THE AEROACOUSTICS OF JET PIPES AND CASCADES

by

ALEXANDER MICHAEL CARGILL

Being an account of research work carried out in the Department of Applied Mathematical Studies, University of Leeds and in the Noise Department, Rolls-Royce Limited, Derby under the supervision of Professor D.G. Crighton.

Submitted in fulfilment of the requirements for the degree of Doctor of Philosophy.

July 1981

ABSTRACT

This thesis is concerned with theoretical studies in two areas related to the noise of modern turbofan aeroengines. In the first case, we are concerned with the propagation of internally generated sound though the propulsion nozzle to the farfield, including the effects of diffraction by the jet pipe, refraction and Doppler amplification by the mean flow and the exchange of energy between acoustic and hydrodynamic modes at the nozzle lip. Both low and high frequencies are discussed. In the low frequency case, the essential aim is the derivation of simple analytical expressions and their interpretation in the context of various forms of acoustic analogy, and the analysis is continued to second order in a frequency parameter or Helmholtz number, giving predictions of the farfield directivity and of the magnitude of reflected waves in excellent agreement with experiments, and yet expressible in simple analytical fashion. An essential element of the solution of these problems is the satisfaction of a Kutta condition at the nozzle lip. In the high frequency case, two approximate theories are formulated, and compared with each other and with exact solutions. The theories we handle are Kirchhoff's approximation and the geometric theory of diffraction. The aim is again to provide a theoretical framework in which as many effects as possible can be handled in a rational manner.

The final two chapters are concerned with a particular aspect of compressor noise - the buzzsaw field generated by blading non-uniformities. This field is determined by the variation of shock strength of a non-uniform cascade. This is achieved by a combination of analysis of the detached shock waves ahead of a non-uniform

cascade and a linear examination of the outflow from the cascade showing this to depend on area alone, at typical operating conditions. The upshot is a relation between the pressure rise across the shock wave ahead of the nth and (n - 1)th blades. This relation is in significantly better agreement with experiment than relations using attached shock waves, and provides a theoretical basis for blade shuffling procedures designed to alleviate buzzsaw noise.

The aim throughout the thesis is to take the calculations as far as is possible and sensible by purely analytical means and to provide simple physical insight into the mechanisms involved.

ACKNOWLEDGEMENTS

I wish to express sincere thanks to my supervisor, Professor David Crighton, for his enthusiastic approach and encouragement during the course of these studies. My thanks are also due to him for his patience and careful guidance during the preparation of this thesis.

I also wish to express my sincere gratitude to the Head of Noise Research at Rolls-Royce Limited, Mr. Brian Barry, for his encouragement and for making it possible for me to undertake these studies.

My thanks must go to my colleagues at Rolls-Royce Limited for many useful discussions.

I gratefully acknowledge the financial support of the Science Research Council through an Industrial Studentship, and my employers Rolls-Royce Limited.

Finally, I am grateful to Mrs. E.S. Sedgewick, Mrs. J.L. Bunn, Miss L. Desmond and Mrs. R.M. Thompson for their careful typing of the thesis and to Mr. John Buckland and his staff in the Rolls-Royce Engineering Illustrations Group for their conscientious preparation of the figures.

To

My Parents

CONTENTS

N.B. Each chapter contains its own conclusions and references, and the figures and sections of each chapter are numbered separately.

CHAPTER 1

INTRODUCTION

INTRODUCTION

The noise of a modern aero-engine is made up of contributions from a large number of different sources. The chief among these are jet mixing noise, internally generated combustion associated noise, turbine noise and fan noise. This thesis addresses itself to two problems that are of importance in the generation and propagation of this noise. They are the propagation of internally generated sound out of the propulsion nozzle and the generation of sound by a supersonic fan. The aim throughout is to take the calculations as far as is feasible and sensible by purely analytical means, and to provide simple physical insight into the mechanisms involved. The purpose of this introduction is to set the work described in the main part of the thesis in some sort of context with respect to both the noise of modern aero-engines and the academic confines of the subject of aeroacoustics.

We begin by discussing the relevance of the work presented here to a modern aero-engine, showing how this differs from that of early engines, for which only jet noise was important. We then discuss the features of the interaction of sound with a vortex sheet such as will exist behind any sharp edge in a mean flow. Examples of such edges lie in the nozzle lip in the jetpipe problems discussed here, and in the trailing edges of turbomachinery blading. There follows a description of some of the important aspects of supersonic compressor performance, and also of the interaction of sound with a cascade. Finally, we list other work done under this Science Research Council Industrial Studentship, but which is not included in this thesis.

When the first jet engines appeared, their noise signature was dominated by sound of the jet mixing with its environment. This was first explained by Lighthill (1952, 1954), who showed the jet to be acoustically equivalent to a distribution of quadrupole sources and thence, by using simple scaling laws for the turbulence intensity, and its associated length and time scales, derived his famous eighth power law. This asserted that the noise increased with the eighth power of the jet velocity, and showed that the most direct way to reduce jet noise was to use engines with a lower jet velocity. When engines with lower jet velocities went into service (initially the low bypass ratio engines like the Rolls-Royce Spey and Conway, and Pratt and Whitney JT3D and JT8D, and later the high bypass RB211, JT9D and CF6), it was found that the jet noise had not been reduced by as large a margin as had been expected. Usually the jet noise is found to vary more nearly as the sixth power of the jet velocity. A useful survey of these differences between engines and model jets was given by Bushell (1971). Collating data from a great number of different sources, he showed that cold model jets did indeed follow a U^8 law, but that hot jets were noisier than this at low jet velocities. This difference has become known as the "excess noise" problem. It can be split conveniently into two parts: the increase in noise in going from a cold to a hot jet, and the difference between the noise of a hot model jet and that of an engine. Considering these in turn, it seems to be generally accepted that the low Mach number noise of a hot jet does scale on a sixth power law, this being caused by the scattering of "pseudo-sound" or turbulence pressure fluctuations, by density gradients. This was first demonstrated in the context of aerodynamic sound by Morfey (1973). Besides this, there is also a much weaker monopole component, scaling as the fourth power of jet velocity, but this can be shown to be small (Kempton, 1976), depending on the variation of the specific heats of the exhaust gases with temperature.

On an engine, a great many other sources have now been discovered which can increase the level of "jet-noise". The first and probably most important of these is combustion noise. Certainly, tests on isolated combustors produce levels which are comparable to those measured on engines (Mathews and Rekos, 1976) and the major component of the excess noise, with around 500Hz peak frequency on the larger engines, appears to correlate well with combustion parameters. As a result of the combustion process a turbulent flow of non-uniform temperature is produced. This can generate noise as it passes through any pressure drop, examples of which are the turbine rotor and stator blade rows (Pickett, 1974, Cumpsty & Marble, 1974), and the final nozzle (e.g. Ffowcs Williams & Howe, 1975). It remains unclear, though, how important such indirect combustion noise is, although Cumpsty (1975) has shown that theoretical predictions of its levels are consistent with observations on engines.

Other internal noise sources that may be important are the radiation of sound when the flow separates off the exhaust struts (e.g. Bryce & Stevens, 1975) causing them to radiate as dipole sources in the manner first described by Curle (1955), and the convection of turbulence past the nozzle lip. While this latter source was once thought to be important, this now seems unlikely, on account of the discovery that no sound is radiated from turbulence convected with the mean flow, when a Kutta condition is satisfied at the trailing edge, Howe (1976). We shall return to the issue of the Kutta condition later on in this introduction, and discuss that noise generation mechanism in Chapter 2 of this thesis.

Outside the nozzle, the jet noise may be further increased by so-called "jet noise amplification" (e.g. Moore, 1977, Bechert & Pfizenmaier, 1975). In this mechanism, the internal noise propagating down the jetpipe interacts with the nozzle lip to produce an instability wave in the jet which grows downstream, and causes the level of turbulence and hence of broadband jet noise to be artifically increased. The detailed mechanism by which energy is transferred to the instability wave will be discussed at length in Chapter 2 of this thesis.

Another mechanism that can increase jet noise in flight is what has come to be known as "installation effects" (see Bryce, 1979, for a general summary and Southern, 1980, for detailed experimental results). These are of two sorts, the first, important in flight, is that aerodynamic disturbances, such as wakes and the nacelle boundary layer, may distort the jet flow and raise the jet noise. The second, is that internally generated noise may be reflected or scattered by the aircraft control surfaces to give an enhanced level of low frequency noise. This level, which is small in relation to the jet noise statically, may be important in flight when the jet noise is reduced by the external flow.

In any event, it is now generally accepted that low frequency excess noise (other than the hot jet noise referred to above) is the result of some internal noise source. Now, any internal noise source must produce propagating sound waves that interact with the propagation nozzle in some way. It is this interaction that forms the subject of the first two chapters of this thesis.

Thus far, we have described the sources of low frequency noise. We now come to the high frequency noise, generated by turbomachinery. The most important element of this is fan noise, followed by turbine noise. The latter, which we shall discuss first, is not well understood. This is largely the result of the large number of stages that are important to the generation mechanism, and also the relatively small amount of research devoted to it as a result of its lesser importance. We only remark here that, like other internal noise fields, the noise of the turbine has to propagate out of the jetpipe.

This topic of high as opposed to low frequency propagation forms the subject of the next chapter of the thesis. As compared with low frequency propagation, we find that the application of the Kutta condition has less bearing on the radiated sound field, and the latter has the same character as it has in the absence of a mean flow, but modified by refraction, according to the rules of ray theory.

The main source of high frequency noise on a modern high by-pass aero-engine, the fan noise, may be split into a number of different components. The first of these is the noise directly associated with the pressure field of the fan (steady in rotor co-ordinates) and is only significant when the fan rotates at supersonic speeds so that the flow field ahead of the fan is composed of shock waves which propagate non-linearly (Hawkings, 1971) away from the fan face. At first sight, this would be expected to give rise to a field whose spectrum consists of tones at multiples of the blade passing frequency only. However, in practice the blades are not identical, so that the shock waves have different strengths from blade to blade, and thus propagate at different speeds. The result of this is that the wave-form is distorted and the initial periodicity is lost, so that its spectrum now contains all the harmonics of the fan rotational frequency. This gives the characteristic sound of buzzsaw noise. On modern engines, it is only significant at the take-off condition, and even there is not terribly important. Besides being radiated into the air outside the aircraft, however, buzzsaw noise can also be an important source of cabin noise. It is this aspect with which the last two chapters of the thesis are concerned.

The other sources of tone noise on aero-engines are termed usually "distortion tones" and "interaction tones". The former is caused mainly by ingested turbulence interacting with the fan, and is most important statically; when the aircraft is in flight, the ingested turbulence is relatively negligible, this source is no longer important (see e.g. Cumpsty & Lowrie, 1973) and interaction tones dominate. These interaction tones are predominantly due to the wakes of the rotor blades impinging on the downstream stators, causing unsteady loading on the stator vanes, and hence sound radiation. The sound radiation is organised into a set of circumferential modes (Tyler & Sofrin, 1962) and may be controlled by choosing the blade and vane numbers so that the induct sound field associated with these modes is axially decaying. Finally, the fan is also a substantial source of broadband noise (Ginder & Newby, 1976), which is highly sensitive to fan aerodynamic loading.

To summarise the situation as regards the importance of the various noise sources, we have reproduced two figures from Barry (1979), showing the breakdown of the noise of an RB211-524 engine at take-off and approach conditions, in terms of the subjectively important Perceived Noise Decibel (PNdB) units. At take-off thrust conditions, Fig. 1, the forward radiated fan noise and the jet noise are the most important noise sources, while at approach thrust, Fig. 2, the jet noise is relatively negligible, lower even than the self-noise of the airframe. It should be pointed out, though, that the relative levels of these sources are still controversial. In particular, the internally generated combustion or "core" noise is often thought to be more important than shown here, especially when evidence from source location procedures, such as the polar correlation technique (Fisher, Glegg & Harper-Bourne, 1975) and its devebpments (Fisher & Tester, 1981) are taken into account.

In the following pages, we discuss some of the issues that are central to the modellings contained in subsequent chapters of the thesis. The first of these is the interaction of sound with vortex sheets and trailing edges. The importance of these lies in the fact that trailing edges and vortex sheets occur in all the interaction problems in this thesis, both at the lip of the propulsion nozzle, and the ensuing jet shear layer (which may be treated as a vortex sheet) and also at the trailing edges of compressor blades.

As befits such a fundamental problem, the interaction of sound with a vortex sheet has a long history. Yet it was not until 1957 that Miles (1957) and Ribner (1957) gave the correct solution for the interaction of plane acoustic waves with a vortex sheet, earlier treatments having been in error as a result of using the wrong boundary conditions on the vortex sheet (i.e. conservation of velocity not sheet displacement). In considering the transmission of harmonic plane waves across a vortex sheet, one complete bypasses the issue of causality and of possible instabilities. It has long been known, of course, that a vortex sheet is unstable (Helmholtz, 1868, Kelvin, 1871) to small perturbations. These instabilities would be expected to arise if the vortex sheet were impulsively excited. The first attempt at treating a source near a vortex sheet was that of Gottlieb (1960). He was, however, only concerned with harmonic sources and his solutions only demonstrated the effects of refraction by the mean flow, important in pure jet noise studies. The time causal problem, with impulsive excitation, was first solved by Friedland & Pierce (1969) and later by Howe (1970). They showed that when a sound pulse interacted with a vortex sheet, in addition to the usual reflected and transmitted sound waves, the vortex sheet became unstable, the instability being convected by the mean flow.

The causal time harmonic problem was solved by Jones & Morgan (1972). They showed that the field of the instability waves was contained within a wedge-shaped region close to the vortex sheet, and that the form of the waves for an impulsive source was highly singular so that the mathematical theory of ultra-distributions (delta functions of complex argument) had to be used.

In practice, of course, the idealisation of the shear layer as a vortex sheet is not really valid, and other models must sometimes be used. For example, one can idealise the shear layer as a laminar flow of continuously variable mean velocity. Then it is found that amplifying instability waves only appear for wavelengths greater than a certain limit. While it might be thought that this thick shear layer result would contain all the significant effects, this is not the case, for three reasons. First, the shear layer grows downstream so that it cannot be idealised as a parallel mean flow, as is usually done; some account must be taken of the increase in width of the shear layer downstream for that eventually causes the instability waves to reach a location at which they cease to amplify and thereafter decay. Second, any instability wave will not, in practice, grow linearly; when it reaches a significant amplitude, so that the velocity perturbations are a significant fraction of the mean, non-linear effects will become important. In the case of a thin vortex layer, this corresponds physically to the point at which the vortex layer starts to roll up, and the displacement of the vortex sheet from its unperturbed position is not small. Lastly, from the point of view of sound propagating through the shear layer, it may not be relevant to treat the vortex sheet as a perturbed steady flow. This has been discussed in the context of jet noise by Crighton (1979), who shows that if the jet flow varies on a time scale less than the time taken for a sound wave to cross it, it can never be regarded as steady, and the problem might best be handled by considering the propagation of sound through an ensemble of possible jet states.

Having discussed the interaction of sound with an infinite vortex sheet, we now go on to discuss what is more important for this thesis, the interaction of sound with a semi-infinite vortex sheet. This has numerous applications in the whole field of aero-acoustics, and unsteady aerodynamics. In particular, we are interested in two cases, that where there are two different flows on either side of the vortex sheet, as in, for example, the lip of a jet pipe, and that where the two flows are the same, as in the trailing edge of the wing or compressor blade. In either case, a critical issue is what sort of condition should be imposed at the trailing edge of the splitter plate dividing the two flows. In the steady aerodynamics of incompressible flows, it has long been conventional to assume that there will be a Kutta condition: i.e., that the fluid leaves the trailing edge tangential to it. This is necessary for two reasons. First, if there were no Kutta condition the fluid would have an infinite velocity at the edge, and this would cause violent shedding of vorticity and flow separation. Second, without the application of trailing edge condition, the flow would be non-unique: a flow with any value of the circulation about the aerofoil would obey all the equations and boundary conditions (see, e.g., Batchelor, 1967). When the unsteady problems, such as the flow about an oscillating aerofoil were first studies (e.g. Theodorsen, 1935), it was natural to apply such a condition. The consequence of a Kutta condition in unsteady flow is that vorticity is now shed periodically at the trailing edge, to keep constant the circulation of the complete system, as required by Kelvin's theorem (see, e.g., Batchelor). This appears to provide a satisfactory mathematical solution to the problem, at least as long as the shear layer and the boundary layer on the splitter plate are thin. When this is not the case, the situation is more complicated as will be shown below.

When the velocities on either side of the splitter plate are different there is more scope for different approaches to the trailing edge problem. Orszag & Crow (1970) discussed this problem and defined three sorts of edge condition; the full Kutta condition, in which the flow leaves the plate with zero gradient, a rectified Kutta condition in which the flow leaves either with zero gradient, or directed into the moving fluid, and the case where there is no Kutta condition at all, implying infinite velocities. A feature of these solutions is that there is usually (except for one special case) an unstable vortex layer downstream, which grows, at least in the linearised thin vortex sheet model, as it travels downstream. This means that any solution containing the instability will be unbounded and special care must be taken of this in the mathematics. One can either assume that the solution is initially bounded, giving no unstable wave but also no Kutta condition, and then add on the part of the solution containing the unbounded instability wave later, or one can solve a complete causal problem in which the solution is only required to be causal, and may or may not be bounded. Which of these is more appropriate is open to doubt. One might at first think that causality was essential to the problem, and this is the approach adopted by Morgan (1974) and by Crighton & Leppington (1974). In their solutions, they found that the causal solution consisted of ordinary acoustic waves, reflected off the plate and reflected and transmitted through the vortex sheet, and also instability wave components which grow downstream. In this <u>causal</u> solution one has the option of satisfying a Kutta condition or not. Only in the former case is the velocity non-singular at the trailing edge.

Another point of view is that of Dowling, Ffowcs Williams & Goldstein (1978), who argue that in any turbulent flow, the question of causality is irrelevant, since the flow is determined by its own past history. This point is taken up in the context of these vortex sheet/ plate problems by Rienstra (1979). From the above discussion it should be clear that the Kutta condition is crucial to the response of a trailing edge to a disturbance, but that this does not necessarily have anything to do with the causality issue (although it may have ramifications as regards boundedness).

One now asks the question: does the real flow obey a Kutta condition? The answer to this appears to be that it does, at least at low frequencies. Crow & Champagne (1971) were the first to study this problem in the context of a jet. They applied a measure of unsteady forcing to the jet flow and observed that a definite coherent structure was formed in the jet. This corresponds to the instability waves predicted by theory and is responsible for the increases in the broadband noise of the jet with internal forcing observed by Moore (1977) and Bechert & Pfizenmaier (1975a). Thus it is clear that the instability waves are produced, consistent with a Kutta condition. Further evidence of the satisfaction of a Kutta condition was provided by Bechert & Pfizenmaier's (1975b) measurements of the actual behaviour at the edge of the shear layer. When interpreted correctly (see Crighton, 1981) these measurements are indeed consistent with the Kutta condition.

This far, we have only discussed the question of a semi-infinite vortex sheet. In practice the splitter plate or nozzle will have a boundary layer. The simplest way of accounting for a boundary layer is to assume that the mean flow, while still unidirectional, will vary with distance from the plate, and to then consider perturbations of that flow. This has been done by Goldstein (1979) who showed that non-zero boundary layer thickness had little effect on the basic physical phenomena. While this solution referred specifically to a splitter plate one could equally well apply his ideas to a nozzle. Then, the major change is that the instability waves are not those of a plug flow jet but of a jet with continuously variable velocity (see Michalke, 1971).

In reality, of course, the comments made earlier about the infinite shear layer, with regard to non-linearity, spatial spreading, and unsteadiness, apply equally well to the semi-infinite shear layer, and we will discuss them no further. In any event, it appears unlikely that these effects have any bearing on the interaction of unsteady flows with the trailing edge. In the thin vortex layer models we are still left with the ambiguity at the edge. This can be resolved at least for laminar flow, by the use of the triple deck type of solution (see, e.g., Daniels, 1978). In these, the flow around the edge is split up into several different regions, the solutions in which are obtained separately and matched together. From this this it appears that except when the unsteady flow is of very low amplitude, the flow <u>outside</u> the boundary layer does obey a Kutta condition in the parameter ranges where the theory applies. The acoustical implications of this theory have been discussed by Rienstra (1979) and more recently by Crighton (1981) who shows it to be wholly consistent with Bechert and Pfizenmaier's (1975b) experiments.

In these nozzle/vortex sheet problems it is always the case that only the simplest geometries can be handled analytically, for example, a cylindrical pipe and top-hat profile jet. Even this analytical solution becomes very cumbersome at high frequencies where many duct modes are propagating. In this regime other approximate methods are appropriate, as explained in the chapter on the propagation of high frequency sound out of a jetpipe. The oldest approximate method is that based on Kirchhoff's theorem, in which the solution to the governing wave equation is expressed in terms of the fields on the surfaces bounding the sound field. This is related to the classical theory of pistons in baffle plates dating back to Rayleigh (1896). We use that Kirchhoff theory to determine the radiation from a pipe, when a mean flow is present. A more accurate method is the geometric theory of diffraction (see e.g. Keller, 1957 & 1962). In this, the solution to a high frequency diffraction problem is shown to be made up of two parts: the geometric acoustics field neglecting scattering, and the fields diffracted by any edges and corners. The essence of the theory is that it is only necessary to treat diffraction on a local basis, so that the diffracted fields may be calculated using the theory of plane wave diffraction. After diffraction, any acoustic waves are assumed to propagate according to ray theory. Applications of this geometric theory of diffraction to aero-acoustics are discussed by Jones (1976) and Broadbent (1976) who show it to be inherently more accurate than the Kirchhoff solution. Fortunately the difference between the two is negligible at the peak angle of radiation, in these duct radiation problems. The geometric theory of diffraction is extensively used in optics and other electromagnetic wave propagation problems.

We turn now to the other main topic of the thesis, dealing with a cascade of non-uniformly staggered supersonic compressor blades, having detached shock waves. Aero-engine fans are usually designed so that they have supersonic tip speeds at the design condition. This means that a system of shock waves exists ahead of the fan over a considerable portion of the flight envelope, in fact everywhere but at the approach condition. The basic aerodynamic features of these fans are described in a recent review article by Kerrebrock (1981). We concentrate on the details of the shock waves. Typical flow patterns in the tip section of a fan are shown in Fig. 3 from

Dunker & Hungenburg (1980). The measurements were made using a laser anemometer. Three conditions are shown, representing different blade loading. When the blade is lightly loaded, the shock waves ahead of the blades are attached to the leading edges, and there is a complicated shock system in the passage. At the maximum efficiency condition, the shock waves are just detached and the in-passage shock stronger. When the fan approaches its highest loading (just below surge) the shock waves have become completely detached from the blade leading edges, and the passage shock is now strong, dividing subsonic and supersonic flows. This is most similar to the fan flow we model in the last two chapters of this thesis.

As we noted above, in practice the blades are non-uniformly staggered as a result of manufacturing tolerances. Thus, the shock wave strengths differ from blade to blade, so that they propagate forward at different speeds. This non-linear propagation can be handled by weak shock wave theory (see Whitham, 1974) as described by Hawkings (1981), and results in a distortion of the waveform seen in the fan duct, with the appearance of engine order tones, and the rapid decay of the harmonic of the blade passing frequency. In the past, the initial amplitude of the blades have been calculated on the basis that they are attached to the blade leading edges (e.g. Fink, 1971). But at the conditions of most interest this is certainly not the case, and as shown by Stratford and Newby (1976), that results in a quite different dependence of the flow on the blading parameters. When the flow is attached, the shock wave strengths only depend on the angles of the fore part of the blades, but when they are detached, the shock strength depends on the spillage of air around the leading edge of the blade which is governed by the mass flow at the exit of the cascade.

In this thesis we produce a model for the flow field in such a compressor with non-uniform blading, showing it to produce results in acceptable agreement with experiment. A key feature of the analysis is a calculation of mass flow out of the cascade. We show that when the flow is steady and the blade to blade differences small, the flows out of the individual blade passages depend on their area alone. The theoretical method we use to tackle this is the Wiener-Hopf technique (Noble, 1958), a method that we have also used in the jetpipe problems earlier in the thesis. It has also been used in other analyses of the unsteady flow in turbomachinery (e.g., Koch, 1971, Mani & Horvay, 1970, and Goldstein et al, 1977). Of these, the Goldstein paper, which is concerned with flutter, is the most relevant. They considered a fan with a supersonic entry flow separated from the subsonic exit flow by a strong shock wave. The blades are perturbed from their initial positions, in a similar manner to that used here, but the flow is unsteady. In addition to the problem examined in this thesis the concept of calculating the exit and entry flows to the cascade separately and then matching them together has more general application to turbomachinery noise (see below).

Besides the topics described in detail in subsequent chapters of this thesis, a number of other topics have been studied during the period of the Science Research Council Industrial Studentship. We now list these, giving a brief description of the work done in each case.

(1) Some existing work (Cargill & Duponchel, 1977) on the acoustics of inverted velocity profile coannular jets has been written up and accepted for publication in the Journal of Sound and Vibration, (to appear provisionally in Volume 78, No. 3). This work was concerned with the claimed advantages of inverted velocity profile jets in

an advanced supersonic transport aircraft. It showed that the benefits were considerably less than had been claimed as a result of various American research programmes.

(2) Work has begun on examination of ways of improving the resolution of source location schemes (e.g. Polar Correlation, Fisher et al, 1977) currently used on aero-engines. This resolution was thought to be limited to around a wavelength, but studies drawing on experience in such fields as spectral analysis, geophysics and radio-astronomy, mostly using the maximum entropy method, have shown that the resolution is, in fact, only limited by the 'noise' in the system. The first stage of the work was a review of the available methods of resolution enhancement, and formed the subject of a presentation to the 1980 British Theoretical Mechanics Colloquium, and in shortened form to the Rolls-Royce Noise Panel. The second stage of the work will involve a trial of the best of the methods using simulated data. It is intended that on completion of this work it will be written up and submitted to the Journal of Sound and Vibration for publication.

(3) The first draft of a comprehensive critical review of aero-engine turbomachinery noise has been written. This review had two main objectives; to guide future Rolls-Royce work on turbomachinery noise and to provide an up-to-date review of the state of aero-engine turbomachinery noise, highlighting weaknesses in theoretical and experimental knowledge. It is intended to publish the review in Progress in Aerospace Sciences, with whom it is currently

under review. Some of the work has also been presented to the Rolls-Royce Noise Panel.

(4) A substantial amount of work has been put in to lay the groundwork for a new attack on the problem of the generation of sound by turbomachinery. This has involved theoretical studies of the generation of sound in rotating flows and a new method of calculating the sound emitted from a cascade has been devised. This has involved solving for the cascade outlet and inlet flows, using the Wiener-Hopf technique, the two flows being matched together using slowly-varying duct theory. This allows the effect of blade loading to be accounted for in an elementary manner. Some three-dimensional effects are allowed for by allowing the blades to have a non-zero span, thus making the model quasi-three-dimensional. This, we argue, is the best method of treating the problem at the moment. All the available analytical methods involve unloaded blades, or complex numerical procedures that are not feasible at high frequencies. We maintain that in a practical turbomachine the effects of loading can never be completely ignored acoustically, as to do so would result in the wrong types of waves propagating at infinity and the erroneous predictions of acoustic cut-off. It is hoped that this method will be programmed in due course and will eventually be written for publication.

(5) Some work has been done to assist with the existing Rolls-Royce fan noise source location programme. This involved experimental work using a multiple microphone array and theoretical studies. It is described in an AIAA paper (Cargill 1980a). The work was instrumental

in highlighting a number of fundamental shortcomings in the technique.

(6) Some work has been done, aimed at understanding further the features of scattering by unsteady jet flows. This built on the work of Howe (1976b) and was presented in preliminary form in an AIAA paper (Cargill 1980b). The material in that paper also forms the basis for the work presented in Chapter 4 of this thesis and shows how energy is scattered into different frequencies by the unsteady flow so that tones are spectrally broadened. It is intended that this work will eventually be written up for publication.

(7) A limited amount of work has also been done to support other Rolls-Royce work on modelling installation effects, applying the geometric theory of diffraction ideas set out in Chapter 4. It is complementary to the experimental work described by Southern (1980).

In addition we note that, of the work in this thesis, the Chapter 2 has been accepted for publication in somewhat abbreviated form by the Journal of Fluid Mechanics, and appeared in summary form at the Conference on the Mechanics of Sound in Flows held in Göttingen in 1979 (Cargill, 1979). The two noise propagation papers have been submitted to the Journal of Sound and Vibration and the two fan noise papers to the Journal of Aircraft.

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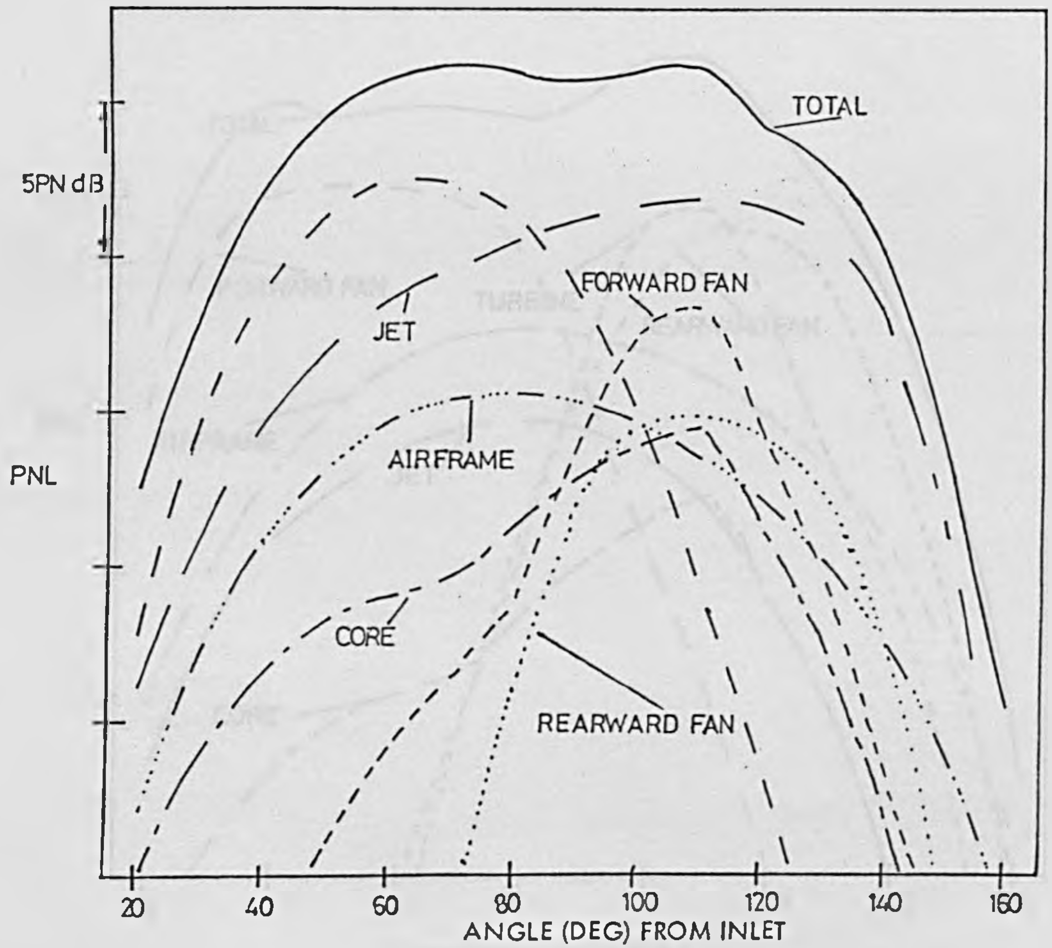

Fig. 1. RB211-524 in-flight noise sources at take-off thrust 800ft altitude.

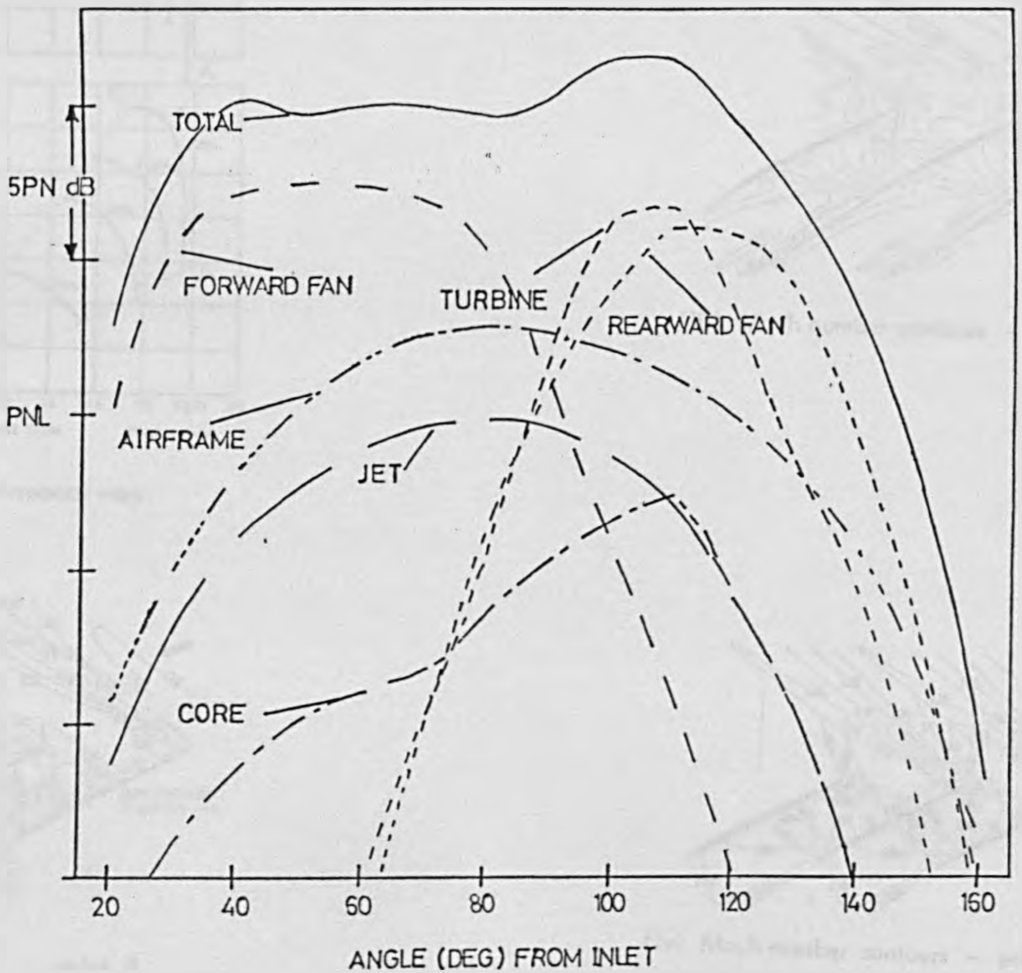

Fig. 2. RB211-524 in-flight noise sources at approach thrust 370ft altitude.

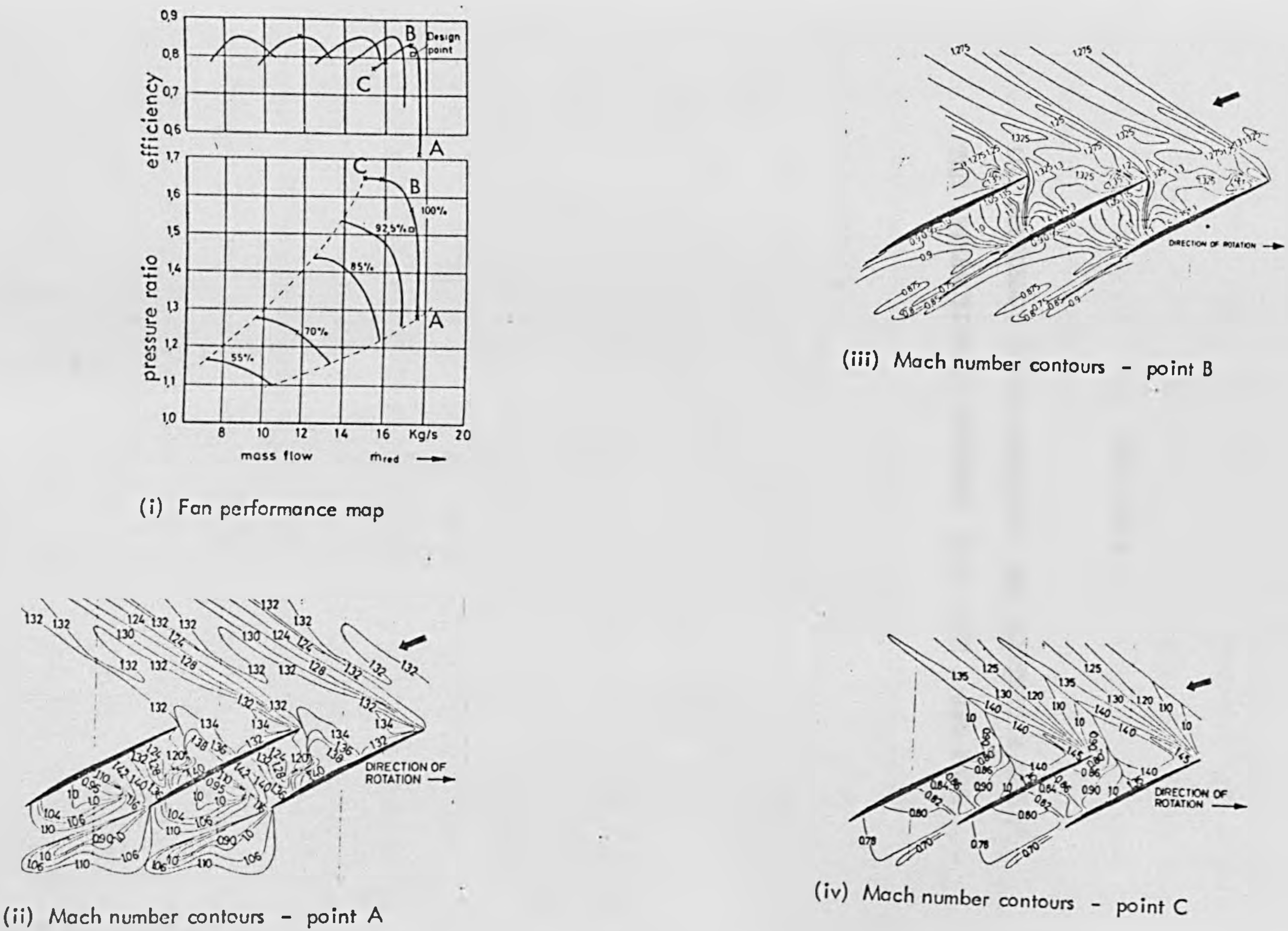

Fig. 3. Fan tip Mach number distributions (after Dunker & Hungenberg, 1980).

CHAPTER 2

LOW FREQUENCY SOUND RADIATION AND GENERATION DUE TO THE INTERACTION OF UNSTEADY FLOW WITH A JET PIPE

LOW FREQUENCY SOUND RADIATION AND GENERATION DUE TO THE INTERACTION OF UNSTEADY FLOW WITH A JET PIPE

ABSTRACT

In this chapter we examine the low frequency sound radiated when various types of unsteady flow interact with a jet pipe. In each case we solve the problem exactly by the Wiener-Hopf technique, producing results valid for arbitrary internal and external Mach numbers and temperatures, discuss the importance of a Kutta condition at the duct exit and provide an interpretation in elementary terms of the radiated sound field using the Lighthill acoustic analogy.

When low frequency sound propagates down the jet pipe, little of it reaches the farfield and the major disturbance outside the pipe is that associated with the jet instability waves. At subsonic jet speeds and low enough Strouhal number these waves transport kinetic energy at a rate precisely balancing the loss of acoustic energy from the pipe, resulting in a net attenuation of the sound power. For supersonic jet conditions a further wave motion, the unsteady flow counterpart of the steady wave structure of an imperfectly expanded jet, is present in addition to the instability wave. We use the Lighthill acoustic analogy to show that for high enough jet Mach number and temperature, the sound radiation is largely caused by quadrupole sources arising from the jet instability waves. An alternative interpretation uses the acoustic analogy incorporating a mean flow due to Dowling, Ffowcs Williams

and Goldstein and expresses the farfield sound as the sum of contributions from monopoles and dipoles distributed over the duct exit. The directivity and power of the calculated farfield sound are in good agreement with experiments.

We also calculate the sound scattered by the jet pipe when there is an incident external sound field, and show a previously published result to be in error. In general, the flow phenomena produced by internal and external incident sound fields are similar.

When vorticity is convected past the pipe exit we find that the imposition of a Kutta condition is of crucial importance in determining the radiation. Indeed, if the vorticity is convected at the speed of the flow, no sound is radiated when a Kutta condition is enforced, confirming for the jet pipe flow a result given by Howe for two-dimensional unsteady flow past a splitter plate. However, in both this and the other problems discussed in this chapter all fields are causal, whether or not a Kutta condition is enforced.

Finally, we discuss the effects of nozzle contraction. We find that the radiated sound field is little changed in character, but that the reflection properties of the nozzle may be drastically altered.

CONTENTS

1. INTRODUCTION

In this chapter we examine the interaction between a number of types of unsteady flow and a jet pipe. The motivation behind this study was the so-called "excess noise" problem on jet engines. It has been found that when the noise of an engine is measured statically, it is somewhat greater than would have been predicted on the basis of tests on model jets. This discrepancy is even greater in flight and has been the subject of a great deal of research (Bryce 1979). In this paper we model some of the possible mechanisms of excess noise: the transmission of internally generated noise out of the jet pipe to the farfield, the scattering of external sound fields by the jet pipe, and the convection of turbulence past the end of the jet pipe. We consider only the low frequency limit, but unlike many other authors we allow the mean flow both outside and inside the pipe to have arbitrary Mach numbers and temperatures. This is important since at the conditions of interest (typically jet Mach number 0.8, internal to external temperature ratio 2.5) the effects of flow may be considerable. For example, Goldstein (1976) shows that placing a low frequency acoustic source inside a jet flow has a dramatic effect on the field-shape of the radiated sound.

The problem we solve first is the propagation of acoustic waves out of the jet pipe to the farfield; in this as in the other problems we idealise the propulsion nozzle as a semi-infinite rigid cylindrical pipe. The mean flow outside the pipe consists of a uniform semi-infinite jet bounded by a vortex sheet. We confine the discussion to low frequencies, where the incident sound field in the pipe is in the form of plane waves. In the absence of a mean flow this problem was first solved by Levine & Schwinger (1948) using the Wiener-Hopf method. Their solution was extended to include the same uniform mean flow both inside and outside the pipe by Carrier (1956). The first attempts to include the effects of different mean flows inside and outside the pipe were made by Mani (1973) and Savkar (1975), for plane and circular pipe geometry respectively, who used an approximate method. The exact solution to the circular pipe problem was found by Munt (1977) who, again using the Wiener-Hopf technique, allowed for arbitrary internal and external Mach numbers and temperatures and obtained fieldshapes for the radiated sound in excellent agreement with experiments. Munt later extended his work to calculate both the amplitude of the sound reflected back up the pipe (1981a) and to examine the variation in the total power radiated with jet conditions (1981b). The power radiation has been studied experimentally by Bechert et al (1977) who observed that the power radiated to the farfield could be substantially less than the net power flow along the pipe, so that there was a net loss of acoustic energy. Munt's paper (1981b) is consistent with these results, as is the work of Howe (1979) who has studied the sound transmission problem in the low Mach number, low frequency approximation. Bechert (1979) also explains this net power loss using a simple theory similar to that of the present work only at very low Mach numbers.

In this chapter we use the low frequency asymptote of Munt's theory to obtain simple expressions for the sound radiated to the far-field, that reflected back up the jet pipe, and for the unsteady motion of the jet column. The latter consists mainly of a spatially growing instability wave. This is an important feature of all problems involving the interaction of unsteady flows and jet pipes. In the limit of vanishingly low frequency this instability wave grows only very slowly and is convected with the mean flow. In addition to the usual low frequency limit we also discuss the case where the jet is very hot compared with its surroundings, so that, as it were, it is hotter than it is acoustically compact. Here, there is a dramatic change in the nature of the radiated sound field, similar to that found by Dowling et al (1978) in their study of jet noise. In Munt's paper it is assumed that the sound radiated is causally related to the incident sound field and that a Kutta condition is obeyed at the duct exit. We discuss the effect of relaxing the Kutta condition while still insisting on causality, and establish that the jet instability wave can then be made to vanish. In that case, there is no loss of acoustic energy, and all the power in the incident wave is reflected back up the duct, apart from an $O(k^2a^2)$ fraction which is lost to the far-field. We further use an idea of Howe (1979) to provide an alternative modelling of the instability waves.

Munt's solution only allows for subsonic jet speeds. We extend his theory to cover supersonic jet conditions, using concepts due to Morgan (1974) and show that an additional physical phenomenon is present at these speeds; the unsteady counterpart of the periodic steady wave structure of an imperfectly expanded supersonic jet.

Using methods similar to those of Munt we determine the sound scattered when an external sound field is incident on the pipe. In the absence of a mean flow the solution is known (see e.g. Noble, 1958) and may be deduced from that for incident internal sound by reciprocal arguments. There is no existing exact solution when a mean flow is present, the only published work being the approximate solution of Jacques (1975). We show that his solution is in error, although the scaling laws he deduces are substantially correct when the incident sound waves are due to some near-by aerodynamic disturbance.

We model the generation of sound when turbulence is convected by the mean flow past the end of the jet pipe, by idealising the turbulence as convected vortex rings. The published work on this low frequency problem is limited to two cases. Leppington (1971) models the turbulence as non-convected quadrupoles, whose near field is scattered as sound by the end of the pipe, resulting in farfield sound levels which scale as the sixth power of the jet velocity. Crighton (1972) models the problem as the scattering of the energy of a jet instability wave by the pipe exit and finds the same overall scaling laws as Leppington. Related to these problems is work by Howe (1976) on the sound generated when vortices are convected past the trailing edge of a flat plate. He finds that the sound radiated depends critically on the imposition of a Kutta condition at the edge of the plate. When a Kutta condition is enforced and the vortices are convected with the mean flow, then no sound is radiated. In our study of the semi infinite cylindrical pipe, we find that a similar result holds.

A useful way of examining sound radiation problems is by use of acoustic analogies. These ascribe the sound radiation to monopole and dipole sources on bounding surfaces, and to quadrupole sources distributed throughout the flow field. We use two different acoustic analogies; that of Lighthill (1952) as reformulated by Ffowcs Williams & Hawkings (1969), which does not explicitly include the fluid shielding effects of any mean flow, and that of Dowling et al (1978) which does. In each case the source terms are determined using the lowest order asymptotic low frequency solutions for the flows in the jet and the pipe. We show that the sound fields determined in this way are precisely the same as those obtained exactly by the Wiener-Hopf method.

Thus far we have idealised the end of the jet pipe by a cylindrical pipe. On real engines the end of the jet pipe contracts to form a nozzle. We analyse the transmission of sound through such a nozzle, and the sound generated when entropy waves are convected through the contraction. We use the methods of Marble & Candel (1977) and Cumpsty & Marble (1977), who were concerned with variable area ducts and the transmission of acoustic waves across turbines, respectively. Our results for the transmission problem are in good agreement with the recent experimental results of Bechert (1979) and our expressions for the sound generated by entropy waves are essentially the same as those obtained by Ffowcs Williams & Howe (1975) using another method.

Finally, we discuss the practical significance of our results and compare them with the limited experimental evidence.

2. RADIATION OF INTERNAL NOISE FROM A JET PIPE

In this section we consider the radiation of low frequency internal noise from a cylindrical pipe with both internal and external flows. We first solve the problem for a subsonic jet in the low frequency limit, subject to the condition that it satisfies a trailing edge Kutta condition. Then, we discuss the implications of relaxing the Kutta condition and finally modify the analysis to allow for supersonic jet conditions.

2.1. Subsonic Jet with Kutta condition

The mathematics in this section largely follows the work of Munt (1977). For convenience, and to aid comparison with his papers, we use his notation. While this problem has been solved in some detail by Munt we repeat the steps in the mathematics since the analysis forms the basis for both the rest of this section and for sections 3 and 4 of this paper. The major difference between our analysis and Munt's is that we choose to work with pressure rather than velocity potential as the fundamental variable.

We consider a cylindrical semi-infinite rigid tube of radius a, from which issues a jet of density ρ_j, speed of sound c_j and velocity $U_j = M\, c_j$, occupying the region $x > 0$, $r < a$. The jet and pipe are assumed to be immersed in an infinite region of velocity $U_o = \alpha\, M\, c_j$, density $\rho_o = \gamma\, \rho_j$ and speed of sound $c_o = c_j/C$. We assume that α covers the range $0 \leqslant \alpha \leqslant 1$.

The non-dimensional quantities γ, C, α express the ratios of the mainstream to jet value of density, reciprocal of sound speed, and velocity. When the jet and mainstream are perfect gases with the same specific heats, we find that $\gamma = C^2$.

The waves in the pipe are assumed to have the time dependence $e^{i\omega t}$, and this factor is suppressed throughout the analysis. The equations satisfied by the pressure fluctuations in cylindrical coordinates are:

$$\frac{\partial^2 p}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial \phi^2} - \left(ik + M\frac{\partial}{\partial x}\right)^2 p = 0, \qquad r < a \tag{2.1}$$

$$\frac{\partial^2 p}{\partial x^2} + \frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial p}{\partial r} + \frac{1}{r^2}\frac{\partial^2 p}{\partial r^2} - C^2\left(ik + M\,\alpha\frac{\partial}{\partial x}\right)^2 p = 0, \qquad r > a \tag{2.2}$$

where $k = \omega/c_j$. From the assumption that the cylinder is rigid, one derives the boundary condition that the normal gradient of pressure vanishes on it,

$$\frac{\partial p}{\partial r}(a,\phi,x) = 0, \qquad x \leqslant 0. \tag{2.3}$$

The boundary conditions on the jet vortex layer are the continuity of pressure, so that

$$p(a^-,\phi,x) = p(a^+,\phi,x), \qquad x \geqslant 0 \tag{2.4}$$

and the kinematic condition of equal particle displacement on both sides of the vortex layer. Let $\eta(x,\phi)$ denote the displacement of the vortex layer from its mean position, $r = a$. Then this latter condition implies that η satisfies

$$\left.\begin{aligned} c_j^2\left(ik + M\frac{\partial}{\partial x}\right)^2 \eta(a,\phi) &= -\frac{1}{\rho_j}\frac{\partial p}{\partial r}(a^-,\phi,x) \\ c_j^2\left(ik + \alpha M\frac{\partial}{\partial x}\right)^2 \eta(a,\phi) &= -\frac{1}{\gamma\rho_j}\frac{\partial p}{\partial r}(a^+,\phi,x) \end{aligned}\right\} \quad (x > 0). \tag{2.5}$$

Two other conditions are important in determining the sound field: causality, and the Kutta condition. Causality is defined to be the requirement that the sound field shall vanish for impulsive excitation before the source is switched on. As Jones & Morgan (1974) have shown, if a time harmonic solution is used, this must then obey certain constraints on its behaviour in the lower half plane for complex k. The Kutta condition concerns the requirement to be satisfied by the displacement of the vortex layer at the edge of the cylinder. The usual Kutta condition is that the layer should leave the end of the pipe with zero gradient. The solution found by Munt satisfies both causality and this Kutta condition. We shall later discuss solutions that are causal but do not satisfy a Kutta condition.

Accordingly we now require for our solution that

$$\frac{\partial \eta}{\partial x}(o^+,\phi) = 0. \tag{2.6}$$

We split the total field into two parts, an incident field which is assumed to be known and the additional term arising from its interaction with the pipe. We assume that the incident field has the form of an acoustic duct mode with

$$p_i(r,\phi,x) = \frac{J_m(j'_{mn}\, r/a)}{J_m\,(j'_{mn})} \exp[-i(\mu_{mn}\, x - m\phi)], \quad r < a$$
$$= 0, \qquad r > a, \tag{2.7}$$

which satisfies (2.1) and (2.3) and where

$$\mu_{mn} = \frac{[k^2 - (1 - M^2)j'^2_{mn}/a^2]^{\frac{1}{2}} - kM}{(1 - M^2)}$$

with Im $\mu < 0$. Here j'_{mn} is the nth zero of $\frac{d}{dy}\, J_m(y)$ and $J_m(y)$ is the Bessel function of order m. Since the primary wave has the dependence $e^{im\phi}$, we further assume that the diffracted field has the same dependence.

To assist the analysis we assume that k has a negative imaginary part, so that any waves produced will decay as $x \to \pm\, \infty$. In particular we define $k = k_r + ik_i = |k| \exp(-\, i\delta)$, where $0 < \delta < \pi$. At the end of the analysis we shall put $\delta = 0$ to obtain the solution for real ω.

We define the half range Fourier transforms of any quantity ψ, say, by the formulae

$$\Psi^{\pm}(u) = \int_{-\infty}^{+\infty} \psi(x) \exp(+\, ikux)\, H(\pm\, x)\, dx, \tag{2.8}$$

where H(x) is the unit step function with

$$H(x) = 1, \qquad x > 0,$$

$$H(x) = 0, \qquad x < 0.$$

The inverse of these transforms is given correspondingly

$$\psi(x) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \Psi(u)\ k \exp(-\ ikux)\ du, \tag{2.9}$$

where

$$\Psi(u) = \Psi^{+}(u) + \Psi^{-}(u).$$

After Fourier transformation the equations of motion, (2.1), (2.2) become

$$\{ \frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} + k^2 v^2(u) - \frac{m^2}{r^2} \}\ \ P(k,r) = 0, \quad r < a, \tag{2.10}$$

$$\{ \frac{1}{r}\frac{\partial}{\partial r}\, r\, \frac{\partial}{\partial r} + k^2 w^2(u) - \frac{m^2}{r^2} \}\ \ P(k,r) = 0, \quad r > a, \tag{2.11}$$

in which we have defined

$$v^2(u) = ((1 - Mu)^2 - u^2), \tag{2.12}$$

$$w^2(u) = ((1 - M\alpha u)^2 - u^2). \tag{2.13}$$

The branches of w, v are taken to be those where $Im(v,w) < 0$ as $u \to +\infty$. The dependence on u of the transform $P(k,r,u)$ will sometimes be omitted, as in (2.10), (2.11), while elsewhere it will be the dependence on (u,r) which is explicitly displayed. The branch cuts are taken to be from

$$u = \frac{1}{(1 + M)} \text{ to } + \infty \text{ and } \frac{-1}{(1 - M)} \text{ to } - \infty \text{ for } v(u),$$

and from

$$u = \frac{C}{(1 + \alpha MC)} \text{ to } + \infty \text{ and } \frac{-C}{(1 - \alpha MC)} \text{ to } - \infty \text{ for } w(u).$$

It therefore follows that the ± Fourier transforms have the regions of regularity, $R^{\pm}$ shown in Fig. 1. In that diagram we have shown the branch cuts drawn with $\frac{C}{(1 + \alpha MC)} > \frac{1}{(1 + M)}$, and $\frac{1}{(1 - M)} > \frac{C}{(1 - \alpha MC)}$. If this condition is not satisfied the order of the branch points on the real u axis should be reversed.

Both half range transforms can be seen to be analytic in the region of overlap between $R^{\pm}$, and the integration path in (2.9) is taken to lie in this strip, and specifically along the line $\arg u = \delta$.

The solutions to (2.10), (2.11) are Bessel functions of order m. We require that the solution be finite at r = 0 and decay as $r \to \infty$ for u in the strip.

Hence

$$\left.\begin{aligned} P(u,r) &= A(u)\, J_m(kvr) \qquad r < a \\ &= B(u)\, H_m^{(2)}(kwr) \qquad r > a \end{aligned}\right\} . \tag{2.13}$$

Defining, further, the half range transforms of the boundary displacement as $Z^{\pm}$, the boundary conditions (2.3), (2.4) become

$$P_j^+(u,a^-) + P_i^+(u,a^-) = P_o^+(u,a^+) \tag{2.14}$$

and

$$Z^-(u) = 0, \tag{2.15}$$

in which P_j, P_o are the transforms of the pressure in $r < a$, $r > a$, and P_i is the transform of the incident pressure

$$P_i^+(u,a^-) = \int_o^\infty \frac{J_m(j'_{mn}a^-)}{J_m(j'_{mn}a)} e^{-i\mu_{mn}x + ikux}\, dx,$$

$$= \frac{1}{i(\mu_{mn} - ku)} , \tag{2.16}$$

for u in R^+.

We solve (2.14) and (2.15) by noting that

$$P_j(u,a^-) - P_o(u,a^+) + P_i^+(u,a^-) = P_j^-(u,a^-) - P_o^-(u,a^-),$$

$$= F^-, \tag{2.17}$$

a function regular in R^-.

Using equation (2.5) we find that

$$P_j(u,a^-) = \frac{Z(u)\ \rho_j c_j{}^2 k^2 . D_j^2\ J_m(kva)}{kv\ J_m'(kva)} , \tag{2.18}$$

$$P_j(u,a^+) = \frac{Z(u)\ \rho_j c_j{}^2 k^2 \gamma D_o^2\ H_m^{(2)}(kwa)}{kw\ H_m^{(2)}\ (kwa)} , \tag{2.19}$$

where $D_j{}^2 = (1 - Mu)^2$, $D_o^2 = (1 - \alpha Mu)^2$. Whence, substituting

(2.18), (2.19), (2.16), in (2.19) and noting that, from (2.5), $Z^-(u) = 0$, we find that

$$K(u)\, Z^+(u) + \frac{1}{i(\mu_{mn} - ku)} = F^-(u), \tag{2.20}$$

where

$$K(u) = \rho_j c_j^{\,2} k^2 \left[\frac{D_j^{\,2} J_m(kva)}{kv\, J_m'(kva)} - \frac{\gamma D_o^{\,2}\, H_m^{(2)}(kwa)}{kw\, H_m^{(2)\prime}(kwa)} \right] . \tag{2.21}$$

We solve (2.20) by the Wiener-Hopf technique, described, for example, in Noble's book (1958).

We factorise $K(u)$ as $K(u) = K^+(u)\, K^-(u)$ where, $K^{\pm}(u)$ are analytic and non-zero in the two half planes R^+ and R^-. Then (2.20) gives

$$Z^+(u)\, K^+(u) + \frac{1}{K^-(u)\, i(\mu - ku)} = \frac{F^-(u)}{K^-(u)} . \tag{2.22}$$

This may be rewritten as

$$Z^+(u)\, K^+(u) + \frac{1}{K^-(\frac{\mu}{k})\, i(\mu - ku)} = \frac{F^-(u)}{K^-(u)}$$

$$- \left(\frac{1}{K^-(u)} - \frac{1}{K^-(\mu/k)}\right) \frac{1}{i(\mu - ku)} . \tag{2.23}$$

The left and right hand sides of this equation are analytic in the respective half planes $R_{\pm}$. By an extension of Liouville's theorem they must therefore both equal some function $\bar{C}(u)$, which is regular over the whole u plane, except at infinity. The only form of $\bar{C}(u)$ which has the required properties is a polynomial in u.

We consider now the edge conditions on η, and the resulting constraints on the behaviour of Z(u). From Munt (1977), we find that if $k \in \Delta_o$, where Δ_o is a region of the k plane determined by the instability zeroes u_o, u_o^* of K(u), then $K^+(u) = O(u^{3/2})$, $K^-(u) = O(u^{-1/2})$ as $|u| \to \infty$, but if $k \notin \Delta_o$, then

$$K^+(u) = O(u^{1/2}), \quad K^-(u) = O(u^{1/2}) \text{ as } |u| \to \infty \quad .$$

The region Δ_o is shown in Fig. 2.

Then if $k \in \Delta$, we find that if $\eta(x)$ is $O(x^n)$, then Z(u) is $O(u^{-(n+1)})$, and the left hand side of (2.23) is $O(x^{-(n-\frac{1}{2})})$. This means that $\bar{C}(u)$ is a polynomial of order $(^1/_2 - n)$. For instance if $n = {}^3/_2$ and the solution obeys the Kutta condition, we are restricted to $\bar{C}(u) = 0$. If the solution is the least singular one <u>not</u> obeying a Kutta condition, then $\bar{C}(u)$ must be a constant. The procedure for obtaining a causal solution in either case is to solve equation (2.23) for some $k \in \Delta$, for example with $\delta = \frac{\pi}{2}$, and then argue the result for real k by analytic continuation.

Hence we find that

$$Z^+(u) = \left\{\frac{-1}{i(\mu - ku)\, K^+(u)\, K^-(\mu/_k)} + \frac{\bar{C}(u)}{K^+(u)}\right\} \quad ; \tag{2.24}$$

as $Z^-(u) = 0$, this is also the value of Z(u). Then hence by (2.9)

$$\eta(x,k) = \int_{-\infty \exp i\delta}^{+\infty \exp i\delta} [Z^+(u)]e^{-ikux}\, k\, du. \tag{2.25}$$

As δ passes from $\delta = \frac{\pi}{2}$ to $\delta = 0$, the pole at $u = u_o$ is passed over by the integration path for u. For a causal solution we require, from a theorem of Jones & Morgan (1974), that $\eta(x,k)$ is analytic in the lower half plane, and that

$$\exp[(ib + d)k]\ \eta(x,k) = O(|k|^p) \text{ as } |k| \to \infty,$$

where b, d are real numbers and $b > 0$. Therefore there must be no discontinuity in η as the contour passes over u_o. Therefore, we must add on a residue contribution from the pole for $\delta < \arg u_o$, and thus for a causal solution we have

$$\eta(x,k) = \frac{1}{2\pi} \int_{-\infty \exp i\delta}^{+\infty \exp i\delta} Z^+(u) e^{-ikux}\, k\,du$$

$$+ H(\arg u_o - \delta) \lim_{u \to u_o} i\, Z^+(u_o)\, (u - u_o) k\, e^{-iku_o x} . \qquad (2.26)$$

With reference to Morgan (1974) it can be seen that the solution is causal whether $\bar{C}(u) = 0$ (Kutta condition) or $C(u)$ is a constant (no Kutta condition).

Then the causal solution subject to a Kutta condition is given by:

$$Z(u) = \frac{i}{K^+(u)\, K^-(\mu/_k)\, (\mu - ku)} , \qquad (2.27)$$

$$P_j(u) = \frac{i\rho_j c_j{}^2 k^2 D_j{}^2 J_m(kvr)}{kv\, J_m'(kva)\, K^+(u)\, K^-(\mu/_k)\, (\mu - ku)} , \qquad (2.28)$$

$$P_o(u) = \frac{i\rho_j c_j^2 \, k^2 \gamma \, D_o^2 \, H_m^{(2)}(kwr)}{kw \, H_m^{(2)\prime}(kwa) \, K^+(u) \, K^-(\mu/_k) \, (\mu - ku)} \quad . \tag{2.29}$$

The general properties of the split functions have been given by Munt. We list them in the low frequency limit in Appendix I.

We consider first, and in most detail, solutions for an incident plane acoustic wave. In this limit the split functions are

$$K^+(u) = -\frac{2M^2 a(u - u_o^*)\,(u - u_o)\rho_j c_j^2 k^2}{(ka)^2 \, v^+}, \quad K^-(u) = \frac{1}{v^-} \quad . \tag{2.30}$$

The pressure perturbation, for $r < a$, is given by

$$p_j(x) = -\frac{1}{2\pi}\int_{-\infty \exp i\delta}^{\infty \exp i\delta} \frac{\rho_j c_j{}^2 k^2 D_j^2 \, J_m(kva) k e^{-ikux} \, du}{kv \, J_m'(kva) \, i \, (\mu - ku) \, K^+(u) \, K^-(\mu/_k)} \, . \tag{2.31}$$

In the limit of very low frequency,

$$\left.\begin{aligned} K^+(u) &= \frac{-2a(1 - Mu)^2 \rho_j c_j^2 k^2}{(ka)^2 \, (1 - (1 + M)u)} \, , \\ \mu/_k &= \frac{1}{(1 + M)} \, , \\ K^-(\mu/_k) &= \frac{(1 + M)}{2} \, , \end{aligned}\right\} \tag{2.32}$$

and therefore,

$$p_j(x) = \frac{1}{\pi}\int_{-\infty}^{+\infty} \frac{e^{-ikux} \, du}{(1 - M^2)\,(u - \frac{1}{(1 + M)})\,(u + \frac{1}{(1 - M)})} \, . \tag{2.33}$$

The value of this integral is equal to one or other of the pole residue contributions according as x is greater than or less than zero. The pole at $u = 1/(1 + M)$ cancels out the incident field for $x > 0$, while the pole at $u = \frac{-1}{(1 - M)}$ gives the reflected field inside the jet pipe. This has a value $p_j = - \exp[ikx/(1 - M)]$, and therefore the reflection coefficient R for incident plane waves is, in the low frequency limit

$$R = -1 \ . \tag{2.34}$$

The result (2.34) is the basis for Bechert's (1979) simple theory of nozzle flow sound attenuation; it is entirely dependent on the satisfaction of a Kutta condition at the nozzle lip (see later).

It is clear that in this low frequency limit there are no contributions to the pressure from instability waves. This follows from our approximation to $K^+(u)$ in which we set the instability poles, u_o, u_o^*, at $^1/_M$, rather than the more exact value (see Appendix) of $\frac{1}{M} \pm i\sigma$. If the latter value had been used in $K^+(u)$, there would have been contributions to the pressure from these two instability waves, growing and decaying exponentially along the jet. The value of the contribution to the pressure from these two poles is $O(\sigma^2)$ which is negligible for low enough frequency. The axial velocity in the jet does, however, contain contributions corresponding to the instability waves, and it is given exactly by

$$u_x = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\rho_j c_j^2 \; k^2 D_j^2 \; J_m(kva) e^{-ikux} \; ku \; du}{kv \; J_m'(kva) \; i(\mu - ku) \; K^+(u) \; K^-(^\mu/_k) \; (1 - Mu)\rho_j c_j} \ , \tag{2.35}$$

which, in the low frequency limit, becomes

$$u_x = \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{u\, e^{-ikux}\, du}{\rho_j c_j (1 - M^2)(u - \frac{1}{(1+M)})(u + \frac{1}{(1-M)})(1 - Mu)} .$$

The contributions from the poles $u = \frac{1}{(1+M)}$, $\frac{-1}{(1-M)}$, are the acoustic waves discussed above. The contribution from the pole at $u = {}^1/_M$, gives the velocity fluctuation

$$u_x = \frac{2}{\rho_j c_j} \exp[-ikx/M]. \tag{2.36}$$

We see that this represents a convected instability wave albeit of vanishingly small growth rate. If we had used the more complete form of $K^+(u)$, u_x would have been given by

$$u_x =$$

$$\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{2u\,(1-Mu)\, e^{-ikux}\, du}{\rho_j c_j\, M^2(1-M^2)(u - \frac{1}{(1+M)})(u + \frac{1}{(1-M)})(u - \frac{(1-i\sigma)}{M})(u - \frac{(1+i\sigma)}{M})}, \tag{2.37}$$

and the instability wave contribution to this gives

$$u_x = \frac{e^{-ikx/M}}{\rho_j c_j} \left[e^{k\sigma x/M} + e^{-k\sigma x/M}\right] ,$$

which tends to the previous result (2.36) for $x \ll {}^1/_\sigma$, but displays clearly the amplification and decay factors of spatial instability waves convected with the jet speed.

We can also derive the jet displacement due to these instability waves: it is

$$\eta(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{-(ka)^2\ (1-(1+M)u)e^{-ikux}\ k\ du}{2M^2a\ (u-\frac{(1-i\sigma)}{M})\ (u-\frac{(1+i\sigma)}{M})\ (\frac{1+M}{2})\rho_j c_j k^2 (\frac{1}{(1+M)}-u)}, \tag{2.38}$$

$$= \frac{ae^{-ikx/M}}{\rho_j c_j{}^2 M^2} \left[\frac{\sinh\ (kx\sigma/M)}{2\sigma/M}\right] . \tag{2.39}$$

For small x, $x \lesssim M/k\sigma$, this expression shows that the shear layer displacement grows linearly with distance from the end of the pipe. This expression is only valid for $x \gg a$, however, since our expression for $K^+(u)$ was only valid for $uka \ll 1$ and the behaviour near $x = 0$ depends on the value of $K^+(u)$ as $u \to \infty$. Despite this, however, the approximate form of $\eta(x)$ does at least give zero displacement at $x = 0$, even if the slope of the shear layer is non-zero; the exact equation for $\eta(x)$ does of course have zero slope at $x = 0$.

The linear growth of the amplitude of the instability waves with distance has a simple physical explanation. Consider a jet with an instability wave of negligible growth rate whose axial velocity fluctuation is $u_x = \hat{u}\exp[i\omega(t - x/U_j)]$. For this wave, the pressure fluctuation is zero. Therefore the continuity equation can be written

$$\frac{1}{r}\frac{\partial vr}{\partial r} + \frac{\partial u}{\partial x} = 0,$$

where v is the radial velocity, and hence

$$v = \frac{i\omega r}{2U_j} \hat{u} \exp[i\omega (t - x/U_j)].$$

Now the displacement of the jet boundary is related to the velocity v, by

$$\frac{\partial\eta}{\partial t} + U_j \frac{\partial\eta}{\partial x} = v.$$

We assume that η is the form $\eta = \hat{\eta}(x) \exp[i\omega(t - x/U_j)]$ and then it is clear that if $\eta(o) = 0$,

$$\hat{\eta} = \frac{\hat{u}}{U_j} \cdot \frac{i\omega r}{2U_j} \cdot x. \tag{2.40}$$

This is precisely the same relation (2.39) obtained from the exact analysis.

