A COMPARATIVE STUDY OF STUDENTS' UNDERSTANDING OF TRIGONOMETRY IN THE UNITED KINGDOM AND THE TURKISH REPUBLIC

ALI DELICE

Submitted in accordance with the requirements for
the degree of Doctor of Philosophy

The University of the Leeds
School of Education

April 2003

Dedicated to my dearest

Grandmother Hatice DELICE

and

Grandfather Mehmet DELICE…

ABSTRACT

Research focus: This is a comparative study of English and Turkish 16-18 year old students' performance in trigonometry: finding unknown lengths or angles from diagrams, simplification of expressions and solving word problems. It is also concerned with the culture of learning because students' learning histories are shaped by curricula and national cultures of education.

Methodology: This is a comparative study with exploratory and descriptive enquiry purposes. It employs an interpretitivist paradigm with a naturalistic mode of enquiry. A wide variety of instruments were used to address two sets of research questions with two different samples: students and teachers. The primary focus of this study is students' performance. Data collected from teachers is used to gain insight into how students learn. Four written tests were used to collect data from the student sample. The foci of these tests were: algebra, simplification of trigonometric expressions, finding unknown quantities in right-angled triangles and solving word problems. Interviews and concurrent verbal protocols were conducted with a subset of the student sample to explore reasoning behind the answers in the tests. Two questionnaires with follow up interviews and classroom observations were employed to collect data from the teacher sample.

Main Findings: Turkish students' performance in the algebra and simplification of trigonometric expressions tests was considerably better than English students' performance: 71% of Turkish students' answers in the algebra test were correct, compared with 44% in the case of English students; 33% of Turkish students' answers in the trigonometry test were correct, compared with 18% in the case of English students. Turkish and English students' performance in the right-angled triangles test were similar, 66% of Turkish and 68% of English answers were correct. English students' performance in the trigonometry word problem test was considerably better than Turkish students' performance: 63% of English students' answers were correct, compared with 46% of Turkish students' answers. Despite these differences, the interviews and verbal protocols revealed a uniformity of approach, from both countries' students, to simplifying trigonometric expressions and answering trigonometry word problems. Document analysis and classroom observations revealed significant differences in the trigonometry curricula and the privileging of techniques, e.g. calculator methods in England and surd forms in Turkey.

Discussion: The Discussion section focuses on three issues: the nature of trigonometry in the two countries, a model of students' manner of simplifying trigonometric expressions and students' methods of solving trigonometry word problems. With regard to the first focus, an analysis of similarities and differences in curricula, teaching approaches and the 'tools' students use suggests that trigonometry in England and Turkey are substantively distinct areas of mathematics. With regard to the second and third foci, models are developed which are, despite the radical differences in the 'trigonometry' in the two countries, independent of the nationality of the students.

ACKNOWLEDGEMENTS

As with any major study in a person's life there are certainly those without whose contribution the completion of that study would have been impossible. It is hard where to begin and I apologise if I omit anyone wrongfully.

I am greatly grateful, therefore, to my supervisors, Dr John D. Monaghan and Mr Tom Roper for their continuous support, guidance (both academically and emotionally) and encouragement throughout my study. They painstakingly read and commented on numerous drafts of the study and offered invaluable advice and stimulating discussions. I could not have asked more from a supervisor, and I consider myself lucky to have the opportunity to work with them.

Many thanks are also due to both the English and the Turkish students and teachers who took part in the study, for responding to the written questionnaires and sharing their thoughts with me during interviews. This research would not have been possible without their co-operation.

I also would like to thank Professor Hikmet Savci, then-Dean of Ataturk Faculty of Education who helped me to have the opportunity to study at Leeds University. Thanks also go to the School of Education and its staff, university of Leeds for providing necessary facilities to conduct my research.

Gratitude and thanks also go to my closest friend Emin Aydin. His friendship, and emotional and academic support have greatly helped me go through this long 'journey'. I also would like to thank Margaret Taylor, Phill Scott, Richard Sykes, Tufan Adiguzel, Erol Karakirik, Ali Riza Coban, Aydin Dogan, Erhan Bingolbali, Mehmet Fatih Ozmantar, Kerem Karaagac, Veysel Kayser, Kerem Altiparmak, Nam Joon Nohkang, Viv Hawkesworth, Joe Hawkesworth, Pauline Bailey, Mohammad Mohammadi, Bagher Yaghoobi, Pumadevi Sivasubramaniam, Ahmed Al-Rabaani for their support and encouragement.

Finally, I want to thank to my wife, Sehnaz Delice and my daughter, Elif Delice for their endless love, patience, understanding, support and encouragement which gave me strength in completing this study. I also would like to thank members of my family grandmother, grandfather, mother, father, my sisters and brothers for their love and support in this long journey. I am forever thankful for the foundation my family provides for my life.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF TABLES

LIST OF ABBREVIATIONS

The following abbreviations have been used in the thesis.

UK	The united Kingdom (England)
TR	The Turkish Republic
TT	Trigonometry test
AT	Algebra test
TWP	Trigonometry word problems test
TORT	Trigonometric functions on right-angled triangles test
CA	Correct answer
IA	Incorrect answer
PA	Partial answer
NAQ	Non-attempted question

CHAPTER 1: INTRODUCTION

Trigonometry is a fascinating world of ratios, angles and transcendental functions and is a part of every high school mathematics curriculum. It is an area of pure mathematics that has important applications in every scientific discipline. It may be developed from considerations of the ratios of the sides of a right-angled triangle or from the rotation of a point on the unit circle.

I was astonished to find that research on students' understanding of trigonometry is virtually non-existent. I was shocked and frightened but my supervisors were shocked and delighted. There are many articles in professional journals around the world which provide teachers with ideas of innovative ways to teach trigonometry but very little that could be called 'research'. Although this was initially daunting, I came to see this as an opportunity.

My interests and research questions developed over time but settled on English (UK) and Turkish (TR) students' performance in both pure and applied trigonometry. I wanted to know how they handled trigonometric identities and formulae and how they solved trigonometry word problems. I was primarily concerned with older school students (16-18 years of age) but decided that to do this properly I must also look at the trigonometry curriculum of younger students (14-16 years of age). As a Turkish student studying in the UK, a comparison between the two countries was clearly of personal interest but, more importantly, provides a very useful research focus. The remainder of this chapter describes: the research questions; methodology; pilot studies; the chapters.

1. Research questions

There are two research questions. My first research question concerns student performance on tasks concerned with trigonometric identities and formulae, students' manner of simplifying trigonometric expressions and their performance in solving trigonometry word problems. My second research question concerns the influence of teaching, the curriculum, examinations and resources on students' performance in this area. I split these two research questions up as follows.

RQ1-i

The focus here is on students' performance of trigonometric identities, trigonometric formulae and their use in 'simplifying' trigonometric expressions:

a- What difficulties do they experience, what errors do they make?

b- How do they use their knowledge of trigonometric identities in their simplifications of trigonometric expressions?

c- How do these performances interact with their knowledge and use of algebraic conventions?

RQ1-ii

The focus here is on students' performance of trigonometric word problems:

a- What 'mental models' do students follow in solving trigonometric word problems?

b- What difficulties do they experience, what errors do they make and what conceptions do they hold?

c- To what extent do the context and the terminology affect the solution of trigonometric word problems?

d- How do visual and symbolic representations interact in the solution process?

RQ2-i

The focus here is on teachers in both countries:

a- How do they teach trigonometry, what resources do they use and not use, how does the curriculum affect their teaching of trigonometry?

b- What emphasis do they place on the foci of the first research question , e.g. how and in what order do they teach these?

RQ2-ii

The focus here is on the curriculum in both countries:

a- "What is it, as in written documents?"

b- How do teachers implement this in terms of classroom activities?

c- What aspects to textbooks 'privilege'? What is examined and how important are these examinations?

2. Methodology

This study is a comparative study with exploratory and descriptive enquiry purposes. It employs an interpretitivist paradigm with a naturalistic enquiry approach in the sense that I have observed, as far as possible, 'what is' in both countries without manipulating the course of teaching and learning in any manner. In terms of the type of the data my research is mainly qualitative but both qualitative and quantitative data collection instruments are used.

My student sample consisted of 55 students doing A-level mathematics from one English college and 65 similar aged students (studying mathematics) in one Turkish school. My teacher sample (for observation and interview) were the mathematics teachers in those schools and a wider set of similar teachers (10 UK and 60 TR).

My research instruments were selected to answer my research questions as best I could. I used a wide variety of instruments (see p. 48). Student data included four different written tests (55 UK and 65 TR students), interviews (7 UK and 9 TR students) and verbal protocols (4 UK and 8 TR students). Teacher data included responses from two different questionnaires (10 UK and 60 TR teachers), interviews (5 UK and 9 TR teachers) and observations (5 UK and 9 TR teachers). My last data set was collected by an analysis of documents (curricula[1], textbooks, schemes of work and examinations) in the UK and the TR. My rationale in collecting student data was that students' performances in tests would allow a large number of students to be sampled over a wide range of items but would not provide detailed reasons why they did what they did. Tests would thus be followed up with interviews with a subset of the students in order to understand reasons for their responses. But interviews only provide 'after the fact' data and I wanted to know what they did/thought as they worked on problems. I thus, on a smaller subset of students, used concurrent verbal protocols as students solved similar problems to gain further insight into the thinking behind their performance. Students were selected for the protocol work to represent a range of attainments (in my tests and in school work) and for their ability to communicate well, based on their teachers' recommendations.

Data collection yielded mostly qualitative data, which was in forms of written accounts or spoken words. To deal with qualitative data I noted patterns and themes and constructed catgeories. Quantitative data were similarly categorised.

3. Pilot studies

Since so little mathematics education research had been conducted in the area of trigonometry, in terms of students' performance or curricula design, I had no opportunity to trial instruments used in other research. I therefore had to create the appropriate instruments to answer my research questions as best as I could. To investigate whether these instruments would answer my research questions or not I went through two stages which I call trial and pilot. In the trial stage I mainly focused on administration of the instruments, clarity of the questions and translation of the questions. In the pilot stage I aimed to determine whether the instruments suited the research questions, clarity of the questions and interpretation of the questions.

For the student instruments, my trial sample was undergraduate and postgraduate students and my pilot sample was 16-18 year old UK and TR students. For the teacher instruments, my trial sample was PGCE students and the UK and the TR teachers. My pilot sample was the UK and the TR teachers. In the sub-sections below I briefly present relevant findings from the trial and pilot of the instruments as follows: written tests, questionnaires, interviews, verbal protocols and observations.

The written tests

The trial written tests for the students were a trigonometry test, a true-false test for trigonometric identities, a trigonometry word problems test and a non-trigonometry word problems test (algebra test). The trial allowed me: to see students' interpretation of the questions, the

[1] I refer to UK and TR students, teachers and curricula and note, p. 1, that UK actually means 'English'. The only complication here is the term UK National Curriclum. This is not correct because the UK is not a nation, but England is. Henceforth I use the term 'UK' in all instances except references to the National Curriculum where I use the term 'English'.

correctness of the Turkish translation, the time taken to answer the questions, to appreciate their difficulties and to clarify my focus, e.g. simplification of expressions or solving equations. After the trial I evaluated the 'worth' of each test and re-wrote (where appropriate) questions for each test, with their rationales, for the pilot. Trial data suggested that data from the true-false test was subsumed in data from the trigonometry test and I decided to abandon the true-false test. The non-trigonometry word problems test provided data on word problem solving but I judged that this was taking me too far away from my research foci and I abandoned it. The items in the other two tests were analysed and this revealed that one paper would be easier to administer as it would require less time and fewer question papers (practical constraints on the number and the duration of the tests being an important consideration if I was to get schools to agree to help). After piloting the tests, I judged that the test would be administered in about 45 minutes. Several criteria were used in selecting questions for the final tests. Facility level: questions that were found to be too difficult or too easy were dropped. Variety: questions that focused on a single aspect of trigonometry and algebra, e.g. $\sin^2\theta+\cos^2\theta=?$ focuses on the Pythagorean property, were, in general, discarded in favour of questions that simultaneously focused on several aspects of trigonometry or algebra. Highlighted ideas: some questions produced a variety of ideas from the students while others produced only either right or wrong answers. After piloting, all instruments were prepared for the main study.

The questionnaires

There were two questionnaires for teachers: a teacher questionnaire, which had two sections, and a textbook questionnaire. Questions in the teacher questionnaire were collected from the research in the literature and adapted to trigonometry but some were created by me. The purpose of the trial was to determine the interpretation of the questions and the time taken to answer the questions. The responses showed that most of the questions were clear and the questions were valid for eliciting the teachers' views. Some of the teachers gave some advice about the language of the items. This was taken into consideration. Some repeated items and irrelevant items to the research question were rejected. Consequently all items were reviewed and then piloted. In the piloting, the results showed that teachers did not want to complete the open question part. Some of them found it time consuming and so did not complete it. But I wanted teachers' manipulation and solutions to some trigonometry problems which I considered an important data to collect. Consequently these problems were not omitted. There seemed to be no problem with other questions in the questionnaires in terms of time, organization and research questions.

Interviews

For both the teacher and the student sample I wanted to use semi-structured interviews. I prepared a set of questions and applied them in the trial. The purpose was to determine the time taken to complete the interview, the effectiveness of the interview schedule and the administration. The semi-structured interviews used Tomlinson's (1989) hierarchical focusing method. The results suggested that the questions were appropriate. Piloting suggested that the language of the interview, interview time, content of the questions (with regard to the research questions), administration of the interview, use of a tape recorder and the hierarchical focusing method were appropriate. The student interviews also used a semi-structure interview style. Trials of the interviews showed that the hierarchical focusing method was not appropriate to use in these interviews because the interview questions were based on students' errors in the tests, i.e. every student would have their own questions. Interviews were tape recorded and then transcribed. The results showed that students' ability to communicate well should be taken into account in the main study. The semi-structured interview format allowed me to discover and probe underlying errors.

Verbal protocols

As mentioned above, p.3, I felt a need to supplement 'after the fact' interview data with 'as students think' concurrent verbal protocol data. I planned my trial according to the phases described in Green (1998). Trials convinced me that selected students must be able to verbalise their thoughts, namely they should be talkative. I then prepared instruments for piloting. The aim of the pilot was to determine the time taken to answer the questions, interpretation of the questions, the effectiveness of the verbal protocol procedure and the administration of the technique with real target students. Piloting revealed four important issues: students should be well trained before the verbal protocol session (particularly on verbalising their thoughts), that there is a difference between 'think aloud' and 'talk aloud' methods, they should take place in a quiet room to avoid interruption and students should not see any of the other questions when they are working on a question.

Observations

In the trial I applied observation instruments to both teachers and students. For teachers a classroom observation schedule was prepared and applied. It was a five way Likert scale schedule with 9 items. However the items were so specific that some of the items could not be answered. It was not satisfactory and generated poor data. I reviewed my work and developed an open observation schedule where I recorded everything relevant to my interests at five minutes intervals. I judged this to be a suitable instrument for my purposes. The trial and pilot of the student observation scheduled revealed that it is very difficult if not impossible to observe all students at the same time in a classroom. Further to this, it was impossible to see what mistakes students made. I was unable to see how they solved trigonometry problems and what they did throughout the solution of the problem. I also discovered that in observing one student in a classroom, the existence of the observer disturbed and distracted the student. I thus discarded any plans to systematically observe students in classrooms.

Finally it should be noted that the trials and pilots, collectively, refined my research questions. The research questions presented on pp.1-2 are 'more focused' than my initial research questions as I learnt that data collection instruments are there to answer, as fully as possible, research questions, not simply to gather data that may be useful.

4. The chapters

The following summarises the remaining chapters, 2 to 6.

Chapter 2, Literature Review, is in four sections. The first section discusses issues in comparing educational systems. The second section provides a literature review of educational studies concerning trigonometry. The third section looks at problem solving and the last section looks at curricula and teachers.

Chapter 3, Methodology, discusses the methodological approach of this research and is presented in six main sections: research questions, the research paradigm (approach), research methods (techniques), design of research instruments, data collection and the data analysis.

Chapter 4 presents the results, raw data and analysed data. Space constraints prevent the presentation of all data collected and analysed. Two criteria were selected for which data to include/omit: (i) all results referred to in the discussion chapter are included; (ii) sufficient results are included for the reader to gain an overall picture of the areas investigated. The results are presented in two parts which concern, respectively, the two research questions: The UK and the TR students' performance in trigonometry and possible factors influencing students' performance. The first part is in six sections, which concern, respectively, students' performance in trigonometry test, algebra tests, comparison of students' performances in trigonometry and algebra tests, trigonometry word problems test, trigonometric functions on right-angled triangle tests and comparison of the students' performance in trigonometry word problems and trigonometric functions on right-angle triangle tests. The second part is in two sections, which concern, respectively, teachers' approach to teaching trigonometry and curriculum resources.

Chapter 5 is the discussion chapter and is, arguably, the heart of my thesis. The first section considers whether trigonometry is really the same topic in each country (other than by name). I then, respectively, discuss two major theoretical constructs that emerged from data analysis: a model of students' manner of simplifying trigonometric expressions and students' methods of solving trigonometric word problems. The chapter concludes with an examination of my research questions in the light of the data collected.

Chapter 6 is the concluding chapter. There are three sections: an overview, educational implications and suggestions for further research.

CHAPTER 2: LITERATURE REVIEW

My area of study is students' performance in trigonometry. My study is basically a comparison of the UK and TR from the students' position, what they do and what they understand. Student understanding is central to all cognitive based mathematics education studies. However, the term 'understanding' has become a problematic term over the last decade and people seem afraid to use it because we do not really understand exactly what understanding is. Given this I focus on students' performance. This does not mean that I have behaviourist tendencies. I remain concerned with understanding but report on observable outcomes. Students do not perform in a vacuum. They have learning histories shaped by the curriculum and the culture of education of their countries. My central research focus is students' performance but because of the importance of learning histories I have a second research focus concerned with the culture of learning.

The literature review has continued throughout my study and has been used to discuss the issues in my research focuses. The literature review was conducted in two stages pre and post data collection. Literature published after 2000 has not informed the construction of the research questions and data collection but has informed the analysis and conclusions. I have discussed the research in the literature in four sections. The importance of the comparison of the UK and TR is discussed in the section, Comparative study. All documentation relevant to trigonometry is briefly reviewed in the section, An overall review of trigonometry literature. A broad understanding of problem solving is discussed in the section, Problem solving. The curriculum and teachers' issues in the teaching and learning of trigonometry are discussed in the section, The curriculum and teaching.

1. Comparative study

My initial approach to trigonometry in both the UK and TR revealed some interesting points, e.g. use of calculators and formulae sheets in the UK but not TR, which showed that although trigonometry was taught to the same age groups in TR and the UK (14-16 and 16-18 years old students), as well as similarities, there seemed to be differences between trigonometry in the two countries. So the comparison of the two countries might reveal some issues, e.g. the similarities and differences between the teaching and learning of trigonometry in the UK and TR, which could broaden concepts of what is and is not possible in the classroom. Moreover, comparing two educational systems can give greater opportunities for understanding the impact of culture, personal and contextual factors, and of educational interventions (Schmidt et al., 1998).

I do not expect this comparison will necessarily lead to improvement (Maeroff, 1991 in ibid.) in teaching and learning mathematics, which is a cultural activity that is very difficult to understand (Gallimore, 1996). Because each culture develops its own norms and expectations, which are widely shared and so familiar, for teaching and learning, these norms and expectations become nearly invisible to members within each culture. So comparing two countries, e.g. in terms of the classroom practices, can reveal these accepted and unquestioned cultural models in trigonometry context (Kawanaka et. al., 1999). In their research Kawanaka et al (ibid.) found that mathematics was indeed taught and learnt differently and the roles of teachers and students were different across three countries.

Some international comparative studies have been conducted in the past into the mathematics attainment of many countries. The International Association for the Evaluation of Educational Achievement has carried out three surveys of mathematics attainment during the last 40 years; the first international mathematics survey in 1964, the second international mathematics survey in 1980-1982, which provided many interesting insights, the third international mathematics and science survey (TIMSS) in 1995, which influenced policy makers' perceptions of the need for change and was the largest and most complex study to date. TIMSS was repeated in 1999, which assessed eighth-grade students in both mathematics and science to measure trends in student achievement since 1995. Moreover there were some additional countries taking part in this study. These studies were conducted with very large populations and yielded interesting results that countries tried to understand and hence make reforms in their education systems. Reports were written on the results gained from these studies, e.g. Harris (1997), who also highlighted the increasing interest in the standards achieved by primary and secondary school pupils in both the UK and in other countries within Europe and beyond. Some of the studies in which the UK was compared with other countries are; Vulliamy and Nikki (1997) compared the educational reform in primary schools in England and Finland and Wolf and Steedman (1998) compared Swedish and English 16 year olds in terms of schooling (the mathematics curriculum, upper secondary school, assessment and certification), achievement, performance. Reynolds and Farell (1996) provide an overview of the performance of pupils in England as compared with that of their counterparts in other countries in earlier international comparative studies.

In terms of international studies, the UK participated to both TIMSS in 1995 and 1999 at the age 13-14 whereas the TR only took part in TIMSS 1999. There were, however, no trigonometry items in the mathematics test of TIMSS, so it was not possible to compare the UK and the TR students' performance in TIMSS in terms of trigonometry. Moreover, there was no study on 16-18 year old the UK and the TR students' performance of further trigonometry in the literature. So comparing the UK and TR in terms of trigonometry would be beneficial in educational studies:

> to know how others stand, that we may know how we ourselves stand; and to know how we ourselves stand , that we may correct our mistakes and achieve our deliverance that is our problem. (Arnold, 1960-1976 in Rapple, 1989)

Rapple (1989) emphasised that comparison is clearly very useful because learning about another society provides people with greater knowledge of their own society. The comparison between the UK and TR may also provide a very useful picture of teaching and learning in TR.

2. Overall view of trigonometry in literature review

Trigonometry is an important area of senior secondary mathematics in every country in the world but surprisingly little mathematics education research has been conducted in this area in terms of students' conceptions, performance or curriculum design. An extensive literature review has been conducted to get this result. Fifteen journals and the proceedings of the International Group for the Psychology of Mathematics Education have been reviewed manually, all relevant databases and electronic journals in Leeds library web page have been reviewed e.g. ERIC (the Educational Resources Information Centre), Education-line, Science Direct, Index to Theses. Also, by using research engines in the World Wide Web, I reached individual academic works. Moreover I also searched through CiteSeer, which is a scientific literature digital library, and AskERIC, which are two of the biggest organisations providing professional and academic works.

In this section I present the overall picture of the available research conducted in trigonometry in the literature. In the following sections, some of the researches will be revisited, e.g. Pritchard and Simpson (1999). First of all I want to highlight that in my study, paper and pencil method is used. Villarreal (2000) stated that paper is a place where thought can develop and an interface that allows thought to be expressed. She called students' paper and pencil a thinking collective. Paper and pencil is a medium mediating student's thought and she also stated that it would be difficult to approach mathematical questions algebraically without paper and pencil medium. Though I am not really interested in the research of trigonometry in computer and algebraic/developed calculator context, I will briefly mention one of the researches that is relevant to my study.

There were mainly two sorts of documentation in the literature, professional journals and research studies. Professional journals occasionally publish articles with ideas for teachers, e.g. Ellery (1980) encourages the use of practical apparatus in introductory trigonometry lessons, Barraclough (1990) suggests unit circle activities that generate basic identities and Ellis (1990) stresses the importance of practical activities which link trigonometry to real world problem solving. In the professional journals, the Mathematics Teacher, Mathematics in School, Mathematics Teaching, teachers highlighted the activities to teach trigonometry, e.g. the real world activities, the relation between geometry, algebra and trigonometry, introducing trigonometry by using different approaches. Of the few research studies available, Blackett and Tall (1991) examined the role of the interactive computer graphics in learning introductory trigonometry and found a greater improvement in experimental students' performance. Kendal (1992) compared ratio and unit circle introductory approaches and found the ratio method more effective in terms of performance and retention of concepts. Pritchard and Simpson (1999) examined the role of pictorial images in students' solutions of trigonometry problems and found that propositional knowledge, that is the network of knowledge students got by their experiences and understanding of the topic, took precedence over imagery in students' thinking. I have found no research linked to students' understanding of post introductory aspects of trigonometry.

3. Problem solving

Mathematical problem solving has an important place in educational studies. Chinnappan (1998) confirms that with his reason:

> problem solving occupies a central role in mathematics learning and teaching and provides a window through which we can view students' grasp of mathematical concepts and procedures. The primacy of problem-solving in mathematics teaching is reflected in major curriculum reform documents including Curriculum Evaluation and Standards (National Council of Teachers of Mathematics, 1989) where it is recommended that 'problem solving should be the central focus of the mathematics curriculum' (p. 23).

Problem solving might be seen as an opportunity for applying mathematical knowledge, but it is more than that. Schoenfeld (1985, p. 14) states students' performance in problem solving is more than to know something:

> ...students' problem-solving performance is not simply a product of what the students know; it is also a function of their perceptions of that knowledge, derived from their experiences with mathematics. Their beliefs about mathematics...establish the psychological context within which they do mathematics

Insights into students' problem solving processes are important in order to better understand students' approaches to problems. Hughes et al. (1997) found out the question-answering model of GCSE exam questions. The model considers the complete process of responding to a task, from when the question is presented through to completion, they claim it has use for learning and teaching, it will also be used for examiners to write guidance material. The model they found was read, recognise, understand, plan, extract, execute, record, check.

Lesh (1981) states that problem solving is a tool and a means of thinking:

> Problem solving is more than obtaining answers. It is a tool, a means of thinking, and a philosophy. It is a predisposition to learn from every available opportunity the most that can be gleaned from that experience

In his belief about classroom teaching of secondary school mathematics teachers, Polya (1987, p. 2) also sees problem solving as a way of thinking:

> We cannot meaningfully discuss teaching, if we do not agree to some extent about the aim of teaching. Let me be specific...I have an old fashioned idea about its aim: first and foremost, it should teach young people to think...such 'thinking' may be identified...with 'problem solving'

According to NCTM, (1989, p. 11) learning to solve problems is the principal reason for studying mathematics:

> problem solving is the process by which students experience the power and usefulness of mathematics in the world around them. It is also a method of inquiry and application...to provide a consistent context for learning and applying mathematics. Problem situations can establish a 'need to know' and foster the motivation for the development of concepts.

There are some factors, which influence problem solving, e.g. prior knowledge, skill. Problem solving is not solely learning rote procedures and arithmetic, but more than that. Prior knowledge was distinguished as a necessary component in successful problem solving by each of Schoenfeld, Mayer and Silver in their studies in 1982. As well as the strategies to be used in trying to solve the problem, prior knowledge affects the problem solver's understanding of the problem. Schoenfeld (1982) explains the problem solving process as a dialogue between the problem solver's prior knowledge, his attempts, and his thoughts along the way. Furthermore, Cockcroft (1982) advocated problem solving as a means of developing mathematical thinking as a tool for daily living. He also said problem-solving ability lies 'at the heart of mathematics' (p.73) because mathematics can be applied to a variety of unfamiliar situations by problem solving. Problem solving is more than a means for teaching, reinforcing mathematical knowledge and helping students to meet everyday challenges. It is also a skill that can develop logical reasoning. Problem solving has a special importance in the study of mathematics, because developing the ability to solve a wide variety of complex mathematics problems is a primary goal of mathematics teaching and learning (Wilson et al., 1993).

Stanic and Kilpatrick (1989) identify three main themes regarding the usage of problem solving in their historical review: "problem solving as context" i.e. problems are employed as vehicles in the service of other curricular goals, "problem solving as skill" and "problem solving as art". The idea of problem solving as art in contrast to the previous two, holds that real problem solving (that is, working problems of the "perplexing" kind) is the heart of mathematics, if not mathematics itself.

Blum and Niss (1991) described what they mean by a problem and collected mathematical problems into two groups:

> By a *problem* we mean a situation which carries with it certain open questions that challenge somebody intellectually who is not in immediate possession of direct methods/procedure/algorithms etc. sufficient to answer the questions...As to *mathematical* problems there are two kinds: It is characteristic of an *applied* mathematical problem that the situation and the question defining it belong to some segment of the real world and allow some mathematical concepts, methods and results to become involved. By *real world* we mean the "rest of the world" outside mathematics ....In contrast, with a *purely* mathematical problem the defining situation is entirely embedded in some mathematical universe.

Thus they define problem solving simply as referring to the entire process of dealing with a problem in attempting to solve it within the two categories. They see problem solving as either applied mathematical problem solving or purely mathematical problem solving. These two problem solving categories seem to correspond to the specific aspects of trigonometry I have been focusing on in my research, simplification of trigonometric expressions (trigonometry in pure mathematics context) and trigonometry word problems (use of trigonometry in the real world context).

Garofola and Lester (1985) suggested that students are mostly unaware of the processes involved in problem solving, so discovering the process in problem solving and highlighting it in instructions may be important for trigonometry word problems and simplifying trigonometric expression.

Wilson et al. (1993) discuss why the NCTM has strongly endorsed the inclusion of problem solving in school mathematics:

- Problem solving is a major part of mathematics.
- Mathematics has many applications and often those applications represent important problems in mathematics.
- There is an intrinsic motivation embedded in solving mathematics problems.
- Problem solving can be fun.
- Problem solving must be in the school mathematics curriculum to allow students to develop the art of problem solving.

Subsequently, problem solving can be seen as a way of thinking, of analysing a situation, of using reasoning skills, applying both past experience and knowledge to the problem. It is important in the teaching and learning of mathematics.

In the following, I will discuss problem solving focusing on two specific aspects of trigonometry, simplifying trigonometric expressions and trigonometry word problems, namely trigonometry in abstract and real world context. These aspects of trigonometry are the examples, which show how algebra, geometry and trigonometry are interrelated. Marjoram (1974) asserts that trigonometry developed later than algebra and geometry, so that to a degree it represents a bridge between the two, so it is worth looking at it in the light of algebra and geometry. Douglis (1970) presented trigonometry as arithmetical methods in geometry.

