BOUNDARY CONTROL PROBLEMS FOR THE MULTI-DIMENSIONAL DIFFUSION EQUATION

by

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ABSTRACT

The temperature distribution of a heated body can be described by a well-known parabolic initial-boundary value problem. In this thesis, we consider some control-theoretic questions arising in connection with such a heating process.

Suppose that we can vary (subject to certain restrictions) the temperature of the medium which surrounds the body. The task is to choose the temperature under the given restrictions in such a manner that the temperature distribution of the body at a time T comes as close as possible to some desired temperature.

Assume we can reach to the mentioned desired temperature at time level T. We consider the optimal control of the above control problem. This problem consists of finding an admissible boundary control, to minimize an objective functional, which in general depends on time, positions of the points of the body and the control variable.

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CHAPTER 1

Introduction

(1.1) Boundary control of the diffusion equation in arbitrary dimensions

The temperature distribution of a heated body ω can be described by a well-known parabolic initial-boundary value problem. In this thesis, we consider some control-theoretic questions arising in connection with such a heating process.

We give a short explanation in technological terms. Suppose that we can vary (subject to certain restrictions) the temperature $u(t,\xi)$ ($t \in [0,T]$, $\xi \in \partial\omega$) of the medium which surrounds the body ω. Here $[0,T]$ is a fixed time interval, $\partial\omega$ the boundary of the domain ω. The task is to choose u under the given restrictions in such a manner that the temperature distribution $Y(T,x)$ ($x \in \omega$) of the body at time T comes as close as possible to some desired temperature $g(x)$, $x \in \omega$.

Problems of this type have been considered by YEGOROV [1], PLOTNIKOV [1], BUTKOVSKIY [1]. Related questions were studied by many authors. Let us mention only FATTORINI [1], [2], LIONS [1] GLASHOFF and WECK [1]. The main distinction between these publications is a measure for the deviation of $Y(T,x)$ from $g(x)$, for example, some use supremum-norm (the norm in $C(\bar{\omega})$ of $Y(T,.) - g(.)$).

(1.2) An optimal control problem for the one-dimensional diffusion equation.

We consider a control system whose evolution in time is described by a function $Y = Y(x,t)$, defined in $(0,1) X (0,T)$, where T is positive, satisfying

$$Y_{xx}(x,t) = Y_t(x,t), \quad (x,t) \in (0,1) X (0,T) \tag{1.1}$$

with boundary conditions

$$Y_x(0,t) = 0, \quad t \in [0,T]$$

$$Y_x(1,t) = u(t), \quad t \in [0,T] \tag{1.2}$$

$$Y(x,0) = 0, \quad x \in [0,1]$$

where $t \in [0,T] \to u(t)$, is the control function. We define the control u to be *admissible* if it is measurable function on $[0,T]$ and

(a) $u(t) \in [-1,1]$, a.e. for $t \in [0,T]$

(b) $Y(x,T) = g(x)$ a.e. for $x \in [0,1]$.

$g \in L_2(0,1)$, is the desired final state.
Let U be the set of admissible controls. In general this set may be empty, there are many control problems, even in one dimensional state space, without solution because the desired final state can not be reached by means of an admissible control. But it is known from FATTORINI and RUSSELL [1] that a control

system described by the one-dimensional heat equation in an interval (say, $0 \le x \le \pi$) is *nullcontrollable* in any time $T > 0$ by a boundary control applied at one endpoint; that is, given $T > 0$ and $y_0(.) \in L_2[0,\pi]$ such that if $Y = Y(x,t)$ denotes the solution of the equation

$$Y_t = Y_{xx}, \qquad 0 < x < \pi,\ 0 < t < T, \tag{1.3}$$

with initial and boundary conditions

$$\begin{aligned} Y(x,0) &= y_0(x), && 0 \le x \le \pi \\ Y(0,t) &= 0, && 0 \le x \le T, \\ Y(\pi,t) &= u(t), && 0 < t < T \end{aligned} \tag{1.4}$$

then Y also satisfies

$$Y(x,T) = 0. \qquad 0 \le x \le \pi. \tag{1.5}$$

It is well known that the problem (1.3)-(1.5) has a unique solution in a sense made explicit in [FATTORINI and RUSSELL] [1]. Furthermore, there is something very special about the diffusion equation: the set of states which can be reached by means of controls in $L_2(0,T)$ is dense in $L_2(0,1)$ (see, for example, MACCAY, MIZEL and SIEDMAN [1]).

We may reduce the above control problem to moment problems. These moment problems will be studied by employing methods developed by KACZMARZ & STENHAUS [1], PALEY & WIENER [1], R.M.REDHEFFER [1], FATTORINI & RUSSELL [1], WILSON & RUBIO [1] and RUBIO & WILSON [1]. Particular cases of the controllability problems described above have been treated by EGOROV [1], [2] and GALCHUK [1] among others.

We define in the following an optimal control problem associated with the above control problem. Let U, the set of admissible controls, be non-empty, and let the optimal control problem consist of finding a $u \in U$ which minimizes the functional

$$I(u) = \int_0^T f^0(t,u(t))dt, \qquad (1.6)$$

where $f^0 \in C(\Omega)$, the space of continuous functions on $\Omega = [0,T] \times [-1,1]$, with uniform topology. In WILSON & RUBIO [1], the problem was first modified to one in which the minimum is sought of a functional defined on a set of Radon measures. They show the existence of a minimizing measure, and it is shown that this measure may be approximated by a piecewise constant control. Finally, conditions were given under which a minimizing measurable control exists for the unmodified problem.

(1.3) Optimal control problem for the n-dimensional diffusion equation.

Let $n \geq 2$ be a positive integer and let ω be a bounded, open, connected domain in R^n with boundary $\partial\omega \in C^1$. Let Δ be the Laplacian operator in R^n and T a positive number. We consider the n-dimensional diffusion equation

$$\Delta Y = Y_t. \tag{1.7}$$

We shall be concerned with the solution $Y(x,t)$ of (1.7) which satisfies the initial condition

$$Y(x,0) = g_0(x), \quad x \in \omega. \tag{1.8}$$

We attempt to influence the evolution of the solution $Y(x,t)$ by means of a control function $u(x,t)$ defined on $\partial\omega \times [0,T]$. We assume

$$Y(x,t) = u(x,t), \quad (x,t) \in \partial\omega \times [0,T]. \tag{1.9}$$

The point in question is the following . If we specify a terminal condition

$$Y(x,T) = g(x), \quad x \in \omega, \tag{1.10}$$

do there exists control function $u(x,t)$ defined on $\partial\omega \times [0,T]$, such that the solution (1.7), (1.8), (1.9), also satisfies(1.10)? Some results have been obtained when ω is a domain with simple geometry, such as a sphere or parallelopipedon (see FATTORINI & RUSSELL [1], FATTORINI [2], GRAHAM [1]). But RUSSELL [1] studied the above controllability question when the specific geometry of ω was not prescribed, and put quite severe restrictions on the final desired function; we consider them in chapter (5).

In the following we consider an optimal control of the above control problem (1.7) - (1.10). We define the control u to be *admissible* if it is a mesurable function on $\partial\omega \times [0,T]$ and

(a) $u(x,t) \in [-1,1]$ a.e. for $(x,t) \in \partial\omega \times [0,T]$

(b) $Y(x,T) = g(x)$ a.e. for $x \in \omega$.

Let V be the set of all admissible controls and let V be non-empty. The optimal control problem consists of finding an admissible control u which minimizes the functional

$$J(u) = \int_0^T \int_{\partial\omega} f^0(\xi,t,u(\xi,t))d\xi dt$$

where $f^0 \in C(\Sigma)$, the space of continuous functions on $\Sigma = \partial\omega \times [0,T] \times [-1,1]$, with the uniform topology.

(1.4) Outline of thesis

Chapter (2) is concerned with an extension to n-dimensions of the paper by WILSON & RUBIO [1]. We consider the existence of an optimal control for the n-dimensional diffusion equation with the same boundary conditions as in section (1.3) except that we assume $Y(x,0) = 0$, $x \in \omega$. We assume the set of all admissible controls is non-empty and we denote it by U.

Our control problem consists of finding a $u(.,.) \in U$ which minimizes the functional

$$J(u) = \int_0^T \int_{\partial\omega} f^0(\xi,t,u(\xi,t))d\xi dt$$

where $f^0 \in C(\Sigma)$, the space of continuous functions on Σ

$= \partial\omega \times [0,T] \times [-1,1]$.

In chapter (3) we consider a linear program for determining an approximation to the optimal control of the diffusion equation in one and two dimensions. In particular we obtain an approximation to the optimal Radon measure μ^*, which was introduced in WILSON & RUBIO [1] and one we introduce in chapter (2), in 2-dimensions. By using linear programming we show a practical way to obtain an approximation for the optimal measure μ^* in one and two dimensions . Also we obtain the optimal controls corresponding to several different final desired functions in one and two dimensions.

Chapter (4) is concerned with the optimal control problem for the one dimensional diffusion equation with a sequence of Radon measures as generalised control variables. The foundation of this work is contained in RUBIO and WILSON [1]. The purpose of the mentioned paper is this: suppose that the state $g(.) \in L_2(0,1)$, is not reachable by an admissible control, nor by a measure; then no minimization can be carried out. Thus, of course, the optimal control problem is meaningless.

RUBIO & WILSON [1] enlarged the set of admissible controls, further than that in WILSON & RUBIO [1]. They put an appropriate topology on this new space. The dual of this new space, say S, contains the space $L_2(0,T)$ as well as other elements. If no control $u(.) \in L_2(0,T)$ exists so as to reach the final state $g(.)$ $\in L_2(0,1)$, it could be that among the elements of the set S there is one, or more elements, which provide a solution to the corresponding moment problem to the diffusion equation. This means that we can reach the final state $g(.)$, by imposing as control the new element or elements of S.

In general we show that the objective function of the optimal control depends on an infinite sequence of Radon measures defined on a closed interval. This problem is an optimisation problem over a set of sequences of Radon measures satisfying an infinite number of constraints. We reduce this problem to an approximation problem, which is an optimisation problem over n-tuples of Radon measures satisfying a finite number of constraints. Then we transfer this problem to one which is a finite dimensional linear programming problem over a subset of R^n. Also we approximate the infimum of the objective function by a finite summation of the norms of discrete measures. Finally we compute the final desired states and we compute their corresponding control functions. The theory is confirmed by computing the desired final states and control functions of several different examples.

In chapter (5) we consider an extension to the paper of RUBIO & WILSON to n-dimensions. In this chapter we consider the optimal control of the diffusion equation in n-dimensions, which we discussed in chapter (2) with the same boundary conditions as in chapter (2). We want to minimize a functional such as

$$J(u) = \int_0^T \int_{\partial\omega} f^0(t,\xi,u(t,\xi))d\xi dt;$$

the control $u(.,.)$ is in the space $L_2(\partial\omega \times [0,T])$, and there are no constraints imposed on its magnitude.

Consider a state $g(.) \in L_2(\omega)$, which is neither reachable by an admissible control nor by a measure; that is, the set Q defined in chapter (2) is empty. Therefore there can be no minimum on Q, and the problem has no solution. In this chapter we generalized RUBIO & WILSON [1] to n-dimensional space. We extend the set of admissible controls beyond the space of measures and we find that there is at least one element in the dual of the new space, which is the solution of the moment problem with

$$Y(x,T) = g(x), \ x \in \omega.$$

Therefore the final state $g(.) \in L_2(\omega)$ can be reached by imposing as controls the above element or elements.

In chapter (6) we consider an optimal control problem for the n-dimensional diffusion equation with an infinite sequence of Radon measures as generalized control variables. This chapter is indeed an extension of chapter (4) to n-dimensional space. We consider an optimal control problem associated with the n-dimensional diffusion equation

$$\Delta Y(x,t) = Y_t(x,t)$$

where $(x,t) \in \omega \times [0,T]$, with the same initial and boundary conditions as in Chapter 2. It is desired to choose $u(.,.) \in L_2(\omega \times [0,T])$, such that $Y(.,T) = g(x)$ in $L_2(\omega)$ and the function

$$u \to J(u)$$

is minimal, where the function J(.) is defined in Chapter 6. As in chapter (4) we show this objective function J(u) depends on the infinite sequence of Radon measures defined on $\partial\omega \times [0,T]$. We show a scheme for determining the infimum of the objective function. We approximate this infimum by a finite summation of the norms of discrete measures, then we transfer this problem to one which is the minimization of a real linear function over a set of linear constraints in finite dimensional space. Finally, by using the sequence of the control functions introduced in chapter (5) it is shown that we can reach the final state with a rather good approximation. The theory is confirmed by computing the desired final state and control function of one example in 2-dimensions.

CHAPTER 2

Existence of an Optimal Control for the Diffusion Equation in n-Dimensions

2.1 Introduction

This chapter contains the extension to n-dimensions of the paper by WILSON and RUBIO [1]. Thus we consider the existence of an optimal control for the n-dimensional diffusion equation

$$\Delta Y = Y_t, \tag{2.1}$$

where $Y=Y(x,t)$, $(x,t) \in \omega \times [0,T]$, with boundary conditions

$$Y(x,t) = u(x,t), \qquad (x,t) \in \partial\omega \times [0,T],$$

$$Y(x,0) = 0, \qquad x \in \omega;$$

here ω is a bounded open subset of R^n, with the boundary $\partial\omega \in C^1$, and $u(x,t)$, $(x,t) \in \partial\omega \times [0,T]$, is the control.

We say that the control u is *admissible* if it is a measurable function on $\partial\omega \times [0,T]$ and

(a) $u(x,t) \in [-1.1]$ a.e. for $(x,t) \in \partial\omega \times [0,T]$

(b) $Y(x,T) = g(x)$ a.e. for $x \in \omega$, so that $g \in L_2(\omega)$ is the desired final state. We assume the set of all admissible controls is nonempty and we denote it by U.

Our control problem consists of finding a $u(.,.) \in U$ which minimizes the functional

$$J(u)=\int_0^T\int_{\partial\omega} f^0(\xi,t,u(\xi,t))d\xi dt$$

where $f^0 \in C(\Sigma)$, the space of continuous functions on $\Sigma = \partial\omega \times [0,T] \times [-1,1]$ with the uniform topology.

2.2 Modified Control Problem

The solution of Eq.(2.1) is

$$Y(x,t) = -\int_0^t \int_{\partial\omega} \frac{\partial}{\partial\nu_y}\left\{K(x,y,t-T)\right\}u(y,T)dydt,$$

(see ROACH [1],page 251);here $\partial[K(x,y,t-T)]/\partial\nu_y$ is the normal derivative with respect to its second variable, and $K(x,y,t) = \sum_{n=1}^{\infty} \exp(-\lambda_n t)a_n(x)a_n(y)H(t)$, where the functions $a_n(x)$, $n = 1,2,\ldots$, are the orthonormal eigenfunctions, with corresponding eigenvalues λ_n, $n = 1,2,\ldots$, defined by the problem

$$\Delta v(x)+\lambda v(x) = 0, x \in \omega; \; v(x) = 0, \; x \in \partial\omega,$$

and H is the Heaviside function and is included to emphasize the fact that the solution is identically zero for $t < 0$. Thus,

$$Y(x,t) = -\int_0^t \int_{\partial\omega} (\partial/\partial\nu_y)\left\{\sum_{n=1}^{\infty} \exp[-\lambda_n(t-T)]a_n(x)a_n(y)\right\}u(y,T)dydt, \tag{2.2}$$

or

$$Y(x,\tau) = -\sum_{n=1}^{\infty} \left\{\int_0^T e^{-\lambda_n(T-t)}\left[\int_{\partial\omega}(\partial a_n(y)/\partial\nu)u(y,t)dy\right]dt\right\}a_n(x)$$

Now let $T-t=\tau$:

$$Y(x,T) = -\sum_{n=1}^{\infty} \left\{ \int_0^T e^{-\lambda_n \tau} \left[\int_{\partial\omega} (\partial a_n(y)/\partial\nu) u(y,T-\tau) dy \right] d\tau \right\} a_n(x),$$

and define $\nu_n(\tau) = \int_{\partial\omega} (\partial a_n(y)/\partial\nu) u(y,T-\tau) dy$. Then

$$Y(x,T) = -\sum_{n=1}^{\infty} \left[\int_0^T e^{-\lambda_n \tau} \nu_n(\tau) d\tau \right] a_n(x).$$

Since the desired final state belongs to $L_2(\omega)$, we can expand it in terms of the sequence of orthonormal eigenfunctions $\{a_n(x)\}$, so we have

$$g(x) = \sum_{n=1}^{\infty} c_n a_n(x).$$

Therefore

$$c_n = -\int_0^T e^{-\lambda_n \tau} \nu_n(\tau) d\tau \qquad ,n = 1,2,\ldots$$

or

$$c_n = -\int_{\partial\omega \times [0,T]} \partial a_n(y)/\partial\nu . e^{-\lambda_n \tau} u(y,T-\tau) dy d\tau \qquad n = 1,2,\ldots .$$

Since we assumed $\partial\omega \in C^1$, let the parametric equation of $\partial\omega$ be in the following form,

$$Y = (\xi_1(s_1,\ldots,s_{n-1}),\ldots,\xi_n(s_1,\ldots,s_{n-1})),$$

where $0 \le s_i \le 1$, for $i = 1,2,\ldots,n-1$, (see CROWELL and WILLIAMSON [1], p. 419). We also define $b_n(y) = \partial a_n(y)/\partial\nu$, so we have

$$c_n = -\int_0^T \int_A b_n(\xi_1(s),\dots,\xi_n(s))e^{-\lambda_n \tau} u(\xi_1(s),\dots,\xi_n(s),T-\tau)$$
$$.B(s)dsdt,$$

where for simplicity let

$$B(s) = \sqrt{\left[\frac{\partial(x_2,\dots,x_n)}{\partial(s_1,\dots,s_{n-1})}\right]^2 + \dots + \left[\frac{\partial(x_1,\dots,x_{n-1})}{\partial(s_1,\dots,s_{n-1})}\right]^2}$$

where $s = (s_1,\dots,s_{n-1})$, $ds = ds_1.ds_2\dots ds_{n-1}$, and $A = \overbrace{[0,1] \times [0,1] \times \dots \times [0,1]}^{(n-1)\text{times}}$. Let $F_n(s,\tau,u) = b_n(\xi_1(s),\dots,\xi_n(s))e^{-\lambda_n \tau} u.B(s)$, then, the control problem reduces to finding a measurable control $u(y,t) \in [-1,1]$, $(y,t) \in \partial\omega \times [0,T]$, which satisfies,

$$c_n = \int_0^T \int_A F_n(s,t,\hat{u}(s,t))dsdt, \quad n = 1,2,\dots . \tag{2.3}$$

where $\hat{u}(s,t)=u(\xi_1(s),\dots,\xi_n(s),t)$, and which minimizes

$$J(u) = \int_0^T \int_A \hat{f}^0(s,t,\hat{u}(s,t))dsdt, \tag{2.4}$$

where $\hat{f}^0(s,t,u(s,t)) = f^0(s,t,\hat{u}(s,t)).B(s)$.

In general, a minimising solution to the problem may not exist; in the following we replace this problem by another one in which the minimumum of a linear functional calculated over a set of

Radon measures on $\Omega \equiv \overbrace{[0,1] \times [0,1] \times \ldots \times [0,1]}^{(n-1)\text{times}} \times [0,T] \times [-1,1]$.

We notice that, for a fixed $\hat{u}$, the following mapping

$$f(.,.,.) \to \int_A \int_0^T f(s,t,\hat{u}(s,t))dsdt$$

defines a positive linear functional on $C(\Omega)$. Thus, by the Riesz representation theorem, there exists a unique positive Radon measure μ, on Ω, such that

$$\int_0^T \int_A f(s,t,\hat{u}(s,t))dsdt = \int_\Omega f d\mu \equiv \mu(f), \qquad (2.5)$$

for all $f \in C(\Omega)$; in particular the above equality is valid for $f = f^0$. Now we replace the minimization problem by one in which we are going to find the minimum of $\mu(f^0)$ over a set Q of positive Radon measures on Ω, to be defined below. Measures in Q, should have some properties which are deduced from the definition of admissible controls. First, from (2.5)

$$|\mu(f)| \le T.\sup_\Omega |f(s,t,u)|,$$

hence, $\int_\Omega d\mu \le T$.

Next, measures in Q must satisfy an abstracted version of Eq. (2.3):

$$\mu(F_n) = c_n, \quad n=1,2,\ldots;$$

note that this is possible, since $F_n \in C(\Omega)$, n=1,2,... .
Finally consider functions $h \in C(\Omega)$, which do not depend on u; that is, for all $(s,t) \in A \times [0,T]$, and all $u_1, u_2 \in [-1,1]$, we have $h(s,t,u_1) = h(s,t,u_2)$.
Then the measures in Q must satisfy

$$\int_{\Omega} h \, d\mu = \int_0^T \int_A h(s,t,u) ds dt = c_h,$$

where u is an arbitrary number in [-1,1] and c_h is the Lebesgue integral of $h(.,.,u)$, which is independent of u. This property of Q will be used in the next section, when we use an extension of a theorem due to Ghouila -Houri, [1]. Let $\mathcal{M}^+(\Omega)$ be the set of positive Radon measures on Ω. The set Q is defined as a subset of $\mathcal{M}^+(\Omega)$:

$$Q = S_1 \cap S_2 \cap S_3$$

where,

$$S_1 = \left\{ \mu \in \mathcal{M}^+(\Omega) \ : \ \mu(1) \le T \right\}$$

$$S_2 = \left\{ \mu \in \mathcal{M}^+(\Omega) \ : \ \mu(F_n) = c_n, \quad n=1,2,\dots \right\}$$

$$S_3 = \left\{ \mu \in \mathcal{M}^+(\Omega) \ : \ \mu(h) = c_h, \ h \in C(\Omega), \text{ and independent of } u \right\}.$$

Now we topologize the space of all Radon measures on Ω, by the weak*- topology. We show in appendix (A.2) that S_1 is compact. Thus the set S_2 can be written as

$$S_2 = \bigcap_{n=1}^{\infty} \left\{ \mu \in \mathcal{M}^+(\Omega) \; : \; \mu(F_n) = c_n \right\} = \bigcap_{n=1}^{\infty} M_n ,$$

where each M_n is closed because it is the inverse image of a closed set on the real line ((the set $\{c_n\}$), under a continuous map. We know that the infinite intersection of closed sets is closed, so S_2 is closed. By a similar argument, it is easy to show that S_3 is closed. Therefore Q is a closed subset of the compact set S_1, and then Q is compact. By definition of a convex set, it is easy to show that the sets S_1, S_2, S_3, are convex; thus, Q, is a compact convex set. By the Krein-Milman theorem (see ROBERTSON, A. and ROBERTSON, W. [1]), it has extreme points. Consider now the functional $I : Q \to R$, defined by

$$I(\mu) = \int_{\Omega} f^0 d\mu, \quad \mu \in Q. \qquad (2.6)$$

This is a continuous linear functional on a convex compact set, Q; it will therefore attain its minimum at least one extreme point; we have shown the following proposition.

Proposition (2.1) The measure-theoretical control problem, which is to find the minimum of the functional I, over the set Q, attains its minimum μ^*, in Q.

2.3 Approximation of the optimal control by a piecewise constant control

With each piecewise constant admissible control $u(.,.)$, we may associate a measure μ_u, in $\mathcal{M}^+(\Omega) \cap S_1 \cap S_2$. Let Q_1 be the set

of all such measures μ_u. The extension to n-dimensions of theorem (1) of GHOUILA-HOURI [1] which is proved in appendix (B2), shows that, when the space of all Radon measures on Ω, has the weak*-topology, Q_1 is dense in $\mathcal{M}^+(\Omega) \cap S_1 \cap S_2$.
A basis of closed neighborhoods in the weak*- topology is given by sets of the form

$$\left\{ \mu : |\mu(G_n)| \le \varepsilon,\ n = 1,2,\dots,k+1 \right\},$$

where k is an integer, $G_n \in C(\Omega)$, $n = 1,2,\dots,k+1$, and $\varepsilon > 0$. In any weak*- neighborhood of μ^* (the minimizing measure of proposition (2.1)), we can find a measure μ_u, corresponding to a piecewise control u. In particular, we choose

$$G_1 = f^0,\quad G_2 = F_1,\ \dots\ ,\ G_{k+1} = F_k;$$

we can then find a piecewise control $\hat{u}_k(.,.)$, such that

$$\left| \int_0^1\int_A f^0[s,t,\hat{u}_k(s,t)]dsdt - \mu^*(f^0) \right| \le \varepsilon$$

$$\left| \int_0^1\int_A F_n[s,t,\hat{u}_k(s,t)]dsdt - c_n \right| \le \varepsilon,\quad n = 1,2,\ \dots\ ,k \tag{2.7}$$

Therefore, by using the piecewise constant control $\hat{u}_k(.,.)$, we can reach within ε of the minimum value $\mu^*(f^0)$. The analysis of the relationship between the desired final state, $g(x)$ and $Y_k(x,T)$, $x \in \omega$, the one attained by the use of the control $\hat{u}_k(.,.)$, is somewhat complicated. Let

$$Y_k(x,T) = \sum_{n=1}^{\infty} d_n(T)a_n(x),$$

where, of course,

$$d_n(T) = \int_0^T\int_A F_n[s,t,\hat{u}_k(s,t)]dsdt, \quad n = 1,2,\dots .$$

By substituting the values of $d_n(T)$, $n = 1,2,\dots,k$, in the inequalities (2.7) we obtain

$$|d_n(T) - c_n| \le \varepsilon, \quad n = 1,2,\dots,k.$$

We can show that by choosing k large enough the distance between g(x) and $Y_k(.,T)$, in $L_2(\omega)$, can be made as small as desired.

Proposition (2.2) Let $\delta \ge 0$ be given. We can choose k and $\varepsilon > 0$ such that

$$\int_\omega\left[Y_k(x,T) - g(x)\right]^2 dx \le \delta. \tag{2.8}$$

Proof We have shown in appendix (C.2) that the Fourier coefficient $d_n(T)$ of $Y_k(.,T)$ satisfies

$$d_n^2(T) \le e_n, \quad \text{if} \quad n \ge M, \tag{2.9}$$

where M is a specified positive integer and $\sum_{n=1}^{\infty} e_n$ is a convergent series. Similarly, since it is assumed that the desired final state g(x) is reachable with an admissible control, c_n^2 satisfies the same inequality as $d_n^2(T)$. Thus,

$$\int_{\omega} [Y_k(x,T) - g(x)]^2 dx = \sum_{n=1}^{L} (d_n - c_n)^2 + \sum_{n=L+1}^{\infty} (d_n - c_n)^2$$

$$\leq \sum_{n=1}^{L} (d_n - c_n)^2 + 4 \sum_{n=L+1}^{\infty} e_n. \qquad (2.10)$$

In the last term we have used the following known inequality, $(x + y)^2 \leq 2(x^2 + y^2)$, for all $x,y \in R$. Now we choose L sufficiently large such that $L \geq M$ and

$$4 \sum_{n=L+1}^{\infty} e_n \leq \delta/2.$$

The integer K can be chosen as one satisfying

$$K \geq \max\{L, (1/2\delta)\} \qquad (2.11)$$

Then

$$4 \sum_{n=K+1}^{\infty} e_n \leq \delta/2; \qquad (2.12)$$

we choose $\varepsilon = \sqrt{\delta/2K}$; thus from (2.11) it follows that $k \geq (1/2K)$, from which in turn it follows that $\varepsilon \leq \delta$. In the neighbourhood defined by choosing ε and K as above, there exists a μ_u, corresponding to a piecewise constant control $u(.,.)$, for which

$$|d_n - c_n| \leq \varepsilon, \quad n = 1,2,\ldots,k;$$

hence,

$$\sum_{n=1}^{L}(d_n - c_n)^2 \le k.\varepsilon^2 = \delta/2.$$

From (2.10), (2.12) and the last relation we have the proof of the relation (2.8); then the proof of the proposition (2.2) is completed.□

2.4 Unmodified Control Problem

In this section, we are looking for some conditions under which the original, classical, unmodified problem has a classical solution. Indeed, we show that any close approximation to the optimal measure is a solution to the classical problem. We show that if the function f^0 is convex, then we can obtain the classical solution to this unmodified control problem.

Proposition (2.3) Suppose that the function f^0, in the following performance criterion

$$J[u(.,.)] = \int_0^T \int_\omega f^0(\xi,t,u(\xi,t))d\xi dt \qquad (2.13)$$

satisfies the conditions :

(*i*) The derivative f_u^0 exists and is uniformly continuous in the interior of $\Sigma = \bar{\omega} \times [0,T] \times [-1,1]$. [where $\bar{\omega}$ is closure of the open set ω.].

(*ii*) The function f^0 is convex in $u \in [-1,1]$ for all $(\xi,t) \in \partial\omega \times [0,T]$.

Then there exists an admissible control $u^*(.,.)$ such that

$$J[u^*(.,.)] = \inf_{u(.,.) \in U} J[u(.,.)] = \rho,$$

where U is the set of admissible controls.

Proof Since $f^0 \in C(\Sigma)$, and Σ is a closed bounded and thus compact subset of R^{n+2}, therefore there exists a real number say m, such that $f^0(\xi,t,u) \geq m$, for all $(\xi,t,u) \in \Sigma$. Thus, by definition of $J(.)$ the functional J defined by (2.13) is bounded below; therefore there exists a sequence of admissible controls, $\{u_i(.,.)\}$ such that

$$\lim_{i\to\infty} J\,[u_i(.,.)] = \rho.$$

Since each control in the sequence $\{u_i(.,.)\}$ is admissible,

$$\int_0^T\int_{\partial\omega} u_i^2(\xi,t)d\xi dt \leq TL,$$

where L is the area of $\partial\omega$. Thus, the L_2- norm of the controls $u_i(.,.)$ in this sequence satisfies $\|u_i(.,.)\| \leq \sqrt{T.L}$, $i = 1,2,...,$ We endow $L_2(\partial\omega \times [0,T])$ with the weak-topology, which means the set $W = \{ u(.,.) : \| u(.,.) \| \leq \sqrt{T.L} \}$ is compact, and $\{u_i(.,.)\}$ has a weakly-convergent subsequence, which we again denote by $\{u_i(.,.)\}$ and whose limit we denote by v; we claim that $v(.,.) \in L_2(\partial\omega \times [0,T])$, a result which follows directly from the weak compactness of W. Also

$$-\int_{\partial\omega \times [0,T]} (\partial a_n(y)/\partial v).e^{-\lambda_n \tau}.v(y,T-\tau)dydt = c_n, \quad n = 1,2,...,$$

since if this equalitly was false for some n, then an $\varepsilon > 0$ would exist with

$$\int_{\partial\omega \times [0,T]} (\partial a_n(y)/\partial v).e^{-\lambda_n \tau} |v(y,T-\tau) - u_i(y,T-\tau)| dydt > \varepsilon$$

for all $i \geq K$; here K is some positive integer. However, since $e^{-\lambda n\tau}.(\partial a_n(y)/\partial \nu) \in L_2(\partial\omega \times [0,T])$, this contradicts the fact that $\{u_i(.,.)\}$ converges weakly to $v(.,.)$. We show that $|v(y,t)| \leq 1$ a.e. on $\partial\omega \times [0,T]$. Suppose that $|v(y,t)| > 1$ on some subset of $\partial\omega \times [0,T]$ having nonzero Lebesgue measure. Let $p(.,.)$ be the function defined on $\partial\omega \times [0,T]$ by

$p(y,t) = 1 \quad (y,t) \varepsilon \{(\xi,s) : v(\xi,s) > 1\}$

$p(y,t) = -1 \qquad (y,t) \varepsilon \{ (\xi,s) : v(\xi,s) < -1 \}$

$p(y,t) = 0 \qquad (y,t) \varepsilon \{(\xi,s) : v(\xi,s) \leq 1\}.$

Since $v(.,.)$ is measurable, $p(.,.) \varepsilon L_2(\partial\omega \times [0,T])$, and

$$\int_{\partial\omega \times [0,T]} p(y,t)v(y,t)dydt > \int_{\partial\omega \times [0,T]} p(y,t)u_i(y,t)dydt$$

for all i. This contradicts the fact that $\{ u_i(.,.) \}$ converges weakly to $v(.,.)$ We have therefore shown that $v(.,.)$ is admissible. We now show

$$J(v) \leq \rho.$$

Consider now the assumption of convexity and differentiability on f^0; they imply that

$$f^0(y,t,v_1) \geq f^0(y,t,v_2) + (v_1 - v_2) f^0_u(y,t,v_2)$$

for every $v_1, v_2 \in [-1,1]$, $(y,t) \in \partial\omega \times [0,T]$. Hence,

$$\int_{\partial\omega \times [0,T]} f^0[y,t,u_i(y,t)]\, dy.dt \geq \int_{\partial\omega \times [0,T]} f^0[y,t,v(y,t)]\, dy.dt + \int_{\partial\omega \times [0,T]} [u_i(y,t)-v(y,t)] f^0_u[y,t,v(y,t)]\, dydt.$$

Therefore

$$\rho = \lim_{i\to\infty} J\,[u_i(.,.)]$$

$$= \lim_{i\to\infty} \int_{\partial\omega \times [0,T]} f^0[y,t,u_i(y,t)]\, dydt \tag{2.14}$$

$$\geq J[v(.,.)] + \lim_{i\to\infty} \int_{\partial\omega \times [0,T]} [u_i(y,t)-v(y,t)] f^0_u[y,t,v(y,t)] dydt.$$

By assumption, f^0_u is uniformly continuous in the interior of Σ, and thus bounded on Σ. Since $v(y,t)$, $(y,t) \in \partial\omega \times [0,T]$ is measurable (see, for example, EWING, G.M.) the function $f^0_u[y,t,v(y,t)] \in L_2(\partial\omega \times [0,T])$. Also since $\{u_i(.,.)\}$ converges weakly to $v(.,.)$, the last limit in (2.14) reduces to zero, and therefore

$$\rho \geq J\,[v(.,.)].$$

From above inequality and by definition of ρ we conclude

$$J(v) = \rho. \square$$

Discussion

We have shown in section (2.1) of this chapter that the set of all admissible controls for the diffusion equation in n-dimensions is nonempty; in section (2.2) we obtained a positive Radon measure μ^* which minimizes the criterion functional $J(u)$. Next, in section (2.3) we found a piecewise constant control $\hat{u}(.,.)$, corresponding to the approximation of the optimal measure μ^*. In proposition (2.2) we showed that we may choose the piecewise constant control $u(.,.)$ such that the solution of the diffusion equation corresponding to the above control $u(.,.)$, at final time T, becomes very close to the final state $g(x)$, $x \in \omega$. Finally in section (2.4), we considered the unmodified control problem and we showed that if we put conditions of differentiability and convexity on f^0 then there exists an admissible control say $\bar{u}(.,.)$, such that if $\rho = \inf_{u \in U} J(u)$, then $J(\bar{u}) = \rho$.

In the next chapter we are going to obtain an approximation to the mentioned optimal Radon measure μ^* by a finite combination of atomic measures; then, by using linear programming, we obtain a piecewise constant optimal control corresponding to the approximation of μ^*, and in some examples in one and two dimensional spaces we obtain the corresponding approximation to optimal measures and optimal controls for different final states.

Appendix (A.2)

In this appendix we prove the following lemma (1)

Lemma (1)

$S_1 = \left\{\mu \in \mathcal{M}^+(\Omega) : \| \mu(1) \| \le T \right\}$ is compact in $\mathcal{M}(\Omega)$, with respect to weak* topology.

Proof First we prove that $\mathcal{M}^+(\Omega)$, is closed in $\mathcal{M}(\Omega)$, with respect to weak*- topology; that is we show $\overline{\mathcal{M}^+(\Omega)} = \mathcal{M}^+(\Omega)$. Let $\mu \in \overline{\mathcal{M}^+(\Omega)}$ and let $f \ge 0$, be any continuous function on Ω, so we have $\mu(f) \ge 0$, since $\mu \in \overline{\mathcal{M}^+(\Omega)}$. Then for every positive integer n

$$\left\{ \nu : |(\nu - \mu)(f)| < 1/n \right\} \cap \mathcal{M}^+(\Omega) \ne \phi,$$

so there exists $\nu_n \in \mathcal{M}^+(\Omega)$, such that

$$|(\nu_n - \mu)(f)| < 1/n,$$

or,

$$|\nu_n(f) - \mu(f)| < 1/n; \qquad (A2.1)$$

from (A2.1) we conclude that $\mu(f) = \lim_n \nu_n(f)$, but by definition we have $\nu_n(f) \ge 0$, for every n, so $\mu(f) \ge 0$, or $\mu \in \mathcal{M}^+(\Omega)$, which proves $M^+(\Omega)$ is closed in $M(\Omega)$.

Corollary 12.7 (CHOQUET (I)[1] p.217) asserts that for $a > 0$, ($a < \infty$), the set

$$\{\mu \in M^+(\Omega): \| \mu \| \leq a\},$$

is compact in $M(\Omega)$, therefore the set $A = \{\mu \in M^+(\Omega): \|\mu\| \leq T\}$, is compact in $M(\Omega)$, thus $A \cap M^+(\Omega)$, is compact in $M(\Omega)$[since $A \cap M^+(\Omega)$ is a closed subset of the compact set A, so it is compact in A, (see for example JAMESON[1] p.84) thus it it is compact in $M(\Omega)$].By definition we have $S_1 = A \cap M^+(\Omega)$, so S_1 is compact in $M(\Omega)$.□

Appendix (B.2)

Extension of theorem 1 of GHOUILA HOURI [1] to n-dimensions

Let Q_T be the set of positive Radon measures μ defined on the space $A \times [0,T] \times U$, where $A = [0,1] \times \ldots \times [0,1]$, n -1 times, here U is a compact set of controls defined on $A \times [0,T]$. Let Q_T^0 be the set of piecewise constant functions defined on $A \times [0,T]$; we assume the measures on $A \times [0,T] \times U$ are projected on $A \times [0,T]$ with respect to the Lebesgue measure. The extension of theorem 1 of Ghouila - Houri to n dimensions is as follows

Note: Let μ be a measure on $A \times [0,T] \times U$, then, the projection of the measure μ, say μ_1, is defined as follows:

$$\mu_1(C) = \mu(C \times U), \text{ for } C \subset A \times [0,T].$$

Theorem (1). Q_T^0 is everywhere dense in Q_T.

Proof Suppose $f_1, f_2, \ldots, f_{m_1}$ are real-valued continuous functions defined on $A \times [0,T] \times U$, and let $\varepsilon > 0$. One can find easily a finite sequence [We show in Lemma (1) in this appendix, a way to obtain this sequence in practice] of numbers

$$t_0 = 0 < t_1 < t_2 < \dots < t_r = T,$$

$$s_0^i = 0 < s_1^i < s_2^i < \dots < s_p^i = 1, \qquad \text{for } i = 1,2,\dots,n-1,$$

and a partition $A_1, A_2, \dots, A_q$, of U which are Borel sets such that for every $i=1,2,\dots,r$, $j=1,2,\dots,n-1$, $m = 1,2,\dots,n-1$, $k = 1,2,\dots,q$, we have

$$\left.\begin{array}{l} t,t' \in [t_{i-1}, t_i) \\ \\ s'_j, s''_j \in [s_{j-1}, s_j) \\ \\ u, u' \in A_k \end{array}\right\} \Rightarrow |f_l(s',t,u) - f_l(s'',t',u')| \le \epsilon \qquad \text{(B2.1)}$$

for all $l = 1,2,\dots,m_1$, where $s' = (s'_1,\dots,s'_{n-1})$ and $s'' = (s''_1,\dots,s''_{n-1})$. [Since f_l is continuous on the compact set $A \times [0,T] \times U$ it is uniformly continuous on it, and (B2.1) is just the definition of uniform continuity].

Suppose μ be an element of Q_T; let

$$K_{ijk} = \int_{[t_{i-1},t_i] \times \prod_{m=1}^{n-1} [s^m_{j-1}, s^m_j) \times A_k} d\mu(t,s,u)$$

where $i = 1,2,\dots,r$, $j = 1,2,\dots,p$, $k = 1,2,\dots,q$.

Therefore we have

$$\sum_{k=1}^{q} K_{ijk} = (t_i - t_{i-1}) . \prod_{m=1}^{n-1} (s_j^m - s_{j-1}^m) \tag{B2.2}$$

where $i = 1,2,\dots,r$ and $j = 1,2,\dots,p$. Let

$$B_{ijk} = \left[t_{i-1} + \sum_{k'<k} K_{ijk'}, t_{i-1} + \sum_{k' \leq k} K_{ijk'}\right]$$

$$\times \prod_{m=1}^{n-1} \left[s_{j-1}^m + \sum_{k'<k} K_{ijk'}, s_{j-1}^m + \sum_{k' \leq k} K_{ijk'}\right].$$

Let $u_1, u_2, \dots, u_q$ be elements of A_1, A_2, $\dots$ $, A_q$ and $U \in Q_T^0$ be defined by

$U(s,t) = u_k$ for $(s,t) \in B_{ijk}$ for $k = 1,2,\dots,q$ and all i and j. Now for every i, j, k, and l we define the following number

$$m_{ijkl} = \int_{B_{ijk} \times U} f_l(s,t,u) d\mu(s,t,u) = \int_{B_{ijk}} f_l(s,t,u_k) dsdt$$

[we again mention that $s = (s_1, s_2, \dots, s_{n-1})$] but, m_{ijkl} satisfies the following inequalities,

$$K^n_{ijk} \cdot \inf\left\{f_l(s,t,u) \mid (s,t) \in C_j \times [t_{i-1}, t_i), u \in A_k\right\}$$

$$\leq m_{ijkl}$$

$$\leq K^n_{ijk} \cdot \sup\left\{f_l(s,t,u) \mid (s,t) \in C_j \times [t_{i-1}, t_i), u \in A_k\right\}$$

where $C_j = \prod_{m=1}^{n-1} [s^m_{j-1}, s^m_j)$.

