step (y- δ, y] is uniformly distributed over the exit period obtained as the difference between the corresponding latest and earliest exit times i.e., [g(y) – g(y-δ)], the number of vehicles allocated to integer exit steps (t-δ), t and (t+δ) respectively are given below:

$$V_y(t-\delta) = \frac{[t-y-\tau(y-\delta)]\delta\, u(y)}{\tau(y)-\tau(y-\delta)+\delta} \qquad (3.22)$$

$$V_y(t) = \frac{\delta^2\, u(y)}{\tau(y)-\tau(y-\delta)+\delta} \qquad (3.23)$$

$$V_y(t+\delta) = \frac{[y+\tau(y)-t]\delta\, u(y)}{\tau(y)-\tau(y-\delta)+\delta}. \qquad (3.24)$$

Interpolation process is graphically represented in Figure 3.3.

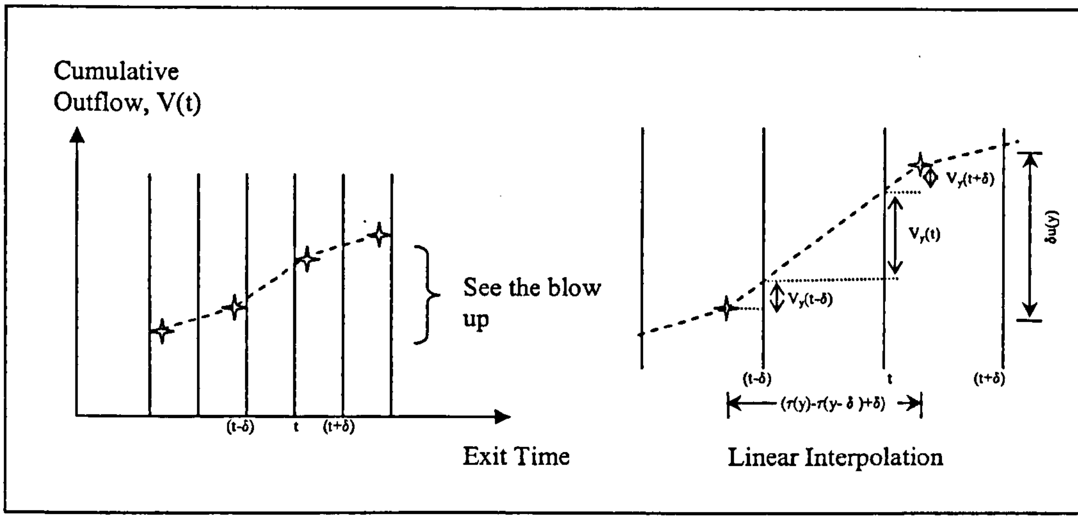

Figure 3.3 Computing Outflow at Integer Exit Time Steps

Finally, when the simulation reaches the final step N, i.e., the end of the analysis period, there will still be some vehicles on the links which need to be allowed to reach their destination. This is achieved by extending the simulation period as set by a parameter called 'cooling period', during which no inflows to the link are allowed and only the vehicles on the link are allowed to exit.

The linear interpolation scheme presented here may seem similar to the published work of others, for example Nie and Zhang (2005). However between the two methods, there are some subtle but important differences, which are described in this paragraph. The aim of the interpolation in this research is to calculate exactly the number of vehicles leaving the link in each discrete time step with the help of the cumulative outflow curve, whereas Nie and Zhang aim to approximate the average outflow rate at the mid-point of each discrete time interval. The approximation in the outflow rate affects the number of vehicles on the link, and in turn the travel time calculation. In their scheme, the bigger the discrete time interval the greater is the error in the average outflow rate, and in turn the travel time. On the other hand, in this research, the travel time is calculated using (3.18) and (3.19) recursively as described previously, and with the discrete time steps mapped back to a continuous time scale, the error in travel time computation is not expected to be significantly affected by the size of the discrete time step. (See Section 3.5.1 for further discussion on this point).

Dynamic link loading model passes back the average travel time in each departure time period to the route choice module. The mean travel time in each departure period on each route is computed as the mean of the travel times at each of the constituent minor time steps in that departure period weighted by their inflow rates. Formally, in case of a single link – path network, the departure time dependent mean travel time for route r with uniform inflow rate in any departure time period T bounded by $(w_{i-1}, w_i]$ may be expressed as,

$$\tau_r^T = \{ \frac{1}{(w_i - w_{i-1})} \} \sum_{t=(i-1)n\delta+1}^{in\delta} \tau(t) \tag{3.25}$$

where, n is the number of minor time steps in major time period T.

3.4.4 Whole Link Travel Time Model for a Network

This section extends the whole link travel time models from a single link case to the case of a network where a path is commonly constituted by a series of links rather than one single link. Now, consider a series of links a_i (for i = 1,2,..,n), such that the sequence of links $[a_1, a_2, ..., a_{n-1}, a_n]$ constitutes any route r. Assuming that whole link travel time models of the form (3.18) are defined on each of the links on route r, the travel time function and exit time functions for any link a_i may be expressed as a nested path cost operator following similar principles as described for a single link. Then the expressions for travel time and the exit time are as given below:

$$\tau_{a_i}(g_{a_{i-1}}(t)) = \alpha_{a_i} + \beta_{a_i}(x_{a_i}(g_{a_{i-1}}(t))) \tag{3.26}$$

$$g_{a_i}(t) = g_{a_{i-1}}(t) + \tau_{a_i}(g_{a_{i-1}}(t)) \tag{3.27}$$

where, $\tau_{a_i}(.)$ is the travel time on the link a_i and $g_{a_i}(.)$ is the exit time from the link a_i.

However, as described in section 3.4.1, the model discretises time into a finite number of minor time steps, and hence we have the knowledge of travel times computed only at the discrete time steps. But this will be insufficient to compute the path travel time correctly especially from the second link onwards on any path with multiple links where the travel time needs to be computed at some real time and not just integers, otherwise the error in the travel time accumulates by the end of the dynamic network loading cycle. This is countered by computing the travel time in equation (3.26) using linear interpolation again by applying similar principles as described in the previous section, which is given below:

$$\tau_{a_i}(t) \approx \hat{\tau}_{a_i}(<t/_{\delta}>.\delta) + \frac{t - <t/_{\delta}>.\delta}{\delta}\left[\ \hat{\tau}_{a_i}((<t/_{\delta}>+1).\delta) - \hat{\tau}_{a_i}(<t/_{\delta}>.\delta)\ \right] \qquad (3.28)$$

for, $(t \geq 0;\ i = 1,2,\ldots,n)$

where, $\hat{\tau}_{a_i}(.)$ = travel time on link a_i at integer time, and

$<t/\delta>$ = integer part of time t.

Then the path travel time for vehicles entering the link a_1 at time t on route r (with a_n being the last link on route r before discharging the vehicles to their destination) is simply given as the difference between the exit time and the entry time at the origin, expressed as $[g_{a_n}(t) - t]$. Finally, the mean travel time for vehicles entering the path at any time t is computed using the equation (3.25).

3.4.5 Computer Program for Whole Link Travel Time Model

Dynamic network loading is a relatively straightforward process (Figure 3.4), and the majority of the programming effort is involved in computing the outflow, transferring the flows at nodes and computing the path travel time. Computation of the outflow at discretised integer exit time steps is based on the method of linear interpolation and has been described in detail in Section 3.4.3. The path travel time is worked out using the nested cost operator (3.26 and 3.27) and also uses linear interpolation to obtain the travel times at real entry times (3.28) used in the nested cost operator. Finally, the flow transfers are based on the link-path incidence relationship and the model requires complete path enumeration. The dynamic network loading module specially programmed using MATLAB, is also designed to remember each path flow progression over the network in time and space. This information will be used in computing the Jacobians of travel time functions, described in Chapter 5.

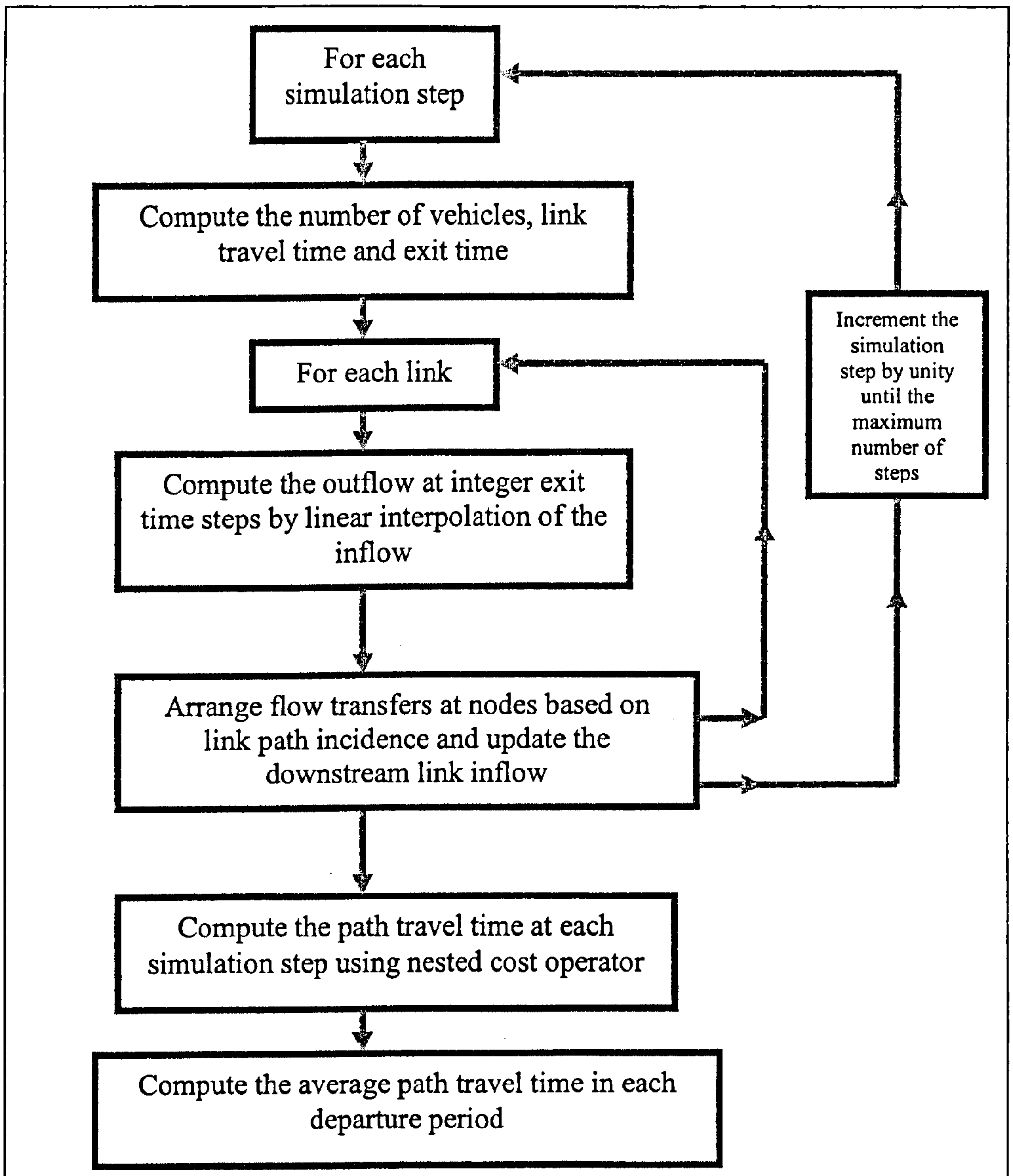

Figure 3.4 Dynamic Network Loading Process

3.5 Numerical Examples

This section discusses the numerical results obtained by implementing the whole link travel time model for single link paths with uniform inflow profiles described in the previous section and analyses some properties such as the capacity outflow rate and the impact of minor time step length on the overall results. It also reports the implementation of the model with more realistic inflow profiles such as the continuous time sinusoidal inflow functions and also extends the implementation process to a network of links where paths constitute a series of links and are not just limited to a single link.

3.5.1 Step Function Inflow Profiles

Consider a two link parallel route network (Figure 3.5) serving one O-D pair with α_1 = 12 minutes and β_1 = 0.025 minutes/vehicle for route 1 and α_2 = 9 minutes and β_2 = 0.035 minutes/vehicle for route 2. It is assumed that there are two departure periods of 15 minutes each with 400 and 700 drivers uniformly spread over each departure period. Typically the route choice module assigns the demand to alternative routes (which is not discussed in this chapter) and works out the route flows. One such flow pattern satisfying stochastic user equilibrium conditions is as shown in Table 3.1:

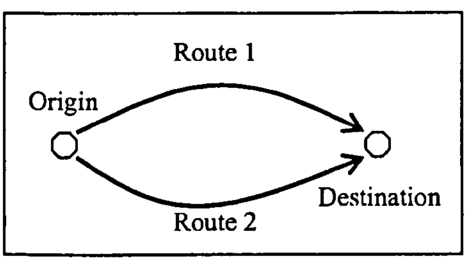

Figure 3.5 Two Link Network

Table 3.1 Demand in Departure Periods

Route/ Departure period	Period 1	Period 2
Route1	143	376
Route2	257	324

A minor step length of 1 minute was used to model the two departure periods and hence N = 30. The dynamic loading process was used to simulate the movement of the vehicles along the road link and compute the travel time profile and outflow profile given the inflow profile (Figure 3.6).

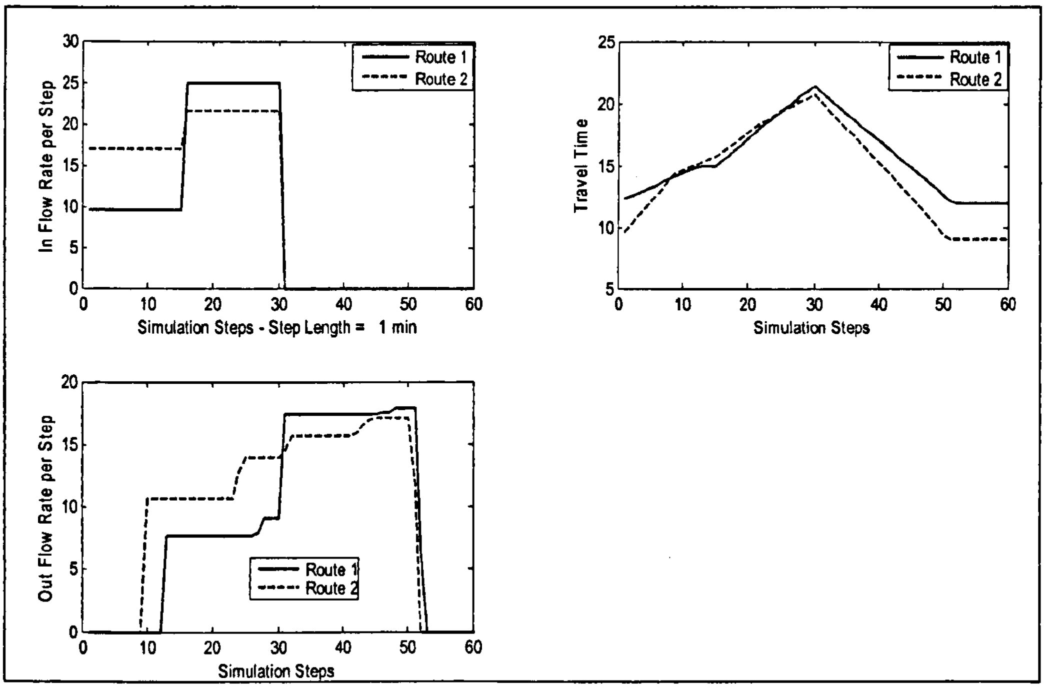

Figure 3.6 Outflow and Travel Time Profiles for Discrete Demand Function

The inflow profiles are in two steps with route 2 absorbing a larger proportion of the total flow in the first departure period and in the second period giving away the first position to route 1. The travel times are computed using the link travel time function in tandem with the flow conservation equation shown above. The travel time profiles peak at about 30 minutes and start reducing from then as the inflow ceases. Steepest gradient of the travel time function appears to be of the order of -0.5 which is well above the limit of -1 to preserve FIFO. In fact, in the simulation program, the derivative of the travel time function has been monitored at each minor time step to ensure FIFO. The corresponding outflow profiles are then worked out and are as shown in Figure 3.6. Thus the flow moves from origin to destination along the paths.

Capacity Outflow: Let us consider a single link serving one O-D pair with a linear travel time function defined as (3.18). At steady state, the inflow and outflow rates are equal and the number of vehicles on the link stays constant. Then we have,

$u = v = x/\tau$

$\tau = \alpha + \beta x$

$$\Rightarrow v = \frac{x}{\alpha + \beta x}. \qquad (3.29)$$

Consider the limit of (3.29) as x → ∞, then we have,

$$\lim_{x\to\infty} v = \lim_{x\to\infty} \frac{x}{\alpha + \beta x} = \frac{1}{\beta}.$$

Now, let $\alpha = 12$ minutes and $\beta = 0.025$ minutes/vehicle, as in the previous example. If we allow the length of the link to be unlimited (so that it can physically accommodate a large number of vehicles) then allowing the inflow rate to increase to a large value will result in a very high number of vehicles on the link. In such a situation, the outflow rate asymptotically tends to the outflow capacity given by 1/ β. The outflow rate in Figure 3.7 tends to 40 vehicles per minute which is equal to 1/0.025 vehicles per minute. This means that the outflow rate from a linear whole link travel time model can never exceed the outflow capacity.

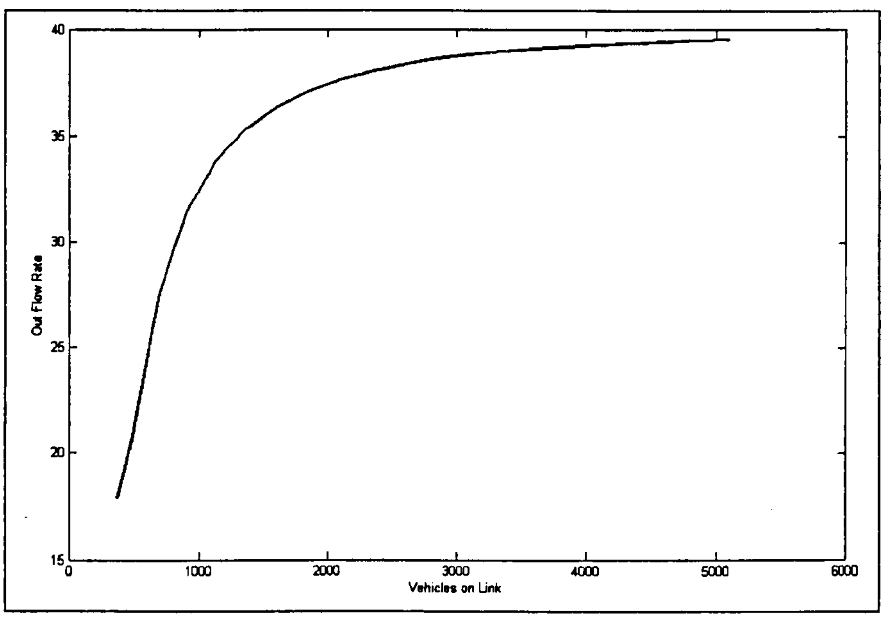

Figure 3.7 Capacity Outflow Rate

Impact of Step Length: The simulation step length was reduced from one minute to 30 seconds, 15 seconds and to one second and its impact on the mean travel time was studied. Table 3.2 shows the modelled mean travel times by departure period for the two routes in the above example.

Table 3.2 Mean Travel Time by Departure Period (in Minutes)

Route	Step Length in Seconds	Mean Travel Time in Minutes	
		Departure Period 1	Departure Period 2
1	60	13.8297	18.4424
	30	13.7797	18.3362
	15	13.7547	18.2831
	1	13.7314	18.2335
2	60	13.2551	18.5793
	30	13.1432	18.4971
	15	13.0872	18.4558
	1	13.0350	18.4172

As the step length reduced the mean travel time also reduced (though marginally) due to the fineness in working out the travel times over smaller steps. Clearly, this is due to the assumption that the travel time for all vehicles in a given simulation step is the same irrespective of their relative position in the group. However, the first vehicle in the group is not delayed by any of the following vehicles in the same group whereas the last vehicle is delayed by all vehicles ahead of it. In other words, the smaller the simulation step length, the better the travel time estimate. Ideal size for a step length should be that it is just sufficient to accommodate one vehicle. However, as the dynamic network loading module runs over a large number of iterations while solving for the equilibrium state, the computer time increases significantly, and hence we need to select an appropriate simulation step length suitable for handling the decision in question.

3.5.2 Continuous Function Inflow Profiles

In the second example keeping the network parameters identical to the previous example, a more general pattern of continuous inflow profiles are considered over a period of one hour divided into four departure periods of 15 minutes each. The inflow profiles for route 1 are indicated as below:

$$u(t) = \begin{cases} 32\ Sin(\pi t/30) & 0 < t \leq 15 \\ 32 & 15 < t \leq 30 \\ 20 + 12\ Sin^5[\pi(t+1.5)/65] & 30 < t \leq 45 \\ 20 + 12\ Sin^5[\pi(t+1.5)/65] & 45 < t \leq 60 \end{cases} . \quad (3.30)$$

Inflow profiles for route 2 are assumed half of the flow rate for route 1. The inflow profiles in this example clearly indicate the typical flow profiles during peak hour with steeply rising inflows during the earlier part and decreasing inflows after a period of stable inflows. The travel time profiles and outflow profiles are as indicated in Figure 3.8. The travel time profiles in Figure 3.8 are fairly smooth unlike the profiles corresponding to the step function inflow profiles shown in Figure 3.6. Continuous time inflow profiles are likely to result in finer estimation of travel time and hence the outflow profiles, as they represent the case of minor time step length $\delta \rightarrow 0$.

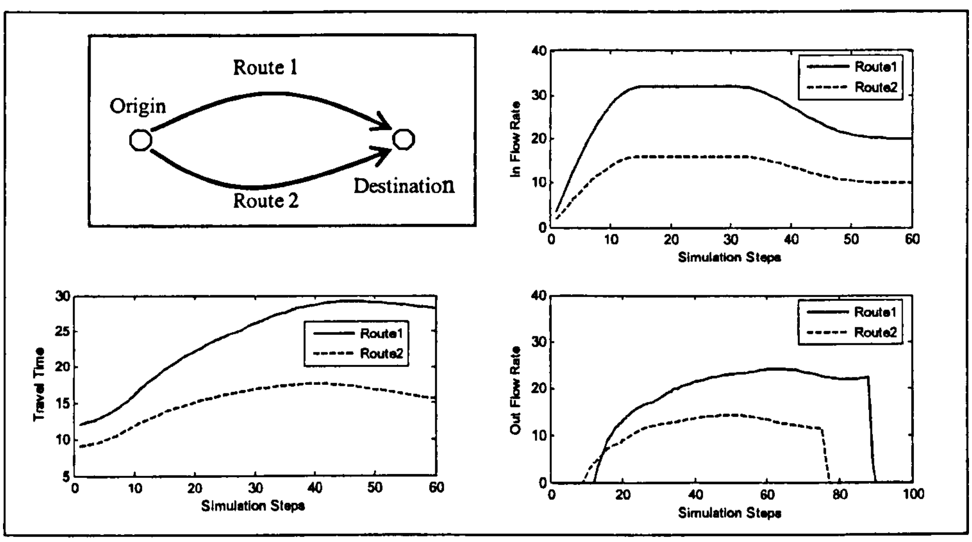

Figure 3.8 Outflow and Travel Time Profiles for Continuous Demand Function

3.5.3 Five Link Network Example

This example illustrates the dynamic loading of vehicles to road links in a five link network specifically highlighting the merging and diverging manoeuvres at the junctions. Although the junctions are modelled as notional points in this example, the principles described here can easily be extended to incorporate the modelling of junction delays. Moreover, in this network, there are overlapping routes using the same road link at any given instance – a feature necessary in larger networks. Consider the five link network as shown in Figure 3.9 with three paths serving a single O-D pair. Links 1 and 4 are used by route 1, links 2,3 and 4 are used by route 2 and route 3 constitutes links 2 and 5. At node 1, flows on routes 2 and 3 diverge whereas, at node 2 flows from routes 1 and 2 merge together to form inflows to link 4.

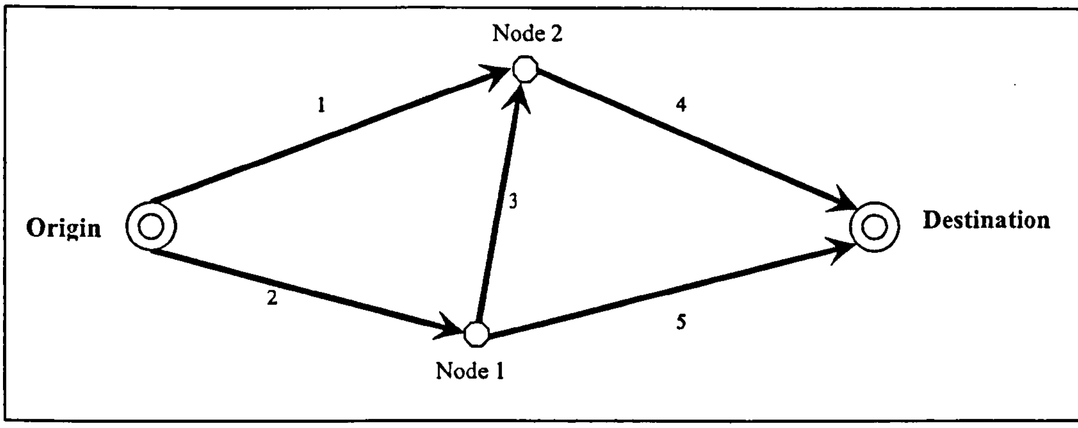

Figure 3.9 Network Topology

On each of the five links whole-link dynamic travel time functions of the general form $\tau_a(t) = \alpha_a + \beta_a x_a^{\gamma_a}(t)$ are defined with parameter values as shown in Table 3.3. In case of links 1,2,4 and 5, linear travel time functions have been defined, and the corresponding values of β have been set so that they reflect the saturation flows. For example, link 1 has been assigned a value of 0.025 min/veh for β, the inverse of which represents a capacity of 40 veh/min i.e., 2400 veh/hr which is similar to the saturation flow of an approach at a junction. In case of link 3, a quadratic relationship has been chosen though the FIFO property is not guaranteed in this case. Throughout the simulation process, the travel times at each step were monitored and ensured that the FIFO is not violated.

Table 3.3 Network Parameters

Link	α_a (minutes)	β_a (minutes/vehicle)	γ_a	Functional Form
1	12	0.025	1	Linear
2	9	0.035	1	Linear
3	10	0.00015	2	Quadratic
4	12	0.025	1	Linear
5	9	0.035	1	Linear

The demand is assumed to spread over four discrete departure periods of 15 minutes each and the departure time dependent inflow on the three routes (output of a route choice model) is as indicated in Table 3.4.

Table 3.4 Departure Time Dependent Demand

Route	Departure Period			
	1	2	3	4
1	148.44	302.71	40.58	32.97
2	49.49	79.79	15.42	16.18
3	202.07	317.50	44.00	50.85

Movement of vehicles on the road network is simulated through the dynamic network loading procedure as described in the earlier example and the link-wise flow and travel time profiles are as indicated in Figure 3.10. As an example, let us consider the flow on route 1. Travel time profiles are worked out using the procedure described earlier and the corresponding outflow profiles are as shown. Thus, the outflow from link 1 reaches node 2 where it joins with the outflow from link 3 resulting in an inflow profile to link 4. Repeating the procedure for moving the vehicles along link 4 yields the necessary outflow profile. Thus the movement of flow from origin to destination via route 1 has been simulated.

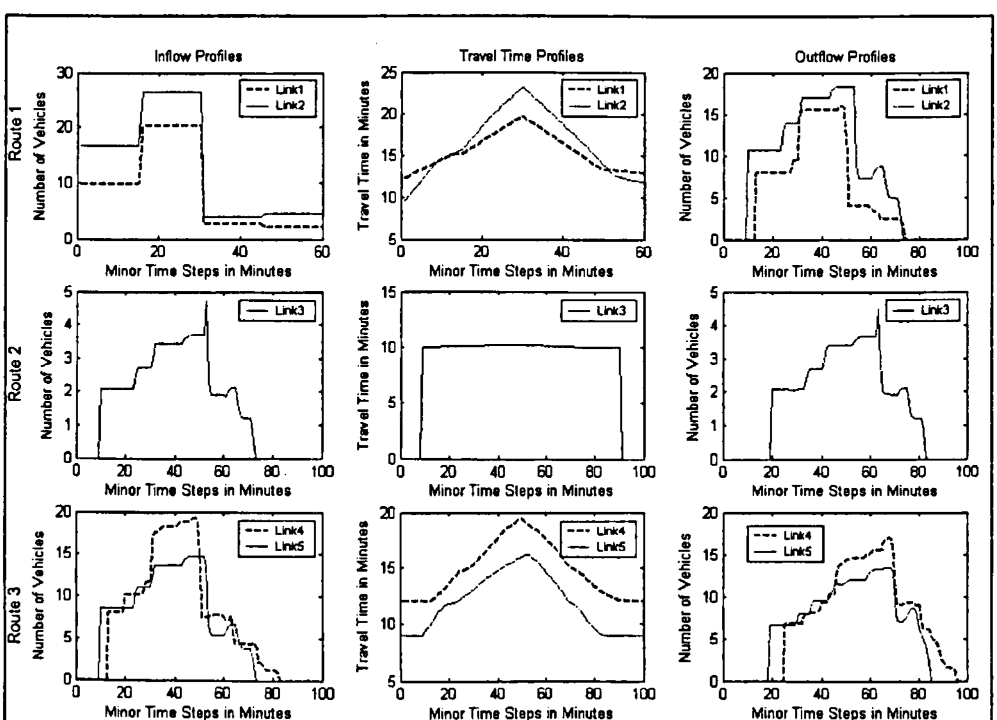

Figure 3.10 Flow and Travel Time Profiles

The travel time profiles in Figure 3.10 are interesting to observe. The travel time profiles for links 1,2,4 and 5 are showing clear peaks indicating that the travel time on them increases and then decreases until the free flow time as set by the parameter α. While the increasing travel time profile is no particular concern, the decreasing profile is important to monitor to check whether the FIFO is violated. For example, on link 2, the travel time for vehicles entering at 30th minute is about 23 minutes and that for vehicles entering at 52nd minute is 12 minutes. That means the travel time has dropped by 11 minutes in an entry span of 22 minutes indicating a slope of -0.5, which is greater than -1, thus satisfying the requirement $\frac{d\tau(t)}{dt} > -1$, in an overall sense. In fact, in the computer program travel times at each minor time step are compared and FIFO has been ensured. In case of link 3, the travel time remained almost unaltered, mainly due to a very low demand to use that route and even perhaps a very low value of the coefficient β chosen.

3.6 Summary

This chapter introduced the dynamic network loading models required for carrying out dynamic traffic assignment and discussed their desirable properties such as flow conservation, FIFO and causality. It was noted that a whole link linear travel time model has the desirable properties besides being differentiable in inflows, and hence was implemented. Some critical issues in its implementation, such as computing the outflow using interpolation method were discussed. Numerical examples were constructed to illustrate the principles described using step functions and continuous functions as inflow profiles. In particular, the impact of the size of the simulation step length on the model performance was investigated. Finally, the results of dynamic network loading over a network with overlapping routes were discussed.

Chapter 4
Variance Approximation Method

4.1 Background

As described in Chapters 2 and 3, there have been a range of techniques based on equilibrium theory and stochastic processes to represent dynamic traffic flow phenomena and their impact on departure time dependent route choice (within day dynamics) and the associated behavioural adaptation, learning and information acquisition process (day-to-day dynamics). This Chapter presents the research that for the first time unifies the advantages of these competing philosophies, by bringing together three elements, namely:

- the representation of both within- and between-day dynamics within the overall framework of a discrete-time stochastic process, as proposed in the unifying framework of Cantarella and Cascetta (1995);
- the representation of within-day traffic flow dynamics by a continuous-time, whole-link, dynamic network loading model of the kind commonly used in the dynamic traffic assignment literature; and
- The solution and estimation of the equilibrium properties of the resulting stochastic process using a combination of conventional within-day dynamic network equilibrium theory and a theoretical approximation result.

Besides providing a practical solution technique, this approach is also able to place the diverse subjects of network equilibrium theory, simulation and stochastic process models within a common theoretical framework, whereby advances in one area may be used to the advantage of other areas.

Specifically, the modelling approach adopted is based on modelling the day-by-day evolution of driver decisions as a discrete-time Markov process, and the evolution of traffic flow dynamics within the day by a continuous-time whole-link model (hence it is termed 'doubly dynamic'). While it is known that the stationary output of such a model, in the form of an equilibrium probability distribution of time-dependent network flows, may in principle be estimated by Monte Carlo simulation, there are many potential drawbacks of such an approach, such as the difficulties of detecting stationarity and the effects of multiple attractors, and of dealing with Monte Carlo error in policy tests (Cantarella & Cascetta, 1995; Watling, 1996; Watling, 2002).

With such considerations in mind, Hazelton & Watling (2004) proposed an analytic approximation method for directly estimating the equilibrium probability distribution of such a model, without the need for simulation, and requiring only knowledge of a conventional Stochastic User Equilibrium (SUE) solution. Utilising a result previously established by Davis and Nihan (1993), namely that this equilibrium distribution is asymptotically multivariate Normal, with mean the SUE flow vector, then all that is required to complete the approximation process is the variance-covariance matrix of the flows, as provided by the approximation result of Hazelton and Watling (2004). However, a significant restriction of their work is that it was restricted to within-day static models. The contribution of the present thesis is to make the important step of extending this work to the within-day dynamic case, in which the origin-destination demand flows are time-sliced by departure time (to any desired resolution), and in which the interactions on the network are handled through a continuous-time, dynamic network loading model (estimated in practice by an arbitrarily fine discretisation).

In the following section, the underlying model is explicitly stated in quite general terms, and an approximation derived for the moments of the equilibrium probability distribution of the process. This approximation in turn requires the computation of various equilibria and Jacobian matrices, and so the next Chapter is devoted to explaining techniques for performing these computations. In particular, one complete section is devoted to describe the derivation of the route travel time Jacobian for a particular form of popular whole-link, dynamic network loading model.

4.2 Earlier work

Davis and Nihan (1993) estimated the mean of a Markov chain assignment in a day-to-day context and demonstrated that it converges to the Stochastic User Equilibrium assignment, hence motivating further research in approximating the properties of equilibrium probability distributions. Hazelton and Watling (2004) attempted to compute the equilibrium co-variance matrix in a day-to-day context (assuming static within day cost flow functions) and proved that the equilibrium probability distributions can be approximated in a fraction of time as compared to that by solving the Davis and Nihan's fixed point equations or by the method of simulating the route choices using Monte Carlo techniques.

Hazelton and Watling (2004) follow similar principles as set out by Cascetta (1989) and the basic stochastic process model can be specified exactly in an identical way as described in Section 2.1.1 of Chapter 2. Let $\mathbf{F}_k^n$ be a random vector of route flows on day n for inter-zonal movement k and a collection of the route flows across all the O-D movements is contained in $\mathbf{F}^n$. Route flows on day n for inter-zonal movement k are obtained conditionally on the costs given up to day n-1 as a multinomial distribution as defined by equation (2.5). As noted by Cascetta (1989), the process $\mathbf{F}^n$ for n = 1,2,...is an m-dependent Markov chain and hence equivalently the combined process $\mathbf{F}^n$, $\mathbf{F}^{n-1}$,.... $\mathbf{F}^{n-m+1}$ for n = 1,2,... is a Markov chain and the probability distribution of the combined process converges to a unique equilibrium provided that there is a non-zero probability of the process switching from one state to any other. The focus of Hazelton and Watling (2004) is on estimating the moments of the marginal equilibrium distribution of $\mathbf{F}^n$.

Davis and Nihan (1993) demonstrated that the limiting probability distribution of **C** is multivariate Normal. This implies that the conditional distribution of $\mathbf{F}^{n+1}$ given $\mathbf{C}^n$ must also be multivariate Normal when the process is stationary. Therefore, the distribution of $\mathbf{F}^n$ is also multivariate Normal when the process is stationary.

Davis and Nihan (1993) also proved that the mean of the probability distribution of **F** tends to the SUE assignment when the demand is large. This means **F** can be approximated by

$$\mathbf{F} \sim \text{Normal}(\mathbf{f}^*, \hat{\Sigma}), \text{ approximately.} \tag{4.1}$$

where, **f*** is the route flow vector when the process is stationary. **f***, the mean route flows, can be estimated by any suitable method such as the method of successive averages and $\hat{\Sigma}$ the variance-covariance of route flows is estimated as described in the following paragraphs. It is important to note that the approximation to a Normal distribution given in (4.1) works well provided the demand is large. This condition is usually satisfied by developing density-based models as in Davis and Nihan (1993) and Hazelton and Watling (2004). For example, Hazelton and Watling consider the case of link capacity growing in proportion to the demand, so that in the limit as the demand tends to infinity the approximation results still remain valid. However, such an approach based on densities (or proportions, equivalently) will result in significant notational complexity and hence the present work uses the notation based on absolute numbers rather than proportions.

Assuming the cost flow function is linear in the neighbourhood of **f***, then

$$\mathbf{c}(\mathbf{F}) = \mathbf{c}(\mathbf{f}^*) + \mathbf{B}(\mathbf{F}-\mathbf{f}^*) \tag{4.2}$$

where, **B** is the Jacobian matrix of the route costs with respect to the route flows. It may be recalled that the dynamic link travel time model as described in section 3.4 of the previous chapter, is defined as a linear function of the number of vehicles on the link at each instant and hence the equation (4.2) remains valid throughout the present modelling framework. Similarly, by assuming that the route choice probability function is linear in the neighbourhood of mean route costs **c*** when the system is stationary, then it follows that,

$$\mathbf{p}(\mathbf{C}) = \mathbf{p}(\mathbf{c}^*) + \mathbf{D}(\mathbf{C}-\mathbf{c}^*) \tag{4.3}$$

where, **D** is the Jacobian matrix of the route choice probability with respect to the route costs. Following from Davis and Nihan (1993), Hazelton and Watling (2004) used a logit-based route choice probability function. It is well known that the driver behaviour as modelled by a logit-based route choice function can be represented by an S-shaped curve and that the shape of the curve depends on the value of the dispersion scaling parameter θ. The lower the value of θ, the higher the dispersion resulting in an almost straight line, conversely, the higher the value of θ, the lower the dispersion resulting in a shape that is similar to the mirror image of the letter, Z. In the light of the above comments, it is important to note that the equation (4.3) and the underlying assumption of the linearity of the route choice probability function remain valid for a certain range of values of θ. Some of the numerical tests reported in Chapter 7 explore the validity of the approximation model for various values of θ.

Hazelton and Watling (2004) estimated the variance-covariance matrix of the route flows, in case of multiple origin-destination pairs, as below:

$$\hat{\mathbf{\Sigma}} = \mathbf{\Theta}^* + s(\lambda)^{-2}\left\{\mathbf{QDB\Theta}^*(\mathbf{QDB})^{\mathrm{T}} + \mathbf{QDMB\Theta}^*(\mathbf{QDMB})^{\mathrm{T}}\right\} \tag{4.4}$$

where, $\mathbf{\Theta}^* = diag(\mathbf{f}^*) - \mathbf{f}^*(\mathbf{f}^*)^{\mathrm{T}}$, the conditional covariance matrix of the distribution in (2.5) evaluated at SUE,

$\mathrm{s}(\lambda) = (1-\lambda^{\mathrm{m}})/(1-\lambda)$, sum to 'm' terms of the memory weight λ for $0 < \lambda < 1$,

$\mathbf{Q}$ = diagonalised demand,

$\mathbf{M} = \mathrm{s}(\lambda)^{-1}\mathbf{BD} + \lambda\mathbf{I}$, an intermediate matrix with **I** representing an identity matrix of appropriate dimensions, and

T represents the transposing operator.

In the above expression for variance-covariance matrix, all parameters on the right hand side are known except the Jacobian matrices **D** and **B**. Jacobian of the route choice probabilities **D**, can be easily computed by assuming a functional form such as logit and evaluating its derivative at SUE. In the case of the Jacobian matrix of the route costs **B**, Hazelton and Watling (2004) could easily evaluate the relationships obtained by differentiating the BPR style static within day cost-flow functions they assumed. However, in the case of dynamic within day cost flow relationships, the route cost is a composite variable and is a function of the route flows not only in one departure period, but also may be a function of the route flows which departed earlier from the reference time period. In general, the route costs are affected by route flows belonging to various O-D pairs using any given link at any given moment. Clearly, all the vehicles from the previous departure periods which are still on the link in consideration will influence the travel cost and hence computing the derivative of route travel cost becomes much more complicated. Therefore, the integrated day-to-day and within day situation (with multiple time periods) poses a major challenge to the researchers in approximating the properties of equilibrium probability distributions. Chapter 5 describes the method of computing the Jacobian matrix of the travel time (cost) which is one of the main contributions of this research project.

4.3 Variance Approximation Method

This model considers the case of multiple origin-destination movements, served by a network of links with overlapping path-flows on which there are time-varying travel time-flow relationships defined in continuous time. On the demand side, the origin-destination flows are assumed to be specified (as is typically the case in practice) in discrete departure periods, though these periods can be made as fine as the modeller desires. Though not a necessary restriction, for notational simplicity let us assume that all origins are discretised into the same number of departure periods. On the network side let us consider the continuous-time dynamic network loading map as a relationship between a given vector of path flows (by departure interval and origin-destination movement) and the vector of resulting mean path travel times (by departure interval and origin-destination movement). Broadly speaking, the model and derivation of the approximation result in this section re-interprets that given in Hazelton and Watling (2004), with origin-destination movements instead interpreted here more generally as *commodities*. In this case, a commodity consists of a triple (origin, destination, departure period), so that the number of commodities will be the product of the number of origins, number of destinations and number of departure periods. It is important to note that the thrust of the approximation method lies in an asymptotic (law of large numbers) argument, as the demand levels become large, but assuming that the network capacity grows in relation to the demand. This allows a large sample approximation to be made of any arbitrary network problem, whether or not the original problem has demands that are "large" in any sense. Below, an outline is provided of the key assumptions and results.

This section introduces the notation and describes the assumptions made in developing the model. The commodity demand flows (i.e. origin-destination demands for each departure time period) are held in a vector **q** of dimension K with elements q_k $(k = 1,2,...,K)$. Each commodity k is served by a set of routes R_k with $|R_k|$ elements; the full route set across all commodities thus has dimension $\rho = \sum_{k=1}^{K} |R_k|$. Note that this notation includes some repetition, since for each O-D movement, each entry time period will be repeated, yet this substantially eases the derivations below. If the ρ-vector of commodity route flows (across all commodities) is contained in the vector **f**, then $\mathbf{c}(\mathbf{f})$ denotes the ρ-vector of mean commodity route costs as a function of the commodity route flows. Let us presume $\mathbf{c}(\mathbf{f}) = \mathbf{b} + \gamma\tilde{\mathbf{c}}(\mathbf{f})$, where $\tilde{\mathbf{c}}(\mathbf{f})$ denotes the mean commodity route travel times (obtained from a suitable dynamic network loading model), γ denotes the value-of-time, and where **b** is a ρ-vector representing the composite effect on commodity route costs of other flow-independent attributes (tolls, distance, etc.). Suppose also that $\tilde{\mathbf{c}}(\mathbf{f})$ is sufficiently smooth to be at least piecewise differentiable in **f**.

It is assumed that all the trip makers of commodity k are rational in their behaviour when choosing their route, in an attempt to minimise their *perceived* cost of travel. For each commodity k and route $r \in R_k$, the perceived travel cost $\hat{C}_r^{(n)k}$ at the start of day k is given by

$$\hat{C}_r^{(n)k} = C_r^{(n-1)k} + \eta_r^{(n)k} \qquad (4.5)$$

where $C_r^{(n-1)k}$ is the population-mean perceived cost for commodity k and route r at the end of day n-1, and $\eta_r^{(n)k}$ is a random variable describing unobserved attributes contributing to the population-dispersion of the perceived attractiveness of route r by commodity k. The ρ-vector $\mathbf{C}^{(n-1)}$ represents the collection of population-mean perceived costs across all routes and commodities. The probability of choosing route r on day n is then given by:

$$p_r^k(\mathbf{C}^{(n-1)}) = \text{Prob}\left(C_r^{(n-1)k} + \eta_r^{(n)k} < C_i^{(n-1)k} + \eta_i^{(n)k} \quad \forall \ i \neq r\right) . \tag{4.6}$$

$\mathbf{p}^k(.)$ then represents the vector (of dimension $|R_k|$) of route choice probabilities for the commodity k, and $\mathbf{p}(.)$ denotes the collection of these choice probability vectors over all the commodities (i.e. $\mathbf{p}(.)$ is a vector of dimension ρ). The functional form of the path choice probabilities depends on the joint probability density function assumed for the residuals $\{\eta_r^{(n)k} : r \in R_k\}$ for each commodity k, resulting (for example) in a logit model if independent Gumbel distributions are assumed, and a probit model for a multivariate Normal distribution.

While the behavioural choice-side of the model is quite conventional, a simple linear learning filter is used to replicate drivers building up their experience of travel costs on a day-by-day basis following the completion of each day's trip. Although several authors including Horowitz (1984), Cascetta (1989), Ben-Akiva et al (1991), Iida et al (1992), Nakayama et al (1999) used similar perception updating models, very few e.g., Iida et al (1992) presented any analysis of the model parameters. While there are other approaches for updating the perceptions such as the Bayesian approach (Jha et al 1998), in this research, the weighted average approach has been used in order to derive analytic results though all other authors used learning models in simulation experiments. Thus following the completion of trips on any day n, the population-mean perceived costs are updated based on a weighted average of costs actually incurred in a finite number of previous days m, using the form:

$$\mathbf{C}^{(n-1)} = s(\lambda)^{-1}\sum_{j=1}^{m}\lambda^{j-1}\mathbf{c}(\mathbf{F}^{(n-j)}) \qquad 0<\lambda<1 \tag{4.7}$$

where $s(\lambda) = \sum_{j=1}^{m}\lambda^{j-1} = (1-\lambda^m)/(1-\lambda)$ is simply a scaling factor to make the weights sum to unity and $\mathbf{c}(.)$ is the commodity route cost-flow function as defined above, and where $\mathbf{F}^{(n)}$ is a vector random variable of dimension ρ denoting the network path flows by commodity on day n. Assuming that for any day n and for each commodity k, all q_k drivers wishing to travel make their travel choices independently, conditional on their experiences in past days, then the number of drivers taking each possible route on day n by each commodity k, conditional on the costs (3) experienced in the past, is obtained as:

$$\mathbf{F}^{(n)k}\,\big|\,\mathbf{C}^{(n-1)} \sim \text{Multinomial}(q_k, \mathbf{p}^k(\mathbf{C}^{(n-1)})) \text{ independently, for } k = 1,2,\ldots K \tag{4.8}$$

where $\mathbf{F}^{(n)k}$ is the vector of route flows on day n by the commodity k.

The route flows on day n are then given conditionally by the partitioned vector as below:

$$\mathbf{F}^{(n)}\,\big|\,\mathbf{C}^{(n-1)} = \begin{pmatrix} \mathbf{F}^{(n)1}\,\big|\,\mathbf{C}^{(n-1)} \\ \hline \mathbf{F}^{(n)2}\,\big|\,\mathbf{C}^{(n-1)} \\ \hline \vdots \\ \hline \mathbf{F}^{(n)K}\,\big|\,\mathbf{C}^{(n-1)} \end{pmatrix} \tag{4.9}$$

where $\mathbf{F}^{(n)k}\,\big|\,\mathbf{C}^{(n-1)}$ is given by (4.8).

Then the expectation and variance-covariance matrix would follow as:

$$E[\mathbf{F}^{(n)}\,\big|\,\mathbf{C}^{(n-1)}] = \begin{pmatrix} E[\mathbf{F}^{(n)1}\,\big|\,\mathbf{C}^{(n-1)}] \\ \hline E[\mathbf{F}^{(n)2}\,\big|\,\mathbf{C}^{(n-1)}] \\ \hline \vdots \\ \hline E[\mathbf{F}^{(n)K}\,\big|\,\mathbf{C}^{(n-1)}] \end{pmatrix} \tag{4.10}$$

where, $E\left[F^{(n)k} \middle| C^{(n-1)}\right] = q_k \, p^k (C^{(n-1)})$ (4.11)

and the conditional covariance matrix has a block-diagonal form:

$$\mathrm{Var}(F^n | C^{(n-1)}) = \begin{pmatrix} \mathrm{Var}(F^{(n)1} | C^{(n-1)}) & 0 & 0 \\ 0 & \ddots & 0 \\ 0 & 0 & \mathrm{Var}(F^{(n)K} | C^{(n-1)}) \end{pmatrix} \quad (4.12)$$

where, by standard results for the multinomial distribution:

$$Var\,(F^{(n)k} | C^{(n-1)}) = q_k \,[\mathrm{diag}\,(p^k (C^{(n-1)})) - p^k (C^{(n-1)})(p^k (C^{(n-1)}))^T] \quad (4.13)$$

where, the superscript T denotes the transposition operator.