3.1. Simplification of trigonometric expressions

Reviewing the literature, I have discovered that there has been no research relevant to the simplification of trigonometric expressions. However, since algebra and trigonometry go hand in hand, some research studies on simplifying algebraic expressions could be very useful. I have found a study, which was published in 2002 by Hall about thought process during simplification of algebraic expression. In this study, Hall explained the reason behind the study and importance of the simplification of algebraic expressions which might have informed this study in terms of simplifying trigonometric expression:

> The simplification of expressions was chosen as an area of study for two reasons. First, this researcher has witnessed many instances of errors in the cancelling of terms. Second, he has found much less literature on the simplification of complex algebraic terms than on the solution of equations...Research into pupils' thoughts while simplifying expressions gains in significance when it is appreciated that algebraic simplification can be viewed in several ways: as being a skill in its own right and as a skill useful in the solution of equations.

In simplifying algebraic/trigonometric expressions basically an equivalent expression is expected to be reached. Research has shown that "making equivalent expressions" as well as "substituting numbers and variables, solving (systems of) linear equations with two or more unknowns" are the algebraic topics with which students have many difficulties (Booth 1988, chapters in Wagner and Kieran 1989; Kieran 1997). Although there are definitions and procedures given in the textbooks relevant to simplifying and finding the simplified form of an algebraic expression, nothing pertinent to trigonometric expressions was met.

In simplification of algebra/trigonometric expression students do symbolic manipulations by using algebraic and trigonometric properties. The rules of manipulating symbols, simplifying expressions as well as solving equations, and artificial application problems are the traditional image of algebra (Kaput, 1999). Manipulations are important in simplification. Arcavi (1996) discussed eight behaviours to describe the symbol sense, the sixth one is flexible manipulation skill in which he said the correct manipulation of symbols consists of much more than a dry obedience of the rules. He gave some aspects of sensing the symbols: the realization of a potential circularity in symbol manipulation and the "gestalt" view of some symbolic expressions. Circularity is the process of symbolic manipulation which results in an obvious or tautological identity, which is uninformative and unproductive. Having a "gestalt" view is sensing the symbols not only as a concatenation of letters, but being able to discern form, as, for example:

> If you can see your way past the morass of symbols and observe that equation #1 [$v\sqrt{u} = 1+2v\sqrt{(1+u)}$ which is required to be solved for v] is linear in v, the problem is essentially solved: an equation of the form $av=b+cv$, has a solution of the form $v=b/(a-c)$, if $a\neq c$, no matter how complicated the expressions a, b, and c may be. Yet students consistently have great difficulty with such problems. They will often perform legal transformations of the equations, but with the result that the equations become harder to deal with; they may go 'round in circles' and after three or four manipulations recreate an equation that they had already derived... Note that in these examples the students sometimes perform the manipulations correctly... Wenger (1987).

The first and second parts of the above quote are an example of gestalt and circularity respectively. Wenger also says that students "often appear to choose their next move almost randomly, rather than with a specific purpose in mind". These three factors of behaviour in flexible manipulation skills might also be seen in simplifying trigonometric expressions.

As well as carrying out the manipulations correctly, students must recognise to use appropriate properties or recognise the forms. Kieran (1997) discussed the literature on simplification of expressions and concluded that:

> ...students may have the basic of manipulation techniques, but are often thwarted because they lack a global view of how to read an expression and of what should be done with it...

Moreover, most of the students carried out very detailed calculations for the exercise $\left[\frac{z^4}{c^2(b-a)}\right]+\left[\frac{z^4}{c^2(a-b)}\right]$, what seems to be missing is the ability to recognise more general laws and forms, so this exercise can be solved by "seeing" its distributivity (Menghini, 1994)). She also highlighted the possible reason behind the long, repetitive and unconscious looking manipulations:

> There are various researches concerned with difficulties connected with algebraic manipulations. The problem of "unconscious manipulation" emerges from nearly all of the investigations on the teaching of algebra in secondary schools ... a problem which is linked to the inability to translate into symbols and interpret formulae. Many students seem to prefer long, monotonous, obviously repetitive process which as they become automatic, require very little concentration or reasoning (but are "guaranteed") instead of brief concise process with few calculations which however require the active distinguishing of similarities and differences and understanding of the rules and the ability to synthesise.

Throughout their simplification, students can get stuck in finding a simplified expression. Discovering the stages where students get stuck in the solving process can be very important for improving their solving processes. Mason et al. (1982, p. 49) suggested that a stuck state is 'an honourable and positive state, from which much can be learned' and it is very useful to become good at using such states in a positive manner.

In the UK and the TR textbooks there were four ways of asking to simplify trigonometric expressions: "prove", "show that", "verify" and "simplify". In the first three, an expression to be reached is given but in the last one no target expression is given. In the last one, how students interpret the expression might be important because students need to decide whether the expression they reach is the simplified form or not. In the literature there are ongoing discussions on interpreting the algebraic expressions. There are mainly two ways of interpreting an algebraic expression or equation: "procedural" (Kieran, 1992) or "operational" (Sfard, 1991), and "structural" (Kieran, 1992; Sfard, 1987) or "conceptual" (Hiebert and Lefevre, 1986). For example, a + 2 can be interpreted as the procedure or operation "add 2 to a"; it can also be interpreted as an object or concept in a mathematical structure, e.g. fractions. "Process and object" (Sfard, 1991), or "process and product" (Tall and Thomas, 1991) or "process and concept" (Gray and Tall, 1991) are the alternative conceptions. So in simplifying trigonometric expressions, students' interpretation of the expressions that they reach might be important. If students recognise the procedure/properties in the expression they reach then they can continue to simplify the expression. However, if they see the expression as an object then they might stop at the expression, as it is a simplified form.

Chazan (1996) discussed that the nature of symbolic expressions and the purpose and goal of the manipulations of symbols are not satisfactorily explained in the traditional algebra curriculum. Moreover instructions in the questions, e.g. 'simplify', do not describe the final version of the expression:

> For example, the first use of symbols encountered in a traditional text is the algebraic expression (e.g., $2x^2+3x+1$), which is thought of as a description of an unknown number. One does not know what number $2x^2+3x+1$ is until someone specifies what number x is. After practicing evaluating such expressions for particular values of x, the instructions for exercises involving manipulation of these expressions read "Simplify," "Expand," "Multiply," or "Factor." These cryptic instructions are meant to suggest the type of manipulation the student must do to the expression, but they are not conceptually sufficient. These instructions do not adequately describe the desired final versions of the expression, do not alert students to properties that are being preserved as the expression is rewritten, and, finally, do not embed rewriting expressions in a larger, purposeful context. It is difficult to understand how doing such exercises can be justified on bases other than the "exercise" of certain mental muscles.

Consequently, simplification seems to be an important operation in trigonometry to use in other topics, e.g. solving equations. Students should have symbolic manipulation skills and be aware of the stages throughout the simplification such as stuck state, simplified form, circularity, recognising the forms or properties. However, instructions are also important in simplifying a trigonometric expression in that students should know what expression to find by 'simplify'. The expression students find should be interpreted by students in a way that they stop or continue to their simplification.

Lastly, although algebra has an important place in trigonometry, I will not discuss algebra in detail because on its own it is a huge area. It might be essential to underscore some issues in algebra and trigonometry: algebraic identities, e.g. difference of two squares, can be used in trigonometric expressions as well, but the main difference which also might affect the simplifying trigonometric expression is trigonometric identities which cannot be used in algebra in the same way, e.g. $\sin^2\theta+\cos^2\theta=1$ however a^2+b^2 is itself as long as no value is assigned for the variables a and b.

3.2. Trigonometry word problems

Word problems have an important place in mathematics education in that they give a basic experience in mathematical modelling and represent the interplay between mathematics and reality (Reusser, 1995). Odvárko et al., (1990) defines word problems as problems in which objects, phenomena and situations (with their diverse properties and relationships) from various non-mathematical domains occur. According to Semadeni, (1995) a characteristic feature of a word problem is the use of words in the description of the problem and a word problem should somehow refer to real-world context, that is, word problems are opposed to purely mathematical problems. In this perspective real world problems are word problems including real world objects and occurrences in a reasonable framework.

In this section I will first discuss the only research done on trigonometry word problems and then I will discuss a broad understanding of word problems including the issues revealed by the first study.

3.2.1 Pritchard and Simpson's research

The only research conducted in real world application of trigonometry mainly looked at the role of pictorial images in answering trigonometry word problems (Pritchard and Simpson 1999). Trigonometry real world problems are the problems embedded into the real world context that often involve information from 'real world' situations: ships, buoys, cliffs, kites and lighthouse abound in trigonometric word problems. Trigonometry word problems can be seen as a topic in which students can flexibly move between visual and symbolic ways of working. This reflects a characteristic of mathematical thought described by Noss et al. (1997) which is the capacity to move freely between the visual and the symbolic, the formal and the informal, the analytic and the perceptual and the rigorous and the intuitive. Pritchard and Simpson state that trigonometry is an excellent place to explore the ways in which pictorial images are used in the solution of problems and explored students diagrams in three categories; the creation of diagrams, the use in solving problems and the use in checking and meaning. As a result they found that propositional knowledge took precedence over imagery in students' thinking.

Pritchard and Simpson's (1999) research highlighted some issues in answering trigonometry word problems: First, all students drew diagrams, even though no diagrams were given and no students were encouraged to produce diagram. Nunokawa (1994) endorsed the importance of the diagrams by review of other studies as well that:

> ...drawing diagrams or pictures is helpful for solving problems in general...for solving mathematical problems in particular...and that this kind of activity is considered to be one of the problem solving strategies...its usefulness lies, we think, in the fact that it can show relationships among elements in the problem clearly. For example Nickerson, Perkins and Smith (1985) state "*The intent of this heuristic is to concretise the problem. Part of such concretisation has to do with visual thinking: once a graph or diagram is drawn, the problem solver can bring perceptual processes to bear on it. Also a visual representation of a problem can make apparent certain relations among parts that might otherwise go unnoticed (pp. 75-76).*

How then might diagrams affect students' performance? Pritchard and Simpson did not state anything about relation between diagrams and students' performance. As well as being important in the answering process of the trigonometry word problems, Doishita et al. (1986, p. 77) said that the more successful students are likely get correct answers because they can draw correct diagrams whereas less successful students cannot solve the problems because they cannot draw the correct diagram. Pritchard and Simpson discussed that throughout the construction of the diagrams students use their mental image/visualisation, which is endorsed by Solano and Presmeg (1995) that it plays an important role in the process of answering problems.

Secondly, Pritchard and Simpson found that students build their answers on their diagrams to solve the problem and use their diagrams for checking and meaning because diagrams can make certain relations between angle and sides apparent. At the end of the construction of the diagrams, students had right-angled triangles and used the definitions of the basic trigonometric functions on it to get the answer. Students' use of trigonometric functions relied on memory and some students used a mnemonic way to remember the functions, e.g. SOHCAHTOA.

Subsequently, in the light of the Pritchard and Simpson's research, students seemed to follow some sort of answering model of trigonometry word problems. There seems to be mainly two parts in the answering process: diagrammatic and symbolic, which shows how geometry, algebra and trigonometry go hand in hand. In their study, Koedinger and Tabachneck (1994) also found algebraic and diagrammatic to be two of four strategies used by students in answering word problems: in the diagrammatic strategy students translated the algebraic problem into a diagrammatic representation. Transformations are performed on the diagram, including annotations, and the diagram supports inferences. These issues reveal that there are some important factors influencing the answering process of trigonometry word problems: drawing diagrams, visualisation in answering problems, working with the diagrams, symbolic manipulations, context, students' background knowledge and experiences.

I will focus on further factors, which seemed to be important in the answering process of trigonometry word problems, in the analysis and discussion chapters.

3.2.2 Discussion of the issues raised by the work of Pritchard and Simpson and others

In this section I will discuss the particular issues in the answering process of word problems, which might be essential in answering trigonometry word problems, as well as those raised by the work of Pritchard and Simpson and others. These issues will be discussed under the headings; Language, reading and comprehension, Context, Visualisation and Description of the process of solving word problems. Each of these titles is relevant to my second part of the first research question (see pp. 1-2).

3.2.2.1. Language, reading and comprehension

Studies have indicated that students have difficulties in solving mathematical word problems (Carpenter, 1985; Verschaffel and de Corte, 1997). One of the difficulties is reading which can affect the performance in problem solving so careful, detailed, and analytical reading is necessary (Collier and Lerch, 1969). Another difficulty was highlighted in the Cockcroft report (1982); that translation of the problem into appropriate mathematical terms presents great difficulty to many students since students do not give sufficient attention to the understanding of the problems (Lester, 1985).

Ellis (1990) sees language and terminology (mathematical language) as an issue in the understanding of trigonometric ratios and their use in solving problems involving triangles

"...both the language and the terminology used and the algorithms for the solutions of word problems and their mathematical model (the triangle) are incredibility difficult...." The semantic comprehension of the text of the problem, the translation phase according to Mayer et al. (1984) requires most of the cognitive processes necessary for the comprehension of every other text, argumentative, narrative etc. plus some special knowledge about the meaning of some mathematical terms, such as *"altogether"*, *"more than"*, *"less than"*, etc. In trigonometry word problems the terms "angle of elevation", "angle of depression", "horizontal", "vertical" could be added to the terms that students should know the meaning of.

3.2.2.2. Context

In their review of the literature Verschaffel et al. (1997) stated how context could be harmful in trying to get the correct answer of word problem:

> pupils learn that relying on commonsense knowledge and making realistic considerations about the problem context–as one typically does in real–life problem situations encountered outside school–is harmful rather than helpful in arriving at the "correct" answer of a typical school world problem.

Boaler (1993) stated that the degree of the influence of the context on students' performance has been underestimated for years in mathematics education, however, after recognising the importance of the context, a belief that mathematics in an 'everyday' context is easier than its abstract equivalent and that learning mathematics in an everyday context can ensure transfer to the 'everyday' lives of students appeared to be a misconception. Lave (1988 in Boaler, 1993) suggested that the specific context within a mathematical task is capable of determining not only general performance but choice of mathematical procedure. So the context might affect students' performance in trigonometry word problems beyond all other factors such as drawing or symbolic manipulations.

Context and mathematical language are two of many sources of difficulties found in GCSE mathematics questions (Fisher–Hoch and Hughes, 1996). The effect of the context on the performance of students was also reported by the APU (1988 in Fisher–Hoch and Hughes, 1996) that "Context has been found to affect success rate from a few percentage points up to 20%". Nickson and Green (1996) identified five elements of context which are pictures, words, numbers, symbols and graphics. They found that the richness of context, in which a mathematical question is set, can affect pupils' selection of the correct mathematical operator. Identifying the degree of contextualisation, which is facilitatory for students of different abilities, seems to be important in the problems asked of students. In their study, Nickson and Green concluded that:

> What seems clear is that placing a mathematical problem in a context has the effect of supplying a goal for pupils in a way that a context–free mathematics problem does not. A context gives pupils something to reason with and also supplies them with a source of reassurance about the reasonableness of their answer. However, underlying the assessment of any mathematics in a presented context there lies the assumption that pupils will bring

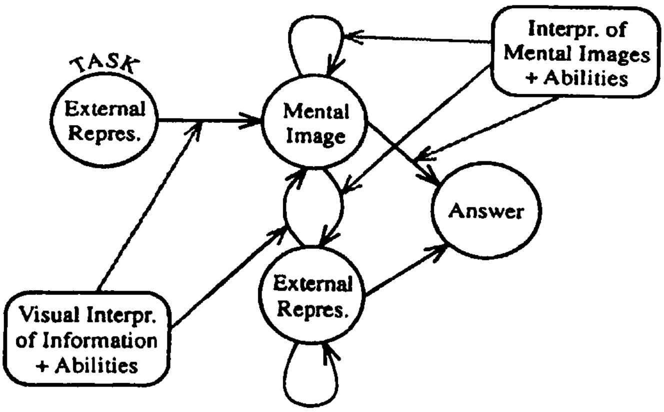
...'prior knowledge' to the situation in order for them to 'hear' or 'see' what they are shown.

3.2.2.3. Visualisation

The representation and visualisation of the word problem structure is important in word problem solving processes for its understanding. Zimmermann & Cunningham (1991, p. 3) describe visualization as understanding in mathematics. According to them mathematical visualization is the process of forming images (mentally, or with pencil and paper, or with the aid of technology) and using such images effectively for mathematical discovery and understanding. So the use of the term visualisation in mathematics is not the same as the everyday use of the term. It does not equate to just forming a mental image, it is about visualising a concept or problem rather than an idea. Toom (1999) claims that many non-word problems are necessary but not exciting technical exercises, whereas many interesting and non-standard problems are in the form of word problems. However, Toom goes on to say understanding of natural language and the ability to translate between different modes of representations: words, symbols, images are needed in word problems.

Although visualization and imagery in the teaching and learning of mathematics are important, the nature and role of them are complex so there are difficulties concerned with them (Dreyfus 1991; Love 1995). If mathematical visualization is taken as Zimmerman and Cunningham (1991, p. 3) describe then such difficulties can relate to the process of forming images as well as using them in solving problems. Similarly, if mental imagery is taken as involving: "constructing an image from pictures, words or thoughts; re-presenting the image as needed; and transforming that image" (Wheatly, 1991), then difficulties can arise from the processes of constructing, representing, and transforming. Nemirovsky and Noble (1997) describe visualisation as the means of travelling between external representations and the learner's mind; then the difficulty is the process of travelling.

Figure 2.1. Main visualisation elements integrating the solution of a mathematical task (Gutierrez, A., 1996)



Visualisation is one of the main bases of cognition. The connection of visualisation to drawing, constructing and handling of 3-dimensional objects is relevant to psychology, mathematics and mathematical education. Gutierrez (1996) made a very brief summary on visualisation and the relevant terms and then described her own definition of visualisation which is a kind of reasoning activity based on the use of visual or spatial elements, either mental or physical, which is performed to solve problems. However, visualisation is the integration of four elements, which are mental images, external representations, processes of visualisation and abilities of visualisation. Mental image is any kind of cognitive representation of a mathematical concept or property by means of visual or spatial elements. An external representation is any kind of verbal or graphical representation of concepts or properties including pictures, drawings, diagrams, etc. that help to create or transform mental images and to do visual reasoning. She defines the process of visualisation as a mental or physical action where mental images are involved. Visual interpretation of information and interpretation of mental images are two processes performed in visualisation. According to her, individuals should acquire and improve a set of abilities of visualisation when they are performing with specific mental images for a given problem, e.g. mental rotation, perception of spatial positions. She also presented a figure, which shows the steps to be followed when using visualisation to solve a problem (Figure 2.1. from ibid pp. 1-11).

In Kosslyn's (1980) theory, mental image consists of a surface representation; the quasi-pictorial entity present in the active memory and a deep representation; the information stored in the long-term memory from which the surface representation is derived. Kosslyn also identifies four processes applicable to visualisation and mental images;

- Generating a mental image from some given information.
- Inspecting a mental image to observe its position or the presence of parts or elements.
- Transforming a mental image by rotating, translating, scaling or decomposing it.
- Using a mental image to answer a question.

By being consistent with the literature, Zazkis, Dubinsky and Dautermann (1996) make a distinction between what is external (on paper or a computer screen) and what is internal (in the mind) in their definition of visualisation which is not restricted to either the learner's 'mind,' or 'some external medium,' but they defined visualization as the means for traveling between these two:

> Visualization is an act in which an individual establishes a strong connection between an internal construct and something to which access is gained through the senses. Such a connection can be made in either of two directions. An act of visualization may consist of any mental construction of objects or processes that an individual associates with objects or events perceived by her or him as external. Alternatively, an act of visualization may consist of the construction, on some external medium such as paper, chalkboard, or computer screen, of objects or events which the individual identifies with object(s) or process(es) in her or his mind.

In his paper discussing 'understanding' Davis (1992) mentions the mental representation in problem solving and claims that building up an appropriate mental representation of the problem situation, in one's mind, is one of the central parts of "problem solving". He also mentions the shift of the focus from what the student writes on paper to the mental representations that the student builds in his or her mind.

Transforming this mental representation might be the spatial ability students have and also students should have the capability to transform their mental representation onto paper, namely they should construct the diagram. The importance of constructing diagrams throughout the answering process is revealed in many studies. In their study Campbell et al. (1995) found that students drew diagrams as an aid to problem solving, the diagrams provided greater clarity and detail than a visual image and students overcame the limitations of working memory. In his research Cox (1999) concluded that constructing one's own external representation assists problem solving in numerous ways and involves a wide range of processes:

- Translating information from one type of representation to another.
- Exploiting both the phonological and visuospatial sketchpad components of working memory.
- Re-ordering information in useful ways.
- Directing attention to unsolved parts of the problem.
- Organising information spatially.
- Keeping track of progress through the problem.
- Providing perceptual assistance.
- The self-explanation effect.
- Transferring information between cognitive subsystems.
- Changing what is recalled.
- Facilitating the inference of motion (mental animation).
- Shifting the subject's mode of reasoning.
- Refining and disambiguating mental images.

He also emphasised that during construction of an external representation, subjects examine their own ideas, re-order information, translate information from one modality into another (re-represent) and keep track of their progress throughout the problem.

One of the familiar references in the literature with regard to drawing diagrams was probably made by George Pólya (1945). Based on his own experience with mathematics, Pólya compiled a list of *heuristic* suggestions for successful problem solving. Among the heuristics Pólya offered his students was to *"draw-a-figure"* which was not only for the topic of geometry but *"even if your problem is not a problem of geometry, you may try to draw-a-figure. To find a lucid geometric representation for your non-geometric problem could be an important step toward the solution"* (p. 108).

3.2.2.4. Description of the process of solving word problems

In addition to these researches, I also want to mention Butler's (1965, p. 509) pattern to solve trigonometry word problems. Although it is not a research paper, the pattern seems to fit very well with other researches and is characteristic of an answering trigonometry word problem process. The order of the occurrence of the steps are logical in the analysis and solution of the problem: drawing and lettering the figure, the selection and indication of a literal symbol to represent the unknown part of the figure, writing the equation, the transition from the ratio concept to the numerical concept of a trigonometric function and the substitution of the numerical value for the ratio, the actual solution of the equation for the unknown part and the reinterpretation of this in terms of the diagram or of the original problem.

After constructing the diagram by labelling the sides of the right-angled triangle, students do the mathematics part of answering trigonometry word problems, e.g. identifying trig function, developing mathematics and doing symbolic manipulation. Thus students need to use the definition of basic trigonometric functions, which could be by the use of memory, either literally or by a mnemonic way (Pritchard and Simpson, 1999), to identify the function. Blackett and Tall (1991) stated that the initial stages of the learning of ideas of trigonometry are fraught with difficulty, requiring the learner to relate pictures of triangles to numerical relationships, to cope with ratios such as sinA=opp/hypotenuse and to manipulate the symbols involved in such relationships. In their study, Owen and Sweller (1985) identified two types of errors, pertinent to use of trigonometric ratios, students committed in their answering process; trigonometric and fundamental. Fundamental errors indicated that students do not understand what a trigonometric ratio means, when or how to apply it. Students were unable to use a trigonometric ratio correctly within a right-angled triangle in which either the wrong side was selected or the inverse of the correct ratio was selected when they committed trigonometric errors. So, difficulty in identifying the sides of the right-angled triangle, e.g. adjacent, or accurately remembering the definitions of the three ratios, e.g. sine, may lead to trigonometric errors. Despite these difficulties a ratio approach is the more effective method in terms of performance and retention of concepts compared with the unit circle method (Kendal, 1992).

Subsequently, in the mathematical education literature there are many studies on word problems even though there is only one on trigonometry word problems. Some of these studies present some models and patterns for solving problems (Butler 1965; Lesh 1981; Chinnappan and Lawson 1996; Fisher-Hoches and Hughes 1996; Hughes et al. 1997; Greer 1997). However, the review reveals a point that answering trigonometry word problems could have a model including important cognitive processes, e.g. 'transformation', and it might help to find out the factors which can affect students' performance, e.g. context, terminology, students' ability.

In answering trigonometry word problems students could progress through some cognitive processes that Mayer (1985; 1987) has analysed in solving mathematical word problems:

translation, in which the student must convert each sentence into an individual mental representation; *integration* in which the student must select and combine information into a coherent representation of the entire problem; *planning* in which the student must break the problem down into a series of steps; *execution* in which the student must carry out mathematical operations. The first two processes seem to be relevant to the constructing diagrams part of answering trigonometry word problems and the last two seem to be pertinent to the symbolic part of answering trigonometry word problems.

Swanson et al. (1993) found that being able to solve word problems was correlated with measures of working memory, problem classification, knowledge of processing operations, reading comprehension and verbatim recall of word problem text. As a result of examining the literature, Lucangeli et al. (1998) formulated a model for the solution of word problems: text comprehension, problem representation that is a construction of a mental model, problem categorization, result estimation, planning the steps towards the solution, self-evaluation of procedure and self-evaluation of the calculation, for a total of seven components.

4. The teaching and curriculum

Students' have their learning histories shaped by the curriculum and the culture of education of their countries. So students' performance in problem solving might be influenced by the teachers and curriculum/curriculum documents, e.g. textbooks, which will be discussed under the headings teaching and curriculum respectively.

4.1. Teaching

Teachers have a very important place in every education system in every country. They might individually have a teaching approach or style or have been influenced by pedagogical factors. They are the ones who teach students in the classroom how to solve trigonometry problems. So I will look into teaching from three perspectives. In this section I will particularly address, teaching to answer trigonometry problems, teaching of problem solving and the influence of different pedagogies.

4.1.2. Teaching to answer trigonometry problems

As seen in the literature review so far there are some important points in simplifying trigonometric expressions and answering trigonometry word problems that teachers should be aware of. For example, in simplifying trigonometric expressions they have to be careful with the instruction 'simplify', that students should know where to stop with a simplified form. They also need to be aware of the students' simplification processes to better understand students' difficulties, furthermore they can focus on these difficulties in their teaching. In this respect Hughes et al. (1997) found out the question-answering model of GCSE exam questions showing the complete process of responding to a task.

In answering trigonometry word problems, students construct diagrams and then do some mathematics to get the result. Constructing diagrams does not happen so easily, it seems to be a complicated process. Students use mental imagery, visualisation and abilities, e.g. spatial ability. In a geometry context, Gorgorio (1998) stated that students' spatial abilities could be developed by different teaching methods and claimed that teachers having better knowledge about the strategies students use, and difficulties they encounter, when solving geometrical tasks, can contribute to the solution of the actual problems of teaching mathematics.

In answering trigonometry word problems, language and terminology might be a problem as it is in word problems but students can be trained. Low et al. (1994) confirm the importance of text comprehension training of their students to detect necessary and sufficient information from algebraic word problems. Training students can improve their problem-solving efficiency (Lucangeli et al., 1998). Teaching students how to represent arithmetic word problems using diagrams or schematic drawings improved their problem-solving performance in the studies of Lewis (1989) and Willis and Fuson (1988).

After constructing the diagram students should build the mathematics part of the answer by utilising the diagram. To do this they need to define trigonometric functions by using right-angled triangles or the unit circle method. These are the two teaching methods of introductory trigonometry debated by Kendal (1992), who found that students using the right-angled triangle method were more successful than the ones who used the unit circle method. But students still seem to have problems with identification of the functions, as observed in the literature. Kendal also found out that the ratio method has a dependency on algebraic skills.

4.1.3. Teaching of problem solving

Reed et al. (1994) highlight the importance of the examples of how people learn to solve problems. When students are given example solutions to algebra word problems, they perform much better on equivalent test problems. In this situation, students can examine and compare the two problems and apply solution procedures that worked for one problem to the new problem. Weaver and Kintsch (1992) viewed word problem solving as a type of problem solving by analogy and found out that students can and do take advantage of clearly represented structural similarities among word algebra problems and they can learn to do so with reasonable effort.

As an influence of the Cockcroft Report (1982, p. 71), practical work and problem solving (including the applications of mathematics to everyday situations) in the teaching of mathematics were seen at all levels in the UK. In the report, teaching practices are discussed and it is suggested that "mathematics teaching at all levels should include opportunities for exposition by the teacher; discussion between teacher and pupils and between pupils themselves; appropriate practical work; consolidation and practice of fundamental skills and routines; problem solving, including the application of mathematics to everyday situations; investigational work". In the same year HMI prepared a report about the general style of teaching in the sixth form "...the teacher presents a topic on the blackboard, works through an example and while the students carry out exercises based on the topic the teacher helps individuals..." That showed what was happening in the classroom. Problem solving exercises of some form are a part of lessons in every country. It is also one of the three major areas of emphasis in teaching mathematics which are facts and skills, understanding, and problem solving (Gadanidis, 1988).

Sigurdson et al. (1994) states that incorporating problem solving into teaching is very difficult for teachers and any attempt to incorporate problem solving into the classroom must address the issue of selecting an approach. They presented five of Blum and Niss's (1991 in Sigurdson et al., 1994) six organizational schemes that can be used for problem solving in the classroom:

- Polya's problem-solving strategies.
- Strategy problems as the basis of a lesson, a different approach to using the "strategy problems" is using them as the basis for a lesson.
- The investigations mode, investigations have long been advocated by mathematics educators from Britain. Mason et al. (1982) claim this approach will develop "mathematical thinking."
- A teaching approach to problem solving, using a heuristics or an investigations approach presents difficulties for the classroom teacher because the activities are seen as being separate from the regular mathematics content.
- A problem-process approach, a final suggestion for a problem-solving focus in the middle grades classroom is for daily exposure to problem solving.

Schoenfeld (1989) suggested that problem solving should be the main activity for engaging the students in learning and developing mathematics. He also highlighted that emphasis should be given to the following four related components, which are important in understanding what aspects of students' processes used for solving mathematical problems require more attention in mathematical instruction, during the analysis of the students' problem solving processes: (a) domain knowledge, which includes definitions, facts, and procedures used in the mathematical domain, (b) cognitive strategies, which include heuristic strategies, such as drawing diagrams, (c) metacognitive strategies, which involve monitoring the selection and use of the strategies while solving the problem and (d) belief systems, which include the ways that students think of mathematics and problem solving.