On the other hand by the definition of K^n_{ijk}, we have

$$K^n_{ijk} \cdot \inf\left\{f_l(s,t,u) \mid (s,t) \in C_j \times [t_{i-1}, t_i), u \in A_k\right\}$$

$$\leq \int_{C_j \times [t_{i-1}, t_i] \times A_k} f_s(s,t,u).d\mu(s,t,u)$$

$$\leq K^n_{ijk} \cdot \sup\left\{f_l(s,t,u) \mid (s,t) \in C_j \times [t_{i-1}, t_i), u \in A_k\right\}.$$

So for every l $(l = 1,2,\ldots,n_1)$

$$\left| m_{ijkl} - \int_{C_j \times [t_{i-1}, t_i) \times A_k} f_l(s,t,u).d\mu(s,t,u) \right| \leq K^n_{ijk}.\varepsilon,$$

or we have

$$\left| \sum_{k=1}^{q} \sum_{j=1}^{p} \sum_{i=1}^{r} m_{ijkl} - \sum_{k=1}^{q} \sum_{j=1}^{p} \sum_{i=1}^{r} \int_{C_j \times [t_{i-1}, t_i] \times A_k} f_l(s,t,u).d\mu(s,t,u) \right|$$

$$\leq \varepsilon \,.\, \sum_{k=1}^{q} \sum_{j=1}^{p} \sum_{i=1}^{r} K^{n}_{ijk} \qquad \text{(B2.3)}$$

or

$$\left| \int_{\bigcup_{k=1}^{q} \bigcup_{i=1}^{r} \bigcup_{j=1}^{p} B_{ijk}} f_l\,(s,t,u_k\quad).ds.dt - \int_{A \times [0,T] \times U} f_l(s,t,u).d\mu(s,t,u) \right|$$

$$\leq \sum_{k=1}^{q} \sum_{j=1}^{p} \sum_{i=1}^{r} K^{n}_{ijk} \qquad \text{(B2.4)}$$

By (B2.2)we have $K_{ijk} \leq T$; Thus,

$$\sum_{k=1}^{q} \sum_{j=1}^{p} \sum_{i=1}^{r} K^{n}_{ijk} \leq T^{n-1} . \sum_{k=1}^{q} \sum_{j=1}^{p} \sum_{i=1}^{r} K_{ijk} \leq$$

$$T^{n-1} \sum_{j=1}^{p} \sum_{i=1}^{r} (t_i - t_{i-1}) \prod_{m=1}^{n-1} (s^m_j - s^m_{j-1}); \qquad \text{(B2.5)}$$

by definition of the partitions $\{s^m_j\}$ of $[0,1]$

$$\prod_{m=1}^{n-1} (s^m_j - s^m_{j-1}) \leq 1 \;;$$

by using the above inequality in (5) we have

$$\sum_{k=1}^{q}\sum_{j=1}^{p}\sum_{i=1}^{r}K^{n}_{ijk} \le T^{n-1}\cdot\left[\sum_{i=1}^{r}(t_i - t_{i-1})\right] \le T^{n-1}.T = T^{n}. \qquad \text{(B2.6)}$$

Therefore from (B2.4) and (B2.6)

$$\left|\int_0^T\int_A f_l(s,t,u_k)dsdt - \int_{A\times[0,T]\times U} f_l\ (s,t,u)d\mu(s,t,u)\right| \le \varepsilon.T^n.$$

Here n-1, is the dimension of the space in which we chose the set A, thus, it is fixed and we change ε to $\varepsilon\left[1/T^n\right]$. Thus

$$\left|\int_0^T\int_A f_l(s,t,u_k)dsdt - \int_{A\times[0,T]\times U} f_l\ (s,t,u)d\mu(s,t,u)\right| \le \varepsilon.\square$$

Lemma (1) of Appendix (B.2)

Let $f_1, f_2, \ldots, f_N$ be N continuous functions defined on a compact set $\Omega \subseteq R^p$ and let $\varepsilon > 0$ be any positive number. In this lemma we show how we can divide Ω into a finite number of subsets say, Ω_j, $j = 1,2,\ldots,m$ of equal volume or measure such that for every x, $x' \in \Omega_j$, $j = 1,2,\ldots,m$

$$|f_i(x) - f_i(x')| < \varepsilon,\ i = 1,2,\ldots,N$$

Proof Let i be any integer satisfying $1 \le i \le N$. By assumption $f_i(.)$ is continuous on the compact set $\Omega \subset R^p$, therefore there exists a positive integer M_i and a partition $P_i = \{\Omega^i_j\}$, $j = 1,\ldots,M_i$, such that

$$\forall x,x' \in \Omega^i_j \Rightarrow |f_i(x) - f_i(x')| < \varepsilon.$$

Thus, for $i = 1,\dots,N$, we have N partitions $P_i = \{\Omega_j^i\}$, $j = 1,\dots,M_i$, $i = 1,\dots,N$, of Ω corresponding to N functions $f_i(.)$, $i = 1,\dots,N$. Let $P = \{\Omega_i\}$, $i = 1,\dots,m$, be a partition of Ω which is finer than any partition P_i, $i = 1,\dots,N$(see the definition in BARTLE [1] p.320), and further so that the volume or measure of the subsets Ω_j are equal. Thus we have for $j = 1,\dots,m$

$$\forall x, x' \in \Omega_j \in P \Rightarrow |f_i(x) - f_i(x')| < \varepsilon,\ i = 1,\dots,N. \square$$

Appendix (C.2)

Let $Y = Y(x,t)$ be the solution of the n-dimensional diffusion equation introduced in the begining of chapter (2) and let $Y(x,T) = \sum_n d_n(T).a_n(x)$, where the functions $a_n(x)$ are orthonormal eigenfunctions with corresponding eigenvalues λ_n defined by the problem

$$\Delta v(x) + \lambda.v(x) = 0,\ x \in \omega\ ; v(x) = 0\ , x \in \partial\omega\ .$$

As we showed in (2.3),

$$d_n = \int_0^T \int_0^1 F_n(s,t,\hat{u}(s,t))ds.dt.$$

In this appendix we show there exists a positive integer M and a sequence $\{e_k\}$ such that for every $n > M$,

$$d_n(T)^2 \le e_n,$$

where $\sum_{n=M}^{\infty} e_n < \infty$.

Proof We find first the solution of the equation

$$\Delta Y - Y_t = 0 \qquad (x,t) \in \omega \times (0,T) \tag{C2.1}$$

with the following boundary conditions

$$Y(x,t) = u(x,t) \qquad (x,t) \in \partial\omega \times [0,T] \tag{C2.2}$$

$$Y(x,0) = 0 \qquad x \in \omega \tag{C2.3}$$

Let

$$Y(x,t) = U(x,t) + v(x,t) \tag{C2.3'}$$

be the solution of the problem (C2.1)-(C2.3), where $v(x,t)$ is a new unknown function and $U(x,t)$ is an arbitrary (sufficiently smooth) function which assumes the value $u(x,t)$ on $\partial\omega \times [0,T]$. Now we substitute (C2.3') in (C2.1) therefore

$$(\Delta U + \Delta v) - (U_t + v_t) = 0,$$

or $\Delta v - v_t = U_t - \Delta U$. We substitute $f(x,t) = U_t - \Delta U$, $(x,t) \in \omega \times (0,T)$, so the problem (C2.1)-(C2.3) is in the following form in terms of $v(x,t)$:

$$\Delta v - v_t = f(x,t) \qquad (x,t) \in \omega \times (0,T) \tag{C2.4}$$

$$v(x,t) = 0 \qquad (x,t) \in \partial\omega \times [0,T] \tag{C2.5}$$

$$v(x,0) = \varphi(x) \qquad x \in \omega \tag{C2.6}$$

where $\varphi(x) = -U(x,0)$. Now let

$$f(x,t) = \sum_{n=1}^{\infty} f_n(t) a_n(x)$$

$$\varphi(x) = \sum_{n=1}^{\infty} \varphi_n a_n(x)$$

where $f_n(t) = \int_\omega f(x,t) a_n(x) dx$ and $\varphi_n = \int_\omega \varphi(x) a_n(x) dx$ and the functions $f_n(t)$ belong to $L_2(0,T)$. By Parseval's equality, $\sum_{n=1}^{\infty} \varphi_n^2 = \|\varphi\|^2_{L_2(\omega)}$, and for all $t \in (0,T)$

$$\sum_{n=1}^{\infty} f_n^2(t) = \int_\omega f^2(x,t).dx = \int_\omega [U_t(x,t) - \Delta U]^2 dx.$$

We call the last integral $F(t)^2$, so that

$$\|f_n\|_{L(0,T)} = \int_0^T f_n^2(t) dt \le \int_0^T F^2(t) dt.$$

Let $\int_0^T F^2(t) dt \equiv D$, thus $\|f_n\|_{L_2(0,T)} \le D \quad n = 1,2,\ldots$. For any $n = 1,2,\ldots$ we consider the functions

$$l_n = \varphi_n . e^{-\lambda_n t} - \int_0^t f_n(\tau) . e^{-\lambda_n (t-\tau)} . d\tau. \tag{C2.7}$$

By differentiating from both sides of (C2.7) we have

$$\dot{l}_n(t) = \varphi_n.(-\lambda_n).e^{-\lambda_n t} - f_n(t) - \int_0^t (-\lambda_n).f_n.e^{-\lambda_n(t-\tau)}.d\tau,$$

or $\dot{l}_n(t) = -\lambda_n.\left[\varphi_n.e^{-\lambda_n t} - \int_0^t f_n(\tau).e^{-\lambda_n(t-\tau)}.d\tau \right] - f_n(t)$, thus

$$\dot{l}_n(t) = -\lambda_n.l_n(t) - f_n(t).$$

We have $l_n(0) = \varphi_n$, so it is easy to check that the function $Z_n(x,t) \equiv l_n(t).a_n(x)$ is a solution of the problem

$$\Delta v - v_t = f_n(t).a_n(x),$$

with the initial condition $v(x,0) = \varphi_n.a_n(x)$. It is seen that

$$v(x,t) = \sum_{n=1}^{\infty} l_n(t).a_n(x) \qquad \text{(C2.7')}$$

is the solution of the problem (C2.4) - (C2.6). But from theorem 3 of MIKHAILOV [1] (page 372)

$$|l_n| \le |\varphi_n|.e^{-\lambda_n t} + \int_0^t |f_n(\tau)|.e^{-\lambda_n(t-\tau)}.d\tau$$

$$\le |\varphi_n|.e^{-\lambda_n t} + \frac{\| f_n \|_{L_2(0,T)}}{\sqrt{2\lambda_n}}, \text{ when } n > 1 \qquad \text{(C2.8)}$$

But we showed $\|f_n\|_{L_2(0,T)} \le D$, where D is a positive constant, and

$$|\varphi_n| \le |\int_\omega \varphi(x).a_n(x)dx| \le \left(\int_\omega |\varphi(x)|^2\right)^{1/2}.\left(\int_\omega a_n^2(x)dx\right)^{1/2}$$

since $\{a_n\}$ is orthonormal. Thus we have $|\varphi_n| \le \|\varphi\|_{L_2}(0,T).1$. Let now $\|\varphi\|_{L_2(0,T)} \equiv B$ where B is a constant, therefore

$$|\varphi_n| < B \quad n = 1,2,\dots .$$

By using this notation we conclude from (C2.8)

$$|1_n(t)| \le B.e^{-\lambda_n t} + \frac{D}{\sqrt{2\lambda_n}} \tag{C2.9}$$

We know that $(x+y)^2 \le 2.x^2 + 2.y^2$, so (C2.9) implies

$$1_n(t)^2 \le 2.B^2.e^{-2\lambda_n t} + \frac{D}{\lambda_n}, \; t \in [0,T] \tag{C2.9'}$$

therefore

$$1_n(T)^2 \le 2.B^2.e^{-2\lambda_n T} + \frac{D}{\lambda_n}, \; n = 1,2,\dots . \tag{C2.10}$$

But all of the $\{\lambda_n\}$ are positive and asymptotically $\lambda_n = c^2.n^2 + O(n)$, for an appropriate constant $c > 0$; (See, e.g. R. COURANT AND HILBERT [1]) so there exists a positive contant K such that

$$|\lambda_n - c^2n^2| \le K.n \qquad \text{for all} \quad n \ge N,$$

or therefore, $c^2n^2 - Kn \le \lambda_n \le c^2n^2 + Kn$, for all $n \ge N$, thus $\dfrac{1}{c^2n^2 + K.n} \le \dfrac{1}{\lambda_n} \le \dfrac{1}{c^2n^2 - K.n}$, for all $n \ge N$. We can choose n sufficiently large such that $\dfrac{1}{c^2n^2 - K.n} \le \dfrac{2}{c^2n^2}$, which simply requires $n \ge \dfrac{2K}{c^2}$. Therefore we choose $M \ge \max\left\{ N, [2K/c^2] + 1 \right\}$, so we have

$$\frac{1}{\lambda_n} \le \frac{2}{c^2n^2}, \text{ if } \quad n \ge M.$$

Thus by using the above inequality we may rewrite (C2.10) in the following form

$$l_n^2(T) \le 2.B^2.e^{-2\lambda_n T} + \frac{2D}{c^2n^2}, \text{ for } n \ge M \qquad \text{(C2.11).}$$

Now let $n \ge M$ so $\dfrac{1}{\lambda_n} \le \dfrac{2}{c^2n^2}$, or $\lambda_n \ge \dfrac{n^2\pi^2}{2}$ when $n \ge M$. Thus $-2.T.\lambda_n \le -T.c^2n^2$, when $n \ge M$. So (C2.11) can be written in the following form

$$l_n^2(T) \le 2.B^2.e^{-c^2n^2T} + \frac{2D}{c^2n^2}, \text{ when } n \ge M. \qquad \text{(C2.12)}$$

Let $f_n = 2.B^2.e^{-c^2n^2T} + \dfrac{2D}{c^2n^2}$, so it is seen that $\sum_{n=1}^{\infty} f_n$ is convergent. Thus from (C2.12) we have $l_n^2(T) \le f_n$ for $n = 1,2,\ldots$ therefore, $\sum_{n=1}^{\infty} l_n^2(T)$ is convergent. But from (3′) we have $Y(x,t) = U(x,t) + v(x,t)$. Let

$$Y(x,t) = \sum_{n=1}^{\infty} d_n(t).a_n(x) \quad \text{for} \quad t \in [0,T] \qquad \text{(C2.13)}$$

Let the expansion of the known function U(x,t) be as follows

$$U(x,t) = \sum_{n=1}^{\infty} k_n(t).a_n(x) \quad \text{for} \quad t \in [0,T]. \qquad \text{(C2.14)}$$

We know that $\sum_{n=1}^{\infty} k_n^2(t)$ is convergent, also in (C2.7′) we assumed $v(x,t) = \sum_{n=1}^{\infty} l_n(t).a_n(x)$ and we showed above that $\sum_{n=1}^{\infty} l_n^2(t)$ is convergent. Now by definition of Y(.,.) we have $d_n(t) = k_n(t) + l_n(t)$, and by using the inequality $(x + y)^2 \le 2(x^2 + y^2)$ we have

$$d_n^2(t) \le 2\left[k_n^2(t) + l_n^2(t)\right] \quad n = 1,2,\dots . \qquad \text{(C2.15)}$$

From (C2.15) we conclude that $\sum_{n=1}^{\infty} d_n^2(t)$ is convergent for $t \in [0,T]$. Let $e_n = 2\left[k_n^2(T) + l_n^2(T)\right]$, thus $d_n(T)^2 \le e_n$, for $n \ge M$, and $\sum_{n=1}^{\infty} e_n(t) < \infty$, because we showed $\sum_{n=1}^{\infty} k_n^2(t)$ and $\sum_{n=1}^{\infty} l_n^2(t)$ are convergent.□

CHAPTER 3

A linear program for determining an optimal control of the diffusion equation in one and two dimensions

3.1 Introduction

In this chapter we construct linear programs for determining optimal controls of the diffusion equation in one and two dimensions. In section (3.2) we obtain an approximation to the optimal Radon measure μ^*, by using linear programming In sections (3.3)-(3.5) we obtain a piecewise constant control function corresponding to a desired final state by using the minimization scheme developed in section (3.2). In section (3.4), we obtain the controls corresponding to two different final states and we show the graphs of these controls in the (t,u)-plane. In sections (3.5) and (3.6) we develop similar results for two dimensional space.

3.2 An approximation to the optimal measure

Let

$$S_1 = \left\{ \mu \in \mathcal{M}(\Omega) : \mu(1) \le T, \ n = 0, 1, \dots \right\}$$

$$S_2 = \left\{ \mu \in \mathcal{M}(\Omega) : \mu(\psi_n) = \alpha_n, \ n = 1, 2, \dots \right\}$$

$$S_3 = \left\{ \mu \in \mathcal{M}(\Omega) : \mu(G) = a_G, \ G \in C(\Omega) \text{ and independent of } u \right\}$$

where $\psi_n(t,u) = 2(-1)^n.\exp\,[-n^2\pi^2(T-t)].u$, $n = 1, 2, \dots \psi_0(t,u) = u$, $t \in [0,T]$,and let $Q = S_1 \cap S_2 \cap S_3$.

Proposition (2.1) of Chapter 2 claims: The measure theoretical control problem, which consists in finding the minimum of the functional

$$\mu \in Q \rightarrow \mu(f_0) \in R$$

over the set Q of $\mathcal{M}^{+}(\Omega)$, possesses a minimizing μ^{*}, say, a measure in Q.

In the following we define $G_i(.,u)$ as monomials in t only, that is, $G_i(t,u) = t^i$, and from Weierstrass Approximation Theorem any continuous function on $[0,T]$, can be uniformly approximated by a finite linear combinations of elements of the set $\{G_i(.,u) : i= 0,1,\dots\}$ (see BARTLE [1]p.183). Now for simplicity we use the notation $\lim_i(a_i)$, instead of $\lim_{i\to\infty}(a_i)$.

Proposition 3.1 Consider the linear program which consists of minimizing the function $\mu \rightarrow \mu(f_0)$ over the set $Q(M_1,M_2)$ of measures in $\mathcal{M}^{+}(\Omega)$ satisfying :

$$\mu(\psi_n) = \alpha_n \qquad n = 0,1,2,\dots,M_1$$

$$\mu(G_i) = a_{G_i} \qquad n = 0,1,2,\dots,M_2.$$

Then, as M_1 and M_2 tend to infinity,

$$\eta(M_1,M_2) = \inf_{Q(M_1,M_2)} \mu(f_0)$$

tends to $\eta = \inf_{Q} \mu(f_0)$.

Proof The proof is the similar to the proof of proposition (iii,1) of RUBIO [1] □.

We conclude the following proposition from a result of ROSENBLOOM [1], (see its proof in RUBIO [1] Theorem A.5), also it is possible to characterise a measure in the set $Q(M_1,M_2)$ at which the linear functional $\mu \to \mu(f_0)$ attains its minimum.

Proposition 3.2. The measure μ^* in the set $Q(M_1,M_2)$ at which the function $\mu \to \mu(f_0)$ attains its minimum has the form

$$\mu^* = \sum_{k=1}^{M_1+M_2} \alpha_k^* . \delta(Z_k^*), \qquad (3.1)$$

with $Z_k^* \in \Omega$ ($\Omega = [0,T] \times [-1,1]$), and the coefficients $\alpha_k^* \geq 0$, $k = 0,1,\ldots,M_1+M_2$; here $\delta(Z)$ is a unitary atomic measure with support the singleton set $\{Z\}$ which is characterized by $\delta(Z)F = F(Z)$, $F \in C(\Omega)$, $Z \in \Omega$. (see RUBIO [1] p.114).

Now let $P(M_1,M_2,\varepsilon) \subseteq R^N$ be the set of all $(a_1,a_2,\ldots,a_N)$ defined by

$$\begin{cases} \alpha_i \geq 0 & i = 0,1,2,\ldots,N \\ -\varepsilon \leq \sum_{j=1}^{N} \alpha_j . \psi_i(Z_i) - a_i \leq \varepsilon & i = 0,1,2,\ldots,M_1 \\ -\varepsilon \leq \sum_{j=1}^{N} \alpha_j . G_i(Z_i) - a_{G_i} \leq \varepsilon & i = 0,1,2,\ldots,M_2 \end{cases}$$

Theorem 3.1. For every $\varepsilon > 0$, the problem of minimizing the function $\sum_{j=1}^{N} \alpha_j.f(Z_j)$, where $Z_j \in \sigma$, $j = 0,1,\dots,N$ with σ a dense subset of Ω, on the set $P(M_1,M_2,\varepsilon)$, has a solution for $N = N(\varepsilon)$ sufficiently large. The solution satisfies

$$\eta(M_1,M_2) + \rho(\varepsilon) \le \sum_{j=1}^{N} \alpha_j.f_0(Z_j) \le \eta(M_1,M_2) + \varepsilon$$

where $\rho(\varepsilon)$ tends to zero as ε tends to zero.

Proof The proof is the same as that of Theorem (*iii*.1) RUBIO [1]. When setting up the linear programming problem akin to proposition 3.2, it was decided to take the parameter ε as zero, at least formally; of course, the error present in the numerical computations will ensure that the solution of the linear programming problem will not satisfy exactly the constraint equations. Now our linear programming problem consists of minimizing the linear form

$$\sum_{j=1}^{N} \alpha_j.f_0(Z_j)$$

over the set of coefficients $\alpha_j \ge 0$, $j = 1,2,\dots,N$ such that

$$
\begin{cases}
\sum_{j=1}^{N} \alpha_j \cdot \psi_i(Z_j) = a_i, & , \quad i = 0,1,2,\dots,M_1 \\
\sum_{j=1}^{N} \alpha_j G_k(Z_j) = a_{G_k}, & i = 0,1,2,\dots,M_2 \\
\mu(1) = \sum_{j=1}^{N} \alpha_j . 1(Z_j) = \sum_{j=1}^{N} \alpha_j \le T;
\end{cases}
\tag{3.2}
$$

or, equivalently, we are looking for a solution of the following linear programming problem which consists of minimizing the linear form

$$\sum_{j=1}^{N+1} \alpha_j . f_0(Z_j)$$

over the set of coefficients $\alpha_j \ge 0$, $j = 1,2,\dots,N+1$ such that

$$
\begin{cases}
\sum_{j=1}^{N+1} \alpha_j \cdot \psi_i(Z_j) = a_i, & , \quad i = 0,1,2,\dots,M_1 \\
\sum_{j=1}^{N+1} \alpha_j G_k(Z_j) = a_{G_k}, & i = 0,1,2,\dots,M_2 \\
\sum_{j=1}^{N+1} \alpha_j = T,
\end{cases}
$$

where we used one slack variable α_{N+1}, to put the last inequality in (3.2) in the form of the following equality,

$$\sum_{j=1}^{N} \alpha_j + \alpha_{N+1} = T \qquad (\alpha_{N+1} \geq 0).$$

Remark Instead of choosing, $F_i(t,u) = t^i$, $(i = 0,1,2,\ldots)$, it is suitable to choose $F_i(t,u)$ as the following functions :

$$F_i(t,u) = 1 \qquad t \in J_i$$
$$= 0 \qquad \text{otherwise}$$

where $J_i = [(i-1)d,\ id)$, $i = 1,2,\ldots,L$ and $d = \frac{\Delta t}{L}$ and L is a positive integer which is in fact the number of the subintervals in the partition of the interval $[0,T]$. The functions F_i are not continuous, however,

(i) Each of the F_i, $i = 1,2,\ldots,L$ is the limit of an increasing sequence of positive continuous functions, $\{F_{ik}\}$, then if μ is any positive Radon measure on Ω,

$$\mu(F_i) = \lim_k \mu(F_{ik}).$$

(ii) Consider now the set of functions F_i, $i = 1,2,\ldots,L$, for all positive integers L. It has been shown in RUBIO [1] that linear combinations of these functions can approximate arbitrary well a function in $C_1(\Omega)$, where $\Omega \equiv [0,T] \times [-1,1]$, [here $C_1(\Omega)$, is the class of all continuous functions depending only on t] in the sense of essential supremum (see Friedman [2]). Thus the error function can be made to tend to zero by choosing in an appropriate manner a sufficient number of terms in the corresponding expansions (see RUBIO [1])□.

3.3 Construction of piecewise constant control functions

In this section we wish to obtain the control function corresponding to a desired final state by using the minimization scheme developed above.

We showed in section (3.1) that our linear programming problem for the control for the diffusion equation in one dimension consists of minimizing the following function

$$\sum_{j=1}^{N+1} \alpha_j . f_0(Z_j)$$

over the set of coefficients $\alpha_j \geq 0$, $j = 1,2,\ldots,N+1$ such that

$$\begin{cases} \sum_{j=1}^{N+1} \alpha_j . \psi_i(Z_j) = a_i & , \quad i = 0,1,2,\ldots,M_1 \\ \sum_{j=1}^{N+1} \alpha_j F_s(Z_j) = b_s & s = 0,1,2,\ldots,M_2 \\ \sum_{j=1}^{N+1} \alpha_j = T & \end{cases} \qquad (3.3)$$

where $Z_k = (t_k, u_k)$, $k = 1, 2, \ldots,$ and $\sigma = \{Z_k, k = 1,2,\ldots\}$ is chosen as being dense in Ω; in practice, the set $\sigma^N = \{Z_k, k = 1,2,\ldots,N+1\} \subset \sigma$, was constructed by dividing the appropriate intervals into a number of equal sub-interevals, defining in this way a grid of points. We explain more about the points Z_k later.

In the following examples we choose $f_0(.)$ a known function and choose for example $T = 1$. We choose two different desired final states belonging to $L_2([0,1])$, where a_i, $i = 0,1,\ldots,M_1$ are the cosine Fourier coefficients of the function $g(.)$, and b_s's

$= 1,2,\dots,M_2$, are defined as the following integral: $b_s = \int_0^1 F_s(t,.).dt$, introduced in section (3.2).

Now we refer to the solution of the diffusion equation in one dimension as introduced in the paper of WILSON and RUBIO [1] as follows

$$\psi_0(t,u) = u$$

$$\psi_n(t,u) = 2.(-1)^n.\exp[-n^2\pi^2].u \qquad n = 1,2,\dots, \qquad t \in [0,T]$$

$$F_s(t) = 1 \quad \text{if} \quad t \in J_s$$

$$= 0, \quad \text{otherwise},$$

where we used the notation $F_s(t,.) = F_s(t)$, and where $J_s = ((s-1)d,\ sd)$ where $d = \dfrac{\Delta t}{L}$ and $\Delta = 1 - 0 = 1$ and we choose $L = 10$ (the number of equi-distance subintervals of the partition of the interval $[0,1]$). Therefore we have $d = (1/10)$ and

$$b_s = \int_0^1 F_s(t).dt = d = 0.1 \quad \text{for all} \quad s = 1,\dots,10.$$

Now we divide the interval $[-1,1]$ on the u axes to 20 equal subintervals therefore we have $N = 200$ and (3.3) will be in the following form

$$\begin{cases} \sum_{j=1}^{201} \alpha \cdot \psi_i(Z_j) = a_i & , \quad i = 0,1,2,\dots,M_1, \qquad (3.4) \\ \sum_{j=1}^{201} \alpha_j F_s(Z_j) = b_s & \quad s = 0,1,2,\dots,M_2, \qquad (3.5) \\ \sum_{j=1}^{201} \alpha_j = 1, & \qquad (3.6) \end{cases}$$

and we define

$$u_1 = u_2 = \dots = u_{10} = -0,95$$

$$\vdots$$

$$u_{191} = u_{192} = \dots = u_{200} = 0.95$$

and also we have

$$t_1 = t_{11} = \dots = t_{181} = t_{191} = 0.05$$

$$\vdots$$

$$t_{10} = t_{20} = \dots = t_{190} = t_{200} = 0.95 \; .$$

Therefore by definition of F_s's we have

$$F_1(t_1) = F_1(t_{11}) = \dots = F_1(t_{191}) = 1$$

and for other t_j's, $F_1(t_j) = 0$ also

$$\vdots$$

$$F_{10}(t_{10}) = F_{10}(t_{20}) = \dots = F_{10}(t_{200}) = 1$$

and for other t_j's, $F_{10}(t_j) = 0$.

Therefore the equations (3.5) may be written in the following form

$$\begin{cases} \alpha_1 + \alpha_{11} + \dots + \alpha_{191} = 0.1 \\ \alpha_2 + \alpha_{12} + \dots + \alpha_{192} = 0.1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{10} + \alpha_{20} + \dots + \alpha_{200} = 0.1. \end{cases} \tag{3.7}$$

We conclude from equations (3.7) that $\sum_{j=1}^{200} \alpha_j = \sum_{j=1}^{10} 0.1 = 1$, so by using this result we have from (3.6) $\alpha_{201} = 0$. Therefore $\sum_{j=1}^{200} a_j \delta(Z_j)$, is an approximation for the optimal measure μ^*, where Z_j, $j = 1,2,\dots,200$, will be specified precisely in section (3.5)

3.4 Two examples with different final states and cost functions.

In this section we obtain the control functions of two examples.

Example (3.1) Let the final state be $g(x) = 0.1$ and we choose the criterion function $f_0(t,u) = u^2$, so our linear programming problem consists of minimizing the following function

$$\sum_{j=1}^{200} \alpha_j . u_j^2$$

over the set of coefficients $\alpha_j \geq 0$, $j = 1,2,\dots,200$ such that

$$\sum_{j=1}^{200} \alpha_j . u_j = 0.1$$

$$\sum_{j=1}^{200} \alpha_j . 2(-1).\exp[-\pi^2(1-t_j)].u_j = 0.0$$

$$\sum_{j=1}^{200} \alpha_j . 2(-1).\exp[-4\pi^2(1-t_j)].u_j = 0.0$$

$$\sum_{j=1}^{200} \alpha_j . 2(-1).\exp[-9.\pi^2(1-t_j)].u_j = 0.0$$

$$\sum_{j=1}^{200} \alpha_j . 2(-1).\exp[-16\pi^2(1-t_j)].u_j = 0.0$$

$$\alpha_1 + \alpha_{11} + \ldots + \alpha_{191} = 0.1$$

$$\alpha_2 + \alpha_{12} + \ldots + \alpha_{192} = 0.1$$

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$$\alpha_{10} + \alpha_{20} + \ldots + \alpha_{200} = 0.1.$$

We used a NAG library program which is based on the Revised simplex method. The computational results are as follows:

$\alpha_{80} = 0.005236$ $\alpha_{99} = 0.09313$ $\alpha_{100} = 0.05378$

$\alpha_{107} = 0.05750$ $\alpha_{108} = 0.1000$ $\alpha_{109} = 0.006872$

$\alpha_{110} = 0.02088$ $\alpha_{111} = 0.1000$ $\alpha_{112} = 0.1000$

$\alpha_{113} = 0.1000$ $\alpha_{114} = 0.1000$ $\alpha_{115} = 0.1000$

$\alpha_{116} = 0.1000$ $\alpha_{117} = 0.04250$ $\alpha_{120} = 0.02011$.

The cost function = 0.01574, CPU time = 1.03 seconds, the number of iterations in phase 1 = 42 and in phase 2 is 1.

Example 3.2 Let the final state be $g_1(x) = 0.01.\cos(\pi x)$ and we choose the criterion function $f_0(t,u) = |u|$ so our linear programming problem consists of minimizing the following function

$$\sum_{j=1}^{200} \alpha_j . |u_j|$$

over the set of coefficients $\alpha_j \geq 0$, $j = 1,2,\ldots,200$, with the number of linear constraints equal to 15 [the same as example 3.1]. Then the results of the computations are as follows:

$\alpha_{39} = 0.02401$ $\alpha_{50} = 0.01571$ $\alpha_{96} = 0.0073$

$\alpha_{97} = 0.1000$ $\alpha_{98} = 0.1000$ $\alpha_{99} = 0.07599$

$\alpha_{100} = 0.02527$ $\alpha_{101} = 0.1000$ $\alpha_{102} = 0.1000$

$\alpha_{103} = 0.1000$ $\alpha_{104} = 0.1000$ $\alpha_{105} = 0.1000$

$\alpha_{106} = 0.0927$ $\alpha_{120} = 0.05733$ $\alpha_{190} = 0.001696$.

Cost function = 0.07935, CPU time = 0.93 second, the number of iterations in phase 1 = 26, the number of iteration in phase 2 is equal to 1.

3.5 Control functions

In this section we intend to construct the control functions for the above two examples.

Here we follow the procedure explained in RUBIO [1] (Chapter 5), suppose that the set $\sigma = \{Z_j \; ; \; j = 1,2,\dots \}$ has been chosen as being dense in the set $[0,1] \times [-1,1]$; in practice, the set $\sigma^N = \{Z_j \; ; \; j = 1,2,\dots,N \}$ was costructed by dividing the intervals $[0,1]$ and $[-1,1]$ into a number of equal subintervals, defining in this way a grid of points. Here we divide the interval $J = [0,1]$ into 10 equal subintervals, also we divide the interval $U = [-1,1]$ into 20 equal subintervals, the values t_j were taken as 0.5, 0.15, ... 0.95, the values u_j were taken -0.95, -0.85, ... -0.05, 0.05, ... 0.85, 0.95. Now by GHOUILA - HOURI's theorem, we have

$$K_{ij} = \int_{[t_{i-1}\,,\; t_i] \times [u_{j-1}\,,\; u_j]} d\mu^*(t,u) = \mu^*([t_{i-1}, t_i) \times [u_{j-1}, u_j))$$

for $1 \le i \le 10$ and $1 \le j \le 20$.

Let $\sigma_{i,j}$ be the i, jth subrectangle of $[0,1] \times [-1,1]$ where $1 \le i \le 10$ $1 \le j \le 20$. The point Z^* has been chosen as the following point $Z^*_{i,j} = (t_i, u_j)$ $1 \le i \le 10$ and $1 \le j \le 20$. Now let λ be the following transformation

$$\lambda : N \times N \to N$$

defined by $\lambda(i,j) = 10(j-1) + i$ $1 \le i \le 10$ and $1 \le j \le 20$, and let $Z_{\lambda(i,j)} = Z^*_{(i,j)}$ for $1 \le i \le 10$ and $1 \le j \le 20$ or

$Z^*_{(i,j)} = Z_{10(j-1)+i}$ for $1 \le i \le 10$ and $1 \le j \le 20$.

Therefore the optimal measure introduced in (3.1) will be in the following form

$$\mu^* = \sum_{k=0}^{M_1+M_2} \alpha_k^*.\delta(Z_k),$$

where M_1 and M_2 are the number of constraints. But if $Z \in \Omega$ then by definition of δ- measure $\int_\Omega F.d\delta(Z) = F(Z)$, $F \in C(\Omega)$.

Therefore for the constant function $F(t) \equiv 1$ and $Z \in \Omega$ we have $\int_\Omega 1 \, d\delta(Z) = 1$. Thus,

$$K_{ij} = \int_{\sigma_{i,j}} d\mu^*(t,u) = \int_{\sigma_{i,j}} d[\sum_{k=1}^{M_1+M_2} \alpha_k.\delta(z_k)\ ] = \alpha_k,$$

where $k = 10(j-1)+i$ and $1 \le i \le 10$ and $1 \le j \le 20$.

Now we construct the control function of example 3.1, again here we use the method introduced in RUBIO [1] (section 1 of chapter 5) and we use the same notations in that section, so we have

$K_{10,8} = \alpha_{80} = 0.005236 \qquad t_{80} = 0.95 \qquad u_{80} = -0.15$

$K_{9,10} = \alpha_{99} = 0.09313 \qquad t_{99} = 0.85 \qquad u_{99} = -0.05$

.

.

.

$K_{10,12} = \alpha_{120} = 0.02011 \qquad t_{120} = 0.95 \qquad u_{120} = 0.15.$

Again by using the notation in (5.1) of the above reference we have $u(t) = u_{10(j-1)+1}$, $t \in B_{i',j'}$ where

$$B_{i'j'} = (t_{i'-1} + \sum_{k<j'} K_{i'k} \, , \, t_{i'-1} + \sum_{k\le j'} K_{i'k})$$

such that $t_{i'}$ are the numbers 0.0, 0.1, ... 0.9, 1. Therefore for example we have $B_{10,8} = (0.9, 0.905236)$ and for $t \in B_{10,8}$ we have $u(t) = u_{80} = -0.15$. We obtain all $B_{i'j'}$ in the same way. Thus we have the control function of example 3.1 in fig (3.1). Similarly we obtain the control function of example 3.2 which is shown in fig (3.2).

Finally we wish to calculate the final state Y(.) corresponding to the desired final state g(.) in example 3.1, by using the piecewise constant control function u(.) in example 3.1. Let $Y(x) = d_0$, where

$$d_0 = \int_0^1 u(t)\, dt = \sum \alpha_k u_k = 0.10072,$$

thus we have $\|Y(.) - g(.)\|_{L_2(0,1)} = 0.00072$. Also let $Y_1(.)$ be the final state corresponding to the desired final state $g_1(.)$ in example 3.2, by using piecewise constant control function u_1 in example 3.2. Therefore let $Y_1(x) = d_1 \cos(\pi x)$, where

$$d_1 = -\int_0^1 \psi_1[(t, u_1)]dt = 2\int_0^1 \exp[-\pi^2(1-t)]\; u_1(t)dt$$

so we have

$$\|Y_1(.) - g_1(.)\|_{L_2(0,1)} = 0.008.$$

3.6 A linear program for determining an optimal control for the diffusion equation in two dimensions

We assume $\Gamma = [0,1] \times J \times [-1,1]$ where $J = [0,T]$ and let

$$S_1 = \left\{ \mu \in \mathcal{M}^+(\Gamma) : \mu(1) \le T \right\},$$

$$S_2 = \left\{ \mu \in \mathcal{M}^+(\Gamma) : \mu(F_n) = c_n , \quad n = 1,2, \ldots \right\}, \qquad (3.8)$$

$$S_3 = \left\{ \mu \in \mathcal{M}^+(\Gamma) : \mu(F) = c_h, \ h \in C(\Omega) \text{ and independent of } u \right\},$$

where c_n, $n = 1,2,\ldots$ are the Fourier coefficients of $g(x), x \in \omega$. In this section ω is an open region in the plane, and $\partial\omega \in C^1$. We have defined c_h, $h \in C(\Gamma)$, in chapter 2, where $h(.,.,u)$ is independent of u, and only depends on (x,t). In chapter 2, we defined $\Gamma = [0,1] \times [0,T] \times [-1,1]$, and assumed $Q' = S_1 \cap S_2 \cap S_3$, and we showed that the control problem of the diffusion equation in two dimensions can be reduced to finding a measurable control, $u(y,t) \in [-1,1]$, $(y,t) \in \partial\omega \times [0,T]$ which satisfies

$$c_n = \int_0^T \int_0^1 F_n(s,t,\hat{u}(s,t))\, ds.dt \qquad n = 1, 2, \ldots$$

where $\hat{u}(s,t) = u(\xi_1(s),\xi_2(s),t)$, $t \in [0,T]$, $s \in [0,1]$, such that $y = (\xi_1(s),\xi_2(s))$ is the parametric equation of $\partial\omega$ and,

$$F_n(s,t,u) = -(\partial a_n / \partial\nu)(\xi_1(s),\xi_2(s)).e^{-\lambda_n.t}.\sqrt{\xi_1'^2+\xi_2'^2}\,.u,$$

for $n = 1,2,\dots$.

Proposition (2.1) in Chapter 2 claims: The measure theoretical control problem, which consists of finding the minimum of the functional

$$\mu \in Q' \to \mu(f_0) \in R,$$

over the subset Q' of $M^+(\Gamma)$, possesses a minimizing μ^*, say, a measure in Q'.

Now we define $h_i(.,.,u)$ on $[0,1] \times [0,T]$ as follows:

$$h_{ij}(s,t,u) = 1, \quad (s,t) \in J_{ij} \qquad (3.9)$$
$$= 0, \quad \text{otherwise}$$

where $J_{ij} = ((i-1)d, id) \times ((j-1)d', jd')$ such that $d = \frac{1}{K}$, $d' = \frac{T}{L}$ and $i = 0,1,\dots,K$, $j = 0,1,\dots,L$. [Here we assumed that the interval [0,1] on s - axis has been divided into K equal subintervals, and the interval $J = [0,1]$ on t - axis into L equal subintervals]

Remark : The linear combinations of the functions h_{ij}, can approximate a function arbitrary well in $C_2(\Gamma)$ where $\Gamma = [0,1] \times [0,T] \times [-1,1]$, [$C_2(\Gamma)$, is the subspace of $C(\Omega)$, which depends only on the variables, s, and t.]. This means that for $h \in C(\Gamma)$ there exists a sequence, $\{h^{ij}\}$, of functions in the subspace spanned by the functions $h_{ij}(.,.,u)$ defined in (3.9), such that

$$h^{ij}(s,t,u) \to h(s,t,u)$$

uniformly when i, and j, tend to infinity; see HEWITT and STROMBERY [1], (page 159). The functions h_{ij}, are not continuous; however, each of the h_{ij}, $i = 0,1,\ldots,j = 0,1,\ldots$, is the limit of an increasing sequence of positive continuous functions $\{h_{ij,k}\}_k$, [this notation means that this is a sequence of the integer variable k and the other integer variables i and j are fixed], then if μ is any positive Radon measure, on Γ, we have

$$\mu(h_{ij}) = \lim_k (h_{ij,k}).$$

Proposition (3.3) Consider the following linear program which consists of minimizing the function $\mu \to \mu(f_0)$, over the set $Q'(M_1,M_2,M_3)$ of measures in $\mathcal{M}^+(\Gamma)$ satisfying

$$\mu(F_n) = c_n \qquad n = 1,2,\ldots,M_1,$$

$$\mu(h_{ij}) = c_{h_{ij}} \qquad i = 0,1,\ldots,M_2,$$
$$j = 0,1,\ldots,M_3.$$

As M_1, M_2, and M_3 tend to infinity

$$\eta(M_1,M_2,M_3) = \inf_{Q'(M_1,M_2,M_3)} \mu(f_0)$$

tends to

$$\eta = \inf_{Q'} \mu(f_0).$$

Proof This proposition is an extension of proposition (3.1), where instead of the set $Q(M_1,M_2)$ in that proposition, here we have $Q'(M_1,M_2,M_3)$ and instead of $\eta(M_1,M_2)$ here we defined $\eta(M_1,M_2,M_3)$, so its proof is similar to the proof of proposition (3.1).□

Remark: It is possible to characterise a measure in the set $Q'(M_1,M_2,M_3)$ at which the linear functional

$$\mu \to \mu(f_0)$$

attains its minimum; it follows from a result of ROSENBLOOM [1], which is shown in RUBIO [1]. The following proposition is an extension of the proposition (3.2), and its proof is similar to the prooof of that proposition.□

Proposition (3.4) The measure μ^*, in the set $Q'(M_1,M_2,M_3)$, at which the functional $\mu \to \mu(f_0)$ attains its minimum has the following form

$$\mu^* = \sum_{k=1}^{M_1+M_2+M_3} \alpha_k.\delta(Z_k),$$

where $Z_k \in \Gamma = [0,1] \times [0,T] \times [-1,1]$, the coefficients $\alpha_k \geq 0$, $k = 1, 2,\dots,M_1+M_2+M_3$, and δ a unitary atomic measure with support the singleton set $\{Z\}$, which we show by $\delta(Z) \in \mathcal{M}^+(\Omega)$ and is characterzed by

$$\delta(Z)F = F(Z), \quad F \in C(\Gamma), \quad Z \in \Gamma$$

Now let $P(M_1, M_2, M_3, \varepsilon)$, be the following sets of inequalities

$$\alpha_i \geq 0, \quad i = 1,2,\dots,N$$

$$-\varepsilon \leq \sum_{j=1}^{N} \alpha_j . F_i(Z_j) - c_i \leq \varepsilon, \; i = 1,2,\dots,M_1$$

$$-\varepsilon \leq \sum_{j=1}^{N} \alpha_j . h^{ik}(Z_j) - c_{\bar{h}^{ik}} \leq \varepsilon, \; i = 1,2,\dots,M_2$$
$$k = 1,2,\dots,M_3.$$

Theorem (3.4) For every $\varepsilon > 0$, the problem of minimizing the function $\sum_{j=1}^{N} \alpha_j . f(Z_j)$, [where $Z_j \in \theta$, $j = 1,2,\dots,N$ and θ, is a dense subset of Γ] on the set $P(M_1, M_2, M_3, \varepsilon)$, has a solution for $N = N(\varepsilon)$, sufficiently large .The solution satisfies

$$\eta(M_1, M_2, M_3) + \rho(\varepsilon) \leq \sum_{j=1}^{N} \alpha_j . f_0(Z_j) \leq \eta(M_1, M_2, M_3) + \varepsilon$$

where $\rho(\varepsilon)$, tends to zero, as ε, tend to zero.