Note that the moments above are all obtained conditionally on the perceived route costs; however, for any sensible prediction the *unconditional* moments are required. Based on standard results, the unconditional first moment is given as:

$$E\left[F^{(n)}\right] = E\left[E\left[F^{(n)} \middle| C^{(n-1)}\right]\right] \quad (4.14)$$

Now, applying the work of Davis and Nihan (1993) to the case of multiple commodities (i.e., reinterpreting the multiple origin-destination movements case of Hazelton and Watling (2004)), the mean of the multinomial distribution in (4.8) converges asymptotically to the solution of the within-day dynamic stochastic user equilibrium (SUE) problem F^{SUE}, as the demand grows to infinity in tandem with the capacities. Since Davis & Nihan also establish asymptotic convergence in distribution of the process to a multivariate Normal, the only remaining piece of information thus required to characterise the full equilibrium distribution is the covariance matrix.

Now, the unconditional second moment is given as (again by standard statistical identities):

$$\mathrm{Var}(\mathbf{F}^{(n)}) = \mathrm{E}\left[\mathrm{Var}(\mathbf{F}^{(n)} \middle| \mathbf{C}^{(n-1)})\right] + \mathrm{Var}(\mathrm{E}\left[\mathbf{F}^{(n)} \middle| \mathbf{C}^{(n-1)}\right]) \qquad (4.15)$$

The first term on the right hand side of (4.15) is given by

$$\mathrm{E}[\mathrm{Var}(\mathbf{F}^{n} | \mathbf{C}^{(n-1)})] = \left(\begin{array}{c|c|c} \mathrm{E}[\mathrm{Var}(\mathbf{F}^{(n)1} | \mathbf{C}^{(n-1)})] & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \ddots & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathrm{E}[\mathrm{Var}(\mathbf{F}^{(n)K} | \mathbf{C}^{(n-1)})] \end{array} \right) \qquad (4.16)$$

where for each commodity k, $\mathrm{E}\left[\mathrm{Var}(\mathbf{F}^{(n)k} \middle| \mathbf{C}^{(n-1)})\right] \rightarrow \mathbf{\Theta}_k^*$ with $\mathbf{\Theta}_k^*$ given by the multinomial covariance matrix evaluated at SUE path flow proportions $(\mathbf{p}^{1SUE}, \mathbf{p}^{2SUE}, \ldots, \mathbf{p}^{KSUE})$:

$$\mathbf{\Theta}_k^* = q_k[\mathrm{diag}(\mathbf{p}^{kSUE}) - \mathbf{p}^{kSUE}(\mathbf{p}^{kSUE})^T] \qquad (4.17)$$

where $\mathbf{p}^{kSUE}$ denotes the SUE path flow proportions for the commodity k.

Based on applying the results established in Hazelton and Watling (2004) to this new application area, the second term on the right hand side of (4.15) can be shown to be in the limit

$$\mathrm{Var}(\mathrm{E}[\mathbf{F}^{(n)} | \mathbf{C}^{(n-1)}]) \rightarrow s(\lambda)^{-2}[\mathbf{QDB\Theta}^*(\mathbf{QDB})^T + \mathbf{QDMB\Theta}^*(\mathbf{QDMB})^T] \qquad (4.18)$$

where $\mathbf{\Theta}^*$ is the collection of the conditional covariance matrices (4.17) across all commodities; $\mathbf{Q} = \mathrm{diag}(\mathbf{\Gamma q})$ is a diagonalised matrix of the demand flow vector $\mathbf{q}$ where $\mathbf{\Gamma}$ is the commodity/route incidence matrix; $\mathbf{D}$ is the Jacobian of commodity/route choice probabilities $\mathbf{p}(\mathbf{C})$ with respect to commodity/route costs $\mathbf{C}$; and $\mathbf{B}$ is the Jacobian of commodity/route travel costs $\mathbf{c}(\mathbf{f})$ with respect to commodity/route flows **f**. The matrix **M** is an intermediate matrix, with $\mathbf{M} = s(\lambda)^{-1}\mathbf{BD} + \lambda\mathbf{I}$ where **I** represents an identity matrix of appropriate dimension.

The implementation of the approximation method above requires the calculation of the two terms on the right hand side of (4.15), namely (4.16) and (4.18). Expression (4.17) is computed by finding the time-dependent SUE by the Method of Successive Averages, with the SUE route flow proportions by commodity then input to (13) along with the time-varying demand profile. Expression (14) requires as input the memory length, memory weights and demand profile, together with the Jacobians **B** and **D** evaluated at SUE. Computing **D** is straightforward for a logit model, and is possible for other approaches such as probit. Computing **B** is potentially more problematic, partially because the route travel time depends on link travel times at the appropriate link entry times for a vehicle following that route, and these entry times themselves depend on the network flows and the travel times on previous links in the route. Chapter 6 shows how this Jacobian may be deduced for a common form of dynamic network loading model from the literature.

4.4 Jacobians of Route Choice Probabilities

Following from equation (4.5), the functional form of route choice probabilities will be a logit model based on the assumption that the random residuals in perceiving the route costs are Gumbel distributed, and the choice probabilities for commodity k choosing route r are given by

$$p_r^k = \frac{e^{-\theta C_r^k}}{\sum_{s \in R_k} e^{-\theta C_s^k}} \tag{4.19}$$

Differentiating (4.19) with respect to route costs of commodity k using route r gives,

$$\frac{\partial p_r^k}{\partial C_r^k} = \frac{\theta\, e^{-\theta C_r^k}\left[e^{-\theta C_r^k} - \sum_{s \in R_k} e^{-\theta C_s^k}\right]}{\left[\sum_{s \in R_k} e^{-\theta C_s^k}\right]^2} \qquad \forall\, r \in R_k \qquad (4.20)$$

Similarly differentiating (4.19) with respect to any $s \neq r$ gives the off diagonal elements of the Jacobian matrix, and is given as,

$$\frac{\partial p_r^k}{\partial C_h^k} = \frac{\theta\, e^{-\theta\left(C_r^k + C_h^k\right)}}{\left[\sum_{s \in R_k} e^{-\theta C_s^k}\right]^2} \qquad h \neq r,\ \forall\, r, h \in R_k \qquad (4.21)$$

The Jacobian matrix will be a block diagonal matrix with as many blocks as there are departure periods, and each block will have diagonal elements given by (4.20) and off diagonal elements defined by (4.21). All elements of the Jacobian matrix other than the block diagonals are zero because the route choice probabilities in any departure period are based on the costs given in that departure period and so are independent of the costs in the other departure periods.

4.5 Implementing the Variance Approximation Method

The variance approximation as specified in sections 4.3 and 4.4 earlier in this chapter, has been implemented by developing purpose-written MATLAB programs. The scheme of implementation is shown in Figure 4.1. The main module of the program is based on a logit choice rule for splitting the trips of each commodity over the set of feasible routes and then the route flows are averaged over the number of iterations completed so far (the method of successive averages) which are passed on to the dynamic network loading module (See Figure 3.4). The dynamic loading process moves the vehicles along the street network in time and space and computes the path travel time in each departure period which is then passed on to the next iteration of the logit model. The number of iterations of the MSA is set very large such as 10000 iterations, to ensure that the route flows are at equilibrium. As noted by Ortuzar and Willumsen (1999), the MSA may be very slow in converging and hence the higher number of iterations. Clearly, the algorithm for solving the DSUE is not central to the present thesis, and perhaps a more efficient algorithm as for example, Han (2000) may be used which might require much fewer iterations. Thus, the route flows obtained are at dynamic stochastic user equilibrium as all the drivers are assigned to their least perceived cost routes within the departure period. The model implemented here assumes that the drivers do not have a choice of departure time period and concentrates only on the route choice.

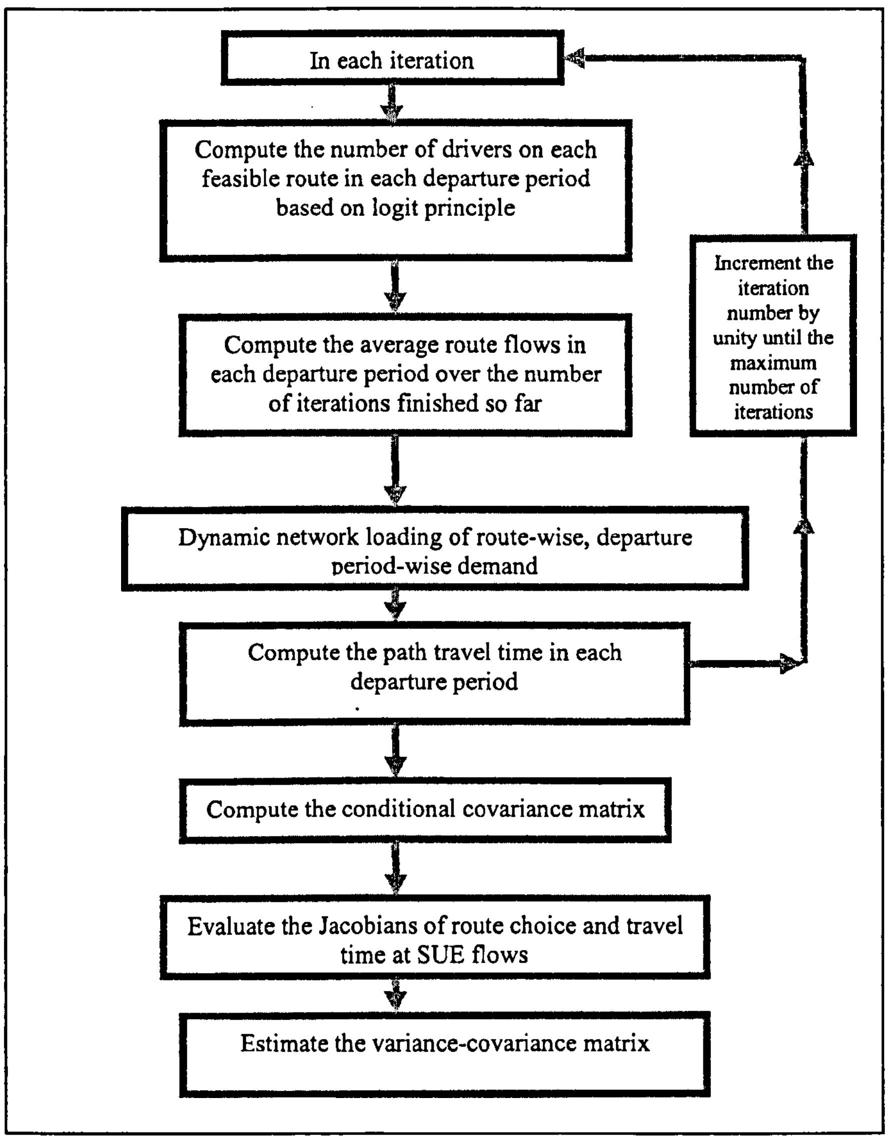

Figure 4.1 Flow Chart for Variance Approximation Method

The balance of the computation is simply evaluating the conditional covariance matrix, and the Jacobian matrices of route choice and travel time, at SUE flows.

The conditional covariance matrix is specified as a block diagonal matrix with each block representing each commodity and is given by equation (4.17), while the route choice Jacobian is also a block diagonal matrix and the blocks are defined with the help of equations (4.20) and (4.21). However, computing the Jacobians of travel time requires a more detailed treatment and will be the focus of Chapter 5.

4.6 Summary

The equilibrium probability distribution of a stochastic process can be approximated by a Normal distribution and its mean converges to SUE, provided the demand is sufficiently large (Davis and Nihan 1993). This chapter reinterprets the work of Hazelton and Watling (2004) and extends the method of approximating the variance-covariance of a stationary probability distribution to the case of dynamic travel time functions, thus bringing together the two moments viz., the mean (as estimated by DSUE) and the variance (as approximated by the variance approximation method), with which the equilibrium probability distribution can be described. However, it is noted that the variance approximation method requires the Jacobian of travel time function, and the specification of this Jacobian forms the focus of Chapter 5.

Chapter 5

Deriving Travel Time Derivatives

5.1 Background

This chapter is aimed at formulating the methodology required for working out the travel time derivatives with respect to the path inflows. Path travel times are defined as the sum of the link travel times based on the nested cost operator as defined by equation (3.26). However, the link travel times in a dynamic network assignment are potentially a function of previous, contemporaneous and future departures along any path, and hence computing the derivatives of path travel times of any given path with respect to its inflows poses a challenge to transport modellers.

There has been very little work published on this topic. Lindveld (2003) computes the analytical derivatives of link flows with respect to O-D flows which followed on from the sensitivity analysis of Tobin and Friesz (1988). Tobin and Friesz compute the derivatives of link flows with respect to perturbations of the cost functions and of the trip table. Lindveld expresses the derivatives of the link flows in terms of the derivatives of the path dependent link flows and in their approximation considers the dynamic link path incidence and route choice proportions and ignores the effects of dynamic network loading and re-routing due to the changes in the trip table.

In this research, the derivatives of travel time with respect to equilibrium path inflows were computed and include the effects of dynamic network loading. Specifically, the Jacobian matrix contained the derivatives of the average route travel time in each departure period with respect to the inflow rate in that or earlier departure periods. As the inflows in later departure periods can not affect the travel times in earlier departure periods, the Jacobian matrix is a block triangular matrix.

5.2 Analytical Travel Time Derivatives

A key element to determine in the approximation method (4.18) presented in Chapter 4, is the Jacobian **B** of the route cost flow function $\mathbf{c(f)}$, which in turn is equivalent (under the stated assumptions) to the Jacobian of the route travel time function (or dynamic network loading map) $\tilde{\mathbf{c}}(\mathbf{f})$. In this section, a method for computing this Jacobian is presented based on a form of whole-link travel time model popular in the dynamic traffic assignment literature.

5.2.1 Link travel time model and notation

Specifically, following Friesz et al (1993), the travel time $\tau_a(t)$ for vehicles entering any link a (with links indexed a = 1,2,..., A) at any continuous time t is related to the number of vehicles $x_a(t)$ on the link at that entry time, given by the relationship

$$\tau_a(t) = \psi_a(x_a(t)) \qquad (t \geq 0) \tag{5.1}$$

for some non-negative, differentiable function $\psi_a(.)$ with derivative $\psi'_a(.)$. Several researchers including Astarita (1996) and Xu et al (1999) also investigated the properties of (5.1) either in linear or non-linear form. Astarita (1996) proves that the linear form of (5.1) limits the outflows to a maximum of the capacity of the link without additional constraints, while Xu et al (1999) show that it satisfies FIFO if the inflows are bounded from above in the linear case, or if the gradient of $\psi_a(x_a(t))$ is bounded from above in the non-linear case. However, as (5.1) assumes that the travel time is independent of the relative position of the vehicles on the link, it would be, in general, suitable for use on congested links.

In order to implement the given model, a fine discretisation is performed. Care should be taken in mapping this to the underlying continuous time axis, and ultimately in aggregating to the typically coarser departure time periods over which route choices are made (for the application of the theory in the previous chapter). In practice, this method makes no premise about the level of discretisation involved, this is under the control of the modeller.

Let δ denote the time increment of this discretisation, and denote the complete analysis period by $(0, N\delta]$ for some positive integer N. The time increments are thus the intervals $(t-\delta, t]$ for $t = \delta, 2\delta, \ldots, N\delta$, which are referred to as *minor* time steps. Below, when a time step (or interval) t is referred, it is to be understood that it refers the period $(t-\delta, t]$. Assume that δ is chosen so as to be smaller than the free flow time to traverse any link, i.e. $\delta < \psi_a(0)$ for all $a = 1,2,\ldots,A$. This is an assumption which will be used implicitly on a number of occasions – implying that a vehicle could not enter and exit a link in the same increment of time. Quite separately to the issue of how to discretise the travel time model, a coarser time-discretisation is assumed for specifying the origin-destination demand rates. In practice, the level of time-discretisation for the O-D demands will be controlled by data availability; in order to get a sensible approximation to the underlying continuous time dynamic network loading model, it is then sensible that the modeller chooses a somewhat finer discretisation for implementing the travel time model. The O-D demand rates are assumed (for notational convenience) to be specified over a common discretisation of the whole analysis period $(0,N\delta]$, divided it into L *major* time periods, also referred to as departure periods $(w_{i-1}, w_i]$ (for i = 1,2,,..L) such that $(w_0, w_1] \cup (w_1, w_2] \cup \ldots\ldots \cup (w_{L-1}, w_L] = (0, N\delta]$. These match exactly the departure periods defined in the previous section, and for convenience are assumed to be of the same duration, i.e. $w_i - w_{i-1} = \kappa$ for all $i = 1,2,\ldots,L$ and some given κ.

Now, unlike conventional network equilibrium theory, a distinction arises here in the flow variables that are being considered. In the stochastic process approach, the commodity route flow vector **f** is an *absolute* flow (measured in units of number of vehicles or drivers). In order to translate to the flow units more conventionally used for dynamic network loading models, consider the flow *rates* **u**, given by a simple scaling of **f** as $\mathbf{u} = \kappa^{-1}\mathbf{f}$. Thus, the route commodity travel time Jacobian computed as a function of **u**, namely $\mathbf{J} = \text{Jac}(\tilde{\mathbf{c}}, \mathbf{u})$, but then it is simple to recover the required route travel time Jacobian (of $\tilde{\mathbf{c}}$ with respect to **f**) as $\kappa^{-1}\mathbf{J}$. Then, finally the route cost Jacobian is given by $\mathbf{B} = \gamma\kappa^{-1}\mathbf{J}$.

In the previous Chapter it turned out to be convenient to merge the notion of an origin-destination movement and departure period into a single entity, referred to as a commodity. In the present Chapter, on the other hand, a slight change of notation will considerably ease the presentation. In particular, now suppose that all R routes across all origin-destination movements (but neglecting departure periods) are indexed r = 1,2,...,R, making the origin-destination movements implicit in the routes. Then the commodity route flow rate vector **u** is written with the route and departure period explicit: thus, for each route and time period referred to in **u** one can identify the corresponding route label r (in the new route labelling system) and departure time period label i, and will thus henceforth refer to the departure time specific route flow rates as $\tilde{u}_{ir}$ for $r = 1,2,...,R$ and i = 1,2,...,L. It will also be convenient to move between time defined on a continuous axis and time defined in terms of the discrete departure intervals. Thus the indicator function $I(t)$ $(0 \le t \le N\delta)$ is introduced which takes the value i if continuous time t refers to departure interval i (for i = 1,2,...,L).

5.2.2 Decomposition of outflows

A key element of the description below is the consideration of the cumulative outflows from each link. In particular, $V_a(t)$ defines the cumulative outflow from link a (a = 1,2,...,A) at any time t $(t \ge 0)$ which it is natural to associate with the end of each simulation time increment $(t-\delta, t]$ since it is a cumulative flow. In particular, $V_a(t)$ is written as the sum of its constituent outflows corresponding to each (minor) inflow period, $t = \delta, 2\delta, \ldots, N\delta$, as:

$$V_a(t) = \sum_{r=1}^{R}\sum_{s=1}^{N} V_{asr}(t) \qquad (a = 1,2,...,A;\ t = \delta, 2\delta, ..., N\delta) \qquad (5.2)$$

where, $V_{asr}(t)$ denotes the cumulative outflow from link a at time t arising from flows on route r *entering the network* before the end of the minor time increment sδ (i.e., before the end of the interval (sδ - δ, sδ]). An important point to note is that the disaggregation is done by the minor time increments, not by the major time increments at which the O-D demands are defined.

Before considering the outflows further, let us turn attention to the cumulative inflows to each link. There are two kinds of contribution, those from vehicles starting their journey on this link (contribution $U_a^1(t)$) and those entering from incident links (contribution $U_a^2(t)$), thus the cumulative inflow to any link a can be written in the form:

$$U_a(t) = U_a^1(t) + U_a^2(t) \qquad (a = 1,2,\ldots,A;\ t = \delta, 2\delta, \ldots, N\delta). \tag{5.3}$$

If ε_{ar} is a 0/1 indicator variable, equal to 1 only if link a is the first link on route r $(a = 1,2,\ldots,A;\ r = 1,2,\ldots,R)$, then based on the notation introduced above, the contribution to the cumulative inflow to any link a from vehicles starting their journey is:

$$U_a^1(t) = \sum_{r=1}^{R} \varepsilon_{ar} \left(\tilde{u}_{I(t),r}(t - w_{I(t)-1}) + \sum_{i=1}^{I(t)-1} \tilde{u}_{ir}(w_i - w_{i-1}) \right) \quad (t = \delta, 2\delta, \ldots, N\delta) \tag{5.4}$$

Then, let us define the 0/1 indicator variable E_{abr} to be 1 only if link a follows link b on path r, then the contribution from flows incident to link a is:

$$U_a^2(t) = \sum_{r=1}^{R} \sum_{b=1}^{A} E_{abr} V_b(t) \qquad (a = 1,2,\ldots,A;\ t = \delta, 2\delta, \ldots, N\delta). \tag{5.5}$$

Note that this notation automatically deals with traffic reaching its destination at the end of a link b that is incident to link a (for a particular route r), since the corresponding E_{abr} would then be zero.

By conservation of flow it follows that the number of vehicles on the link at time t is the difference between the cumulative inflow and cumulative outflow:

$$x_a(t) = U_a(t) - V_a(t) \qquad (a = 1,2,\ldots,A;\ t = \delta,2\delta,\ldots,N\delta). \tag{5.6}$$

Combining equations (5.2)-(5.6), the number of vehicles on the link at any minor time increment is expressed as a linear combination of the route inflow rates starting on that link in the current time period, and the exit flows from incident links which are decomposed according to (5.2).

5.2.3 Relationships between route and link travel time derivatives

Now, let us turn attention to the travel times. Recall that the travel time on any link a for vehicles entering at time t is denoted $\tau_a(t)$. Now, the ultimate interest is in the path travel times from the dynamic network loading model. Supposing that the link travel times *at any continuous time* t are already known, then one would simply trace along the links of the route in the relevant time trajectory, following the notion of a nested cost operator introduced by Friesz (1993). Thus if the n(r) links used in sequence by any route r have the link indices $a_{1r} \rightarrow a_{2r} \rightarrow \ldots \rightarrow a_{n(r),r}$, then define g_{krj} as the time that drivers departing on route r exit link a_{kr} if they begin their journey at time $j\delta$ for j= 1,2,...,N. These intermediate link exit times are built up recursively according to:

$$g_{1rj} = j\delta + \tau_{a_{1r}}(j\delta)\,; \qquad g_{krj} = g_{k-1,rj} + \tau_{a_{kr}}(g_{k-1,rj}) \ \text{ for } k = 2,3,\ldots,n(r) \tag{5.7}$$

with the desired route travel time for the complete journey given by the difference between the departure time $j\delta$ and the final exit time at the end of recursion (5.7), namely a travel time of $g_{n(r),rj} - j\delta$.

However, there is a difficulty with (5.7) in that practical implementation of the dynamic network loading model requires a time discretisation, and therefore the link travel times at any continuous time are not known in general. Therefore some interpolation is required; in fact, the aim is to differentiate this interpolation directly (rather than its underlying continuous time system) and so there is a need to specify this more precisely in the notation. Specifically, if it is supposed now that $\tau_a(t)$ as previously defined refers to any continuous entry time $t \geq 0$, and $\hat{\tau}_a(t)$ denotes the travel times known only at the discrete entry periods $t = 0, \delta, 2\delta, \ldots, N\delta$, then if for any real number x, the notation <x> denotes the integer part of x, a linear interpolation yields:

$$\tau_a(t) \approx \hat{\tau}_a(<t/\delta>.\delta) + \frac{t - <t/\delta>.\delta}{\delta}\left[\ \hat{\tau}_a((<t/\delta>+1).\delta) - \hat{\tau}_a(<t/\delta>.\delta)\ \right] \qquad (5.8)$$

$$(t \geq 0;\ a = 1,2,\ldots,A)$$

Combining (5.7) and (5.8) thus allows the intermediate link exit times on any route (and the complete route travel time) to be calculated for any departure time, based on knowledge of link travel times at the minor time increments. Clearly, $\hat{\tau}_a(t)$ can be related to the number of vehicles on the link for discrete time intervals through (5.1), in just the same way as $\tau_a(t)$ would be in continuous time. The ultimate interest is in the mean route travel time during each of the L *major* entry time periods (each of which consists of κ/δ minor time increments, in terms of the notation already defined), namely:

$$\tilde{c}_{ir} = \frac{\delta}{\kappa}\sum_{j=1}^{N}(g_{n(r),rj} - j\delta)\, I(j\delta)\ . \qquad (5.9)$$

Having now built up an expression for the route travel time by departure period, from (5.9) and earlier expressions, let us now start with the differentiation (with respect to the route flow rates by departure time). It is trivial to see from (5.9) that:

$$\frac{\partial \tilde{c}_{ir}}{\partial \tilde{u}_{hs}} = \frac{\delta}{\kappa} \sum_{\substack{j=1 \\ \text{s.t. } w_{i-1} < j\delta \leq w_i}}^{N} \frac{\partial g_{n(r),rj}}{\partial \tilde{u}_{hs}}\ . \qquad (5.10)$$

Now, some care is needed in differentiating the path travel times above derived from the nested cost operator (5.7). To understand this point, consider a network consisting of a single path (path r = 1) made up of two links in series (n(1) = 2), the path using first link 1 then link 2 (i.e. $a_{11} = 1,\ a_{21} = 2$), and suppose that $\delta = 1$. For a vehicle departing at minor time step j, the nested operator (5.7) would give a complete path travel time of

$$g_{21j} - j = j + \tau_1(j) + \tau_2(j + \tau_1(j)) - j = \tau_1(j) + \tau_2(j + \tau_1(j)). \qquad (5.11)$$

Now suppose that an infinitesimal perturbation is made to the flow rate $\tilde{u}_{h1}$ on route 1 for some major time period h that is before or includes the current minor time step j (i.e. $h \leq I(j)$). Then the time-profile of the number of vehicles on each link will be perturbed, and clearly through (5.1), there will be a direct impact on the time-profile of travel times on each link. However, since the argument of $\tau_2(.)$ in (5.11) is itself a function of $\tau_1(.)$ there will also be an impact on the 'census time' at which the relevant travel time is picked out on link 2 that contributes to this path travel time.

Returning to the general case, the interpolation (5.8) in fact allows the decomposition of these two effects. Since the case k = 1 in (5.7) is straightforward (the first link on a path does not have the 'census time' problem), let us restrict the attention to the cases $k = 2,3,\ldots,n(r)$, substituting (5.8) into (5.7):

$$g_{krj} = g_{k-1,rj} + \hat{\tau}_{a_{kr}}(<{}^{g_{k-1,rj}}/_{\delta}>.\delta) + \frac{g_{k-1,rj} - <{}^{g_{k-1,rj}}/_{\delta}>.\delta}{\delta}\left[\ \hat{\tau}_{a_{kr}}((<{}^{g_{k-1,rj}}/_{\delta}>+1)\delta) - \hat{\tau}_{a_{kr}}(<{}^{g_{k-1,rj}}/_{\delta}>.\delta)\ \right] \qquad (5.12)$$

Now an infinitesimal perturbation to any of the (earlier) route flow rates will directly impact on the two travel time terms involved, but they are now evaluated at discretised census times that will not be affected by a small perturbation. The impact on the census times is captured, on the other hand, through the interpolation term, which is a function of the continuous exit time $g_{k-1,rj}$ from the previous link. Thus differentiating (5.12) yields:

$$\frac{\partial g_{krj}}{\partial \tilde{u}_{hs}}=\frac{\partial g_{k-1,rj}}{\partial \tilde{u}_{hs}}+\frac{\partial \hat{\tau}_{a_{kr}}(<^{g_{k-1,rj}}/_{\delta}>.\delta)}{\partial \tilde{u}_{hs}}+\frac{g_{k-1,rj}-<^{g_{k-1,rj}}/_{\delta}>.\delta}{\delta}\left[\frac{\partial \hat{\tau}_{a_{kr}}((<^{g_{k-1,rj}}/_{\delta}>+1).\delta)}{\partial \tilde{u}_{hs}}-\frac{\partial \hat{\tau}_{a_{kr}}(<^{g_{k-1,rj}}/_{\delta}>.\delta)}{\partial \tilde{u}_{hs}}\right]$$
$$+\frac{1}{\delta}\frac{\partial g_{k-1,rj}}{\partial \tilde{u}_{hs}}\left[\hat{\tau}_{a_{kr}}((<^{g_{k-1,rj}}/_{\delta}>+1).\delta)-\hat{\tau}_{a_{kr}}(<^{g_{k-1,rj}}/_{\delta}>.\delta)\right] \quad (k=2,3,\ldots,n(r)). \tag{5.13}$$

Expression (5.13) therefore defines a recursive method for computing the link exit time derivatives along any path, with the recursion initiated by the first link on the path (k = 1) for which:

$$\frac{\partial g_{1rj}}{\partial \tilde{u}_{hs}}=\frac{\partial \hat{\tau}_{a_{1r}}(j\delta)}{\partial \tilde{u}_{hs}}. \tag{5.14}$$

Thus the required derivatives on the right hand side of (5.10) may be obtained as the limit of the recursive process (5.13)/(5.14), as the link exit times are traced through to the path's destination. It may be seen that (5.13) assumes prior knowledge of the link travel time profile derivatives at any *given* discrete entry to a link. An important point is that these link travel time derivatives (at given entry times) can be independently derived link-to-link, without the concern for the 'census time' impacts, since the latter impacts are subsequently captured by tracing recursion (5.13) along the relevant paths. Let us now focus on the process by which the link travel time derivatives may be computed.

Now, combining equations (5.1), (5.3), (5.5) and (5.6), and then differentiating by the chain rule, yields for any given $t=\delta,2\delta,\ldots,N\delta$:

$$\frac{\partial \hat{\tau}_a(t)}{\partial \tilde{u}_{hs}}=\psi'_a\left(U^1_a(t)+\sum_{m=1}^{R}\sum_{b=1}^{A}E_{abm}V_b(t)-V_a(t)\right).\left(\frac{\partial U^1_a(t)}{\partial \tilde{u}_{hs}}+\sum_{m=1}^{R}\sum_{b=1}^{A}E_{abm}\frac{\partial V_b(t)}{\partial \tilde{u}_{hs}}-\frac{\partial V_a(t)}{\partial \tilde{u}_{hs}}\right). \tag{5.15}$$

It is straightforward to deduce from (5.4) that:

$$\frac{\partial U_a^1(t)}{\partial \tilde{u}_{hs}} = \begin{cases} t - w_{I(t)-1} & \text{if } \varepsilon_{as} = 1 \text{ and } h = I(t) \\ w_h - w_{h-1} & \text{if } \varepsilon_{as} = 1 \text{ and } h < I(t) \\ 0 & \text{otherwise} \end{cases} . \qquad (5.16)$$

Then the only unknowns in (5.15) are the link exit flow derivatives, which by differentiating (5.2) are decomposed according to:

$$\frac{\partial V_a(t)}{\partial \tilde{u}_{hs}} = \sum_{r=1}^{R}\sum_{i=1}^{N} \frac{\partial V_{air}(t)}{\partial \tilde{u}_{hs}} \quad (a = 1,2,...,A;\ t = \delta, 2\delta, ..., N\delta;\ h = 1,2,...,L;\ s = 1,2,...,R). \qquad (5.17)$$

5.2.4 Computational process for determining decomposed link exit flow derivatives

The process to determine all the relevant $V_{air}(t)$, $\hat{\tau}_a(t)$, $\partial V_{air}(t)/\partial \tilde{u}_{hs}$ *and* $\partial \hat{\tau}_a(t)/\partial \tilde{u}_{hs}$ terms in (5.15) and (5.17) operates chronologically, advancing all links/paths across the network by one (minor) time increment (t-δ, t] before moving on to the next time increment. At the same time, the link exit times g_{krj} for each path and departure increment are computed from (5.7) and (5.8) as they become available. Importantly, the fact that all these terms (exit flows and travel times, their derivatives, and the link exit times) are calculated in time order means that these values are all known for all links at all time increments *earlier* than the current one. The steps followed for time increment (t-δ, t] (for each $t = \delta, 2\delta, ..., N\delta$) are as follows:

For each link a = 1,2,…,A, each major period $h = 1,2,...,L$, and each route $r = 1,2,...,N$ that uses link a (for routes not using link a the flows and derivatives are clearly all zero):

Step 1: In this step we project the path flows entering the network in earlier time increments into outflows exiting link a in the current increment (t-δ, t] (recall the assumption made earlier, that δ is sufficiently small that any vehicles exiting a link in one time increment could not have entered the link in the same time increment). Suppose that the current link a is the k^{th} link of route r. For each such earlier minor time increment $i\delta$ for $i = 1,2,\ldots, 1/\delta - 1$, let us first determine the entry time to link a, the k^{th} link along route r, which will simply be the exit time from the $(k-1)^{th}$ link, namely $g_{k-1,ri}$ (extending the definition (5.7) so that $g_{0ri} = i\delta$). These link exit times will be known from the application of this procedure in previous iterations. For each $i = 1,2,\ldots, 1/\delta - 1$ there are then three possible cases (assuming FIFO to hold):

Case A: No flow on route r entering the network in period $((i-1)\delta, i\delta]$ exits link a before the end of the current increment $(t-\delta, t]$. This occurs if the first vehicle from the entry interval arrives after the end of the current increment, i.e. if $g_{k-1,r,i-1} + \tau_a(g_{k-1,r,i-1}) > t$, and then:

$$V_{air}(t) = 0 \quad \text{and} \quad \partial V_{air}(t) \big/ \partial \tilde{u}_{hs} = 0 \quad (\text{for } s = 1,2,\ldots,N).$$

Case B: All the flow on route r entering the network in period $((i-1)\delta, i\delta]$ exits link a before the end of the current increment $(t-\delta, t]$. This occurs if the last vehicle from the entry interval arrives before the end of the current increment, i.e. if $g_{k-1,r,i} + \tau_a(g_{k-1,r,i}) \le t$, and then:

$V_{air}(t) = \delta \tilde{u}_{I(i)r}$ and $\partial V_{air}(t) \big/ \partial \tilde{u}_{hs} = \delta$ if $h = I(i)$ and $r = s$, and is 0 otherwise ($s = 1,2,\ldots,N$).

Case C: Some of the flow on route r entering the network in period $((i-1)\delta, i\delta]$ exits link a before the end of the current increment $(t-\delta, t]$, but not all. This occurs if the first vehicle from the entry interval arrives before the end of the current increment, and if the last entering vehicle arrives after the end of the current increment, i.e. if $g_{k-1,r,i-1} + \tau_a(g_{k-1,r,i-1}) \le t$ and $g_{k-1,r,i} + \tau_a(g_{k-1,r,i}) > t$. In this case, the path r inflow $\delta \tilde{u}_{I(i)r}$ over the entry period $((i-1)\delta, i\delta]$ is translated to an outflow from link a spread (uniformly, it is assumed) over a period from the earliest to the latest exit from this link, namely $(g_{k-1,r,i-1} + \tau_a(g_{k-1,r,i-1}), g_{k-1,r,i} + \tau_a(g_{k-1,r,i})]$.

However, the end of this interval is after the end-time t of the current interval, and so only a proportion of this translated flow actually exits by the end of the current increment, namely:

$$V_{air}(t) = \frac{\left(t - (g_{k-1,r,i-1} + \tau_a(g_{k-1,r,i-1}))\right)\delta\tilde{u}_{I(i)r}}{g_{k-1,r,i} + \tau_a(g_{k-1,r,i}) - (g_{k-1,r,i-1} + \tau_a(g_{k-1,r,i-1}))}. \tag{5.18}$$

Now, when differentiating the expression above, let us treat the exit times from the previous link, the g terms, as constants (recall the discussion following equation (5.14)). Thus for s = 1,2,...,N, it follows that:

$$\frac{\partial V_{air}(t)}{\partial \tilde{u}_{hs}} = \left(g_{k-1,r,i} + \tau_a(g_{k-1,r,i}) - (g_{k-1,r,i-1} + \tau_a(g_{k-1,r,i-1}))\right)^{-2} \times$$

$$\left\{\left(g_{k-1,r,i} + \tau_a(g_{k-1,r,i}) - (g_{k-1,r,i-1} + \tau_a(g_{k-1,r,i-1}))\right)\times\right.$$

$$\left[\left(t - (g_{k-1,r,i-1} + \tau_a(g_{k-1,r,i-1}))\right)\frac{\partial \tilde{u}_{I(i)r}}{\partial \tilde{u}_{hs}}\delta - \frac{\partial \tau_a(g_{k-1,r,i-1})}{\partial \tilde{u}_{hs}}\delta\tilde{u}_{I(i)r}\right]$$

$$\left. - \left(t - (g_{k-1,r,i-1} + \tau_a(g_{k-1,r,i-1}))\right)\delta\tilde{u}_{I(i)r}\left[\frac{\partial \tau_a(g_{k-1,r,i})}{\partial \tilde{u}_{hs}} - \frac{\partial \tau_a(g_{k-1,r,i-1})}{\partial \tilde{u}_{hs}}\right]\right\} \tag{5.19}$$

where $$\frac{\partial \tilde{u}_{I(i)r}}{\partial \tilde{u}_{hs}} = \begin{cases} 1 & \text{if } h = I(i) \text{ and } r = s \\ 0 & \text{otherwise} \end{cases}. \tag{5.20}$$

Note, however, that the link travel time derivatives in the expression above are evaluated at a continuous time instant that may not match with the time discretisation chosen, and so they must be interpolated: differentiating (5.8) at a given continuous time g yields

$$\frac{\partial \tau_a(g)}{\partial \tilde{u}_{hs}} \approx \frac{\partial \hat{\tau}_a(<\frac{g}{\delta}>.\delta)}{\partial \tilde{u}_{hs}} + \frac{g-<\frac{g}{\delta}>.\delta}{\delta}\frac{\partial \hat{\tau}_a((<\frac{g}{\delta}>+1).\delta)}{\partial \tilde{u}_{hs}}. \tag{5.21}$$

and the discrete time derivatives above are known from the application of this procedure to earlier time increments.

Step 2: Using equations (5.1)-(5.6) and $V_{air}(t)$ computed in Step 1, hence compute the travel times $\hat{\tau}_a(t)$ on all links $a = 1,2,...,A$ for the current time step t. Using the derivatives $\partial V_{air}(t) / \partial \tilde{u}_{hs}$ for the current time increment t, for all links a, all routes r and s, and all entry periods h, calculate the derivatives $\frac{\partial \hat{\tau}_a(t)}{\partial \tilde{u}_{hs}}$ by (5.15) for all a, s and h at the current time increment t. Calculate link exit times from (5.7) and (5.8) as they become available from the information computed so far.

Having carried out the computations above for all time increments, the path travel time derivatives are then computed from recursion (5.13)/(5.14), and the results substituted into (5.10) to give the required Jacobian of the dynamic network loading model.

5.2.5 Computer Program for Analytical Travel Time Derivatives

The main idea of calculating the travel time derivatives is based on decomposing the cumulative outflows from any link into its constituent inflows at the origin, i.e., at each exit step, the cumulative outflows are traced back to the inflows from all the contributing inflow time steps and then the cumulative outflow derivatives with respect to the route inflows are computed (Figure 5.1). As the travel time derivatives are evaluated at SUE, it is sufficient to run the derivatives sub-program only once during the final iteration of the variance approximation method (See Figure 4.1). Therefore, in the final iteration of MSA at each time step, if the outflow exists, then for each earlier entry time step, the inflow is classified as no contribution (Case A), full contribution (Case B) or partial contribution (Case C) and then the decomposed outflow derivative for the current time step corresponding to the inflow from each entry time is computed as defined in Case A or B or C in section 5.2.4. Then the cumulative outflow derivative for the current time is computed by summing all the decomposed derivatives for that time step. Then the travel time derivative for the current time step is computed using equation (5.15). Alternatively, if there is no outflow, the travel time derivative increases at a constant rate and uses only the first part of equation (5.15), as the outflows and their corresponding derivatives are equal to zero. Finally, the path time derivatives are computed by using equations (5.13) and (5.14). Thus the analytical derivatives of travel time so obtained are compared with numerically obtained travel time derivatives which are worked out as described in the ensuing section.

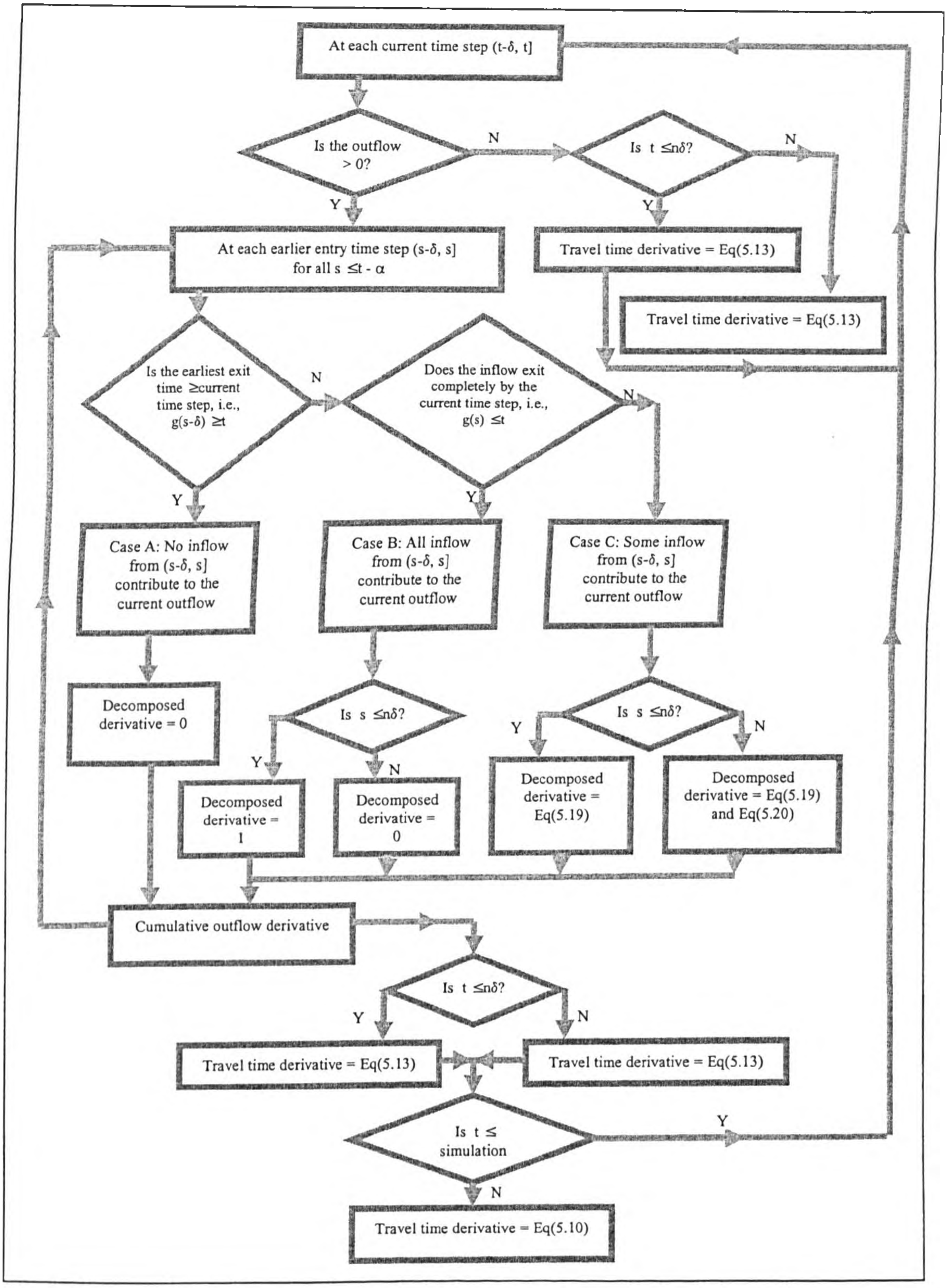

Figure 5.1 Flow Chart for Analytical Travel Time Derivatives

5.3 Finite Difference Travel Time Derivatives

Travel time derivatives can also be computed based on the perturbations of path inflows in any given departure period. This involves working out the differences in travel times from the dynamic network loading map before and after the perturbations, divided by the amount of perturbed path inflow. This results in estimates of the derivatives of travel time, which are called finite difference derivatives. Finite difference derivatives are useful in checking the analytical derivatives computed from the specification section 5.2. A simple scheme of implementation is shown in Figure 5.2.

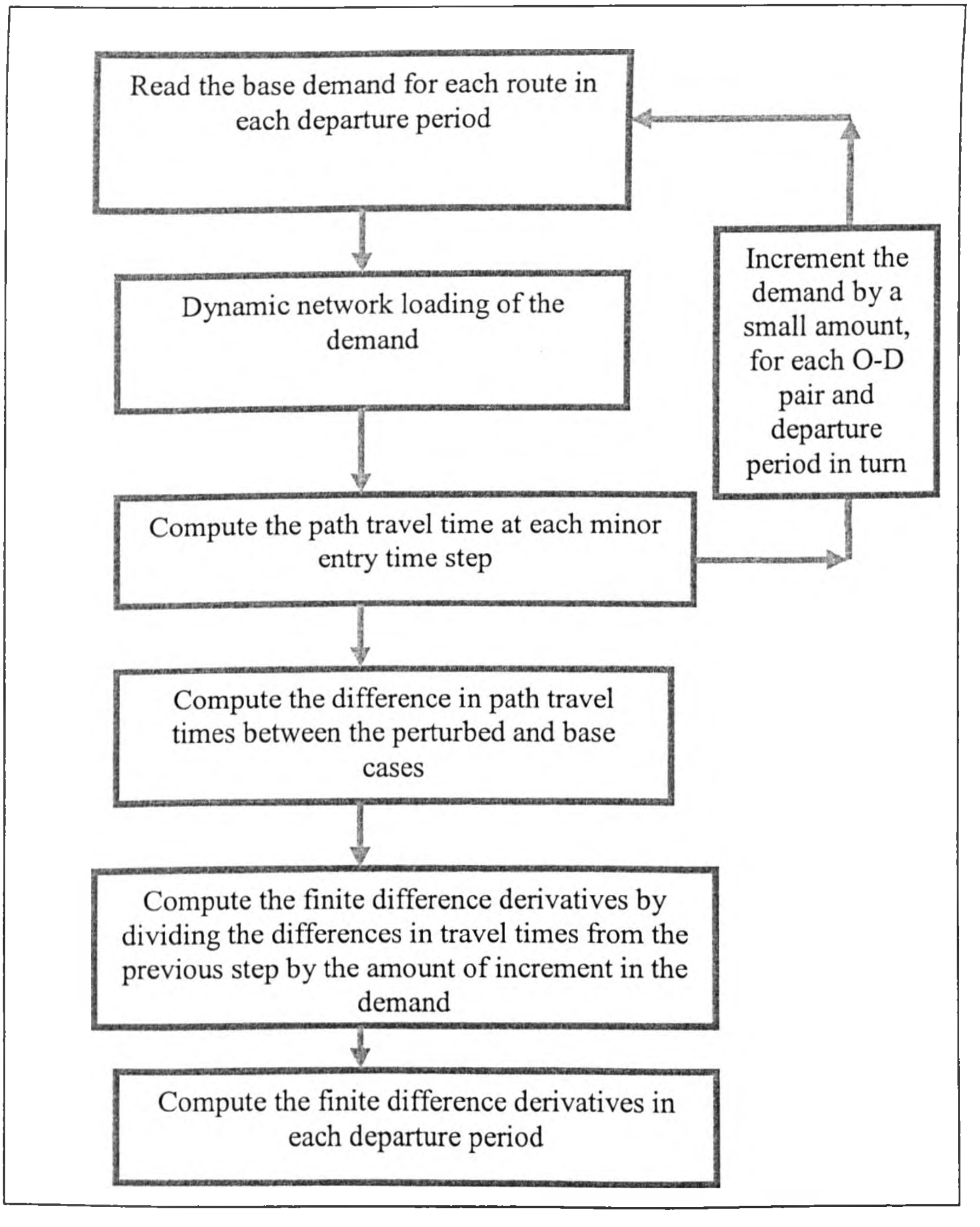

Figure 5.2 Flow Chart for Computing Finite Difference Derivatives

5.4 Summary

This chapter specified the framework for computing the Jacobian matrix of path travel times with respect to the path inflows for a network serving multiple O-D pairs with overlapping paths, in particular the link travel times which are captured by dynamic travel time relationships. The main idea of differentiating the travel time relationships relies on the fact that the outflows from any link can be decomposed into their constituent inflows at an earlier time step thus leaving only the inflow terms on the right hand side of the travel time equation. Then it is straightforward to proceed with the differentiation to compute the analytical Jacobian. This chapter has specified the steps involved in decomposing the outflow into the constituent inflows and eventually computing the analytical Jacobian. Finite differencing method can also be used to compute the Jacobian of travel time function, for which a simple computational framework has been described. As described in Chapter 4, the main use of variance approximation method is to describe an equilibrium probability distribution, and hence it is important to create a stationary probability distribution by a standard method, which can be used as a benchmark for comparing the results of the Variance Approximation. Therefore, the next chapter focuses on setting up a doubly dynamic simulation model, and discusses the issues in identifying the stationary probability distribution.