We consider next the sound field outside the jet. The pressure fluctuation is, from (2.29),

$$p_o = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{\rho_j c_j^2 k^2 \gamma D_o^2 H_m^{(2)} (kwr) e^{-ikux} kdu}{kw H_m^{(2)\prime} (kwa)(\mu-ku) K^+(u) K^-(\mu/_k)} \tag{2.41}$$

and in the farfield this expression is best evaluated by the method of stationary phase. The stationary point is at

$$u = C \cos\theta/(1 + \alpha MC \cos\theta), \tag{2.42}$$

so that

$$p_o = \frac{e^{-ikc_j R/c_o}}{4\pi R(1+\alpha MC\cos\theta)}\left[\frac{4}{i}\cdot\frac{k\,\rho_j c_j{}^2\,k^2\,\gamma\,D_o{}^2}{kw\,H_m^{(2)}(kwa)(\mu-ku)K^+(u)K^-(\mu/_k)}\right],$$

$$\text{evaluated at}\quad u = \frac{C\cos\theta}{(1+M\alpha C\cos\theta)}, \qquad (2.43)$$

where $(R\ ,\ \theta)$ is the position of the farfield observer in so-called "emission-time" coordinates (Fig. 3). The bracketed term becomes, on substituting for $K^{\pm}$ from Appendix 1,

$$\left[\frac{4\ k\rho_j c_j{}^2 k^2\gamma(1-\alpha Mu)^2 a(1-(1+M)u)}{i\ (\frac{2}{\pi ia})\ .\ 2(1-Mu)^2\ (\frac{1}{(1+M)}-u)\ k\ \frac{(1+M)}{2}}\right]$$

$$= \frac{2\ ka.\pi a\ \gamma}{(1-MC(1-\alpha)\cos\theta)^2}\ . \qquad (2.44)$$

By substituting for γ, C, M, we can then rewrite p_o as

$$p_o = \frac{A_j}{4\pi R}\ \frac{i\omega\ \rho_o.(2p_i/\rho_j c_j)\ \exp[-\ i\omega R/c_o]}{(1+\alpha MC\cos\theta)\ (1-MC(1-\alpha)\cos\theta)^2}\ , \qquad (2.45)$$

where A_j is the duct area, and $(2p_i/\rho_j c_j)$ may be recognised as the velocity fluctuation at the exit of the duct, u_N. The significant features of this formula lie in the scaling of the level (for a given u_N) on the farfield density rather than the jet density, and the field shape. The field shape is determined by the product of a Doppler factor based on the external flow velocity and the square of a Doppler factor based on relative flow velocity. This latter dependence is characteristic of low frequency acoustic sources placed inside infinite jets (Goldstein (1976), Dowling et al (1978)). For angles close to 90°, ($M\cos\theta \ll 1$) the effect of "flight" (i.e., external flow) is represented by a factor $(1 - \alpha MC\cos\theta)^{-6}$ on the intensity which is identical to that found experimentally by Pinker & Bryce (1976).

We now discuss the flows of energy in this problem. In the jet pipe the net power flow is given by

$$W_N = \frac{A_j p_i^2}{\rho_j c_j} \left[(1 + M)^2 - |R|^2 (1 - M)^2\right], \tag{2.46}$$

where p_i is the strength of the incident wavefield, and R is the reflection coefficient. In the limit of low frequency we have shown that $R = -1$, so that

$$W_N = 4M.\frac{A_j p_i^2}{\rho_j c_j} \quad . \tag{2.47}$$

This implies that there is a net flux of acoustic energy along the pipe, proportional to the Mach number, and independent, to a first order, of frequency. This is in contrast to the case of zero flow where the net energy flux is of order

$$W_N = \frac{A_j p_i^2}{\rho_j c_j} (ka)^2 \ .$$

In the jet the only significant motion is associated with the convected instability waves. Since there is negligible fluctuation in pressure associated with them, the only energy flow is a flux of kinetic energy. This is the product of the fluctuations in kinetic energy and mass flow so that the net energy flux in the jet is

$$W_j = (\rho_j\, u_N'\, A_j)\,(U_j\, u_N') ,$$

$$= 4M\,\frac{A_j\, p_i^{\,2}}{\rho_j c_j} \qquad (2.48)$$

where we have taken the velocity fluctuation as $u_N = 2p_i/\rho_j c_j$. It is clear that in this low frequency limit there is a <u>total conversion of acoustic energy into kinetic energy associated with the instability waves</u>. We shall show later that this conversion of energy is critically dependent on the imposition of a Kutta condition.

The power radiated to the farfield is found to be (Munt 1979)

$$W_R = \int_0^{\pi} 2\pi\, R^2\, \frac{p^2(\theta,R)}{\rho_o c_o}\,.\,\sin\theta(1 + \alpha MC\cos\theta)^2\, d\theta\ , \qquad (2.49)$$

$$= \int_0^{\pi} \frac{2\pi R^2}{(4\pi R)^2}\,.\,\frac{\sin\theta\ i\omega\,\rho_o\,(2p_i/\rho_j c_j)^2\, A_j^{\,2} d\theta}{\rho_o\, c_o\,(1 - \alpha MC(1-\alpha)\cos\theta)^4}\ , \qquad (2.50)$$

for a farfield pressure level of $p(\theta,R)$.

Putting $A_j = \pi a^2$ and integrating gives

$$W_R = \left(\frac{p_i}{\rho_j c_j}\right)^2 \frac{\omega^2 \rho_o a^2 (1 + M_R^2/3)}{c_o^{\,2}\,(1 - M_R^{\,2})^3}\ ,$$

$$= \frac{A_j p_i^{\,2}}{\rho_j c_j}\left(\frac{\omega a}{c_o}\right)^2 .\ \left(\frac{\rho_o\, c_o}{\rho_j\, c_j}\right) .\ \frac{(1 + M_R^{\,2}/3)}{(1 - M_R^{\,2})^3}\ , \qquad (2.51)$$

where

$$M_R = MC(1 - \alpha) = (U_j - U_o)/c_o.$$

This expression is the product of the net incident energy in the pipe in the absence of flow, the square of the compactness ratio ka of the jet, the ratio of the impedances of the jet and ambient medium, and a factor which depends on the velocity of the jet relative to the surrounding fluid and which causes the power radiated to increase rapidly as $M_R \to 1$. When the jet and ambient fluid have the same velocity the power radiated is unaffected by Mach number. The singularity in the radiated power when $M_R = 1$ could have been avoided by using a more accurate expression for $K^+(u)$; then the Doppler factors $[1 - MC(1 - \alpha)\cos\theta]$ would have been replaced by $[(1 - MC(1 - \alpha)\cos\theta)^2 + \sigma^2 M^2]^{\frac{1}{2}}$ and the singularity at the Mach angle removed.

In their study of jet noise, Dowling, Ffowcs Williams & Goldstein (1978) have shown that when a low frequency acoustic source is placed in a jet, the radiation from it changes dramatically if the temperature of the jet increases to such an extent that "it is hotter than it is compact", that is when $\frac{\rho_o}{\rho_j} \gg ((ka)^2 \ln(ka))$. We analyse the propagation of sound out of a jet pipe in this limit.

The sound pressure outside the jet is, from (2.41),

$$p_o = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{\rho_j c_j^2 \, k^2 \gamma D_o{}^2 H_m^{(2)} (kwr) \, k e^{-ikux} \, du}{(kw) \, H_m^{(2)\prime} (kwa) \, i(\mu - ku) \, K^+(u) \, K^-(\mu/_k)} \quad . \tag{2.52}$$

When the jet is very hot, we have shown (Appendix 1) that the form of the split functions changes:

$$K^{+}(u) \to \frac{\pi a^{2}(-\gamma)\ \ell n(ka)(ka)^{2}(u + i\varepsilon)k^{2}\rho_{j}c_{j}}{(ka)^{2}\ (1 - (1 + M)u)} , \tag{2.53}$$

$$K^{-}(u) \to \frac{(u - i\varepsilon)}{(1 + (1 - M)u)} , \qquad \varepsilon = \frac{\sqrt{2/\gamma}}{ka[\ell n(\frac{kaC}{2}) + \beta + \frac{\pi}{2i}]^{\frac{1}{2}}} .$$

Therefore, the sound in the farfield is equal to its previous value multiplied by the factor

$$\frac{2(1 - Mu)^{2}}{\pi\ \gamma(ka)^{2}\ \ell n(kaC).(u + i\varepsilon)\ (\frac{1}{(1+M)} + i\varepsilon)} ,$$

evaluated at

$$u = C\cos\theta/(1 - M\alpha C\cos\theta) ,$$

that is, multiplied by

$$\frac{2\ \rho_{j}\ (1 - MC(1-\alpha)\cos\theta)^{2}\ (1+MC\alpha\cos\theta)(1+M)}{\pi\ \rho_{o}\ (1 - MC\alpha\ \cos\theta)^{2}\ (ka)^{2}\ \ell n(kaC)\ \cos\theta} . \tag{2.54}$$

Therefore, the farfield pressure is given by

$$p_{o} = \frac{A_{j}\ i\omega\ \rho_{j}\ u_{N}\ (1 + M)\ e^{-i\omega R/c_{o}}}{2\pi^{2}R\ (1 + M\alpha C\cos\theta)^{2}\ (ka)^{2}\ \ell n(kaC)\ \cos\theta} . \tag{2.55}$$

Compared with the previous result, the field shape for the light jet does not show the downstream Doppler amplification, but is infinite in the sideline (90°) direction. We see, therefore, that the "light jet" condition always fails at this position, which is indeed the Mach angle for disturbances transmitted along the jet.

The field in the jet is given by (2.31)

$$p_j = \frac{-1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_j c_j^{\,2} k^2 \, D_j^{\,2} \, J_m(kvr) \, k e^{-ikux} \, du}{(kv) \, J_m'(kva) \, i(\mu - ku) \, K^+(u) \, K^-(\mu/_k)} \; ; \qquad (2.56)$$

substituting for $K^+(u)$, $K^-(\mu/_k)$ from (2.53), we find that

$$p_j = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{2\,\varepsilon^2 e^{-ikux} \, (1-Mu)^2 \, du}{\pi(\frac{1}{1+M} + i\varepsilon)(u+i\varepsilon)(1-M^2)(u + \frac{1}{(1-M)})(u - \frac{1}{(1+M)})} . \qquad (2.57)$$

The integral is thus the sum of the three pole contributions at $u = \frac{1}{(1+M)}$, $\frac{-1}{(1-M)}$, $-i\varepsilon$. The first of these, does not, unlike the case considered earlier, cancel the incident field, which now continues to propagate along the jet. In this limit the jet behaves, to first order, as a semi-infinite rigid tube. The pressure due to this pole is given by

$$p_j \sim \frac{1}{\pi} \cdot \varepsilon^2 (1 + M) \, e^{-ikx/(1+M)} \qquad (2.58)$$

and the level in the jet due to the pole at $u = -i\varepsilon$ is given by

$$p_j \sim \frac{2}{\pi} \varepsilon^2 \;\; e^{-\varepsilon kx} (1+M) . \qquad (2.59)$$

Both these fields (which arise for $x > 0$) are small in the light jet limit (ε small).

The reflected sound field in the jet pipe is given by the contribution from the pole at $u = \frac{-1}{(1-M)}$, namely

$$p_j = \frac{(1 - M^2)\varepsilon^2}{\pi} e^{ikx/(1-M)} \quad . \tag{2.60}$$

Clearly then, the reflection coefficient is of order $[\gamma(ka)^2 \ln (ka)]^{-1}$, which is small in this light jet limit.

We have thus shown that if the jet is sufficiently hot, then there is a radical change in the acoustic behaviour of the jet pipe system. The majority of the sound is no longer reflected back up the pipe, but continues trapped inside the jet. The reason for this is seen by examining the relation between the pressure gradient and displacement on the jet boundary, $\frac{D^2\eta}{Dt^2} + \frac{1}{\rho_o} \frac{\partial p}{\partial r} = 0$. From this it is clear that if ρ_o is greatly increased, and tends to infinity, then for a given value of pressure gradient, the boundary displacement must tend to zero. In the farfield the radiation is reduced compared with that for the non-light jet case, except for angles close to the sideline direction. At this 90° position (corresponding to the Mach angle), the compactness condition of the light jet does not hold, as already observed. An additional feature of the light jet limit is that the instability waves on the jet column are suppressed.

We now consider, in less detail, the radiation from higher order spiral duct modes. In the low frequency limit, it is well known that in the absence of a mean flow sound radiates very inefficiently. Substituting for the split functions $K^{\pm}(u)$, (Appendix 1, A1.16) in the expression for the sound field outside the jet, we find that

$$p_o = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{4\rho_j c_j^2\, k^2\gamma D_o^2\, (\frac{i}{4} H_m^{(2)}(kwr))ak\, e^{-iku}\, du}{(k\omega a)\, \frac{mi}{2\pi}\, .\, (\frac{2}{kwa})^{m+1}\quad i(\mu_m - ku)\rho_j c_j^2\, k^2 a(D_j^2+\gamma D_o^2)} \quad . \qquad (2.61)$$

Using stationary phase we find that the farfield sound pressure is

$$p_o = \frac{ae^{-i\omega R/c_o}}{4\pi R\,(1+\alpha MC\cos\theta)} \left\{\left[\frac{\gamma D_o^2}{D_j^2+\gamma D_o^2}\right] \frac{(kwa)^m k\; 8\pi}{i(\mu_m - ku)m}\right\}_{u = \frac{C\cos\theta}{(1+\alpha MC\cos\theta)}} \quad . \qquad (2.62)$$

For these spiral modes,

$$\frac{\mu_{mn}}{k} = \frac{-M + \sqrt{(1-(1-M^2)(j'^2_{mn}/(ka)^2}}{(1-M^2)},$$

so that the bracketed factor becomes

$$\frac{\gamma\,(ka)^m(\sin\theta)^m\; 8\pi/m}{((1 - MC(1-\alpha)\cos\theta)^2 + \gamma)(1 + \alpha MC\cos\theta)^{m-1}\,(\frac{\mu}{k}\,(1 - M\alpha C\cos\theta) - \cos\theta)},$$

and hence the complete radiation field is given by

$$p_o = \frac{a\, e^{-i\omega R/c_o}\,(ka)^m\,(\sin\theta)^m\,(8\pi/m)\,\gamma}{4\pi R(1+\alpha MC\cos\theta)^m\,(\frac{\mu}{k}(1+\alpha MC\cos\theta) - \cos\theta)[(1-MC(1-\alpha)\cos\theta)^2 + \gamma]} \quad .$$

(2.63)

This formula shows that as the mode number m increases, the power radiated at a given (low) frequency progressively decreases. The radiation is predominantly in the sideline direction, and the effect of flight is largely that of the Doppler factors $(1+\alpha MC\cos\theta)$ which shift the field further forwards for higher values of α. For sufficiently low frequencies, μ/k becomes $ij'_{mn}/ka(1-M^2)^{1/2}$, which is much greater than unity, so that the farfield pressure can be written

$$p_o = \frac{a\, e^{-i\omega R/c_o} (ka)^{m+1} (\sin\theta)^m\, 8\pi\sqrt{(1-M^2)}}{4\pi R\, (1+\alpha MC\cos\theta)^{m+1}\, mi\, j'_{mn}} \left[\frac{\gamma}{((1-MC(1-\alpha)\cos\theta)^2+\gamma)}\right]. \tag{2.64}$$

The radiation from all of these higher order modes varies as a higher power of ka than that from the plane wave mode. Further, for a <u>given</u> pressure level at infinity upstream, the pressure level at the nozzle, on which this radiation scales, is exponentially small.

The field in the jet and pipe, $r < a$, is given by

$$p_j = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_j c_j^2\, k^2 D_j^2\, J_m(kvr)(-1)e^{-ikux}\, k\, du}{kv\, J_m'(kva)\, (\mu/k-u)k\rho_j c_j^2 k^2 a(D_j^2+\gamma D_o^2)}\,. \tag{2.65}$$

In this expression the pole at $kva = j'_{mn}$ cancels out the incident field. Expanding the Bessel functions yields

$$p_j = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \left(\frac{r}{a}\right)^m \frac{D_j^2}{D_j^2+\gamma D_o^2} \cdot \frac{e^{-ikuz}}{(\mu/k-u)}\, du, \tag{2.66}$$

and noting that the instability poles of $(D_j^2 + \gamma D_o^2)^{-1}$ are $u = \frac{(1\pm i\gamma)}{M(1\pm i\alpha\gamma)}$, this gives

$$p_j = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \left(\frac{r}{a}\right)^m \frac{e^{-ikux}\, du}{[\mu/k-u]\left[u - \frac{1}{M}\frac{(1+i\gamma)}{(1+i\alpha\gamma)}\right]\left[u + \frac{1}{M}\frac{(1-i\gamma)}{(1-i\alpha\gamma)}\right][M^2(1+\alpha^2\gamma^2)]}\,. \tag{2.67}$$

The contribution to p_j from the instability pole $u = \frac{1}{M}\frac{(1 + i\gamma)}{(1 + i\alpha\gamma)}$ is

$$p_j = \left(\frac{r}{a}\right)^m \frac{e^{-ik(1+i\gamma)x/M(1-i\alpha\gamma)}}{\left[\frac{j'_{mn}}{ka\sqrt{(1-M^2)}} - \frac{1}{M}\frac{(1+i\gamma)}{(1+i\alpha\gamma)}\right][M^2(1+\alpha^2\gamma^2)]\left[\frac{1}{M}\frac{(1+i\gamma)}{(1+i\alpha\gamma)} + \frac{1}{M}\frac{(1-i\gamma)}{(1-i\alpha\gamma)}\right]}, \tag{2.68}$$

and for low Strouhal number ($\frac{ka}{M} << 1$) this is

$$p_j = \left(\frac{r}{a}\right)^m \frac{e^{-ik(1+i\gamma)x/(1+i\alpha\gamma)M}\sqrt{1 - M^2}\ ka}{2\, j'_{mn}\, M\,(1 + \alpha\gamma^2)} \ . \tag{2.69}$$

Unlike the plane wave case considered earlier, the instability wave does now have a pressure disturbance associated with it, which increases in proportion to the Strouhal number for a given initial amplitude. The growth rate is given by $\frac{k\gamma(1 - \alpha)}{M(1 + \alpha^2\gamma^2)}$, which vanishes when there is no velocity difference across the jet boundaries. Further, these non-axisymmetric waves are amplified with distance downstream even for the lowest frequencies, although there the initial disturbance level is very small, due to the aforementioned dependence on frequency, and the exponentially small level of the sound at the pipe exit due to the spiral acoustic modes being cut off in the pipe.

2.2. Subsonic Jet with No Kutta Condition

We describe here two types of problem relating to a jet with no Kutta condition. First, we consider the case where a jet eigensolution is added on to the Kutta condition solution, by taking $\overline{C}(u)$ in (2.24) as constant. Second we adopt an approach due to Howe (1979). Instead of assuming the existence of an instability wave, in the jet, he assumes that the jet motion consists of some other neutrally stable wave which is convected at a Mach number νM, less than the jet Mach number M. He then finds the field due to this and matches it to the nozzle flow. He adds this field to that found with no Kutta condition, and forces the total to satisfy a Kutta condition.

We examine first the case where $\bar{C}(u)$ is a constant C_o, say. Then equation (2.27) to (2.29) become

$$Z(u) = \frac{1}{K^+(u)} \quad [E + C_o] \quad , \tag{2.70}$$

$$P_j(u) = \frac{-\rho_j c_j{}^2 k^2 D_j^2 J_m (kvr)}{kv J_m' (kva) K^+(u)} \quad [E + C_o] \quad , \tag{2.71}$$

$$P_o(u) = \frac{-\rho_j c_j^2 k^2 D_o^2 H_m^{(2)} (kwr)}{kw H_m^{(2)'} (kwa) K^+(u)} \quad [E + C_o], \tag{2.72}$$

where we have defined

$$E = \frac{-1}{i(\mu - ku)K^-(\mu/_k)} \quad .$$

By choosing different (non-zero) values of C_o, we can obtain a whole range of solutions, none of which obeys a Kutta condition. One of these is of special interest: that where C_o is chosen to cancel the instability pole u_o. We choose

$$C_o = \frac{1}{K^-(\mu/_k) \; i(\mu - ku_o)} \tag{2.73}$$

and then

$$E + C_o = \frac{-k(u - u_o)}{iK^+(\mu/k_o)(\mu - ku_o)(\mu - ku)} \qquad (2.74)$$

Clearly, the net effect of this is to multiply the Kutta condition solutions in u by $(\frac{u - u_o}{\mu/_k - u_o})$. In the farfield, where $u = C\cos\theta/(1 - \alpha MC\cos\theta)$, and for the plane wave case $(\frac{\mu}{k} = \frac{1}{1 + M}$, $u_o = \frac{1}{M})$, this factor is

$$\frac{[\frac{C\cos\theta}{(1+\alpha MC\cos\theta)} - \frac{1}{M}]}{[\frac{1}{(1+M)} - \frac{1}{M}]} = \frac{(1 - (1 - \alpha)MC\cos\theta)(1 + M)}{(1 + \alpha MC\cos\theta)} \qquad (2.75)$$

The formula for the farfield sound, (2.45), then becomes

$$p_o = \frac{\rho_o u_N}{4\pi R}\frac{i\omega}{c_o} \frac{A_j (1 + M)e^{-i\omega R/c_o}}{(1 + \alpha MC\cos\theta)^2 (1 - MC(1 - \alpha)\cos\theta)} . \qquad (2.76)$$

The major difference between this and the previous result is the removal of one of the <u>relative</u> velocity based Doppler factors, which results in a considerably less directional sound field.

The corresponding multiplier for the field inside the pipe is obtained using $u = {}^{-1}/(1 - M)$, giving

$$\frac{(-\frac{1}{(1-M)} - {}^{1}/_{M})}{({}^{-1}/(1+M) - {}^{1}/_{M})} = \frac{(1 + M)}{(1 - M)} .$$

Therefore, the reflection coefficient has changed from -1 to $-\frac{(1 + M)}{(1 - M)}$. It is of particular interest that the power flow in the duct, which is proportional to $[(1 + M)^2 - |R|^2 (1 - M)^2]$, is now precisely zero. Therefore, we conclude that the imposition of a Kutta condition is essential for a transfer of power from acoustic to hydrodynamic fields to take place. With no Kutta condition, and no generation of growing instability waves, almost all the incident energy is reflected back up the duct, a negligible $O(k^2a^2)$ fraction being diffracted to the farfield.

Howe proceeds by adding on to this non-Kutta-condition solution the field due to a jet motion convected at a speed νMc_j. We assume that in the absence of the pipe this wave would induce a jet displacement $Z_1 e^{-ikx/\nu M}$. Then the transform of this over $x < 0$ is Z_o^-, where

$$Z_o^- = \frac{Z_1}{ik(u - {}^1/\nu M)} \quad . \tag{2.78}$$

Now from (2.14) and (2.13) with $P_o = 0$ (since there is no driving pressure) we have

$$P_j(u,a^-) - P_o(u,a^+) = P_j(u,a^-) - P_o(u,a^+) = F^-, \tag{2.79}$$

and

$$Z^- + Z_o^- = 0. \tag{2.80}$$

Hence

$$K(u)\,(Z^+(u) - Z_o^-) = F^-, \tag{2.81}$$

so that

$$K^+(u)\ Z^+(u) - \frac{Z_1\ (K^+(u) - K^+(^1/_{\nu M}))}{ik\ (u - {}^1/_{\nu M})} = \frac{F^-(u)}{K^-(u)} + \frac{Z_1\ K^+(^1/_{\nu M})}{ik(u - {}^1/_{\nu M})} . \quad (2.82)$$

The left and right hand sides are regular in the regions R^+ and R^- and are, therefore, equal to the same constant C_1, say. Then the displacement Z(u) is

$$Z(u) = \frac{1}{K^+(u)} \left\{ \frac{-Z_1\ K^+(^1/_{\nu M})}{ik(u - {}^1/_{\nu M})} + C_1 \right\} = \frac{E_1}{K^+(u)} \quad \text{, say,} \quad (2.83)$$

which must now be combined with the non-Kutta condition solution, which has

$$Z(u) = \frac{(u - u_o)}{K^+(u)\ ik(\mu/_k - u)\ (\mu/_k - u_o)\ K^-(\mu/_k)} . \quad (2.84)$$

We require that the total Z(u) due to (2.83), and (2.84) must not have any instability pole, there being instead a real pole at $u = {}^1/_{\nu M}$ representing the neutrally stable convected wave in the jet.

To remove the instability pole at $u = u_o$ we set

$$C_1 = \frac{Z_1\ K^+(^1/_{\nu M})}{ik\ (u_o - {}^1/_{\nu M})} \quad (2.85)$$

and then

$$Z(u) = \frac{1}{K^+(u)} \left[\frac{(u - u_o)}{ik(\frac{\mu}{k} - u)(\frac{\mu}{k} - u_o)\, K^-(\mu/_k)} - \frac{Z_1\, K^+(\frac{1}{\nu M})(u - u_o)}{ik(u_o - {}^1/_{\nu M})(u - {}^1/_{\nu M})} \right].$$

(2.86)

The correct behaviour at the edge then requires that the bracketed factor is $O({}^1/_u)$ as $u \to \infty$. This may be obtained by choosing

$$\frac{Z_1\, K^+({}^1/_{\nu M})}{(u_o - {}^1/_{\nu M})} = \frac{1}{K^-\,(\mu/_k)\,(\mu/_k - u_o)}\,, \tag{2.87}$$

so that now

$$Z(u) = \frac{1}{K^+(u)\, K^-(\mu/_k) ik(\mu/_k - u)} \left[\frac{(u - u_o)(1 - \nu M\mu/k)}{(\mu/_k - u_o)(1 - \nu M u)}\right]. \tag{2.88}$$

The square bracketed term is clearly the factor multiplying the original solution obeying a Kutta condition.

In the farfield, therefore, the radiation field is multiplied by this factor with

$$u = C\cos\theta/(1 - \alpha MC\cos\theta), \quad u_o = {}^1/_M, \; \mu = {}^1/_{(1 + M)}\,,$$

giving a factor

$$(1 + M(1 - \nu))\; \frac{(1 - MC(1 - \alpha)\cos\theta)}{(1 - MC\,(\nu - \alpha)\cos\theta)}\,.$$

Thus we see that as ν is varied between 0 and 1 the solution changes continuously from the non-Kutta-condition solution to the Kutta condition solution. The major change in the farfield, as compared with the Kutta condition solution, lies in the replacement of the $(1 - MC(1 - \alpha)\cos\theta)$ Doppler factor by one based on the convection speed of the waves. This results in a less directional radiation field, with a corresponding reduction in the acoustic power radiated.

The reflection coefficient is obtained by substituting $u = -1/_{(1-M)}$, giving a multiplier

$$1. \frac{(-1/_{1+M} - 1/_M)(1 - \frac{\nu M}{1+M})}{(1/_{1+M} - 1/_M)(1 + \frac{\nu M}{(1-M)})} = \frac{(1 + (1 - \nu)M)}{(1 - (1 - \nu)M)} \quad . \tag{2.89}$$

The reflection coefficient varies again from its Kutta to non-Kutta-condition values as ν is altered. This is only to be expected, since with $\nu = 1$, the convected waves are indistinguishable from the instability waves, while with $\nu = 0$ there is no spatial variation, and the wave is effectively absent.

In this derivation of the radiation the existence of the waves convected at speed νMc_j is only an assumption. There are grave doubts over its validity, since thewave is not in fact the solution of any equations governing the motion of the fluid. Nevertheless, the idea remains a plausible means of representing in some way the characteristics of the real jet flow.

2.3. Supersonic Jet

When we come to the supersonic jet the basic equations are the same; the difference in the solution concerns only the position of the branch cuts of the u plane. When $M \to 1$, the branch point at $\frac{-1}{1 - M}$ goes to $-\infty$ for subsonic flow, but when $M > 1$ it reappears on the other side of the diagram at $^1/_{(M-1)}$. This is a consequence of the impossibility of waves propagating upstream against the supersonic flow. The resultant branch cuts and positions of R^+, R^- for $M > 1$, are essentially as described by Morgan (1974) and are shown in Fig. 3. It is assumed in this diagram that $\frac{1}{(M-1)} > \frac{C}{(1-\alpha M)} > \frac{1}{(M+1)}$. If this is not so, the order of these points on the real axis is correspondingly changed.

We consider the case of an incident plane wave propagating down the jet pipe towards the exit,

$$p_i = \exp[-i\mu x], \quad \mu = k/(1 + M), \tag{2.90}$$

and confine the analysis to plane waves purely for simplicity. There is no other reason for doing so here since all modes are cut on in supersonic flow. The derivation of the field due to the higher order modes follows in an altogether similar fashion.

As $ka \to 0$, it is shown in Appendix 2 that (unless $u >> {}^1/ka$, $\alpha \neq 0$); $K^-(u) = 1$, $K^+(u) = K(u)$. Then the formulae used previously may be applied, with the previous value of $K^+(u)$ multiplied by $\frac{J_o(kva)}{v}$, the factor $K^-(\mu/k)$ multiplied by $v^-(\mu/k)$. Consequently, all the Fourier transformed quantities $P_j(u)$, $P_o(u)$, $Z(u)$, are multiplied by

$$\frac{v^-(u)}{v^-(\mu/k)} \cdot \frac{1}{J_o(kva)} = Q \quad \text{(say)}. \tag{2.91}$$

In the above discussion we did not mention the edge conditions to be satisfied near $x = 0$, and on which there was previously so much emphasis. At high kva ($u \to \infty$) the kernel $K(u)$ has a behaviour similar to its two dimensional vortex sheet equivalent. The latter case has been examined in detail by Morgan, who finds that $K^+(u)$ is $O(u)$; $u \to \infty$, resulting in a displacement $\eta(x) \sim x$; $x \to 0^+$. The displacement is, therefore, continuous across $x = 0$, but its slope is not, so that a Kutta condition in the subsonic sense cannot be applied. However, one would not expect it to hold for this <u>unsteady</u> supersonic flow, any more than it does in <u>steady</u> supersonic flows, and it cannot because of the impossibility of the downstream motion of the jet affecting the edge. Besides that described above, further solutions corresponding to the subsonic non-Kutta condition solutions could be obtained. These would be even more singular at the edge, and are physically implausible (displacement at the edge must at least be discontinuous).

We determine next the farfield sound level. This is given by its subsonic value with a Kutta condition, multiplied by the above factor (2.91), evaluated at $u = C\cos\theta/(1 + \alpha MC\cos\theta)$; that, is multiplied by

$$\left(\frac{1 + M}{2}\right) \frac{(1 - C\cos\theta\,(M(1 - \alpha) - 1))}{(1 + \alpha MC\cos\theta)} . \tag{2.92}$$

The farfield pressure is accordingly

$$p_o = \frac{i\omega A_j u_N \rho_o e^{-i\omega R/c_o}}{4\pi R} \left[\frac{(1 + M)\ (1 - C\cos\theta(M(1 - \alpha) - 1))}{2(1 - M(1 - \alpha)\cos\theta)\ 2(1 + M\alpha C\cos\theta)}\right]. \tag{2.93}$$

The interesting features of this formula are the large values of forward arc amplification (the exponent of $(1 + \alpha M \cos\theta)$ is increased) and the factor $(1 - C\cos\theta(M(1 - \alpha) - 1))$. The latter causes the field to have a zero if $C(M(1 - \alpha) - 1) > 1$.

The reflected field inside the pipe is now precisely zero, since the pole at $u = {}^{-1}/(1 - M)$ is no longer present. In fact the field inside the pipe is precisely zero everywhere, since all the poles representing cut-off waves inside the pipe are now in R^-, and cannot contribute for $x < 0$.

We consider the field in the jet in more detail; it is given by

$$p_j = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{J_o(kvr)\, e^{ikux}\, du}{J_o(kva)(\frac{1}{(1 + M)} - u)} \quad . \tag{2.94}$$

The pole at $u = {}^{1}/_{(1 + M)}$ cancels the incident field. The other poles at $J_o(kva) = 0$ are the unsteady flow analogue of the steady wave structure of an imperfectly expanded supersonic jet. (We did not consider them for the subsonic jet, since there they represented fields which decayed exponentially along the jet axis.) These poles occur at $kva = j'_{om}$, that is, where

$$u_m = \frac{- M + \sqrt{(1 - (1 - M^2)\, j'^{\,2}_{om}/(ka)^2}}{(1 - M^2)} , \tag{2.95}$$

$$= \pm \frac{j'_{om}}{ka(M^2 - 1)^{\frac{1}{2}}} + \frac{M}{(M^2 - 1)} + O(\frac{ka}{j'_{om}}). \tag{2.96}$$

Only the first term needs to be taken for ka sufficiently small, since for small enough ka these poles occur at large u. Then with $\beta^2 = M^2 - 1$

$$p_j = -\frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{1}{u}\,\frac{J_o(kr\beta u)}{J_o(ka\beta u)} \cdot e^{-ikx}\,du, \tag{2.97}$$

$$= \sum_{n=1}^{\infty} \frac{-2\,J_o(j_n r/a)}{j_n\,J_o'(j_n)} \cos\left(\frac{j_n x}{\beta a}\right) \quad . \tag{2.98}$$

This formula, which is composed of contributions from the quasi-periodic wave structure, is valid for $x \ll {}^{1}/_{ka}$. A form valid over the whole distance x is only obtained by use of more exact approximations for the poles. In addition to these contributions to the pressure in the jet, there is again an instability wave present, which only affects the velocity (see Section 2.1). Compared with that for the subsonic jet this wave has a velocity amplitude $v^{-}({}^{1}/_{M})/v^{-}(\frac{1}{1 + M})$ greater; and the velocity fluctuation at the exit is now $u_N = \frac{(1 + M)}{M} \cdot \frac{p_i}{\rho_j\,c_j}$, for an incident pressure wave amplitude p_i. The contribution (2.98) alone will be referred to as "the coherent wave structure", and denoted by p_c, with corresponding velocity u_c.

We next consider the energy flows involved: inside the duct the energy flux is that in the incident sound wave alone and is, accordingly,

$$W_o = \frac{p_j^{\,2} A_j}{\rho_j c_j}(1 + M)^2. \tag{2.99}$$

The energy in the instability wave is

$$W_i = (\rho A u')\,(u' U) = \frac{(1 + M)^2}{M} \cdot \frac{p_i^{\,2}\,A}{\rho_j c_j} \quad . \tag{2.100}$$

Clearly, the energy in this wave is not equal to the net energy in the jet pipe. However, there are now two additional contributions to the energy - those due to the coherent wave structure alone, and due to the interference field between it and the instability wave. This may be contrasted with the work of Ffowcs Williams & Howe (1978) who considered the scattering of the coherent wave by random shear layer turbulence. They found that all of its energy was scattered into sound, there being no coupling of the coherent structure and instability waves.

The energy flux across a section of the jet is given by $W = \int dA\, h_o'(\rho v_n')$, where h_o' is stagnation enthalpy. Splitting this into components due to the coherent wave structure, p_c, u_c, and the instability wave u_i ($p_i = 0$), we obtain

$$W = \int dA \left(\frac{p_c}{\rho} + Uu_c + Uu_i\right)\left(\frac{Up_c}{c^2} + \rho u_c + \rho u_i\right). \qquad (2.101)$$

Now the fluctuation due to the coherent wave structure is to a first order quasi-static, so that $\frac{p_c}{\rho} + u_c U = 0$. Therefore, the contribution to the integral due to this wave structure alone is negligible, and the only important term (apart from the instability term already calculated in (2.100) is the interference term

$$W_{int} = \int u_i\, U \left(\frac{p_c U}{c^2} + u_c \rho\right) dA$$

$$= \int M\, p_c\, u_i \frac{(M^2 - 1)}{M}\, dA. \qquad (2.102)$$

Substituting for p_c, u_i this becomes

$$W_{int} = \int dA\ M \sum_{\substack{n \neq o \\ n=-\infty}}^{n+\infty} \frac{M^2-1}{M} \cdot \frac{1}{c_j} \frac{2\ J_o(j_n\ r/_a)}{j_n\ J_o'\ (j_n)} \cos(\frac{j_n\ x}{\beta a})\ e^{i\omega x/U_c} \frac{(1+M)}{M} \ . \tag{2.103}$$

Integrating over the area of the jet, and noting that

$\int_o^a J_o(j_n\ r/_a)\ rdr = \frac{a^2}{j_n{}^2} \cdot j_n\ J_1(j_n)$,

$$W_{int} = 4A_j \frac{(M^2-1)}{M\ c_j} \sum_{-\infty}^{+\infty} \frac{\cos(j_n\ x/\beta a)}{j_n{}^2}\ e^{i\omega x/U_j} \ . \tag{2.104}$$

For finite x, the cosine terms dominate, and the exponential can be ignored.

We now consider the flux of energy through the walls of the jet. This is given by $W_N = \int dSdx(Uu_i)\ v_c\rho$ where v_c is the radial velocity in the coherent field, and x is the length of the jet, perimeter S. The velocity in the coherent field is given by $U \frac{\partial v_c}{\partial x} = - \frac{1}{\rho} \frac{\partial p}{\partial x}$,

so that

$$\frac{\partial v}{\partial x} = \frac{1}{M} \sum_{-\infty}^{+\infty} \frac{2\ J_o'(j_n\ r/_a)}{J_o'\ (j_n)\ j_n} \cdot \frac{j_n}{a} \cdot \cos\ (j_n x/\beta a)$$

$$\therefore\ v = \frac{\beta}{M} \sum_{-\infty}^{+\infty} \frac{2\ \sin\ (x\ j_n/\beta a)\ J_o'(j_n\ r/_a)}{j_n\ J_o'(j_n)} \ . \tag{2.105}$$

Hence

$$W_n = \sum_{-\infty}^{+\infty} \int_0^x dx\ 4\pi a \cdot \frac{\beta}{M} \cdot \frac{(1+M)}{M} \frac{p_i}{\rho_j c_j} \cdot \frac{\sin(xj_n/\beta a)M}{j_n}$$

$$= \sum_{-\infty}^{+\infty} 4\pi a \frac{(M^2-1)(M+1)}{Mj_n^2} [\cos(j_n x/\beta a) - 1] \frac{p_i}{\rho_j c_j} \quad . \tag{2.106}$$

This contribution, when added to the previous one, gives the total power through the walls and cross section of the jet up to a certain distance x as

$$W_J = \sum_1^{\infty} \frac{4Ap_i^2}{\rho_j c_j} \cdot \frac{(M^2-1)(M+1)}{M_j\ \ j_n^2} \quad . \tag{2.107}$$

Now, it can be shown that $\sum_1^{\infty} \frac{1}{j_n^2} = \frac{1}{4}$. Therefore, W_J is equal to

$$\frac{Ap_i^2}{\rho_j c_j} \cdot \frac{(M^2-1)(1+M)}{M} \quad . \tag{2.108}$$

Adding W_J to the net power in the jet instability wave, W_i, we obtain a total power of

$$\frac{Ap_i^2}{\rho_j c_j} \cdot \frac{(1+M)^2}{M} ((M - 1) + 1) = \frac{Ap_i^2}{\rho_j c_j} (1 + M)^2 \quad . \tag{2.109}$$

This is precisely equal to the power in the incident wave. We have therefore shown that, with a supersonic jet, all the power in the incident jet pipe acoustic wave is converted into either hydrodynamic kinetic energy or into the power in the interference field between the quasi-steady wave structure and the instability wave. Indeed, in this case all the incident acoustic energy is in some sense absorbed, and the basic phenomenon found by Bechert et al (1977) for a subsonic jet applies equally for a supersonic jet.

3. SCATTERING OF AN EXTERNAL SOUND FIELD BY A CYLINDRICAL PIPE WITH FLOW

We consider the same system as in Section 2.1. In this case, instead of assuming a plane acoustic wave incident from inside the duct, we consider an externally incident plane wave, which is then scattered by the pipe and jet. A more realistic problem would perhaps have involved a point acoustic source in the ambient fluid. But if such a source is in the farfield of the pipe it just generates plane waves at the nozzle and the problems are equivalent. To some extent the alternative case of a source near the nozzle is really the subject of the next section, §4, where the 'source' takes the form of ring vortices convected past the nozzle exit.

The present problem has also been solved approximately by Jacques (1975). He, however, finds a formula for the farfield scattered sound that is different from ours. We shall show in Section 5 that his result is incorrect because, in his application of the acoustic analogy, he omits certain source terms.

3.1. Subsonic Jet

We consider an incident plane acoustic wave with pressure $e^{iku_1x-kv_1y}$ in which, if the wave vector is at an angle β to the jet axis $u_1 = C\cos\beta/(1 + \alpha MC\cos\beta)$.

We first split this up into its circumferential modes using the result (Abramowitz & Stegun (1965))

$$e^{-ikv_1r\cos\phi} = \sum_{o}^{\infty} \varepsilon_m(-i)^m J_m(kv_1r) \cos m\phi \tag{3.1}$$

where $\varepsilon_m = 1, \quad m = 0$

$$\varepsilon_m = 2, \quad m \neq 0,$$

to find that the incident pressure field is

$$p_i = \sum_0^\infty \varepsilon_m (-i)^m J_m(kv_1 r) \cos m\phi \, e^{-iku_1 x} , \tag{3.2}$$

where $v_1 = v(\beta)$.

To apply the theory of the previous section we require the pressure that would have existed on the wall of the pipe had the pipe been infinite. To find this, we add on to each modal term an extra term $A_m H_m^{(2)}(kv_1 r)$ and apply $\frac{\partial\phi}{\partial r} = 0$ on $r = a^+$, to get

$$\varepsilon_m (-i)^m = - A_m H_m^{(2)\prime}(kv_1 a)$$

so that

$$p = \sum_0^\infty \varepsilon_m (-i)^m \frac{[H_m^{(2)\prime}(kv_1 a) J_m(kv_1 r) - J_m'(kv_1 a) H_m^{(2)}(kv_1 r)]e^{-iku_1 x}}{H_m^{(2)\prime}(kva)}. \tag{3.3}$$

To the lowest order in ka we find that the plane wave (m = 0) component is $p = e^{-iku_1 x}$, while the 1st spiral mode component is

$$p = 2e^{-iku_1 x} \cdot (-ikv_1 a) \left(\frac{r}{a} + \frac{a}{r}\right) ,$$

$$= -4i\, kv_1 a \, e^{-iku_1 x} \quad \text{on} \quad r = a. \tag{3.4}$$

Component (3.4), and the other components with $m > 0$ are smaller, by a factor at least $(ka)^m$ smaller than the $m = 0$ component as $ka \to 0$, and can accordingly be neglected. We therefore concentrate on the plane wave component only. We briefly repeat for this external forcing the derivation of the Wiener-Hopf equation, presented in Section 2.1.

The equation (2.17) for the pressure difference across the boundary is now

$$P_j(u,a^-) - P_o(u,a^+) - P_i^+(u,a^-) = F^- . \qquad (3.5)$$

The only difference between this equation and (2.17) lies in the source of P_i, an external, not an internal, field. Here P_i^+ is given by $\frac{1}{-ik(u - u_1)}$. Consequently, examining the rest of the theory, we see that the Fourier transformed pressures and displacements have their previous values (2.27-2.29) multiplied by the factor

$$-\frac{(u - \mu/k)}{(u - u_1)} \cdot \frac{K^-(\mu/k)}{K^-(u_1)} . \qquad (3.6)$$

In the farfield this must be evaluated at

$$u = C\cos\theta/(1 - \alpha MC\cos\theta),$$

and then it becomes

$$\frac{(1 - C(1 + M(1 - \alpha)\cos\theta))(1 + C\cos\beta(1 - M(1 - \alpha)))}{2(\cos\theta - \cos\beta)C} . \qquad (3.7)$$

Compared with the field shape of internal noise this has two interesting features. First, it is singular at $\theta = \beta$. This singularity is spurious, and is similar to that found in half plane diffraction problems on geometrical optics boundaries. It can be removed using an improved evaluation of the stationary phase integral, taking account of the fact that when $\theta \approx \beta$, the integrand has a pole near the stationary phase point. Second, the field is zero at the angle $\theta = \cos^{-1}[C/(1 + (1 - \alpha)M)]$. This is the "cone of silence" angle for sound waves passing from the jet to the farfield.

In addition to the scattered field discussed above there is an additional field present due to the pole at $u = u_1$. This cancels, in a manner entirely familiar in diffraction theory, the portion of the incident field that represented sound reflected off the duct walls, but it only exists for angles less than β to the jet axis.

To obtain the fields in the jet and pipe we again use the previous solution, multiplied by (3.6). The pole at u_1 represents the sound waves inside the jet due to the incident field. These are pressure waves of amplitude equal to that of the incident field, i.e., $p_j = p_i e^{-iu_1kx}$. The field reflected up the pipe is given by the pole at $u = {}^{-1}/(1 - M)$, for which the above multiplier is equal to -1. Therefore the amplitude of the reflected wave is equal to that of the incident wave. The pole at $u = {}^1/_M$ once again gives the instability waves, whose effects are felt only as an axial velocity surging, the pressure perturbation being absent. Then the above multiplier, (3.6), is

$$-\frac{1}{2}\left(1 - \frac{u_1}{(1 - Mu_1)}\right) , \qquad (3.8)$$

so that the instability wave has an axial velocity fluctuation

$$u_x = -\frac{p_i}{\rho_j c_j}\left(1 - \frac{u_1}{(1 - Mu_1)}\right) \qquad (3.9)$$

This completes our evaluation of the sound scattered when low frequency plane waves are incident upon a cylindrical pipe with internal and external flows. It is of interest to compare our results with those of Jacques (1975). In his paper, he first derives the 'zero order' fields in the jet and the pipe neglecting the secondary sound radiation. Then he applies the acoustic analogy to determine the latter. It is clear that the zero order fields which we derive here are identical to his approximate solutions. Our result differs only in the field shape of the radiated sound field, which is more complicated than his. The two results are unequal even in the low Mach number limit. We pursue the application of the acoustic analogy to this problem in some detail in Section 5.

In discussing the relevance of his model, Jacques supposes that the incident sound waves are caused by some nearfield turbulent pressure fluctuation from, say, a nearby jet. Therefore he takes the incident pressure to scale as $p_i \sim \rho U^2$ where U is some turbulence velocity. Inserting that into either our or his formulae for the farfield sound gives sound levels scaling as $p \sim \rho U^2 . \frac{U}{c_o} . \frac{a}{R}$, where it is assumed that the incident pressure has frequencies proportional to this velocity U. We feel his modelling to be inappropriate. If the pressure fluctuations do scale in this way, and are further the result of some nearby aerodynamic disturbance, then the incident sound field cannot be modelled as plane waves. In that case the modelling we have used in Section 4 would be more suitable. There we model turbulence by quadrupole sources, or by convected vortices. In spite of our misgivings about Jacques' problem as a modelling of these disturbances it does appear, however, that his scaling laws are correct.

In our treatment of the scattering of external sound waves we have neglected such factors as the finite growth rate of the instability waves, and the light jet issue. In this problem, though, the basic phenomena they represent are no different from those with incident internal noise and the corresponding results could easily be derived.

We consider briefly the effect of relaxing the Kutta condition at the exit of the pipe. Generally the changes, compared with the case where a Kutta condition does apply, are similar to those for internal noise. As before we can relax the Kutta condition, by choosing some constant value of $\overline{C}(u)$ in the Wiener-Hopf equation. When we choose $\overline{C}(u)$ so as to extinguish all the unstable waves in the jet, then the overall effect is to multiply all the P(u), Z(u) by $\dfrac{(u - u_o)}{(u_1 - u_o)}$, where u_o is the instability pole. For the farfield this factor, with $u = C\cos\theta/(1 + \alpha MC\cos\theta)$, becomes

$$\frac{(1 + M\alpha C\cos\beta)\;(1 + MC(1 - \alpha)\cos\theta)}{(1 + M\alpha C\cos\theta)(1 + MC(1 - \alpha)\cos\beta)} \;. \qquad (3.10)$$

The effect on the field shape is to replace one of the relative jet velocity based Doppler factors by one based on the relative velocity and incidence angle β. There is increased Doppler amplification in the upstream arc due to external flow.

The field transmitted up the pipe is given by the above factor with $u = -1/(1-M)$, that is,

$$\frac{\left(-\frac{1}{1-M} - \frac{1}{M}\right)}{\left(\frac{C\cos\beta}{(1 + M\alpha C\cos\beta)} - \frac{1}{M}\right)} = \frac{(1 + \alpha MC\cos\beta)}{(1-M)(1-M(1-\alpha)C\cos\beta)} \; . \tag{3.11}$$

This field is usually larger than that for the Kutta condition case, and may become very large as $M \rightarrow 1$. The field in the jet arises now only from the pole at $u = {}^1/(1 + M)$, since the pole at ${}^1/_M$ representing the instability wave is cancelled; with $u = {}^1/(1 + M)$ we have to multiply the Kutta condition solution by the factor

$$\frac{(1 + \alpha MC\cos\beta)}{(1 + M)(1 - M(1 - \alpha)C\cos\beta)} \; . \tag{3.12}$$

We could now continue as in §2 and deal with jet waves convected at a speed νMc_j, following Howe (1979). The phenomena induced by this are, however, little different than those for incident internal noise, and we do not pursue the possibility further.

In the above discussion we have not considered the energy flows involved, as we did for incident internal noise. The issue is felt to be unimportant here, since there is no clear "incident" energy flow to act as a reference point. The only useful such reference quantity is the net acoustic energy flow inside the jet, directly due to the incident waves. Then there is an interesting counterpart to the acoustic energy conversion discussed earlier, in that some energy is converted to kinetic energy which is carried away by the jet instability waves.

3.2. Supersonic Jet

One of the most interesting aspects of Jacques' work is the prediction that the sound scattered vanishes when the jet is sonic. We have shown in §3.1 that in our solution this does not occur. We now examine the supersonic jet problem (M > 1). Then, with the same incident wave as in §3.1, and with the same modifications to the internal noise theory, we can use the theory of §2.3. Then the functions P(u), Z(u) are thus multiplied by the factor

$$-\frac{(u - \mu_0/k)\, K^-(\mu/k)}{(u - u_1)\, K^-(u_1)}\,, \tag{3.13}$$

which is equal to $-\dfrac{(u - \mu_o/k)}{(u - u_1)}$, since $K^- = 1$ for supersonic flows at low frequency.

The farfield sound level is, therefore, for a given p_i, multiplied by this factor evaluated at $u = C\cos\theta/(1 + \alpha MC\cos\theta)$ giving a multiplier

$$-\frac{(1 - (1 + (1 - \alpha)MC\cos\theta))(1 + \alpha MC\cos\beta)}{C(\cos\theta - \cos\beta)(1 + M)}\,. \tag{3.14}$$

We notice that these are factors similar to those in the subsonic case, giving a sound field zero at the cone of silence angle, $\theta = \cos^{-1} \dfrac{1}{(1 + (1 - \alpha)M)C}$ and the stationary phase calculation failing at $\theta = \beta$. There is no hint of the field becoming zero when the Mach number approaches one.

The field in the pipe is still zero, as it was for internal noise, since the factor $-\frac{(u - \mu_o/k)}{(u - u_1)}$ is still finite with $u = {}^{-1}/(1 - M)$. The pressure fields in the jet are also multiplied by this factor. For the cellular wave structure, with $u \sim O({}^{1}/ka)$, the pressure amplitude is changed only in sign. The value of the instability wave axial velocity is multiplied by the factor with $u = {}^{1}/_{M}$, namely

$$-\frac{(1 + M\alpha C\cos\beta)}{(1 + M)\,(1 - (1 - \alpha)C\cos\beta)} \quad . \tag{3.15}$$

In all this, the field representing the incident wave is, of course, $p = p_i\, e^{iku_1x}$, as it was for the subsonic case.

4. SOUND GENERATION DUE TO THE CONVECTION OF VORTICITY PAST THE END OF A CYLINDRICAL PIPE WITH FLOW

In this section we consider the convection of vorticity past the exit of a cylindrical pipe with flow. We confine ourselves initially to subsonic flow, and derive results that apply with or without a Kutta condition enforced at the duct exit.

4.1. Internal Convected Vorticity

Here we assume that the vorticity is convected at some speed $\nu M c_j$; the pressure field induced by this vorticity is described in Appendix 3. The motion due to the termination of the pipe is driven, as it was in sections 2, 3 by the pressure difference that would have existed across the jet shear layer if there were no termination to the pipe. This difference may be written in the form

$$p = \frac{M - M_c}{M_c} \cdot Q \cdot \exp[- ik x/M_c], \tag{4.1}$$

where the convection speed is $U_c = M_c\, c_j$ and Q is given in Appendix 3. In equation (2.16) we now have

$$P_o^+ (u,a) = \left(\frac{M - M_c}{M_c}\right) Q \cdot \frac{1}{ik(u - u_c)}, \tag{4.2}$$

where $u_c = {}^1/_{M_c}$.

Then, compared with the internal noise case (Section 2) the quantities P(u) and Z(u) are multiplied by the factor

$$\frac{(u - u_o) \cdot K^-(u_o)}{(u - u_c) K^-(u_c)} \quad , \quad \text{with} \qquad u_o = \frac{1}{(1 + M)} \tag{4.3}$$

The farfield sound, for a given pressure level, is then multiplied by this factor evaluated at the stationary phase point $u = C\cos\theta/(1 + \alpha M\cos\theta)$; with $K^-(u) = (1 + (1 - M)u)^{-1}$ and $M_c = \nu M$, we have

$$\frac{(u - u_o) K^-(u_o)}{(u - u_c) K^-(u_c)} = \frac{(1 - (1 + M(1 - \alpha))C\cos\theta)(1 - M(1 - \nu))}{2\,(1 - M(\nu - \alpha)\,C\cos\theta)} \tag{4.4}$$

so that

$$p_o(\omega) = \frac{Q(1-\nu)\ \rho_o\ 2\pi a\ \gamma\ ka\ [1 - (1+M(1-\alpha))C\cos\theta][1 - M(1-\nu)]e^{-i\omega R/c}}{4\pi R.(1+\alpha M\cos\theta).2\ [1+M(1-\alpha)C\cos\theta]^2[1 + M(\nu - \alpha)C\cos\theta]^2} \ . \tag{4.5}$$

A principal feature of this radiation field is that when $\nu = 1$ no pressure signal is radiated (Q does not alter much with ν (see Appendix 3)). This is in agreement with the results of Howe (1976), who found a corresponding result for the two dimensional case when a Kutta condition was enforced. The velocity dependence of the sound field is given by $p \sim \rho\, U^2 M$, since Q scales as ΓU which is proportional to U^2, and the frequency will be proportional to U. There is a zero in the field at the cone of silence angle $\theta = \cos^{-1}[(1 + (1 - \alpha)M)C]^{-1}$ as there was for the external sound field, and there is an extra Doppler factor based on the convection speed νMc_j.

From Appendix 3, we see that

$$Q = \frac{-i}{2\pi} \sum_{m=1}^{\infty} \rho_j \Gamma \left[\frac{J_o'(\mu_m r_o)}{J_o(\mu_m a)}\right] \cdot \frac{\mu_m r_o}{\left[\frac{\omega^2 a^2 \beta^2}{U_c^2} + \mu_m^2 a^2\right]} . \qquad (4.6)$$

In this expression, the value of Q is made up of contributions from the various radial eigenmodes of the pipe, the vortex ring being situated at a radius r_o. Substituting this value of Q into (4.6) gives

$$p_o(\omega) = \sum_{m=1}^{\infty} \frac{(1 - \nu)\ \pi a\ \rho_o\ ka\Gamma\ \ J_o'(\mu_m r_o)\ \mu_m r_o\ e^{i\omega R/c}\ F(\theta)}{J_o\ (\mu_m a)\ \left(\frac{\omega^2 a^2 \beta^2}{U_c^2} + \mu_m^2 a^2\right)} . \qquad (4.7)$$

where $F(\theta)$ is a directional factor independent of ω. This expression can be Fourier transformed to give the pressure level as a function of time,

$$p_o(t)$$

$$= \sum_{m=1}^{\infty} \frac{(1-\nu)}{4\pi R} \cdot \frac{a\rho_o}{R} \frac{\Gamma U_c}{a\beta} \cdot \frac{a}{c_j} \frac{\partial}{\partial t} \left[\exp\left\{- \frac{U_c \mu_m (t-R/c)}{\beta}\right\}\right] F(\theta)\ \frac{r_o}{a} \cdot \frac{J_o'(\mu_m r_o)}{J_o(\mu_m a)} . \qquad (4.8)$$

which reveals more clearly the scaling of the pressure on U^3.

In Appendix 3 we also derive an expression for the pressure field of a convected distribution of vorticity. Comparing that case with this, we see that there is very little change in the scaling of the farfield sound level with velocity.

The fields in the jet and pipe due to the convected vortices are again obtained from the internal noise case by the multiplicative factor

$$Q(1 - \nu) \frac{(u - u_o) K^-(u_o)}{(u - u_c) K^-(u_c)} \quad . \tag{4.9}$$

The sound field propagating up the pipe is obtained when $u = \frac{-1}{(1 - M)}$, at which condition this factor is simply

$$Q(1 - \nu). \tag{4.10}$$

Thus the sound field transmitted up the pipe is again (-1) times the pressure on the walls of the pipe in the absence of the duct termination. It also varies as $(1 - \nu)$, vanishing when the vortices are convected with the mean flow.

The pressure in the jet given by the pole at $u = u_c$ cancels the incident pressure field on the jet boundary. The instability wave is given by the pole at $u = {}^1/_M$, when the above multiplying factor becomes

$$-\frac{1}{2} Q \,(1 - (1 - \nu)M), \tag{4.11}$$

that is, the velocity fluctuation is given by

$$U_x = \frac{Q}{\rho_j c_j} (1 - (1 - \nu)M) e^{-ikx/_M} \quad . \tag{4.12}$$

Unlike any of the other fields discussed so far, this does not decrease to zero for vortices convected at the mean flow speed. A similar result has again been obtained by Howe (1976) for two dimensions. He found that there was a vortex wake present, which cancelled the field due to the convected incident vortex, and this enabled a Kutta condition to be satisfied.

We next consider the sound field generated when the Kutta condition is not enforced. As before we take the value of $\bar{C}(u)$ in the Wiener-Hopf equation as a constant. The value of $\bar{C}(u)$ is chosen to cancel the instability wave completely. Then all the Fourier transform quantities above are multiplied by

$$\frac{(u - {}^1/_M)}{(u_c - {}^1/_M)} \, . \tag{4.13}$$

In the farfield we take $u = C\cos\theta/(1 + \alpha MC\cos\theta)$ as usual, and then the factor (4.13) becomes

$$\frac{\nu}{(1 - \nu)} \cdot \frac{(1 - MC(1 - \alpha)\cos\theta)}{(1 + \alpha MC\cos\theta)} \, . \tag{4.14}$$

This factor has two important features. First, for all u (except ${}^1/_{M\nu}$), there is a multiplication by $\frac{\nu}{(1 - \nu)}$ removing the previous dependence on $(1 - \nu)$. Sound is thus generated even when the vortices convect with the flow. This effect of the removal of the Kutta condition is again in complete agreement with Howe's results for two dimensions. Second, the sound field is less directional, one of the Doppler factors $[1 - (1 - \alpha)MC\cos\theta]$ being removed and replaced by the external flow Doppler factor $(1 + \alpha MC\cos\theta)$.

The field in the pipe is accordingly multiplied by the factor

(4.14) with $u = {}^{-1}/(1 - M)$, that is, by

$$\frac{\nu M}{(1-\nu)M} \cdot \frac{1}{(1-M)} \quad . \tag{4.15}$$

The field in the jet due to the instability wave is obviously zero.

An alternative theory related to that just described has been given by Crighton (1972). He considered a simple instability wave whose energy is scattered by the duct exit. This is essentially given by the field due to the constant $\bar{C}(u)$ in the Wiener-Hopf equation. With $\bar{C}(u)$ = constant = C_o we obtain a jet displacement

$$Z(u) = \frac{C_o}{K^+(u)} \quad . \tag{4.16}$$

Substituting this into the formula for the pressure in the jet gives

$$p_j = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{\rho_j c_j^2 k^2 D_j^{\,2} . C_o (ka)^2 (1 - (1 + M)u) e^{-ikxu}\, k\, du}{kv\, J_m'(kva).\ -2a\, \rho c_j^{\,2} k^2 D_j^{\,2}} \quad , \tag{4.17}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{C_o\, e^{-ikux}\, k\, du}{(1 + (1 - M)u)} \quad . \tag{4.18}$$

The only pole in this expression is that representing a sound wave propagating up the duct, with amplitude

$$- \frac{k\, C_o\, e^{+ikx/(1-M)}}{(1 - M)} \quad . \tag{4.19}$$

Additionally, we find that there is an instability wave, which produces a velocity fluctuation

$$u_x = -\frac{kC_o\, e^{-ikx/M}}{\rho_j c_j M} , \tag{4.20}$$

arising from the pole at $^1/_M$. In Crighton's analysis this instability wave essentially drives the motion.

The farfield sound level is then given by

$$p_o = \frac{1}{2\pi i}\int \frac{\rho_j c_j{}^2 k^2 \gamma D_o^2\ (ka)^2(1-(1+M)u)C_o\ e^{-ikux}\ H_o^{(2)}(kwr)kdu}{kw\ H_o^{(2)'}(kwa).\ -2\rho_j c_j{}^2 k^2 D_j{}^2 a} . \tag{4.21}$$

Following Crighton, we scale the results on the value of the velocity fluctuation

$$u_N = -\frac{C_o k\ e^{+ikx/(1-M)}}{(1-M)} , \quad \text{at the nozzle exit,}$$

to give

$$p_o = \frac{A_j \rho_o}{4\pi R}\left[\frac{\partial u_N}{\partial t}\right]\frac{(1-M)\ (1-(1+M(1-\alpha))C\cos\theta)}{(1+M\alpha C\cos\theta)^2(1-MC(1-\alpha)\cos\theta)^2} . \tag{4.22}$$

This sound field has several interesting characteristics. In the low Mach number limit, $M \to 0$, it reduces to Crighton's value. While at first sight this appears to scale on $M_j{}^2$, assuming u_N to scale as U_j, Crighton however, adopts a different procedure, noting that u_N is linearly related to the net force on the nozzle. He then assumes that this force is driven by the unsteady flow at the nozzle and must scale as $U_j{}^2$, so that the sound field scales as $U_j{}^3$. The difference between our result and Crighton's is in the fieldshape. In the low frequency limit his field shape is proportional to $(1 - \cos\theta)$ and he suggests that forward speed U_o will increase this sound level by two inverse powers of the Doppler factor $(1 + \alpha MC\cos\theta)$. We see that this is incorrect, and the dependence on angle is considerably more complicated, except in the limit of vanishing Mach number. The complicated effect of external flow is, however, in keeping with other studies of forward speed effects (e.g. Dowling (1975), and Sections 2 and 3).

The field discussed above does not obey a Kutta condition, and any attempt to make it do so through choosing $\bar{C}(u) = C_o + C_1 u$ (say) is doomed to failure, since adding the additional instability wave solution will just result in the pressure vanishing everywhere if a Kutta condition is applied.

4.2. External Convected Vorticity

The formula for the excess pressure in this case may be written as $p_i = \frac{(\alpha M - M_c)}{M_c} Q_\alpha \exp[-ikx/M_c]$, where $M_c c_j$ is the convection velocity. This velocity should clearly be scaled with the external velocity, so that $M_c = \alpha\nu M$. Then we can rewrite p_i as

$$p_i = \frac{(1 - \nu)}{\nu} Q_\alpha e^{-ikx/\nu M} ,$$

and the corresponding value of $P_o^+(a)$ is

$$\frac{(1 - \nu)}{\nu} \frac{Q_\alpha}{ik(u - 1/M\alpha\nu)} . \tag{4.23}$$

This yields a multiplying factor, as introduced earlier, and relating the present problem to that with internal noise, equal to

$$\left(\frac{1-\nu}{\nu}\right) \cdot Q_\alpha \frac{\left[u - \frac{1}{(1+M)}\right]\left[1 + (1-M)/\alpha M\nu\right]}{\left[u - {}^{1}/_{\alpha M\nu}\right]\left[1 + (1-M)/(1+M)\right]} \quad . \tag{4.24}$$

In the farfield this factor becomes (setting $u = C\cos\theta/(1 + \alpha MC\cos\theta)$)

$$\left(\frac{1-\nu}{\nu}\right) Q_\alpha \cdot [1 - M(1 - \alpha\nu)] \frac{[1 - (1 + M(1-\alpha))C\cos\theta]}{[1 - (1 + M\alpha(1-\nu))C\cos\theta]} \quad . \tag{4.25}$$

The major effect on the fieldshape, as compared with the case of internally convected vorticity, is that the Doppler factor in the denominator is much reduced in effect because of the lower vortex convection velocity.

Again, for the fields in jet and pipe we find that the pressure transmitted upstream is unchanged (for a given ν), while the amplitude of the instability wave is correspondingly given by

$$2Q_\alpha \frac{(1-\nu)}{\nu} \cdot \frac{1}{M(1-\alpha\nu)} \quad .$$

Therefore, that amplitude, for given Q_α, and ν, is relatively bigger by the factor $\frac{(1-\nu)}{(1-\alpha\nu)}$ than for the internally convected vorticity.

The effect of relaxing the Kutta condition is again to multiply the amplitude of the field by $\frac{(1 - Mu)}{(1 - Mu_1)}$, which for the farfield is given by $\frac{M_c}{M_c - M} \cdot \frac{(1 - M(1-\alpha)\ C\cos\theta)}{(1 + \alpha MC\cos\theta)}$. In this case the relaxation of the Kutta condition gives a large increase in the sound field for vortices convected with the velocity of the external flow, so that the sound level is again non-zero when the vortices are convected with the flow.

In this section we have shown that when axisymmetric vortex flows interact with the jet pipe, sound is generated which scales in amplitude in the farfield on U_j^3. This is the same dependence as that found by Leppington (1971) who modelled the turbulence as non-convecting point quadrupoles close to the end of the pipe: we have, though, also shown that when a Kutta condition is enforced the sound field vanishes. This is because this sound is essentially driven by the pressure fluctuations induced on the pipe wall by the vortices, this vanishing to a linear approximation when the vortices are convected with the flow.

We have not considered supersonic flows here. The phenomena produced are not expected to be any different from those for internal noise, while some of the features of the flow found by Howe (1976) for vortices convected past a two dimensional plate, are also expected to be present.

5. ACOUSTIC ANALOGIES

In this section we use two forms of acoustic analogy to derive equations for the sound field. These enable the farfield sound to be ascribed to various monopole, dipole and quadrupole sources. The results are of interest for several reasons. In the past these analogies have been used alone to determine the farfield sound. In most cases this has been done incorrectly, ignoring the quadrupole sources. We show that at high Mach numbers, these quadrupole sources are responsible for most of the farfield sound. Further, we show how the O(1) fields induced in the pipe and jet may be deduced by simple reasoning in the low frequency limit, without reference to the Wiener-Hopf solution to the complete problem. We then use these zero order fields to evaluate the source terms.

We consider two forms of the acoustic analogy: that derived from the Lighthill (1952) equations and a different analogy, due to Dowling, Ffowcs Williams & Goldstein (1978), incorporating a mean flow. An alternative analogy is that of Howe (1975) which relates the sound field to unsteady vorticity. Howe (1979) has used it to discuss the transmission of sound out of a pipe with flow with results similar to those of our analysis, but restricted to low Mach numbers.

5.1. The Lighthill Analogy

Ffowcs Williams & Hawkings (1969) have shown that the equation governing the sound field created by a moving surface defined by

$f(\underline{x}) = 0$ and moving at a speed v is:

$$\left(\frac{\partial^2}{\partial t^2} - c_o^2 \frac{\partial^2}{\partial x_i^2}\right) H(f)\,(\rho - \rho_o) = \frac{\partial^2\, T_{ij}\, H(f)}{\partial^2\, x_i\, \partial x_j}$$

$$- \frac{\partial}{\partial x_i} \left[(p_{ij} + \rho u_i(u_j - v_j))\delta(f) \frac{\partial f}{\partial x_j}\right]$$

$$+ \frac{\partial}{\partial t} \left[(\rho_o v_i + \rho(u_i - v_i))\delta(f) \frac{\partial f}{\partial x_i}\right],$$

(5.1)

where $\underline{u}$ is the fluid velocity, $T_{ij} = \rho u_i u_j + p_{ij} - c_o^2 \delta_{ij}$ is the Lighthill acoustic stress, and p_{ij} the compressive stress tensor. We apply this to a surface which encloses the end of the nozzle and the outer walls of the pipe. To take account of the external flow, we express the solution to this equation in convected coordinates such that the nozzle is fixed relative to the observer. Then, the Green's function G satisfying

$$\left[\frac{D^2}{Dt^2} - c_o^2 \frac{\partial^2}{\partial x_i^2}\right] G = \delta(t - t_o)\, \delta(x - x_o), \quad \frac{D}{Dt} = \frac{\partial}{\partial t} + U_o \frac{\partial}{\partial x},$$

(5.2)

is

$$G = \frac{\delta(t - t_o - R/_c + \hat{\underline{r}} \cdot \underline{x}_o/(1 - \underline{M}.\hat{\underline{r}}))}{4\pi R\,(1 + \underline{M}_o.\hat{\underline{r}})c_o^2}, \qquad (5.3)$$

where $\hat{\underline{r}}$ is the direction of the observer relative to the source, and the result is valid in the farfield.