4.1.4. The influence of different pedagogies

The performances of students from different participating countries in the international studies were different. There might be many factors behind these performances but one of the them could be the pedagogies in different countries. Research indicates that there are distinctive national pedagogies. Significantly, one of the supplementary projects to the TIMSS confidently reported distinctive national, at least in respect of France, Switzerland, Spain, Norway, Japan and the United States, traditions described as characteristic pedagogical flow (Schmidt et al., 1996). Pepin (1999) also found that teachers' classroom practices in England, France and Germany reflected their beliefs and conception of mathematics and its teaching and learning. Pepin stated that the main determinant for different pedagogies practiced in different countries is educational systems which is supported by a powerful argument in the literature and Pepin also argued that there are 'non-visible' forces in classrooms which are often unvoiced principles, philosophies and beliefs that penetrate the educational setting. Moreover, Pepin goes on to say pedagogical styles are a personal response to a set of; institutional and societal constraints, educational and philosophical traditions, and assumptions about the subject and its teaching and learning. Thus, it is argued that teachers' pedagogies need to be analyzed and understood in terms of a larger cultural context and in relation to teachers' conceptions and beliefs.

Woodrow (1997) states that English mathematics teachers do not now take any responsibility for their curriculum and they are less interested in taking part in debates on teaching trigonometry:

> ...they are only concerned with delivering the given curriculum...there is little doubt that they have become increasingly expert at delivering what they are given but have little interest or expertise in creating or recognising what they (are) delivering. Debates about trigonometric functions as opposed to trigonometric ratios are irrelevant to (the) modern-day mathematics teacher.

4.2. Curriculum

Curriculum is the core in every country's education system. There are many important studies in the UK to develop the curriculum compared with TR.

In the UK there have been many reforms of the curriculum since the Second World War. Some examples are the Jefferey Report of 1944, development in modern mathematics, e.g. School Mathematics Project, Cockcroft Report, the Mathematical Curriculum and the Mathematical Numeracy Strategy. In Turkey very little has changed in the past 40 years (see www.meb.gov.tr). Integration of the subjects, which are arithmetic, algebra and geometry, was suggested. Mathematics is linked more closely with real life and less emphasis is placed on formal work. There was a considerable amount of trigonometry in the syllabi[2]; it is recommended that trigonometry should be more closely associated with geometry and practical applications (Cornelius, 1985). Trigonometry is a discipline that utilizes the techniques of both the algebra and geometry that students have previously learned.

[2] Trigonometry for the 16-19 year old students in the UK is outside the National Curriculum but in the syllabii.

One of the strongest features of English mathematics education has been the way in which attempts have been made to present mathematics in context and to encourage its use and application (Howson 1991, p. 35). One of the defining features of the syllabus 'Schools Mathematics Project' 11-16 was the relation between mathematics and the real world:

> considerable (but not exclusive) emphasis at all levels on the relationship of mathematics to the real world (MEG 1994, p. 3 in Fisher-Hoch and Hughes, 1996).

This can be observed in the UK curriculum, real world application of mathematics is emphasised, whereas in the TR curriculum, although application of the topics are emphasised almost all of them are technical, abstract applications of pure mathematics in terms of trigonometry.

In the research it has been observed that teachers were firmly following the curriculum and that they seemed concerned with covering the content of the curriculum (Pepin, 1999). The textbooks are the written documents including the content of the curriculum that teachers can utilise in their teaching. So textbooks have an important place in teaching and learning. Then what makes a textbook useful and usable in teaching and learning? Presentation and content seem to be two factors. In his study, Pepin (ibid.) analysed the use of textbooks in the UK and found out that textbooks were usually presented with brief explanations, cartoons and pictures in the introduction followed by exercises. He also concluded that:

> It was difficult to find a textbook in England which promoted the kind of cognitive activities that might help teachers to teach their lessons 'investigatively' (investigations are given at the end of chapters, as side-aspects of the main content teaching).

However there is still not enough research done on textbooks as Usiskin (1999) pointed out, there are few studies comparing mathematics textbooks in recent times. "Only 5 of 627 of mathematics education studies in 1995 and only 3 of 529 published studies in 1996 are textbook comparisons at any level." More studies need to be done.

With regard to the research on textbooks in Turkey, there appears to be no study conducted to date. Therefore it is difficult to give a detailed account of the state and use of textbooks. Nevertheless it should be noted that textbooks are published in essence by the Ministry of Education of Turkey (MET) and some private publishers. Private publishers are required to fulfill some conditions to produce textbooks that MET sets. These conditions are usually related to the physical appearance of the books e.g. size and to the linguistics e.g. compatibility of the texts with Turkish grammar. The textbooks also must follow the curriculum. However there are no limitations as to the presentation and the selection of problems and texts in the books.

Exams are the place where students are assessed in terms of teaching and learning. In terms of their literature review Fisher-Hoch and Hughes (1996) find out key issues which could effect the difficulty of examination questions in GCSE. These are the language of the questions, the capacity of working memory, the level of contextualisation, mathematical (technical) language, the development of mathematical understanding. Then they developed a table of the sources of difficulties in exam questions which might help teachers to work on them throughout their teaching and problem solving.

CHAPTER 3: RESEARCH DESIGN AND ISSUES

This chapter presents the research methodology adopted in this study. It discusses the methodological approach which seems to be the most appropriate to the nature of the research topic and to the research questions. However, it has been recognised that it is necessary to have a clear understanding of philosophical frameworks in social sciences in order to set the selected methodology in context. This chapter will be presented in six main sections: research questions, the research paradigm (approach), research methods (techniques), design of research instruments, data collection and data analysis.

1. Research questions

Two main research questions (RQs) in this research are asked. My first research question concerns student performance on tasks concerned with trigonometric identities and formulae, their use in the manner of 'simplifying' trigonometric expressions and their performance in solving trigonometry word problems. My second research question concerns the influence of teaching, the curriculum, examinations and resources on students' performance in this area. I split these two RQs up as the following.

***RQ1-i*[3]**

The focus here is on students' performance of trigonometric identities, trigonometric formulae and their use in the manner of 'simplifying' trigonometric expressions:

a- What difficulties do they experience, what errors do they make?

b- How do they use their knowledge of trigonometric identities in their simplifications of trigonometric expressions?

c- How do these performances interact with their knowledge and use of algebraic conventions?

RQ1-ii

The focus here is on students' performance of trigonometric word problems:

a- What 'mental models' do students follow in solving trigonometric word problems?

b- What difficulties do they experience, what errors do they make and what conceptions do they hold?

c- To what extent do the context and the terminology affect the solution of trigonometric word problems?

d- How do visual and symbolic representations interact in the solution process?

RQ2-i

The focus here is on teachers in both countries:

a- How do they teach trigonometry, what resources do they use and not use, how does the curriculum affect their teaching of trigonometry?

[3] Repeated as an aid to reader

b- What emphasis do they place on the foci of the first research question , e.g. how and in what order do they teach these?

RQ2-ii

The focus here is on the curriculum in both countries:

a- "What is it, as in written documents?"

b- How do teachers implement this in terms of classroom activities

c- What aspects do textbooks 'privilege'? What is examined and how important are these examinations?

2. What kind of research is this study?

Research can be defined as a systematic enquiry with the aim of producing knowledge (Ernest, 1994, p. 8). That raises the importance of the enquiry in research, which may be classified by the form of enquiry it employs. Enquiries may be categorized in terms of their purposes; exploratory, descriptive, and explanatory (Robson, 1993, p. 42). There is no restriction on the number of the purposes a study might have, it might be any combination of these purposes which are used; one, two or possibly three of them might be used. However, one of the purposes often predominates. When exploratory purposes are used in a study, by the researcher, what is happening is discovered, new insights are searched for and phenomena are assessed in a new light. Research using exploratory purpose of the enquiry is usually qualitative rather than quantitative (ibid. p. 42). An explanation of a situation or problem, usually in the form of a causal relationship is what a researcher is searching for in a research with explanatory purpose of enquiry. Research with explanatory purposes may be qualitative and/or quantitative as well as research with descriptive purposes. In a research with descriptive purpose an accurate profile of persons, events or situations is portrayed by the researcher. This purpose needs extensive previous knowledge of the situation to be researched or described, so that the researcher knows the appropriate aspects on which to gather information (ibid.). Descriptive and exploratory purposes predominate in this study: it is descriptive because it deals with perspectives of trigonometry in the UK and TR, it is exploratory because it tries to find out what is happening. It is not explanatory because it, in no way, seeks to provide explanations of cause and effect.

This study is conducted in the UK and TR with descriptive and exploratory purposes to the enquiry. It is a comparative study. The trigonometry aspect of mathematics education is being included in this study for the sake of comparing the UK and TR. By 'compare' (in terms of Graf and Leung, 2000) I mean to identify similarities and differences, and to interpret and explain the similarities and differences identified. Given two things or concepts, there may exist infinitely many aspects of similarities and differences, and hence in a comparative study, the comparison is always narrowed down to a particular theme or to a few particular themes. For this study, therefore, the UK and TR from the students' position (what they do and what they understand) are compared in terms of their performance in trigonometry. Important aspects of mathematics education that are of interest to this study can be found in studying mathematics education in these countries (ibid.). Bruhn (1995) emphasized the importance of the comparative studies in mathematics education stating that it is an unavoidable integral part of mathematics education, e.g. the insight into different dimensions of attitudes to school mathematics and how they are related to other factors, especially achievement.

3. Research paradigm

Before the methodological approach applied in this study is described and presented, there is an important issue of carrying out the study, which ought to be taken into account. This issue is deciding the paradigm, the theory of research methods, that underpins this research (see p. 54 in Romberg, 1992 and Ernest 1994,p. 18). The relation between research methods and paradigms is discussed by Guba and Lincoln (1994, p.105) who think paradigms are superior to methods of enquiry in a research:

> both qualitative and quantitative methods may be used appropriately with any research paradigm. Questions of methods are secondary to questions of paradigm, which we define as the basic belief system or world view that guides the investigator, not only in choices of method but in ontologically and epistemologically fundamental ways.

They also continue to assert that (ibid p.116) paradigm issues are imperative and a researcher should have a clear understanding about what paradigm informs and guides his or her approach. From the philosophical point of view Hughes (1990, p.11) aptly states the significance of defining a paradigm and its relation to the instruments and methods. Its significance comes from the fact that every research instrument is inextricably embedded in dedications to particular versions of the world. None of the methods of the enquiry are self-validating so that their effectiveness depends on epistemological justifications and, most importantly, research instruments and methods cannot be divorced from theory.

In summary, before the research methods, I should decide on the research paradigm which is the most appropriate approach to inform and guide my research. This is, nevertheless, a very difficult task. The terminology seems very confusing that the paradigms are sometimes so closely interrelated or even overlapping that it is difficult to establish clearly what the differences are. Blaikie (1993, p. 215) states these complexities and the limitation:

> In adopting an approach to social enquiry, the researcher is buying into a set of choices with far-reaching implications....No one approach or strategy, and its accompanying choices on these issues, provides a perfect solution for the researcher; there is no ideal way to gain knowledge of the social world....all involve assumptions, judgements and compromises; all are claimed to have deficiencies. However, depending on where one stands, it is possible to argue their relative merits.

By taking the difficulties, complexities and constraints mentioned so far into account, I have attempted to simplify the understanding of research paradigms by examining the relevant paradigms in the literature to decide on the one, which could support this study.

In the light of the literature review, it has been observed that there are three broad well-known paradigms in mathematics education, sociological and social science. Furthermore a distinction quite commonly made between different research approaches in the literature is: a) the empirical-analytical, logical positivist, behaviourist or positivist (or scientific or normative, see Cohen and Manion, 1994) paradigm; b) the interpretive (or naturalistic, constructivist, alternative paradigms research and qualitative approach to educational research (see Robson, 1993 and Ernest, 1994) hermeneutic, phenomenological or symbolic paradigm; and c) the critical paradigm (Vithal 1999, Ernest 1994, Romberg 1992, p.54, Kilpatrick 1988). Overall, "a" is called the positivist approach and "b" and "c" are called the anti-positivist approach where these two paradigms are also called normative and interpretive respectively (Cohen et al., 2000). Understanding these two approaches should be considered as a basis for the choice of philosophical stance to be adopted in this study. 'Behaviour' is a key concept in normative paradigm whereas interpretative approaches focus on 'action'. Going back to the research questions it can be seen that they are concerned with the interpretation of actions, and meanings placed on those actions. The positivist paradigm is criticised by Guba and Lincoln (1994, p. 106) who stated that:

> Human behaviour, unlike that of physical objects, cannot be understood without reference to the meanings and purposes attached by human actors to their activities

With a non-positivist base, this study has encompassed an interpretivist perspective. The main concern of this paradigm is human understanding, interpretation and intersubjectivity (Ernest 1994, p. 24). In recent years, this paradigm has gained much ground with the strong emergence of constructivism (Vithal, 1999). The reasons for this choice can be justified both by the nature of the research project and by the research questions. In adopting an interpretivist paradigm the objective is not to gather data and facts to measure how often a certain pattern occurs, but instead to appreciate different constructions and meanings through people's experience. The 'how' and 'what' types of research questions also lead to this approach where the explanation, interpretation and the construction of meanings and motives, will be taken from the perspective of the 'social actors' who are students and teachers in this study (Nickson, 1992, pp. 107-108). So it is possible to discover and understand what is happening. Kilpatrick (1988) points out that this paradigm sets out to "capture and share the understanding that participants in an educational encounter have of what they are teaching and learning". It seems therefore, appropriate to say that this study has adopted an interpretativist approach. So my overall approach to this study may be called 'naturalistic' in the sense that I have observed, as far as possible, 'what is' in both countries without manipulating the course of teaching and learning in any manner. So the study has been conducted in the original settings without any interfering action (Robson 1993, pp. 60-61).

Research can be categorised in terms of the data the researcher is seeking. These categories are quantitative research and qualitative research. Quantitative research seeks to establish facts, make predictions, and test hypotheses that have already been stated by using a deductive approach. The data analysis of quantitative research is mainly statistical, striving to show that the world can be looked at in terms of one reality; this reality, when isolated in context, can be measured and understood, a perspective known as positivism (Gay and Airasian, 1999).

In contrast to the statistical nature of quantitative research, "Qualitative research is multi-method in focus, involving an interpretative, naturalistic approach to its subject matter" (Denzin and Lincoln, 1994). This means "qualitative researchers study things in their natural settings, attempting to make sense of, or interpret, phenomena in terms of the meanings people bring to them" (ibid.).

I will not discuss deeply the difference between the quantitative and qualitative research. However I will try to draw out the important differences for the sake of my research. The difference between these two forms of research is that quantitative research is often positivist in its outlook and qualitative research usually has a non-positivist perspective. Qualitative research is a method of naturalistic enquiry which is usually less obtrusive than quantitative investigations and does not manipulate a research setting. It aims to study people in their natural social settings and to collect naturally occurring data. While quantitative researchers work mostly with numerical data, qualitative researchers use mainly "non-numerical data such as observations, interviews, and other more discursive sources of information" (Gay and Airasian, 1999). Another difference between the two types of research is that where quantitative research seeks to find evidence which supports or does not support an existing hypothesis, "qualitative designs allow the hypotheses to emerge from patterns of recurring events" (Huysamen, 1997). Quantitative research is based on manipulation and control, results verified by sense data (by the researcher) whereas qualitative research is based on insights and understandings about individual perception of events (by the subjects). Qualitative research can provide rich, illustrative examples and its findings are often more accessible than those obtained via the purely quantitative approach.

Hoepfl (1997) suggests that educators should "engage in research that probes for deeper understanding rather than examining surface features." She posits that qualitative methodologies are powerful tools for enhancing our understanding of teaching and learning, and that they have "gained increasing acceptance in recent years". This research is mainly qualitative rather than quantitative, because "choice of research practices depends upon the questions that are asked and the questions depend on their context" (Denzin and Lincoln, 1994). Moreover, there has been little previous research on the topic, there is no recognized theory which has been developed relevant to the topic, the study is exploratory as it is aforementioned, the individual experiences of the sample subjects are at least partly the product of individual interpretation. My interest in how students simplify trigonometric expression by using trigonometric identities and answering trigonometry word problems and what is the influence of the teachers and the curriculum on students' performance leads me to the use of qualitative research (and methods). The focus is on the meanings the students and teachers in the study setting attach to their social world. Its strength is the ability to study people in the 'field', i.e. in their natural settings because of the belief that "the phenomena of study take their meaning as much from their contexts as they do from themselves"(Guba and Lincoln, 1994).

In summary, this study is a comparative study with exploratory and descriptive purposes of its enquiry. It uses an anti-positivist paradigm which is interpretitivist with a naturalistic enquiry approach. In terms of the type of the data, my research is mainly qualitative but it is a combination of both qualitative and quantitative data collection tools.

4. Research methods

The choice of the method of inquiry depends on the research paradigm, which is the philosophical approach chosen to apply in the study. The 'positivist' paradigm is about objectivity, prediction, replicability and the discovery of generalisation. Research methods based on the 'positivist' paradigm are concerned basically with testing existent theories. Measuring variables and working on large samples are usually preferred to make statistical generalisations in the positivist paradigms. On the other hand, non-positivist paradigms are concerned with human understanding, interpretation and so on. The methods based on these paradigms take completely different assumptions into account. So the research methods on the base of different paradigms might be different. These differences are asserted by Bogdan and Taylor (1984, p. 2) in terms of the positivism and phenomenology. The researchers using these two paradigms undertake different kinds of problems and search for different kinds of answers, therefore their research requires different methodologies. These differences might be pertinent to what these paradigms are concerned with and that in contrast to the scientific approach, the phenomenologist endeavours to understand the motives and beliefs behind people's actions on a personal level.

Some researchers believe quantitative and qualitative methods represent different paradigms of inquiry (Creswell, 1994). The difference between these methods might be research strategies and data collection procedures, but Bryman (1992, p. 105) states that there are more than these in quantitative and qualitative methods:

> These approaches represent fundamentally different epistemological frameworks for conceptualising the nature of knowing, social reality, and procedures for comprehending these phenomena.

Different strategies and methods have been suggested in order to understand and interpret the phenomena in the social sciences. Questionnaires, inventories, demography (Bogdan and Taylor 1984, p. 2), survey, comparative experimental and quasi-experimental methods (Ernest 1994, p. 22) are some of the methods used in a positivist research to produce data amenable to statistical analysis, so it seems to yield quantitative data but qualitative data can also be used. In an anti-positivist approach, however, in contrast to positivist research, mostly qualitative methods are used although quantitative methods could be used as well. Participant observation, in-depth interviewing (Bogdan and Taylor 1984, p. 2), ethnographic case study, triangulation (Ernest 1994, p. 24) are some of the methods used in anti-positivist research which yields rich, descriptive and contextually situated data (King, 1996).

After the theoretical framework of the research, the research questions can be answered through employing many appropriate methods. For this purpose, both quantitative and qualitative research methods are drawn on in this study. In other words, a multiple-method as a combination of the methods is employed in this study. This approach has many advantages even though it is time consuming. The initial and obvious benefit of using a multiple-method is that it involves more data, thus being likely to improve the quality of the research (Denscombe, 1998). The multiple-method approach also reduces inappropriate certainty. That is to say, finding a definite result by using a single method may mislead the researcher into believing that they found the 'right' answer. Using additional methods may point to conflicting answers, which remove specious certainty (Robson 1993, p. 290). They may help to answer complimentary questions rather then specific questions, it may also be used in enhancing interpretability, for instance in a quantitative study, statistical analysis can be enhanced by interviews or a narrative account (Robson 1993, pp. 290-291). The multiple-method approach seems to be identical with Denzin's 'between methods' triangulation (Cohen and Manion, 1994). This method is important and suitable when the complex phenomena needs elucidation such as a comparative study, for holistic view of educational outcomes, at evaluating different teaching methods, for full evaluation of controversial aspects of education, for complementary purposes if approaches give a limited picture and for case studies. There are many research methods presented in the literature (e.g. Cohen et al. 2000 p.77, Robson 1993, Dyer 1995, Oppenheim 1992). In this study, the appropriate methods are selected to apply in terms of the characteristic of this research and research paradigm. The research methods used in this study are diagnostic tests, interviews, verbal protocol, questionnaire, observation, and document analysis which I now deal with.

4.1. Diagnostic tests (written tests)

To explore the RQ1 students' explicit manipulations are needed. In this way it is possible to discover the flaws[4], weaknesses and difficulties students experience throughout their answering process. Diagnostic testing is an appropriate instrument to obtain written evidence that exhibits the particular strengths, weaknesses and difficulties in the performance of students. To discover the difficulties that students have, several items testing for the same features were used in the

[4] Flaws are the mistakes in incorrect answers, the obstacles in partial answers

tests. Constructed test items should reflect a range of very specific difficulties students might experience (Cohen et al. 2000, p. 322).

4.2. Interviews

Although diagnostic tests help to identify the difficulties students meet and the errors students make, they do not give the reason behind their answers. So these tests are followed up with interviews in order to understand the reasons for the students' answers. Teachers are also interviewed for the purpose of getting their thoughts and views of all topics pertinent to the teaching and learning of trigonometry.

Interviews are a very widely used instrument in social research. Using interviews in research is critical to gathering deeper information about the responses given by students or teachers. Cohen et al. (2000) has given three purposes to use interviews in research; to gather information having direct bearing on the research objectives, testing a new hypothesis or suggesting a new hypothesis and using interviews in conjunction with other methods (multiple-method, see also Dyer 1995, p. 64). There are many different types of interviews which are given by Cohen et al. (2000, p. 270) in detail. The choosing of an appropriate type of interview depends on the purpose of the inquiry. I will focus on the types of interviews which are distinguished by a commonly used typology.

Dyer 1995 (pp. 58-59) sees different forms of interviews as lying along a continuum. Structured and unstructured interviews are at the two ends of this continuum. Structured interviews have an accurate form and direction of the questioning determined in advance of actually meeting for the interview. At the other end of the continuum there is no prepared list of questions, the interviewer is free to ask questions depending on the information received from the informant. Dyer points outs that a combination of these two forms, which is called a semi-structured format, is probably the most successful approach to use in interviewing:

> A semi-structured format, in which the interviewer works from a number of prepared questions, while allowing the respondent plenty of opportunity to expand answers, and pursue individual lines of thought seems to offer the best approach

Robson (2002, p. 270) explicitly categorised interviews under three titles; fully structured, semi structured and unstructured interviews based on their structure and standardisations. These are identical with the use of Dyer. Semi-structured and unstructured interviews are referred as qualitative research interviews by King (1994 cited in Robson, 2002). These interviews are most appropriate for exploratory purposes of the research enquiry.

So the most appropriate interview method for this study seems to be a semi-structured interview. However, within the semi-structured interview, I have applied Tomlinson's (1989) hierarchical focusing interview method. There are some advantages which distinguish the hierarchical focusing interview from the structured and unstructured interview methods. The hierarchical interview provides the interviewer flexibility to change the wording or makes it possible to eliminate some questions in line with the conversation flow as the topic may be covered in responses to earlier questions even though both hierarchical and structured interviews determine the aims and required information to be gathered. There is no schedule in an unstructured interview and informants have the freedom to talk about the topic in which they are interested in relation to the main purpose of the interview. The structured interview is more frequently used in gathering descriptive data whereas the hierarchical and unstructured interviews are frequently used in exploratory research (Robson, 1993). The hierarchical interview is pointed out as a suitable way to elicit the research data in an effective way by Novak et al. (1984) and Tomlinson (1994).

Furthermore, open-ended questions are used in hierarchical interviews, starting with general questions and gradually moving to more specific questions in the aim of getting points the interviewer needs to address in the study (Drever, 1995). Starting with general questions allows the nervous interviewees to settle down and prepares them for the further specific questions in terms of comfort, speech, confidence and getting familiar with the interviewer.

4.3. Verbal protocol

Verbal protocol is a self-report of a person about what she/he is doing or about to do or what they expect to achieve with regard to a particular task (Johnson and Briggs 1994, p. 61). Green (1998, p.1) sees 'verbal protocol' as a label representing the data gathered from an individual under special conditions, which could be a task with either 'think aloud' or 'talk aloud'. Green (1995) briefly describes verbal protocols (or verbal reports as Ericsson and Simon, 1993, used as gathering data on how people approach a problem and the mental process they adopt in their approach to the problem. Green and Gilhooly (1996) define verbal protocols as transcription which is derived from recordings of the participant's speech during the task they are carrying out under thinking-aloud instruction. Verbal protocol analysis is a methodology that is frequently used in cognitive psychology and education. For qualitative researchers interested in getting a rich source of data, the verbal protocol analysis method is an excellent choice (Branch, 2000).

In this research, the written tests will provide broad indicators of student performance and relative strengths and weaknesses in the two countries. They are not, however, likely to provide information on why students respond in the way they do or reveal their thought processes. Verbal protocol analysis aims to find cognitive processes while solving a problem, verbal reporting is bringing thoughts into consciousness, making the ideas verbal if needed and then verbalizing them (Ericsson and Simon, 1993). This methodology was used by Schoenfeld (1983) and Schoenfeld and Herrmann (1982) in mathematical problem solving (cited in Green 1998, p. 2). In this study, verbal protocols were used as students solved similar problems to gain further insight into the thinking behind their performance. In the spirit of qualitative research for exploratory purposes this method, it is hoped, will release hidden information behind students' performance which cannot be obtained by written tests or interviews. Furthermore, one of this method's important features is being a sequential process so that utterances are the reflection of the natural thought process as Ericsson and Simon (1993, p. xxxii) discussed:

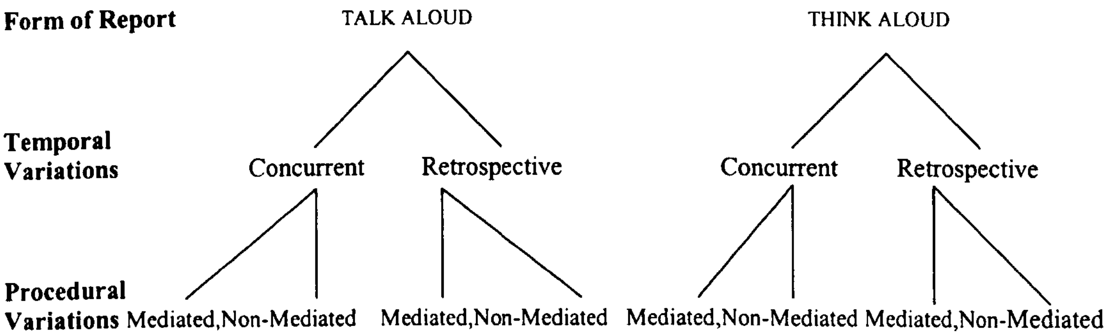
> subjects can generate verbalizations, subordinate to task-driven cognitive processes (think aloud), without changing the sequence of their thoughts, and slowing down only moderately due to the additional verbalization.

Verbal protocol is different from other research methods employing verbal data, such as interviewing which focuses primarily on linguistic content and structure. In the case of verbal protocol, inferences are in fact made about the cognitive processes that produced the verbalisation (Green 1998, p. 1). In written tests, students' manipulations on paper is the only evidence which can be seen in the answering process and there is nothing about their cognitive involvement. It is not possible to read students' minds between the steps in their manipulations, but they cannot write everything they think onto paper. Students cannot directly report their own cognitive processes. The verbal protocols will help to infer cognitive processes and attendant information from the student. The important thing is that protocol analysis wants subjects to utter their thoughts. This might help to infer the form of mental processes from verbal reports. The advantages and the value of the verbal protocol method is emphasised by Crutcher (1994):

> Beyond the increased acceptance of verbal reports as behavioural data, the unique advantages of verbal report data are increasingly apparent to many people. In particular, researchers have emphasised the advantage of verbal reports as protocol data, providing a sequence of observations over time rather than just a single observation at the end of the process..........In addition, verbal protocols provide many more observations of a phenomenon over a given time period than other methodologies, increasing the information yield of studies..........Verbal reports can provide information difficult to obtain by other means...in particular, information about types of knowledge accessed in task processing.....

There are slightly different ways to gather verbal protocols under changeable circumstances according to research questions, paradigm and purposes of the enquiry of the study. I will discuss the appropriate approaches given by Green (1998, p. 5), as presented in Figure 3.1., to find the most appropriate approach for my study.

Figure 3.1. Some variations on the verbal protocol procedure.



The first step in choosing the appropriate approach is to determine whether the report is to be a "talk aloud" or "think aloud" report. In talk aloud, student attend to phonemic information that can be vocalized directly, and simply. The information contained in these sorts of reports roughly corresponds to words in the mind, or thoughts that might be spoken. In think aloud, student recodes verbally and utters thoughts that can be or have been held in memory in some other form (e.g. visually), orally encoded information and all kinds of thoughts which can be simple or complex, perceived, generated through cognitive processes or recalled from long term memory (see also Ericsson and Simon, 1993, p. 226).

The difference between think aloud and talk aloud is quite delicate, nevertheless it is essential that researchers should be aware of the slight distinction between these two approaches. Everything the students verbalise is information that is already coded in verbal form in the talk aloud approach. They may also encode non-verbal visual or auditory information. In the think aloud approach it is clear that what is verbalised is all heeded information. Information may have to be recoded prior to verbalisation because they are not already in verbalisable form. (Green, 1998, p. 7). The aim in using verbal protocol is to gain further insight into the students thinking behind their performance by answering similar problems. So, think aloud more than talk aloud gives me the information that cannot originally have been encoded in verbal form and so the think aloud protocol was chosen in the research. This method helped to reveal the reasons and the ways for missing steps and errors in students' solutions.