Chapter 6

Doubly Dynamic Simulation Model

The variance approximation method described in Chapter 4 is aimed at estimating the variance-covariance of a stationary probability distribution of a stochastic process. Now, in order to assess the quality of the approximation method, a stochastic process needs to be realised by using some standard method. For this reason, Monte Carlo simulation method, a commonly followed procedure in the literature (e.g., Cascetta and Cantarella 1991, Watling 1996, Hazelton 2002) has been used and a doubly dynamic simulation experiment has been set up. Summaries of the stochastic process have been worked out and compared with the results of the Variance Approximation Method. However, it is important to note that identifying the stationarity of a stochastic process is not easy. Therefore, necessary conditions have been identified to test the stationarity of a stochastic process.

6.1 Background

Simulation models offer a fairly transparent framework to model stochastic processes. Developing a *doubly dynamic* simulation assignment model involves specifying a *day-to-day* route choice model as a stochastic process combined with a driver learning and adjusting model, and a *within day* dynamic network loading model for moving the vehicles along the links of the network while capturing the interactions amongst vehicles that departed in the same/successive departure periods. The model specification included in this chapter follows identical principles described in the earlier chapters 2 and 4, however, the method for solving the assignment is based on Monte Carlo simulation experiments. Numerical tests in this chapter aim to demonstrate the stationarity of the stochastic process, while also illustrating the consistency of the link flow model with properties such as First-In, First-Out (FIFO), in case of a simple two route network serving a single origin-destination pair, for which the O-D demand varies over two departure periods.

Cascetta and Cantarella (1991) developed a doubly dynamic simulation model in which they defined the route flows on any day as a stochastic process, and included a queuing model to capture the delays on the links. In this chapter, the main aim is to develop a doubly dynamic simulation model capable of handling the variations in day-to-day and within day traffic. In particular the dynamic link travel time needs to be differentiable so that the results are comparable with those obtained from the variance approximation method.

6.2 Methodology

As noted in the previous section, the mathematical principles involved in setting up a simulation model of a stochastic process are identical to the description given in Chapter 2 – section 2.1.1. The weighted average approach described in section 2.1.3.1 has been used to model the drivers' learning and adjusting model. Similarly, the dynamic network loading method has been based on the principles set out in Chapter 3 – section 3.4.

Experimental set up

Although the specification of the simulation model has been identical to the principles described in the previous chapters, the method of solving for assignment is entirely different, in the sense that the route choices are here simulated using a Monte Carlo method. This means that the drivers are allocated to the routes based on random numbers generated from a probability distribution with the expected values given by the route choice probabilities. The steps in the simulation are listed below:

- Initialise the route choice probabilities based on free flow costs;
- Allocate the drivers in various departure periods to routes based on random drawings from a distribution mimicking the process of each driver choosing a route, and repeat this step for all drivers in all departure periods;
- Sum up all the drivers in each departure period on each route to feed the dynamic network loading model with the departure period dependent route flow;

- Work out the departure period dependent experienced route costs based on a dynamic network loading map;
- At the end of day n-1, the population perceived mean route costs are updated using the learning model and the costs fed back to the first step above;
- Repeat the steps above over a predetermined number of days of simulation; and
- Compute the summaries viz., mean and variance of route flows at the end of the realisation.

A computer program has been developed in MATLAB based on the sequence of steps described above and an outline flow chart is shown in Figure 6.1. The main module of the program focuses on computing the route choice probabilities using the logit principle, and then allocating the drivers to routes in various departure periods. Then the dynamic loading sub-program is called in, which then facilitates updating the memorised route costs combining the historic costs with the experienced costs from the current run. The updated departure period dependent route costs are then fed to the following day's simulation. Finally, when all the days of simulation are completed, then the simulation summaries are computed viz., the average route flows in departure periods and variance-covariance matrix.

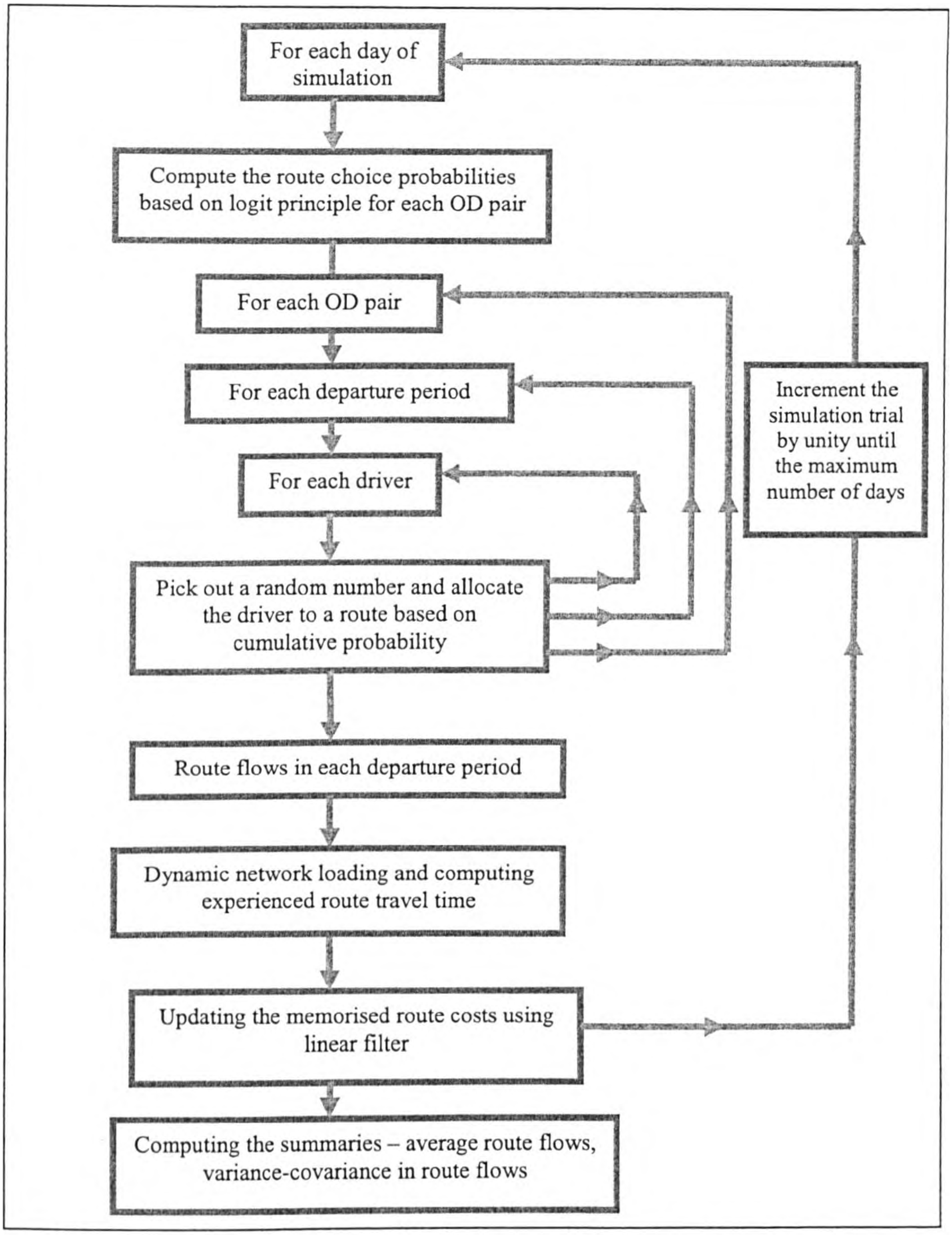

Figure 6.1 Flow Chart for Doubly Dynamic Simulation Model

6.3 Numerical Examples

Two networks of different sizes have been used to illustrate the implementation of the simulation model described earlier. The properties of day-to-day and within day models, including stationarity, autocorrelations and probability distribution of route flows will be discussed.

6.3.1 Two link network

In order to illustrate the principles described in the previous section, a simple two link network is used where routes themselves are the links serving a single origin-destination pair (Figure 6.2). It is assumed that dynamic linear travel time functions with α and β parameters shown below are defined on both the links of the network. A demand of q = (650 1150) drivers each is assumed to spread over two departure periods (L = 2) of 15 minutes each respectively. Drivers are assumed to remember experiences over a two day period so that m = 2, and the memory weight was assumed at λ = 0.5. The route choice is assumed to follow the logit principle with the dispersion parameter $\theta = 0.1$ minutes^{-1}, unless otherwise mentioned. For dynamic loading purposes, it is further assumed that each departure period is sub-divided into 15 minor time steps of $\delta = 1$ minute each. This means that the simulation is carried over a total number of steps N = 30 steps of a minute duration each.

$$\alpha = \begin{bmatrix} 12 \\ 9 \end{bmatrix} \text{ minutes, and } \beta = \begin{bmatrix} 0.025 \\ 0.035 \end{bmatrix} \text{ minutes/vehicle.}$$

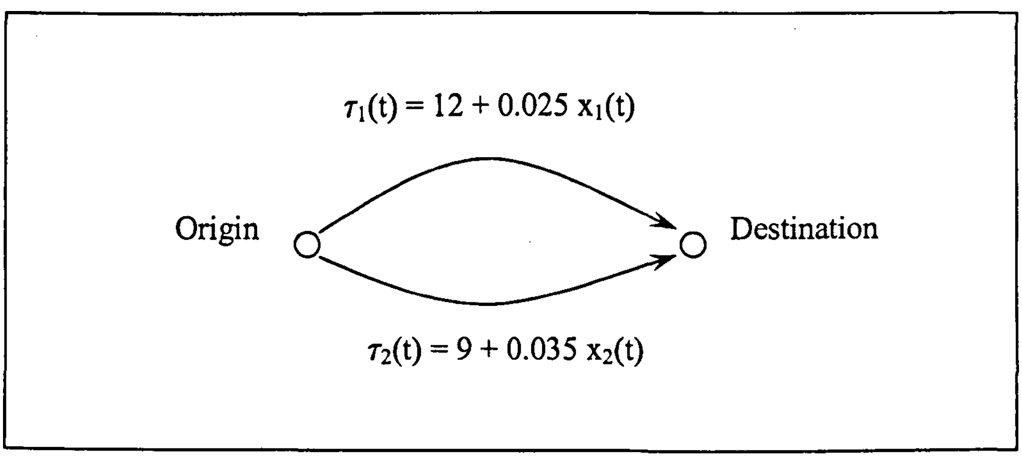

Figure 6.2 Test Network

Then as described in section 6.2, the simulation experiment was run over a realisation of 400 days long and the summaries were computed. While computing the summaries, each driver is assumed to have occupied one vehicle, i.e., at an occupancy rate of 1 person per vehicle. The following section discusses the main results from this exercise.

6.3.2 Total travel time

Total travel time measured by the vehicle-hours on the network indicates the intensity of travel over the network, and if monitored over the period of simulation, will indicate the day-to-day evolution of the intensity of travel. Figure 6.3 shows the plot of total travel on the network over a 400-day long realisation.

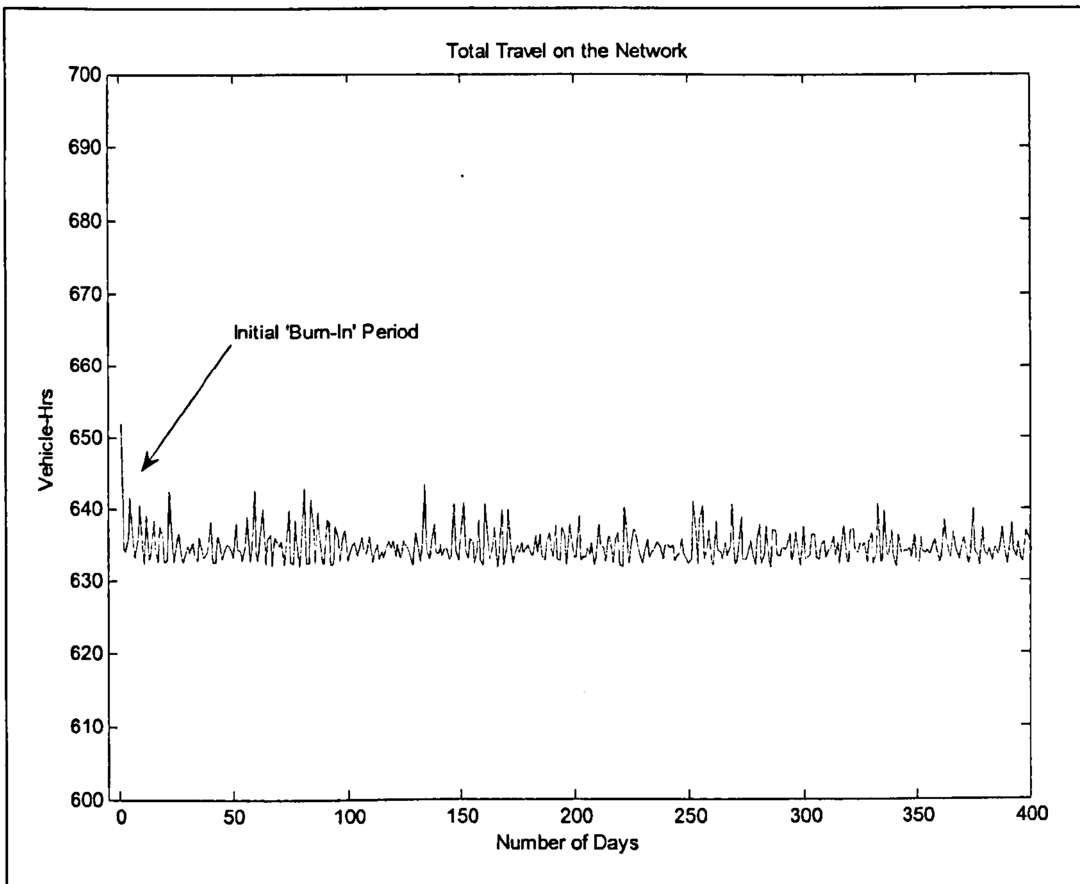

Figure 6.3 Total Daily Travel Time (400-day realisation)

It indicates that in a realisation of 400 days, the total travel time on the network settles down to about its mean value (= 634.7 veh-hrs), with a standard deviation of 2.37 veh-hrs. Based on the initial condition that the network is empty at the beginning of the simulation, the first few days of simulation experience high volatility of travel conditions on the network, and is as indicated by the 'Burn-In' period on Figure 6.3. Hence, it is common to ignore some of the initial period of simulation as the 'Burn-In' when computing the summaries such as means and variances. In order to check whether it makes any difference on the settling down of the stochastic process, a longer simulation run over a period of 1000 days was carried out. Figure 6.4 shows the day-to-day evolution of travel over 1000 days and provides a sustained visual reassurance that the process is stable. Total travel time shown in both the Figures 6.3 and 6.4 exhibits positive skewness with sharp increases above the level of mean total travel time than below. This suggests that the network under consideration is operating close to the minimum total travel time i.e., the system optimum due to the low level of congestion at the given demand. It is important to note that unlike a deterministic variable, a random variable still shows some variability even when it is stationary, resulting in the kind of fluctuations shown in Figures 6.3 and 6.4. This property of random variables makes it difficult to identify the stationarity, and hence more rigorous tests are needed to confirm the stationarity. The following section introduces a larger network with multiple OD pairs with overlapping paths, and then discusses the stationarity issue in detail.

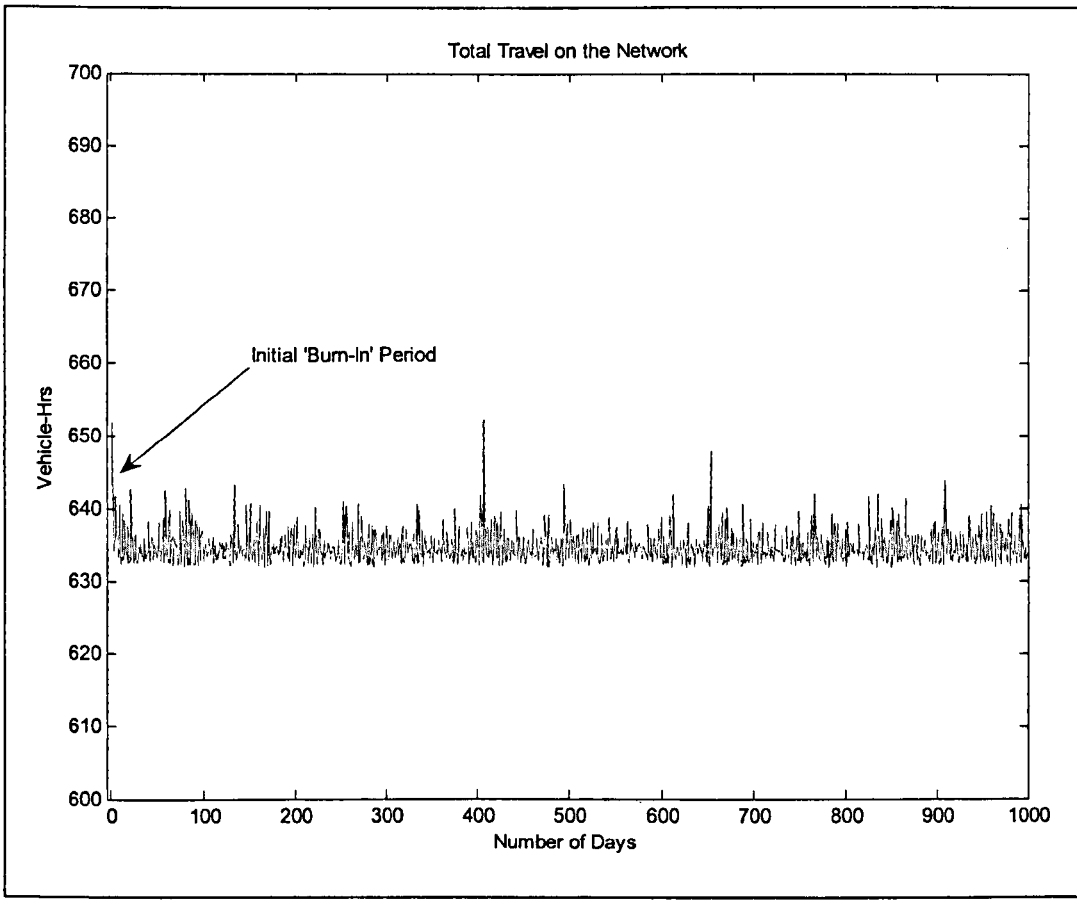

Figure 6.4 Total Daily Travel Time (1000-day realisation)

6.3.3 Grid Network Example

The second numerical example involving a grid network of 12 links serving two origins and three destinations (Figure 6.5) has been constructed to illustrate the properties of day-to-day and within day traffic flows in a general situation. Note that all the links are one-way, and there are 14 routes in all and the link-path incidence is shown in Table 6.1. Dynamic linear travel time functions with parameters shown in Table 6.2 are defined on all the links of the network. The demand for each of the six possible O-D pairs is assumed to be known in each of the four discrete departure periods of 15 minutes duration, and is as shown in Figure 6.6. The route choice is based on the logit principle with the dispersion parameter θ = 0.1 minutes^{-1}. In this example, it is further assumed that each departure period is subdivided into 15 minor time steps with one minute duration each. Drivers were assumed to remember up to two days (m = 2), and the memory weight λ was taken to be 0.5.

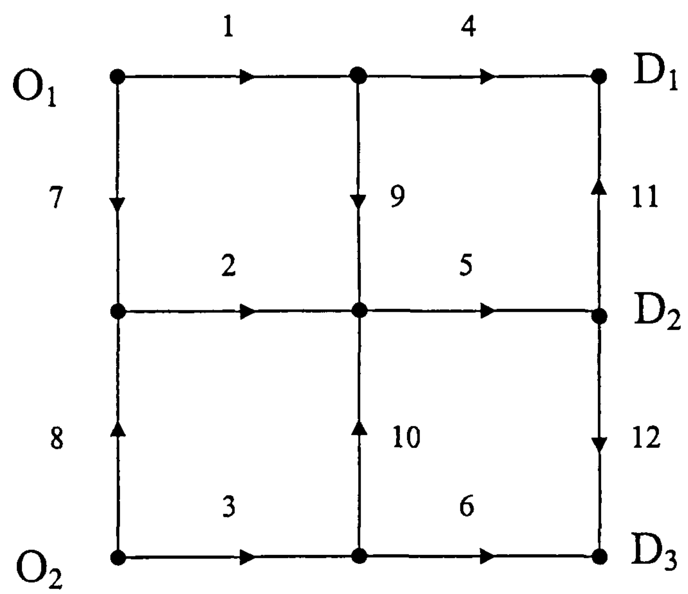

Figure 6.5 Grid Network

Table 6.1 Link – Path Incidence

OD Pair	Path	Links Used by Path
O_1-D_1	1	1-4
	2	1-9-5-11
	3	7-2-5-11
O_1-D_2	4	1-9-5
	5	7-2-5
O_1-D_3	6	1-9-5-12
	7	7-2-5-12
O_2-D_1	8	8-2-5-11
	9	3-10-5-11
O_2-D_2	10	8-2-5
	11	3-10-5
O_2-D_3	12	3-6
	13	8-2-5-12
	14	3-10-5-12

Table 6.2 Network Link Parameters

Link	Free flow time, α_a minutes	Service Rate, β_a minutes/vehicle	Exit Capacity, Vehicles/hour
1	6	0.025	2400
2	4	0.040	1500
3	5	0.029	2069
4	4	0.021	2857
5	5	0.015	4000
6	5	0.030	2000
7	3	0.018	3333
8	2	0.024	2500
9	4	0.019	3158
10	3	0.022	2727
11	6	0.01	6000
12	5	0.01	6000

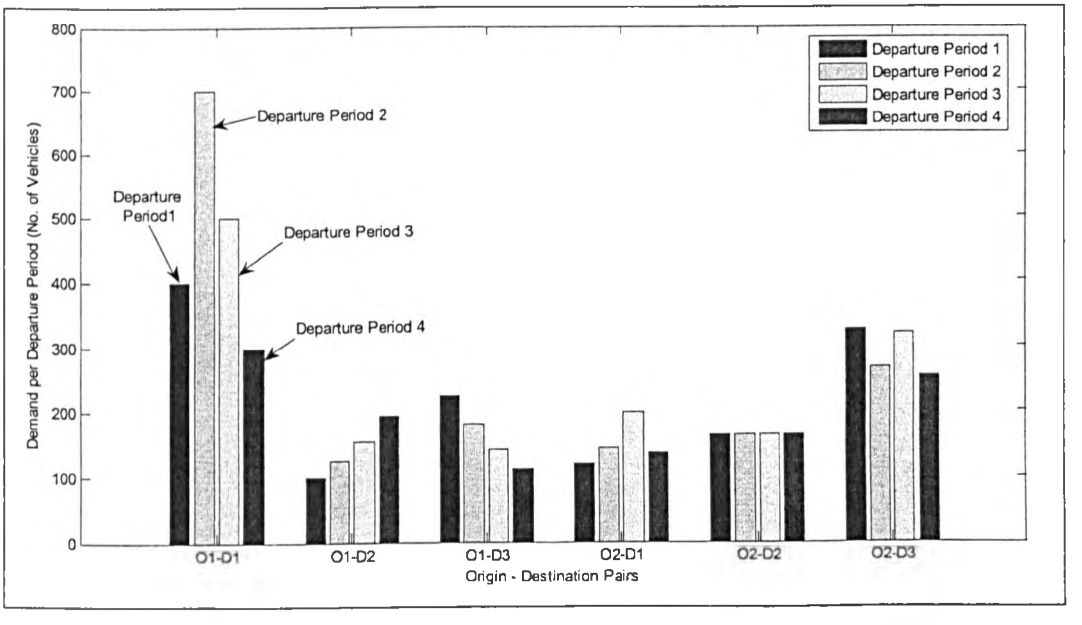

Figure 6.6 Origin-Destination Demand Profiles

The following sections illustrate the properties of the day-to-day model (e.g., stationarity) and the within day model (e.g., FIFO).

6.3.4 Stationarity of the Stochastic Process

A stochastic process is said to be strictly stationary if its properties remain unaffected by a change of time origin, or in other words, the joint probability distribution of m observations made at any set of times **t** (for t = 1,2,..,m) is the same as that associated with m observations separated by an integer k made at a set of times **t**+k (for t = 1,2,..,m and k is an integer) where k is called the lag (Box and Jenkins 1970).

To illustrate this, let us consider the flow on route 1 on the grid network described in section 6.3.3 over a period of 300 days from 201 to 500 in a realisation of 1000 days. Let us also consider that another set of 300 observations is also picked up from the same realisation from days 426 to 725. Figure 6.7 shows the joint probability distribution of flows on route 1 for each of the four departure periods, for each of the two sets of observations as described above.

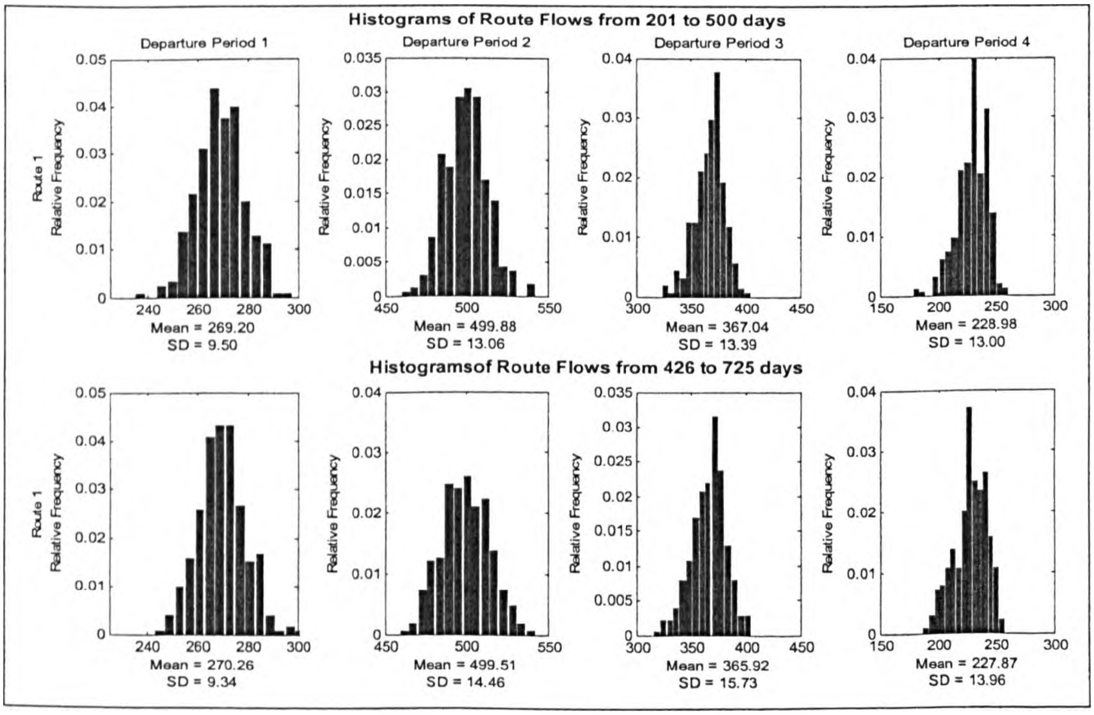

Figure 6.7 Histograms of Flows on Route 1

Visual observation of Figure 6.7 reveals that the distribution of the flows on route 1 in each departure period are similar in each case of the two sets of observations. Moreover, in each case the mean and standard deviation of the route flows are nearly identical to each other indicating that the stochastic process being considered is stationary. It is important to note that this is a necessary condition for a stochastic process to be stationary, but not a sufficient condition.

Figures 6.8 and 6.9 show the histograms of flows on routes 2 and 3 for the two sets of observations as described in the case of route 1. The mean route flows and their standard deviations are similar in each case during various departure periods. These observations about routes 2 and 3 support the earlier comments on route 1 and reassure the stationarity of the process once again. The ensuing section presents a statistical procedure to test the stationarity of the stochastic processes.

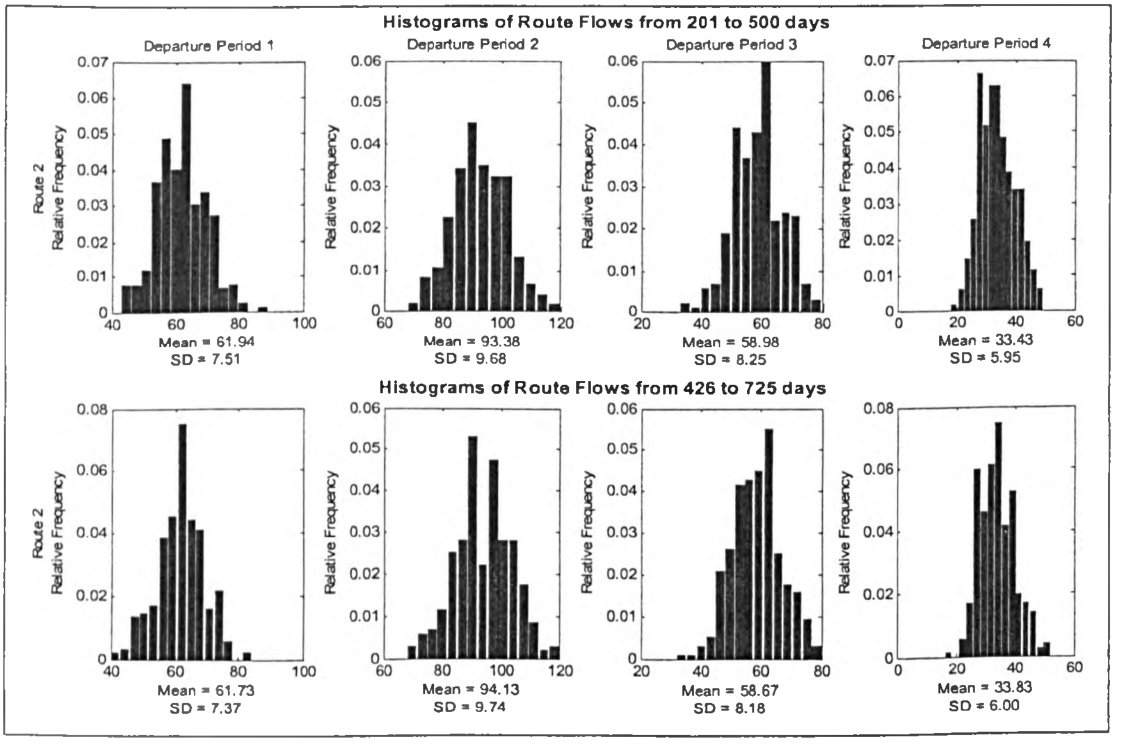

Figure 6.8 Histograms of Flows on Route 2

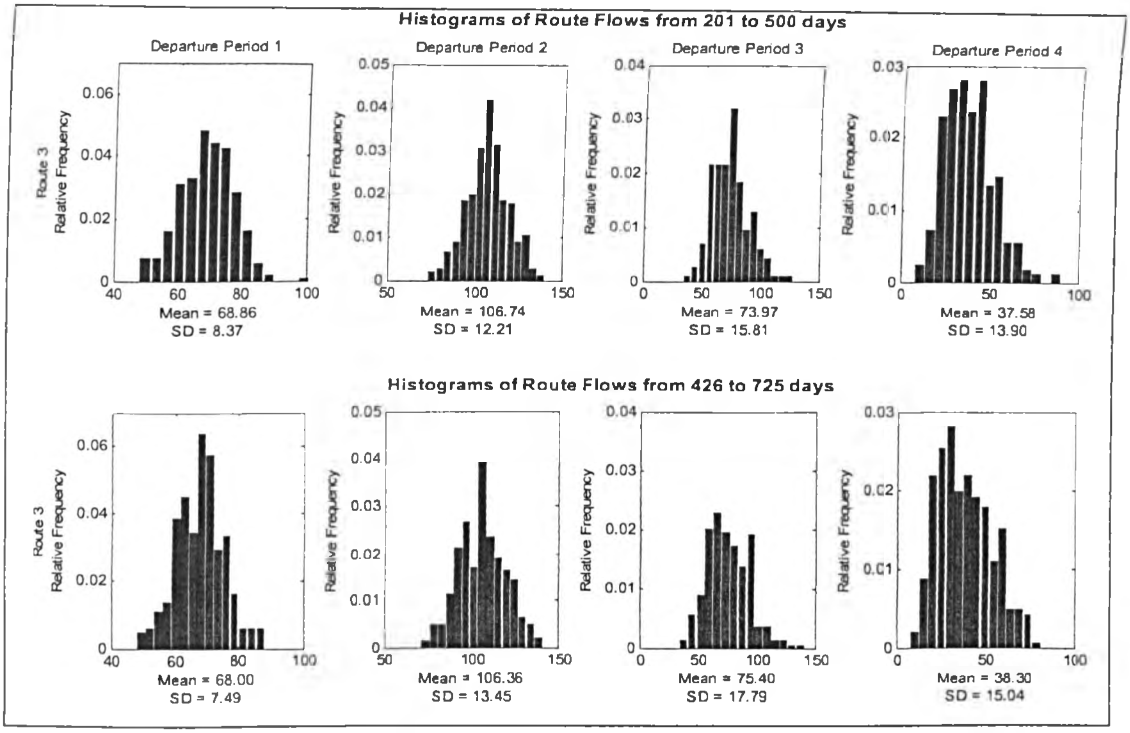

Figure 6.9 Histograms of Flows on Route 3

6.3.5 Autocorrelations of route flows

As noted from various sections in Chapter 2, Chapter 4 and earlier in this chapter, analysis of stationarity of stochastic processes is of critical importance because only then one can summarise the process by its moments such as mean and variance. A simple t-test comparing the two samples considered in the previous section could have been applied, but due to the critical nature of the problem, use of a more robust test for identifying the stationarity of the process has been explored. The proposed method of identifying stationarity analyses the autocorrelations of a stochastic process. A necessary condition that a stochastic process is stationary is that, the autocorrelations are expected to die down with larger lags, which indicates that the random variable under consideration is stable about its mean value (Bartlett, 1946).

Variance of estimated autocorrelations for any lag k, greater than some hypothesized value $\tilde{k}$ beyond which the theoretical autocorrelations may be deemed to have died out is given by Bartlett (1946) as below:

$$Var[\hat{\rho}_k] \approx \frac{1}{N} \left\{ 1 + 2 \sum_{v=1}^{\tilde{k}} \rho_v^2 \right\} \tag{6.1}$$

where, $\hat{\rho}_k$ is an estimator of autocorrelation at lag k

ρ_v is the true theoretical autocorrelation at lag v

N is the length of the time series i.e., the number of observations in one realization.

In practice, the estimated autocorrelations are substituted for the theoretical autocorrelations and then the square root of the variance in estimated autocorrelations is referred to as the *large lag standard error*. On the assumption that the theoretical autocorrelations are zero beyond some hypothesized lag $k = \tilde{k}$, the large lag standard error then approximates the standard deviation of $\hat{\rho}_k$ for lags $k > \tilde{k}$.

In order to further ensure that the stochastic process is stable, the autocorrelations of route flows have been tested for large lag standard errors by applying the Bartlett's criterion, based on a 1000-day long simulation. Figure 6.10 shows the autocorrelations in path flows and the corresponding error bars for hypothesised lags for routes 1,2 and 3 up to 15 days of lag over a realisation of 1000 days.

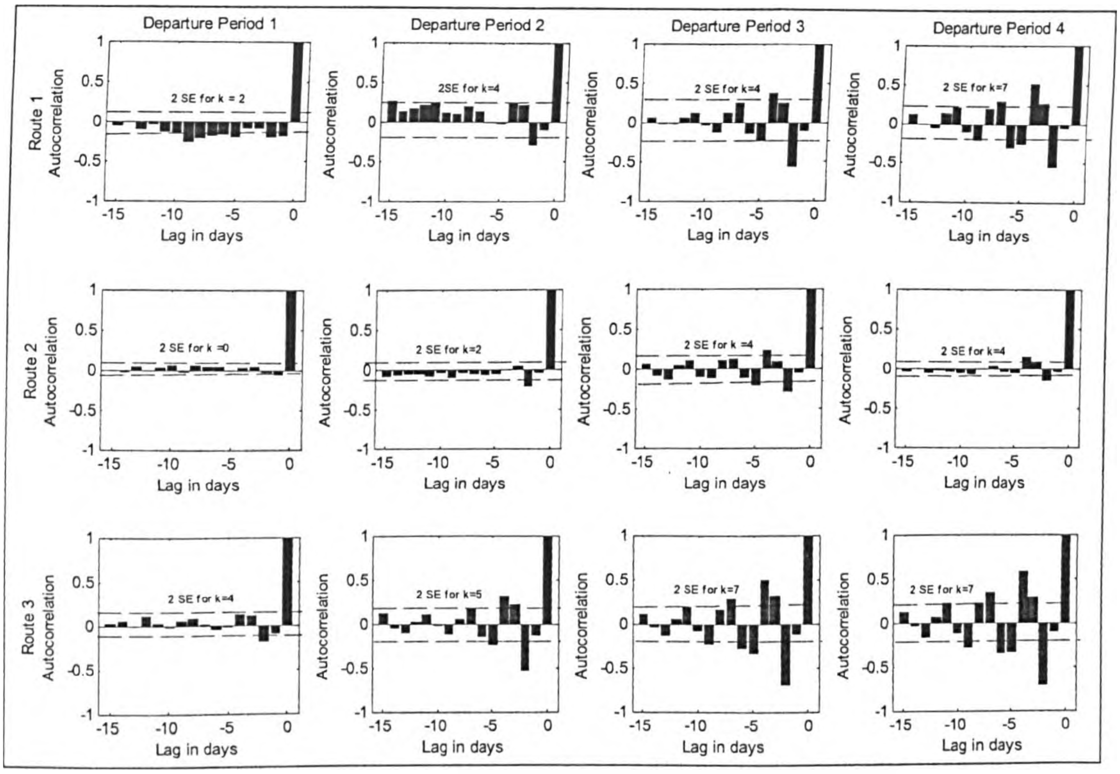

Figure 6.10 Correlogram of Flows on Routes 1,2 and 3 (θ = 0.1)

As the correlation of the flows with themselves is unity, the first bar (with '0' lag) reflects the same. From then on, the autocorrelations can be observed to reduce with increasing lags. Figure 6.10 includes error bars based on Bartlett's formula for large lag standard error, given by equation (6.1), for each of the routes 1,2 and 3, for some lag $k > 0$ beyond which the theoretical autocorrelation function is deemed to have died out. Insignificant autocorrelations, compared to standard errors at some lag $k > 0$, indicate that the flows on any route do not depend on the flows on the same route beyond k days, during the same departure period. This condition also implies that the process is stationary.

It can also be commented that the route flows in departure period 1 settle down relatively quickly, compared to the flows over the rest of the departure periods. This is intuitively supportive to the notion that the delays in later time periods are affected by the flows from the earlier time periods, and hence the flows in later time periods settle down much slower compared to the flows in earlier departure periods.

Quite differently from the above discussion, it will be interesting to see how the autocorrelations change with a change in dispersion of the perceived costs which is parameterised by the logit choice parameter θ. As described earlier, autocorrelations in Figure 6.10 were based on a value of θ = 0.1 minutes^{-1}. Figure 6.11 illustrates the autocorrelations of route flows with θ = 0.01. As θ decreases, the assumed dispersion of the perceived costs increases indicating that the drivers almost ignore the experienced costs and choose routes at random. In this case, the route flows on any day do not depend on any other day's flows implying that the autocorrelations should be smaller compared to those in Figure 6.10. In the limit, the autocorrelation bars will vanish with even lower values of θ.

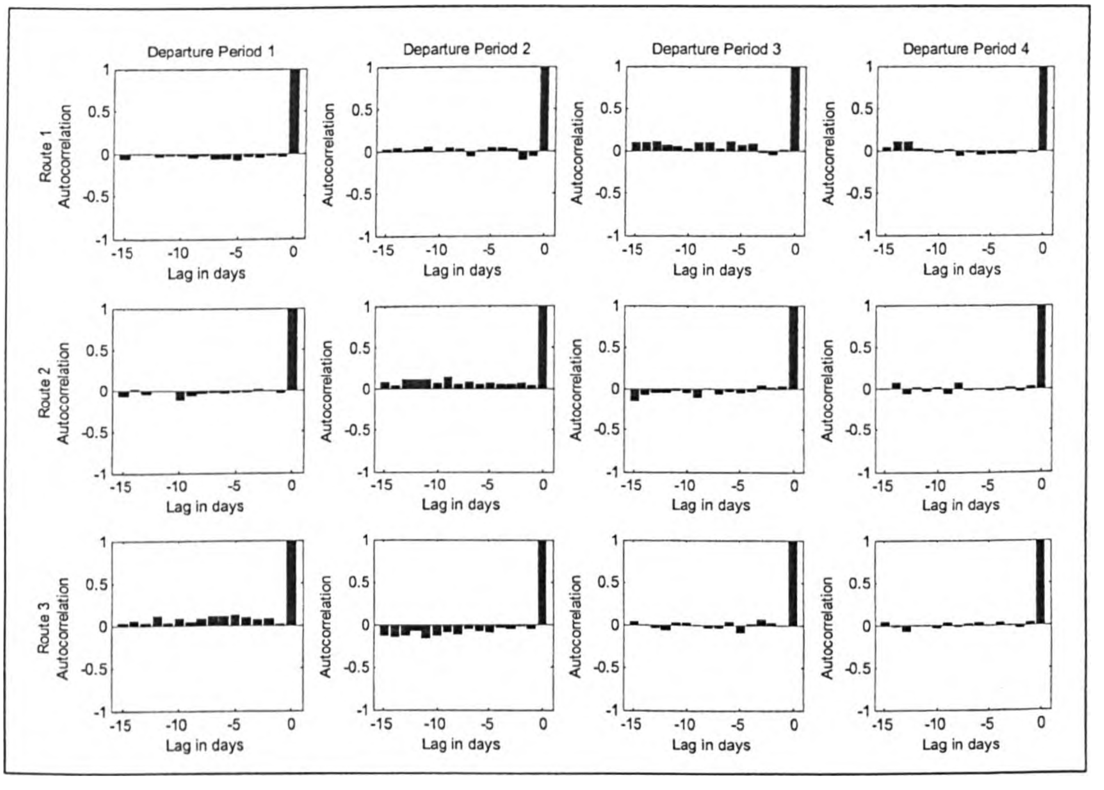

Figure 6.11 Correlogram of Flows on Routes 1,2 and 3 (θ = 0.01)

On the other hand, increasing values of θ will reduce the dispersion of the perceived costs and then the drivers start thinking alike while perceiving the route costs and making route choices. In this case, when most of the drivers choose the same route on any given day, there is very little probability that they choose the same route on the following day because they experience high cost of travelling on that day. This means that the route flows tend to be negatively correlated as shown in Figure 6.12. In the limit with higher values of θ, the autocorrelations will be equal to -1.

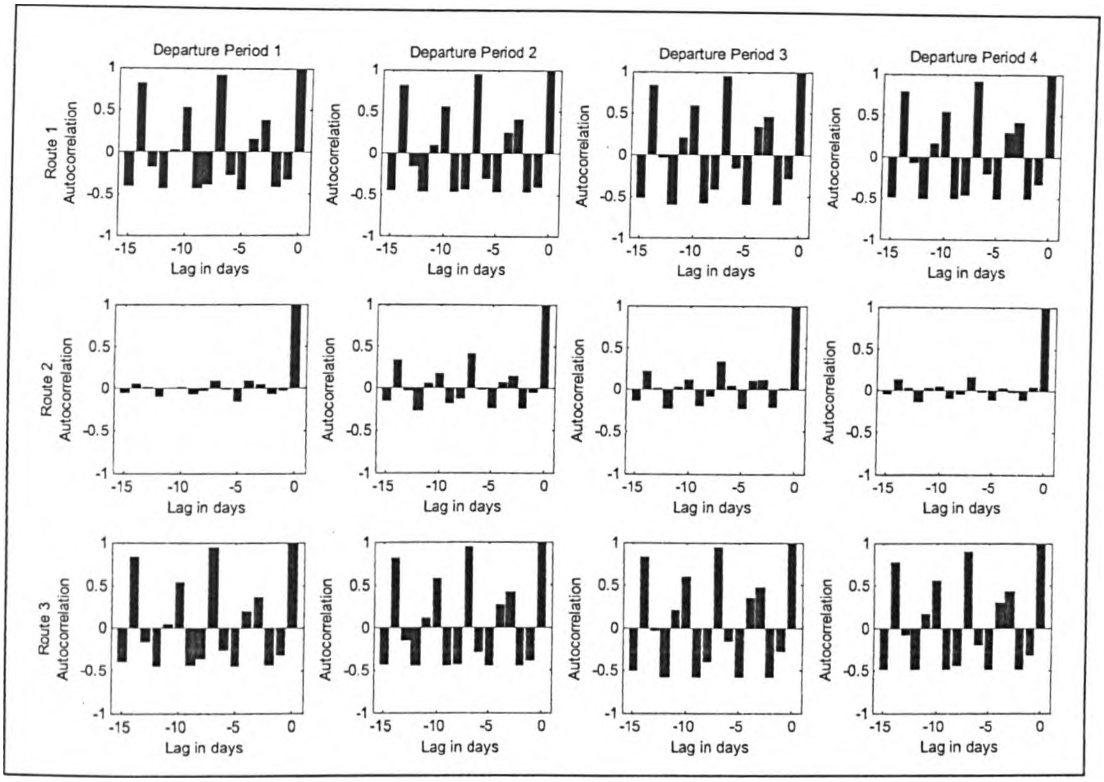

Figure 6.12 Correlogram of Flows on Routes 1,2 and 3 ($\theta = 0.5$)

6.3.6 Assigned flows and Travel Times

Table 6.3 shows the average assigned route flows in each departure period. As the stochastic process exhibits variance unlike deterministic processes, the standard deviations of route flows are also computed for the 1000-day long realisation, after ignoring the first 10% of the simulated days to account for the empty network conditions. In the table, average path travel times and their standard deviations are also shown, which are based on the outputs of the dynamic network loading model, summarised over non-discarded number of trials of the simulation model. It may be observed that the path travel times are relatively stable earlier during the peak hour and start to vary during the later departure periods as indicated by the higher standard deviations in departure periods 3 and 4 compared to departure periods 1 and 2. This is one of the reflections of the dynamic nature of the traffic flows as witnessed in the real world situation. The travel times vary significantly during the peak hour due to the building up and dissipation of the congestion in time and space. In the present model the route choice of drivers is purely based on the average path travel times memorised over a number of days, although the standard deviation in travel time can also influence the choice process as modelled in reliability type applications.

Table 6.3 Assigned Flows and Travel Times

Route	Average Route Flow and Standard Deviation				Average Path Travel Time (Minutes) Standard Deviation			
	Departure Period				Departure Period			
	1	2	3	4	1	2	3	4
1	269.61	499.67	365.99	227.79	16.30	26.04	32.39	32.56
	9.59	14.21	15.67	14.41	0.17	0.45	0.99	1.80
2	61.88	93.71	59.04	33.84	30.98	42.79	50.61	51.68
	7.31	9.87	8.43	6.04	0.29	0.71	1.54	2.19
3	68.51	106.62	74.97	38.37	30.03	41.55	48.45	51.04
	8.14	13.58	18.03	15.48	0.45	1.17	2.61	4.20
4	47.81	58.48	69.17	93.94	24.46	35.79	43.51	44.85
	5.09	6.89	12.33	22.20	0.28	0.69	1.56	2.26
5	52.19	66.52	85.83	101.06	23.54	34.57	41.34	44.20
	5.09	6.89	12.33	22.20	0.44	1.15	2.63	4.29
6	106.89	85.18	63.35	54.77	30.08	41.41	49.12	50.42
	7.86	8.97	11.31	12.81	0.30	0.70	1.57	2.25
7	118.11	96.82	78.65	57.23	29.12	40.19	46.94	49.76
	7.86	8.97	11.31	12.81	0.44	1.14	2.61	4.25
8	60.45	59.84	76.93	54.13	27.80	38.99	46.34	49.69
	5.90	6.79	12.69	14.07	0.41	0.99	2.31	4.02
9	60.55	84.16	122.07	82.87	27.86	35.65	41.60	45.28
	5.90	6.79	12.69	14.07	0.29	0.55	1.12	2.02
10	82.67	69.06	63.55	65.05	21.35	32.04	39.22	42.82
	6.65	7.68	10.78	17.07	0.41	0.98	2.32	4.10
11	82.33	95.94	101.45	99.95	21.40	28.76	34.48	38.30
	6.65	7.68	10.78	17.07	0.28	0.53	1.11	2.06
12	193.67	184.49	232.85	189.90	16.10	20.96	25.26	28.06
	8.88	7.88	8.81	9.57	0.19	0.36	0.67	1.34
13	65.91	34.57	33.30	26.10	26.89	37.68	44.83	48.39
	7.37	5.89	8.32	10.39	0.41	0.97	2.31	4.07
14	65.42	47.94	52.86	39.00	26.97	34.41	40.10	43.90
	7.34	6.58	7.14	6.61	0.29	0.54	1.12	2.06

Figure 6.13 shows the link-time plot for routes 1,2 and 3 in each of the departure periods. They indicate the travel times are fanning out in general, meaning that the congestion builds up as we progress with the dynamic loading of vehicles over the network. Especially on links 1 and 2, this phenomenon is very clear. On the other hand, parallel travel time lines indicate that the links are uncongested and operate below the capacity, as is the case with most of the links on routes 1,2 and 3. Figure 6.13 also indicates that the model results are consistent with FIFO property as we do not have any intersecting link travel time lines. The figure is also indicative of satisfying the FIFO property at the path level.