Proof The proof is similar to the proof of theorem (III.1), (RUBIO, [1]).□

As we described in section 3.2 we decide to take the parameter ε in Proposition (3.4) as zero. Therefore the linear programming problem consists of minimizing the linear form,

$$\sum_{j=1}^{N} \alpha_j . f_0(Z_j),$$

over the set of coefficients $\alpha_j \geq 0$, $j = 1,2,\dots,N$, such that

$$\sum_{j=1}^{N} \alpha_j F_i(Z_j) = c_i, \quad i = 1,2,\ldots,M_1,$$

$$\sum_{j=1}^{N} \alpha_j h^{kl}(Z_j) = c_{h^{kl}}, \quad k = 1,2,\ldots,M_2, \qquad (3.10)$$

$$l = 1,2,\ldots,M_3,$$

$$\mu'(1) = \sum_{j=1}^{N} \alpha_j . 1(Z_j) = \sum_{j=1}^{N} \alpha_j \le T,$$

or equivalently, we are looking for a solution of the following linear programming problem, which consists of minimizing the linear form,

$$\sum_{j=1}^{N+1} \alpha_j f_0(Z_j)$$

over the set of coefficients $\alpha_j \ge 0$, $j = 1,2,\ldots,N+1$, such that

$$\sum_{j=1}^{N+1} \alpha_j F_i(Z_j) = c_i, \quad i = 1,2,\ldots,M_1,$$

$$\sum_{j=1}^{N+1} \alpha_j h^{kl}(Z_j) = c_{h^{kl}}, \quad k = 1,2,\ldots,M_2,$$

$$, \ l = 1,2,\ldots,M_3,$$

$$\sum_{j=1}^{N+1} \alpha_j = T,$$

where

$$F_i(Z) = F_i(s,t,u) = \frac{\partial a_i}{\partial \nu}(\xi_1(s),\xi_2(s)).e^{-\lambda_n t}.\sqrt{\xi_1'^2 + \xi_2'^2}\ .u.$$

here $x = \xi_1(s)$, $y = \xi_2(s)$ ($s \in [0,1]$) is the equation of the boundary of the region ω, in the xy-plane, we have chosen one slack variable α_{N+1}, so as to put all linear constraints in equality form; that is, the last inequality in (3.10) changes to

$$\sum_{j=1}^{N} \alpha_j + \alpha_{N+1} = T, \quad (\alpha_{N+1} \geq 0).$$

Example 3.3 In this example let ω be the rectangle $[0,\pi] \times [0,\pi]$, in the xy-plane; we assume $T = 1$, so that $\Gamma = [0,1] \times [0,1] \times [-1,1]$. We divide the set $[0,1] \times [0,1]$, in the st-plane into 64 equal subrectangles and we divide the set $[-1,1]$, into 15 equal subintervals, so the set Γ is divided into 960 equal subsets. Our linear programing problem then consists of minimizing the following real function

$$\sum_{j=1}^{960} \alpha_j f_0(Z_j)$$

over the set of coefficients $\alpha_j \geq 0$, $j = 1,2,\ldots,960$, such that

$$\sum_{j=1}^{961} \alpha_j F_i(Z_j) = c_i, \quad i = 1,2,\ldots M_1,$$

$$\sum_{j=1}^{961} \alpha_j f_{kl}(Z_j) = c^{kl}, \quad k = 1,2,\ldots M_2, \qquad (3.11)$$

$$, \quad l = 1,2,\ldots,M_3,$$

$$\sum_{j=1}^{961} \alpha_j = T.$$

We prove that the slack variable α_{961} is zero. Next we choose the criterion function, f_0, a known function, and also we specify the final state function g(x). Let c_i, i = 1,2,..., be the Fourier coefficients of the function g(x) in terms of the orthonormal eigenfunctions $a_k(x)$. That is let $g(x) = \sum_{k=1}^{\infty} c_k.a_k(x)$.

We define the functions f'_{kl}s as follows,

$$f_{kl}(s,t,u) = 1, \quad (s,t) \in J_{kl}$$
$$= 0, \quad \text{otherwise},$$

where J_{kl} = ((k-1)d, kd) X ((l-1)d', ld'), $d = \frac{1}{K}$, $d' = \frac{1}{L}$, k = 1,2,...,K, l = 1,2,...,L. We choose for example K = L = 8, therefore $d = d' = \frac{1}{8}$ so k = 1,2,...,8, l = 1,2,...,8 and c^{kl}, is the integral of the functions f_{kl}, over J_{kl}, that is

$$c^{kl} = \int_0^1\int_0^1 f_{kl}(s,t)ds.dt = \int_{(l-1)d'}^{ld'}\int_{(k-1)d}^{kd} 1ds.dt = d'd,$$

or

$$c^{kl} = (1/8)(1/8) = 1/64, \quad k = 1,2,\dots,8 \text{ and } l = 1,2,\dots,8.$$

As we mentioned before we divide the interval [-1,1] on the u axis into 15 equal subintervals, and we choose the u_k,s as follows:

$u_1 = u_2 = \dots = u_{64} = -14/15$,

.

.

.

$u_{65} = u_{66} = \dots = u_{128} = -12/15$;

in general we define

$$u_{64k+1} = u_{64k+2} = \dots = u_{64(k+1)} = (2k-14)/15, \quad k = 0,1,\dots,14.$$

Also we choose s_k as follows

$$s_1 = s_2 = \dots = s_8 = s_{65} = s_{66} = \dots = s_{72} = \dots = s_{897} = \dots = s_{904}$$
$= 1/16$.

In general we have

$$s_{64k+8i+j} = (2i+1)/16. \quad \text{for} \quad i = 0,1,\dots,7, \; j = 1,2,\dots,8 \text{ and}$$
$$k = 0,1,\dots,14.$$

We choose

$$t_1 = t_9 = t_{17} = \dots = t_{64k+8i+1} = \dots = 1/16, \quad i = 0,1,\dots,7$$
$$k = 0,1,\dots,14$$

or in general

$$t_{64k+8i+j} = (2j-1)/16, \text{ for } j = 1,2,\dots,8, \; i = 0,1,\dots,7,$$
$$k = 0,1,\dots,14.$$

By definition of the functions f_{kl} and due to the choice of the s_i' s and t_j' s we have for $k = 1,\dots,8$ and $l = 1,\dots,8$ the following correspondence,

$$(s_l,\ t_k) \;\ast\; (s_{64n+8(l-1)+k}\ ,\ t_{64n+8(l-1)+k})$$

for all $n = 0,1,\ldots,14$, $k = 1,2,\ldots,8$, and $l = 1,2,\ldots,8$. Therefore we have

$$f_{lk}(s_l,\ t_k) = f_{lk}(s_{64n+8(l-1)+k}\ ,\ t_{64n+8(l-1)+k}) = 1,$$

$$f_{lk}(.,.) = 0, \quad \text{otherwise.}$$

So we have the following equations

$$\alpha_1 f_{1,1}(s_1,t_1) + \alpha_2 f_{1,1}(s_2,t_2) + \ldots + \alpha_{960} f_{1,1}(s_{960},\ t_{960}) = 1/64$$

.
.
.

$$\alpha_1 f_{kl}(s_1,t_1) + \alpha_2 f_{kl}(s_2,t_2) + \ldots + \alpha_{960} f_{kl}(s_{960},\ t_{960}) = 1/64$$

.
.
.

$$\alpha_1 f_{8,8}(s_1,t_1) + \alpha_2 f_{8,8}(s_2,t_2) + \ldots + \alpha_{960} f_{8,8}(s_{960},\ t_{960}) = 1/64,$$

where $k = 1,2,\ldots,8$ and $l = 1,2,\ldots,8$. We substitute the value of f_{kl} in the above system of linear equations giving

$$\alpha_1 + \alpha_{65} + \alpha_{129} + \ldots + \alpha_{64n+1} + \ldots + \alpha_{897} = 1/64,$$

$$\alpha_{8(l-1)+k} + \alpha_{64+8(l-1)+k} + \ldots + \alpha_{64n+8(l-1)+k} + \ldots + \alpha_{64.14+8(l-1)+k} = 1/64,$$

.
.
.

$$\alpha_{64} + \alpha_{128} + \alpha_{129} + \ldots + \alpha_{64n} + \ldots + \alpha_{960} = 1/64.$$

If we add all the equations in the above system we obtain

$$\sum_{j=1}^{960} \alpha_j = \sum_{j=1}^{64} 1/64 = 1.$$

Now we compare the above equality with the constraint $\sum_{j=1}^{961} \alpha_j = 1$ in (3.11); we conclude, $\alpha_{961} = 0$, that is, the only one slack unknown α_{961}, is zero and the other 960, variables are nonnegative. Now in this example we choose $g(x,y) = 0.1((2/\pi).\sin x.\sin y)$; in the next section we calculate the functions $F_i(.,.,.)$, and we assume $f_0(s,t,u) = u^2$, so our linear programming problem is as follows:

Minimize

$$\sum_{j=1}^{960} \alpha_j u_j^2,$$

over the set of coefficients $\alpha_j \geq 0$, $j = 1,2,\dots,960$, such that,

$$\sum_{j=1}^{960} \alpha_j . 8.\sin(\pi s_j).e^{-(4/\pi)t_j}.\ u_j = 0.1$$

$$\sum_{j=1}^{960} \alpha_j .[4(3\sin(\pi s_j) + \sin(3\pi s_j)].e^{-(20/\pi)t_j} = 0.0$$

$$\sum_{j=1}^{960} \alpha_j . 24.\sin(3\pi s_j).e^{-(36/\pi)t_j}.\ u_j = 0.0$$

$$\sum_{j=1}^{960} \alpha_j .8.[2\sin(2\pi s_j) - \sin(4\pi s_j)].e^{-(40/\pi)t_j} = 0.0$$

$$\alpha_1 + \alpha_{65} + \alpha_{129} + \dots + \alpha_{64n+1} + \dots + \alpha_{897} = 1/64,$$

.
.
.

$$\alpha_{8(l-1)+k} + \alpha_{64+8(l-1)+k} + \dots + \alpha_{64n+8(l-1)+k} + \dots + \alpha_{64.14+8(l-1)+k} = 1/64,$$

.
.
.

$$\alpha_{64} + \alpha_{128} + \alpha_{129} + \dots + \alpha_{64n} + \dots + \alpha_{960} = 1/64.$$

In example (3.3) we have assumed the region ω to be the rectangle $[0,\pi] \times [0,\pi]$. Now we find a sequence of functions $\{a_n(x,y)\}$, n = 1,2,..., the set of orthonormal eigenfunctions of the problem

$$u_{xx} + u_{yy} + \lambda u = 0, \ (x,y) \in \omega; \quad u(x,y) = 0, \quad (x,y) \in \partial\omega.$$

The corresponding eigenvalues λ_n are

$\lambda_1 = (2/\pi)(1^2 + 1^2) = 4/\pi$ $\qquad \lambda_5 = (2/\pi)(3^2 + 2^2) = 26/\pi$

$\lambda_2 = (2/\pi)(1^2 + 2^2) = 10/\pi$ $\qquad \lambda_6 = (2/\pi)(1^2 + 4^2) = 34/\pi$

$\lambda_3 = (2/\pi)(2^2 + 2^2) = 16/\pi$ $\qquad \lambda_7 = (2/\pi)(3^2 + 3^2) = 36/\pi$

$\lambda_4 = (2/\pi)(1^2 + 3^2) = 20/\pi$ $\qquad \lambda_8 = (2/\pi)(2^2 + 4^2) = 40/\pi$

.
.
.

Now we define the functions $a_k(.,.)$, k = 1,2,..., corresponding to the above eigenvalues λ_k as follows

$a_1(x,y) = (2/\pi).\sin(x).\sin(y)$ $\qquad a_2(x,y) = (2/\pi).\sin(x).\sin(2y)$

$a_3(x,y) = (2/\pi).\sin(2x).\sin(y)$ $\quad$ $a_4(x,y) = (2/\pi).\sin(x).\sin(3y)$

$a_5(x,y) = (2/\pi).\sin(3x).\sin(2y)$ $\quad$ $a_6(x,y) = (2/\pi).\sin(x).\sin(4y)$

$a_7(x,y) = (2/\pi).\sin(3x).\sin(3y)$ $\quad$ $a_8(x,y) = (2/\pi).\sin(2x).\sin(4y)$

.
.
.

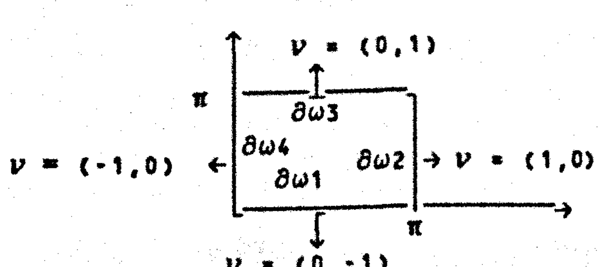
It is easily seen that the above functions $a_n(.,.)$, are solutions of the following problem

$$u_{xx} + u_{yy} + \lambda_n u = 0, \ (x,y) \in \omega; \ u(x,y) = 0, \ (x,y) \in \partial\omega,$$

with the corresponding eigenvalues λ_n, $n = 1,2,\ldots$. It is easy to show the functions $a_n(.,.)$, $n = 1,2,\ldots$, are orthonormal. Now we calculate the following functions on $\partial\omega$

$$F_n(s,t,u) = \left(\frac{\partial a_n}{\partial \nu}\right)(\xi_1(s),\xi_2(s)).e^{-\lambda_n(T-t)}.\sqrt{\xi_1'^2 + \xi_2'^2}\,.u.$$

First we calculate the functions $\left(\frac{\partial a_n}{\partial \nu}\right)(x,y)$, $n = 1,2,\ldots$, $(x,y) \in \partial\omega$, where ω is the following rectangle:



Therefore we have $\partial\omega = \partial\omega1 \cup \partial\omega2 \cup \partial\omega3 \cup \partial\omega4$, where $\partial\omega i$,

$i = 1,2,3,4$, have been shown in the above figure; the equations of $\partial\omega i$, $i = 1,2,3,4$, are:

$$\partial\omega 1:\begin{cases}\xi_1(s) = \pi s\\ \xi_2(s) = 0\end{cases}, \qquad \partial\omega 2:\begin{cases}\xi_1(s) = \pi\\ \xi_2(s) = \pi s\end{cases},$$

$$\partial\omega 3:\begin{cases}\xi_1(s) = \pi(1-s)\\ \xi_2(s) = \pi\end{cases}, \qquad \partial\omega 4:\begin{cases}\xi_1(s) = 0\\ \xi_2(s) = \pi(1-s)\end{cases},$$

where $s \in [0,1]$.

Now we may calculate the functions, $\left[\frac{\partial a_n}{\partial \nu}\right](\xi_1(s),\xi_2(s))$, $s \in [0,1]$, for $n = 1,2,\dots$, on $\partial\omega$, as follows

$$\left[\frac{\partial a_1}{\partial \nu}\right](\xi_1(s),\xi_2(s)) = [(2/\pi)\sin(\pi s).\cos(0)](-1)$$

$$+ [(2/\pi)\cos(\pi s).\sin(\pi s)](1) + [(2/\pi)\sin(\pi s).\cos(\pi)](1)$$

$$+ [(2/\pi)\cos(0).\sin(\pi s)](-1)$$

or $\left[\frac{\partial a_1}{\partial \nu}\right](\xi_1(s),\xi_2(s)) = (-8/\pi)\sin(\pi s)$, $s \in [0,1]$, also we have $\sqrt{\xi_1'^2 + \xi_2'^2} = \pi^2$, thus

$$F_1(s,t,u) = \left[\frac{\partial a_1}{\partial \nu}\right](\xi_1(s),\xi_2(s))\, e^{-\lambda_1(T-t)}.\sqrt{\xi_1'^2 + \xi_2'^2}\,.u$$

$$= -8\sin(\pi s).e^{-(4/\pi)(T-t)}.u, \quad (s,t) \in [0,1] \times [0,1].$$

In general we have $a_k(x,y) = (2/\pi)\sin(mx)\sin(ny)$, for some positive integers m, and n. Henceforth we assume $b_k(.,.) = \left[\frac{\partial a_k}{\partial \nu}\right](.,.)$, where $k = 1,2,\dots,$ and $\nu = (\nu_1,\nu_2)$, is the outward normal derivative to $\partial\omega$, therefore we have

$$b_k(\xi_1(s),\xi_2(s)) = \partial[(2/\pi)\sin(mx)\sin(ny)]/\partial\nu$$

$$= [(2m/\pi)\cos(mx)\sin(ny)]\nu_1 + [(2n/\pi)\sin(mx)\cos(ny)]\nu_2.$$

By substituting the values of ν_1, ν_2, on $\partial\omega i$, $i = 1,2,3,4$, we may write

$$b_k(\xi_1(s),\xi_2(s)) = -(2n/\pi)\sin(m\pi s)[1+(-1)^{n+m}]$$

$$+ (2m/\pi)\sin(n\pi s)[(-1)^m+(-1)^n].$$

Note:

(I) If both m and n, are odd or even and $n \neq m$, then

$b_k(\xi_1(s),\xi_2(s)) \neq 0.$

(II) If n and m are odd, and $n = m$, then $b_k(\xi_1(s),\xi_2(s)) \neq 0.$

In other cases $b_k(\xi_1(s),\xi_2(s)) = 0.$

Therefore we conclude:

(1) $F_1(s,t,u) = -8\sin(\pi s).e^{-(4/\pi)(T-t)}.u$

(2) $F_2(s,t,u) = F_3(s,t,u) = F_5(s,t,u) = F_6(s,t,u) = 0$

(3) $F_4(s,t,u) = -4[3\sin(\pi s)+\sin(3\pi s)].e^{-(20/\pi)(T-t)}.u$

(4) $F_7(s,t,u) = -24\sin(3\pi s).e^{-(36/\pi)(T-t)}.u$

(5) $F_8(s,t,u) = -8[2\sin(2\pi s) - \sin(4\pi s)].e^{-(40/\pi)(T-t)}.u$

Now we have the following computational results:
Cost function = 0.1463, and CPU time = 18.08 seconds. We construct a piecewise constant control function similar to the one-dimension case for this example. The graph of control function in su-plane is shown in Fig (3.3).

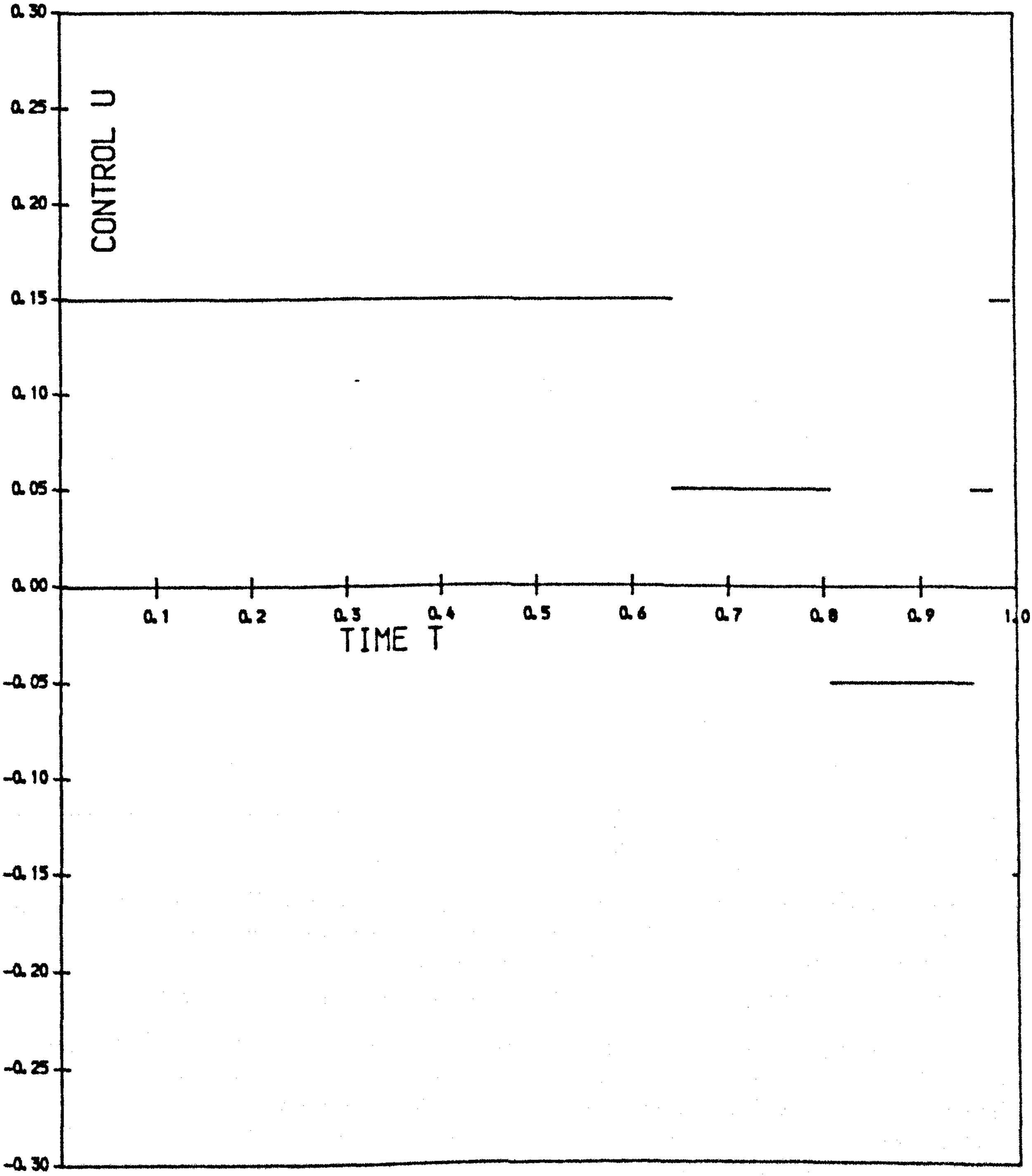


FIG (3. 1) -CONTROL FUNCTION FOR EXAMPLE (3. 1)

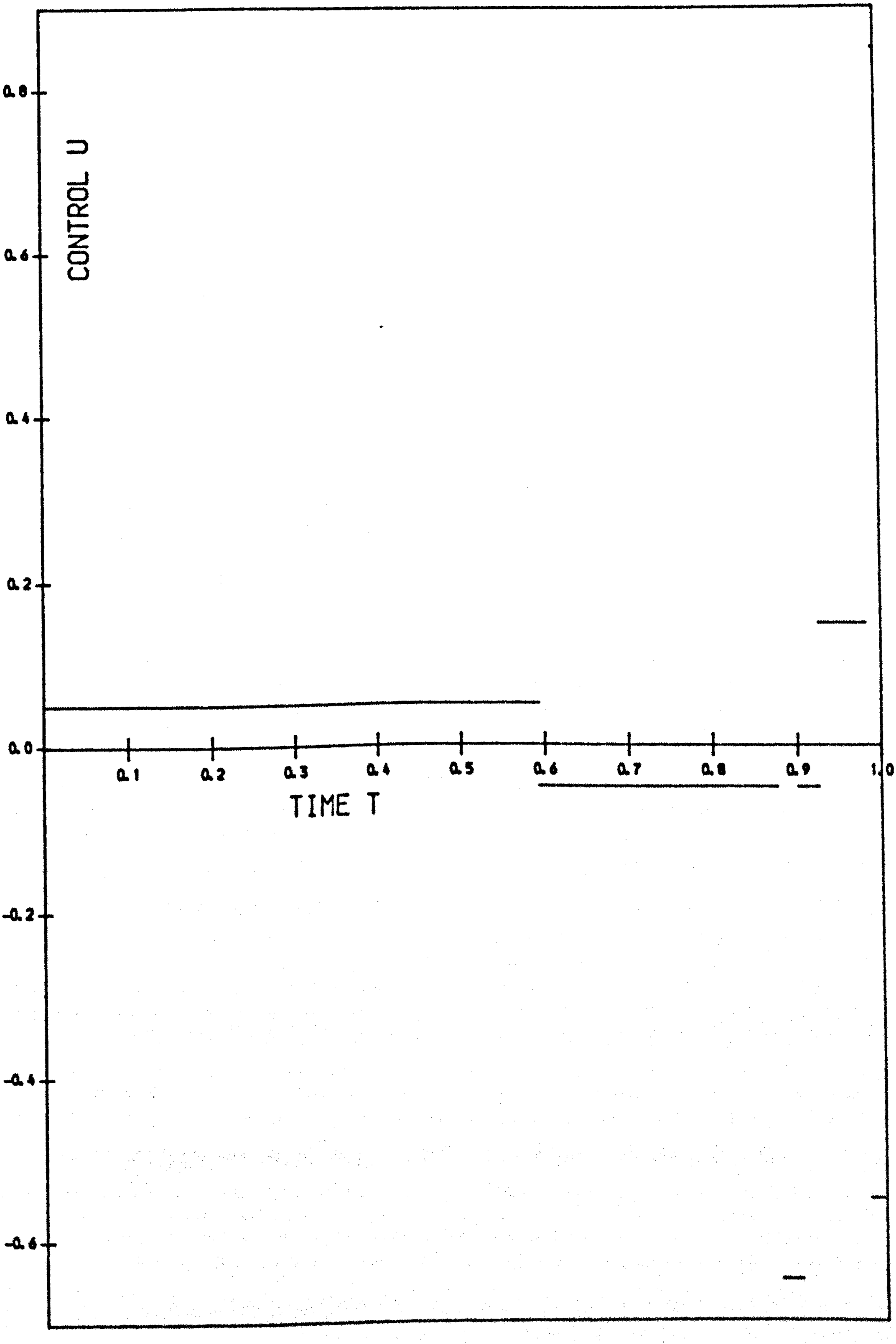


FIG (3. 2) -CONTROL FUNCTION FOR EXAMPLE (3. 2)

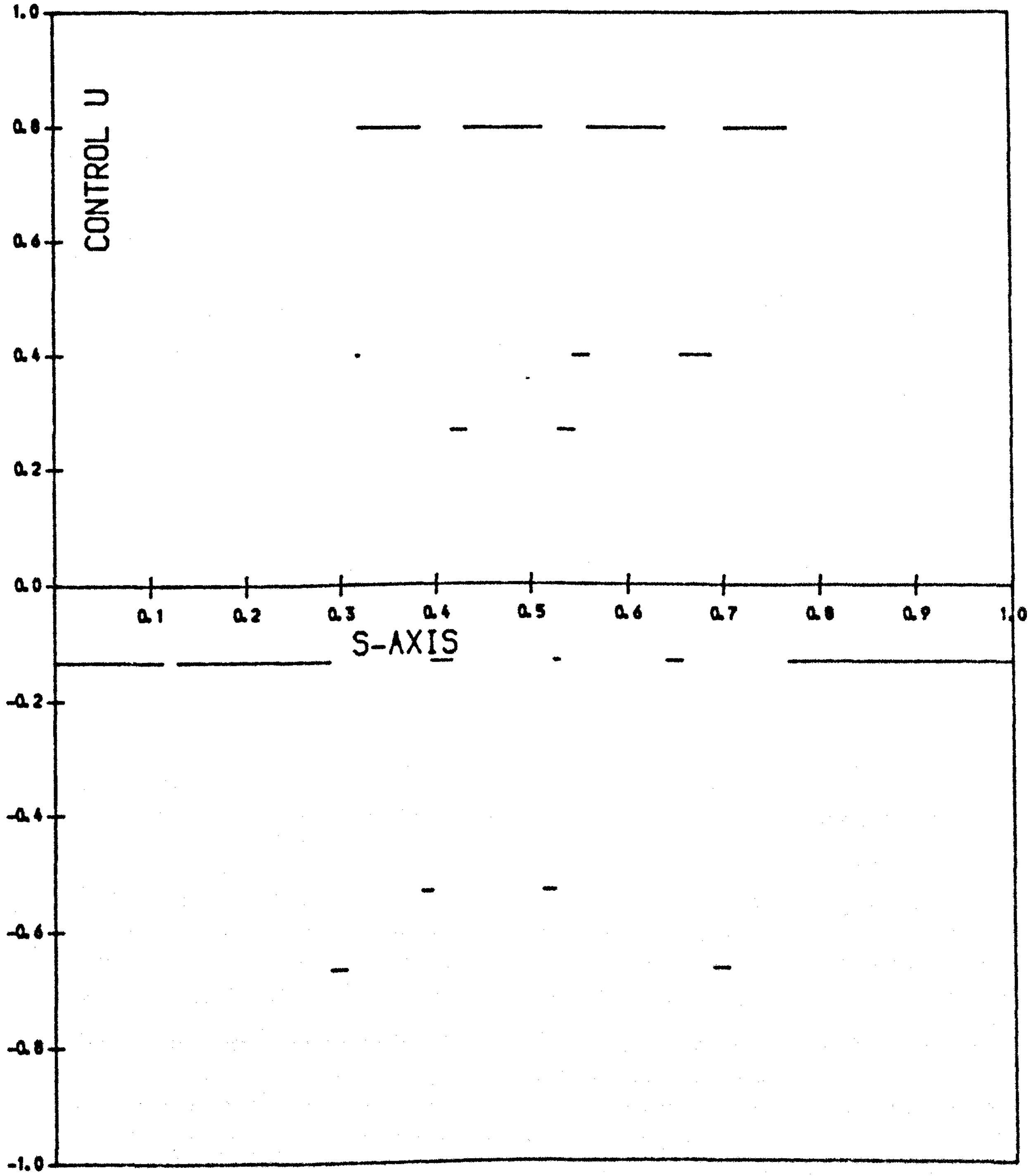


FIG (3. 3) -CONTROL FUNCTION FOR EXAMPLE (3. 3)

CHAPTER 4

Optimal control problem for the one-dimensional diffusion equation with generalised control variables.

4.1 Introduction

In this chapter we consider again the one dimensional diffusion equation

$$Y_{xx}(x,t) = Y_t(x,t), \quad (x,t) \in (0,1) \times (0,T) \tag{4.1}$$

with boundary conditions

$$Y_x(0,t) = 0, \quad t \in [0,T]$$

$$Y_x(1,t) = u(t), \quad t \in [0,T] \tag{4.2}$$

$$Y(x,0) = 0, \quad x \in [0,1]$$

where $u(.)$ is the control function.

It is desired to choose $u(.) \in L_2(0,T)$, such that $Y(.,T) = g(x)$ in $L_2(0,1)$ and the function

$$u \to J(u)$$

is minimized (we specify the function $J(.)$ later). Let $g(.) \in L_2(0,1)$ be the desired final state, with the half-range Fourier series as follows:

$$g(x) = \sum_{n=0}^{\infty} \alpha_n \cos(n\pi x).$$

RUBIO and WILSON [1] have shown that a solution of (4.1) with boundary conditions (4.2) corresponding to a control $u(.) \in L_2(0,T)$, satisfing the terminal condition $Y(.,T) = g(x)$ in $L_2(0,1)$ also satisfies

$$\int_0^T \phi_n(t)u(t)dt = \alpha_n, \quad n = 0,1,\ldots \tag{4.3}$$

with

$$\phi_n(t) = \exp[-n^2\pi^2(T-t)], \quad t \in [0,T], \quad n = 0,1,\ldots$$

It is apparent from (4.3) that the problem of attaining a given state $g(.)$ at time T can be studied by considering the *moment problem* (4.3). From results of Fattorini and Russell [1], it can be shown that there is a control $u(.) \in L_2(0,T)$ satisfying (4.3) if there is a constant C so that the moments α_n satisfy

$$|\alpha_n| \le C \exp(-n^2\pi^2). \quad n = 0,1,\ldots \tag{4.4}$$

There are, however, many functions in $L_2(0,1)$ whose moments do not decrease with n as rapidly as this condition requires, for example

$$f(x) = \sum_{n=0}^{\infty} (1/n^2)\cos(n\pi x)$$

RUBIO and WILSON [1] assumed $\mathcal{F}$ be the space of real-valued functions infinitely differentiable on $[0,T]$ such that

$$\sup_{[0,T]} |D^j\phi(t)| \le C L^j,$$

for some constants C, L. $\mathcal{F}$ was endowed with the LF-toplogy and S was assumed to be the dual of the space $\mathcal{F}$. In S a solution was found to the problem of moments. Proposition (VIII.4) of RUBIO and WILSON [1] shows that the following linear functional s on $\mathcal{F}$ defined in (4.5) is in S

$$s(\phi) = \sum_{k=0}^{\infty} (-1)^k \mu(D^k\phi), \tag{4.5}$$

for all $\phi \in \mathcal{F}$, where μ_k, $k = 0,1,\ldots$ are Radon measures on [0,T] such that

$$\sum_{k=0}^{\infty} L^k \int_{[0,T]} d|\mu_k| < \infty, \tag{4.6}$$

for all $L \ge 0$. If s satisfies (4.5) for all $\phi \in \mathcal{F}$, it is denoted by

$$s = \sum_{k=0}^{\infty} D^k\mu_k. \tag{4.7}$$

In proposition (VIII.5) in the above reference, they find an atomless $s \in S$ such that $s(\phi_n) = \alpha_n$, $n = 0,1,\ldots$. In other words, it was proved that the set

$$S_g = \Big\{ s \in S : s(\phi_n) = \alpha_n,\ n = 0,1,\ldots,\ s \text{ is of the form (4.5), the measures } \mu_k,\ k = 0,1,\ldots \text{ associated with it are atomless and } \sum_{k=0}^{\infty} L^k \int_{[0,T]} d|\mu_k| < \infty \Big\}$$

is non-empty, where α_n, $n = 0,1,\ldots$ are the cosine Fourier coefficients of the desired final state. RUBIO and WILSON showed then that the problem of moments has a solution in S; they also introduced a sequence of controls in $L_2(0,T)$ which approximate $s \in S$.

We consider now the following correspondence

$$s = \sum_{k=0}^{\infty} D^k\mu_k \in S \leftrightarrow (\mu_0,\ \mu_1,\ \ldots\ ,\mu_k,\ \ldots), \tag{4.8}$$

where

$$\sum_{k=0}^{\infty} L^k \int_{[0,T]} d|\mu_k| < \infty, \tag{4.9}$$

for all $L \geq 0$. We show in appendix A.4 that the correspondence (4.8) is in fact an injection. Thus, from (4.8) it is possible to indentify any element of S with a sequence $\{\mu_k\}$, of Radon measures on $[0,T]$. So henceforth we use the notation $s = \{\mu_k\} \in S$. [We mention here that it is easier to use the inequality (4.9) in the following equivalent form

$$\sum_{k=0}^{\infty} L^k \|\mu_k\| < \infty \tag{4.10}$$

for all $L \geq 0$. Note that we used the fact that $\|\mu_k\| = |\mu_k|(1) = \int_{[0,T]} d|\mu_k|$; see for example (CHOQUET [1] p.215)]. By Theorem 13-21 of APOSTOL [1] we can obtain an equivalent condition to the condition (4.10) as follows,

$$\lim_k \sup \|\mu_k\|^{1/k} = 0,$$

which in turn is equivalent to the following:

$$\lim_k \|\mu_k\|^{1/k} = 0, \qquad (4.11)$$

since by definition 12.2 of the above reference, $\lim_k \sup \|\mu_k\|^{1/k} = 0$ if and only if for every $\varepsilon > 0$ there exists an integer K such that $k > K$ implies $\|\mu_k\|^{1/k} < \varepsilon$; this condition is the definition of $\lim_k \|\mu_k\|^{1/k} = 0$.

By definition of s_g we have, then, that $s = (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in s_g$, if and only if

(1) $s(\phi_n) = \sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) = \alpha_n, \quad n = 0,1,\ldots$

(2) $\lim_k \|\mu_k\|^{1/k} = 0.$

We have shown in appendix (B.4) that the space S is a linear space. Now let J be an objective functional on S. For example let

$$J(s) = \sum_{k=0}^{\infty} \|\mu_k\|,$$

where $s = (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in S$. It is apparent that J is well defined, since $\sum_{k=0}^{\infty} \|\mu_k\| < \infty$.

The reason that we have chosen J(.) in the above form is that, if the classical control problem consists of finding a control u(.)

which minimizes the functional

$$I[u(.)] = \int_{[0,T]} |u(t)|dt \equiv \|u\|_{L_1(0,T)}$$

then we can define the associated measure μ_u:

$$\int_{[0,T]} \psi d\mu_u \equiv \int_{[0,T]} \psi(t)u(t)dt, \ \psi \in \mathcal{F};$$

thus,

$$\|\mu_u\| = |\mu_u|(1) = \int_{[0,T]} d|\mu_u| = \int_{[0,T]} |u(t)|dt = \|u\|_{L_1(0,T)}.$$

Therefore the above objective functional J(.) is indeed a true extension of the functional I(.).

We show in this chapter that:

(i) For positive integers K, N, L, and a positive number ε we have

$$\inf_{Q(K,N,L)} \sum_{k=1}^{KL} \|\nu_k\| = \lim_{\varepsilon \to 0+} \inf_{Q_\varepsilon(K,N,L)} \sum_{k=0}^{KL} \|\mu_k\|$$

where $Q_\varepsilon(K,N,L)$ is the set of all $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$ [we define $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0}) \equiv (\mu_0, \mu_1, \ldots, \mu_{KL}, 0, 0, \ldots)$] such that

(a) $\left| \sum_{k=1}^{KL} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) - \alpha_n \right| < \varepsilon, \ n = 0, 1, \ldots, N$

(b) $\|\mu_k\|^{1/k} < 1/K, \ (K-1)L < k \le KL.$

and Q(K,N,L) is the set of all $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$ of Radon measures such that

(c) $\sum_{k=0}^{KL} (-1)^k(n^2\pi^2)^k \mu_k(\phi_n) = \alpha_n$, for $n = 0,1,\dots,N$

(d) $\|\mu_k\|^{1/k} < 1/k$, for $(K-1)L < k \le KL$.

(ii) For positive integers K, N, L,

$$\inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\| = \lim_{K\to\infty} \inf_{Q(K,N,L)} \sum_{k=0}^{\infty} \|\mu_k\|$$

where $Q(N)$ is the set of all $(\mu_0,\mu_1,\dots,\mu_k,\dots)$ such that

$$\sum_{k=1}^{\infty} (-1)^k(n^2\pi^2)^k\mu_k(\phi_n) = \alpha_n, \quad n = 0,1,\dots,N,$$

$$\lim_{k\to\infty} \|\mu_k\|^{1/k} = 0.$$

(iii) We show next

$$\inf_{S_g} \sum_{k=0}^{\infty} \|\mu_k\| = \lim_{N\to\infty} \inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\|$$

(iv) Let $\theta = \inf_{S_g} \sum_{k=0}^{\infty} \|\mu_k\|$. We approximate θ by $\inf_{P_\varepsilon(K,N)} \sum_{k=0}^{K} \|\mu_k\|$,

where $P_\varepsilon(K,N)$ is the set of all $(\mu_0,\mu_1,\dots,\mu_K,\bar{0})$, such that each of μ_k is a discrete measure on $[0,T]$, and

$$\left| \sum_{k=1}^{KL} (-1)^k(n^2\pi^2)^k\mu_k(\phi_n) - \alpha_n\right| < \varepsilon, \quad n = 0,1,\dots,N, \qquad (4.12)$$

[ε is agiven positive number and K and N are sufficiently large positive integers].

(v) We transfer the above problem to the one which consists of the minimization of a linear function over a set of linear constraints in finite dimensional space.

(vi) Finally, we use the sequence of control functions introduced in chapter (5), and show, through several examples, that we can reach different final states with a rather good approximation.

(4.2) A scheme for determining the infimum of the objective function

In this section we show that

$$\inf_{Q} \sum_{k=0}^{KL} \|\mu_k\| = \lim_{\varepsilon \to 0+} \left[\inf_{Q_K(\varepsilon)} \sum_{k=0}^{KL} \|\mu_k\| \right],$$

where K, L are positive integers, N is a nonnegative integer, $\varepsilon > 0$, and $Q_K(\varepsilon) \equiv Q(K,N,L,\varepsilon)$ is the set of all sequences $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$ of Radon measures such that

(i) $\left|\sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) - \alpha_n\right| < \varepsilon$ for $n = 0, 1, \ldots, N$.

(ii) $\|\mu_k\|^{1/k} < 1/k$, for $(K-1)L < k \le KL$,

and $Q = Q(K,N,L)$ is the set of all sequences $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$ of Radon measures such that

(iii) $\sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) = \alpha_n$, for $n = 0, 1, \ldots, N$

(iv) $\|\mu_k\|^{1/k} < 1/k$, for $(K-1)L < k \le KL$.

First we show in lemma (4.1) the sequence $\{Q(K,N,L,\varepsilon)\}_K$, is non-decreasing.

Lemma (4.1) $Q(K,N,L,\varepsilon) \subset Q(K+1,N,L,\varepsilon)$

Proof It is seen that any element $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0}) \in Q(K,N,L,\varepsilon)$

is an element of $Q(K+1,N,L,\varepsilon)$:

(i) $\left|\sum_{k=0}^{L(K+1)} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) - \alpha_n\right|$

$$= \left|\sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) - \alpha_n\right| < \varepsilon \quad \text{for } n = 0,1,\dots,N.$$

(ii) $\|\mu_k\|^{1/k} = 0 < 1/k+1$, for $(K-1)L < k \le KL$.□

Remark It is shown in appendix (C.4) that $Q(K,N,L,\varepsilon)$ is non-empty for any nonnegative integers K, N, L, and $\varepsilon > 0$.

Now we define

$$\xi_{K,N,L}(\varepsilon) = \inf_{Q_K(\varepsilon)} \sum_{k=0}^{\infty} \|\mu_k\|, \tag{4.13}$$

where $Q_K(\varepsilon) = Q(K,N,L,\varepsilon)$. Now in lemma (4.2) we show the sequence $\{\xi_{K,N,L}(\varepsilon)\}_K$, is non-increasing:

Lemma (4.2) $\xi_{K+1,N,L}(\varepsilon) \le \xi_{K,N,L}(\varepsilon)$, when N, L, and ε are fixed and K takes value 1,2,... .

Proof In lemma (4.1) we showed $Q(K,N,L,\varepsilon) \subset Q(K+1,N,L,\varepsilon)$, therefore we have

$$\inf_{Q(\varepsilon)_{K+1}} \sum_{k=0}^{\infty} \|\mu_k\| \le \inf_{Q(\varepsilon)_K} \sum_{k=0}^{\infty} \|\mu_k\|, \tag{4.14}$$

where $Q_K(\varepsilon) = Q(K,N,L,\varepsilon)$, for K = 1,2,... . But by definition (4.13) we conclude from (4.14) that

$$\xi_{K+1,N,L}(\varepsilon) \le \xi_{K,N,L}(\varepsilon) \tag{4.15}$$

We show in appendix (C.4) that $Q_K \equiv Q(K,N,L)$ is non-empty so the following definition is meaningful:

$$\xi_{K,N,L} = \inf_{Q_K} \sum_{k=0}^{\infty} \|\mu_k\| \tag{4.16}$$

In the following we intend to obtain a relationship between $\xi_{K,N,L}$ and $\xi_{K,N,L}(\varepsilon)$; therefore we prove the following lemma.

Lemma (4.3) $Q(K,N,L) = \bigcap_{\varepsilon>0} Q(K,N,L,\varepsilon)$.

Proof It follows from the definitions of $Q(K,N,L)$ and $Q(K,N,L,\varepsilon)$ that for any $\varepsilon > 0$,

$$Q(K,N,L) \subset Q(K,N,L,\varepsilon), \tag{4.17}$$

since if $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0}) \in Q(K,N,L)$ then we have

(i) $\left|\sum_{k=0}^{KL} (-1)^k(n^2\pi^2)^k \, \mu_k(\phi_n) - \alpha_n\right| = 0 < \varepsilon$; for any $\varepsilon > 0$

(ii) $\|\mu_k\|^{1/k} < 1/K$, for $L(K-1) < k \le LK$,

then $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0}) \in Q(K,N,L,\varepsilon)$ for any $\varepsilon > 0$. Now we prove $\bigcap_{\varepsilon>0} Q(K,N,L,\varepsilon) \subset Q(K,N,L)$, so let $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0}) \subset \bigcap_{\varepsilon>0} Q(K,N,L)$; then we have

(a) $\left|\sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \ \mu_k(\phi_n) - \alpha_n\right| < \varepsilon$ for any $\varepsilon > 0$, and all $n = 0, 1, \ldots, N$,

(b) $\|\mu_k\|^{1/k} < 1/K$, for $L(K - 1) < k \le LK$,

We conclude that $\left|\sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \ \mu_k(\phi_n) - \alpha_n\right| = 0$. If not, let $\left|\sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \ \mu_k(\phi_n) - \alpha_n\right| = \varepsilon > 0$, then the above condition (a) also should be valid for $\varepsilon/2$, this means that $\varepsilon = \left|\sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \ \mu_k(\phi_n) - \alpha_n\right| < \varepsilon/2$, for all $n = 0, 1, \ldots, N$ or $\varepsilon < \varepsilon/2$, which is a contradiction. So the above conditions (a) and (b) can be written in the following equivalent conditions (c) and (d):

(c) $\sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \ \mu_k(\phi_n) = \alpha_n$; $n = 0, 1, \ldots, N$

(d) $\|\mu_k\|^{1/k} < 1/K$, for $L(K - 1) < k \le LK$,

therefore by definition of $Q(K,N,L)$ we conclude that $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0}) \in Q(K,N,L)$.□

Now we define another set $Q(N)$, which we will use later, as the set of all $(\mu_0, \mu_1, \ldots, \mu_k, \ldots)$ of Radon measures such that

(a) $\sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \ \mu_k(\phi_n) = \alpha_n$; $n = 0, 1, \ldots, N$

(b) $\lim_k \|\mu_k\|^{1/k} = 0$.