The sound field is, from (5.1), given by

$$\rho - \rho_o = \int_V G \frac{\partial^2}{\partial y_i \, \partial y_j}(H(f)\, T_{ij})\, dV - \int_S \frac{\partial G}{\partial y_i}(p_{ij} + \rho u_i(u_j - v_j))\ell_j \, dS$$

$$+ \int \frac{DG}{Dt}(\rho_o \, v_i + \rho(v_i - u_i))\, \ell_i \, dS \qquad (5.4)$$

where ℓ_i is the normal to the surface S. In the problem under consideration, the derivatives may be taken outside the integrals to give the farfield result

$$\rho - \rho_o = \frac{1}{4\pi R \, c_o^2 \, (1 + M_o \cos\theta)^3} \frac{\partial^2}{\partial t^2} \int_V [H(f)\, T_{rr}]\, dV$$

$$+ \frac{1}{4\pi R \, c_o^2 \, (1 + M_o \cos\theta)^2} \frac{\partial}{\partial t} \int_S p_{rn} + \rho u_r \, (u_n - v_n)\, dS$$

$$+ \frac{1}{4\pi R \, c_o^2 \, (1 + M_o \cos\theta)^2} \frac{\partial}{\partial t} \int_S \rho_o \, u_n + \rho(u_n - v_n)\, dS$$

$$(5.5)$$

where the square brackets signify that the integrals are to be evaluated at the retarded time $t - {}^R/_c + \dfrac{y_r}{(1 + M_o \cos\theta)}$, $M_o = U_o/c_o$ and r and n denote the radiation direction and the normal to the surface S.

We now examine the quadrupole term in more detail. The stress tensor can be written in the form $T_{rr} = (\bar{T}_{rr} + T'_{rr})\, H(g)$ where $g = 0$ is the boundary of the jet, and $\bar{T}_{rr}$ and T'_{rr} are respectively the mean and fluctuating components of T_{rr}. We now take the time derivatives inside the integral, and split g into $\bar{g} + g'$, its mean and fluctuating positions. Then

$$\frac{\partial^2}{\partial t^2} H(f)\, H(g)\, T_{rr} = \frac{\partial}{\partial t} \left(H(f) \frac{\partial T_{rr}}{\partial t} H(g) + T_{rr}\, \delta(g) \frac{\partial g}{\partial t}\right), \qquad (5.6)$$

and if g moves at speed v_g,

$$\frac{\partial g}{\partial t} + v_i \frac{\partial g}{\partial x_i} = 0,$$

so that

$$\frac{\partial^2}{\partial t^2}[H(f)\, H(g)\, T_{rr}] = \frac{\partial}{\partial t} \left(H(f) \frac{\partial T'_{rr}}{\partial t}\; H(\bar{g}) - \bar{T}_{rr}\, \delta(\bar{g})\, v_i \frac{\partial g'}{\partial x_i}\right), \qquad (5.7)$$

where we have ignored terms of second order in the fluctuating quantities. Thus the sound due to the quadrupole sources can be written:

$$(\rho - \rho_o)_Q = \frac{1}{4\pi\, R\, c_o^2\, (1 + M_o \cos\theta)^3} \frac{\partial}{\partial t} \int [H(f) \frac{\partial T'_{rr}}{\partial t} H(\bar{g})$$

$$- \bar{T}_{rr}\, H(f)\, \delta(\bar{g})\, v_i \frac{\partial g'}{\partial x_i}]\, dV. \qquad (5.8)$$

It is clear that the source term due to the steady part of T_{ij}, acting over a variable volume, is equivalent to a surface source. We may also write the sound from this quadrupole as

$$(\rho - \rho_o)_Q = \frac{1}{4\pi R\, c_o^2\, (1 + M_o \cos\theta)^3} \frac{\partial}{\partial t} \int_{S_J} [v_n\, \bar{T}_{rr}]\, dS_J, \qquad (5.9)$$

where S_J is the exterior surface of the jet, which moves at speed $\underline{v}$. This is the velocity measured in free space. It is convenient to convert this into a velocity in the jet flow. To do this we write $v_n = \frac{\partial \eta}{\partial t}$, where η is the radial displacement of the jet boundary. Then the velocity inside the jet, u_n, is related to the displacement by

$$\frac{\partial \eta}{\partial t} + U_J \frac{\partial \eta}{\partial x} = u_n,$$

so that

$$(\rho - \rho_o)_Q = \frac{1}{4\pi R\, c_o^2 (1 + M_o \cos\theta)(1 - M_R \cos\theta)^2} \frac{\partial}{\partial t} \int [u_n\, \bar{T}_{rr}]\, dS_J. \qquad (5.10)$$

In this expression the change to the fluid velocity in the jet has caused one of the Doppler factors based on the external flow Mach number M_o to be replaced by one based on the relative flow velocity $(U_j - U_o) = M_R\, c_o$.

For many purposes it is useful to relate the integrand of (5.10) to the pressure and velocity in the jet, since the radial velocity of the fluid in the jet is not a quantity easily calculated in our problems. The equation of continuity is, after linearisation,

$$\frac{Dp'}{Dt} + \rho_j c_j^2 \quad \nabla . \underline{u}' = 0, \tag{5.11}$$

so that if we consider a section of the jet flow, we find that

$$\int_V \frac{Dp}{Dt}\, dV + \rho_j c_j{}^2 \int_{S_x} u_x \, dS + \int_{S_n} \rho_j c_j^2 \, v_n \, dS = 0, \tag{5.12}$$

where S_x, S_n are the axial cross section, and outer surface of the jet. Hence we find that for a section of the jet of length dx,

$$\int_{S_x} \frac{Dp}{Dt}\, dS_x \; dx + \rho_j c_j{}^2 \int_{S_n} (\frac{\partial u}{\partial x})\, dS_x \; dx + \rho_j c_j{}^2 \int_{S_n} v_n \; r \; d\theta \; dx = 0 \tag{5.13}$$

For an axisymmetric motion of the jet it follows that the 'steady' quadrupole term is

$$(\rho - \rho_o)_Q$$

$$= \frac{-\,\bar{T}_{rr}}{4\pi R\, c_o^2\, (1 + M_o \cos\theta)^2 (1 - M_R \cos\theta)} \frac{\partial}{\partial t} \int [\frac{Dp}{Dt} \cdot \frac{1}{\rho_j c_j{}^2} + \frac{\partial u}{\partial x}]\, dV, \tag{5.14}$$

where the region of integration is the volume of the jet. We now apply the above results to a number of practical cases.

i) Internal Noise Propagating down a Pipe with Internal and External Flow

We consider initially the situation described in §3.1. and first estimate the relevant source terms. Clearly, the pressure and normal velocity on the surface of the pipe are both small and zero respectively, so that the sources on the outer wall of the pipe are negligible. At the nozzle exit the pressure fluctuations are similarly negligible, as the flow cannot respond to low amplitude fluctuations in velocity. Setting the pressure fluctuation at the nozzle equal to zero, and assuming that the radiation at low frequencies is relatively small ($O(k^2a^2)$ in energy), we find that the reflected amplitude within the pipe is (-1) times the incident amplitude, in agreement with the exact solution. The axial velocity fluctuation at the nozzle is then given by $u_N = 2p_i'/\rho_j c_j$, where u_N is assumed constant across the nozzle exit.

The motion in the jet is assumed to consist of the simple convected neutrally stable wave of axial velocity fluctuation and zero pressure fluctuation; this is the limit of the cylindrical vortex sheet eigenfunction for very low frequencies. We now use these zero order fields to evaluate the individual source terms, and sound fields.

The monopole is

$$(\rho - \rho_o)_M = \frac{1}{4\pi R\, c_o^2(1 + M_o\cos\theta)^2}\frac{\partial}{\partial t}\int_S [\rho_o v_n + \rho\,(u_n - v_n)]\, dS. \tag{5.15}$$

Here v_n is the velocity of the end of the pipe relative to the external fluid and is, therefore, equal to $- U_o$; the velocity u_N is then $u_N + (U_j - U_o)$, and with $\rho - \rho_j$ zero at the nozzle the monopole term is

$$(\rho - \rho_o)_M = \frac{\rho_j A_j}{4\pi R c_o^2 (1 + M_o \cos\theta)^2} \frac{\partial u_N}{\partial t} , \qquad (5.16)$$

where A_j is the exit area of the nozzle.

The dipole term is

$$(\rho - \rho_o)_D = \frac{1}{4\pi R c_o^3 (1 + M_o \cos\theta)^2} \frac{\partial}{\partial t} \int_S [p_{nr} + \rho u_r (u_n - v_n)] \, dS. \qquad (5.17)$$

The quantities on the nozzle exit are the same as those used for the monopole source so that $p_{nr} = 0$, $\rho u_r(u_n - v_n)$ $= \rho_j (u_N + u_j - u_o))(u_N - u_j)$. Accordingly, the dipole term is

$$(\rho - \rho_o)_D = \frac{\rho_j A_j (2M_R + M_o)\cos\theta}{4\pi R c_o^3 (1 + M_o \cos\theta)^2} [\frac{\partial u_N}{\partial t}] , \qquad (5.18)$$

while the unsteady quadrupole term is:

$$(\rho - \rho_o)_{UQ} = \frac{A_j}{4\pi R c_o^3 (1 + M_o \cos\theta)^3} \frac{\partial^2}{\partial t^2} \int_0^\infty [\rho u_r u_r + (p - c_o^2 \rho)]' \, dy. \qquad (5.19)$$

Since the motion in the jet is dominated by the instability wave, p', ρ' are zero. Then with $U = u_N + (U_j - U_o)$ we find that if $u_N \equiv u_N (t - y/U_j)$ the quadrupole term takes the form

$(\rho - \rho_o)_{UQ}$

$$= \frac{\rho_j \; A_j \; 2M_R \cos^2\theta}{4\pi R \; c_o{}^3 \; (1+M_o\cos\theta)^3} \int_o^\infty \frac{\partial^2}{\partial t^2} \left[U_N\left(t - \frac{y(1-(M_j-M_o)\cos\theta)}{U_j(1+M_o\cos\theta)}\right)\right] dy \qquad (5.20)$$

Integrating with respect to y gives

$$(\rho - \rho_o) = \frac{\rho_j \; A_j \; . \; 2M_R M_j \cos^2\theta}{4\pi R \; (1+M_o\cos\theta)^2(1-M_R\cos\theta)} \left[\frac{\partial u}{\partial t}\right]_{y=o}^{y=\infty} \quad , \qquad (5.21)$$

and the contribution from the point at infinity must vanish for a causal solution, since the disturbance will not have reached infinity in a finite time. It follows that the contribution to the sound field from this unsteady quadrupole source is

$$(\rho - \rho_o)_{UQ} = \frac{- \rho_j \; A_j \; . \; 2M_R \; M_j \cos^2\theta}{4\pi R \; c_o{}^2(1 + M_o\cos\theta)^2 \; (1 - M_R\cos\theta)} \left[\frac{\partial u_N}{\partial t}\right] . \qquad (5.22)$$

We now evaluate the steady quadrupole term. Since there is no pressure fluctuation in the jet this is

$$(\rho - \rho_o)_{SQ} = \frac{\bar{T}_{rr}}{4\pi R \; c_o{}^2(1+M_o\cos\theta)^2(1-M_R\cos\theta)} \frac{\partial}{\partial t} \int \left[\frac{\partial u}{\partial y}\right] dy \; dS, \quad (5.23)$$

Now the value of $\bar{T}_{rr}$ is $\rho_j \cos^2\theta \, (u_j - u_o)^2 - a_o{}^2 \, (\rho_j - \rho_o)$, so that

$(\rho - \rho_o)_{SQ} =$

$$\frac{(\rho_j \cos^2\theta M_R^2 - (\rho_j - \rho_o))}{4\pi R \; c_o{}^2(1+M_o\cos\theta)^2(1-M_R\cos\theta)} \int \left[\frac{\partial u}{\partial x}\left(t - \frac{y(1-M_R\cos\theta)}{u_j(1 + M_o\cos\theta)}\right)\right] dy \; dS,$$

$$(5.24)$$

and evaluating the integral as for the unsteady quadrupole, we have this sound field in the form

$$(\rho - \rho_o)_{SQ} = \frac{\rho_j A_j \ (M_R^2\cos^2\theta - (\rho_j/\rho_o - 1))}{4\pi R\ c_o^2(1+M_o\cos\theta)(1-M_R\cos\theta)^2}\ [\frac{\partial u_N}{\partial t}]\,. \tag{5.25}$$

Adding the four source terms we find that the total sound field is then

$$(\rho - \rho_o) = \frac{\rho_j A_j}{4\pi R c_o^3}\ [\frac{\partial u_N}{\partial t}]\ [\underbrace{\frac{1}{(1+M_o\cos\theta)^2}}_{\text{monopole}} + \underbrace{\frac{(2M_R+M_o)\cos\theta}{(1+M_o\cos\theta)^2}}_{\text{dipole}}$$

$$- \underbrace{\frac{2M_R(M_R + M_o)\cos^2\theta}{(1 + M_o\cos\theta)^2(1-M_R\cos\theta)}}_{\text{unsteady quadrupole}} + \underbrace{\frac{M_R^2\cos^2\theta-(\rho_j/\rho_o-1)}{(1+M_o\cos\theta)(1-M_R\cos\theta)^2}}_{\text{steady quadrupole.}}\ ] \tag{5.26}$$

and simplification of the bracketed term gives precisely the sound field obtained earlier by the Wiener-Hopf method, namely

$$(\rho - \rho_o) = \frac{(\rho_o A_j)}{4\pi R c_o^2(1+M_o\cos\theta)(1-M_R\cos\theta)^2}\ [\frac{\partial u_N}{\partial t}]\ . \tag{5.27}$$

For high density ratios ρ_o/ρ_j and for high Mach numbers this total field comes mainly from the steady quadrupole term. In particular, this is responsible for the scaling (for a given u_N) on the farfield density ρ_o rather than the jet density ρ_j, and for the high convective amplification observed on the fieldshape. Further, it shows that in problems of this kind involving coupled unstable wave motion, it is never permissible to neglect the instability wave when calculating the sound radiation. Indeed the sound from these unstable (albeit neutrally stable at low frequencies) waves apparently dominates the farfield sound for high enough Mach numbers. In some senses this last conclusion is not really surprising as the dominance of quadrupole sources would seem to be a universal feature of high speed flow.

We now consider, in much less detail, the radiation from a very hot jet. From the results of §3.1 we find that all the sound energy is transmitted out of the jet pipe. The fields on the exit plane are obviously $p' = p_i'$, $u = p_i/\rho_j c_j$. These give dipole and monopole sound sources as described above and both can be neglected here since they are proportional to the jet density ρ_j, which is by assumption very small. Of the fields in the jet, that due to the propagating guided acoustic wave is very small (proportional again to ρ_j) and can be neglected. Because the density ratio is enormous, the boundary displacement is small (the jet boundary appears as if almost rigid). Therefore, the steady quadrupole source is negligible. The remaining term is the quadrupole due to the pressure wave

$$p = p_i \cdot \frac{2\varepsilon^2}{\pi} \exp(-\varepsilon kx), \qquad \varepsilon^2 = \frac{1}{[\frac{\rho_o}{\rho_j} (ka)^2 \ \ell n \ ka]} \quad . \tag{5.28}$$

For the essentially illustrative purpose of this section, we consider only the low Mach number case. Then the quadrupole element is dominated by the term $(p - c_o{}^2\rho)$ which in the limit $\rho_o/\rho_j \to \infty$ is simply p'. Therefore, the quadrupole field becomes

$$\rho - \rho_o = \frac{1}{4\pi R c_o^4} \frac{\partial^2}{\partial t^2} \int \frac{2}{\pi} \varepsilon^2 p_i e^{i\omega t + iy \frac{\omega\cos\theta}{c} - \varepsilon kx - \omega R/c} dy, \tag{5.29}$$

$$\sim \frac{p_i}{4\pi R c_o^2} \cdot \frac{i\omega c_o}{\cos\theta} \cdot \frac{2\varepsilon^2}{\pi} \text{ as } \varepsilon \to 0. \tag{5.30}$$

Substituting for ε, we obtain the result

$$\rho - \rho_o = \frac{A_j}{4\pi R} \cdot \frac{i\omega p_i}{c_o^3} \frac{(2/\pi)}{[\rho_o/\rho_j \cdot (ka)^2 \ln (ka)]}, \tag{5.31}$$

which is precisely equal to the field calculated exactly in section 3.1. We have shown further that this sound arises from the isotropic unsteady quadrupole term.

In the above account we have only touched on the subsonic jet with a Kutta condition. However, since the purpose of this section was mainly to illustrate the principles involved, there seems little point in proceeding with the cases of a jet with no Kutta condition or of a supersonic jet.

ii) Scattering of an Externally Incident Sound Field by a Jet Pipe

This problem has been attempted by Jacques (1975) using an acoustic analogy. He, however, considered only the monopole and dipole terms on the nozzle exit. We shall show that many more source terms should be included; dipole sources on the outside wall of the pipe, and steady and unsteady quadrupole sources due to both the instability wave and the portion of the incident sound field which propagates along the jet. For simplicity we confine the analysis to a jet of the same temperature as its surroundings, and no external flow. We consider the various source terms in turn.

The amplitudes of the various sources are derived using the following low frequency asymptotes to the unsteady flows in the jet and pipe. On the outer wall of the duct, the pressure is equal to the ambient pressure, $p = p_i\, e^{-iku_1x}$, for this compact jet ($ka << 1$). The jet itself is surrounded by a pressure fluctuation $p_i\, e^{-iku_1x}$, the incident sound field, so that there is a pressure wave of this magnitude inside the jet. The pressure also sends a wave of amplitude p_i up the pipe, so that the pressure in the pipe is $p_i\, e^{ikx/(1-M)}$. Clearly, these pressure waves provide an imbalance in velocity on either side of the nozzle exit plane. This is balanced by the convected instability wave (which has zero pressure fluctuation) and is accordingly described by

$$u = -\frac{p_i}{\rho_j c_j}\left[1 + \frac{u_1}{(1 + Mu_1)}\right] e^{-ikx/M} \quad . \tag{5.32}$$

We now consider each of the source terms.

The dipole source on the outside wall of the cylinder gives rise to the density field

$$(\rho - \rho_o)_{DW} = \int \frac{(\underline{n}.\hat{\underline{r}})}{4\pi R\, c_o^{\,2}} \cdot \frac{\partial}{\partial t}\, p_n \left(t - \frac{R}{c_o} + \frac{\underline{y}.\underline{r}}{c_o}\right) dS, \tag{5.33}$$

where S is the surface of the jet pipe. If the incident wave is of the form

$$p = p_i \exp[ikx \cos\theta_o + iky \sin\theta_o]$$

we can write this dipole field as

$$(\rho-\rho_o)_{DW} = \frac{p_i\, i\omega e^{-i\omega R/c_o}}{4\pi R\, c_o^{\,2}} \int_o^{2\pi} d\phi \int_o^{\infty} 2\pi a\, dx \sin\theta \cos(\phi-\phi_o) e^{-i(\cos\theta-\cos\theta_o)kx}$$

$$\cdot e^{i(\sin\theta_o - \sin\theta\cos\phi)ka} , \qquad (5.34)$$

and to evaluate this integral for ka → 0 we simply expand the exponentials for small ka. Then

$$(\rho-\rho_o)_{DW} = \frac{p_i . i\omega e^{-i\omega R/c_o}}{4\pi R\, c_o^{\,2}} \int_o^{2\pi} d\phi \int_o^{\infty} dx\ 2\pi a \sin\theta \cos(\phi-\phi_o) e^{ik(\cos\theta_o - \cos\theta)x}$$

$$. \ [1 + ika(\sin\theta_o - \sin\theta\cos(\phi - \phi_o))],$$

$$(5.35)$$

and the only axisymmetric term is

$$(\rho-\rho_o)_{DW} = \frac{p_i . i\omega\ \pi a}{4\pi R\, c_o^{\,2}}\, e^{-i\omega R/c_o} \int_o^{\pi} ika \sin^2\theta\, e^{i(\cos\theta_o - \cos\theta)kx}\, dx,$$

$$= - \frac{p_i\ .\ i\omega A_j \sin^2\theta\, e^{-i\omega R/c_o}}{4\pi R\, c_o\, (\cos\theta - \cos\theta_o)} , \qquad (5.36)$$

where we have assumed that k has a small imaginary part to ensure convergence at infinity. This is the most important of the terms neglected by Jacques, and is important even for vanishingly small Mach numbers.

The monopole on the jet pipe exit plane has strength

$$\rho u_N = - \frac{p'}{c} (1 - M), \tag{5.37}$$

resulting in a sound field

$$(\rho - \rho_o)_M = \frac{i\omega\ A_j\, p_i}{4\pi R\ c_o^{\ 2}} (1 - M) e^{-i\omega R/c_o}. \tag{5.38}$$

The dipole strength is

$$p + (\rho U^2)' = p'(1 - M)^2 . \tag{5.39}$$

and therefore, the radiation field from the dipoles on the exit of the duct is

$$(\rho - \rho_o)_{DE} = \frac{i\omega A\ p_i}{4\pi R\ c_o^{\ 2}} (1 - M)^2 \cos\theta\ e^{-i\omega R/c_o} . \tag{5.40}$$

We consider next the unsteady quadrupole due to the instability wave; for an instability wave amplitude u_i this is given by

$$(\rho - \rho_o)_{UQI} = \frac{i\omega A\ \rho_j}{4\pi R\ c_o^{\ 2}} \frac{M^2 \cos^2\theta\ u_i}{(1 - M\cos\theta)} e^{-i\omega R/c_o} . \tag{5.41}$$

Here we have

$$u_i = - \frac{p_i}{\rho_j c_j} \left(1 + \frac{\cos\theta_o}{(1 - M\cos\,\theta_o)}\right) \quad , \tag{5.42}$$

so that this quadrupole field is

$$(\rho - \rho_o)_{UQI} = \frac{i\omega\, A_j P_i \, e^{-i\omega R/c_o}}{4\pi R\, c_o^{\,2}} \cdot \frac{M^2 \cos^2\theta}{(1 - M\cos\theta)} \cdot \left(1 + \frac{\cos\theta_o}{(1 - M\cos\theta_o)}\right). \tag{5.43}$$

Correspondingly, the sound radiation from the steady quadrupoles excited by the instability wave is given by the previous result (5.25), with the new amplitude of the instability wave substituted and with $\rho = \rho_j$, giving

$$(\rho - \rho_o)_{SQ} = \frac{i\omega A_j\, p_i \, e^{-i\omega R/c_o}}{4\pi R\, c_o^{\,2}} \cdot \frac{M^2\cos^2\theta}{(1 - M\cos\theta)^2} \left(1 + \frac{\cos\theta_o}{(1 + M\cos\theta_o)}\right) . \tag{5.44}$$

The unsteady longitudinal quadrupole due to the incident wave existing in the jet flow has strength

$$((\rho u^2)' + p' - c_o^{\,2}\rho')_{rr} = p'_i \, M^2\cos^2\theta + \frac{2M \cos^2\theta \cos\,\theta_o}{(1 - M\cos\theta_o)} . \tag{5.45}$$

Then using the earlier results we see that the sound radiation from this source is given by

$$(\rho - \rho_o)_{UQJ} = \frac{A_j \, e^{-i\omega R/c_o} \, i\omega \, p_o}{4\pi R\, c_o} \left(M^2\cos^2\theta + \frac{2M \cos^2\theta\cos\theta_o}{(1 - M\cos\theta_o)}\right). \tag{5.46}$$

On the other hand, the steady quadrupole due to the wave in the jet has strength

$$\left(\frac{1}{\rho_j c_j^2}\frac{Dp}{Dt} + \frac{\partial u}{\partial x}\right)\bar{T}_{rr} \quad ,$$

and the bracketed factor becomes, on substituting for p′ and u,

$$\frac{i\omega p_i}{\rho_j c_j^2}\left(\frac{\cos^2\theta_o}{(1 - M\cos\theta_o)} - (1 - M\cos\theta_o)\right) \quad ,$$

while $\bar{T}_{rr}$ is again just equal to $M^2\cos^2\theta$. Then the radiation from this steady quadrupole is given by

$$(\rho - \rho_o)_{SQJ}$$

$$= \frac{i\omega p_i A_j e^{-i\omega R/c_o}}{4\pi R} \cdot \frac{M^2\cos^2\theta(\cos^2\theta_o - (1-M\cos\theta_o)^2)}{(1-M\cos\theta)(1-M\cos\theta_o)} \int_o^\infty e^{ikx(\cos\theta_o - \cos\theta)} dx,$$

$$= \frac{A_j\, i\omega\, p_i\, e^{-i\omega R/c_o} M^2\cos^2\theta(\cos^2\theta_o - (1-M\cos\theta_o)^2)}{4\pi R\, c_o\, (1-M\cos\theta)(1-M\cos\theta_o)(\cos\theta - \cos\theta_o)} \quad . \qquad (5.47)$$

Addition of these quantities, (5.36), (5.38), (5.40), (5.43), (5.44), (5.46), (5.48), yields the radiation field derived exactly in the low frequency limit (section 3). Comparison of this result with Jacques' shows that he has neglected all the quadrupole sources and also the dipoles on the duct wall. In the low Mach number limit our radiation field is

$$(\rho - \rho_o) = \frac{A\, p_i\, i\omega e^{-i\omega R/c_o}}{4\pi R\, c_o} \left(\cos\theta - 1 + \frac{\sin^2\theta}{(\cos\theta - \cos\theta_o)}\right), \qquad (5.48)$$

of which the first two terms are those used by Jacques while the last is the duct wall dipole. Adding these up gives the low frequency low Mach number scattered field

$$(\rho - \rho_o) = \frac{A\, p_i\, i\omega(1 - \cos\theta_o)(1 - \cos\theta)}{4\pi R\, c_o\, (\cos\theta - \cos\theta_o)} \quad . \qquad (5.49)$$

In this result, unlike that of Jacques, there is a reciprocal relation between the incident and scattered fields.

iii) <u>Source Terms due to Convected Vortices</u>

The wave transmitted up the pipe due to the vortices is taken to be $- p_o\, e^{ik_1 x/(1-M)}$. This allows us to estimate (again for $\rho_j = \rho_o$ and no external flow) the magnitude of the dipole and monopole sources on the duct exit; the monopole source has strength

$$\rho_j\, u' + \rho' U_j = - (1 - M)\, p_i/c_o, \qquad (5.50)$$

so that the monopole sound field is

$$(\rho - \rho_o)_M = \frac{A_j\, i\omega}{4\pi R\, c_o}\, p_i\, (M - 1)\, e^{-i\omega R/c_o} \quad . \qquad (5.51)$$

Similarly the contribution from the internal pressure field to the exit plane dipole is given by the dipole strength $p_i\, (1 - M)^2$, resulting in a radiated field

$$(\rho - \rho_o)_E = \frac{A_j \, i\omega \, (1 - M)^2 \, p_i \, \cos\theta \, e^{-i\omega R/c_o}}{4\pi R \, c_o} \, . \tag{5.52}$$

(There is no contribution to this source from the incident pressure due to the convected vortices since it integrates to zero.) The amplitude of the waves in the jet due to the incident pressure convected with the vortices is

$$p = p_i \, e^{-ikx/M\nu}$$

which has an associated axial velocity fluctuation

$$u_x = \frac{-\, p_i \, e^{-ikx/M\nu}}{\rho_j c_j \, M(1 - \nu)} \tag{5.53}$$

where the convection speed $U_c = M\nu c_j$.

Matching the velocity fluctuations at the nozzle exit we find that the magnitude of the instability wave is given by

$$u_i = \frac{p_i}{\rho_j c_j} \left(-\, 1 - \frac{1}{M(1-\nu)}\right) = \frac{-\, p_i}{\rho_j c_j} \, \frac{(1 + M(1 - \nu))}{M(1 - \nu)} \, . \tag{5.54}$$

The strength of the radiation due to the instability wave unsteady quadrupole is, using (5.54) and the earlier results,

$$(\rho - \rho_o)_{UQI} = \frac{-\, A_j \, i\omega \, p_i \, e^{-i\omega R/c_o} \, M^2 \cos^2\theta (1 + M(1 - \nu))}{4\pi R \, c_o \, M(1 - \nu) \, (1 - M\cos\theta)} \, . \tag{5.55}$$

Similarly, it follows that the strength of the steady quadrupole driven field associated with the instability wave is

$$(\rho - \rho_o)_{SQI} = - \frac{A\, i\omega\, p_i\, e^{-i\omega R/c_o}\, M \cos^2\theta\, (1 + M(1 - \nu))}{4\pi R\, c_o \cdot 2(1 - \nu)\, M(1 - M\cos\theta)^2} \,. \qquad (5.56)$$

The unsteady quadrupole due to the waves convected with the incident vorticity is given by substituting for the pressure and axial velocity in the usual formula, to give

$$(\rho - \rho_o)_{SQO}$$

$$= \frac{A_j\, e^{-i\omega R/c_o}}{4\pi R\, c_o^{\,2}} \frac{\partial^2}{\partial t^2} \int_o^\infty p_i(-M^2\cos^2\theta + \frac{2M\cos^2\theta}{(1 - \nu)})\, e^{i\omega t - ikx(1 - M\nu\cos\theta)/M\nu} dx,$$

$$= \frac{A_j\, e^{-i\omega R/c_o}}{4\pi R\, c_o^{\,2}} (i\omega)\, p_i\, (- M^2\cos^2\theta + \frac{2M\cos^2\theta}{(1 - \nu)}) \frac{\nu M}{(1 - \nu M\cos\theta)} \,. \qquad (5.57)$$

Finally, the sound radiation from the unsteady quadrupole is given by

$$(\rho - \rho_o)_{UQ}$$

$$= \frac{A_j\, p_i\, M^2\cos^2\theta\, e^{-i\omega R/c_o}}{4\pi R\, c_o^{\,2}(1 - M\cos\theta)} \frac{\partial}{\partial t} \int_o^\infty (\frac{\partial v}{\partial x} + \frac{1}{\rho_j c_j^{\,2}} \frac{Dp}{Dt})\, dx. \qquad (5.58)$$

Substituting for $p' = p_i$ and v this becomes

$$(\rho - \rho_o)_{UQ} = \frac{A_j\, e^{-i\omega r/c_o}}{4\pi R\, c_o^{\,2}} \frac{M^2\cos^2\theta}{(1 - M\cos\theta)} \cdot i\omega\, p_i\, [- \frac{1}{M\nu} \cdot \frac{1}{M(1 - \nu)}$$

$$+ \frac{(\nu - 1)}{\nu}] \cdot \frac{M\nu}{(1 - M\nu\cos\theta)} \,,$$

$$= - \frac{A_j \, M^2 \cos^2\theta \, (1 - (1 - \nu)^2 M^2) \, i\omega p_i \, e^{-i\omega R/c_o}}{4\pi R \, c_o^{\,2} \, (1 - M\cos\theta)(1 - M\nu\cos\theta) \, M \, (1 - \nu)} \; . \tag{5.59}$$

Adding the various source terms we recover the original formula (4.6) for the sound radiation. At low Mach numbers it has the $(1 - \cos\theta)$ directivity similar to that for a scattered externally incident sound wave. For high Mach numbers the quadrupole terms progressively dominate and are responsible for the appearance of Doppler factors based on the vortex convection velocity.

5.2. The Dowling, Ffowcs Williams, Goldstein Analogy

Dowling et al (1978) consider sources of sound (quadrupoles, surface dipoles and monopoles) immersed in a jet flow and show how the acoustic analogy introduced in §5.1 must be modified to account for both the propagation of sound through the mean flow and for the presence of flow in the acoustic environment of the source. They do this by using a non-causal Green's function, free from troublesome instabilities.

Specifically, they show that for sources in a jet flow the farfield sound level is given by

$$\rho - \rho_o = \beta \int_V \frac{\partial^2 G^+}{\partial y_i \partial y_j} (H(f) \, T_{ij}) \, dV$$

$$- \beta \int \frac{\partial}{\partial y_i} \; (\delta(f) \, \nabla_j f (p_{ij} - \rho u_i \, (u_j - v_j))) G^+ \, \ell_j \, dS$$

$$- \beta \int \delta(f) \, \nabla f \, \frac{DG^+}{D\tau} \, (\rho_o \, v_i + (u_i - v_i)\rho) \, \ell_i \, dS. \tag{5.60}$$

In this equation, G^+ is what Dowling et al call the "reciprocal Green's function", representing an incoming wave (reverse time) solution and β is $((1 - M_r)^2 \rho_j/\rho_o)^{-1}$ where M_r is the Mach number in the radiation direction; but they show that βG^+ is equal to the more usual Green's function for a source in the jet flow with outgoing waves. In the expressions for the source strengths all the velocities and pressures are measured relative to their mean value in the medium in which they are situated.

We consider only the case of incident plane waves in the pipe. Then at the nozzle $p' = 0$, and $u_i' - v_i = u_N = 2p_i/\rho_c$ while the quadrupole sources vanish since they are of second order in fluctuating quantities.

The monopole strength is then given by

$$(\rho - \rho_o) = \int - \frac{DG}{D\tau} (\rho_j u_N + \rho_o (- U_o)) dS \tag{5.61}$$

and the second term vanishes. The axial dipole has strength $p' + \rho_j u_k (u_k - v_k)$. This is given by $\rho U_j u_N'$, so that the dipole source leads to the field

$$(\rho - \rho_o)_D = - \int \frac{\partial G}{\partial y_k} U_k \rho u_N dS. \tag{5.62}$$

Adding the two sources we note that the $U_k \frac{\partial G}{\partial y_k}$ terms cancel leaving

$$(\rho - \rho_o) = \int \frac{\partial G}{\partial \tau} \rho_j u_N' dS. \tag{5.63}$$

Now for these low frequencies, it has been shown by Dowling et al that with no external flow

$$G = \frac{(\rho_o/\rho_j)}{4\pi R\ (1 - M\cos\theta)^2}\quad \delta(t - \tau - {}^{R}/_{c} + \frac{\underline{y}\cdot\hat{\underline{r}}}{c})\ . \qquad (5.64)$$

Substitution of this in the above formula (5.64) leads to the far-field density fluctuation

$$(\rho - \rho_o) = \frac{\rho_o\ A_j\ \partial u_N/\partial t}{4\pi R\ c_o^{\ 2}\ (1 - M\cos\theta)^2}\ . \qquad (5.65)$$

This result is valid for no external flow. When external flow is present, the only change is that the Green's function is multiplied by $(1 + M_o\cos\theta)^{-1}$ and the original result is quickly recovered.

In applying this analogy which explicitly incorporates a mean flow we have removed the quadrupole sources, which are now included implicitly in the Green's function which then accounts for all propagation effects. We have only given this one example for the purpose of illustration. The sound fields for the other cases discussed earlier could equally be derived with equal facility using this analogy. In particular, the light jet result follows easily if the appropriate Green's function is used.

6. THE EFFECTS OF NOZZLE CONTRACTION

In this section we examine the change in reflection coefficient and sound radiation (section 2) when a contracting nozzle is connected to the pipe. Additionally we determine the radiation produced when a slug of fluid of different entropy from the mean flow convects through the duct.

The method of analysis we use is to assume that the nozzle is sufficiently short that the flow through it is quasi-static with no instantaneous storage of mass or energy in the nozzle. We need therefore only consider the conservation of mass flow or energy flux across the nozzle. Our method is then identical to that employed by Cumpsty & Marble (1977) for turbine disks and by Candel & Marble (1977) for variable area ducts. It is also similar to an analysis of the nozzle problem by Ffowcs Williams (1971). That analysis though contains an error (see Mani (1981)). We further assume that at these low frequencies the boundary condition at the end of the nozzle is that the pressure fluctuation p' is zero, (cf. section 2). For higher frequencies the theory could still be used but some sort of impedance condition at the nozzle exit would have to be used.

The equation of continuity of mass flow, applied at the two ends of the nozzle, at stations 1 and 2, say, is $(\rho UA)_1 - (\rho UA)_2 = 0$. Linearising this in the fluctuations in density and velocity gives

$$\frac{\rho_1'}{\rho_1} + \frac{u_1'}{U_1} = \frac{\rho_2'}{\rho_2} + \frac{u_2'}{U_2} \quad , \qquad (6.1)$$

an equation exact for low enough frequencies. At higher frequencies it should be augmented by a term $\int_1^2 \frac{A}{(\rho UA)} \frac{\partial \rho}{\partial t} dx$ representing the instantaneous storage of mass in the nozzle. For a frequency ω, this term is of order $\omega L/c$ smaller than the others, where L is a typical nozzle length, and may be neglected here. (Of course the argument is only valid for fixed values of M and in particular is not expected to be uniformly valid as $M \to 1$.) Since $p_2' = 0$ and entropy is conserved, (6.1) may be rewritten

$$\left(\frac{p'}{\rho c^2} + \frac{u'}{U}\right)_1 = \left(\frac{u'}{U}\right)_2 \quad . \tag{6.2}$$

The other equation we use is the energy equation. This states that across the nozzle the specific stagnation enthalpy is conserved, so that

$$(C_p T' + Uu')_1 = (C_p T' + Uu')_2 \tag{6.3}$$

where T' is the temperature fluctuation. Since entropy S is conserved and $TdS = C_p dT - dp/\rho$, it follows that with $p_2' = 0$ and $s_2' = s_1'$

$$\frac{p_1'}{\rho_1} + Uu_1' + (T_1 - T_2)\, s_1' = U_2 u_2' . \tag{6.4}$$

In this equation, as with the continuity equation, we have neglected a term of relative order $\frac{\omega L}{c}$ representing the unsteady storage of energy in the nozzle.

Assume now that upstream of the nozzle there are incident and reflected waves $p_i\, e^{-i\omega x/(U+c)}$ and $R\, p_i\, e^{i\omega x/(c-U)}$. Downstream of the nozzle there is a convected neutrally stable wave $u_2\, e^{-i\omega x/U_2}$. We substitute these forms into our mass flow and energy conservation equations giving, respectively,

$$\frac{p_i}{\rho_1 c^2}\,[(1 + R)M_1 + (1 - R)] = \frac{u_2}{c_2} \cdot \frac{M_1}{M_2} \quad , \tag{6.5}$$

and

$$\frac{p_i}{\rho_1 c_1{}^2}\,[(1 + R) + M_1\,(1 - R)] = M_2\,\frac{c_2{}^2}{c_1{}^2} \cdot \frac{U_2}{c_2} \quad . \tag{6.6}$$

Solving these two equations (6.5), (6.6) we find that the velocity is

$$u_2 = \frac{2M_2\, c_2}{M_1 c_1} \cdot \frac{(1 + M_1)}{(1 + M_2{}^2\, c_2{}^2/M_1 c_1{}^2)} \cdot \frac{p_i}{\rho_1\, c_1} \quad , \tag{6.7}$$

and the reflection coefficient is

$$R = -\,\frac{(1 + M_1)}{(1 - M_1)}\;\frac{(1 - M_2{}^2 c_2{}^2/M_1 c_1{}^2)}{(1 + M_2{}^2 c_2{}^2/M_1 c_1{}^2)} \quad . \tag{6.8}$$

In these expressions, we can use

$$\frac{c_2{}^2}{c_1{}^2} = \frac{(1 + (\gamma - 1)M_1{}^2/2)}{(1 + (\gamma - 1)\, M_2{}^2/2)} \quad , \tag{6.9}$$

for isentropic flow, γ here denoting the adiabatic exponent while for small Mach numbers, $M_2/M_1 = A_1/A_2$ (the area ratio of the nozzle), so that then

$$R = -\frac{(1 + M_2A_2/A_1)}{(1 - M_2A_2/A_1)}\ \frac{(1 - M_2A_1/A_2)}{(1 + M_2A_1/A_2)}\ . \tag{6.10}$$

It is clear from this expression that the reflection coefficient is zero when $M_2c_2{}^2/M_1c_1{}^2 = 1$, that is, when $M_2 = A_2/A_1$ for low enough M_2. The result is in agreement with the recent experimental results of Bechert (1979). In that paper Bechert presents a theory for this phenomenon which is similar to ours, except that it does not include the effects of compressibility and is, therefore, restricted to low Mach numbers.

A consequence of the above theory is that since both the radiation field and the instability wave amplitude depend only on the velocity u_2 at the nozzle exit, the ratio of their net energy fluxes is unchanged and quite independent of the nozzle contraction. It is, nevertheless, of interest to express the radiated sound in terms of the upstream pressure wave p_i. The radiated sound power is

$$W_R = \frac{\rho_o\ \omega^2 a^2 A_j\ u_N\ (1 + M_R{}^2/_3)}{c_o{}^3\quad (1 - M_R{}^2)^3}\ .$$

Substituting for $u_N = u_2$ this becomes

$$W_R = \frac{(1+M_R{}^2/_3)}{(1-M_R{}^2)^3}\cdot \rho_o(\frac{\omega a}{c_o})^2\ c_o\ (\frac{p_i}{\rho_1 c_1})^2\ \frac{(1+M_1)^2.2M_2c_2A_2}{(1+M_2{}^2c_2{}^2/M_1c_1{}^2)M_1c_1}\ , \tag{6.11}$$

$$= \frac{(1+M_R{}^2/_3)}{(1-M_R{}^2)^3}\cdot (\frac{\omega a}{c_o})^2\ (\frac{\rho_o c_o}{\rho_2 c_2})\ (\frac{c_2}{c_1})\ \frac{W_I}{(1+M_2{}^2c_2{}^2/M_1c_1{}^2)}\ . \tag{6.12}$$

where W_I is the power flux in the incident wave in the pipe. Therefore the ratio of the farfield to the incident power, (W_R/W_I), is increased in the ratio $\dfrac{c_2}{c_1(1 + M_2{}^2c_2{}^2/M_1c_1{}^2)}$ by the contraction. This ratio is less than unity which shows that there is always less power radiated due to the addition of nozzle contraction, even at the condition when the reflection coefficient is zero. In that case, all the incident power is, to first order, transferred to the <u>instability wave</u>.

We consider next the transmission of sound out of a choked nozzle. Instead of assuming as the boundary condition that there is an instability wave downstream with zero pressure, we use a condition of constant non-dimensional mass flow through the choked nozzle. This condition is the same as that introduced by Cumpsty & Marble for a choked turbine. The choked nozzle condition is that

$$(m\sqrt{T_{o1}}/Ap_{o1}) = \text{constant}$$

where m is the mass flow, A the area, and T_{o1} and p_{o1} the stagnation temperature and pressure. In this case the energy equation cannot be used to determine the unsteady flows since the choked flow is not isentropic.

Cumpsty & Marble linearise the constant mass flow condition to obtain an extra equation relating the pressure, temperature and velocity at the entrance to the nozzle. In our case there is no need to do this. We note that, since both the choked and subsonic values of the reflection coefficient must be the same when $M_2 = 1$, we can obtain the choked flow reflection coefficient for arbitrary upstream Mach number M_1 by simply setting $M_2 = 1$ in (6.8). Then with $c_2^2/c_1^2 = (1 + \frac{\gamma-1}{2} M_1^2)/(\frac{\gamma+1}{2})$, the reflection coefficient is

$$R = - \frac{(1 + M_1)((\gamma + 1) M_1 - 2 - (\gamma - 1) M_1^2)}{(1 - M_1)((\gamma + 1) M_1 + 2 + (\gamma - 1) M_1^2)} , \tag{6.13}$$

$$= \frac{(1 - (\gamma - 1)M_1/2)}{(1 + (\gamma - 1)M_1/2)} . \tag{6.14}$$

For subsonic M_1 (this is always the case), this reflection coefficient is always positive and less than unity. This may be compared with the negative value obtained for a non-contracting nozzle. If the full anlysis with the constant mass flow relation is used, the same result is obtained.

Another interesting result that can be obtained from the above theory is the reflection coefficient of a duct inlet. This result is obtained by reversing the sign of the Mach numbers in the formula (setting $M_1 = - M_2$, $M_2 = - M_1$) and putting $A_2/A_1 = 0$, for a "bellmouth" inlet. Then we find that the magnitude of the reflection coefficient is equal to $(1 - M)/(1 + M)$. This is in good agreement with the experimental value of Ingard & Singhal (1975) who obtain a value of $[(1 - M)/(1 + M)]^{1.33}$. Further, it corresponds to total reflection of the sound energy incident on the end of the tube.

Finally, consider an entropy wave incident on the nozzle from upstream. Again we take $p_2' = 0$ at the nozzle exit. We assume that the pipe contains a wave of form $p_1 e^{-i\omega x/(c_1-U_1)}$, and that entropy is conserved across the nozzle. The continuity equation then reads

$$-\frac{p_1(1 - M_1)}{\rho_1 c_1^{\,2}\, M_1} = \frac{u_2}{c_2 M_2} \quad , \tag{6.15}$$

and the energy equation becomes

$$\frac{p_1}{\rho_1}(1 - M_1) + (T_1 - T_2)\, s' = U_2 u_2 \quad . \tag{6.16}$$

Hence

$$(T_1 - T_2)\, s' = u_2(U_2 + M_1 c_1^{\,2}/c_2), \tag{6.17}$$

so that

$$u_2 = \frac{(T_1 - T_2)s'\, c_2 M_2}{\left(1 + \frac{c_2^{\,2} M_2^{\,2}}{c_1^{\,2} M_1}\right) c_1^{\,2} M_1} \quad , \quad p_1 = \frac{(T_1 - T_2)s'\, \rho_1}{(1 - M_1)\left(1 + \frac{c_2^{\,2} M_2^{\,2}}{c_1^{\,2}\, M_1}\right)} \quad . \tag{6.18}$$

We can now determine the sound radiated to the farfield, using

$$\rho - \rho_o = \frac{\rho_o A_2\, i\omega\, u_2}{4\pi R\,(1 - M_R \cos\theta)^2\,(1 + M_o \cos\theta) c_o^{\,2}} = \frac{\rho_o\, A_2\, i\omega\, u_2\, F(\theta)}{4\pi R} \text{ ,say,}$$

to give

$$\rho - \rho_o = \frac{\rho_o\, A\, i\omega\, F(\theta)\, (T_1 - T_2)s'\, c_2\, M_2}{4\pi R\,(1 + c_2^{\,2} M_2^{\,2}/c_1^2 M_1)\, c_1^{\,2} M_1} \quad , \tag{6.19}$$

and the energy reflected up the pipe is

$$\frac{p_1^{\,2}\,(1 - M_1^{\,2})A_1}{\rho_1 c_1} = \frac{A_1(T_1 - T_2)^2\, s'^2\, \rho_1^{\,2}}{\rho_1 c_1\,(1 + c_2^{\,2} M_2^{\,2}/c_1^{\,2} M_1)} \quad . \tag{6.20}$$

In the above formulae $T_1 - T_2$ and s' may easily be related to other flow parameters. For small Mach numbers, $T_1 - T_2 \sim \Delta P/\rho\ C_p$ where Δp is the pressure difference across the nozzle. (More generally, $T_1 - T_2 = (u_2^2 - u_1^2)/2C_p$ and $s' = - C_p\ (\frac{\rho'}{\rho})$ where ρ' is the density fluctuation due to the entropy wave). From this we see that, if this density varies on a time scale $\sim \ell/u$ where ℓ is a typical length scale, then the scaling of the sound field is, for low Mach number, as

$$p^2 \sim \frac{a^2}{R^2} \cdot \Delta\rho_j^{\,2}\ U_j^{\,4}\ (\frac{\Delta\rho}{\rho})^2\ \frac{U_1^{\,2}a^2}{c_1^{\,2}c^2} \quad . \tag{6.21}$$

More generally, in this low frequency limit where additionally terms of order M^2 are neglected, we can show that with $\rho_2 = \rho_o$ the radiated field is

$$p_2' = \frac{(1+2M\cos\theta)}{4\pi R}\ \frac{A_2\rho_o}{c_o}\ \frac{\partial}{\partial t}\ [\frac{\Delta\rho .\ \Delta P\ A_1}{\rho\ P\ A_2(1 + MA_1/A_2)}]. \tag{6.22}$$

For $M \to 0$ this result reduces to

$$p' = \frac{A}{4\pi R} \cdot \frac{\rho_o}{\rho} \cdot \frac{1}{c}\ \frac{\partial}{\partial t}\ (\frac{\rho - \rho_o}{\rho})\ \ \Delta P,$$

which is in precise agreement with the results derived by more sophisticated means by Ffowcs Williams & Howe (1975). Their analysis assumes that a sharp fronted slug of gas of density different from that of the mean flow is convected through the nozzle, and determines the farfield sound by a rather more elegant application of the acoustic analogy.

7. DISCUSSION AND COMPARISON WITH EXPERIMENTAL RESULTS

The purpose of this section is to discuss the overall features of the results obtained in sections 2-6 and to compare them with such experimental results as are available.

There are a number of comparisons with published data that can be made for incident internal noise. Figure 5 compares our low frequency field shapes (2.45) with the exact calculations of Munt (1977) for the same problem, for cold jet conditions ($C = 1$); the two agree beyond 60° to the jet axis. Near the jet axis there is a discrepancy which increases with frequency and Mach number. This might have been expected since our predicted power levels increase very rapidly as M nears unity, and would be expected to exceed the exact values. We note that, in the theory, as the jet nears sonic velocity one of the branch points tends to infinity and then the approximate factorisations which we have used are not uniformly valid as $M \to 1$. That would accord with expectations that the reflection coefficient should actually decrease near $M = 1$, so that at $M = 1$ it changes gradually to its zero value for a supersonic jet. Therefore our solution is expected to be invalid for Mach numbers close to one. In Munt's (1977) paper theory is compared with the experimental results of Pinker & Bryce (1976) for both hot and cold jets. In the latter case the agreement is good, as it is for our theory for low enough Mach numbers. For the hot jet Munt's results are much lower than the experimental points close to the jet axis, and show a dip consistent with refraction of sound by the jet. A possible reason for this disagreement is the incomplete modelling of the jet instability waves. In the model problem these grow exponentially as along the jet and have no conventional acoustic farfield. In reality, however, the growth is limited by the spreading of the mean flow downstream of the nozzle and by non-linear effects. Further, in Munt's theory, the region of the discrepancy is the one where the direct field of the instability is present, and limiting the growth of this instability would probably result in an extra farfield, dependent on the growth and decay rates of the instability wave, but confined essentially to the angular region in which the direct field of the original instability wave was present. In our theory, there is no such farfield outside the jet associated with the instability waves, since this angular sector is vanishingly small for these low frequency waves which grow at negligible rate.

The reflection coefficient we have determined is in agreement with both the limited experimental data of Schlinker (1977) and Munt's computations (1979). However, it would appear to be valid over only a limited frequency range. At non-zero frequency it is found that for non-zero Mach numbers the reflection coefficient initially rises to give a peak at a nearly constant Strouhal number and then decreases as more sound is radiated, in accordance with the established theory without flow (Levine & Schwinger, 1948). We note though that this behaviour does not violate conservation of energy, since $|R|$ is always less than $(1 + M)/(1 - M)$. In a subsequent paper (Cargill, 1981; chapter 3 of this thesis) we carry out the low frequency calculation initiated here to higher order in ka, with results that adequately reproduce the entire behaviour observed experimentally and computed by Munt.

In our theory the effect of external flow on intensity has been shown to be nearly as $(1 + M_o \cos\theta)^{-6}$ near $\theta = 90^o$. This is in excellent agreement with the results of Pinker & Bryce (1976), which covered higher frequencies. The highly directional field-shape we obtain is, further, characteristic of sources immersed in jet flows at low frequency (Goldstein (1975), Mani (1974)).

Of great interest is the comparison between the net power in the pipe and the power radiated to the far field. Figures 6-8 compare our results with Munt's exact theory (1979), Howe's low Mach number theory (1979) and Bechert, Michel & Pfizenmaier's (1977) experiments. For the lowest frequencies all four are in good agreement. As might be expected, our theory diverges from the experiments and Munt's theory for higher frequencies, and agreement is only obtained over reduced frequency ranges as the Mach number is increased, which is consistent with overprediction of the far field sound levels. We have shown, further, that the conversion from acoustic to hydrodynamic energy implicit in these relations is critically dependent on the existence of a Kutta condition at the pipe exit. When the Kutta condition is relaxed, and no jet instability wave is produced, we find that there is no such energy conversion, in agreement with Howe (1979). Further, we find that then all the incident energy is reflected up the duct and the reflection coefficient is $-(1 + M)/(1 - M)$. We have also shown, again in agreement with Howe, that if the instability wave is replaced by some sort of neutral wave convected at a speed νMc_j, then the radiation changes with ν from the Kutta condition value ($\nu = 1$) to the non-Kutta-condition value ($\nu = 0$).

An alternative way of looking at the power transmission ratio is as a function of Mach number. In figure 9 we compare our results with Moore's (1976). We find that at Mach numbers between 0.2 and 0.8 agreement is good despite the relatively high ka value (0.46) of Moore's experiments. At low Mach numbers our result fails because the Strouhal number of his experiment is no longer low, while at high Mach numbers we probably over-estimate the farfield radiation.

A further corollary to this energy loss mechanism concerns the resonances in a tube with flow. We have shown that energy is lost from such a tube, and this loss would result in the elimination of any resonant peaks. This has been demonstrated by Ingard & Singhal (1975). Their results, as reproduced in Figure 10, do indeed show a significant reduction in the relative amplitude of the resonant peaks of the frequency response when a mean flow is present.

When the jet is "hotter than it is compact" we find that a quite different set of phenomena occurs. Then, all the sound escapes from the pipe (the reflection coefficient is zero) and is channelled along the jet, which in this limit behaves as a rigid walled tube. There is no jet instability wave. Further, the pressure in the farfield is reduced relative to its normal ($\frac{\rho_o}{\rho_j} \sim O(1)$) value by a factor $\sim [\frac{\rho_o}{\rho_j} (ka)^2 \ln (ka)]^{-1}$. This factor is by definition large in the light jet condition. We find, though, that for a jet composed of a perfect gas, the condition always fails around the 90^o position in the farfield, where there is a peak in the field shape corresponding to the Mach angle for disturbances transmitted along this very hot jet. These results are entirely consistent with those established by Dowling et al (1978) for jet noise. Interesting though this result is, it appears to have little relevance in an aeronautical context, as the temperatures required to achieve the light jet condition are far too high ($\sim$ 10,000K).

Examination of our results for a supersonic jet shows phenomena similar to those for the subsonic jet. Again there is a conversion from acoustic to hydrodynamic energy. But compared with the subsonic jet, the reflection coefficient is now zero, since sound cannot propagate upstream against the flow, and there is an additional motion of the jet which corresponds to the steady wave structure of an imperfectly expanded supersonic jet. The energy in the pipe splits itself between the instability wave and these quasi-periodic waves. The field shape of the radiated sound is also somewhat changed as compared with the subsonic case.

Our result for the scattering of an externally incident sound wave by the pipe may be compared with the theory of Jacques. He deduces the radiated sound from an application of the acoustic analogy. We show this to be incorrect, firstly because he neglects the sources on the wall of the pipe, and secondly because he neglects the quadrupole sources in the jet. Our results do however agree with his for the "zero order" fields in the pipe and jet column. An interesting feature of the field shape of the radiated sound is the appearance of a zero at the cone of silence angle for waves propagating out of the jet and into the ambient fluid.

We have discussed the sound generated when vortices are convected past the end of the pipe. This sound is shown to scale as $p^2 \sim \rho^2 U^4 M^2 \ell^2 / R^2$ which is in agreement with other theories, for example that of Leppington (1971) who modelled turbulence by point quadrupoles. The sound source due to convection of vortices past the end of the pipe only exists when there is an external flow over the jet, and could be one of the "installation effects" of Bryce (1979) which raise the noise level of an aircraft in flight above the level predicted for pure jet noise. An important feature of our result is that, when a Kutta condition is enforced, no sound is radiated when the vortices are convected at the speed of the mean flow. This is similar to a result obtained by Howe (1976) for the convection of line vortices past a flat plate. In our model it arises because the sound field is essentially driven by the pressure that would exist on the wall of the duct if it were infinite, and in our linear approximation this is proportional to the convection speed of the vortices relative to the mean flow. When no Kutta conditions are enforced, the response of the sound field to this pressure is increased and the dependence on the velocity of slip removed.

We have re-examined Crighton's (1972) theory for the scattering of an instability wave by the pipe. We find a result which agrees with his in the zero Mach number limit, but differs somewhat otherwise, where the field shape is altered due to the internal and external flows. Then the effect of flight is more complicated than the four powers of Doppler factor assumed by Crighton.

In all these problems which we have solved by the Wiener-Hopf method in the low frequency limit, we have implicitly assumed that both Strouhal number (ka/M) and Helmholtz number are small. This limits the usefulness of the solutions. In an aeronautical context, Strouhal numbers of order one are important. Examination of all our formulae shows however, that as the frequency is changed the field-shapes are all changed by the same factor ($\sim 1/K^+(u)$). To obtain the behaviour at these higher frequencies all we have to do, therefore, is use Munt's results for the internal noise at higher frequency and scale the other results appropriately. Subject to the comments we have already made about Munt's results compared with ours, our results for these other mechanisms may be directly read across to higher frequencies.

We have used Lighthill's acoustic analogy to deduce a set of equivalent sources for these sound fields and we find that there are usually four types of source: dipoles and monopoles on the duct exit and side walls, and two types of quadrupole in the jet flow. The quadrupoles involve the unsteady part of the Lighthill stress tensor acting over a fixed volume, and the steady part of the stress tensor acting over the variable volume of the jet, the latter reducing to a surface source on the outer surface of the jet. At higher Mach numbers and for high density ratios, the sound from the steady quadrupole dominates the far field and is responsible for the high convective amplification on the field shape of internal noise radiation. It is also responsible for the sound field being proportional not to the jet density as one might expect, but to the farfield density (for a given velocity fluctuation at the nozzle exit). We have also shown, in consequence, that in problems such as these the instability wave is an essential feature of the unsteady motion of the jet. In the low frequency limit, the instability wave degenerates to a neutral convected vorticity pattern on the jet boundary.

We have also used another analogy due to Dowling et al (1978), which incorporates, explicitly, the effects of fluid shielding by the mean flow. Then the only sources are those dipoles and monopoles on the duct exit alone, while the quadrupole sources are negligible, being now of second order in fluctuating quantities. Thus the field shape and density dependence appear as an artifact of the particular Green's function used and not of the quadrupole sources.

We have produced a simple theory for the effects on these sound radiation problems of the contraction of the nozzle. In the low frequency limit we find that this contraction has no effect on the transfer of power from acoustic to hydrodynamic energy, but does have a large effect on the reflection coefficient. Indeed, as the Mach number increases from zero, the reflection coefficient decreases instead of remaining constant, reaching zero when the Mach number is equal to the area ratio of the nozzle. This behaviour is found in recent experimental results of Bechert (1979), and figure 11 compares our result with his. The position of the minimum in the reflection coefficient is well predicted. For a choked supersonic nozzle we find the reflection coefficient is always positive, and less than unity.

We have also used this theory to study the sound produced where 'hot spots' or entropy waves are convected out of the nozzle. Our results are in excellent agreement, for low Mach number, with those of Ffowcs Williams & Howe (1975) and show the sound field to depend on the temperature drop across the nozzle.

The author wishes to thank his employers, Rolls-Royce Limited, for permission to publish this paper and the Science Research Council for an Industrial Studentship.

He also wishes to acknowledge the extensive and helpful advice given by his supervisor Professor D.G. Crighton.

APPENDIX I

PROPERTIES OF THE WIENER-HOPF KERNEL - SUBSONIC FLOW

The purpose of this appendix is to set out the properties of the kernel K(u) of the Wiener-Hopf equation (2.20);

$$K(u) = \rho_j c_j k^2 \frac{[D_j^2 J_m(kva) kw H_m^{(2)\prime}(kwa) - \gamma D_o^2 H_m^{(2)}(kwa)\, kv\, J_m'(kva)]}{kv.kw\,.\, J_m'(kva)\, H_m^{(2)\prime}(kwa)}\,. \tag{A.1.1}$$

We consider first the axisymmetric case, m = 0. Then as ka → 0 the denominator

$$kv\, kw\, J_m'(kva)\, H_m^{(2)}(kwa) \sim \frac{i}{\pi} k^2 v^2. \tag{A.1.2}$$

This has the factorisation $\frac{ik^2}{\pi} v^+ . v^-$,

where $$v^+ = ((1 - Mu) - u),$$

$$v^- = ((1 - Mu) + u). \tag{A.1.3}$$

The quantity

$$[D_j^2\, J_m\,(kva)\; kw\, H_m^{(2)\prime}(kwa) - \gamma\, D_o^2\, H_m^{(2)}\,(kwa)\; vk\; J_m'(kva)] = Q, \text{ (say)}$$

is, to second order in ka,

$$D_j^2 (1 - \frac{(kva)^2}{4}) \; kw \; [- \frac{2i}{\pi \; kwa} + kwa(\frac{i}{\pi} \ln (\frac{wka}{2}) - \frac{\pi}{2} - \frac{i}{2\pi} - \frac{i\gamma_E}{\pi})$$

$$- \gamma \; D_o^2 \; kv \; (- \; kva) \; (- \frac{2i}{\pi} \ln \frac{(kwa)}{2} + 1 - \frac{2i\gamma E}{\pi})], \qquad (A.1.4)$$

where γ_E is Euler's constants, $\gamma_E = \cdot 57721 \ldots$.

Multiplying out the bracketed term:

$$Q = \frac{2i \; D_j^2}{\pi a} [1 - \tfrac{1}{2} [\frac{D_j^2 (kwa)^2 - \gamma D_o^2 (kva)^2}{D_j^2}] \; [\ln (\frac{kwa}{2}) + \gamma_E - \frac{\pi i}{2} - \tfrac{1}{2}]].$$

(A.1.5)

The zeros of this expression depend on the ranges of the parameters involved. We distinguish between the two cases, that in which γ is $O(1)$ as $ka \to 0$, and the light jet case of Dowling et al (1978) where $\gamma \gg \frac{1}{(ka)^2 \ln ka}$ as $ka \to 0$. For the former case the zeros are near $u = {}^1/_M$, at $u = u_o$, $u_o^* = \frac{1}{M} (1 \pm i\sigma)$ where

$$\sigma = \sqrt{\frac{\gamma}{2}} \cdot D_o \; kva \; [\ln(\frac{wka}{2}) + \gamma_E - \tfrac{1}{2} - \frac{\pi i}{2}] \quad , \qquad (A.1.6)$$

which is to be evaluated with $u = 1/M$.

Then it is clear that Q may be factorised as $Q^+ . Q^-$, where, for $k \in \Delta$ (Fig. 2)

$$Q^+ = \frac{2iM^2}{a\pi} (u - u_o) \; (u - u_o^*), \qquad Q^- = 1, \qquad (A.1.7)$$

with

$$u_o, u_o^* = \frac{1}{M} \pm \frac{i\sqrt{\gamma}\,(1-\alpha)}{\sqrt{2}\,M^2}\,[\ell n\,(\frac{ka}{2M}\,(1 - C^2(1-\alpha)^2M^2)^{\frac{1}{2}}) + \gamma_E - \frac{\pi i}{2} - \frac{1}{2}]\,. \tag{A.1.8}$$

This expression differs from Munt's, because we have included terms of $O(k^2a^2)$ and not just $O((ka)^2\ \ell n\ ka)$ to obtain the correct normalisation for the ℓn ka term.

When γ is sufficiently large it is clear that σ is no longer small and this approximation breaks down. This is the light jet limit. There, the second term in Q dominates. In Q, we have $\gamma = C^2$ for a perfect gas, and therefore

$$D_j^{\,2}(kwa)^2 - \gamma\, D_o^{\,2}(kva)^2 = -\,(kau)^2\ (D_j^{\,2} - \gamma\, D_o^{\,2}), \tag{A.1.9}$$

$$Q = \frac{2i}{\pi a}\,(D_j^{\,2} + \frac{k^2a^2u^2}{2}\ \ (D_j^{\,2} - \gamma\, D_o^{\,2})\ (\ell n\,(\frac{kwa}{2}) + \gamma_E - \frac{i\pi}{2} - \frac{1}{2})$$

$$+\ O(k^3a^3). \tag{A.1.10}$$

Then the zeros of Q are near $u = 0$, at $\pm\, i\varepsilon$, say, where ε satisfies $Q(\pm\, i\varepsilon) = 0$, or

$$1 + k^2a^2\varepsilon^2\gamma\ \ [\ell n\,(\frac{kaC}{2}) + \gamma_E - \frac{\pi i}{2} - \frac{1}{2}] = 0, \tag{A.1.11}$$

so that to a first approximation

$$\varepsilon = \sqrt{2}\ /\ \sqrt{\gamma}\ ka|\ell n\,(\frac{kaC}{2})|^{\frac{1}{2}} \tag{A.1.12}$$

Then the factorisation is

$$Q^+ = \frac{2i}{\pi a \varepsilon^2}(u + i\varepsilon), \quad Q^- = (u - i\varepsilon). \tag{A.1.13}$$

We now consider the case of other azimuthal modes, at low frequency. In the limit of small ka, u finite, the mth azimuthal kernel function $K_m(u)$ may be expanded for small ka to give

$$K_m(u) \sim$$

$$\frac{\rho_j c_j^2 k^2 a^2 [D_j^2 (\frac{kva}{m!})(\frac{kva}{2})^m \frac{i}{2\pi} (\frac{2}{kwa})^{m+1} + \gamma D_o^2 \frac{m}{m!} (\frac{kva}{2})^{m-1} (\frac{kva}{2}) \frac{i}{\pi} (\frac{2}{kwa})^m]}{kva.kwa\ (\frac{im}{2\pi})\ (\frac{2}{kwa})^{m+1}\ (\frac{1}{2} \frac{m}{m!}\ (\frac{kva}{2})^{m-1})}$$

$$= \rho_j c_j^2 k^2 a^2\ (D_j^2 + \gamma\ D_o^2). \tag{A.1.14}$$

The zeros of this factor are both in R^- so that we obtain the result

$$\left.\begin{aligned} K_m^+ &= \rho_j c_j^2 (ka)^2\ (D_j^2 + \gamma D_o^2), \\ K_m^- &= 1. \end{aligned}\right\} \tag{A.1.15}$$

This factor $(D_j^2 + \gamma\ D_o^2)$ will be recognised as the dispersion relation describing the instabilities of a plane two-dimensional vortex sheet in compressible flow. The zeros are where

$$(1 - Mu)^2 = \pm\ i\gamma(1 - M\alpha u)^2, \tag{A.1.16}$$

i.e., at

$$u_o, u_o^* = \frac{1}{M}\frac{(1 \pm i\gamma)}{(1 \pm i\alpha\gamma)} , \qquad \text{(A.1.17)}$$

where the upper and lower signs refer to the stable and unstable modes of the jet. The factor $(D_j^2 + \gamma D_o^2)$ may then be written as

$$D_j^2 + \gamma D_o^2 = M^2 (1 + \alpha^2\gamma^2) (u - u_o) (u - u_o^*). \qquad \text{(A.1.18)}$$

APPENDIX II

PROPERTIES OF THE WIENER-HOPF KERNEL - SUPERSONIC FLOW

This appendix examines the properties of the Wiener-Hopf kernel K(u) for supersonic conditions. As before, K(u) is given by

$$K(u) = \frac{\rho_j c_j^2 k^2 [D_j^2 J_m(kva)\, kw\, H_m^{(2)\prime}(kwa) - \gamma\, D_o^2\, H_m^{(2)}(kwa) kv\, J_m'(kva)]}{kv.kw.J_m'(kva)\, H_m^{(2)\prime}(kwa)} \tag{A.2.1}$$

For convenience we consider only the m = 0 mode and ignore the light jet condition.

In the subsonic case, the only poles of the numerator of (A.2.1), that were important at low frequencies were those representing instability waves. The other poles near the zeros of $J_m(kva)$ represented waves in the jet decaying as $\exp[-j_{mn}/a\sqrt{(1-M^2)}$ where $J_m(j_{mn}) = 0$, and were unimportant. For supersonic jet speeds, these poles produce non-decaying waves, which are the analogue of the wave structure of an imperfectly expanded jet in steady flow. We divide the range of u into two regimes for the factorisation of K(u). First, where $u \ll 1/ka$ these poles are of no consequence, and we can again approximate K(u) as

$$K(u) = \frac{-2\rho_j c_j^2\, k^2\, D_j^2}{(kv)^2\, a} \quad . \tag{A.2.2}$$

For supersonic flow this is a plus function. This is because K(u) depends only on the jet, not ambient, conditions, and because no waves can propagate against the flow. Therefore we can take $K^-(u) = 1$ and $K^+(u) = K(u)$.

For values of $u >> 1/ka$ the poles of the numerator of (A.2.1) become significant. We only deal with the case of no external flow, where $\alpha = 0$. Then, we can approximate $K(u)$ as

$$K(u) = \left[\frac{2\rho_j c_j^2 k^2 D_j^2 \; J_m(kva)}{kv.J_m'(kva)}\right] \quad . \tag{A.2.3}$$

Again, we can take $K^-(u) = 1$ and $K^+(u) = K(u)$.

If there is an external flow present, we have to consider the full numerator, and $K^-(u)$ is no longer unity.