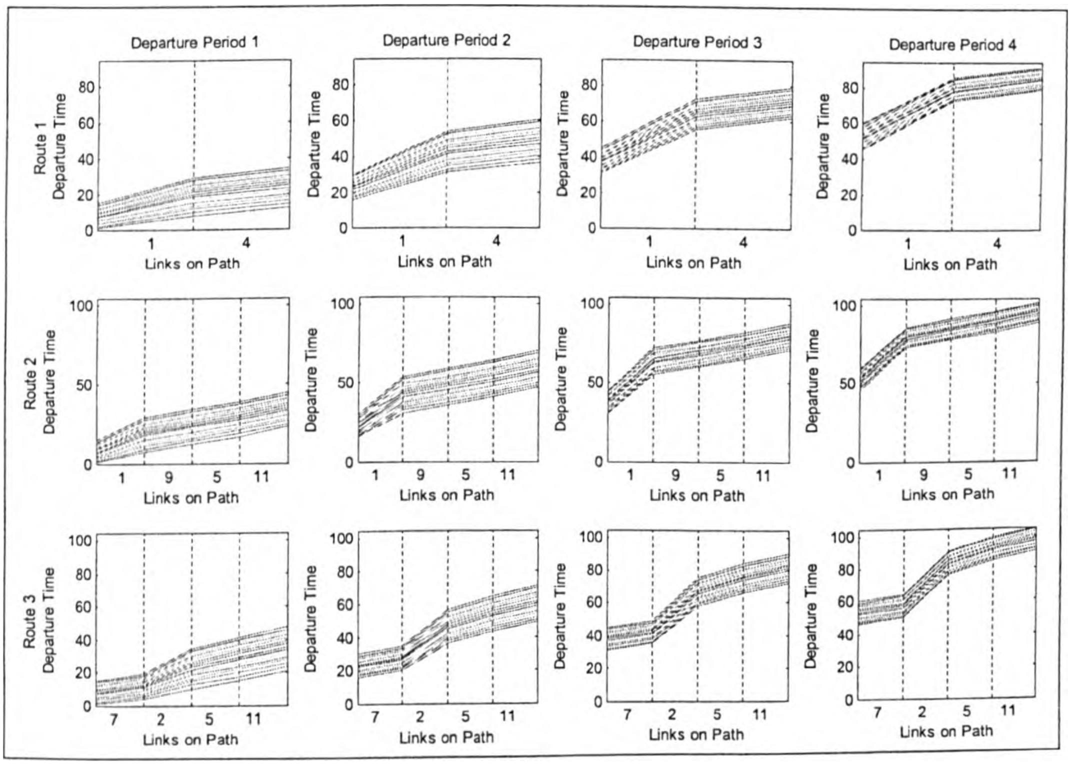

Figure 6.13 Link Time Plots

As link 2 is used by several paths (see Table 6.1), in order to illustrate the dispersion of outflows over larger periods than the inflow periods the link inflow and outflow profiles have been drawn (Figure 6.14). Figure 6.14 indicates that the vehicles on link 2 no longer operate under free flow speeds and have started experiencing higher travel times due to the increase in the level of congestion. However, link 2 has still some spare capacity.

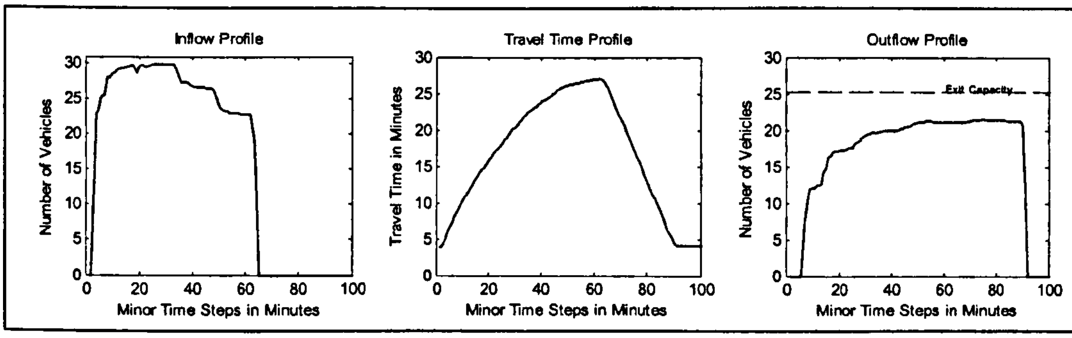

Figure 6.14 Travel Time and Outflow on Link 2

6.4 Summary

In this chapter, a doubly dynamic simulation model has been specified and implemented. Two implementations of the simulation model, starting with a simple two link network and a grid network serving multiple OD pairs, have been reported. Various properties of the model including day-to-day variation of total travel time, stationarity of the stochastic process, and autocorrelations have been discussed. An important outcome of this chapter is that practically useable tests based on necessary conditions were developed to detect the stationarity of a stochastic process. In Chapter 4, the variance approximation method developed computes the variance-covariance of a stationary stochastic process. Therefore, it is important to confirm that a stochastic process is stationary before assessing the quality of the variance approximation method. The next chapter compares the results obtained by variance approximation method and simulation method and assesses the quality of the approximation. Besides, it also tests the sensitivity of variance-covariance to the input model parameters.

Chapter 7
Numerical Experiments

This chapter compares the results obtained by the Variance Approximation Method described in Chapter 4 with the alternative simulation method described in Chapter 6. In particular, the variance approximation method requires the computation of Jacobians of the route choice probability function, path travel time function, and the theoretical framework for deriving each one of them has been discussed in Chapters 4 and 5 respectively. Based on these expositions, bespoke computer programs were developed using MATLAB and in particular two parallel streams of programs – one each for the variance approximation method and simulation method - were developed. An outline flow chart for each one of them is indicated in Figures 4.1 and 6.1 respectively.

Both the variance approximation method and simulation method have an inbuilt dynamic network loading module each, for working out the departure-time-dependent route travel times experienced, and the basic principles of the method have been described in Chapter 3.

For each of the two methods described in Figures 4.1 and 6.1, separate schemes involving networks of various sizes starting from the simplest two link/route network serving a single O-D pair, to a slightly larger five link network with links shared by paths but still serving a single O-D pair, and to an even larger more general kind of network serving six O-D pairs with 12 links via 14 paths have been developed. The following sections describe each one of the network schemes just mentioned, and compare the results obtained by the method of variance approximation and the simulation method.

7.1 Results with Two Link Network

In this simple numerical example, the theory described in the previous chapters 4 and 5 is illustrated by considering the two-link parallel route network servicing one O-D pair (used in the previous chapter), but having the demand spread over a single departure period, for simplicity. As before, travel time relationships are assumed to be linear functions of the number of vehicles on the link as given by $\tau_a(t) = \alpha_a + \beta_a x_a(t)$, with parameters α_a = 12 minutes and β_a = 0.025 minutes/vehicle for route 1, and α_a = 9 minutes and β_a = 0.035 minutes/vehicle for route 2. A minor time increment of δ = 1 minute was used for implementing the travel time model over a 15 minute period (N = 15 minor time increments). A single demand period is modelled (L = 1 major time intervals) with a demand of 400 drivers over the single within-day time period of $\kappa = 15$ minutes (so the O-D demand matrices **q** and **Q** reduce to the scalar 400).

Drivers' dispersion in travel cost perceptions in choosing their routes on any day, conditionally on their past experience, are assumed to be explained by a logit model: $p_r(\mathbf{C}) = e^{-\theta C_r}\left\{\sum_s e^{-\theta C_s}\right\}^{-1}$ where $\theta > 0$ is the logit choice scaling parameter. The value of time is assumed to be $\gamma = 1$, with travel time the only component of travel cost. Within-day Dynamic Stochastic User Equilibrium flows are obtained using the Method of Successive Averages (MSA) based on 25000 iterations. Each iteration of MSA calls the dynamic link loading model by feeding in the departure time dependent route flows, and receives back the updated departure time dependent route travel times, and the process continues until the number of iterations is exhausted. In the drivers' learning model, assume m = 5 days and λ = 0.5. Here the means and variances by three methods viz., simulation, naïve and the approximation, are compared as explained below.

Simulation: The assignment process was simulated over a period of 40000 days and an initial period of 4000 days was discarded as burn-in time to allow the network to reach sufficient levels of loading (i.e. discarding the transient states of the process). Summary statistics are computed from the non-discarded days. Based on the law of large numbers, it is well known that the relative frequency of any event will tend to its probability, if repeated over a large number of independent trials (Olofsson 2005). Hence, the length of the realisation was set equal to an arbitrarily large number of days such as 40000, with the expectation that the relative frequencies of route choices are then approximately equal to their probabilities.

That is to say, in this experiment the large number of trials as represented by the length of the realisation in 'days', indirectly assures the stationarity of the probability distribution of route choices. The results of this long simulation are treated as a benchmark against which the results of the approximation method viz., the mean route flows and the variance-covariance matrices of route flows are compared. However, in other experiments such as the sensitivity analysis (See section 7.3), where the stationarity of the probability distributions is not so critical, use of shorter realisations over 1000 days has been explored.

Naïve Method: This method assumes that the conditional covariance matrix (4.17) is sufficient to explain the variance in flows, and ignores the day-to-day variability in choice probabilities and travel costs.

Approximation: The variance approximation method is applied, giving an estimate of variance using the expression (4.18), and the mean of the route flows is estimated using the MSA based on 25000 iterations.

In each case, when applying the approximation method, appropriate Jacobian matrices of choice probability and travel times during the single major time period must be computed; for example, in the case of θ=0.1 in the two link network specified, these are given by:

$$D = \begin{bmatrix} -0.0248 & 0.0248 \\ 0.0248 & -0.0248 \end{bmatrix} \quad B = \begin{bmatrix} 0.0129 & 0 \\ 0 & 0.0172 \end{bmatrix}.$$

The Jacobian matrix of travel time was computed following the analytical procedure described in section 5.2 and compared with the derivatives obtained by the finite differencing method described in section 5.3, which were found to be identical to four decimal places as shown in Table 7.1. As the travel time on each route is not affected by the inflows on the other route in the two link parallel route network, the corresponding derivatives are both zeros.

Table 7.1 Comparison of Jacobians by Analytical and Finite Differencing Methods

Cell Reference	Analytical Jacobian	Finite Difference Jacobian
Travel time on route 1 wrt inflows on route 1, $\frac{\partial \tau_1}{\partial u_1}$	0.0129	0.0129
Travel time on route 2 wrt inflows on route 1, $\frac{\partial \tau_2}{\partial u_1}$	0	0
Travel time on route 1 wrt inflows on route 2, $\frac{\partial \tau_1}{\partial u_2}$	0	0
Travel time on route 2 wrt inflows on route 2, $\frac{\partial \tau_2}{\partial u_2}$	0.0172	0.0172

Returning to the main discussion, Table 7.2 compares the means and variances for Route 1 by the three methods introduced earlier, for various values of the logit choice parameter θ.

Table 7.2 Comparison of Estimates of Mean and Variance on Route 1

Method	$\theta = 0.005$		$\theta = 0.1$		$\theta = 1$	
	Mean	Variance	Mean	Variance	Mean	Variance
Simulation	198.88	99.66	182.49	102.84	170.13	26964
Naïve	198.88	100.00	182.52	99.24	143.71	92.08
Approximation	198.88	100.00	182.52	102.20	143.71	451.70

Considering the results in Table 7.2, it can be seen that when the logit choice parameter is relatively low (θ=0.005), the random dispersion element in drivers' travel costs is very high making them choose routes almost at random regardless of the experienced (mean) route costs. Thus approximately equal numbers of drivers are expected to select each route, and the variation is approximately binomial. In this case as would be expected, the naïve approximation is sufficient to explain the variance in traffic flows, because by definition, the conditional covariance ignores the variation due to the variation in costs. Consequently, the approximation method adds little to the conditional covariance matrix in this case. It is noted also that the mean flows by all the three methods are identical to two decimal places.

In case of θ=0.1, the mean flows by all the three methods are again very close. However, the naïve variance underestimates the benchmark (simulation) variance. This underestimation of the variance is expected because the naïve method assumes fixed multinomial choice probabilities as opposed to the real case of day-to-day varying choice probabilities, i.e. it neglects one source of variability. The approximation method successfully corrects this underestimation by inflating the variance towards the benchmark simulation variance.

As an example, the matrices of variance-covariance from the simulation, naïve and approximation methods in the case of $\theta = 0.1$, are shown respectively as:

$$\Sigma = \begin{bmatrix} 102.84 & -102.84 \\ -102.84 & 102.84 \end{bmatrix} \quad \Theta^* = \begin{bmatrix} 99.24 & -99.24 \\ -99.24 & 99.24 \end{bmatrix} \quad \hat{\Sigma} = \begin{bmatrix} 102.20 & -102.20 \\ -102.20 & 102.20 \end{bmatrix}.$$

When the logit parameter is relatively large such as $\theta = 1$, the dispersion in perceived costs is very small and hence on any given day all drivers tend to behave in a similar manner. In this case, the summary statistics presented in Table 7.2 for the simulation method are produced by a rather different trajectory of the process, in which a large number of drivers choose one route on any given day and the following day they almost all shift their choice towards the other route, having experienced congestion on the first route. This *en masse* behaviour causes a large variance in the system and the probability mass gets concentrated towards the two extremes (all-or-nothing solutions), resulting in a bimodal distribution. Thus, the variance is measuring some kind of system instability (in the same sense that one could calculate a variance for a deterministic periodic system) rather than random variation of a stationary variable. To elaborate, consider the limiting case of all 400 drivers choosing route 1 on any given day and all of them shifting *en masse* to route 2 the following day, resulting in none getting assigned to route 1, and so on, over a number of days. Then, the mean of the flows on route 1 will be equal to 200 and the variance will be $200^2 = 40000$. Just to confirm the argument, a quick simulation run was carried out with $\theta = 10$ over a 1000-day long realisation which resulted in an average route flow of 178 and a variance of 39396 for route 1. In terms of the route flows, the resulting assignment is highly unstable because of the high fluctuations between the days, however in terms of the probability distribution, it is stable because of the bimodal distribution with equal probability of choosing route 1 by all drivers or none.

The naïve variance is not expected to explain such behaviour and hence is too small. Indeed, neither is the approximation method designed to explain such a variation, and the corrected variance is seen to be far lower than the simulation variance. This is not surprising as we are estimating the variance of a bimodal distribution with an approximate Normal distribution, which is clearly not an ideal choice; nevertheless, it is partially successful in correcting the variance in the appropriate direction. In the light of the above observations, the case of $\theta = 1$ has not been pursued further in the rest of the analyses presented in this chapter. The results presented in this section are consistent with the observations made by Hazelton and Watling (2004) in their day-to-day dynamic, but static within day examples.

7.2 Results with Five Link Network

Consider the five link network described in Chapter 3, which has three routes serving one O-D pair (See Figure 7.1). Route 1 uses links 1 and 4, route 2 uses links 2,3 and 4, and route 3 uses links 2 and 5. In this scheme, the links 2 and 4 are shared by two paths each, and hence require the flows to diverge and merge in space and time which is a common requirement in dynamic traffic assignment over networks serving multiple O-D pairs. Moreover, in this example the demand considered is spread over four departure periods of 15 minutes each, and in this sense, this example generalises and extends the simple case described in the previous section. Thus, even though this particular example deals with a single O-D pair, the principles described here can easily be extended to the networks serving multiple O-D pairs.

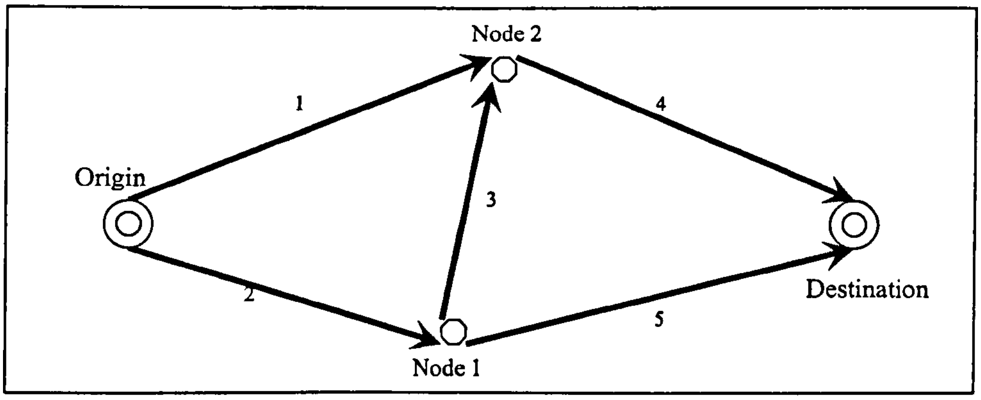

Figure 7.1 Five Link Test Network

On each of the five links whole-link type dynamic travel time functions of the general form $\tau_a(t) = \alpha_a + \beta_a x_a^{\gamma_a}(t)$ are defined with parameter values as shown in Table 7.3.

Table 7.3 Network Parameters

Link	α_a (minutes)	β_a (minutes/vehicle)	γ_a	Functional Form
1	12	0.025	1	Linear
2	9	0.035	1	Linear
3	10	0.00015	2	Quadratic
4	12	0.025	1	Linear
5	9	0.035	1	Linear

It may be noted here that FIFO compliance is one of the important features of any dynamic traffic assignment model, as well as being an assumption of the approximation method, and so to confirm this assumption, FIFO has been monitored closely during the execution of the program and has been found satisfactory. It is assumed that the demand is spread over four discrete departure periods giving $L = 4$, and each departure period of $\kappa = 15$ minutes duration has been modelled, with a varying demand profile thus indicated by,

$\mathbf{q}_k = [400 \ \ 700 \ \ 100 \ \ 100]$.

As before, for the dynamic network loading purposes a minor time step of $\delta = 1$ minute has been assumed. In addition, a memory length of $m = 5$ days with a memory weighting of $\lambda = 0.5$ has been retained. The route choice is assumed to be based on a logit principle, and Dynamic Stochastic User Equilibrium (DSUE) flows are solved for using the Method of Successive Averages together with the dynamic network loading method as before. As before the variance of the route flows by a similar set of alternative methods, viz., simulation, naïve and the method of approximation, run to similar specifications as indicated previously, are compared.

7.2.1 Comparison of Mean Flows

Table 7.4 compares the mean route flows estimated in each departure period by the methods of simulation, naïve and approximation as before for the case of logit choice parameter θ = 0.01. For relatively low values of θ such as 0.01, the dispersion in the perceived costs will be higher and hence the drivers choose routes randomly ignoring the experienced costs. Therefore, in the limit, the drivers get equally distributed among the feasible set of routes in each case of departure period. Mean route flows displayed in the table indicate the same trend, although they are not exactly same. Even lower values of θ should produce an equal share of route flows in each departure period. The naïve and approximation methods produce nearly identical mean route flows compared to those of simulated route flows.

Table 7.4 Comparison of Mean Route Flows (θ = 0.01)

Method	Time Period 1			Time Period 2			Time Period 3			Time Period 4		
	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
Simulation	136.90	121.13	141.97	245.24	207.62	247.13	34.81	30.10	35.10	33.78	30.69	35.53
Naïve	136.93	121.06	142.00	245.24	207.61	247.16	34.76	30.14	35.10	33.78	30.70	35.52
Approximation	136.93	121.06	142.00	245.24	207.61	247.16	34.76	30.14	35.10	33.78	30.70	35.52

With a higher value of θ (= 0.1), the means are well estimated by the naïve and approximation methods as shown in Table 7.5. However, the route flow patterns are different from those reported in Table 7.4. This is due to the higher value of θ, which reduces the dispersion in the perceived costs of the drivers making them choose routes sensitive to the mean route costs experienced on the previous set of days. As a result, more number of drivers are allocated to quicker routes (example, routes 1 and 3) and slower routes experience reduction in assignment (example, route 2).

Table 7.5 Comparison of Mean Route Flows ($\theta = 0.1$)

Method	Time Period 1			Time Period 2			Time Period 3			Time Period 4		
	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
Simulation	149.37	48.92	201.71	302.08	81.24	316.68	40.71	15.11	44.19	35.11	15.99	48.89
Naïve	149.39	48.88	201.73	302.07	81.27	316.66	40.62	15.15	44.23	35.07	15.99	48.93
Approximation	149.39	48.88	201.73	302.07	81.27	316.66	40.62	15.15	44.23	35.07	15.99	48.93

Figures 7.2 and 7.3 graphically compare the mean route flows in each of the two cases described above, and it is easy to see that the naïve and approximation methods are able to estimate the simulation means in each case. Considering the nature of the results with $\theta = 0.01$ and $\theta = 0.1$, and by analogy with the two link network example, it can be stated that with even higher values of θ, (such as 1) the simulation mean and the naïve and approximation means as estimated by SUE, all will converge to a deterministic mean due to the all-or-nothing type solutions realised by a deterministic periodic system.

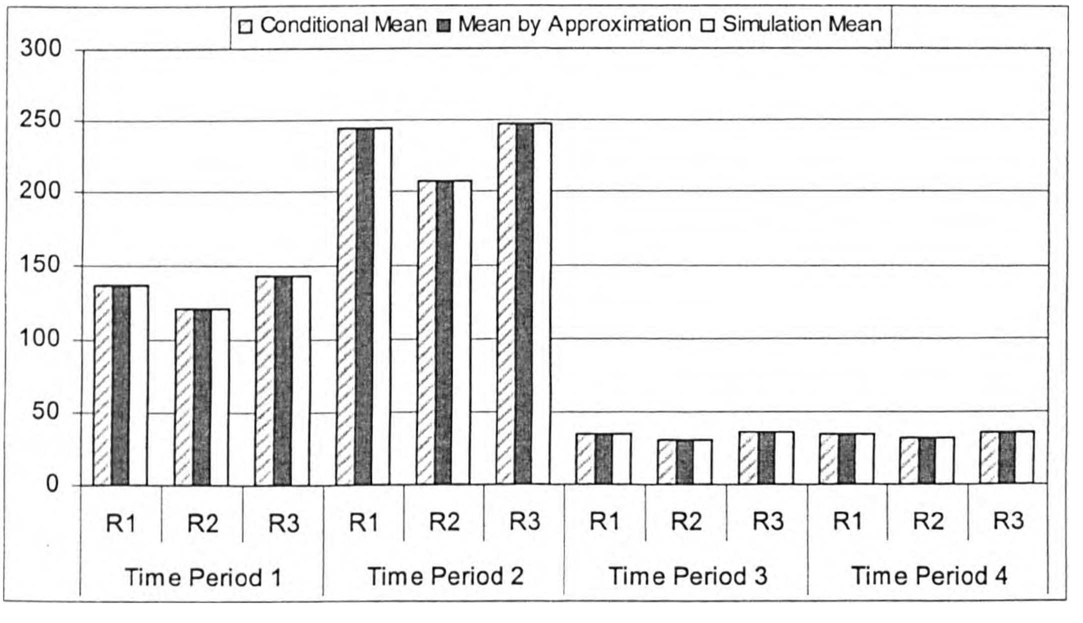

Figure 7.2 Comparison of Mean Route Flows ($\theta = 0.01$)

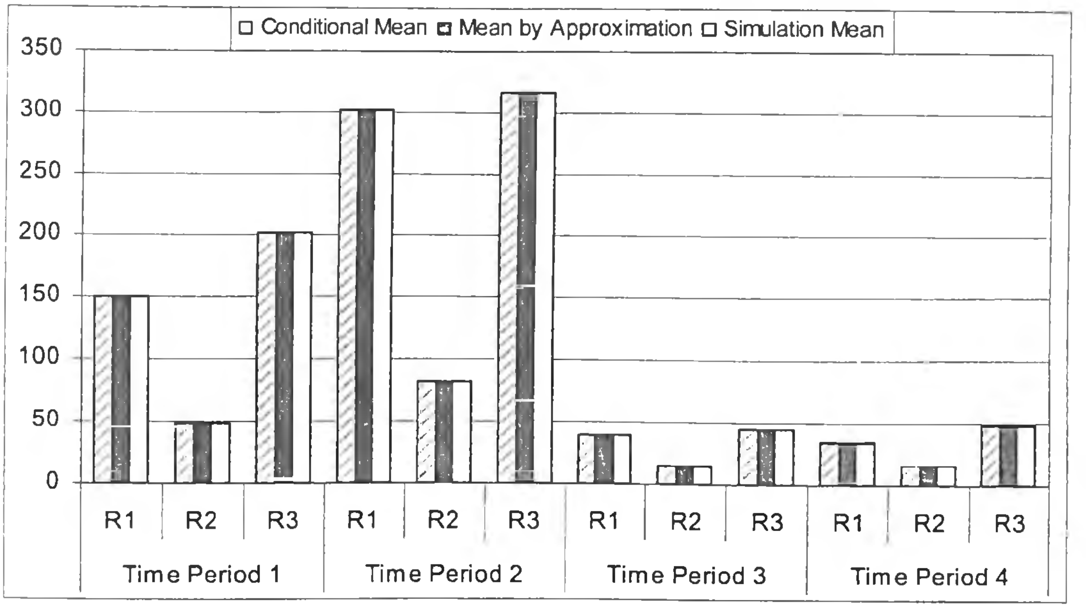

Figure 7.3 Comparison of Mean Route Flows (θ = 0.1)

7.2.2 Comparison of Variance

With four departure periods and three routes being considered, the variance-covariance matrix and other matrices such as the Jacobians of route choice and travel times will be of dimension 12 × 12 and as an example, these matrices are presented in Figures 7.4 – 7.8 with appropriate titles, just for the case of θ = 0.1, although the comparison of variance and covariance has been carried out for the case of θ = 0.01 as well based on approximated and simulated variance-covariance matrices included in Appendix B.

The Jacobian matrix of route choice probability with respect to the route costs (Figure 7.4) is a block diagonal matrix with each block representing each departure period and the rows in each block represent the number of routes. Off diagonal values in this matrix are zeros as the route choices are calculated based on the given costs in each departure period and hence are independent of the costs in other departure periods.

		Departure Period 1			Departure Period 2			Departure Period 3			Departure Period 4		
		Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
Departure Period 1	Route 1	-0.0234	0.0046	0.0188	0	0	0	0	0	0	0	0	0
	Route 2	0.0046	-0.0107	0.0062	0	0	0	0	0	0	0	0	0
	Route 3	0.0188	0.0062	-0.025	0	0	0	0	0	0	0	0	0
Departure Period 2	Route 1	0	0	0	-0.0245	0.005	0.0195	0	0	0	0	0	0
	Route 2	0	0	0	0.005	-0.0103	0.0053	0	0	0	0	0	0
	Route 3	0	0	0	0.0195	0.0053	-0.0248	0	0	0	0	0	0
Departure Period 3	Route 1	0	0	0	0	0	0	-0.0241	0.0062	0.018	0	0	0
	Route 2	0	0	0	0	0	0	0.0062	-0.0129	0.0067	0	0	0
	Route 3	0	0	0	0	0	0	0.018	0.0067	-0.0247	0	0	0
Departure Period 4	Route 1	0	0	0	0	0	0	0	0	0	-0.0228	0.0056	0.0172
	Route 2	0	0	0	0	0	0	0	0	0	0.0056	-0.0134	0.0078
	Route 3	0	0	0	0	0	0	0	0	0	0.0172	0.0078	-0.025

Figure 7.4 Jacobian of Route Choice Probability Function (θ = 0.1)

The Jacobian matrix of route travel times (Figure 7.5) represents the partial derivatives of route travel times with respect to the route inflows. For example, the first cell value 0.3688 represents the derivative of the travel time on route 1 with respect to the route inflows in departure period 1. The second value in the first column, 0.1688 represents the derivative of travel time on route 2 in departure period 1 with respect to the route 1 inflows in departure period 1 and so on. Given the structure of the Jacobian, based on the principle of causality, one may think that the route travel times in the earlier departure periods are not affected by the route inflows in later departure periods, and hence may arrive at the conclusion that the Jacobian matrix should be lower triangular. However, this is not strictly true in the case of links being shared by multiple paths. For example, travel time on route 2 in departure period 1 seems to be affected by the inflows on route 1 in departure period 2 as indicated by the value of 0.0390 in the second row, fourth column of the matrix, which may appear to be a causality violation. This is due to the spatio-temporal interaction of the vehicles belonging to route 1 and route 2 which share link 4. To elaborate, the travel time on link 4 at any given instant is a function of the number of vehicles on the link at that time which may constitute the flows from route 2 and route 1. It may be noted that the flows on route 2 traversing through links 2 and 3 need a longer time to reach node 2, whereas the flows on route 1 reach node 2 earlier, which just need to traverse through link 1, thus leading to an interaction between the flows originating from different departure periods (1 and 2 as in this case) on link 4. Therefore, the travel time at any instant is strictly dependent on the traffic ahead of it thus satisfying the definition of causality. This discussion illustrates one of the common features of the dynamic networks in real world situation.

		Departure Period 1			Departure Period 2			Departure Period 3			Departure Period 4		
		Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
Departure Period 1	Route 1	0.3688	0.0577	-0.0045	0	0	0	0	0	0	0	0	0
	Route 2	0.1688	0.5211	0.3026	0.0390	0	0	0	0	0	0	0	0
	Route 3	0	0.2249	0.4456	0	0	0	0	0	0	0	0	0
Departure Period 2	Route 1	0.3809	0.2277	-0.0229	0.3573	0.0296	-0.0038	0	0	0	0	0	0
	Route 2	0.1221	0.5175	0.3080	0.2470	0.4409	0.2099	0.1466	0	0	0	0	0
	Route 3	0	0.3209	0.5144	0	0.1852	0.4290	0	0	0	0	0	0
Departure Period 3	Route 1	0.2061	0.1569	-0.0114	0.5228	0.1905	-0.0275	0.3443	0.0102	-0.0025	0	0	0
	Route 2	0.0806	0.2805	0.1910	0.2030	0.6421	0.3483	0.3484	0.5714	0.1584	0.2269	0	0
	Route 3	0	0.2659	0.3395	0	0.3451	0.6824	0	0.0213	0.4832	0	0	0
Departure Period 4	Route 1	0.0981	0.1050	0.0133	0.2653	0.2654	-0.0087	0.5242	0.2059	-0.0330	0.3471	0.0091	-0.0021
	Route 2	0.0332	0.1593	0.1272	0.0832	0.3820	0.2426	0.1765	0.7942	0.3287	0.3438	0.0072	0. 2008
	Route 3	0	0.1894	0.2388	0	0.2695	0.4792	0	0.2004	0.7786	0	0.1090	0.4707

Figure 7.5 Jacobian Matrix of Route Travel Times (θ = 0.1)

The conditional covariance matrix (Figure 7.6) is also a block diagonal matrix with as many blocks as there are departure periods and each block is a square matrix of size equal to the number of feasible routes within that departure period. All off-diagonal elements are equal to zero because the route flow in any departure period on any given day is defined as being from an independent multinomial distribution. The conditional variance-covariance within each departure period has been worked as the multinomial variance based on standard properties using (4.17).

		Departure Period 1			Departure Period 2			Departure Period 3			Departure Period 4		
		Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
Departure Period 1	Route 1	93.60	-18.25	-75.34	0	0	0	0	0	0	0	0	0
	Route 2	-18.25	42.91	-24.65	0	0	0	0	0	0	0	0	0
	Route 3	-75.34	-24.65	99.99	0	0	0	0	0	0	0	0	0
Departure Period 2	Route 1	0	0	0	171.72	-35.07	-136.65	0	0	0	0	0	0
	Route 2	0	0	0	-35.07	71.84	-36.76	0	0	0	0	0	0
	Route 3	0	0	0	-136.65	-36.76	173.41	0	0	0	0	0	0
Departure Period 3	Route 1	0	0	0	0	0	0	24.12	-6.15	-17.97	0	0	0
	Route 2	0	0	0	0	0	0	-6.15	12.85	-6.7	0	0	0
	Route 3	0	0	0	0	0	0	-17.97	-6.7	24.67	0	0	0
Departure Period 4	Route 1	0	0	0	0	0	0	0	0	0	22.77	-5.61	-17.16
	Route 2	0	0	0	0	0	0	0	0	0	-5.61	13.44	-7.83
	Route 3	0	0	0	0	0	0	0	0	0	-17.16	-7.83	24.99

Figure 7.6 Conditional Covariance Matrix (θ = 0.1)

The approximated variance-covariance matrix has been worked out based on equation (4.18). The main diagonal of Figure 7.7 represents the variance and the off-diagonal elements represent the covariance and it has the same structure as that of the conditional covariance matrix. As the conditional covariance ignores the variations due to cost variation, equation (4.18) inflates the values in Figure 7.6 and pushes them towards the true variance-covariance.

		Departure Period 1			Departure Period 2			Departure Period 3			Departure Period 4		
		Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
Departure Period 1	Route 1	105.45	-16.56	-88.89	25.13	2.59	-27.72	2.17	0.36	-2.53	1.41	0.34	-1.75
	Route 2	-16.56	44.78	-28.22	3.79	3.23	-7.03	0.32	0.39	-0.71	0.23	0.28	-0.51
	Route 3	-88.89	-28.22	117.11	-28.92	-5.82	34.75	-2.49	-0.75	3.24	-1.64	-0.61	2.26
Departure Period 2	Route 1	25.13	3.79	-28.92	288.03	-23.72	-264.31	18.08	2.54	-20.62	11.79	2.50	-14.29
	Route 2	2.59	3.23	-5.82	-23.72	83.53	-59.80	1.60	2.30	-3.91	1.17	1.66	-2.83
	Route 3	-27.72	-7.03	34.75	-264.31	-59.80	324.11	-19.68	-4.84	24.52	-12.96	-4.16	17.12
Departure Period 3	Route 1	2.17	0.32	-2.49	18.08	1.60	-19.68	27.54	-5.72	-21.83	2.37	0.46	-2.83
	Route 2	0.36	0.39	-0.75	2.54	2.30	-4.84	-5.72	13.40	-7.69	0.33	0.44	-0.77
	Route 3	-2.53	-0.71	3.24	-20.62	-3.91	24.52	-21.83	-7.69	29.52	-2.70	-0.90	3.60
Departure Period 4	Route 1	1.41	0.23	-1.64	11.79	1.17	-12.96	2.37	0.33	-2.70	24.68	-5.22	-19.46
	Route 2	0.34	0.28	-0.61	2.50	1.66	-4.16	0.46	0.44	-0.90	-5.22	13.86	-8.64
	Route 3	-1.75	-0.51	2.26	-14.29	-2.83	17.12	-2.83	-0.77	3.60	-19.46	-8.64	28.10

Figure 7.7 Approximated Variance - Covariance Matrix (θ = 0.1)

Figure 7.8 summarises the simulated traffic flows and specifically, represents the variance-covariance of the route flows of the system which is directly comparable to the values in Figure 7.7.

		Departure Period 1			Departure Period 2			Departure Period 3			Departure Period 4		
		Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
Departure Period 1	Route 1	102.04	-18.82	-83.22	18.55	-1.19	-17.36	2.11	-0.08	-2.04	1.36	-0.28	-1.08
	Route 2	-18.82	42.67	-23.85	0.54	-0.16	-0.37	0.09	0.16	-0.25	0.28	0.06	-0.34
	Route 3	-83.22	-23.85	107.06	-19.09	1.36	17.73	-2.20	-0.08	2.29	-1.64	0.22	1.42
Departure Period 2	Route 1	18.55	0.54	-19.09	285.37	-32.67	-252.70	20.61	-0.10	-20.51	14.08	0.39	-14.47
	Route 2	-1.19	-0.16	1.36	-32.67	72.52	-39.86	1.24	-0.06	-1.18	0.82	0.04	-0.87
	Route 3	-17.36	-0.37	17.73	-252.70	-39.86	292.56	-21.84	0.16	21.68	-14.91	-0.43	15.34
Departure Period 3	Route 1	2.11	0.09	-2.20	20.61	1.24	-21.84	28.59	-6.11	-22.48	3.15	0.03	-3.18
	Route 2	-0.08	0.16	-0.08	-0.10	-0.06	0.16	-6.11	12.97	-6.85	0.08	-0.08	0.00
	Route 3	-2.04	-0.25	2.29	-20.51	-1.18	21.68	-22.48	-6.85	29.33	-3.23	0.05	3.18
Departure Period 4	Route 1	1.36	0.28	-1.64	14.08	0.82	-14.91	3.15	0.08	-3.23	25.02	-5.39	-19.63
	Route 2	-0.28	0.06	0.22	0.39	0.04	-0.43	0.03	-0.08	0.05	-5.39	13.33	-7.94
	Route 3	-1.08	-0.34	1.42	-14.47	-0.87	15.34	-3.18	0.00	3.18	-19.63	-7.94	27.56

Figure 7.8 Simulated Variance - Covariance Matrix (θ = 0.1)

Figure 7.9 compares the variance of route flows obtained by the simulation method with that obtained by the method of approximation. In the case of $\theta = 0.01$, the drivers are relatively insensitive to the costs and choose routes almost completely at random. In this case, the naïve variance is sufficient to explain the simulation variance (Figure 7.9), and hence the approximation method leaves them almost unaltered. In contrast, in the case of $\theta = 0.1$ (Figure 7.10), it is noticed that the naïve variance is insufficient to explain the simulation variance and the approximation method adds a correction term and lifts the flow variances close to the simulation variance.

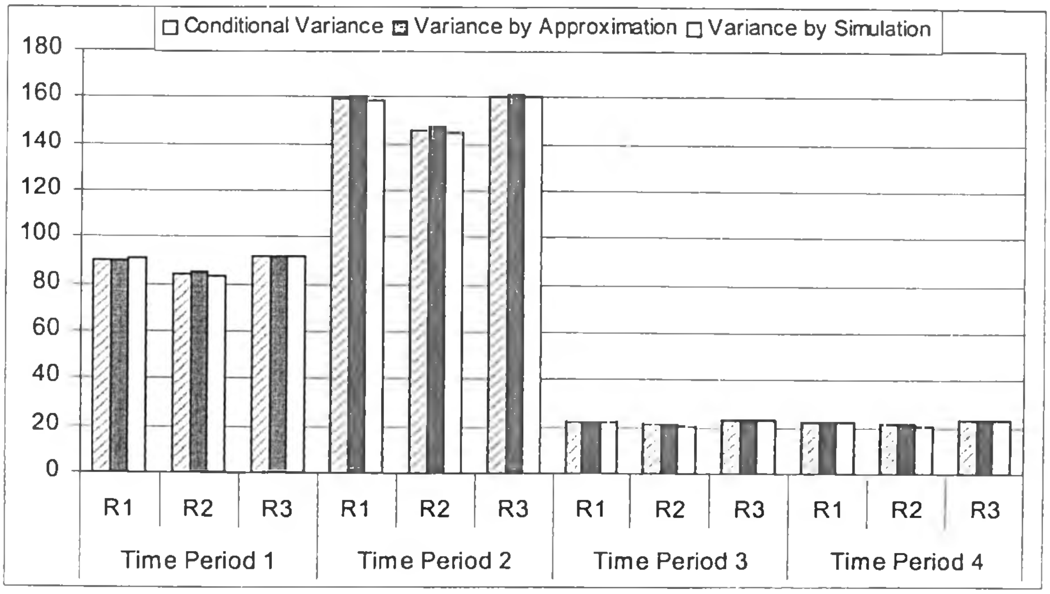

Figure 7.9 Comparison of Variance (θ = 0.01)

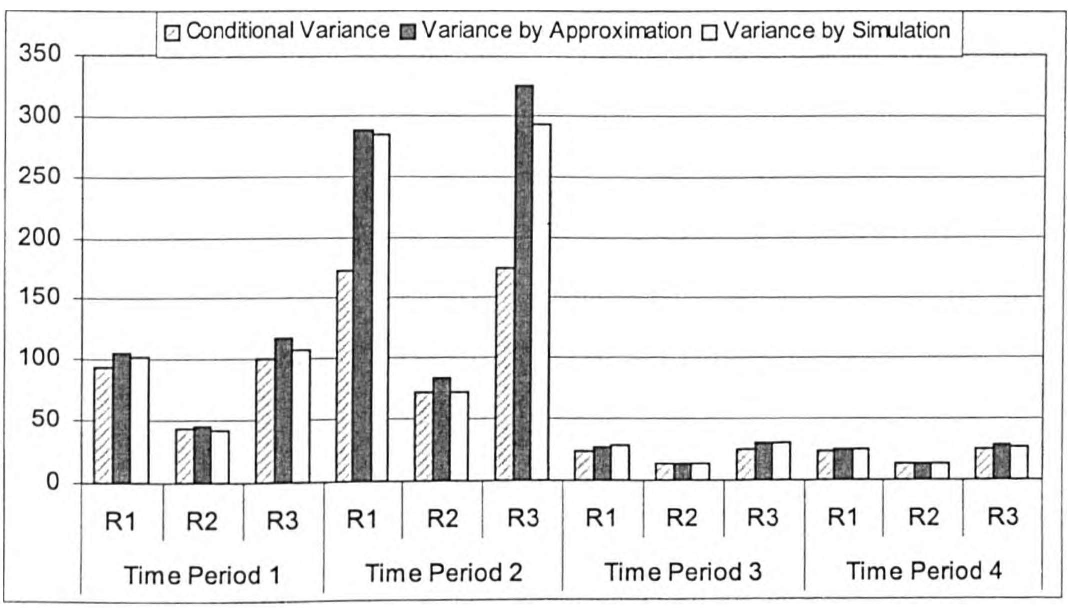

Figure 7.10 Comparison of Variance (θ = 0.1)

7.2.3 Comparison of Covariance

The route flow covariance represents the drivers' reactions to the interactions between the drivers departing in different time periods, but using the network during the same time period. To elaborate the meaning of the term interactions, consider, for example, a driver choosing (say) route 1 in the major time period 2 on any given day, who may interact with the drivers who already chose the same route during the earlier major time period but not yet completed their journey. If the carried over demand from the earlier period is high, the drivers from the later period would incur an increased travel time and this might make the driver in consideration from the later period, choose another route on the following day which is accounted for through the drivers learning and adjusting process. Thus, there are potentially complex chains of cause and action across the days, routes and time periods. Covariance of flows between various time periods on any given route precisely represents this reaction. The off-diagonal elements of Figures 7.7 and 7.8 are the covariance of route flows for the case of θ=0.1, for the cases of the approximation and simulation methods respectively.

Figure 7.11 summarises the comparison of diagonal and off diagonal elements of variance-covariance matrices generated by the approximation and simulation methods for the case of θ=0.01 (See Appendix B). Similarly, Figure 7.12 plots the variance-covariance from both methods for the case of θ=0.1 (See Figures 7.7 and 7.8). It may be observed that in the case of θ=0.01, the approximation method converges well with the simulation method. This is because the conditional variance-covariance was sufficient to explain the variance in traffic flows, whereas in the case of θ=0.1, the conditional variance-covariance was inflated by adding a correction term. In both the cases, the approximation method performed extremely well, as indicated by an R^2 value over 0.99 in each case.

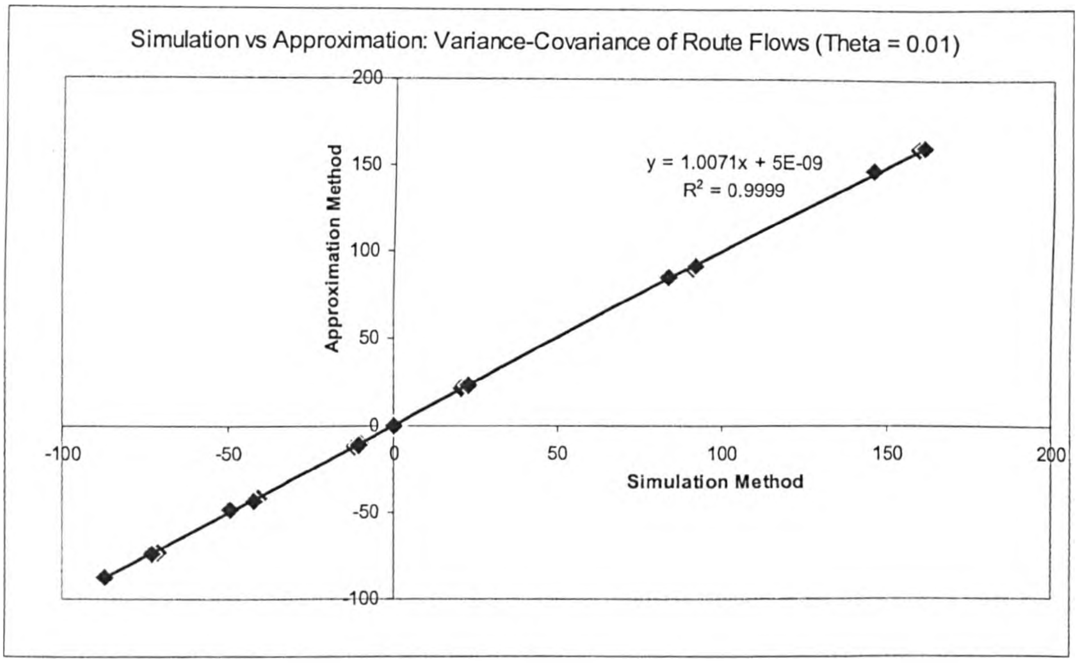

Figure 7.11 Comparison of Variance – Covariance ($\theta = 0.01$)

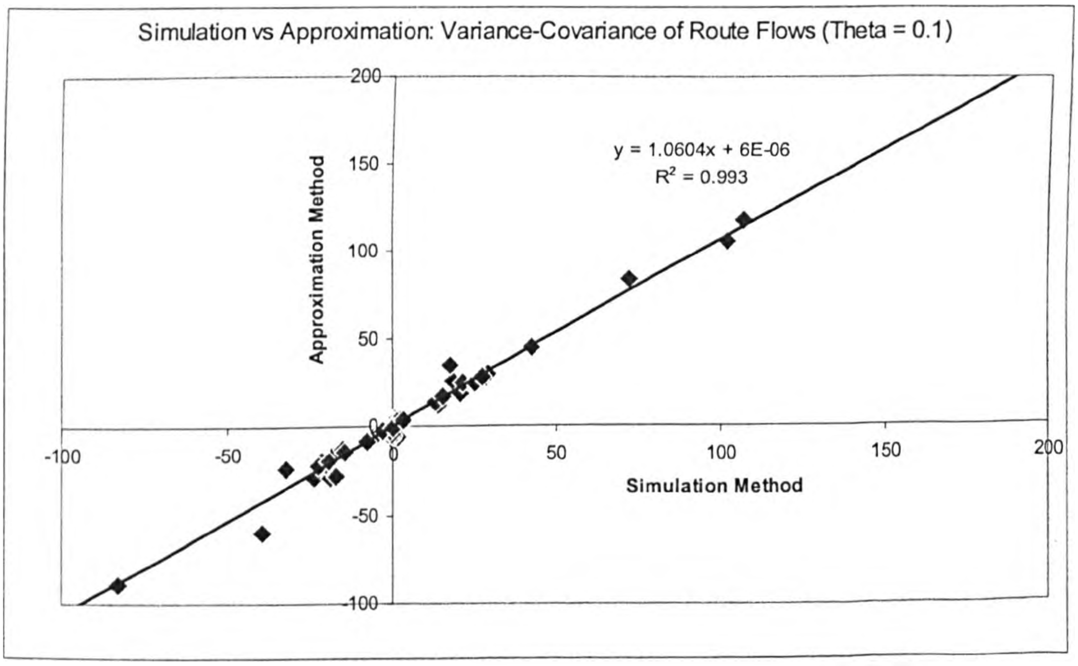

Figure 7.12 Comparison of Variance – Covariance ($\theta = 0.1$)

7.3 Sensitivity Analysis

The aim of this analysis is to investigate the sensitivity of the model outputs viz., stationary variance-covariance of the route flows to the model inputs viz., memory length m, and memory weighting λ. Estimates of variance-covariance are generated for a realisation of 1000 days by simulating the route choices as described in

Chapter 6, and the variance-covariance matrices are also approximated using the variance approximation method with the MSA operated over 10000 iterations, to estimate the DSUE flows. In this analysis, the five link network described earlier has been used with the demand spread over four discrete departure periods of 15 minutes each, $\mathbf{q} = [400\ 700\ 350\ 100]$.

There are two tests in this analysis –each one studying the impact of varying memory length m and memory weighting λ respectively, and the testing schemes are described as below:

Test 1: studies the impact of varying the memory length, m.
Base Case: defines the benchmark case and the input model parameters are $\theta = 0.1$, $\lambda = 0.5$ and m = 5 days.
Case 1: defines the system with lower memory than the base case and the input parameters are $\theta = 0.1$, $\lambda = 0.5$ and m = 1 day.
Case 2: defines the system with higher memory than the base case and the input parameters are $\theta = 0.1$, $\lambda = 0.5$ and m = 12 days.

Test 2: studies the impact of varying the memory weighting, λ.
Base Case: defines the benchmark case identical to the one in Test 1 above and the input model parameters are $\theta = 0.1$, m = 5 days and $\lambda = 0.5$.
Case 1: defines the system with lower memory weighting than the base case and the input parameters are $\theta = 0.1$, m = 5 days and $\lambda = 0.1$.
Case 2: defines the system with higher memory weighting than the base case and the input parameters are $\theta = 0.1$, m = 5 days and $\lambda = 0.9$.

The following paragraphs analyse how the simulation model performed in each test, followed by the approximation method, and then the overall comparison of the model performance by comparing the simulation model with the variance approximation method is provided.