We have the following lemma:

Lemma (4.4) $Q(K,N,L) \subset Q(N)$ for all positive integers K, L, and all non-negative integers N.

Proof Let $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0}) \in Q(K,N,L)$, then $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0}) \in Q(N)$, since

$$(I)\ \sum_{k=0}^{\infty} (-1)^k(n^2\pi^2)^k\ \mu_k(\phi_n) = \sum_{k=0}^{KL} (-1)^k(n^2\pi^2)^k\ \mu_k(\phi_n) = \alpha_n,$$

$$n = 0,1,\ldots,N$$

$$(II)\ \lim_k \|\mu_k\|^{1/k} = 0, \quad \text{because} \quad \mu_k \equiv 0, \text{ for all } \quad k > KL. \square$$

In the next lemma we show that

$$\theta_{K,N,L} \equiv \lim_{\varepsilon\to 0+} \xi_{K,N,L}(\varepsilon), \qquad (4.18)$$

exists. Then we show that $\theta_{K,N,L} = \xi_{K,N,L}$. $\Big[\xi_{K,N,L}$, and $\xi_{K,N,L}(\varepsilon)$, were defined in (4.13) and (4.16)$\Big]$.

Lemma (4.5) $\theta_{K,N,L} \equiv \lim_{\varepsilon\to 0+} \xi_{K,N,L}(\varepsilon)$ exists and $\theta_{K,N,L} = \xi_{K,,N,L}$, where $\theta_{K,N,L}$, has been defined in (4.18).

Proof We showed in (4.17) that $Q(K,N,L) \subset Q(K,N,L,\varepsilon)$ for all positive integers K, N, L, and $\varepsilon > 0$. Therefore

$$\inf_{Q_k(\varepsilon)} \sum_{k=0}^{\infty} \|\mu_k\| \le \inf_{Q_k} \sum_{k=0}^{\infty} \|\mu_k\|;$$

thus by definition we have

$$\xi_{K,N,L}(\varepsilon) \le \xi_{K,N,L} \qquad (4.19)$$

for all $\varepsilon > 0$. Now let $0 < \varepsilon_1 < \varepsilon_2$, then

$$Q(K,N,L,\varepsilon_1) \subset Q(K,N,L,\varepsilon_2), \qquad (4.20)$$

(see definition of $Q(K,N,L,\varepsilon)$, in the begining of section (4.2)) therefore from (4.20) and the definition of $\xi_{K,N,L}(\varepsilon)$ we conclude

$$0 \le \xi_{K,N,L}(\varepsilon_2) \le \xi_{K,N,L}(\varepsilon_1).$$

Therefore $\xi_{K,N,L}(\varepsilon)$, as a function of ε, is non-increasing; since it bounded from above, it has a limit as $\varepsilon \to 0+$; that is $\theta_{K,N,L} = \lim_{\varepsilon \to 0+} \xi_{K,N,L}(\varepsilon)$, exists. Now we prove $\theta_{K,N,L} = \xi_{K,N,L}$. From (4.19) we have $\xi_{K,N,L}(\varepsilon) \le \xi_{K,N,L}$ for all $\varepsilon > 0$, therefore $\lim_{\varepsilon \to 0+} \xi_{K,N,L}(\varepsilon) \le \xi_{K,N,L}$, for all K, N and L, or

$$\theta_{K,N,L} \le \xi_{K,N,L}. \qquad (4.21)$$

Since $\xi_{K,N,L}(\varepsilon)$ as a function of ε is non-increasing, then for any $\varepsilon > 0$, we have

$$\xi_{K,N,L}(\varepsilon) \le \theta_{K,N,L},$$

but by (4.16) we defined $\xi_{K,N,L}(\varepsilon) = \inf_{Q_K(\varepsilon)} \sum_{k=0}^{\infty} \|\mu_k\|$, where $Q_K(\varepsilon) = Q(K,N,L,\varepsilon)$, therefore

$$\inf_{Q_K(\varepsilon)} \sum_{k=0}^{\infty} \|\mu_k\| \le \theta_{K,N,L}, \qquad (4.22)$$

for all $\varepsilon > 0$, thus from (4.22) we conclude $\inf_{\bigcap_{\varepsilon>0} Q_K(\varepsilon)} \sum_{k=0}^{\infty} \|\mu_k\|$ $\le \theta_{K,N,L}$, where by definition, $Q_K(\varepsilon) \equiv \bigcap_{\varepsilon>0} Q(K,N,L,\varepsilon)$. We showed in lemma (4.3) that $\bigcap_{\varepsilon>0} Q(K,N,L,\varepsilon) = Q(K,N,L)$, therefore

$$\inf_{Q_K} \sum_{k=0}^{\infty} \|\mu_k\| \le \theta_{K,N,L},$$

where $Q_K \equiv Q(K,N,L)$. Or by definition

$$\xi_{K,N,L} \le \theta_{K,N,L}. \tag{4.23}$$

From (4.21) and (4.23) we conclude

$$\xi_{K,N,L} = \theta_{K,N,L}. \square \tag{4.24}$$

4.3 Approximation of the infimum of the objective function when the cosine Fourier series of the desired final state is a finite summation.

Let $g(x) = \sum_{n=0}^{N} \alpha_n \cos(n\pi x)$, where N is an arbitrary non-negative integer. We will show that for any positive integer L

$$\inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\| = \lim_{K} \inf_{Q_K} \sum_{k=0}^{\infty} \|\mu_k\|, \qquad (4.25)$$

where $Q_K \equiv Q(K,N,L)$, Q(N) was defined as the set of all sequences $(\mu_0, \mu_1, \ldots, \mu_k, \ldots)$ of Radon measures such that

(a) $\sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) = \alpha_n; \quad n = 0,1,\ldots,N$

(b) $\lim_{k} \|\mu_k\|^{1/k} = 0.$

We show in appendix (C.4) that Q(N) is nonempty, so $\inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\|$ is meaningful. First we show that $\lim_{K} \xi_{K,N,L}$ exists.

Lemma (4.6) For arbitrary K, N and L

$$\xi_{K+1,N,L} \le \xi_{K,N,L},$$

where $\xi_{K,N,L}$ was defined in (4.16).

Proof: From lemma (4.2) we have $\xi_{K+1,N,L}(\varepsilon) \le \xi_{K,N,L}(\varepsilon)$, for all $\varepsilon > 0$. Therefore

$$\lim_{\varepsilon \to 0+} \xi_{K+1,N,L}(\varepsilon) \le \lim_{\varepsilon \to 0+} \xi_{K,N,L}(\varepsilon),$$

or, by definition

$$\theta_{K+1,N,L} \le \theta_{K,N,L}. \tag{4.26}$$

But we showed in lemma (4.5) that $\theta_{K,N,L} = \xi_{K,N,L}$, therefore we may express (4.26) in the following form

$$\xi_{K+1,N,L} \le \xi_{K,N,L}. \square \tag{4.27}$$

In lemma (4.6) we showed that the sequence $\left\{\xi_{K,N,L}\right\}_K$ is non-increasing when N and L are fixed, and it is bounded from below by 0. Now let

$$\eta_{N,L} = \lim_K \xi_{K,N,L}, \tag{4.27'}$$

$$\theta_N = \inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\|. \tag{4.27''}$$

With the above notations we can restate (4.25) in the following equivalent form

$$\theta_N = \eta_{N,L}.$$

We remember that $\xi_{K,N,L} = \inf_{Q_K} \sum_{k=0}^{\infty} \|\mu_k\|$. It may appear that $\eta_{N,L} = \lim_K \xi_{K,N,L}$ depends on the value of L, but we shall show that it is independent of the value $L \ge 1$; in other words we shall show that for any fixed $L \geq 1$, we have $\theta_N = \eta_{N,L}$. First we prove the following lemma which we will use later in this section.

Lemma (4.7) Let $\phi_n(t) = e^{-n^2\pi^2(T-t)}$, where $n = 1,2,\ldots,N$ (N is a fixed positive integer) and $t \in [0,T]$. Let $t_1 < t_2 < \ldots < t_N$, be defined by $t_i = i.\Delta$, where $\Delta = T/N$, and $i = 1,2,\ldots,N$. Then the following matrix

$$\phi = \begin{bmatrix} \phi_1(t_1) & \phi_1(t_2) & \ldots & \phi_1(t_N) \\ \phi_2(t_1) & \phi_2(t_2) & \ldots & \phi_2(t_N) \\ \vdots & \vdots & & \vdots \\ \phi_N(t_1) & \phi_N(t_2) & \ldots & \phi_N(t_N) \end{bmatrix}, \qquad (4.28)$$

is nonsingular.

Proof It is appearent that the functions $\phi_n(t) = e^{-n^2\pi^2(T-t)}$, $n = 1,2,\ldots,N$ are linearly independent since if we let $e^{-\pi^2(T-t)} = X$, then $\phi_n(t) = X^{n^2}$, $n = 1,2,\ldots,N$, are linearly independent; therefore the above matrix ϕ is nonsingular.□

Note Let $D = \phi^{-1}$ have elements d_{ij}. We define

$$C_k = \sum_{i=1}^{N} \sum_{j=1}^{N} \left[|d_{ij}| / j^{2k} \right], \qquad (4.29)$$

which we will use in the following lemma.

Lemma (4.8) Let a_n, $n = 1,2,\ldots,N$ be N, real numbers such that $|\alpha_n| < 1$, for $n = 1,2,\ldots,N$, and let k be a fixed positive integer. Then there exists a Radon measure ν_k such that

$$\nu_k(\phi_n) = a_n \,/\, (n^2\pi^2), \quad \text{for } n = 1,2,\ldots,N,$$

where $\phi_n(t) = e^{-n^2\pi^2(T-t)}$, and $\|\nu_k\| \le C_k \,/\, \pi^{2k}$.

Proof We look for ν_k as a measure of the form $\nu_k = \sum_{i=1}^{N} b_i\delta(t_i)$, where $t_i = i\Delta$, $i = 1,2,\ldots N$, $\Delta = T/N$, and b_i, $i = 1,2,\ldots,N$, are unknowns to be determined. By assumption we must have $\nu_k(\phi_n) = a_n/(n^2\pi^2)$, $n = 1,2,\ldots,N$, or $\left[\sum_{i=1}^{N} b_i\delta(t_i)\right](\phi_n) = a_n/(n^2\pi^2)^k$, $n = 1,2,\ldots,N$. Therefore we have $\sum_{i=1}^{N} b_i\phi_n(t_i) = a_n/(n^2\pi^2)^k$, for $n = 1,2,\ldots,N$. Then we have the following system of linear equations

$$b_1\phi_n(t_1) + b_2\phi_n(t_2) + \ldots + b_N\phi_n(t_N) = a_n \,/\, \pi^{2k}, \; n = 1,2,\ldots,N,$$

or we have

$$\begin{bmatrix} \phi_1(t_1) & \phi_1(t_2) & \ldots & \phi_1(t_N) \\ \phi_2(t_1) & \phi_2(t_2) & \ldots & \phi_2(t_N) \\ \vdots & \vdots & & \vdots \\ \phi_N(t_1) & \phi_N(t_2) & \ldots & \phi_N(t_N) \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_N \end{bmatrix} = \begin{bmatrix} a_1/\pi^{2k} \\ a_2/(2\pi)^{2k} \\ \vdots \\ a_N/(N\pi)^{2k} \end{bmatrix}. \qquad (4.30)$$

or $\phi\, \underline{b} = \underline{c}$, where $\underline{b}$ and $\underline{c}$ are respectively column vectors $(b_1,\ldots,b_N)$ and $(a_1/\ (2\pi)^{2k},\ldots,a_N/\ (N\pi)^{2k})$.

We showed the matrix ϕ to be non-singular and we defined $D = \phi^{-1} = (d_{ij})$, so $\underline{b} = D.\underline{c}$, or

$b_l = \sum_{j=1}^{N} d_{lj}\left[a_j/(j\pi)^{2k}\right] = (1/\pi^{2k}) \sum_{j=1}^{N}\left[d_{lj} \quad a_j/j^{2k}\right]$, for $l = 1,2,\ldots,N$,

Thus $|b_l| \le (1/\pi^{2k}) \sum_{j=1}^{N}\left[|d_{lj}| \quad |a_j|/j^{2k}\right]$. But by assumption $|a_j| < 1$, for $j = 1,2,\ldots,N$, so $|b_l| \le (1/\pi^{2k}) \sum_{j=1}^{N}\left[|d_{lj}|/j^{2k}\right]$, for $l = 1,2,\ldots,N$. Therefore by using the above inequality we have $\|\nu_k\| \le (1/\pi^{2k}) \sum_{l=1}^{N}\sum_{j=1}^{N}\left[|d_{lj}|/j^{2k}\right]$, or since $C_k = \sum_{=1}^{N}\sum_{j=1}^{N}\left[|d_{lj}|/j^{2k}\right]$,

$$\|\nu_k\| \le C_k/\pi^{2k}.\square$$

Now we define a norm on $\prod_{i=1}^{\infty} M$, where M is the set of all Radon measures on $[0,T]$, and we define a new set, which will be used for defining a topology which will be introduced in the following lemmas. Let

$$\mathcal{L} = \left\{(\mu_0,\mu_1,\ldots,\mu_k,\ldots) \in \prod_{i=1}^{\infty} M \;;\; \sum_{k=0}^{\infty} \|\mu_k\| < \infty\right\};$$

we show in appendix (B.4) that $\mathcal{L}$ is a linear space. We define the function $\|.\|_1$, on $\mathcal{L}$, as follows

$$\|.\|_1 : \mathcal{L} \to R^+,$$

such that for any $z = (\mu_0,\mu_1,\ldots,\mu_k \;,\ldots) \in \mathcal{L}$, $\|z\|_1 = \sum_{k=1}^{\infty} \|\mu\|$. We have shown in appendix (C.4) that the function $\|.\|_1$, is a norm on $\mathcal{L}$.

Note It is apparent that $Q(N) \subset \mathcal{L}$, to see this let $(\mu_0,\mu_1,\ldots,\mu_k,\ldots) \in Q(N)$, $\lim_k \|\mu_k\|^{1/k} = 0$, or equivalently

$\sum_{k=0}^{\infty} L^k \|\mu_k\| < \infty$, for all $L \geq 0$; so for $L = 1$, we conclude $\sum_{k=0}^{\infty} \|\mu_k\| < \infty$, and $Q(N) \subset \mathcal{L}$.

In lemma (4.4) we showed that $Q(K,N,L) \subset Q(N)$, for all K, N, and L, so $\bigcup_{k=0}^{\infty} Q(K,N,L) \subset Q(N)$, but in general $Q(N)$ is not a subset of $\bigcup_{k=0}^{\infty} Q(K,N,L)$. As an example we mention the element $(\mu_0, \mu_1, \ldots, \mu_k, \ldots)$, introduced in the Proposition VIII.5, of RUBIO and WILSON [1], where they have chosen the following measures μ_k, $k = 0, 1, \ldots$:

$$\int_{[0,T]} \phi d\mu_k \equiv \int_0^T \phi(t)(1/k!)(-1)^k u(t)dt, \quad \phi \in F.$$

We know prove the following lemma which we will use in the final lemma of this section.

Lemma (4.9) For any integer $L \geq 0$, $Q(N) \subset \overline{\bigcup_{K=0}^{\infty} Q(K,N,L)}$.

Proof Let $\bar{A}$ denote the closure of the set A with respect to norm-I topology. Also let $z_0 = (\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_k, \ldots) \in Q(N)$; then any neighbourhood of z_0 contains the following open set $O_\varepsilon(z_0) = \left\{ z \in \prod_{i=1}^{\infty} \mathcal{M} : \|z - z_0\|_I < \varepsilon \right\}$, for some $\varepsilon > 0$. If we show that

$$O_\varepsilon(z_0) \cap Q(K,N,L) \neq \phi, \tag{4.31}$$

for at least one $K \geq 1$ and for arbitrary but fixed L, then since

$$O_\varepsilon(z_0) \cap Q(K,N,L) \subset O_\varepsilon(z_0) \cap \left[\bigcup_{k=0}^{\infty} Q(K,N,L) \right],$$

we have

$$O_\varepsilon(z_0) \cap \left[\bigcup_{k=0}^{\infty} Q(K,N,L) \right] \neq \phi.$$

Which shows that z_0 is in the closure of $\bigcup_{k=0}^{\infty} Q(K,N,L)$.

We proceed to prove (4.31). Indeed, let $0 < \varepsilon < 1$. Since $z_0 = (\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_k, \ldots) \in Q(N)$, so $\sum_{k=0}^{\infty} \|\mu_k\| < \infty$; then, there exists a positive integer $M_1 = M_1(\varepsilon)$ such that

$$\sum_{k=KL+1}^{\infty} \|\dot{\mu}_k\| < \varepsilon/2, \tag{4.32}$$

for $K \geq M_1$, and arbitrary but fixed L. Since $(\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_k, \ldots) \in Q(N)$, by definition of Q(N), we have

$$\sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \dot{\mu}_k(\phi_n) = \alpha_n. \text{ for } n = 0,1,\ldots,N \tag{4.33}$$

We consider the above series as the following limit

$$\lim_{K} \sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \dot{\mu}_k(\phi_n) = \alpha_n, \; n = 0,1,\ldots,N,$$

where L is an arbitrary but fixed positive integer. Therefore there exists $K_n = K_n(\varepsilon)$, $n = 0,1,\ldots,N$, such that for $K \geq K_n$, $n = 0,1,\ldots,N$, we have

$$\left| \sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \dot{\mu}_k(\phi_n) - \alpha_n \right| < \varepsilon. \text{ for } n = 0,1,\ldots,N \tag{4.34}$$

By using (4.33) we conclude

$$\left|\sum_{k=KL+1}^{\infty}(-1)^k(n^2\pi^2)^k\,\dot{\mu}_k(\phi_n)\right|$$

$$=\left|\sum_{k=0}^{\infty}(-1)^k(n^2\pi^2)^k\,\dot{\mu}_k(\phi_n) - \sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\,\dot{\mu}_k(\phi_n)\right|$$

$$=\left|\alpha_n - \sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\,\dot{\mu}_k(\phi_n)\right|. \text{ for } n = 0,1,\dots,N.$$

From the above equality and using (4.34) we have for $K > K_n$,

$$\left|\sum_{k=KL+1}^{\infty}(-1)^k(n^2\pi^2)^k\,\dot{\mu}_k(\phi_n)\right| < \varepsilon, \text{ when } n = 0,1,\dots,N, \tag{4.35}$$

where $0 < \varepsilon < 1$. Let $B(K,n) = \sum_{k=KL+1}^{\infty}(-1)^k(n^2\pi^2)^k\,\dot{\mu}_k(\phi_n)$, for $n = 0,1,\dots,N$. It is apparent from (4.35) that $|B(K,n)| < 1$, for $n = 0,1,\dots,N$. Let $M' = \mathrm{Max}\{K_n(\varepsilon);\ n = 0,1,\dots,N\}$, and $K \ge M'$. From the results in lemma (4.8) we can determine a Radon measure ν_{KL+1}, as follows:

(i) $\nu_{KL+1}(\phi_n) = \dfrac{A(K,n)}{(n^2\pi^2)^{KL+1}}$, $n = 0,1,\dots,N$,

(ii) $\|\nu_{KL+1}\| \le C_{KL+1}/\pi^{2(KL+1)}$,

where $A(K,n) = (-1)^{KL+1}\,B(K,n)$, and where C_{KL+1} was defined in (4.29). We define now Radon measures ν_k, $k = 0,1,\dots,KL$, as follows: $\nu_k = \dot{\mu}_k$, for $k = 0,1,\dots,KL$, and $\nu_k \equiv 0$, for $k = KL+2, KL+3, \dots$; we show $(\nu_0,\nu_1,\dots,\nu_{KL},\nu_{KL+1},\bar{0}) \in Q(K+2,N,L)$. Indeed, from (4.33) we conclude

$$\nu_0(\phi_0) = \nu_0(1) = \dot{\mu}_0(1) = \alpha_0. \tag{4.36}$$

But for n = 1,2,...,N, we have

$$\sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \, \nu_k(\phi_n) = \sum_{k=0}^{KL+1} (-1)^k (n^2\pi^2)^k \, \nu_k(\phi_n)$$

$$= \sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \, \nu_k(\phi_n) + (-1)^{KL+1} (n^2\pi^2)^{KL+1} \, \nu_{KL+1}(\phi_n).$$

Now by definition of ν_k, k = 0,1,...,KN+1,... we have for n = 1,2,...,N

$$\sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \, \nu_k(\phi_n) = \sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \, \dot{\mu}_k(\phi_n)$$

$$+ (-1)^{KL+1} (n^2\pi^2)^{KL+1} \left[\frac{A(K,n)}{(n^2\pi^2)^{KL+1}} \right]$$

where $A(K,n) = \dfrac{B(K,n)}{(-1)^{KL+1}}$, and we defined $B(K,n) = \sum_{k=KL+1}^{\infty} (-1)^k (n^2\pi^2)^k \, \dot{\mu}_k(\phi_n)$, so we have from the above equality for n = 1,2,...,N

$$\sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \, \nu_k(\phi_n) = \sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \, \dot{\mu}_k(\phi_n)$$

$$+ \sum_{k=KL+1}^{\infty} (-1)^k (n^2\pi^2)^k \, \dot{\mu}_k(\phi_n)$$

$$= \sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \, \dot{\mu}_k(\phi_n). \tag{4.37}$$

But we know $(\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_k, \ldots) \in Q(N)$, so

$$\sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \, \dot{\mu}_k(\phi_n) = \alpha_n, \quad n = 1,2,\ldots,N. \tag{4.38}$$

Therefore from (4.36), (4.37) and (4.38) we conclude

$$\sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \, \nu_k(\phi_n) = \alpha_n, \quad n = 0,1,\ldots,N. \tag{4.39}$$

But we defined $\nu_k \equiv 0$, for $k = KL+2,\ KL+3,\ \ldots,$ so

$$\|\nu_k\|^{1/k} = 0 < 1/K+2, \text{ for } L(K+1) < k \le L(K+2). \tag{4.40}$$

From (4.39) and (4.40) we have by definition of $Q(K+2,N,L)$

$$(\nu_0, \nu_1, \ldots, \nu_{(K+2)L}, \bar{0}) = (\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_{KL}, \nu_{KL+1}, \bar{0}) \in Q(K+2,N,L). \tag{4.41}$$

We defined ν_{KN+1} such that

$$\|\nu_{KL+1}\| \le C_{KL+1}/\pi^{2(KL+1)}. \tag{4.41'}$$

where $C_{KL+1} = \sum_{i=1}^{N} \sum_{j=1}^{N} \left[|d_{ij}| / j^{2(KL+1)} \right]$. But $j^{2(KL+1)} \ge 1$, for all $j \ge 1$, thus

$$\|\nu_{KL+1}\| \le \left[\sum_{i=1}^{N} \sum_{j=1}^{N} |d_{ij}| \right] / \pi^{2(KL+1)}.$$

Now we choose $M_2 = M_2(\varepsilon)$, such that for $K \geq M_2$

$$\|\nu_{KL+1}\| \leq \left[\sum_{i=1}^{N} \sum_{j=1}^{N} |d_{ij}|\right]\left[1/\pi^{2(KL+1)}\right] < \varepsilon/2,$$

therefore for $K \geq M_2$

$$\|\nu_{KL+1}\| < \varepsilon/2. \qquad (4.42)$$

Now let $M = M(\varepsilon) = \text{Max}\ \{M_1(\varepsilon),\ M_2(\varepsilon),\ K_0(\varepsilon),\ K_1(\varepsilon),\ \ldots,\ K_N(\varepsilon)\}$, then for $K \geq M$, we have $K \geq K_n(\varepsilon)$, for $n = 0, 1, \ldots, N$; therefore (4.35) valid, that is

$$\left|\sum_{k=KL+1}^{\infty} (-1)^k (n^2\pi^2)^k\ \dot{\mu}_k(\phi_n)\right| < \varepsilon < 1, \text{ for } n = 0, 1, \ldots, N.$$

Since we assumed $K \geq M$, so $K \geq M_1$, therefore (4.32) is valid, that is

$$\sum_{k=KL+1}^{\infty} \|\dot{\mu}_k\| < \varepsilon/2, \qquad (4.43)$$

and since $K \geq M$, $K \geq M_2$, we have (4.42); that is

$$\|\nu_{KL+1}\| < \varepsilon/2 \qquad (4.44)$$

Finally let $K \geq M$, and let $z = (\nu_0, \nu_1, \ldots, \nu_k, \ldots) \equiv (\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_{KL}, \nu_{KL+1}, \bar{0})$. We have shown in (4,41) that $(\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_{KL}, \nu_{KL+1}, \bar{0}) \in Q(K+2, N, L)$, therefore

$$z = (\nu_0, \nu_1, \ldots, \nu_k, \ldots) \in Q(K+2, N, L) \qquad (4.45)$$

Now for $z_0 = (\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_k, \ldots) \in Q(N)$, we have

$$\|z_0 - z\|_I = \|(\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_{KL}, \mu_{KL+1}, \ldots) - (\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_{KL}, \nu_{KL+1}, \bar{0})\|_I$$

$$= \|(0, 0, \ldots, 0, \dot{\mu}_{KL+1} - \nu_{KL+1}, \dot{\mu}_{KL+2}, \ldots, \dot{\mu}_k, \ldots)\|_I$$

$$= \|\dot{\mu}_{KL+1} - \nu_{KL+1}\| + \sum_{k=KL+2}^{\infty} \|\dot{\mu}_k\|$$

$$\leq \|\nu_{KL+1}\| + \|\dot{\mu}_{KL+1}\| + \sum_{k=KL+2}^{\infty} \|\dot{\mu}_k\|,$$

or

$$\|z_0 - z\|_I \leq \|\nu_{KL+1}\| + \sum_{k=KL+1}^{\infty} \|\dot{\mu}_k\|.$$

By using (4.43) and (4.44) we have

$$\|z_0 - z\|_I < \varepsilon/2 + \varepsilon/2$$

therefore $\|z_0 - z\|_I < \varepsilon$.

We showed in (4.45) that $z = (\nu_0, \nu_1, \ldots, \nu_k, \ldots) \in Q(K+2, N, L)$, therefore $z \in \bigcup_{K=1}^{\infty} Q(K+2, N, L)$, thus

$$z_0 \in \overline{\bigcup_{K=1}^{\infty} Q(K+2, N, L)}$$

or we conclude

$$Q(N) \in \overline{\bigcup_{K=1}^{\infty} Q(K+2,N,L)}.\square$$

We can now prove the main lemma in this section that is, $\eta_{N,L} = \theta_N$. We prove half of it in the following lemma.

Lemma (4.10) Let L be an arbitrary positive fixed integer Then for any integer $N \geq 0$,

$$\eta_{N,L} \leq \theta_N.$$

Proof We have defined $\theta_N = \inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\|$, and $\eta_{N,L} = \lim_K \xi_{K,N,L}$, where $\xi_{K,N,L} = \inf_{Q_K} \sum_{k=0}^{\infty} \|\mu_k\|$. Let now $z_0 = (\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_k, \ldots)$ be an arbitrary element of Q(N). In the proof of the last lemma we showed $Q(N) \subset \overline{\bigcup_{K=1}^{\infty} Q(K+2,N,L)}$, therefore we conclude that $z_0 \in \overline{\bigcup_{K=1}^{\infty} Q(K+2,N,L)}$, so there exists a sequence $\{z_m\}$, in $\bigcup_{K=1}^{\infty} Q(K+2,N,L)$, with $z_m = (\mu_0^m, \mu_1^m, \ldots, \mu_{K_m \cdot L}^m, \bar{0}) \in Q(K_m,N,L)$, such that $\lim_m z_m = z_0$, with respect to norm-I topology. In lemma (4.6) we have shown that the sequence $\{\xi_{K,N,L}\}_K$, is a non-increasing sequence with respect to K, and since $\eta_{N,L} = \lim_K \xi_{K,N,L}$, therefore $\eta_{N,L} \leq \xi_{K,N,L}$, for all K and for fixed N and L. Thus we have for all $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0}) \in Q(K,N,L)$, $\eta_{N,L} \leq \sum_{k=0}^{KL} \|\mu_k\|$, since $\xi_{K,N,L} = \inf_{Q_K} \sum_{k=0}^{\infty} \|\mu_k\| = \inf_{Q_K} \sum_{k=0}^{KL} \|\mu_k\|$. Since for each m $= 1,2,\ldots,$ have $z_m = (\mu_0^m, \mu_1^m, \ldots, \mu_{K_m \cdot L}^m, \bar{0}) \in Q(K_m,N,L)$, we conclude

$$\eta_{NL} \le \sum_{k=0}^{K_m \cdot L} \|\mu_k^m\|, \quad m = 1,2,\ldots \,. \tag{4.46}$$

Thus, for $z_m = (\mu_0^m, \mu_1^m, \ldots, \mu_{K_m \cdot L}^m, \bar{0})$, $m = 1,2,\ldots$ we have $\|z_m\|_I = \sum_{k=0}^{K_m \cdot L} \|\mu_k^m\|$, for $m = 1,2,\ldots$; therefore from (4.46), we conclude that

$$\eta_{NL} \le \lim_m \|z_m\|_I. \tag{4.47}$$

But the norm- function is continuous with respect to norm-topology, so $\|.\|_I$, is continuous on $\mathcal{L} \subset \prod_{i=1}^{\infty} M$;

$$\lim_m \|z_m\|_I = \| \lim_m z_m\|_I = \|z_0\|_I. \tag{4.48}$$

since $z_0 = \lim_m z_m$. From (4.47) and (4.48) we have

$$\eta_{NL} \le \|z_0\|_I, \tag{4.49}$$

where $z_0 = (\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_k, \ldots)$, and $\|z_0\|_I = \sum_{k=0}^{\infty} \|\dot{\mu}_k\| < \infty$. Therefore from (4.49) we have

$$\eta_{NL} \le \sum_{k=0}^{\infty} \|\dot{\mu}_k\|. \tag{4.50}$$

We recall that $z_0 = (\dot{\mu}_0, \dot{\mu}_1, \ldots, \dot{\mu}_k, \ldots)$ was chosen as an arbitrary element of Q(N); thus we have from (4.50)

$$\eta_{NL} \le \inf_{Q(N)} \sum_{k=0}^{\infty} \|\dot{\mu}_k\|, \tag{4.51}$$

where we defined $\theta_N = \inf_{Q(N)} \sum_{k=0}^{\infty} \|\dot{\mu}_k\|$; thus, (4.51) implies that

$$\eta_{NL} \le \theta_N . \square \tag{4.52}$$

Lemma (4.11) For N = 1,2,... and any fixed L ≥ 1

$$\eta_{NL} = \theta_N .$$

Proof We have shown in lemma (4.4) that Q(K,N,L) ⊂ Q(N); therefore for all K = 1,2,..., we have

$$\inf_{Q(N)} \sum_{k=0}^{\infty} \|\dot{\mu}_k\| \le \inf_{Q_K} \sum_{k=0}^{\infty} \|\dot{\mu}_k\|, \ K = 1,2,... \tag{4.53}$$

where $Q_K \equiv Q(K,N,L)$, and each element of Q(K,N,L), is of the form $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$, Thus (4.53) implies that

$$\theta_N \le \xi_{K,N,L}, \ K = 1,2,\ldots . \tag{4.54}$$

In lemma (4.6) we showed that the sequence $\{\xi_{K,N,L}\}_K$, converges as K tends to infinity, and we defined $\eta_{NL} = \lim_K \xi_{K,N,L}$; from (4.54) we conclude $\theta_N \le \lim_K \xi_{K,N,L}$, or

$$\theta_N \le \eta_{NL} \tag{4.55}$$

From (4.52) and (4.55), we conclude

$$\theta_N = \eta_{NL} . \square$$

Note : In lemma (4.11) we showed $\eta_{NL} = \theta_N$, for any L ≥ 1; that is, η_{NL} does not actually depend on L.

(4.4) Approximation of the infimum of the objective function when the desired final state belongs to $L_2(0,1)$.

In section (4.1), we mentioned that the set s_g, defined in section (4.1), is non-empty. Thus, $\inf_{s_g} \sum_{k=0}^{\infty} \|\mu_k\|$ is meaningful. Let

$$\theta = \inf_{s_g} \sum_{k=0}^{\infty} \|\mu_k\|.$$

In this section we are going to show that

$$\theta = \lim_{N} \inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\|.$$

where $Q(N)$ is the set of all $(\mu_0, \mu_1, \ldots, \mu_k, \ldots)$ such that

$$\sum_{k=1}^{\infty} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) = \alpha_n, \; n = 0,1,\ldots,N,$$

$$\lim_{k\to\infty} \|\mu_k\|^{1/k} = 0.$$

The set s_g may be characterized as follows

$s_g = \Big\{ s \in S : s(\phi_n) = \alpha_n, \quad n = 0,1,\ldots, \quad s$ is of the form (4.5), the measures μ_k, $k = 0,1,\ldots$ associated with it are atomless and $\lim_k \|\mu_k\|^{1/k} = 0 \Big\}$; the set $S \subset \prod_{i=1}^{\infty} M$, has been defined in section (4.1).

We assumed in section (4.1) that $g(.) \in L_2(0,1)$, and $g(x) = \sum_{n=0}^{\infty} \alpha_n \cos(n\pi x)$. Define

$$g_N(x) = \sum_{n=0}^{N} \alpha_n \cos(n\pi x)$$

and define

$$s_{g_N} = \left\{ s \in s_g : s(\phi_n) = \alpha_n,\ n = 0,1,\dots,N \right\},$$

where $s = (\mu_0, \mu_1, \dots, \mu_k, \dots)$, and $s(\phi_n) = \sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n)$.

Lemma (4.12) $g_N \to g$, with respect to $L_2(0,1)$-norm.

Proof Its proof is well known.□

We see that indeed $s_{g_N} = Q(N)$, because we can describe the set s_g as the set of all $(\mu_0, \mu_1, \dots, \mu_k, \dots)$, of Radon measures such that

(a) $\sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) = \alpha_n$, $n = 0,1,\dots,$

(b) $\lim_k \|\mu_k\|^{1/k} = 0$.

Similarly we can describe s_{g_N}, as the set of all $(\mu_0, \mu_1, \dots, \mu_k, \dots)$, of Radon measures such that

(c) $\sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) = \alpha_n$, $n = 0,1,\dots,N$,

(d) $\lim_k \|\mu_k\|^{1/k} = 0.$

Comparing the definitions of the sets Q(N) and s_{g_N} we conclude

$$Q(N) = s_{g_N}. \tag{4.56}$$

It is clear from the definition of s_{g_N} that

$$s_g \subset \ldots \subset s_{g_{N+1}} \subset s_{g_N} \subset \ldots \subset s_{g_1} \subset s_{g_0}. \tag{4.57}$$

But we defined in (4.27'')

$$\theta_N = \inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\|,$$

or, by using (4.56), we have

$$\theta_N = \inf_{s_{g_N}} \sum_{k=0}^{\infty} \|\mu_k\|. \tag{4.58}$$

From (4.57) and the above result, we conclude that

$$\theta_0 \le \theta_1 \le \ldots . \le \theta_N \le \theta_{N+1} \le \ldots \le \theta; \tag{4.59}$$

therefore the sequence $\{\theta_N\}_N$, is a non-decreasing bounded sequence, thus convergent. Let $\xi = \lim_N \theta_N$; from (4.59)

$$\xi \le \theta. \tag{4.60}$$

Lemma (4.13) $\xi = \theta$.

Proof Let $P = \bigcap_{N=0}^{\infty} s_{g_N}$. Since $s_g \subset \ldots \subset s_{g_{N+1}} \subset s_{g_N} \subset \ldots \subset s_{g_1} \subset s_{g_0}$,

$$P \supset s_g. \tag{4.61}$$

But we defined in (4.58) $\theta_N = \inf_{s_{g_N}} \sum_{k=0}^{\infty} \|\mu_k\|$, thus

$$\xi = \lim_N \theta_N = \inf_P \sum_{k=0}^{\infty} \|\mu_k\| \tag{4.62}$$

(see RUBIO [1], p.27). Now we show that $P \subset s_g$. Let $s = (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in P$, then it is clear that $s \in s_{g_N}$, for all $N = 0, 1, \ldots$; this means that

(a) $s(\phi_n) = \alpha_n$, $n = 0, 1, \ldots$

(b) $\lim_k \|\mu_k\|^{1/k} = 0$.

Therefore by definition of s_g, we conclude that $s = (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in s_g$, thus $P \subset s_g$, since we showed in (4.61) that $P \supset s_g$, we conclude

$$P = s_g. \tag{4.63}$$

From (4.62) and (4.63), we have

$$\xi = \inf_{S_g} \sum_{k=0}^{\infty} \|\mu_k\|; \tag{4.64}$$

since

$$\theta = \inf_{S_g} \sum_{k=0}^{\infty} \|\mu_k\|, \tag{4.65}$$

thus from (4.64) and (4.65) we have

$$\xi = \theta. \square \tag{4.66}$$

(4.5) Approximation of the infimum of the objective function by a finite summation of the norms of discrete measures.

In this section we show that for any $\varepsilon > 0$ and for any fixed integer $L \geq 1$ there exists non-negative integers K and N such that

$$\left| \theta - \sum_{k=0}^{KL} \|\nu_k\| \right| < \varepsilon,$$

where $\theta = \inf_{S_g} \sum_{k=0}^{\infty} \|\mu_k\|$, and ν_k, $k = 0,1,\ldots,KL$, are discrete measures on $[0,T]$. We also show that there exists δ_0 $(0 < \delta_0 < \varepsilon/5)$ such that

$$|\theta - \theta_{KNL}(\delta_0)| < \varepsilon,$$

where $\theta_{KNL}(\delta_0) = \inf_{P_K} \sum_{k=0}^{\infty} \|\mu_k\|$; $P_K \equiv P(K,N,L,\delta_0)$ is the set of all $(\nu_0,\nu_1,\ldots,\nu_{KL},\bar{0})$ so that each of ν_k, $k = 0,1,\ldots,KL$, is a discrete measure defined on $[0,T]$ such that

$$\left|\sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) - \alpha_n\right| < \delta_0.$$

In the following lemma we show that for a finite number of Radon measures defined on $[0,T]$ there are corresponding discrete measures on $[0,T]$ so that their norms are close to the norms of the initial Radon measures.

Lemma (4.14) Let ε, $(\varepsilon < 1)$ and δ_0 be fixed positive numbers such that $\delta_0 < \varepsilon/5$. Further let K, N, and L, be non-negative integers.

Assume that

$$(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0}) \in Q(K,N,L,\delta_0/2)$$

Then there exists an element

$$(\nu_0,\nu_1,\ldots,\nu_{KL},\bar{0}) \in Q(K,N,L,\delta_0),$$

where ν_k, $k = 0,1,\ldots,KL$, are discrete measures on $[0,T]$ such that

$$\left|\sum_{k=0}^{KL}\|\mu_k\| - \sum_{k=0}^{KL}\|\nu_k\|\right| < \varepsilon/5.$$

Proof Let $\mu_k = \mu_k^+ - \mu_k^-$, be the decomposition of the Radon measures μ_k, $k = 0,1,\ldots,KL$, where μ_k^+, μ_k^-, are positive Radon measures defined on $[0,T]$. Then by a theorem of approximation (see CHOQUET [1],p.221), there exist discrete measures $\nu_k'^+$, $k = 0,1,\ldots,KL$, corresponding to the positive measures μ_k^+, $k = 0,1,\ldots,KL$, as follows

$$\nu_k'^+ = \sum_{i=0}^{M_k}\beta_i'^k\,\delta(y_i^k),\ k = 0,1,\ldots,KL,$$

where $\beta_i'^k \geq 0$, for $k = 0,1,\ldots,KL$, $i = 0,1,\ldots,M_k$ and $A_k = \{\, y_i^k;\ i = 0,1,\ldots,M_k\}$, $k = 0,1,\ldots,M_k$, are partitions of $[0,T]$, [usually, a partition P, of [a,b], is specified as a finite set of real numbers $\{x_0,x_1,\ldots,x_n\}$, such that $a \leq x_0 \leq x_1 \leq \ldots \leq x_n = b$. See BARTLE [1], P.275], such that

$$|(\nu_k'^+ - \mu_k^+)\phi_n| < \gamma_0\,/4R,\ n = 0,1,\ldots,N,\ k = 0,1,\ldots,KL$$

here $R \equiv R_{KNL} = \sum_{k=0}^{KL} (N^2\pi^2)^k$, $\gamma_0 < \min\left\{\delta_0, \ (1/K - \omega_0)^{KL}\right\}$, and $\omega_0 = \text{Max} \left\{\|\mu_k\|^{1/k}; \ (K-1)L < k \le KL\right\}$.

By the above definition we have $\omega_0 < 1/K$, since $\|\mu_k\|^{1/k} < 1/K$, for $(K-1)L < k \le KL$. Similarly there exist discrete measures $\nu_k'^{-}$, corresponding to the measures μ_k^{-}, $k = 0,1,\dots,KL$, as follows;

$$\nu_k'^{-} = \sum_{i=0}^{N_k} \gamma_i'^{k} \ \delta(z_i^k), \ k = 0,1,\dots,KL,$$

where $\gamma_i'^{k} \ge 0$, for $i = 0,1,\dots,N_k$, and $k = 0,1,\dots,KL$, such that

$$|(\nu_k'^{-} - \mu_k^{-})\phi_n| < \gamma_0 /4R, \text{ for } n = 0,1,\dots,N, \text{ and } k = 0,1,\dots,KL.$$

Let $B_k = \{z_i^k; \ i = 0,1,\dots,N_k\}$, $k = 0,1,\dots,KL$, be partitions of $[0,T]$. Now let $P = \bigcup_{k=0}^{KL} A_k \ \bigcup_{k=0}^{KL} B_k$; P is a partition of $[0,T]$. We reindex the elements of P, and let $P = \{t_i; \ i = 0,1,\dots,M\}$. We claim for this new partition of $[0,T]$, that there exist $\beta_i^k \ge 0$, $k = 0,1,\dots,KL$, $i = 0,1,\dots,M$, such that if we define $\nu_k^{+} = \sum_{i=0}^{M} \beta_i^k \delta(t_i)$, $k = 0,1,\dots,KL$, then we have

$$|(\nu_k^{+} - \mu_k^{+})\phi_n| \le |(\nu_k'^{+} - \mu_k^{+})\phi_n| < \gamma_0 /4R. \qquad (4.67)$$

For example we can choose β_i^k's as follows:

$\beta_i^k = \beta_i'^{k}$, if $t_i = y_j^k \in A_k$

$\beta_i^k = 0$, if $t_i \notin A_k$.