APPENDIX III

THE PRESSURE FIELD OF CONVECTED VORTICES

(a) Internal Vortices

In this appendix we determine the pressure fluctuations induced on the wall of a pipe by convected vorticity. Specifically, we consider a distribution of convected ring vortices $\underline{\omega}(r,x - U_c t)$ in which the vortices move at speed U_c. We first solve for the perturbation in stagnation enthalpy due to these vortices. This quantity is used rather than pressure since the latter vanishes when the vortices are convected with the flow.

The equation governing the perturbations in stagnation enthalpy due to vorticity is (Howe (1975)),

$$\left(\frac{D}{Dt}\left(\frac{1}{c^2}\frac{D}{Dt}\right) + \frac{1}{c^2}\frac{D\underline{v}}{Dt}\cdot\nabla - \nabla^2\right)B = \nabla\cdot(\underline{\omega}\wedge\underline{v}) - \frac{1}{c^2}\frac{D\underline{v}}{Dt}\cdot(\underline{\omega}\wedge\underline{v}), \qquad (A.3.1)$$

for isentropic flow. For the present case, with a uniform mean flow and convected vorticity, this equation becomes

$$\left(\frac{1}{c^2}\frac{D}{Dt^2} - \nabla^2\right)B = \frac{1}{r}\frac{\partial}{\partial r}\left(r\ \ \omega_\theta(r,x - U_c t)\ U_c\right)\ . \qquad (A.3.2)$$

We transform this equation into co-ordinates moving with the vortices. Then, with $x' = x - U_c t$, we have

$$\left(\frac{1}{r}\frac{\partial}{\partial r}\frac{r\partial}{\partial r} + \beta^2\frac{\partial^2}{\partial x^2}\right)B = -\frac{U_c}{r}\frac{\partial}{\partial r}\left(r\ \omega_\theta(r,x')\right), \qquad (A.3.3)$$

where $\beta^2 = (1 - \frac{(U_j - U_c)^2}{c_j^2})$ accounts for the effect of compressibility. The Green's function for this problem, satisfying

$$\nabla_r^2 G + \beta^2 G_{x'x'} = \delta(x' - x_o)\,\delta(r - r_o)/r_o, \qquad (A.3.4)$$

is

$$G = \sum_{m=1}^{\infty} \frac{J_o(\mu_m r_o)\, J_o(\mu_m r)\, e^{-\mu_m|x-x_o|/\beta}}{2\pi a^2\, \mu_m \beta\, J_o^2(\mu_m a)}, \qquad (A.3.5)$$

where

$$J_o'(\mu_m a) = 0.$$

Applying (A.3.5) to (A.3.3), we obtain the result

$$B = \int_o^a \sum_1^{\infty} - \frac{U_c}{2\pi a^2 \mu_m \beta} \cdot \frac{J_o(\mu_m r)}{J_o(\mu_m a)} \cdot \frac{J_o(\mu_m r_o)}{J_o(\mu_m a)}\, e^{-\mu_m|x-x_o|/\beta}\, \frac{\partial}{\partial r_o}(r_o \omega(r_o,x))dr_o. \qquad (A.3.6)$$

If we assume that the vorticity vanishes at the walls of the duct, $\omega(a,x) = 0$, then

$$B = \sum_1^{\infty} \int \frac{U_c}{2\pi a^2 \beta} \cdot \frac{J_o'(\mu_m r)}{J_o(\mu_m a)} \cdot \frac{J_o(\mu_m r_o)}{J_o(\mu_m a)}\, e^{-\mu_m|x-x_o|\beta}\, \omega(r_o,x) r_o dr_o\, dx. \qquad (A.3.7)$$

To use this result we need some specific form for $\omega(x,r_o)$, and we consider two cases - convected ring vortices and a convected distribution of vorticity.

i) <u>Convected ring vortices</u>

Here we take

$$\omega = \Gamma\delta(r - r_o)\ \delta(x') \qquad (A.3.8)$$

representing a vortex with circulation Γ and radius r_o. For convected turbulence, the appropriate magnitude of Γ is $u'\ell$, where u' is a turbulence velocity fluctuation and ℓ is a length scale. From (A.3.7) the fluctuating stagnation enthalpy is

$$B = \sum_1^\infty \frac{U_c\ \Gamma r_o\ J_o'(\mu_m r_o)\ J_o(\mu_m r)\ e^{-\mu_m|x-U_c t|/\rho}}{2\pi a^2\beta[J_o(\mu_m a)]^2} . \qquad (A.3.9)$$

In this expression we have reverted to the original co-ordinate system. To convert to the perturbation in pressure, p', we note that here

$$\frac{1}{\rho}\ \frac{\partial p}{\partial t} = \frac{DB}{Dt} ,$$

so that

$$p = \sum_1^\infty \frac{\rho_j\ \Gamma r_o\ (U_c - U_j)\ J_o'(\mu_m r_o)\ J_o(\mu_m r)\ e^{-\mu_m|x-U_c t|/\beta}}{2\pi a^2\beta\ [J_o(\mu_m a)]^2} . \qquad (A.3.10)$$

We observe that the pressure fluctuations are dependent on the velocity of slip of the vortex relative to the mean flow, in this linear approximation. If non-linear terms had been included there would be an additional component of order $p \sim \rho_j u'^2$ which would not vanish for $U_c = U_j$.

In the problems we solve in this paper we need the time harmonic components of this pressure. Defining the Fourier time transform of p(t) by

$$p(\omega) = \int_{-\infty}^{+\infty} p(t)\, e^{-i\omega t}\, dt \tag{A.3.11}$$

we have

$$p(\omega) =$$

$$\sum_{1}^{\infty} \frac{\rho_j\, \Gamma r_o (U_c - U)\, J_o'(\mu_m r_o) J_o(\mu_m r) i\beta \mu_m r_o\, e^{-i\omega x/U_c}}{2\pi a^2\, [J_o(\mu_m a)]^2 \qquad U_c\left(\frac{a^2\omega^2\beta^2}{U_c^{\,2}} + \mu_m^2 a^2\right)} . \tag{A.3.12}$$

In practice we require the pressure fluctuations on the duct wall, which are

$$p(\omega, a) =$$

$$-\frac{i}{2\pi} \sum_{M=1}^{\infty} \rho_j \Gamma \left[\frac{J_o'(\mu_m r_o)}{J_o(\mu_m a)}\right] \left[\frac{U_j - U_c}{U_c}\right] \frac{\mu_m r_o\, e^{-i\omega x/U_c}}{\left(\frac{\omega^2 a^2 \beta^2}{U_c^{\,2}} + \mu_m^2 a^2\right)} . \tag{A.3.13}$$

An interesting facet of this expression is that there is no pressure fluctuation due to the zero order radial mode of the vorticity distribution.

ii) Convected vorticity distribution

We next consider a distribution of ring vortices $\omega(r, x - U_c t)$; from equation (A.3.16) we find that

$$p(t) = \sum_{1}^{\infty} \int \frac{(U_c - U_j)\ \rho_j\ J_o'(\mu_m r_o) J_o(\mu_m r)\ r_o \omega(x_o', r_o)\ e^{-\mu_m |x - x_o'|}}{2\pi a^2 \beta\ [J_o\ (\mu_m r_o)]^2}\ dr_o\ dx_o \qquad \text{(A.3.14)}$$

We now Fourier analyse this using equation (A.3.11), splitting ω into its wave number components. Setting

$$\omega(x, r_o) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \omega(k, r_o)\ e^{-ikx_o}\ dk \qquad \text{(A.3.15)}$$

we have

$$p(\omega) = \int_{-\infty}^{+\infty} \sum_{1}^{\infty} \frac{(U_c - U_j)}{(2\pi)^2 a^2 \beta} \cdot \rho_j \frac{J_o'(\mu_m r_o) J_o(\mu_m r)\ r_o \omega(k, r_o) e^{-ik(x - U_c t) - i\omega t}\ (-2i\mu_m/\beta)\ dk\ dt\ dr_o}{[J_o(\mu_m a)]^2\ \ (k^2 + \mu_m^2/\beta^2)} . \qquad \text{(A.3.16)}$$

Integrating with respect to t and R, and noting that

$$\int F(k)\ e^{-ikx - i(\omega - U_c k)t}\ dt\ dk = \frac{2\pi}{U_c} F\left(\frac{\omega}{U_c}\right) \qquad \text{(A.3.17)}$$

we find that

$$p(\omega) = \sum_{1}^{\infty} \frac{i(U_j - U_c)}{2\pi U_c} \cdot \rho_j \frac{J_o'(\mu_m r_o) J_o(\mu_m r)}{[J_o(\mu_m a)]^2}\ \omega\left(\frac{\omega}{U_c}, r_o\right) \frac{e^{-i\omega x/U_c}\ \ i\mu_m r_o}{\left(\frac{\omega^2 a^2 \beta^2}{U_c^{\,2}} + \mu_m^2 a^2\right)} . \qquad \text{(A.3.18)}$$

This is then the pressure fluctuation due to the vorticity $\omega(r,x)$ convected along the duct. It again vanishes when the pattern is convected with the flow.

iii) Convection of vorticity along the outside of the duct

We consider the same situation as above, except that vorticity is convected along the outside of the circular pipe. The Green's function satisfied by (A.3.4), with $\frac{\partial G}{\partial r} = 0$ on the outside of the tube is first derived. We define the Fourier transform of the Green's function as

$$\hat{G}(k) = \int G(x)\, e^{ikx}\, dx, \qquad (A.3.19)$$

so that $\hat{B}$, the Fourier transform of B, must satisfy

$$\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial \hat{B}}{\partial r} - \beta^2 k^2 \hat{B} = -\frac{U_c}{r_o}\frac{\partial}{\partial r_o}\, \omega(r_o,k). \qquad (A.3.20)$$

The Green's function, with $r < r_o$, can then be written

$$\hat{G} = \frac{i}{4} H_o^{(2)}\,(i\beta k r_o)\, J_o\,(i\beta k r) + A\, H_o^{(2)}\,(i\beta k r), \qquad (A.3.21)$$

where A is a constant to be determined from the boundary condition. Setting $\frac{\partial \hat{G}}{\partial r} = 0$ on $r = a$, gives

$$A = -\frac{i}{4} H_o^{(2)}\,(i\beta k r_o)\, J_o'\,(i\beta k r_o)/H_o^{(2)'}\,(i\beta k a). \qquad (A.3.22)$$

Hence for $r < r_o$

$$\hat{G} = \frac{i}{4} \frac{H_o^{(2)}(i\beta k r_o)}{H_o^{(2)\prime}(i\beta ka)} [H_o^{(2)\prime}(i\beta ka)\, J_o(i\beta ka) - H_o^{(2)}(i\beta ka)\, J_o'(i\beta ka)], \tag{A.3.23}$$

or equivalently,

$$\hat{G} = \frac{1}{2\pi} \frac{K_o(\beta k r_o)}{K_o'(\beta k r_o)} [K_o'(\beta ka)\, I_o(\beta kr) - K_o(\beta kr)\, I_o'(\beta ka)] \,. \tag{A.3.24}$$

For the duct wall r = a, we find, using the Wronskian for Bessel functions,

$$\hat{G} = \frac{1}{2\pi} \frac{K_o(\beta k r_o)}{\beta ka\, K_o'(\beta k r_o)} \,, \tag{A.3.25}$$

and hence the pressure on the duct wall is given by

$$p = \frac{\rho_j}{(2\pi)^2} \int e^{-ik(x-U_c t)+ikx_o} \frac{K_o(\beta kr)}{\beta ka\, K_o(\beta ka)} \cdot (U_o - U_c) \frac{\partial}{\partial r_o} (r_o \omega(x, r_o)) \, dx_o \, dr_o \, dk. \tag{A.3.26}$$

If we again assume that $\omega(x,r_o)$ is zero on r = a, (A.3.31) gives

$$p = -\frac{\rho_j}{2\pi} \int \frac{r_o}{a} \frac{K_o'(\beta\omega r_o/U_c)}{K_o'(\beta\omega a/U_c)} \cdot \frac{(U_o - U_c)}{U_c} e^{-i\omega x/U_c} \times \int \omega(x', r_o)\, e^{i\omega x'/U_c} \, dx' d\, r_o. \tag{A.3.27}$$

We consider two cases as before:

(a) Convected ring vortices

Here

$$\omega = \Gamma \quad \delta(x) \frac{\delta(r - r_o)}{r_o}$$

gives

$$p(\omega,a) = \frac{-\rho_j \; r_o \; K_o'(\beta\omega r_o/U_c)(U_o - U_c)\Gamma \quad e^{i\omega x/U_c}}{2\pi \quad a \quad K_o'(\beta\omega a/U_c) \quad U_c} \quad . \qquad (A.3.28)$$

For low frequencies, where our acoustic theory is valid,

$$p(\omega) = \frac{\rho\Gamma}{2\pi} \quad \frac{(U_o - U_c)}{U_c} \quad . \qquad (A.3.29)$$

This is a much simpler result than for the interior of the circular pipe. It is caused by the non-appearance of duct modes for an infinite medium.

(b) Distributed Vorticity

Here we take the vorticity as $\omega(r_o,x_o)$. We then find that

$$p(\omega,a) = - \frac{\rho_j}{2\pi} \int_{-\infty}^{+\infty} \frac{r_o \; K_o'(\beta\omega r_o/U_c) \; (U_c - U_o)\omega(\omega/U_c,r_o)e^{i\omega x/U_c} dr_o}{a \quad K_o'(\beta\omega a/U_c) \quad U_c} \; .$$

(A.3.30)

Again, the pressure fluctuations take on a simple form for $\omega\beta r_o/U_c \ll 1$,

$$p(\omega,a) = \frac{\rho_j}{2\pi} \; (\frac{U_c - U_o}{U_c}) \int_a^{\infty} \omega(\frac{\omega}{U_c} \; , \; r_o) \; e^{-i\omega x/U_c} \; dr_o. \qquad (A.3.31)$$

All the results in this appendix result in pressure fluctuations of the form

$$p(\omega) \sim \left(\frac{U_c - U_o}{U_c}\right) Q(\omega) \exp(- i\omega x/U_c) \quad ,$$

where Q is a constant depending on the strength of the vorticity in the pipe. The pressure fluctuations accordingly scale on the velocity of the vortices relative to the mean flow.

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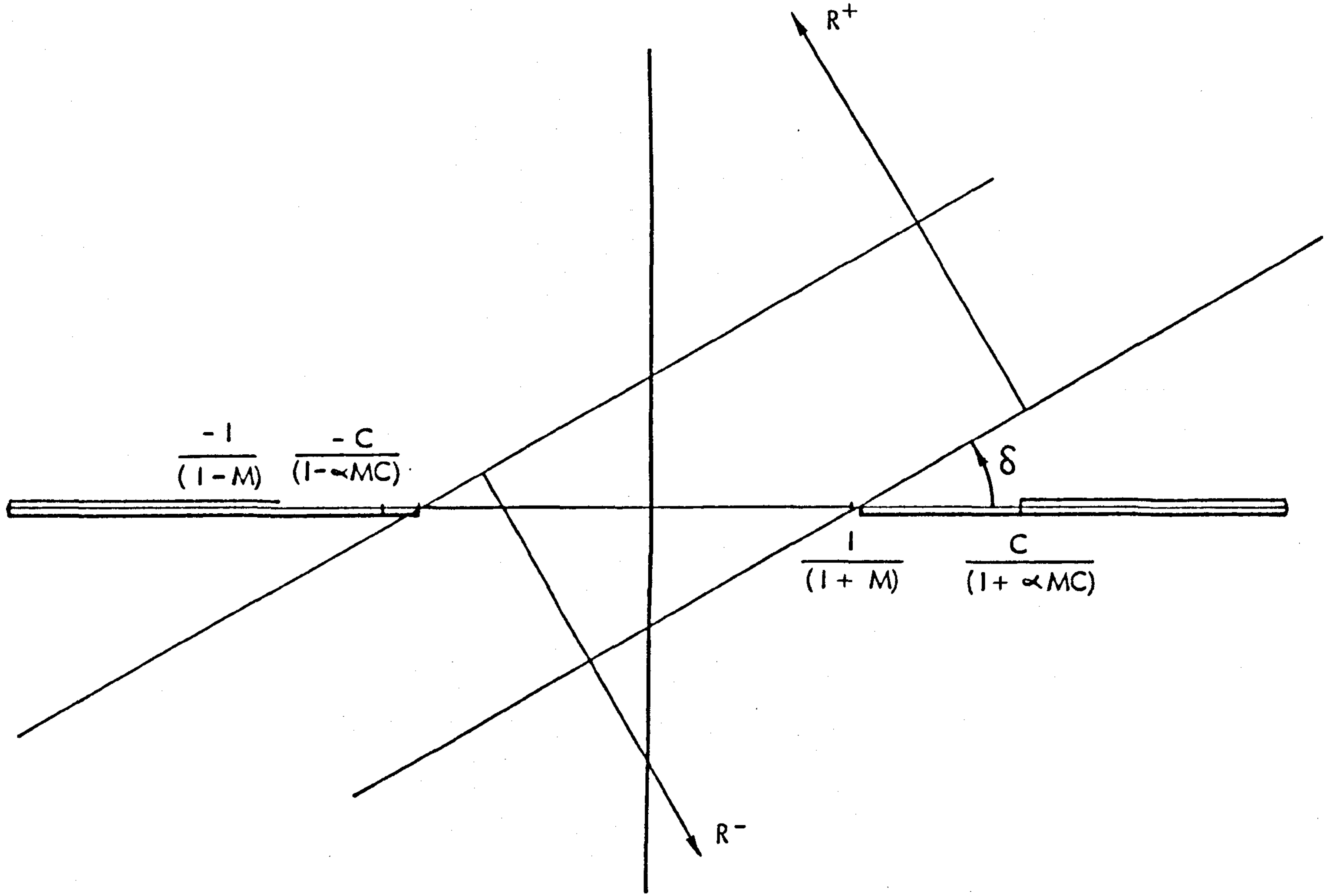


Fig. 1. Position of branch cuts and regions of regularity $R^{\pm}$ in the complex u plane for subsonic flow.

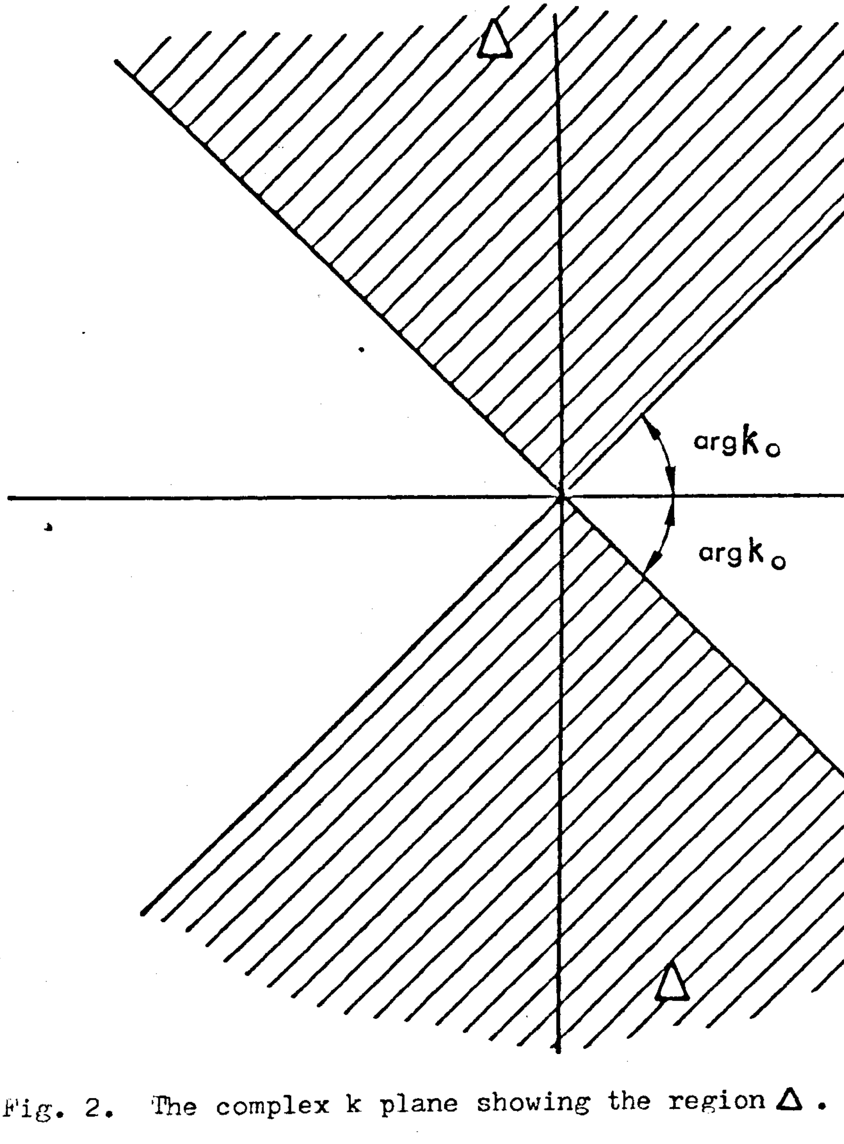
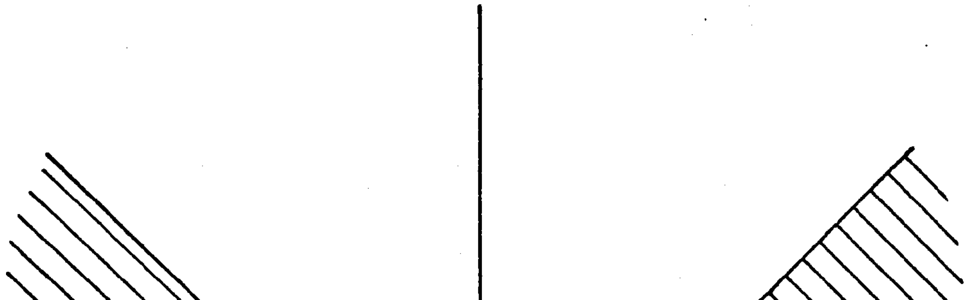


Fig. 2. The complex k plane showing the region Δ .



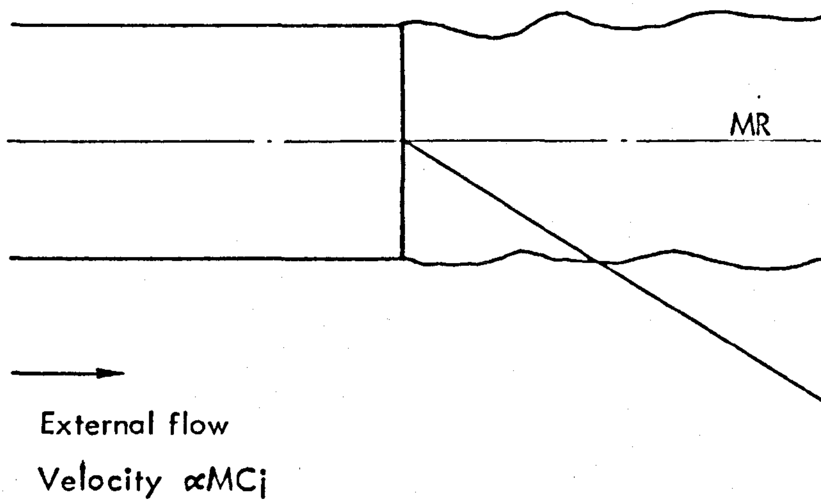
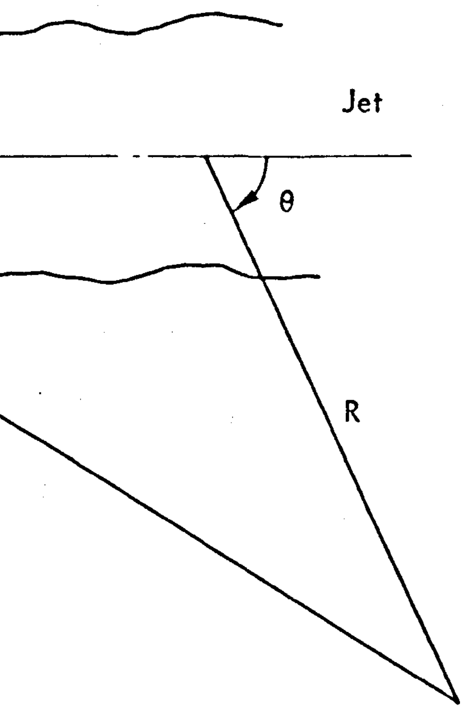


Fig. 3. The emission time co-ordinates R,θ.


Jet
θ
R

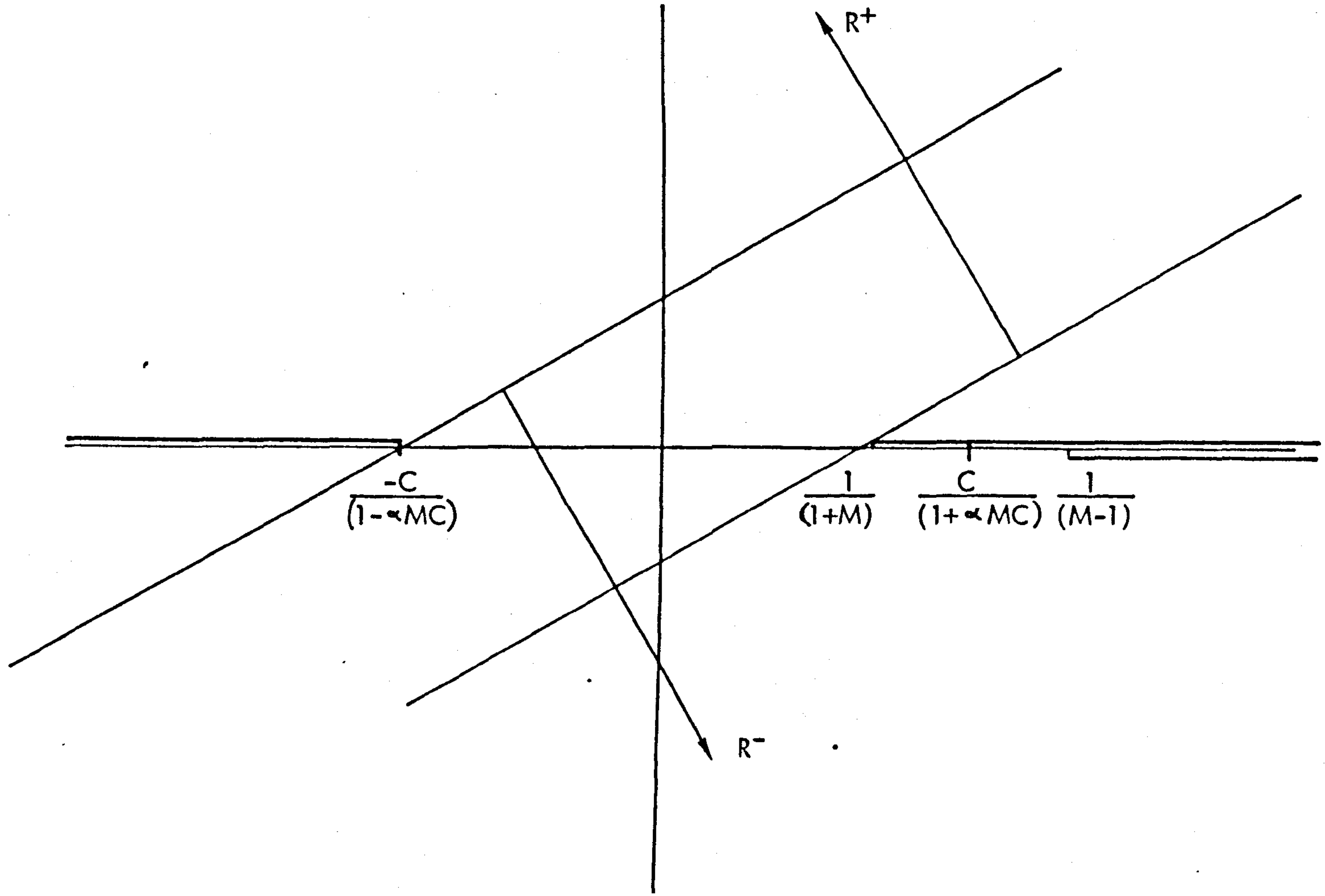


Fig. 4. Position of branch cuts and regions of regularity $R^{\pm}$ in the complex u plane for supersonic flow.

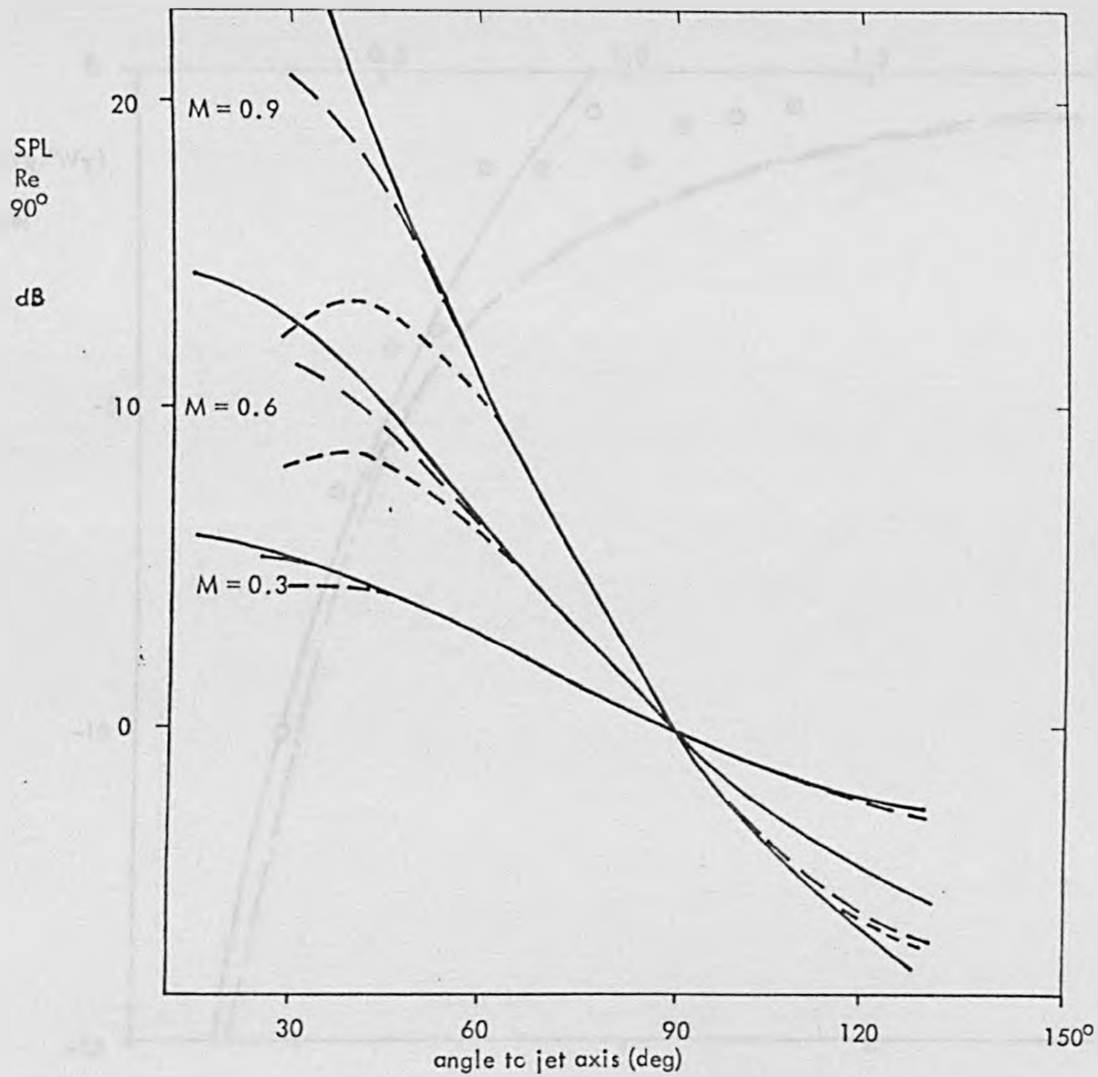


Fig. 5. Comparison of low frequency fieldshapes with Munt's calculations. X = 0, C = 1.0, ——— ka → 0, — — ka = 0.24 (Munt 1977), - - - - ka = 0.6 (Munt, 1977).

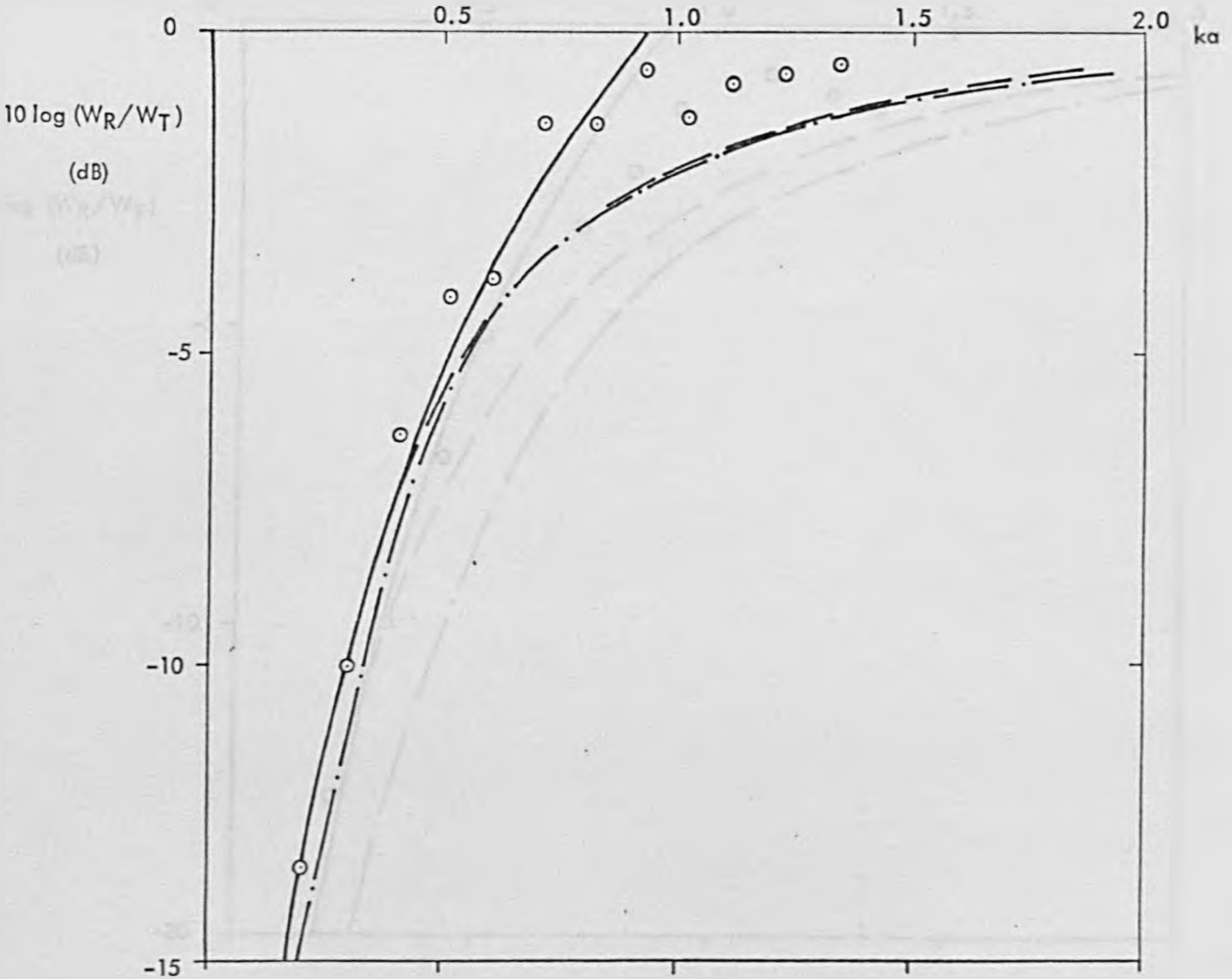


Fig. 6. Comparison of the ratio (radiated power/net power induct) with measurements and Munt's theory, $M = 0.3$, $X = 0$, $C = 1.0$ — — exact theory (Munt, 1981b), ——— low frequency theory —— · —— approximate theory (Howe, 1978).

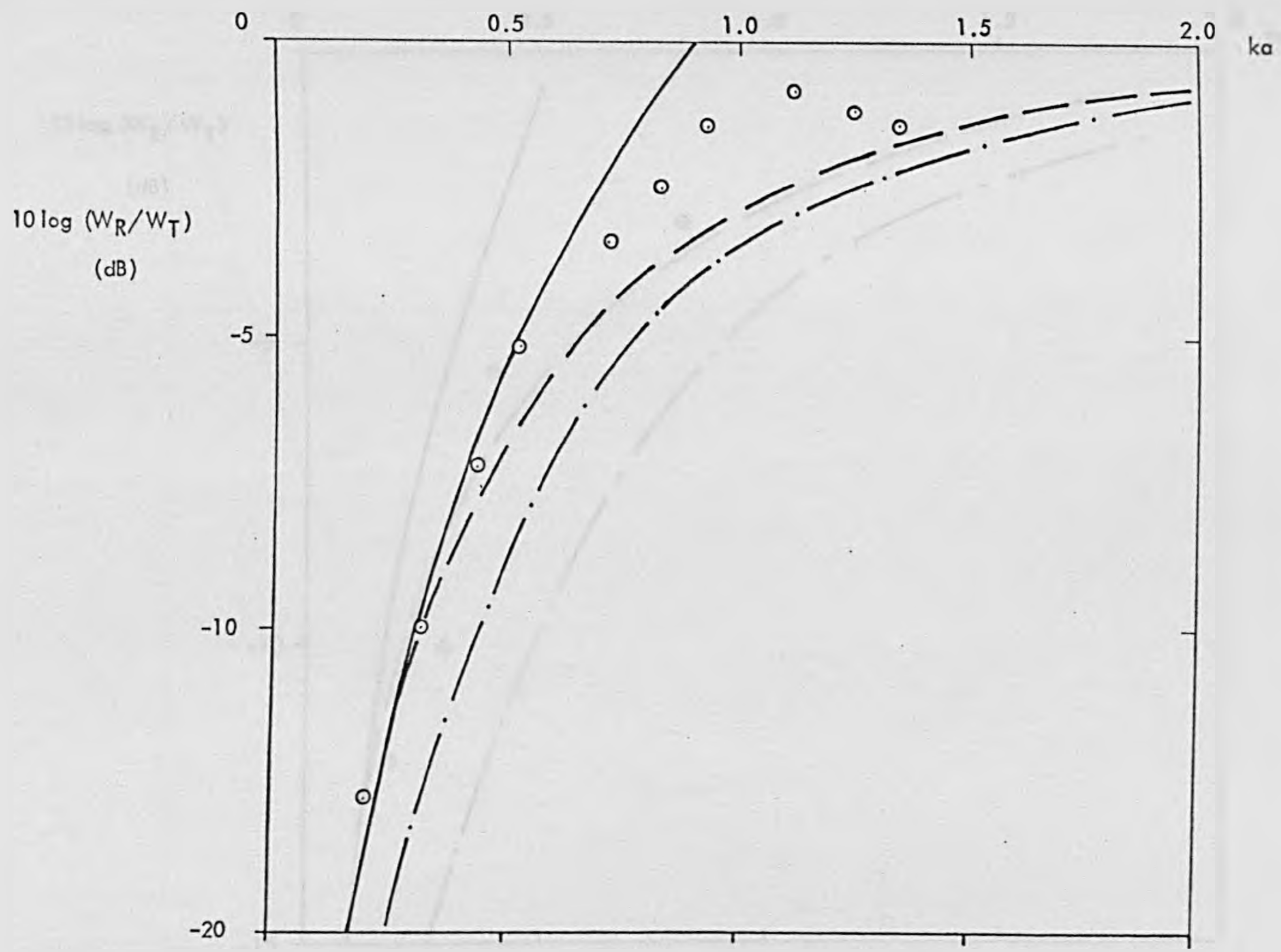


Fig. 7. Comparison of the ratio (radiated power/net power induct) with measurements and Munt's theory, M = 0.5, X = 0, C = 1.0 — — exact theory (Munt, 1981b), —— low frequency theory, —·—·- approximate theory (Howe, 1978).

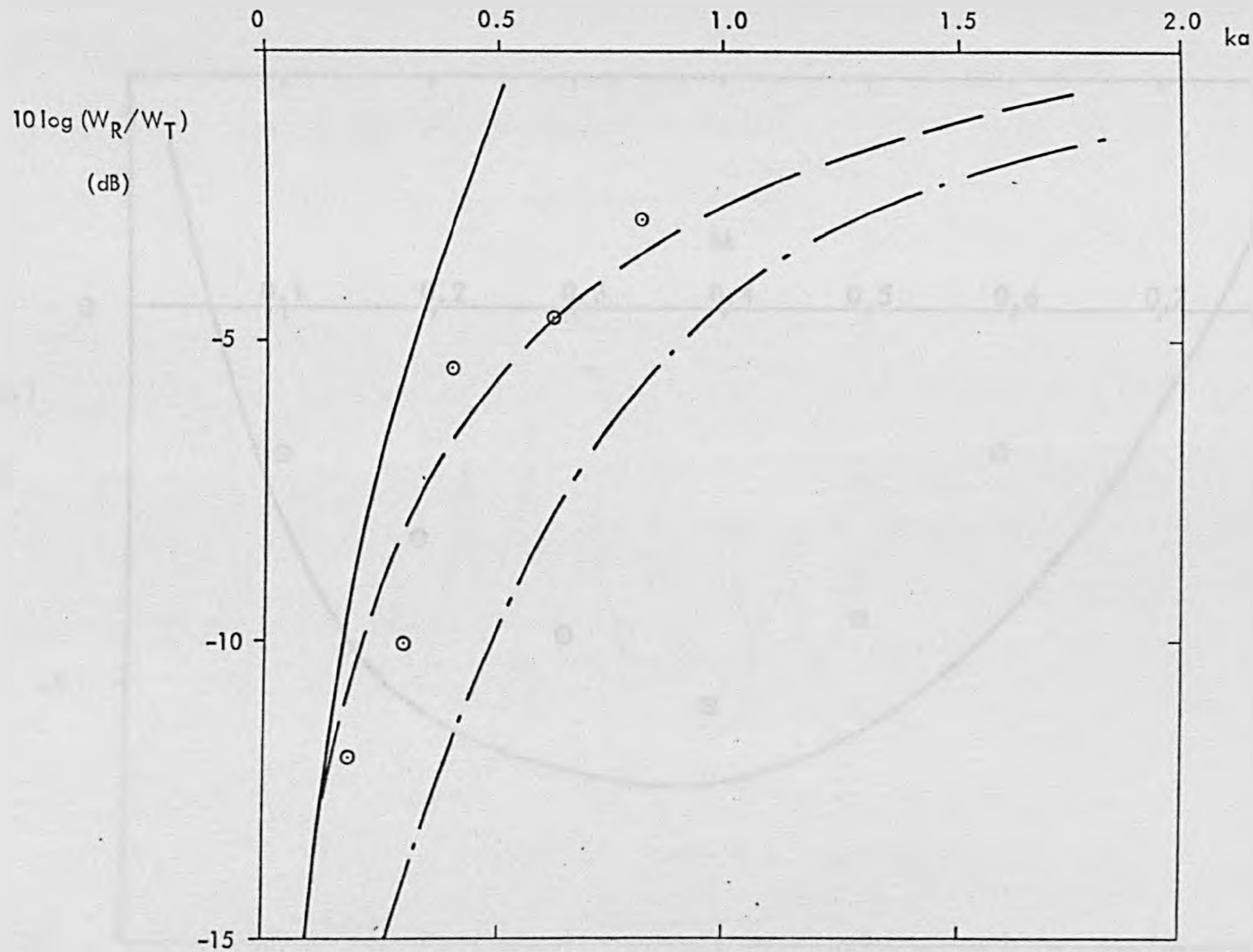


Fig. 8. Comparison of the ratio (radiated power/net power induct) with measurements and Munt's theory, M = 0.7, X = 0, C = 1.0 — — exact theory (Munt, 1981b), —— low frequency theory, —·— approximate theory (Howe, 1978).

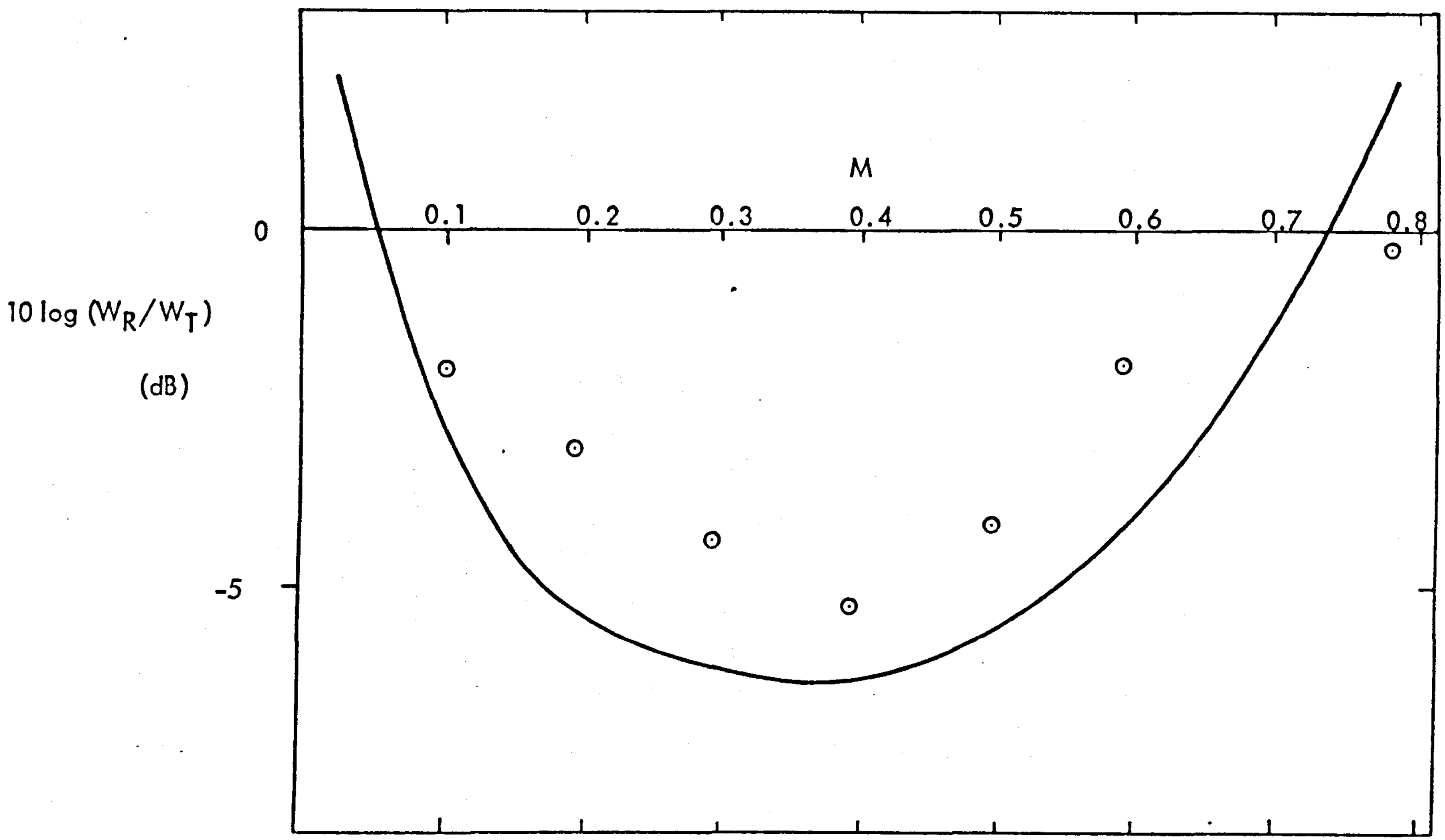


Fig. 9. Comparison of the ratio (radiated power/net power induct) with measurements by Moore (1977): $\alpha = 0$, C = 1.0, ka = 0.46: ——— low frequency theory, ⊙ measurements.

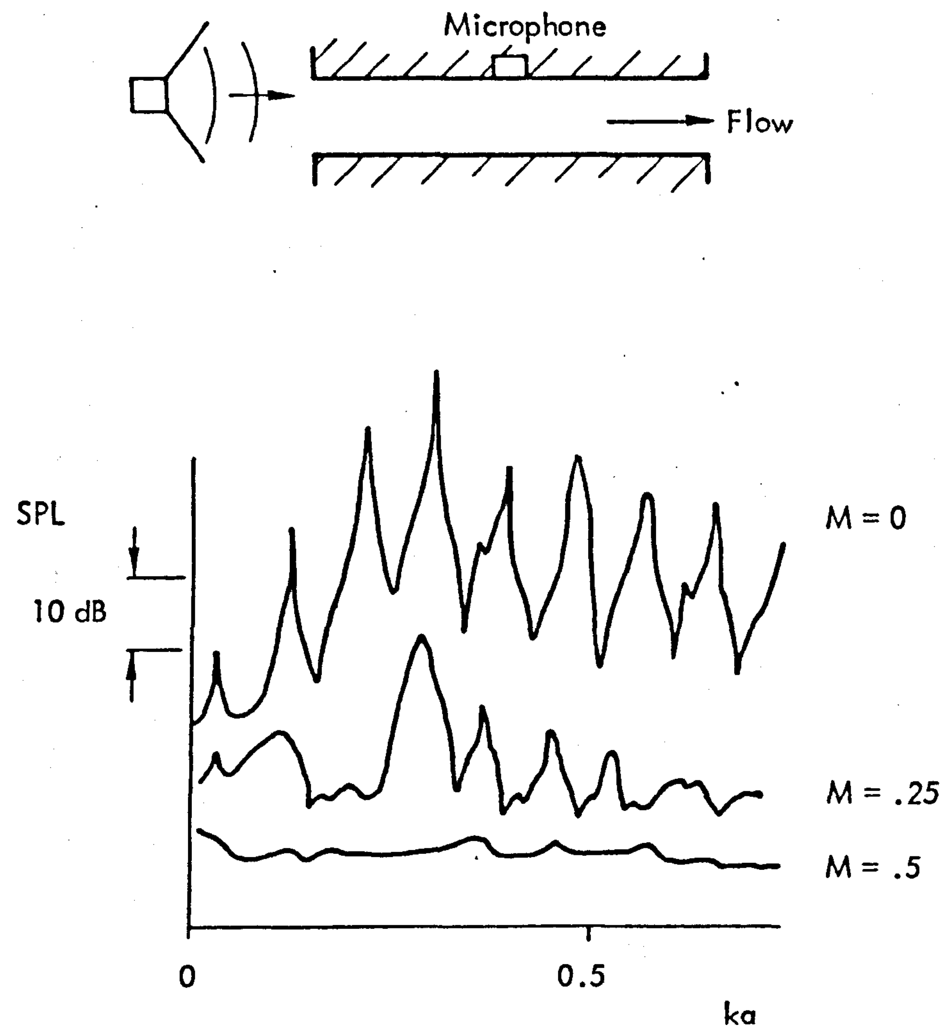


Fig. 10. Effect of flow on duct resonances (after Ingard & Singhal, 1975).

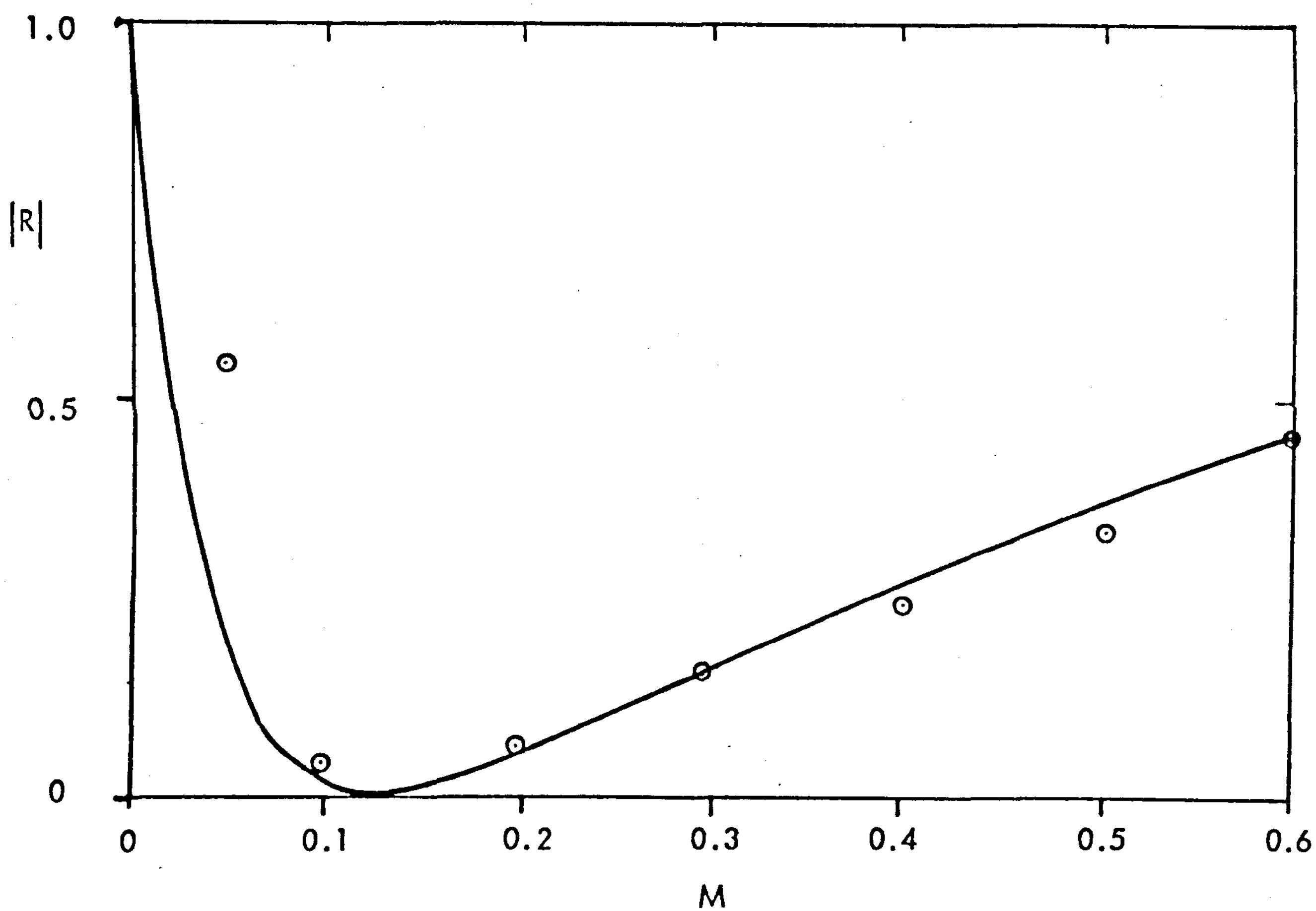


Fig. 11. Sound power reflection coefficient - comparison with Bechert (1979)
——— low frequency theory, ⊙ Bechert (1979).

CHAPTER 3

LOW FREQUENCY ACOUSTIC RADIATION FROM A JET PIPE — A SECOND ORDER THEORY

SUMMARY

In several recent papers, Munt has solved the problem of the radiation of sound out of a jet pipe by the Wiener-Hopf technique. This paper extends his work, giving explicit formulae for both the farfield radiation, and the sound reflected back up the pipe for an incident plane wave. These formulae, which are valid to second order in the ratio of duct diameter to wavelength, are shown to be in excellent agreement with Munt's exact numerical computations.

1. INTRODUCTION

An important problem in the study of aero-engine noise is the propagation of internally generated sound out of the jet pipe to the farfield. This paper presents theory relating to that problem, for the case of low frequency plane wave internal noise.

The problem of sound propagation out of pipes has a long history, its modern era beginning with Levine and Schwinger[1]. Using the Wiener-Hopf technique[2], they were able to obtain exact formulae for the radiated and reflected sound waves in the pipe, in the absence of any mean flow. At low frequencies, they found that the magnitude of the reflection coefficient had the form $(1 - \alpha(a/\lambda)^2)$ where a is the duct radius, λ the wavelength and α is a constant. The radiation field was found to be omnidirectional for these low frequencies, the pipe behaving as a simple source. At higher frequencies the farfield sound became beamed along the pipe axis. The first attempt at incorporating a mean flow into the analysis was that of Carrier[3]; he showed that provided the flow was the same everywhere, both inside and outside the pipe, then a solution could be obtained by a relatively simple modification of Levine and Schwinger's analysis. Candel[4] has given a similar solution for a two-dimensional duct.

The first attempts at including different mean flows inside and outside the duct were made by Mani[5], for a two-dimensional duct, and by Savkar[6], for a cylindrical duct. Unfortunately, their solutions were only approximate and did not consider the all-important instabilities of the jet shear layer, because they used an approximate but inappropriate factorisation of the Wiener-Hopf kernels. Recognition of the importance of these instabilities, is crucial to a correct solution of the problem, since it has a bearing on the condition assumed where the flow leaves the pipe, which in turn affects the farfield. This was originally demonstrated by Morgan[7] and Crighton and Leppington[8], who showed how the correct allowance for the vortex sheet motion in the related problem of a two-dimensional splitter plate offered one a choice of whether or not a Kutta condition was satisfied.

The first complete solution to the problem under discussion here was given by Munt[9]. He again used the Wiener-Hopf technique and obtained a solution that both obeyed a Kutta condition at the end of the pipe and satisfied the requirement that the radiation be causally related to the incident field in the pipe. Despite the considerable simplifications in the theoretical model to make the analysis tractable, very good agreement was obtained with experimental measurements of the farfield directivity pattern by Pinker and Bryce[10]. Extensions to this work were given by Howe[11] and Cargill[12,13]. Howe solved the problem in the low Mach number limit and was able to give analytic results for the reflected field inside the pipe. In that low Mach number limit, he showed that the end-correction of the pipe was equal to its value in the absence of the mean flow plus an additional part due to the flow. He also discussed the behaviour of the reflection coefficient, showing it to decrease from unity at low frequencies. A feature of this and other analyses in which a Kutta condition is assumed to hold at the pipe exit, is that most of the acoustic energy in the incident wave is converted to hydrodynamic energy associated with the instabilities of the jet column. This loss of energy was first discovered by Bechert et al [14] and by Moore [15] and is correctly predicted by the theories. The two papers by Cargill of which [12] is a summary and [13] the detailed analysis, considered Munt's theory in a low frequency limit which does not require the Mach number to be small and in that important respect differ from the theory of Howe. With these assumptions, Cargill was able to give specific formulae for the low frequency radiated field shapes, and showed the low frequency limit of the reflection coefficient magnitude to be unity. An additional feature of his analysis was the inclusion of alternative solutions in which a trailing edge Kutta condition either did or did not hold. He found that in the absence of the Kutta condition and in the absence of shed vorticity, the reflection coefficient was increased to a value of $(1+M)/(1-M)$. In a similar vein, is work by Rienstra [16]. A new feature of his limit was that he obtained a value for the pipe end-correction in the low frequency ($\frac{a}{\lambda}\to 0$, with Mach number fixed) limit, showing it to be different from the low Mach number value (Howe [11]). Recently, Ting [17] has tackled the low frequency problem, for a two-dimensional duct, in a quite different way, using matched asymptotic expansions: his approach really relies, however, on the use of conformal mapping to solve the "inner" incompressible problem and cannot be adapted to deal with the axisymmetric pipe problem.

Work complementary to that of this paper is presented in two papers by Munt [18,19]. The first of these deals with the reflection coefficient, for a cold jet, and the second with the radiated power. Munt shows that when a mean flow is present, the pressure reflection coefficient does not initially decrease with frequency but increases, reaching a maximum value at a Strouhal number of 0.5 and thereafter decreasing, but still remaining higher than the value it had in the absence of the mean flow. This increase is reduced by the addition of an external flow. It might be thought that the increase violates energy conservation but this is not so and the energy reflection coefficient is always less than unity. This increase in the reflection coefficient is well substantiated by experiment [20,21,22], at least for the lower Mach numbers. Munt also shows [19], that the previously mentioned power absorption phenomenon is well predicted theoretically, in agreement with experiments [14].

The aim of the present paper is to model the above phenomena in more detail. One of the features of Munt's work is that only computed results are given, which do not necessarily enable the physical basis for the results to be understood. The aim here is to obtain explicit formulae for the reflection coefficient and radiation field to second order in the frequency parameters, thus showing in detail how the interesting features of the reflection coefficient arise. In the author's opinion, this paper is likely to represent the limit of what it is useful to do with this problem analytically. To go further than the present approximation would produce very complicated formulae which, as a result, would do little to aid physical understanding.

The paper begins by summarising the Wiener-Hopf theory of the sound transmission problem following Munt [9] and Cargill [13]. The results are derived both with and without the application of a Kutta condition, and also with the jet instability assumed to consist of a neutrally stable convected wave following Howe [10]. The next section determines the reflection coefficient. This is done by expanding the Wiener-Hopf split functions to second order in the frequency parameter, ka. A solution is thus obtained that is valid as $ka \to 0$ with the Mach number, M, fixed. It appears that this solution is valid, at fixed (ka) over the whole range of Mach numbers, so that it is not restricted to low Strouhal numbers (ka/M). As a result the theory is able to predict the whole range of the results presented by Munt, using a simple analytic formula. Results are given both with and without Kutta conditions, and, unlike Munt [18,19], they include the possibility of different temperatures inside and outside the jet. The effect of the Strouhal number on the end-correction is also discussed. Finally, the paper derives formulae for the radiated field in the same limit. This enables some assertions made by Cargill [13], concerning the field at high Mach numbers (jet velocity greater than ambient sound speed), to be proved.

The paper concludes with a discussion of where further work is required on this problem. In particular, the possibility of modelling some features of the real jet excluded by the present idealisation is highlighted.

2. BASIC THEORY

The purpose of this section is to summarise the theory contained in Munt[9] and Cargill[13], which forms the basis of the work in later sections. The following problem is considered. A rigid cylindrical pipe of radius a, containing a fluid of density ρ_j, sound speed c_j and mean velocity U_j is placed in an ambient medium of density ρ_o, sound speed c_o and mean velocity U_o. The boundary between the two fluids, $r=a$, $x>0$ takes the form of a vortex sheet, which is attached to the pipe at $x=0$ (Fig. 1). The pipe contains an incident sound field having a pressure perturbation

$$p = p_i \exp\left[ikc_j t - ik u_i x\right], \tag{1}$$

in which ku_i is the wave number of the incident sound field, and for the plane waves considered here is $ku_i = k/(1+M_j)$, where $M_j = U_j/c_j$. The frequency kc_j is assumed to be of the form $c_j|k|\exp[-i\delta]$, where δ is between 0 and π, and will be set to zero at the end of the analysis. To account for the presence of the pipe termination, we assume correction pressure fields p_j, p_o to exist in the jet and in the ambient fluid respectively. These fields satisfy the convected wave equations

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} - \left(ik - M_j\frac{\partial}{\partial x}\right)^2\right] p_j = 0, \tag{2}$$

and

$$\left[\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} - C_1^2\left(ik - M_j\alpha\frac{\partial}{\partial x}\right)^2\right] p_o = 0, \tag{3}$$

where $C_1 = c_j/c_o$, $\alpha = U_j/U_o$.

Since the problem involves incident plane waves and the geometry is axisymmetric, the dependence on the azimuth angle ϕ may be dropped. These equations (2) and (3) are solved subject to the requirements that the particle displacement η is zero on the walls of the pipe $(r=a,\ x<0)$ and is conserved across the jet $(r=a, x>0)$ and that the pressure is constant across the jet boundary $(r=a, x>0)$.

To proceed with the solution one introduces the Fourier transforms

$$P^{\pm}(u) = \int_{-\infty}^{+\infty} p(x)\, H(\pm x)\, e^{ikux}\, dx, \tag{4}$$

for the pressures, and similarly $Z^{\pm}(u)$ are defined as the Fourier transforms of $\eta(x)H(\pm x)$, where H is the unit step function. In these transforms, the $\pm$ parts are analytic in the regions $R^{\pm}$ (Fig. 2), which overlap in a strip of regularity. Then, the correction fields P_j, P_o, satisfy the equations

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2 v^2\right) P_j = 0, \tag{5}$$

$$\left(\frac{1}{r}\frac{\partial}{\partial r} r \frac{\partial}{\partial r} + k^2 w^2\right) P_o = 0, \tag{6}$$

where $v^2 = \left((1-Mu)^2 - u^2\right)$ and $w^2 = C^2(1-\alpha M u)^2 - u^2$,

in which the branch cuts of v, w are taken as shown on Fig. 2, and the values of v, w taken so as to ensure regular behaviour within the strip of overlap between R^+ and R^-. Solving these equations, it is obvious that P_o, P_j have the forms

$$P_j = A\, J_o(kvr), \tag{7}$$

$$P_o = B\, H_o^{(2)}(kwr), \tag{8}$$

where J_o is a zero order Bessel function of the first kind and $H_o^{(2)}$ is a zero order Hankel function of the second kind. Fourier transforming the boundary conditions, it is found that continuity of pressure gives

$$P_j^+(a^-) - P_o^+(a^+) - \frac{p_i}{ik(u-u_i)} = 0, \tag{9}$$

continuity of displacement across the jet shear layer similarly gives

$$Z_j^+(a^-) - Z_o^+(a^+) = 0, \tag{10}$$

and the rigidity of the pipe implies that

$$Z_j^-(a^-) - Z_o^-(a^+) = 0. \tag{11}$$

Solving these equations, one finds that

$$K(u)Z^+(u) - \frac{P_i}{ik(u-u_i)} = P_j^-(a^-) - P_o^-(a^+) = F^- \text{ (say)}, \tag{12}$$

where

$$K(u) = -k^2 c_o^2 \rho_o \left[\frac{(C^2/\gamma)kw\, H_o^{(2)\prime}(kwa)\, J_o(kva) D_j^2 - D_o^2 kv\, J_o'(kva)\, H_o^{(2)\prime}(kwa)}{k^2 vw\, J_o'(kva)\, H_o^{(2)}(kwa)}\right],$$

and $\gamma = (\rho_o/\rho_j)$. (13)

The solution of equation (12) is described in some detail by Munt[9]. Essentially the technique is to split K into the product of two factors K^+, K^-, regular in R^+ and R^- respectively, so that we obtain the functional equation

$$Z^+(u)K^+(u) - \frac{P_i}{ik(u-u_i)K^-(u_i)} = F^-(u) - \frac{P_i}{ik(u-u_i)}\left[\frac{1}{K^-(u_i)} - \frac{1}{K^-(u)}\right]. \tag{14}$$

One initially solves this equation with the edge condition, that the vortex sheet must leave the pipe with zero gradient. i.e. the usual Kutta condition. Initially δ is taken as greater than $\arg u_o$, where u_o is the instability pole of $K(u)$. Then, as $u \to \infty$, $K^+(u) \sim u^{3/2}$, $K^-(u) \sim u^{-1/2}$ (see e.g. Munt[9]). Since it is assumed that $\eta \sim x^{3/2}$, $Z(u) \sim u^{-5/2}$ as $u \to \infty$, so that $K^+(u)Z^+(u) \sim u^{-1}$. By Liouville's theorem, this implies that both sides of equation (14) must be equal to zero, so that

$$Z(u) = \frac{P_i}{ik(u-u_i)K^+(u)K^-(u_i)}. \tag{15}$$

To obtain a causal solution with real k (i.e. $\delta = 0$), one deforms the Fourier inversion contour continuously to give $\delta = 0$. In so doing the pole at $u = u_0$ is encountered, and a contribution due to this must be added to (15) to ensure that the field is regular when $\mathrm{Im}\, k < 0$, a requirement for causality (Morgan[7]). Thus

$$Z(u) = \frac{P_i}{ik(u-u_i)K^+(u)K^-(u_i)} + H(\arg k+\delta)\frac{2\pi i\, P_i}{ik(u_0-u_i)K^{+\prime}(u_0)K^-(u_i)} \quad (16)$$

In the above expression, $K^\pm$ are taken as the analytic continuation for $\delta < |\arg u_0|$ of $K^\pm$ defined with $\delta > |\arg u_0|$. In practice it will be convenient to work with $\delta = 0$, and in that case the split of $K(u)$ is $K^{*\pm}(u)$ (say), in which

$$K^{*+}(u) = \frac{K^+(u)}{(u-u_0)}, \quad K^{*-}(u) = K^-(u)(u-u_0). \quad (17)$$

It thus follows from (15) that the fields inside the jet and in the ambient medium are given by

$$P_j(u) = \frac{k^2\rho_j c_j^2 D_j^2 J_0(kvr)\, P_i}{kv\, J_0'(kvr)\, ik(u-u_i)K^+(u)K^-(u_i)}, \quad (18)$$

$$P_0(u) = \frac{k^2 c_0^2 \rho_0 D_0^2 H_0^{(2)}(kwr)\, P_i}{kw\, H_0^{(2)\prime}(kwr)\, ik(u-u_i)K^+(u)K^-(u_i)} \quad (19)$$

The fields in the ambient medium and in the jet pipe are then obtained by Fourier inversion in the usual way.