After choosing think aloud protocol, the temporal step of verbal protocol should be considered (Figure 3.1). There are two reports, concurrent and retrospective, for think aloud. Concurrent (simultaneous) reports are produced simultaneously as the student is working on the task, whereas Retrospective (subsequent) reports are created after the student has finished working on the task. The time interval between task completion and the start of the retrospective verbal report is important. If the interval is short then information will still be present in working memory. If the distance is long then the retrieval process must be considered (Green, 1998, p.6). Concurrent reports are used in research, because they are far less susceptible to influences from unwanted variables than are retrospective reports. The steps of carrying out the task can be sorted out by verbalisation at the same time. This provides what students think, between and behind the steps, when they give their information verbally. In a retrospective report, in the researcher's opinion, the time interval, either short or long, affects negatively the student's verbalisation. The student may not remember what they used in the solution, because the questions require some formulae, equations and their use properly in the solution.

It was decided to use think-aloud and concurrent reports in this research. It is imperative that throughout carrying out the task in verbal protocol students should be prompted to talk when they stop for a designated period of time. Non-intrusive words, for example "keep talking" and "think aloud" are used as prompts in "Non-mediated verbalisation". If students skip some details in verbalising between steps in answering the questions, then the questions "why did you do that?" and " what were you up to just then?" are asked, this is "Mediated verbalisation". To conclude, "Non-mediated verbalisation" throughout carrying out the task was used in the concurrent verbal protocol.

4.4. Questionnaire

To explore the teacher's influence on students' performance, interviews and observations are used. However, not all necessary points can be explored in an interview nor be seen in an observation. So to gain access to the broader picture of how teachers influence students' performance, data was gathered from teachers by means of a questionnaire.

Questionnaires have advantages and disadvantages. The most important advantages of the questionnaire are its simplicity, versatility and low cost as a method of data gathering. Questionnaires can give good quality data towards the answering of research questions (Fife-Schaw, 1995). However questionnaires are a kind of list of written or printed questions which should have the least possibility for misunderstandings and ambiguities to occur in order to explore more information on respondent's attitudes, behaviour, beliefs or experiences (Dyer, 1995).

Questionnaires have their own constraints and it is of basic importance to the researcher to be aware of them. Questionnaires are a good way of collecting certain types of information quickly and relatively cheaply as long as subjects are sufficiently literate and as long as the researcher is sufficiently disciplined to abandon questions that are superfluous to the main task (Bell, 1987). In the case of the present inquiry, the subjects are certainly sufficiently literate to expose their own ideas in written form. Questions judged to be unnecessary or repetitive will not be included in the questionnaire. Another constraint on the use of questionnaires is that "Many people will fabricate favourable comments if their names are disclosed. The same people may well give most enlightening unfavourable comments if they are permitted to remain anonymous" (Schofield, 1972). It is essential that teachers should write their answers sincerely and their true comments are needed to show what they really think.

Types of questions of the questionnaire

A questionnaire can consist of closed questions or open questions or both open and closed questions. A closed question is one in which the respondents are offered a choice of alternative replies and may be asked to tick their chosen answer. Open questions do not have any alternative replies, so it is not followed by any kind of choice. Their answer should be written in sentences.

Closed Questions

Closed questions are easier and quicker to answer, respondents do not have anything to write, they just tick the statements. They are easy to code and analyse and can readily be compared with each other across different respondents. However, the loss of expressiveness and spontaneity is a disadvantage of closed questions. It is not known what the respondent thinks when she/he answers, even if we force them to choose one of statements, because the range of possible responses to a statement is entirely determined by the researcher. Teachers give their answers at the time of reading and very quickly. These are inherent weaknesses of closed questions which are difficult to overcome.

Dyer (1995) states that closed questions are best used for collecting straightforward factual information such as for establishing baseline data on behaviour which can later be explored using more open ended questions such as in an interview. The interview allows the researcher to go into more depth if she/he chooses and can clear up any misunderstanding of the questionnaire. So the interview helps to find out more information than is obtained from the questionnaire. Moreover, the observation of teachers in classroom settings allows the researcher to discover whether what teachers actually do is what they say in questionnaire.

Graded response questions are used, in which the respondent will be completing the questionnaire to express degrees of magnitude in their answers. There are three kinds of graded response questions: attitude statement questions, Likert-type questions and semantic differential questions. With attitude statement questions, the informant is provided with a statement which reflects an attitude to a particular issue or topic and is then invited to select an answer from a continuum which most closely matches their response to that statement. Likert-type questions are similar to attitude statement questions in structure, apart from that they have an ordinal scale which enables the informant's response to be expressed directly. Semantic differential questions use a technique to provide a way of quantifying feelings, emotions, perceptions and similarly subjective variables that are difficult to express numerically. Informants are asked to choose from the continuum, that has extreme ends, which best describes their feelings (Dyer 1995). Attitude statement questions are used when_the informant is provided with a statement which reflects an attitude towards trigonometry lessons or students in trigonometry lessons and is then invited to select one of the continuum which most closely matches their response to that statement. In this research respondents were offered four points along the continuum at which to make their responses, and not five. They were not given the choice of selecting a neutral middle option. This was done so that the respondents were forced to make a choice within one half or the other of the continuum. This is given as a suggestion by Risnes (1998).

Open Questions

Teachers' teaching approaches are influenced and shaped by their academic or professional background and experiences. Some aspects of the teaching approaches of teachers in trigonometry lessons can be discovered by their attitudes towards trigonometry lessons, much more specifically trigonometric identities and the simplification of trigonometric expressions by using trigonometric identities. Moreover, their attitudes towards students also helped me to explore their teaching styles, the resources they use and the written information they follow as sources can have an effect on their teaching styles. But still it is necessary to explore how they solve trigonometry word problems and simplify the trigonometric expressions by using trigonometric identities. Then their ways, methods, steps to answer such questions should be seen, in the case of missing their approach to answer these questions in the observation. Closed questions cannot help to find answer to that question, because each teacher has her/his own style, so full and verbatim answers are needed. Therefore open questions were used to explore their teaching style in the second section of the teacher questionnaire. This also helped me to see the correlation between teachers' and students' answers to specific questions.

The open questions give the respondents freedom to answer in their own ways, namely they enable a respondent to answer a question wholly in her/his own words rather than having them provided for her/him. "The open question is effective in showing the respondent's attitude to a situation, since the words she/he uses to answer the question are her/his own...there is the possibility of her/his not understanding the question or the situation to which it refers, but this will be revealed in the answer" (Schofield, 1972).

Nevertheless, even though open questions are often easy to ask, they can be difficult to answer and even more difficult to analyse. Because people are free to answer in their own words, the range of possible responses is infinitely wide and this should be taken into account throughout the data analysis (Dyer, 1995; Oppenheim, 1992). The data, as a result of open questions, is in the form of sentences written by respondent. One of the most common ways to analyse data from open questions is to convert the appearance of the data into a form that allows the researcher to do some calculations to interpret the data. Generally this can be done by a classification system. The process of classifying responses in this way is known as coding into numerical form (Oppenheim, 1992). Coding of responses involves combining the detailed information included in the response into a limited number of categories that enable simple descriptions of the data and allow for data analysis. This process brings about loss of information (Robson, 1993). General aspects of the teaching style of teachers in trigonometry lessons in the UK and TR are the main points to be found out rather than the very individual details of the teachers' answers. So, although time consuming time and the need for a coding system are two disadvantages of open questions, classifying these answers properly and carefully may reduce the loss of information. As an open question, teachers can be given some of the questions from the students' written tests and be asked the way they expect students to answer these questions.

4.5. Observation

Observation is a research instrument that allows researchers to gather 'live' data from the 'live' situations and in which the investigator systematically watches, listens and records the phenomenon of interest. After administering questionnaires and interviews to the teachers, observations enable the researcher to see things missed in written or spoken data, to find out things which could not be expressed by teachers in the interview, compare whether what they say and think is actually what they do with respect to a relevant focus of the research. Moreover, contrary to questionnaires and tests, observations are less predictable (Cohen et al., 2000, p. 305). Observation is used to validate or corroborate the information obtained in teachers' interviews and questionnaires. So it can be used as a supportive or supplementary data technique. It helps to address different but complementary research questions rather than focusing on a single specific research question. Observation is a distinct and direct way to collect data. It is more direct than asking people about their views, feelings or attitudes or obtaining what people say they do, or what they say they think. It is watching people to witness events first hand, to find out what they do and to listen to what they say (Robson, 1993).

Morrison (1993, p. 80) argued that by using observation the researcher gathers data on physical settings, the human settings, the interactional setting and the programme setting. The physical environment and its organisation, in which trigonometry is taught in the UK and TR, is observed and also the teachers' teaching approach to trigonometry and the implementation of the official resources can be observed so as to have complimentary data on the research questions in addition to the main instruments which are the interview and questionnaire data collected from the teachers.

Cohen et al. (2000, p. 305) point out that there is a continuum for observations, but it is in terms of the structure, from a highly structured observation to an unstructured observation. Semi-structured observation is in the middle. In highly structured observation the researcher knows in advance what she/he is looking for and has observation categories worked out. Although there is an agenda of issues in semi-structured observation, in a far less pre-determined or systematic way observation data is gathered to illuminate these issues. In unstructured observation however, the researcher is not limited by any prior assumption. She/he tries to gather all the information which is available in the situation. The first type of observation is for hypothesis-testing whilst the second and third ones are hypothesis-generating. My study is exploratory so hypothesis-testing is the most suitable type as observation (Robson, 1993).

It seems that, almost always, the researcher takes her/his place in gathering observation data as a participant. So observations may also be categorised based on the researcher role in a continuum; complete participant, participant-as-observer, observer-as-participant and complete observer (Gold, 1958). Other researchers followed or modified Gold's categorisations. Dyer (1995) placed the different form of observations on a continuum as he did with the interviews.

The two extreme ends of this continuum are non-participant/ 'observation only'/or 'pure' observation and participant respectively (see also Robson 2002, pp. 313-319). In participant observation, the observer takes an active role and is fully engaged in the picture that the research describes. In non-participant observation, on the other hand, the observer does not play a direct part in the picture of observation, she/he is an outsider who is not involved in the activities taking place and merely observes. In participant-as-observer role, the researcher makes the group aware that she/he will observe, tries to be accepted as a full member of the group and so tries to establish close relationships with the members of the groups she/he is working with. An observer-as-participant role is very close to the non-participant end of the continuum. The researcher does not have a part in the activities (this is questionable see Robson 2002, p. 319) but is still known as the researcher. The researcher's fundamental concern is collecting data although she/he seems to be involved as a participant in any on-going activity. Dyer (1995) presented systematic observation approach for non-participant observation which associated the role of the researcher with observer-as-participant. In advance of actual observations, the researcher decides what is to be observed and under what conditions. So the researcher knows the purpose, which is to record behaviours under defined conditions so that any other behaviour can be ignored. Since one of the aims in the research is to explore how teachers teach trigonometry, what resources they use and how the official resources implemented in the classroom activities, systematic observation with the time-interval sampling seems to be an appropriate approach in which to observe teachers in the classroom in terms of trigonometry. I have employed the non-participant observation approach in my study. I simply observed and did not participate.

4.6. Document analysis

To explore whether the curriculum is one of the factors on students' performance in trigonometry and its relation to the teachers' aspect of this research, an important factor in this research is to discover what is written in all possible documents which are primarily, curriculum, textbooks, schemes of works and exam papers. In my analysis, I used 'written documents' which is referred to as documents by Robson (2002, p. 348). Document analysis techniques differ from other research methodology techniques. Although observing, interviewing, or questionnaire methods are directly used for the aim of the researcher's enquiry, documents are prepared for some other purposes. They are not affected by the fact that the researcher uses them in her/his research. They are therefore non-active. Robson (1993, p. 272; 2002, p. 349) discusses content analysis as a common approach to document analysis. The main advantages are that existing documents are used in a way that researchers observe without being observed, namely, the nature of the documents are non-reactive. They are not affected by the fact that a researcher is using it, data is in permanent form and can be reached in the original form at any time. Since the topic is specifically students' understanding of trigonometry and possible factors on their performance, solely trigonometry parts of the documents are collected.

They are written to be used in teaching trigonometry which affects students' performance. That shows how close the purposes of the documents I have collected are to my study. Since the documents are used in the multiple-method approach in conjunction with the other instruments, causal relationships may be explored to some extent.

TIMSS (Schmidt et al., 1997) is used in the document analysis. Official government documents such as the curriculum almost always define at the national level statements of student learning goals, topics to be taught, textbooks to be used. These documents, however, differ in the degree of detail with which the learning goals, topics to be taught, textbooks to be used are specified. These are the facts that might have direct influence on students' performance and on teachers 'privilege' in different countries.

5. Reliability and Validity

Reliability and validity are two important keys to the research. What are reliability and validity used for in research; for instruments, data or analysis? An analysis is a procedure that is used to interpret data gathered by instruments. So calculations are made on the data for the reliability and validity. Reliability and validity of an instrument cannot be defined without the data collected from it. They contribute to the research whether or not the instruments and the data are really competent to provide a precise and meaningful answer to the research questions which are being worked at. Basically, reliability is a synonym for consistency, in other words, it is an extent to which a test or any measuring procedure yields the same result on repeated applications (see Cohen et al 2000, p. 117). Validity refers to the degree to which a study accurately reflects or assesses the specific concept which the researcher is attempting to measure, namely, a particular instrument should measure what it is supposed to measure (ibid. p. 105).

There are different ways of assessing the reliability and validity of research instruments for data collection, as Cohen et al. (2000, p. 105) stated, these two concepts are multi faceted. Although how reliability and validity are addressed in qualitative and quantitative research varies, they are both applicable in these two types of researches. My research is mainly qualitative and I will use the reliability and validity concepts in this context. However, reliability and validity are not easy to analyse in qualitative research. Seale (2002) points out the difficulty that even a sentence such as 'reliability and validity in qualitative research' could be confusing in terms of paradigms and he presents how Lincoln and Guba (1985) transformed the questions which are asked in the quantitative research to the ones in qualitative research. They call reliability 'dependability' and (external) validity 'transferability'.

Reliability is dependent upon stability, consistency and predictability (Lincoln and Guba, 1985, p. 296). Reliability is a necessary precondition of the validity. So a research instrument such as a particular experiment or questionnaire is said to be reliable if it is consistent, and this is generally deemed to be a good thing as far as research is concerned (Denscombe, 1998). There are some methods of assessing the reliability of an instrument in quantitative research (Dyer 1995) which cannot be simply workable for qualitative research (Cohen et al., 2000, p. 119). In qualitative research, reliability can be regarded as a fit between what researchers record as data and what actually occurs in the natural setting, namely, it is a degree of accuracy and comprehensiveness of coverage (ibid.). Criteria for reliability in qualitative research are different from the quantitative research. Reliability in qualitative research includes fidelity to real life, comprehensiveness, detail, honesty, depth of response and meaningfulness to the respondents, which could also be gained by the triangulation (multiple-method) approach. Ratcliff, (1995) points out that multiple reviewing of the data by the researcher or by many people is a way of ensuring reliability. As a result of that, high reliability could be found and it may suggest a systematic bias at work in data, a bias shared by multiple researchers. This is why many qualitative researchers emphasize validity rather than reliability; documenting what occurs in an accurate manner may reveal inconsistencies. Low reliability could be consistent with high validity if the social situation is constantly in flux, or people might see things differently because they are seeing different aspects, different levels, different perspectives, of the whole which is far more complex than any single perspective/person might see. So multiple-method will be used to approve the consistency of my data and will help to show how the data is real, true and on target.

In a broad sense, validity means that the data and the methods are right. In terms of research data, the notion of validity hinges around whether or not the data reflect the truth, reflect reality and cover the crucial matters. The idea of validity hinges around the extent to which research data and the methods for obtaining the data are deemed accurate, honest and on target (Denscombe, 1998). In line with recommendations by Lincoln and Guba (1985), transferability or external validity for this study may be maintained through use of multiple data sources and "thick" descriptions. Use of the multiple-method, which is the approach used in this study (see p. 48), is one of the characteristics in qualitative research that provides rich and 'thick' data and increases the confidence in their validity (see also Robson 1993; p. 69, Cohen et al., 2000, pp. 112-113; Smith 1996, p. 193). The technique of multiple- method holds that a robust assertion is established when more than one source of data coincides (Yin, 1994). In addition to this approach, face and content validity are also used in a research study. Face validity is about an instrument, whether it measures what it is designed to measure at the face level. Content validity is defined as the agreement among professionals that a scale logically appears to accurately reflect what it intends to measure (Zikmund, 1991, p. 262). It is a technique involving a step that the judges check the meanings of the items and their acceptability for the purpose of measurement or the study.

6. Sample

Not merely appropriateness of paradigm and research methods but also the suitability of the sampling strategy is important to increase the quality of research (Cohen et al. 2000, p. 92). The sample will be constructed from students and then teachers in this research. Sample size and sampling methods are two important issues in research. Dyer (1995) emphasised that the bigger the sample size is, then generally the better. In that way, the effect of any extreme values in the data gained from a few people would be buried within the remaining moderate data values. Therefore, the number of the participants will be as big as possible, although this is an exploratory research with a naturalistic paradigm for which small groups are usually used.

There are two basic approaches to sampling: probabilistic and non-probabilistic. If the purpose of the research is to draw conclusions or make predictions affecting the population as a whole (as most research usually is), then a probabilistic sampling approach is used. On the other hand, if the research is interested in only seeing how a small group, perhaps even a representative group, is doing for purposes of illustration or explanation, then a non-probabilistic sampling approach is used. The sampling approach in this study will be non-probabilistic. Among the types of the sampling which are convenience, quota, purposeful, dimensional and snowball (Cohen et al. 2000, p. 104), in non-probabilistic approach purposeful sampling is the most suitable one.

Purposeful sampling is also the dominant strategy in qualitative research. Purposeful sampling seeks information-rich cases which can be studied in depth (Patton, 1990, pp. 182-183). Patton identifies and describes 16 types of purposeful sampling (Patton, 1990, pp. 169-183). The most appropriate purposeful sampling strategy to my study is convenience sampling in which available individuals are taken or the cases are taken as they occur (ibid.).

The UK college had good reports overall for mathematics and good A-level results. It is well known in the West Yorkshire locality for this. Ethical considerations made it important to choose a school with a positive ethos and good academic results. For example, I considered that it would be improper of me to go into classrooms where I might expect teaching to be poor. Further to this, there is a sense in which it would have been wrong of me to give students tests where I expected them to do poorly. The opportunity to make an in-depth study of a good college presented itself and I decided to limit my UK data collection to this college. There is then a sense in which my UK data applies only to this college or perhaps two colleges where the ethos and performance are similar. Having chosen this UK college I sought a TR college with a similar positive ethos and good student performance. Whatever the limitations of my sampling technique, I am confident that I have selected institutions which are comparable relative to the conditions within each country.

7. Design of the Research Instruments

A literature review was undertaken to find instruments used in similar studies and then to design the most appropriate instruments for this research. But this survey revealed that trigonometry is an almost forgotten area of study in mathematics education. So very few studies relevant to the topic and research questions could be found. However, all methods and methodologies used in other studies of understanding and problem solving, and their rational connections with research questions of this study are analysed throughout the literature review. Then, research instruments to be used in the study are created. To explore the students' understanding of the trigonometric identities and their use in simplifying trigonometric expressions and answering trigonometry word problems and the possible factors which can have an effect on the students' performance are planned to be explored by written tests, questionnaire, interview, observation, verbal protocol, document analysis. Using many instruments produces different kinds of data on the same topic. The initial and obvious benefit of using the multiple-method is that it involves more data, thus being likely to improve the quality of the research (Denscombe, 1998) and it reduces inappropriate certainty. For the purpose of an overall glance, research questions, the sample and relevant instruments are presented in Table 3.1. In the remaining section I will present the design of instruments with respect to the research question and sample.

Table 3.1. Linking research instruments to sample and research questions.

Research Questions	Sample	Instruments
RQ1i	Students	• Written tests; trigonometry test, algebra test • Interview • Verbal protocol
RQ1ii	Students	• Written tests; trigonometric word problems, trigonometric functions on right-angled triangles test • Interview • Verbal protocol
RQ2i	Teachers	• Teachers questionnaire • Observation • Interview
RQ2ii	Documents	• Curriculum • Textbook questionnaire • Exam papers • Scheme of works • Textbooks

7.1. Exploring students' understanding

My research instruments are selected to answer my research questions as best I can. I used a wide variety of tools (see Table 3.1) to obtain data on students' performance of trigonometric identities, trigonometric formulae and their manner of 'simplifying' trigonometric expressions and answering trigonometry word problems I initially constructed written tests. This enabled a large number of students to be sampled over a wide range of items. I followed up these tests with interviews with a subset of the students in order to understand reasons for students' responses. I also, on a smaller subset of students, used concurrent verbal protocols as students solved similar problems to gain further insight into the thinking behind their performance.

7.1.1. Written tests

The main purpose of this part of the study was to develop the items that were to be used on the written tests and to obtain data on the students' performance, difficulties and the mistakes that they make. For this purpose, four tests, which are a trigonometry test, an algebra tests, a trigonometry word problems test and a trigonometric functions on right-angled triangles test, were constructed. To construct test items, in addition to the literature review, over 40 English and Turkish textbooks which are utilised in schools to teach trigonometry to 14-16 and 16-17 year old students were examined to construct questions. These questions were piloted and discussed with the teachers from both countries about their difficulty level and their relation to the research questions.

Trigonometry and algebra tests

The trigonometry tests contained 16 questions (see Appendix A) which were about the simplification of the trigonometric expressions by using trigonometric and algebraic skills. These questions were chosen at the end of the trial and piloting process with the designated rationales. There were many trigonometric identities in trigonometry which could be used in the simplification of trigonometric expressions. I discarded some of the trigonometric identities. The remainder of the trigonometric identities were used as trigonometry skills students use in the trigonometry test. The rationales for choosing items in the trigonometry test were; trigonometry skills, algebra skills, difficulty level and familiarity which are presented below:

Trigonometry skills; Addition identities.
Double-angle identities.
Pythagorean identities.
Quotient identities.
Reciprocal identities.

Algebraic skills;

1-) Simplify

1-Trigonometric expressions.
- 1-Addition and subtraction.
- 2-Multiplication and division.
- 3-Expanding Brackets.
- 4-Gathering within brackets.

2-Trigonometric fractions.
- 1-Cancelation (Reduction).
- 2-Addition and subtraction.
- 3-Multiplication and division.

2-) Factors

1-Difference of two squares.

2-Common factors.

3-Trinomials.

3-) Power

1-Squares of brackets.

2-Squaring linear functions (Squaring both sides of an equation).

3-Completing the square.

4-Roots.

4-) Algebraic substitution.

5-) Simultaneous equations (sum of two equations).

Difficulty level; In the piloting, the difficulty level of the items was calculated. Hence, the most difficult and the easiest questions were not included in order to encourage students to attempt questions and so to observe the difficulties they meet and mistakes they make. All selected items for the main study were replaced in difficulty order in the tests from the easiest one to the most difficult one in terms of the results gained in the preliminary studies.

Familiarity; The familiarity of all the items in the tests were examined in the piloting so that all items of the tests in the main study are the types of questions with which students would be familiar. The questions were shown to teachers beforehand to make sure that students did not feel alienated when they met the questions and leave them without any attempt at answering them. Being unfamiliar with the items might increase the difficulty level so students' familiarity with the items was essential.

In simplification of trigonometric expressions, algebraic skills are also sought as well as the trigonometric skills. Therefore, to observe how students' performances in the trigonometry test interacted with their knowledge and use of algebraic conventions, an algebra test was constructed. This test also helped to explore students' difficulties and mistakes in algebra, which might have an influence on students' performance in the trigonometry test. The rationales behind the preparation of the algebraic test are, by and large, taken into consideration whilst the trigonometry test was prepared. To discover the interaction of algebra and trigonometry in students' answers, exactly parallel items would be very useful. By parallel items, I mean, replacing the trigonometric functions with a letter to convert a trigonometric expression into an algebraic expression. However, that was difficult because of those trigonometric expressions that would create algebraic expressions which were in unsimplifiable forms, e.g. finding greatest common divisor 1 in a polynomial fraction. Therefore only two trigonometry test items were appropriate to have an exact parallel form of algebraic test items. Other algebraic test items included the algebraic skills which were needed in trigonometry test items. There are 16 items, one of which has two sub-sections, in the algebra test (see Appendix B).

Trigonometry word problem and trigonometric functions on right-angled triangles tests

The trigonometry word problems test was aimed at exploring students' performance in answering trigonometry word problems; particularly the difficulties that they meet and the mistakes that they make. There are six questions in the trigonometry word problems test (see Appendix C). Each question is in the context of a real world application of trigonometry, which can be solved by using basic trigonometric functions. No diagram is given in the trigonometry word problems test so the students could draw their own diagrams if they needed to do so. Not giving a diagram is a common way to ask trigonometry word problems. In this way, it might be possible to observe students' reflection of a 'mental model' on the paper and the strategy that illustrates whether they need to draw a diagram or not. The rationales in deciding the questions in trigonometry word problems test were; the number of right-angled triangles which could possibly be drawn in answering questions, the characteristic of the unknown (the number of unknowns and whether the unknown is an angle or a side length), dimensions of the diagram which could possibly be drawn (2-D or 2-D of 3-D situation) and variety of the questions which will show the richness of the context students can work with in trigonometry word problems.

The trigonometry word problems test also reflected the effect of context and the terminology in answering trigonometric word problems, but it seems that on its own the trigonometry word problems test was not enough to say something about the effect of the context and terminology on students' performance. Subsequently, for this purpose there should be more evidence and strong evidence which could be the context free form of the trigonometry word problems test and students' performance on them. So the trigonometric functions on right-angled triangles test was constructed as a context-free form of the trigonometry word problems test (see Appendix D). There are 5 questions, three of which have sub-questions. Every trigonometry word problem has its parallel context-free form question in the trigonometric functions on right-angled triangles test. The characteristics of the questions in the trigonometric functions on right-angled triangles test which becomes a context-free form of the problems in trigonometry word problems test are diagrams which are possibly needed to solve the corresponding word problems, the slight difference was the numerical values which were different so as not to make students aware of the parallel questions.

7.1.2. Interview

Written tests provide the written manipulations as concrete evidence, which show the difficulties and mistakes students experience throughout answering the questions in tests. But it does not give any reason behind their answers. A more complete understanding of students' answers and their strategies may be gained through interaction with students. The rationale for the interview method employed in this study is to gain a deeper understanding of students' reasons behind their answers and observe their strategies. This might also provide supportive data to answer the first research question.

A semi-structured interview with a hierarchical focus was constructed. So, open-ended questions were used, starting with very general questions and gradually moving to more specific ones until the points the interviewer needed to address were covered. General questions were about their school, education life and future plans and their attitude towards mathematics and trigonometry. Starting with general questions allows the nervous students to calm down and provides the researcher with insight into their personalities, allowing adjustment to accommodate the respondents' idiosyncrasies. After the general questions every student has their own questions because the specific questions are relevant to the student's own answers given in the tests. The specific questions are prepared as soon as the written tests were conducted and analysed. Every student has the questions from any combination of the four written test or all of them. The initial four questions were common in all the interviews. The remaining four questions, which are dependant on the student taking the interview, are drawn from the difficulties or mistakes the student experienced in the written tests. As Tomlinson suggested, a construal interview agenda was prepared to be in my hand during the interview in the aim of managing the interview, changing wording if necessary, and organising questions in a way to get rid of the ones already answered.

7.1.3. Verbal protocols

In addition to written tests and interviews, on a smaller subset of students, concurrent verbal protocols were used as students solved similar problems to gain further insight into the thinking behind their performance. This technique would also help to gain data on how students used their knowledge of trigonometric identities in their simplifications of trigonometric expressions and how visual and symbolic representations interacted in their answering process as well as giving rich and supportive data in conjunction with other instruments. This method is also basically presenting students' mental processes in the performance of tasks because it consists of having students say what is going on in their minds as they go about solving a problem.

Four verbal protocols were created; these protocols were trigonometry (see Appendix M), algebra (see Appendix N), trigonometry word problems (see Appendix O) and trigonometric functions on right-angled triangles (see Appendix P). There were five questions in each of the trigonometry and algebra protocols and four in each of the trigonometry and trigonometric functions on right-angled triangles protocols. There were mainly two types of questions, which were exactly the same questions from the written tests and a modified form of some questions from the written tests, in these protocols. The rationales for choosing the items in protocols were; students' overall performance in the written test that the chosen questions were placed in order of their difficulties from the easiest one to the most difficult one, questions should push students to reveal their thoughts, strategies and cognitive actions. Each question in all the protocols was printed on a blank sheet of paper so as not to distract the student's mind by looking at the other questions. Instruction to the students in verbal protocol is vital, it should be clear and encouraging. For this purpose, by using the specified instructions, an example of the question type in each protocol was prepared, these examples were simple and aimed to push and encourage students to answer and think aloud throughout the protocol.

Moreover, at the end of the verbal protocol, a very short unstructured interview was conducted to highlight some points behind their overall performance. The aim for this interview was to gain more data to explore the student's understanding as deeply as possible.

7.2. Investigating the influence of the teachers on students' performance at trigonometry

Teachers are likely to have an effect on students' performance at trigonometry. This could be in many ways but the focus in this study will be the relevant ones such as teachers' teaching approaches, teachers' view of trigonometry, the resources they use in teaching trigonometry, the effect of the curriculum in their teaching of trigonometry, the emphasis of teaching of simplifying trigonometric expressions by using trigonometric identities and trigonometry word problems. To serve these purposes, questionnaires, interviews and observation methods were designed.

7.2.1. Questionnaire

In the light of the piloting study, the teacher questionnaire was constructed (see Appendix E). Piloting suggested the questionnaire should be in two sections. The first section includes open questions (mathematics questions). The second section includes closed questions with four way attitude statement questions.

Open questions

The first section of the teacher questionnaire was aimed at broadly outlining the end products that a teacher in each country is aiming for with their students. This was done by asking them to show the way in which they simplify trigonometric and algebraic expressions and answer the trigonometry word problems (and trigonometry on right-angled triangles) in class. This section consisted of 10 open questions. These were 3 trigonometry word problems, 3 trigonometric functions on right-angled triangles, 3 trigonometry questions and the question which was asking for the definition of 'simplification' in a trigonometry context. Except the last question, all questions were chosen from the written tests. That also made it possible to compare the students and the teachers answer.