Therefore we have $\nu_k^+ = \sum_{i=0}^{M} \beta_i^k \ \delta(t_i) = \sum_{i=0}^{M} \beta_i^k \ \delta(y_i^k)$; for n = 0,1,...,N, and k = 0,1,...,KL

$$|(\nu_k^+ - \mu_k^+)\phi_n| = |(\nu_k'^+ - \mu_k^+)\phi_n| < \gamma_0 /4R.$$

Similarly, there exist $\gamma_i^k \geq 0$, i = 0,1,...,M, and k = 0,1,...,KL, such that if we define $\nu_k^- = \sum_{i=0}^{M} \gamma_i^k \ \delta(t_i)$, k = 0,1,...,KL, then we have for n = 0,1,...,N, and k = 0,1,...,KL,

$$|(\nu_k^- - \mu_k^-)\phi_n| \leq |(\nu_k'^- - \mu_k^-)\phi_n| < \gamma_0 /4R. \qquad (4.68)$$

Now define

$$\nu_k = \nu_k^+ - \nu_k^-, \text{ for } k = 0,1,\ldots,KL, \qquad (4.68')$$

then

$$|(\mu_k - \nu_k)\phi_n| = |[(\mu_k^+ - \mu_k^-) - (\nu_k^+ - \nu_k^-)]\phi_n|$$

$$= |(\mu_k^+ - \nu_k^+)\phi_n - (\mu_k^- - \nu_k^-)]\phi_n|$$

$$\leq |(\mu_k^+ - \nu_k^+)\phi_n| + |(\mu_k^- - \nu_k^-)]\phi_n|$$

$$< \gamma_0/4R + \gamma_0/4R.$$

or

$$|(\mu_k - \nu_k)\phi_n| < \gamma_0/2R, \qquad (4.69)$$

for $n = 0,1,\ldots,N$, $k = 0,1,\ldots,KL$. But we have $\|\mu_k\| = (\mu_k^+ + \mu_k^-)(1)$, and $\|\nu_k\| = (\nu_k^+ + \nu_k^-)(1)$, (see CHOQUET [1], p.215). Thus by using (4.67′) and (4.68), and since $\phi_0(t) \equiv 1$ on $[0,T]$, we have

$$\left|\sum_{k=0}^{KL}\|\mu_k\| - \sum_{k=0}^{KL}\|\nu_k\|\right| = \left|\sum_{k=0}^{KL}(\mu_k^+ + \mu_k^-)(1) - \sum_{k=0}^{KL}(\nu_k^+ + \nu_k^-)(1)\right|$$

$$\leq \sum_{k=0}^{KL}\left|(\mu_k^+ - \nu_k^+)(1)\right| - \sum_{k=0}^{KL}\left|(\mu_k^- - \nu_k^-)(1)\right|$$

$$< \sum_{k=0}^{KL}\gamma_0/4R + \sum_{k=0}^{KL}\gamma_0/4R$$

$$\leq (KL+1)\gamma_0/2R. \tag{4.70}$$

Also we have by definition $R = \sum_{k=0}^{KL}(N^2\pi^2)^k \geq KL+1$, for $N \geq 0$; therefore $KL+1/\,R \leq 1$, so from this result we may write (4.70) in the following form

$$\left|\sum_{k=0}^{KL}\|\mu_k\| - \sum_{k=0}^{KL}\|\nu_k\|\right| < \gamma_0/2. \tag{4.71}$$

Since $\gamma_0 < \min\left\{\delta_0, \ (1/K - \omega_0)^{n}\right\}$, we have $\gamma_0 < \delta_0$; since by hypothesis we assumed $\delta_0 < \varepsilon/5$, so we have

$$\left|\sum_{k=0}^{KL}\|\mu_k\| - \sum_{k=0}^{KL}\|\nu_k\|\right| < \varepsilon/10 < \varepsilon/5.\square \tag{4.72}$$

By proving lemmas (4.15) and (4.16) below, we intend to show $(\nu_0,\nu_1,\ldots,\nu_{KL},\bar{0}) \in Q(K,N,L,\delta_0)$. First in lemma (4.15) we prove

(I) $\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ \nu_k(\phi_n) - \alpha_n\right| < \delta_0$, for $n = 0,1,\dots,N$;

in lemma (4.16) we show that

(II) $\|\nu_k\| < 1/K$, for $(K-1)L < k \le KL$.

Lemma (4.15) $\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ \nu_k(\phi_n) - \alpha_n\right| < \delta_0$, for $n = 0,1,\dots,N$.

Proof For $n = 0,1,\dots,N$, we have

$$\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ \mu_k(\phi_n) - \alpha_n\right|$$

$$= \left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ [(\nu_k - \mu_k) + \mu_k](\phi_n) - \alpha_n\right|$$

$$= \left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ [(\nu_k - \mu_k)(\phi_n) + \sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ \mu_k(\phi_n) - \alpha_n\right|$$

$$\le \left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ (\nu_k - \mu_k)\phi_n\right| + \left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ \mu_k(\phi_n) - \alpha_n\right|,$$

By the assumption of the lemma (4.14) we know that $(\mu_0,\mu_1,\dots,\mu_{KL},\bar{0}) \in Q(K,N,L,\delta_0/2)$, so by definition we have

$$\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ \mu_k(\phi_n) - \alpha_n\right| < \delta_0/2, \tag{4.73}$$

also we have

$$\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ (\nu_k - \mu_k)\phi_n\right| \le \sum_{k=0}^{KL}(n^2\pi^2)^k\left|(\nu_k - \mu_k)\phi_n\right|.$$

Now, (4.69) implies that $\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ (\nu_k - \mu_k)\phi_n\right| < \sum_{k=0}^{KL}(n^2\pi^2)^k.(\gamma_0\ /2R)$, but $\sum_{k=0}^{KL}(n^2\pi^2)^k \le \sum_{k=0}^{KL}(N^2\pi^2)^k = R$, for $n = 0,1,\ldots,N$. Therefore $\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ (\nu_k - \mu_k)\phi_n\right| < R.(\gamma_0\ /2R)$. But since $\gamma_0 < \delta_0$, we conclude

$$\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ (\nu_k - \mu_k)\phi_n\right| < \gamma_0/2 < \delta_0/2, \tag{4.74}$$

for $n = 0,1,\ldots,N$. By (4.73) and (4.74) we have

$$\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ \nu_k(\phi_n) - \alpha_n\right| \le \left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ (\nu_k - \mu_k)\phi_n\right|$$

$$+ \left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ \mu_k(\phi_n) - \alpha_n\right|$$

$$< \delta_0\ /2 + \delta_0\ /2$$

or

$$\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ \nu_k(\phi_n) - \alpha_n\right| < \delta_0,\ n = 0,1,\ldots,N. \square$$

Lemma (4.16) Let ν_k, $k = 0,1,\ldots,KL$, to be defined as in (6.68′); then, $\|\nu_k\| < 1/K$, for $(K-1)L < k \le KL$.

Proof For $k = 0,1,\ldots,KL$, we have from inequalities (4.67) and (4.68) of lemma (4.14)

$$|(\nu_k^+ - \mu_k^+)\phi_n| < \gamma_0 /4R, \; n = 0,1,\ldots,N, \qquad (4.75)$$

$$|(\nu_k^- - \mu_k^-)\phi_n| < \gamma_0 /4R, \; n = 0,1,\ldots,N. \qquad (4.76)$$

From the definitions of $\phi_n(t)$, and $R \equiv R_{KNL}$, we conclude $\phi_0(t) \equiv 1$, $(0 \le t \le T)$, and $R \ge 1$. Therefore from (4.75) and (4.76), we conclude

$$|\nu_k^+(1) - \mu_k^+(1)| < \gamma_0 /4R \le \gamma_0 /4 \qquad (4.77)$$

$$|\nu_k^-(1) - \mu_k^-(1)| < \gamma_0 /4R \le \gamma_0 /4 \qquad (4.78)$$

Now we have

$$\begin{aligned} |\|\nu_k\| - \|\mu_k\|| &= |\nu_k^+(1) + \nu_k^-(1) - [\mu_k^+(1) + \mu_k^-(1)]| \\ &\le |[\nu_k^+(1) - \mu_k^+(1)] + [\nu_k^-(1) - \mu_k^-(1)]| \\ &\le |\nu_k^+(1) - \mu_k^+(1)| + |\nu_k^-(1) - \mu_k^-(1)|. \end{aligned}$$

By (4.77) and (4.78), we have $|\|\nu_k\| - \|\mu_k\|| < \gamma_0 /4 + \gamma_0 /4$, or $|\|\nu_k\| - \|\mu_k\|| < \gamma_0$, for $k = 0,1,\ldots,KL$,. Thus $\|\nu_k\| - \|\mu_k\| \le |\|\nu_k\| - \|\mu_k\|| < \gamma_0$, finally for $k = 0,1,\ldots,KL$, we have

$$\|\nu_k\| < \|\mu_k\| + \gamma_0 \qquad (4.79)$$

At this point we need to use the following inequality, which is valid for all $a \ge 0$, $b \ge 0$, and $k = 1,2,\ldots$:

$$(a + b)^{1/k} \le a^{1/k} + b^{1/k}.$$

Thus we conclude from (4.79) that

$$\|\nu_k\|^{1/k} < (\|\mu_k\| + \gamma_0)^{1/k} \le \|\mu_k\|^{1/k} + \gamma_0^{1/k}. \qquad (4.80)$$

Let $(K-1)L < k \le KL$; then we conclude that $0 < k \le KL$, [since $0 \le (K-1)L$] thus

$$1/k \ge 1/KL \qquad (4.81)$$

From the proof of lemma (4.14) we have

$$\gamma_0 < \min\left\{\delta_0, \ (1/K \ - \omega_0)^{KL}\right\}, \qquad (4.82)$$

where $0 < \delta_0 < 1$. Therefore $\gamma_0 < 1$, so $1/\gamma_0 > 1$, so from (4.81) we have $(1/\gamma_0)^{1/k} \ge (1/\gamma_0)^{1/KL}$, or

$$\gamma_0^{1/k} \le \gamma_0^{1/KL}. \qquad (4.83)$$

From (4.82) we have $\gamma_0 < (1/K \ - \omega_0)^{KL}$, or

$$\gamma_0^{1/KL} < (1/K \ - \omega_0), \qquad (4.84)$$

where we defined $\omega_0 = \text{Max}\left\{\|\mu_k\|^{1/k}; \ (K-1)L < k \le KL\right\}$, therefore from (4.80) we have $\|\nu_k\|^{1/k} \le \|\mu_k\|^{1/k} + \gamma_0^{1/k} \le \omega_0 + \gamma_0^{1/k}$, and by using (4.83) we have $\|\nu_k\|^{1/k} \le \omega_0 + \gamma_0^{1/KL}$, for $(K-1)L < k \le KL$; thus (4.84) implies that, for $(K-1)L < k \le KL$,

$$\|\nu_k\|^{1/k} \leq \omega_0 + \gamma_0^{1/KL} < \omega_0 + (1/K - \omega_0)$$

$$\|\nu_k\|^{1/k} < 1/K, \text{ for } (K-1)L < k \leq KL. \square$$

Proposition (4.1) For every $\varepsilon > 0$, there exists a positive number δ_0 such that $\delta_0 < \varepsilon/5$, and there exists an element $(\nu_0, \nu_1, \ldots, \nu_{KL}, \bar{0}) \in Q(K,N,L,\delta_0)$, where ν_k, $k = 0,1,\ldots,KL$, are discrete Radon measures on $[0,T]$ (where L, is an arbitrary but fixed positive integer and K, N, are two non-negative integers which will be determined later), such that

$$\left|\theta - \sum_{k=0}^{KL} \|\nu_k\|\right| < \varepsilon;$$

here $\theta = \inf_{S_g} \sum_{k=0}^{\infty} \|\nu_k\|$.

Proof In lemma (4.13) we showed that $\theta = \lim_N \theta_N$, where $\theta_N = \inf_{S_{g_N}} \sum_{k=0}^{\infty} \|\nu_k\|$. Therefore for any $\varepsilon > 0$, there exists an integer $N \geq 0$, such that

$$|\theta - \theta_N| < \varepsilon/5. \tag{4.85}$$

We assumed in the hypothesis that the positive integer L was arbitrary but fixed. Let the number N, which was obtained above, be fixed. We have shown in lemma (4.13) that $\theta_N = \eta_{NL}$, where $\eta_{NL} = \lim \xi_{KNL}$, and $\xi_{KNL} = \inf_{Q_K} \sum_{k=0}^{KL} \|\mu_k\|$; here $Q_K = Q(K,N,L)$, is the set of all $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$, such that

(a) $\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\ \mu_k(\phi_n) = \alpha_n,\quad n = 0,1,\ldots,N,$

(b) $\|\mu_k\|^{1/k} < 1/K,$ for $(K-1)L < k \le KL.$

Thus $\theta_N = \lim_K \xi_{KNL}$; therefore there exists a non-negative integer K_0 such that for every $K \ge K_0$, we have

$$|\theta_N - \xi_{KNL}| < \varepsilon/5. \tag{4.86}$$

Assume that K is a fixed positive integer. In lemma (4.5) we showed $\xi_{KNL} = \theta_{KNL}$, where $\theta_{KNL} = \lim_{d\to 0+} \xi_{KNL}(d)$. Therefore $\xi_{KNL} = \lim_{d\to 0+} \xi_{KNL}(d)$, where K, N, L, be fixed integers defined above. But by definition $\xi_{KNL}(d) = \inf_{Q_K(d)} \sum_{k=0}^{KL} \|\mu_k\|$, where $Q_K(d) \equiv Q(K,N,L,d)$, therefore there exists $\delta_0 > 0$, (we assume $\delta_0 < \varepsilon/5$, and $\delta_0 < 1$), such that

$$|\xi_{KNL} - \xi_{KNL}(d)| < \varepsilon/5, \text{ for } 0 < d < \delta_0. \tag{4.87}$$

But $(\delta_0/2) < \delta_0$, thus

$$|\xi_{KNL} - \xi_{KNL}(\delta_0/2)| < \varepsilon/5. \tag{4.88}$$

Therefore by definition of infimum, there exists $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0}) \in Q(K,N,L,\delta_0/2)$, such that

$$\left|\sum_{k=0}^{KL} \|\mu_k\| - \xi_{KNL}(\delta_0/2)\right| < \varepsilon/5. \tag{4.89}$$

From the definition of $Q(K,N,L,\varepsilon)$ we have

$$Q(K,N,L,\delta_0/2) \subset Q(K,N,L,\delta_0) \subset Q(K,N,L,\varepsilon/5),$$

since $\delta_0 < \varepsilon/5$, therefore the specified $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0})$ is an element of $Q(K,N,L,\delta_0)$. In lemma (4.14) we showed that there exists an element $(\nu_0,\nu_1,\ldots,\nu_{KL},\bar{0}) \in Q(K,N,L,\delta_0)$, where each of ν_k, $k = 0,1,\ldots,KL$, is a discrete measure on $[0,T]$, such that

$$\Big| \sum_{k=0}^{KL} \|\mu_k\| - \sum_{k=0}^{KL} \|\nu_k\| \;)\Big| < \varepsilon/5. \tag{4.90}$$

Now by using (4.85)-(4.90), we have

$$|\theta - \sum_{k=0}^{KL} \|\nu_k\|| = |\theta - \theta_N + \theta_N - \xi_{KNL} + \xi_{KNL} - \xi_{KNL}(\delta_0/2) +$$

$$\xi_{KNL}(\delta_0/2) - \sum_{k=0}^{KL} \|\mu_k\| + \sum_{k=0}^{KL} \|\mu_k\| - \sum_{k=0}^{KL} \|\nu_k\||$$

$$\le |\theta - \theta_N| + |\theta_N - \xi_{KNL}| + |\xi_{KNL} - \xi_{KNL}(\delta_0/2)|$$

$$+ |\xi_{KNL}(\delta_0/2) - \sum_{k=0}^{KL} \|\mu_k\|| + |\sum_{k=0}^{KL} \|\mu_k\| - \sum_{k=0}^{KL} \|\nu_k\||$$

$$< \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5 + \varepsilon/5$$

or $|\theta - \sum_{k=0}^{KL} \|\nu_k\|| < \varepsilon$. □

We mention now some corollaries which will be useful later in section (4.6).

Corollarly (4.1) For K, (K $\geq K_0$, defined in inequality (4.86)) there exists $(\mu_0, \mu_1, \ldots, \mu_{(K+1)L}, \bar{0}) \in Q(K+1, N, L, \delta_0)$, such that $|\theta - \sum_{k=0}^{(K+1)L} \|\nu_k\|| < \varepsilon$.

Proof we change K, $(K \geq K_0)$, in inequality (4.86), to K+1, then with similar proof we obtain $(\mu_0, \mu_1, \ldots, \mu_{(K+1)L}, \bar{0}) \in Q(K+1, N, L, \delta_0)$, where each of the μ_k's is a discrete measure and

$$|\theta - \sum_{k=0}^{(K+1)L} \|\nu_k\|| < \varepsilon. \square$$

Now we introduce another set which will be useful in corollary (4.2).

Let K, N, and L, be positive integers and $\delta \geq 0$. We define $Q'(K,N,L,\delta)$, as the set of all $(\nu_0, \nu_1, \ldots, \nu_{KL}, \bar{0})$, such that each of the ν_k,s is a discrete measure. We show in appendix (C.4), that for all positive integers K,N,L, and positive δ, $Q'(K,N,L,\delta) \neq \phi$. Therefore the following definition is meaningful

$$\xi'_{K,N,L}(\delta) = \inf_{Q'_K(\delta)} \sum_{k=0}^{KL} \|\nu_k\|,$$

where $Q'_K(\delta) = Q'(K,N,L,\delta)$.

Corollary (4.2) $|\theta - \xi'_{K,N,L}(\delta_0)| < (7/5)\varepsilon$.

Proof From (4.89) and (4.90), we have

$$|\xi_{K,N,L}(\delta_0 /2) - \sum_{k=0}^{KL} \|\nu_k\|| = |\xi_{K,N,L}(\delta_0 /2) - \sum_{k=0}^{KL} \|\mu_k\|$$

$$\sum_{k=0}^{KL} \|\mu_k\| - \sum_{k=0}^{KL} \|\nu_k\|| \le |\xi_{K,N,L}(\delta_0 /2) - \sum_{k=0}^{KL} \|\mu_k\|| +$$

$$+ |\sum_{k=0}^{KL} \|\mu_k\| - \sum_{k=0}^{KL} \|\nu_k\|| < \varepsilon/5 + \varepsilon/5$$

or

$$|\xi_{K,N,L}(\delta_0 /2) - \sum_{k=0}^{KL} \|\nu_k\|| < 2\varepsilon/5. \qquad (4.91)$$

Also by using (4.87) of prop.(4.1), we have

$$|\xi_{K,N,L}(\delta_0) - \xi_{K,N,L}(\delta_0 /2)| = |\xi_{K,N,L}(\delta) - \xi_{K,N,L}$$

$$+ \xi_{K,N,L} - \xi_{K,N,L}(\delta_0 /2)| \le |\xi_{K,N,L}(\delta) - \xi_{K,N,L}|$$

$$+ |\xi_{K,N,L} - \xi_{K,N,L}(\delta_0 /2)| < \varepsilon/5 + \varepsilon/5,$$

thus

$$|\xi_{K,N,L}(\delta_0) - \xi_{K,N,L}(\delta_0 /2)| < 2\varepsilon/5. \qquad (4.92)$$

We have from (4.91) and (4.92)

$$|\xi_{K,N,L}(\delta_0) - \sum_{k=0}^{KL} \|\nu_k\|| = |\xi_{K,N,L}(\delta_0) - \xi_{K,N,L}(\delta_0 /2)$$

$$+ \xi_{K,N,L}(\delta_0 /2) - |\sum_{k=0}^{KL} \|\nu_k\||$$

$$\leq |\xi_{K,N,L}(\delta_0) - \xi_{K,N,L}(\delta_0 /2)| + |\xi_{K,N,L}(\delta_0 /2) - |\sum_{k=0}^{KL} \|\nu_k\||$$

$$< 2\varepsilon/5 + 2\varepsilon/5$$

or

$$|\xi_{K,N,L}(\delta_0) - \sum_{k=0}^{KL} \|\nu_k\|| < 4\varepsilon/5. \qquad (4.93)$$

By definition we have $Q'(K,N,L,\delta_0) \subset Q(K,N,L,\delta_0)$, so

$$\inf_{Q_K(\delta_0)} \sum_{k=0}^{KL} \|\nu_k\| \leq \inf_{Q'_K(\delta_0)} \sum_{k=0}^{KL} \|\nu_k\|, \qquad (4.94)$$

where $Q_K(\delta_0) \equiv Q(K,N,L,\delta_0)$, and $Q'_K(\delta_0) \equiv Q'(K,N,L,\delta_0)$, or by definition we have $\xi_{K,N,L}(\delta_0) \leq \xi'_{K,N,L}(\delta_0)$; from Proposition (4.1) we have $(\nu_0,\nu_1,\ldots,\nu_{KL},\bar{0}) \in Q'(K,N,L,\delta_0)$, so

$$\xi_{K,N,L}(\delta_0) \leq \xi'_{K,N,L}(\delta_0) \leq \sum_{k=0}^{KL} \|\nu_k\|$$

We can conclude from the above inequalities that

$$|\xi_{K,N,L}(\delta_0) - \xi'_{K,N,L}(\delta_0)| < |\xi_{K,N,L}(\delta_0) - \sum_{k=0}^{KL} \|\nu_k\||. \qquad (4.95)$$

By (4.93) and (4.94)

$$|\xi_{K,N,L}(\delta_0) - \xi'_{K,N,L}(\delta_0)| < 4\varepsilon/5. \qquad (4.96)$$

Also by formulas (4.85) and (4.87) of proposition (4.1) we have

$$|\theta - \xi_{K,N,L}(\delta_0)| = |\theta - \theta_N + \theta_N - \xi_{K,N,L} + \xi_{K,N,L} - \xi_{K,N,L}(\delta_0)|$$

$$\le |\theta - \theta_N| + |\theta_N - \xi_{K,N,L}| + |\xi_{K,N,L} - \xi_{K,N,L}(\delta_0)|$$

$$< \varepsilon/5 + \varepsilon/5 + \varepsilon/5;$$

thus

$$|\theta - \xi_{K,N,L}(\delta_0)| < 3\varepsilon/5. \qquad (4.97)$$

Finally by using (4.96) and (4.97) we conclude

$$|\theta - \xi'_{K,N,L}(\delta_0)| = |\theta - \xi_{K,N,L}(\delta_0) + \xi_{K,N,L}(\delta_0) - \xi'_{K,N,L}(\delta_0)|$$

$$\le |\theta - \xi_{K,N,L}(\delta_0)| + |\xi_{K,N,L}(\delta_0) - \xi'_{K,N,L}(\delta_0)|$$

$$< 3\varepsilon/5 + 4\varepsilon/5$$

or

$$|\theta - \xi'_{K,N,L}(\delta_0)| < 7\varepsilon/5.\square \qquad (4.98)$$

Note If we change ε, in proposition (4.1), by $5\varepsilon/7$, then the result (4.98) is in the following standard form:

$$|\theta - \xi'_{K,N,L}(\delta_0)| < \varepsilon.$$

Corollarly (4.3) $|\theta - \xi'_{K+1,N,L}(\delta_0)| < \varepsilon.$

Proof In corollary (4.2) we change K to K+1 then we have

$$|\theta - \xi'_{K+1,N,L}(\delta_0)| < \varepsilon.\square$$

Note In fact corollary (3) expresses that if $|\theta - \xi'_{K,N,L}(\delta_0)| < \varepsilon$, then $|\theta - \xi'_{K+1,N,L}(\delta_0)| < \varepsilon$.

We defined $Q'(K,N,L,\delta)$, as the set of all $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0})$, such that each of μ_k, is a discrete measure on $[0,T]$, and

(a) $\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\,\mu_k(\phi_n) - \alpha_n\right| < \delta, \quad n = 0,1,\ldots,N,$

(b) $\|\mu_k\|^{1/k} < 1/K$, for $(K-1)L < k \le KL$.

Now we define $P(K,N,L,\delta)$ as the set of all $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0})$, where each of the μ_k's is a discrete measure, satisfying only the condition (a) above; that is:

(a) $\left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\,\mu_k(\phi_n) - \alpha_n\right| < \delta, \quad n = 0,1,\ldots,N.$

It is apparent that $Q'(K,N,L,\delta) \subset P(K,N,L,\delta)$, and it is easy to show that $P(K,N,L,\delta) \subset Q'(K+1,N,L,\delta)$; indeed, let $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0}) \in P(k,N,L,\varepsilon)$, then

(a) $\left|\sum_{k=0}^{(K+1)L}(-1)^k(n^2\pi^2)^k\,\mu_k(\phi_n) - \alpha_n\right|$

$$= \left|\sum_{k=0}^{KL}(-1)^k(n^2\pi^2)^k\,\mu_k(\phi_n) - \alpha_n\right| < \varepsilon, \quad n = 0,1,\ldots,N.$$

(b) $\|\mu_k\|^{1/k} = 0 < 1/K+1$, for $(K-1)L < k \le KL$,

thus we conclude that $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0}) \in Q'(K+1,N,L,\varepsilon)$, or $P(K,N,L,\delta) \subset Q'(K+1,N,L,\delta)$. Therefore we have

$$Q'(K,N,L,\delta) \subset P(K,N,L,\delta) \subset Q'(K+1,N,L,\delta). \tag{4.99}$$

From (4.99) we conclude

$$\inf_{Q'_{K+1}(\delta)} \sum_{k=0}^{KL} \|\nu_k\| \le \inf_{P_K(\delta)} \sum_{k=0}^{KL} \|\nu_k\| \le \inf_{Q'_K(\delta)} \sum_{k=0}^{KL} \|\nu_k\|, \tag{4.100}$$

where $Q'_K(\delta) \equiv Q'(K,N,L,\delta)$ and $P_K(\delta) \equiv P(K,N,L,\delta)$. Now let $\theta_{K,N,L}(\delta) \equiv \inf_{P_K(\delta)} \sum_{k=0}^{KL} \|\nu_k\|$ and since $\xi'_{K,N,L}(\delta) = \inf_{Q'_K(\delta)} \sum_{k=0}^{KL} \|\nu_k\|$, (4.100) we have

$$\xi'_{K+1,N,L}(\delta) \le \theta_{K,N,L}(\delta) \le \xi'_{K,N,L}(\delta). \tag{4.101}$$

Lemma (4.17) Let $a \le b \le c$, and ζ, be any real number then $|\zeta - b|$, is less than of at least one of $|\zeta - a|$ or $|\zeta - c|$.

Proof Case (1) $\zeta \le b$, then $|\zeta - b| = b - \zeta \ge 0$. But we know $b \le c$, so we have $b - \zeta \le c - \zeta$, or $|\zeta - b| \le |\zeta - c|$.
Case (2) $\zeta \ge b$, then $|\zeta - b| = \zeta - b \ge 0$. But we know $a \le b$, so $\zeta - b \le \zeta - a$, or $|\zeta - b| \le |\zeta - a|$.□

Lemma (4.18) If ζ, is any real number, then $|\zeta - \theta_{K,N,L}(\delta)|$, is less than of at least one of $|\zeta - \xi'_{K,N,L}(\delta)|$, or

$|\zeta - \xi'_{K+1,N,L}(\delta)|$.

Proof By lemma (4.17) the proof is clear.□

We use the following note in section (4.6).

Note We showed in corollary (4.3) that if $|\theta - \xi'_{K,N,L}(\delta_0)| < \varepsilon$, then $|\theta - \xi'_{K+1,N,L}(\delta_0)| < \varepsilon$. Therefore by applying lemma (4.18) we have

$$|\theta - \theta_{K,N,L}(\delta_0)| < \varepsilon.\square$$

Discussion In this section we showed that the infimum of the objective function on a set of infinite sequences of Radon measures satisfying to an infinite number of constraints can be approximated by the infimum of the objective function on a set of finite sequences of discrete Radon measures satisfying a finite number of constraints.

In section 4.6 we intend to compute the approximation of the infimum of the objective function by using the results of this section.

4.6 Computation of the infimum of the objective function and Control Functions

In this section we apply the results of section (4.5), and approximate the problem by one which consists of the minimization of a real linear function defined on R^k, for some positive integer k, over a finite set of linear constraints. Then we construct the control functions by the help of the paper of RUBIO and WILSON [1]. Finally, we show that the theory is confirmed, by solving numerically several problems with different final states. In section (4.5) we showed that for any $\varepsilon > 0$ and for arbitrary but fixed positive integer L, there exist non-negative, sufficiently large integers K and N and a real number δ_0, $(0 < \delta_0 < \varepsilon/5)$, such that

$$|\theta - \theta_{K,N,L}(\delta_0)| < \varepsilon.$$

This fact asserts that $\theta_{K,N,L}(\delta_0)$ tends to $\theta \equiv \inf_{S_g} \sum_{k=0}^{\infty} \|\mu_k\|$ when K and N are sufficiently large, $\theta_{K,N,L}(\delta_0) \equiv \inf_{P_K(\delta_0)} \sum_{K=0}^{\infty} \|\nu_k\|$, and $P_K(\delta) \equiv P(K,N,L,\delta_0)$.

We know from previous section that our problem is to calculate

$$\theta_{K,N,L}(\varepsilon) = \inf_{P_K(\varepsilon)} \sum_{K=0}^{\infty} \|\nu_k\|;$$

in other words our problem is to minimize

$$\sum_{K=0}^{\infty} \|\nu_k\|$$

on the set $P(K,N,L,\varepsilon)$, which we defined in section (4.5), as the set of all $(\nu_0,\nu_1,\ldots,\nu_{KL},\bar{0})$ such that each of ν_k, $k = 0,1,\ldots,KL$, is a discrete measure with support in the set $[0,T]$ and

$$\left| \sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \nu_k(\phi_n) - \alpha_n \right| < \varepsilon, \quad n = 0,1,\ldots,N,$$

where K, N, and L, are positive integers and $\varepsilon > 0$.

Proposition (4.2) Let Γ be a countable dense subset of $[0,T]$, and let $\varepsilon > 0$. Further, let μ be a Radon measure on $[0,T]$. Then there exists a discrete measure ν whose support is in Γ and

$$|(\mu - \nu)\phi_n| < \varepsilon, \quad n = 0,1,\ldots,N.$$

Proof Since μ is a Radon measure on $[0,T]$, then by a theorem of approximation (see CHOQUET [1] p.221) there exists a discrete measure $\mu_1 = \sum_{p=1}^{L} \gamma_p \; \delta(z^p)$, where $z^p \in [0,T]$ and L is a positive integer, such that

$$|(\mu - \mu_1)\phi_n| < \varepsilon/2, \quad n = 0,1,\ldots,N. \qquad (4.103)$$

Let now $\Gamma = \{t_p\}$, be a countable dense subset of $[0,T]$ and let $\mathcal{M}'$, be the space of discrete measures on Γ. We show there exists a discrete measure ν in $\mathcal{M}'$ such that

$$|(\mu_1 - \nu)\phi_n| < \varepsilon/2, \quad n = 0,1,\ldots,N.$$

Also let $\nu = \sum_{p=1}^{L} \gamma_p \; \delta(t_p)$, where $t_p \in \Gamma$, and γ_p and L were defined above. Then we have

$$(\mu_1 - \nu)\phi_n = \mu_1(\phi_n) - \nu(\phi_n) = \left[\sum_{p=1}^{L} \gamma_p\ \delta(z^p)\right]\phi_n - \left[\sum_{p=1}^{L} \gamma_p\ \delta(z^p)\right]\phi_n$$

$$= \sum_{p=1}^{L} \gamma_p\ \phi_n(z^p) - \sum_{p=1}^{L} \gamma_p\ \phi_n(t_p) = \sum_{p=1}^{L} \gamma_p[\phi_n(z^p) - \phi_n(t_p)],$$

therefore,

$$|(\mu_1 - \nu)\phi_n| = |\sum_{p=1}^{L} \gamma_p[\phi_n(z^p) - \phi_n(t_p)]|$$

$$\le \sum_{p=1}^{L} |\gamma_p||\phi_n(z^p) - \phi_n(t_p)|$$

or

$$|(\mu_1 - \nu)\phi_n| \le (\sum_{p=1}^{L} |\gamma_p|) \operatorname*{Max}_{n,p} \left\{|\phi_n(z^p) - \phi_n(t_p)|\right\}, \tag{4.104}$$

for $n = 0,1,\ldots,N$. But $\|\mu_1\| = |\mu_1|(1) = \sum_{p=1}^{L} |\gamma_p|$, so from (4.104) we have

$$(\mu_1 - \nu)\phi_n| \le \|\mu_1\| \operatorname*{Max}_{n,p} \left\{|\phi_n(z^p) - \phi_n(t_p)|\right\}, \tag{4.105}$$

where $n = 1,2,\ldots,N$. By choosing t_p, $p = 1,2,\ldots,L$, sufficiently close to z_p, the $\operatorname*{Max}_{n,p}$, can be made less than $\epsilon/2(\|\mu_1\| + 1)$ [because for each $n = 1,2,\ldots,N$, $\phi_n(t) \equiv \exp[-n^2\pi^2(T - t)]$, $t \in [0,T]$, is continuous]; therefore we have

$$|(\mu_1 - \nu)\phi_n| \le \|\mu_1\|\ [\epsilon/2(\|\mu_1\| + 1)]$$

or

$$|(\mu_1 - \nu)\phi_n| < \varepsilon/2, \ n = 0,1,\dots,N \tag{4.106}$$

By (4.103) and (4.106) we have

$$|(\mu - \nu)\phi_n| = |(\mu - \mu_1 + \mu_1 - \nu)\phi_n| \le |(\mu - \mu_1)\phi_n| + |(\mu_1 - \nu)\phi_n|$$

$$< \varepsilon/2 + \varepsilon/2, \ n = 0,1,\dots,N$$

or

$$|(\mu - \nu)\phi_n| < \varepsilon, \ n = 0,1,\dots,N. \square$$

We remind the reader that the set $\Gamma = \{t_p;\ k = 1,2,\dots\}$ was chosen as being dense in $[0,T]$; in practice, we choose the set $\Gamma^M = \{t_k; k = 1,2,\dots,M\} \subset \Gamma$, which is constructed by dividing the interval $[0,T]$ into M equal subintervals $[t_k, t_{k+1}]$, $k = 1,2,\dots,M$.

In the following we apply proposition (2) in the special case when the Radon measure μ is a discrete measure on $[0,T]$.

Let us remember that our problem is to minimize

$$\sum_{k=0}^{KL} \|\nu_k\|$$

over $(\nu_0,\nu_1,\dots,\nu_{KL},\bar{0}) \in P(K,N,L,\varepsilon)$, where each of ν_k, $k = 0,1,\dots,Kl$, is a discrete measure with support in $[0,T]$, and where K, and L can be any positive integer; therefore KL can be any positive integer, so let K be an arbitrary positive integer. Thus our problem is to minimize

$$\sum_{k=0}^{K} \|\nu_k\|$$

over $(\nu_0, \nu_1, \ldots, \nu_{KL}, \bar{0}) \in P(K,N,\varepsilon) \equiv P(K,N,1,\varepsilon)$, which means

$$\left| \sum_{k=0}^{K} (-1)^k (n^2\pi^2)^k \nu_k(\phi_n) - \alpha_n \right| < \varepsilon, \quad n = 0,1,\ldots,N.$$

Now let $\nu_k^+ = \sum_{i=1}^{M} \beta_i^k\ \delta(t_i)$, and $\nu_k^- = \sum_{i=1}^{M} \gamma_i^k\ \delta(t_i)$, where $\beta_i^k \geq 0$ and $\gamma_i^k \geq 0$, for $i = 1,2,\ldots,M$, $k = 0,1,\ldots,K$, [as we introduced in lemma (4.14)]. Therefore we have for $k = 0,1,\ldots,K$

$$\nu_k^+(\phi_n) = \left[\sum_{i=1}^{M} \beta_i^k\ \delta(t_i) \right] \phi_n = \sum_{i=1}^{M} \beta_i^k\ \phi_n(t_i), \quad n = 0,1,\ldots,N,$$

$$\nu_k^-(\phi_n) = \left[\sum_{i=1}^{M} \gamma_i^k\ \delta(t_i) \right] \phi_n = \sum_{i=1}^{M} \gamma_i^k\ \phi_n(t_i), \quad n = 0,1,\ldots,N.$$

Also we have $\nu_k^+(\phi_0) \equiv \nu_k^+(1) = \sum_{i=1}^{M} \beta_i^k$, and $\nu_k^-(\phi_0) \equiv \nu_k^-(1) = \sum_{i=1}^{M} \gamma_i^k$, for $k = 0,1,\ldots,K$. Therefore we conclude

$$\sum_{k=0}^{K} \|\nu_k\| = \sum_{k=0}^{K} |\nu_k|(1) = \sum_{k=0}^{K} [\nu_k^+(1) + \nu_k^-(1)] = \sum_{k=0}^{K} \left(\sum_{i=0}^{K} \beta_i^k + \sum_{i=0}^{K} \gamma_i^k \right)$$

or

$$\sum_{k=0}^{K} \|\nu_k\| = \sum_{k=0}^{K} \sum_{i=0}^{K} \left[\beta_i^k + \gamma_i^k \right]. \tag{4.106}$$

Also we have

$$\sum_{k=0}^{K} (-1)^k (n^2\pi^2)^k \nu_k(\phi_n) = \sum_{k=0}^{K} (-1)^k (n^2\pi^2)^k [\nu_k^+(\phi_n) - \nu_k^-(\phi_n)]$$

$$= \sum_{k=0}^{K} (-1)^k (n^2\pi^2)^k \left[\sum_{i=0}^{K} \beta_i^k \phi_n(t_i) - \sum_{i=0}^{K} \gamma_i^k \phi_n(t_i) \right]$$

$$= \sum_{k=0}^{K} \sum_{i=1}^{M} (-1)^k (n^2\pi^2)^k \left[\beta_i^k - \gamma_i^k\right] \phi_n(t_i) \qquad (4.107)$$

Therefore from (4.106) and (4.107) we conclude that our problem is to minimize

$$\sum_{k=0}^{K} \sum_{i=1}^{M} \left[\beta_i^k + \gamma_i^k\right] \qquad (4.108)$$

on the subset of $R^{2M(K+1)}$, say, $S(K,N,M,\varepsilon)$, defined by

$\beta_i^k \geq 0$, and $\gamma_i^k \geq 0$, $i = 1,2,\ldots,M$, $k = 0,1,\ldots,K$, and

$$\left| \sum_{k=0}^{K} \sum_{i=1}^{M} (-1)^k (n^2\pi^2)^k \left[\beta_i^k - \gamma_i^k\right] \phi_n(t_i) - \alpha_n \right| < \varepsilon, \qquad (4.109)$$

for $n = 0,1,\ldots,N$.

Now we rename the variables β_i^k and γ_i^k, as follows:

$(\beta_1^0, \beta_1^1, \ldots, \beta_1^K) \leftrightarrow (x_1, x_2, \ldots, x_{K+1})$

$(\beta_2^0, \beta_2^1, \ldots, \beta_2^K) \leftrightarrow (x_{K+2}, x_{K+3}, \ldots, x_{2K+2})$

$\vdots$

$(\beta_i^0, \beta_i^1, \ldots, \beta_i^K) \leftrightarrow (x_{(i-1)K+i}, x_{(i-1)K+i+1}, \ldots, x_{i(K+1)})$

$\vdots$

$(\beta_M^0, \beta_M^1, \ldots, \beta_M^K) \leftrightarrow (x_{(M-1)K+M}, x_{(M-1)K+M+1}, \ldots, x_{M(K+1)})$

or in general $\beta_i^k = x_{(i-1)K+k+i}$, $i = 1,2,\ldots,M$, $k = 0,1,\ldots,K$.

Let for simplicity $J_{KM} = M(K+1)$ and we rename the variables γ_i^k, in a similar way to β_i^k above, we assume $\gamma_i^k = x_{J_{KM}+(i-1)K+k+i}$, i = 1,2,...,M, k = 0,1,...,K. Therefore from (4.108) we have

$$\sum_{k=0}^{K} \sum_{i=0}^{K} \left[\beta_i^k + \gamma_i^k\right] = \sum_{i=1}^{M(K+1)} x_i + \sum_{i=M(K+1)+1}^{M(K+1)} x_i = \sum_{i=1}^{2M(K+1)} x_i . \tag{4.110}$$

Also from (4.107) we have

$$\sum_{k=0}^{K} (-1)^k (n^2\pi^2)^k \nu_k(\phi_n) = \sum_{k=0}^{K} \sum_{i=1}^{M} (-1)^k (n^2\pi^2)^k \left[\beta_i^k - \gamma_i^k\right] \phi_n(t_i)$$

$$= \sum_{k=0}^{K} \sum_{i=1}^{M} (-1)^k (n^2\pi^2)^k \left[x_{(i-1)K+k+i} - x_{J_{KM}+(i-1)K+k+i}\right] \phi_n(t_i) \tag{4.111}$$

where $J_{KM} = M(K+1)$. Finally, from (4.108)-(4.111), our problem is to minimize

$$\sum_{i=1}^{2M(K+1)} x_i , \tag{4.112}$$

on the set $S(K,N,M,\varepsilon)$ in $R^{2M(K+1)}$, defined by

$x_i \geq 0$, i = 1,2,...,2M(K+1)

$$\left| \sum_{k=0}^{K} \sum_{i=1}^{M} (-1)^k (n^2\pi^2)^k \left[x_{(i-1)K+k+i} - x_{J_{KM}+(i-1)K+k+i}\right] \phi_n(t_i) - \alpha_n \right|$$

$$< \varepsilon, \quad n = 0,1,\ldots,N, \tag{4.113}$$

here the only unknowns are x_i, i = 1,2,...,2M(K+1), while the t_i, i = 1,2,...,M are fixed points of a partition of the interval [0,T]. The number of inequalities in this linear programming problem is 2(N+1).

The parameter ε appearing in the constraints (4.113), can be considered as the error present in numerical computations of the expressions involved in constraints (4.113), so we can choose the parameter ε as zero. Although we know the errors present in the numerical computations of the solution of the linear programming problem, will not satisfy exactly the constraint equations. Finally, from (4.112) and (4.113) and the above discussion, our problem is to minimize

$$\sum_{i=1}^{2M(K+1)} x_i \quad ,$$

on the set S(K,M,N) in $R^{2M(K+1)}$, defined by

$x_i \geq 0, \ i = 1,2,\dots,2M(K+1)$

$$\sum_{k=0}^{K} \sum_{i=1}^{M} (-1)^k (n^2\pi^2)^k \left[x_{(i-1)K+k+i} - x_{J_{KM}+(i-1)K+k+i}\right] \phi_n(t_i) = \alpha_n$$

$$n = 0,1,\dots,N,$$

where $J_{KM} = M(K+1)$. Supposing that this problem has been solved, we intend now to construct practically the sequence of control functions u_j^K, defined in RUBIO and WILSON [1]. They have proved that $u_j^K \to L_t^K$, in $D'(\omega)$ strongly, where $\omega = (-1,T+1)$, and L_t^K has been defined as follows, L_t^K: $C_c^\infty(\omega) \to R$, by

$$\langle L_t^K, \phi \rangle = \sum_{k=0}^{K} (-1)^k \int_0^t D^k \phi(\xi) \, d\mu_k(\xi),$$

for $\phi \in C_c^\infty(\omega)$, K a fixed positive integer and $t \in [0,T]$. In appendix (E.4), we have shown that

$$u_j^K(\tau) = \sum_{k=0}^{K} (-1)^k \int_0^T D^k \rho_{1/j}(\tau - \xi)\, d\mu_k(\xi), \quad 0 \le \tau \le T, \qquad (4.114')$$

where $\rho_{1/j}(x) = j^n \rho(jx)$, when

$$\rho(x) = \begin{cases} a \exp[-1/(1-|x|^2)], & \text{for } |x| < 1 \\ 0 & \text{for } |x| \ge 1 \end{cases}$$

and $a = \left[\int_{|x|} \exp\,[-1/(1 - |x|^2)]\, dx\right]^{-1}$.