When the Kutta condition does not hold, then the displacement $\eta(x) \sim x^{1/2}$ as $x \to \infty$, so that $Z(u) \to u^{-3/2}$ as $u \to \infty$. Therefore, the left hand side of (14) is $O(1)$ as $u \to \infty$. Thus by Liouville's theorem, both sides of the equation must be equal to a constant, E (say). That is, from (15),

$$Z(u) = \frac{P_i}{ik(u-u_i)K^+(u)K^-(u_i)} + \frac{E}{K^+(u)}. \quad (20)$$

Clearly, there is an infinite number of possible solutions to this equation, each corresponding to a different value of E. Only one of these is of interest, though, and that is the one for which there is no instability wave pole produced, so that $Z(u_0)=0$. This is termed the non-Kutta condition solution, and with

$$E = \frac{-P_i}{K^-(u_i)\, ik\,(u_0-u_i)}\,, \tag{21}$$

is precisely the same as the result that would have been obtained if the causality argument had been ignored, and δ taken as zero from the beginning. That is

$$\Phi_{No\ Kutta}(u) = \Phi_{Kutta}(u)\cdot\frac{(u-u_0)}{(u_i-u_0)}\,, \tag{22}$$

where Φ is any field variable.

A third solution that will be required later is that introduced by Howe [11]. He argues that at high Strouhal numbers the jet boundary may well be thick on the scale of the hydrodynamic wavelength, $(2\pi U_j/c_j k)$, while remaining thin on the scale of the acoustic wavelength $(2\pi/k)$, at low Mach numbers. He therefore proposes that to the non-Kutta condition solution one should add another solution, corresponding to a neutral convected motion at wave number ku_H, say, with the requirement that the Kutta condition is restored. That is, we set

$$Z(u) = \frac{P_i(u-u_0)}{ik(u-u_i)K^+(u)K^-(u_i)(u_0-u_i)} + \frac{A(u-u_0)}{ik(u-u_H)K^+(u)K^-(u_H)(u_0-u_H)}\,, \tag{23}$$

where A is a constant. Adding these so as to obtain the required growth of $\eta \sim x^{3/2}$ at $x\to 0$, one obtains the relation analogous to (22)

$$\Phi_H = \Phi_{Kutta}\cdot\frac{(u-u_0)(u_i-u_H)}{(u_i-u_0)(u-u_H)}\,. \tag{24}$$

Some of the implications of these solutions for the farfield sound and the reflected field are discussed by Cargill [13]. The discussion of the deductions from these solutions is split into two sections. In the first, the reflected field in the pipe is discussed and in the second, the farfield radiation. In each case, the analysis is taken to second order in the frequency parameter ka, as opposed to the first order analysis of [13].

3. THE REFLECTED FIELD INSIDE THE JET PIPE

Using (18), the reflected field inside the jet pipe is given by

$$P_j(x) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{k^2 c_j^2 \rho_j D_j^2 J_0(kvr)\, e^{-ikux}\, du}{kv_0\, J_0'(kva)(u-u_i)K^+(u)K^-(u_i)} \qquad (25)$$

For $x<0$, the contour may be closed in the upper half plane, giving pole contributions from the zeros of $J_0'(kva)$, which represent duct acoustic modes travelling upstream away from the nozzle. Of these, only one does not decay exponentially at low frequencies. This is the plane wave mode given by $u = u_r = \frac{-1}{(1-M)}$ leading to

$$P_j(x) = \frac{c_j^2 D_j^2(u_r)\, e^{-iku_r x}}{a\, K^+(u_r) K^-(u_i)(u_r-u_i)} \quad . \qquad (26)$$

Clearly what is required now is the asymptotic behaviour of $K^+(u_r), K^-(u_i)$. Now $K^\pm(u)$ are defined in the usual way by the integrals

$$K^\pm(u) = \exp\left[\frac{\pm 1}{2\pi i}\int \frac{\ln K(s)\, ds}{(s-u)}\right], \qquad (27)$$

in which the contour is taken within the region of regularity so as to run below u for K^+ and above for K^-. Since

$$K(u) = -k^2 c_0^2 \rho_0 \left[\frac{(G^2/\gamma)\, kw\, H_0^{(2)\prime}(kwa) J_0(kva) D_j^2 - D_0^2\, kv J_0'(kva)\, H_0^{(2)}(kwa)}{k^2 vw\, J_0'(kva) H_0^{(2)\prime}(kwa)}\right], \tag{28}$$

it is convenient to split K into three factors,

$$K(u) = -\frac{2k^2 c_j^2 \rho_j D_j^2 a}{(k^2 v^2 a^2)} \cdot S \cdot T, \tag{29}$$

where

$$S = \left[-\frac{J_0(kva)\, kva}{2 J_0'(kva)}\right], \tag{30}$$

and

$$T = \left[\frac{kw H_0^{(2)\prime}(kwa) J_0(kva) D_j^2 - (\gamma/G^2)\, kv\, H_0^{(2)}(kwa) J_0'(kva) D_0^2}{kw H_0^{(2)\prime}(kwa)\, J_0(kva)\, D_j^2}\right]. \tag{31}$$

These factors have been chosen so that $S, T \to 1$ as $ka \to 0$, with u, M fixed. Substituting in (26), with $S = S^+ S^-$, and $T = T^+ T^-$, noting that the first factor in (29) becomes

$$\left[\frac{-2k^2 c_j^2 \rho_j D_j^2 a}{k^2 a^2 (1-(1+M)u)}\right]\left[\frac{1}{(1+(1-M)u)}\right],$$

and evaluating the integrals in (25) and (27) with $\delta = 0$, so that one must add the term due to the instability pole, one obtains for the reflected pressure wave

$$p = p_i\, e^{-iku_r x} \frac{(1+M)(u_0 - u_i)}{(1-M)(u_0 - u_r) S^+(u_r) S^-(u_i) T^+(u_r) T^-(u_i)}. \tag{32}$$

In this integral the split functions are defined using (27) with $\delta = 0$, and the $(u_0 - u_i)/(u_0 - u_r)$ factor arises because of the need to make this field continuous with that obtained when $\delta > \arg u_0$, when the contour in (27) runs the other side of u_0. The functions S, T will now be factorised in turn. To calculate the modulus of the reflection coefficient, one only needs the magnitude of the split function; this is given by

$$|K^+(u)| = |K(u)|^{1/2} \exp\left[\frac{1}{2\pi} \int \frac{\arg K(s)\, ds}{s - u}\right] \tag{33}$$

the value of the integral for by noting that only the combined value $\left[T^{+}(-1/1+M)\,T^{-}(1/1+M)\right]$ is required. This, using (35) and (33), is equal to

$$T^{+}\left(\frac{-1}{(1-M)}\right)T^{-}\left(\frac{1}{(1+M)}\right) = \exp\int_{-C/(1-\alpha MC)}^{C/(1+\alpha MC)} \frac{(kv_a)^2}{8}\frac{\gamma}{C^2}\cdot\frac{D_o^2}{D_j^2}\left[\frac{1}{(s+1/1-M)} - \frac{1}{(s-1/1+M)}\right]ds, \quad (36)$$

$$= \exp\left[\frac{\gamma(ka)^2}{4C^2}\int_{-C/(1-\alpha MC)}^{C/(1+\alpha MC)}(D_o^2/D_j^2)\,ds\right], \quad (37)$$

$$= \exp\left[\frac{\gamma(ka)^2}{4C^2}\Psi(M,\alpha,C)\right]. \quad \text{say } (38)$$

Noting that $D_o^2 = (1-\alpha Mu)^2$, $D_j^2 = (1-Mu)^2$, the integral for Ψ may be performed by making the substitution $(1-Ms) = t$ to give

$$\Psi = \frac{1}{M}\int_{(1-MC/(1+\alpha MC))}^{(1+CM/(1-\alpha MC))}\left(\frac{\alpha}{t} - (1-\alpha)\frac{1}{t^2}\right)dt, \quad (39)$$

$$= \frac{(1-\alpha)C}{(1-M^2C^2(1-\alpha)^2)} + \frac{\alpha}{M}\ln\left[\frac{(1+\alpha MC)(1+MC(1-\alpha))}{(1-\alpha MC)(1-MC(1-\alpha))}\right]. \quad (40)$$

Substituting (37) into (32), and using (40), one finds that correct to second order in (ka), the magnitude of the reflection coefficient is

$$|R|_k = \frac{(u_o - 1/(1+M))(1+M)}{(u_o + 1/(1-M))(1-M)}\left(1 - \frac{\gamma(ka)^2}{2C^2}\Psi\right), \quad (41)$$

where the subscript k refers to the fact that a Kutta condition holds. When the Kutta condition does not hold, one has, using (22),

$$|R|_{NK} = \frac{(1+M)}{(1-M)}\left(1 - \frac{\gamma(ka)^2}{2C^2}\Psi\right). \quad (42)$$

where $K(u)$ is any kernel function, and the integral is interpreted as a principal value. Taking S first one notes that since $v=0$ at u_r, u_i, $|S(u_r)|, |S(u_i)| = 1$.

Examining the integrand of (33) one sees that between the branch points, on the path of integration, v is real, so that $\arg S=0$. On the parts of the path from $-1/(1-M)$ to ∞ and $1/(1+M)$ to ∞, $v=-i|v|$ so that $\arg S$ is again zero. Thus there is no contribution to $|S^{\pm}|$ from the integration path. Nor is there any contribution from the portion of the curve that is indented round the branch points.

Considering the factor T, one first notes that on the part of the path outside the branch cuts, $w=-i|w|$, and $v=-i|v|$. It then follows that on this part of the integration path $\arg T=0$. There is similarly no contribution from the indentation round the branch points. From the appendix, the low frequency expansion of $T(u)$, obtained by expanding $T(u)$ as $ka \to 0$, u, M fixed, is

$$T(u) = \left[1 + \left(\frac{kva}{2}\right)^2 \frac{\gamma}{C_1^2} \cdot \frac{D_0^2}{D_j^2}\left(\ln\left(\frac{kva}{2}\right) + \gamma_E + \frac{i\pi}{2}\right)\right]. \qquad (34)$$

Examining this formula, it is clear that $T(u) \to 1$ when $v \to 0$, at the branch points $u=u_i, u_r$ so that there is a unit contribution to $|T^{\pm}(u)|$ from $|T(u)|^{1/2}$. Furthermore, it is also clear that $\arg T(u)$ is only non-zero on the part of the integration contour between the branch points at $u = C_1/(1+\alpha M C_1)$ and $-C_1/(1-\alpha M C_1)$, and there

$$\arg T(u) = \frac{(kva)^2}{2} \cdot \frac{\gamma}{C_1^2} \cdot \frac{D_0^2}{D_j^2} \cdot \frac{\pi}{2}, \qquad (35)$$

correct to second order in $(kva)^2$. Now, it is easiest to find

Examining these in detail, one first notes that at sufficiently low Strouhal number $u_0 \to 1/M$, so that $|R|_K \to 1$. In the same limit, $|R|_{NK} \to (1+M)/(1-M)$. These are in agreement with Cargill[13]. At high Strouhal number u_0 is a zero of the plane vortex sheet dispersion relation

$$\left(\frac{D_j^2 \rho_j}{k v} + \frac{D_0^2 \rho_0}{k w}\right) = 0 . \tag{43}$$

In the low Mach number limit, one can take $v, w, = -iu$ in (43), so that

$$u_0 = \frac{1}{M}\frac{(1 + i\gamma^{1/2})}{(1 + i\kappa\gamma)^{1/2}} . \tag{44}$$

It is clear then, that when the two flows have the same speed u_0 is again $(1/M)$ and the factor

$$\left[(1+M)/(1-M)\right]\left[\left(u_0 - 1/(1+M)\right)/\left(u_0 + 1/(1-M)\right)\right]$$

appearing in (41), is unity. When the two flows have different speeds, this augmentation factor, that increases the reflection coefficient above unity, is given by

$$\left|\frac{1 + i\gamma^{1/2}(1 + M(1-\kappa))}{1 + i\gamma^{1/2}(1 - M(1-\kappa))}\right| . \tag{45}$$

This clearly illustrates how this augmentation increases for high Strouhal number to a constant value, which increases with Mach number and density ratio, but is unity when the speeds of the two streams are equal.

It is of interest to compare these results with Munt's[19] exact computations. The important feature of the latter is that as the frequency is increased from zero, the reflection coefficient rises above unity, reaching its maximum value at a Strouhal number $(ka/2\pi M)$ of about 0.5 and thereafter decreasing. It is easiest to examine the present formulae in the low Mach number limit $M^2 \ll 1$, and for a cold jet, so that $\psi^\wedge = 1$.

Then in the absence of external flow

$$|R|_K = \left|\frac{(1+M)u_0 - 1}{(1-M)u_0 + 1}\right|\left(1 - \frac{(ka)^2}{2}\right). \tag{46}$$

The last part of this expression is the Levine and Schwinger 1 reflection coefficient, so that the effect of flow is, to first order contained in the first factor, which we shall refer to as A. Writing $u_0 = (u_1 + iu_2)/M$, where in this low Mach number limit u_1, u_2 are functions only of (ka/M), and expanding for small M one finds

$$|A| = \left[1 + 2M\left(1 - \frac{u_1}{u_1^2 + u_2^2}\right)\right]. \tag{47}$$

In the low Mach number limit, values of u_1, u_2 have been given by Crow and Champagne [24] using a theory due to Batchelor and Gill [25]. These values have been used to calculat $|A|$ with results which are compared in Fig. 3 with those derived from Munt's paper. As can be seen, the two are in excellent agreement over the whole Strouhal number range. Figure 4 compares the present results with those of Munt for Mach numbers of 0 and 0.3. The agreement is excellent, except at high frequencies where the approximation $ka \ll 1$ fails to hold. Furthermore, these results are in good agreement with experiments ([20], [21], [22], see [18]).

At higher Strouhal numbers, still for a cold jet, it is convenient to look at the effect of external flow, since here (Fig. 5) Munt shows the effect of the external flow to be considerable. For $(ka/M) \to \infty$, $u_0 \to \left(\frac{1+i}{1+i\alpha}\right)/M$, so that

$$|A| \sim \left(1 + M(1-\alpha)\right). \tag{48}$$

This demonstrates the reduction in reflection coefficient with external flow.

Examining now the non-Kutta-condition solution, it is clear that here the initial lift in the reflection coefficient is completely absent, although, of course, the absolute value is always greater than unity. Indeed, in the same low Mach number approximation as was introduced earlier,

$$|R|_{NK} = (1+2M)\left(1 - \frac{(ka)^2}{2}\right). \tag{49}$$

There is thus no dependence on Strouhal number through the instability poles.

The most interesting aspect of the two solutions, represented by $|R|_K$ and $|R|_{NK}$ lies in the energy fluxes involved. In the case of the Kutta condition solution, acoustic energy in the jet pipe is converted to hydrodynamic kinetic energy in the jet instability wave. No such transfer occurs when the Kutta condition does not hold [13]. It does not appear possible to produce a formula for that energy transfer that displays explicitly its dependence on the Strouhal number, since the value of u_0 is not known as a function of Strouhal number. The energy aspects of the non-Kutta solution can, however, be ascertained as follows.

In the non-Kutta-condition solution, the net energy flux in the pipe is clearly given by

$$W = \frac{p_i^2 A_j}{\rho_j c_j}\left((1+M)^2 - (1-M)^2\, |R|_{NK}^2\right), \tag{50}$$

in which the usual formulae for the energy fluxes in positive- and negative-going waves have been used (see, e.g., Morfey [26]). Substituting for $|R|_{NK}$, one obtains

$$W = \frac{p_i^2 A_j (1+M)^2}{\rho_j c_j} \cdot \frac{\gamma}{2 C^2} \cdot (ka)^2 \cdot \Psi . \tag{51}$$

Now from Cargill[9], the radiation field of the non-Kutta-condition flow is

$$p = \frac{(\rho_0/\rho_j)\, k\, \pi a^2 (1+M)\, p_i}{4\pi R (1+\alpha M C\cos\theta)^2 (1 - M C(1-\alpha)\cos\theta)} . \tag{52}$$

Therefore the radiated power is

$$W_R = \int_0^{\pi} \frac{2\pi R p^2}{\rho_0 c_0} (1+\alpha M C\cos\theta)^2 \sin\theta \, d\theta \tag{53}$$

$$= \left(\frac{\rho_0 c_j}{\rho_j c_0}\right) \frac{(ka)^2}{4} (1+M)^2 (\pi a^2) \frac{p_i^2}{\rho_j c_j} \int_0^{\pi} \frac{\sin\theta \, d\theta}{(1+\alpha M\cos\theta)^2 (1+M C(1-\alpha)\cos\theta)} . \tag{54}$$

One now makes the substitution $C\cos\theta/(1+\alpha M C\cos\theta) = t$, so that

$$W_R = \frac{(ka)^2}{4} (1+M)^2 \frac{A_j}{\rho_j c_j} p_i^2 \frac{\gamma}{C^2} \int_{-C/(1-\alpha M C)}^{C/(1+\alpha M C)} \frac{(1-\alpha M t)^2}{(1-Mt)^2} \, dt . \tag{55}$$

The integral here is of the same form as that in the integral for the reflection coefficient. Using this fact and observing the result for the total energy in the pipe, it becomes clear that, to this approximation, the acoustic energy is conserved, <u>all the net energy flow in the pipe reaching the farfield, with none being lost to any vortical disturbances</u>. This is entirely what one might expect, since there is, by definition, no vorticity generated here. Furthermore, this result applies at all Strouhal numbers.

The Howe model[11], with the neutrally convected wave in the jet, is also of interest. Here the reflection coefficient becomes, on using (24),

$$|R|_H = \frac{(u_H(1+M)-1)}{(u_H(1-M)+1)}\left(1 - \frac{\gamma(ka)^2}{2C^2}\Phi\right). \tag{56}$$

In the case considered by Howe $u_H = 1/\nu M$, where ν is, by definition, the ratio of the assumed convection speed to jet speed. One has then

$$|R|_H = \frac{(1+M(1-\nu))}{(1-M(1-\nu))}\left(1 - \frac{\gamma(ka)^2}{2C^2}\Phi\right). \tag{57}$$

As ν is varied, the augmentation varies from $(1+M)/(1-M)$ when $\nu = 0$, and the wave is effectively absent, to 1 when $\nu = 1$, and the wave is convected with the flow. Clearly, as a means of representing the real flow due to the instability wave, this analysis is incomplete, and could possibly be extended to include other forms of u_H. It should be emphasised, though, that such work can only be of an approximate nature. If it is desired to represent the true velocity profile, then the equations should be solved as a linear perturbation of that profile. The Howe type of approach does not really contain the true effect of such a profile.

Finally, the end-correction will be discussed. It would be very useful if one could, as a result of the above analysis, obtain similar formulae for the end-correction. Unfortunately, this cannot be done, except in the low and high Strouhal number limits. In the former case, Rienstra [16] has shown, by expanding the results for low ka with M fixed, that the Kernel may be approximated by the value

$$K(u) = -k^2 c_0^2 \rho_0 \frac{C^2}{\gamma} \frac{J_0(kva)\, D_j^2}{J_0'(kva)\, va}, \tag{58}$$

which can be factorised as an infinite product. The resulting end-correction is somewhat different from that in the absence of flow (Levine and Schwinger[1]). For high Strouhal numbers the factorisation has been given by Howe[11], who finds that the Levine and Schwinger value must be augmented by an additional term scaling with M; since $M = ka(M/ka)$ this part may be thought of as a correction varying inversely with the Strouhal number. For intermediate values of Strouhal number nothing can be done, beyond obtaining an integral expression for the end-correction. This is because the integrals involved in $K^{\pm}(u)$ have paths which produce contributions from their whole length, not just from a finite part like those for $|K^{\pm}(u)|$.

4. THE FARFIELD RADIATION

In this section, the farfield radiation produced by the jet will be determined, again to second order in ka. The lowest order result has been given by Cargill[12,13], who shows that the principal effect of flow is to alter the previously omnidirectional directivity pattern to $(1+M\alpha\cos\theta)^{-1}(1-(1-\alpha)M\varsigma\cos\theta)^{-2}$. This field is clearly singular at the Mach angle, where $\cos\theta = ((1-\alpha)M\varsigma)^{-1}$. In [13], it was asserted that at that angle one should employ the more accurate forms for the instability poles (i.e. not just $u_0 = 1/M$) and that this would remove the singularity. One of the main aims of this section is to substantiate that claim. Another feature of the exact result is that a refraction valley appears around the axis $\theta = 0$ as the frequency is increased. Examination of this will also feature in the succeeding analysis.

Generally, the analysis proceeds in a manner similar to that for the reflection coefficient, except that one now needs to evaluate the split functions for arguments other than those corresponding to the branch points. This considerably complicates their evaluation.

From [19], one obtains the field in the ambient medium as

$$p_o(\underset{\sim}{x}) = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{k^2 c_o^2 \rho_o D_o^2 H_o^{(2)}(kwv)\, e^{-ikux}\, du}{kw\, H_o^{(2)\prime}(kwa)(u-u_i)K^+(u)K^-(u_i)}, \tag{59}$$

which may be evaluated by the method of stationary phase, to give

$$p_o = \frac{e^{ikRC}\, p_i}{4\pi R\,(1+\alpha M C\cos\theta)}\left[\frac{k^2 c_o^2 \rho_o D_o^2}{kw\, H_o^{(2)\prime}(kwa)(u-u_i)K^+(u_f)K^-(u_i)(u_f-u_i)}\right], \tag{60}$$

in which the term in square brackets is evaluated at $u = u_f = C\cos\theta/(1+\alpha M C\cos\theta)$, and the details of the stationary phase evaluation can be found in, for example, Munt [9]. Therefore to evaluate (60) one requires the values of the product $K^+(u_f)K^-(u_i)$. To obtain this, $K(u)$ is again split according to (29). In that expression, the first factor is, by inspection, as described earlier; however S and T require a somewhat different treatment from that in Section 3.

The function S is factorised by the infinite product theorem. Again, only the moduli of S^+, S^- are required, and are obtained by using (33), and $\arg S$ is zero over the whole of the integration path, so that $|S^+(u)| = |S^-(u)| = |S(u)|^{1/2}$. Furthermore, since $v(u_i) = 0$, it automatically follows that $S^-(u_i) = 1$.

The factorisation of T also proceeds in the same manner as in Section 3. As noted there, there is no contribution to the integral except from the region between the branch points of w, so that

$$\left|T^{\pm}(u)\right| = \left|T(u)\right|^{1/2}\exp\left[\frac{\pm 1}{2\pi}\int_{-C/(1-M\alpha C)}^{C/(1+\alpha MC)} \frac{\arg\, T(s)\, ds}{(s-u)}\right]. \qquad (61)$$

substituting for $\arg T(s)$ from (35), it is clear that one needs to evaluate

$$I = \int_{-C/(1-\alpha MC)}^{C/(1+\alpha MC)} \frac{v^2 D_0^2\, ds}{Dj^2(s-u)} = \int_{-C/(1-\alpha MC')}^{C/(1+\alpha MC')} \frac{((1-Ms)^2-s^2)(1-\alpha Ms)^2 ds}{(1-Ms)^2(s-u)}. \qquad (62)$$

This integration is described in detail in Appendix II. As that appendix points out, the result for general values of is very complicated. To simplify matters, without significant loss of understanding, it suffices here to give only the result for $\alpha=0$ which is

$$I = \left[\frac{1}{M^2}\left(1-\frac{M^2u^2}{(1-Mu)^2}\right)\ln\left(\frac{(1+MC)}{(1-MC)}\right) - \frac{1}{M^2}\cdot\frac{2MC'}{(1-Mu)(1-M^2C'^2)} - \left(1-\frac{u^2}{(1-Mu)^2}\right)\ln\left(\frac{(C'+u)}{(C'-u)}\right)\right]. \qquad (63)$$

If one is only interested in the directivity of the radiated sound, one only requires the term in (60) depending on u, not those depending on u_i. Thus the directivity of the pressure field is proportional to

$$p \sim \left[\frac{D_0^2\,(1-(1+M)u)\ \exp-(k^2a^2\gamma I/8C^2)}{(u_f-u)\ kw\ H^{(2)\prime}(kwa)\ Dj^2\,|u-u_0|\,|S|^{1/2}\,|T|^{1/2}}\right], \qquad (64)$$

evaluated with $u = C\cos\theta$. Two areas of interest will be examined in detail, namely the behaviour near the axis as $\theta\to 0$, and that near the Mach angle, $\cos\theta = (1/MC)$. In the former case, it can easily be seen that to $O((ka)^2)$, the only important terms are those in $|T|^{1/2}$ and in I. Now from

Appendix I,

$$T(u) = \left[1 + \frac{(kva)^2 \gamma \mathcal{D}_0^2}{2C^2 \quad \mathcal{D}_j^2} \left(\ln\left(\frac{ikwa}{2}\right) + \gamma_E + i\frac{\pi}{2} \right) \right].$$

Thus as $\theta \to 0$, $w \to 0$, and $T \to \infty$. Also, from Appendix II, it is clear that $I \to \infty$ as $u \to 0$. Therefore, it is seen that the field is always zero at the point $\theta = 0$, and that this appears to occur whenever the mean flow is present. In the absence of a mean flow, $V = W$, and the term giving rise to this non-uniform behaviour then disappears, in the case of I, or tend to zero in the case of T. It is thus seen that the refraction dip on axis is an essential feature of the directivity whenever a mean flow is present.

The other point of interest is the behaviour in the neighbourhood of the Mach angle, $\cos\theta \sim 1/MC$. Here the low frequency limit is singular [12,13] on account of the $\mathcal{D}_j^2$ terms that appear in $K^+(u)$. When terms of $O((ka)^2)$ are included, this singularity vanishes, being cancelled by the terms in $T^+(u)$. The resulting effect of including the order $(ka)^2$ terms is that, near the Mach angle, the resulting split functions are the same as those obtained by less rigorous arguments in [13].

In this section, attention has been concentrated on the solution in which a Kutta condition holds. It would have been equally possible to have discussed the field shape when there is no Kutta condition imposed. But the results then do not seem to have any particular interest and in any event, differ only from the Kutta condition solutions in the way described in [13] and in equation (22).

5. DISCUSSION AND CONCLUSIONS

This paper has discussed the interaction of internally incident sound waves with a jet pipe in the low frequency limit. Results have been given that are correct to second order in the frequency parameter (ka). In particular, it has been shown that the low frequency reflection coefficient modulus has a simple analytic form, that is valid for all Strouhal numbers (with $ka \ll 1$) and agrees with both Munt's computations [18], and experiments [20,21,22]. The radiation field has also been examined and it has been shown how the presence of a mean flow inevitably involves a refraction valley in the farfield near the jet axis.

In the author's opinion, this paper is likely to represent the limit of what it is useful to do analytically. As has been indicated, to go beyond the present analysis produces very complicated results that do not seem to be particularly useful. It does not appear possible to derive simple formulae for the end-correction, for example, or for the form of the instability wave.

There are a number of respects in which the idealisation implied in this analysis may be too extreme. The most important of these concerns the jet shear layers. Here is has been assumed that the jet shear layer does not grow with distance downstream, and is, furthermore, infinitely thin. Neither of these is true in practice. Therefore, some method of modelling these effects is required. This is likely to be important at the higher Strouhal numbers.

One approach to this problem, used by Howe [11], is to add on the field due to the jet instability wave separately and to then assumed that wave to be convected at a velocity different from the jet velocity. That is only partially satisfactory, however, as that wave does not represent a solution to any relevant wave equation. Another approach is to assume that there is a finite thickness velocity profile, both inside the jet and the pipe. This should show some additional effect of Strouhal number on the reflection coefficient, beyond that calculated here, and might perhaps agree even better with experiments. The defect of that analysis is that the flow is not allowed to be variable in an axial direction. But that might not be too important, as the flow does at least diverge slowly. In any event, one might anticipate that any shortcomings of the representation of the jet flow used here would principally affect the farfield directivity, and would have only an insignificant effect on the reflected field properties (which are essentially locally determined at the pipe lip).

ACKNOWLEDGEMENT

The author would like to thank the Directors of Rolls-Royce Limited for permission to publish this paper, and the Science Research Council for an Industrial studentship. He would also like to thank Professor D.G. Crighton for his comments on the draft of the paper.

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APPENDIX I

LOW FREQUENCY EXPANSION OF T(u)

In this appendix, the low frequency expansion of the function $T(u)$ will be given, where

$$T(u) = \left[\frac{kw\, H_0^{(2)\prime}(kwa)\, J_0(kva) D_j^2 - (\gamma/C^2) H_0^{(2)}(kwa) J_0'(kva) D_0^2 kv}{kw\, H_0^{(2)\prime}(kwa)\, J_0(kva)\, D_j^2}\right]. \quad \text{(A1)}$$

This expansion is obtained by expanding the Bessel functions to second order in ka, being careful to retain the terms of both $O((ka)^2)$ and $O((ka)^2 \ln(ka))$. Then using the formulae in Abramowitz and Stegun [23], the numerator of $T(u)$, A (say), is

$$A = -\frac{2iD_j^2}{\pi a}\left[1 - \frac{(kva)^2}{4} - \frac{(kwa)^2}{2}\left[\ln\left(\frac{kwa}{2}\right) + \gamma_E - 1 + \frac{i\pi}{2}\right] + \frac{D_0^2\,\gamma\,(kva)^2}{D_j^2\,C^2}\left[\ln\left(\frac{kwa}{2}\right) + \gamma_E + \frac{i\pi}{2}\right]\right], \quad \text{(A2)}$$

where γ_E is Euler's constant $(\sim 0{\cdot}5722)$.

Similarly the denominator of $T(u)$, B (say), becomes

$$B = -\frac{2iD_j^2}{\pi a}\left(1 - \frac{(kva)^2}{4} - \frac{(kwa)^2}{2}\left(\ln\left(\frac{kwa}{2}\right) + \gamma_E - 1 + \frac{i\pi}{2}\right)\right). \quad \text{(A3)}$$

Hence dividing A by B, it follows that, to order $(ka)^2$,

$$T(u) = \left[1 + \frac{\gamma\,(kva)^2 D_0^2}{C^2\; 2\;\; D_j^2}\left[\ln\left(\frac{kwa}{2}\right) + \gamma_E + \frac{i\pi}{2}\right]\right]. \quad \text{(A4)}$$

APPENDIX II

EVALUATION OF I(u)

From equation (62), $I(u)$ is defined by

$$I(u) = \int_{-C/(1-\alpha MC)}^{C/(1+\alpha MC)} \frac{((1-Ms)^2-s^2)(1-\alpha Ms)^2\,ds}{(1-Ms)^2(s-u)} . \qquad \text{(A5)}$$

In general this integration may be found by splitting the denominator up as partial fractions. Each of the resulting integrals may then be re-arranged into a form that can be integrated in a straightforward fashion. Unfortunately the results obtained are exceedingly complicated and will not be quoted here. In the main part of the text only the result for $\alpha = 0$ is required. In that case the integrand may be easily re-arranged in partial fractions to give

$$I(u) = \int_{-C/(1-\alpha MC)}^{C/(1+\alpha MC)} ds\left[-\frac{1}{M(1-Ms)}\left(1-\frac{M^2u^2}{(1-Mu)^2}\right)-\frac{1}{M(1-Ms)^2(1-Mu)}+\frac{1}{(s-u)}\left(1-\frac{u^2}{(1-Mu)^2}\right)\right] . \qquad \text{(A6)}$$

This may be integrated to give

$$I(u) = \left[\frac{1}{M^2}\left(1-\frac{M^2u^2}{(1-Mu)^2}\right)\ln\frac{(1+MC)}{(1-MC)} - \frac{1}{M^2(1-Mu)}\frac{2MC}{(1-M^2C^2)} - \left(1-\frac{u^2}{(1-Mu)^2}\right)\ln\frac{(C+u)}{(C-u)}\right] . \qquad \text{(A7)}$$

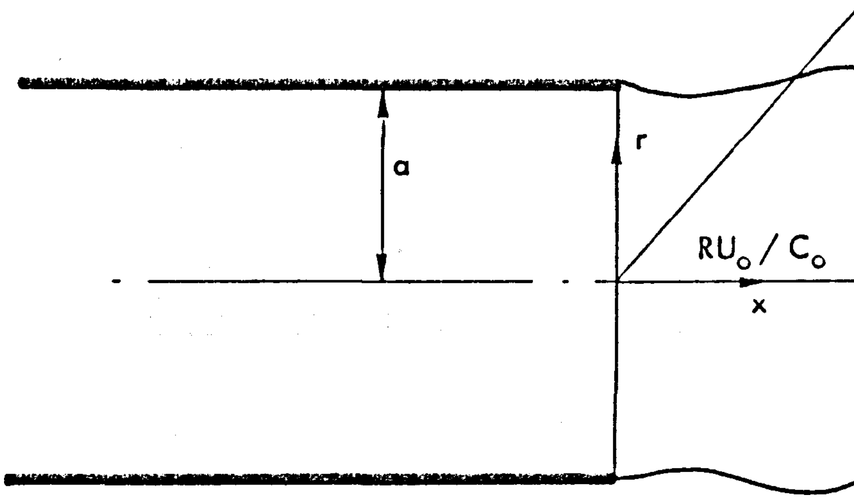
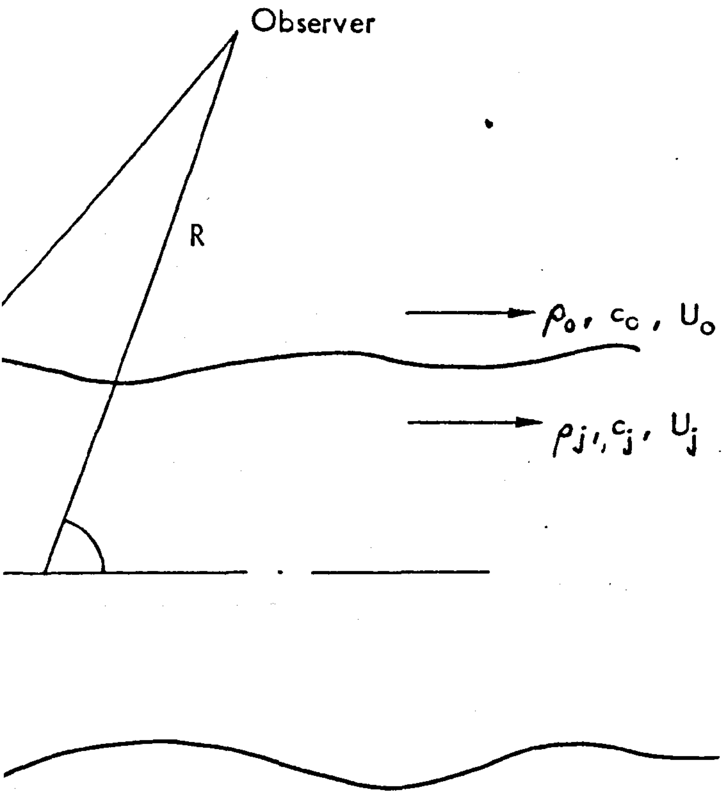


Fig. 1. Notation and co-ordinate system.


Observer
R
ρ₀, c₀, U₀
ρⱼ, cⱼ, Uⱼ

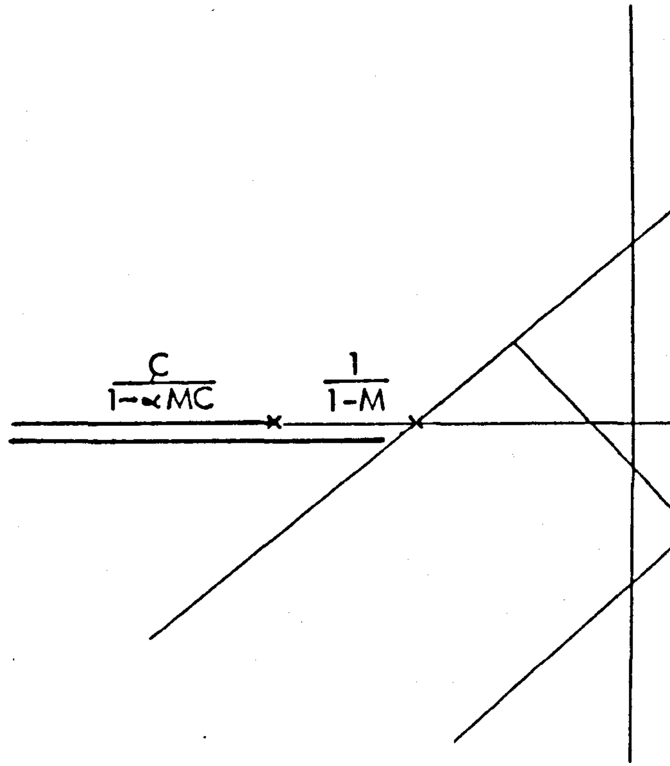
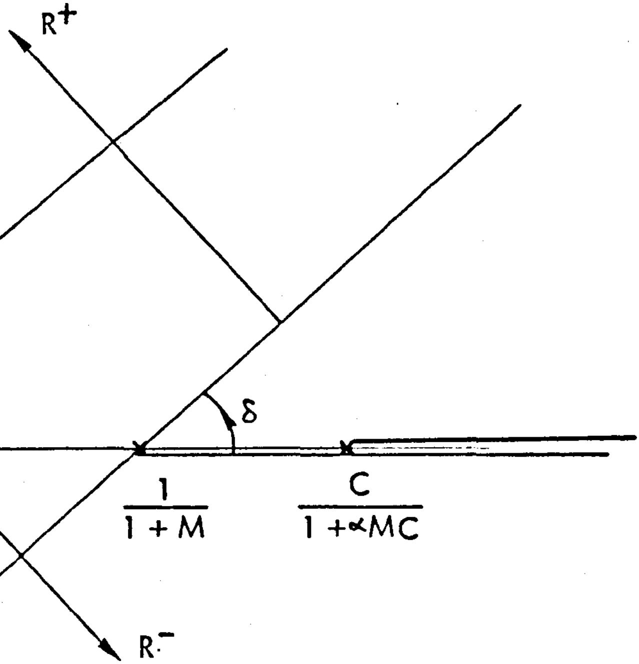


Fig. 2. The complex u-plane.



R^+
δ
$\frac{1}{1+M}$
$\frac{C}{1+\alpha MC}$
R^-

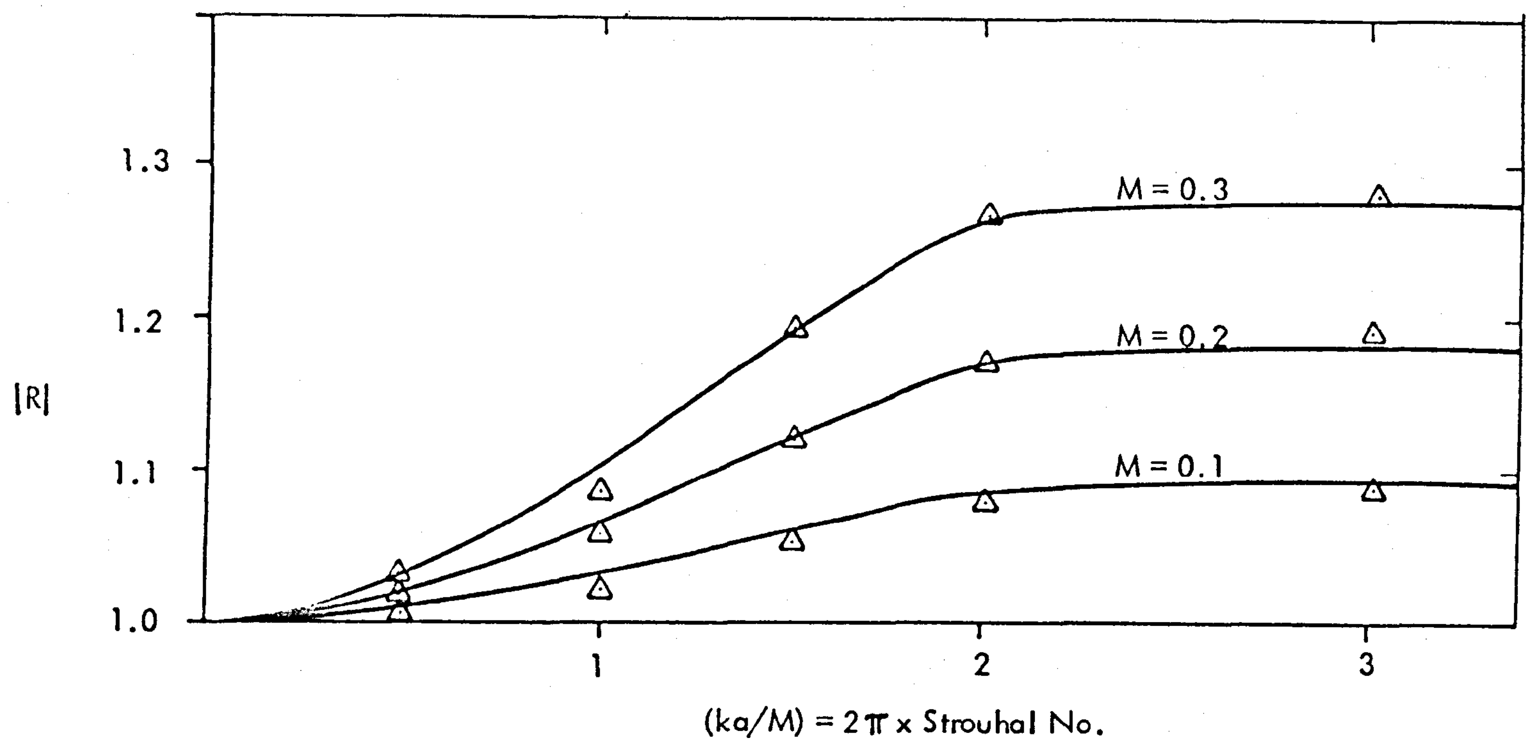


Fig. 3. Increase in reflection coefficient due to instability - comparisons of present theory with Munt [18]. ———— present theory, Δ Munt.

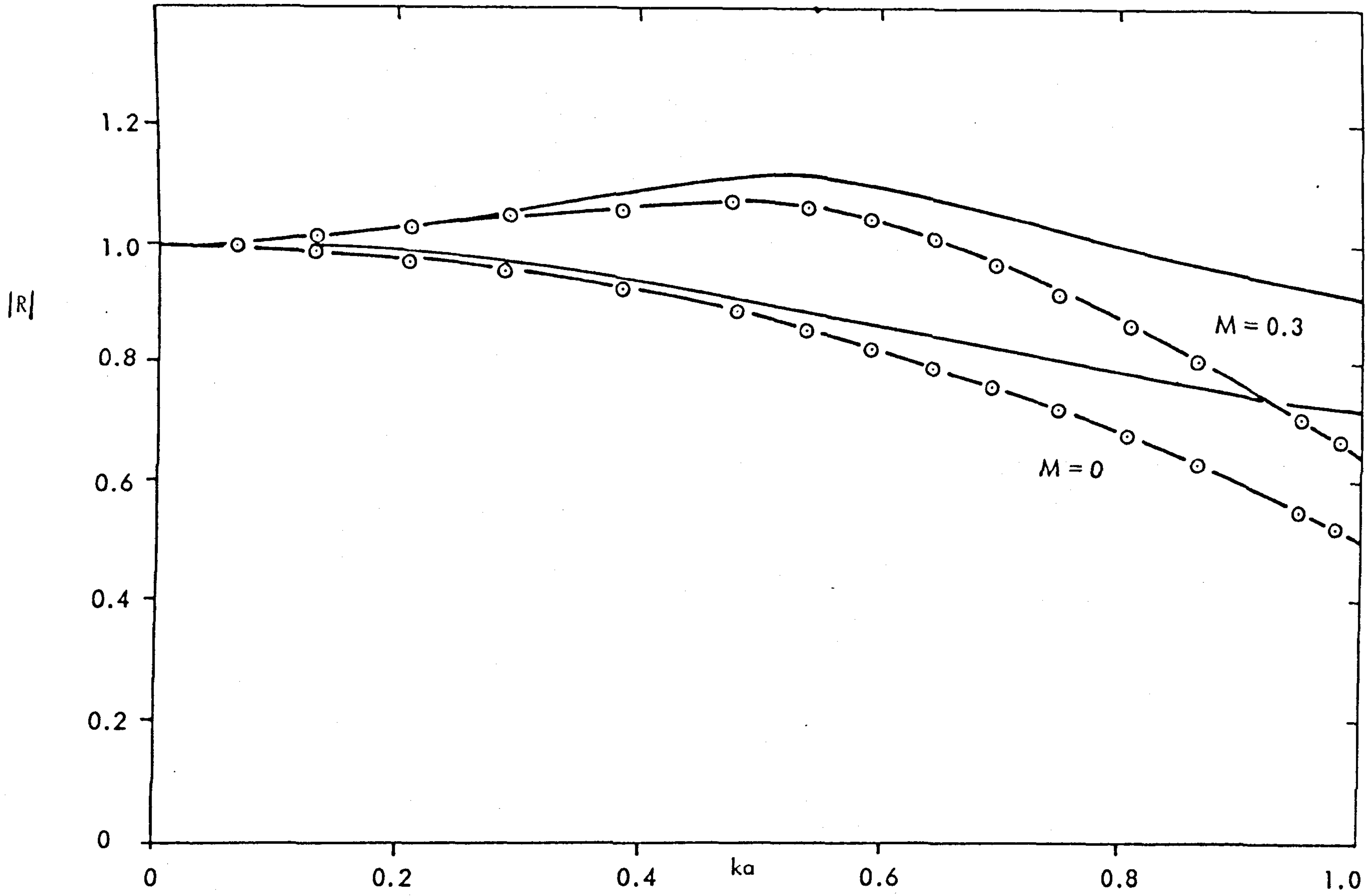


Fig. 4. Comparison of low frequency reflection coefficient with Munt's [18] computations. ——— present theory —⊙— Munt.

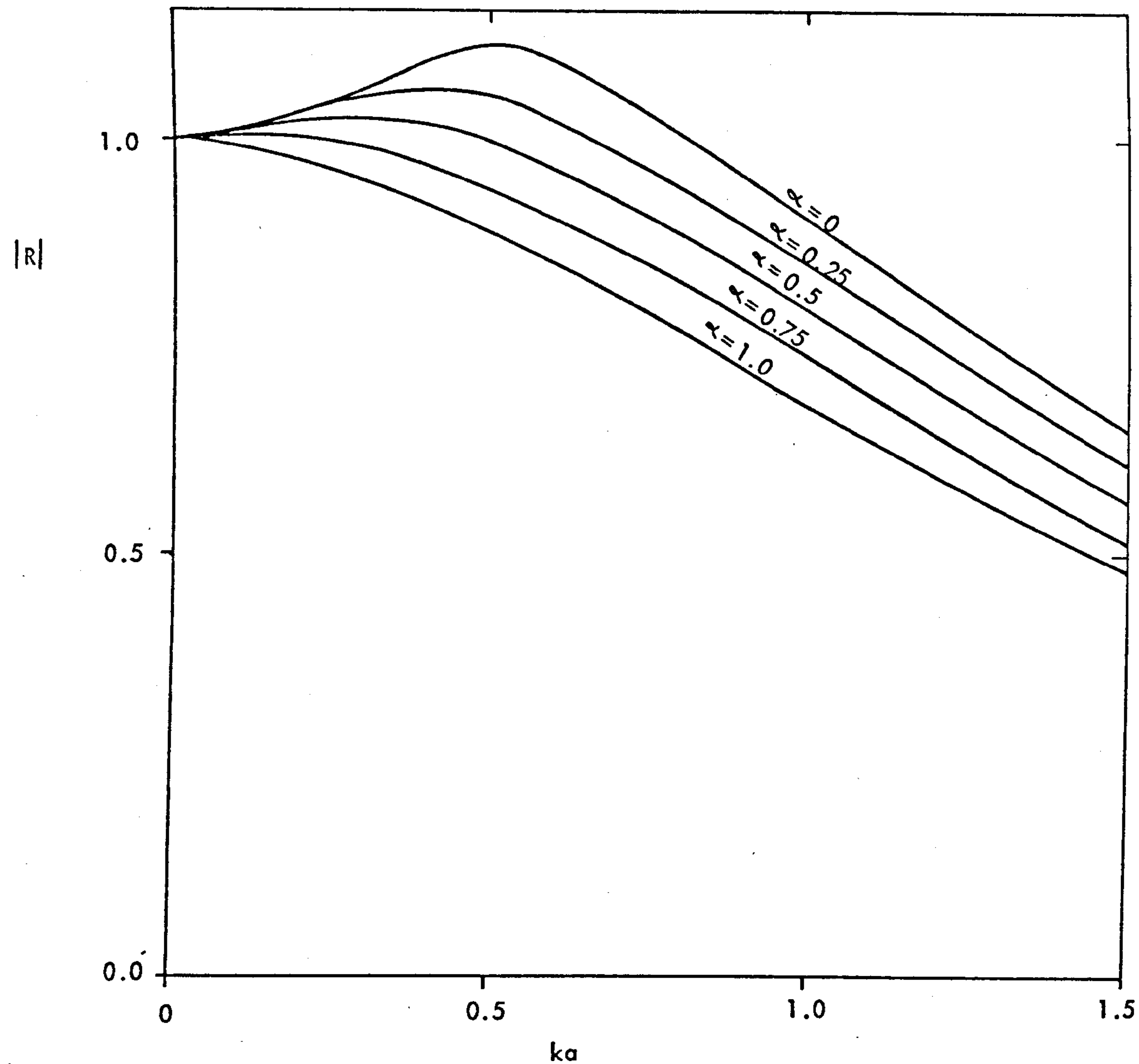


Fig. 5. Effect of external flow on reflection coefficient (after Munt [18]).

CHAPTER 4

THE RADIATION OF HIGH FREQUENCY SOUND OUT OF A JET PIPE

THE RADIATION OF HIGH FREQUENCY SOUND OUT OF A JET PIPE*

By

A.M. Cargill

Noise Department Rolls-Royce Ltd., Derby

And

Department of Applied Mathematical Studies,
University of Leeds

ABSTRACT

The chapter begins by discussing a simple model problem: the radiation of sound out of a semi-infinite cylindrical pipe, with internal and external flows. Two approximate high frequency solutions are presented, one based on Kirchhoff's approximation, and the other in the spirit of the geometrical theory of diffraction, and are compared with Munt's [1] exact solution by the Wiener-Hopf technique. The radiation from a jet emerging from an orifice in a baffle plate is also discussed. Next, the paper considers the differences between this simple model and an aero engine configuration, showing how the results are modified by the presence of a secondary flow (e.g.the fan stream on a turbo-fan engine), by the contraction of the final nozzle, and by the presence of many duct modes in the pipe.

*Much of the material in this paper appeared in preliminary form as "The Radiation of High Frequency Sound from a Jet Pipe" 2 , presented as paper 80-0970 at the A.I.A.A. 6th Aeroacoustics Conference held at Hartford, Connecticut, June 4 - 6 1980.

1. INTRODUCTION

This study is concerned with the propagation of high frequency sound out of the jet-pipes of aero engines. Typically, this sound has a ka value ($2\pi \times$ pipe radius / wavelength) greater than 20, and may be both tonal and broad-band in nature. The mean flow out of the nozzle is typically of a high subsonic Mach number (~ 0.8) and may be heated. The aim of this paper is to account for all the features of the propagation of this noise by using relatively simple approximate methods.

The paper begins by idealising the flow as a semi-infinite top-hat jet issuing from a cylindrical pipe in which there is a uniform mean flow. For a given incident duct mode, this problem has been solved exactly by Munt [1], using the Wiener-Hopf technique. His solution, while complete, is rather unwieldy and the hope here is to show that the main features of the radiation field can be adequately calculated using simpler approximate theories.

The first of these is Kirchhoff's approximation (see for example, Jones [3]). In this, the radiation is first expressed as a function of the field on the duct exit plane. This exit plane field is then determined by using the assumption that for sufficiently high frequencies (above the cut-off frequency of the incident duct mode), there is no sound reflected from the termination, so that the fluctuations on the exit plane are then those due to the incident field alone. Here it is shown that this Kirchhoff solution is identical to Munt's Wiener-Hopf solution at the peak angle of the radiation field. Elsewhere, agreement is less good, the Kirchhoff solution failing completely in the forward arc. But there, the radiation from these high frequency sources is in any event negligible, that this discrepancy has little practical importance. It may therefore be expected that our approximate solution will be as useful as the corresponding baffled duct solutions have been in the field of forward radiated compressor noise [4].

An additional feature of many real situations is that the jet pipe is shrouded by a secondary flow, from the fan stream in a turbofan engine for example. This may be handled by assuming that it is thick on a wavelength scale, so that the propagation of sound through it may be calculated by using geometric acoustics. Comments will also be given on the possibility of solving the radiation problem in other configurations, for instance that where sound radiates from the secondary jet pipe.

Following on from the Kirchhoff solution, the effect of a "baffle" (or "flange") around the pipe exit is next considered. The solution presented here corresponds to the well-known solution in the absence of a mean flow [4]. While the baffled duct solution, with a mean flow present, has no immediate application to aero engines, it may have considerable application to small scale test rigs.

The cylindrical pipe radiation problem is then tackled by a different method. This is the geometric theory of diffraction, originally due to Keller [5,6] and worked out for a pipe in the absence of flow by Felsen and Yee [7].Here, it is argued that at high enough frequencies, the radiated field is that due to the fields diffracted by the lip of the duct. There, diffracted fields are calculated by assuming that the pipe walls are locally plane. Compared with the Kirchhoff approximation, this method has the advantage that it is a formal high frequency limit to the exact solution and is therefore valid over a much wider range of far field angles. It will be used here mainly to illuminate the failings of the much simpler Kirchhoff solution.

All of the above solutions have concentrated on the radiation due to a single incident duct mode in the pipe.In practice, of course, many modes are present. For broadband noise these are uncorrelated and result in a field shape with none of the lobular character of the modal solution. It is therefore necessary to examine the summation of the fields from many such modes and discuss the effects of different types of sound source on the radiated sound. Finally,when the sound propagates through the jet turbulence some of its energy will be scattered and radiated in different directions. This was discussed in detail by Cargill [2]. The matter will receive no further discussion here, but will form the topic of a future paper.

2. KIRCHHOFF'S APPROXIMATION

The purpose of this section is to apply what is often known as Kirchhoff's approximation to the problem of the radiation of sound from a cylindrical duct with external and internal mean flows. In the absence of a mean flow, the principles of the method have been described by Jones [3]. Briefly, the method is applied as follows: Kirchhoff's theorem relates the acoustic field at any point to the fields on any surfaces. Therefore, to estimate the radiation from a pipe all we need to do is to estimate the pressures on these surfaces, which are taken as the pipe exit plane and the outer walls of the duct. The pressures are estimated by assuming that they are those due to the incident wave alone, so that the processes of diffraction and reflection by the pipe are neglected. Then Kirchhoff's theorem is applied and the far field radiation obtained. A useful comparison of the Kirchhoff approximation and the exact solution has been made by Butler [8] for the scattering of plane waves by a half plane. He shows that they are in good agreement in directions close to that of the incident waves, but in much less good agreement elsewhere.

In the situation here, illustrated in Figure 1, the above considerations must be modified somewhat to account for the presence of a mean flow. In this case, the counterpart of Kirchhoff's theorem has been given for a general flow by Ffowcs Williams and Hawkings [9] and (as in the jet here) for one containing vortex sheets by Dowling, Ffowcs Williams and Goldstein [10]. Clearly, one way of proceeding would be to apply their analogies, with the field variables on the surfaces in question determined from the incident field, having first linearised the equations in the fluctuating quantities, so that all the quadrupole sources vanish. Here, we shall adopt a different approach. The standard approach of Dowling et al [10] is inconvenient in two respects. First, it is written in terms of the pressure and velocity on the surfaces, rather than in a single field variable, and second, it was originally formulated to deal with surfaces that move with the flow and requires modifications here, where the surfaces are fixed. The initial formulation here will be of an analogue of Kirchhoff's theorem for a surface in a uniform mean flow, and use of this for the surfaces inside or outside the jet, together with an appropriate Green's function which will describe the propagation effects of the jet shear layer.

Before going on to do this it is important to discuss the importance of causality, and of the Kutta condition, vis à vis our solution. In determining the exact solution, Munt requires that his solution both obey a Kutta condition and be related to the incident duct mode. Here neither requirement will be enforced. They are in fact both irrelevant over the range of angles in which the Kirchhoff approximation is applicable. The exact solutions with and without a Kutta condition have been compared by Cargill [11,12]. He shows that they are identical near the angle of peak radiation. Here, there is no real means by which a Kutta condition may be applied as the Kutta condition is concerned with the behaviour of the field close to the edges of the pipe and that is precisely the region that is poorly approximated by the Kirchhoff solution. The causality requirement is also involved; in Munt's exact solution, it is argued that causality is important, and the solution is determined subject to that requirement. In the analogy of Dowling et al, it is argued, however, that for a turbulent jet causality is irrelevant, since the sound field of the turbulence itself is in a state determined by its own past history. In the case of sound propagation out of a pipe however, causality must presumably be obeyed, in some sense at least, as if one turns off the sound source, sound radiation must eventually cease. But from the point of view of the radiated sound field, the question of causality is academic in the Kirchhoff solution. Its only consequence there, is whether instability waves are included or not in the Green's function, and these instability waves do not radiate as a sound field decaying as $(1/r)$, so that they are totally irrelevant to the far field sound.

The first step is to develop an equation for the sources due to a surface moving in a uniform fluid. Linearising about the mean properies of the fluid, the pressure obeys the convected wave equation

$$\left[\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U\frac{\partial}{\partial x}\right)^2\right] p = 0 \qquad (1)$$

in the usual notation. One wishes to obtain an equation for the sound outside a surface. This surface is defined as $f=0$, with $f>0$ outside it and $f<0$ inside it. Therefore one multiplies (1) by $H(f)$ where H is the Heaviside unit step function ($H=1, f>0$; $H=0, f<0$) and obtains, on transferring H inside the derivatives,

$$\left(\nabla^2 - \frac{1}{c^2}\frac{D^2}{Dt^2}\right)(pH) = \nabla(p\nabla H) + \nabla H.\nabla p - \frac{1}{c^2}\frac{D}{Dt}\left(p\frac{DH}{Dt}\right) - \frac{1}{c^2}\frac{DH}{Dt}\frac{Dp}{Dt} \tag{2}$$

where $\frac{D}{Dt} = \frac{\partial}{\partial t} + U\frac{\partial}{\partial x}$.

This equation will now be used to obtain the far field radiation from the pipe in the Kirchhoff approximation. A surface S (Figure 1) is considered which is composed of the outer wall of the pipe, and the exit plane of the pipe. In the Kirchoff approximation, the fields on this surface are determined by assuming that they are the same as would exist on that surface in the absence of the termination. Thus on the wall of the pipe, the normal derivative of p is zero by definition, but the pressure itself is also zero by assumption. Therefore all sources on this surface can be neglected. On the part of S that forms the exit plane of the pipe, the values of p, and its derivatives are precisely those that occur in the incident wave. In detail, the following problem, as illustrated in Figure 1, will be examined. A pipe of radius a, contains a mean flow of density ρ_j sound speed c_j and velocity U_j. Outside the pipe, the corresponding mean flow parameters are ρ_o, c_o, U_o. The pipe contains an incident sound field of the form

$$p_i = J_m(j'_{mn} r/a)\exp(-ik_o x - im\theta + i\omega t), \tag{3}$$

where $J_m(j'_{mn}) = 0$ and J_m is the Bessel function of the first kind and order m. A Green's function $G(\underset{\sim}{x}, t \mid \underset{\sim}{x}_o, t_o)$ is defined by

$$\left(\nabla^2 - \frac{1}{c^2}\left(\frac{\partial}{\partial t} + U_j\frac{\partial}{\partial x}\right)^2\right)G = \delta(\underset{\sim}{x} - \underset{\sim}{x}_o)\,\delta(t-t_o) \tag{4}$$

for $r < a$, inside the jet, and

$$\left[\nabla^2 - \frac{1}{c_o^2}\left(\frac{\partial}{\partial t} + U_o\frac{\partial}{\partial x}\right)^2\right] G = 0 \tag{5}$$

outside the jet. In the derivation of the Green's function G it is assumed that the particle displacement and pressure are conserved across the jet shear layer. Then applying G to equation (2) one obtains

$$p = \int_V \left[(G\nabla p - p\nabla G).\nabla H - \frac{1}{c^2}\left(G\frac{Dp}{Dt} - p\frac{DG}{Dt}\right)\frac{DH}{Dt}\right] dV. \tag{6}$$

In this equation, ∇H and DH/Dt become, in the co-ordinate system of Figure 1,

$$\nabla H = \hat{\underset{\sim}{x}}\,\delta(x)\ , \quad \frac{DH}{Dt} = U_j\,\delta(x)\ , \tag{7}$$

where $\hat{\underset{\sim}{x}}$ is the unit vector in the x direction. From Appendix 1, the Green's function for a source $\exp[i\omega t - im\theta]\delta(x-x_o)\delta(r-r_o)/r$ can be written in Fourier integral form as

$$G = -\frac{i}{4}\int_{-\infty}^{+\infty} J_m(\nu r_o)\,T(k)\,H_m^{(2)}(wr)\exp[-ik(x-x_o)]\,dk. \tag{8}$$

Substituting for G in (6), and with the values of p and its derivatives taken as those in the incident wave (3), it is found that

$$p = -\frac{i}{4}\int_{-\infty}^{+\infty} T(k)\left[-i(k+k_o) + \frac{U_j}{c_j^2}\left((\omega - U_jk) - (\omega - U_jk_o)\right)\right] \cdot J_m(\nu r_o)H_m^{(2)}(wr)\exp[-ikx]\,r_o\,dr_o\,dk. \tag{9}$$

Integrating with respect to r_o using a standard integral relation [13, p.484] and then with respect to k using stationary phase (as in the description in the appendix) one obtains the final result of

$$p = \frac{-\pi a}{4\pi R}\exp\left[-\frac{i\omega R}{c_o} + i\omega t + im\frac{\pi}{2} - im\theta\right].D.I.T, \tag{10}$$

where $D = -ia(\chi + \chi_o)$; $\chi = k + U_j(\omega - U_jk)/c_j$,

$$I = \frac{-2\nu a\ J_m'(\nu a)\,J_m(j_{mn}')}{\left((\nu a)^2 - (j_{mn}')^2\right)}\ , \quad T = T(k)\ .$$

These factors are all evaluated at $k = \omega\cos\theta/(1+M_o\cos\theta)c_o$
They represent the following physical processes:
I describes the interference between waves emitted from different parts of the aperture, and is equivalent to the $(\sin x/x)$ pattern familiar in the radiation from a two-dimensional piston. Its principal features are that it has a strong peak at $va = j'_{mn}$, the mode-ray angle of the incident mode, and is zero at $va = j'_{mp}$, $p \neq n$ the mode ray angles of the other modes of the same azimuthal mode set.
T(k) represents the transmission properties of the jet. When there is no flow mismatch, T = 1, and at low frequency T tends to

$$(\rho_o/\rho_j)\left(1-(U_j-U_o)\cos\theta/c_o\right)^{-2}$$ (see [11, 12]).

Two other features are of interest. As the angle θ is varied the terms in the denominator of T(k) cause T(k) to oscillate. Physically this is the result of the interference between reflected sound waves inside the jet. For angles inside the cone of silence $\left(\theta < \cos^{-1}[c_o/(c_j+U_j-U_o)]\right)$
T becomes exponentially small as $ka\,(=\omega a/c_o) \to \infty$.
The poles of T(k) represent the instability waves of the jet.
D is a directivity function which varies relatively smoothly with angle. The principal point of interest here is that for the incident mode

$$\chi_o = \frac{\omega}{c_j}\left[1-(1-M_j^2)j'^2_{nm}c_j^2/\omega^2a^2\right] \tag{11}$$

so that χ_o is precisely zero when the incident mode is just cut on. The properties of the radiation field will now be examined in two stages. First, the result (9) will be compared with Munt's exact solution and then the variation in the fieldshape with jet conditions and frequencies will be discussed in detail.

Munt has solved the present problem exactly, and in integral form his solution can be written as

$$p = \frac{1}{2\pi i}\int\frac{\rho_o D_o^2 H_m^{(2)\prime}(wr)e^{-ikx}\,dk}{(k-k_o)wa\,H_m^{(2)\prime}(wa)\,K^+(k)\,K^-(k_o)} \tag{12}$$

where K^+K^- is the Wiener-Hopf split of the function.

$$K(k) = \frac{\rho_j\,wa\,J_m(va)D_j^2H_m^{(2)\prime}(wa) - D_o^2\rho_o\,va\,J_m'(va)H_m^{(2)}(wa)]}{\left(wa.va.\,H_m^{(2)\prime}(wa)\,J_m'(va)\right)}. \tag{13}$$

With some algebraic manipulation, it can be shown that the Kirchhoff solution presented here is simply Munt's Wiener-Hopf solution multiplied by $K^-(k)/K^-(k_o)$. Thus the two solutions are identical at the mode-ray angle. Now one notes that this factor, $K^-(k)/K^-(k_o)$ will not be strongly frequency dependent. This is because at high frequencies, it can be shown (see Section 4) that $K^- \sim \left(\frac{\omega}{c} + k\right)^{-1/2}$, in the absence of a mean flow and when the sound speeds and densities of the two media are identical. Then it follows that the factor $K^-(k)/K^-(k_o)$ departs from unity only slowly and on a scale independent of frequency, so that at these high frequencies the first few lobes of the radiation pattern will have levels identical to those predicted by the Kirchhoff theory. Beyond the first few lobes, the two solutions will not agree so well. But the solution there is of little practical interest since it will, in practice, be swamped by radiation from other modes.

Figure 2 gives a comparison of the two solutions, using a result from Munt's paper. This is for an $(m,n) = (4,1)$ mode at a ka of 11.7, and the relatively low Mach number of 0.14. We see that the agreement is exceptionally good until an angle of 100° is reached, where the curves diverge. Indeed there is, at higher angles, an extra zero in the fieldshape predicted by the Kirchhoff method. This additional zero may be explained as being associated with the point $\nu a = -j'_{mn}$ and would actually be a physically correct zero for an inlet radiating in the opposite direction. It is entirely spurious, as will be shown in the next section. For the case of no mean flow, such zeros are present over the whole of the $90^{\circ} - 180^{\circ}$ region, but when a flow is present, they are limited to a much smaller region.

Figure 3 demonstrates the powerful effect of refraction on single mode field shapes for an $(m,n) = (2,2)$ mode. An interesting feature of this and other results in that at the angle corresponding to the cone of silence (equivalent to $\theta = 0$ with no flow and the same densities), where $\nu a = 0$, the sound pressure level is finite — rather than zero, as it is in the absence of a mean flow. In Figure 3, the curves have been terminated where the method becomes invalid (see above). Figure 4 demonstrates the typical effect of the external flow. The change in the position of the lobes is predicted by simple ray refraction arguments, but more interesting is the apparent reduction in level everywhere, with no hint of the Doppler amplification phenomenon usually associated with flight effects on internal noise radiation. Lastly, Figure 5 demonstrates the effect of increasing the frequency for a particular mode at constant jet conditions. The main effect is that as the frequency is increased the number of lobes increases and the principal lobe approaches the cone of silence angle.

It is interesting to deduce some features of these field shapes from the analytic results. First, at the mode-ray angle, one clearly has $\mathcal{X} = \mathcal{X}_o$, $\mathcal{D} = -2i\mathcal{X}_o$; the interference term is $I = [J_m(j'_{mn})]^2(1 - m^2/j'^2_{mn})$ and the transmission factor T is

$$T = \frac{2i}{\pi \nu a} \cdot \frac{D_o^2 \rho_o}{D_j^2 \rho_j H_m^{(2)\prime}(wa)} . \tag{14}$$

Thus

$$p = \frac{-\pi a}{4\pi R(1 + M_o \cos\theta)} \cdot \left[\frac{2i}{\pi w a H_m^{(2)\prime}(wa)} \cdot \frac{D_o^2 \rho_o}{D_j^2 \rho_j} \cdot J_m^2(j'_{mn}) \left(1 - m^2/j'^2_{mn}\right) \right]_{k=k_o} \tag{15}$$

The presence of the factor D_j^2 indicates that for a collection of modes, the peak angles experience Doppler amplification in the same manner as the low frequency sound [11, 12]. At angles between these (where $J_m(\nu a) = 0$) it can be seen by inspection of T (A11) that this Doppler amplification is absent.

At the cone of silence angle, I becomes zero, with $\nu a J_m'(\nu a)$. But at this condition, T is infinite as $(J_m(\nu a))^{-1}$. These items, when multiplied together, give a finite result, so that the resulting sound field is finite here despite the fact that the interference factor becomes zero. Inside the cone of silence, one might at first expect that the same field would be exponentially small on account of the behaviour of T, giving the usual exponential decay found in jet noise studies, (see, for example, [14]). Here, this is not the case, as I becomes exponentially large at the same rate. Thus, there is <u>no sharp cut off of the sound</u> in that region, but a more gradual reduction in level. Physically,the reason for this is that while the radiation from a source inside a jet may decay exponentially inside the cone of silence, that from a pipe does not because the radiation field is dominated by the sound from source elements situated at the edge of the duct exit. Thus there is no exponential decay.