Section 1 asked what happens with trigonometry lessons, students, resources and written information. But teachers' teaching approach cannot be seen or observed by closed statements. And in case of failure to observe how the teachers solve these questions from observation, I used an open question asking teachers to give the type of solution they would suggest to students as one of the better ways to solve each of the questions and also to show all the steps they would expect to see on a student paper. In that way, it was possible to see their manipulations, strategy and approach in answering trigonometry questions.

The piloting highlighted that 'simplification' seems a problematic word in the context of trigonometry. All the UK and the TR teachers found the same answer as a simplified form of the initial expression given in the questions, but despite this most of their students' answers fell short in their tests in the piloting. Simplification also seems to be an important terminology in trigonometry. To clarify the meaning of simplification, therefore, teachers are asked for the meaning of "simplification" in the last question.

Closed Questions

The second section of the questionnaire was designed to attempt to provide indicators of the view teachers have on areas in trigonometry such as:

- Trigonometric identities.
- Trigonometry word problems.
- Students in trigonometry lessons.
- Resources (textbooks, calculators, etc.).
- Syllabi.
- The things they follow to plan a trigonometry lesson.

The first section consisted of 7 main questions. The questions 1, 2 and 3 were information gathering questions, which dealt with some of the teachers' personal information such as their educational background, years of teaching experience and the level they were teaching at the time of the given teacher questionnaire respectively. The main questions 4, 5, 6 and 7 were closed and attitude statement questions having 7, 35, 4 and 6 statements respectively. Some items of this section of the questionnaire were selected from some studies in the literature and some of them were written by the researcher. Most of the items were changed by the researcher to adapt them to the research in terms of themes.

The third International Mathematics and Science Study, which was investigating the mathematics and science achievement in over fifty educational systems around the world, was one of the studies I have taken some items from but they were modified with respect to my research topic. The only modification was replacing the words "mathematics", "mathematics lesson" and other relevant words with the "trigonometry", "trigonometry lesson" and relevant words. Then the items were rephrased if it was needed. Some items, moreover, were drawn from some official and unofficial web sites, one of which was the Local Systematic Change through Teacher Enhancement web page prepared by Mariani (see TS1 and TS2 in bibliography). Mariana stated that this web page is specifically devoted to the research and action to promote learner's autonomy. Another web site was prepared by the Indiana State University, Faculty Computing Resource Centre (see TS3 in references). The items, however, were modified according to the focus of the research if it was needed. Furthermore, a few items were also prepared by the researcher with the inspiration of the extensive literature review and the study.

7.2.2. Interview

I used a semi-structured interview schedule using a hierarchical focusing (Tomlinson 1989). This allowed the teachers to introduce themes that they saw as important while simultaneously ensuring that my interests were addressed in the interviews. My interests were also my rationales and may be expressed in four themes;

- Curriculum resources and assessment.
- The development of trigonometry at GCSE and at A-level.
- The structure of trigonometry lessons.
- Simplification of trigonometric expressions.

So there were four questions in the interview, each question has sub-questions (see Appendix G). I also prepared a construal interview agenda (ibid.) to use in the interview (see Appendix H). That helped me to control the interview, to save time and to keep teachers focused on the questions. The questions were prepared to get rich and deep data about the themes above. The teachers were also supporting and explaining the answer they gave in the questionnaire and other issues that emerged during the interview.

To discover teachers' teaching order of the trigonometry topics, a jigsaw was prepared to administer at the end of the interview (see Appendix I). The jigsaw contained all the topics of trigonometry taught to student at age 14-16 and 16-18. For the clarity of purpose, every topic was illustrated by an example. All topics were randomly replaced on a piece of paper. All topics and the illustrating examples in the jigsaw were prepared as a result of reading 30 textbooks and curricula of the two countries.

7.2.3. Observation

As well as to gain data about whether what teachers say is actually what they do, observation was also conducted to enrich the data on teachers. The role the researcher had in observation was a non-participant observant in the classroom. In that way, the teaching or any activity in the classroom was not interfered with by the researcher so that everything occurred in its natural flow. Hence fresh, immediate and less predictable data could be gathered by observing what was going on in the classroom. The most appropriate approach was a systematic observation with time-interval sampling which was used to observe teachers in the classroom when they were teaching. A time activity table was prepared for this purpose (see Appendix J). The time interval was five minutes. Everything that occurred in the classroom in terms of teachers and teaching was to be manually recorded by the researcher. In this way, neither teachers nor students would be influenced by the existence of the researcher compared with using a video camera. The aim of using this technique was to freely observe teachers in a classroom context when they were teaching trigonometry.

7.2.4. Documents

The full study of curricular documents would include ministerial policy documents, curriculum guidelines, course syllabi, textbooks, syllabi for national examinations, teacher pedagogical plans as they interpret broader requirements, tests (Schmidt et. al. 1997). Documents, however, were limited with curriculum guides, textbooks, scheme of works and trigonometry exams because these documents are central and common to both countries. These documents might help me to discover their effect on trigonometry lessons in schools. Because:

- The curriculum guides are official documents and they most clearly reflect the intentions, visions and aims of curriculum makers
- Textbooks are less official than curriculum guides and supply partial reflections of intentions, but they have an essential place in some countries in terms of intentions and aims.
- Scheme of works are created by teachers in the light of curriculum guides, school programs or a mixture of these combined with their experiences. They show their plan for trigonometry.
- Exams provide me with the extent to which teachers fit into the curricula, textbooks and schemes of work, the questions they asked and their assessment.

Supplementary materials, except central documents, such as annotated teachers' and students' editions of textbooks can give additional data. Nevertheless, since they are less consistently used, I did not include them in this study. For the document analysis no specific instrument was constructed, to observe whether the curriculum, schemes of work and examinations could be the factors on students' performance of trigonometry. A questionnaire was constructed to find out the most used textbooks in the UK and TR by teachers (see Appendix F). Teachers were asked if they used any textbook and if they do, what textbook they used and why they used it. If they did not use one, again the reason was asked. That question was aimed to help find the most used textbook for the document analysis. Schemes of work and examinations papers were all collected to analyse.

8. Data Collection and analysis

8.1. Sample

The purposeful convenience sampling strategy is used in this study. The main focus of this study is to compare the understanding of specific aspects of trigonometry of 16-17 year old students from both the UK and TR. Moreover, the second focus is the possible factors, which can have an effect on students' understanding, such as teachers. Therefore the participants in this study were the teachers and the students of both countries. 16-17 year old UK students were from A-level classes and the TR students were from the second year of high school.

There was a wide range of teachers sampled, including the teachers from the schools where the student sample was taken. Five A-level classes of a school in the wider Leeds area were selected for the study in the summer term of 2000 in England. The 5 classes had 55 students involved in this research. Furthermore, the mathematics teachers of the chosen classes and 5 more teachers, who teach trigonometry to A-level students, were selected. So there are 10 teachers in the UK sample.

Similarly, two tenth grade mathematics classes of a school in the city of Istanbul were selected in the study in the first term of 2000-2001 education season, which is end of the year 2000, in Turkey. The two classes had 65 students involved in this research. Moreover, including the mathematics teachers of chosen classes, who teach trigonometry to 16-17 year old students, there were 60 TR teachers involved in this study.

Based on the TWP and trigonometry questions seven UK and nine TR students were interviewed to follow up on mistakes. Four UK and eight TR students provided (verbal) protocols as well to follow up on their errors and mistakes.

Based on the teacher questionnaire five UK and nine TR teachers from each country would be interviewed and observed in trigonometry classes to follow up on their response to items in the teacher questionnaire.

8.2. Data collection

After designing the appropriate research instruments in terms of the paradigm of the research, purposes of the inquiry and defining the sample, data was collected by administering all research instruments to the designated sample in the UK and TR. The data was first collected in the UK then in TR. In this section, data collection is presented with respect to samples, which are students and teachers respectively. Moreover, at the end of the section, collecting necessary documents for document analysis is presented as well.

8.2.1. Students' data

The data from students was collected by written tests, interviews and verbal protocols. The timetable of administration of these instruments in the two countries is given in Table 3.2. After getting the official permission to do the students' part of the research in the schools, I attended a meeting with the mathematics teacher (and principle of the school in TR) and discussed the timetable, availability of the classes and students for written tests and interviews. As a consequence of these meetings, the timetable in Table 3.2 was revealed. The time allocated for the trigonometry (TT) and algebra tests (AT) was 45 minutes and the time allocated for the trigonometry word problem (TWP) and trigonometric functions on right-angled triangle tests (TORT) was 30 minutes. The time allocated for the interviews (Int.) and verbal protocols (VP) was 30-45 minutes. VP-TT stands for trigonometry verbal protocol and the numbers in the brackets shows the size of the sample.

Table 3.2. Administration of the written tests and interviews with students.

Written test administration to students

Country and class	Day (x*)	Day (x+1)	Day (x+2)	Day (x+3)
UKclass1	TT, TWP		AT, TORT	
UKclass2	TT, TWP		AT, TORT	
UKclass3	TT, TWP		AT, TORT	
UKclass4		TT, TWP		AT, TORT
UKclass5		TT, TWP		AT, TORT
TRclass1	TT, TWP		AT, TORT	
TRclass2	TT, TWP		AT, TORT	

* 'x' merely represents the school days. The data in the UK and TR was collected at different times.

Interviews and verbal protocols with students

Country	Day (x+4)	Day (x+5)	Day (x+6)
UK	Int. (3)	Int (2) VP-TT (2)	Int (2) VP-TT (2)
TR	Int. (3)	Int (3) VP-TT (4)	Int (3) VP-TT (4)

The written tests

The written tests, which are the trigonometry test, algebra test, trigonometry word problems test and trigonometric function on right angle triangle test, were administered to 65 students doing A-level mathematics from one UK college and 85 similar aged students (studying mathematics) in one Turkish school. As explained before, the trigonometry and algebra tests were parallel as were the trigonometry word problems test and trigonometric functions on right-angled triangle tests. So every student who has taken one of the parallel tests should take the other test as well. Furthermore, every student should take all four tests for the spirit of the comparative study. Therefore, during the data collection process, the students from both countries who did not take one of the parallel tests and were absent during the administration of the tests were not taken into consideration. Subsequently only 55 UK and 65 TR students completed all four tests.

In the administration of the written tests, the instructions were given clearly, the aim of the study and the written tests were explained to students. Students were encouraged by the explanations and the fact that these tests would not affect their marks in the mathematics class, so they were asked to do their best in answering all the questions. In addition to the tests, calculators and formulae sheets in the UK and trigonometry tables in the TR were given to the students. In order to collect data from students in their natural teaching and learning environment, the UK students were allowed to use calculators and formula sheets whilst the TR students were only allowed to use trigonometric tables. No problem except 'time' was met during the tests. Some students found the time a bit short for the trigonometry test. None of the students from either country asked any questions during the written tests and, furthermore, it was observed that all students concentrated well on the tests. Administering the written tests enabled a large number of students to be sampled over a wide range of items. I followed up these tests with interviews with a subset of the students in order to understand the reasons for the students' responses.

Interview

I analyzed all (55 UK and 65 TR) papers. As a result of the analysis, 7 UK and 9 TR students were selected to conduct interviews with. Interviews were conducted right after the written tests (see Table 3.2), because the main part of the interview was based on the students' answers in the written tests and it was judged to be important to gain access to what they had done in the test and their thinking as soon as possible. In both the UK and TR, the interviews were all held in schools and they took place during school hours. Students were taken out from their lessons by the permission of the principle and teachers. A quiet room was provided by the principals of the schools. None of the interviews were interrupted by outside factors. Every possible instrument, which could be used by students and interrupt the flow of the interview, was prepared and kept ready by the researcher. These instruments were a pen, pencil, eraser, calculator, formulae sheet, batteries for the tape-recorder, cassettes and tissues. Before the interview started, a few minutes introduction section was held to make students feel more comfortable. I introduced myself and explained the aim of the research again. Students were told that their names would be kept anonymous and that the tapes would only be listened to by the researcher. Students were asked to introduce themselves. After the warming up and preparation section, the interview with the student was conducted by the researcher. During the interview sections, the tape recorder was kept out of students' sight to reduce the distraction of it to a minimum level. Interviews were completed with almost no problem. The only problem was that one of the UK students was not talkative, productive and tried to answer all of the questions with either 'yes', 'no' or 'I do not know' without giving any explanations. After the written tests, in addition to interviews, on a smaller subset of students, concurrent verbal protocols were also used by the researcher as students solved similar or the same problems on the written tests to gain further insight into the thinking behind their performance.

Verbal protocol

Green's (1998, p. 15) series of distinct phases relevant to collecting the concurrent verbal protocol was taken as a guideline. These phases of collecting data procedures were selecting subjects, training subjects, collecting verbal protocols and collecting supplementary data. The students who were chosen for the verbal protocol were different from those interviewed. Students were selected for the protocol work to represent a range of attainments (in my tests and in school work) and for their ability to communicate well, based on their teachers' recommendations. There were basically two tasks for the concurrent verbal protocols. These tasks were based on the pairs of parallel tests, which were trigonometry/algebra tests and trigonometry word problems/trigonometric functions on right-angled triangles tests. A student of protocol took either of these pairs of tasks. Each of the trigonometry/algebra test and trigonometry word problems/trigonometric functions on right-angled triangles test protocol were conducted with two UK and four TR students. Although verbal protocol is a very useful technique to gather data about what is going on inside the student's head during the problem solving, students should know what concurrent verbal protocol is and what to do during the protocol to yield better data. Therefore students were trained by the researcher before the concurrent verbal protocol was conducted. In the training session, the aim of the study and concurrent verbal protocol technique was explained verbatim. By training, students became aware of the protocol technique itself and the reasons for conducting the study. Students were given initial warm-up questions with the instructions (see Appendix L) illustrating the difference between 'talk aloud' and 'think aloud' approaches (see Appendix K), which is very important for the aim of the study, to make students clear about the 'think aloud' approach. The aim of explaining this difference is that students should recognise the difference between concurrent and retrospective reports to generate good reports. Warm-up questions were easy questions that students could confidently answer and alter their thoughts immediately. The training session also made students accustomed to the microphones, tape recorders, atmosphere, style, instructions and researcher. After the training session, concurrent verbal protocol was conducted with the students. A quiet room was arranged by principles of the schools that prevented students from being affected by environmental causes. The researcher sat next to the students to reduce the amount of social interaction taking place and the amount of intrusion and also the tape recorder was placed out of the students' sight to reduce distraction as much as possible. No limitation was given for the time so that students would not feel the pressure of it. Each question was printed on a blank paper and was given to the student in a faced down position so that the students did not see the next question before completing the first one. In the spirit of naturalistic enquiry, students used the resources they used for trigonometry in their school work: calculators and formula sheets for the UK students but only trigonometric tables for the TR students. Whenever the students fell silent, they were prompted to talk with phrases "keep talking" and "think aloud" and words were chosen carefully to be non-intrusive. After the concurrent verbal protocol session was completed, an unstructured interview was conducted with the students for the aim of providing supplementary data.

8.2.2. Teachers' data

My teacher sample were the teachers from the schools in the Leeds area of the UK and the Istanbul area of TR. The questionnaire, interview and observation techniques were used to gather data from teachers' perspective of the research. My teacher sample for the questionnaires was wider than for the interviews and observations. My teacher sample for observations and interviews were a subset of the mathematics teachers in the schools of the students' sample of the research and a wider set of similar teachers.

Questionnaire

Two sorts of questionnaire, which were a teacher questionnaire and a textbook questionnaire, were administered to the UK and the TR teachers. The teacher questionnaire in the UK was administered to a total of 60 mathematics teachers. The questionnaire was either personally given to the teachers to be taken away or posted to them with all envelopes and stamps provided. In TR, teacher questionnaires, were personally given to just over 300 teachers in their schools to be completed and returned at any time within a month. Both the UK and the TR teachers were not given a short time limit to provide them with a more comfortable time and less pressure to complete the questionnaire. The questionnaire was 5 pages long. The front page presented the aim of the study and instructions clearly. The second and third pages contained problems which the teachers were asked to solve in the way that they would encourage their students to solve them. The fourth and fifth pages presented questions about attitudes towards and views about trigonometry, and asked for responses on a four point Likert scale. But, unfortunately, at the end of a long wait the number of the questionnaires which were completed and returned was only 10 in the UK and 60 in TR. Particularly in TR, although the researcher visited all the teachers weekly for 3 months he did not get many questionnaires back. During the administering of the teacher questionnaire, it revealed that some teachers did not want to show their answering style of the questions in section one, although they were optimistic and completed the Likert scale sections of the questionnaire. Some teachers also wrote their thoughts and comments abut the research behind the questionnaire, all of them were supportive and highlighting the necessity and importance of this research. The other questionnaire which was conducted with teachers was a textbook questionnaire and it was conducted in a slightly different style. The textbook questionnaire was conducted in two parts, which were piloting and main study respectively. In piloting, the textbook questionnaire was given to 35 mathematics teachers in the UK and 20 mathematics teachers in TR. All piloting questionnaires were completed by teachers and gathered successfully. In the main study, it was given to the same sample who took the teacher questionnaire. The same sample who completed the teacher questionnaire in both countries completed the textbook questionnaire. Subsequently there were 10 UK and 60 TR textbook questionnaires more. Consequently, a total of 45 UK and 80 TR textbook questionnaires were gathered to find the most used mathematics textbooks in both countries. After the questionnaire, the interview with a subset of the teachers, who took the questionnaire, was also conducted to get 'thick' and 'rich' data and get more insights behind teachers' answers in the questionnaire.

Interview

I conducted an interview with every teacher who was teaching in the schools of the students' sample involved in this research in the UK and TR and a subset of teachers from the wider set in TR. The TR teachers who were not teaching in the schools from where students' data was collected were chosen in terms of availability and being willing to be interviewed, because a main difficulty met in TR was that some teachers did not want to be recorded. They made strong objections to the researcher. Therefore it was difficult to find teachers to interview that were available and volunteer teachers were taken as the interview sample. Namely, there were 5 UK and 9 TR teachers with whom an interview was conducted. In both the UK and TR, the interviews were all held in schools and they took place during school hours as the teacher wanted. A quiet room was provided by principals of the schools. None of the interviews were interrupted by outside factors. A semi-structured interview schedule using hierarchical focusing was used. All questions were printed on one side of the paper and teachers were allowed to read them before the interview commenced. The aims of the study and the interview were also explained to the teachers by the researcher. The teachers had a copy of the questions to prevent any misunderstanding of the questions during the interview. Furthermore, the researcher used both interview questions and a construal interview agenda which helped the researcher to manage time, control the interview and keep to the focus of the interview. At the end of the interviews, teachers were given a jigsaw including all topics of trigonometry for the age group 14-16 and 16-18 with an illustrating example of each topic. Then teachers were asked to reorder them in terms of teaching order. Interview sessions took 30-45 minutes. A tape recorder was used during the sessions and teachers were very comfortable with the tape-recorder. So as to gain data about whether what teachers say is actually what they do, observation was also conducted to enrich the data on teachers.

Observation

The same teachers' sample for interviews, 5 UK and 9 TR, were observed in the main study. Teachers arranged their 45 minutes mathematics lesson for the researcher. The teachers were again told the aim of the study and asked to teach in the way they always taught. One disadvantage of the observation was that some lessons were not about the simplification of trigonometric expressions or the answering of trigonometry word problems. That was the limitation of the study that the researcher did not have any chance to choose because of the time restrictions and the topic itself. Simplification of trigonometric expressions did not have any separate lesson but was used in every topic after it was taught, so it was difficult to find a lesson specifically on the simplification of trigonometric expressions. But observing teachers in a mathematics lesson was important to get an idea about their teaching approach, lesson structure, and resources they used in the lesson. That observation would not be radically different from trigonometry lessons. In the observed lessons, the role of the researcher was being a non-participant observant in the classroom. He sat at the very back of the classroom to be out of students' sight and prevent them from being distracted by his presence which could affect the teachers' performance. By the same reason, video recorders or tape-recorders were not used. The teaching approaches and everything teachers did in the classroom were recorded manually by the researcher every five minutes. In the light of the experience the researcher got in the piloting he was very familiar with this technique. This way of recording the observation gave an immediate and fresh account available and a full picture of teachers at the time of observation in the lesson to the researcher.

8.3. Documents Data

There was no human sample but documents in both the UK and TR was gathered in collecting the data from documents. These documents were Curricula/syllabi, textbooks, schemes of work and exam papers. Curricula and syllabi, which were applicable to the 14-16 and 16-18 age group of both the UK and the TR students, were collected from the official resources. As explained in the questionnaires which were conducted with the teachers, there was a textbook questionnaire to find the most used mathematics textbook at the age group of 14-16 and 16-18. By the help of the textbook questionnaire, the most used the TR and the UK mathematics textbooks in these age ranges were collected to be analysed. Schemes of works were collected from the teachers with whom interview and observation were conducted. Moreover, some exam papers of the UK and the TR students sample of this study were collected from their teachers who took part in interviews and observations.

8.4. Data analysis

After data collection, the next phase of research commenced by the determination of the meaning of the data through analysis. However analysing the data, particularly the qualitative data compared with the quantitative data, was not easy work to do (also see Robson 1993 and 2002). Data analysis technique was decided depending on the research design, research instruments and the method of data collection of this research. In this study, the research instruments used in data collection yielded mostly qualitative data which were in forms of written accounts or spoken words (see Table 3.3.). So I had to deal with qualitative data analysis to make sense of this data in terms of the written accounts of teachers, students and documents about the situation, noting patterns, themes, categories and regularities (Cohen at al. 2000, p. 147).

Table 3.3. Data and corresponding sample and instruments.

	Sample and corresponded research instruments		
Data form	**Students**	**Teachers**	**Documents**
Written accounts	- Written tests - Verbal protocol	- Teacher questionnaire - Textbook questionnaire - Observation	- Curriculum/syllabi - Textbooks - Schemes of works - Exam papers
(Tape-recorded and transcribed account) spoken words	- Interview - Verbal protocol	- Interview	

Coding is one of the ways to analyse qualitative data. That technique seems to convert qualitative data into quantitative data. That reveals an issue of analysis of the qualitative and quantitative data, which should be clarified at this stage, to discuss; could this approach affect the 'thick' and 'rich' data gathered in the qualitative research? The answer might be given by arguments done by Silverman (1993), Miles and Huberman (1998) and Behrens and Smith (1996); using numbers or words in analysing the data are not the main matter. Numbers or words are utilised to ascribe properties to the data; furthermore, they are also symbols that have underlying referents. In both cases the referents are connected to the symbol by an entire series of inferences and arguments. The main issue is the actual occurring and existing properties we are concerned with. Although qualitative data can be categorised and so be converted into numbers for the purpose of using descriptive statistics, it does not mean that qualitative data loses its feature of being 'thick' and 'rich' data, it still stays as qualitative data. Qualitative researchers like Silverman (2000) and Miles and Huberman (1998) assure that the presentation of numbers does not disqualify a study from being qualitative in nature. Silverman (2000, p. 185) argues that simple counting techniques by using the created categories can offer a way to analyse the whole amount of data ordinarily lost in intensive qualitative research. In this way, the reader has an opportunity to gain a sense of the flavour of the data as a whole instead of taking the researcher's word for it. Therefore, quantification can supplement, extend and enhance qualitative analysis (Ely et al, 1997 p. 194). This discussion between the analyses of qualitative and quantitative data is briefly summarised by Ely et al. (1991) who argue that the important thing is qualitative/quantitative researchers' decision-making process in establishing findings that should be elucidated by researchers themselves. To deal with the qualitative data in a reasonable way, to overcome with the overwhelming amount of the qualitative data, to organise data and make analysis as practical as possible, all qualitative data were categorised and coded.

Coding

My data were mostly qualitative in the form of written accounts, tapes of interviews, tapes and written work of verbal protocols, notes made during observation and documents. The only quantitative data was the second section of the teachers' questionnaire. The real issue was analysing the qualitative data, which were categorised in terms of themes relevant to research questions and then these categorisations were coded. Robson (1993, p. 385) defines a code as symbols to classify or categorise a group of words and moreover he highlights them as retrieval and organising devices to find and then bring all occurrences of a particular kind together. Coding qualitative data was helpful to comment on the overall picture in terms of the categories created in the light of the research questions and also it also gave a tidy and structured view of massive data. Coffey and Atkinson (1996) stated that coding is a procedure which tries to link all related fragments under a key idea or concept. Coding techniques require reading and rereading of the collected data to become familiar with them and to get a clearer idea of an appropriate category (Hammersley and Atkinson 1983, pp. 177-178). Since there was a big gap between the data collection in the UK and TR in terms of the time, I had already started to analyse the UK data. Hence, there was ongoing analysis throughout my data collection, which seems to be a suggested and typical approach to qualitative data analysis (see Robson 1993, p. 384, Cohen and Manion 1994, p. 147). In analysing data, Driver and Erickson (1983) nomothetic and ideographic approaches and Miles and Huberman's (1984) first and second level coding notion were utilised to make categorisations. These two different approaches will be explained in the remainder of this section.

8.4.1. Students' data

The multiple-method approach was used in collecting data from students. This approach, particularly in the naturalistic studies and qualitative research, is positive for improving the reliability and the validity of the data. The gathered data were all in qualitative form. The approaches to the analysis of the written tests, interview, and verbal protocol are presented in this sub-section.

Written tests

In the written tests, students answered the questions by showing their manipulations on the paper. Classifying this data into certain categories allows comparison to be made between the UK and the TR students. I was inspired by Driver and Erickson (1983) to analyse data obtained from written tests. Driver and Erickson defined the characteristic of data analysis as nomothetic or ideographic. In a nomothetic approach, students' answers are analysed against a group of predetermined accepted categories that might emerge from a view of what constitutes the incorrect answer to a question. Examples might include analysis against the three distinct categories such as mathematically 'correct', 'incorrect' and 'partial'. In the ideographic approach, however, the students' answers are analysed in their own terms rather than categorising them into predetermined groups of categories as is the case of the nomothetic approach.

The categories were developed throughout the data analysis. In this study, students' manipulations in written tests were analysed by both nomothetic and ideographic approaches since one of the aims of the research question is to find out the difficulties students met and the errors students made in answering questions. The coding procedure, which is, in other words, categorising students' written answers to questions, started by categorising the students' answers into four mutually exclusive categories as 'correct answer', 'incorrect answer', 'partial answer' and 'non-attempted questions' so as mainly to find out both the UK and the TR students' performance in simplifying trigonometric expressions and answering trigonometry word problems and also the connection between these performances and students' performances in simplifying algebraic expressions and using trigonometric functions on right-angled triangles respectively. After this first iteration, all incorrect and partial answers were classified to discover the students difficulties and mistakes. Since there was no response or repetition of the questions and students gave expected and valid responses, the coding of correct answers and non-attempted questions, respectively, was not done. The nomothetic approach was followed by an ideographic approach where students' answers were examined in their own manipulations in each written test. These categorisations were made in two steps: first in terms of the mathematics knowledge and skill they needed but failed to show they had; secondly, what types of mathematical knowledge and skill caused students to reach the correct answer. First step categorisations were; in trigonometry tests 'basic manipulation, algebraic prerequisites and recognition of trigonometric identities'; in algebra tests 'basic manipulation and algebraic prerequisites'; in trigonometry word problems tests 'reading, misuse of terminology, draw, match & labelling, identify function, develop mathematics and symbolic manipulation'; in trigonometric functions on right-angled triangles test 'reading, terminology, identifying function, developing math and doing symbolic manipulation'; in trigonometric functions on right-angled triangles tests 'reading, misuse of terminology, identify functions, develop mathematics and symbolic manipulations'. Second step categorisations are presented in more detail with the first step categorisations in the next chapter. After the categorisation, descriptive statistics, which were particularly percentages and means, were used to illustrate the findings.

Reliability and validity of written tests: For the reliability of the coding procedure two steps were followed by the researcher. First, the researcher recoded the data twice in two months with the constructed categories. Secondly, inter-rater reliability (Cohen et al.2000, p. 119) was determined for independent codings on the students' answers to all the questions by six judges. These judges were four research students, who were doing research in mathematics education and were mathematics teachers who taught trigonometry as well, and my two supervisors. The judges, except my supervisors, were informed of the aims of the study in general and aims of the analysis. Then all judges were given some copies of the incorrect and partial answers of the students written tests, category sets for these answers and detailed notes for interpretation of the category sets. After the judges had finished coding independently, a comparison was made between the codings of the judges and the researcher to address any inconsistencies in codes and coding procedure. So that consistency of coding decisions were increased by modifying the problematic codings.

Validity of an instrument is also an important feature in a research. Among the various forms, the face validity and content validity were used to examine the validity of the written test. Face validity was established by judgements made by the researcher throughout the reviewing literature, by asking the students and mathematics teachers about the appearance of the tests, whether they tested what they were designed to test, throughout the piloting and the main study. Content validity is about coverage and representativeness rather than patterns of answers, in other words it is a matter of judgement rather than measurement (Kerlinger, 1986 quoted in Cohen et al. 2000, p. 131). Cohen et al. (ibid.) stated that content validity is achieved by professional judgements on the content of the instruments whether it represents the content it is supposed to. After the creation of the written tests, as a result of extensive literature review and piloting of every item in them, they were discussed with the two supervisors of the researcher. Furthermore, four research students and ten mathematics teachers were also used to render intelligent judgement as to whether the content of the written tests represented the content they expected to see. Subsequently the content validity of the written tests was established by the judgements of the professionals.