7.3.1 Sensitivity Analysis of Simulation Model

Test 1: Figure 7.13 compares the variance of the route flows in base case with that in Test 1-Case 1, both having been generated by the simulation method, for various routes over various departure periods, and shows that the shorter memory systems have higher variance than the higher memory systems. This is intuitively supportive to the general notion that if a driver chooses routes based on only one experience, then that driver is likely to be less informative and such decisions may result in highly volatile traffic system. To verify this notion, correlograms were drawn which indicated that the route flows are highly negatively correlated similar to a deterministic periodic behaviour of the system.

On the other hand, when the memory length is increased (Test1-Case2), the flow variance remains relatively unaltered, although the routes 1 and 3 exhibit some growth in variance in departure periods 2 and 3 (Figure 7.14). In this experiment, although the memory length is increased to 12 days, the relative distribution of memory weight remained almost similar to that in the base case. This can be verified through the normalised memory weights given in Table 7.6. From the table, it is clear that in Test1-Case2, nearly 97% of the cumulative memory weight is accounted for by the first 5-days, thus explaining the similarity of the variance in Base Case and Test1-Case2.

Table 7.6 Normalised Values of Memory Weight, λ

Memory Length, days	1	2	3	4	5	6	7	8	9	10	11	12	Sum
Base Case 5-days	0.5161	0.2581	0.1290	0.0645	0.0323								1.0000
Test1-Case2 12-days	0.5001	0.2501	0.1250	0.0625	0.0313	0.0156	0.0078	0.0039	0.0020	0.0010	0.0005	0.0002	1.0000

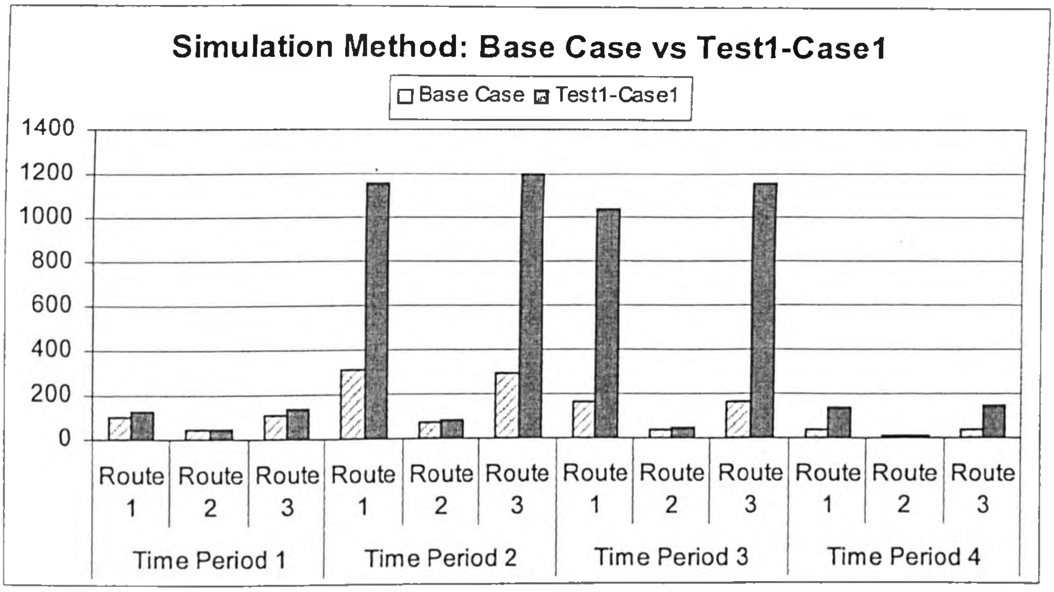

Figure 7.13 Comparison of Variance by Simulation Method (Test1-Case1)

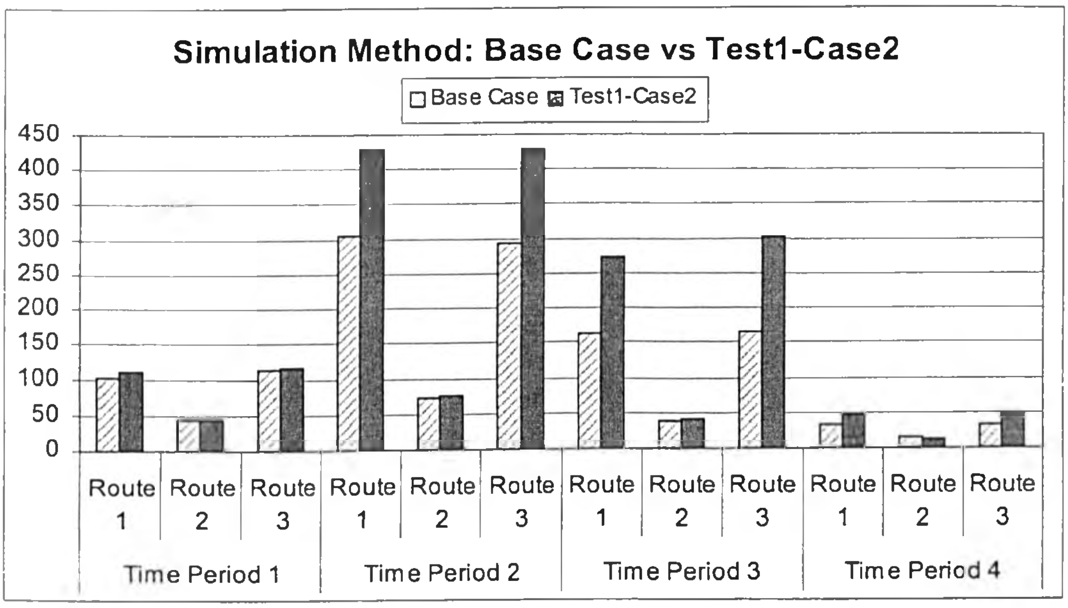

Figure 7.14 Comparison of Variance by Simulation Method (Test1-Case2)

Test 2: Figure 7.15 shows higher variance bars with lower memory weight (Test2-Case1) compared to the base case. Lower memory weight indicates that the drivers place highest weight on the most recent experiences than the older experiences and almost ignore the rest of the experiences other than the most recent one. In this case the system behaves like a shorter memory system and hence we expect a higher system variance just as in Test1-Case1 described in the previous paragraph. On the other hand, increased memory weight (Test2-Case2) indicates that the drivers consider the most recent experience in addition to the older experiences in developing the perceptions for the following day and hence the system replicates the behaviour of a longer memory system. Therefore, it is not surprising that the variances have slightly reduced compared to the base case in Figure 7.16. Both the tests 1 and 2 support the idea that highly sensitive systems largely depending only on the most recent experience are likely to be more unstable. These results are consistent with the observations made by Cantarella and Cascetta (1995).

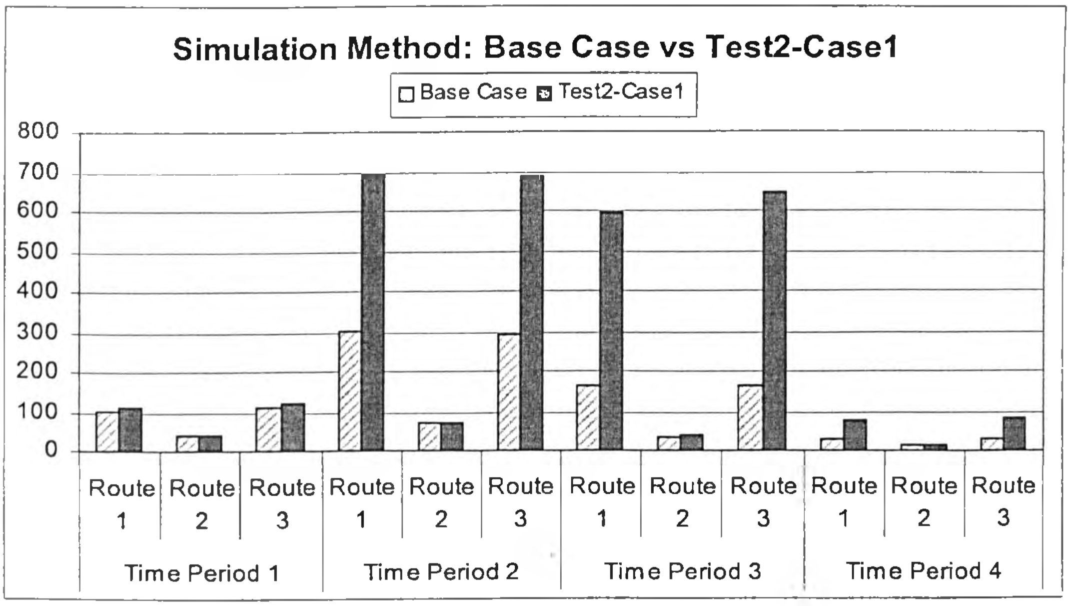

Figure 7.15 Comparison of Variance by Simulation Method (Test2-Case1)

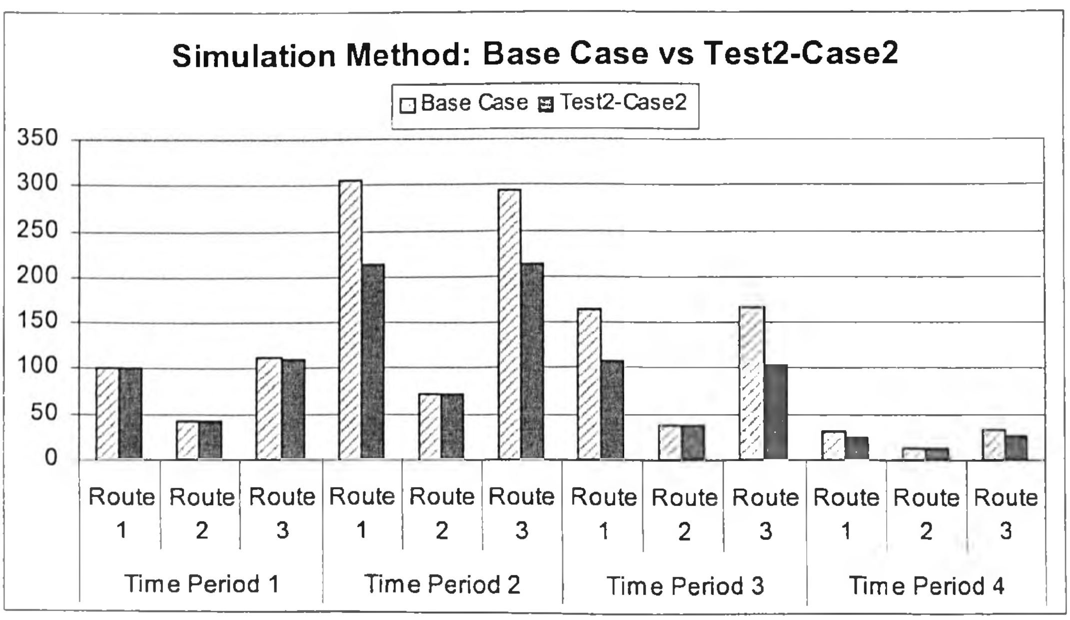

Figure 7.16 Comparison of Variance by Simulation Method (Test2-Case2)

7.3.2 Sensitivity Analysis of Variance Approximation Model

Test 1: The variance approximation model also produced similar results to that of the simulation model and hence the same arguments are applicable here as well. In fact, the variance approximation method produced highly consistent results, with the reduction in memory length (Test1-Case1) producing higher route flow variance (Figure 7.17), and increasing memory length (Test1-Case2) producing exactly opposite effect reducing the route flow variance (Figure 7.18). It is interesting to note that in the case of longer memory systems (Test1-Case2), the route flow variance appears to be just as insensitive as in the case of its simulation counterpart, and seems to be affected by the distribution of normalised weights as discussed earlier.

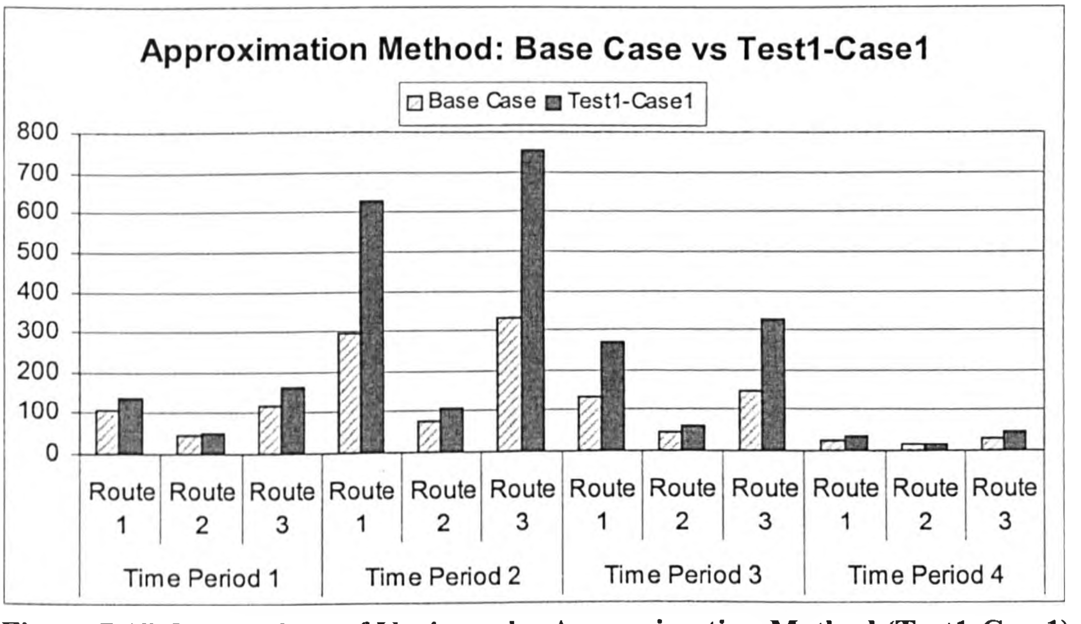

Figure 7.17 Comparison of Variance by Approximation Method (Test1-Case1)

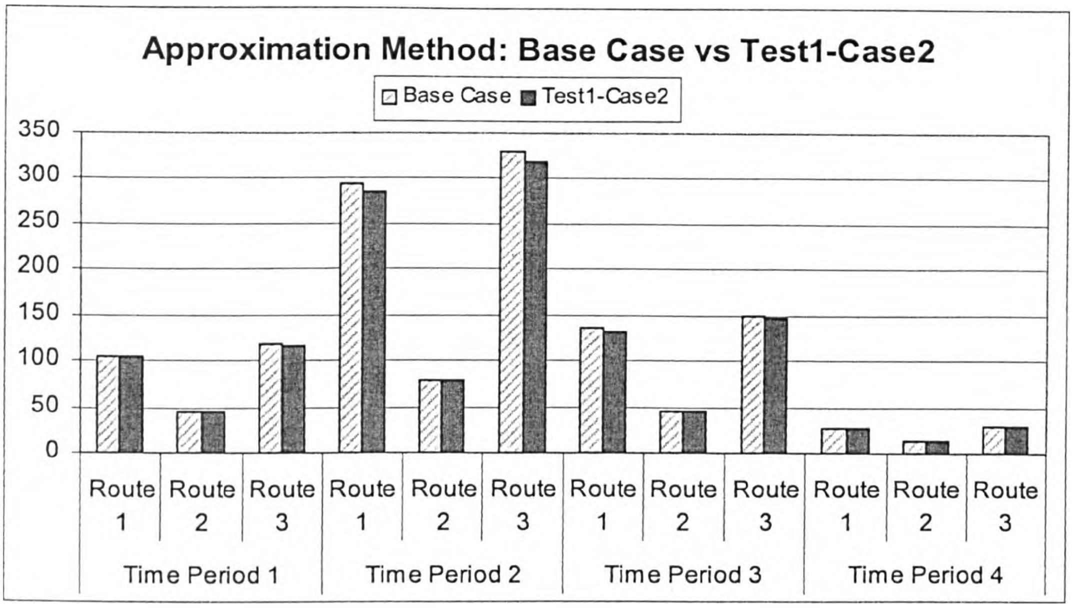

Figure 7.18 Comparison of Variance by Approximation Method (Test1-Case2)

Test 2: In the case of varying memory weights, the approximation method performed well and produced intuitively supportive results just as the simulation model did. Lower memory weight indicates that the drivers place higher weight on the most recent experience and hence the system behaves akin to a shorter memory system and hence has higher variances (Figure 7.19). On the other hand, a higher memory weight produces the effect of considering the experiences over a longer period, and hence behaves similar to longer memory systems. Therefore, the system exhibited lower variance with higher memory weight (Figure 7.20).

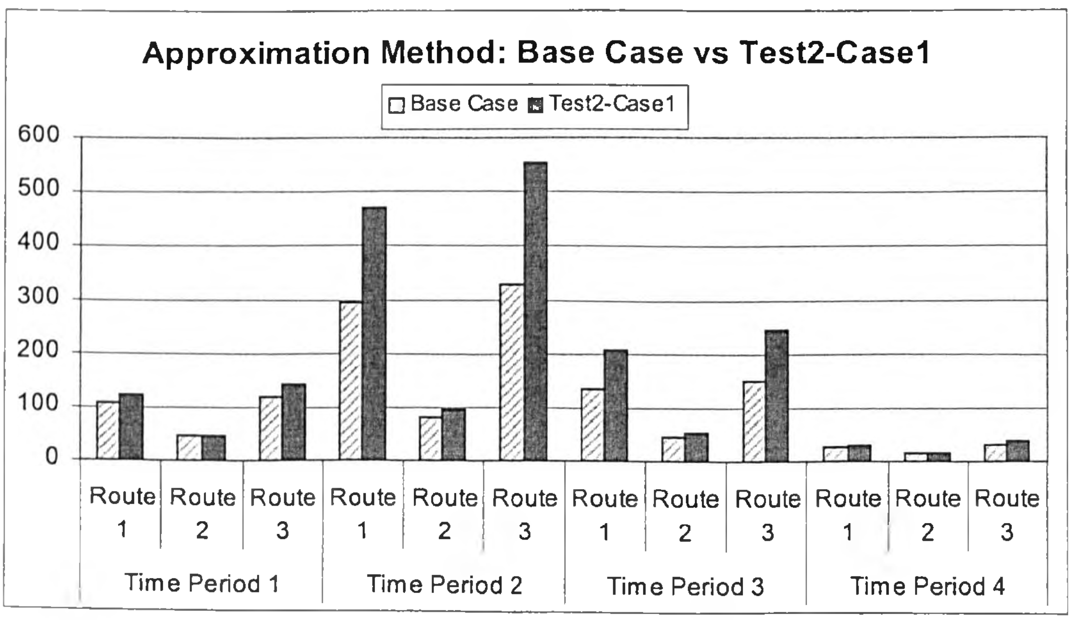

Figure 7.19 Comparison of Variance by Approximation Method (Test2-Case1)

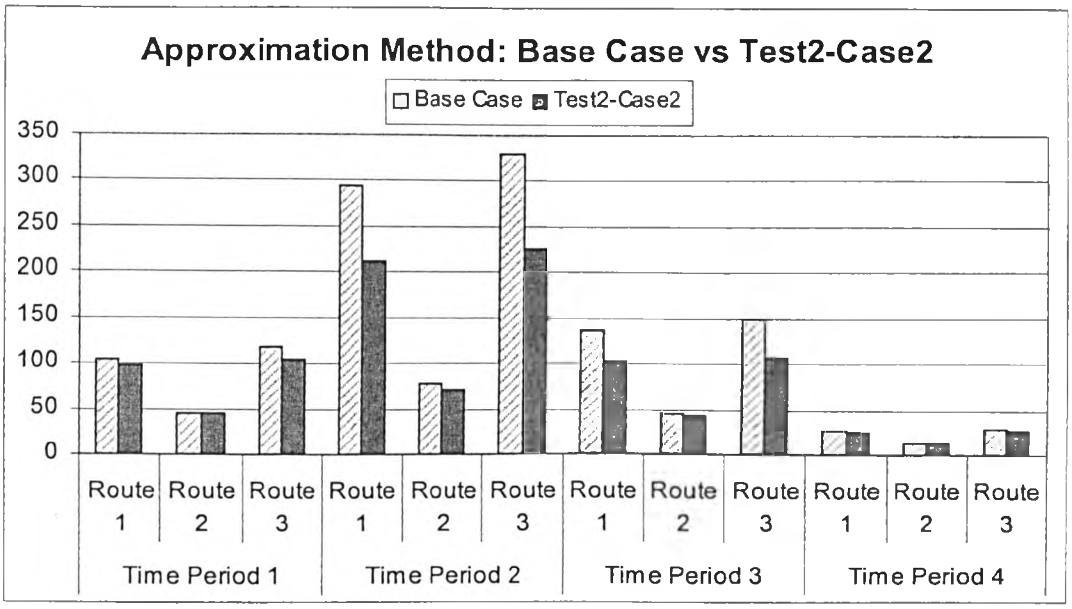

Figure 7.20 Comparison of Variance by Approximation Method (Test2-Case2)

7.3.3 Overall Model Performance

This section analyses the overall model performance of the variance approximation model against the simulation model and compares the variance-covariance matrices generated by the simulation model and the approximation model (Figures 7.21 – 7.25) in each of the cases described in section 7.3.2. All the comparisons indicate a high level of agreement of the approximation results with the corresponding results obtained by the simulation method. However, Figure 7.22 shows slightly lower convergence (R^2 = 0.822) which is expected because Test1-Case1 appears to destabilise the system towards a deterministic periodic system as described in section 7.3.1 and hence the variance-covariance approximated by the method overestimates the simulation variance-covariance. On the other hand, longer memory systems and those systems with higher memory weight seem to be well estimated by the variance approximation method.

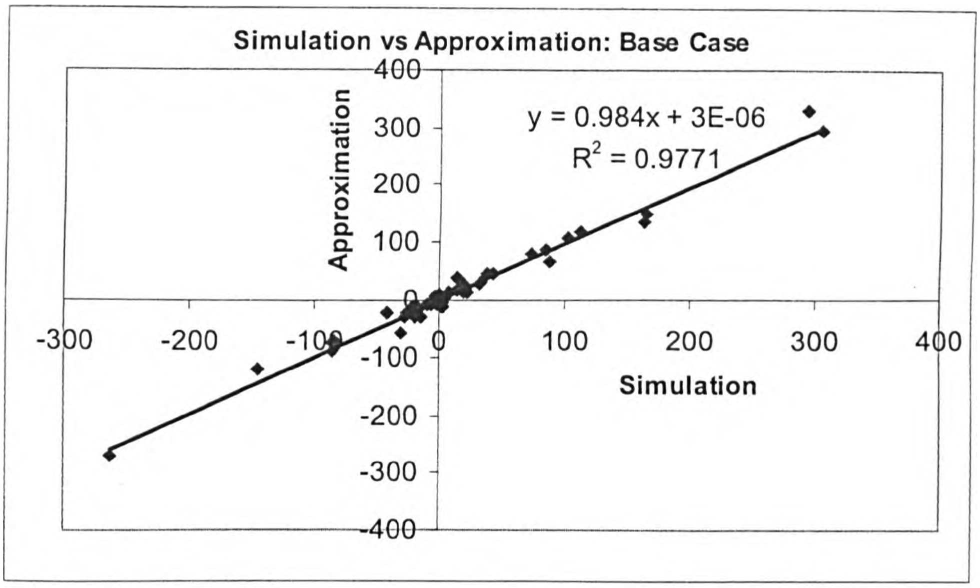

Figure 7.21 Comparison of Variance – Covariance (Base Case)

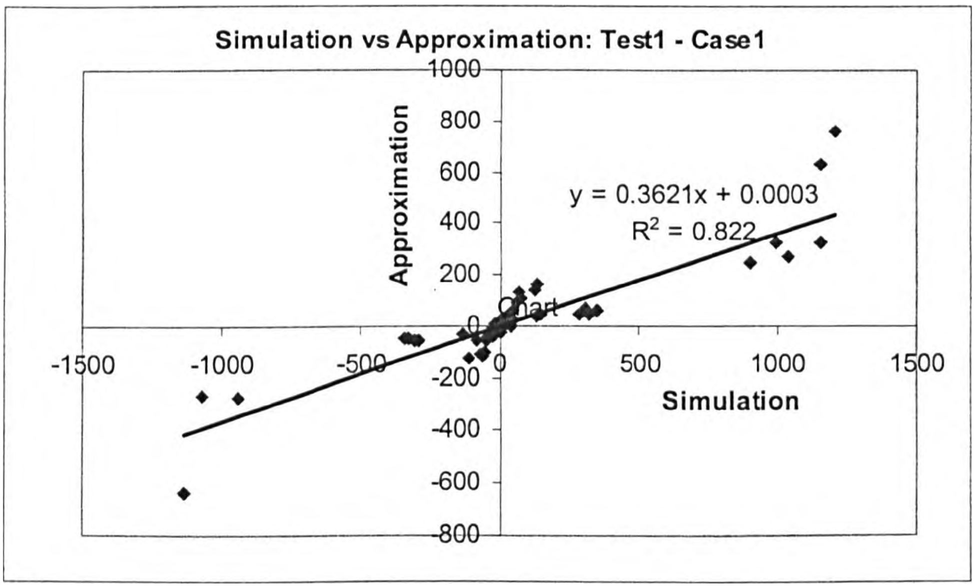

Figure 7.22 Comparison of Variance – Covariance (Test1-Case1)

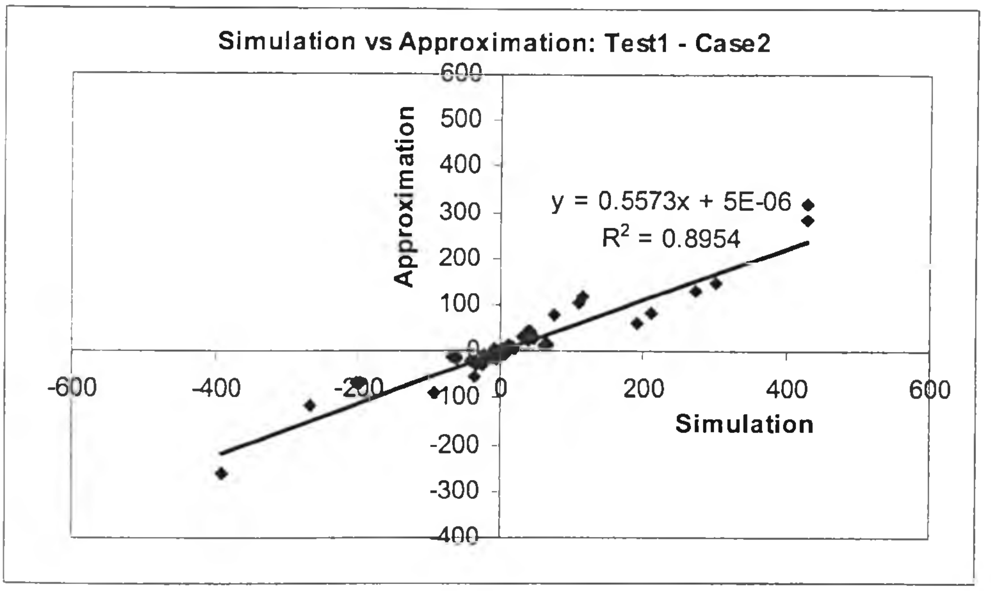

Figure 7.23 Comparison of Variance – Covariance (Test1-Case2)

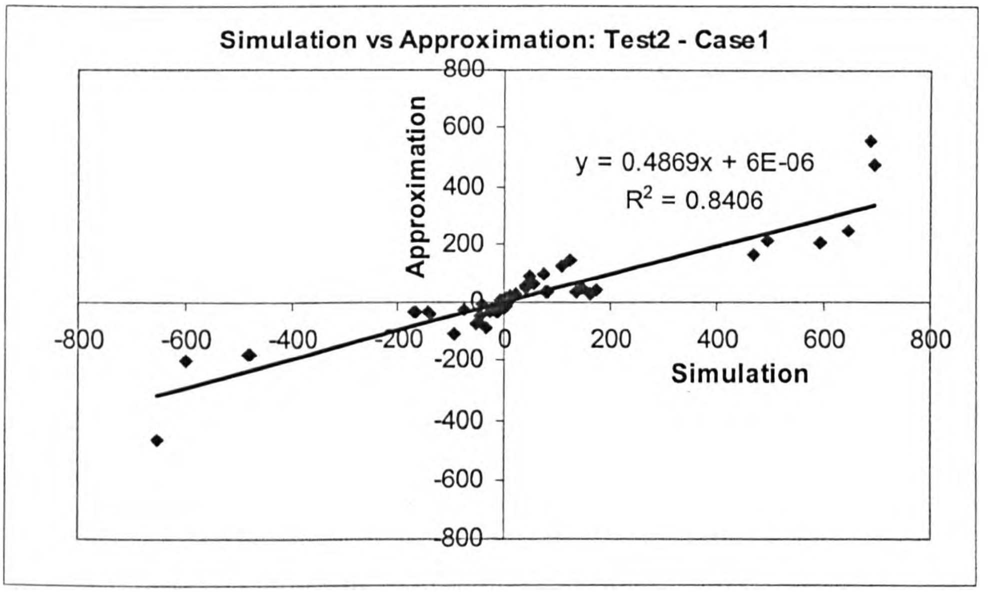

Figure 7.24 Comparison of Variance – Covariance (Test2-Case1)

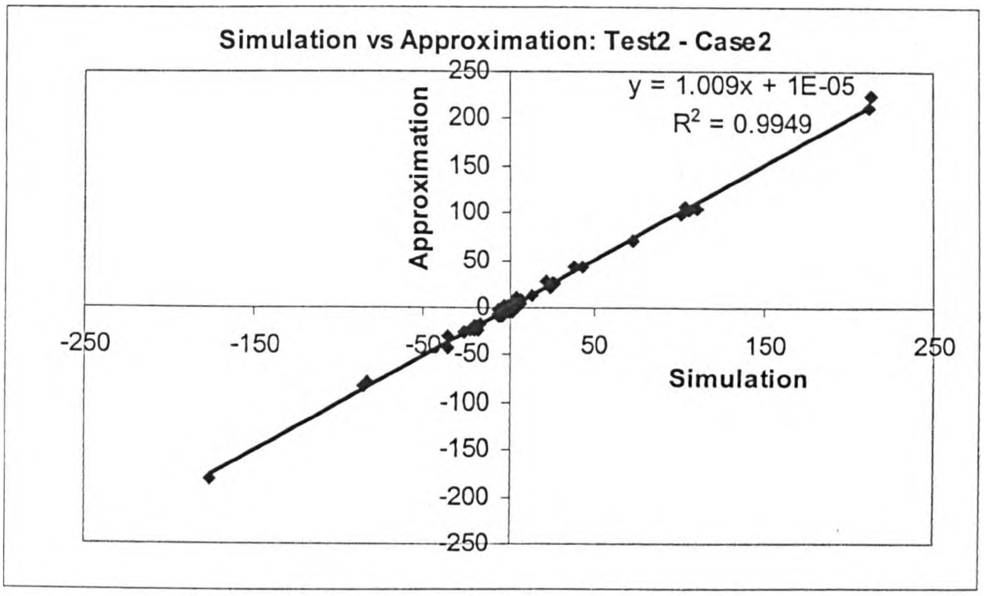

Figure 7.25 Comparison of Variance – Covariance (Test2-Case2)

7.4 Grid Network

This section is aimed at identifying the procedure while analysing the variance in flows from a doubly dynamic traffic assignment thus utilising various techniques such as the autocorrelation function described earlier. For this purpose, a grid network of 12 links serving six O-D pairs using 14 paths has been used. The demand is assumed to spread over four discrete departure periods of 15 minutes duration each. The basic attributes of the network links and O-D demand are as described in section 6.3.3 of the previous chapter.

In fact, setting up the doubly dynamic traffic assignment problem over a larger network did not require any special inputs other than those required for simpler networks such as the five link network described earlier, however the dimensions of the matrices used within the model are proportionately larger. For example, on the grid network under consideration, as there are 14 paths used in four departure periods, the conditional covariance matrix and the Jacobian matrices of route choices and travel times will be of size (14 × 4=) 56 × 56.

In this example, the conditional covariance matrix is a block diagonal matrix of 14 × 14 in each departure period, with each block being defined by the number of routes used by each O-D pair. Clearly, in this example network, O-D pair 1 uses three routes and hence the size of the block is 3 × 3, and O-D pair 2 uses 2 routes therefore will be a 2 × 2 block and so on. The elements of each block are defined by equation (4.17). All the other elements in each block diagonal matrix (in each departure period) will be equal to zero based on the assumption that conditional on past experiences, the drivers belonging to each O-D pair choose routes independent to those of the decisions made by the drivers from the rest of the O-D pairs and departure periods. However, this assumption may be far from reality as in the real world situation, drivers interact with other drivers belonging to different O-D pairs and even departure periods. The effect of such interactions is captured by the approximation method through the variance-covariance inflation factor as defined by equation (4.18).

Similarly, the Jacobian matrix of route choices in this example is of size 56 × 56, composed of four blocks corresponding to as many departure periods and each block is of 14 × 14 as there are 14 routes in all being used by the O-D demand. In each departure period the matrix is a block diagonal matrix composed of six blocks (= number of O-D pairs). The diagonal elements of each block are defined by equation (4.20) and the off-diagonal elements are given by equation (4.21). Similar to the conditional covariance matrix, all other elements are zeros.

The Jacobian matrix of travel time is computed with the help of the specification described in Chapter 5.

The simulation model and variance approximation method were carried out on the grid network and the results are compared. The route choices were simulated over 10000 days and summarised after ignoring the first 1000 days to account for the empty network conditions. Similarly the stochastic user equilibrium assignment was carried out based on an MSA run of 10000 iterations, after which the Jacobian matrices were evaluated to complete the variance approximation. The simulation and variance approximation were repeated for three different values of θ, the logit choice parameter mainly to identify the regimes over which the approximation method is effective. In this exercise, θ was set equal to 0.01, 0.1 and 0.9 and the models were repeated. As an example, the conditional covariance matrix and Jacobians of route choice and travel time functions for the case of θ = 0.1, have been included in Appendix C. Finally, the corresponding variance-covariance matrices computed by the approximation method and simulation method are also included in the same Appendix.

Figure 7.26 compares the variance in route flows (i.e., the diagonal elements of the variance-covariance matrices) by the simulation and approximation methods, with θ = 0.01. It shows an almost perfect comparison indicating a very high degree of convergence between the results obtained by the two methods. Figure 7.27 compares the variance-covariance in route flows (i.e., all elements of the matrices including the diagonal elements of the matrices) by the two methods and again confirms a very high degree of convergence. In order to check whether or not the drivers consider the experienced costs while making the route choices, correlograms (Figure 7.28) were drawn for routes 1,2 and 3. These revealed that the route flows are almost independent on any given day and do not depend on any of the previous days' flows. This observation supports the notion that under the given conditions, the drivers' route choices are independent of other drivers across other routes/and departure periods and hence the conditional covariance matrix is sufficient to describe the simulation variance, thus resulting in the kind of greater degree of convergence as shown in Figures 7.26 and 7.27.

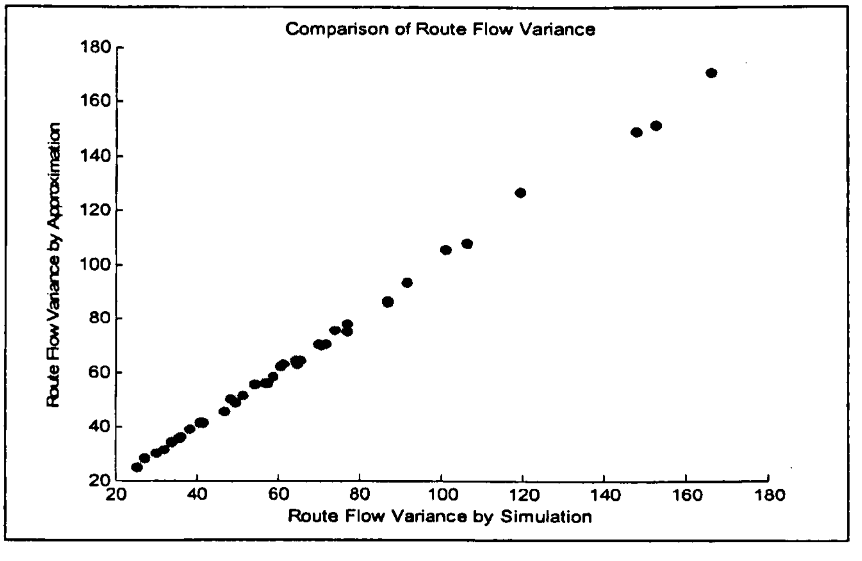

Figure 7.26 Comparison of Route Flow Variance (θ = 0.01)

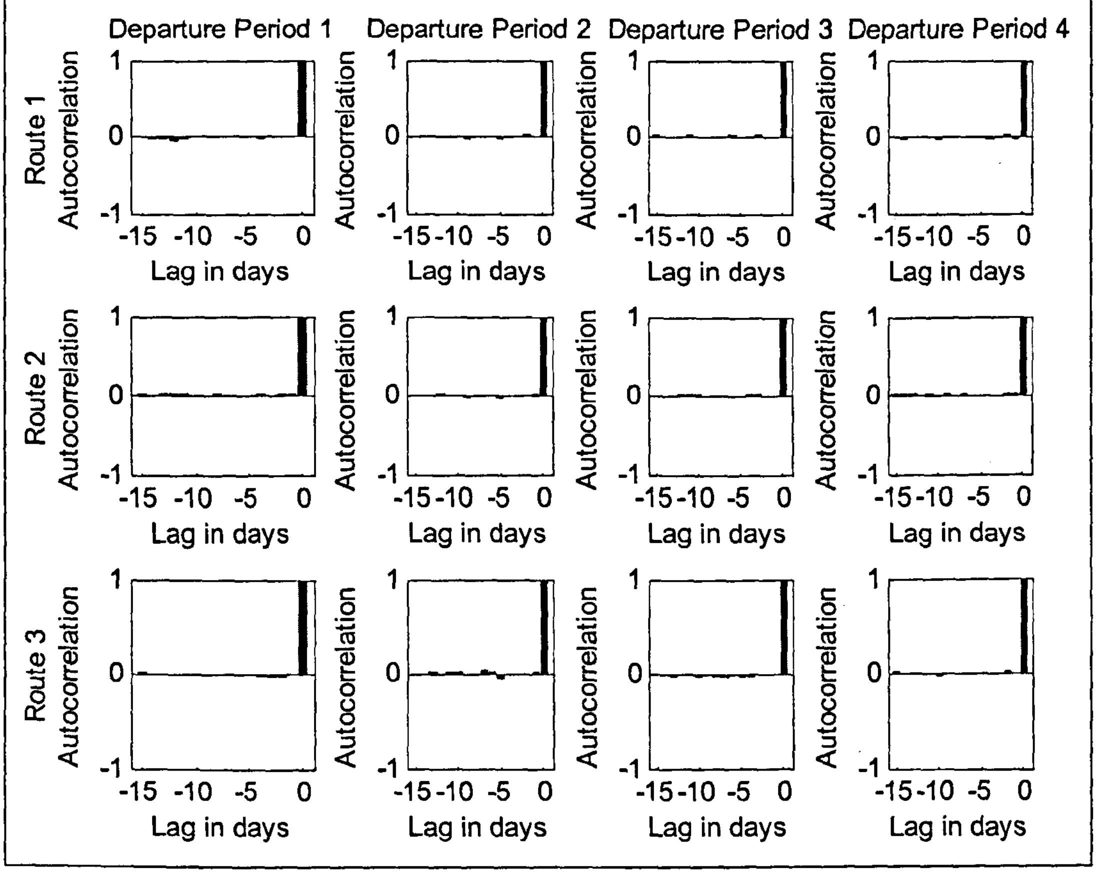

Figure 7.28 Autocorrelation of Route Flows, $\theta = 0.01$

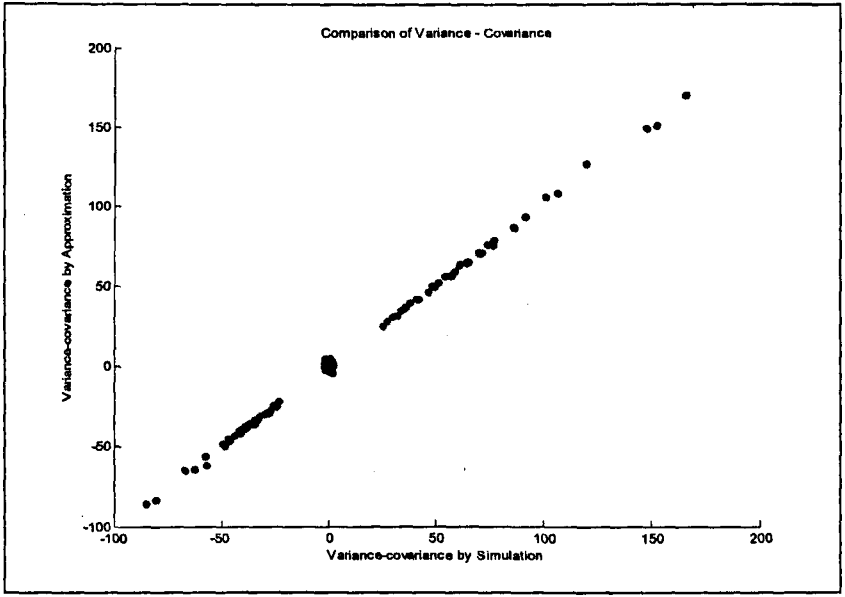

Figure 7.27 Comparison of Route Flow Variance and Covariance (θ = 0.01)

On the other hand, comparison of variance with θ = 0.1 puts up a more mixed picture (Figure 7.29). While the majority of the points are on or close to the perfect convergence line, the rest of the points are below the line indicating that the approximation method under-estimates the simulation variance in some cases. A closer look at the numerical results revealed that the approximation variances in departure periods 1 and 2 are closer to the simulation variances, compared to those in departure periods 3 and 4. In order to understand the results better, correlograms were drawn (Figure 7.30) which show that during departure periods 1 and 2, the route flows are fairly independent resulting in greater convergence between the simulation and approximation methods, but during departure periods 3 and 4 the route flows are significantly negatively correlated to the previous memorised days' flows, indicating deterministic periodic type systems described in section 7.1 earlier in this chapter. This observation is pointing towards the logit parameter producing different dispersion effects in different departure periods due to the overlapping paths present in the network. It is suggested that the logit choice parameters may be lowered for departure periods 3 and 4 to increase the dispersion in perceived costs which will then push the system towards a stationary state, which can be approximated by a multivariate Normal distribution.

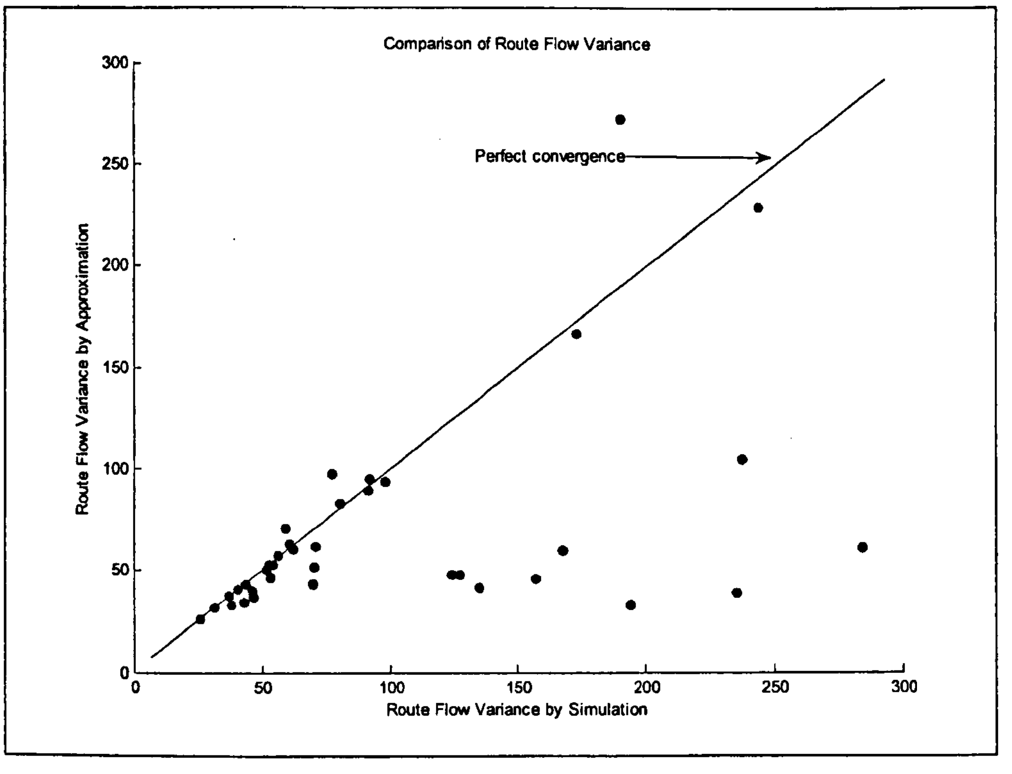

Figure 7.29 Comparison of Route Flow Variance (θ = 0.1)

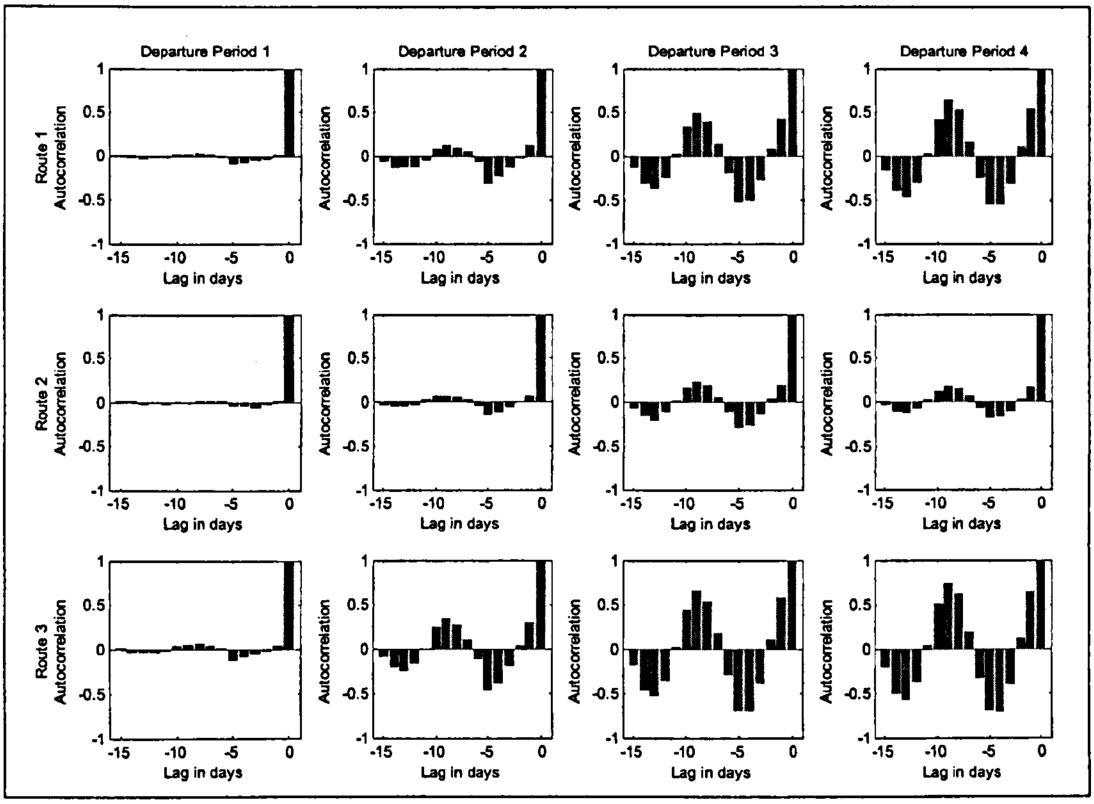

Figure 7.30 Autocorrelation of Route Flows, θ = 0.1

Finally, given the previous results with $\theta = 0.01$ and 0.1, the variance in route flows will be even higher with higher values of θ. The third leg of the analysis included a value of $\theta = 0.9$, and as expected the approximation variance grossly under estimates the simulation variance due to the deterministic periodic kind of shifting of flows from one route to the other arising due to the all-or-nothing type assignment. Just to confirm, the correlograms were drawn again (Figure 7.31) which indicated the drivers shifted in a group between routes 1 and 3 while route 2 is more expensive, and could not attract any significant proportion of the flow. These observations are consistent with the results obtained earlier with two link and five link networks. Through these workings, it is clear that there are three regimes – one where the system operates completely at random, second where the variance approximation method works well, and the final where the system operates like a deterministic periodic one. Besides, the variance approximation method can also be used to identify highly autocorrelated flow systems which are important to identify while simulating the traffic assignment process.