3. RADIATION FROM A BAFFLED DUCT WITH FLOW

Unlike the unbaffled duct, the baffled duct configuration of Figure 6 cannot be solved exactly. Indeed, even in the absence of a mean flow, no solution is yet known (Noble [15]). In that case, though, a number of approximate solutions can easily be obtained. First and most accurate, is the variational method of Levine and Schwinger [16 ,17], which is good at low frequencies, but converges slowly at high frequencies. At these high frequencies, an approximate solution may be obtained using the geometric theory of diffraction [5], with the edge diffraction calculated from two dimensional theory. This then provides an exact high frequency limit. Finally, there is the well known "baffled duct" theory (e.g.[4]), with the velocity on the aperture determined from the incident field alone. It is this theory whose counterpart, with a mean flow present, is to be given here.

The difference between this and the Kirchhoff solution lies in the choice of Green's function. In the absence of flow, one chooses a Green's function giving $\frac{\partial G}{\partial n} = 0$ over the whole of the baffle plate, and the aperture. One way of proceeding (at least in principle) would be to use such a Green's function here. Another is to require that $\frac{\partial G}{\partial n} = 0$ only on the baffle plate. The choice between the two is somewhat arbitrary, and only goes to show the ambiguity inherent in the Kirchhoff procedure. Of these two the latter is by far the easier to apply, as the Green's function cannot be easily obtained with $\partial G/\partial n = 0$ on both aperture and baffle. This difficulty is illustrated in Figure 7, where rays are shown from a hypothetical source point in the jet. Those rays that are reflected off the baffle plate are calculated by the Green's function with $\partial G/\partial n = 0$ on the baffle alone (at least in the high frequency limit), the ray direction corresponding to the reverse flow properties of the Green's function. But for rays that were reflected off the aperture plane, there is no simple determination of the reflected direction, which is why the Green's function is itself difficult to determine. However, the main difference between the results from the two Green's functions occurs inside the zone of silence, where the sound level is, anyway, small.

To use the solution, equation (6) is again employed. This time f is taken as the whole of the baffle and the aperture. Since the baffle is assumed rigid there can be no mean flow outside the jet, so that $\frac{DH}{Dt} = 0$ on the baffle. Also, (since the baffle is rigid) and by construction, $\partial G/\partial n = 0$ on the baffle. Thus, all the source terms on the baffle plate vanish, and it follows that

$$p = \int\left(\left[G\,\nabla p - p\,\nabla G\right] - \frac{1}{c^2}\left[G\frac{Dp}{Dt} - p\frac{DG}{Dt}\right]U_j\right)\ell_j\,dS, \tag{16}$$

where ℓ_j is the normal to the aperture, G is the Green's function defined in Appendix 2: it is of the form $G_o(M) + G_o(-M)$, where G_o is the previously defined unbaffled duct Green's function. Thus it follows that the radiation field will be

$$p = \frac{-\pi a}{4\pi R}\exp\left[i\omega t - i\omega R/c_o - im\theta\right]\left[D^+T(M)\,I(M) + D^-T(-M)\,I(-M)\right], \tag{17}$$

where $D^+ = -ia\left((k+k_o)(1-M_j^2) + 2\omega U_j/c_j^2\right)$

$D^- = -ia\left((k-k_o)(1-M_j^2) + 2\omega U_j/c_j^2\right).$

In the absence of a mean flow, it is clear that the (M) and (-M) components are equal, and further that $(D^+ + D^-) = -2iak_o$. When there is a mean flow present, it is clear that the principal lobes of the (+M) and (-M) components are in different places, and that at the mode ray angle D^- is small (O(M)), so that the previous solution is again obtained. In Section 2, it was found that the Kirchhoff solution was invalid in the forward arc $\left(\cos\theta < -M_j c_o/c_j(1-M_j^2)\right)$. A similar effect pertains here, but this time at angles close to the <u>rear arc axis</u>. This is a direct result of only having $(\partial G/\partial n) = 0$ on the baffle.

This completes our discussion of the baffled duct solution. Clearly, there is much further work that could be done on this topic, but unfortunately progress is likely to be very difficult, since exact solutions cannot be obtained. It is fortunate indeed that this solution has only limited application to practical situations.

4. A GEOMETRIC THEORY OF DIFFRACTION APPROACH TO DUCT RADIATION

At very high frequencies, it is well established that problems of diffraction may be treated by what has become known as the geometric theory of diffraction. The principles of this theory, as originally expounded by Keller [5,6] are that the only significant contributions to the far field sound from any scatterer are those from the direct field of the incident sound, and the waves scattered by the edges on which that field impinges. These scattered waves may be calculated as if the surface were flat with straight edges and the incident waves plane. Keller applied this analysis to the diffraction of sound by an aperture. He found that the direct field from the incident wave was cancelled out so that the only contributions to the far field were, to first order, those from the edges. Now in this simple case, it can also be proved that the geometric theory of diffraction gives results equal to the high frequency limit of the exact theory (see e.g[15]) . Thus unlike the Kirchhoff theory, the geometric theory does contribute a formal high frequency limit to the diffraction problem. A useful comparison between the geometric and Kirchhoff theories as applied to a half plane has been given by Butler [8]. He finds that the theories agree well near the edge of the shadow zone but that they disagree in the deep shadow zone.

The radiation from a pipe does not appear to have been treated in this way in the literature, but there is a relatively large body of work on the associated problem of the reflection of sound from the pipe termination. Felsen and Yee [7] treat the problem using a pure geometric acoustics method. This has the disadvantage that the solution may become singular on shadow boundaries. To overcome this limitation, Boersma [18,19] has developed the theory so that it is valid along these shadow boundaries and so that it accounts for the multiple diffraction of sound by the pipe edge.

The major difference between the present and previous analysis is the inclusion of a mean flow. This has two consequences. First, to generate a geometric theory of diffraction with a mean flow present, we must use canonical solutions with the mean flow included [20], [21] . Second, we have to account for the reflection of sound off the jet shear layer.

There are two ways in which we can proceed in order to derive the high frequency solution to the duct radiation problem. The first, and more accurate, involves the Fourier transform of all quantities; thus the perturbed pressure in the pipe is set equal to

$$p_j = A H_m^{(2)}(vr) + B H_m^{(1)}(vr) , \tag{18}$$

and that in the ambient medium to

$$p_o = C H_m^{(2)}(wr). \tag{19}$$

The mixed (pipe plus shear layer) boundary value problem is then solved by the Wiener-Hopf technique. Matching pressures across the interface $r = a$, and with an incident pressure on the wall of the pipe $p = p_i \exp\left[-ik_o x + i\omega t\right]$, one obtains the equation

$$\left[A H_m^{(2)}(va)\right]^+ + \left[B H_m^{(1)}(va)\right]^+ - \frac{p_i}{i(k-k_o)} = \left[C H_m^{(2)}(wa)\right]^+ \tag{20}$$

where the + signs indicate half range Fourier transforms as described in Appendix III. The way to proceed from here is to note that if the "scattered fields" $AH^{(2)}(vr)$, $CH^{(2)}(wr)$, propagating away from the edge of the pipe, are small, so is the"reflected" field, $B H_m^{(2)}(wr)$. This field is now rewritten as a Fourier integral

$$\left[BH^{(1)}(va)\right]^+ = \frac{1}{2\pi}\int_0^\infty dx \int_{-\infty}^{+\infty} B(k')H_m^{( )}(v(k')a)e^{-ik'x+ikx}\, dk'\, dx. \tag{21}$$

$$= \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{B(k')\, H_m^{(1)}(v(k')a)\, dk'}{(k-k')} ; \tag{22}$$

that is, it may be expressed as an integral of quantities similar to that arising from the incident wave. The equation must then be solved iteratively. First, the approximation B = 0 is made and a solution obtained, from which is calculated the next value of B, and so on.

There is, however, an alternative way of looking at the problem; one observes that at high frequencies, the waves will be travelling outwards. Thus to a good approximation they may be treated as two dimensional. Therefore one may find the amplitudes of scattered waves close to the edge by application of two-dimensional diffraction theory. This is described in Appendix III. For larger radii one uses the principle of the conservation of energy in a radial direction, which dictates that the amplitude is proportional to $(a/r)^{1/2}$. The secondary diffracted fields, that arise when these first order diffracted fields strike the edge, may be calculated in a similar manner.

Thus in this example one has from the incident mode in the pipe (3) an incident pressure on the wall of the pipe

$$p_i = J_m(j'_{mn}) \exp\left[-ik_0 x - im\phi + i\omega t\right]. \tag{23}$$

For the top edge, this will give a scattered field (using the results of Appendix III)

$$p_0 = \frac{J_m(j'_{mn})\, e^{-iw(y-a)} \cdot 2L^-(k)}{2i(k-k_0)\,w(k)\,L(k)\,L^-(k_0)} . \tag{24}$$

In the far field one must add the sound from the top and bottom edge plus any waves reflected off the jet shear layer (Figure 7). The effects of multiple diffraction will be ignored since the corresponding fields are $O(ka)^{-1}$ smaller than the primary diffracted fields. For angles well beyond the geometric acoustics shadow of the pipe exit, only sound from the upper edge is present, since the wave from the bottom edge will be shielded by the top edge. This explains the spurious zeros seen in the Kirchhoff solution (e.g.Figure 2) obtained earlier, and show that solution to be grossly inappropriate in that region.

For a plane wave incident on the jet interface, the reflection coefficient $\mathbb{R}$ and the transmission coefficient $\mathbb{T}$ are related, by continuity of pressure ,through

$$1 + \mathbb{R} = \mathbb{T}, \tag{25}$$

and by continuity of particle displacement through

$$1 - \mathbb{R} = \alpha \mathbb{T}, \tag{26}$$

where $\alpha = (w\rho_j D_j^2 / v\rho_o D_o^2)$

Thus

$$\mathbb{R} = (1-\alpha)/(1+\alpha), \tag{27}$$

and $$\mathbb{T} = 2/(1+\alpha). \tag{28}$$

One is now in a position to add up all the waves that reach the far field. In doing so one must assume that there is a phase change of $\frac{\pi}{2}$ each time a wave passes through the axis of the jet. There is therefore in crossing the jet a total phase change of $\frac{\pi}{2} + 2va$. Also it should be noted that at emission the phases of the waves at the "top" and "bottom" of the duct differ by $m\pi$. Thus in the far field, the pressure will be, with the phase referenced to that of the direct waves from the upper edge,

$$p = p_o \Big[1 - \underbrace{\alpha \mathbb{T} e^{im\pi} \Big(1 + \sum_{m=1}^{\infty} \mathbb{R}^{2m} \beta^{2m}\Big)}_{\text{upper edge}} - \underbrace{\alpha \mathbb{T} \mathbb{R} e^{2im\pi} \beta^2 \Big(1 + \sum_{m=1}^{\infty} \mathbb{R}^{2m} \beta_{2m}\Big)}_{\text{lower edge}} \Big] \tag{29}$$

where $\beta = \exp[-2iwa + i\pi/2]$.

Summing the geometric series, and substituting for $\mathbb{R}$ and using (27), (28), it follows that

$$p = p_o \frac{(1 - \beta e^{im\pi})}{(1 - \mathbb{R}\beta e^{im\pi})}. \tag{30}$$

This summation of terms is rather similar to that employed by Ffowcs Williams and Berman [22] who studied instabilities of a two-dimensional jet. In our treatment, any instability waves generated on reflection have been ignored, since they are, for this semi-infinite jet, of no consequence to the far field radiation.

Now substituting $\mathbb{R} = (1-\alpha)/(1+\alpha)$, one has

$$p = p_o \frac{(1+\alpha)\sin(va - m\pi/2 - \pi/4)}{\cos(va - m\pi/2 - \pi/4) - i\alpha\sin(va - m\pi/2 - \pi/4)}. \tag{31}$$

Next, at high frequencies, one may replace all the sinusoidal components by their Bessel function equivalents (e.g. $\cos(z - m\pi/2 - \pi/4) \to \sqrt{\frac{\pi z}{2}} J_m(z)$. Substituting for p_o, it is found, after some algebra, that the pressure in the far field is just $L^-(k)/L^-(k_o)$ times that derived using the Kirchhoff approximation. Thus, when compared with the exact solution, this one, with the Kirchhoff solution, is precisely correct at the mode ray angle. Unlike the Kirchhoff solution, it should be valid everywhere, as $(\omega a/c_o) \to \infty$, except perhaps at the points where $\nu a = 0$. There, a more sophisticated analysis must be used, as it must at the shadow line in the rear arc. Such an analysis would account correctly for any apparent discontinuities in the field found in the first approximation. The final alteration to the analysis would be the addition of terms depending on the second order diffracted fields. These are, though, of order $(\omega a/c_o)^{-1}$ smaller than the first order fields and (see e.g. Keller [5,6])are not of any great significance at high frequencies.

An unfortunate feature of this analysis, which perhaps limits its application, is that the functions $L^+(k)$, $L^-(k)$ cannot be determined analytically, for the usual case where jet and ambient conditions are different. Nevertheless, in the limit of $U_j, U_o \to 0$, $c_j \to c_o$, $\rho_j \to \rho_o$, we can determine $L^-(k)/L^-(k_o)$. In that limit, $L^-(k) \sim \sqrt{\frac{\omega}{c_o} + k}$ so that at k = $\cos\theta$

$$\left(L^-(k)/L^-(k_o)\right) = \left((1+\cos\theta)/(1+\cos\theta_o)\right)^{1/2}; \tag{32}$$

where θ_o is the mode ray angle of the incident mode in the pipe. Rewriting (32) as $\left(\cos(\theta/2)/\cos(\theta_o/2)\right)$ it is clear that this ratio is close to unity when $\theta \sim \theta_o$, and only becomes very large or small as $\theta, \theta_o \to 180°$ where, as has been pointed out, the theory is in any event invalid. It does however, illustrate the assertion made earlier, that the ratio of the split functionsis not strongly frequency dependent at very high frequencies.

The reflected field inside the pipe will now be derived to the same approximation. Adding the contributions from the two edges, together with the associated reflections off the duct walls gives, using the notation of (29),

$$p = \alpha p_o \Big[\Big(e^{-i\nu(a-y)} + e^{i\pi - i\nu(4a-(a-y))} + e^{i\pi - i\nu(4a+(a-y))} \ldots \Big) + e^{-im\pi}\Big(e^{-i\pi/2 - i\nu(2a-(a-y))} + e^{-i\pi/2 - i\nu(2a+(a-y))} + \ldots\ldots \Big) \Big], \tag{33}$$

in which the first term refers to waves originally emitted from the top edge, and reflected off the walls of the pipe, the individual terms representing sound from the images of the edge; and the second term likewise refers to waves from the bottom edge. Re-arranging the terms in (33) gives

$$p = \alpha p_0 \left[e^{-iva - i\left(\frac{m\pi}{2} + \frac{\pi}{4}\right)} \cos\left(vy + \frac{m\pi}{2} + \frac{\pi}{2}\right) \right] \left[1 + \sum_{1}^{\infty} e^{-im\pi - 2inva} \right], \tag{34}$$

and summing the series, the result

$$p = \frac{\alpha p_0 \cos\left(vy - (m\pi/2 + \pi/4)\right)}{\sin\left(va - (m\pi/2 + \pi/4)\right)} \tag{35}$$

is obtained.

To a good approximation ,(see above) the cosines may be replaced by Bessel functions, and then substituting for p_o and α gives

$$p = \frac{J_m(j'_{mn}) . 2L^-(k)}{2i\, L(k)\, L^-(k_0)(k - k_0)\, w(k)} \left\{ \frac{w\, D_j^2 \rho_j}{v\, D_0^2 \rho_0} \right\} \frac{J_m(vy)}{J_m'(va)} . \tag{36}$$

The Fourier transform may be inverted with contributions coming only from the poles at $J_m(va) = 0$. Noting that, near v = 0,

$$va\, J_m'(va) \doteq J_m(va)\left(1 - \frac{m^2}{(va)^2}\right) \chi a\, (k - k_1)\, a \, , \tag{37}$$

the summation of the residue terms gives

$$p = \sum_{n=1}^{\infty} \frac{e^{ik_n x}}{2L(k_n)} \left\{ \frac{L^-(k_n)}{L^-(k_0)} \right\} \frac{1}{(k_n - k_0)} \left\{ \frac{D_j^2 \rho_j}{D_0^2 \rho_0} \right\} \frac{1}{\chi_n a} \frac{J_m(v_n a)}{(1 - m^2/(va)^2_n)\, J_m((va)_n)} . \tag{38}$$

Several features of this result are noteworthy. First, the reflected field is apparently infinite when $k_n = k_0$. This is seen in other theories of this type (e.g. Felsen and Yee [7] or Boersma [18,19]) and is the result of ignoring multiple diffraction. The other main feature is that as frequency is increased, the reflected field becomes progressively lower, as $(ka)^{-1}$, so that, away from the cut-off frequencies $(k = k_n)$ one can safely ignore reflections at high frequencies.

Clearly in this section we have only hinted at the possibilities of this technique. Indeed its main merit would seem to be that it can produce the correct high frequency limit of the field diffracted by the jet pipe in situations where an exact solution is not just difficult but unobtainable analytically. Such a case might be the important practical one of an aero engine with twin stream exhausts. In that case an exact closed form solution by the Wiener Hopf technique is impossible (unless the exists of the inner and outer streams are coplanar), but a good approximation can be found by present methods. Additionally, it should be easy to incorporate into this model the effects of the nozzle shape, and the exact flow profiles, because these do not affect the canonical diffraction problem governed by the $L^{\pm}(k)$ functions but only the propagation of the signals after scattering. Finally, it should be re-emphasised that this solution is a formal high frequency limit to the exact solution and should therefore be superior to the Kirchhoff solution everywhere.

5. MULTIMODE FIELD SHAPES

In the preceding section, attention was concentrated on the field shapes of single modes. In practice the signal at any one frequency will be composed of a large number of modes, so that one should be interested in the field shapes of combinations of modes. Typically, a smooth field shape is produced (e.g Figure 8) which for realistic jet conditions peaks at around 70° to the jet axis, just outside the cone of silence.

To illustrate the effects of mode averaging, the field due to a uniform distribution of uncorrelated monopole sources in a duct will first be considered. Then consideration in less detail will be given to the field of other sources (for example a tip source), the effects of changes in duct area for a given upstream source, and the possible effects of source correlation.

Consider a source, strength $Q(r,\phi)\,\delta(x)$, situated in a rigid-walled duct of radius a. It is easily shown (by methods described, for example in Morse and Ingard [23]) that the field at a distance x from the source is

$$p_d = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{\infty} \frac{\exp[-ik_{mn}x - im\phi]\, J_m(j'_{mn}r/a)}{2i\chi_{mn}\cdot\pi a^2 J_m^2(j'_{mn})(1-m^2/j'^2_{mn})^2} \cdot \int_0^{2\pi}\int_0^a Q(r_0,\phi_0)\, J_m(j'_{mn}r/a)\, e^{-im\phi} r_0\, dr_0\, d\phi_0 \,. \tag{39}$$

This result is derived by, for example, splitting the source up into its circumferential and radial modal components, finding the field due to each and summing the results. Assume now that the source distribution consists of uncorrelated monopoles, so that

$$\langle Q(r_1,\phi_1)\,Q(r_2,\phi_2)\rangle = Q\,\delta(\phi_1-\phi_2)\,\delta(r_1-r_2)/r_1 \tag{40}$$

where the brackets $\langle\ \rangle$ indicate an ensemble average. To obtain the far field radiation, use the previously derived formula (10) in the Kirchhoff approximation, to find the far field radiation pressure as

$$p = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{\infty} \frac{-\pi a \quad D_{mn} T_{mn} \quad I_{mn} \exp[-i\omega R/c_0 - i k_{mn} L + i\omega t + im\pi/2 - im\theta]}{4\pi R(1+M_0\cos\theta).2i\chi_{mn}.\pi a^2 J_m^2(j'_{mn})(1-m^2/j'^2_{mn})}$$

$$\cdot \int_0^{2\pi}\int_0^a Q(r_0,\phi_0) J_m(j'_{mn} r_0/a) e^{im\phi_0} r_0 dr_0 d\phi_0. \qquad (41)$$

for a duct of length L. Next, form the mean square of the pressure, which by virtue of (40) comes from uncorrelated modes,

$$|p|^2 = \sum_{m=-\infty}^{+\infty}\sum_{n=1}^{\infty} \frac{(\pi a)^2 |D_{mn}|^2 |I_{mn}|^2 |T_m|^2 Q \exp[-2(I_m k_{mn})L]}{(4\pi R)^2 4|\chi_{nm}|^2 \pi a^2 J_m^2(j'_{mn})(1-m^2/j'^2_{mn})(1+M_0\cos\theta)^2}. \qquad (42)$$

Summation of this expression over all the modes is rather complicated, and the details are relegated to the Appendix IV. In expression (42), the summation need only be taken over the propagating modes, as all the other modes are exponentially small and anyway, for the cut-off modes the Kirchhoff radiation formula must provide a poor representation of the field. Summing then over the radial modes, one obtains, from Appendix IV,

$$|p|^2 = \sum_{m=0}^{M_{max}} \frac{(\pi a)^2 \; |T_m|^2 \quad 2\left[(J_m'(va))^2 - J_m''(va)J_m(va)\right]}{(4\pi R)^2(1+M_0\cos\theta)^2 \pi a^2}. \qquad (43)$$

The major point of difficulty in this summation is how to account for the variation in angle and m of T_m, where, to recapitulate, T_m is given by

$$T_m = \frac{-2}{\pi va} \cdot \frac{1}{(J_m'(va) H_m^{(2)}(wa) - \alpha H_m^{(2)\prime}(wa) J_m(va))}. \qquad (44)$$

The general features of the terms in the summation are clear. The T_m term oscillates as the angle θ is varied for a given m. The Bessel functions in the numerator of (43) have the property that for a given m, the corresponding term is only significant beyond an angle such that $(va/m) \gg 1$. As there is no way of summing the series exactly, approximations must be made. First, note that in the range where the terms are significant one can approximate T_m by expanding the Bessel function in the high frequency limit. Thus

$$|T_m|^2 = \frac{4wa}{va\left[\cos^2(va - m\pi/2 - \pi/4) + \alpha^2 \sin^2(va - m\pi/2 - \pi/4)\right]}. \qquad (45)$$

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Next, average over the ripples in $|T_m|$ so that as the average of $(\cos^2 x + \alpha \sin^2 x)^{-1} = \alpha^{-1/2}$, one obtains

$$|T_m|^2 = \frac{4\,va.wa.\pi^2}{\pi^2 v^2 a^2 .\, 4\alpha} \tag{46}$$

$$= \left(D_o^2 \rho_o / D_j^2 \rho_j \right). \tag{47}$$

This has reduced the summation (43) to

$$|p|^2 = \sum_{1}^{M_{max}} \frac{(\pi a)^2 \quad D_o^2 \rho_o \quad Q \,.\, 2}{(4\pi R(1+M_o\cos\theta))^2 \, D_j^2 \rho_j \,.\, \pi a^2} \left[(J_m'(va))^2 - J_m''(va) J_m(va) \right] \tag{48}$$

Finally, as m_{max} tends to infinity, the Bessel functions can be summed to give ½, and as in further problems m_{max} is proportional to ka, that limit can be used to yield

$$|p|^2 = \frac{\pi a^2 \quad Q}{(4\pi R)^2 (1+M_o\cos\theta)^2 \left(1-(U_j-U_o)\cos\theta/c_o\right)^2} . \tag{49}$$

Now this result is identical to that which can be obtained, with much less effort, by a simple application of geometric acoustics (see,e.g.,studies of sources in free jet wind tunnels by Tester and Morfey [24]. In addition to the usual geometric acoustics field, it has been shown here (Appendix IV)that the field will fall off within a region of angular width of the rear arc shadow line of the jet pipe.

Consider now the effect of other sources in the jet pipe, and the effects of area variation. The general rule should be obvious from the result proved in the appendix, namely that the mode eigenvalue may be directly translated into the radial wave number va. Thus if one considers an uncorrelated tip source with

$$\left\langle Q(r_1,\phi_1)\, Q(r_2,\phi_2) \right\rangle = Q \,\frac{\delta(r-a)}{r}\, \frac{\delta(r_1-r_2)}{r_2}\, \delta(\phi_1-\phi_2) \tag{50}$$

the difference from the previous result is that the amplitude of the (m, n)th mode changes from

$$\int_0^a J_m(j_{mn}' r/a)\, J_m(j_{mn}' r/a)\, r\,dr = \left(1-\frac{m^2}{j_{mn}'^2}\right) J_m^2(j_{mn}')\, a^2/2 \tag{51}$$

to

$$\int_0^a J_m(j_{mn}' r/a)\, J_m(j_{mn}' r/a)\, \delta(r-a)\, dr = J_m^2(j_{mn}'). \tag{52}$$

Thus the amplitudes are, for a given mode set m, approximately $\left(1-\left(m^2/(\nu a)^2\right)\right)$ times those from the previous source. Similar rules apply to other sources, in that those producing predominantly high radial order modes will produce far field radiation patterns that peak at a large angle to the jet axis. Furthermore, when liners are fitted inside a jet pipe, the attenuation is strongly dependent on the mode cut on ratio $\left((1-M_j^2)\,j_{mn}'^2\,c_j^2/\omega^2 a^2\right)$ (see,e.g. Rice 25). But by the rules just described, this translates directly into νa and thence to the angle of radiation. Similarly, non monopole sources will give field shapes depending on the source type; for example, an axial dipole source will produce a field shape biased in the downstream direction. An alternative explanation of these effects in the context of ray theory has been given by Kempton [26,27].

The question of area change must now be discussed. Here, the principal feature of transmission is that energy is conserved if the duct area varies sufficiently slowly(see,e.g.[28,29]). In this case the energy flux for a uniform axial flow is given by the formulae of Blokhintsev [30]. Applying these to one mode

$$p = A\,J_m\left(j_{mn}'\,r/a\right)\exp\left(-ik_{mn}x\right), \qquad (53)$$

one obtains an energy flux in the axial direction of

$$E = \frac{A^2 J_m^2(j_{mn}')\,\pi a^2\left(1-(m^2/j_{mn}'^2)\right)\chi_{mn}\left(1-M_j^2\right)}{\rho_j c_j\left(1-(M_j\chi_{mn}c_o/\omega)\right)}. \qquad (54)$$

In this formula, one can interpret the j_{mn}' as described above, in the sense that $j_{mn}=\nu a$. To illustrate the effect of the area variation, consider the case where a mode travels down a converging nozzle. If the mode is fairly near cut-on at the source, it will become (unless the flow nears sonic speed) less well cut on (χ decreasing) when it reaches the nozzle exit. Thus, the pressure amplitude will be correspondingly increased and the amplitude in the far field increased. An additional effect will be that as the mode ray angles of the cut-on modes are increased by the nozzle convergence, then the field shape will be biased towards higher angles.

Obviously the question of modal addition is very complex and there are a lot of items that have not been discussed, but which may be important - for example the effects of secondary nozzles and flows. However these can all be handled using the methods described above.

6. CONCLUSIONS

In this paper, a number of approximate methods for calculating the radiation of high frequency noise from aero-engine exhaust systems have been presented. While this has involved a fairly complete description of the principal methods there are clearly many items not treated at all; for example, the practical effects of the velocity profile and exhaust systems found on real engines. These can, however, all be treated using the methods described.

Two particularly important topics which have not found mention here are the effects of axial variation in velocity profile and scattering by jet turbulence. The former is very difficult and can probably only be handled conveniently by true ray theory. A start on this has been made by Candel [31]. The topic of scattering is also complex, and was discussed in the original version of this paper [2]. We hope to return to it in a later paper.

ACKNOWLEDGEMENT

The author wishes to thank the Directors of Rolls-Royce Limited for permission to publish this paper, and the Science Research Council for an Industrial Studentship. He also wishes to thank Professor D.G. Crighton for his comments on the draft and Mr. S. Bashforth for computing the results in Section 2.

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APPENDIX I

THE GREENS FUNCTION FOR A CIRCULAR JET

In this appendix, the Green's function for a circular jet will be determined. While similar Green's functions have been derived elsewhere, in jet noise studies (see for example: Dowling et al [10]), the derivation is given in full to establish the notation used throughout the paper.

The Green's function, G, satisfies

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{m^2}{r^2} - \frac{1}{c_j^2}\left(i\omega + U_j\frac{\partial}{\partial x}\right)^2\right)G = \frac{\delta(r-r_0)}{r}\delta(x), \quad \text{(A1)}$$

inside the jet and

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{m^2}{r^2} - \frac{1}{c_0^2}\left(i\omega + U_0\frac{\partial}{\partial x}\right)^2\right)G = 0, \quad \text{(A2)}$$

outside the jet. Defining the Fourier transform of G by

$$\bar{G}(k) = \int_{-\infty}^{+\infty} e^{ikx} G(x)\, dx , \quad \text{(A3)}$$

one finds that $\bar{G}$ satisfies

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{m^2}{r^2} + v^2\right)\bar{G} = \frac{\delta(r-r_0)}{r_0}, \quad \text{(A4)}$$

inside the jet and

$$\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r} + \frac{m^2}{r^2} + w^2\right)\bar{G} = 0, \quad \text{(A5)}$$

outside the jet, where

$$v^2 = \frac{(\omega - U_j k)^2}{c_j^2} - k^2 , \quad w^2 = \frac{(\omega - U_0 k)^2}{c_0^2} - k^2 , \quad \text{(A6)}$$

the branches of v, w, being chosen to ensure that $\text{Im}\, v, \text{Im}\, w < 0$. Note that in the absence of any jet boundaries, (A4) may be solved to give

$$\left.\begin{aligned} \bar{G} &= -i\frac{\pi}{2} H_m^{(2)}(vr_0)\, J_m(vr) , \quad r < r_0 \\ &= -i\frac{\pi}{2} H_m^{(2)}(vr)\, J_m(vr_0), \quad r > r_0 \end{aligned}\right\} \quad \text{(A7)}$$

(see, for example, Morse and Ingard [23]).

To obtain the complete solution with the jet boundaries present, one adds on an additional part, $-\frac{i\pi}{2} R(k) J_m(\nu r_0) J_m(\nu r)$ inside the jet and $-\frac{i\pi}{2} T(k) \cdot J_m(\nu r_0) J_m(\nu r)$ outside the jet. In these expressions, the solutions have been chosen to give finite pressures on the axis, and outgoing decaying waves at infinity, and R,T, may be regarded as reflection and tranmission coefficients for the shear layer. They are determined by matching the two solutions to conserve pressure and particle displacement across the jet boundary. Hence matching the pressures, it follows that

$$H_m^{(2)}(\nu a) + R J_m(\nu a) = T H_m^{(2)}(wa), \tag{A8}$$

and matching displacements

$$H_m^{(2)\prime}(\nu a) + R J_m^{\prime}(\nu a) = \alpha T H_m^{(2)\prime}(wa), \tag{A9}$$

where

$$\alpha = \frac{\rho_j D_j^2 w}{\rho_0 D_0^2 \nu}, \quad D_0 = (\omega - U_0 k), \; D_j = (\omega - U_j k). \tag{A10}$$

Solving equations (A8), (A9), one obtains

$$T(k) = \frac{-2}{\pi \nu a}\left[\frac{1}{J_m^{\prime}(\nu a) H_m^{(2)}(wa) - \alpha J_m(\nu a) H_m^{(2)\prime}(wa)}\right]. \tag{A11}$$

The complete Green's function is obtained by inverting the Fourier transform as

$$G = -\frac{i}{4}\int_{-\infty}^{+\infty} J_m(\nu r_0) H_m^{(2)}(wr) T(k) e^{-ikx} dk, \quad r > a. \tag{A12}$$

The far field pressure is then obtained by evaluating (A12) by the method of stationary phase to give

$$G = -\frac{e^{-i(\omega R/c_0) + im\pi/2}}{4\pi R(1+M_0\cos\theta)}\Big[2\pi T(k) J_m(\nu r_0)\Big]_{k=\frac{\omega\cos\theta}{c_0(1+M_0\cos\theta)}}. \tag{A13}$$

The Green's function when the jet has a continuous velocity profile will also be discussed. In particular, there is interest in the case where the source is situated in a region of uniform mean flow, which is itself surrounded by another region in which the mean flow varies slowly on a wavelength scale. There are two ways of solving this problem. First, note that each wave number component G(k) must obey an equation of the form

$$\left(\frac{1}{rc^2}\frac{\partial}{\partial r}rc^2\frac{\partial}{\partial r} + \frac{2(dU/dr)k}{(\omega - Uk)^2}\frac{\partial}{\partial r} + W^2(r) - \frac{m^2}{r^2}\right)\bar{G} = 0. \quad \text{(A14)}$$

A number of authors have studied this equation in the context of jet noise. At high frequencies, it may be solved (see for example Balsa [14]) by the WKB method. Writing the equation as (with ρc^2 constant)

$$\left(\frac{\rho D^2}{r}\frac{\partial}{\partial r}\frac{r}{\rho D^2}\frac{\partial}{\partial r} + \frac{m^2}{r^2} + W^2\right)\bar{G} = 0, \quad \text{(A15)}$$

one sets

$$\tilde{G} = \left(r/\rho D^2\right)^{1/2}\bar{G}, \quad \text{(A16)}$$

and finds that with an error or order $(1/k^2)$, $\bar{G}$ satisfies

$$\left(\frac{\partial^2}{\partial r^2} + W^2\right)\tilde{G} = 0. \quad \text{(A17)}$$

Thus by the WKB method, one obtains

$$\bar{G} = \left(\frac{\rho D^2}{r}\right)^{1/2} exp\left[-i\int W(r)\,dr\right]. \quad \text{(A18)}$$

Matching this to the previously derived Green's function, and using a suffix 'a' to denote the conditions at the edge of the region of constant mean flow, it is clear that the effect of the variation in the mean flow is to cause the Green's function to be multiplied by a correction factor, for each wavenumber component

$$\left[\frac{(W/\rho D^2)_a}{(W/\rho D^2)_\infty}\right]^{1/2} exp\left[-i\int_a^\infty \left(W(r) - W_\infty\right)dr\right]. \quad \text{(A19)}$$

Thus the effect of the mean flow variation is to change slightly the amplitude of the radiated sound and to alter its phase.

As an alternative to the above analysis, note that when sound of sufficiently high frequency propagates through a variable mean flow it does so according to geometrical acoustics. A general discussion of the conditions under which this remains true has been given by Lighthill [32]. The correct expression for the energy flux per unit area, in this situation, is that given by Blokhintsev [30], and may be written as

$$E_r = (p + \rho U u_x) u_r , \tag{A20}$$

where u_x, u_r are the x, r components of the velocity perturbation quantities, and observing that any wave always travels at the local phase speed, one obtains precisely the law of propagation derived earlier, (A18), for a wave of given axial wavenumber k. Thus the two approaches give exactly the same result.

APPENDIX II

THE GREEN'S FUNCTION FOR A BAFFLED DUCT

In this appendix, a discussion is given of the Green's function for a baffled duct, that is for a point source in the jet, and with the condition $(\partial G/\partial n)=0$ on $x=0$. Note that this condition is only imposed for $r>a$. To do so for the whole of the r plane seems unnecessary for the problem under consideration. As the mean flow cannot penetrate the baffle, the solution is restricted to $M_0 = 0$.

From Appendix 1, the Green's function in the absence of a flange is, for $r>a$

$$G_0 = -\frac{i}{4}\int_{-\infty}^{+\infty} H_m^{(2)}(wr)\, J_m(vr_0)\, T(k)\, e^{-ik(x-x_0)}\, dk \,. \tag{A21}$$

To obtain a Green's function with zero gradient normal to the $x=0$ surface, one must add another part to G_0 of the form

$$G^* = -\frac{i}{4}\int_{-\infty}^{+\infty} H_m^{(2)}(wr)\, A(k)\, e^{-ikx}\, dk \,. \tag{A22}$$

If this is done naively one obtains $G=0$ everywhere, which is clearly wrong. To get the correct result, one must reverse the sign of k in (A21), and then the condition $(\partial G/\partial n)=0$ on $r>a$ leads to

$$A(k) = J_m(v(-k)r_0)\, T(-k)\, e^{-ikx_0}, \tag{A23}$$

so that

$$G^* = -\frac{i}{4}\int_{-\infty}^{+\infty} J_m(v(-k)r_0)\, T(-k)\, e^{-ik(x+x_0)}\, dk. \tag{A24}$$

Now in v, and T, k always appears as either k^2, or kM, so that

$$V(-k,M) = V(k,-M)\;,\; T(-k,M) = T(k,-M)\,. \tag{A25}$$

Thus the complete Green's function is

$$G = G_0(x,r|x_0,r_0,M) + G_0(x,r|-x_0,r_0,-M), \tag{A26}$$

where G_0 is the unbaffled Green's function.

At this stage it is important to emphasise that this Green's function only satisfies $\partial G/\partial n = 0$ on $r > a$, not on $r < a$. The Green's function satisfying $(\partial G/\partial n) = 0$ on all parts of x = 0 is much more difficult to obtain.

If one had allowed the mean flow in $r > a$ to be non-zero, the problem would have been solved by replacing the change in variable from k to $-k$ by one from k to k^* which ensures that $w(k) = w(k^*)$. The required change is $k^* = -k - 2M_0 k_0/(1-M_0^2)$. The problem is then solved in the manner described above, and a slightly different value of G obtained.

APPENDIX III

DIFFRACTION OF A PLANE WAVE

BY A SEMI-INFINITE VORTEX SHEET

In this appendix, the theoretical work of Crighton and Leppington [20] and of Morgan [21] is summarised. Both these authors use the Wiener-Hopf technique [15] to find the diffracted field when a plane wave is incident upon a semi-infinite vortex sheet, Figure 10.

A plane wave

$$p = p_i \exp\left[-ik_o x + i\omega t\right], \tag{A27}$$

is incident from the lower side of the half plane. The scattered pressures on the two sides of the plate are taken as p_o, p_j, and the half range Fourier transforms

$$P^{\pm}(k) = \int_{-\infty}^{+\infty} p(x)\, e^{ikx} H(\pm x)\, dx \tag{A28}$$

are introduced.

Since the pressures on each side of the vortex sheet obey convected wave equations, it follows that will have the forms

$$\left.\begin{aligned} P_o &\propto \exp\left[-iwy\right] & y &> 0 \\ P_j &\propto \exp\left[+ivy\right] & y &< 0 . \end{aligned}\right\} \tag{A29}$$

Here the suffixes 0, j are used to signify the regions where $U = U_o$, U_j and w, v are defined as in (A6). The branch cuts of w, v, are chosen so as to ensure that all waves decay as $|y| \rightarrow \infty$.

The boundary conditions are that pressure is continuous across the vortex sheet, so that

$$P_i^+ + P_j^+ - P_o^+ = 0 \quad ; \tag{A30}$$

and that the displacement of the sheet is zero on the plate, so that if the Fourier transform of this displacement is Z(k), then

$$Z^-(k) = 0 . \tag{A31}$$

From the momentum equation, Z and P are related by

$$\left.\begin{aligned} -(\omega - U_0 k)^2 Z &= \frac{i w}{\rho_0} P_0 \\ -(\omega - U_j k)\, Z &= -\frac{i v}{\rho_j} . P_j , \end{aligned}\right\} \tag{A32}$$

so that (A30) can be rewritten as

$$K(k) Z^-(k) + \frac{-P_i}{i(k-k_0)} = (P_i^- - P_j^-) , \tag{A33}$$

where

$$K(k) = -(D_j^2 \rho_j + D_0^2 \rho_0)/wv . \tag{A34}$$

Then K(k) may be split as a product of two factors K^+, K^-, analytic in appropriate half planes. If this splitting is done with w nearly real (ie. w = w+i0), then by the usual Wiener-Hopf arguments the solution may be obtained as

$$Z^-(k) = \frac{+ \quad P_i}{i(k-k_0)\, K^+(k) K^-(k_0)} . \tag{A35}$$

This solution does not obey a trailing edge Kutta condition, or satisfy causality (see [20], [21]). If these are to be satisfied, one must add on an extra part, incorporating an instability wave proportional to exp $(-ik_I x)$, where k_I is the wavenumber of the Kelvin-Helmholtz instability of the vortex sheet. Adding this to the above solution, and cancelling the part giving singular behaviour near the edge, one obtains

$$Z^-(k) = \frac{P_i (k_0 - k_I)}{i(k-k_0)(k-k_I)\, K^+(k) K^-(k_0)} . \tag{A36}$$

It is convenient to introduce the notation

$$L^+(k) = K^+(k)(k-k_I) , \quad L^-(k) = -K^-(k)/(k-k_I) . \tag{A37}$$

Then one can write the pressure in the region $y > 0$ as

$$P_0 = \frac{P_i . 2L^-(k) e^{-iwy}}{2iw(k-k_0) L(k) L^-(k)} , \tag{A38}$$

and in the region $y < 0$ as

$$P_j = -\alpha P_0 ; \tag{A39}$$

where $\alpha = (w D_j^2 \rho_j / v D_0^2 \rho_0)$.

APPENDIX IV

MODAL SUMMATIONS

It is required to sum (from (42)) an expression for the mth circumferential mode , of the form

$$\sum_{n=1}^{N} \left| \frac{\chi_{mn} + \chi(k)}{\chi(k)} \right|^2 \frac{1}{(1 - m^2/j'^2_{mn})\left[(va)^2 - j'^2_{mn}\right]} . \tag{A40}$$

Consider an integral of the form

$$I = \frac{1}{2\pi i} \oint \left| \frac{\chi(\mu) + \chi(y)}{\chi(\mu)} \right|^2 \frac{J_m(\mu)\, d\mu}{J_m'(\mu)\, (y^2 - \mu^2)^2} . \tag{A41}$$

taken round the contour shown on Figure 11. In the subsequent analysis, one may ignore the contributions from the $[(\chi(\mu) + \chi(y))/\chi(\mu)]$ term. This is because the major part of the integral comes from the region around $\mu = y$ and further, the error due to ignoring that term is only the same as that arising from differences between the Kirchhoff method and the exact solution.

One can evaluate the integral (A42) in two ways; from the poles of $(J_m'(\mu)(y-\mu))$ and from the contributions from the arcs of the contour. Taking the poles first, take(for definiteness)the contour to pass through the zeros of $J_m(\mu)$ between the Nth and N+1th zeros of $J_m'(\mu)$. Denote the zeros of $J_m'(\mu)$ by μ_n. Then with

$$J_m'(\mu) \sim - J_m(\mu_n)\left(1 - m^2/\mu_n^2\right)(\mu - \mu_n) , \tag{A42}$$

the contribution to I from these poles is

$$I_1 = \sum_{n=1}^{N} \frac{1}{(1 - m^2/\mu_n^2)(y^2 - \mu_n^2)} . \tag{A43}$$

The contributions from the poles at $y = \mu_n$ are then given by

$$I_2 = \frac{d}{d\mu}\left[\frac{J_m(\mu)}{J_m'(\mu)\,(y+\mu)^2}\right]_{\mu=y} + \frac{d}{d\mu}\left[\frac{J_m(\mu)}{J_m'(\mu)(y-\mu)}\right]_{\mu=-y} \tag{A44}$$

$$= \frac{1}{4y^2}\left[1 - \frac{J_m''(y)\,J_m(y)}{(J_m'(y))^2} - \frac{J_m(y)}{y\, J_m'(y)}\right] + \frac{1}{4y^2}\left[1 - \frac{J_m''(-y)J_m(-y)}{(J_m'(-y))^2} + \frac{J_m(-y)}{y\,J_m'(-y)}\right] \tag{A45}$$

Now using the well known continuation formulae

$$J_m(y) = e^{im\pi} J_m(-y), \; J_m''(y) = e^{im\pi} J_m''(-y), \tag{A46}$$

one finds that this part of the integral gives

$$I_2 = \frac{1}{2y^2}\left[\frac{(J_m'(y))^2 - J_m''(y)J_m(y)}{(J_m'(y))^2}\right] . \tag{A47}$$

One can now proceed to show that to leading order in $(1/ka)$ the value of the integral taken along the contour is negligible, so that the series (A43) simply sums to I_2. That is,

$$I = -\sum_{1}^{N} \frac{1}{(1-m^2/\mu_n^2)(y^2-\mu_n)^2} + \frac{1}{4y^2}\left[1 - \frac{J_m''(y)J_m(y)}{(J_m'(y))^2}\right] . \tag{A48}$$

If the four arcs in the integral those at infinity do not contribute, and the other two are equal, giving, with as the value of the intercept with the real axis,

$$I = \frac{1}{2\pi i}\int_{-\infty}^{+\infty} \frac{2i\, J_m(\alpha+ix)\, dx}{J_m'(\alpha+ix)(y^2-(\alpha-ix)^2)} . \tag{A49}$$

In this analysis it is assumed that the frequency is high, so that one can replace the Bessel functions by their high frequency expansions;for example,

$$J_m(\alpha+ix) = \sqrt{\frac{2}{\pi(\alpha+ix)}} \cos\left(\alpha+ix-\frac{m\pi}{2}-\frac{\pi}{4}\right) . \tag{A50}$$

Thus, with α chosen to give $J_m(\alpha)=0$,

$$I \sim \frac{1}{\pi i}\int_{-\infty}^{+\infty} \frac{\tanh x\, dx}{((x-i\alpha)^2+Y^2)^2} . \tag{A51}$$

In this integral, one can take $\tanh x = 1$, with an error of order exp (-2α) The integral can be expressed in the form

$$I = S(\alpha) - S(-\alpha) , \tag{A52}$$

where

$$S(\alpha) = \frac{1}{\pi i}\int_0^{\infty} \frac{dx}{((x-i\alpha)^2+y^2)^2} \tag{A53}$$

$$= \frac{1}{2\pi i y}\frac{d}{dy}\int_0^{\infty} \frac{dx}{((x-i\alpha)^2+y^2)} , \tag{A54}$$

$$= \frac{1}{2\pi i y^3}\left[\frac{\pi}{2} - \tan^{-1}\left(-\frac{i\alpha}{y}\right) + \frac{i\alpha y}{(y^2-\alpha^2)}\right]. \qquad \text{(A55)}$$

Two cases, must now be considered;where $(y/\alpha) \gg 1$, which corresponds to angles of radiation well away from the shadow line where y (or $Va(k)$) $= j'_{mN_{max}}$, and $(y/\alpha) \sim 1$, corresponding to the case of a radiation angle very near the limiting angle. Write

$$\tan^{-1}\left(-\frac{i\alpha}{y}\right) = Z, \qquad \text{(A56)}$$

so that after some algebra

$$Z = \frac{1}{2i}\operatorname{Log}\left(\frac{\alpha+y}{\alpha-y}\right) + \frac{\pi}{2}, \qquad \text{(A57)}$$

and

$$S = \frac{1}{2\pi y^3}\left[\operatorname{Log}\left(\frac{\alpha+y}{\alpha-y}\right) - \frac{\alpha y}{(\alpha^2-y^2)}\right]. \qquad \text{(A58)}$$

Thus,

$$I = S(\alpha) - S(-\alpha)$$
$$= \frac{1}{2\pi y^3}\left[\operatorname{Log}\left(\frac{(\alpha+y)}{(\alpha-y)}\right)^2 - \frac{2\alpha y}{(\alpha^2-y^2)}\right]. \qquad \text{(A59)}$$

When y is not too near α, then this component is of order $(1/ka)$ smaller (i.e. $1/y$) than that from the poles. When $y \to \alpha$, the second term of (A59) is large so that

$$I = \frac{-1}{2\pi\alpha^3}\left(\frac{\alpha}{\alpha-y}\right). \qquad \text{(A62)}$$

Clearly this becomes singular near $\alpha = y$ The reason for this can be seen by looking at the integral. When $\alpha \to y$ it is clear that one of the poles will, at some stage, intersect the contour. This results in the integral along the contour becoming very large. The correct value of the summation at this boundary must then be taken as (1/2) that given above, and this summation will change from its typical value over a region of angular width $O(1/ka)$.

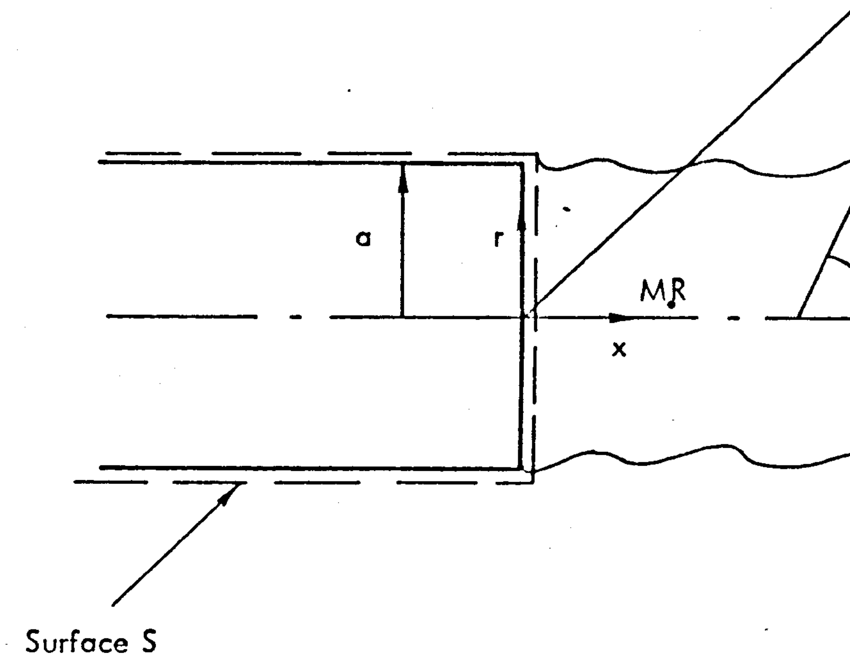
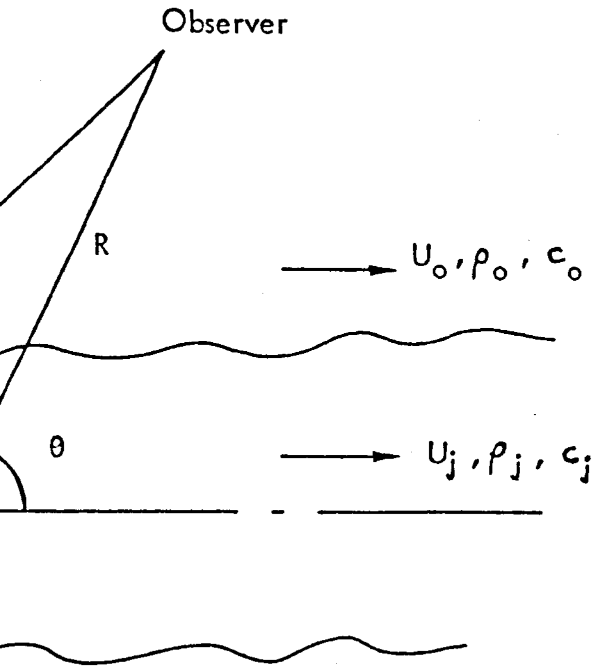


Fig. 1. Radiation from unbaffled duct.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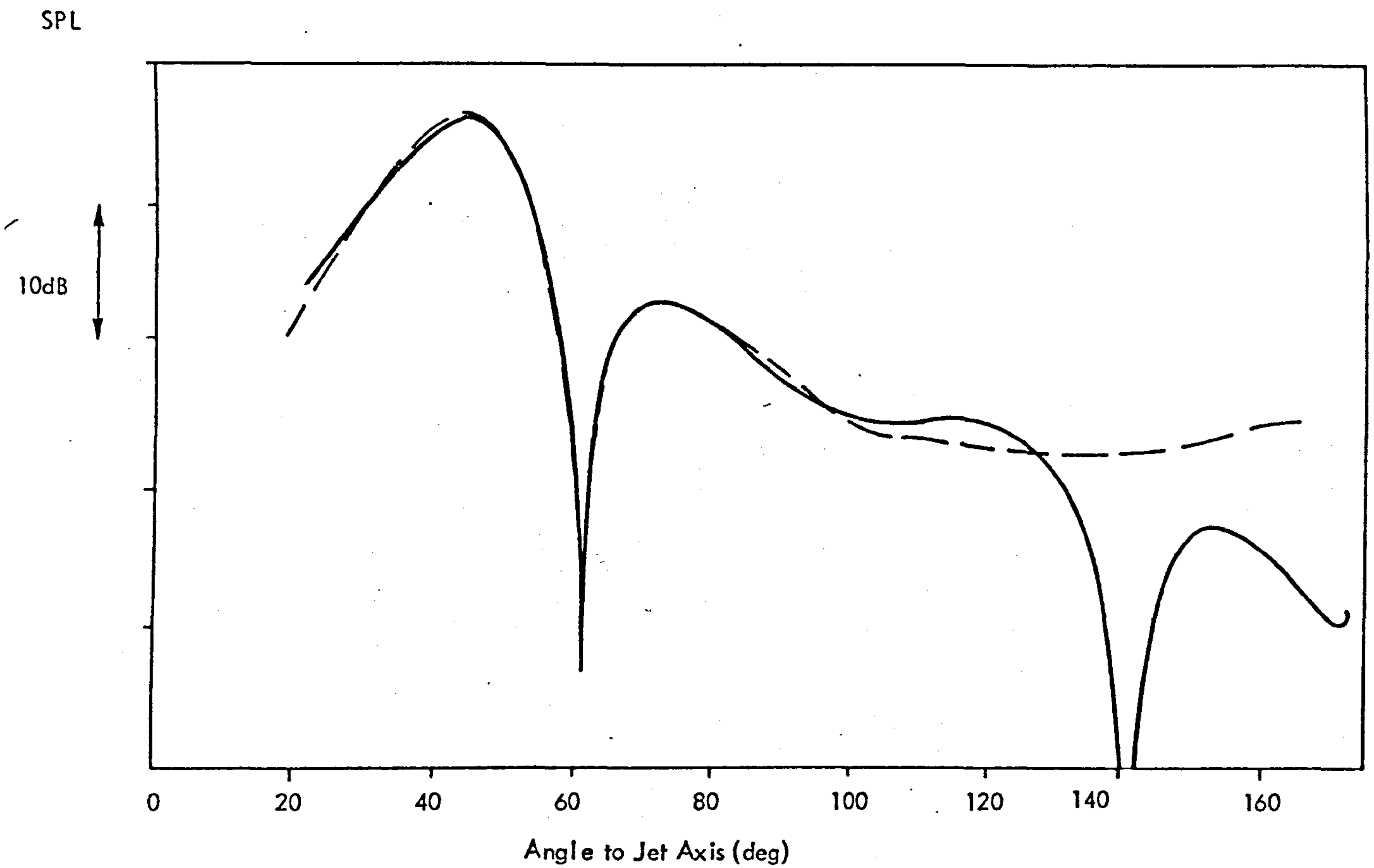


Fig. 2. Comparison of Kirchhoff and exact solutions - (4,1) mode; $M_o = 0$ $M_j = 0.4$, $c_j/c_o = 1$, $ka = 11.7$.

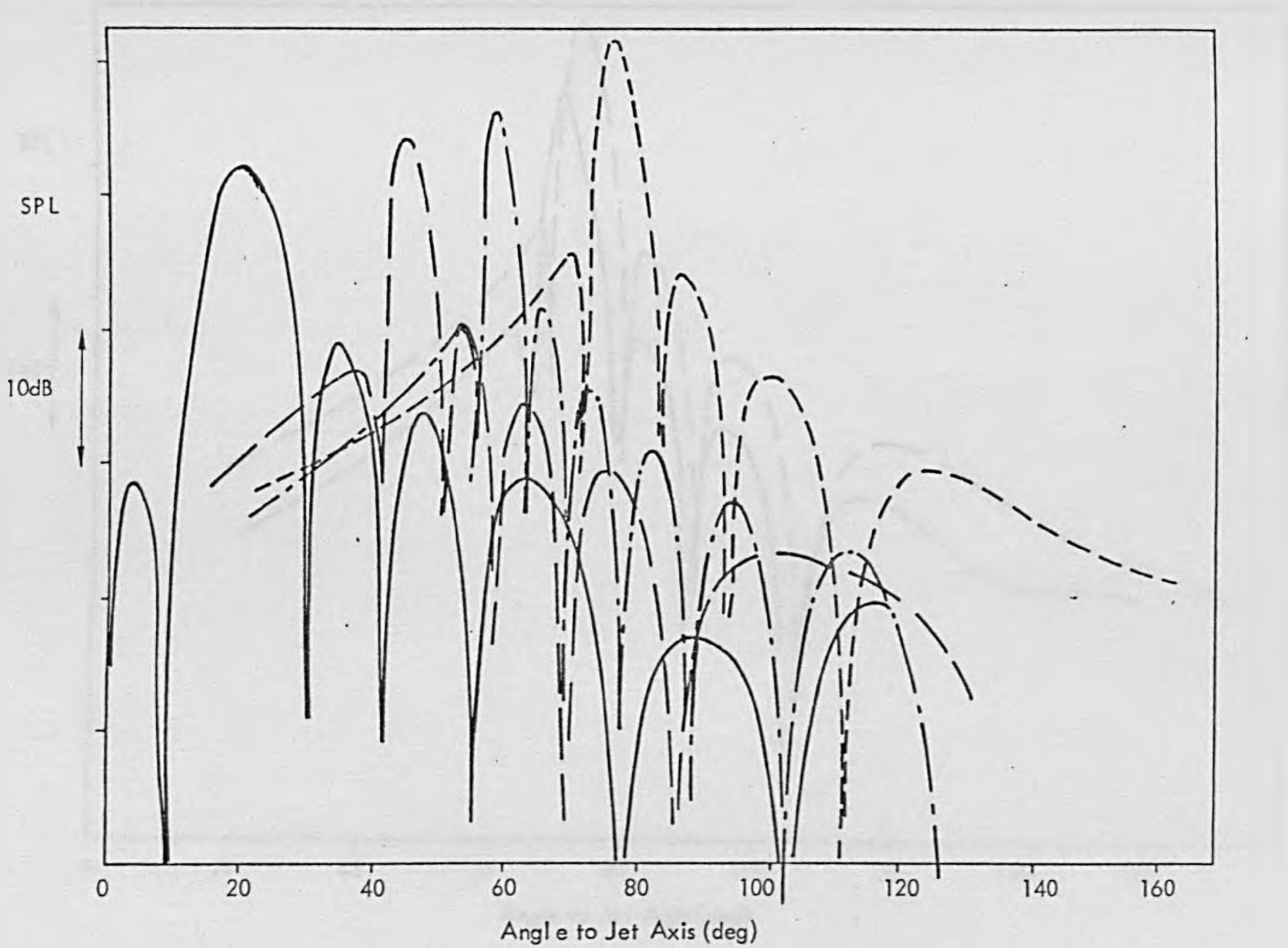


Fig. 3. Effect of jet Mach number and temperature ratio - (2,2) mode $M_o = 0$, $ka = 20$.
——— $M_j = 0$, $c_j/c_o = 1.0$; — — $M_j = 0.3$; $c_j/c_o = 1.0$
—·—·— $M_j = 0.7$, $c_j/c_o = 1.0$; - - - - $M_j = 0.7$, $c_j/c_o = 1.5$

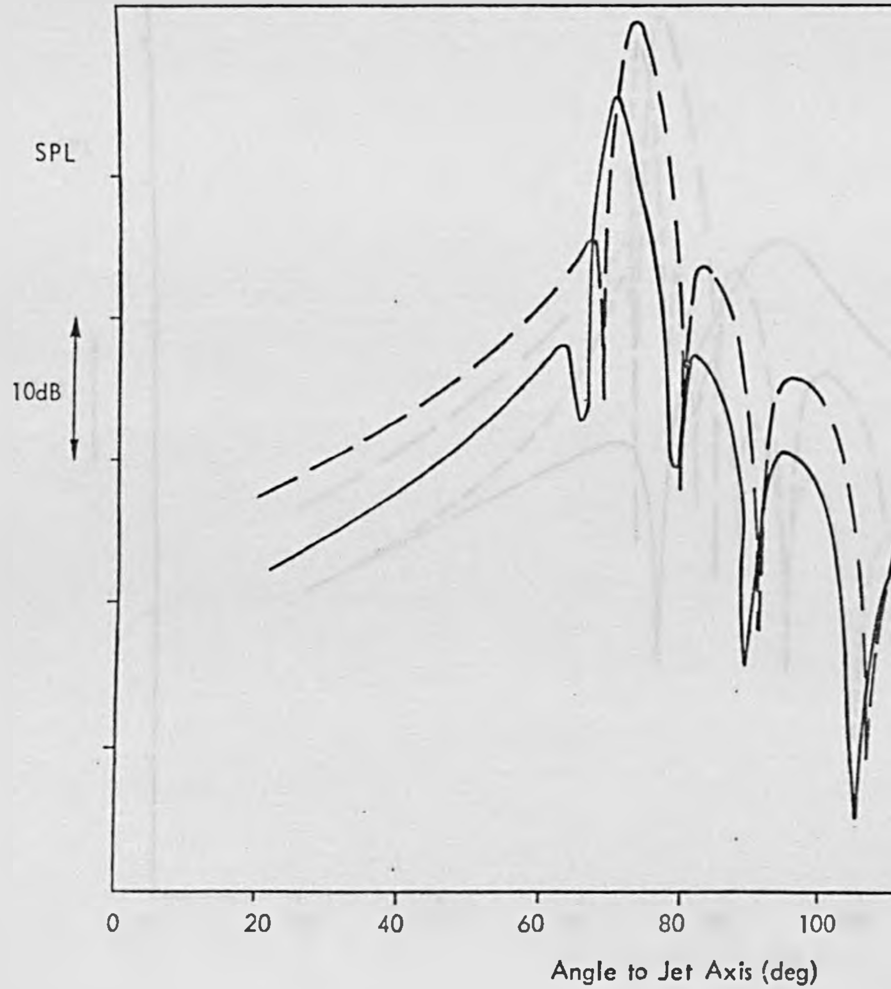


Fig. 4. Effect of external flow - (2,2) mode, M_j = 0.
—— —— $M_o = 0$, ———— $M_o = 0.25$.

7, c_j/c_o = 1.5, ka = 20

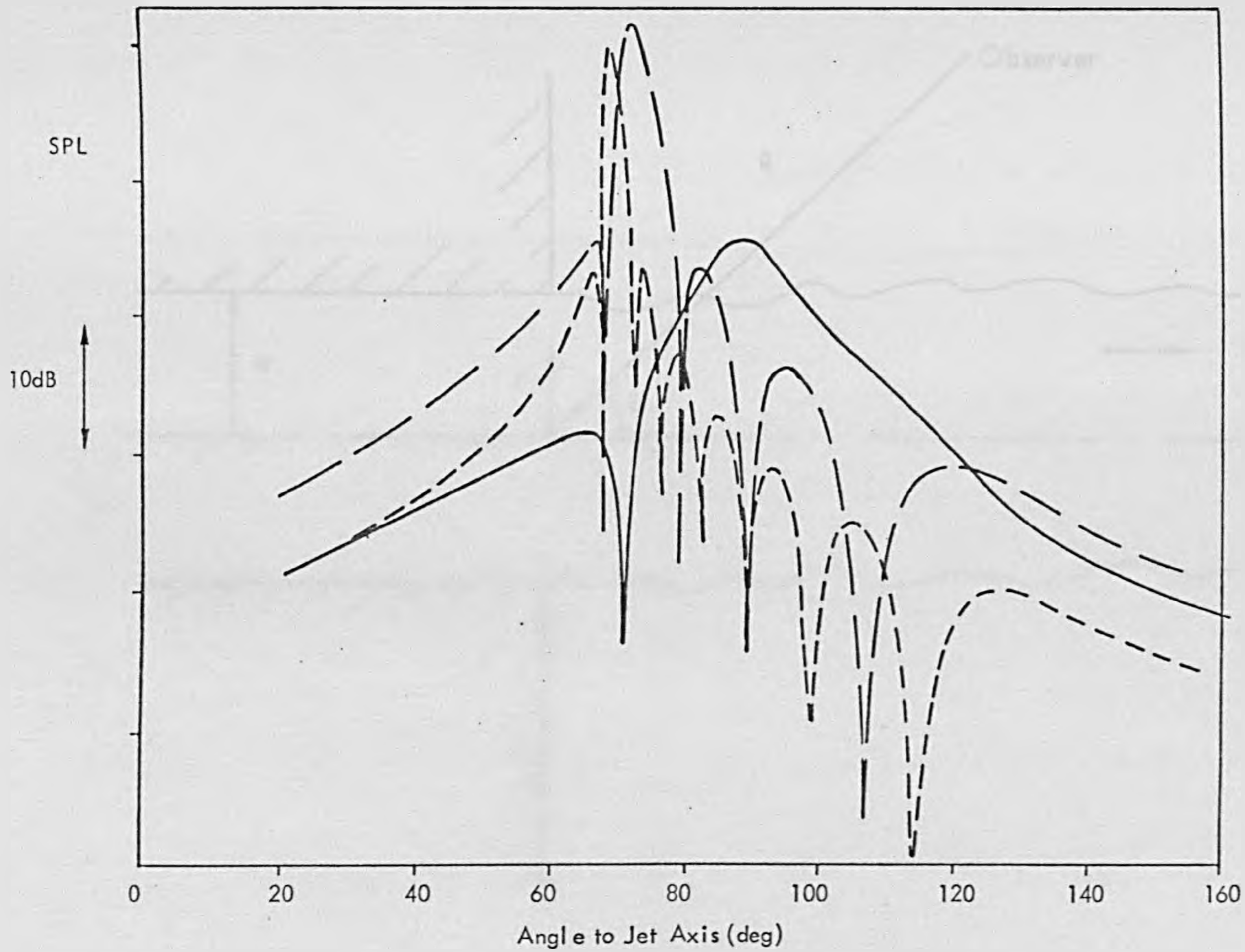


Fig. 5. Effect of frequency - (2,2) mode, $M_o = 0$, $M_j = 0.7$, $c_j/c_o = 1.5$
——— ka = 0, — — ka = 20, - - - - - , ka = 30.