Interviews

After the interviews with the UK and the TR students, each one was fully transcribed for analysis by the researcher. The TR students' interviews were first transcribed in Turkish then translated into English. The transcripts were not analysed independently of the written answers. After analysing the written responses, supportive and challenging data from the interviews were extracted to answer research questions and used to expand findings from the written responses. Quotations from the interviews were used to show and clarify the written responses which were not possible to be interpreted simply by analysing the written responses. In analysing the interviews, all transcriptions were read several times by the researcher to capture the themes involved in the transcriptions pertinent to the research questions. In this way, it was also aimed to categorise interview data for the purpose of organising to use them as quotations and/or supportive data for the written tests. The categorisations made in the interviews of corresponding written tests were; algebraic prerequisites, trigonometric identities, formulae sheets and memorising the identities, and simplification of trigonometric expressions in the trigonometry tests interview; basic manipulations, algebraic prerequisites and simplification of algebraic expressions in the algebra test interview; reading, terminology, drawing, matching and labelling, identifying function, mnemonics, developing mathematics, symbolic manipulations and procedure-how to solve TWP in trigonometry word problems tests interview. There was no need to make categorisations in the trigonometric functions on right-angled triangles test, because the students' performances were very high in this test.

By using the suggestions of Silverman (1993), to improve and increase the reliability of the interview, the interview schedule was carefully piloted several times with ten students, interviewees were trained just before the interview sessions commenced. For the reliability of the categories, the researcher himself recoded the interview data three times in designated time intervals. Furthermore, inter-rater reliability was used to enhance the reliability of categorisations that the five research students and the two supervisors of the researcher coded the two students transcriptions after the instructions and the aim of the study was explained by the researcher. The notion of valid interview data is problematic in that the facts from the world of the students' social and school experience and the existence of multiple influences on their experiences might affect their answers in the interviews. But to try to overcome this problem, the triangulation method was used as well as trying to minimize 'bias' because of the researcher, respondent and questions, by discussing with two supervisors. Typically, interview data are considered valid when triangulation confirms that what the different parties say about an event coincide (Partington, 1998). For the validity of the instruments, face and content validity approaches were used as in the written tests.

Verbal protocols

In verbal protocols there were two sorts of data, which were written manipulations and spoken/recorded information gathered by the think aloud method. The main data to analyse was the spoken data/recorded data, the written data was used to illustrate and support this data. Therefore, after the tape-recording of students' simplification of trigonometric expressions, algebraic expressions, answering trigonometry word problems and trigonometric functions on right-angled triangles questions, the concurrent verbal protocols were transcribed and then fully analysed. First of all, students answers were categorised by the nomothetic approach as it was done in the written tests analysis as correct, incorrect, partial answers. Then percentages were used to show students' performances in the protocols. After this initial analysis process, every transcript of the students' correct, incorrect and partial answers were segmented and encoded (Green 1998, Green and Gilhooly 1996, p.62). Coding reliability was checked by six research students and two supervisors of the researcher on two different students transcripts. The level of inter-coder agreement on the category validation was found to be very high and that gave a confidence about the categories' feasibility and consistency. Furthermore, intra-coder reliability approach was also used by the researcher to confirm consistency within his coding. In the coding of the protocols, it was aimed to capture the information needed as the verbal protocol was produced. it was necessary to keep all spoken information to code individual segments of protocols. After the think aloud transcripts were coded into segments, schematics were drawn to show the flow of the protocols in terms of the coding made throughout the answering process in the protocol. The interviews which were conducted at the end of the protocols were transcribed as well and merely used as a supplementary data and for quotation.

8.4.2. Teachers' data

The multiple-method approach was used in collecting data from teachers as well as students. This approach, particularly in the naturalistic studies and qualitative research, is positive for improving the reliability and the validity of the data. The approaches to the analysis of the teacher questionnaires, interviews, observations and textbook questionnaires is presented in this sub-section.

Questionnaire

As mentioned before, there were two main sections in the teacher questionnaire. These sections yielded both qualitative and quantitative data respectively. In the first section of the questionnaire since teachers were asked to show the steps of the simplifying trigonometric and algebraic expressions and also answering trigonometry word problems, the data were teachers written accounts or manipulations. The data gathered from these written accounts were categorised in two ways. First, answers were categorised as students' data of written test were done by using nomothetic approach to observe whether teachers' answers were correct, incorrect or partial for the characteristic of their answering style of these questions and also to make a comparison between students performance and written data. Percentages were used to describe it. Secondly, every answer given by teachers was categorised in terms of the themes relevant to students' data. Descriptive statistics, i.e. percentages, frequencies, were used to illustrate the data. Furthermore, in the last questions of the first section of the teacher questionnaire, teachers gave their written accounts for the 'simplification' in trigonometry context. First, the percentages of the teachers in terms of the attempt was presented, and then all given definitions were compared in terms of similar and different words by using the percentages. Since the questions were from the written tests of the students, their validity was already examined and reliability of the codings was examined by using the inter-rater approach again. By explaining the aim of the study and questionnaire and also the descriptions of the coding sets, five teachers' written accounts were given to five research students from the education department and they coded the teachers data. The reliability of the coding was found to be high.

In the second-section of the questionnaire, a 4-way Likert scale was used. There were seven main questions. Every question was coded in a different way and only the sixth and seventh questions were coded in the same way. Except the initial three questions, all questions' responses were coded with respect to the Likert scale, for example, the responses given to the items in the fourth question were coded as 'not important', 'less important', 'somewhat important' and 'very important' (see the section 2.1.1.2. in the result chapter p. 132). Although this section seemed to be quantitative data, it was treated as qualitative data. Because of the sample size, inferential statistics were not used, so data gathered from the second section of the teacher questionnaire was expressed by using basic descriptive statistics. Most of the items in the second section of the questionnaire were inspired or taken from the studies in which their reliability and validity were already examined and improved. However, since they were used in the aim of this study they were re-examined. As Oppenheim (1992) said, to avoid time and condition problems the internal consistency method was used. The scale reliability was established by using Cronbach's alpha coefficient, which is one of the most widely used reliability measures (Bryman and Cramer, 1997). Cronbach's alpha coefficient was computed for reliability and it ranged from 0.55 to 0.75. Although this range is not very low and acceptable it could be explained by the exploratory nature of the purpose of the study. To examine the validity of the instruments, first face validity was examined by judgements made by the researcher throughout the reviewing of the literature, by asking the mathematics teachers about the appearance of the items in the questionnaire, whether they test what they were designed to test throughout the piloting and the main study. But face validity is not really good enough (Oppenheim, 1992). Churchill (1987) notes that the validity of an instrument in research can be assessed by looking for evidence of its content and construct validity. Content validity refers to the agreement among professionals that a scale logically appears to accurately reflect what it intends to measure (Zikmund, 1991), although its determination is subjective and judgmental (Emory, 1980). Oppenheim (1992) points out that the researcher concentrates on content validity, because it is more difficult to find a suitable external criterion, construct system or some other method of validation. Therefore content validity was used in the second section of the teacher questionnaire. Furthermore, initially an extensive literature review to create questionnaire items was done by the researcher and then several drafts were discussed by two supervisors, some research students and teachers. This procedure also improved the validity of the instruments (Munby, 1997)

Interviews

All interviews with the UK and the TR teachers were recorded by the researcher to prevent missing some information and also to avoid the interviewer being biased. After the interviews with the UK and the TR teachers were conducted, they were all fully transcribed for analysis by the researcher. The TR teachers' interviews were first transcribed in Turkish and then translated into English. When all transcriptions were completed the researcher listened to the interviews and read the transcriptions several times to get a sense of the whole (Hycner, 1985). This helped the researcher to identify general and unique themes pertinent to this research. Then, the interview data were categorised by using first and second level of coding approaches as given by Miles and Huberman (1984). They make a distinction between these two approaches; first-level coding is concerned with attaching labels to groups of words. Second level, which is also called pattern coding, groups the initial codes into a smaller number of themes or patterns. In first level coding, the main themes the researcher was interested in were identified and categorised as curriculum resources and assessment, the development of trigonometry at GCSE and at A-level, the structure of trigonometry lessons and simplification of trigonometric expressions. Then as a second level of coding, under each heading, sub-themes, which arose from the teachers responses, i.e. were 'grounded in', were introduced. Then selected quotes from the interviews were utilised to illustrate arguments/opinion of the teachers on these themes and sub-themes. For the reliability and the validity of the data (coding) and instruments respectively, the same approaches as in the students' interview were used. The jigsaw which is given at the end of the interviews was analysed by a counting system and so a basic descriptive statistic was used to explain the data, which is the teaching order of the trigonometry topics.

Observation

Observation was used a complementary instrument data for teacher questionnaire and interview in collecting data from teachers. All observations in the classroom were recorded by the researcher manually in five minute intervals, so that there were at least two A4 pages of written accounts. In recording eight out of nine dimensions from descriptive observation were also used (Spradley 1980). These dimensions were space, actors (only teachers), activities (teaching and pertinent activities), objects, acts, events, time and goals. Instead of creating categories and codings as other qualitative data was gathered, the UK and TR were descriptively compared in terms of teachers' teaching approaches in lessons, resources, and lesson structure, which were relevant to the second research question. To do that, a several times reading procedure was executed so that the main themes were highlighted and taken as notes. As well as teachers in teaching, the school, classroom, resources (tools), more generally the physical environment relevant to the teaching of trigonometry was also observed and descriptively compared.

Textbook questionnaire

Although the aim of this questionnaire was to discover the most used mathematics textbooks in both the UK and TR for document analysis, it was administered to both the UK and the TR teachers. Therefore the analysis of the textbook questionnaire is presented here. It was a very short questionnaire which asked about the author, textbook title, appearance of the textbook, percent, rank, publisher, publication date, resources and reasons to use textbooks. At the end, since the aim was to find the most used textbook and then to have a general view, a basic descriptive statistic was used to interpret data gathered from the questionnaire. The frequency of the textbook selection was described in terms of percentages. After the frequencies were tallied, textbook selections were also ranked in an order. The mean number of books cited by each respondent was calculated. The chosen books from the UK and TR were analysed under some criteria with respect to trigonometry which will be discussed in the document analysis section. For the validity and reliability of the textbook questionnaire, data was examined in the same way as for the written tests.

8.4.3. Document analysis

Documents for curriculum analysis were limited to the curriculum guides, textbooks, exams and scheme of work, which were all official documents. These documents were central and common to both countries and they might help the researcher to discover their effect on trigonometry lessons in schools and so in students' performance. For document analysis existing documents such as, the curriculum/syllabus, textbooks, exam papers and teachers' scheme of works were collected from both countries to be analysed for their characteristics. The curriculum was the main official existing document in the two countries as it is in most of the countries. So the relation between curriculum and other documents was also analysed.

Curriculum

Both the UK and the TR curriculum/syllabus were descriptively compared by the similarities and the differences between them in terms of content, topic and curriculum objectives.

Textbooks

As a result of the textbook questionnaire completed by teachers, representative textbooks of both the UK and TR for the 14-16 and 16-18 year old age groups were chosen to be analysed. Textbooks analysis was made on criteria such as, the order of the topics of trigonometry, physical appearance of the book, presentation of trigonometry in the book, pictures, formulae, worked examples and questions. Basic descriptive statistics were used to interpret data gathered from the textbooks. They were also compared with the curriculum in terms of the trigonometry topics.

Schemes of works

In both the UK and TR, from the teachers involved in the sample (including the ones from the school the students data was gathered from) schemes of works were collected to analyse. The contents of the schemes of works were compared to the curriculum descriptively. Furthermore, the physical appearance of the schemes of work was also compared descriptively.

Exam papers

In both the UK and TR, the high-stakes examinations, which are the university entrance examination papers in the TR and pure mathematics papers containing trigonometry questions in AS/A examination papers in the UK, for the years 1999, 2000 and 2001, were collected to analyse the trigonometry questions. All trigonometry questions were first categorised in terms of the topic, question types. Then basic descriptive statistics were used to interpret data.

CHAPTER 4: RESULTS

In this chapter, raw data is presented alongside its analysis. Space constraints prevent the presentation of all data collected and analysed. Two criteria have been selected for which data to include/omit: (i) all results referred to in the discussion chapter are included; (ii) sufficient results for the reader to gain an overall picture of the areas investigated are included. The results are presented with two sections in respect to the research questions (see pp. 1-2): The UK and the TR students' performance in trigonometry tasks and possible factors influencing students' performance. The first research question concerns the UK and the TR students' performance in trigonometry. All relevant data yielded by students is presented in the first section below. The second research question is concerned with possible factors influencing students' performance and the relevant data is presented in the second section of this chapter.

1. The UK and the TR students' performance in trigonometry tasks

In this section, the data collected from both the UK and the TR students by written tests, interviews and verbal protocols are analysed and the results are presented in six sections with respect to the focus of the research questions:

- Students' understanding of trigonometric identities and their manner of simplifying trigonometric expressions.
- Students' algebra knowledge and use of algebraic conventions.
- Comparison of the UK and the TR students' performance in trigonometry and algebra tests in general.
- The use of trigonometry in real world contexts.
- The use of trigonometry in the context free questions.
- Comparison of the UK and the TR students' performance in trigonometry word problems, and trigonometric functions on right-angled triangles tests.

Sections 1, 2, 4 and 5 follow a written test-interview-verbal protocol format. Section 3 and 6 compare the UK and TR results and follow a test result-flaws-parallel question format.

1.1. Students' understanding of trigonometric identities and their manner of simplifying trigonometric expressions

In this section, results of trigonometry tests, interviews and verbal protocols on simplification of trigonometric expressions are analysed.

1.1.1. Trigonometry test

The trigonometry test (see Appendix A) consisted of 16 questions, 14 of which asked students to simplify given trigonometric expressions. Of the other two, one required the use of a given substitution in an expression and the other required the use of the sine addition formula. Seven questions were in the form of fractions, four questions required the manipulation of fractions and the rest were non-fractional expressions. In the majority of the questions, 10, there were implicit or explicit trigonometric identities, which might be used in simplifying the initial expression. In the rest of the questions there were no trigonometric identities which could be seen implicitly or explicitly. The trigonometry test aimed:

- At revealing students' errors, misconceptions and difficulties in items.
- To see whether students know trigonometric identities, double-angle and the addition formulae, how they use them to simplify trigonometric expressions.
- To investigate students' use of algebraic notation and conventions.
- To investigate students' solution strategies and algebraic/trigonometric competencies.
- To determine how students use trigonometric identities in simplifying expressions.

The trigonometry test was conducted with 55 UK and 65 TR students.

1.1.1.1. Categorisations of students' answers

As explained in the Methodology chapter (see p. 65), students' answers were categorised at two stages: initial categorisation and further categorisation. The analysis started through reading each student's answers and noting the kind of response identified. I initially divided the answers into four groups: correct answers (CA), incorrect answers (IA), partial answers (PA) and non-attempted questions (NAQ) (see Table 4.1.).

Table 4.1. Initial categorisation of student responses to the items in the trigonometry test.

Groups of Answers	Abbr.	Description
Correct answers	**CA**	Appropriate trigonometric identities are applied to yield a valid solution; manipulations resulted in students writing the expected response.
Incorrect answers	**IA**	Inappropriate use of algebraic and/or trigonometric transformation rules.
Partial answer	**PA**	The student approached the question in a correct manner but stopped short of the expected simplification.
Non-attempted questions	**NAQ**	No response or simply a repetition of the question.

After this first iteration, all IAs and PAs were further classified. Since some students did not provide any response (NAQ) or provided the correct response (CA), further coding of these two categories was not done. The further classification of IAs and PAs began with reading through each student's answers and analysing the flaws/reasons behind the results. The classification of the IAs and PAs were shaped by the algebraic and trigonometric methods used in simplifying trigonometric expressions.

In the exploration of the IAs and PAs, a 'working model' of student flaws in trigonometry tasks was constructed: an analysis of the mathematics involved in the questions and an analysis and categorisation of the types of flaws in incorrect and partial answers that students made in the trigonometry test were also included. The model has three basic quasi-hierarchical levels. The presentation of these, below, is from the most basic to the most complex:

- Basic manipulations includes errors that may occur in number, algebra and trigonometry work, e.g. incorrect manipulation of negative numbers, incorrect manipulation of expressions involving square roots.
- Algebraic prerequisites concerns purely algebraic flaws, i.e. errors that do not involve trigonometry, e.g. incorrect cancellation of algebraic fractions.
- Recognition of trigonometric forms concerns errors due purely to trigonometric expressions, e.g. incorrect manipulation of expressions involving double angle formulae.

All flaws committed by the UK and the TR students were picked up and then coded and replaced under the relevant title (see Table 4.2.).

Table 4.2. Further categorisation of students' incorrect and partial answers in the trigonometry test.

CODE		CATEGORY
Basic manipulations	**MR**	Misreading the question
	MI	Misinterpreting the question
	DA	Direct answer (without operation)
	EAS	Simplify trigonometry expressions, addition and subtraction
	EMD	Simplify trigonometry expressions, multiplication and division
	EEB	Simplify trigonometry expressions, expanding Brackets
	ETB	Simplify trigonometry expressions, gathering within brackets
	FAS	Simplify trigonometry fractions, addition and subtraction
	FMD	Simplify trigonometry fractions, multiplication and division
Algebraic prerequisites	**FC**	Simplify trigonometry fractions, cancellation (reduction)
	SE	Simultaneous equations (sum of two equations)
	FDS	Factors, difference of two squares
	CF	Factors, common factors
	SB	Power, squares of brackets
	SLF	Power, squaring linear functions (taking square of both sides of an equation)
	CS	Power, completing square
	R	Power, roots
Recognition of trigonometric	**AI**	The addition identities
	DAI	Double-angle identities
	PI	Pythagorean identities
	QI	Quotient identities
	RI	Reciprocal identities

1.1.1.2. The UK and the TR students' performance in the trigonometry test

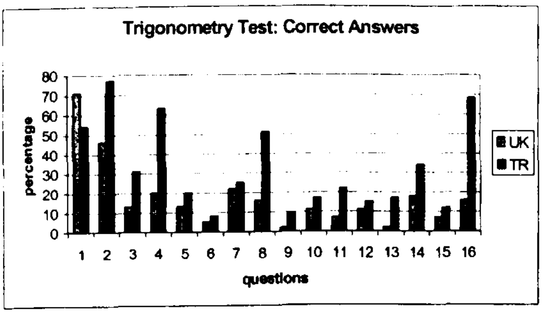
In this section, first the results of initial categorisations are presented and then the flaws committed by both the UK and the TR students in incorrect and partial answers are exemplified.

Initial categorisation of the answers in the trigonometry tests

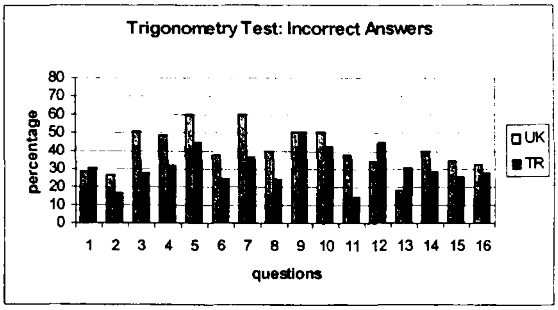
Every answer given by students was identified as correct (CA), incorrect (IA), partial (PA) or non-attempted (NAQ). The results are presented in Figure 4.1. A-B below.

Figure 4.1. The percentages of the UK and the TR students' initial categorisations in the trigonometry test.

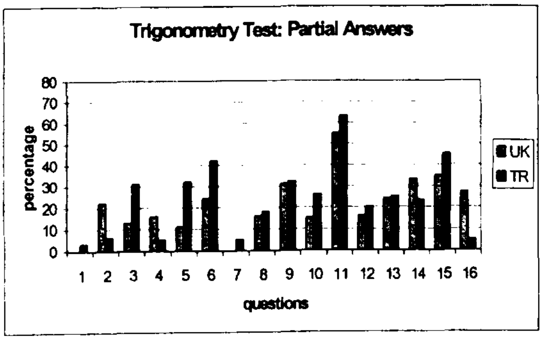
A



B



C



D

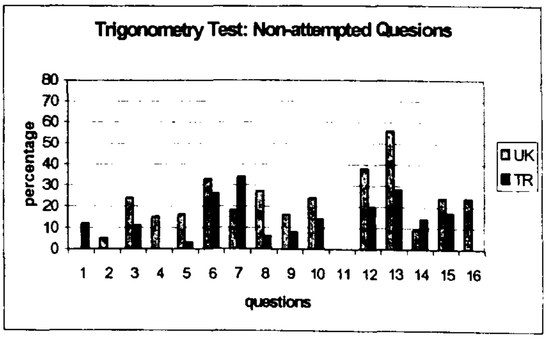


Figure 4.1-A indicates that the percentage of correct answers given by the TR students is, to some extent, greater than the UK students in every question, except question 1. The percentage of the UK students who answered question 1 correctly was far greater than the TR students, 71% and 54% respectively. The percentage of correct answers given by the TR students is far greater than the UK students in questions 2, 3, 4, 8, 14 and 16.

The results in Figure 4.1-B reveal that in a majority of the questions, the UK students give more incorrect answers than the TR students do. However, the percentage of incorrect answers given by the TR students is greater in the questions 1, 12 and 13. The data shows that incorrect answers given by the UK students is far greater than the TR students in questions 3, 4, 5, 7 and 11.

Figure 4.1-C shows that the TR students gave more partial answers than the UK students did. More than half of the UK and the TR students gave partial answers to question 11, 63% and 55% respectively and the percentage of partial answers given by the TR students to the questions 3, 5 and 6 is far greater than the UK students . Moreover, almost 5% of the TR students gave partial answers to questions 1 and 7, while no partial answers to these questions were given by the UK students.

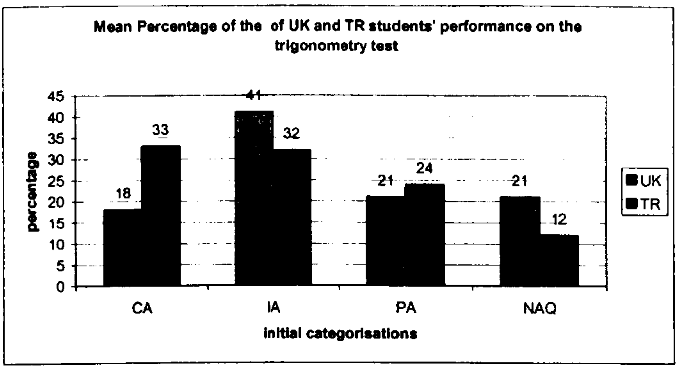
Figure 4.1-D shows that a greater percentage of the UK students did not attempt questions. Questions 1, 7 and 14 were the only questions more TR students than the UK students did not attempt to answer. Interestingly, all UK and TR students attempted question 11. However, there are some questions which were not attempted by either the UK or the TR students. All UK

students attempted question 1, whilst no TR students did. On the other hand, all TR students attempted questions 2, 4 and 16, whilst no UK student did. In questions 3, 5, 8, 12, and 13 a much higher percentage of the UK students, than the TR students, did not attempt an answer.

Both questions 1 and 7 are similar in that, an addition identity is explicitly presented in the initial expression. However, there is a big gap between students' performance on these two questions. The percentage of the UK and the TR students who gave a correct answer to question 1 is more than twice that of question 7. The percentage of incorrect answers given by students from both countries in question 1 is less than that in question 7. Nearly 5% of the TR students and no UK students gave partial answers to questions 1 and 7. Question 7 appears to have daunted students and a high percentage of students did not attempt it. Overall, students did not attempt the questions with long and complex expressions, e.g. questions 6, 12, 13, 15.

Figure 4.2. displays the mean results of students' overall performance. Whilst the TR students clearly gave more correct answers, the UK students gave more incorrect answers. The TR students also attempted more questions than the UK students. In addition, the TR students gave slightly more partial answers than the UK students.

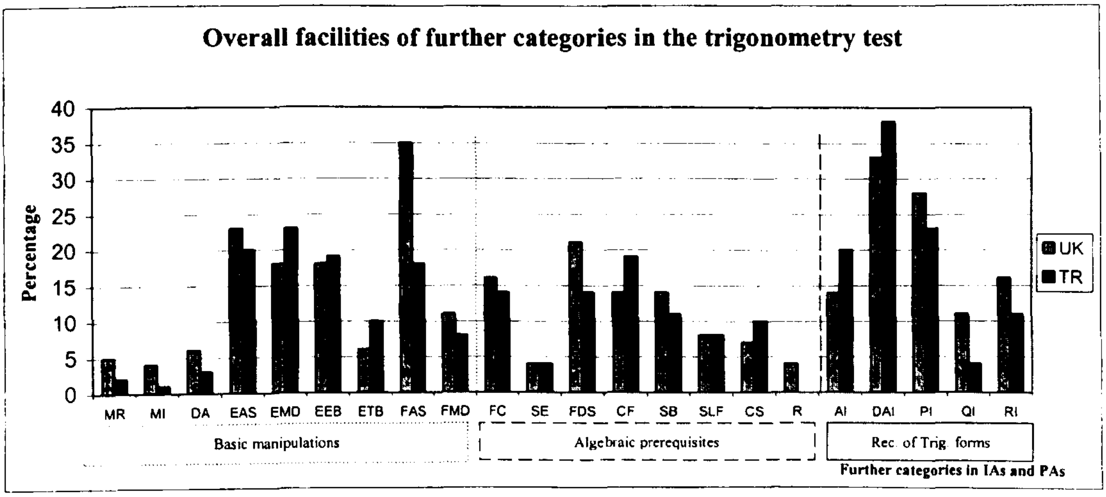
Figure 4.2. The UK and the TR students' overall performances in the trigonometry test.



Further categorisation of the incorrect and partial answers

After the initial categorisation of the students' answers, the reasons behind students' IAs and PAs were explored, all flaws committed by the UK and the TR students were coded (Table 4.2.). The flaw categories in IAs and PAs are displayed in the bar chart (Figure 4.3.). Each student represented here made a flaw once in each question. That is, the number in the flaw categories represents the number of students as well. Moreover, since the categories appeared in IAs and PAs but not in the CAs and NAQs, the flaw percentages were only calculated for the IAs and PAs.

Figure 4.3. The UK and the TR students' further categories in incorrect and partial answers in the trigonometry test.



χ^2 test statistics, df=8, 7, 4 and 21 were produced for basic manipulations, algebraic prerequisites, recognition of trigonometric expressions and the entire flaws respectively. In each case the table had the countries as rows and the flaw categories as columns. $p<0.0001$ was obtained for all basic manipulations, algebraic prerequisites, recognition of trigonometric expressions and the entire flaws, which provides strong evidence that differences in the performance of students from the two countries did not arise by chance.

In simplifying trigonometric expressions, both countries' students committed almost twice as many algebraic flaws as trigonometric flaws. Almost the same percentage of the UK and the TR students committed flaws in basic manipulations, algebraic prerequisites and recognition of trigonometric forms. Overall the most common flaws in the UK and the TR students' answers were:

- Addition/subtraction of non-fractional expressions, multiplication/division of non-fractional expressions, expanding brackets and addition/subtraction of fractions in basic manipulations.
- Cancellation, difference of two squares, common factors and squares of brackets in algebraic prerequisites.
- Addition identity, double angle identity, Pythagorean identity and reciprocal identity in recognition of trigonometric forms.

Flaws in basic manipulations, addition/subtraction of fractions (FAS) was remarkable in that it was the most common flaw the UK students made, the second highest percentage across any of the questions, and almost twice that of the TR students. This flaw was committed in questions which included fractions, such as 3, 4, 5 and 7 in which more UK students committed the flaws and 2, 9 and 10, in which more TR students committed this flaw. In the algebraic prerequisites, more UK students committed flaws in the difference of two squares and more TR students committed flaws in common factors. Flaws with the difference of two squares were observed in questions, 8, 10 and 14, in which more UK students committed flaws than TR students. Flaws with common factors were seen in questions 3, 9, 12 and 13, and this was a common flaw for the UK students in questions 9 and 12. Recognition of trigonometric expressions, the double angle identity, was the most common source of flaws for both the UK and the TR students. The TR students committed the highest percentage of the flaws in this category, which was slightly more than the UK students. The results show that the percentage of flaws made by the TR students is to some extent greater than the UK students in all questions except 7, 9, 12 and 13. There were no double angle identity flaws committed in question 1 but 60% of all students (both countries) committed a double angle identity flaw in at least one of questions 9, 13, and 16. The second most common source of flaws by students from both countries concerned the Pythagorean identity.[5]

1.1.2. Interviews with students

The trigonometry test was followed up with a semi-structured interview with seven UK and nine TR students in order to understand the reasons for the students' responses. These students were chosen with respect to their answers in the tests, their level of success in the class and their verbal ability in interview (from teachers' recommendations). Interviews focused on what students did and did not do in the tests, with particular emphasis placed on students' flaws. Interview data is organised in four themes:

- Algebraic prerequisites.
- Trigonometric identities.
- Formulae sheets and memorising the identities.
- Simplification of trigonometric expressions.

Under each heading selected quotes are used from the interviews to illustrate arguments/opinion of the students on these themes.

1.1.2.1. Algebraic prerequisites

Both the UK and the TR students experienced difficulties at times with algebraic notations and manipulations. The interview extracts below illustrate the range of algebraic flaws.

Some UK students had difficulties with the expression sin ((a+b)/2):

> I didn't know what to do because it looked different to how it is on a formulae sheet because I am used to having sin (a+b) for example but when it was sin((a+b)/2) and you start to say that looks a bit strange, I'll do a different one.

Both the UK and the TR students experienced difficulties in using powers in trigonometric identities:

$_{UK\_}(\sin^2 x)^2-(\cos^2 x)^2=(\sin^2 x-\cos^2 x)^2$.

---

[5] As mentioned in the opening of this chapter, space constraints prevent me displaying all data analysed. All results in this paragraph come from data not displayed in this thesis.

TR_...(sinx+cosx)($\sin^2$x+$\cos^2$x)=(sinx+cosx)^3...it is a rule that I cannot remember clearly...if one of the signs in the bracket is minus others should be plus...something like that there is a rule...when you expand the brackets...there are some extra terms...it should be a cube of a bracket.

Some TR students made mistakes in taking common factors out:

...(I:what is $\cos^2$x+$\sin^2$x)...1...(I:what is $\cos^2$x-$\sin^2$x)...if I take -1 out of the parenthesis it is equal to 1 then $\cos^2$x-$\sin^2$x becomes -1.