Let now $\mu_k = \sum_{i=0}^{M-1} \zeta_i^k(t_i)$, $k = 0,1,\ldots,K$, where $(\mu_0,\mu_1, \ldots,\mu_K,\bar{0})$, is the optimal measure found in our problem. We have from (4.114')

$$u_j^K(\tau) = \sum_{k=0}^{K} (-1)^k \int_0^T D^k \rho_{1/j}(\tau - \xi)\; d\left[\sum_{i=0}^{M-1} \zeta_i^k.\delta(t_i)\right]$$

or

$$u_j^K(\tau) = \sum_{k=0}^{K} (-1)^k \sum_{i=0}^{M-1} \zeta_i^k \int_0^T D^k \rho_{1/j}(\tau - \xi) d\delta(t_i)$$

but by definition of unitary atomic measure for $t \in [0,T]$, $\int_0^T F d\delta(t) = F(t)$, thus

$$u_j^K(\tau) = \sum_{k=0}^{K} \sum_{i=0}^{M-1} (-1)^k \zeta_i^k\, D^k \rho_{1/j}(\tau - t_i), \qquad (4.115)$$

here $0 \le \tau \le T$ and $\{t_i; i = 0,1,\ldots,M\}$, is a partition of $[0,T]$.

Now let $\zeta_i^k = \beta_i^k - \gamma_i^k$, $i = 0,1,\ldots,M-1$, $k = 0,1,\ldots,K$. Therefore from (4.115) we have for $j = 1,2,\ldots$

$$u_j^K(\tau) = \sum_{k=0}^{K} \sum_{i=0}^{M-1} (-1)^k(\beta_i^k - \gamma_i^k)D^k \rho_{1/j}(\tau - t_i), \tag{4.116}$$

Again we use the old notations

$$\beta_i^k = x_{(i-1)K+k+i}, \; k = 0,1,\ldots,K, \quad i = 0,1,\ldots,M-1,$$

$$\gamma_i^k = x_{J_{KM}+(i-1)K+k+i}, \; k = 0,1,\ldots,K, \quad i = 0,1,\ldots,M-1,$$

where $J_{KM} = M(K + 1)$. Now we deduce from (4.116)

$$u_j^K(\tau) = \sum_{k=0}^{K} \sum_{i=0}^{M-1} (-1)^k\Big[x_{(i-1)K+k+i} - x_{J_{KM}+(i-1)K+k+i}\Big].$$

$$.D^k \rho_{1/j}(\tau - t_i), \; 0 \le \tau \le 1.$$

In the following we consider some examples with different final states.

Example(4.1) Let the final state be $g_1(x) = 0.01 + 0.1\cos(\pi x)$. We choose $K = 2$, $M = 10$ and let $T = 1$, therefore $J_{KM} = 30$. Thus by (4.114) our problem is, minimize

$$\sum_{i=1}^{60} x_i$$

on a subset of R^{60}, defined by

$x_i \geq 0,\ i = 1,2,\dots,60$

$$\sum_{i=0}^{9} (x_{3i+1} - x_{3i+31}) = 0.01$$

$$\sum_{k=0}^{2} \sum_{i=0}^{9} (-1)^k \pi^{2k} \left[x_{3i+k+1} - x_{3i+k+31}\right] e^{-\pi^2[1-(i/10)]} = 0.1$$

and the results are:

Cost function = 0.01085

$x_1 = 0.01$, $x_6 = 0.00085$ and all other $x_i = 0$. Therefore we have

$$u_j^2(\tau) = 0.01\ \rho_{1/j}(\tau - 0.05) + 0.00085\ D^2\ \rho_{1/j}(\tau - 0.95), \quad (4.117)$$

for $0 \leq \tau \leq 1$.

Note For a given integer $N > 0$ and $0 \leq t \leq T$, we have

$$\int_0^t u_j^K(\xi)\ \phi_n(\xi,T)d\xi \to a_n(t,S_K),\ n = 0,\ 1,\ 2,\ \dots,\ N,$$

[see RUBIO and WILSON [1], (VIII.32)], where $\phi_n(\xi,t) = e^{-n^2\pi^2(t-\xi)}$, and

$$a_n(t;s_K) = \sum_{k=0}^{K} (-1)^k \int D^k\phi_n(\xi,t)d\mu_k(\xi).$$

where $s_K(\phi) = \sum_{k=0}^{K} (-1)^K\mu_k(D^k\phi)$, $\phi \in \mathfrak{F}$, and $0 \leq t \leq T$. Finally we conclude from proposition (VIII.10) of the above reference that for given any $\varepsilon > 0$

$$|\alpha_n - a_n(T,s)| < \varepsilon, \ n = 0,\ 1,\ 2,\ \ldots,\ N.$$

We may conclude from the above inequalities, that $a_n(T,s_K)$, is an approximation for α_n, $n = 0, 1, \ldots, N$.

By using note (1) of example (1) and (4.117) we have

$$a_0(T,s_2) = a_0(1,s) = \int_0^1 u_j^2(t)dt = 0.01 \int_0^1 \rho_{1/j}(t - 0.05)dt$$

$$- 0.00085 \int D^2\rho_{1/j}(t - 0.95)dt \qquad (4.118)$$

$$a_1(T,s_2) = +0.01 \int_0^1 e^{-\pi^2(1-t)}\rho_{1/j}(t - 0.05)dt$$

$$+ 0.0017 \int_0^1 e^{-\pi^2(1-t)} D^2\rho_{1/j}(t - 0.95)dt. \qquad (4.119)$$

Note Lemma (15.2) of TREVES [1] claims that:

Let f be a continuous function with support in R^n, then for any $\varepsilon > 0$, the functions

$$f_\varepsilon(x) = \int_{R^n} \rho_\varepsilon(x - y)f(y)dy$$

converge uniformly to f, when $\varepsilon \to 0$. Also

$$(\partial/\partial x)^p f_\varepsilon = \int D^p \rho_\varepsilon(x - y)f(y)dy \to (\partial/\partial x)^p f \ (\text{as } \varepsilon \to 0).$$

Now by using Note (2) in (4.118) and (4.119) we have

$$a_0(1,s_2) = 0.01,\ a_1(1,s_2) = 0.10073$$

Therefore we reach the state $G_1(x) = a_0(1,s_2) + a_1(1,s_2)\cos(\pi x)$

$= 0.01 + 0.10073\cos(\pi x)$, by imposing the computed controls. Fig (4.1) shows the desired final state $g_1(x)$ and $G_1(x)$ computed one.

Example (4.2) Let

$$g_2(x) = \begin{cases} 2x & 0 \le x \le 1/2 \\ 2(1-x) & 1/2 \le x \le 1 \end{cases}$$

The half range Fourier expansion of $g_2(.)$ is

$$g_2(x) = 1/2 - (4/\pi^2)\cos(2\pi x) - (4/9\pi^2)\cos(6\pi x)$$

$$- (4/25\pi^2)\cos(10\pi x) - \ldots .$$

Put in (4.112) and (4.113) $T = 1$, $M = 10$, $K = 6$ and $N = 6$ thus $J_{KM} = M(K+1) = 70$. So our problem is to minimize

$$\sum_{i=1}^{140} x_i$$

over a subset of R^{140} defined by

$$x_i \ge 0, \text{ for } i = 1,2, \ldots ,140$$

$$\sum_{k=0}^{6} \sum_{i=1}^{10} (-1)^k (n^2\pi^2)^k (x_{7i+k-6} - x_{7i+k+64}) e^{-n^2\pi^2(1-t_i)} = \alpha_n,$$

for $n = 0,1, \ldots,6$ and $t_i = (2i-1)/20$ where $\alpha_0 = 1/2$, $\alpha_2 = -4/\pi^2$, $\alpha_6 = -4/9\pi^2$ and $\alpha_1 = \alpha_3 = \alpha_4 = \alpha_5 = 0$. The result of the computation is:

cost function = 0.5081

$x_{49} = \beta_6^6 = 0.2675E-02$ $\qquad$ $x_{138} = \gamma_9^4 = 0.2874E-03$

$x_{62} = \beta_8^5 = 0.4987E-02$ $\qquad$ $x_{139} = \gamma_9^5 = 0.2881E-05$

$x_{63} = \beta_8^6 = 0.1350E-03$ $\qquad$ $x_{140} = \gamma_9^6 = 0.6995E-08$

$x_{64} = \beta_9^0 = 0.5000E+00$

and all other $x_i = 0$. Therefore from (4.116) we conclude

$$u_j^6(\tau) = \beta_6^6 D^6\rho_{1/j}(\tau - 0.65) - \beta_8^5 D^5\rho_{1/j}(\tau - 0.85) + \beta_8^6 D^6 \rho_{1/j}(\tau - 0.85) + \beta_9^0 \rho_{1/j}(\tau - 0.95) - \gamma_9^4 D^4 \rho_{1/j}(\tau - 0.95) + \gamma_9^5 D^5\rho_{1/j}(\tau - 0.95) - \gamma_9^6 D^6\rho_{1/j}(\tau - 0.95).$$

Finally, with the same notation as in note in example (4.1) the final state produced by imposing the above controls when $j \to \infty$, is the following

$$G_2(x) = \sum_{n=0}^{6} a_n(1,s_6) = 0.5 + 0.0087\cos(\pi x) - 0.445\cos(2\pi x) - 0.00685\cos(3\pi x) - 0.00019\cos(4\pi x) + 0.00977\cos(5\pi x) - 0.0226\cos(6\pi x).$$

Fig (4.2) shows the desired final state $g_2(.)$ and the state $G_2(.)$, which we obtained.

Discussion Although the state $G_2(.)$ is close to the state $g_2(.)$ a better state could be achieved by introducing more than seven constraints, and also by choosing more measures; that is, by increasing K and N. When we choose N > 6, however, there is a difficulty, because the coefficients of the linear constraints are of the following form

$$(-1)^k(n^2\pi^2)^k e^{-(n^2\pi^2)(1-t_i)}, \quad n = 0,1,\ldots, \text{ and } k = 0,1,\ldots,$$

where $t_i = (2i-1)/20$, $i = 1,2,\ldots,10$.

It is apparent that when n is chosen large, then the absolute value of the above functions are very small, even less than 10^{-78}, which is the limitation imposed by the word -size of the computer; all of such numbers are treated as zero. This causes ill-conditioning (see ROMAN [1] p.448) in the matrix that we use in the *revised simplex method* (see Gass [1] p.96). Although we reach the final solution, the accuracy is poor. So it seems that it is an open problem to find a way to overcome this difficulty despite the limitation of the computer.

Example (4.3) Let

$$g_3(x) = \begin{cases} 0 & 0 < t < 1/2 \\ 1 & 1/2 < t < 1 \end{cases},$$

then its half-range Fourier expansion of the function $g_3(x)$, is the following

$$g_3(x) = 1/2 - (2/\pi)\cos(\pi x) + (2/3\pi)\cos(3\pi x)$$

$$- \ldots + [2(-1)^{k+1}/(2k+1)]\cos[(2k+1)\pi x] + \ldots$$

In this example we choose K = 6, N = 4, M = 10 so $J_{KN} = 70$ and our problem is to minimize

$$\sum_{i=1}^{140} x_i$$

$x_i \geq 0$, for $i = 1,2, \ldots, 140$

$$\sum_{k=0}^{6} \sum_{i=1}^{10} (-1)^k (n^2\pi^2)^k \, (x_{7i+k-6} - x_{7i+k+64}) e^{-n^2\pi^2(1-t_i)} = \alpha_n,$$

for $n = 0,1, \ldots,4$ and $t_i = (2i-1)/20$ where $\alpha_0 = 1/2$, $\alpha_1 = -2/\pi$, $\alpha_2 = 0$, $\alpha_3 = 2/3\pi$ and $\alpha_4 = 0$. The results of the computation are:

cost function = 0.5000

x_{56} = 0.2153E-06 x_{64} = 0.5000E+00

x_{69} = 0.3922E-08 x_{133} = 0.7531E-08.

As in example (4.2), the computed final state is the following

$$G_3(x) = 1/2 - 0.61948\cos(\pi x) - 0.090\cos(2\pi x) + 0.222\cos(3\pi x)$$

$$+ 0.002\cos(4\pi x).$$

Discussion In this example as in example (4.2), the value of N cannot be taken to be very large; N can not exceed 4, because of ill-conditioning for N > 4. Fig(4.3) shows the desired final state $g_3(x)$ and the computed one $G_3(x)$.

Appendix (A.4) The following mapping is one to one

$$s = \sum_{k=0}^{\infty} D^k \mu_k \leftrightarrow (\mu_0, \mu_1, \ldots, \mu_k, \ldots)$$

Proof Let σ be the following mapping

$$\sigma : S \rightarrow \prod_{i=1}^{\infty} N$$

where we defined S in section (4.2) and where $\mathcal{M}$ is the set of all Radon measures defined on [0,T]. Let now $\gamma = \sum_{k=0}^{\infty} D^k \mu_k \in S$. We define

$$\sigma(\gamma) = \sigma\left[\sum_{k=0}^{\infty} D^k \mu_k\right] = (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in \prod_{i=1}^{\infty} \mathcal{M}.$$

Assume $\gamma_1 = \sum_{k=0}^{\infty} D^k \mu_k$ and $\gamma_2 = \sum_{k=0}^{\infty} D^k \nu_k$ belong to S, then we have

$$\sigma(\gamma_1) = \sigma\left[\sum_{k=0}^{\infty} D^k \mu_k\right] = (\mu_0, \mu_1, \ldots, \mu_k, \ldots)$$

$$\sigma(\gamma_2) = \sigma\left[\sum_{k=0}^{\infty} D^k \nu_k\right] = (\nu_0, \nu_1, \ldots, \nu_k, \ldots).$$

Let $\sigma(\gamma_1) = \sigma(\gamma_2)$, then we have

$$(\mu_0, \mu_1, \ldots, \mu_k, \ldots) = (\nu_0, \nu_1, \ldots, \nu_k, \ldots)$$

we have $\mu_k = \nu_k$, for $k = 0, 1, \ldots$, (by definition of equality of two sequences) which shows $\gamma_1 = \gamma_2$ or σ is an injection.□

Appendix (B.4) The following space is a linear subspace of $\prod_{i=1}^{\infty} \mathcal{M}$

$$\mathcal{L} = \left\{ (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in \prod_{i=1}^{\infty} \mathcal{M} : \sum_{k=0}^{\infty} \|\mu_k\| < \infty \right\}.$$

Proof Let $s = (\mu_0, \mu_1, \ldots, \mu_k, \ldots)$ and $r = (\rho_0, \rho_1, \ldots, \rho_k, \ldots)$, be two element of $\mathcal{L}$, then by definition we have $\sum_{k=0}^{\infty} \|\mu_k\| < \infty$ and $\sum_{k=0}^{\infty} \|\rho_k\| < \infty$. Let α, $\beta \in R$. It is easy to show that $\alpha s + \beta r \in \mathcal{L}$, because, we have

$$\sum_{k=0}^{\infty} \|\alpha\mu_k + \beta\rho_k\| \le \alpha \sum_{k=0}^{\infty} \|\mu_k\| + \beta \sum_{k=0}^{\infty} \|\rho_k\|,$$

since $\|\alpha\mu_k + \beta\rho_k\| \le \alpha\|\mu_k\| + \beta\|\rho_k\|$. Therefore we conclude

$$\sum_{k=0}^{\infty} \|\alpha\mu_k + \beta\rho_k\| < \infty.$$

Appendix (C.4) The function $\|.\|_I : \mathcal{L} \to \mathbf{R}$ defined below is a norm on $\mathcal{L}$.

$$\|s\|_I = \sum_{k=0}^{\infty} \|\mu_k\|,$$

where $s = (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in \mathcal{L}$, and $\mathcal{L}$ was defined in appendix (B.4).

Proof (I) Let $s = (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in \mathcal{L}$ and let $\|s\|_I = 0$, or $\sum_{k=0}^{\infty} \|\mu_k\| = 0$, then $\|\mu_k\| = 0$, for all $k = 0, 1, \ldots$, so $\mu_k \equiv 0$, for $k = 0, 1, \ldots$, therefore $s = (0, 0, \ldots, 0, \ldots) \equiv 0$.

(II) Let $s = (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in \mathcal{L}$ and $r = (\nu_0, \nu_1, \ldots, \nu_k, \ldots) \in \mathcal{L}$, then we have

$$\|s + r\|_I = \|(\mu_0, \mu_1, \ldots, \mu_k, \ldots) + (\nu_0, \nu_1, \ldots, \nu_k, \ldots)\|_I =$$

$$= \|(\mu_0 + \nu_0, \mu_1 + \nu_1, \ldots, \mu_k + \nu_k, \ldots)\|_I = \sum_{k=0}^{\infty} \|\mu_k + \nu_k\|.$$

$$\le \sum_{k=0}^{\infty} (\|\mu_k\| + \|\nu_k\|) = \sum_{k=0}^{\infty} \|\mu_k\| + \sum_{k=0}^{\infty} \|\nu_k\| = \|s\|_I + \|r\|_I$$

or $\|s + r\|_I \le \|s\|_I + \|r\|_I$.

(III) Let λ be a real number, then

$$\|\lambda s\|_1 = \|(\lambda\mu_0, \lambda\mu_1, \ldots, \lambda\mu_k, \ldots)\|_1 = \sum_{k=0}^{\infty} \|\lambda\mu_k\| = \sum_{k=0}^{\infty} |\lambda|\|\mu_k\|$$

$$= |\lambda| \sum_{k=0}^{\infty} \|\mu_k\|$$

therefore $\|\lambda s\|_1 = |\lambda|\|s\|_1$.□

Appendix (D.4) The sets $Q(K,N,L,\varepsilon)$, $Q(k,N,L)$, $Q(N)$, $Q'(K,N,L,\varepsilon)$ and $P(K,N,L,\varepsilon)$ defined below, are non-empty; here K, N and L are positive integers.

(1) The set $Q(k,N,L,\varepsilon)$, is the set of all $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$, such that

(a) $\left| \sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) - \alpha_n \right| < \varepsilon$, for $n = 0, 1, \ldots, N$

(b) $\|\mu_k\|^{1/k} < 1/K$, for $(K - 1)L < k \le KL$.

(2) The set $Q(K,N,L)$, is the set of all $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$, such that

(c) $\sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) = \alpha_n$, for $n = 0, 1, \ldots, N$

(d) $\|\mu_k\|^{1/k} < 1/K$, for $(K - 1)L < k \le KL$.

(3) The set $Q(N)$, is the set of all $(\mu_0, \mu_1, \ldots, \mu_k, \ldots)$, such that

(e) $| \sum_{k=0}^{\infty} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) = \alpha_n$, for $n = 0, 1, \ldots$

(f) $\lim \|\mu_k\|^{1/k} = 0$.

(4) The set $Q'(K,N,L,\varepsilon)$, is a subset of $Q(K,N,L,\varepsilon)$, such that

each of the measures μ_k, is a discrete measure.

(5) The set $P(K,N,L,\varepsilon)$, is the set of all $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$, such that each of the measures μ_k is a discrete measure and

$$\text{(g)} \quad \left| \sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) - \alpha_n \right| < \varepsilon, \text{ for } n = 0, 1, \ldots, N.$$

Now we intend to define a set which is a subset of all the sets defined above. Let $P(K,N,L)$, be the set of all $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$, such that

$$\text{(h)} \quad \sum_{k=0}^{KL} (-1)^k (n^2\pi^2)^k \mu_k(\phi_n) = \alpha_n, \text{ for } n = 0, 1, \ldots, N,$$

where each of the measures μ_k, is a discrete measure.
It is obvious from the definitions that

$$P(K,N,L) \subset P(K,N,L,\varepsilon) \subset Q'(K,N,L,\varepsilon) \subset Q(K,N,L,\varepsilon), \qquad \text{(D.4.1)}$$

$$P(K,N,L) \subset Q(K+1,N,L), \qquad \text{(D.4.2)}$$

$$P(K,N,L) \subset Q(N). \qquad \text{(D.4.3)}$$

From (D.4.1)-(D.4.3) we conclude it is sufficient to prove $P(K,N,L) \neq \phi$ for arbitrary positive integers N, L and nonnegative integer K. We intend to find an element of $P(K,N,L)$, of the form $(\mu_0, \bar{0})$; it is then sufficient to show

$$\mu_0(\phi_n) = \alpha_n, \; n = 0,1,\ldots,N.$$

Let $\mu_0 = \sum_{i=0}^{N} \zeta_i . \delta(t_i)$, where $t_i = i(T/M)$; then, there exist ζ_0, $\xi_1, \ldots, \xi_N$, such that

$$\mu_0(\phi_n) = \sum_{i=0}^{N} \zeta_i \phi_n(t_i) = \alpha_n, \; n = 0, 1, \ldots, N.$$

Indeed,

$$\zeta_0\phi_0(t_0) + \zeta_1\phi_0(t_1) + \ldots + \zeta_N\phi_0(t_N) = \alpha_0$$

$$\zeta_0\phi_1(t_0) + \zeta_1\phi_1(t_1) + \ldots + \zeta_N\phi_1(t_N) = \alpha_1$$

$$\vdots$$

$$\zeta_0\phi_N(t_0) + \zeta_1\phi_N(t_1) + \ldots + \zeta_N\phi_N(t_N) = \alpha_N;$$

the contention follows since the determinant of the coefficients of the above system is non-zero; the proof is similar to that in the proof of lemma (4.7).□

Appendix (E.4) In this section we remind the reader of some aspects of the sequence of control functions defined in proposition (VIII) of RUBIO and WILSON [1].

Let $\omega = (-1, T+1)$, and let $C_c^\infty(\omega)$ be the space of infinitely differentiable functions with compact support in ω, with the LF-topology. For any $\phi \in C_c^\infty(\omega)$ and a fixed integer $K > 0$, they defined the functional $L_t^K: C_c^\infty(\omega) \to R$, by

$$\langle L_t^K, \phi \rangle = \sum_{k=0}^{K} (-1)^k \int_0^t D^k \phi(\xi) d\mu_k(\xi)$$

for any $t \in [0, T]$. In the above reference it has been shown that

L_t^K is continuous, that is, in $D'(\omega)$, so by theorem 24.2 of TREVES [1] the support of L_t^K is compact, and from TREVES [1] page 302, there is a sequence $\{u_j^K\}_j$, of functions in $L_2(\omega)$ such that

$$u_j^K = \rho_{1/j} * L_t^K, \text{ for } 1/j < d(\text{supp } T, C\omega),$$

and u_j^K converges to L_t^K where $C\omega = (-\infty, 1] \cup [T+2, \infty)$, and $\rho_\varepsilon(.)$ is defined as follows: $\rho_\varepsilon(x) = \varepsilon^{-n} \rho(x/\varepsilon)$, for $\varepsilon > 0$, where

$$\rho(x) = \begin{cases} a \exp[-1/(1-|x|^2)], & \text{for } |x| < 1 \\ 0 & \text{for } |x| \geq 1 \end{cases}$$

and $a = \left[\int_{|x|} \exp\ [-1/(1 - |x|^2)]\ dx\right]^{-1}$. From page 288 of TREVES [1], we have

$$u_j^K(\tau) = (L_t^K * \rho_{1/j})(\tau) = \langle L_\xi^K, \rho_{1/j}(\tau-\xi)\rangle$$

$$= \sum_{k=0}^{K} (-1)^k \int_0^t D^k \rho_{1/j}(\tau-\xi) d\mu_k(\xi), \quad 0 \leq \tau \leq T$$

Therefore $u_j^K(\tau) = \sum_{k=0}^{K} (-1)^k \int_0^t D^k \rho_{1/j}(\tau-\xi) d\mu_k(\xi)$, $0 \leq \tau \leq T$.

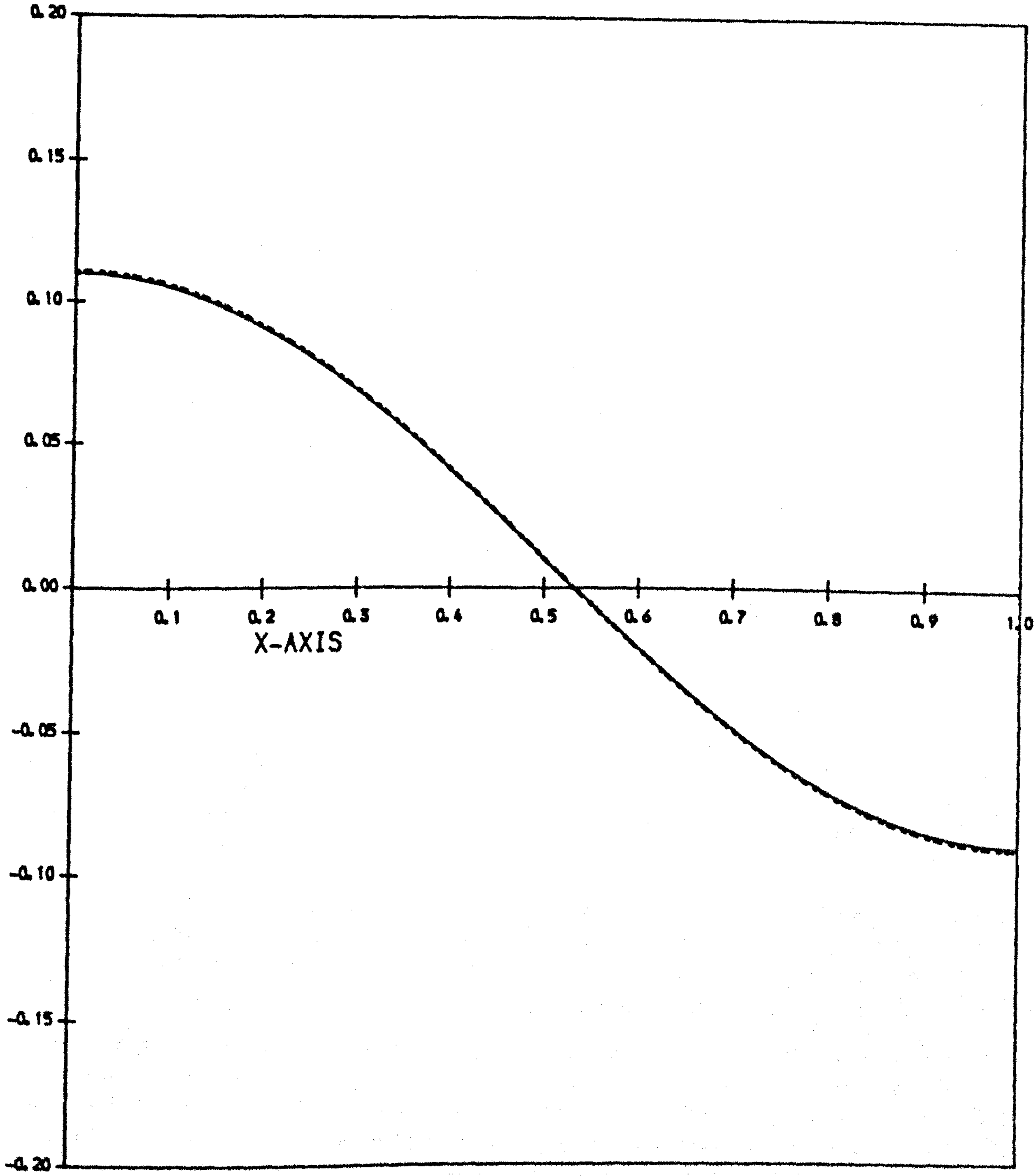


FIG (4. 1) -FINAL STATE FOR EXAMPLE (4. 1)

SOLID-DESIRED FINAL STATE FOR EXAMPLE (4. 1)

BROKEN-COMPUTED FINAL STATE FOR EXAMPLE (4. 1)

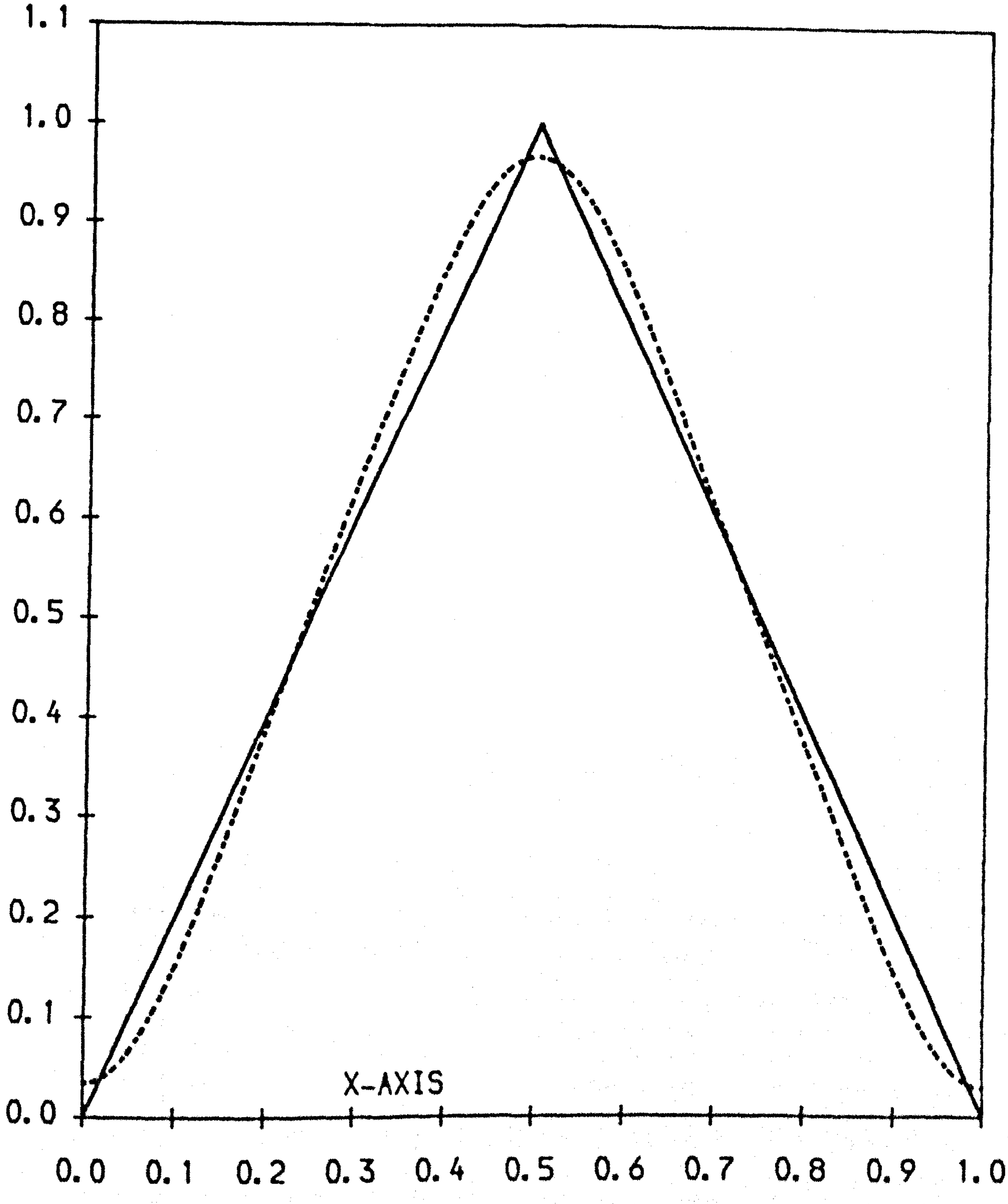


FIG (4.2) -FINAL STATE FOR EXAMPLE (4.2)

SOLID-DESIRED FINAL STATE FOR EXAMPLE (4.2)

BROKEN-COMPUTED FINAL STATE FOR EXAMPLE (4.2)

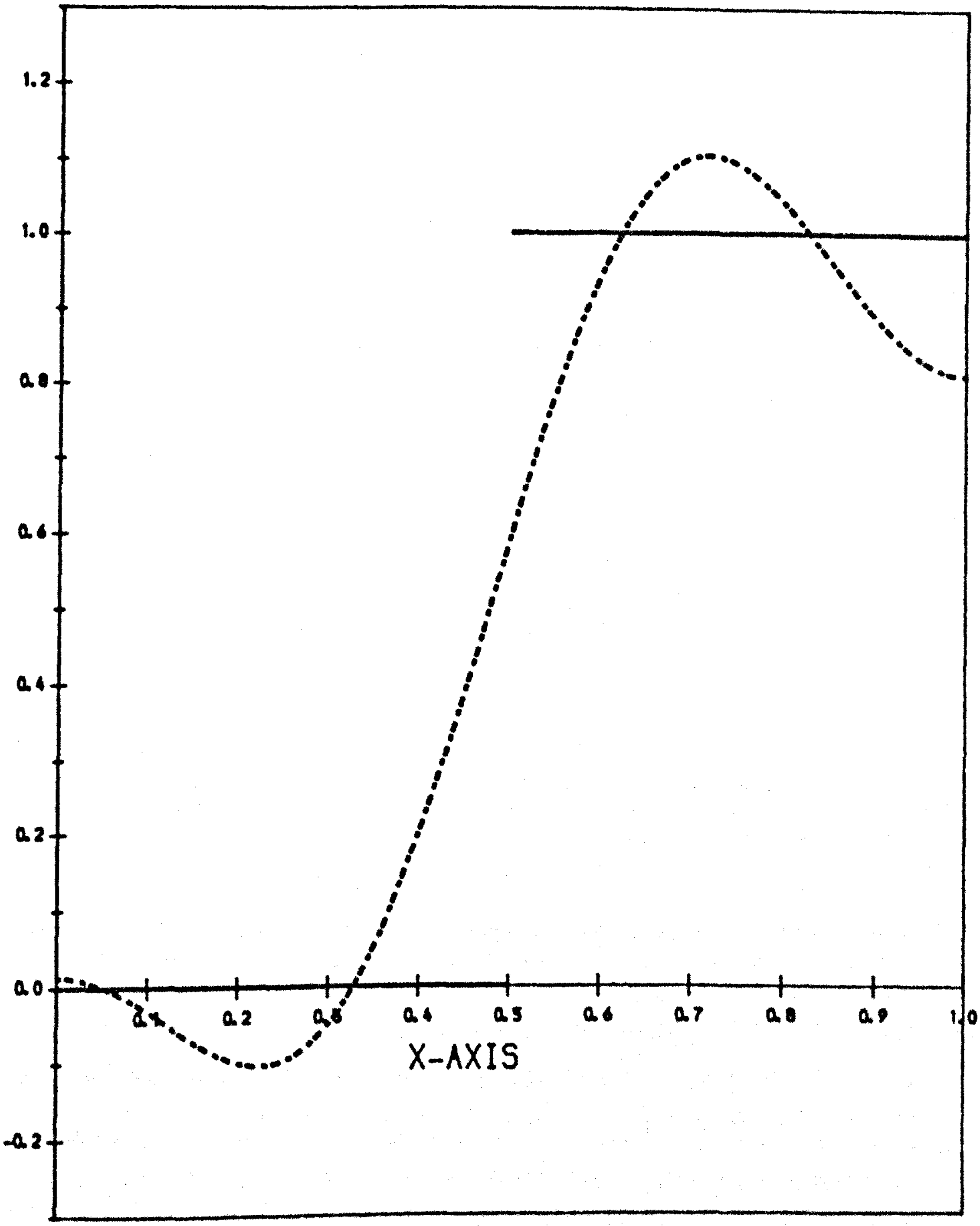


FIG (4.3) -FINAL STATE FOR EXAMPLE (4.3)

SOLID-DESIRED FINAL STATE FOR EXAMPLE (4.3)

BROKEN-COMPUTED FINAL STATE FOR EXAMPLE (4.3)

CHAPTER 5

Strong Controllability of the Diffusion Equation in n-Dimensions.

5.1 Introduction

This chapter contains an extension of the paper of RUBIO and WILSON [1] to n-dimensions.

Consider first the optimal control of the diffusion equation in n-dimensions which we discussed in chapter 2. That is, consider

$$\Delta Y(x,t) = \dot{Y}(x,t), \quad (x,t) \in \omega \times [0,T] \qquad (5.1)$$

with the same boundary conditions as in the beginning of chapter (2); we want to minimize a functional such as

$$J(u) = \int_0^T \int_{\partial\omega} f^0(t,\xi,u(t,\xi))d\xi dt;$$

where the control is in the space $L_2(\partial\omega \times [0,T])$, and there are no constraints imposed on its magnitude. Consider a state $g(.) \in L_2(\omega)$ which is neither reachable by an admissible control nor by a measure; that is, assume that the set of measures Q defined in chapter 2 is empty. Therefore there can be no minimum on Q, and the problem has no solution.

There are many optimal control problems, even in one dimensional state spaces, without solution because the desired final state can not be reached by imposing an admissible control (see RUBIO and WILSON [1]). But MACCAMY, MIZEL and SEIDMAN [1]) proved that the set of states which can be reached by means of controls in $L_2(\partial\omega \times [0,T])$ is dense in $L_2(\omega)$. The above fact is not helpful if the set of measures Q is empty, but it suggests that we may arrange things so that every state in $L_2(\omega)$ is reachable from the origin (see RUBIO and WILSON [1]). In this chapter we extend the set of admissible controls, beyond the set of measures.

(5.2) Defining a set larger than the set of measures.

Let $\{a_k(x);\ k = 1,2,\dots\}$, be a sequence of eigen-functions corresponding to the sequence of eigen-values $\{\lambda_k;\ k = 1,2,\dots\}$ of the Laplacian operator Δ in ω (for more detail see section (4.2)). Let the expansion of $g(.) \in L_2(\omega)$ in terms of the eigen-functions be $g(x) = \sum_{k=1}^{\infty} c_k\, a_k(x)$. From chapter (2) the solution of (5.1) with the boundary conditions corresponding to a control $u(.,.) \in L_2(\partial\omega \times [0,T])$ satisfies the terminal condition $y(.,T) = g(.)$ in $L_2(\omega)$, if and only if

$$c_k = \int_{\partial\omega \times [0,T]} (\partial a_k(\xi)/\partial\nu)\ \exp(-\lambda_k t)\ u(\xi,\ T-t)d\xi dt. \qquad (5.2)$$

for $k = 1,2,\dots,$ where ν is the unit outward normal at ξ(undefined on the subset (assumed negligible) at which $\partial\omega$ is not smooth; see MACCAMY [1]). From (5.2) it is apparent that the problem of attaining a given state g(.) at time T can be achieved by considering the moment problem (5.2). From the results of RUSSELL and FATTORINI [1], it can be shown that there is a control $u(.,.) \in L_2(\partial\omega \times [0,T])$ satisfying (5.2) if there are two constants η_1 and η_2, such that for all $k = 1,2,\dots,$

$$|c_k| \le \eta_1 \exp[-(\eta_2 + \varepsilon)\omega_k], \qquad (5.3)$$

where η_1 is any positive number and η_2 satisfies the following inequalities: $\eta_2 > M_1$ and

$$\eta_2 > \tilde{M}\, e^{1/2}\, [2c^{1/2} + 2c^{-1/2}(1 + \log(3c)^{1/2}],$$

$c = \log 3/2$, $\tilde{M} > \hat{M}$ and $\hat{M}$ is a number such that $|F(z)| \le \exp(\hat{M}|z|^{1/\beta})$, for all complex z, where $F(z) = \prod_{k=1}^{\infty} (1 + z^2/\lambda_k^{\,2})$. More details for M_1, $\hat{M}$ and $\tilde{M}$ can be found in FATTORINI & RUSSELL p.287-291. Here $\{\lambda_k\}$ is the sequence of eigen values defined above, satisfying the following inequalities

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \lambda_{n+1} < \ldots;$$

also these eigenvalues have of course the property that $\lim_n \lambda_n = \infty$, where $\omega_k = \sqrt{\lambda_k}$.

It is possible, however, to think of many functions in $L_2(\omega)$ whose moments do not decrease with k as the condition (5.3) requires, for example the following function g(.) does not satisfy the condition (5.3).

$$g(x) \equiv \sum_{(m)} \sin(m_1x_1 + m_2x_2 + \ldots + m_nx_n)$$

where (m) under Σ, indicates the whole combination of n-tuple integers $m = (m_1, m_2, \ldots, m_n)$ and the n-tuple $(x_1, x_2, \ldots, x_n)$ belongs to $(0,2\pi) \times \ldots \times (0,2\pi)$ (see MIZOHATA [1] p.32).

In the following we intend to introduce a new set such that among its element we can find at least one element providing a solution to the moment problem.

We change the variable t to T-t in formula (2.2) of chapter (2), thus we have

$$c_k = \int_{\partial\omega \times [0,T]} [\partial a_k(\xi)/\partial\nu] \exp[-\lambda_k(T-t)]\, u(\xi,t)d\xi dt, \quad (5.4)$$

for k = 0,1,... . With the above change of variable the formula (2.3) becomes

$$c_n = \int_0^T \int_A b_n(\xi_1(s),\dots,\xi_n(s))\ e^{-\lambda_n \tau}\ u(\xi_1(s),\dots,\xi_n(s),T-\tau)\ J(s)dsdt \quad (5.5)$$

where $s = (s_1,s_2,\dots,s_{n-1})$ and $A = [0,1] \times \dots \times [0,1]$ and $\left[\frac{\partial(x_2,\dots,x_n)}{\partial(s_1,\dots,s_{n-1})}\right]^2$, is the Jacobian determinant, $J(s)$

$$= \sqrt{\left[\frac{\partial(x_2,\dots,x_n)}{\partial(s_1,\dots,s_{n-1})}\right]^2 + \dots + \left[\frac{\partial(x_1,\dots,x_{n-1})}{\partial(s_1,\dots,s_{n-1})}\right]^2}$$ and

$$\hat{b}_k(s) \equiv b_k(\xi_1(s),\dots,\xi_n(s)).J(s) \quad (5.5')$$

$$\phi_k(t,s) \equiv \hat{b}_k(s)\ e^{-\lambda_k(T-t)} \quad (5.5'')$$

for k = 1,2,.... Also let $\hat{u}(t,s) \equiv u(\xi_1(s),\dots,\xi_n(s),T-\tau)$.

We can now write (5.5) in the following form

$$c_k = \int_{[0,T] \times A} \phi_k(t,s)\ \hat{u}(t,s)dsdt,\ k = 1,2,\dots. \quad (5.6)$$

We are looking for a continuous linear functional in the space $L_2([0,T] \times A)$ such as to satisfy (5.6). This functional is defined by the control $\hat{u}(.,.)$. But we can say that the functions ϕ_k are elements of a subspace of $L_2([0,T] \times A)$, such as the space of infinitely differentiable functions with respect to the t variable on $[0,T] \times A$, where the derivatives satisfy the condition (5.7) below (we explain more about this space in (5.3)). Then we put an appropriate topology on that such that its dual contains the space $L_2([0,T] \times A)$, as well as other elements. If there is no control $\hat{u}(.,.) \in L_2([0,T] \times A)$ that satisfies (5.6), there is a chance that we find one or more elements in the dual space to provide a solution to the moment problem.

(5.3) Defining a new space

Let F be the space of real-valued functions on $[0,T] \times A$ infinitely differentiable with respect to the variable t, on $[0,T] \times A$ (that is, these functions have uniformly continuous derivatives on $(0,T) \times A$ of all orders with respect to the variable t) such that

$$\sup_{(t,s) \in [0,T] \times A} |D_1^j\phi(t,s)| = \sup_{(t,s) \in [0,T] \times A} |(\partial^j/\partial t^j)\phi(t,s)| \le c\, L^j \qquad (5.7)$$

for some constants c, L dependent in general on the function $\phi \in F$; here $D^{(j,0,\dots,0)}\phi(t,s) \equiv D_1^j\phi(t,s)$. We define a topology on F as follows. Let $L > 0$, and F_L be the space of infinitely differentiable functions with respect to t on $[0,T] \times A$ which satisfy the inequality (5.7) for this particular L, and for some c which may be depend on ϕ. Then we have

(i) $F_{L_1} \subset F_{L_2}$, when $L_1 < L_2$.

(ii) We define below a real valued function $\|.\|$ on F_L and we show in appendix (B.5) that it is a norm on F_L

$$\|\phi\|_L = \sup_{j,(t,s) \in [0,T] \times A} (1/L^j)|D_1^j\phi(t,s)|.$$

With the topology induced by the above norm, F_L is a Banach space. (see appendix B.5).