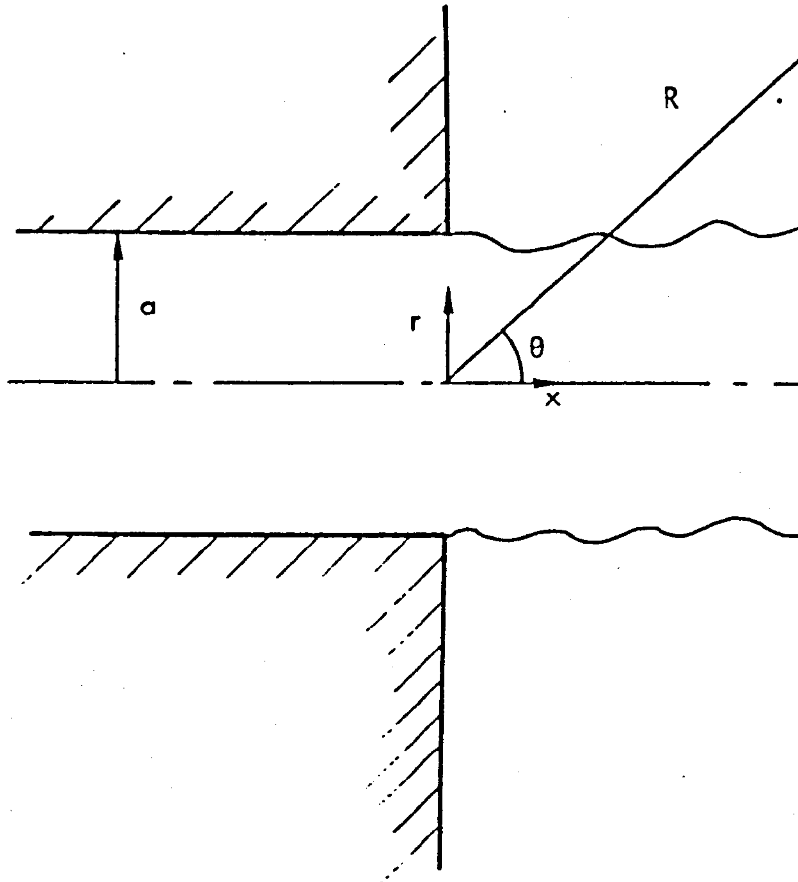
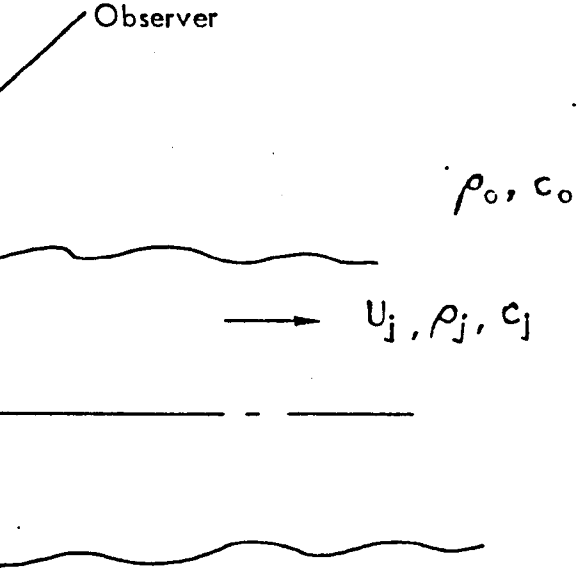


Fig. 6. Radiation from a baffled duct.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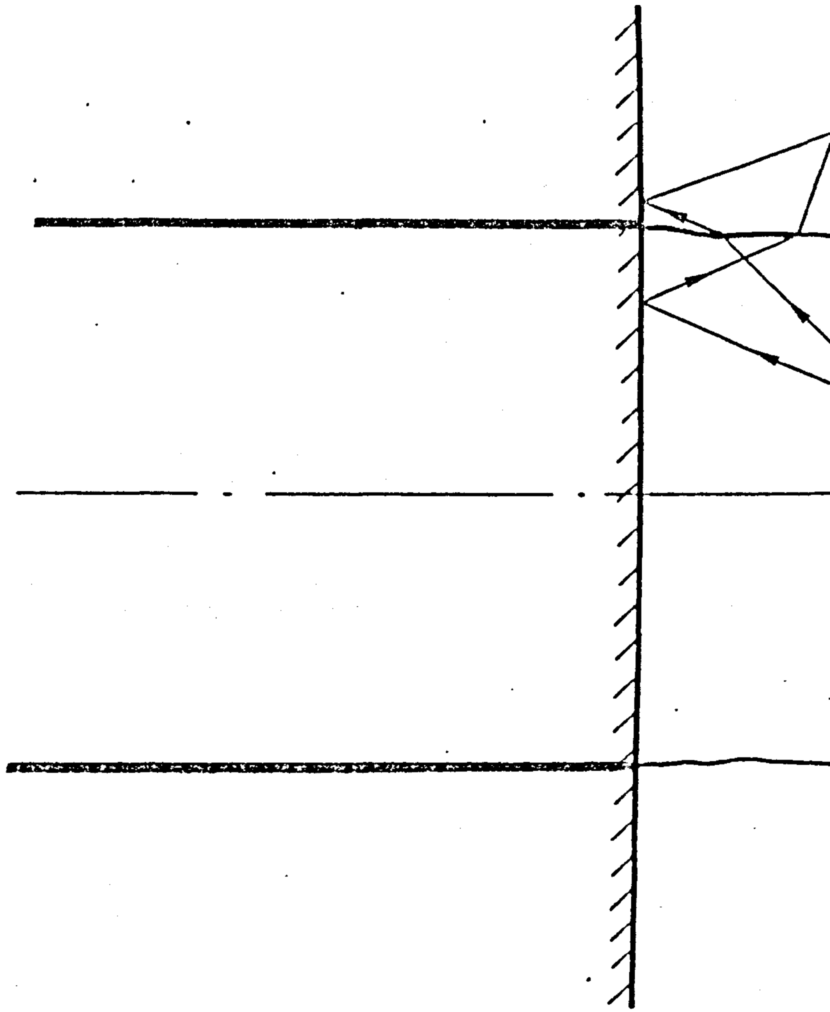
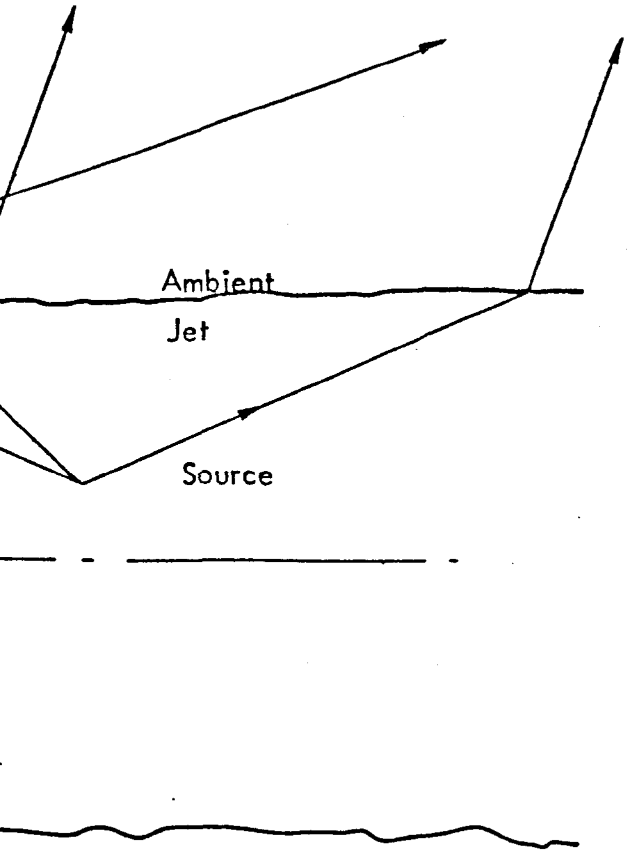


Fig. 7. Ray paths with baffle across jet.


Ambient
Jet
Source

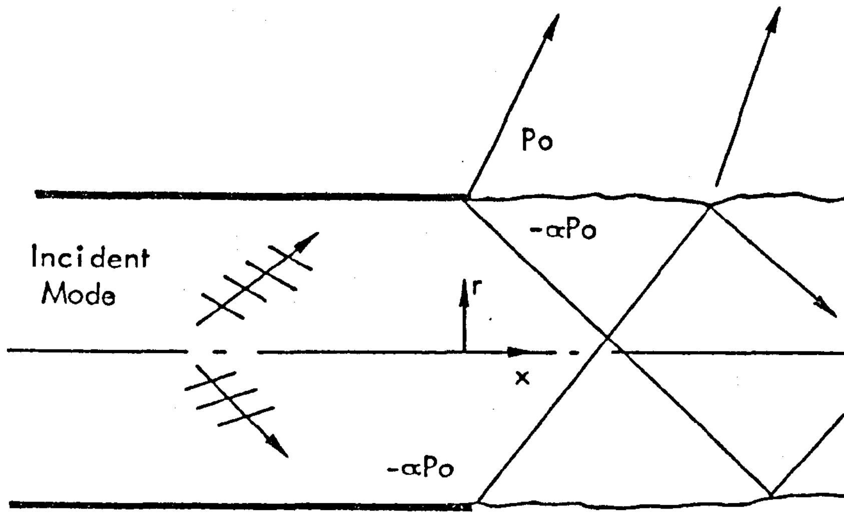
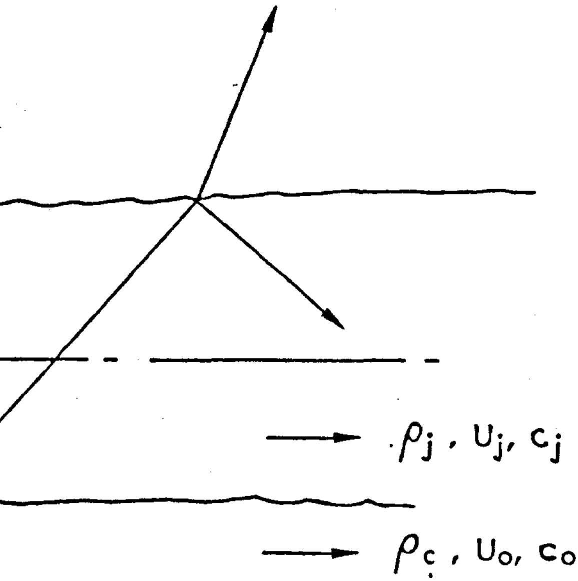


Fig. 8. Edge scattering solution.


ρj , Uj, cj
ρc , Uo, co

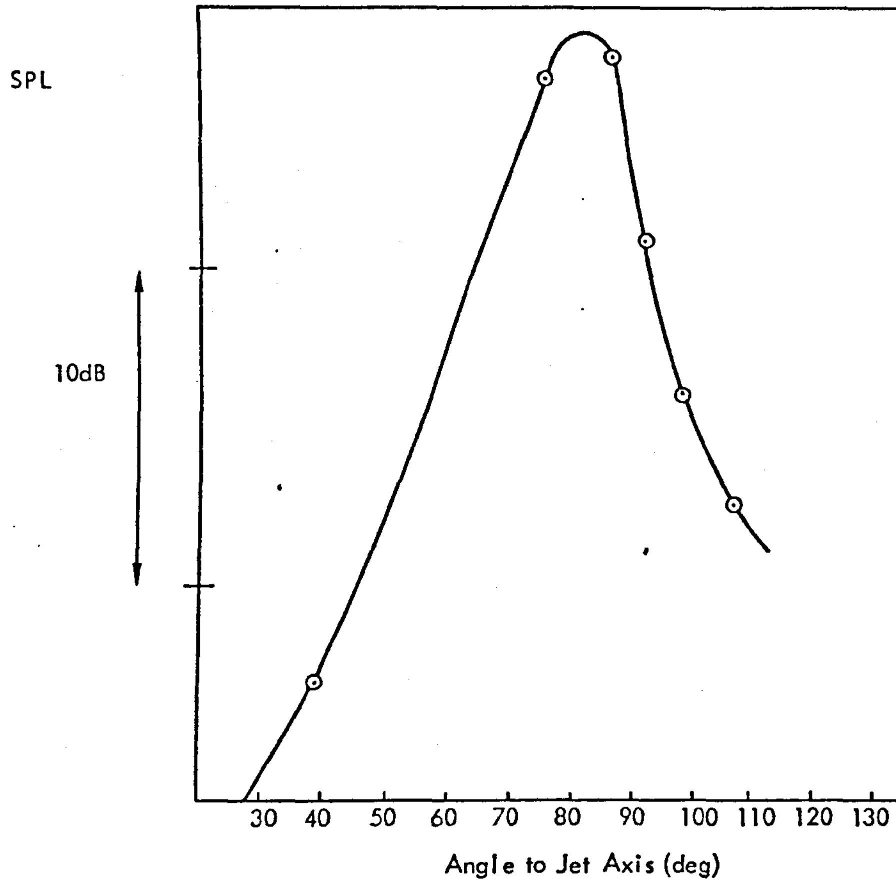
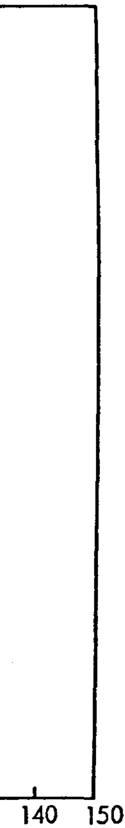


Fig. 9. Typical turbine tone fieldshape.


140
150

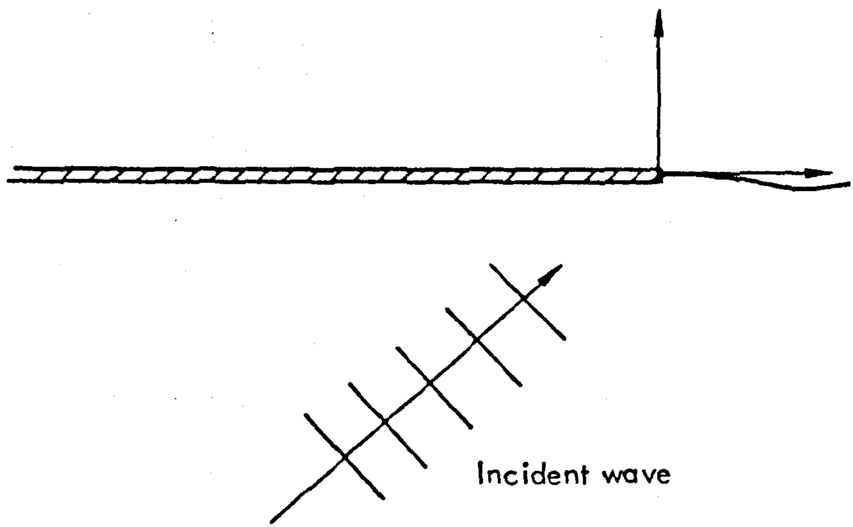
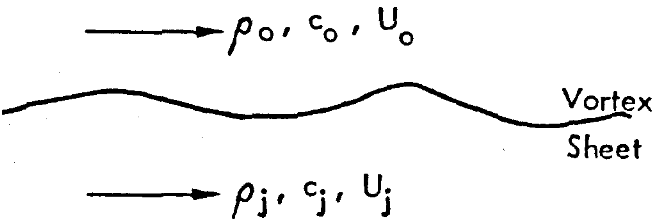


Fig. 10. Notation for diffraction by a half plane.


ρ_o, c_o, U_o
Vortex
Sheet
ρ_j, c_j, U_j

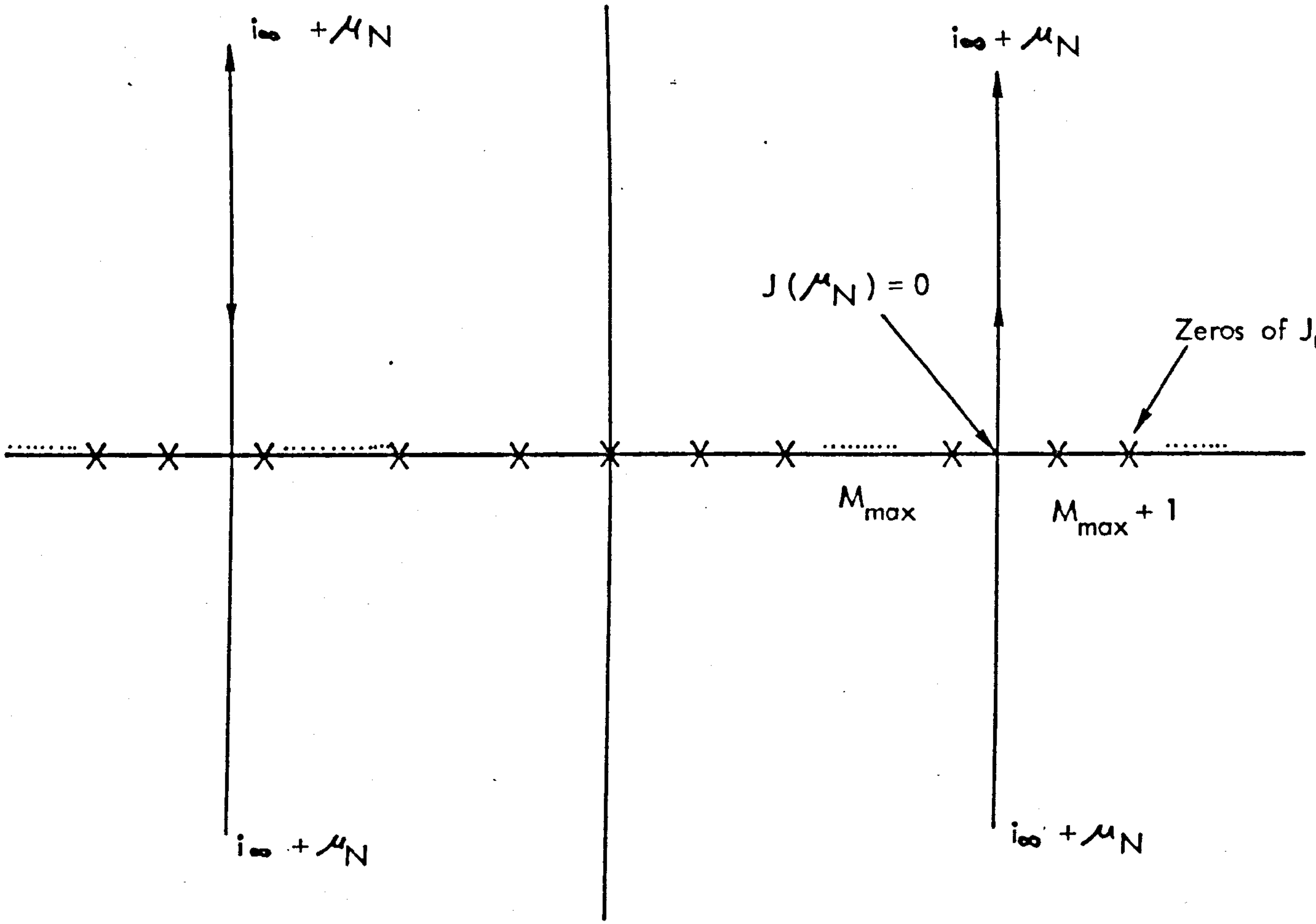


Fig. 11. Integration contour for modal summation in complex u plane.

CHAPTER 5

THE SHOCK WAVES AHEAD OF A FAN WITH NON-UNIFORM BLADES

PART I: PRELIMINARY ANALYSIS

ABSTRACT

When a fan operates at supersonic tip speeds, shock waves are generated ahead of the blades. If these blades are non-uniform, then the shock waves are also non-uniform, and tones at harmonics of the fan rotational frequency are generated. This chapter presents a simple theory for the relation between the strengths of the individual shock waves, the blade stagger angles and the blade thicknesses. That relation which is derived in Part II of the paper, depends on a number of assumptions about the flow. The justification of these assumptions forms the subject of this Part I. First, the general features of detached shock waves ahead of a blunt nosed body are discussed and the method of analysis used in Part II established. Next the outflow from a cascade of unevenly staggered blades is discussed. Using the Wiener-Hopf method, it is shown that the mass flow from a blade passage depends only on the area of that passage, for the nearly sonic Mach numbers of interest.

1. INTRODUCTION

This paper discusses the generation of "buzzsaw" or "combination-tone" noise by supersonic compressors with non-uniform blading. If such a compressor has uniform blading, any tones generated are, by symmetry, at multiples of blade passing frequency only. The low harmonics of rotational frequency, that do in practice exist, are an important component of the aft cabin noise on the Lockheed L1011 TriStar. The causes of this buzzsaw noise in modern turbofan engines are the shock waves which appear ahead of the fan. These shock waves are generated with different amplitudes, depending on the blade parameters, and they then propagate non-linearly along the fan inlet duct. Recent work by Stratford and Newby [1] has shown that while the strengths of the shock waves change during this propagation process, the levels of the low order harmonics of rotational frequency do not. The levels of these harmonics are, therefore, set by their levels at the fan disc. Stratford and Newby found a linear relation between shock amplitude and the differences in the stagger of adjacent blades. This is used [2] as the basis for controlling shock strengths and hence the low order harmonics upstream of the fan. The order of the fan blades is changed to minimise the second and third harmonics of the stagger angle pattern. While this has been broadly successful in reducing the sound level in these harmonics, an additional factor is thought to be significant. This is the thickness of the blades. Recently, much work has been done to correlate the shock strengths with blade-to-blade thickness variations [2]. As a result, when the blades are shuffled, thickness as well as stagger angle variations are taken into account. In the Stratford and Newby paper, the experimental relation between blade angle and shock strength was at variance with existing theories. In these (see e.g. Kurosaka [3]) the shock waves were assumed to be attached to the fan blades, and the flow pattern as in Fig. 1. If we neglect the losses through the shocks, the Mach number at a point is directly related to the local flow angle. The flow field may then be solved by the method of characteristics. There, the Mach number and flow angle are constant along "characteristic" lines, at an angle $\mu = \cos^{-1}(1/M)$ to the flow. Using this property, and the conservation of mass flow, energy and momentum across the shock waves, the complete flow field ahead of the fan can be constructed. In practice, it is much easier to analyse the flow beyond the first blade spacing ahead of the fan, using one-dimensional non-linear acoustic theory (see Hawkings [4]).

Since the blades are only slightly cambered, they may be taken as flat, to a first approximation, and with one blade twisted from its original position the flow pattern close to the fan is as depicted in Fig. 2. Over the section AB, the flow ahead of the shock attached to the nth blade is uniform; all the changes in the flow behind it are caused by the expansion fan shed from its leading edge, where the flow is turned. Hence the shock strength only depends on the blade angles and is, therefore, easily calculated. Since the shock waves are attached, they are uninfluenced by the flow downstream of the fan. Consider now the effect of changing the incidence of one blade; if the stagger angle of the nth blade is decreased, the Mach number ahead of the (n + 1)th blade is reduced, and the Mach number behind the nth blade reduced. Consequently, the strength of the shock from the nth blade is increased and that from the (n + 1)th blade reduced. Linearising, we may then expect a relation of the form $\Delta P_n = A\alpha_n - B\alpha_{n-1}$ where ΔP_n is the shock amplitude, α_n the increase in stagger angle of the nth blade, and A and B are positive constants of the same order of magnitude.

This is in contradiction to the observed relation [1] $\Delta P_n = -0{\cdot}194\,(\alpha_n - \alpha_{n-1})$, for which the slope has the opposite sign. Stratford and Newby argued that the difference arises because in practice the shock waves are detached from the blade leading edges. When a uniform fan or cascade is running, there is only one value of flow direction (for a given Mach number) for which the shock waves are attached - the so-called "unique incidence condition" (see, e.g., [5]). In reality fans are operated away from this condition, at higher values of blade incidence. As Stratford and Newby demonstrate, this means that some of the inlet flow to a blade passage is spilled around the leading edge of the blade. This spillage is necessary to match the mass flows through the passages with those far upstream of the fan and it makes the blade behave, in effect, as a thick blunt-nosed body which has a detached shock ahead of it. A further consequence of shock wave detachment is that the shock waves are affected by the flow downstream of the fan. Stratford and Newby assert that this is the primary factor controlling the shock amplitudes. As the blade angles are altered, the change in swallowing capacity of a blade passage is proportional to the change in angle. This change in swallowing capacity then changes the spillage around the blade and alters the shock strengths accordingly. It is, therefore, plausibly responsible for the observed relation between shock strength and blade angles.

The aim of this paper is to model this phenomenon in a relatively simple way, and to justify quantitatively the observed experimental relations between shock strength, blade angles and thicknesses. To do this we assume that the blades are essentially thin and uncambered and we consider only two-dimensional flow, such as persists near the tips of the blades where the experimental measurements are made. Further, we consider only the flow close to the fan since its upstream development is easily modelled by existing non-linear weak shock theory. Three-dimensional effects in both the generation and propagation of the shock waves may, however, be important, particularly over the full length of the 'S' duct on the Lockheed L1011 TriStar. This will form the subject of another paper.

The chapter is split into two parts: Part II derives the relation between the shock strengths and the blading non-uniformities, while Part I is devoted to some preliminary analyses to justify the assumptions on which Part II is based. In this Part I we begin by discussing the mechanism of shock wave detachment from isolated two-dimensional blunt-nosed bodies in supersonic flows. We then derive a simple theory that agrees well with exact calculations. This theory forms the basis of the work on non-uniform fans presented in Part II. An assumption made in Part II is that the outflow from the non-uniform cascade is, for these nearly sonic Mach numbers, only dependent on the blade passage areas. We justify this in Part I by solving a linearised problem involving a semi-infinite cascade with the blades moved from their mean position. In solving a linear problem it may be objected that the flow is inherently non-linear at nearly sonic Mach numbers. We maintain, nonetheless, that the linear solutions will be adequate so long as the flow is not singular at $M = 1$. The analysis proceeds by the Wiener-Hopf technique [6]. We start by solving the problem for a perfectly general set of perturbations, and generate a set of simultaneous equations, of number equal to the number of blades. We then specialise the analysis to harmonic variations in the stagger angles and displacements. These harmonics are essentially responsible for the harmonics of shaft rotational frequency in the resulting engine order tones. An essential feature of this analysis is that we initially solve for time harmonic variations, and then let the frequency tend to zero. In that respect, the analysis might have some relevance to the flutter problem, and certainly has some similarities to the supersonic flutter work of Goldstein et al [7]. It is also related to much published work on the transmission of sound through blade rows (see e.g. Mani and Hòrvay [8]).

2. DETACHED SHOCK-WAVES ON BLUNT-NOSED BODIES

We consider a semi-infinite body of thickness t and nose radius r immersed in a uniform mean flow of Mach number M. The flow pattern is shown in Fig. 3 in both the physical and hodograph planes. The flow decelerates through the shock from C to C', B to B' and A to A'. It then accelerates up a stream line B'F, say, to the point where it meets the sonic line CE. It then accelerates supersonically until it eventually approaches the free stream Mach number M.

These changes can be conveniently shown on a hodograph diagram, which is a plot of the flow in the (U,V) plane, where U,V are the two components of the velocity. On this diagram the flow passing through the shock goes from A, B, C on the V = 0 axis to points A', B', C' on the shock polar.

There are a number of ways of estimating the shock detachment distance for such isolated blunt-nosed bodies. First, there is direct calculation using a numerical solution of the flow equations. This is feasible but complicated. Furthermore, it is unlikely to yield either the degree of understanding required here, or some simple relation that could be fitted to an acoustic theory. Additionally, since, as we shall see later, the outflow from the cascade can only be crudely modelled, great accuracy seems unwarranted.

A second method is that of matched expansions. A number of solutions are available (see e.g. Van Dyke [9]). They have two disadvantages. First, they arise from expansions of the flow variables in powers of $1/M_\infty$, and are accordingly excellent for hypersonic speeds $(M_\infty > 4)$, but much less suitable for the Mach numbers relevant here $(M_\infty < 1{\cdot}7)$. Second, they mostly solve only the so-called "indirect problems", in which the shape of the shock wave is specified (a hyperbolic shape is frequently assumed) and the body profile then determined.

Our approach is based on the much simpler method due to Möckel, as described in detail by Shapiro [10]. He assumes a shock shape, and finds its position by balancing the mass flows through the sonic line with the upstream mass flow. We adopt his method with a number of simplifications, and obtain a result which agrees well with more exact calculations and therefore also, as shown by Shapiro, with experiments for the

We balance the mass flow through the sonic line, (see Fig. 4), with that through the characteristic AB. Downstream of the sonic line we assume that the flow is isentropic, and that the characteristics are all of one family. This means that we are in some sense ignoring the rise in entropy through the shock. Since the rise is $O(M - 1)^3$ and here $M = 1.5$, this is not too important. Since only one characteristic family is present, the position of the sonic line on the body E, may be determined. The sonic line is further assumed to be straight (see Shapiro [1C]) and at an angle $\left(\frac{\pi}{2} - \eta\right)$ to the mean flow.

The mass flow through the characteristic AB is

$$\dot{m}_\infty = \left(\frac{\dot{m}\sqrt{T_0}}{A\,P_0}\right) \cdot \frac{P_0\,A_{AB}}{\sqrt{T_0}} , \qquad (1)$$

where $\left(\dot{m}\sqrt{T_0}/A\,P_0\right)$ is the one-dimensional mass flow function, P_0, T_0 the stagnation pressure and temperature, and A_{AB} the area of AB perpendicular to the flow (per unit span). This is set equal to the flow through the sonic line,

$$\dot{m}_c = \left(\frac{\dot{m}\sqrt{T_0}}{A\,P_0}\right) \frac{A_c\,P_{0,c}}{\sqrt{T_{0,c}}} . \qquad (2)$$

Writing $\left(\dot{m}\sqrt{T_0}/AP_0\right)$ as Q, where $Q = M\left(1 + \left(\frac{\gamma-1}{2}\right)M^2\right)^{-(\gamma+1)/(2(\gamma-1))}$, we find that with $A_{AB} = r + d\cos\eta$ and $A_c = d$, (per unit span)

$$\left(r + d\cos\eta\right) Q_\infty = R Q_c d , \qquad (3)$$

where R is the average ratio of stagnation pressures across the shock. Hence

$$\frac{d}{r} = \frac{1}{\left(R(Q_c/Q_\infty) - \cos\eta\right)} . \qquad (4)$$

Figure 5 compares this result with that of Möckel's calculation procedure quoted by Shapiro [10]. We see that the two are in reasonable agreement, except above a Mach number of 2, where the pressure losses become significant. At very low excess Mach numbers, the shock stand-off distance tends to infinity. This region is probably not well modelled either, since the geometric assumptions are of doubtful validity when the shock is far enough detached. However, the agreement is quite acceptable in the region of real interest.

We examine the form of the result (4) for marginally supersonic flow. It can be easily shown that, with $(M-1)=\zeta$,

$$(Q/Q_c)=\left(1+2\zeta^2/(\gamma+1)\right),\quad \cos\eta=1-O(\zeta^2),\quad R=O(\zeta^3). \tag{5}$$

Hence the detachment for small excess Mach numbers is

$$\frac{d}{r}=\frac{\gamma+1}{2(M-1)^2}. \tag{6}$$

This is plotted on Fig. 5, and agrees well with (4) as evaluated exactly.

3. THE STEADY OUTFLOW FROM A CASCADE OF NON-UNIFORM BLADES

We consider a semi-infinite two-dimensional cascade, as shown in Fig. 6, containing a mean flow of Mach number M, and with its blades displaced from their nominal positions by angles θ_n and distances δ_n. We initially solve the time dependent problem in which the blades displacements have the time dependence $\exp[ikct]$ where kc is the frequency, and k is assumed to have a small imaginary part such that $\mathrm{Im}\,k=-\varepsilon,\ \varepsilon>0$. We obtain the steady flow solution by setting k equal to zero at the end of the analysis.

In each passage we define the pressure perturbation relative to the mean to be $p_n(x_n', y_n')\exp[ikct]$, where the (x_n', y_n') coordinates are fixed in the nth blade. We define general coordinates (x, y) such that $x_n' = x - ns$, $y_n' = y - nh$. These pressures, p_n, satisfy a convected wave equation

$$\left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{c^2}\left(ikc + U\frac{\partial}{\partial x}\right)^2\right] p_n = 0 . \tag{7}$$

The problem is solved with the boundary conditions that on the blade surfaces $(y_n = 0, x_n < 0)$ the fluid particle displacements η_n shall be equal to the blade displacements, so that

$$\eta_n(0, x_n') = \eta_{n-1}(h, x_n') = \delta_n + \theta_n x_n' . \tag{8}$$

On the streamline leaving the nth blade, we assume that the pressure is continuous, so that

$$p_n(0, x_n') = p_{n-1}(h, x_n'), \tag{9}$$

and similarly the displacement is continuous, with

$$\eta_n(0, x_n') = \eta_{n-1}(h, x_n') . \tag{10}$$

We also assume that the flow leaves the trailing edge of the blades smoothly; that is, a Kutta condition is assumed to hold.

We now introduce the Fourier transforms of the perturbation quantities; for a general perturbation ϕ these are defined as

$$\Phi_n^{\pm}(u) = \int_{-\infty}^{+\infty} \varphi_n(x_n') H(\pm x_n') e^{iux_n'} dx_n', \tag{11}$$

where H is the unit step function, and the $\pm$ transforms are regular in the regions $R^{\pm}$ of Fig. 7. The transformed pressure thus satisfies the equation

$$\left(\frac{\partial^2}{\partial y^2} + \Gamma^2\right)P_n = 0\ , \tag{12}$$

where $\Gamma(u) = D^2 - u^2$, $D = (k - Mu)^2$, and the branches of $\Gamma(u)$ are chosen such that $Im\,\Gamma < 0$ as $k \rightarrow \infty$ in the common strip of regularity of the regions . Transforming the boundary conditions and defining the transform of $\eta_n(x_n)$ as Z_n, we have

$$Z_{b,n}^- = \frac{\delta_n}{i(u-\epsilon)} + \frac{\theta_n}{(u-\epsilon)^2}\ , \tag{13}$$

where b refers to the blade, rather than the fluid, and in which (to ensure convergence) we have set the blade displacements as $(\delta_n + \theta_n x_n)\exp(-i\epsilon\ )$ where $Im\,\epsilon > 0$ and ϵ will be set equal to zero at the end of the analysis. Fourier transforming equation (9), we have

$$P_n^+(0) = P_{n-1}^+(h)\,e^{-ius}\ , \tag{14}$$

which can be re-written in the form

$$\left(P_n(0) - P_{n-1}(h)e^{-ius}\right) = \left(P_n^-(0) - P_{n-1}^-(h)e^{-ius}\right) - F_n^-\text{, say} \tag{15}$$

where F^- is regular in R^-. On the blade we have, similarly, for $x > 0$

$$Z_n^+(0) - Z_{n-1}^+(h) = 0\ . \tag{16}$$

This may be combined with the boundary condition on the blade surface to yield

$$Z_{b,n}^- + Z_n(0) = Z_{bn}^- + Z_{n-1}(h)\,e^{-ius} = 0. \tag{17}$$

From equation (12), we can take the pressure to be of the form

$$P_n = A_n e^{i\Gamma y} + B_n e^{-i\Gamma y}\ , \tag{18}$$

and then using the equation of momentum conservation in the y direction it follows that

$$Z_n(y) = \frac{i\Gamma}{\rho c^2 D^2}\left[A e^{i\Gamma y} - B e^{-i\Gamma y}\right]. \tag{19}$$

Now the condition that the particle displacement is continuous across $y=0$ implies that

$$Z_n(0) = Z_{n-1}(h)e^{-ius}, \quad Z_{n+1}(0) = Z_n(h)\, e^{-ius}. \tag{20}$$

Thus it follows from (19) and (20) that

$$A_n = \left[\rho c^2 D^2 / 2\Gamma \sin\Gamma h\right]\left[Z_n(0)e^{-i\Gamma h} - Z_{n+1}(0)e^{ius}\right], \tag{21}$$

$$B_n = \left[\rho c^2 D^2 / 2\Gamma \sin\Gamma h\right]\left[Z_n(0)e^{i\Gamma h} - Z_{n+1}(0)e^{ius}\right], \tag{22}$$

so that substituting in (15) we have

$$A_n + B_n - A_{n-1}e^{-ius + i\Gamma h} - B_{n-1}e^{-ius - i\Gamma h} = F_n^-, \tag{23}$$

and substituting for A_n, B_n,

$$-\left(\rho c^2 D^2/\Gamma \sin\Gamma h\right)\left[Z_{n+1}\, e^{ius} - 2Z_n \cos\Gamma h + Z_{n-1}\, e^{-ius}\right] = F_n^-. \tag{24}$$

One way of proceeding from here would be to take (24) and substitute for Z_n, using (17) to give

$$Z_n = -Z_{bn}^- + Z_n^+. \tag{25}$$

This would then give the following set of equations:-

$$\sum_{m=1}^{B} -\left(\rho c^2 D^2/\Gamma \sin\Gamma h\right)\left[K_{mn}\right]\left[-Z_{bn}^- + Z^+\right] = F_n^-. \tag{26}$$

A set of simultaneous equations such as this cannot, in general, be solved by the Wiener-Hopf technique, since there is no general method of factorising the matrix K_{mn} into parts regular in the two half planes. However, in this case K_{mn} always has the form K_{m-n} . and the equations may indeed be solved. This is done by splitting all the components into their harmonic components, so that

$$Z_n(u) = \sum_{\ell=1}^{B} \bar{Z}_\ell(u) \exp\left[2\pi i n\ell/B\right], \tag{27}$$

where we have assumed that the cascade is periodic every B blades, though we emphasise that this assumption is not necessary for the solution of the equation (26). Making this substitution in (26) we obtain

$$K_\ell(u)\left(\bar{Z}_\ell^+ - \bar{Z}_{b\ell}^-\right) = F_\ell^-, \tag{28}$$

where $K_\ell(u)$ is the function

$$K_\ell(u) = -2\rho c^2 D^2\left(\cos(us + 2\pi\ell/B) - \cos\Gamma h\right)\Big/\left(\Gamma \sin\Gamma h\right). \tag{29}$$

Equation (28) can now be solved by the Wiener-Hopf technique. Substituting for $\bar{Z}_{\ell b}$ and splitting $K = K^+K^-$ where the $K^\pm$ are regular in $R^\pm$, we obtain

$$K_\ell^+\bar{Z}_\ell^+ - \left[\frac{K^+(u) - K^+(\varepsilon)}{i(u-\varepsilon)}\right]\bar{\delta}_\ell - \frac{\bar{\theta}_\ell}{(u-\varepsilon)}\left[-\frac{K^{+\prime}(\varepsilon)}{(u-\varepsilon)} + \frac{(K^+(u) - K^+(\varepsilon))}{(u-\varepsilon)}\right]$$

$$= \frac{K^+(\varepsilon)\,\bar{\delta}_\ell}{i(u-\varepsilon)} + \frac{\bar{\theta}_\ell K^+(\varepsilon)}{(u-\varepsilon)^2} + \frac{\bar{\theta}_\ell K^{\prime +}(\varepsilon)}{(u-\varepsilon)^2} + F_\ell^- . \tag{30}$$

The two sides of this equation are regular in R^+ and R^- respectively. Therefore, by Liouville's theorem they must be equal to a polynomial. If we choose to apply a trailing edge Kutta condition, then $Z^+(u) \sim u^{-5/2}$ as $u \to \infty$ and from the Appendix, $K(u) \sim u^{3/2}$ as $u \to \infty$; so we find that the polynomial must be zero. Therefore, noting that $\bar{Z}_\ell^- = -\bar{Z}_{b\ell}^-$, we have

$$\bar{Z}_\ell = -\frac{K_\ell^+(\varepsilon)\,\bar{\delta}_\ell}{K_\ell^+(u)\, i(u-\varepsilon)} - \frac{\bar{\theta}_\ell}{K_\ell^+(u)}\frac{d}{d\varepsilon}\left[\frac{K_\ell^+(\varepsilon)}{(u-\epsilon)^2}\right]. \tag{31}$$

The factorisation of the split functions is discussed in the Appendix.

We are now in a position to calculate the mass flows in the blade passages. Using the above value of $\bar{Z}_e$, (31), and with (21), (22) and (13), it follows that the pressure $\bar{p}_e$ is given by

$$\bar{p}_e = \frac{\rho c^2 D^2}{2\Gamma \sin \Gamma h}\left[e^{i\Gamma y}\left(e^{-i\Gamma h} - e^{2\pi i e/\beta + ius}\right) + e^{-i\Gamma y}\left(e^{i\Gamma h} - e^{2\pi i e/\beta + ius}\right)\right]$$

$$\cdot \left[\frac{-K_e^+(\varepsilon)\bar{\delta}_e}{i(u-\varepsilon)K_e^+(u)} - \frac{\bar{\theta}_e}{K_e^+(u)}\frac{d}{d\varepsilon}\left(\frac{K^+(\varepsilon)}{u-\varepsilon}\right)\right]. \qquad (32)$$

For the sake of simplicity we drop the suffix e and the overbar from K, θ, δ etc. We deal separately with the mass flows due to the displacement δ, and angular perturbation θ. Dealing with the displacement first, we find, on Fourier transforming (32) and substituting for $K^+(u)$ from the Appendix, that

$$p = \int_{-\infty}^{+\infty} \frac{\rho c^2 D^2 \left[e^{i\Gamma y}\left(e^{-i\Gamma h} - e^{2\pi i e/\beta + ius}\right) + e^{-i\Gamma y}\left(e^{i\Gamma h} - e^{2\pi i e/\beta + ius}\right)\right]}{2\Gamma \sin \Gamma h\,(u-\varepsilon)}$$

$$\cdot \frac{(k-(1+M)u)\ D(\varepsilon)^2\,(1+q_0\varepsilon)e^{-iux}}{(k+(1-M)u)\ D(u)^2\,(1+q_1 u)}\,du. \qquad (33)$$

As we only require the field in $x<0$, this consists of contributions from the pole at $u=\varepsilon$, that at $u=-k/(1-M)$ and from an infinite number of poles arising from the $\sin\Gamma h$ term. The latter represent cut-off acoustic duct modes whose amplitudes decay exponentially away from the exit of the cascade. They do not contribute to the mass flow since they integrate to zero. Considering the two important poles in turn, that at $u=\varepsilon$ gives

$$p = \frac{-\delta}{h}\frac{\rho c^2 D^2(\varepsilon)\ \Gamma(\varepsilon)h}{(D^2(\varepsilon)-\varepsilon^2)\sin\Gamma(\varepsilon)h}\left[2\cos\Gamma(\varepsilon)(h-y) - 2\cos\Gamma(\varepsilon)y\, e^{2\pi i e/\beta + i\varepsilon s}\right]. \qquad (34)$$

To get the field in the steady flow case we take the limit of (34) as $k \to 0$ and then $\varepsilon \to 0$. The order of these limits is most important, only this order giving the correct result. That ordering of the limits is physically equivalent to first solving the problem of the steady flow from blades with displacement decreasing exponentially with distance from the exit plane. Taking the limits the other way round gives the solution for low frequency vibrations of perfectly flat plates and is irrelevant and incorrect here. The correct result is, for the pole at $u = \varepsilon$,

$$p = \frac{\delta M^2 \rho c^2}{h(1-M^2)}\left(1 - e^{2\pi i \ell/B}\right). \tag{35}$$

Similarly, the contribution from the pole at $u = -\varepsilon/(1-M)$ gives

$$p = \frac{\delta \rho c^2 D^2(\varepsilon)}{h\,(D^2(\varepsilon) - \varepsilon^2)}\left[\frac{\Gamma(0)}{\sin \Gamma(0) h}\right]\left(1 - O(k)\, e^{2\pi i \ell/B}\right) \tag{36}$$

$$= -\frac{\delta \rho c^2 M^2}{h\,(1-M^2)}\left(1 - e^{2\pi i \ell/B}\right). \tag{37}$$

Clearly, these two parts, (35) and (37), cancel, so that as $k \to 0$, there is no pressure or velocity perturbation. Thus the only perturbation in mass flow is that due to the area change. This is entirely what one would expect for a set of blades that are displaced parallel to themselves.

The pressure perturbation due to the change in stagger angle is found by Fourier transforming (32) and substituting for $K^+(u)$ as $k \to 0$ from the Appendix to give

$$p = \frac{-i\theta}{2\pi i}\frac{d}{d\varepsilon}\int_{-\infty}^{+\infty} \frac{e^{-iux}\,du \;\; \rho c^2 D^2 (1 + q(\varepsilon - u))}{2(\sin \Gamma h/\Gamma h)}\left[e^{i\Gamma y}\left(e^{-i\Gamma h} - e^{\frac{2\pi i \ell}{B} + ius}\right) + e^{-i\Gamma y}\left(e^{i\Gamma h} - e^{2\pi i \ell/B + ius}\right)\right] \tag{38}$$

The pole at $u = \varepsilon$ gives

$$p = -i\theta \frac{d}{d\varepsilon} \frac{\rho c^2 D^2(\varepsilon)^2 \left(2\cos \Gamma(\varepsilon)(h-y) - 2e^{2\pi i\ell/\beta + i\varepsilon s}\cos\Gamma(\varepsilon)y\right)}{2\left(\sin \Gamma(\varepsilon)h / \Gamma(\varepsilon)h\right)\left(D(\varepsilon)^2 - \varepsilon^2\right)}, \quad (39)$$

and letting $h \to 0$, followed by $\varepsilon \to 0$ we obtain

$$p = -\frac{i\theta s \rho c^2 M^2 e^{2\pi i\ell/\beta}}{h(1-M^2)} + \frac{i\theta(1-e^{2\pi i\ell/\beta})(-ix)\rho c^2 M^2}{(1-M^2)}. \quad (40)$$

A similar evaluation for the pole at $u = -k/(1-M)$ gives

$$p = -\frac{i\theta M^2 \rho c^2}{(1-M^2)h} \cdot q\left(1-e^{2\pi i\ell/\beta}\right) + \frac{i\theta s \rho c^2 e^{2\pi i\ell/\beta}}{(1-M^2)}. \quad (41)$$

We now require the values of the density and axial velocity perturbations. The latter are obtained from the linearised momentum equation, which in the steady flow limit integrates to give $u_x = p'/\rho U$ plus a constant which may be taken as zero. Now for these perturbations with no y' variation, the mass flow per passage is given by

$$\dot{m} = h\left(U\rho' + u_x'\rho\right) = \frac{hp'}{c}\frac{(M^2-1)}{M}. \quad (42)$$

Combining (41) and (42), we find that the mass flow per passage is given by

$$\dot{m} = i\theta\rho U q\left(1 - \exp(2\pi i\ell/\beta)\right), \quad (43)$$

and substituting for q from the Appendix we obtain

$$\dot{m} = i\theta\rho U\left(1-e^{2\pi i\ell/\beta}\right)\left[\frac{B}{2\pi\ell}(s - i\beta h) - \frac{i\beta h}{4\pi}\ln\left(\frac{\beta^2h^2+s^2}{4\beta^2h^2}\right) + s\tan^{-1}(s/\beta h) - \beta h/2\right]. \quad (44)$$

Inasmuch as the full solution can be constructed from the above equation by summation over the harmonic components ℓ, this completes the formal solution of the problem. From the point of view of the buzzsaw cabin noise problem, we are only interested in the solution for small mode numbers ℓ.

Expanding (44), with (ℓ/β) small, we obtain

$$\dot{m} = \theta\rho\mathcal{U}(s - i\beta h) + \frac{\ell}{\beta}\, i\theta\rho\mathcal{U}\Big[-\pi(s - i\beta h)$$

$$- 2i\pi\left(\frac{i\beta h}{2\pi}\ln\left(\frac{\beta^2h^2+s^2}{4\beta^2h^2}\right) + \frac{s}{\pi}\tan^{-1}\left(\frac{s}{\beta h}\right) - \frac{\beta h}{2}\right)\Big]. \qquad (45)$$

Now the $\theta\rho\mathcal{U}s$ term arises from the geometry (it represents the area increase relative to h, depending on θ). But as $M \to 1, \beta \to 0$. Therefore, we have shown that to first order in ℓ/β, the mass flow perturbations depend on area alone. Furthermore, it can be shown that the second order terms in (45) are non-singular as $\beta \to 0$. This means that it is unlikely that the theory will be greatly affected by non-linearities.

Part II of this study continues with the application of the ideas of Section 2 to non-uniform cascades in which a crucial simplification is the dependence of mass flow on area alone, as has been justified in linearised form in this work.

APPENDIX

THE FACTORISATION OF K(u)

Previous factorisations of the function $K(u)$ in (29) have been given, in different notation, by Mani and Horvay[8] and Goldstein et al.[9] Of these, the former is only concerned with the module of $K^{\pm}(u)$, which is not good enough for our purposes, while in the latter the result is quoted in full but with a minimum of explanations. In this Appendix we both give that result in the notation of the present paper, and deduce the limiting form as $k \to 0$. Briefly, the function $K(u)$ may be split into infinite products as described in Noble.[6] Thus:

$$K^{+}(u) = \frac{(k-Mu)^2 e^{\chi(u)}(1-(u/\lambda_0^+))}{(k-(1+M)u)} \frac{\prod_1^\infty (1-(u/\lambda_n^+))(1-(u/\nu_n^+))}{\prod_1^\infty (1-(u/\mu_n^+))}, \qquad (A1)$$

where

$$\lambda_0^+ = [a(0) + b(0)]/((1-M^2)+(s^2/h^2)),$$

$$\lambda_n^+ = [a(n) + b(n)]/((1-M^2) + (s^2/h^2)),$$

$$\nu_n^+ = [a(-n) + b(-n)]/((1-M^2) + (s^2/h^2)),$$

$$\mu_n^+ = \left[-Mk + \sqrt{(k^2-(1-M^2)n^2\pi^2/h^2)}\right]/(1-M^2).$$

with

$$a(n) = -(Mk - 2\pi(n-\ell B)s/h^2),$$

$$b(n) = \left[k^2(1+s^2/h^2) - (2Mks/h^2)2\pi(n-\ell/B) + (1-M^2).4\pi^2(n-\ell/B)^2/h^2\right]^{1/2},$$

and

$$\chi(u) = u\left[\frac{s}{\pi}\tan^{-1}\left(\frac{s}{\beta h}\right) - \frac{\beta h}{2} - \frac{i\beta h}{2\pi}\ln\left(\frac{\beta^2h^2+s^2}{\beta^2h^2}\right)\right].$$

and is required to make $K^{+}(u)$ have algebraic behaviour as $k \to \infty$. For the calculations in the main part of the paper, we only require the values of $K^{+}(u)$, as $k, u \to 0$.

In this limit, with (k/u) fixed, $K^+(u)$ may be written

$$K^+(u) = \frac{(k - \Pi u)^2}{(k-(1+\Pi)u)}\left[1 + q.u\right], \qquad \text{(A2)}$$

where

$$q = \left[-\frac{i\beta}{2\pi} \ln\left(\frac{\beta^2 h^2 + s^2}{4\beta^2 h^2}\right) + \frac{s}{\pi} \tan^{-1}\left(\frac{s}{\beta h}\right) - \frac{\beta h}{2} + \frac{Bs}{2\pi e} - \frac{i\beta h B}{2\pi e}\right].$$

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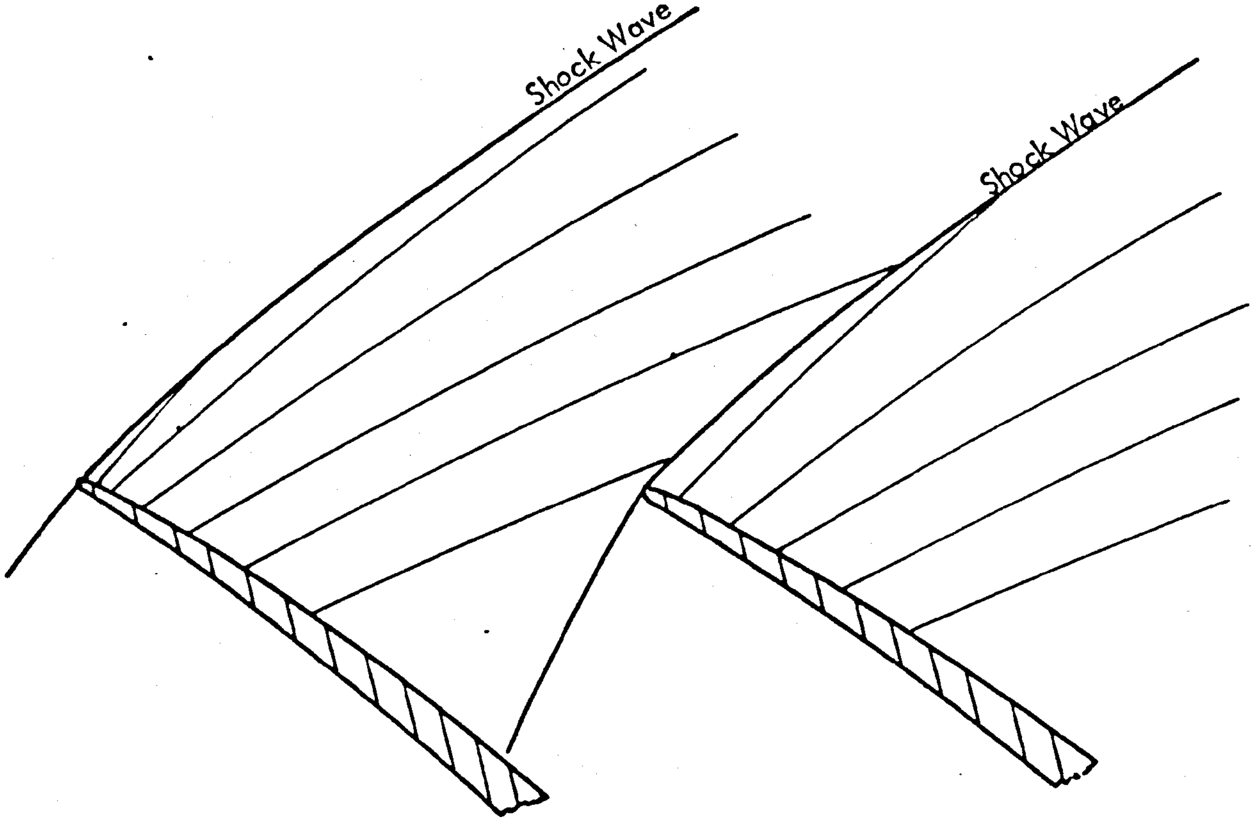

Fig. 1. Typical flow pattern - attached shock waves.

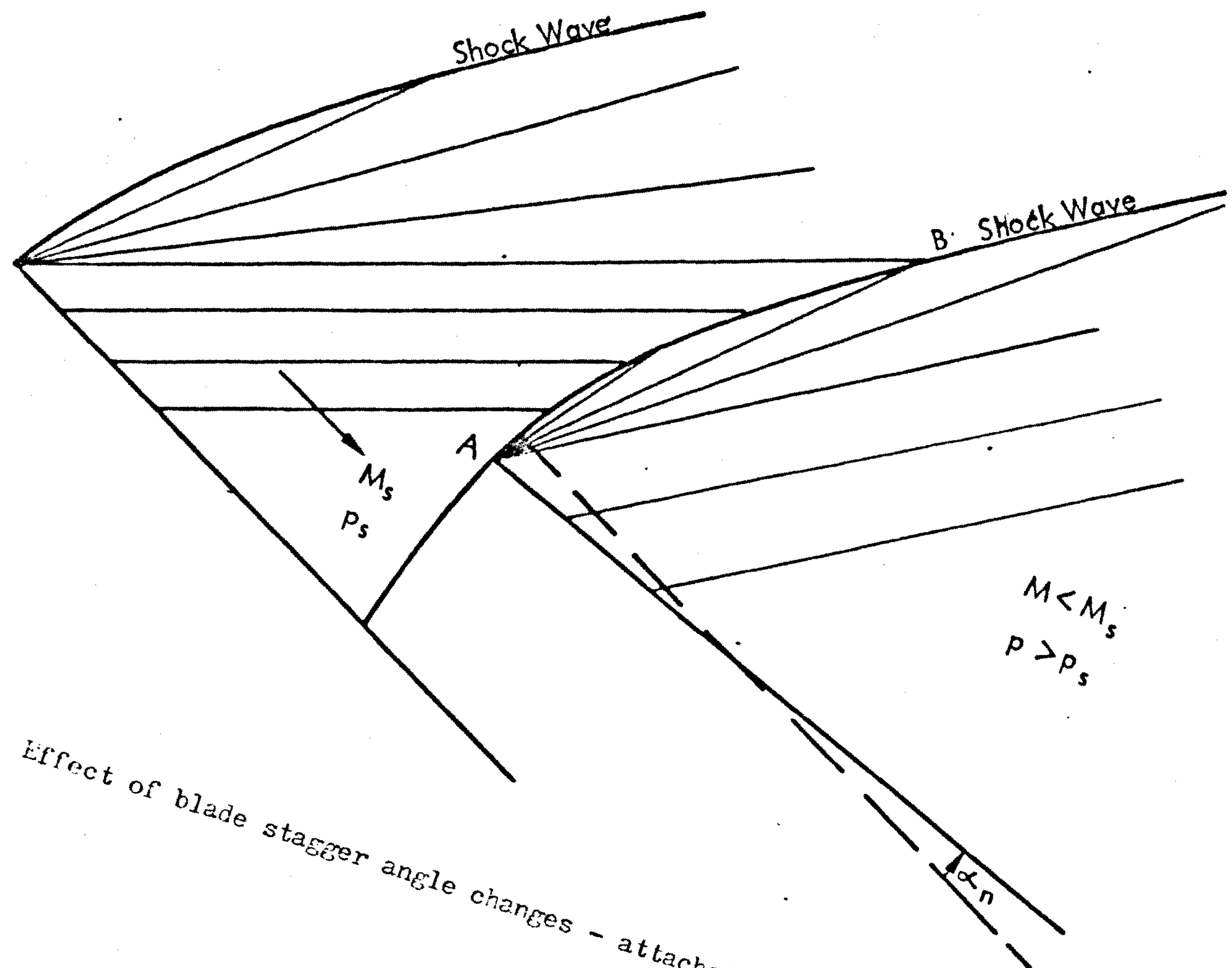

Fig. 2. Effect of blade stagger angle changes - attached shock waves.

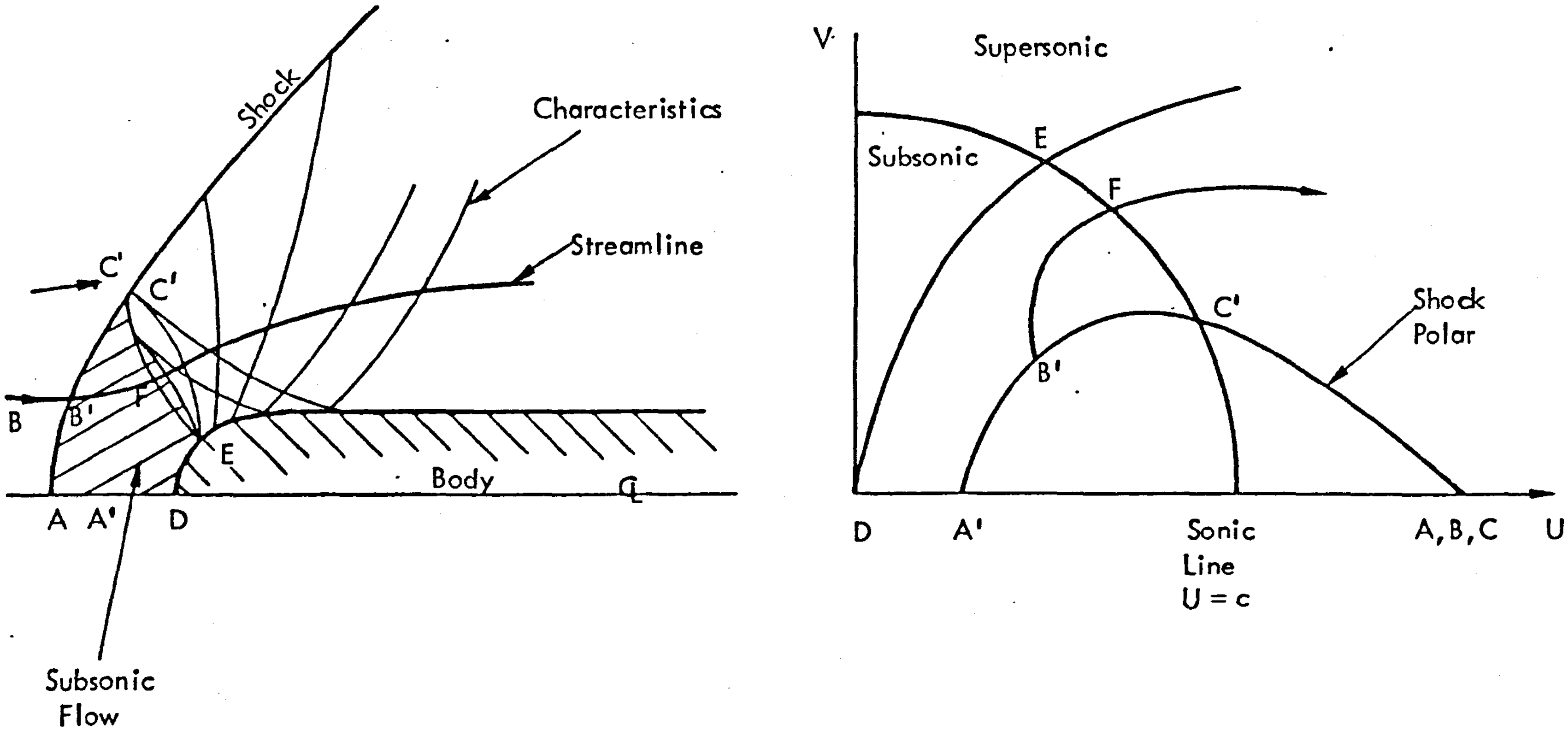

Fig. 3. Shock wave detachment for blunt-nosed body (a) physical plane, (b) hodograph plane.

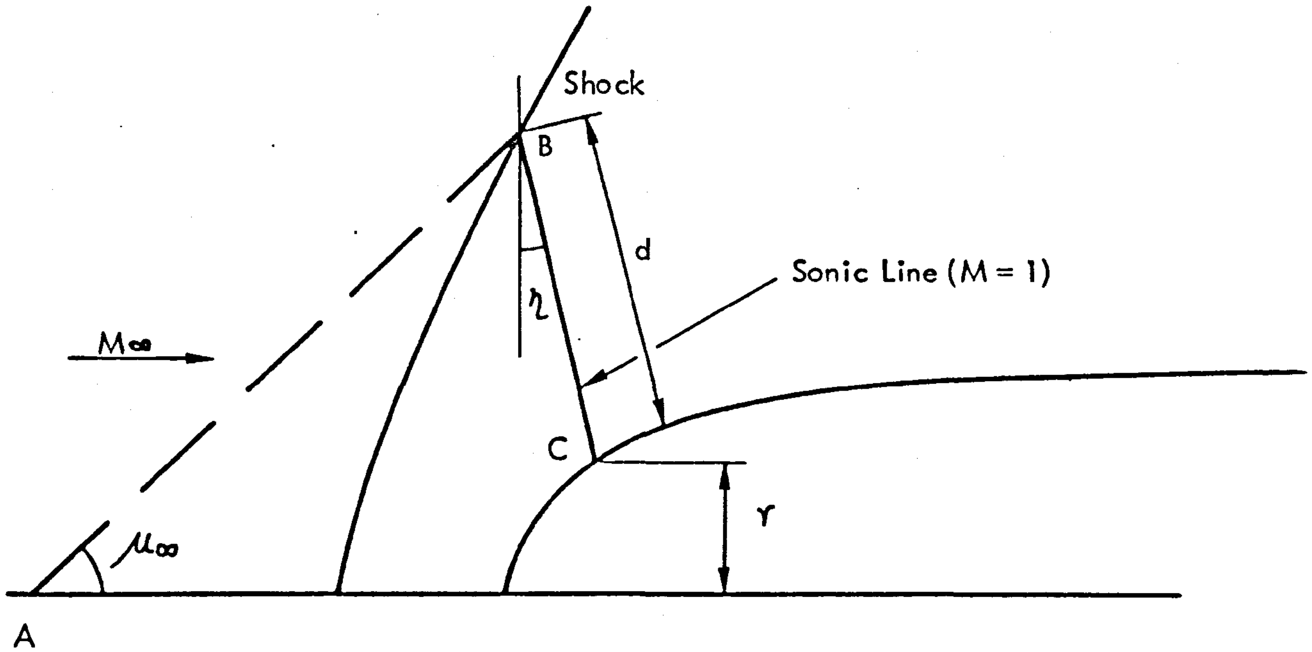

Fig. 4. Notation for calculation of blunt-nosed body shock wave stand-off distance.

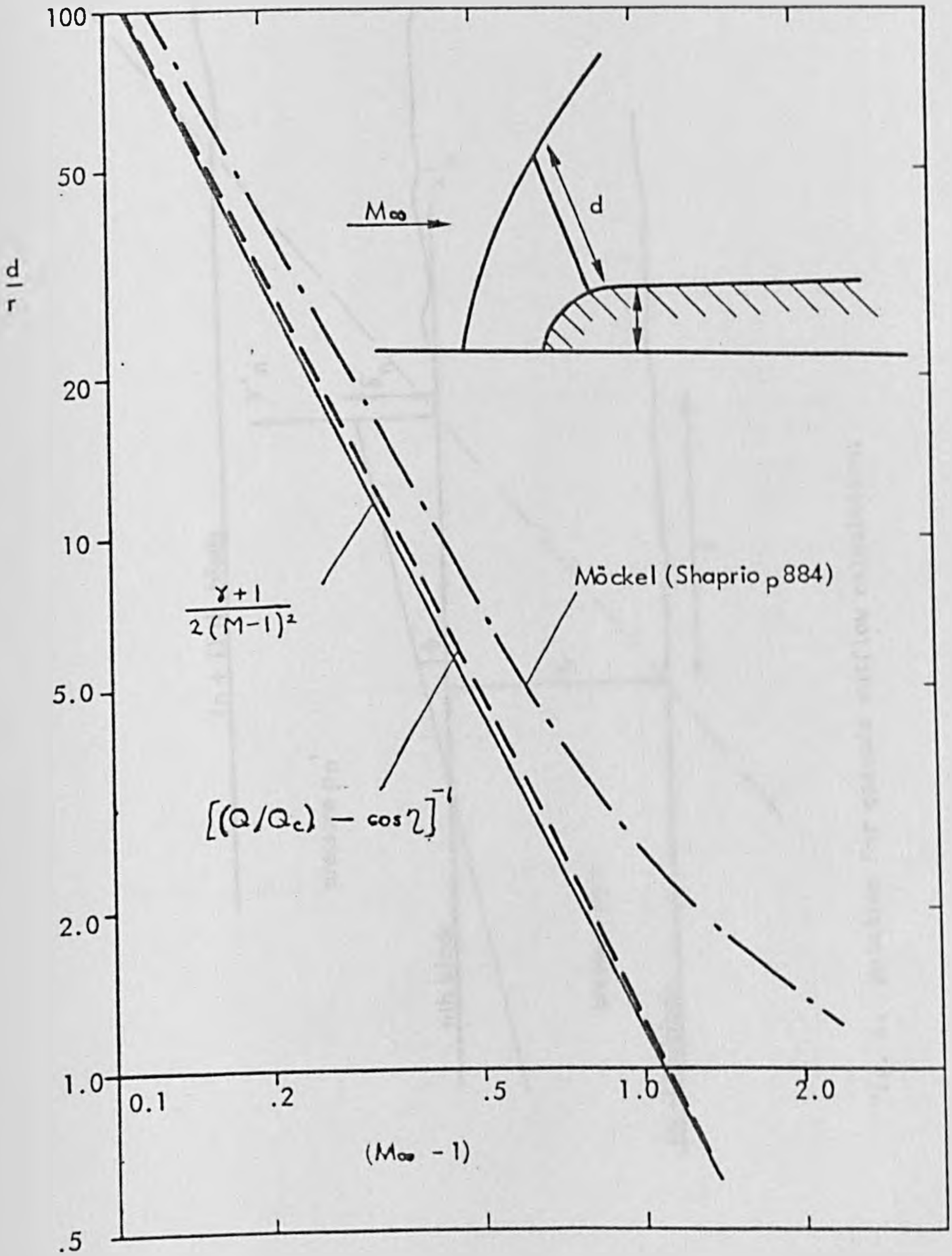

Fig. 5. Shock wave detachment - comparison of theories.

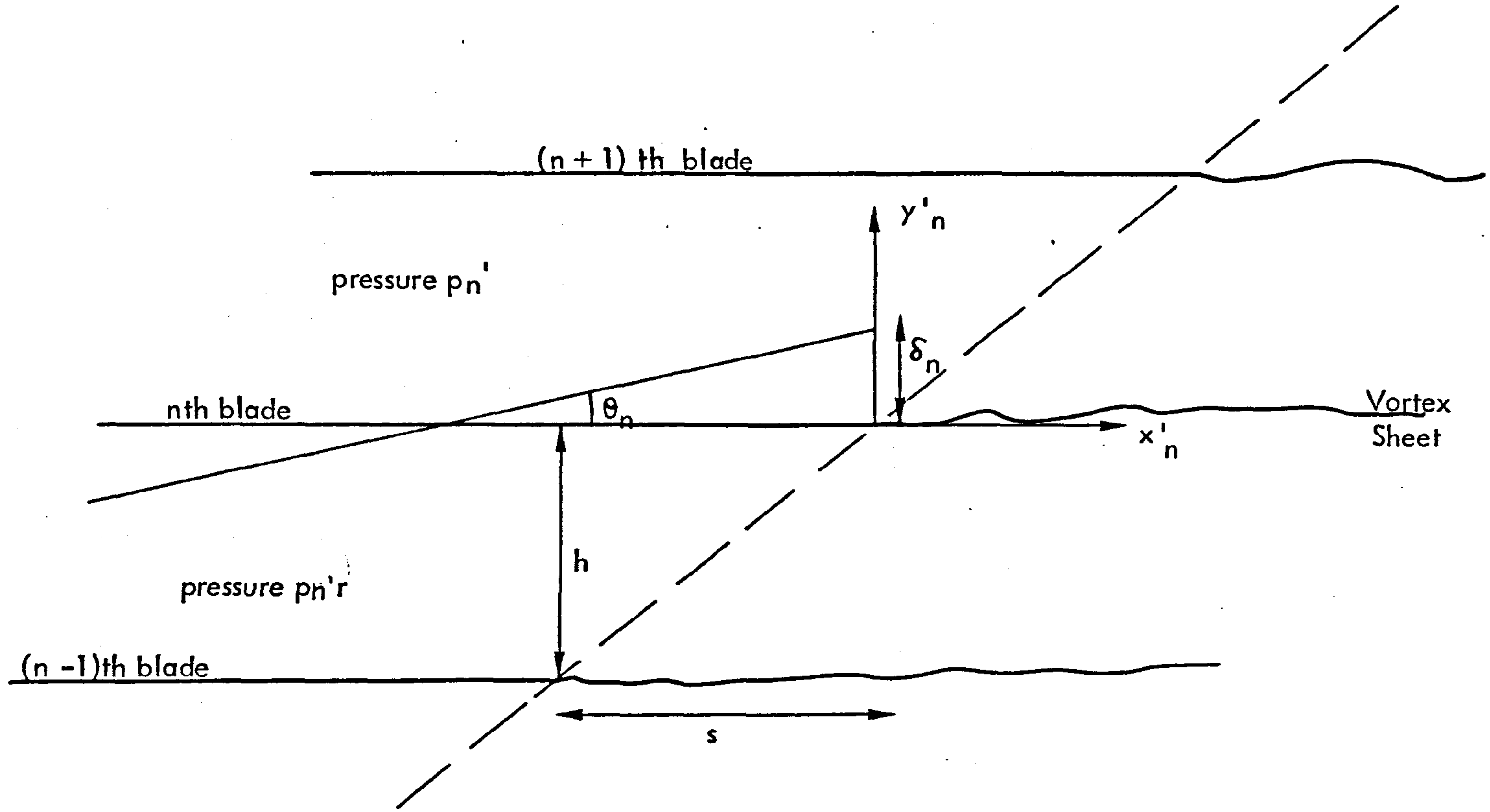

Fig. 6. Notation for cascade outflow calculation.