Both the UK and the TR students experienced difficulties with the power four in the expression ($\sin^4$x-$\cos^4$x)/(sinx-cosx) in question 8 in the trigonometry test to see the difference of two squares, although they did manage it in algebraic expressions:

UK_difference of two squares is less obvious here because it is 4.

TR_..$\sin^4$x-$\cos^4$x is $\sin^2$x $\sin^2$x - $\cos^2$x $\cos^2$x then by using Pythagorean identity $\sin^2$x is equal to 1- $\cos^2$x....($\sin^2$x - $\cos^2$x)/ sinx – cosx by using difference of two squares...

Some UK students, but no TR students transformed trigonometric expressions into algebraic expressions by substituting the sine and cosine functions by letters e.g. a and b. The reason UK students stated for using this method is to reduce the algebraic complexity of very long expressions:

I sometimes use substitution but it depends how many components there are like with that one there's cosine, q, x, sine all this kind of stuff and it was just a massive equation to try and get your head round...just thought it would be easier to simplify it like that...then afterwards I would have replaced 'x's with cos

1.1.2.2. Trigonometric identities

The UK students displayed a tendency to refer to their formula sheets in choosing the appropriate identity to simplify an expression. Instead of looking for the appropriate identity they looked for the identity they are used to or that they are familiar with or which is similar to the one in the expression. It seemed they did not know exactly which one to use and this made their manipulations longer. For example in the expression $\cos^2$x-$\sin^2$x, some UK students checked their formula sheet to find a familiar identity and used the Pythagorean identity instead of the double angle identity:

I have been used to doing this in math as I've been used to when seeing this $\cos^2$ and $\sin^2$.

They sometimes used trial and error methods:

when I use trigonometric identity and then when I feel am are getting stuck I go back and change this identity.

In simplifying trigonometric expressions some of the UK and the TR students experienced difficulties recognising trigonometric identities. Moreover, if they did not recognise any further trigonometric identity they stopped and recognised their stopping as the simplified form:

UK_1/sinxcosx...I cannot go further...I'd leave it like that.

$_{TR\_}$...(1+tan^{2}x)/tanx...I do not see anything....this is the simplified form...

$_{TR_...}$1/sinxcosx...there is nothing to do in this expression...for instance if there was a square root in the denominator I would multiply it by same expression and get rid off the square root...but I do not see anything in here...

Some UK and TR students said they tried to choose the appropriate trigonometric identity to simplify the expression:

$_{UK_}$If I've got sine squared x for example just look down the identity sheet and find one with a sine squared x and see if there's a couple and the most appropriate one.

$_{TR_}$I wanted to cancel 1 out so I have used cos^{2}x-1...I could use others but then 1 would not disappear...I use the most appropriate identity, which makes cancellations possible, in expression.

1.1.2.3. Formulae sheet and memorising the identities

All UK students either explicitly or implicitly utilised formula sheets to find familiar identities to use in the simplification of the trigonometric expressions in the interview. Sometimes some of them focused on the formula sheet rather than the expression:

...I just want to...to get something that looks like something on the formula sheet really...I just do anything that...makes...it look different and then look to see if it looks like anything and then change anything else and then if it is wrong start again and then change something different...

Some UK students pointed out that they could answer the question without the help of a formulae sheet, because they had already memorised some of them:

...I know most of the formulae...I memorised them...I could have a go without...

Although the UK students were given a formulae sheet, some of them did not use it:

well some of them you just know from past exam papers and things like that...because not all of them are on formula sheets in the exam. You got to learn some of them.

On the other hand the TR students handled all simplifications with the identities they had in their mind. They said they solved many questions, so they automatically memorise them, they got used to the trigonometric identities:

trigonometric identities are in my mind...it is not a memorisation...it can be called logic that I gain after solving many questions...before I use them I visualise them...and see whether expression can be simplified further or not.

1.1.2.4. Simplification of trigonometric expressions

Most UK and TR students in the trigonometry test and interviews stopped at different expressions as the simplified form of the initial expressions in the same question. Students were asked what they understood by simplification or why the expressions they stopped at is the simplified form. They usually said they stop whenever they are not able to apply any more algebraic or trigonometric properties:

$_{UK_}$it should be shorter...when you cannot use any more function, when you cannot go further...it is the simplified form.

TR_I use cancellation and some other methods...it is reducing the number of the terms...I simplify the expression as much as I can do and then the different terms remain in the expression...in that case I use trial and error method...and I stop at the expression where I cannot use any trig identity, any factorisation and cancellation and so I cannot go further...it becomes my simplified form.

Some TR students underscored the importance of solving various and many questions to be familiar with trigonometric identities and procedures to be used in the simplification of trigonometric expressions:

if I do not solve many question with the identity or the procedure there could be questions where I might forget how to use them.

Some UK and TR students said they go back and try other possible identities and procedures when they got stuck in reaching a simplified form:

UK_If I get stuck and I don't think it's right I'll go back and see if there's anything else obviously I can do but umm normally I can't.

TR_I always check everything I have done to ensure I got the right expression...if the expression gets longer and complicated I go back and check manipulations and the identities I used to find what is wrong...

Some TR students explained what they do when they get a trigonometric expression to simplify:

...when I saw the expression (question 8 in trigonometry test)...from the difference of squares...I factorise...it is like life style you know it whenever you see...algebraic properties...in general I am looking at the expression if there is any algebraic rules I know like common factors, factorising, difference of two squares or trigonometric identities...then I am looking for cancellation if I can do...

1.1.3. Verbal protocol with students

Concurrent verbal protocols were used to obtain data on students' manner of simplifying trigonometric expressions as the UK and the TR students solved simplification items, to gain insight into their thinking. As discussed in the methodology chapter, two UK and four TR students provided protocols (see also pp. 37-40). In this section, firstly the questions drawn on in the protocol are introduced, then the coding of protocols is detailed and finally results of the verbal protocol are presented.

The problems used in the concurrent verbal protocol

There were five questions in the trigonometry protocol (see Appendix M). The fourth and fifth questions were taken verbatim from the trigonometry test. The remaining questions were very similar to questions in the trigonometry test in terms of the identities and procedures to be used in the simplifying process. All questions were collected from currently used textbooks at the data collection time. They were also piloted and it was observed that students were familiar with these types of questions. The first and third questions contained non-fractional expressions. The second question required the subtraction of two fractions. Although the fourth question looked like a non-fractional expression there were fractions in the brackets. The fifth question is a bit complicated compared to the others and is a fraction.

The coding of the protocols

The concurrent protocols revealed a uniformity of approach with regard to approaches to simplification. The analysis of verbal protocol data from the UK and the TR students on trigonometry test items suggested that successful solutions have a common pattern. I created a template (model) of this pattern (see Table 4.3.) and conducted a moderation exercise with my two supervisors. I explained the model and clarified my interpretation of the terms. My supervisors then applied this model to a complete verbal protocol. Percentages were not calculated but agreement between my two supervisors' and my allocation of stages in the model to protocol segments was very high. It was agreed that I could proceed to work with this model. The components of the model and their relevance to the protocols are explained below.

The coding

The coding for the protocols was developed in tandem with the model. Coding aimed to capture all information produced in the protocols. The coding provided accounts for all utterances made. However, although students were trained to 'think aloud' it was observed that they had some difficulties verbalising what was going on in their minds when working on the task.

Students did not continuously verbalise in the verbal protocol sessions. There were times when they were quiet for two to five seconds. They were clearly thinking but not thinking out loud. As discussed on page 60 this presents a problem for the researcher. My way out of this problem was to make reasonable inferences as to what students were thinking. Some coding, then, is implicit, as explained in Table 4.3. These implicit codings are, however, based on observable actions such as reading and symbolic manipulation.

Table 4.3. Coding categories of the concurrent verbal protocols for the trigonometry test.

Coding category	Description
Read	Reads the problem completely or partially
Recognise	Verbal response clearly indicates that the students sees a property or relationship
Recall	Student brings a property or relationship from their memory store
Symbolic manipulation	Mathematical operations and manipulations student did
Rewritten form	Equivalent form of the initial expressions or equivalent form of the terms of the initial expressions
Result	The final answer given by students

NB read, recognise, recall, rewritten form, symbolic manipulation, rewritten form and result are coded 'implicit' if they are not verbalised but it is clear from subsequent verbal or written protocol data that reading, recognising, recalling, doing symbolic manipulation, rewriting the form or reaching the result has taken the place.

The results of the protocol analysis

Protocols were analysed in two steps. First, students' answers were categorised by using the initial categorisations, which were used in the analysing of trigonometry test. Then the protocol of every answer students gave was divided into segments and then coded.

Initial categorisation of the UK and the TR students' answers in protocols are presented in the Table 4.4.

Table 4.4. The results of the UK and the TR students' performance on verbal protocol task.

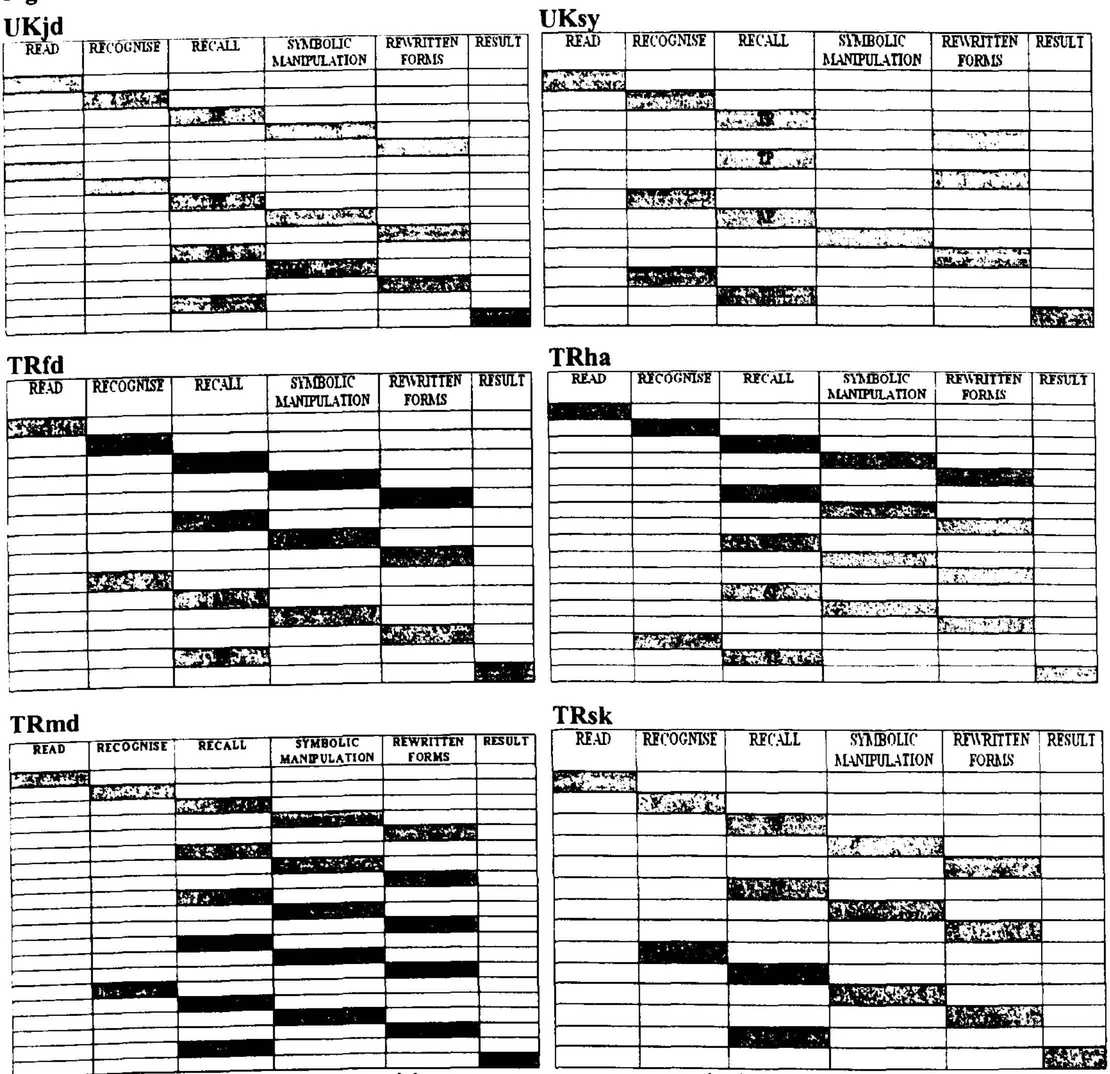
Students	Trigonometry questions in protocol				
	TT1	**TT2**	**TT3**	**TT4**	**TT5**
UKjd	CA	CA	PA	CA	PA
UKsy	CA	CA	PA	CA	PA
TRfd	CA	CA	PA	IA	IA
TRha	CA	IA	PA	IA	IA
TRmd	CA	CA	PA	IA	PA
TRsk	CA	CA	PA	IA	PA

Individual protocols were divided into coded segments. All protocols, regardless of whether or not the answer was correct, were analysed in the same manner. The results of the coding of the verbal protocols are presented in the next section.

The results of the coding categories in concurrent verbal protocols

Figure 4.4. shows the coded segments of the UK and the TR students' protocols in answering the first trigonometry question. Due to space restrictions only the segment analysis of one student, TRfd, is presented (see Table 4.5.). This student's work is then illustrated (see Figure 4.5.).

Figure 4.4. The schematic of the UK and the TR students' protocol for the first trigonometry question.

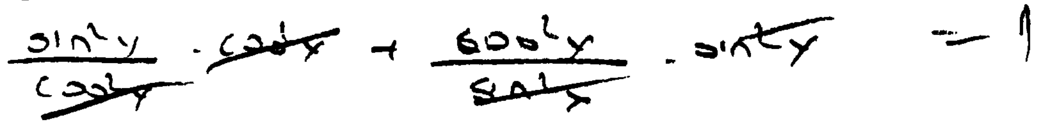


NB TP and AP in the recall column stand for 'trigonometric property' and 'algebraic property' respectively.

Table 4.5. The segment analysis of TRfd's verbal protocol.

Stages in problem solving	***Segments of verbal protocol***
read	Simplify...
recognise	
recall (trigonometric property)	if tan squared x is written as sin squared x over cos squared x
symbolic manipulation	then times cos squared x plus
rewritten form	
recall (trigonometric property)	if cot squared x is written as sin...cos squared x over sin squared x
symbolic manipulation	then times sin squared x,
rewritten form	
recognising	
recall (algebraic property)	by using of the cancellations
symbolic manipulation	
rewritten form	
recall (trigonometric property)	
result	the result is 1

Figure 4.5. The manipulation of TRfd.



The student read the question and recognised the terms. This is implicit but her subsequent work justifies my inference that she did this. She then focused on $\tan^2 x$, rewrote this as $\frac{\sin^2 x}{\cos^2 x}$. She then did the same for $\cot^2 x$, getting $\frac{\cos^2 x}{\sin^2 x}$. The rewritten form suggested (by recognition) cancellation, which she did (recall, symbolic manipulation). The rewritten form with cancellation lines is a new rewritten form which suggests (recall) the Pythagorean identity and produces the answer 1.

1.2. Students' algebra knowledge and use of algebraic conventions.

In this section, results of algebra tests, interviews and verbal protocols on simplification of algebraic expressions are analysed. I follow the format used in the previous section.

1.2.1. Algebra test

The algebra test (see Appendix B) consisted of 16 questions. Question 5 of which had two sub-questions. Two questions asked students to use given substitutions to find answers. Two questions asked students to solve given equations and the remaining questions asked students to simplify given algebraic expressions. Four questions involved fractions, five questions involved the manipulation of fractions and the rest were non-fractional expressions. The algebra test aimed:

- At revealing students' errors, misconceptions and difficulties in items.
- To investigate students' use of algebraic notation and conventions.
- To investigate students' solution strategies and their algebraic competencies.

The algebra test was conducted with 55 UK and 65 TR students.

1.2.1.1. Categorisations of students' answers

Students' answers were categorised as described on page 74 with the exception that I work with algebra (see Table 4.6.).

Table 4.6. Initial categorisation of the algebra test.

Groups of Answers	Abbr.	Description
Correct answers	**CA**	Appropriate algebraic identities or operations are applied to yield a valid solution; manipulations resulted in students writing the expected response.
Incorrect answers	**IA**	Inappropriate use of algebraic properties
Partial answer	**PA**	The student approached the question in a correct manner but stopped short of the expected simplification
Non-attempted questions	**NAQ**	No response or simply a repetition of the question.

In the exploration of the IAs and PAs, a 'working model' of student flaws, similar to the trigonometry one, in algebra tasks was constructed. It is based on an analysis of the mathematics involved in the questions and an analysis and categorisation of the types of flaws in incorrect answers and partial answers students made in the algebra test. It has two basic quasi-hierarchical levels, which are similar to the trigonometry model. Basic manipulations include errors that may occur in numerical and algebraic work, e.g. incorrect manipulation of negative numbers, incorrect manipulation of expressions. Algebraic prerequisites concern purely algebraic flaws, e.g. incorrect cancellation of algebraic fractions.

All flaws made by the UK and the TR students were noted, and then coded and placed under the relevant title (Table 4.7.). Since there was no use of trigonometric identities in the algebra test, the last level of the trigonometry flaw model was not in the algebra flaw model. However, flaws AS and FT below are slightly different from those in the trigonometry flaw model.

Table 4.7. Further categorisation of the UK and TR students' incorrect and partial answers in the algebra test.

CODE		CATEGORY
Basic manipulations	**MR**	Misreading the question
	MI	Misinterpreting the question
	DA	Direct answer (without operation)
	EAS	Simplify algebraic expressions, addition and subtraction
	EMD	Simplify algebraic expressions, multiplication and division
	EEB	Simplify algebraic expressions, expanding brackets
	ETB	Simplify algebraic expressions, gathering within brackets
	FAS	Simplify algebraic fractions, addition and subtraction
	FMD	Simplify algebraic fractions, multiplication and division
Algebraic prerequisites	**FC**	Simplify algebraic fractions, cancellation (reduction)
	AS	Algebraic substitution
	SE	Simultaneous equations (sum of two equations)
	FDS	Factors, difference of two squares
	CF	Factors, common factors
	FT	Factors, trinomials
	SB	Power, squares of brackets
	SLF	Power, squaring linear functions (taking square of both sides of an equation)
	CS	Power, completing square
	R	Power, roots

1.2.1.2. Students' performance in the algebra test

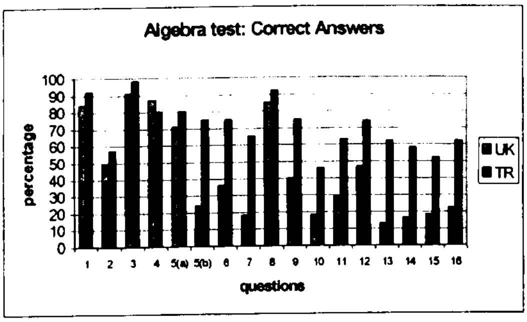
In this section first the results of the initial categorisations are presented and then the flaws committed by students in incorrect and partial answers are exemplified.

Initial categorisation of the all answers in the algebra tests

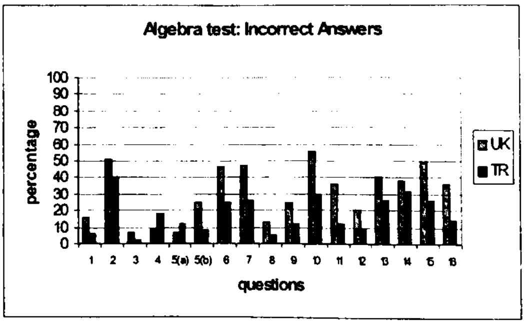
Figure 4.6. A-D presents the percentage of the UK and the TR students in each initial categorisation for each question.

Figure 4.6. The percentages of the UK and the TR students' initial categorisations in the algebra test.

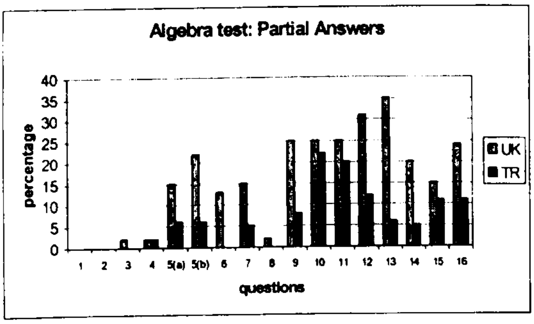
A



B



C



D

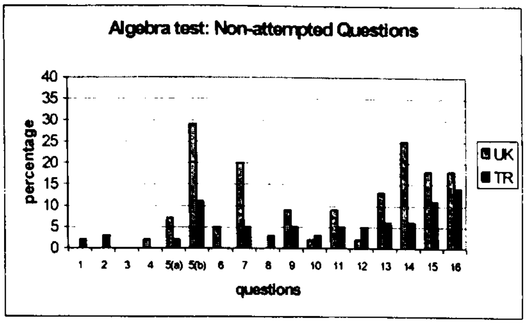


Figure 4.6.-A shows that more than half of the TR students answered all questions except question 10 correctly. On the other hand, less than 30% of the UK students gave correct answers to half of the questions. The most notable differences between the percentage of the UK and the TR students were in questions 5b-7 and 9-16, where TR students, gave more correct answers than the UK students.

Figure 4.6.-B illustrates that the percentage of the UK students giving incorrect answers was greater than the percentage of the TR students except for questions 4 and 5b, which were the non-fractional questions. The difference between students from the two countries is considerable in most questions, and is most pronounced in questions 5b, 6, 7, 10, 11, 13, 15 and 16.

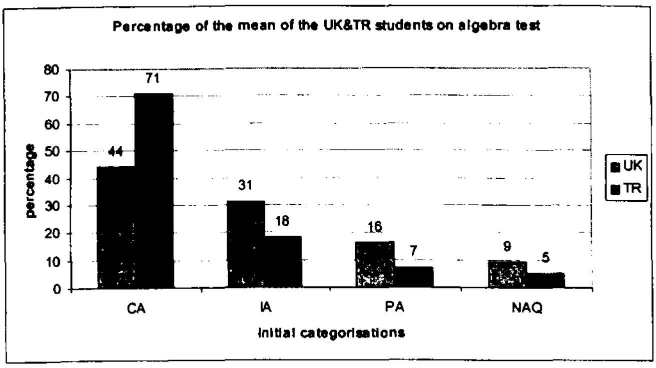
The results in Figure 4.6.-C show that the percentage of both the UK and the TR students answering questions partially was less than 36%. However, the percentage of the UK students giving partial answers was more than the percentage of the TR students. A considerable difference exists between the percentages of the UK and the TR students in questions 9, 12, 13, 14 and 16. In question 13, 35% of the UK students gave partial answers compared to only 5% of the TR students. No partial answers were given to the non-fractional questions 1 and 2.

Figure 4.6.-D shows the percentage of the students from both countries who did not attempt questions was less than 30%. The percentage of the UK students who did not attempt questions is greater than the TR students, particularly in questions 5b, 7 and 14.

Questions 3, 5 and 9 are similar substitution questions. The percentage of correct responses declines over these questions in the order 3, 5, and 9. It appears that students' performance in substitution questions is sensitive to changes in question format. When the students worked with numbers they performed well but when they worked with variables they did not perform as well as they did when working with numbers.

Figure 4.7. displays the mean results of students' overall questions. There is a clear declining pattern in the mean percentage of the students across the initial categories of both the UK and the TR students. Although this is trivial at one level (because this is categorical data) it is interesting that the pattern is the same for both countries. The TR students clearly gave more correct answers and the UK students more incorrect answers. The UK students also answered more questions partially than the TR students. There were very few non-attempted questions by the UK and the TR students.

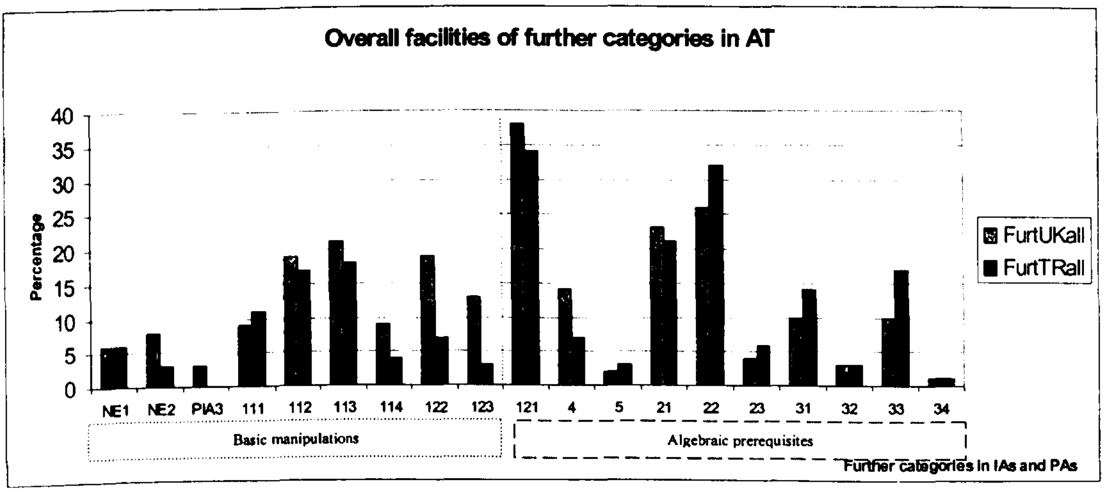
Figure 4.7. The UK and the TR students' overall performances in the algebra test.



Further categorisation of the incorrect and partial answers

The percentages for the further categories in incorrect and partial answers of the algebra test is presented in Figure 4.8.

Figure 4.8. UK and the TR students' further categories in incorrect and partial answers in the algebra test.



χ^2 tests statistics, df=8 and 9, were produced for basic manipulations and algebraic prerequisites respectively. In each case the table had the countries as rows and the flaw categories as columns. $p<0.0001$ for basic manipulations and $p<0.013$ for algebraic prerequisites provides strong evidence that differences in the performance of students from the two countries did not arise by chance.

As seen in Figure 4.6. (see p. 87), the TR students performed far better in simplifying algebraic expressions than UK students and gave less IAs and PAs than the UK students. Students from both countries, particularly the TR students, committed the flaws in algebraic prerequisites more than the flaws in basic manipulations. Overall, in the algebra test, the UK students made more basic manipulation flaws than the TR students whereas the TR students committed more algebraic prerequisites flaws than UK students. Overall the most common flaws in the UK and the TR students' answers were:

- Multiplication/division of non-fractional expressions, expanding brackets and addition/subtraction of fractions in basic manipulation.
- Cancellation, difference of two squares, common factors and completing squares in algebraic prerequisites.

The most commonly committed flaws by both the UK and the TR students were flaws in basic manipulation multiplication/division of non-fractional expressions and expanding brackets: where the UK students committed slightly more than the TR students. The flaw of expanding brackets was mostly committed in question 1 (more TR students committed this flaw) and question 2, in which more UK students committed this flaw. The flaw of multiplication/division of non-fractional expressions was mostly committed in questions 1, 3, 6 and 9, in which more TR students made this flaw and question 2, in which more UK students committed this flaw. Regarding flaws in basic manipulations, flaws of addition/subtraction of fractions were largely committed by the UK students. The flaw of addition/subtraction of fractions was mostly seen in questions 6, 7, 14 and 15. The most common source of flaws in both the UK and the TR students' answers were cancellation, difference of two squares and common factors. The UK students had the highest percentage of flaws of cancellation across all questions. Both the UK and the TR students made flaws of cancellation in almost every question. The TR students committed the flaw of common factors more than the UK students. However, students from both countries committed the flaw of common factors in a majority of the questions, particularly in questions 5a and 11, in which more TR than UK students made this flaw and in questions 13 and 10 in which more UK than TR students committed this flaw.[6]

[6] As mentioned in the opening of this chapter, space constraints prevent me displaying all data analysed. All results in this paragraph come from data not displayed in this thesis.

1.2.2. Interviews with students

As described on page 79, a semi-structured interview was conducted with seven UK and nine TR students following the algebra test to understand the reasons for their responses. These students were the same students with whom trigonometry interviews were conducted. I report on basic manipulations, algebraic prerequisites, and the simplification of algebraic expressions.

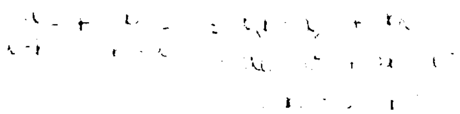
1.2.2.1. Basic manipulations

Students from each country committed the most common basic manipulation flaws, seen in the algebra test, in the interviews. For UK students this was in the addition/substraction of fractions and expanding brackets for TR students:

> UK_...(in simplifying a/(a-b)+b/(b-a) see Figure 4.9.)...if you multiply every term by (a-b)(b-a) then you get that...so I didn't need the denominator because if you multiply that by that and that, that is I already there say you don't write it...If you multiply a/(a-b) by a-b that disappears and by b-a so b-a is up here they should multiply that by b-a disappears and by that so that comes there that is what I thought that is why I did that...

Figure 4.9. The UK student's answer to question 6 in the trigonometry test.

6-) Simplify $\frac{a}{a-b}+\frac{b}{b-a}$



> TR_...after the cancellation I got (a-b)(a^2+b^2)...that is...if I multiply one by one and expand the brackets I got a^3-ab^2-ba^2+b^3 that is the equivalent form of the a^3+b^3 ...

1.2.2.2. Algebraic prerequisites

Both UK and the TR students committed flaws of taking common factors out. Interestingly, thinking aloud helped UK students to find answers without taking the common factor –1 out:

> UK_...(Given that a-b=5 simplify $(b-a)^2$-from test)...I didn't understand that because it was the other way around. So I thought I need to find a-b somewhere in this...It is negative it is the other way around so like well yeah I know how you do now it should be b-a equals to –5.

> TR_...(($(2ab)^2$- $4a^2$)/$(2ab+2a)^2$)...I first wrote the equivalent form of the difference of two squares...then I expand the square of the bracket...then I take common factors out and cancelled out the common term in numerator and denominator...(2ab-2a)(2ab+2a)= 2a(b-1)(b+1)...2a belongs to both of the brackets...and I cancelled 4 out...