(iii) For $L_2 > L_1$, the topology induced by F_{L_2} on F_{L_1} is the same as the topology given on F_{L_1}, and the norms $\|.\|_{L_1}$ and $\|.\|_{L_2}$ are equivalent on F_{L_1}. Indeed for all $(t,s) \in [0,T] \times A$ and all j, and all $\phi \in F_{L_1}$, it is apparent that

$$(1/L_2^j)|D_1^j\phi(t,s)| \le (1/L_1^j)|D_1^j\phi(t,s)|.$$

Therefore we have $\|\phi\|_{L_2} \le \|\phi\|_{L_1}$. Thus, the injection from F_{L_1} into F_{L2} is continuous and then an isomorphism into; therefore the two topologies are equivalent.

(iv) The space F, defined above, is the union of all spaces F_L, that is:

$$F = \bigcup_{L>0} F_L.$$

Therefore we can put on F the LF structure generated by the space F_L. Let $\mathcal{L}$ be the dual of the space F, this space with LF topology, $\mathcal{L}$ is the new space in which we will find a solution to the problem of moments. In the following proposition an important linear functional is defined.

Proposition (5.1) The linear functional γ on F, defined by (5.8), is in $\mathcal{L}$, that is, it is continuous:

$$\gamma(\phi) = \sum_{k=0}^{\infty} (-1)^k \mu_k(D_1^k\phi), \tag{5.8}$$

for all $\phi \in F$; μ_k, k = 0,1,..., are Radon measures on $[0,T] \times A$ such that

$$\sum_{k=0}^{\infty} L^k \int_{[0,T] \times A} d|\mu_k| < \infty, \tag{5.9}$$

for all $L \geq 0$. If γ satisfies (5.8) for all $\phi \in F$, we write

$$\gamma = \sum_{k=0}^{\infty} D_1^k\mu_k. \tag{5.10}$$

Proof The proof is similar to that of proposition (VIII.4) in RUBIO and WILSON [1].□

According to this proposition, $\mathcal{M}(\Omega)$ and $L_2([0,T] \times A)$ are subsets of $\mathcal{L}$, in the latter case we identify a function $\hat{u} \in L_2([0,T\} \times A)$ with a Radon measure μ_u, so that for every $\phi \in F$, we define $\mu_u(\phi) = \int_{[0,T] \times A} \phi\, \hat{u}(t,s)dtds$.

In the following we are going to show that the functions in (5.5''), are in F. The eigenfunctions $a_k(x)$, k = 1,2,... are known to be of class C^∞ in the closure of ω (that is, these functions have uniformly continuous derivatives on ω of all orders) [see for example RUSSELL [1] p.200]. So $\hat{b}_k(s) = \partial a_k(x)/\partial\nu$ is continuous and bounded in the closure of ω; $\hat{b}_k(s) = b_k(\xi_1(s),\dots,\xi_n(s)).J(s)$, where we defined J(s) below (5.5), and $\partial\omega$ is assumed to be continuously differentiable. We conclude that J(s) is continuous on A = [0,T] X ... X [0,T], and then J(s) is bounded on A. Also $b_k(\xi_1(s),\ldots,\xi_n(s))$ are continuous on A. Let $X = (\xi_1,\ldots,\xi_n)$, then $b_k(\xi_1(s),\ldots,\xi_n(s)) = (b_k o X)(s)$; since X is continuous on A and b_k is continuous on $\partial\omega$ [$\partial\omega$ is the image of A under X], the composition of b_k and X is continuous on A. Therefore $\hat{b}_k(s) = b_k(\xi_1(s),\ldots,\xi_n(s))$ is continuous and thus bounded on A. Let d_k, k = 1,2,... be constants such that

$$|\hat{b}_k(s)| \le d_k, \; k = 1,2,\ldots, \quad s \in A, \tag{5.11}$$

then we have

$$D_1^j\phi_k(t,s) = D_1^j[\hat{b}_k(s)e^{-\lambda_k(T-t)}] = \hat{b}_k(s).\lambda_k^j.e^{-\lambda_k(T-t)}.$$

By using (5.11) we deduce from the above equalities

$$|D_1^j\phi_k(t,s)| \le \lambda_k{}^j|\hat{b}_k(s)| \le d_k\lambda_k{}^j. \tag{5.12}$$

Comparing (5.12) with (5.7), we conclude that we can take $c = d_k$, $L = \lambda_k$ for an arbitrary k = 1,2,... so $\phi_k(t,s) \in F$, for all k = 1,2,....

Notation: In (5.3) we substitute $\beta = \eta_2+\varepsilon$ (ε is any positive number), therefore the condition (5.3) becomes

$$|c_k| \le \eta_1 \exp(-\beta\omega_k) \tag{5.13}$$

where $\omega_k = \sqrt{\lambda_k}$, k = 1,2,....

Proposition (5.2) Let $g \in L_2(\omega)$, then the following set $\mathcal{L}_g$ is nonempty:

$$\mathcal{L}_g = \Big\{\gamma \in \mathcal{L}: \gamma(\phi_n) = c_n,\ n = 1,2,\ldots,\ \gamma, \text{ is of the form (5.8), the measures } \mu_k,\ k = 1,2,\ldots, \text{ associated with it are atomless}\Big\}.$$

Proof We divide the proof of this proposition in (a) and (b).

(a)† We obtain an uppper bound for the sequence $\{-\theta\lambda_k\}$. By definition $\omega_k \to \infty$ when $k \to \infty$, since $\lambda_k \to \infty$, as $k \to \infty$. Therefore there is a positive integer N such that for every $k > N$ $\omega_k \geq \beta$, (β was defined above), or $\omega_k\omega_k \geq \beta\omega_k$, thus $\lambda_k \geq \beta\omega_k$, for $k > N$. Now we specify a positive number $\theta \geq 1$, such that $\theta\lambda_k \geq \beta\omega_k$, for $k = 1,2,\ldots,N$. Let $\theta_1 = \sup\{\beta/\omega_k,\ k = 1,2,\ldots,N\}$, and let $\theta = \mathrm{Max}\{\theta_1, 1\}$; then we have $\theta\omega_k \geq \beta$, for $k = 1,2,\ldots,N$ or $\theta\lambda_k \geq \beta\omega_k$. We showed above that $\lambda_k \geq \beta\omega_k$ for $k > N$; therefore we conclude that $\theta\lambda_k \geq \beta\omega_k$ for all $k = 1,2,\ldots$; thus

$$-\theta\lambda_k \leq -\beta\omega_k,\ k = 1,2,\ldots \tag{5.14}$$

(b) We consider the following problem of moments

$$\int_{[0,T]\times A} \phi_k(t,s)\hat{u}(t,s)\,dt\,ds = c_k\, e^{-\theta\lambda_k},\ k = 1,2,\ldots\ . \tag{5.15}$$

The moments c_k of $g \in L_2(\omega)$, satisfy $|c_k| \leq M$, for some constant $M > 0$. We use the inequality in (5.14) so that we have

$$|c_k \exp(-\theta\lambda_k)| \leq M \exp(-\beta\omega_k),\ k = 1,2,\ldots\ ,$$

so condition (5.13) or (5.3) is satisfied. Therefore from the results of RUSSELL and FATTORINI [1], we see that there exists a control $\hat{u}(.,.)$ satisfying (5.15). Now we define the element γ by

† Let θ be a new number to be defined below.

$$\gamma(\phi) = \sum_{k=0}^{\infty} \int_{[0,T] \times A} D_1^k \phi(t,s) \frac{\theta^k}{k!} \hat{u}(t,s)dtds \qquad (5.15')$$

for $\phi \in F$. According to proposition (5.1), this function belongs to $\mathcal{L}$; indeed, we have chosen the measures μ_k, $k = 1,2,\dots$, as follows

$$\int_{[0,T] \times A} \phi d\mu_k = \int_{[0,T] \times A} \phi(t,s) \ (-1)^k \frac{\theta^k}{k!} \ \hat{u}(t,s)dtds, \ \phi \in F.$$

These measures satisfy the condition (5.9), since for any $L \geq 0$, we have

$$\sum_{k=0}^{\infty} L^k \int_{[0,T] \times A} d|\mu_k| = \sum_{k=0}^{\infty} L^k \frac{\theta^k}{k!} \int_{[0,T] \times A} |\hat{u}(t,s)|dtds$$

$$= \sum_{k=0}^{\infty} \frac{(L\theta)^k}{k!} \int_{[0,T] \times A} |\hat{u}(t,s)|dtds$$

$$= \exp(L\theta) \int_{[0,T] \times A} |\hat{u}(t,s)|dtds < \infty,$$

since $\hat{u}(.,.) \in L_2([0,T] \times A) \subset L_1([0,T] \times A)$. Therefore $\gamma \in \mathcal{L}$, is of the form (5.8), and the measures μ_k, $k = 0,1,\dots$, associated with it are atomless.

Now we can compute the moments $\gamma(\phi_n)$, since $\hat{u}(.,.)$ is a solution of the associated problem of moments, so we have

$$\gamma(\phi_n) = \sum_{k=0}^{\infty} \int_{[0,T] \times A} D_1^k \phi_n(t,s) \frac{\theta^k}{k!} \hat{u}(t,s)dtds$$

$$= \sum_{k=0}^{\infty} \int_{[0,T] \times A} \lambda_n^k . \phi_n(t,s) \frac{\theta^k}{k!} \hat{u}(t,s)dtds$$

$$= \sum_{k=0}^{\infty} \left[(\theta\lambda_n)^k/k!\right] \int_{[0,T] \times A} \phi_n(t,s) \frac{\theta^k}{k!} \hat{u}(t,s)dtds$$

$$= e^{\theta\lambda_n} . c_n e^{-\theta\lambda_n} = c_n, \; n = 1,2,\ldots .$$

Therefore, the functional γ defined by (5.15) is in $\mathcal{L}_g$, thus $\mathcal{L}_g$ is nonempty.□

We have solved in some sense the problem of moments associated with the diffusion equation. At this point we are going to define the action of the functional $\gamma \in \mathcal{L}$, not only at the final time T, but also at all time t where $t \in (0,T)$, then we approximate this action by a sequence of controls $\hat{u}(.,.)$ in $L_2([0,T] \times A)$, or equivalently by controls $u(.,.) \in L_2([0,T] \times \omega)$.

Let $\hat{u}(.,.) \in L_2([0,T] \times A)$, then the corresponding solution of (5.1) can be written as follows:

$$Y(x,t) = \sum_{k=1}^{\infty} c_k(t,\hat{u})a_k(x), \; (x,t) \in \omega \times (0,T),$$

where

$$c_k(t,\hat{u}) = \int_{[0,T] \times A} \phi_k(\xi,t,s)\hat{u}(t,s)dtds, \; k = 1,2,\ldots , \quad (5.16)$$

and $\phi_k(\xi,t,s) = \hat{b}_k(s)e^{-\lambda_k(t-\xi)}$. If $t = T$, $\xi = t$, we have $\phi_k(t,T,s) = \hat{b}_k(s) \; e^{-\lambda_k(T-t)} = \phi_k(t,s)$, ($\phi_k(t,s)$, defined in

(5.5'').

Let a functional $\gamma \in \mathcal{L}$ of the form (5.8) be defined in the following form

$$c_n(t,\gamma) = \sum_{k=0}^{\infty} (-1)^k \int_{[0,T] \times A} D_1^k \phi_n(\xi,t,s) d\mu_k(\xi,s), \qquad (5.17)$$

for n = 1,2,... .

Before we prove that the expressions for $c_n(.,\gamma)$, n = 1,2,... are well defined, we note that, if γ is a functional γ_u corresponding to a control $u \in L_2([0,T] \times \partial\omega)$, or corresponding to a control $\hat{u}(.,.) \in L_2([0,T] \times A)$, then the expressions (5.17) become identical with those in (5.16); therefore the expressions in (5.17) are true extensions of the previous ones. We prove now that the functions $c_n(.,\gamma)$ are well defined and we show some fundamental properties of these functions.

Proposition (5.3) The functions $c_n(.,\gamma)$, n = 1,2,... , defined in (5.17), are well defined and continuous on [0,T].

Proof The proof is similar to that of proposition (VIII.8) of RUBIO and WILSON [1].□

The proof of proposition (5.3) suggests an approximation scheme: it seems that we can approximate the action of the functional γ by the truncated functional γ_K, defined below, at every $t \in [0,T]$.

$$\gamma_K(\phi) = \sum_{k=0}^{K} (-1)^k \mu_k(D_1^k \phi), \quad \phi \in F.$$

Now we define

$$c_n(t,\gamma_k) = \sum_{k=0}^{K} (-1)^k \int_{[0,T] \times A} D_1^k \phi_n(\xi,t,s) d\mu_k(\xi,s)$$

$$\sum_{k=0}^{K} (-1)^k \lambda_n^k \int_{[0,T] \times A} \phi_n(\xi,t,s) d\mu_k(\xi,s). \qquad (5.18)$$

In the following we develop this scheme.

Define $\kappa = (-1,T+1)$ and $A' = \overbrace{(-1/2,3/2) \times \ldots \times (-1/2,3/2)}^{n-1}$. Let $C_c^\infty(\kappa \times A')$ be the space of infinitely differentiable functions with respect to first variable t, such that each of these partial derivatives is continuous on $\kappa \times A'$. Also assume that the supports of these functions are compact in $\kappa \times A'$. We put the LF-topology on $C_c^\infty(\kappa \times A')$. Now for every $\phi \in C_c^\infty(\kappa \times A')$ and a fixed integer $K > 0$, we define the functional L_t^K: $C_c^\infty(\kappa \times A') \to R$ by

$$\langle L_t^K, \phi\rangle = \sum_{k=0}^{K} (-1)^k \int_{[0,T] \times A} D_1^k \phi(\xi,t,s) d\mu_k(\xi,s) \qquad (5.19)$$

for any $t \in [0,T]$. Now we have the following proposition

Proposition (5.4) The functional L_t^K defined by (5.19) is continuous, that is, in $D'(\omega \times A')$. There exists a sequence of functions in $L_2(\omega \times A')$, $\{u_j^K\}_j$, such that

$$u_j^K \to L_t^K \text{ in } D'(\omega \times A') \text{ strongly} \qquad (5.20)$$

Proof Let $\tau \times \eta$ be a compact subset of $\kappa \times A'$, and let the support of ϕ be in $\tau \times \eta$. Then we have

$$|\langle L_t^K, \phi\rangle| \le K\left(\sup_{0 \le k \le K} \int_{[0,t] \times A} d|\mu_k| \right) \sup_{0 \le k \le K} \sup_{\tau \times \eta} |D_1^k \phi(t,s)|,$$

which implies that L_t^K is in $D'(\kappa \times A')$. For the definition of the sequence $\{u_j^K\}_j$, see appendix (C.5).□

In the following we are going to approximate the action of the family of functionals L_t^K, $0 \le t < T$, by the corresponding restriction of u_j^K on Ω_t, where $\Omega_t = (0,t) \times A'$, $0 \le t < T$. Then, we have

Proposition (5.5) Let u_{jt}^K be the restriction of u_j^K to Ω_t, for $j = 1,2,\ldots$. Then

$$u_{jt}^K \to L_t^K \text{ in } D'(\Omega) \text{ strongly as } j \to \infty. \qquad (5.21)$$

Proof (i) Let Λ_t^K be the restriction of L_T^K to Ω_t, that is, we assume that the test functions ϕ are elements of $C_c^\infty(\Omega_t)$ and define

$$\langle \Lambda_t^K, \phi \rangle = \sum_{k=0}^{K} (-1)^k \int_{[0,T] \times A} D_1^k \phi(\xi,t,s)\, d\mu_k(\xi,s), \quad \phi \in C_c^\infty(\Omega_t)$$

From here on the proof is similar to the proof of proposition (VIII.8) of RUBIO and WILSON.□

We want now to use this result in solving our problem of approximation. First we must extend the functions $(\xi,s) \to \phi_n(\xi,t,s)$ to $\kappa \times A'$. We have

Proposition (5.6) Let N be a positive integer, and let B′ be the set of all functions of the form

$$(\xi,s) \to \phi_n^*(\xi,t,s) = \hat{\phi}_n(\xi,t,s)\gamma'(\xi,s), \quad (\xi,s) \in \kappa \times A', \quad n$$

$= 1,2,\ldots,N.$

$t \in [0,T]$. The functions $\hat{\phi}_n$ is the extension of ϕ_n to $\kappa \times A'$ and $\gamma' \in C_c^{\infty}(\kappa \times A')$is such that $\gamma'(t,s) = 1$ on $[0,T] \times A'$. Then the set B' is bounded in $C_c^{\infty}(\kappa \times A')$.

Proof The Proof is similar to that of proposition (VIII.9) of RUBIO and WILSON [1].□

From propositions (5.5) and (5.6) we have

$$\sup_{t\in[0,T]} \sup_{1\leq n\leq N} |\langle u_{jt}^{K} - L_{t}^{K}, \phi_n(.,.,t)\rangle| \to 0$$

as $j \to \infty$. The above expression is equivalent to

$$\int_{[0,T] \times A} u_j^K(\xi,s)\phi_n(\xi,s,t)d\xi ds \to c_n(t,\gamma_K),\ n = 1,2,\ldots,N,$$

uniformly on $[0,T]$. We can therefore approximate uniformly the action of γ_K at every $t \in [0,T]$ on a finite, but arbitrary number of functions $(\xi,s) \to \phi_n(\xi,t,s)$. Finally we have the final approximation scheme.

Proposition (5.7) Given an integer $N > 0$ and any $\varepsilon > 0$, there exists a control $u(.,.) \in L_2((0,T) \times \omega)$ such that

$$\sup_{t\in[0,T]} \sup_{1\leq n\leq N} |c_n(t,u) - c_n(t,\gamma)| < \varepsilon.$$

In particular ,

$$|c_n - c_n(T,u)| < \varepsilon,\ n = 1,2,\ldots,N.$$

Proof The proof is similar to that of proposition (VIII.10) of RUBIO and WILSON [1].□

We could approximate the action of γ, in the sense that any finite number of functions $c_n(.,\gamma)$, $n = 1,2,\dots,N$, associated with the functional γ may be uniformly approximated on $[0,T]$ to within any accuracy by using a control function from $L_2((0,T) \times \omega)$.

Discussion In this chapter we have found that any state in $L_2(\omega)$ can be reached at time T. Thus the set of admissible controls is not empty. In the next chapter we consider an optimal control for the n-dimensional diffusion equation, which is based on this chapter.

Appendix (A.5)

The function $\|.\|_L : F_L \to R$ defined below is a norm on F_L

$$\|\phi\|_L = \sup_{j,(t,s) \in [0,T] \times A} (1/L^j)|D_1^j\phi(t,s)|, \quad \phi \in F_L$$

Proof (i) It is obvious that $\|\phi\|_L \geq 0$, for all $\phi \in F_L$.

(ii) If $\|\phi\|_L = 0$, then we have $(1/L^j)|D_1^j\phi(t,s)| = 0$, for all $j = 0,1,\ldots$, and $(t,s) \in [0,T] \times A$, so $D_1^j\phi(t,s) = 0$ for all $j = 0,1,\ldots$. Specifically for $j = 0$ we have $\phi(t,s) = 0$ for all $(t,s) \in [0,T] \times A$, this means $\phi \equiv 0$ on $[0,T] \times A$.

(iii) Let ϕ and ψ be two elements of F_L, first we show that $\phi + \psi \in F_L$. Since by definition there are two constants c_1 and c_2 such that $\sup_{(t,s) \in [0,T] \times A} |D_1^j\phi(t,s)| \leq c_1L^j$; $\sup_{(t,s) \in [0,T] \times A} |D_1^j\psi(t,s)| \leq c_2L^j$. therefore we have

$$\sup_{(t,s) \in [0,T] \times A} |D_1^j(\phi + \psi)(t,s)|$$

$$= \sup_{(t,s) \in [0,T] \times A} |D_1^j\phi(t,s) + D_1^j\psi(t,s)|$$

$$\leq \sup_{(t,s) \in [0,T] \times A} |D_1^j\phi(t,s)| + \sup_{(t,s) \in [0,T] \times A} |D_1^j\psi(t,s)|$$

$$\leq c_1L^j + c_2L^j \leq (c_1 + c_2)L^j \leq c_3L^j,$$

where $c_3 = c_1 + c_2$. Therefore from this last inequality we conclude $\phi + \psi \in F_L$.

It is easy to show that $\|\phi + \psi\|_L \leq \|\phi\|_L + \|\psi\|_L$

(iv) Let $\phi \in F_L$ and $\alpha \in R$, then $\alpha\phi \in F_L$. Since

$$\sup_{(t,s) \in [0,T] \times A} |D_1^j(\alpha\phi)(t,s)| = \sup_{(t,s) \in [0,T] \times A} |\alpha||D_1^j\phi(t,s)|$$

$$= |\alpha| \sup_{(t,s) \in [0,T] \times A} |D_1^j \phi(t,s)| \le |\alpha| c_1 L^j \le c_4 L^j,$$

where $c_4 = |\alpha| c_1$ so $\alpha\phi \in F_L$.

Now we have

$$\|\alpha\phi\|_L = \sup_{j,(t,s) \in [0,T] \times A} (1/L^j)|D_1^j(\alpha\phi)(t,s)|$$

$$= |\alpha| \sup_{j,(t,s) \in [0,T] \times A} (1/L^j)|D_1^j\phi(t,s)| = |\alpha| \|\phi\|_L . \square$$

Appendix (B.5)

Define on F_L the following norm

$$\|\phi\|_L = \sup_{j,(t,s) \in [0,T] \times A} (1/L^j)|D_1^j\phi(t,s)|, \quad \phi \in F_L.$$

With the topology induced by this norm, F_L is a Banach space. (i.e, it is complete).

Proof Let $\{\phi_n\}$ be a Cauchy sequence in F_L, then, by definition

$\lim_{m,n \to \infty} \|\phi_n - \phi_m\|_L = 0$, so for any $\varepsilon > 0$, there exists a positive integer N, such that for all n, m $\ge$ N, we have

$$\|\phi_n - \phi_m\|_L = \sup_{j,(t,s) \in [0,T] \times A} (1/L^j)|D_1^j[\phi_n(t,s) - \phi_m(t,s)| < \varepsilon$$

so for every j and every $(t,s) \in [0,T] \times A$, we have

$$(1/L^j)|D_1^j[\phi_n(t,s) - \phi_m(t,s)]| < \varepsilon, \quad n, m \ge N,$$

or

$$|D_1^j\phi_n(t,s) - \phi_m(t,s)| < \varepsilon L^j, \qquad (c.5.1)$$

for all n, m ≥ N and every $(t,s) \in [0,T] \times A$ and all $j \geq 0$. Let now $j = 0$ in (c.5.1), then we have

$$|\phi_n(t,s) - \phi_m(t,s)| < \varepsilon \qquad (c.5.2)$$

for all m, n ≥ N and $(t,s) \in [0,T] \times A$. Thus the sequence $\{\phi_n(t,s)\}$, is a Cauchy sequence in R for every $(t,s) \in [0,T] \times A$, since R is complete, there exists a function of (t,s) like $\phi(t,s)$ such that for every $(t,s) \in [0,T] \times A$ we have

$$\phi_n(t,s) \to \phi(t,s), \text{ as } n \to \infty.$$

Let n be fixed in (c.5.2) and $m \to \infty$, Then we have

$$|\phi_n(t,s) - \phi(t,s)| \leq \varepsilon, \text{ for every } n \geq N \text{ and } (t,s) \in [0,T] \times A,$$

which shows $\phi_n(t,s) \to \phi(t,s)$, *uniformly on* $[0,T] \times A$. Now we choose $j = 1$ in (c.5.1) so we have

$$|D_1^1\phi_n(t,s) - D_1^1\phi_m(t,s)| \leq \varepsilon L, \text{ for every } n,\ m \geq N$$

and $(t,s) \in [0,T] \times A$. This shows that the sequence $\{D_1^1\phi_n(t,s)\}$, converges *uniformly on* $[0,T] \times A$ (with similar proof given for $\{\phi_n(t,s)\}$). Then by theorem 7.17 in RUDIN [1], we have

$$D_1^1\phi_n(t,s) \to D_1^1\phi(t,s), \textit{ uniformly } \text{on } [0,T] \times A.$$

Using a similar proof we have for any $j \geq 0$

$$D_1^j\phi_n(t,s) \to D_1^j\phi(t,s), \textit{ uniformly} \text{ on } [0,T] \times A. \qquad \text{(c.5.3)}$$

Now let $\varepsilon = L^j$, (for an arbitrary but fixed $j \geq 0$). From (c.5.3), there exists a positive integer N_1 such that for all $n \geq N_1$ and all $(t,s) \in [0,T] \times A$

$$|D_1^j\phi_n(t,s) - D_1^j\phi_{N_1}(t,s)| < L^j,$$

so $|D_1^j\phi_n(t,s)| < |D_1^j\phi_{N_1}(t,s)| + L^j$, for all $(t,s) \in [0,T] \times A$,

or

$$(1/L^j)|D_1^j\phi_n(t,s)| < (1/L^j)|D_1^j\phi_{N_1}(t,s)| + 1$$

$$\leq \sup_{j,\,(t,s) \in [0,T] \times A} (1/L^j)|D_1^j\phi_{N_1}(t,s)| + 1 = \|\phi_{N_1}\|_L + 1.$$

So for every j and every $(t,s) \in [0,T] \times A$, from the above inequalities we have $|D_1^j\phi_n(t,s)| < L^j(\|\phi_{N_1}\|_L + 1)$. Let now $n \to \infty$ in the last inequality, so we have

$$\sup_{(t,s) \in [0,T] \times A} |D_1^j\phi(t,s)| < L^j(\|\phi_{N_1}\|_L + 1),$$

but $\|\phi_{N_1}\|_L + 1$, is a constant, say c, so we have from the above inequality

$$\sup_{(t,s) \in [0,T] \times A} |D_1^j \phi(t,s)| < cL^j$$

which shows $\phi(t,s) \in F_L$ □

Appendix (C.5)

In this appendix we intend to construct the sequence of control functions introduced in Proposition (5.4).

For constructing the above sequence we take two steps.

(1) We approximate an arbitrary distribution in $\Sigma = \kappa \times A' = (-1,T+1) \times (-1/2,3/2) \times \ldots \times (-1/2,3/2)$, by a sequence of distributions in Σ, which have compact supports.

(2) By using lemma 28.1 of TREVES [1], we approximate any distribution with compact support by a sequence of test functions.

We consider now the convolution $V*\phi$ of distribution V with a C_c^∞ function ϕ. We may regard ϕ as a distribution $[\psi \to \int\psi(x)\phi(x)dx]$ so $V*\phi$ is the following distribution

$$\psi \to \langle V*\phi,\ \eta\rangle = \langle V,\ \hat{\phi}*\psi\rangle$$

where $\psi \in C_c^\infty$ and $\hat{\phi}(x) = -\phi(x)$.

But $V*\phi$ is, in fact, a C^∞-function, precisely by definition 27.1 of TREVES [1]

$$x \to \langle V_y,\ \phi(x-y)\rangle,$$

where the right - hand side is called the convolution of ϕ and V and is denoted by $V*\phi$ or $\phi*V$. [When V is a locally integrable function f, we have $\langle V_y,\ \phi(x-y)\rangle = \int f(y)\phi(x-y)dy$].

Let V be a distribution in Σ, then from Theorem 28.1 of TREVES [1], there is a sequence of distributions with compact support, $\{V_k\}$ ($k = 0,1,\dots$), such that, given any relatively compact open subset [that is, its closure is compact] Σ' of Σ, there is an integer $k(\Sigma') \geq 0$ such that, for all $k \geq k(\Sigma')$, the restriction of V_k to Σ', $V_k|\Sigma'$, is equal to restriction of V to Σ', $V|\Sigma'$. Then from Theorem 28.2, TREVES [1], the test functions $\phi_k = \rho_{1/j_k} * V$ converge to V in $D'(\Sigma)$ as $k \to \infty$, where we select the integer j_k as follows: for each k we select $j_k \geq k$ (in order to ensure that $j \to \infty$), sufficiently large so that the neighborhood of order $1/j_k$ of supp V_k is a compact subset of Σ. We have defined the function ρ_ε in chapter (4) and V_k, $k = 0,1,\dots$ are distributions with compact suport.

By definition L_t^K is a distribution with compact support, so we can choose the sequence V_k, $k = 0,1,\dots$ as $V_k = L_t^K$ for all $k = 1,2,\dots$ and we define the test functions $\phi_k^K = \rho_{1/k} * L_t^K$ (see TREVES [1] p.302). By theorem 28.2 of the above reference, ϕ_k converges to L_t^K in $D'(\Sigma)$ [strongly], where $1/k < d([0,t] \times A, C(\tau \times A'))$; d is the distance between two sets and $C(\tau \times A')$ is the complement of $\tau \times A'$. Now we rename ϕ_k^K, $k = 1,2,\dots$ by u_k^K or

$$u_k^K = \rho_{1/k} * L_t^K$$

for all k such that $1/k < d([0,t] \times A, C(\tau \times A'))$.

Now we calculate the elements of the sequence $\{u_j^K\}$ $j = 1,2,\dots$

By definition

$$(\eta,\sigma) \to \langle L^K_{t,(\xi,s)}, \rho_{1/j}(\eta - \xi, \sigma - s)\rangle$$

is the convolution of $\rho_{1/j}$ and L_t^K, denoted by $L_t^K * \rho_{1/j}$ so we have

$$u_j^K(\eta,\sigma) = (L_t^K * \rho_{1/j})(\eta,\sigma)$$

$$= \sum_{k=0}^{K} (-1)^k \int_{[0,t] \times A} D_1^k \rho_{1/j}(\eta - \xi, \sigma - s) d\mu_k(\xi,s)$$

But $\rho_{1/j}(\eta - \xi, \sigma - s) = j^n \rho(j(\eta - \xi), j(\sigma - s))$ therefore

$$u_j^K(\eta,\sigma) = \sum_{k=0}^{K} (-1)^k j^n \int_{[0,t] \times A} D_1^k \rho(j(\eta - \xi), j(\sigma - s)) d\mu_k(\xi,s)$$

or by definition of the function ρ we have

$$u_j^K(\eta,\sigma) = \sum_{k=0}^{K} (-1)^k j^n \int_{[0,t] \times A}$$

$$D_1^k a \exp[-1/\{1 - [j^2(\eta - \xi)^2 + j^2|\sigma - s|^2]\} d\mu_k(\xi,s).$$

CHAPTER 6

Optimal control problem for the n-dimensional diffusion equation with a generalised control variable.

(6.1) Introduction

We consider an optimal control problem associated with the following n-dimensional diffusion equation

$$\Delta Y(x,t) = Y_t(x,t), \qquad (6.1)$$

where $(x,t) \in \omega \times [0,T]$, with boundary conditions

$$Y(x,t) = u(x,t), \ (x,t) \in \partial\omega \times [0,T];$$

$$Y(x,0) = 0, \ x \in \omega, \qquad (6.2)$$

$$u \to J(u)$$

is minimum (we will specify the function J(.) later).

Let $\{a_k(x)\}$, $k = 1,2,\ldots$ be the normalized eigenfunctions corresponding to eigenvalues $\{\lambda_k\}$, $k = 1,2,\ldots$ satisfying

$$1 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots,$$

defined by the problem

$$\Delta v(x) + \lambda v(x) = 0, \ x \in \omega; \ v(x) = 0, \ x \in \partial\omega.$$

In the following we give more details for the sequnce $\{\lambda_k\}$.

Remark There exists a sequence of eigenvalues $\{\Lambda_k\}$, $k = 1,2,\ldots$, where

$$1 < \Lambda_1 < \ldots < \Lambda_k < \Lambda_{k+1} < \ldots$$

and a sequence of eigenfunctions $\{\bar{a}_k(\xi)\}$, $\xi \in \omega$, corresponding to the above eigen-values where

$$\Delta\bar{a}_k(\xi) + \lambda_k\bar{a}_k(\xi) = 0,\ \xi \in \omega;\ \bar{a}_k(\xi) = 0,\ \xi \in \partial\omega.$$

Proof Let $\{\lambda_k\}$, be a sequence of the eigenvalues of the Laplacian operator Δ, in ω satisfying

$$0 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots, \tag{6.2'}$$

as we defined in chapter (2). If $\lambda_1 > 1$, we choose $\{\Lambda_k\} \equiv \{\lambda_k\}$ and the proof is finished. But if $0 < \lambda_1 \leq 1$, we choose the constant P, a fixed real number such that $P^2\lambda_1 > 1$. We define the set $\Omega \subset R^n$, as follows:

$$\Omega = \{P\xi:\ \xi \in \omega\}$$

Let $\{a_k(x)\}$, $k = 1,2,\ldots$, be the eigen-functions corresponding to the eigen-values $\{\lambda_k\}$, $k = 1,2,\ldots$, in Ω and let $x_i = P\xi_i$, $i = 1,2,\ldots, n$. Let define $a_k(x) \equiv \bar{a}_k(\xi)$, where $x = (x_1,\ldots,x_n)$, and $\xi = (\xi_1,\ldots,\xi_n)$. Thus we have

$$\partial^2 a_k(x) / \partial x_i^2 = (1/P^2)\partial^2\bar{a}_k(\xi) / \partial\xi_i^2.$$

Therefore

$$\Delta a_k(\xi) + \lambda_k a_k(\xi) = (1/P^2)\Delta\bar{a}_k(\xi) + \lambda_k\bar{a}_k(\xi) \qquad (6.2'')$$

By definition of $\{a_k(x)\}$, we have

$$\Delta a_k(\xi) + \lambda_k a_k(\xi) = 0,\ x \in \Omega;\ a_k(x) = 0,\ x \in \partial\Omega \qquad (6.2''')$$

But it is obvious that

$$x \in \Omega \Leftrightarrow \xi \in \omega$$

$$x \in \partial\omega \Leftrightarrow \xi \in \partial\omega$$

Thus from (6.2″) and (6.2‴), it is clear that

$$(1/P^2)\Delta\bar{a}_k(\xi) + \lambda_k\bar{a}_k(\xi) = 0,\ \xi \in \omega;\ \bar{a}_k(\xi) = 0,\ \xi \in \partial\omega;$$

or

$$\Delta\bar{a}_k(\xi) + P^2\lambda_k\bar{a}_k(\xi) = 0,\ \xi \in \omega;\ \bar{a}_k(\xi) = 0,\ \xi \in \partial\omega.$$

Let now $\Lambda_k = P^2\lambda_k$, $k = 1,2,\ldots$. But we defined P, such that $\Lambda_1 = P^2\lambda_1 > 1$, therefore from (6.2′) we conclude

$$1 < \Lambda_1 < \ldots < \Lambda_k < \Lambda_{k+1} < \ldots . \square$$

Note Henceforth, we denote the sequences $\{\Lambda_k\}$ and $\{\bar{a}_k(x)\}$, by $\{\lambda_k\}$ and $\{a_k(x)\}$ respectively.

Let $g(.) \in L_2(\omega)$ be the desired final state, and let

$$g(x) = \sum_{k=1}^{\infty} c_k a_k(x), \quad x \in \omega.$$

In chapter (2) we have shown that the solution of (6.1) with the boundary conditions corresponding to a control $u(.,.) \in L_2(\partial\omega \times [0,T])$ satisfying the terminal conditionn $Y(.,T) = g(.)$ in $L_2(\omega)$, satisfies

$$c_k = -\int_{\partial\omega \times [0,T]} [\partial a_k(\xi)/\partial\nu]\exp(-\lambda_k t)u)\xi,\ T-t)d\xi dt \qquad (6.3)$$

where $k = 1,2,\ldots$ and ν is the unit outward normal at ξ. It is apparent from (6.3) that the problem of attaining a given state $g(.)$ at time T can be studied by considering the moment problem (6.3). We have shown in chapter (5) that there is a control $u(.,.) \in L_2(\partial\omega \times [0,T])$ satisfying (6.3) if there are two non-negative constants η_1 and η_2 such that for all $k = 1,2,\ldots$

$$|c_k| \le \eta_1 \exp[-(\eta_2+\varepsilon)\omega_k], \qquad (6.4)$$

where η_1 is any positive constant, η_2 has been defined in detail in chapter (5), $\omega_k = \sqrt{\lambda_k}$, and ε is an arbitrary positive number.

There are, however, many functions in $L_2(\omega)$ whose moments do not decrease with n as rapidly as this condition requires, for example

$$f(x,y) = \sum_{n=1}^{\infty} (1/n^2)\cos[n\pi(x+y)].$$

In chapter (5) we defined a space F of real-valued functions ψ on $(0,T) \times [0,1] \times \ldots \times [0,1] \equiv (0,T) \times A$, infinitely differentiable with respect to the first variable $t \in (0,T)$, such that each of the partial derivatives $(\partial^j/\partial t^j)\psi(t,s)$, $j = 0,1,\ldots$ is continuous on $[0,T] \times A$, and

$$\sup_{(t,s) \in [0,T] \times A} |(\partial^j/\partial t^j)\psi(t,s)| \le CL^j, \tag{6.5}$$

for some constant C, L. We put the LF topology on F, and we considered $\mathcal{L}$, the dual of F; here we found a solution to the problem of moments (see chapter (5)).

Proposition (5.1) in chapter (5) shows that the linear functional γ on F defined in (6.6) is in $\mathcal{L}$

$$\gamma(\phi) = \sum_{k=0}^{\infty} (-1)^k \mu_k(D_1^k\phi), \tag{6.6}$$

for all $\phi \in F$, and μ_k, $k = 0,1,\ldots,$ are Radon measures on $[0,T] \times A$, such that

$$\sum_{k=0}^{\infty} L^k \int_{[0,T] \times A} d|\mu_k| < \infty, \tag{6.7}$$

for all $L \ge 0$. If γ satisfies (6.6) for all $\phi \in F$, we write it as

$$\gamma = \sum_{k=0}^{\infty} D_1^k \mu_k. \tag{6.8}$$

In Proposition (5.2) in chapter (5), we have shown the existence of a $\gamma \in \mathcal{L}$ such that $\gamma(\psi_n) = c_n$, $n = 1,2,\ldots$, where

$$\psi_n(t,s) = \hat{b}_n(s)e^{-\lambda_n(T-t)}, \quad (t,s) \in [0,T\} \times A,$$

and $\hat{b}_n(s)$ was defined in chapter (5). Also in chapter (5) we proved that the following set $s_g \subset \mathcal{L}$, is non-empty:

$s_g = \{\gamma \in \mathcal{L}$: $\gamma(\psi_n) = c_n$, $n = 1,2,\ldots$; γ is of the form (6.6), the measure μ_k, $k = 1,2,\ldots$ associated with it are atomless and satisfying (6.7)}.

It was shown in chapter (5) that a sequence of controls in $L_2(\omega)$ exists which approximate $\gamma \in \mathcal{L}$.

As in the chapter (4) we can show the following map is one to one

$$\gamma = \sum_{k=0}^{\infty} D_1^k\mu_k \in \mathcal{L} \leftrightarrow (\mu_0,\mu_1,\ldots,\mu_k,\ldots),$$

where $\sum_{k=0}^{\infty} L^k \int_{[0,T] \times A} d|\mu_k| < \infty$, for all $L \geq 0$. We can identify every element of $\mathcal{L}$ with a sequence of the form $\{\mu_k\}$, $k = 0,1,\ldots$. Thus, we denote $\gamma = \{\mu_k\} \in \mathcal{L}$. As in chapter (4) an equivalent condition to condition (6.7) is $\lim_{k\to\infty} \|\mu_k\|^{1/k} = 0$.

Therefore $\gamma = (\mu_0,\mu_1,\ldots,\mu_k,\ldots) \in s_g$ if and only if

(a) $\gamma(\psi_n) = \sum_{k=0}^{\infty} (-1)^k\lambda_n^k\mu_k(\psi_n) = c_n$, $n = 1,2,\ldots$

(b) $\lim_{k\to\infty} \|\mu_k\|^{1/k} = 0$

where $\psi_n(t,s) = \hat{b}_n(s)\, e^{-\lambda_n(T-t)}$, $(t,s) \in [0,T] \times A$.

Let the objective function J be a function defined on $\prod_{k=0}^{\infty} \mathcal{M}$, by

$J(\gamma) = \|\gamma\|_I = \sum_{k=0}^{\infty} \|\mu_k\|$, where $\gamma = (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in \mathcal{L} \subset \prod_{k=0}^{\infty} M$, therefore the function J is well defined since $\sum_{k=0}^{\infty} \|\mu_k\| < \infty$. The reason that we have chosen J(.) in the above form is that if the classical control problem consists of finding a control u(.,.), which minimizes the functional

$$I[\hat{u}(.,.)] = \int_{\partial\omega \times [0,T]} |u(\xi,t)| d\xi dt \equiv \|u\|_{L_1(\partial\omega \times [0,T])}$$

then, defining the measure μ_u as

$$\int_{\partial\omega \times [0,T]} \psi \, d\mu_u \equiv \int_{\partial\omega \times [0,T]} \psi(\xi,t) u(\xi,t) d\xi dt, \quad \psi \in F.$$

Thus we have

$$\|\mu_u\| = |\mu_u|(1) = \int_{\partial\omega \times [0,T]} d|\mu_u| = \int_{\partial\omega \times [0,T]} |u(\xi,t)| d\xi dt$$

$$= \|u\|_{L_1(\partial\omega \times [0,T])}.$$

Therefore the above objective functional J(.) is indeed a true extension of the functional I(.).

We show in this chapter:

(i) For positive integers K, N, L, and a positive number ε we have

$$\inf_{Q(K,N,L)} \sum_{k=0}^{KL} \|\mu_k\| = \lim_{\varepsilon \to \infty} \inf_{Q_\varepsilon(K,N,L)} \sum_{k=0}^{KL} \|\mu_k\|$$

where $Q_\varepsilon(K,N,L)$ is the set of all $(\mu_0, \mu_1, \ldots, \mu_{KL}, \bar{0})$ [we define

$(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0}) \equiv (\mu_0,\mu_1,\ldots,\mu_{KL},0,0,\ldots)]$ such that

(a) $\left| \sum_{k=0}^{KL} (-1)^k \lambda_n^k \mu_k(\psi_n) - c_n \right| < \varepsilon,\ n = ,1,\ldots,N$

(b) $\|\mu_k\|^{1/k} < 1/K,\ (K-1)L < k \le KL.$

(ii) For positive integers K, N, L,

$$\inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu\|_k = \lim_{K\to\infty} \inf_{Q(K,N,L)} \sum_{k=0}^{KL} \|\mu\|_k$$

where Q(N) is the set of all $(\mu_0,\mu_1,\ldots,\mu_k,\ldots)$ such that

$$\sum_{k=0}^{\infty} (-1)^k \lambda_n^k \mu_k(\psi_n) = c_n,\ n = 1,2,\ldots,N,$$

$$\lim_{k\to\infty} \|\mu_k\|^{1/k} = 0.$$

(iii) We show next

$$\inf_{S_g} \sum_{k=0}^{\infty} \|\mu_k\| = \lim_{N\to\infty} \inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\|$$

(iv) Let $\theta = \inf_{S_g} \sum_{k=0}^{\infty} \|\mu_k\|$, then we approximate θ by $\inf_{P_\varepsilon(K,N)} \sum_{k=0}^{K} \|\mu_k\|$, where $P_\varepsilon(K,N)$ is the set of all $(\mu_0,\mu_1,\ldots,\mu_K,\bar{0})$, such that each of μ_k is a discrete measure on $[0,T] \times A$, and

$$\left| \sum_{k=0}^{K} (-1)^k \lambda_n^k \mu_k(\psi_n) - c_n \right| < \varepsilon,\ n = 1,2,\ldots,N$$

[K and N are sufficiently large and ε is a given positive number].

(v) We transfer the above problem to one which involves minimizing of a linear function over a set of linear constraints in finite space.

(vi) Finally, by using the sequence of the control functions introduced in chapter (5), we show practically that we can reach different final states to a good approximation.