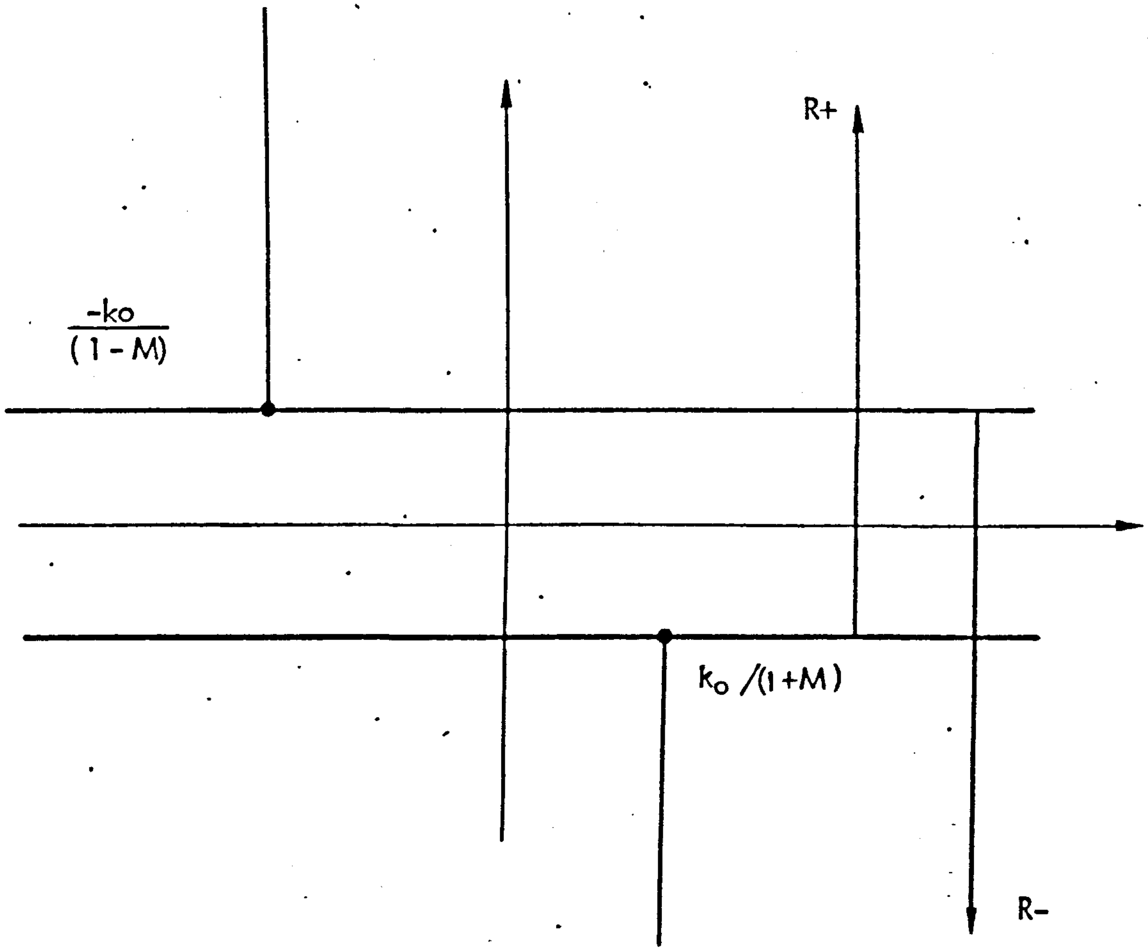

Fig. 7. The complex u-plane.

CHAPTER 6

THE SHOCK WAVES AHEAD OF A FAN WITH NON-UNIFORM BLADES

PART II: THE RELATION BETWEEN SHOCK STRENGTH AND BLADING PARAMETERS

ABSTRACT

This is the second part of an investigation into the relation between shock wave strength and blading non-uniformities for a supersonic aero engine fan in which the shock waves are detached from the blade leading edges. Part I of the paper contained some preliminary analyses which we use here to derive a simple relation between the strengths of the individual shock waves, the blade stagger angles and thicknesses. This theory is in good agreement with experiments, and so provides a theoretical basis for the blade shuffling procedures used to minimise blade-to-blade variation and to control shaft order tone generation by these variations.

1. INTRODUCTION

This is the second part of an investigation into the relation between the shock wave strengths and blading non-uniformities for a fan in which the shock waves are detached from the blade leading edges. Part I [1] contained two analyses which were necessary to the analysis in this part. The first of these was a general simplified analysis of the features of the detached shock wave ahead of a blunt-nosed body. This produced a relation between shock detachment and the dimensions of the body that is in good agreement with experiments [2] and forms the basis for the analysis that follows here. Second, Part I gave an analysis of the outflow from a cascade with non-uniformly staggered blades. This showed that at the nearly sonic Mach numbers that occur in the tip section blade passages of supersonic aero-engine fans, the perturbations in mass flow, due to the non-uniformities, only depend on the exit area of each blade passage. This established the validity of the approximation for the cascade outflow used in the present paper.

In this part we begin by considering a uniform cascade. A simple analysis, using conservation of mass flow alone, is used to derive a simple relation between the shock wave strength and the blade incidence and thickness. In the following section we look at perturbations of this condition, due to blading non-uniformities. The analysis is in two parts: first, the shock wave detachment at the leading edge of the blades is determined and then the propagation of the shock waves forward of the fan is calculated. As a result a simple relation between shock wave strength and the blading non-uniformities is obtained, which is in good agreement with experiments.

2. SHOCK WAVE DETACHMENT FOR A UNIFORM CASCADE

Shock Detachment Due to Blade Incidence

We consider a cascade of blades of stagger angle α and incidence i, as shown in Fig. 1. The suction surface Mach number is M_{ss}, and the upstream Mach number M_∞. The blades are assumed to be thin.

The mass flow through the characteristic AB, $\dot{m}_{AB}$, must be equal to the sum of the mass flow through the blade passage and the mass flow through the sonic line. The former is equal to the upstream mass flow per blade passage. With a blade spacing s (c. f. Part I equation 3),

$$Q_\infty s\cos(\alpha+i) = (d\cos\eta + s\cos\alpha)Q_{ss} - R_c Q_c d\,, \tag{1}$$

where the notation is defined in Part I, with Q the one-dimensional mass flow function, and R_c the presssure loss ratio. Transforming this equation, we find that

$$\frac{d}{s} = \frac{Q_{ss}\cos\alpha - Q_\infty\cos(\alpha+i)}{R_c Q_c - Q_{ss}\cos\eta}\,. \tag{2}$$

Since the incidence is always small in practice, we can expand the formula for small i. Then, changes in Mach number are small and Q_{ss} may also be expanded about Q_∞.

From the appendix, we find that

$$Q_{ss} = Q_\infty - i\sqrt{M_\infty^2-1}\,Q_\infty\,, \tag{3}$$

and hence

$$\frac{d}{s} = \frac{Q_\infty\cos\alpha - i\sqrt{M_\infty^2-1}\,Q_\infty\cos\alpha - Q_\infty\cos\alpha - Q_\infty i\sin\alpha}{R_c Q_c - Q_\infty\cos\eta}, \tag{4}$$

$$= i\left[\frac{\sin\alpha - \sqrt{M_\infty^2-1}\cos\alpha}{(R_c Q_c/Q_\infty) - \cos\eta}\right]. \tag{5}$$

This formula shows that shock detachment measured along the sonic line increases almost linearly with incidence, and is zero for zero incidence, which here is the "unique incidence condition".

In the design of turbofan compressors, a quantity often used is the capture area ratio (CAR). This is defined as the ratio of the mass flow through the characteristic AB to the net mass flow, i.e.

$$CAR = \frac{\dot{m}_{AB}}{\dot{m}_\infty} = \frac{\dot{m}_\infty + \dot{m}_c}{\dot{m}_\infty}, \qquad (6)$$

$$= 1 + \frac{d\,Q_c R_c}{s\,Q_\infty \cos(\alpha+i)}, \qquad (7)$$

$$= 1 + \frac{Q_c R_c\left(Q_{ss}\cos\alpha - Q_\infty \cos(\alpha+i)\right)}{Q_\infty \cos(\alpha+i)\left(R_c Q_c - Q_{ss}\cos\eta\right)}. \qquad (8)$$

Linearising with respect to i, this becomes

$$CAR = 1 + i\left[\frac{(Q_c R_c/Q_\infty)\left(\sin\alpha - \sqrt{M_\infty^2-1}\,\cos\alpha\right)}{\left((Q_c R_c/Q_\infty) - 1\right)\cos\alpha}\right]. \qquad (9)$$

For the RB211 at 92% speed, the condition where buzzsaw noise is important, we have $M_\infty = 1.4$, $\alpha = 60^\circ$, $i = 4^\circ$, $\eta = 13.4^\circ$ and $M_{ss} = 1.55$. To estimate R_c, we note that across a normal shock $R_c = 0.913$ at $M_\infty = 1.55$. This must apply to the air which is just spilled over the top of the aerofoil. For an oblique shock wave with a downstream Mach number of 1.4, the pressure ratio is $R_c = 0.964$. Accordingly, we estimate the average value of R_c as 0.964. Then using (9), we find that $d/s = 0.142$.

This is similar to the value* calculated by Stratford and Newby. Furthermore, it corresponds to a value of shock pressure rise $(\Delta P/P_\infty) = 0.35$, at half a blade spacing ahead of the fan. This is acceptably close to the measured value of 0.4 in view of both the approximation in the theory and the difficulties of making measurements of this pressure rise, when the flow is unsteady, and there are large blade to blade variations.

Shock Detachment Due to Thickness

We consider a blade which has a thickness t at the tip, and a thickness δ at the position where the characteristic AB (Fig. 2) meets the blade S.

The method of analysis, as before, is to match the mass flows across the sonic line and at infinity to the mass flow through the characteristic AB.

The mass flow through the characteristic AB is

$$\dot{m}_{AB} = \left(s\cos\alpha - \delta - (d+t)\right) Q_{ss} \,, \tag{10}$$

the mass flow through the sonic line is

$$\dot{m}_c = d R_c Q_c \,, \tag{11}$$

and the mass flow at infinity is

$$\dot{m}_\infty = s\cos(\alpha + i)\, Q_{ss} \,. \tag{12}$$

Hence, since $\dot{m}_\infty + \dot{m}_c = \dot{m}_{AB}$, we have

$$Q_{ss}\cos(\alpha+i) + \frac{d}{s} Q_c R_c = \left(\cos\alpha - \frac{\delta}{s} + \frac{t}{s}\cos\eta\right) Q_{ss} + \frac{d}{s}\cos\eta\, Q_{ss} \,, \tag{13}$$

and therefore

$$\frac{d}{s} = \frac{\left[Q_{ss}\cos\alpha - Q_c\cos(\alpha+i)\right] + \left[(t/s)\cos\eta - (\delta/s)\right] Q_{ss}}{\left[R_c Q_c - Q_{ss}\cos\eta\right]} \,. \tag{14}$$

This shows that if the tip alone is increased in thickness without altering the downstream thickness δ, the change in

* Actual Value 0.42

shock detachment, (d/s), is then given by

$$\Delta\left(\frac{d}{s}\right) = \Delta\left(\frac{t}{s}\right) \cdot \frac{\cos\eta}{\left[(R_c Q_c / Q_{ss}) - \cos\eta\right]} . \qquad (15)$$

This is essentially the same formula as that for an isolated aerofoil (Part I equation 4). An interesting feature of the formula (14), is that if both the tip and upstream thickness increase together (for instance if there is a uniform thickening of the whole blade), then there is virtually no change in shock detachment.

3. SHOCK DETACHMENT FOR NON-UNIFORM BLADING

Effect of Blade Stagger Angle Changes

We consider the geometry shown in Fig. 3, where the blades have perturbations α_n and α_{n-1} respectively, in stagger angle. We assume for simplicity that these perturbations occur about the same point in each blade. There is no real basis for making this assumption except that if the blades are twisted by different amounts during manufacture, the twist will be about the same axis for each blade.

The basis of the solution is to consider the mass flows through a box formed by the characteristic AB, the sonic line and the outflow from the blade passage (Fig. 3). These mass flows are balanced by continuity, and are then expanded about their mean values to first order in the perturbation quantities $\alpha, \Delta d$. The shock detachment distance can then be determined.

We consider first the change in mass flow through the exit of the cascade. This is assumed to depend only on the area. The reasons for making this assumption are as follows. Physically, since the cascade discharges into what is effectively a constant pressure sink, the outlet pressure on DE must be constant, and unaffected by the non-uniformity, and since the flow is nearly sonic the mean flow will be relatively insensitive to any variations in Mach number. As the stagnation temperature and pressure are constant, and (dQ/dM) is $O((1-M)^2)$, as $M \to 1$, the mass flow will be proportional to area. Furthermore, the linearised analysis of Part I has shown that this is indeed asymptotically true (as $M \to 1$) for low harmonic order variations in the blade positions. The area to be used in these calculations is an effective area (rather than the true, geometric, area), and should account for the presence of the blade boundary layer. We will ignore the variation in the latter with blading non-uniformity.

If these arguments are accepted we have, in the usual notation, the increase in area per unit span as $(c-x)(\alpha_n - \alpha_{n-1})$ Hence the change in mass flow is

$$\Delta \dot{m}_{ED} = (c-x)(\alpha_n - \alpha_{n-1}) R_p A_p Q_p \, , \qquad (16)$$

where R_p accounts for the loss in total pressure between the inlet and exit of the box, A_p accounts for the contraction in stream tube height that occurs on a real fan, and represents, therefore, some attempt to take three-dimensional effects into account.

The change in mass flow through the sonic line is

$$\Delta \dot{m}_{BC} = \Delta d \, Q_c R_c \, . \qquad (17)$$

Here there is no stream tube contraction, and, furthermore, we assume that the change in the pressure loss is negligible. This is reasonable, since the loss is in any event a small quantity.

We now consider the mass flow change through AB, which is

$$\Delta \dot{m}_{AB} = \left[Q_{ss}+\Delta Q_{ss}\right]\left[(d+\Delta d)\cos(\eta-\alpha_n) + \left(\left(x(\alpha_{n-1}-\alpha_n)/\cos\alpha\right)+s\right) \cdot \cos(\alpha+\alpha_{n-1})\right] - Q_{ss}(d\cos\eta + s\cos\alpha). \tag{18}$$

Expanding in the perturbation quantities ΔQ, α_n, α_{n-1}, Δd, we obtain

$$\Delta \dot{m}_{AB} = \Delta Q_{ss}(d\cos\eta + s\cos\alpha) - Q_{ss}\left[\alpha_{n-1} s\sin\alpha - \Delta d\cos\eta - \alpha_n d\sin\eta - x(\alpha_{n-1}-\alpha_n)\right].$$

We now use the results in the appendix, to express ΔQ in terms of α_{n-1}, giving

$$\frac{\Delta Q_{ss}}{Q_{ss}} = \alpha_{n-1}\beta_{ss}, \tag{20}$$

where $\beta_{ss} = \sqrt{M_{ss}^2 - 1}$.

We then find that

$$\Delta \dot{m}_{AB} = Q_{ss}\left(\alpha_{n-1}\beta_{ss}(d\cos\eta + s\cos\alpha) - \alpha_{n-1} s\sin\alpha + \Delta d\cos\eta + x(\alpha_{n-1}-\alpha_n) + \alpha_n d\sin\eta\right). \tag{21}$$

Since the net mass in-flow into ABCDE, must vanish, we have

$$\Delta \dot{m}_{AB} = \Delta \dot{m}_{BC} + \Delta \dot{m}_{DE}, \tag{22}$$

so that

$$Q_{ss}\left(\alpha_{n-1}\beta_{ss}(d\cos\eta + s\cos\alpha) - \alpha_{n-1} s\sin\alpha + \Delta d\cos\eta + x(\alpha_{n-1}-\alpha_n) + \alpha_n d\sin\eta\right) = (c-x)(\alpha_n-\alpha_{n-1})R_p A_p Q_p + \Delta d R_c Q_c, \tag{23}$$

and therefore

$$\frac{\Delta d}{d} = \left[\frac{Q_p}{Q_c}\frac{c}{s}R_p A_p(\alpha_n-\alpha_{n-1}) + \frac{x}{s}(1-A_pR_p)(\alpha_{n-1}-\alpha_n)(\alpha_{n-1}-\alpha_n) + \alpha_{n-1}\beta_{ss}\left(\frac{d}{s}\cos\eta + \cos\alpha\right) - \alpha_{n-1}\sin\alpha + \frac{d}{s}\alpha_n\sin\alpha\right] \cdot \left[\frac{R_cQ_c}{Q_{ss}} - \cos\eta\right]^{-1}. \tag{24}$$

There are several noteworthy features to this formula. First, it has the multiplying factor $\left[(R_c Q_c/Q_{ss}) - \cos\eta\right]^{-1}$ found in all these problems. Second, considering the terms in the numerator, $(1 - R_p A_p)x/s$ is much less than $(c/s)(R_p A_p)$ and may be neglected. Third, $(d/s)\,\alpha_n \sin\eta$ is also small (both (d/s) and $\sin\eta$ are small) and is neglected.

For the RB211, at the conditions of interest, we estimate the following quantities:

$R_pA_p = 0.9$, $x/s = 0.6$, $d/s = 0.15$, $c/s = 1.25$

$R_c = 0.95$, $\quad = 13^{\circ}$. $Q_c/Q_{ss} = 1.21$, $M_s = 1.55$, $\cos y = 0.98$

$x = 60^{\circ}$, $\quad_{ss} = 1.18$. The typical value of $\quad_n = 0.2^{\circ}$

Then

$$\frac{\Delta d}{d} = 48\left(\alpha_n - \alpha_{n-1}\right) - 4\,\alpha_{n-1}, \tag{25}$$

where α is in radians. Converting α into degrees gives

$$\frac{\Delta d}{d} = 0{\cdot}84\left(\alpha_n - \alpha_{n-1}\right) - 0{\cdot}07\,\alpha_{n-1}. \tag{26}$$

Change in Shock Detachment Due to Blade Thickness Changes

In this section, we discuss the change in shock detachment due to the change in blade thickness. We assume that the cascade of blades is uniform and therefore that thickness and stagger are uncoupled. This is a reasonable assumption and likely to be valid, at least for the small changes encountered in practice. The method of analysis is essentially that used above. We balance the mass flow through a box bounded by the blades, sonic line, trailing edge plane and the characteristic from the blade to the sonic line: see Fig. 4.

The change in mass flow through AB is again dependent on areachange, which is $-(\delta_n + \delta_{n-1})$. The change in mass flow is simply

$$\Delta \dot{m}_{AB} = -\,Q_p A_p R_p\left(\delta_n + \delta_{n+1}\right). \tag{27}$$

The result in this form is open to two objections. First, the thicknesses ought really to incorporate the boundary layer thicknesses, which may not be negligible for these transonic flows. Second, because the streamlines on either side of the blade converge as the flow leaves the blade, the area of the flow should account for the wake and its downstream mixing. But nevertheless, we believe that the assumptions leading to (27) are sufficiently good for our purposes.

The change in mass flow through the sonic line is

$$\Delta \dot{m}_{AB} = R_c Q_c \Delta d \tag{28}$$

and the change in mass flow through AB is

$$\Delta \dot{m}_{AB} = \left(-h_{n-1} + (\Delta d_n + \Delta t_n)\cos\eta\right) Q_{ss}. \tag{29}$$

Therefore, since

$$\Delta \dot{m}_{AB} = \Delta \dot{m}_{BC} + \Delta \dot{m}_{ED} , \tag{30}$$

$$Q_{ss}\left[(\Delta d_n + \Delta t_n)\cos\eta - h_{n-1}\right] = \Delta d R_c Q_c - Q_p A_p R_p (\delta_n + \delta_{n-1}), \tag{31}$$

so that

$$\Delta d = \frac{\left[\Delta t_n \cos\eta + (Q_p A_p R_p / Q_{ss})(\delta_n + \delta_{n-1}) - h_{n-1}\right]}{\left[(Q_c R_c / Q_{ss}) - \cos\eta\right]}. \tag{32}$$

This has the usual denominator $\left[(Q_c R_c / Q_s) - \cos\eta\right]$, and otherwise depends on four thicknesses: the actual leading edge thickness, Δt_n, the trailing edge thicknesses $(\delta_n , \delta_{n+1})$, and the upstream thickness of the preceding blade h_{n-1}. Because of this dependence on four variables, it is difficult to apply the relation to a fan. Of these four, only the leading edge thickness is important, since its percentage variation is the largest, the blade leading edges being very thin and hand-finished. Then

$$\frac{\Delta d}{d} = \frac{\Delta t_n}{d} \cdot \frac{\cos\eta}{\left((Q_c R_c / Q_{ss}) - \cos\eta\right)} \quad . \tag{33}$$

In this relation, we note from Section 2 that substitution for d gives

$$\frac{\Delta d}{d} = \frac{\Delta t_n}{s} \cdot \frac{\cos\eta}{\left(\cos\alpha - \cos(\alpha + i) Q_\infty / Q_{ss}\right)} \quad . \tag{34}$$

This formula applies when the uniform component of shock stand-off distance is dominated by spillage rather than thickness. Using the same values for the parameters as in Section 2 we find that

$$\frac{\Delta d}{d} = 33 \cdot 5 \, \frac{\Delta t_n}{s} \; .$$

For typical blades, s = 8" at the tip, and, typically, $\Delta t_n = 0 \cdot 01''$ (standard deviation of measurements). This corresponds to $\left(\frac{\Delta d}{d}\right) = 0 \cdot 04$, which is a much smaller percentage change than that due to blade stagger angle variations. We note in passing that it has been found that it is the leading edge thickness which correlates well with shock strength.

This completes the calculation of shock detachment due to blading non-uniformity.

4. VARIATION OF SHOCK STRENGTH WITH SHOCK DETACHMENT

The objective of this section is to determine the variation in the positions of the shock waves as the detachment distance is altered. To do this we consider a shock wave from a single blade, as shown in Fig. 5. In this figure, the shock wave propagates forward into a uniform flow whose characteristics are at a constant angle $\bar{\mu}_s$ to the downstream flow. The co-ordinates of a point on the shock are r, ψ.

We analyse the problem using weak shock theory [3]. In that theory, the shock is shown to bisect the characteristics intersecting it from upstream and downstream. For the Mach numbers encountered here this approximation should be adequate since at no point (except perhaps close to the sonic line) are the shock waves in any sense strong.

Then from Fig 5

$$\tan\left(\frac{\pi}{2} - \psi + \left(\frac{\psi + \bar{\mu}_s}{2}\right)\right) = -\frac{1}{r}\frac{dr}{d\psi} \, , \qquad (35)$$

which can be rewritten as

$$\cot\left(\frac{\psi - \bar{\mu}_s}{2}\right) = -\frac{1}{r}\frac{dr}{d\psi} \, . \qquad (36)$$

Integrating from $\psi = \psi_0$, the sonic line, to ψ and from $r = d$ to r gives

$$\int_{\psi_0}^{\psi} \cot\left(\frac{\psi - \bar{\mu}_s}{2}\right) d\psi = -\int_d^r \frac{dr}{r} \, , \qquad (37)$$

or

$$\int_{(\psi - \bar{\mu}_s/2}^{(\psi - \bar{\mu}_s)/2} 2\cot\theta \, d\theta = \ln(d/r) \, . \qquad (38)$$

Thus

$$\ln\left[\frac{\sin \frac{1}{2}(\psi - \bar{\mu}_s)}{\sin \frac{1}{2}(\psi_0 - \bar{\mu}_s)}\right] = \frac{1}{2}\ln\left(\frac{d}{r}\right) ; \qquad (39)$$

that is, the shape of this part of the shock wave may then be written

$$\frac{r}{d} = \left[\frac{\sin \frac{1}{2}(\psi_0 - \bar{\mu}_s)}{\sin \frac{1}{2}(\psi - \bar{\mu}_s)}\right]^2 . \qquad (40)$$

Using this relation, it is relatively straightforward to calculate the change in shock positions with $\bar{\mu}_s$ (which depends on the upstream Mach number), ψ_o (which depends on the downstream Mach number) and d.

Examining the above formula, we find that for the fan under discussion, the maximum value of $(\psi - \bar{\mu}_s)$ occurs at the sonic line, where $\psi = 103^\circ$, $\bar{\mu}_s = 40^\circ$ ($M_s = 1.55$). Then $|(\psi - \mu_s)/2| < 32^\circ$ and we can approximate (40) as

$$\frac{r}{d} = \left[\frac{\psi_o - \bar{\mu}_s}{\psi - \bar{\mu}_s}\right]^2 . \tag{41}$$

This formula may be expressed in terms of θ, the angle between the characteristic and the sonic line Since the sum of this angle and ψ is the same for all Mach numbers, and at the upstream Mach number $\psi_s = \bar{\mu}_s$, it follows that

$$(\bar{\mu}_s + \theta_s) = (\psi + \theta) .$$

Then substituting in (41), and noting that $\theta_o = 0$,

$$\frac{r}{d} = \left(\frac{\theta_s}{\theta_s - \theta}\right)^2 \tag{42}$$

Since θ is a function of Mach number alone, this formula may be used to calculate the Mach number, and hence the static pressure rise at any position on the shock.

For a uniformly bladed fan, we choose a co-ordinate system such that $X = r \sin(\psi + \alpha)$, $Y = -r\cos(\psi + \alpha)$. Then X, Y are along the blade leading edge line and perpendicular to the cascade, α is the stagger angle.

The calculated shape of the initial portion of the shock wave is shown in Fig.6. This calculation covers the region of the shock wave before it meets the expansion fan from the leading edge of the preceding blade. It is clear then that we now know the pressure rise as a function of Y/X. This is plotted on Fig.7, for the usual fan conditions, and demonstrates the expected rapid decay of pressure behind the shock.

An alternative plot is of the variation of shock amplitude with distance away from the fan face, ie. of $\Delta P/P_\infty$ as a function of X/d, Fig.8. This shows the expected decay of the shock strength ahead of the fan disc.

To relate the pressure rise at the shock to the changes in the detachment distance d, we use a small perturbation analysis. To analyse the problem properly we have to account for the changes in both blade orientation and upstream Mach number. However, it can be shown that the effects just referred to are small compared with that of the change in the shock detachment distance. If, therefore, we only account for the change in shock detachment distance d, we can write $(\Delta P/P_\infty) = f(Y/d)$ and then for small changes we can write

$$\Delta\left(\frac{\Delta P}{P_\infty}\right) = -f'(Y/d)\cdot\frac{Y}{d}\cdot\left(\frac{\Delta d}{d}\right) \quad . \tag{43}$$

For the fan here, Y/s = 0.5, d/s = 0.125, giving Y/d = 4 and f' = 0.1. Then $\left(\frac{\Delta P}{P_\infty}\right) = 0{\cdot}4\left(\frac{\Delta d}{d}\right)$,

and substituting for $\left(\Delta d/d\right)$ from equation (25) we obtain the required relation between pressure rise and blade incidences,

$$\Delta\left(\frac{\Delta P}{P_\infty}\right)_n = -0.4\left[0.84\left(\alpha_n - \alpha_{n-1}\right) - 0.07\,\alpha_{n-1}\right]. \tag{44}$$

This is to be compared with the relation determined experimentally by Stratford and Newby [2].

$$\Delta\left(\frac{\Delta P}{P_\infty}\right)_n = -0.194\left(\alpha_n - \alpha_{n-1}\right) \tag{45}$$

Two things are clear about these results. First, in our relation, there is an extra weak dependence on α_{n-1}, for fixed $(\alpha_n - \alpha_{n-1})$. Second, we have greatly over-estimated the rate of change of $(\Delta P/P_\infty)_n$ with $(\alpha_n - \alpha_{n-1})$. There are a number of possible reasons for this. The prime one is the extreme sensitivity of the result to the steady detachment distance. Substituting in (45) for Δd and d, we find that

$$\Delta\left(\frac{\Delta P}{P_\infty}\right)_n = -f'\left(Y/d\right)\cdot\left(\frac{\Delta d}{d}\right)\left(\frac{Y}{d}\right) \tag{46}$$

$$= -f'\left(Y/d\right)\frac{sY}{d^2}\,\frac{\left[\frac{Q_p}{Q_{ss}}\cdot\frac{c}{s}R_pA_p + \frac{\tau}{s}\left(1 - A_pR_p\right)\left(\alpha_n - \alpha_{n-1}\right)\right]}{\left[\left(Q_cR_c/Q_{ss}\right) - \cos\eta\right]}. \tag{47}$$

In this expression we have neglected a part proportional to α_{n-1} and in the numerator, only the $\left(\frac{Q_p}{Q_{ss}}\cdot\frac{c}{s}\cdot R_pA_p\right)$ term is significant. Now, since f' is relatively insensitive to the actual value of d chosen and

$$d \propto \left(\left(Q_c/Q_{ss}\right)R_c - \cos\eta\right)^{-1}$$

the result is, in effect, proportional to 1/d.

Substituting for d from Section 2 gives

$$\Delta\left(\frac{\Delta P}{P_\infty}\right)_n = -f'\left(\frac{Y}{d}\right)\cdot\frac{Y}{s}\,\frac{\left(\left(Q_c R_c/Q_{ss}\right)-1\right)\left(\left(Q_p/Q_{ss}\right)\left(\frac{c}{s}\right)R_p A_p + \frac{x}{s}\left(1-R_p A_p\right)\left(\alpha_n-\alpha_{n-1}\right)\right).}{\left(\cos\alpha - \left(Q_{ss}/Q_{ss}\right)\cos(\alpha+i)\right)}$$

Now we note that the numerator and denominator in this expression are both small differences between larger quantities, and the result is therefore very sensitive to small changes in the parameters used. In particular we note the sensitivity to R_c . If R_c were reduced to 0.9, $\left(\frac{Q_c R_c}{Q_{ss}} - \cos\eta\right)$ would be reduced from 0.17 to 0.112 and the result (44) would be more similar to Stratford and Newby's experimental relation. Also noteworthy is the approximate variation as i^{-2}, indicating sensitivity to this quantity also.

CONCLUSIONS

In this paper, we have devised a relatively simple theory for the strengths of the shock waves found ahead of a transonic compressor having non-uniform blading. The theory shows that the shock amplitudes are proportional to the differences between successive blade stagger angles, in agreement with the experimental results of Newby and Stratford. These shock strengths are also dependent on the changes in the thickness of the blades at a number of different points on the blades. Of these thioknesses, that at the leading edge is probably the most important. For typical variations in each, the effect of stagger angle variation is four times that of thickness variation.

While the general dependence of shock amplitude on stagger angle is correctly predicted, the rate of change is not. This is possibly due to the use of inadequate numerical data rather than any defect in the theory itself. The slope of the shock amplitude/stagger curve depends on factors that are very sensitive to the conditions used. In particular, it is sensitive to mean blade incidence and to the losses assumed. Neither of these is accurately known in the present context. The strong dependence on incidence does, however, suggest a method of controlling the source of buzz-saw noise. As incidence increases, so does the average shock detachment. But this causes the <u>extra</u> detachment due to the blade non-uniformity to decreases as a percentage of its mean value, with a resulting decrease in the shock amplitude,

There are several ways in which the analysis could be improved. First, it is clear that the correct values of incidence and loss factors are critical, and some way must be found of accurately determining them. Second, and despite the analysis of Part I, one of the most questionable assumptions in the theory is that for the outflow from the cascade. While Part I justified the assumption that it depended an area alone, it did so on the basis of a linearised analysis, which may be somewhat in error at these high Mach numbers. However, this is likely to be a smaller effect than that of the boundary layer. As the shock strength and position change, so will the boundary layer thickness and this in turm will alter the effective outlet area. To calculate this effect properly would be most difficult.

But in any event the analysis here does provide a description of the flow that is consistent with the observed relationship between the shock strengths and the blading non-uniformity. As such it is about as far as it is worth going with purely analytical means.

ACKNOWLEDGEMENT

The author wishes to thank the Directors of Rolls-Royce Limited for permission to publish the paper, and the Science Research Council for an Industrial Studentship. He wishes to acknowledge the helpful advice of Professor D G Crighton.

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Small perturbations of compressible flow

(a) Two-dimensional flow

We are concerned with small perturbations to two-dimensional compressible flow, and thence with determining the changes in Mach number, and flow per unit area due to changes in flow angle.

We consider a pressure wave of form $p(x-\beta y)$ where $\beta = (M^2-1)^{1/2}$; then, from the linearised flow equations

$$U_j \frac{\partial u_i}{\partial x_j} + \frac{1}{\rho}\frac{\partial p}{\partial x_i} = 0 , \quad \text{(A1)}$$

$$U_i \frac{\partial \rho}{\partial x_i} + \rho \frac{\partial u_i}{\partial x_i} = 0 , \quad \text{(A2)}$$

where u_i is the velocity perturbation, we find, for a flow angle perturbation α , that

$$\frac{u}{U} = -\frac{\alpha}{\beta} , \quad \frac{v}{U} = \alpha , \quad \frac{p}{p} = \gamma M^2 \alpha , \quad \frac{\Delta c}{c} = \frac{\gamma-1}{2}\frac{M^2\alpha}{\beta} ,$$

$$\frac{\Delta \rho}{\rho} = \frac{M^2 \alpha}{\beta} , \quad \text{(A3)}$$

where u , v are the velocity perturbations parallel and perpendicular to the mean flow. The change in Mach number ΔM is determined from

$$\frac{\Delta M}{M} = \frac{\Delta u}{u} - \frac{\Delta c}{c} = -\frac{\alpha}{\beta} - \frac{\gamma-1}{2}\frac{M^2\alpha}{\beta} \quad \text{(A4)}$$

or

$$\frac{\Delta M}{M} = -\frac{\alpha}{\beta}\left(1+\frac{\gamma-1}{2}M^2\right). \qquad \text{(A5)}$$

Then, since the one-dimensional flow function which describes flow along a stream tube satisfies

$$\frac{\Delta Q}{Q} = \frac{\Delta M}{M} \cdot \frac{(1-M^2)}{\left(1+\frac{\gamma-1}{2}M^2\right)} \qquad \text{(A6)}$$

we find that

$$\frac{\Delta Q}{Q} = \alpha\beta. \qquad \text{(A7)}$$

(b) Prandtl-Meyer Relations

We use the definitions of Houghton and Brock [4] ; the notation is illustrated in Fig.9.

Then with the Mach angle $\mu = \sin^{-1}(1/M)$, we find that

$$\Delta\mu = \frac{-\Delta M}{M}\frac{1}{\sqrt{(M^2-1)}}. \qquad \text{(A8)}$$

By definition, the angle ν is equal to the flow angle α plus a constant. Therefore

$$\Delta\nu = \frac{(M^2-1)^{1/2}}{\left(1+\frac{\gamma-1}{2}M^2\right)} \cdot \frac{\Delta M}{M}. \qquad \text{(A9)}$$

We also define another angle θ by (see Houghton and Brock 4)

$$\theta = \sqrt{\frac{\gamma+1}{\gamma-1}} \cos^{-1}\left(\frac{(\gamma+1)}{2+(\gamma-1)M^2}\right)^{1/2} . \qquad \text{(A10)}$$

Then

$$\Delta\theta = \sqrt{\frac{\gamma+1}{\gamma-1}} \frac{\gamma+1}{\left(2+(\gamma-1)M^2\right)^{3/2}} \frac{M\,\Delta M\,(\gamma-1)}{\left(1-(\gamma+1)/(2+(\gamma-1)M^2)\right)^{1/2}} \qquad \text{(A11)}$$

$$= -\frac{\Delta M}{2M} \frac{(\gamma+1)}{\sqrt{(M^2-1)}} \cdot \frac{M^2}{\left(1+\frac{\gamma-1}{2}M^2\right)} . \qquad \text{(A12)}$$

As a check we note that

$$\Delta\nu - \Delta\theta = \frac{\Delta M}{M} \frac{\left(M^2-1-(\gamma+1)M^2/2\right)}{(M^2-1)^{1/2}\left(1+(\gamma-1)M^2/2\right)} \qquad \text{(A13)}$$

$$= -\frac{\Delta M}{M} \frac{1}{\sqrt{(M^2-1)}} . \qquad \text{(A14)}$$

Since $\theta + \mu - \nu = \frac{\pi}{2}$, this is the expected result.

Also of interest is the angle of the sonic line to the free stream: this is $\frac{\pi}{2} - \eta$ in the notation used for the estimation of the shock detachment distance. Now

$$\frac{\pi}{2} + \eta = \theta + \mu ; \qquad \text{(A15)}$$

therefore

$$\Delta\eta = \Delta\theta + \Delta\mu \quad , \tag{A16}$$

$$= \frac{\Delta M}{M\sqrt{M^2-1}}\left[\frac{(\gamma+1)M^2/2}{(1+(\gamma-1)M^2/2)} - 1\right]. \tag{A17}$$

We expand some of these quantities for $M=(1+\varepsilon)$: from

$$\Delta\mu = -\frac{\Delta\varepsilon}{\sqrt{2\varepsilon}} \quad , \tag{A18}$$

we have

$$\mu = \frac{\pi}{2} - \sqrt{2(M^2-1)} + O(\varepsilon^{3/2}). \tag{A19}$$

Similarly,

$$\Delta\theta = \frac{\Delta\varepsilon}{\sqrt{2\varepsilon}} \quad , \tag{A20}$$

and

$$\theta = \sqrt{2(M^2-1)} + O(\varepsilon^{3/2}) \, , \tag{A21}$$

and since

$$\Delta\nu = \Delta(\theta+\mu) \, , \quad \nu \sim O(\varepsilon^{3/2}). \tag{A22}$$

Further $\Delta\eta = \dfrac{\Delta\varepsilon\sqrt{2\varepsilon}}{(\gamma+1)/2}$, so that $\eta \sim O(\varepsilon^{3/2})$ also.

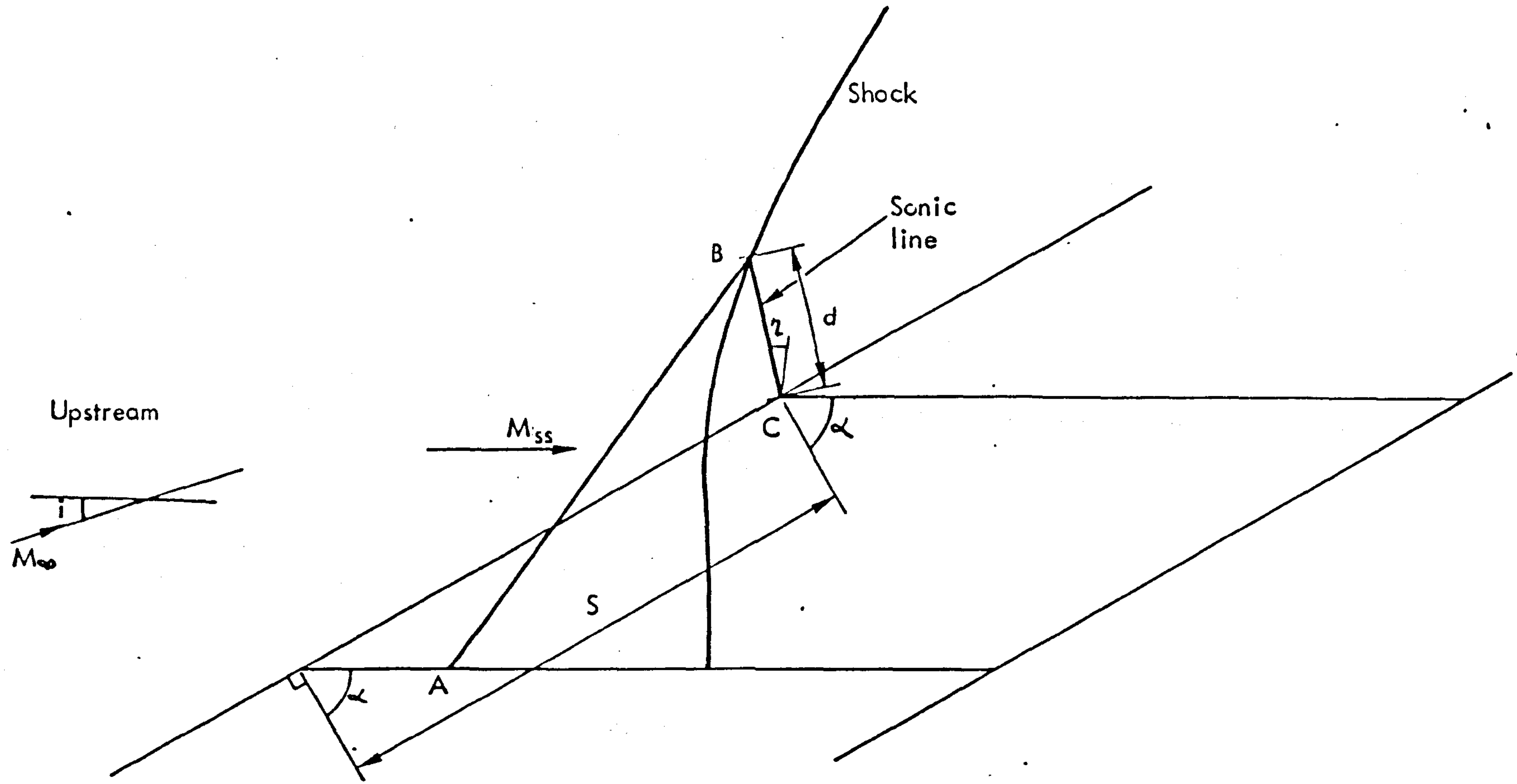


Fig. 1. Notation for calculation of shock wave detachment due to blade incidence angle - uniform blades.

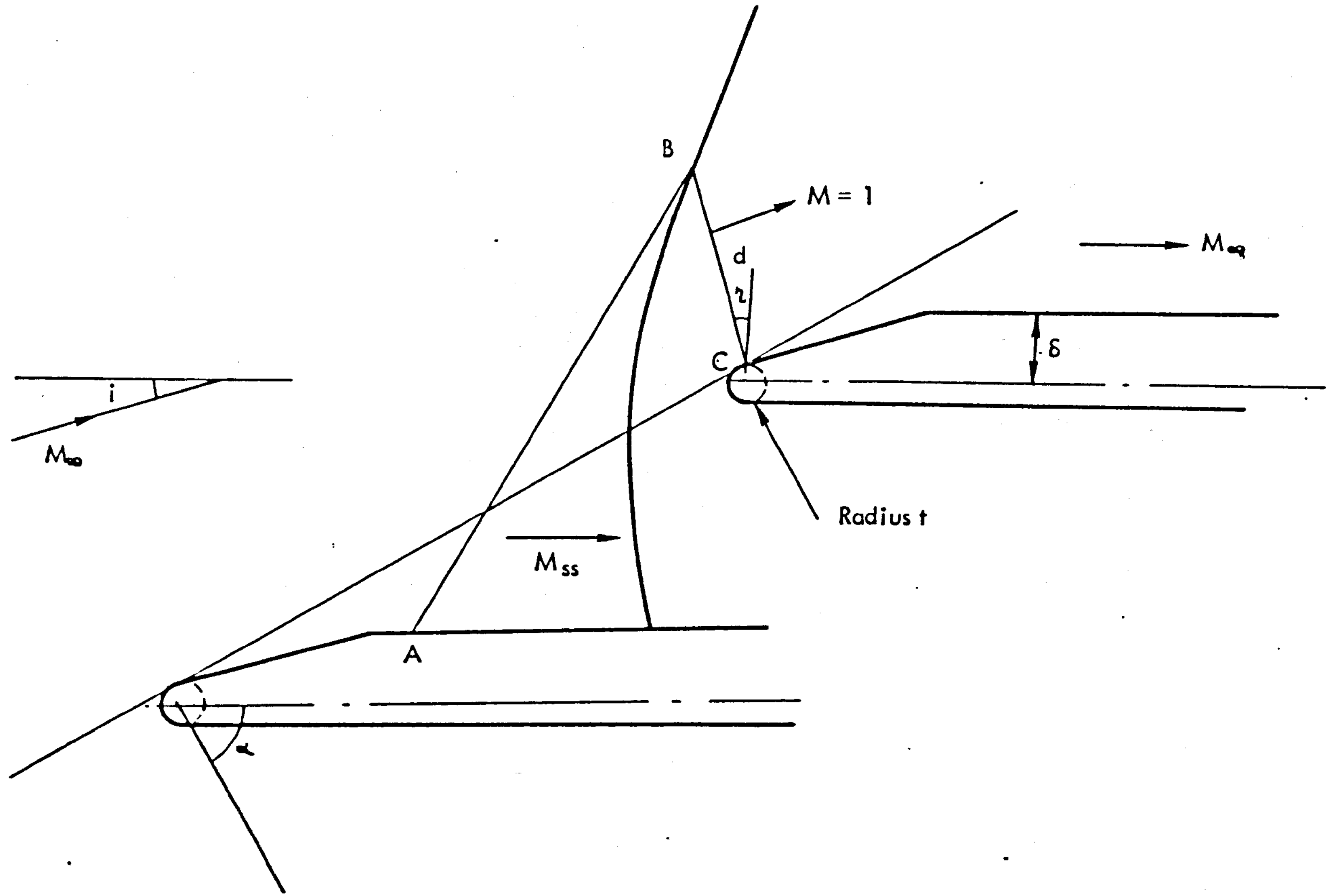


Fig. 2. Notation for calculation of shock wave detachment due to thickness variations - uniform blades.

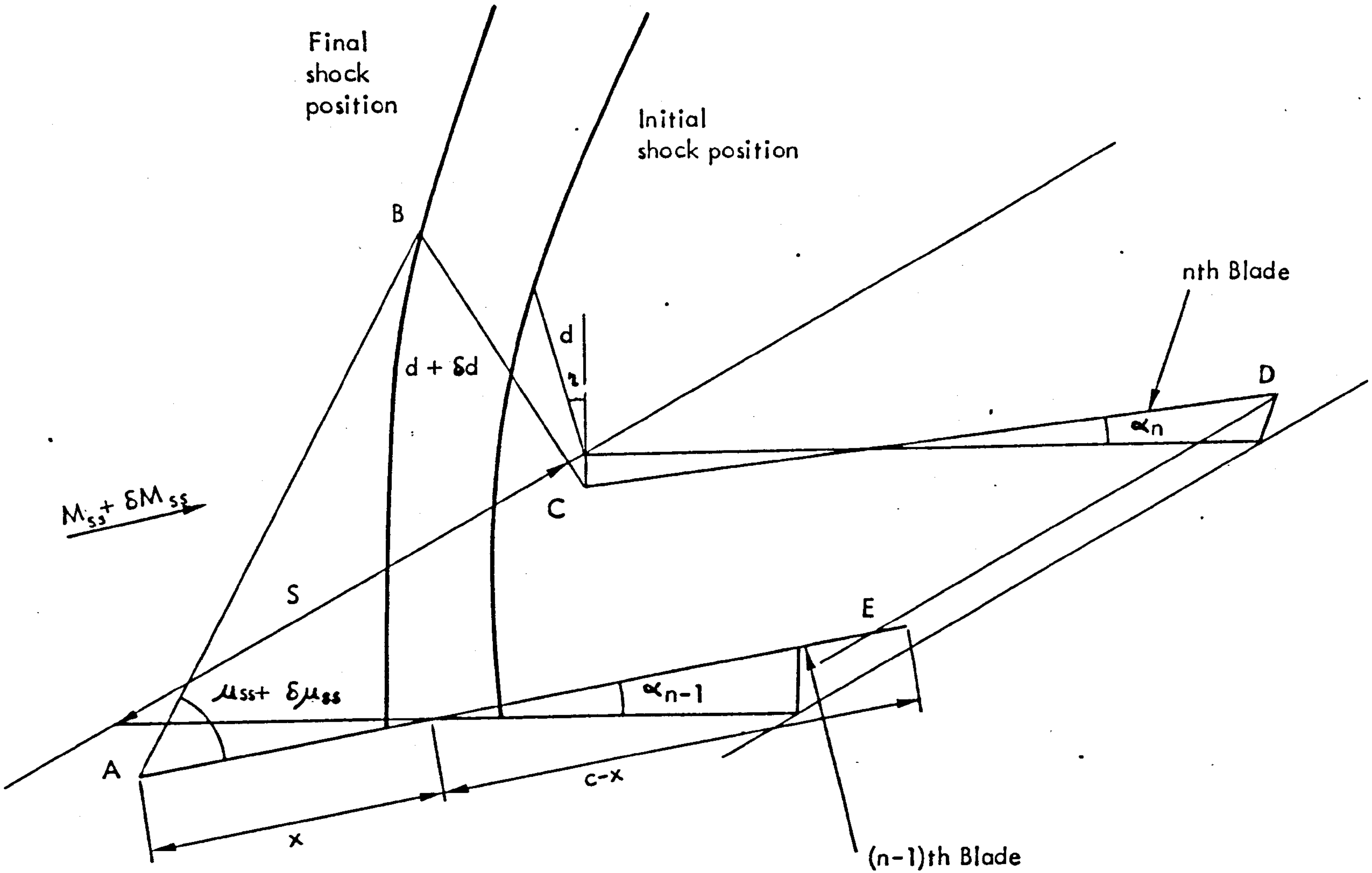


Fig. 3. Notation for calculation of shock wave detachment due to stagger angle variations.

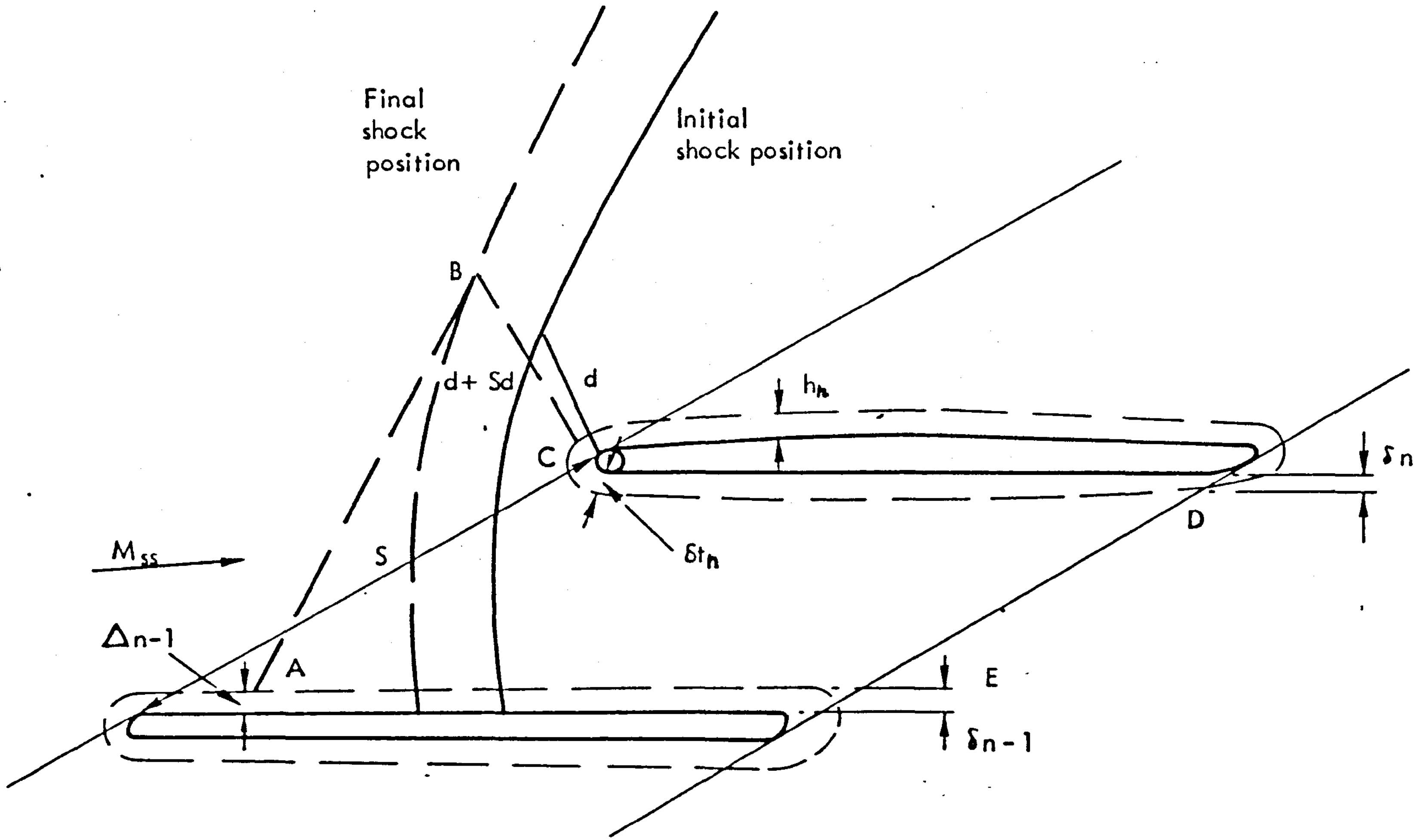


Fig. 4. Notation for calculation of shock wave detachment due to thickness variations - non-uniform blades.

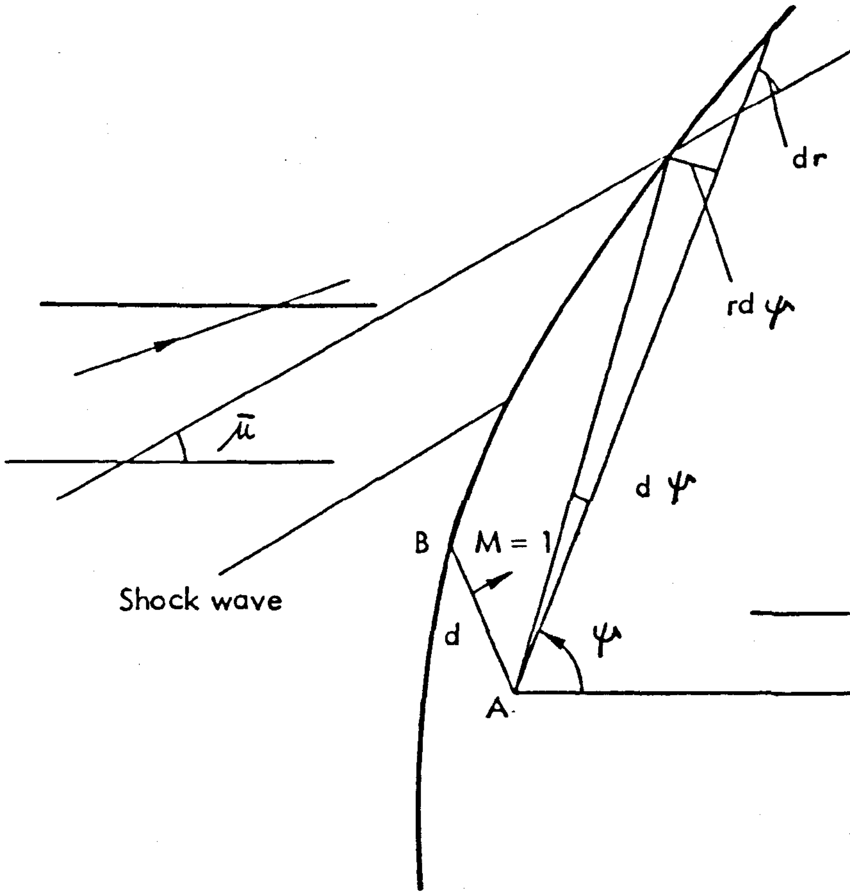
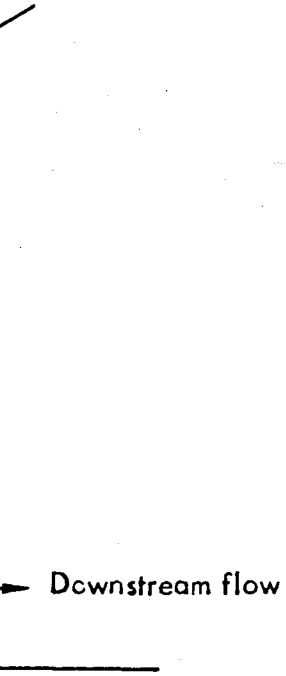


Fig. 5. Notation for shock wave shape calculation.


Downstream flow

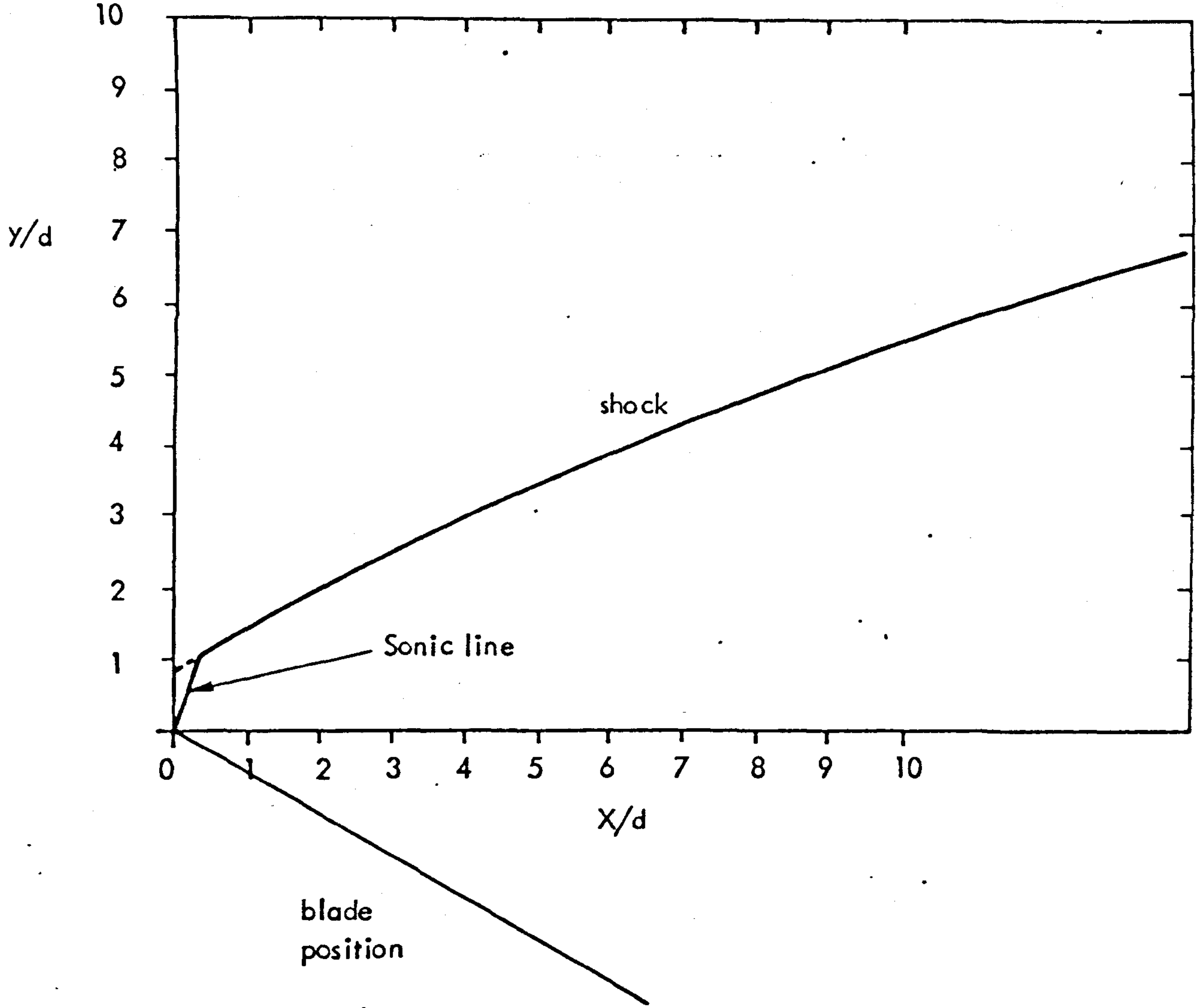


Fig. 6. Calculated shock wave position for uniformly bladed fan.

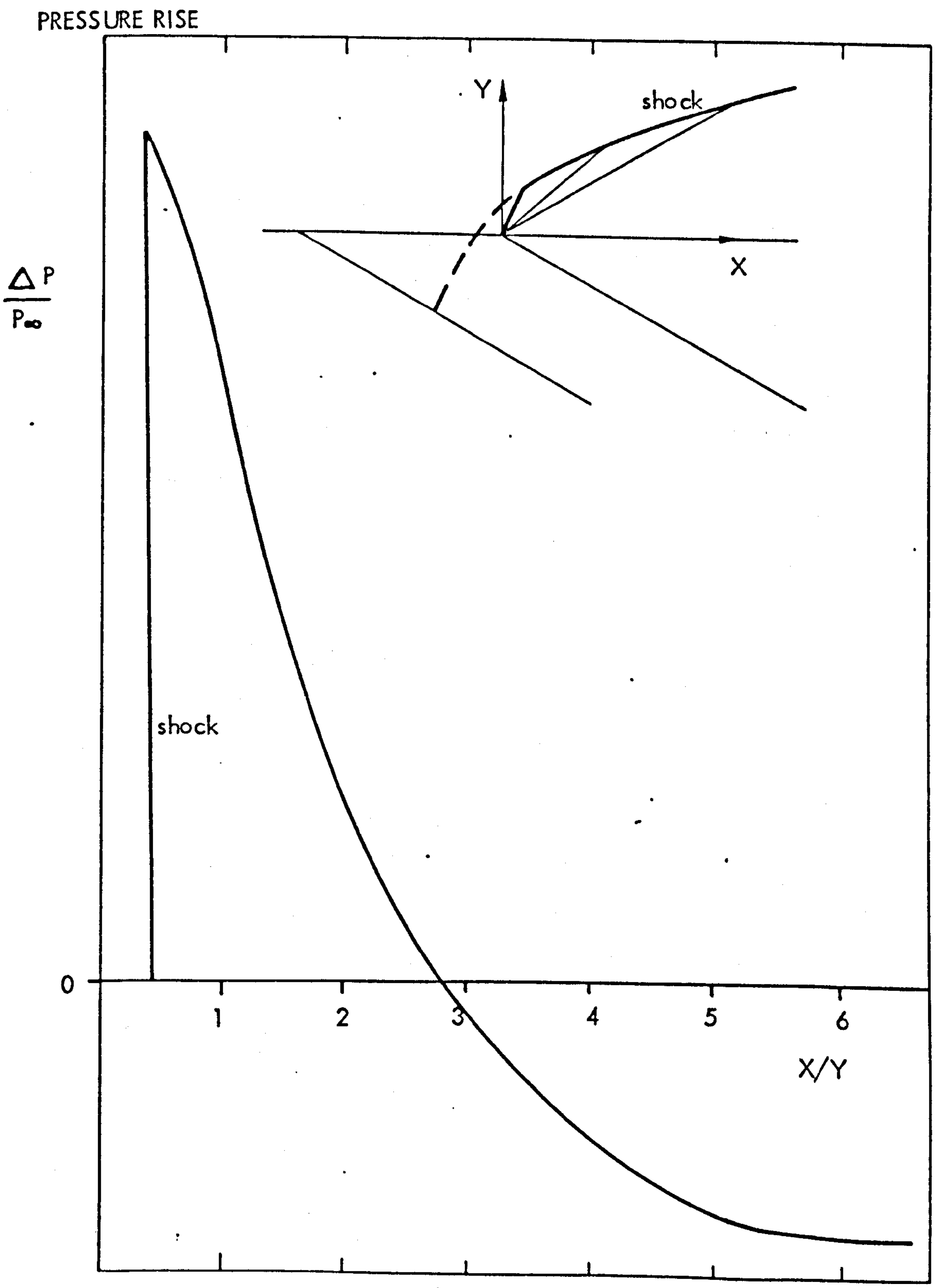


Fig. 7. Pressure rise signature.

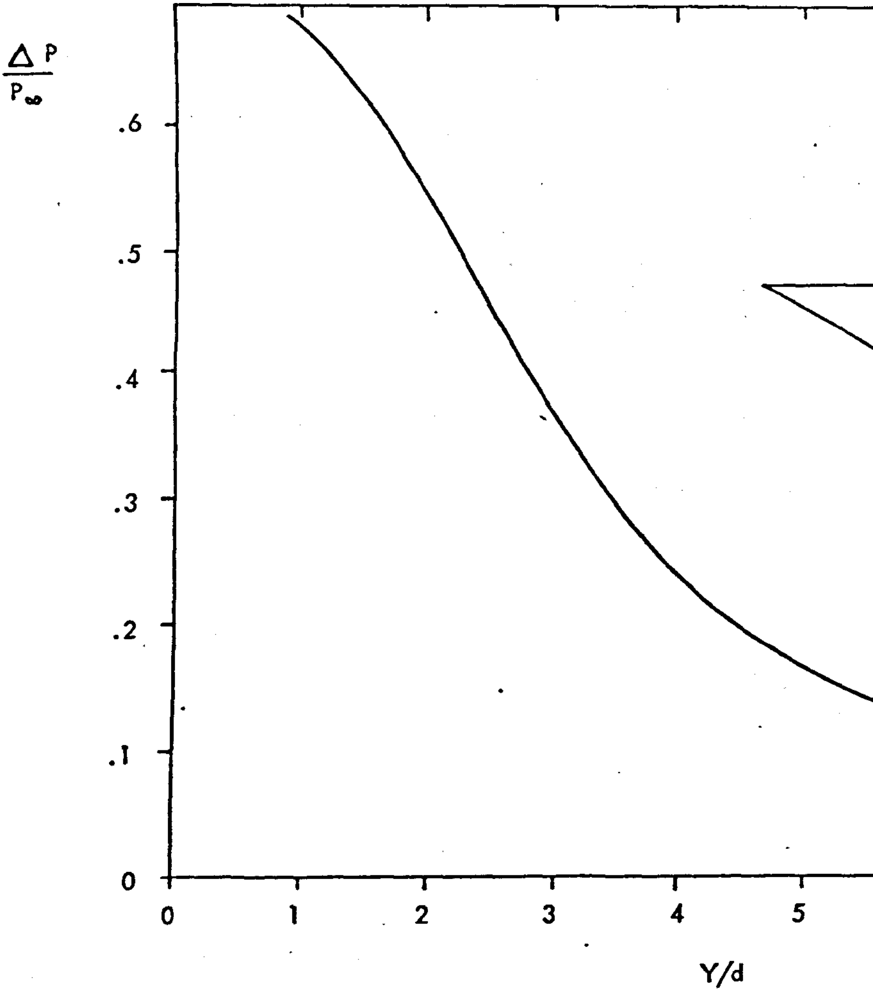
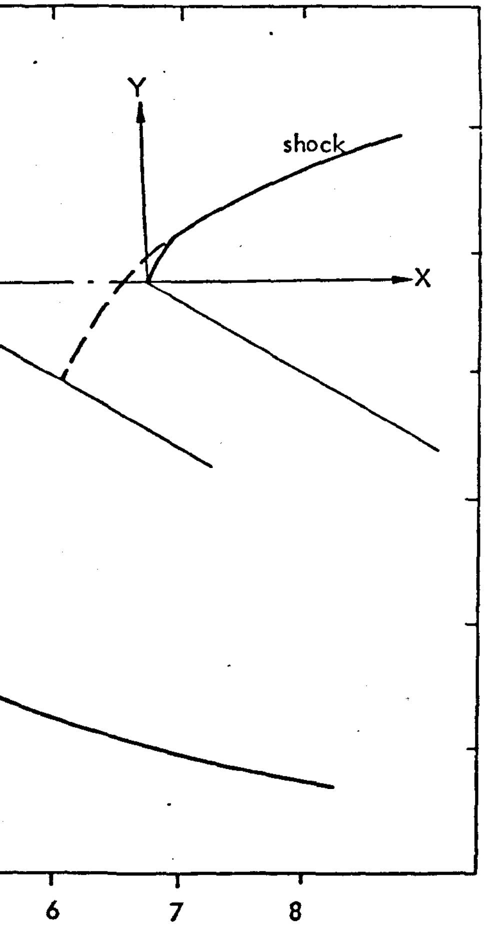


Fig. 8. Decay of shock strength ahead of fan.


Y
shock
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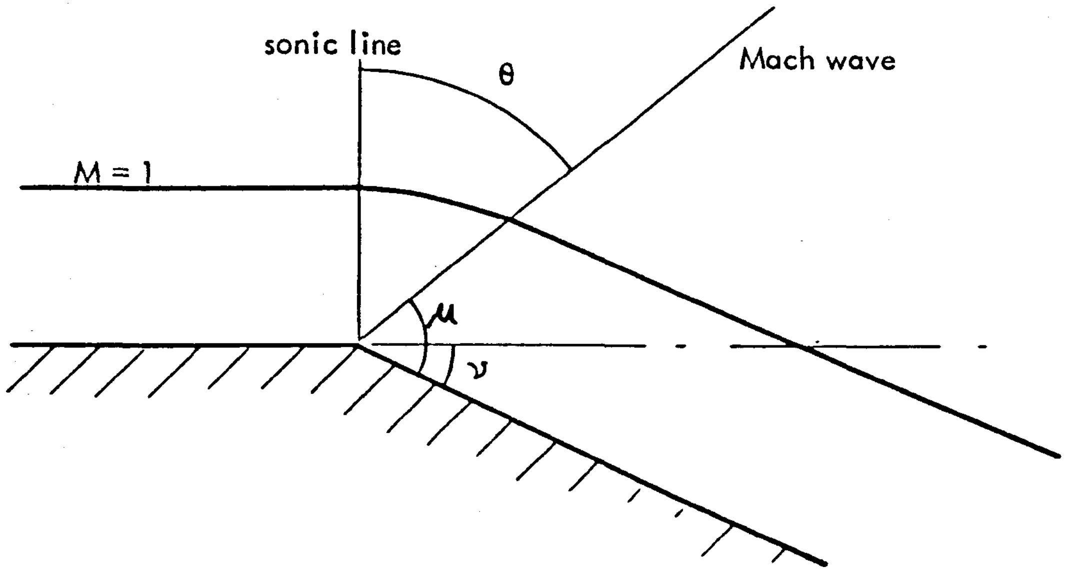

Fig. 9. Notation for Prandtl-Meyer relations - Appendix.