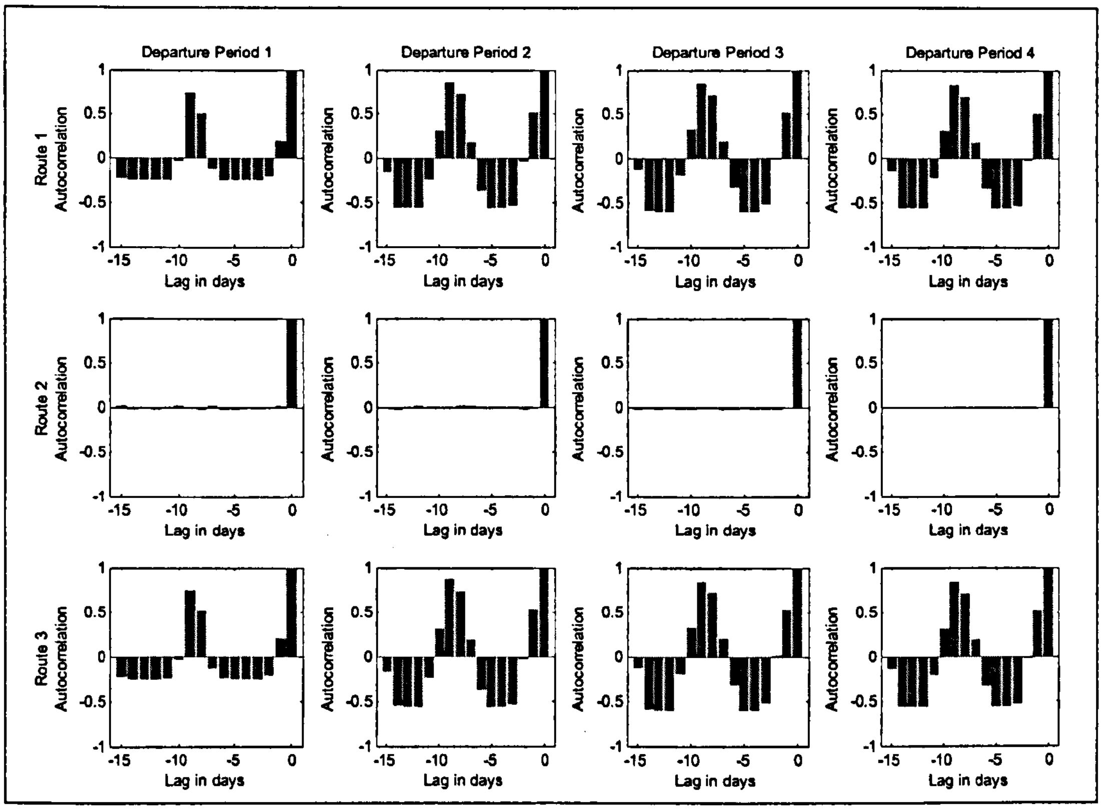

Figure 7.31 Autocorrelation of Route Flows, θ = 0.9

7.5 Computational Performance

This section compares the computational performance of the Variance Approximation Method over the simulation method in terms of the cpu time required for running each model for a given range of networks (Table 7.7). In this exercise three networks viz., two link parallel route network, five link network and 12-link grid network, which were described in sections 7.1, 7.2 and 7.4 were considered. The network and demand specifications were retained as earlier. The computing time has been based on 10000 iterations of MSA in the Variance Approximation Method, and 10000-day long realisation in the simulation method. A fast modern computer with Pentium D processor (3.00 GHz, 1GB RAM) has been used in all the runs. Although the computer programs are not optimised, it is expected that this exercise illustrates the relative time savings in using the Variance Approximation Method over the traditional simulation method. In this exercise the naïve method of computing the variance has not been included explicitly, as it needs approximately the same time as that of the Variance Approximation Method.

Table 7.7 Comparison of Computational Performance (Time in seconds)

Network	Variance Approximation Method	Simulation Method	Time Savings (%) over Simulation Method
Two link network	54.91	205.74	73.31
Five link network	240.62	655.57	63.30
Grid network	1308.49	2986.12	56.18

From Table 7.7 it can be observed that the Variance Approximation Method is much quicker than the simulation method and actually saves substantial computing time. It is to be noted that the Variance Approximation Method specified uses MSA for solving the DSUE, which may not be highly efficient and is known for its slow convergence. The performance of the Variance Approximation Method can be improved further by adopting an efficient algorithm for solving DSUE as in Han (2000). On the other hand, the simulation method will be even more time consuming for networks of practical significance. That is to say, the simulation time will be proportionately higher with the increase in the demand as simulating each marginal driver will incur additional computing time. Hence, the time savings could be even higher for networks of real world.

7.6 Summary

The overall framework of this research has explored alternative methods of carrying out dynamic traffic assignment using stochastic modelling procedures, in particular it has developed a methodology for estimating the properties of an equilibrium probability distribution as an alternative to the usual simulation procedure based on Monte Carlo methods. This chapter compared the stationary variance-covariance matrices of route flows obtained by the variance approximation method and the simulation method, and used a simple two link/route network and a five link network as illustrations. Overall, the variance approximation method performed satisfactorily over different sizes of networks. Variance and covariance of route flows are sensitive to the logit route choice input parameter, and the higher the parameter value, the lower the convergence with the simulation outputs. Outputs of variance approximation method are also sensitive to other model input parameters such as the memory length and the memory weight, however not as sensitive to the logit choice parameter which in some sense defined the boundaries of the utility of the variance approximation method. In general, shorter memory systems produce highly variant route traffic flows and smaller memory weight also produces a similar effect; on the other hand, longer memory systems result in low variance in route flows. This chapter also compared the computational performance of the Variance Approximation Method with the simulation method, and noted that the former method saves significant computing time over the latter method for the networks described earlier. The next and final chapter of this thesis summarises the overall research carried out, draws conclusions and identifies the issues for further research.

Chapter 8

Summary and Conclusions

8.1 Summary

The main focus of this research has been to explore the stochastic process models for carrying out dynamic traffic assignment, and aimed at specifying frameworks which are very general, addressing both day-to-day and within day variations in traffic flows, and even capturing individual drivers' adaptation of route choices based on experienced route costs. Stochastic processes consider route flows as random variables, and their evolution over a period of time (a number of days) is modelled based on probabilistic laws. Stochastic processes under certain mild conditions admit a unique stationary probability distribution which can be solved by using simulation techniques. Alternatively, the moments (e.g., mean and variance) of the equilibrium probability distribution can be estimated. This research has advanced the idea of estimating the properties of such an equilibrium probability distribution, with a particular contribution in formulating the methodology for computing the Jacobians of route travel times with respect to the route inflows in a doubly dynamic context using an analytical procedure, which are necessary for estimating the variance-covariance matrices of stationary route flows.

The overall framework of the models include three modules – (i) a day-to-day route choice model defined as a discrete time stochastic process, (ii) a continuous time dynamic network loading of the route flows considering the complete spatio-temporal effects of the traffic flows that use the road links at the same time, and (iii) the drivers' learning and adjusting model based on a linear filter. The main idea of estimating the properties of a stationary probability distribution in this research builds on two earlier results. The first is a convergence theorem that established, when the demand is sufficiently large, that the equilibrium probability distribution converges approximately to a multivariate Normal distribution and that its mean coincides with the SUE flows (Davis and Nihan 1993). The second result is that the variance can be estimated by an approximation procedure based on the principles set out by Hazelton and Watling (2004). This research also closely follows the results of Cantarella and Cascetta (1995) and unifies the deterministic and stochastic approaches of carrying out traffic assignment.

Alternatively, the equilibrium probability distribution can also be estimated using a relatively straightforward Monte Carlo simulation technique, which involves simulating the route choice process as a Multinomial probability distribution over a number of days and then summarising the properties e.g., the mean and the variance, of the stationary probability distribution. This procedure, though simple, is time consuming and the main difficulty lies in detecting the stationarity of the process. Practically useable tests have been introduced and implemented to test the stationarity of a stochastic process satisfying the necessary conditions.

The stationary variance-covariance of route flows obtained by the variance approximation model was compared with that computed by the simulation procedure. Overall, the variance approximation model performs satisfactorily. The variance-covariance of route flows has been found sensitive primarily to the input logit choice parameter, which defines the boundaries of the validity of the model. The variance-covariance is also affected by the memory length with the shorter memory models essentially producing highly variant systems. Similarly, the variance-covariance of route flows is also sensitive to the memory weight, and the lower memory weight ($0 < \lambda << 1$) produces the same effect as that of shorter memory systems.

8.2 Conclusions

The main conclusions from this research are listed below.

a. This research has been successful in bringing together the deterministic and the stochastic approaches of carrying out traffic assignment and established a procedure for a deterministic approximation of the stationary probability distribution of a stochastic process. By this method, transport modellers can now work out not only the average flows on the network, but also the variance in route and link flows, which have considerable significance in calibrating and validating the traffic models in practice.

b. Jacobians of the travel time functions with respect to the route inflows worked out in the variance approximation method have much wider applications, e.g., dynamic road pricing exercises, evaluating the cost of externalities. To elaborate further, currently road pricing exercises use either externally input gradients or gradients based on finite differencing as part of their optimising scheme, while solving for an optimal solution such as the system optimal. In such cases the quality of the solution directly depends on the quality of the gradients supplied. It is anticipated that the gradients based on analytical procedures such as the one described in this thesis result in high quality, less time consuming optimal solutions.

c. The uniqueness of this research is that it addresses a wide range of issues in traffic assignment including day-to-day dynamics, within day dynamics and drivers learning/adjusting their route choice, all within the framework of stochastic process models which are very general in nature with deterministic approaches as a special case. This approach enables the development of a comprehensive traffic assignment model in the future.

d. In this research, a continuous time dynamic loading process has been adopted for propagating the vehicles in various departure periods belonging to various O-D pairs. In order to be able to capture the interactions between the vehicles that use the links simultaneously, each departure period has been subdivided into several minor time steps. The shorter the minor time step the better the travel time estimate is. In the limit the minor time step should be just sufficient to accommodate one vehicle so that the travel time estimate is perfect. However, the computing time may increase substantially with reducing step sizes as the program needs to iterate through many more minor steps.

e. Detecting the stationarity of the stochastic process could be tricky. Correlograms can be used as a way of detecting the stationarity based on satisfying the necessary condition that the autocorrelations should fade out at some large lags which are called large lag standard errors. On the other hand, the variance approximation method completely avoids the need to go through the process of identifying the stationarity of a stochastic process, and in fact provides a direct method to work out the mean and variance of the equilibrium probability distribution.

f. The stationary variance and covariance of the route flows are sensitive to the input parameters such as the logit choice parameter θ, the memory length m, and the memory weight λ. Firstly, the variance approximation method works well at relatively lower values of θ, and at higher values of θ the system may behave like a deterministic periodic system at which the stationary probability distribution could be significantly different from a Normal distribution and hence the approximation by a Normal distribution may not be suitable. Secondly, the shorter memory systems produce highly variant traffic flows on the network. On the other hand, the longer memory systems will help achieving lightly variant route and link flows, however both cases are still stable as the stochastic process is regular. Finally, the lower memory weight also produces the same kind of effect as that of a shorter memory system by placing the highest weight on the most recent experience and almost ignoring the earlier experiences.

8.3 Further Research

There are various strands in this research which can be further investigated, as given below:

a. In this model, the O-D demand is assumed fixed within each departure period, which in reality keeps varying from day-to-day. To reflect this, the case of stochastic O-D demand may be considered which will need to model the demand as a random variable by assuming a suitable probability distribution function.

b. Another extension to the above case will be to consider the departure time choice of the drivers within a day. This means that the drivers will also have the departure period choice in addition to the route choice, and hence affecting the Jacobians of the route choice probabilities and travel times. Possibly the departure time choice can be nested into the modelling framework.

c. The present model may be extended to accommodate the case of multiple user classes. This can possibly be done by extending the definition of the commodity to include multiple user classes, which presently includes the O-D movement and departure period. However, computing the Jacobians, especially that of travel time can get quite complicated.

d. The driver learning model in this research has been based on a linear learning filter, but there is an excellent scope for looking at other learning models and revising the expressions for the approximation of variance – covariance of the route flows. In addition, the driver learning models can be extended to include the habituated drivers who will carry route specific perception error with them across the days.

e. The route choice models based on logit principle are not free from the independence of irrelevant alternative problem. That is to say, a very small insignificant deviation in route structure will identify that route as a potential route and will allocate a sizeable number of trips to it as well in addition to the main route ('red bus', 'blue bus' problem). One strategy to avoid this problem could be to follow a route choice model based on probit by assuming link-based Normal random error terms to make up the perceived cost.

f. To make the assignment model truly comprehensive, we need to consider extending the model to public transport assignment. As the public transport operates usually along predefined routes, the same can perhaps be treated as a fixed load.

g. Dynamic link travel times were estimated based on whole link type linear travel time functions. These models do not assume any specific physical length of the links and hence the spill back effects can not be modelled. Therefore, there is a wide scope for enhancing the link travel time models to accommodate the effects of spilling back.

h. Although there are no restrictions on the type of dynamic link model, the present approach uses linear link travel time functions due to their simplicity. Hence, there is a need to investigate the application of polynomial functions of higher order to model the link travel time and the delays at the junctions. Moreover, currently the delays at the junctions are only assumed to be a function of the number of vehicles on that link alone (the case of separable functions) and hence need to be extended to the case of non-separable functions by incorporating the flows from other approaching links into the delay functions.

i. The doubly dynamic simulation model analysed the necessary conditions to identify the stationary probability distribution. There is a need to extend this analysis further by identifying the sufficient conditions, as then one can identify the stationary probability distribution positively, which is a critical requirement in the overall analysis of stochastic processes.

j. Finally, further research can also explore the range of values of θ for which the approximation method produces results sufficiently close to the simulation. As an indicator, one can consider the range of values from 0.01 to 0.1 and perhaps slightly beyond, but can clearly avoid higher values of θ such as 1 which lead the system towards highly unstable and even unrealistic states as illustrated in section 7.1.

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Appendix A

Refereed Publications

- **Balijepalli, N.C.** and Watling, D.P. (2005) Doubly Dynamic Equilibrium Approximation Model for Dynamic Traffic Assignment, in *Transportation and Traffic Theory: Flow, Dynamics and Human Interaction,* (Ed. H.Mahmassani), Elsevier, Oxford, UK. Pp 741-760

- **Balijepalli, N.C.**, Watling, D.P. and Liu, R. (2006) Doubly Dynamic Traffic Assignment: Simulation Modelling Framework and Experimental Results, *Transportation Research Record* (Submitted)

Conference Presentations

- **Balijepalli, N.C.**, Watling, D.P. and Liu, R. (2007) Doubly Dynamic Traffic Assignment: Simulation Modelling Framework and Experimental Results, in the proceedings of *the 86th Annual Conference of Transportation Research Board,* Washington D.C. January 2007 (forthcoming)

- **Balijepalli, N.C.**, Watling, D.P. and Liu, R. (2006) Doubly Dynamic Simulation Model for Traffic Assignment, in the proceedings of the *First International Symposium on Dynamic Traffic Assignment,* University of Leeds, England

- **Balijepalli, N.C.** (2006) A Stochastic Process Model for Doubly Dynamic Traffic Assignment, in the proceedings of *the 38th Universities Transport Study Group Annual Conference*, UTSG, Dublin, Ireland

- **Balijepalli, N.C.** and Watling, D.P. (2005) Doubly Dynamic Equilibrium Approximation Model for Dynamic Traffic Assignment, in the proceedings of *the 16th International Symposium on Transportation and Traffic Theory (ISTTT16),* Maryland, USA

- **Balijepalli, N.C.** (2005) Doubly Dynamic Equilibrium Approximation Model for Dynamic Traffic Assignment, in the proceedings of *the 37th Universities Transport Study Group Annual Conference*, UTSG, Bristol, UK

- **Balijepalli, N.C.** (2004) Dynamics of Traffic Flow in Day-to-day and With-in Day Contexts, in the proceedings of *the International Workshop on Modelling Link Flows and Travel Times for Dynamic Traffic Assignment*, Queens University, Belfast, UK.

- **Balijepalli, N.C.** (2004) Stochastic Process Modelling Approach for Dynamic Traffic Assignment, in the proceedings of *the 36th Universities Transport Study Group Annual Conference*, UTSG, Newcastle Upon Tyne, UK

Appendix B

		Departure Period 1			Departure Period 2			Departure Period 3			Departure Period 4		
		Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
Departure Period 1	Route 1	90.1294	-41.4904	-48.639	0.1513	-0.0877	-0.0636	0.0142	-0.0089	-0.0053	0.0098	-0.0062	-0.0036
	Route 2	-41.4904	84.5665	-43.0761	-0.0813	0.2684	-0.1871	-0.0079	0.028	-0.0202	-0.0055	0.0199	-0.0143
	Route 3	-48.639	-43.0761	91.7152	-0.07	-0.1807	0.2507	-0.0063	-0.0191	0.0255	-0.0043	-0.0136	0.0179
Departure Period 2	Route 1	0.1513	-0.0813	-0.07	159.9672	-73.0423	-86.925	0.1028	-0.0472	-0.0556	0.0706	-0.0335	-0.0372
	Route 2	-0.0877	0.2684	-0.1807	-73.0423	147.0781	-74.0359	-0.0455	0.1692	-0.1238	-0.0319	0.1232	-0.0913
	Route 3	-0.0636	-0.1871	0.2507	-86.925	-74.0359	160.9609	-0.0573	-0.122	0.1793	-0.0388	-0.0897	0.1285
Departure Period 3	Route 1	0.0142	-0.0079	-0.0063	0.1028	-0.0455	-0.0573	22.6978	-10.4858	-12.212	0.0144	-0.006	-0.0084
	Route 2	-0.0089	0.028	-0.0191	-0.0472	0.1692	-0.122	-10.4858	21.0884	-10.6026	-0.0059	0.0248	-0.0189
	Route 3	-0.0053	-0.0202	0.0255	-0.0556	-0.1238	0.1793	-12.212	-10.6026	22.8146	-0.0085	-0.0187	0.0273
Departure Period 4	Route 1	0.0098	-0.0055	-0.0043	0.0706	-0.0319	-0.0388	0.0144	-0.0059	-0.0085	22.3822	-10.3752	-12.007
	Route 2	-0.0062	0.0199	-0.0136	-0.0335	0.1232	-0.0897	-0.006	0.0248	-0.0187	-10.3752	21.2955	-10.9203
	Route 3	-0.0036	-0.0143	0.0179	-0.0372	-0.0913	0.1285	-0.0084	-0.0189	0.0273	-12.007	-10.9203	22.9273

Approximated Variance - Covariance Matrix ($\theta = 0.01$)

		Departure Period 1			Departure Period 2			Departure Period 3			Departure Period 4		
		Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3	Route 1	Route 2	Route 3
Departure Period 1	Route 1	90.7238	-41.1306	-49.5933	0.0805	0.0643	-0.1448	-0.1627	0.3209	-0.1582	0.0383	-0.2946	0.2564
	Route 2	-41.1306	83.5549	-42.4243	0.9953	-0.3978	-0.5974	0.0921	-0.0258	-0.0663	0.2011	0.3433	-0.5445
	Route 3	-49.5933	-42.4243	92.0176	-1.0758	0.3335	0.7423	0.0707	-0.2951	0.2245	-0.2394	-0.0487	0.2881
Departure Period 2	Route 1	0.0805	0.9953	-1.0758	158.3957	-71.5481	-86.8476	-0.1806	-0.0462	0.2268	-0.0227	0.2556	-0.2329
	Route 2	0.0643	-0.3978	0.3335	-71.5481	144.8227	-73.2746	0.0969	0.0171	-0.114	0.2165	-0.2378	0.0213
	Route 3	-0.1448	-0.5974	0.7423	-86.8476	-73.2746	160.1222	0.0837	0.0291	-0.1127	-0.1939	-0.0178	0.2116
Departure Period 3	Route 1	-0.1627	0.0921	0.0707	-0.1806	0.0969	0.0837	22.6387	-10.3675	-12.2712	0.141	-0.1587	0.0176
	Route 2	0.3209	-0.0258	-0.2951	-0.0462	0.0171	0.0291	-10.3675	20.9876	-10.6201	-0.0078	-0.0438	0.0516
	Route 3	-0.1582	-0.0663	0.2245	0.2268	-0.114	-0.1127	-12.2712	-10.6201	22.8912	-0.1333	0.2024	-0.0692
Departure Period 4	Route 1	0.0383	0.2011	-0.2394	-0.0227	0.2165	-0.1939	0.141	-0.0078	-0.1333	22.302	-10.2089	-12.0931
	Route 2	-0.2946	0.3433	-0.0487	0.2556	-0.2378	-0.0178	-0.1587	-0.0438	0.2024	-10.2089	20.9206	-10.7117
	Route 3	0.2564	-0.5445	0.2881	-0.2329	0.0213	0.2116	0.0176	0.0516	-0.0692	-12.0931	-10.7117	22.8049

Simulated Variance - Covariance Matrix ($\theta = 0.01$)

Appendix C

C1. Conditional Covariance Matrix

Columns 1-12
Rows1-56

	1	2	3	4	5	6	7	8	9	10	11	12
1	88.094	-41.99	-46.104	0	0	0	0	0	0	0	0	0
2	-41.99	52.6906	-10.7006	0	0	0	0	0	0	0	0	0
3	-46.104	-10.7006	56.8045	0	0	0	0	0	0	0	0	0
4	0	0	0	24.9479	-24.9479	0	0	0	0	0	0	0
5	0	0	0	-24.9479	24.9479	0	0	0	0	0	0	0
6	0	0	0	0	0	56.125	-56.125	0	0	0	0	0
7	0	0	0	0	0	-56.125	56.125	0	0	0	0	0
8	0	0	0	0	0	0	0	30.2499	-30.2499	0	0	0
9	0	0	0	0	0	0	0	-30.2499	30.2499	0	0	0
10	0	0	0	0	0	0	0	0	0	41.2499	-41.2499	0
11	0	0	0	0	0	0	0	0	0	-41.2499	41.2499	0
12	0	0	0	0	0	0	0	0	0	0	0	78.3303
13	0	0	0	0	0	0	0	0	0	0	0	-39.2429
14	0	0	0	0	0	0	0	0	0	0	0	-39.0875
15	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0	0	0
49	0	0	0	0	0	0	0	0	0	0	0	0
50	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0

C1. Conditional Covariance Matrix

Columns 13-24
Rows1-56

	13	14	15	16	17	18	19	20	21	22	23	24
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	-39.2429	-39.0875	0	0	0	0	0	0	0	0	0	0
13	52.5842	-13.3414	0	0	0	0	0	0	0	0	0	0
14	-13.3414	52.4288	0	0	0	0	0	0	0	0	0	0
15	0	0	142.7806	-66.9024	-75.8782	0	0	0	0	0	0	0
16	0	0	-66.9024	81.1062	-14.2039	0	0	0	0	0	0	0
17	0	0	-75.8782	-14.2039	90.082	0	0	0	0	0	0	0
18	0	0	0	0	0	31.1293	-31.1293	0	0	0	0	0
19	0	0	0	0	0	-31.1293	31.1293	0	0	0	0	0
20	0	0	0	0	0	0	0	45.3242	-45.3242	0	0	0
21	0	0	0	0	0	0	0	-45.3242	45.3242	0	0	0
22	0	0	0	0	0	0	0	0	0	35.0004	-35.0004	0
23	0	0	0	0	0	0	0	0	0	-35.0004	35.0004	0
24	0	0	0	0	0	0	0	0	0	0	0	40.1421
25	0	0	0	0	0	0	0	0	0	0	0	-40.1421
26	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0	0	0
49	0	0	0	0	0	0	0	0	0	0	0	0
50	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0

C1. Conditional Covariance Matrix

Columns 25-36
Rows 1-56

	25	26	27	28	29	30	31	32	33	34	35	36
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0
24	-40.1421	0	0	0	0	0	0	0	0	0	0	0
25	40.1421	0	0	0	0	0	0	0	0	0	0	0
26	0	57.0702	-23.8793	-33.1909	0	0	0	0	0	0	0	0
27	0	-23.8793	30.1069	-6.2276	0	0	0	0	0	0	0	0
28	0	-33.1909	-6.2276	39.4185	0	0	0	0	0	0	0	0
29	0	0	0	0	97.606	-43.3853	-54.2206	0	0	0	0	0
30	0	0	0	0	-43.3853	52.1163	-8.731	0	0	0	0	0
31	0	0	0	0	-54.2206	-8.731	62.9516	0	0	0	0	0
32	0	0	0	0	0	0	0	38.2681	-38.2681	0	0	0
33	0	0	0	0	0	0	0	-38.2681	38.2681	0	0	0
34	0	0	0	0	0	0	0	0	0	35.0589	-35.0589	0
35	0	0	0	0	0	0	0	0	0	-35.0589	35.0589	0
36	0	0	0	0	0	0	0	0	0	0	0	47.073
37	0	0	0	0	0	0	0	0	0	0	0	-47.073
38	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0	0	0
49	0	0	0	0	0	0	0	0	0	0	0	0
50	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0

C1. Conditional Covariance Matrix

Columns 37-48
Rows1-56

	37	38	39	40	41	42	43	44	45	46	47	48
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0
36	-47.073	0	0	0	0	0	0	0	0	0	0	0
37	47.073	0	0	0	0	0	0	0	0	0	0	0
38	0	39.0364	-39.0364	0	0	0	0	0	0	0	0	0
39	0	-39.0364	39.0364	0	0	0	0	0	0	0	0	0
40	0	0	0	62.6749	-24.0835	-38.5914	0	0	0	0	0	0
41	0	0	0	-24.0835	29.5305	-5.447	0	0	0	0	0	0
42	0	0	0	-38.5914	-5.447	44.0385	0	0	0	0	0	0
43	0	0	0	0	0	0	52.4555	-25.1823	-27.2732	0	0	0
44	0	0	0	0	0	0	-25.1823	29.0025	-3.8202	0	0	0
45	0	0	0	0	0	0	-27.2732	-3.8202	31.0933	0	0	0
46	0	0	0	0	0	0	0	0	0	48.6685	-48.6685	0
47	0	0	0	0	0	0	0	0	0	-48.6685	48.6685	0
48	0	0	0	0	0	0	0	0	0	0	0	27.9535
49	0	0	0	0	0	0	0	0	0	0	0	-27.9535
50	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0

C1. Conditional Covariance Matrix

Columns 49-56
Rows1-56

	49	50	51	52	53	54	55	56
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0
48	-27.9535	0	0	0	0	0	0	0
49	27.9535	0	0	0	0	0	0	0
50	0	32.561	-32.561	0	0	0	0	0
51	0	-32.561	32.561	0	0	0	0	0
52	0	0	0	39.1327	-39.1327	0	0	0
53	0	0	0	-39.1327	39.1327	0	0	0
54	0	0	0	0	0	47.4025	-18.3404	-29.0621
55	0	0	0	0	0	-18.3404	22.0249	-3.6845
56	0	0	0	0	0	-29.0621	-3.6845	32.7466

C2. Jacobian Matrix of Route Choice Probability Function

Columns 1-12
Rows1-56

	1	2	3	4	5	6	7	8	9	10	11	12
1	-0.022	0.0105	0.0115	0	0	0	0	0	0	0	0	0
2	0.0105	-0.0132	0.0027	0	0	0	0	0	0	0	0	0
3	0.0115	0.0027	-0.0142	0	0	0	0	0	0	0	0	0
4	0	0	0	-0.0249	0.0249	0	0	0	0	0	0	0
5	0	0	0	0.0249	-0.0249	0	0	0	0	0	0	0
6	0	0	0	0	0	-0.0249	0.0249	0	0	0	0	0
7	0	0	0	0	0	0.0249	-0.0249	0	0	0	0	0
8	0	0	0	0	0	0	0	-0.025	0.025	0	0	0
9	0	0	0	0	0	0	0	0.025	-0.025	0	0	0
10	0	0	0	0	0	0	0	0	0	-0.025	0.025	0
11	0	0	0	0	0	0	0	0	0	0.025	-0.025	0
12	0	0	0	0	0	0	0	0	0	0	0	-0.0241
13	0	0	0	0	0	0	0	0	0	0	0	0.0121
14	0	0	0	0	0	0	0	0	0	0	0	0.012
15	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0	0	0
49	0	0	0	0	0	0	0	0	0	0	0	0
50	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0

C2. Jacobian Matrix of Route Choice Probability Function

Columns 13-24
Rows1-56

	13	14	15	16	17	18	19	20	21	22	23	24
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0.0121	0.012	0	0	0	0	0	0	0	0	0	0
13	-0.0162	0.0041	0	0	0	0	0	0	0	0	0	0
14	0.0041	-0.0161	0	0	0	0	0	0	0	0	0	0
15	0	0	-0.0204	0.0096	0.0108	0	0	0	0	0	0	0
16	0	0	0.0096	-0.0116	0.002	0	0	0	0	0	0	0
17	0	0	0.0108	0.002	-0.0129	0	0	0	0	0	0	0
18	0	0	0	0	0	-0.0249	0.0249	0	0	0	0	0
19	0	0	0	0	0	0.0249	-0.0249	0	0	0	0	0
20	0	0	0	0	0	0	0	-0.0249	0.0249	0	0	0
21	0	0	0	0	0	0	0	0.0249	-0.0249	0	0	0
22	0	0	0	0	0	0	0	0	0	-0.0243	0.0243	0
23	0	0	0	0	0	0	0	0	0	0.0243	-0.0243	0
24	0	0	0	0	0	0	0	0	0	0	0	-0.0243
25	0	0	0	0	0	0	0	0	0	0	0	0.0243
26	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0	0	0
49	0	0	0	0	0	0	0	0	0	0	0	0
50	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0

C2. Jacobian Matrix of Route Choice Probability Function

Columns 25-36
Rows1-56

	25	26	27	28	29	30	31	32	33	34	35	36
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0
24	0.0243	0	0	0	0	0	0	0	0	0	0	0
25	-0.0243	0	0	0	0	0	0	0	0	0	0	0
26	0	-0.0214	0.0089	0.0124	0	0	0	0	0	0	0	0
27	0	0.0089	-0.0113	0.0023	0	0	0	0	0	0	0	0
28	0	0.0124	0.0023	-0.0148	0	0	0	0	0	0	0	0
29	0	0	0	0	-0.0195	0.0087	0.0108	0	0	0	0	0
30	0	0	0	0	0.0087	-0.0104	0.0017	0	0	0	0	0
31	0	0	0	0	0.0108	0.0017	-0.0126	0	0	0	0	0
32	0	0	0	0	0	0	0	-0.0247	0.0247	0	0	0
33	0	0	0	0	0	0	0	0.0247	-0.0247	0	0	0
34	0	0	0	0	0	0	0	0	0	-0.0247	0.0247	0
35	0	0	0	0	0	0	0	0	0	0.0247	-0.0247	0
36	0	0	0	0	0	0	0	0	0	0	0	-0.0237
37	0	0	0	0	0	0	0	0	0	0	0	0.0237
38	0	0	0	0	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0	0	0	0	0
48	0	0	0	0	0	0	0	0	0	0	0	0
49	0	0	0	0	0	0	0	0	0	0	0	0
50	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0

C2. Jacobian Matrix of Route Choice Probability Function

Columns 37-48
Rows1-56

	37	38	39	40	41	42	43	44	45	46	47	48
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0	0	0	0	0
36	0.0237	0	0	0	0	0	0	0	0	0	0	0
37	-0.0237	0	0	0	0	0	0	0	0	0	0	0
38	0	-0.0237	0.0237	0	0	0	0	0	0	0	0	0
39	0	0.0237	-0.0237	0	0	0	0	0	0	0	0	0
40	0	0	0	-0.0196	0.0075	0.0121	0	0	0	0	0	0
41	0	0	0	0.0075	-0.0093	0.0017	0	0	0	0	0	0
42	0	0	0	0.0121	0.0017	-0.0138	0	0	0	0	0	0
43	0	0	0	0	0	0	-0.022	0.0105	0.0115	0	0	0
44	0	0	0	0	0	0	0.0105	-0.0132	0.0027	0	0	0
45	0	0	0	0	0	0	0.0115	0.0027	-0.0142	0	0	0
46	0	0	0	0	0	0	0	0	0	-0.0249	0.0249	0
47	0	0	0	0	0	0	0	0	0	0.0249	-0.0249	0
48	0	0	0	0	0	0	0	0	0	0	0	-0.0249
49	0	0	0	0	0	0	0	0	0	0	0	0.0249
50	0	0	0	0	0	0	0	0	0	0	0	0
51	0	0	0	0	0	0	0	0	0	0	0	0
52	0	0	0	0	0	0	0	0	0	0	0	0
53	0	0	0	0	0	0	0	0	0	0	0	0
54	0	0	0	0	0	0	0	0	0	0	0	0
55	0	0	0	0	0	0	0	0	0	0	0	0
56	0	0	0	0	0	0	0	0	0	0	0	0

C2. Jacobian Matrix of Route Choice Probability Function

Columns 49-56
Rows1-56

	49	50	51	52	53	54	55	56
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0
30	0	0	0	0	0	0	0	0
31	0	0	0	0	0	0	0	0
32	0	0	0	0	0	0	0	0
33	0	0	0	0	0	0	0	0
34	0	0	0	0	0	0	0	0
35	0	0	0	0	0	0	0	0
36	0	0	0	0	0	0	0	0
37	0	0	0	0	0	0	0	0
38	0	0	0	0	0	0	0	0
39	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0
41	0	0	0	0	0	0	0	0
42	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0
44	0	0	0	0	0	0	0	0
45	0	0	0	0	0	0	0	0
46	0	0	0	0	0	0	0	0
47	0	0	0	0	0	0	0	0
48	0.0249	0	0	0	0	0	0	0
49	-0.0249	0	0	0	0	0	0	0
50	0	-0.025	0.025	0	0	0	0	0
51	0	0.025	-0.025	0	0	0	0	0
52	0	0	0	-0.025	0.025	0	0	0
53	0	0	0	0.025	-0.025	0	0	0
54	0	0	0	0	0	-0.0241	0.0121	0.012
55	0	0	0	0	0	0.0121	-0.0162	0.0041
56	0	0	0	0	0	0.012	0.0041	-0.0161

C3. Jacobian Matrix of Route Travel Time

Columns 1-12
Rows1-56

	1	2	3	4	5	6	7	8	9	10	11	12
1	0.016	0.0109	0	0.0109	0	0.0109	0	0	0	0	0	0
2	0.0119	0.0244	0.0172	0.0221	0.0154	0.0221	0.0154	0.018	0.0063	0.0161	0.0039	-0.0018
3	-0.0012	0.0058	0.0479	0.0036	0.0462	0.0036	0.0462	0.0405	0.0059	0.0387	0.0037	-0.0017
4	0.0117	0.022	0.0146	0.022	0.0146	0.022	0.0146	0.0153	0.0042	0.0153	0.0042	-0.0016
5	-0.0011	0.0039	0.0448	0.0039	0.0448	0.0039	0.0448	0.0375	0.004	0.0375	0.004	-0.0015
6	0.0117	0.0218	0.0156	0.0218	0.0156	0.0239	0.0172	0.0164	0.004	0.0164	0.004	-0.0017
7	-0.0013	0.0035	0.0462	0.0035	0.0462	0.0055	0.0477	0.0388	0.0038	0.0388	0.0038	-0.0016
8	-0.0011	0.0051	0.0326	0.0031	0.0311	0.0031	0.0311	0.0461	0.0055	0.0445	0.0033	-0.0015
9	-0.0011	0.005	0.0132	0.003	0.0117	0.003	0.0117	0.0139	0.0257	0.0122	0.0234	0.0135
10	-0.001	0.0033	0.0301	0.0033	0.0301	0.0033	0.0301	0.0432	0.0036	0.0432	0.0036	-0.0013
11	-0.001	0.0033	0.011	0.0033	0.011	0.0033	0.011	0.0115	0.0234	0.0115	0.0234	0.0134
12	0	0	0	0	0	0	0	0	0.0118	0	0.0118	0.0209
13	-0.0011	0.003	0.0312	0.003	0.0312	0.0048	0.0326	0.0446	0.0033	0.0446	0.0033	-0.0014
14	-0.0011	0.0029	0.0119	0.0029	0.0119	0.0047	0.0133	0.0124	0.0234	0.0124	0.0234	0.0135
15	0.0187	0.0173	0	0.0173	0	0.0173	0	0	0	0	0	0
16	0.0181	0.0251	0.0552	0.0248	0.0548	0.0248	0.0548	0.0579	0.0044	0.0577	0.0042	-0.0007
17	-0.0007	0.0059	0.0935	0.0055	0.0931	0.0055	0.0931	0.0948	0.0045	0.0946	0.0044	-0.0008
18	0.0179	0.0242	0.051	0.0242	0.051	0.0242	0.051	0.0536	0.004	0.0536	0.004	-0.0006
19	-0.0006	0.0053	0.0887	0.0053	0.0887	0.0053	0.0887	0.09	0.0042	0.09	0.0042	-0.0007
20	0.0178	0.0244	0.0537	0.0244	0.0537	0.0246	0.054	0.0565	0.0042	0.0565	0.0042	-0.0006
21	-0.0006	0.0056	0.0912	0.0056	0.0912	0.0059	0.0915	0.0926	0.0043	0.0927	0.0043	-0.0007
22	-0.0009	0.0066	0.0883	0.0059	0.0877	0.0059	0.0877	0.0943	0.0049	0.0939	0.0047	-0.0011
23	-0.0013	0.0073	0.0465	0.0063	0.0456	0.0063	0.0456	0.0494	0.0274	0.0488	0.0269	0.0194
24	-0.0008	0.0058	0.0834	0.0058	0.0835	0.0058	0.0835	0.0893	0.0045	0.0893	0.0045	-0.001
25	-0.0011	0.0062	0.0425	0.0062	0.0425	0.0062	0.0425	0.0454	0.0263	0.0454	0.0263	0.0191
26	0	0	0	0	0	0	0	0	0.0183	0	0.0183	0.0225
27	-0.0008	0.006	0.0859	0.006	0.0859	0.0065	0.0865	0.0919	0.0047	0.0919	0.0047	-0.001
28	-0.0011	0.0064	0.0452	0.0064	0.0452	0.0073	0.046	0.0482	0.0264	0.0482	0.0264	0.0189
29	0.0142	0.0141	0	0.0141	0	0.0141	0	0	0	0	0	0
30	0.0146	0.0169	0.0679	0.0169	0.0679	0.0169	0.068	0.0692	0.0024	0.0692	0.0024	0.0006
31	0	0.0025	0.0985	0.0025	0.0985	0.0025	0.0985	0.0997	0.0026	0.0997	0.0026	0.0006
32	0.0147	0.0168	0.064	0.0168	0.064	0.0168	0.064	0.0651	0.0023	0.0652	0.0023	0.0005
33	-0.0001	0.0023	0.0945	0.0023	0.0945	0.0023	0.0945	0.0956	0.0024	0.0957	0.0024	0.0005
34	0.0147	0.0169	0.0661	0.0169	0.0661	0.0169	0.0661	0.0673	0.0023	0.0673	0.0023	0.0005
35	0	0.0024	0.0967	0.0024	0.0967	0.0024	0.0967	0.0979	0.0024	0.0979	0.0024	0.0005
36	0	0.0027	0.0983	0.0027	0.0983	0.0027	0.0983	0.0997	0.0027	0.0998	0.0027	0.0005
37	0	0.0033	0.0659	0.0033	0.066	0.0033	0.066	0.0676	0.0159	0.0676	0.0159	0.0132
38	0	0.0025	0.0942	0.0025	0.0943	0.0025	0.0943	0.0956	0.0025	0.0956	0.0025	0.0004
39	0	0.003	0.0617	0.003	0.0617	0.003	0.0617	0.0633	0.0156	0.0633	0.0157	0.0132
40	0	0	0	0	0	0	0	0	0.0124	0	0.0124	0.0126
41	0	0.0026	0.0966	0.0026	0.0966	0.0026	0.0966	0.098	0.0026	0.098	0.0026	0.0005
42	0	0.0032	0.0642	0.0032	0.0642	0.0032	0.0642	0.0658	0.0157	0.0658	0.0157	0.0132
43	0.0115	0.0115	0	0.0115	0	0.0115	0	0	0	0	0	0
44	0.0104	0.0117	0.0552	0.0117	0.0552	0.0117	0.0553	0.0557	0.0027	0.0557	0.0027	0.0017
45	0.0002	0.0015	0.0811	0.0015	0.0811	0.0015	0.0812	0.0815	0.0026	0.0815	0.0026	0.0016
46	0.0106	0.0119	0.053	0.0119	0.053	0.0119	0.053	0.0535	0.0025	0.0535	0.0025	0.0015
47	0.0001	0.0014	0.0797	0.0014	0.0797	0.0014	0.0797	0.0801	0.0025	0.0801	0.0025	0.0014
48	0.0106	0.0119	0.0543	0.0119	0.0543	0.0119	0.0543	0.0548	0.0026	0.0548	0.0026	0.0016
49	0.0001	0.0014	0.0808	0.0014	0.0809	0.0014	0.0809	0.0812	0.0025	0.0812	0.0025	0.0015
50	0.0002	0.0016	0.0846	0.0016	0.0846	0.0016	0.0846	0.0849	0.0025	0.0849	0.0025	0.0014
51	0.0002	0.0018	0.0634	0.0018	0.0634	0.0018	0.0635	0.0643	0.0118	0.0643	0.0118	0.0105
52	0.0001	0.0014	0.0826	0.0014	0.0826	0.0014	0.0826	0.0829	0.0024	0.0829	0.0024	0.0013
53	0.0001	0.0016	0.0604	0.0016	0.0604	0.0016	0.0604	0.0612	0.0118	0.0612	0.0118	0.0106
54	0	0	0	0	0	0	0	0	0.0096	0	0.0096	0.0096
55	0.0001	0.0015	0.084	0.0015	0.084	0.0015	0.084	0.0843	0.0024	0.0843	0.0024	0.0013
56	0.0001	0.0017	0.062	0.0017	0.062	0.0017	0.062	0.0629	0.0119	0.0629	0.0119	0.0106

C3. Jacobian Matrix of Route Travel Time

Columns 13-24
Rows1-56

	13	14	15	16	17	18	19	20	21	22	23	24
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0.0161	0.0039	0	0	0	0	0	0	0	0	0.0003	0
3	0.0387	0.0037	0	0	0	0	0	0	0	0.0003	0.0003	0.0002
4	0.0153	0.0042	0	0	0	0	0	0	0	0	0.0001	0
5	0.0375	0.004	0	0	0	0	0	0	0	0.0002	0.0002	0.0002
6	0.018	0.0061	0	0	0	0	0	0	0	0	0.0001	0
7	0.0404	0.0058	0	0	0	0	0	0	0	0.0002	0.0001	0.0002
8	0.0445	0.0033	0	0	0	0	0	0	0	0	0.0001	0
9	0.0122	0.0234	0	0	0	0	0	0	0	0	0	0
10	0.0432	0.0036	0	0	0	0	0	0	0	0	0.0001	0
11	0.0115	0.0234	0	0	0	0	0	0	0	0	0	0
12	0	0.0118	0	0	0	0	0	0	0	0	0	0
13	0.0461	0.0053	0	0	0	0	0	0	0	0	0	0
14	0.0139	0.0255	0	0	0	0	0	0	0	0	0	0
15	0	0	0.0171	0.0104	0	0.0104	0	0.0104	0	0	0	0
16	0.0577	0.0042	0.0119	0.0249	0.0166	0.0223	0.014	0.0223	0.014	0.0225	0.0076	0.0198
17	0.0946	0.0044	-0.0016	0.0065	0.0419	0.0039	0.0392	0.0039	0.0392	0.0449	0.0078	0.042
18	0.0536	0.004	0.0119	0.0227	0.0138	0.0227	0.0138	0.0227	0.0138	0.0192	0.0055	0.0192
19	0.09	0.0042	-0.0014	0.0045	0.0389	0.0045	0.0389	0.0045	0.0389	0.0413	0.0055	0.0413
20	0.0566	0.0043	0.0117	0.0223	0.0139	0.0223	0.0139	0.0244	0.0161	0.0196	0.0052	0.0196
21	0.0928	0.0044	-0.0015	0.0041	0.0389	0.0041	0.0389	0.0062	0.0411	0.0416	0.0051	0.0416
22	0.0939	0.0047	-0.0013	0.0054	0.0256	0.0032	0.0232	0.0032	0.0232	0.0446	0.0073	0.0419
23	0.0488	0.0269	-0.0009	0.0041	0.0061	0.0022	0.0042	0.0022	0.0042	0.0095	0.0278	0.0072
24	0.0893	0.0045	-0.0012	0.0037	0.0232	0.0037	0.0232	0.0037	0.0232	0.0414	0.0049	0.0414
25	0.0454	0.0264	-0.0009	0.0027	0.0045	0.0027	0.0045	0.0027	0.0045	0.0073	0.0251	0.0073
26	0	0.0183	0	0	0	0	0	0	0	0	0.0121	0
27	0.0922	0.0049	-0.0013	0.0033	0.0231	0.0033	0.0231	0.0052	0.0251	0.0414	0.0045	0.0414
28	0.0488	0.0268	-0.0009	0.0023	0.0043	0.0023	0.0043	0.004	0.0059	0.0072	0.0245	0.0072
29	0	0	0.0255	0.0217	0	0.0217	0	0.0217	0	0	0	0
30	0.0692	0.0024	0.0222	0.0324	0.0548	0.0317	0.0544	0.0317	0.0544	0.0613	0.0054	0.0611
31	0.0997	0.0026	-0.0023	0.0079	0.094	0.0068	0.0933	0.0068	0.0933	0.0984	0.0057	0.098
32	0.0652	0.0023	0.0224	0.0317	0.0514	0.0317	0.0514	0.0317	0.0514	0.0576	0.005	0.0576
33	0.0957	0.0024	-0.0022	0.0067	0.0906	0.0067	0.0906	0.0067	0.0906	0.0948	0.0052	0.0948
34	0.0673	0.0023	0.0223	0.0318	0.0532	0.0318	0.0532	0.0321	0.0533	0.0596	0.0051	0.0596
35	0.0979	0.0025	-0.0023	0.0067	0.0923	0.0068	0.0923	0.0074	0.0927	0.0968	0.0054	0.0968
36	0.0998	0.0027	-0.0024	0.0083	0.0877	0.0069	0.0866	0.0069	0.0866	0.0968	0.006	0.0962
37	0.0677	0.0159	-0.0025	0.009	0.0405	0.0069	0.0387	0.0069	0.0387	0.0486	0.0336	0.0472
38	0.0956	0.0025	-0.0023	0.0069	0.0842	0.0069	0.0842	0.0069	0.0842	0.0931	0.0055	0.0931
39	0.0633	0.0157	-0.0024	0.0071	0.0368	0.0071	0.0368	0.0071	0.0368	0.0446	0.0326	0.0446
40	0	0.0124	0	0	0	0	0	0	0	0	0.0229	0
41	0.098	0.0026	-0.0024	0.0069	0.0858	0.0069	0.0858	0.0078	0.0864	0.095	0.0056	0.095
42	0.0658	0.0158	-0.0025	0.007	0.0381	0.007	0.0381	0.0084	0.0393	0.0463	0.0327	0.0463
43	0	0	0.0206	0.0201	0	0.0201	0	0.0201	0	0	0	0
44	0.0558	0.0027	0.0175	0.0227	0.0528	0.0227	0.0528	0.0227	0.0528	0.056	0.0057	0.056
45	0.0815	0.0026	-0.0012	0.0042	0.0867	0.0042	0.0867	0.0042	0.0867	0.0889	0.0055	0.0889
46	0.0535	0.0025	0.0179	0.0229	0.0507	0.023	0.0507	0.023	0.0507	0.0537	0.0053	0.0538
47	0.0801	0.0025	-0.0013	0.0039	0.0855	0.0039	0.0855	0.0039	0.0855	0.0877	0.0052	0.0877
48	0.0548	0.0026	0.0179	0.023	0.0519	0.023	0.0519	0.023	0.0519	0.055	0.0054	0.055
49	0.0812	0.0025	-0.0013	0.004	0.0866	0.004	0.0866	0.004	0.0866	0.0888	0.0053	0.0888
50	0.085	0.0025	-0.0013	0.0044	0.0899	0.0044	0.0899	0.0044	0.0899	0.0924	0.0053	0.0924
51	0.0643	0.0118	-0.0015	0.0051	0.0589	0.005	0.0589	0.005	0.0589	0.0632	0.0243	0.0632
52	0.083	0.0024	-0.0014	0.004	0.0882	0.004	0.0882	0.004	0.0882	0.0906	0.005	0.0906
53	0.0612	0.0118	-0.0016	0.0047	0.056	0.0047	0.056	0.0047	0.0561	0.0601	0.0243	0.0601
54	0	0.0096	0	0	0	0	0	0	0	0	0.0193	0
55	0.0843	0.0024	-0.0014	0.0042	0.0896	0.0042	0.0896	0.0042	0.0896	0.092	0.0051	0.092
56	0.0629	0.0119	-0.0016	0.0048	0.0576	0.0048	0.0576	0.0048	0.0576	0.0618	0.0243	0.0618