(6.2) A scheme for determining the infimum of the objective function

In this section we show

$$\inf_{Q(k,N,L)} \sum_{k=0}^{KL} \|\mu_k\| = \lim_{\varepsilon\to 0+} \left[\inf_{Q_K(\varepsilon)} \sum_{k=0}^{KL} \|\mu_k\|\right],$$

where K, L are positive integers, N is a nonnegative integer, $\varepsilon > 0$, and $Q_K(\varepsilon) \equiv Q(K,N,L,\varepsilon)$ is the set of all $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0})$ of Radon measures such that

(i) $\left|\sum_{k=0}^{KL} (-1)^k \lambda_n^k \mu_k(\psi_n) - c_n\right| < \varepsilon$, $n = 0,1,\ldots,N$

(ii) $\|\mu_k\|^{1/k} < 1/K$, $(K-1)L < k \le KL$,

Q(K,N,L) is the set of all $(\mu_0,\mu_1,\ldots,\mu_{KL},\bar{0})$ of Radon measures such that

(iii) $\sum_{k=0}^{KL} (-1)^k \lambda_n^k \mu_k(\psi_n) \equiv c_n$, $n = 1,2,\ldots,N$

(iv) $\|\mu_k\|^{1/k} < 1/K$, $(K-1)L < k \le KL$.

First in lemma (6.1) we note that the sequence $\{Q(K,N,L,\varepsilon)\}_K$ is nondecreasing.

Lemma (6.1) $Q(K,N,L,\varepsilon) \subset Q(K+1,N,L,\varepsilon)$.

Proof The proof is similar to that of lemma (4.1).□

As in the appendix (C.4) in chapter (4), we conclude that

$Q(K,N,L,\varepsilon)$ is non-empty for all K, N, L, and ε. We now define

$$\xi_{K,N,L}(\varepsilon) = \inf_{Q_K(\varepsilon)} \sum_{k=0}^{\infty} \|\mu_k\|. \tag{6.9}$$

In lemma (6.2) we note that the sequence $\{\xi_{K,N,L}(\varepsilon)\}_K$ is non-increasing.

Lemma (6.2) $\xi_{K+1,N,L}(\varepsilon) \le \xi_{K,N,L}(\varepsilon)$, when N, L, and ε are fixed but K takes value 1, 2, ...

Proof The proof is similar to that of lemma (4.2).□

As in appendix (C.4) we conclude $Q_K \equiv Q(K,N,L)$ is nonempty so the following definition is meaningful

$$\xi_{K,N,L} = \inf_{Q_K} \sum_{k=0}^{\infty} \|\mu_k\|. \tag{6.10}$$

In lemma (6.3) we obtain the relation between $\xi_{K,N,L}$ and $\xi_{K,N,L}(\varepsilon)$.

Lemma (6.3) $Q(K,N,L) = \bigcap_{\varepsilon>0} Q(K,N,L,\varepsilon)$

Proof The proof is similar to that of lemma (4.3).□

Lemma (6.4) $Q(K,N,L) \subset Q(N)$, for any positive integers K, N and L.

Proof The proof is similar to that of lemma (6.3).□

Now we define

$$\xi_{K,N,L} = \inf_{Q(K,N,L)} \sum_{k=0}^{\infty} \|\mu_k\|.$$

[As in appendix (C.4) in chapter (4) we can show that Q(K,N,L) $\neq \phi$.]

Lemma (6.5) $\theta_{K,N,L} = \lim_{\varepsilon \to 0+} \xi_{K,N,L}(\varepsilon)$, exists and $\theta_{K,N,L} = \xi_{K,N<L}$.

Proof The proof is similar to that of lemma (4.5).□

(6.3) **Approximation of the infimum of the objective function when the cosine Fourier series of the desired final state is a finite summation.**

Let $g(x) = \sum_{n=1}^{N} c_n a_n(x)$, where N is an arbitrary non-negative integer. We will show that for any positive integer L

$$\inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\| = \lim_{K} \inf_{Q_K} \sum_{k=0}^{\infty} \|\mu_k\| \tag{6.11}$$

where $Q_K \equiv Q(K,N,L)$; $Q(N)$ was defined as the set of all $(\mu_0, \mu_1, \ldots, \mu_k, \ldots)$ of Radon measures such that

(a) $\sum_{k=0}^{\infty} (-1)^k \lambda_n^k \, \mu_k(\psi_n) = c_n$, $n = 1,2,\ldots,N$

(b) $\lim_{k \to \infty} \|\mu_k\|^{1/k} = 0.$

As in appendix (C.4), we can show that $Q(N) \neq \phi$, so $\inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\|$ is meaningful. First we note that $\lim_{K} \xi_{K,N,L}$ exists.

Lemma (6.6) For arbitrary K, N, and L

$$\xi_{K+1,N,L} \le \xi_{K,N,L},$$

where $\xi_{K,N,L}$, was defined in (6.10).

Proof The proof is similar to that of lemma (4.6).□

In lemma (6.6) we showed that the sequence $\{\xi_{K,N,L}\}_K$ is non-increasing, when N and L are fixed, and it is bounded from below by 0. We define now

$$\eta_{N,L} = \lim_{K} \xi_{K,N,L}, \qquad (6.12)$$

$$\theta_N = \inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\| \qquad (6.13)$$

by using the above definition we may write (6.11) in the following form

$$\theta_N = \eta_{N,L}.$$

Remark It appears that $\eta_{N,L} = \lim_{K} \xi_{K,N,L}$ depends on the value of L, but we will show that it is independent of the value $L \geq 1$; in other words, we show that for any fixed $L \geq 1$, $\theta_N = \eta_{N,L}$. In the following we note a lemma which will be used later.

Lemma (6.7) Let $\psi_n(t) = e^{-\lambda_n(T-t)}$, $0 \leq t \leq T$, $n = 1,2,\ldots,N$, where N is a fixed integer such that $N \geq 1$, and let $t_i = i\Delta$, $i = 1,2,\ldots,N$ and $\Delta = T/N$, then the following matrix is non-singular.

$$\psi = \begin{bmatrix} \psi_1(t_1) & \psi_1(t_2) & \ldots & \psi_1(t_N) \\ \psi_2(t_1) & \psi_2(t_2) & \ldots & \psi_2(t_N) \\ \vdots & & & \\ \psi_N(t_1) & \psi_N(t_2) & \ldots & \psi_N(t_N) \end{bmatrix}$$

Proof The proof is similar to that of lemma (4.7).□

Note : Let $G = \psi^{-1} = (g_{ij})$ and let $G_N = \sum_{i=1}^{N} \sum_{j=1}^{N} |g_{ij}|$.

In the following lemma we will assume that there exists at least one $s_0 \in A = \overbrace{[0,T] X \ldots [0,T]}^{n-1 \text{ times}}$, such that $\hat{b}_j(s_0) \neq 0$, for $j = 1,2,\ldots,N$, [N is any fixed positive integer]. This matter will be left until appendix (A.6).

Lemma (6.8) Suppose there exists at least one $s_0 \in A$, such that $\hat{b}_j(s_0) \neq 0$, for $j = 1,2,\ldots,N$ and let a_n, $n = 1,2,\ldots,N$ be N real numbers such that $|a_n| < 1$, $n = 1,2,\ldots,N$. If k is any fixed positive integer, then there exists a Radon measure ν_k such that

(i) $\nu_k(\psi_n) = a_n / \lambda_n^k$, $n = 1,2,\ldots,N$

where $\psi_n(t,s) = \hat{b}_n(s)e^{-\lambda_n(T-t)}$, $(t,s) \in [0,T] X A$.

(ii) $\|\nu_k\| < c_N / \lambda_1^{2k}$;

here $c_N = (1/b)G_N$, where G_N was defined above, and $\lambda_1 > 1$ [since we assumed that all the eigenvalues λ_n are greater than 1], and $b = \min\{|\hat{b}_j(s_0)|: j = 1,2,\ldots,N\}$.

Proof We define the discrete measure ν_k as follows

$$\nu_k = \sum_{i=1}^{N} e_i \delta(t_i, s_0),$$

where $t_i = i\Delta$, $i = 1,2,\ldots,N$, $\Delta = T/N$, and e_i, $i = 1,2,\ldots,N$ are N unknowns to be determined below. Since we must have

$\nu_k(\psi_n) = a_n / \lambda_n^k$, $n = 1,2,\ldots,N$, then $\left[\sum_{i=1}^{N} e_i\delta(t_i,s_0)\right]\psi_n = a_n \,/\, \lambda_n^k,\ n = 1,2,\ldots,N,$

or

$$\sum_{i=0}^{N} e_i\psi_n(t_i,s_0) = a_n \,/\, \lambda_n^k,\ n = 1,2,\ldots,N.$$

But $\psi_n(t,s) = \hat{b}_n(s)\ e^{-\lambda_n(T-t)} \equiv \hat{b}_n(s)\phi_n(t)$, where $\phi_n(t) = e^{-\lambda_n(T-t)}$, so we have

$$\sum_{i=1}^{N} e_i\hat{b}_n(s_0)\phi_n(t_i) = a_n \,/\, \lambda_n^k,\ n = 1,2,\ldots,N$$

or

$$\hat{b}_n(s_0)\left[\sum_{i=1}^{N} e_i\phi_n(t_i)\right] = a_n \,/\, \lambda_n^k,\ n = 1,2,\ldots,N.$$

By assumption $\hat{b}_n(s_0) \neq 0$ for $n = 1,2,\ldots,N$ so we have

$$\sum_{i=1}^{N} e_i\phi_n(t_i) = a_n \,/\, [\lambda_n^k.\hat{b}_n(s_0)],\ n = 1,2,\ldots,N. \tag{6.14}$$

We have called ψ the matrix of coefficients of the above system (6.14) of N linear equations in the unknowns e_i, $i = 1,2,\ldots,N$; ψ was defined in lemma (6.7) and we showed there that ψ is non-singular and $G = \psi^{-1} = (g_{ij})$, so we have

$$\begin{bmatrix} e_1 \\ e_2 \\ \vdots \\ e_N \end{bmatrix} = \begin{bmatrix} g_{11} & g_{12} \cdots & g_{1N} \\ g_{21} & g_{22} \cdots & g_{2N} \\ \vdots & & \\ g_{N1} & g_{N2} \cdots & g_{NN} \end{bmatrix} \begin{bmatrix} a_1/\lambda_1^k \, \hat{b}_1(s_0) \\ a_2/\lambda_2^k \, \hat{b}_2(s_0) \\ \vdots \\ a_N/\lambda_N^k \, \hat{b}_N(s_0) \end{bmatrix} \tag{6.15}$$

Therefore we have from (6.15)

$$e_i = \sum_{j=1}^{N} \{g_{ij}[a_j/\lambda_j^k \, \hat{b}_j(s_0)]\}, \; j = 1,2,\ldots,N$$

or

$$|e_i| \le \sum_{j=1}^{N} \{|g_{ij}||a_j|/\lambda_j^k|\hat{b}_j(s_0)|\}, \; i = 1,2,\ldots,N \tag{6.16}$$

Let $b = \min\{|\hat{b}_j(s_0)|; j = 1,2,\ldots,N\}$. Then $b > 0$, because we have assumed that $\hat{b}_j(s_0) \ne 0$, $n = 1,2,\ldots,N$. Also we assumed without loss of generality that the set of eigenvalues $\{\lambda_j\}$, satisfy

$$1 < \lambda_1 < \lambda_2 < \ldots < \lambda_n < \ldots .$$

Therefore $\lambda_j^k > \lambda_1^k$, for $j = 1,2,\ldots$ and $|\hat{b}_j(s_0)| \ge b$, for $j = 1,2,\ldots N$. Further, by assumption, $|a_j| < 1$, $j = 1,2,\ldots,N$; thus we have from (6.16),

$$|e_i| \le \sum_{j=1}^{N} |g_{ij}|/\lambda_1^k b. \tag{6.17}$$

Since $\nu_k = \sum_{i=1}^{N} e_i \delta(t_i, s_0)$,

$$\|\nu_k\| = |\nu_k|(1) = \sum_{i=1}^{N} |e_i|. \tag{6.18}$$

By (6.17) and (6.18) we have $\|\nu_k\| < \sum_{i=1}^{N} \sum_{j=1}^{N} |g_{ij}|/\lambda_1^k b$, or

$$\|\nu_k\| < \sum_{i=1}^{N} \sum_{j=1}^{N} |g_{ij}|/\lambda_1^k b. \qquad (6.19)$$

But since $c_N = (1/b)G_N$, where $G_N = \sum_{i=1}^{N} \sum_{j=1}^{N} |g_{ij}|$, from (6.19) we conclude

$$\|\nu_k\| < c_N/\lambda_1^k. \square$$

To prove the lemma (6.9) we need to define a norm on $\prod_{i=1}^{\infty} M$, where M is the set of all Radon measures on $[0,T] \times A$. Let

$$W = \{(\mu_0,\mu_1,\ldots,\mu_k,\ldots) \in \prod_{i=1}^{\infty} M;\ \sum_{k=0}^{\infty} \|\mu_k\| < \infty\};$$

We define the function $\|.\|_{II}$ on W as follows

$$\|.\|_{II} : W \to R^+$$

such that for any $w = (\mu_0,\mu_1,\ldots,\mu_k,\ldots)$, $\|w\|_{II} = \sum_{k=0}^{\infty} \|\mu_k\|$.

We show below that $\overline{\bigcup_{k=0}^{\infty} Q(K,N,L)} \subset Q(N)$ [where $\bar{A}$ means the closure of the set A, with respect to norm-II, topology] By using the definition of Q(N), it is apparent that $Q(N) \subset W$, because for $(\mu_0,\mu_1,\ldots,\mu_k,\ldots) \in Q(N)$, $\lim_{k\to\infty} \|\mu_k\|^{1/k} = 0$, or equivalently $\sum_{k=0}^{\infty} L^k\|\mu_k\| < \infty$, for all $L \geq 0$; therefore $Q(N) \subset W$. But in general Q(N) is not a subset of $\bigcup_{k=0}^{\infty} Q(K,N,L)$, since in Chapter (5) we showed that the element γ of $s_g \subset Q(N)$ defined as follows

$$\gamma(\phi) = \sum_{k=0}^{\infty} \int_{[0,T] \times A} D_1^k \phi(t,s)(-1)^k [\theta^k/k!]\hat{u}(t,s)dtds, \quad \phi \in F,$$

where we have chosen the measures μ_k, $k = 0,1,\ldots$ as the following

$$\int_{[0,T] \times A} \phi d\mu_k = \int_{[0,T] \times A} \phi(t,s)(-1)^k[\theta^k/k!]\hat{u}(t,s)dtds, \phi \in F.$$

It is seen that $\gamma = (\mu_0,\mu_1,\ldots,\mu_k,\ldots) \notin \bigcup_{k=0}^{\infty} Q(K,N,L)$. Therefore $Q(N)$ is not a subset of $\bigcup_{K=0}^{\infty} Q(K,N,L)$. In lemma (6.9) we show $Q(N) \subset \overline{\bigcup_{K=0}^{\infty} Q(K,N,L)}$.

Lemma (6.9) $Q(N) \subset \overline{\bigcup_{K=0}^{\infty} Q(K,N,L)}$, for any fixed positive integer L.

Proof The proof is similar to that of Lemma (4.9).□

Now we intend to prove $\eta_{N,L} = \theta_N$, but first we prove $\eta_{N,L} \le \theta_N$:

Lemma (6.10) For any fixed $L \ge 1$, and any $N \ge 1$

$$\eta_{NL} \le \theta_N$$

Proof The proof is similar to that of lemma (4.10).□

Lemma (6.11) For any fixed $L \ge 1$, and any $N \ge 1$

$$\eta_{NL} = \theta_N$$

Proof The proof is similar to that of lemma (4.11).□

(6.4) Approximation of the infimum of the objective function when the desired final state belongs to $L_2(\omega)$.

In Proposition (5.2) of Chapter (5) we showed $s_g \equiv \mathcal{L}_g \neq \phi$. Thus, $\inf_{s_g} \sum_{k=0}^{\infty} \|\mu_k\|$, is meaningful. Let

$$\theta = \inf_{s_g} \sum_{k=0}^{\infty} \|\mu_k\|$$

In this section we show $\theta = \lim_{N\to\infty} \inf_{Q(N)} \sum_{k=0}^{\infty} \|\mu_k\|$, where Q(N) and s_g, where defined above. Now define for any positive N

$$g_N(x) = \sum_{n=1}^{N} c_n a_n(x), \quad x \in \omega,$$

and let $s_{g_N} = \left\{\gamma \in s_g: \ s(\psi_n) = c_n, \ n = 1,2,\dots,N\right\}$, where $\gamma = (\mu_0,\mu_1,\dots,\mu_k,\dots)$ and $\gamma(\psi_n) = \sum_{k=0}^{\infty} (-1)^k \lambda_n^k \mu_k(\phi_n)$. First we prove the following lemma

Lemma (6.12) $g_N \to g$, with respect to $L_2(\omega)$-norm.

Proof The proof is similar to that of lemma (4.12).□

By using the definition of θ_N we conclude $\{\theta_N\}$, is a non-increasing bounded sequence satisfying

$$\theta_1 \le \theta_2 \le \dots \le \theta_N \le \dots \le \theta \qquad (6.20)$$

[for more detail see lemma (4.12) and following material in chapter (4)]. Thus $\{\theta_N\}$converges. Let $\xi = \lim_{N} \theta_N$; from (6.20) we have

$$\xi \leq \theta. \tag{6.21}$$

Lemma (6.13) $\xi = \theta$.

Proof Its proof is similar to the proof of lemma (4.13).□

(6.5) Approximation of the infimum of the objective function by a finite summation of the norms of discrete measures.

In this section we consider that for any fixed integer $L \geq 1$ there exists non-negative integers K and N such that

$$|\theta - \sum_{k=0}^{KL} \|\nu_k\|| < \varepsilon,$$

where ν_k, $k = 0,1,\ldots,KL$, are discrete measures on $[0,T] \times \partial\omega$. We also show that there exists $\delta_0 (0 < \delta_0 < \varepsilon/5)$ such that

$$|\theta - \theta_{KNL}(\delta_0)| < \varepsilon,$$

where $\theta_{KLN}(\delta_0) = \inf_{P_K} \sum_{k=0}^{\infty} \|\mu_k\|$, and $P_K \equiv P(K,N,L,\delta_0)$ is the set of all $(\nu_0,\nu_1,\ldots,\nu_{KL},\bar{0})$ where each of ν_k, $k = 0,1,\ldots,KL$, is a discrete measure defined on $[0,T] \times \partial\omega$ such that

$$|\sum_{k=0}^{KL} (-1)^k \lambda_n^k \mu_k(\psi_n) - c_n| < \delta_0.$$

The prcofs of the above claims are similar to the proofs of the same claims in one-dimensional space, given in section (4.5) of Chapter (4).

(6.6) Computations for obtaining an approximation for the infimum of the objective function and control Functions

In this section we apply the claims of section (6.5), and transform the problem into one which consists of minimizing of a real linear function defined on R^k, for some positive integer k, over a finite set of linear constraints. Then we construct the control functions with the help of the results of chapter (5). Finally, the theory is confirmed, by solving numerically one problem.

Let $\Omega = \{Y_k; k = 1,2,\dots\}$. This set can be chosen as one which is dense in $[0,T] \times A$. In practice, however, we choose the set $\Omega^M = \{Y_k; k = 1,2,\dots M\} \subset \Omega$, by dividing the appropriate intervals into a number of equal subintervals, defining in this way a grid of points; thus, in a manner similar to chapter (4) we conclude that our problem is to minimize

$$\sum_{k=0}^{K} \|\nu_k\|$$

over $(\nu_0, \nu_1, \dots, \nu_{KL}, \bar{0}) \in P(K,N,\varepsilon) \equiv P(K,N,1,\varepsilon)$, which means

$$\left| \sum_{k=0}^{K} (-1)^k \lambda_n^k \mu_k(\psi_n) - c_n \right| < \varepsilon, \quad n = 1,2,\dots,N.$$

Now let $\nu_k = \nu_k^+ - \nu_k^-$, $\nu_k^+ = \sum_{i=1}^{M} \beta_i^k \delta(t_i)$, $\nu_k^- = \sum_{i=1}^{M} \gamma_i^k \delta(t_i)$, where $\beta_i^k \geq 0$, and $\gamma_i^k \geq 0$, for $i = 1,2,\dots,M$, $k = 0,1,\dots,K$. Therefore, our problem is to minimize

$$\sum_{k=0}^{K} \sum_{i=1}^{M} \left(\beta_i^k + \gamma_i^k\right) \tag{6.22}$$

on the subset of $R^{2M(K+1)}$, say $S(K,N,M,\varepsilon)$, defined by

$\beta_i^k \geq 0$, and $\gamma_i^k \geq 0$, for $i = 1,2,\ldots,M$, $k = 0,1,\ldots,K$;

$$\left|\sum_{k=0}^{K} \sum_{i=1}^{M} (-1)^k \lambda_n^k \left[\beta_i^k - \gamma_i^k\right] \psi_n(Y_i) - c_n\right| < \varepsilon, \; n = 1,2,\ldots,N. \qquad (6.23)$$

Now we rename the variables β_i^k as follows:

$$(\beta_1^0, \beta_1^1, \ldots, \beta_1^K) \leftrightarrow (x_1, x_2, \ldots, x_{K+1})$$

$$(\beta_2^0, \beta_2^1, \ldots, \beta_2^K) \leftrightarrow (x_{K+2}, x_{K+3}, \ldots, x_{2K+2})$$

$$\vdots$$

$$(\beta_M^0, \beta_M^1, \ldots, \beta_M^K) \leftrightarrow (x_{(M-1)K+M}, x_{(M-1)K+M+1}, \ldots, x_{M(K+1)})$$

or in general $\beta_i^k = x_{(i-1)K+k+i}$, $i = 1,2,\ldots,M$, $k = 0,1,\ldots,K$. For simplicity let $J_{KM} = M(K+1)$; we rename the variables $\gamma_i^k = x_{J_{KM}+(i-1)K+k+i}$, $i = 1,2,\ldots,M$, $k = 0,1,\ldots,K$. Finally from (6.22) and (6.23) and the above notation we conclude that our problem is to minimize

$$\sum_{i=1}^{2M(K+1)} x_i, \qquad (6.24)$$

on the set $S(K,N,M,\varepsilon)$, in $R^{2M(K+1)}$, defined by

$x_i \geq 0$, $i = 1,2,\ldots,2M(K+1)$

$$\left|\sum_{k=0}^{K} \sum_{i=1}^{M} (-1)^k \lambda_n^k \left[x_{(i-1)K+k+i} - x_{J_{KM}+(i-1)K+k+i}\right] \psi_n(Y_i) - c_n\right| < \varepsilon,$$

where $n = 1,2,\dots,N$, and $J_{KM} = M(K+1)$.

Supposing that this problem has been solved, we intend now to obtain numerically the sequence of control functions $\{u_j^k\}_j$, $j = 1,2,\dots$ defined in appendix (C.5) of chapter (5). In appendix (C.5) we showed that

$$u_i^K(\eta,\sigma) = \sum_{k=0}^{K} (-1)^k j^n \int_{[0,T]\ X\ A} D_1^k\ \rho(j(\eta - \xi) + J(\sigma - s))d\mu_k(\xi,s).$$

Let $\mu_k(\xi,\sigma) = \sum_{i=1}^{M} \zeta_i^k \delta(Y_i)$, where $Y_i = (t_i,s_i) \in [0,T]\ X\ A$, $i = 1,2,\dots,M$, therefore we have

$$u_i^K(\eta,\sigma) = \sum_{k=0}^{K} \sum_{i=1}^{M} (-1)^k j^n\ \zeta_i^k D_1^k\ \rho(j(\eta - t_i) + J(\sigma - s_i))$$

We consider now an example in two dimensions.

Example (6.1) $g(x) = (2/\pi)\ \sin(x)\sin(y) + (2/\pi)\sin(x)\sin(3y)$, $(x,y) \in \omega$, where $\omega = \{(x,y);\ 0 \le x \le 1,\ 0 \le y \le 1\}$. Without loss of generality let $T = 1$, $K = 2$, $M = 100$, $J_{KM} = 300$, and $R^{2M(K+1)} = R^{600}$ so our problem is to minimize

$$\sum_{i=1}^{600} x_i$$

on the set $S(2,100)$ in R^{600}, defined by

$x_i \ge 0$, $i = 1,2,\dots,600$;

$$\sum_{k=0}^{2} \sum_{i=1}^{100} (-1)^k \lambda_n^k (x_{3i+k-2} - x_{3i+k+296}) \psi_n(t_i,s_i) = c_n,\ n = 1,2,3,4,$$

where $\psi_n(t,s) = \hat{b}_n(s)e^{-\lambda_n(1-t)}$; here λ_1, $\lambda_2, \ldots,$ are the same as those defined in examples in 2 dimensions in chapter 3. Therefore by the above reference, we conclude that $c_1 = c_4 = 1$, $c_2 = c_3 = 0$.

Thus the results of computations are:

cost function = 0.1344E+00

x_{148} = 0.8421E-01 x_{446} = 0.4823E-01

x_{450} = 0.1761E-02 x_{480} = 0.1973E-03

with $x_i = 0$, for all other i ($1 \le i \le 600$).

Let $\mu_k = \sum_{i=1}^{100} \zeta_i^k \delta(Y_i)$, k = 0,1,2 and $\zeta_i^k = \beta_i^k - \gamma_i^k$, i = 1,2,...,100, k = 0,1,2. Therefore by the one to one correspondence defined below (6.22), we have

$\zeta_{50}^0 = 0.08421$ $\zeta_{50}^2 = 0.001762$ $\zeta_{49}^1 = -0.04823$ $\zeta_{50}^0 = -0.0001973$

with all other $\zeta_i^k = 0$.

According to the formula above Example (6.1), the sequence of control functions $\{u_j^K\}$, is as follows

$$u_j^K(\tau,\sigma) = u_j^2(\tau,\sigma) = \sum_{k=0}^{2} \sum_{i=1}^{200} (-1)^k \zeta_i^k D_1^k \rho_{1/j}(\tau - t_i, \sigma - s_i)$$

$$= 0.08421\rho_{1/j}(t - 0.95, s - 0.45) + 0.04823D_1\rho_{1/j}(t - 0.85, s - 0.45)$$

$$- 0.001762D_1^2\rho_{1/j}(t - 0.95, s - 0.45)$$

$$- 0.000197D_1^2\rho_{1/j}(t - 0.95, s - 0.55),\ (t,s) \in [0,1] \times [0,1].$$

But we showed below Proposition (5.6) of chapter (5), that when $j \to \infty$, then

$$\int_{[0,T] \times A} u_j^K(\xi,s)\phi_n(\xi,s)d\xi ds \to c_n(1,\gamma_K)$$

where $\phi_n(\xi,s) \equiv \hat{b}_n(s)e^{-\lambda_n(1-s)}$. But $c_n(1,\gamma_k)$, is close to $c_n(1,\gamma)$ for large K (by definition); also, according to proposition (5.7) of chapter (5), $c_n(1,\gamma_K)$, is close to c_n, n = 1,2,...,(for large K), where c_n, n = 1,2,..., are the Fourier coefficients of the final state function. In this example (for K = 2) in the desired final state g(.,.) the coefficients of the eigen-functions $a_1(.,.)$ and $a_4(.,.)$ are 1. The corresponding computed coefficients are 1.00166, 1.0089. Therefore the computed final state g'(.,.), corresponding to final state g(.,.), is $g'(x,y) = 1.00166(2/\pi)\sin(x)\sin(y) + 1.0089(2/\pi)\sin(x)\sin(3y)$. Thus we have $|g' - g|_{L_2} = 0.00905$. (See figures (6.1) and (6.2)).

CONCLUSIONS AND RECOMMENDATIONS FOR FUTURE RESEARCH.

The suggestions for further work are as follows:

(1) In Chapters 4 and 6 we studied an optimal control problem for the n-dimensional diffusion equation with a sequence of generalized control variables. There we defined the objective function J, a function defined on $\prod_{k=0}^{\infty} M$, as $J(\gamma) = \|\gamma\|_1 = \sum_{k=0}^{\infty} \|\mu_k\|$, where $\gamma = (\mu_0, \mu_1, \ldots, \mu_k, \ldots) \in \prod_{k=0}^{\infty} M$. We showed in Chapters 4 and 6 that this is a true extension of the classical control problem which consists of finding a control u(.) or u(.,.) on [0,T] or on $\partial\omega \times [0,T]$ respectively, which minimizes the functionals

$$I[u(.)] = \int_{[0,T]} |u(t)|dt = \|u\|_{L_1(0,T)}$$

or

$$I[u(.,.)] = \int_{\partial\omega \times [0,T]} |u(\xi,t)|d\xi dt = \|u\|_{L_1(\partial\omega \times [0,T])}$$

respectively. The first problem is to choose the objective function J, as a function of (x,u,t), where $x \in [0,1]$ or $x \in \omega$ (the region defined in chapter 6, which is in R^n), $u \in V$ (the set of admissible controls defined in chapter 6) and $t \in [0,T]$.

(2) In Chapter 4, we needed to compute functions as follows:

$$(-1)^k(n^2\pi^2)^k \, e^{-(n^2\pi^2)(1-t_i)}, \; n = 0,1,\ldots, \text{ and } k = 0,1,\ldots,$$

where $t_i = (2i - 1)/20$; $i = 1,2,\ldots,10$, for large values of k and n. It is apparent that when n is chosen large, then the absolute value of the above functions are very small, even less than 10^{-78} which is the limitation imposed by the word-size of the computer; all of such numbers are then treated as zero. This causes ill-conditioning in the matrix that we use in the revised simplex method (see for example Gass [1], p.96). So it seems that it is an open problem to find a way to overcome this difficulty despite the limitation of the computer.

(3) In Chapters 2 and 6 we assumed the boundary of the defined open set $\omega \subset R^n$, is differentiable or $\partial\omega \in C^1$. so the third problem could be to assume there are infintely many points on $\partial\omega$ where $\partial\omega$ is not differentiable.

(4) In Chapters 2, 5 and 6 we considered the n-dimentional linear diffusion equation. The fourth problem could be to consider the following linear diffusion equation with variable coefficients:

$$a(x)\, Y_{xx} + b(x)\, Y_x + c(x)\, Y = Y_t$$

where $Y = Y(x,t)$; $(x,t) \in \omega \times [0,T]$, and a, b, c are continuous functions on ω. We can assume the same boundary conditions as in Chapter 2 and define the set of admissible controls and the objective functional as in Chapter 2.

Appendix (A.6)

In lemma (6.8) we assumed that for a fixed positive integer N there exists at least one $s_0 \in A = \overbrace{[0,T] \times \ldots \times [0,T]}^{n\text{-}1 \text{ times}}$, such that $\hat{b}_j(s_0) \neq 0$, for $j = 1,2,\ldots,N$. In this appendix we intend to obtain conditions which guarantee the existence of s_0.

Let

$$B_j = \{s \in A; \hat{b}_j(s) = 0\}, \ j = 1,2,\ldots,N$$

Indeed, B_j is the set of all solutions of the equation $\hat{b}_j(s) = 0$ on A. It is apparent that $\bigcup_{j=1}^{N} B_j \subset A$, but if $\bigcup_{j=1}^{N} B_j \neq A$, then the set $A - \bigcup_{j=1}^{N} B_j \neq \phi$, so there exists at least one $s_0 \in A - \bigcup_{j=1}^{N} B_j$; that is, $s_0 \in A$ and $s_0 \notin B_j$, for $j = 1,2,\ldots,N$, or by definition of B_j, $\hat{b}_j(s_0) \neq 0$, $j = 1,2,\ldots,N$. If all B_j, $j = 1,2,\ldots,N$, are countable sets, then, that is a sufficient condition for the existence of s_0, since if it happens then $\bigcup_{j=1}^{N} B_j$, is countable, thus $A - \bigcup_{j=1}^{N} B_j$, is uncountable so there exists $s_0 \in A - \bigcup_{j=1}^{N} B_j$.

Another equivalent condition to the above sufficient condition is that the sets $B'_j = \{x \in \partial\omega;\ b_j(x) = 0\}$ be countable; here $b_j(x) = \partial a_j(x)/\partial\nu$, $j = 1,2,\ldots$ and ν is the outward normal to $\partial\omega$ (for more detail see the begining of chapter (2)).

Note By definition, $\hat{b}_j(s) = (b_j o x)(s)$, where $x = x(s) = (\xi_1(s),\ldots,\xi_n(s))$, $s \in A$, is the parametric equation of $\partial\omega$. Thus $\hat{b}_j(s) \neq 0$, for some $s \in A$, if and only if $b_j(x) \neq 0$, for some $x \in \partial\omega$.

In the following we show that if ω is an open interval in R^n (for definition see RUDIN [1] p.229) then the sets B_j, $j = 1,2,\ldots,N$, defined above, are countable, which guarantee the existence of

$s_0 \in A$.

Example: Let

$$\omega = \{x = (\xi_1, \dots, \xi_n) \in R^n;\ 0 < \xi_i < a_i,\ i = 1,2,\dots,n\}, \qquad (A6.1)$$

and let $\{(m_1^k, \dots, m_n^k)\}_{k=1,2,\dots}$, be a given sequence in R^n, where the $M_i = \{m_i^k\}_k$, $i = 1,2,\dots,n$ are strictly increasing sequences of integers such that

$$\sum_{i=1}^{n} (m_i^k)^2 [\pi^2/a^2] > 1; \qquad (A6.2)$$

here $a = \max\{a_1, \dots, a_n\}$. Let $l_i = \pi/a_i$, $i = 1,2,\dots,n$,. Now we compute the sequence of eigenvalues $\{\lambda_k\}$, as follows:

$$\lambda_k = \sum_{i=1}^{n} (m_i^k)^2 l_i^2 = \sum_{i=1}^{n} (m_i^k)^2 [\pi^2/a_i^2]. \qquad (A6.3)$$

By definition of m_i^k, we have

$$\lambda_{k+1} = \sum_{i=1}^{n} (m_i^{k+1})^2 [\pi^2/a_i^2] > \sum_{i=1}^{n} (m_i^k)^2 [\pi^2/a_i^2] = \lambda_k,$$

since the sequences M_i, $i = 1,2,\dots,N$, are strictly increasing. By definition of a we conclude $0 < a_i^2 \le a^2$, $i = 1,2,\dots,n$, so we have

$$\sum_{i=1}^{n} (m_i^k)^2 [\pi^2/a_i^2] \ge \sum_{i=1}^{n} (m_i^k)^2 [\pi^2/a^2], \qquad (A6.4)$$

Thus by using (A6.3) and (A6.4), for $k = 1,2,\dots$ we have

$$\lambda_k \geq \sum_{i=1}^{n} (m_i^k)^2[\pi^2/a^2]. \tag{A6.5}$$

From (A6.5) and (A6.2)

$$\lambda_k > 1, \ k = 1,2,\ldots .$$

Therefore we conclude that the sequence $\{\lambda_k\}$ satisfies the following condition

$$1 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \ldots . \tag{A6.6}$$

We define now the sequence of eigenfunctions. Let $x = (\xi_1,\ldots,\xi_n)$, and let

$$a_k(x) = \prod_{i=1}^{n} \sin(m_i^k l_i \xi_i), \ k = 1,2,\ldots,$$

therefore we have

$$\partial a_k(x)/\partial \xi_j = \left[\prod_{\substack{i=1 \\ i\neq j}}^{n} \sin(m_i^k l_i \xi_i)\right](m_j^k\, l_j)\cos(m_j^k\, l_j\, \xi_j)$$

$$\partial^2 a_k(x)/\partial \xi_j^2 = -\left[\prod_{\substack{i=1 \\ i\neq j}}^{n} \sin(m_i^k l_i \xi_i)\right](m_j^k\, l_j)^2 \sin(m_j^k\, l_j\, \xi_j)$$

$$= -(m_j^k\, l_j)^2 \prod_{i=1}^{n} \sin(m_i^k l_i \xi_i) = -(m_j^k\, l_j)^2\, a_k(x)$$

or $\partial^2 a_k(x)/\partial \xi_j^2 = -(m_j^k\, l_j)^2\, a_k(x)$. Therefore we have

$$\Delta a_k(x) = \sum_{j=1}^{n} \partial^2 a_k(x)/\partial \xi_j^2 = \sum_{j=1}^{n} -(m_j^k\, l_j)^2\, a_k(x) \qquad (A6.7)$$

By using (A6.3) we can write (A6.7) in the following form

$$\Delta a_k(x) = -\lambda_k\, a_k(x),\ k = 1,2,\dots\ ,\ x \in \omega. \qquad (A6.8)$$

Now we are going to solve the equation $a_k(x) = 0$, $x \in \partial\omega$. Let $x = (\xi_1,\dots,\xi_n) \in \partial\omega$, thus by definition of $\partial\omega$, x is in one of the following two forms

(i) $x = (\xi_1,\dots,\xi_{j-1},0,\xi_{j+1},\dots,\xi_n)$;

(ii) $x = (\xi_1,\dots,\xi_{j-1},a_j,\xi_{j+1},\dots,\xi_n)$.

We show $a_k(x) = 0$, $x \in \partial\omega$. But for $x = (\xi_1,\dots,\xi_n)$ we have

$$a_k(x) = \prod_{i=1}^{n} \sin(m_i^k l_i \xi_i)\ ,\ k = 1,2,\dots$$

Thus in case (i) it is apparent that

$$a_k(\xi_1,\dots,\xi_{j-1}^{\ k},0,\xi_{j+1},\dots,\xi_n) = 0.$$

In case (ii) we have

$$a_k(\xi_1,\dots,\xi_{j-1},a_j,\xi_{j+1},\dots,\xi_n) = \left[\prod_{\substack{i=1\\ i\neq j}}^{n} \sin(m_i^k l_i \xi_i)\right]\sin(m_j^k\, l_j\, a_j),$$

but we defined $l_j = \pi/a_j$, so we have

$$\sin(m_j^k\, l_j\, a_j) = \sin[m_j^k(\pi/a_j)\, a_j] = \sin(m_j^k\, \pi),$$

but $\sin(m_j^k\ \pi) = 0$, because m_j^k is an integer, thus

$$a_k(\xi_1,\ldots,\xi_{j-1},a_j,\xi_{j+1},\ldots,\xi_n) = 0.$$

Therefore

$$a_k(x) = 0,\ x \in \partial\omega. \qquad \text{(A6.9)}$$

From (A6.8) and (A6.9) we conclude that the sequence $\{a_k(x)\}$ is a sequence of eigenfunctions of the Laplacian operator Δ, corresponding to the sequence of eigenvalues $\{\lambda_k\}$, where $1 < \lambda_1 < \lambda_2 < \ldots < \lambda_k < \lambda_{k+1} < \ldots$.
Now in the following we show that the sets $B'_j = \{x \in \partial\omega;\ b_j(x) = 0\}$, are countable (indeed they are finite). By definition of $b_k(x)$, $k = 1,2,\ldots$, we have $b_k(x) = \partial a_k(x)/\partial\nu = \sum_{i=1}^{n} [\partial a_k(x)/\partial\xi_i]\nu_i$, where $x = (\xi_1,\ldots,\xi_n) \in \partial\omega$, and $\nu = (\nu_1,\ldots,\nu_n)$, the outward normal to $\partial\omega$. Since ω is an interval in R^n, then at any point on its boundary there is some j, $j = 1,2,\ldots,n$, where $\nu_j = +1$ or $\nu_j = -1$ and for other $i \neq j$, $\nu_i = 0$. Thus without loss of generality let $\nu_j = 1$; thus

$$b_k(x) = \sum_{i=1}^{n} [\partial a_k(x)/\partial\xi_i]\nu_i = \partial a_k(x)/\partial\xi_j = \partial\left[\prod_{i=1}^{n}\sin(m_i^k l_i \xi_i)\right]/\partial\xi_j$$

$$= \left[\prod_{\substack{i=1\\ i\neq j}}^{n}\sin(m_i^k l_i \xi_i)\right](m_j^k\, l_j)\cos(m_j^k\, l_j\, \xi_j).$$

Therefore $b_k(x) = 0$ if and only if

$$\left[\prod_{\substack{i=1\\ i\neq j}}^{n} \sin(m_i^k l_i \xi_i)\right](m_j^k l_j)\cos(m_j^k l_j \xi_j) = 0, \tag{A6.10}$$

but for any k, k = 1,2,...,N, (where N is an arbitrary but fixed integer) the number of solutions of the equations (A6.10) is finite, since $0 \le \xi_i \le a_i \le a$, for i = 1,2,...,n. Thus the sets $B'_j = \{x \in \partial\omega;\ b_j(x) = 0\}$, j = 1,2,...,N, are countable.□

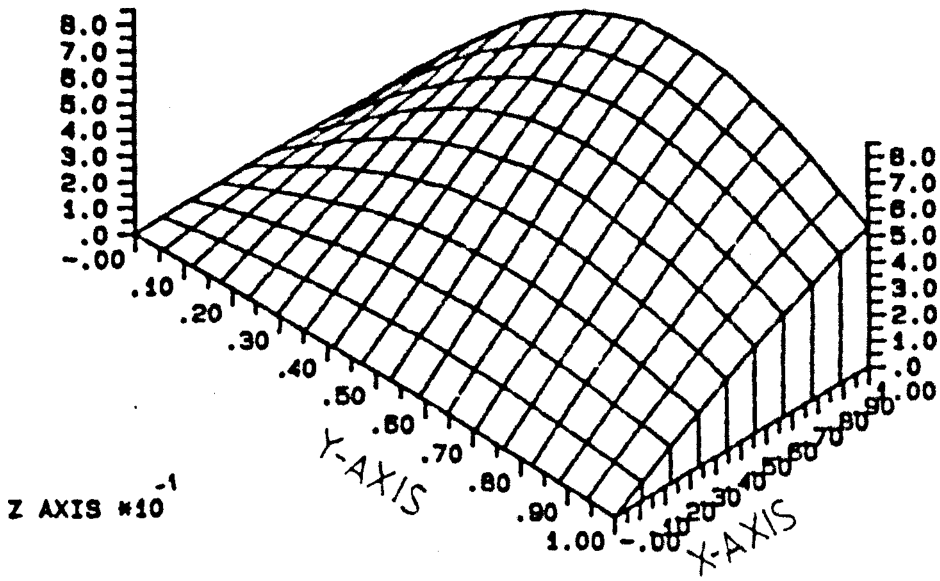


FIG. 6.1 DESIRED FINAL STATE FOR EXAMPLE 6.1

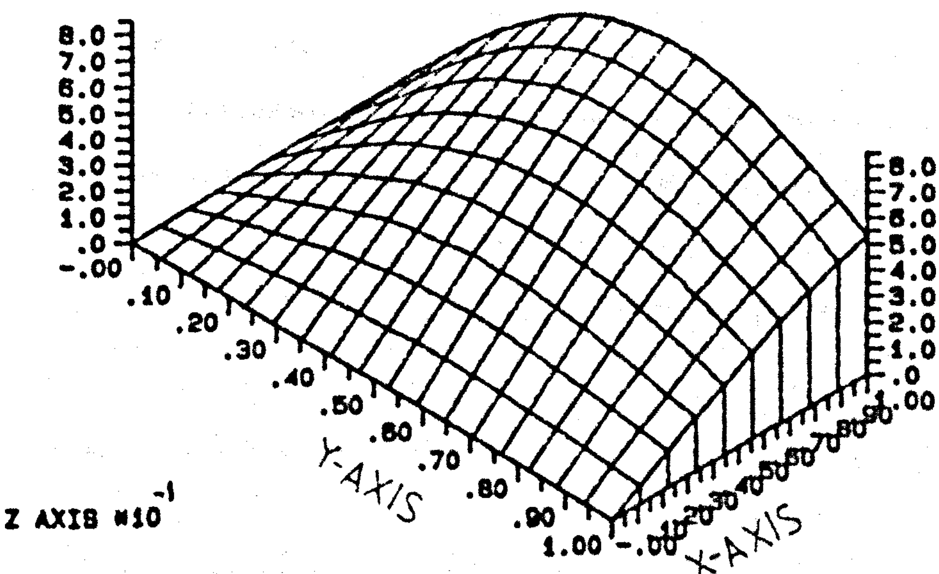


FIG. 6.2 COMPUTED FINAL STATE FOR EXAMPLE 6.1

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