C3. Jacobian Matrix of Route Travel Time

Columns 25-36
Rows1-56

	25	26	27	28	29	30	31	32	33	34	35	36
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0.0001	-0.0001	0	0.0001	0	0	0	0	0	0	0	0
3	0.0001	-0.0001	0.0002	0.0001	0	0	0	0	0	0	0	0
4	0.0001	-0.0001	0	0.0001	0	0	0	0	0	0	0	0
5	0.0002	-0.0001	0.0002	0.0002	0	0	0	0	0	0	0	0
6	0.0001	-0.0001	0	0.0003	0	0	0	0	0	0	0	0
7	0.0001	-0.0001	0.0003	0.0003	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0.0001	0	0	0.0001	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0.0001	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0
16	0.0048	-0.0034	0.0198	0.0048	0	0	0.0002	0	0	0	0	0.0005
17	0.0047	-0.0035	0.042	0.0047	0	0	0	0	0	0	0	0.0004
18	0.0055	-0.0031	0.0192	0.0055	0	0	0.0001	0	0.0001	0	0.0001	0.0002
19	0.0055	-0.0032	0.0413	0.0055	0	0	0	0	0	0	0	0.0004
20	0.0052	-0.0031	0.0218	0.0074	0	0	0	0	0	0	0.0002	0.0002
21	0.0051	-0.0032	0.0439	0.0076	0	0	0	0	0	0	0	0.0003
22	0.0041	-0.0032	0.0419	0.0041	0	0	0	0	0	0	0	0
23	0.0246	0.0123	0.0072	0.0246	0	0	0	0	0	0	0	0
24	0.0049	-0.0029	0.0414	0.0049	0	0	0	0	0	0	0	0
25	0.0251	0.0123	0.0073	0.0251	0	0	0	0	0	0	0	0
26	0.0121	0.0236	0	0.0121	0	0	0	0	0	0	0	0
27	0.0045	-0.0029	0.0437	0.0071	0	0	0	0	0	0	0	0
28	0.0245	0.0122	0.0092	0.0272	0	0	0	0	0	0	0	0
29	0	0	0	0	0.0176	0.009	0	0.009	0	0.009	0	0
30	0.0053	-0.0014	0.0611	0.0053	0.0105	0.0265	0.0146	0.0231	0.0112	0.0231	0.0112	0.0193
31	0.0056	-0.0017	0.098	0.0056	-0.0017	0.0065	0.0382	0.0037	0.0351	0.0037	0.0351	0.0398
32	0.005	-0.0014	0.0576	0.005	0.0107	0.0241	0.0116	0.0241	0.0116	0.0241	0.0116	0.016
33	0.0052	-0.0017	0.0948	0.0052	-0.0016	0.0044	0.0357	0.0044	0.0357	0.0044	0.0357	0.0369
34	0.0051	-0.0014	0.0597	0.0051	0.0106	0.0235	0.0114	0.0235	0.0114	0.0263	0.0141	0.016
35	0.0054	-0.0017	0.0969	0.0054	-0.0017	0.0039	0.0354	0.0039	0.0354	0.0062	0.0379	0.0368
36	0.0058	-0.0021	0.0962	0.0058	-0.0013	0.0053	0.0242	0.0029	0.0215	0.0029	0.0215	0.0402
37	0.0328	0.0224	0.0472	0.0326	-0.0007	0.0029	0.0044	0.0014	0.0026	0.0014	0.0026	0.0067
38	0.0055	-0.002	0.0931	0.0055	-0.0013	0.0035	0.022	0.0035	0.022	0.0035	0.022	0.0377
39	0.0326	0.0225	0.0446	0.0324	-0.0007	0.0018	0.003	0.0018	0.003	0.0018	0.003	0.005
40	0.0229	0.0281	0	0.0229	0	0	0	0	0	0	0	0
41	0.0056	-0.002	0.0954	0.0057	-0.0014	0.0031	0.0218	0.0031	0.0218	0.0051	0.0239	0.0376
42	0.0327	0.0224	0.0472	0.033	-0.0007	0.0015	0.0027	0.0015	0.0027	0.0028	0.0043	0.0048
43	0	0	0	0	0.0277	0.0216	0	0.0216	0	0.0216	0	0
44	0.0057	0.0018	0.056	0.0057	0.0191	0.0339	0.0247	0.0327	0.024	0.0327	0.024	0.0295
45	0.0055	0.0016	0.0889	0.0055	-0.0037	0.0096	0.0635	0.0082	0.0625	0.0082	0.0625	0.067
46	0.0053	0.0016	0.0538	0.0053	0.0197	0.0333	0.0232	0.0333	0.0232	0.0333	0.0232	0.0279
47	0.0052	0.0014	0.0877	0.0052	-0.0038	0.0082	0.063	0.0082	0.063	0.0082	0.063	0.0665
48	0.0054	0.0016	0.0551	0.0054	0.0196	0.0333	0.0237	0.0333	0.0237	0.0339	0.0241	0.0286
49	0.0053	0.0014	0.0888	0.0053	-0.0038	0.0081	0.0633	0.0081	0.0633	0.009	0.0639	0.0669
50	0.0053	0.0012	0.0924	0.0053	-0.0038	0.0101	0.0633	0.0082	0.0619	0.0082	0.0619	0.0699
51	0.0243	0.0194	0.0632	0.0243	-0.0036	0.011	0.0249	0.008	0.0225	0.008	0.0225	0.0303
52	0.005	0.001	0.0906	0.005	-0.0039	0.0083	0.0621	0.0083	0.0621	0.0083	0.0621	0.0688
53	0.0243	0.0197	0.0601	0.0243	-0.0036	0.0084	0.0221	0.0084	0.0221	0.0084	0.0221	0.0275
54	0.0193	0.0197	0	0.0193	0	0	0	0	0	0	0	0
55	0.0051	0.001	0.092	0.0051	-0.004	0.0082	0.0626	0.0082	0.0626	0.0094	0.0634	0.0694
56	0.0243	0.0196	0.0618	0.0243	-0.0038	0.0081	0.0224	0.0081	0.0224	0.0102	0.0241	0.028

C3. Jacobian Matrix of Route Travel Time

Columns 37-48
Rows1-56

	37	38	39	40	41	42	43	44	45	46	47	48
1	0	0	0	0	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0	0	0	0	0
16	0.0014	0.0001	0.0004	-0.0006	0.0001	0.0004	0	0	0	0	0	0
17	0.0009	0.0003	0.0002	-0.0003	0.0003	0.0002	0	0	0	0	0	0
18	0.0007	0.0002	0.0007	-0.0005	0.0002	0.0007	0	0	0	0	0	0
19	0.0004	0.0004	0.0004	-0.0003	0.0004	0.0004	0	0	0	0	0	0
20	0.0006	0.0002	0.0006	-0.0005	0.0005	0.0014	0	0	0	0	0	0
21	0.0003	0.0003	0.0003	-0.0003	0.0004	0.0009	0	0	0	0	0	0
22	0.0004	0	0.0001	-0.0002	0	0.0001	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0	0	0	0	0
24	0.0002	0	0.0002	-0.0001	0	0.0002	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0	0	0	0	0
27	0.0001	0	0.0001	-0.0001	0	0.0004	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0	0	0	0	0
30	0.0084	0.0158	0.0054	-0.0043	0.0158	0.0054	0	0	0.0001	0	0	0
31	0.008	0.0365	0.0048	-0.004	0.0365	0.0048	0	0	0	0	0	0
32	0.0061	0.016	0.0061	-0.0041	0.016	0.0061	0	0	0	0	0	0
33	0.0057	0.0369	0.0057	-0.0038	0.0369	0.0057	0	0	0	0	0	0
34	0.0058	0.016	0.0058	-0.0041	0.0186	0.008	0	0	0	0	0	0
35	0.0053	0.0368	0.0053	-0.0038	0.0394	0.0077	0	0	0	0	0	0
36	0.0076	0.0372	0.0042	-0.0037	0.0372	0.0042	0	0	0	0	0	0
37	0.0262	0.0045	0.0231	0.0113	0.0045	0.0231	0	0	0	0	0	0
38	0.0051	0.0377	0.0051	-0.0035	0.0377	0.0051	0	0	0	0	0	0
39	0.0241	0.005	0.0241	0.0116	0.005	0.0241	0	0	0	0	0	0
40	0.0118	0	0.0118	0.0232	0	0.0118	0	0	0	0	0	0
41	0.0048	0.0376	0.0048	-0.0035	0.04	0.0073	0	0	0	0	0	0
42	0.0236	0.0048	0.0236	0.0115	0.0066	0.0262	0	0	0	0	0	0
43	0	0	0	0	0	0	0.0182	0.0089	0	0.0089	0	0.0089
44	0.009	0.029	0.0088	-0.0002	0.029	0.0088	0.0077	0.0238	0.0051	0.0198	0.001	0.0198
45	0.0088	0.0664	0.0086	-0.0006	0.0664	0.0086	-0.0019	0.0071	0.0259	0.0035	0.0221	0.0035
46	0.0083	0.0279	0.0083	-0.0004	0.0279	0.0083	0.008	0.0211	0.0019	0.0211	0.0019	0.0211
47	0.0082	0.0665	0.0082	-0.0008	0.0665	0.0082	-0.0019	0.0041	0.0237	0.0041	0.0237	0.0041
48	0.0084	0.0286	0.0084	-0.0005	0.0287	0.0085	0.0079	0.0203	0.0011	0.0203	0.0011	0.0238
49	0.0083	0.0669	0.0083	-0.0008	0.0672	0.0083	-0.002	0.0035	0.0228	0.0035	0.0228	0.0066
50	0.0086	0.069	0.0083	-0.0013	0.069	0.0083	-0.0016	0.0058	0.0176	0.0027	0.0144	0.0027
51	0.0371	0.0283	0.0361	0.0239	0.0283	0.0362	-0.0007	0.0027	0.0019	0.0011	-0.0001	0.0011
52	0.0079	0.0688	0.0079	-0.0014	0.0688	0.0079	-0.0016	0.0033	0.0155	0.0033	0.0155	0.0033
53	0.0365	0.0275	0.0365	0.0244	0.0275	0.0365	-0.0007	0.0014	0.0004	0.0014	0.0004	0.0014
54	0.0249	0	0.0249	0.0319	0	0.0249	0	0	0	0	0	0
55	0.008	0.0694	0.008	-0.0015	0.0699	0.0081	-0.0017	0.0027	0.0148	0.0027	0.0148	0.0054
56	0.0365	0.028	0.0365	0.0243	0.0293	0.0371	-0.0007	0.0011	0	0.0011	0	0.0025

C3. Jacobian Matrix of Route Travel Time

Columns 49-56
Rows1-56

	49	50	51	52	53	54	55	56
1	0	0	0	0	0	0	0	0
2	0	0	0	0	0	0	0	0
3	0	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	0
5	0	0	0	0	0	0	0	0
6	0	0	0	0	0	0	0	0
7	0	0	0	0	0	0	0	0
8	0	0	0	0	0	0	0	0
9	0	0	0	0	0	0	0	0
10	0	0	0	0	0	0	0	0
11	0	0	0	0	0	0	0	0
12	0	0	0	0	0	0	0	0
13	0	0	0	0	0	0	0	0
14	0	0	0	0	0	0	0	0
15	0	0	0	0	0	0	0	0
16	0	0	0	0	0	0	0	0
17	0	0	0	0	0	0	0	0
18	0	0	0	0	0	0	0	0
19	0	0	0	0	0	0	0	0
20	0	0	0	0	0	0	0	0
21	0	0	0	0	0	0	0	0
22	0	0	0	0	0	0	0	0
23	0	0	0	0	0	0	0	0
24	0	0	0	0	0	0	0	0
25	0	0	0	0	0	0	0	0
26	0	0	0	0	0	0	0	0
27	0	0	0	0	0	0	0	0
28	0	0	0	0	0	0	0	0
29	0	0	0	0	0	0	0	0
30	0	0.0004	0.0015	0.0001	0.0004	-0.0006	0.0001	0.0004
31	0	0.0004	0.001	0.0003	0.0002	-0.0004	0.0003	0.0002
32	0	0.0002	0.0008	0.0002	0.0008	-0.0005	0.0002	0.0008
33	0	0.0004	0.0005	0.0004	0.0005	-0.0003	0.0004	0.0005
34	0.0001	0.0001	0.0006	0.0001	0.0006	-0.0005	0.0004	0.0016
35	0	0.0004	0.0004	0.0004	0.0004	-0.0003	0.0004	0.0011
36	0	0	0.0005	0	0.0001	-0.0002	0	0.0001
37	0	0	0	0	0	0	0	0
38	0	0	0.0002	0	0.0002	-0.0002	0	0.0002
39	0	0	0	0	0	0	0	0
40	0	0	0	0	0	0	0	0
41	0	0	0.0002	0	0.0002	-0.0002	0	0.0006
42	0	0	0	0	0	0	0	0
43	0	0	0	0	0	0	0	0
44	0.001	0.0062	0.0127	0.002	0.0089	-0.0018	0.002	0.0089
45	0.0222	0.0262	0.012	0.0222	0.0081	-0.0018	0.0222	0.0081
46	0.0019	0.0029	0.0095	0.0029	0.0095	-0.0018	0.0029	0.0095
47	0.0237	0.0238	0.0089	0.0238	0.0089	-0.0019	0.0238	0.0089
48	0.0046	0.0021	0.0091	0.0021	0.0091	-0.0018	0.0057	0.012
49	0.026	0.023	0.0084	0.023	0.0084	-0.0019	0.0263	0.0114
50	0.0144	0.0281	0.011	0.0244	0.007	-0.0021	0.0244	0.007
51	-0.0001	0.0029	0.0272	0.0006	0.0236	0.0109	0.0006	0.0236
52	0.0155	0.0259	0.0078	0.0259	0.0078	-0.0021	0.0259	0.0078
53	0.0004	0.0012	0.025	0.0012	0.025	0.0113	0.0012	0.025
54	0	0	0.0113	0	0.0113	0.0244	0	0.0113
55	0.0176	0.0252	0.0073	0.0252	0.0073	-0.0021	0.0283	0.0105
56	0.0017	0.0007	0.0245	0.0007	0.0245	0.0113	0.0027	0.0275

C4. Variance - Covariance Matrix by Variance Approximation Method

Columns 1-12
Rows1-56

	1	2	3	4	5	6	7	8	9	10	11	12
1	94.9622	-42.4544	-52.5078	2.1097	-2.1097	4.7949	-4.7949	-2.2411	2.2411	-3.0044	3.0044	4.8265
2	-42.4544	52.9077	-10.4533	-0.0036	0.0036	-0.0078	0.0078	0.183	-0.183	0.2453	-0.2453	-0.4335
3	-52.5078	-10.4533	62.9611	-2.106	2.106	-4.7871	4.7871	2.0581	-2.0581	2.7591	-2.7591	-4.393
4	2.1097	-0.0036	-2.106	25.6999	-25.6999	1.7096	-1.7096	-0.6644	0.6644	-0.8907	0.8907	1.4015
5	-2.1097	0.0036	2.106	-25.6999	25.6999	-1.7096	1.7096	0.6644	-0.6644	0.8907	-0.8907	-1.4015
6	4.7949	-0.0078	-4.7871	1.7096	-1.7096	60.0115	-60.0115	-1.5095	1.5095	-2.0236	2.0236	3.1832
7	-4.7949	0.0078	4.7871	-1.7096	1.7096	-60.0115	60.0115	1.5095	-1.5095	2.0236	-2.0236	-3.1832
8	-2.2411	0.183	2.0581	-0.6644	0.6644	-1.5095	1.5095	31.3836	-31.3836	1.5189	-1.5189	-1.9844
9	2.2411	-0.183	-2.0581	0.6644	-0.6644	1.5095	-1.5095	-31.3836	31.3836	-1.5189	1.5189	1.9844
10	-3.0044	0.2453	2.7591	-0.8907	0.8907	-2.0236	2.0236	1.5189	-1.5189	43.2851	-43.2851	-2.6591
11	3.0044	-0.2453	-2.7591	0.8907	-0.8907	2.0236	-2.0236	-1.5189	1.5189	-43.2851	43.2851	2.6591
12	4.8265	-0.4335	-4.393	1.4015	-1.4015	3.1832	-3.1832	-1.9844	1.9844	-2.6591	2.6591	82.6011
13	-4.8554	0.4164	4.439	-1.4246	1.4246	-3.2361	3.2361	2.2274	-2.2274	2.9846	-2.9846	-43.5399
14	0.0288	0.0171	-0.046	0.0231	-0.0231	0.0529	-0.0529	-0.243	0.243	-0.3254	0.3254	-39.0612
15	26.5624	-1.269	-25.2934	8.5518	-8.5518	19.4355	-19.4355	-9.8018	9.8018	-13.1379	13.1379	19.7306
16	-6.7369	0.4641	6.2728	-2.0623	2.0623	-4.6863	4.6863	2.6698	-2.6698	3.5781	-3.5781	-5.2844
17	-19.8255	0.805	19.0206	-6.4895	6.4895	-14.7492	14.7492	7.132	-7.132	9.5598	-9.5598	-14.4461
18	3.5196	-0.0806	-3.4391	1.1988	-1.1988	2.7249	-2.7249	-1.1859	1.1859	-1.5897	1.5897	2.442
19	-3.5196	0.0806	3.4391	-1.1988	1.1988	-2.7249	2.7249	1.1859	-1.1859	1.5897	-1.5897	-2.442
20	5.0929	-0.117	-4.9759	1.7344	-1.7344	3.9423	-3.9423	-1.716	1.716	-2.3004	2.3004	3.5339
21	-5.0929	0.117	4.9759	-1.7344	1.7344	-3.9423	3.9423	1.716	-1.716	2.3004	-2.3004	-3.5339
22	-4.2332	0.251	3.9822	-1.3261	1.3261	-3.0131	3.0131	1.8747	-1.8747	2.5123	-2.5123	-3.4851
23	4.2332	-0.251	-3.9822	1.3261	-1.3261	3.0131	-3.0131	-1.8747	1.8747	-2.5123	2.5123	3.4851
24	-4.7442	0.2829	4.4613	-1.485	1.485	-3.3741	3.3741	2.0992	-2.0992	2.8133	-2.8133	-3.903
25	4.7442	-0.2829	-4.4613	1.485	-1.485	3.3741	-3.3741	-2.0992	2.0992	-2.8133	2.8133	3.903
26	8.6699	-0.5429	-8.127	2.6947	-2.6947	6.1234	-6.1234	-3.4384	3.4384	-4.6082	4.6082	6.8782
27	-5.9899	0.3674	5.6226	-1.8674	1.8674	-4.2433	4.2433	2.4833	-2.4833	3.328	-3.328	-4.819
28	-2.68	0.1755	2.5045	-0.8273	0.8273	-1.8801	1.8801	0.9551	-0.9551	1.2801	-1.2801	-2.0591
29	20.2113	-0.8182	-19.3931	6.6172	-6.6172	15.0385	-15.0385	-7.5599	7.5599	-10.1333	10.1333	15.0267
30	-5.9817	0.2876	5.6942	-1.9243	1.9243	-4.3731	4.3731	2.3358	-2.3358	3.1308	-3.1308	-4.58
31	-14.2296	0.5307	13.6989	-4.6928	4.6928	-10.6653	10.6653	5.2241	-5.2241	7.0025	-7.0025	-10.4467
32	3.5044	-0.0883	-3.416	1.1875	-1.1875	2.699	-2.699	-1.1951	1.1951	-1.6021	1.6021	2.4493
33	-3.5044	0.0883	3.416	-1.1875	1.1875	-2.699	2.699	1.1951	-1.1951	1.6021	-1.6021	-2.4493
34	3.218	-0.0814	-3.1366	1.0903	-1.0903	2.478	-2.478	-1.098	1.098	-1.4719	1.4719	2.2499
35	-3.218	0.0814	3.1366	-1.0903	1.0903	-2.478	2.478	1.098	-1.098	1.4719	-1.4719	-2.2499
36	-4.127	0.218	3.9089	-1.3128	1.3128	-2.9829	2.9829	1.7519	-1.7519	2.3481	-2.3481	-3.3141
37	4.127	-0.218	-3.9089	1.3128	-1.3128	2.9829	-2.9829	-1.7519	1.7519	-2.3481	2.3481	3.3141
38	-3.4399	0.182	3.258	-1.0941	1.0941	-2.4859	2.4859	1.4594	-1.4594	1.956	-1.956	-2.7617
39	3.4399	-0.182	-3.258	1.0941	-1.0941	2.4859	-2.4859	-1.4594	1.4594	-1.956	1.956	2.7617
40	11.3646	-0.5482	-10.8164	3.6546	-3.6546	8.3053	-8.3053	-4.4443	4.4443	-5.9568	5.9568	8.7138
41	-6.1454	0.3047	5.8407	-1.97	1.97	-4.4767	4.4767	2.4626	-2.4626	3.3007	-3.3007	-4.7765
42	-5.2192	0.2435	4.9757	-1.6846	1.6846	-3.8286	3.8286	1.9817	-1.9817	2.6561	-2.6561	-3.9374
43	11.0354	-0.4221	-10.6133	3.6315	-3.6315	8.2531	-8.2531	-4.0988	4.0988	-5.4942	5.4942	8.1569
44	-3.435	0.1516	3.2833	-1.1152	1.1152	-2.5343	2.5343	1.3193	-1.3193	1.7683	-1.7683	-2.5978
45	-7.6004	0.2704	7.3299	-2.5163	2.5163	-5.7188	5.7188	2.7795	-2.7795	3.7258	-3.7258	-5.5591
46	3.7316	-0.1012	-3.6304	1.2591	-1.2591	2.8618	-2.8618	-1.2969	1.2969	-1.7385	1.7385	2.6375
47	-3.7316	0.1012	3.6304	-1.2591	1.2591	-2.8618	2.8618	1.2969	-1.2969	1.7385	-1.7385	-2.6375
48	2.1343	-0.0579	-2.0764	0.7202	-0.7202	1.6368	-1.6368	-0.7417	0.7417	-0.9944	0.9944	1.5085
49	-2.1343	0.0579	2.0764	-0.7202	0.7202	-1.6368	1.6368	0.7417	-0.7417	0.9944	-0.9944	-1.5085
50	-1.964	0.1065	1.8574	-0.6227	0.6227	-1.4148	1.4148	0.8372	-0.8372	1.1221	-1.1221	-1.5827
51	1.964	-0.1065	-1.8574	0.6227	-0.6227	1.4148	-1.4148	-0.8372	0.8372	-1.1221	1.1221	1.5827
52	-2.48	0.134	2.3461	-0.7867	0.7867	-1.7875	1.7875	1.0545	-1.0545	1.4133	-1.4133	-1.9955
53	2.48	-0.134	-2.3461	0.7867	-0.7867	1.7875	-1.7875	-1.0545	1.0545	-1.4133	1.4133	1.9955
54	10.6102	-0.4821	-10.1281	3.4342	-3.4342	7.8043	-7.8043	-4.1442	4.1442	-5.5547	5.5547	8.1038
55	-6.8508	0.3246	6.5262	-2.2074	2.2074	-5.0162	5.0162	2.7295	-2.7295	3.6585	-3.6585	-5.2958
56	-3.7594	0.1576	3.6018	-1.2268	1.2268	-2.7881	2.7881	1.4147	-1.4147	1.8962	-1.8962	-2.808

C4. Variance - Covariance Matrix by Variance Approximation Method

Columns 13-24
Rows1-56

	13	14	15	16	17	18	19	20	21	22	23	24
1	-4.8554	0.0288	26.5624	-6.7369	-19.8255	3.5196	-3.5196	5.0929	-5.0929	-4.2332	4.2332	-4.7442
2	0.4164	0.0171	-1.269	0.4641	0.805	-0.0806	0.0806	-0.117	0.117	0.251	-0.251	0.2829
3	4.439	-0.046	-25.2934	6.2728	19.0206	-3.4391	3.4391	-4.9759	4.9759	3.9822	-3.9822	4.4613
4	-1.4246	0.0231	8.5518	-2.0623	-6.4895	1.1988	-1.1988	1.7344	-1.7344	-1.3261	1.3261	-1.485
5	1.4246	-0.0231	-8.5518	2.0623	6.4895	-1.1988	1.1988	-1.7344	1.7344	1.3261	-1.3261	1.485
6	-3.2361	0.0529	19.4355	-4.6863	-14.7492	2.7249	-2.7249	3.9423	-3.9423	-3.0131	3.0131	-3.3741
7	3.2361	-0.0529	-19.4355	4.6863	14.7492	-2.7249	2.7249	-3.9423	3.9423	3.0131	-3.0131	3.3741
8	2.2274	-0.243	-9.8018	2.6698	7.132	-1.1859	1.1859	-1.716	1.716	1.8747	-1.8747	2.0992
9	-2.2274	0.243	9.8018	-2.6698	-7.132	1.1859	-1.1859	1.716	-1.716	-1.8747	1.8747	-2.0992
10	2.9846	-0.3254	-13.1379	3.5781	9.5598	-1.5897	1.5897	-2.3004	2.3004	2.5123	-2.5123	2.8133
11	-2.9846	0.3254	13.1379	-3.5781	-9.5598	1.5897	-1.5897	2.3004	-2.3004	-2.5123	2.5123	-2.8133
12	-43.5399	-39.0612	19.7306	-5.2844	-14.4461	2.442	-2.442	3.5339	-3.5339	-3.4851	3.4851	-3.903
13	57.1594	-13.6195	-20.5457	5.5513	14.9944	-2.5132	2.5132	-3.6368	3.6368	3.7854	-3.7854	4.239
14	-13.6195	52.6807	0.8152	-0.2669	-0.5482	0.0711	-0.0711	0.1029	-0.1029	-0.3003	0.3003	-0.336
15	-20.5457	0.8152	271.3143	-97.2723	-174.042	18.4307	-18.4307	26.6819	-26.6819	-22.4624	22.4624	-25.1381
16	5.5513	-0.2669	-97.2723	89.2763	7.996	-3.7387	3.7387	-5.4103	5.4103	5.35	-5.35	5.9873
17	14.9944	-0.5482	-174.042	7.996	166.046	-14.692	14.692	-21.2717	21.2717	17.1124	-17.1124	19.1509
18	-2.5132	0.0711	18.4307	-3.7387	-14.692	34.1539	-34.1539	4.3801	-4.3801	-3.1966	3.1966	-3.5773
19	2.5132	-0.0711	-18.4307	3.7387	14.692	-34.1539	34.1539	-4.3801	4.3801	3.1966	-3.1966	3.5773
20	-3.6368	0.1029	26.6819	-5.4103	-21.2717	4.3801	-4.3801	51.6672	-51.6672	-4.6278	4.6278	-5.179
21	3.6368	-0.1029	-26.6819	5.4103	21.2717	-4.3801	4.3801	-51.6672	51.6672	4.6278	-4.6278	5.179
22	3.7854	-0.3003	-22.4624	5.35	17.1124	-3.1966	3.1966	-4.6278	4.6278	39.7532	-39.7532	5.3228
23	-3.7854	0.3003	22.4624	-5.35	-17.1124	3.1966	-3.1966	4.6278	-4.6278	-39.7532	39.7532	-5.3228
24	4.239	-0.336	-25.1381	5.9873	19.1509	-3.5773	3.5773	-5.179	5.179	5.3228	-5.3228	46.1036
25	-4.239	0.336	25.1381	-5.9873	-19.1509	3.5773	-3.5773	5.179	-5.179	-5.3228	5.3228	-46.1036
26	-7.1857	0.3075	39.6009	-9.9246	-29.6763	5.3267	-5.3267	7.7118	-7.7118	-7.2024	7.2024	-8.0657
27	5.1154	-0.2964	-29.1148	7.136	21.9788	-4.0169	4.0169	-5.8154	5.8154	5.6716	-5.6716	6.3518
28	2.0703	-0.0111	-10.4862	2.7886	7.6976	-1.3098	1.3098	-1.8964	1.8964	1.5308	-1.5308	1.7139
29	-15.7516	0.7248	117.0291	-24.6221	-92.407	18.6704	-18.6704	27.0378	-27.0378	-21.9896	21.9896	-24.6057
30	4.8354	-0.2554	-32.6814	7.4252	25.2562	-4.8732	4.8732	-7.0558	7.0558	6.2581	-6.2581	7.0041
31	10.9161	-0.4694	-84.3477	17.1969	67.1508	-13.7972	13.7972	-19.982	19.982	15.7316	-15.7316	17.6016
32	-2.527	0.0777	22.5958	-4.1043	-18.4916	4.0079	-4.0079	5.8057	-5.8057	-4.1113	4.1113	-4.5986
33	2.527	-0.0777	-22.5958	4.1043	18.4916	-4.0079	4.0079	-5.8057	5.8057	4.1113	-4.1113	4.5986
34	-2.3215	0.0715	20.7223	-3.7676	-16.9547	3.6733	-3.6733	5.321	-5.321	-3.7706	3.7706	-4.2176
35	2.3215	-0.0715	-20.7223	3.7676	16.9547	-3.6733	3.6733	-5.321	5.321	3.7706	-3.7706	4.2176
36	3.5664	-0.2523	-28.2318	5.6695	22.5624	-4.6745	4.6745	-6.7702	6.7702	6.2456	-6.2456	6.9936
37	-3.5664	0.2523	28.2318	-5.6695	-22.5624	4.6745	-4.6745	6.7702	-6.7702	-6.2456	6.2456	-6.9936
38	2.9714	-0.2097	-23.4767	4.7256	18.7511	-3.8803	3.8803	-5.6199	5.6199	5.1835	-5.1835	5.8042
39	-2.9714	0.2097	23.4767	-4.7256	-18.7511	3.8803	-3.8803	5.6199	-5.6199	-5.1835	5.1835	-5.8042
40	-9.2	0.4862	62.6486	-13.9547	-48.6939	9.5139	-9.5139	13.7747	-13.7747	-12.1046	12.1046	-13.5502
41	5.0719	-0.2955	-36.245	7.809	28.436	-5.6696	5.6696	-8.2096	8.2096	7.3399	-7.3399	8.2173
42	4.1281	-0.1907	-26.4036	6.1456	20.258	-3.8443	3.8443	-5.5651	5.5651	4.7647	-4.7647	5.3329
43	-8.5451	0.3882	65.9602	-13.3776	-52.5827	10.8338	-10.8338	15.6895	-15.6895	-12.4851	12.4851	-13.9712
44	2.7366	-0.1387	-19.5231	4.2026	15.3205	-3.0557	3.0557	-4.4248	4.4248	3.7475	-3.7475	4.1943
45	5.8085	-0.2494	-46.4371	9.175	37.2621	-7.7781	7.7781	-11.2647	11.2647	8.7376	-8.7376	9.7769
46	-2.732	0.0945	24.3648	-4.444	-19.9208	4.3108	-4.3108	6.2438	-6.2438	-4.5046	4.5046	-5.0393
47	2.732	-0.0945	-24.3648	4.444	19.9208	-4.3108	4.3108	-6.2438	6.2438	4.5046	-4.5046	5.0393
48	-1.5626	0.054	13.9365	-2.5418	-11.3947	2.4658	-2.4658	3.5715	-3.5715	-2.5766	2.5766	-2.8825
49	1.5626	-0.054	-13.9365	2.5418	11.3947	-2.4658	2.4658	-3.5715	3.5715	2.5766	-2.5766	2.8825
50	1.7038	-0.121	-13.3672	2.6753	10.6919	-2.2189	2.2189	-3.213	3.213	2.9426	-2.9426	3.2955
51	-1.7038	0.121	13.3672	-2.6753	-10.6919	2.2189	-2.2189	3.213	-3.213	-2.9426	2.9426	-3.2955
52	2.147	-0.1515	-16.801	3.3692	13.4318	-2.7847	2.7847	-4.0323	4.0323	3.6878	-3.6878	4.1301
53	-2.147	0.1515	16.801	-3.3692	-13.4318	2.7847	-2.7847	4.0323	-4.0323	-3.6878	3.6878	-4.1301
54	-8.5681	0.4643	63.0563	-13.2464	-49.8099	10.0734	-10.0734	14.586	-14.586	-12.4608	12.4608	-13.9484
55	5.6225	-0.3267	-42.0081	8.7238	33.2843	-6.7744	6.7744	-9.8092	9.8092	8.5319	-8.5319	9.5517
56	2.9456	-0.1376	-21.0482	4.5226	16.5256	-3.299	3.299	-4.7768	4.7768	3.9289	-3.9289	4.3967

C4. Variance - Covariance Matrix by Variance Approximation Method

Columns 25-36
Rows1-56

	25	26	27	28	29	30	31	32	33	34	35	36
1	4.7442	8.6699	-5.9899	-2.68	20.2113	-5.9817	-14.2296	3.5044	-3.5044	3.218	-3.218	-4.127
2	-0.2829	-0.5429	0.3674	0.1755	-0.8182	0.2876	0.5307	-0.0883	0.0883	-0.0814	0.0814	0.218
3	-4.4613	-8.127	5.6226	2.5045	-19.3931	5.6942	13.6989	-3.416	3.416	-3.1366	3.1366	3.9089
4	1.485	2.6947	-1.8674	-0.8273	6.6172	-1.9243	-4.6928	1.1875	-1.1875	1.0903	-1.0903	-1.3128
5	-1.485	-2.6947	1.8674	0.8273	-6.6172	1.9243	4.6928	-1.1875	1.1875	-1.0903	1.0903	1.3128
6	3.3741	6.1234	-4.2433	-1.8801	15.0385	-4.3731	-10.6653	2.699	-2.699	2.478	-2.478	-2.9829
7	-3.3741	-6.1234	4.2433	1.8801	-15.0385	4.3731	10.6653	-2.699	2.699	-2.478	2.478	2.9829
8	-2.0992	-3.4384	2.4833	0.9551	-7.5599	2.3358	5.2241	-1.1951	1.1951	-1.098	1.098	1.7519
9	2.0992	3.4384	-2.4833	-0.9551	7.5599	-2.3358	-5.2241	1.1951	-1.1951	1.098	-1.098	-1.7519
10	-2.8133	-4.6082	3.328	1.2801	-10.1333	3.1308	7.0025	-1.6021	1.6021	-1.4719	1.4719	2.3481
11	2.8133	4.6082	-3.328	-1.2801	10.1333	-3.1308	-7.0025	1.6021	-1.6021	1.4719	-1.4719	-2.3481
12	3.903	6.8782	-4.819	-2.0591	15.0267	-4.58	-10.4467	2.4493	-2.4493	2.2499	-2.2499	-3.3141
13	-4.239	-7.1857	5.1154	2.0703	-15.7516	4.8354	10.9161	-2.527	2.527	-2.3215	2.3215	3.5664
14	0.336	0.3075	-0.2964	-0.0111	0.7248	-0.2554	-0.4694	0.0777	-0.0777	0.0715	-0.0715	-0.2523
15	25.1381	39.6009	-29.1148	-10.4862	117.0291	-32.6814	-84.3477	22.5958	-22.5958	20.7223	-20.7223	-28.2318
16	-5.9873	-9.9246	7.136	2.7886	-24.6221	7.4252	17.1969	-4.1043	4.1043	-3.7676	3.7676	5.6695
17	-19.1509	-29.6763	21.9788	7.6976	-92.407	25.2562	67.1508	-18.4916	18.4916	-16.9547	16.9547	22.5624
18	3.5773	5.3267	-4.0169	-1.3098	18.6704	-4.8732	-13.7972	4.0079	-4.0079	3.6733	-3.6733	-4.6745
19	-3.5773	-5.3267	4.0169	1.3098	-18.6704	4.8732	13.7972	-4.0079	4.0079	-3.6733	3.6733	4.6745
20	5.179	7.7118	-5.8154	-1.8964	27.0378	-7.0558	-19.982	5.8057	-5.8057	5.321	-5.321	-6.7702
21	-5.179	-7.7118	5.8154	1.8964	-27.0378	7.0558	19.982	-5.8057	5.8057	-5.321	5.321	6.7702
22	-5.3228	-7.2024	5.6716	1.5308	-21.9896	6.2581	15.7316	-4.1113	4.1113	-3.7706	3.7706	6.2456
23	5.3228	7.2024	-5.6716	-1.5308	21.9896	-6.2581	-15.7316	4.1113	-4.1113	3.7706	-3.7706	-6.2456
24	-46.1036	-8.0657	6.3518	1.7139	-24.6057	7.0041	17.6016	-4.5986	4.5986	-4.2176	4.2176	6.9936
25	46.1036	8.0657	-6.3518	-1.7139	24.6057	-7.0041	-17.6016	4.5986	-4.5986	4.2176	-4.2176	-6.9936
26	8.0657	70.2335	-33.4086	-36.8249	33.99	-9.9019	-24.0881	6.0819	-6.0819	5.5816	-5.5816	-8.2538
27	-6.3518	-33.4086	37.2634	-3.8548	-26.5268	7.6431	18.8837	-4.8476	4.8476	-4.4474	4.4474	6.9546
28	-1.7139	-36.8249	-3.8548	40.6798	-7.4632	2.2588	5.2044	-1.2343	1.2343	-1.1342	1.1342	1.2992
29	24.6057	33.99	-26.5268	-7.4632	228.2827	-76.9922	-151.291	28.6691	-28.6691	26.2676	-26.2676	-36.528
30	-7.0041	-9.9019	7.6431	2.2588	-76.9922	61.4974	15.4948	-6.5026	6.5026	-5.9614	5.9614	9.0965
31	-17.6016	-24.0881	18.8837	5.2044	-151.291	15.4948	135.7956	-22.1665	22.1665	-20.3063	20.3063	27.4315
32	4.5986	6.0819	-4.8476	-1.2343	28.6691	-6.5026	-22.1665	45.5843	-45.5843	6.6994	-6.6994	-8.3722
33	-4.5986	-6.0819	4.8476	1.2343	-28.6691	6.5026	22.1665	-45.5843	45.5843	-6.6994	6.6994	8.3722
34	4.2176	5.5816	-4.4474	-1.1342	26.2676	-5.9614	-20.3063	6.6994	-6.6994	41.1934	-41.1934	-7.6676
35	-4.2176	-5.5816	4.4474	1.1342	-26.2676	5.9614	20.3063	-6.6994	6.6994	-41.1934	41.1934	7.6676
36	-6.9936	-8.2538	6.9546	1.2992	-36.528	9.0965	27.4315	-8.3722	8.3722	-7.6676	7.6676	59.7453
37	6.9936	8.2538	-6.9546	-1.2992	36.528	-9.0965	-27.4315	8.3722	-8.3722	7.6676	-7.6676	-59.7453
38	-5.8042	-6.8627	5.7771	1.0856	-30.2796	7.545	22.7346	-6.9344	6.9344	-6.3508	6.3508	10.4929
39	5.8042	6.8627	-5.7771	-1.0856	30.2796	-7.545	-22.7346	6.9344	-6.9344	6.3508	-6.3508	-10.4929
40	13.5502	19.0561	-14.7462	-4.3098	65.7369	-17.6466	-48.0903	13.5446	-13.5446	12.4153	-12.4153	-18.4148
41	-8.2173	-10.8756	8.6602	2.2154	-40.9851	10.6963	30.2889	-8.8093	8.8093	-8.072	8.072	12.529
42	-5.3329	-8.1804	6.0861	2.0944	-24.7518	6.9503	17.8015	-4.7352	4.7352	-4.3433	4.3433	5.8857
43	13.9712	18.8899	-14.8935	-3.9964	79.5548	-19.4303	-60.1245	18.6786	-18.6786	17.1082	-17.1082	-23.444
44	-4.1943	-5.7861	4.5181	1.2681	-21.3119	5.7214	15.5905	-4.3934	4.3934	-4.0262	4.0262	5.9625
45	-9.7769	-13.1038	10.3755	2.7283	-58.2429	13.7089	44.534	-14.2852	14.2852	-13.082	13.082	17.4815
46	5.0393	6.5783	-5.2763	-1.3021	33.9481	-7.2348	-26.7133	9.22	-9.22	8.4403	-8.4403	-10.6535
47	-5.0393	-6.5783	5.2763	1.3021	-33.9481	7.2348	26.7133	-9.22	9.22	-8.4403	8.4403	10.6535
48	2.8825	3.7627	-3.018	-0.7447	19.4184	-4.138	-15.2804	5.2742	-5.2742	4.8282	-4.8282	-6.0943
49	-2.8825	-3.7627	3.018	0.7447	-19.4184	4.138	15.2804	-5.2742	5.2742	-4.8282	4.8282	6.0943
50	-3.2955	-3.9179	3.2893	0.6286	-19.4761	4.4038	15.0723	-4.9948	4.9948	-4.5722	4.5722	7.4235
51	3.2955	3.9179	-3.2893	-0.6286	19.4761	-4.4038	-15.0723	4.9948	-4.9948	4.5722	-4.5722	-7.4235
52	-4.1301	-4.9262	4.1289	0.7972	-24.363	5.5248	18.8381	-6.2292	6.2292	-5.7023	5.7023	9.2412
53	4.1301	4.9262	-4.1289	-0.7972	24.363	-5.5248	-18.8381	6.2292	-6.2292	5.7023	-5.7023	-9.2412
54	13.9484	18.5544	-14.7406	-3.8138	74.7306	-18.7869	-55.9437	16.914	-16.914	15.4926	-15.4926	-23.0109
55	-9.5517	-12.3497	9.9464	2.4034	-52.5827	12.8425	39.7402	-12.35	12.35	-11.31	11.31	17.2813
56	-4.3967	-6.2047	4.7943	1.4104	-22.1478	5.9444	16.2035	-4.5639	4.5639	-4.1826	4.1826	5.7296

C4. Variance - Covariance Matrix by Variance Approximation Method

Columns 37-48
Rows1-56

	37	38	39	40	41	42	43	44	45	46	47	48
1	4.127	-3.4399	3.4399	11.3646	-6.1454	-5.2192	11.0354	-3.435	-7.6004	3.7316	-3.7316	2.1343
2	-0.218	0.182	-0.182	-0.5482	0.3047	0.2435	-0.4221	0.1516	0.2704	-0.1012	0.1012	-0.0579
3	-3.9089	3.258	-3.258	-10.8164	5.8407	4.9757	-10.6133	3.2833	7.3299	-3.6304	3.6304	-2.0764
4	1.3128	-1.0941	1.0941	3.6546	-1.97	-1.6846	3.6315	-1.1152	-2.5163	1.2591	-1.2591	0.7202
5	-1.3128	1.0941	-1.0941	-3.6546	1.97	1.6846	-3.6315	1.1152	2.5163	-1.2591	1.2591	-0.7202
6	2.9829	-2.4859	2.4859	8.3053	-4.4767	-3.8286	8.2531	-2.5343	-5.7188	2.8618	-2.8618	1.6368
7	-2.9829	2.4859	-2.4859	-8.3053	4.4767	3.8286	-8.2531	2.5343	5.7188	-2.8618	2.8618	-1.6368
8	-1.7519	1.4594	-1.4594	-4.4443	2.4626	1.9817	-4.0988	1.3193	2.7795	-1.2969	1.2969	-0.7417
9	1.7519	-1.4594	1.4594	4.4443	-2.4626	-1.9817	4.0988	-1.3193	-2.7795	1.2969	-1.2969	0.7417
10	-2.3481	1.956	-1.956	-5.9568	3.3007	2.6561	-5.4942	1.7683	3.7258	-1.7385	1.7385	-0.9944
11	2.3481	-1.956	1.956	5.9568	-3.3007	-2.6561	5.4942	-1.7683	-3.7258	1.7385	-1.7385	0.9944
12	3.3141	-2.7617	2.7617	8.7138	-4.7765	-3.9374	8.1569	-2.5978	-5.5591	2.6375	-2.6375	1.5085
13	-3.5664	2.9714	-2.9714	-9.2	5.0719	4.1281	-8.5451	2.7366	5.8085	-2.732	2.732	-1.5626
14	0.2523	-0.2097	0.2097	0.4862	-0.2955	-0.1907	0.3882	-0.1387	-0.2494	0.0945	-0.0945	0.054
15	28.2318	-23.4767	23.4767	62.6486	-36.245	-26.4036	65.9602	-19.5231	-46.4371	24.3648	-24.3648	13.9365
16	-5.6695	4.7256	-4.7256	-13.9547	7.809	6.1456	-13.3776	4.2026	9.175	-4.444	4.444	-2.5418
17	-22.5624	18.7511	-18.7511	-48.6939	28.436	20.258	-52.5827	15.3205	37.2621	-19.9208	19.9208	-11.3947
18	4.6745	-3.8803	3.8803	9.5139	-5.6696	-3.8443	10.8338	-3.0557	-7.7781	4.3108	-4.3108	2.4658
19	-4.6745	3.8803	-3.8803	-9.5139	5.6696	3.8443	-10.8338	3.0557	7.7781	-4.3108	4.3108	-2.4658
20	6.7702	-5.6199	5.6199	13.7747	-8.2096	-5.5651	15.6895	-4.4248	-11.2647	6.2438	-6.2438	3.5715
21	-6.7702	5.6199	-5.6199	-13.7747	8.2096	5.5651	-15.6895	4.4248	11.2647	-6.2438	6.2438	-3.5715
22	-6.2456	5.1835	-5.1835	-12.1046	7.3399	4.7647	-12.4851	3.7475	8.7376	-4.5046	4.5046	-2.5766
23	6.2456	-5.1835	5.1835	12.1046	-7.3399	-4.7647	12.4851	-3.7475	-8.7376	4.5046	-4.5046	2.5766
24	-6.9936	5.8042	-5.8042	-13.5502	8.2173	5.3329	-13.9712	4.1943	9.7769	-5.0393	5.0393	-2.8825
25	6.9936	-5.8042	5.8042	13.5502	-8.2173	-5.3329	13.9712	-4.1943	-9.7769	5.0393	-5.0393	2.8825
26	8.2538	-6.8627	6.8627	19.0561	-10.8756	-8.1804	18.8899	-5.7861	-13.1038	6.5783	-6.5783	3.7627
27	-6.9546	5.7771	-5.7771	-14.7462	8.6602	6.0861	-14.8935	4.5181	10.3755	-5.2763	5.2763	-3.018
28	-1.2992	1.0856	-1.0856	-4.3098	2.2154	2.0944	-3.9964	1.2681	2.7283	-1.3021	1.3021	-0.7447
29	36.528	-30.2796	30.2796	65.7369	-40.9851	-24.7518	79.5548	-21.3119	-58.2429	33.9481	-33.9481	19.4184
30	-9.0965	7.545	-7.545	-17.6466	10.6963	6.9503	-19.4303	5.7214	13.7089	-7.2348	7.2348	-4.138
31	-27.4315	22.7346	-22.7346	-48.0903	30.2889	17.8015	-60.1245	15.5905	44.534	-26.7133	26.7133	-15.2804
32	8.3722	-6.9344	6.9344	13.5446	-8.8093	-4.7352	18.6786	-4.3934	-14.2852	9.22	-9.22	5.2742
33	-8.3722	6.9344	-6.9344	-13.5446	8.8093	4.7352	-18.6786	4.3934	14.2852	-9.22	9.22	-5.2742
34	7.6676	-6.3508	6.3508	12.4153	-8.072	-4.3433	17.1082	-4.0262	-13.082	8.4403	-8.4403	4.8282
35	-7.6676	6.3508	-6.3508	-12.4153	8.072	4.3433	-17.1082	4.0262	13.082	-8.4403	8.4403	-4.8282
36	-59.7453	10.4929	-10.4929	-18.4148	12.529	5.8857	-23.444	5.9625	17.4815	-10.6535	10.6535	-6.0943
37	59.7453	-10.4929	10.4929	18.4148	-12.529	-5.8857	23.444	-5.9625	-17.4815	10.6535	-10.6535	6.0943
38	-10.4929	47.7251	-47.7251	-15.2779	10.3861	4.8918	-19.4144	4.942	14.4724	-8.8135	8.8135	-5.0417
39	10.4929	-47.7251	47.7251	15.2779	-10.3861	-4.8918	19.4144	-4.942	-14.4724	8.8135	-8.8135	5.0417
40	18.4148	-15.2779	15.2779	97.4108	-45.3619	-52.0489	38.8787	-10.9293	-27.9494	15.5416	-15.5416	8.8903
41	-12.529	10.3861	-10.3861	-45.3619	43.1011	2.2608	-25.0269	6.7656	18.2613	-10.5552	10.5552	-6.038
42	-5.8857	4.8918	-4.8918	-52.0489	2.2608	49.7881	-13.8518	4.1637	9.6881	-4.9865	4.9865	-2.8524
43	23.444	-19.4144	19.4144	38.8787	-25.0269	-13.8518	104.183	-37.9093	-66.2737	24.3902	-24.3902	13.9511
44	-5.9625	4.942	-4.942	-10.9293	6.7656	4.1637	-37.9093	32.7855	5.1239	-4.6669	4.6669	-2.6695
45	-17.4815	14.4724	-14.4724	-27.9494	18.2613	9.6881	-66.2737	5.1239	61.1498	-19.7233	19.7233	-11.2817
46	10.6535	-8.8135	8.8135	15.5416	-10.5552	-4.9865	24.3902	-4.6669	-19.7233	62.9002	-62.9002	8.1404
47	-10.6535	8.8135	-8.8135	-15.5416	10.5552	4.9865	-24.3902	4.6669	19.7233	-62.9002	62.9002	-8.1404
48	6.0943	-5.0417	5.0417	8.8903	-6.038	-2.8524	13.9511	-2.6695	-11.2817	8.1404	-8.1404	32.6097
49	-6.0943	5.0417	-5.0417	-8.8903	6.038	2.8524	-13.9511	2.6695	11.2817	-8.1404	8.1404	-32.6097
50	-7.4235	6.1422	-6.1422	-9.4422	6.8221	2.6202	-14.1499	2.8593	11.2906	-7.9447	7.9447	-4.5446
51	7.4235	-6.1422	6.1422	9.4422	-6.8221	-2.6202	14.1499	-2.8593	-11.2906	7.9447	-7.9447	4.5446
52	-9.2412	7.6462	-7.6462	-11.8123	8.5147	3.2976	-17.6435	3.5781	14.0655	-9.8798	9.8798	-5.6515
53	9.2412	-7.6462	7.6462	11.8123	-8.5147	-3.2976	17.6435	-3.5781	-14.0655	9.8798	-9.8798	5.6515
54	23.0109	-19.0639	19.0639	37.6888	-24.3878	-13.301	47.755	-12.0949	-35.6601	21.8084	-21.8084	12.4746
55	-17.2813	14.3108	-14.3108	-26.2164	17.5114	8.705	-34.9042	8.2819	26.6223	-17.0823	17.0823	-9.7713
56	-5.7296	4.7531	-4.7531	-11.4724	6.8763	4.596	-12.8508	3.8129	9.0378	-4.7261	4.7261	-2.7033

C4. Variance - Covariance Matrix by Variance Approximation Method

Columns 49-56
Rows1-56

	49	50	51	52	53	54	55	56
1	-2.1343	-1.964	1.964	-2.48	2.48	10.6102	-6.8508	-3.7594
2	0.0579	0.1065	-0.1065	0.134	-0.134	-0.4821	0.3246	0.1576
3	2.0764	1.8574	-1.8574	2.3461	-2.3461	-10.1281	6.5262	3.6018
4	-0.7202	-0.6227	0.6227	-0.7867	0.7867	3.4342	-2.2074	-1.2268
5	0.7202	0.6227	-0.6227	0.7867	-0.7867	-3.4342	2.2074	1.2268
6	-1.6368	-1.4148	1.4148	-1.7875	1.7875	7.8043	-5.0162	-2.7881
7	1.6368	1.4148	-1.4148	1.7875	-1.7875	-7.8043	5.0162	2.7881
8	0.7417	0.8372	-0.8372	1.0545	-1.0545	-4.1442	2.7295	1.4147
9	-0.7417	-0.8372	0.8372	-1.0545	1.0545	4.1442	-2.7295	-1.4147
10	0.9944	1.1221	-1.1221	1.4133	-1.4133	-5.5547	3.6585	1.8962
11	-0.9944	-1.1221	1.1221	-1.4133	1.4133	5.5547	-3.6585	-1.8962
12	-1.5085	-1.5827	1.5827	-1.9955	1.9955	8.1038	-5.2958	-2.808
13	1.5626	1.7038	-1.7038	2.147	-2.147	-8.5681	5.6225	2.9456
14	-0.054	-0.121	0.121	-0.1515	0.1515	0.4643	-0.3267	-0.1376
15	-13.9365	-13.3672	13.3672	-16.801	16.801	63.0563	-42.0081	-21.0482
16	2.5418	2.6753	-2.6753	3.3692	-3.3692	-13.2464	8.7238	4.5226
17	11.3947	10.6919	-10.6919	13.4318	-13.4318	-49.8099	33.2843	16.5256
18	-2.4658	-2.2189	2.2189	-2.7847	2.7847	10.0734	-6.7744	-3.299
19	2.4658	2.2189	-2.2189	2.7847	-2.7847	-10.0734	6.7744	3.299
20	-3.5715	-3.213	3.213	-4.0323	4.0323	14.586	-9.8092	-4.7768
21	3.5715	3.213	-3.213	4.0323	-4.0323	-14.586	9.8092	4.7768
22	2.5766	2.9426	-2.9426	3.6878	-3.6878	-12.4608	8.5319	3.9289
23	-2.5766	-2.9426	2.9426	-3.6878	3.6878	12.4608	-8.5319	-3.9289
24	2.8825	3.2955	-3.2955	4.1301	-4.1301	-13.9484	9.5517	4.3967
25	-2.8825	-3.2955	3.2955	-4.1301	4.1301	13.9484	-9.5517	-4.3967
26	-3.7627	-3.9179	3.9179	-4.9262	4.9262	18.5544	-12.3497	-6.2047
27	3.018	3.2893	-3.2893	4.1289	-4.1289	-14.7406	9.9464	4.7943
28	0.7447	0.6286	-0.6286	0.7972	-0.7972	-3.8138	2.4034	1.4104
29	-19.4184	-19.4761	19.4761	-24.363	24.363	74.7306	-52.5827	-22.1478
30	4.138	4.4038	-4.4038	5.5248	-5.5248	-18.7869	12.8425	5.9444
31	15.2804	15.0723	-15.0723	18.8381	-18.8381	-55.9437	39.7402	16.2035
32	-5.2742	-4.9948	4.9948	-6.2292	6.2292	16.914	-12.35	-4.5639
33	5.2742	4.9948	-4.9948	6.2292	-6.2292	-16.914	12.35	4.5639
34	-4.8282	-4.5722	4.5722	-5.7023	5.7023	15.4926	-11.31	-4.1826
35	4.8282	4.5722	-4.5722	5.7023	-5.7023	-15.4926	11.31	4.1826
36	6.0943	7.4235	-7.4235	9.2412	-9.2412	-23.0109	17.2813	5.7296
37	-6.0943	-7.4235	7.4235	-9.2412	9.2412	23.0109	-17.2813	-5.7296
38	5.0417	6.1422	-6.1422	7.6462	-7.6462	-19.0639	14.3108	4.7531
39	-5.0417	-6.1422	6.1422	-7.6462	7.6462	19.0639	-14.3108	-4.7531
40	-8.8903	-9.4422	9.4422	-11.8123	11.8123	37.6888	-26.2164	-11.4724
41	6.038	6.8221	-6.8221	8.5147	-8.5147	-24.3878	17.5114	6.8763
42	2.8524	2.6202	-2.6202	3.2976	-3.2976	-13.301	8.705	4.596
43	-13.9511	-14.1499	14.1499	-17.6435	17.6435	47.755	-34.9042	-12.8508
44	2.6695	2.8593	-2.8593	3.5781	-3.5781	-12.0949	8.2819	3.8129
45	11.2817	11.2906	-11.2906	14.0655	-14.0655	-35.6601	26.6223	9.0378
46	-8.1404	-7.9447	7.9447	-9.8798	9.8798	21.8084	-17.0823	-4.7261
47	8.1404	7.9447	-7.9447	9.8798	-9.8798	-21.8084	17.0823	4.7261
48	-32.6097	-4.5446	4.5446	-5.6515	5.6515	12.4746	-9.7713	-2.7033
49	32.6097	4.5446	-4.5446	5.6515	-5.6515	-12.4746	9.7713	2.7033
50	4.5446	38.3515	-38.3515	7.1841	-7.1841	-13.7949	11.3899	2.405
51	-4.5446	-38.3515	38.3515	-7.1841	7.1841	13.7949	-11.3899	-2.405
52	5.6515	7.1841	-7.1841	48.0473	-48.0473	-17.1981	14.1737	3.0245
53	-5.6515	-7.1841	7.1841	-48.0473	48.0473	17.1981	-14.1737	-3.0245
54	-12.4746	-13.7949	13.7949	-17.1981	17.1981	93.3292	-52.0521	-41.2771
55	9.7713	11.3899	-11.3899	14.1737	-14.1737	-52.0521	47.7411	4.3111
56	2.7033	2.405	-2.405	3.0245	-3.0245	-41.2771	4.3111	36.966

C5. Variance-Covariance Matrix by Simulation Method

Columns 1-12
Rows 1-56

	1	2	3	4	5	6	7	8	9	10	11	12
1	91.9629	-42.1214	-49.8415	1.2119	-1.2119	3.2179	-3.2179	-0.6844	0.6844	-1.9653	1.9653	2.2655
2	-42.1214	52.9029	-10.7815	0.4061	-0.4061	1.4873	-1.4873	-0.9909	0.9909	-0.5088	0.5088	-0.8095
3	-49.8415	-10.7815	60.623	-1.618	1.618	-4.7052	4.7052	1.6752	-1.6752	2.4742	-2.4742	-1.456
4	1.2119	0.4061	-1.618	25.9886	-25.9886	2.8693	-2.8693	-0.9118	0.9118	-1.1335	1.1335	0.8402
5	-1.2119	-0.4061	1.618	-25.9886	25.9886	-2.8693	2.8693	0.9118	-0.9118	1.1335	-1.1335	-0.8402
6	3.2179	1.4873	-4.7052	2.8693	-2.8693	62.0779	-62.0779	-1.9431	1.9431	-2.4068	2.4068	1.9621
7	-3.2179	-1.4873	4.7052	-2.8693	2.8693	-62.0779	62.0779	1.9431	-1.9431	2.4068	-2.4068	-1.9621
8	-0.6844	-0.9909	1.6752	-0.9118	0.9118	-1.9431	1.9431	31.5713	-31.5713	1.6797	-1.6797	-0.6331
9	0.6844	0.9909	-1.6752	0.9118	-0.9118	1.9431	-1.9431	-31.5713	31.5713	-1.6797	1.6797	0.6331
10	-1.9653	-0.5088	2.4742	-1.1335	1.1335	-2.4068	2.4068	1.6797	-1.6797	43.6062	-43.6062	-1.1195
11	1.9653	0.5088	-2.4742	1.1335	-1.1335	2.4068	-2.4068	-1.6797	1.6797	-43.6062	43.6062	1.1195
12	2.2655	-0.8095	-1.456	0.8402	-0.8402	1.9621	-1.9621	-0.6331	0.6331	-1.1195	1.1195	80.4303
13	-2.8408	0.3827	2.4582	-0.936	0.936	-3.7005	3.7005	1.6204	-1.6204	2.3648	-2.3648	-41.236
14	0.5753	0.4269	-1.0022	0.0957	-0.0957	1.7385	-1.7385	-0.9873	0.9873	-1.2454	1.2454	-39.1943
15	8.3224	3.5142	-11.8366	5.8939	-5.8939	11.1799	-11.1799	-4.6179	4.6179	-6.6175	6.6175	2.3624
16	2.3598	0.5866	-2.9464	1.3992	-1.3992	5.1329	-5.1329	-2.6093	2.6093	-1.8401	1.8401	0.7974
17	-10.6822	-4.1009	14.7831	-7.2931	7.2931	-16.3128	16.3128	7.2272	-7.2272	8.4576	-8.4576	-3.1598
18	3.9434	1.4081	-5.3516	2.1632	-2.1632	6.2129	-6.2129	-1.8138	1.8138	-2.7718	2.7718	1.3599
19	-3.9434	-1.4081	5.3516	-2.1632	2.1632	-6.2129	6.2129	1.8138	-1.8138	2.7718	-2.7718	-1.3599
20	5.0872	1.7197	-6.8069	2.9898	-2.9898	9.0725	-9.0725	-3.7124	3.7124	-4.4856	4.4856	1.7265
21	-5.0872	-1.7197	6.8069	-2.9898	2.9898	-9.0725	9.0725	3.7124	-3.7124	4.4856	-4.4856	-1.7265
22	-4.651	-1.4123	6.0633	-1.8849	1.8849	-5.8355	5.8355	3.4649	-3.4649	3.7894	-3.7894	-2.1449
23	4.651	1.4123	-6.0633	1.8849	-1.8849	5.8355	-5.8355	-3.4649	3.4649	-3.7894	3.7894	2.1449
24	-4.3616	-1.4547	5.8164	-2.9052	2.9052	-6.4323	6.4323	3.3572	-3.3572	3.3214	-3.3214	-1.3538
25	4.3616	1.4547	-5.8164	2.9052	-2.9052	6.4323	-6.4323	-3.3572	3.3572	-3.3214	3.3214	1.3538
26	0.8632	1.2292	-2.0923	1.3796	-1.3796	2.2563	-2.2563	-1.4942	1.4942	-1.9346	1.9346	1.9424
27	-2.4072	-0.7736	3.1808	-1.6	1.6	-3.6362	3.6362	2.0511	-2.0511	2.7973	-2.7973	-0.7348
28	1.544	-0.4555	-1.0885	0.2204	-0.2204	1.3799	-1.3799	-0.557	0.557	-0.8626	0.8626	-1.2076
29	8.4123	3.5217	-11.9339	6.2693	-6.2693	14.3584	-14.3584	-6.0897	6.0897	-6.3988	6.3988	2.4692
30	2.2467	1.3128	-3.5594	1.2194	-1.2194	4.7878	-4.7878	-2.2016	2.2016	-1.0514	1.0514	2.4094
31	-10.659	-4.8344	15.4934	-7.4887	7.4887	-19.1462	19.1462	8.2913	-8.2913	7.4502	-7.4502	-4.8787
32	6.4662	3.1677	-9.6339	4.1891	-4.1891	12.1537	-12.1537	-5.7034	5.7034	-5.3317	5.3317	3.4152
33	-6.4662	-3.1677	9.6339	-4.1891	4.1891	-12.1537	12.1537	5.7034	-5.7034	5.3317	-5.3317	-3.4152
34	6.4209	2.3748	-8.7957	3.9965	-3.9965	12.0584	-12.0584	-5.0201	5.0201	-4.5509	4.5509	3.1217
35	-6.4209	-2.3748	8.7957	-3.9965	3.9965	-12.0584	12.0584	5.0201	-5.0201	4.5509	-4.5509	-3.1217
36	-7.4391	-2.9285	10.3676	-4.9608	4.9608	-13.5282	13.5282	5.7865	-5.7865	7.0748	-7.0748	-4.215
37	7.4391	2.9285	-10.3676	4.9608	-4.9608	13.5282	-13.5282	-5.7865	5.7865	-7.0748	7.0748	4.215
38	-6.6292	-2.5624	9.1916	-3.3728	3.3728	-10.1613	10.1613	4.8097	-4.8097	4.9958	-4.9958	-3.38
39	6.6292	2.5624	-9.1916	3.3728	-3.3728	10.1613	-10.1613	-4.8097	4.8097	-4.9958	4.9958	3.38
40	2.7893	0.6712	-3.4605	1.9789	-1.9789	5.9432	-5.9432	-2.5539	2.5539	-1.8757	1.8757	0.9056
41	-5.5586	-0.5285	6.0871	-2.6658	2.6658	-8.2852	8.2852	3.7451	-3.7451	3.3284	-3.3284	-1.2769
42	2.7693	-0.1427	-2.6266	0.6869	-0.6869	2.342	-2.342	-1.1912	1.1912	-1.4527	1.4527	0.3713
43	6.5522	3.5896	-10.1418	4.1101	-4.1101	10.2978	-10.2978	-4.7611	4.7611	-5.2348	5.2348	2.1373
44	1.2143	-0.3481	-0.8662	0.7201	-0.7201	2.3911	-2.3911	-0.4486	0.4486	-0.2642	0.2642	0.2473
45	-7.7665	-3.2415	11.008	-4.8302	4.8302	-12.689	12.689	5.2097	-5.2097	5.4989	-5.4989	-2.3846
46	11.0019	3.5619	-14.5638	7.2614	-7.2614	19.659	-19.659	-7.3228	7.3228	-6.5311	6.5311	2.5945
47	-11.0019	-3.5619	14.5638	-7.2614	7.2614	-19.659	19.659	7.3228	-7.3228	6.5311	-6.5311	-2.5945
48	6.1722	3.1134	-9.2856	4.1312	-4.1312	11.7012	-11.7012	-3.511	3.511	-3.8617	3.8617	1.5893
49	-6.1722	-3.1134	9.2856	-4.1312	4.1312	-11.7012	11.7012	3.511	-3.511	3.8617	-3.8617	-1.5893
50	-6.5184	-3.5606	10.079	-4.3272	4.3272	-12.3978	12.3978	5.7205	-5.7205	5.2218	-5.2218	-2.3943
51	6.5184	3.5606	-10.079	4.3272	-4.3272	12.3978	-12.3978	-5.7205	5.7205	-5.2218	5.2218	2.3943
52	-9.164	-3.7071	12.8711	-5.3598	5.3598	-15.1998	15.1998	6.4408	-6.4408	5.7553	-5.7553	-2.2845
53	9.164	3.7071	-12.8711	5.3598	-5.3598	15.1998	-15.1998	-6.4408	6.4408	-5.7553	5.7553	2.2845
54	3.9174	1.2112	-5.1286	2.0191	-2.0191	6.779	-6.779	-3.3677	3.3677	-2.6715	2.6715	0.2575
55	-4.7941	-2.4976	7.2918	-3.0096	3.0096	-8.487	8.487	4.3267	-4.3267	3.3309	-3.3309	-0.8653
56	0.8767	1.2864	-2.1631	0.9905	-0.9905	1.7079	-1.7079	-0.959	0.959	-0.6594	0.6594	0.6078

C5. Variance-Covariance Matrix by Simulation Method

Columns 13-24
Rows 1-56

	13	14	15	16	17	18	19	20	21	22	23	24
1	-2.8408	0.5753	8.3224	2.3598	-10.6822	3.9434	-3.9434	5.0872	-5.0872	-4.651	4.651	-4.3616
2	0.3827	0.4269	3.5142	0.5866	-4.1009	1.4081	-1.4081	1.7197	-1.7197	-1.4123	1.4123	-1.4547
3	2.4582	-1.0022	-11.8366	-2.9464	14.7831	-5.3516	5.3516	-6.8069	6.8069	6.0633	-6.0633	5.8164
4	-0.936	0.0957	5.8939	1.3992	-7.2931	2.1632	-2.1632	2.9898	-2.9898	-1.8849	1.8849	-2.9052
5	0.936	-0.0957	-5.8939	-1.3992	7.2931	-2.1632	2.1632	-2.9898	2.9898	1.8849	-1.8849	2.9052
6	-3.7005	1.7385	11.1799	5.1329	-16.3128	6.2129	-6.2129	9.0725	-9.0725	-5.8355	5.8355	-6.4323
7	3.7005	-1.7385	-11.1799	-5.1329	16.3128	-6.2129	6.2129	-9.0725	9.0725	5.8355	-5.8355	6.4323
8	1.6204	-0.9873	-4.6179	-2.6093	7.2272	-1.8138	1.8138	-3.7124	3.7124	3.4649	-3.4649	3.3572
9	-1.6204	0.9873	4.6179	2.6093	-7.2272	1.8138	-1.8138	3.7124	-3.7124	-3.4649	3.4649	-3.3572
10	2.3648	-1.2454	-6.6175	-1.8401	8.4576	-2.7718	2.7718	-4.4856	4.4856	3.7894	-3.7894	3.3214
11	-2.3648	1.2454	6.6175	1.8401	-8.4576	2.7718	-2.7718	4.4856	-4.4856	-3.7894	3.7894	-3.3214
12	-41.236	-39.1943	2.3624	0.7974	-3.1598	1.3599	-1.3599	1.7265	-1.7265	-2.1449	2.1449	-1.3538
13	56.4369	-15.2009	-6.8946	-2.3276	9.2222	-3.202	3.202	-6.0158	6.0158	4.8641	-4.8641	4.7911
14	-15.2009	54.3951	4.5322	1.5302	-6.0624	1.8421	-1.8421	4.2893	-4.2893	-2.7193	2.7193	-3.4373
15	-6.8946	4.5322	190.2352	-54.4383	-135.797	22.4785	-22.4785	32.2755	-32.2755	-19.3436	19.3436	-21.1164
16	-2.3276	1.5302	-54.4383	91.7544	-37.3161	9.5497	-9.5497	13.4663	-13.4663	-8.5426	8.5426	-8.0237
17	9.2222	-6.0624	-135.797	-37.3161	173.113	-32.0282	32.0282	-45.7418	45.7418	27.8862	-27.8862	29.1401
18	-3.202	1.8421	22.4785	9.5497	-32.0282	43.18	-43.18	17.2748	-17.2748	-10.3894	10.3894	-10.8835
19	3.202	-1.8421	-22.4785	-9.5497	32.0282	-43.18	43.18	-17.2748	17.2748	10.3894	-10.3894	10.8835
20	-6.0158	4.2893	32.2755	13.4663	-45.7418	17.2748	-17.2748	70.2933	-70.2933	-15.495	15.495	-16.1704
21	6.0158	-4.2893	-32.2755	-13.4663	45.7418	-17.2748	17.2748	-70.2933	70.2933	15.495	-15.495	16.1704
22	4.8641	-2.7193	-19.3436	-8.5426	27.8862	-10.3894	10.3894	-15.495	15.495	46.1535	-46.1535	11.3745
23	-4.8641	2.7193	19.3436	8.5426	-27.8862	10.3894	-10.3894	15.495	-15.495	-46.1535	46.1535	-11.3745
24	4.7911	-3.4373	-21.1164	-8.0237	29.1401	-10.8835	10.8835	-16.1704	16.1704	11.3745	-11.3745	53.0848
25	-4.7911	3.4373	21.1164	8.0237	-29.1401	10.8835	-10.8835	16.1704	-16.1704	-11.3745	11.3745	-53.0848
26	-2.4966	0.5542	8.9172	3.0667	-11.9838	4.9139	-4.9139	6.7387	-6.7387	-4.4241	4.4241	-4.7048
27	3.2073	-2.4725	-14.469	-5.1679	19.6369	-7.4959	7.4959	-10.8822	10.8822	8.0586	-8.0586	8.9471
28	-0.7107	1.9183	5.5519	2.1013	-7.6531	2.5821	-2.5821	4.1435	-4.1435	-3.6345	3.6345	-4.2423
29	-9.4153	6.946	66.8502	30.7599	-97.6101	37.8892	-37.8892	53.5695	-53.5695	-34.2204	34.2204	-34.3251
30	-2.7849	0.3755	20.6481	10.9647	-31.6128	12.004	-12.004	17.9458	-17.9458	-10.6173	10.6173	-11.2592
31	12.2002	-7.3215	-87.4983	-41.7246	129.223	-49.8932	49.8932	-71.5153	71.5153	44.8376	-44.8376	45.5842
32	-8.4159	5.0007	56.9535	28.8891	-85.8426	32.8238	-32.8238	47.5026	-47.5026	-29.2616	29.2616	-30.1172
33	8.4159	-5.0007	-56.9535	-28.8891	85.8426	-32.8238	32.8238	-47.5026	47.5026	29.2616	-29.2616	30.1172
34	-7.3988	4.2772	52.6768	26.6535	-79.3303	30.3263	-30.3263	44.4917	-44.4917	-27.9578	27.9578	-27.258
35	7.3988	-4.2772	-52.6768	-26.6535	79.3303	-30.3263	30.3263	-44.4917	44.4917	27.9578	-27.9578	27.258
36	9.9609	-5.7459	-58.8495	-29.6662	88.5157	-34.0788	34.0788	-49.7783	49.7783	31.5733	-31.5733	32.1275
37	-9.9609	5.7459	58.8495	29.6662	-88.5157	34.0788	-34.0788	49.7783	-49.7783	-31.5733	31.5733	-32.1275
38	8.0687	-4.6887	-48.1543	-25.0911	73.2455	-27.8841	27.8841	-40.5104	40.5104	26.1699	-26.1699	27.4015
39	-8.0687	4.6887	48.1543	25.0911	-73.2455	27.8841	-27.8841	40.5104	-40.5104	-26.1699	26.1699	-27.4015
40	-2.8623	1.9568	20.5438	10.1189	-30.6627	11.1408	-11.1408	17.3458	-17.3458	-10.177	10.177	-11.4384
41	5.0343	-3.7574	-33.8085	-16.8816	50.6901	-18.3841	18.3841	-28.1108	28.1108	17.8684	-17.8684	18.3026
42	-2.1719	1.8006	13.2647	6.7627	-20.0273	7.2434	-7.2434	10.765	-10.765	-7.6915	7.6915	-6.8641
43	-7.5128	5.3755	58.0275	30.2999	-88.3274	33.7772	-33.7772	47.9902	-47.9902	-29.204	29.204	-30.3954
44	-0.5433	0.2961	9.9353	5.6056	-15.541	6.0995	-6.0995	9.2716	-9.2716	-5.4677	5.4677	-4.8825
45	8.0561	-5.6715	-67.9628	-35.9055	103.8683	-39.8767	39.8767	-57.2618	57.2618	34.6716	-34.6716	35.2779
46	-11.9693	9.3749	96.5367	52.6388	-149.176	58.279	-58.279	84.1172	-84.1172	-50.1078	50.1078	-50.4376
47	11.9693	-9.3749	-96.5367	-52.6388	149.1755	-58.279	58.279	-84.1172	84.1172	50.1078	-50.1078	50.4376
48	-7.0701	5.4808	55.019	30	-85.019	33.6957	-33.6957	47.7029	-47.7029	-29.2309	29.2309	-28.8327
49	7.0701	-5.4808	-55.019	-30	85.019	-33.6957	33.6957	-47.7029	47.7029	29.2309	-29.2309	28.8327
50	7.9319	-5.5376	-61.9868	-32.9091	94.8959	-36.5419	36.5419	-53.3245	53.3245	32.4151	-32.4151	32.991
51	-7.9319	5.5376	61.9868	32.9091	-94.8959	36.5419	-36.5419	53.3245	-53.3245	-32.4151	32.4151	-32.991
52	9.8339	-7.5495	-75.7703	-40.4997	116.27	-45.7311	45.7311	-65.5209	65.5209	41.5338	-41.5338	40.4594
53	-9.8339	7.5495	75.7703	40.4997	-116.27	45.7311	-45.7311	65.5209	-65.5209	-41.5338	41.5338	-40.4594
54	-3.4135	3.156	30.6863	15.7364	-46.4227	17.5917	-17.5917	25.2597	-25.2597	-15.7159	15.7159	-15.294
55	5.2772	-4.4119	-44.1354	-22.724	66.8594	-26.4039	26.4039	-37.1313	37.1313	22.7438	-22.7438	22.7612
56	-1.8637	1.2559	13.4491	6.9876	-20.4367	8.8122	-8.8122	11.8716	-11.8716	-7.0279	7.0279	-7.4672

C5. Variance-Covariance Matrix by Simulation Method

Columns 25-36
Rows 1-56

	25	26	27	28	29	30	31	32	33	34	35	36
1	4.3616	0.8632	-2.4072	1.544	8.4123	2.2467	-10.659	6.4662	-6.4662	6.4209	-6.4209	-7.4391
2	1.4547	1.2292	-0.7736	-0.4555	3.5217	1.3128	-4.8344	3.1677	-3.1677	2.3748	-2.3748	-2.9285
3	-5.8164	-2.0923	3.1808	-1.0885	-11.9339	-3.5594	15.4934	-9.6339	9.6339	-8.7957	8.7957	10.3676
4	2.9052	1.3796	-1.6	0.2204	6.2693	1.2194	-7.4887	4.1891	-4.1891	3.9965	-3.9965	-4.9608
5	-2.9052	-1.3796	1.6	-0.2204	-6.2693	-1.2194	7.4887	-4.1891	4.1891	-3.9965	3.9965	4.9608
6	6.4323	2.2563	-3.6362	1.3799	14.3584	4.7878	-19.1462	12.1537	-12.1537	12.0584	-12.0584	-13.5282
7	-6.4323	-2.2563	3.6362	-1.3799	-14.3584	-4.7878	19.1462	-12.1537	12.1537	-12.0584	12.0584	13.5282
8	-3.3572	-1.4942	2.0511	-0.557	-6.0897	-2.2016	8.2913	-5.7034	5.7034	-5.0201	5.0201	5.7865
9	3.3572	1.4942	-2.0511	0.557	6.0897	2.2016	-8.2913	5.7034	-5.7034	5.0201	-5.0201	-5.7865
10	-3.3214	-1.9346	2.7973	-0.8626	-6.3988	-1.0514	7.4502	-5.3317	5.3317	-4.5509	4.5509	7.0748
11	3.3214	1.9346	-2.7973	0.8626	6.3988	1.0514	-7.4502	5.3317	-5.3317	4.5509	-4.5509	-7.0748
12	1.3538	1.9424	-0.7348	-1.2076	2.4692	2.4094	-4.8787	3.4152	-3.4152	3.1217	-3.1217	-4.215
13	-4.7911	-2.4966	3.2073	-0.7107	-9.4153	-2.7849	12.2002	-8.4159	8.4159	-7.3988	7.3988	9.9609
14	3.4373	0.5542	-2.4725	1.9183	6.946	0.3755	-7.3215	5.0007	-5.0007	4.2772	-4.2772	-5.7459
15	21.1164	8.9172	-14.469	5.5519	66.8502	20.6481	-87.4983	56.9535	-56.9535	52.6768	-52.6768	-58.8495
16	8.0237	3.0667	-5.1679	2.1013	30.7599	10.9647	-41.7246	28.8891	-28.8891	26.6535	-26.6535	-29.6662
17	-29.1401	-11.9838	19.6369	-7.6531	-97.6101	-31.6128	129.223	-85.8426	85.8426	-79.3303	79.3303	88.5157
18	10.8835	4.9139	-7.4959	2.5821	37.8892	12.004	-49.8932	32.8238	-32.8238	30.3263	-30.3263	-34.0788
19	-10.8835	-4.9139	7.4959	-2.5821	-37.8892	-12.004	49.8932	-32.8238	32.8238	-30.3263	30.3263	34.0788
20	16.1704	6.7387	-10.8822	4.1435	53.5695	17.9458	-71.5153	47.5026	-47.5026	44.4917	-44.4917	-49.7783
21	-16.1704	-6.7387	10.8822	-4.1435	-53.5695	-17.9458	71.5153	-47.5026	47.5026	-44.4917	44.4917	49.7783
22	-11.3745	-4.4241	8.0586	-3.6345	-34.2204	-10.6173	44.8376	-29.2616	29.2616	-27.9578	27.9578	31.5733
23	11.3745	4.4241	-8.0586	3.6345	34.2204	10.6173	-44.8376	29.2616	-29.2616	27.9578	-27.9578	-31.5733
24	-53.0848	-4.7048	8.9471	-4.2423	-34.3251	-11.2592	45.5842	-30.1172	30.1172	-27.258	27.258	32.1275
25	53.0848	4.7048	-8.9471	4.2423	34.3251	11.2592	-45.5842	30.1172	-30.1172	27.258	-27.258	-32.1275
26	4.7048	59.2841	-27.6389	-31.6452	11.4977	3.2355	-14.7332	10.57	-10.57	9.1502	-9.1502	-12.1752
27	-8.9471	-27.6389	36.8211	-9.1822	-23.3007	-6.9499	30.2506	-20.5882	20.5882	-18.4755	18.4755	22.6195
28	4.2423	-31.6452	-9.1822	40.8274	11.803	3.7144	-15.5174	10.0183	-10.0183	9.3253	-9.3253	-10.4443
29	34.3251	11.4977	-23.3007	11.803	243.7474	6.4262	-250.174	128.4681	-128.468	118.9479	-118.948	-130.246
30	11.2592	3.2355	-6.9499	3.7144	6.4262	70.6063	-77.0326	48.316	-48.316	43.1933	-43.1933	-46.6576
31	-45.5842	-14.7332	30.2506	-15.5174	-250.174	-77.0326	327.2062	-176.784	176.7841	-162.141	162.1412	176.9035
32	30.1172	10.57	-20.5882	10.0183	128.4681	48.316	-176.784	157.1124	-157.112	110.0036	-110.004	-119.803
33	-30.1172	-10.57	20.5882	-10.0183	-128.468	-48.316	176.7841	-157.112	157.1124	-110.004	110.0036	119.8026
34	27.258	9.1502	-18.4755	9.3253	118.9479	43.1933	-162.141	110.0036	-110.004	134.9977	-134.998	-110.679
35	-27.258	-9.1502	18.4755	-9.3253	-118.948	-43.1933	162.1412	-110.004	110.0036	-134.998	134.9977	110.6789
36	-32.1275	-12.1752	22.6195	-10.4443	-130.246	-46.6576	176.9035	-119.803	119.8026	-110.679	110.6789	167.5631
37	32.1275	12.1752	-22.6195	10.4443	130.2459	46.6576	-176.904	119.8026	-119.803	110.6789	-110.679	-167.563
38	-27.4015	-9.1299	18.0271	-8.8972	-107.224	-39.8447	147.0686	-99.2494	99.2494	-92.0006	92.0006	101.5925
39	27.4015	9.1299	-18.0271	8.8972	107.2239	39.8447	-147.069	99.2494	-99.2494	92.0006	-92.0006	-101.593
40	11.4384	4.5622	-7.9257	3.3636	46.0348	14.8378	-60.8726	39.5391	-39.5391	35.6946	-35.6946	-40.3074
41	-18.3026	-6.6942	13.0408	-6.3466	-75.4521	-27.0352	102.4873	-68.1961	68.1961	-62.4517	62.4517	70.0052
42	6.8641	2.132	-5.1151	2.983	29.4173	12.1974	-41.6147	28.657	-28.657	26.7571	-26.7571	-29.6978
43	30.3954	10.1856	-20.1829	9.9973	152.7934	53.1415	-205.935	133.7259	-133.726	121.6932	-121.693	-134.757
44	4.8825	1.1416	-3.46	2.3184	27.1086	11.7146	-38.8233	27.9381	-27.9381	25.8926	-25.8926	-27.0294
45	-35.2779	-11.3272	23.6429	-12.3157	-179.902	-64.8561	244.7582	-161.664	161.664	-147.586	147.5858	161.7866
46	50.4376	17.0724	-34.3319	17.2595	249.7528	98.2855	-348.038	240.376	-240.376	219.848	-219.848	-236.792
47	-50.4376	-17.0724	34.3319	-17.2595	-249.753	-98.2855	348.0383	-240.376	240.376	-219.848	219.848	236.7923
48	28.8327	10.0003	-19.7824	9.7821	143.4597	55.9459	-199.406	137.3228	-137.323	125.7103	-125.71	-134.421
49	-28.8327	-10.0003	19.7824	-9.7821	-143.46	-55.9459	199.4056	-137.323	137.3228	-125.71	125.7103	134.4214
50	-32.991	-10.4795	21.9023	-11.4228	-160.351	-61.0157	221.3662	-150.436	150.4355	-137.97	137.9696	150.8878
51	32.991	10.4795	-21.9023	11.4228	160.3505	61.0157	-221.366	150.4355	-150.436	137.9696	-137.97	-150.888
52	-40.4594	-14.9025	28.8457	-13.9431	-199.227	-74.8729	274.0997	-186.369	186.3688	-171.021	171.0211	187.0607
53	40.4594	14.9025	-28.8457	13.9431	199.2269	74.8729	-274.1	186.3688	-186.369	171.0211	-171.021	-187.061
54	15.294	6.0885	-11.1007	5.0122	78.9451	25.8725	-104.818	67.853	-67.853	62.6366	-62.6366	-68.8243
55	-22.7612	-8.0761	15.9718	-7.8957	-116.095	-41.4411	157.5358	-104.422	104.4215	-95.7425	95.7425	105.8151
56	7.4672	1.9876	-4.871	2.8835	37.1496	15.5685	-52.7182	36.5686	-36.5686	33.1059	-33.1059	-36.9908

C5. Variance-Covariance Matrix by Simulation Method

Columns 37-48
Rows 1-56

	37	38	39	40	41	42	43	44	45	46	47	48
1	7.4391	-6.6292	6.6292	2.7893	-5.5586	2.7693	6.5522	1.2143	-7.7665	11.0019	-11.0019	6.1722
2	2.9285	-2.5624	2.5624	0.6712	-0.5285	-0.1427	3.5896	-0.3481	-3.2415	3.5619	-3.5619	3.1134
3	-10.3676	9.1916	-9.1916	-3.4605	6.0871	-2.6266	-10.1418	-0.8662	11.008	-14.5638	14.5638	-9.2856
4	4.9608	-3.3728	3.3728	1.9789	-2.6658	0.6869	4.1101	0.7201	-4.8302	7.2614	-7.2614	4.1312
5	-4.9608	3.3728	-3.3728	-1.9789	2.6658	-0.6869	-4.1101	-0.7201	4.8302	-7.2614	7.2614	-4.1312
6	13.5282	-10.1613	10.1613	5.9432	-8.2852	2.342	10.2978	2.3911	-12.689	19.659	-19.659	11.7012
7	-13.5282	10.1613	-10.1613	-5.9432	8.2852	-2.342	-10.2978	-2.3911	12.689	-19.659	19.659	-11.7012
8	-5.7865	4.8097	-4.8097	-2.5539	3.7451	-1.1912	-4.7611	-0.4486	5.2097	-7.3228	7.3228	-3.511
9	5.7865	-4.8097	4.8097	2.5539	-3.7451	1.1912	4.7611	0.4486	-5.2097	7.3228	-7.3228	3.511
10	-7.0748	4.9958	-4.9958	-1.8757	3.3284	-1.4527	-5.2348	-0.2642	5.4989	-6.5311	6.5311	-3.8617
11	7.0748	-4.9958	4.9958	1.8757	-3.3284	1.4527	5.2348	0.2642	-5.4989	6.5311	-6.5311	3.8617
12	4.215	-3.38	3.38	0.9056	-1.2769	0.3713	2.1373	0.2473	-2.3846	2.5945	-2.5945	1.5893
13	-9.9609	8.0687	-8.0687	-2.8623	5.0343	-2.1719	-7.5128	-0.5433	8.0561	-11.9693	11.9693	-7.0701
14	5.7459	-4.6887	4.6887	1.9568	-3.7574	1.8006	5.3755	0.2961	-5.6715	9.3749	-9.3749	5.4808
15	58.8495	-48.1543	48.1543	20.5438	-33.8085	13.2647	58.0275	9.9353	-67.9628	96.5367	-96.5367	55.019
16	29.6662	-25.0911	25.0911	10.1189	-16.8816	6.7627	30.2999	5.6056	-35.9055	52.6388	-52.6388	30
17	-88.5157	73.2455	-73.2455	-30.6627	50.6901	-20.0273	-88.3274	-15.541	103.8683	-149.176	149.1755	-85.019
18	34.0788	-27.8841	27.8841	11.1408	-18.3841	7.2434	33.7772	6.0995	-39.8767	58.279	-58.279	33.6957
19	-34.0788	27.8841	-27.8841	-11.1408	18.3841	-7.2434	-33.7772	-6.0995	39.8767	-58.279	58.279	-33.6957
20	49.7783	-40.5104	40.5104	17.3458	-28.1108	10.765	47.9902	9.2716	-57.2618	84.1172	-84.1172	47.7029
21	-49.7783	40.5104	-40.5104	-17.3458	28.1108	-10.765	-47.9902	-9.2716	57.2618	-84.1172	84.1172	-47.7029
22	-31.5733	26.1699	-26.1699	-10.177	17.8684	-7.6915	-29.204	-5.4677	34.6716	-50.1078	50.1078	-29.2309
23	31.5733	-26.1699	26.1699	10.177	-17.8684	7.6915	29.204	5.4677	-34.6716	50.1078	-50.1078	29.2309
24	-32.1275	27.4015	-27.4015	-11.4384	18.3026	-6.8641	-30.3954	-4.8825	35.2779	-50.4376	50.4376	-28.8327
25	32.1275	-27.4015	27.4015	11.4384	-18.3026	6.8641	30.3954	4.8825	-35.2779	50.4376	-50.4376	28.8327
26	12.1752	-9.1299	9.1299	4.5622	-6.6942	2.132	10.1856	1.1416	-11.3272	17.0724	-17.0724	10.0003
27	-22.6195	18.0271	-18.0271	-7.9257	13.0408	-5.1151	-20.1829	-3.46	23.6429	-34.3319	34.3319	-19.7824
28	10.4443	-8.8972	8.8972	3.3636	-6.3466	2.983	9.9973	2.3184	-12.3157	17.2595	-17.2595	9.7821
29	130.2459	-107.224	107.2239	46.0348	-75.4521	29.4173	152.7934	27.1086	-179.902	249.7528	-249.753	143.4597
30	46.6576	-39.8447	39.8447	14.8378	-27.0352	12.1974	53.1415	11.7146	-64.8561	98.2855	-98.2855	55.9459
31	-176.904	147.0686	-147.069	-60.8726	102.4873	-41.6147	-205.935	-38.8233	244.7582	-348.038	348.0383	-199.406
32	119.8026	-99.2494	99.2494	39.5391	-68.1961	28.657	133.7259	27.9381	-161.664	240.376	-240.376	137.3228
33	-119.803	99.2494	-99.2494	-39.5391	68.1961	-28.657	-133.726	-27.9381	161.664	-240.376	240.376	-137.323
34	110.6789	-92.0006	92.0006	35.6946	-62.4517	26.7571	121.6932	25.8926	-147.586	219.848	-219.848	125.7103
35	-110.679	92.0006	-92.0006	-35.6946	62.4517	-26.7571	-121.693	-25.8926	147.5858	-219.848	219.848	-125.71
36	-167.563	101.5925	-101.593	-40.3074	70.0052	-29.6978	-134.757	-27.0294	161.7866	-236.792	236.7923	-134.421
37	167.5631	-101.593	101.5925	40.3074	-70.0052	29.6978	134.7572	27.0294	-161.787	236.7923	-236.792	134.4214
38	-101.593	124.125	-124.125	-34.512	58.3672	-23.8552	-111.731	-22.607	134.3375	-196.033	196.0328	-111.816
39	101.5925	-124.125	124.125	34.512	-58.3672	23.8552	111.7305	22.607	-134.338	196.0328	-196.033	111.8158
40	40.3074	-34.512	34.512	77.15	-47.7054	-29.4446	46.4359	8.0658	-54.5018	75.5688	-75.5688	43.2984
41	-70.0052	58.3672	-58.3672	-47.7054	69.969	-22.2636	-78.4359	-14.8839	93.3198	-133.447	133.4469	-76.1986
42	29.6978	-23.8552	23.8552	-29.4446	-22.2636	51.7082	32	6.818	-38.818	57.8781	-57.8781	32.9001
43	134.7572	-111.731	111.7305	46.4359	-78.4359	32	237.1154	4.5178	-241.633	283.2857	-283.286	160.6208
44	27.0294	-22.607	22.607	8.0658	-14.8839	6.818	4.5178	38.1909	-42.7087	62.1356	-62.1356	35.3819
45	-161.787	134.3375	-134.338	-54.5018	93.3198	-38.818	-241.633	-42.7087	284.3419	-345.421	345.4214	-196.003
46	236.7923	-196.033	196.0328	75.5688	-133.447	57.8781	283.2857	62.1356	-345.421	567.4258	-567.426	295.841
47	-236.792	196.0328	-196.033	-75.5688	133.4469	-57.8781	-283.286	-62.1356	345.4214	-567.426	567.4258	-295.841
48	134.4214	-111.816	111.8158	43.2984	-76.1986	32.9001	160.6208	35.3819	-196.003	295.841	-295.841	194.041
49	-134.421	111.8158	-111.816	-43.2984	76.1986	-32.9001	-160.621	-35.3819	196.0027	-295.841	295.841	-194.041
50	-150.888	125.0443	-125.044	-48.5406	85.4582	-36.9177	-181.628	-37.8812	219.5092	-322.65	322.6495	-183.734
51	150.8878	-125.044	125.0443	48.5406	-85.4582	36.9177	181.628	37.8812	-219.509	322.6495	-322.65	183.7344
52	-187.061	155.2053	-155.205	-60.7742	106.6202	-45.846	-225.04	-46.5551	271.5953	-399.365	399.3652	-227.814
53	187.0607	-155.205	155.2053	60.7742	-106.62	45.846	225.0402	46.5551	-271.595	399.3652	-399.365	227.8137
54	68.8243	-57.6183	57.6183	25.4321	-41.4416	16.0095	93.7301	14.2289	-107.959	140.8988	-140.899	81.0079
55	-105.815	87.627	-87.627	-35.7603	61.2256	-25.4653	-135.225	-24.4882	159.7134	-221.656	221.6557	-126.26
56	36.9908	-30.0087	30.0087	10.3282	-19.784	9.4558	41.495	10.2593	-51.7543	80.7568	-80.7568	45.2523

C5. Variance-Covariance Matrix by Simulation Method

Columns 49-56
Rows 1-56

	49	50	51	52	53	54	55	56
1	-6.1722	-6.5184	6.5184	-9.164	9.164	3.9174	-4.7941	0.8767
2	-3.1134	-3.5606	3.5606	-3.7071	3.7071	1.2112	-2.4976	1.2864
3	9.2856	10.079	-10.079	12.8711	-12.8711	-5.1286	7.2918	-2.1631
4	-4.1312	-4.3272	4.3272	-5.3598	5.3598	2.0191	-3.0096	0.9905
5	4.1312	4.3272	-4.3272	5.3598	-5.3598	-2.0191	3.0096	-0.9905
6	-11.7012	-12.3978	12.3978	-15.1998	15.1998	6.779	-8.487	1.7079
7	11.7012	12.3978	-12.3978	15.1998	-15.1998	-6.779	8.487	-1.7079
8	3.511	5.7205	-5.7205	6.4408	-6.4408	-3.3677	4.3267	-0.959
9	-3.511	-5.7205	5.7205	-6.4408	6.4408	3.3677	-4.3267	0.959
10	3.8617	5.2218	-5.2218	5.7553	-5.7553	-2.6715	3.3309	-0.6594
11	-3.8617	-5.2218	5.2218	-5.7553	5.7553	2.6715	-3.3309	0.6594
12	-1.5893	-2.3943	2.3943	-2.2845	2.2845	0.2575	-0.8653	0.6078
13	7.0701	7.9319	-7.9319	9.8339	-9.8339	-3.4135	5.2772	-1.8637
14	-5.4808	-5.5376	5.5376	-7.5495	7.5495	3.156	-4.4119	1.2559
15	-55.019	-61.9868	61.9868	-75.7703	75.7703	30.6863	-44.1354	13.4491
16	-30	-32.9091	32.9091	-40.4997	40.4997	15.7364	-22.724	6.9876
17	85.019	94.8959	-94.8959	116.27	-116.27	-46.4227	66.8594	-20.4367
18	-33.6957	-36.5419	36.5419	-45.7311	45.7311	17.5917	-26.4039	8.8122
19	33.6957	36.5419	-36.5419	45.7311	-45.7311	-17.5917	26.4039	-8.8122
20	-47.7029	-53.3245	53.3245	-65.5209	65.5209	25.2597	-37.1313	11.8716
21	47.7029	53.3245	-53.3245	65.5209	-65.5209	-25.2597	37.1313	-11.8716
22	29.2309	32.4151	-32.4151	41.5338	-41.5338	-15.7159	22.7438	-7.0279
23	-29.2309	-32.4151	32.4151	-41.5338	41.5338	15.7159	-22.7438	7.0279
24	28.8327	32.991	-32.991	40.4594	-40.4594	-15.294	22.7612	-7.4672
25	-28.8327	-32.991	32.991	-40.4594	40.4594	15.294	-22.7612	7.4672
26	-10.0003	-10.4795	10.4795	-14.9025	14.9025	6.0885	-8.0761	1.9876
27	19.7824	21.9023	-21.9023	28.8457	-28.8457	-11.1007	15.9718	-4.871
28	-9.7821	-11.4228	11.4228	-13.9431	13.9431	5.0122	-7.8957	2.8835
29	-143.46	-160.351	160.3505	-199.227	199.2269	78.9451	-116.095	37.1496
30	-55.9459	-61.0157	61.0157	-74.8729	74.8729	25.8725	-41.4411	15.5685
31	199.4056	221.3662	-221.366	274.0997	-274.1	-104.818	157.5358	-52.7182
32	-137.323	-150.436	150.4355	-186.369	186.3688	67.853	-104.422	36.5686
33	137.3228	150.4355	-150.436	186.3688	-186.369	-67.853	104.4215	-36.5686
34	-125.71	-137.97	137.9696	-171.021	171.0211	62.6366	-95.7425	33.1059
35	125.7103	137.9696	-137.97	171.0211	-171.021	-62.6366	95.7425	-33.1059
36	134.4214	150.8878	-150.888	187.0607	-187.061	-68.8243	105.8151	-36.9908
37	-134.421	-150.888	150.8878	-187.061	187.0607	68.8243	-105.815	36.9908
38	111.8158	125.0443	-125.044	155.2053	-155.205	-57.6183	87.627	-30.0087
39	-111.816	-125.044	125.0443	-155.205	155.2053	57.6183	-87.627	30.0087
40	-43.2984	-48.5406	48.5406	-60.7742	60.7742	25.4321	-35.7603	10.3282
41	76.1986	85.4582	-85.4582	106.6202	-106.62	-41.4416	61.2256	-19.784
42	-32.9001	-36.9177	36.9177	-45.846	45.846	16.0095	-25.4653	9.4558
43	-160.621	-181.628	181.628	-225.04	225.0402	93.7301	-135.225	41.495
44	-35.3819	-37.8812	37.8812	-46.5551	46.5551	14.2289	-24.4882	10.2593
45	196.0027	219.5092	-219.509	271.5953	-271.595	-107.959	159.7134	-51.7543
46	-295.841	-322.65	322.6495	-399.365	399.3652	140.8988	-221.656	80.7568
47	295.841	322.6495	-322.65	399.3652	-399.365	-140.899	221.6557	-80.7568
48	-194.041	-183.734	183.7344	-227.814	227.8137	81.0079	-126.26	45.2523
49	194.041	183.7344	-183.734	227.8137	-227.814	-81.0079	126.2602	-45.2523
50	183.7344	234.9809	-234.981	252.672	-252.672	-92.6553	142.534	-49.8786
51	-183.734	-234.981	234.9809	-252.672	252.672	92.6553	-142.534	49.8786
52	227.8137	252.672	-252.672	350.4059	-350.406	-114.504	176.0223	-61.5187
53	-227.814	-252.672	252.672	-350.406	350.4059	114.5036	-176.022	61.5187
54	-81.0079	-92.6553	92.6553	-114.504	114.5036	98.0856	-89.4239	-8.6617
55	126.2602	142.534	-142.534	176.0223	-176.022	-89.4239	127.2315	-37.8076
56	-45.2523	-49.8786	49.8786	-61.5187	61.5187	-8.6617	-37.8076	46.4693