Some UK students had difficulties with equivalent forms of the difference of two squares:

> ...y^2-x^2 is equal to y-x times x-y...(I-do you know the difference of two squares)...yeah it's b^2 minus 2ab that is squared must be squared...and b^2.

Some UK students performed flaws in cancellation in algebraic fractions:

(see Figure 4.10.)...I thought because they are on the top and the bottom you can cancel them ... I think that's what I did... I cancelled everything because it was a common kinda of term.

Figure 4.10. The UK student's answers to questions 14 and 16 in the algebra test.

14-) Simplify $\left[\frac{a-b}{a+b}-\frac{a+b}{a-b}\right]\cdot\frac{a-b}{ab}$

$$= \frac{(a-b)-(a+b)}{(a+b)-(a-b)} \times \frac{a-b}{ab}$$

$$= \frac{a-b}{ab}$$

16-) Simplify $\frac{a^3b-ab^3}{a^3b+2a^2b^2+ab^3}$

$$\frac{-1}{2a^2b^2}$$

1.2.2.3. Simplification of algebraic expressions

Some UK students were not clear about simplification and simplified forms in algebra:

> I dunno I think that is the problem with simplifying I don't really know what I'm trying to find. That is why I found some of the question like the questions are different because I prefer questions what I know I'm going to find like when it just says simplify unless I end up with something like a-b over c or whatever I don't really I dunno I think it is how to know when you finished...I suppose it is like if I end up with something I can't see what to do with it. Then I just stop...

Simplifying algebraic expression was seen as using algebraic properties but was sometimes a feeling for some UK students:

> sometimes I do not know what to do ...but I just try to keep on looking for algebraic properties and identities
>
> ...I do not really know it's just what I feel like doing...

Both UK and the TR students highlighted the use of trigonometric identities as a difference between the simplification of trigonometric and algebraic expressions. Students from both countries, particularly the TR students, found simplifying algebraic expressions easier than simplifying trigonometric expressions and, moreover, some TR students thought trigonometric expressions look more complicated than algebraic expressions:

> $_{UK_}$I expect to have to use to identities and things whereas if you give me an algebra question with *a*s and *b*s and I know the all you can be expected to do and it is it is like a^2+b^2 and things like that and like that trigonometry one I thought that you would have to use identities and things. I dunno it is just like if it is just as and bs it is like saying well it isn't a way I can do with it algebra seems simpler to me
>
> $_{TR\_}$...I feel comfortable in simplifying the algebraic expressions...a hundred percent I trust my answer in algebra but I can be fifty fifty sure in trigonometry...when I find a result a-b it looks all right like everything is finished...but when you see (1-cosx)/sinx it is just disturbing ...actually same rules are applied in algebra and trigonometry, like factorising difference of two squares, adding fractions, taking common factor out. The only difference is instead of a and b sine, cosine, tangent and cotangent are used and they are frightening...for example that question (students points to trigonometry test question 13)...trigonometry looks much more complicated...

1.2.3. Verbal protocols with students

Concurrent verbal protocols of algebra were conducted, as described on page 82, to gain insight into students' thinking. The students were the same students used in the trigonometry protocols. In this section, first the questions which are used in the protocol are introduced, then the coding of protocols is detailed and then the results of the verbal protocol are presented.

The problems used in the concurrent verbal protocol

Five questions were used in the algebra protocol (see Appendix N). The fourth and fifth questions were taken verbatim from the algebra test. The remaining questions were similar to the questions in the algebra test. The first and second questions contained non-fractional expressions. The fourth question was an addition of two fractions. The third and fifth questions were fractions.

The coding of the protocols

The coding of the protocols followed the same procedure described on page 83, so I directly present the results of the protocols.

The results of the protocol analysis

Students' answers were first categorised by using the initial categorisations, the results are presented in the Table 4.8.

Table 4.8. The results of the UK and TR students' performance on verbal protocol task.

Students	Algebra questions in protocol					
	AT1	**AT2a**	**AT2b**	**AT3**	**AT4**	**AT5**
UKjd	CA	CA	CA	CA	CA	CA
UKsy	CA	CA	CA	CA	CA	CA
TRfd	CA	CA	CA	CA	CA	CA
TRha	CA	CA	CA	IA	CA	CA
TRmd	IA	IA	IA	CA	CA	CA
TRsk	CA	CA	CA	IA	CA	CA

Protocols of every answer students gave were then divided into segments and coded.

The results of the coding categories in concurrent verbal protocol

Figure 4.11. show the coded segments of the UK and the TR students' protocols in answering the first algebra question. Due to space restriction only the segment analysis of TRfd is presented (see Table 4.9.). This, and also student's work is illustrated (see Figure 4.12).

Figure 4.11. The schematic of the UK and the TR students' protocol for the first algebra question.

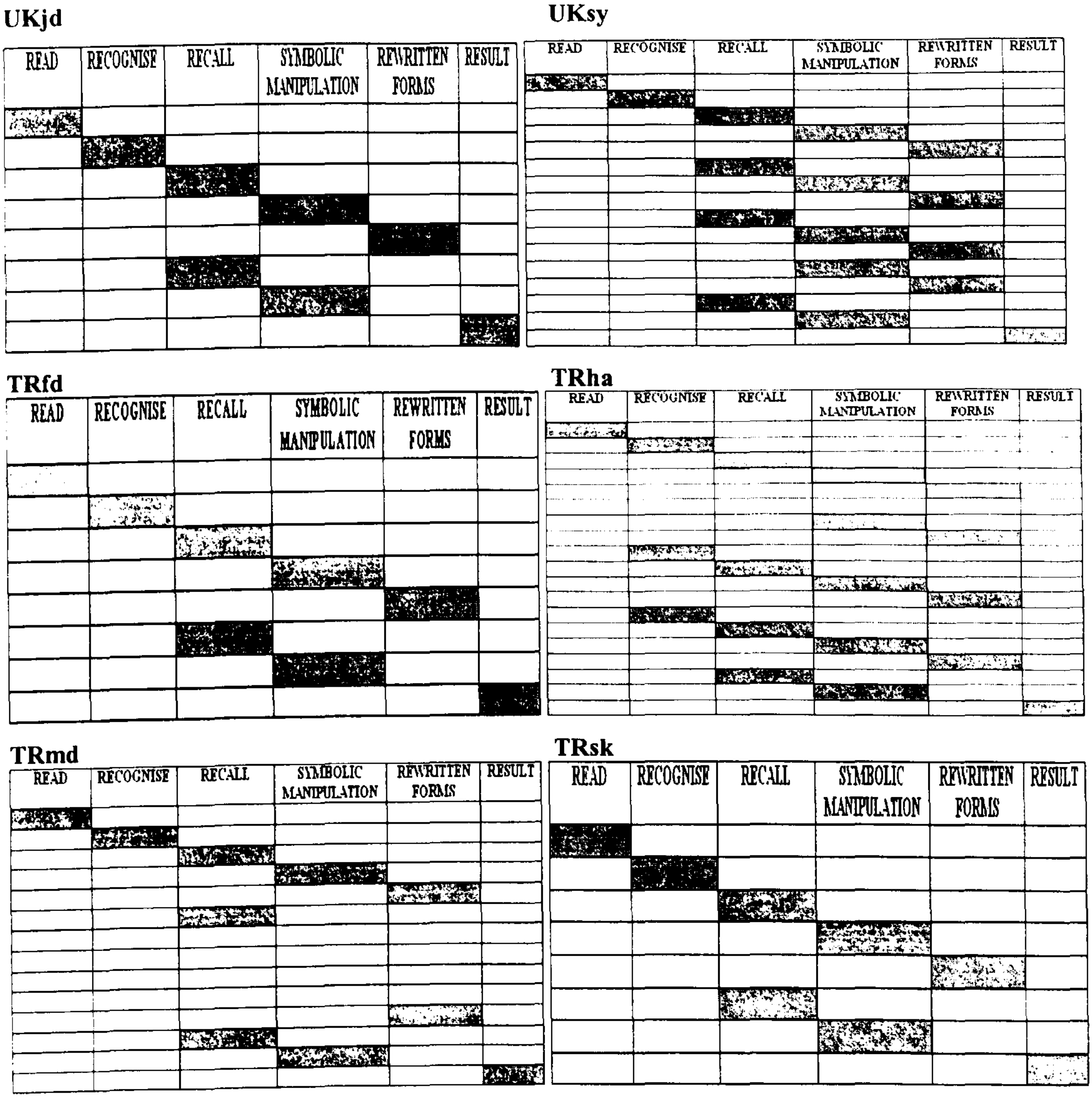


Table 4.9. The segment analysis of TRfd's verbal protocol.

Stages in problem solving	***Segments of verbal protocol***
read	Simplify...
recognise	
recall (algebraic property)	
symbolic manipulation	
rewritten form	12a plus 36 minus 12a plus 8
recall (algebraic property)	
symbolic manipulation	12a and 12a are cancelled out, 36..
rewritten form	44
result	

Figure 4.12. The manipulation of TRfd.

$$\cancel{12a}+36-\cancel{12a}+8 \quad = \underline{44}$$

The student clearly read the statement and recognised the algebraic properties to be used. She then focused on the brackets and did mental manipulations of expanding brackets to rewrite the expression as 12a+36-12a+8. She then cancelled 12 (recalled algebraic property, symbolic manipulation) and mentally added 36 and 8, which is the written form and the result.

1.3. Comparison of students' performance in trigonometry and algebra tests

Students' algebraic competencies and use of notation and conventions are likely to affect their performance in simplifying trigonometric expressions. To observe the interplay between trigonometry and algebra a comparison is made in this section between the trigonometry and algebra tests in terms of general perspective and common flaws made by students in the two tests. I also report on what I call the parallel questions. These are the 7th question of the algebra test, the 8th question of the trigonometry test, the 16th question of the algebra test, and 13th question of the trigonometry test. These questions have the same algebraic form (for each pair) and provide a useful focus to explore algebraic and trigonometric difficulties.

1.3.1. General comparison

Both the UK and the TR students attempted more questions in the algebra test than in the trigonometry test and both sets of students obtained more than twice as many correct answers in the algebra test than they did in the trigonometry test. The percentage of questions in the trigonometry test answered incorrectly by both the UK and the TR students was much greater than in the algebra test. Both countries' students gave more partial answers to questions in the trigonometry test than in the algebra test. The percentage of the UK and the TR students' correct answers was very low compared with incorrect answers, partial answers and non-attempted questions in the trigonometry test. On the other hand the difference between the percentages of questions answered correctly by students from both countries was greater than that of incorrect answers, partial answers and non-attempted questions in the algebra test. The UK and the TR students' performance in both the trigonometry test and the algebra test are compared in terms of initial categorisations in Table 4.10.

Table 4.10. Comparison of the UK and the TR students in the trigonometry test and the algebra test in terms of initial categorisations.

Initial categories	
Correct answers	The TR students gave more correct answers than the UK students did in both trigonometry and algebra tests.
Incorrect answers	The UK students gave more incorrect answers than the TR students did in both the trigonometry and algebra tests.
Partial answers	The TR students gave more partial answers than the UK students did in the trigonometry test while the UK students gave more partial answers than the TR students did in the algebra tests.
Non-attempted question	There were more non-attempted items in both the UK and the TR students' trigonometry test compared with algebra test. Moreover, the TR students attempted more items in both the trigonometry and algebra test compared with the UK students

1.3.2 Comparison of the flaws

The flaw categories of both the UK and the TR students' incorrect and partial answers in the trigonometry and algebra tests were presented in the sections 1.1.1.2 and 1.2.1.2. of this chapter. There is some overlap in the flaw categories of the two tests, which is shown in Table 4.11.

Table 4.11. Comparison of the flaws seen in the trigonometry test and the algebra test.

	Tests		
	TT-AT	**TT**	**AT**
Flaw categories	Basic manipulations; MR,MI,DA,EAS,EMD,EEB, ETB,FAS,FMD Algebraic prerequisites; FC,SE,FDS,CF,SB,SLF,CS,R	Recognition of trigonometric forms; AI,DAI,PI,QI,RI	Algebraic prerequisites; AS,FT

Basic manipulations and algebraic prerequisites are flaw categories presenting in both tests. In basic manipulations, common sources of flaws committed by both the UK and the TR students were addition/subtraction of non-fractional expressions, multiplication/division of non-fractional expressions, expanding brackets and addition/subtraction of fractions. The number of flaws in addition/subtraction of non-fractional expressions made by both the UK and the TR students in the trigonometry test was almost twice that of the algebra test. There was not a big difference in the number of questions in which the UK and the TR students committed the flaws of multiplication/division of non-fractional expressions and expanding brackets. Students from both countries committed more flaws in addition/subtraction of fractions in the trigonometry test than in the algebra test. In both tests, the UK students committed this flaw more than the TR students did.

In algebraic prerequisites, cancellation, difference of two squares, and common factors were the most commonly committed flaws by the UK and the TR students. Contrary to flaws in basic manipulations, flaws in algebraic prerequisites were seen in more algebra questions than trigonometry questions. The percentage of questions in which both the UK and the TR students committed cancellation flaws in the algebra test was more than twice that of the trigonometry test and it was the highest percentage across any of the questions in both tests. Moreover, the percentage of the questions in the algebra test in which both the UK and the TR students committed the flaw of common factors was bigger than the trigonometry test.

1.3.3. Parallel questions in the trigonometry test and the algebra test

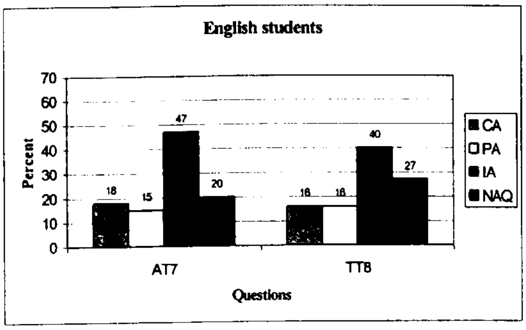
1.3.3.1. Parallel questions

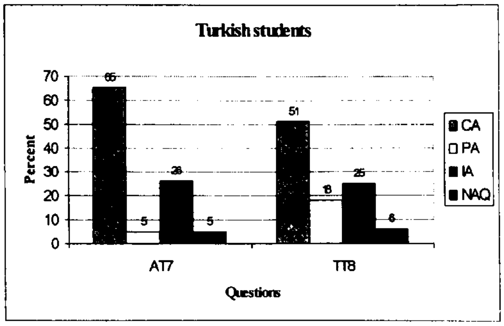
There were two parallel questions in the trigonometry and algebra tests. These parallel questions and both the UK and the TR students' performance in these questions in terms of the initial categorisations are presented in Figure 4.13.

Figure 4.13. The UK and the TR students' performance in parallel questions.

i-Parallel questions 1

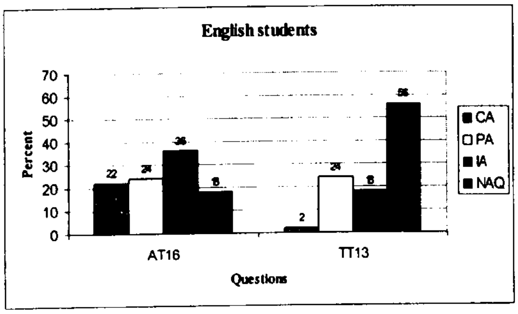
AT7-) Simplify $\dfrac{x^4 - y^4}{x - y}$ **TT8-) Simplify** $\dfrac{\sin^4 x - \cos^4 x}{\sin x - \cos x}$

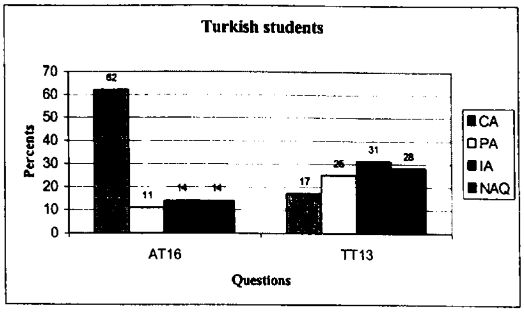




ii-Parallel questions 2

AT16-) Simplify $\dfrac{a^3b - ab^3}{a^3b + 2a^2b^2 + ab^3}$ **T13-) Simplify** $\dfrac{\cos^3 x\sin x - \cos x\sin^3 x}{\cos^3 x\sin x + 2\cos^2 x\sin^2 x + \cos x\sin^3 x}$





Parallel questions 1

The TR students clearly did much better in both questions compared with the UK students. Both the UK and the TR students did slightly better in the algebra questions than in the trigonometry questions. Almost one half of the UK students solved the algebra and trigonometry questions incorrectly whereas about one quarter of the TR students produced incorrect solutions. At the other extreme, more than one half of the TR students gave correct answers to both algebra and trigonometry questions whereas less then one fifth of the UK students solved both questions correctly.

The majority of the TR students attempted both trigonometry and algebra questions. The percentages of the UK students who did not attempt the algebra and trigonometry questions were more than four times the percentage of TR students. The percentages of both the UK and the TR students giving partial answers to trigonometry and algebra questions were less than 20%. The TR students gave more partial answers to the trigonometry question than the algebra question. However, surprisingly, the percentages of the UK students, who partially answered the trigonometry and algebra questions, were the same.

Parallel questions 2

As with parallel question 1, the TR students did much better in both questions compared with the UK students. The percentage of the correct answers given by both the UK and the TR students to the algebra question was greater than that of the trigonometry question. The percentages of correct answers to the two questions from students from the two countries ranged from 2% to 62%. Students performed better on the algebra test. More than half of the UK students did not attempt the trigonometry question whereas a quarter of the TR students did not. Almost same percentage of the UK and the TR students did not attempt the algebra question. A quarter of both the UK and the TR students answered the trigonometry question partially. Moreover, the percentage of the UK students giving partial answers to algebra questions was more than twice that of the TR students.

1.3.3.2. The flaws in parallel questions

The flaws in parallel questions 1

In the trigonometry question the most common source of flaws was the difference of two squares, which was made by more than half of the UK and the TR students, 75% and 55% respectively. The other flaws were committed by less than 30% of both the UK and the TR students. Although the difference of two squares was the most common source of flaws performed by both countries' students, flaws in expanding brackets, adding and subtracting fractions, cancellation and completing the square were common in the algebra test. Some comparable results are worth noting. The percentage of UK students making flaws in adding and subtracting fractions, in the difference of two squares and in cancellation were greater than that of the TR students: in the case of adding and subtracting fractions the percentage of the UK students committing that flaw was seven times greater than that of the TR students. Flaws in expanding brackets and completing the square in the algebra question were committed by over 35% of the TR students, almost four times that of the UK students. As it is in the algebra question, the percentages of the UK students who committed flaws in multiplying and dividing non-fractional expression, expanding brackets, taking common factors out and completing the square were greater than the TR students in the trigonometry question. A quarter of the UK students made flaws in giving the direct answer in both questions whereas this flaw was made by 4% of the TR students in only the trigonometry question.

The flaws in parallel questions 2

Basic manipulation flaws were committed by a low percentage of students. The only notable result concerns adding and subtracting fractional and non-fractional expressions. Less than 25% of students from both countries committed these flaws but only in the trigonometry question. Both the UK and the TR students committed flaws in algebra prerequisites in both questions. Flaws in cancellation were the common flaws made by the UK and the TR students, 72% and

56% respectively in the algebra question. In the trigonometry question, common factors was the most common flaw made by the UK and the TR students and was most conspicuous in the UK students' answers. Taking squares of both sides of an equation, difference of squares and cancellation were also common flaws. In the algebra question nearly twice as many UK students, than TR students, committed cancellation flaws.

In both the algebra and trigonometry questions, difference of two squares, common factors and completing the square were the main flaws in both the UK and the TR students' answers. The percentage of the UK students making flaws in the difference of two squares was almost twice that of the TR students. However, although the flaws in taking common factors out was performed mostly by the UK students in trigonometry questions, the percentage of the TR students making this flaw was more than twice that of the UK students.

1.4. The use of trigonometry in real world contexts

In this section, results of the trigonometry word problems (TWP) tests, interviews and verbal protocols of answering trigonometry word problems are analysed.

1.4.1. Trigonometry word problems test

The TWP test consisted of 6 questions (see Appendix C). All but one of the questions could be solved by using a two-dimensional (2-D) diagram. Question 4, could be solved by the help of a 2-D diagram of the 3-D situation. Only one right-angled triangle was required in the solution process in questions 1, 5 and 6. However, questions 2, 3 and 4 required more than one right-angled triangle. The TWP test had three basic aims:

- To reveal students' errors, misconceptions and difficulties in items.
- To investigate students' use of terminology and context.
- To investigate the interaction of visual and symbolic representations in answering trigonometry word problems.

Further investigation by interviews and protocol analysis of students' performance in TWP had two further aims:

- To investigate students' transformation of their mental representation.
- To investigate students' mental model in the solutions.

The trigonometry word problems test was conducted with 55 UK and 65 TR students.

1.4.1.1. Categorisations of students' responses

Students' answers were categorised as described on page 74 with the exception that they now apply to trigonometry word problems (see Table 4.12.).

Table 4.12. Initial categorisation of the trigonometry word problems test.

Groups of Answers	Abbr.	Description
Correct answers	**CA**	Appropriate diagrams and operations are applied to yield a valid solution; manipulations resulted in students writing the expected response.
Incorrect answers	**IA**	Inappropriate use of trigonometric functions or drawing
Partial answer	**PA**	The student approached the question in a correct manner but stopped after drawing diagrams
Non-attempted questions	**NAQ**	No response or simply a repetition of the question.

After this first iteration, all IAs and PAs were classified. The classification of IAs and PAs began with reading through each student's answers and analysing the flaws/reasons behind the results. In exploration of the IAs and PAs, a 'working model' of student flaws in answering trigonometry word problems was constructed. It is based on an analysis and categorisation of types of flaws in incorrect and partial answers students made in the TWP. The model has seven basic quasi-hierarchical levels, which were also utilised in coding the flaws seen in incorrect and partial answers of the trigonometry word problems test, reading, misuse of terminology, draw, match&labelling, identify function, develop mathematics and symbolic manipulation (see Table 4.13.).

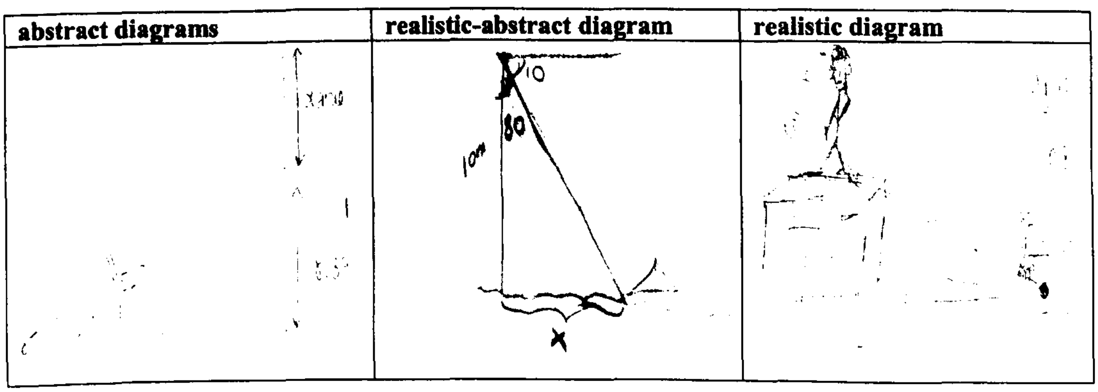
Table 4.13. Further categorisation of students' incorrect and partial answers in the TWP test.

Categories	Description	Code
Reading	Flaws that occurs in reading	**MR**
Misuse of Terminology	Concerns the misuse of words in TWP, e.g. vertical, elevation.	**MT**
Draw	Involves flaws which are done in the construction of diagrams, e.g. incorrect drawings.	**DNM***
		D
Match&labeling	Flaws that occurs in matching and labelling the diagram.	**ML**
Identify function	Flaws which are made in the symbolic part of the solution. Flaw concerning the misuse of trigonometric functions or trigonometric ratios values e.g. $\sin 30^0 = 1/2$ or finding them on right-angled triangle respectively	**TF**
		TR
Develop mathematics	Regarding the symbolic part of the solution, Flaws made during the construction of equations or other mathematical operations and equations depending on the diagram.	**DM**
Symbolic manipulation	Flaws which occurs throughout the application of the operations and manipulations, e.g. incorrect calculation, addition or multiplication.	**SM**
		MND*

** DNM is Diagram but no manipulation; MND is Manipulation but no diagram*

In their answers, students drew three types of diagrams which I call abstract, realistic-abstract and realistic diagrams. These diagrams are illustrated by students' drawings in Figure 4.14.

Figure 4.14. Abstract, realistic-abstract and realistic diagrams from students' answers.



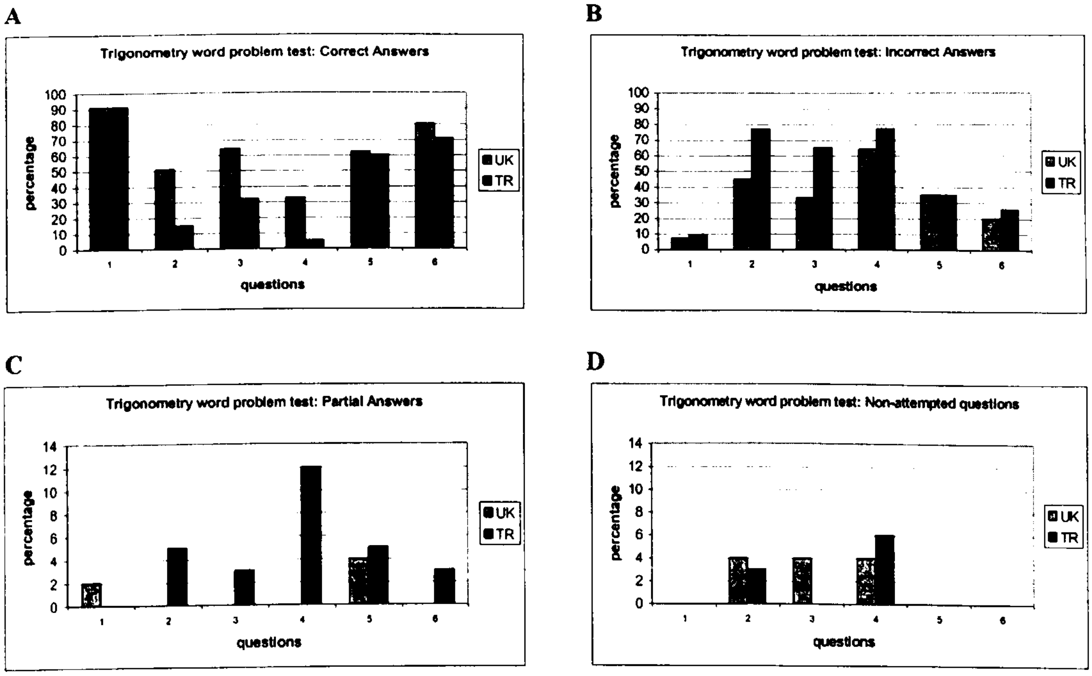
1.4.1.2. Students' performance in the trigonometry word problems test

This section first presents the results of the initial categorisations and then the flaws committed by students in incorrect and partial answers.

Initial categorisation of the all answers in trigonometry word problems tasks

The percentages of initial categorisations per question are illustrated in Figure 4.15.

Figure 4.15. The percentages of the UK and the TR students' initial categorisations in the TWP test.



The percentage of UK students giving correct answers was greater than the TR students in every question except question 1 (Figure 4.15.-A). Questions 1, 5 and 6 were answered correctly by 60% over of all students. In question 1, 91% of both the UK and the TR students gave correct answers. Questions 2, 3 and 4 required students to construct more than one right-angled triangle and to draw 2-D or 3-D diagrams. There were notable differences in the percentage of correct answers of the UK and the TR students. In question 3, the percentage of the UK students giving correct answers was twice that of the TR students, 64% and 32% respectively. The TR students' performance in questions 2,3, and 4 was poor compared to other questions. In question 4 only 5% of the TR students gave correct answers.

Figure 4.15.-B shows that the percentage of the TR students answering questions incorrectly was greater than the percentage of the UK students. In question 3, the percentage of the TR students giving incorrect answers to the question was nearly twice that of the UK students. 77% of the TR and 67% of the UK students answered question 4 incorrectly.

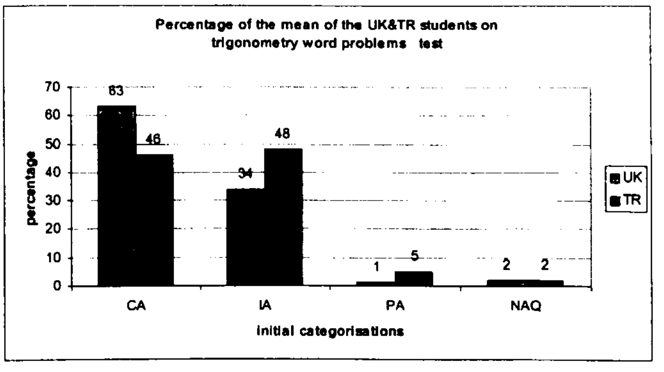
There were few partially answered or non-attempted questions (Figure 4.15.-C and D). Question 4 had the highest percentage, 12%, of partial answers and these were all from the TR students. Question 5 was the only question which both the UK and the TR students answered partially.

All students attempted questions 1, 5 and 6 and the remaining questions were attempted by at least 94% of each countries' students.

In general, the results reveal that nearly all of the UK and the TR students answered all TWP questions, that the UK students' performance was better that the TR students and that students from both countries gave less correct answers to questions 2, 3 and 4 than questions 1, 5 and 6.

Figure 4.16. displays the mean results of students over all questions. Nearly all of the questions were attempted by students from both countries. Similarly, few questions were partially answered by the UK and the TR students. The UK students gave more correct answers than the TR students.

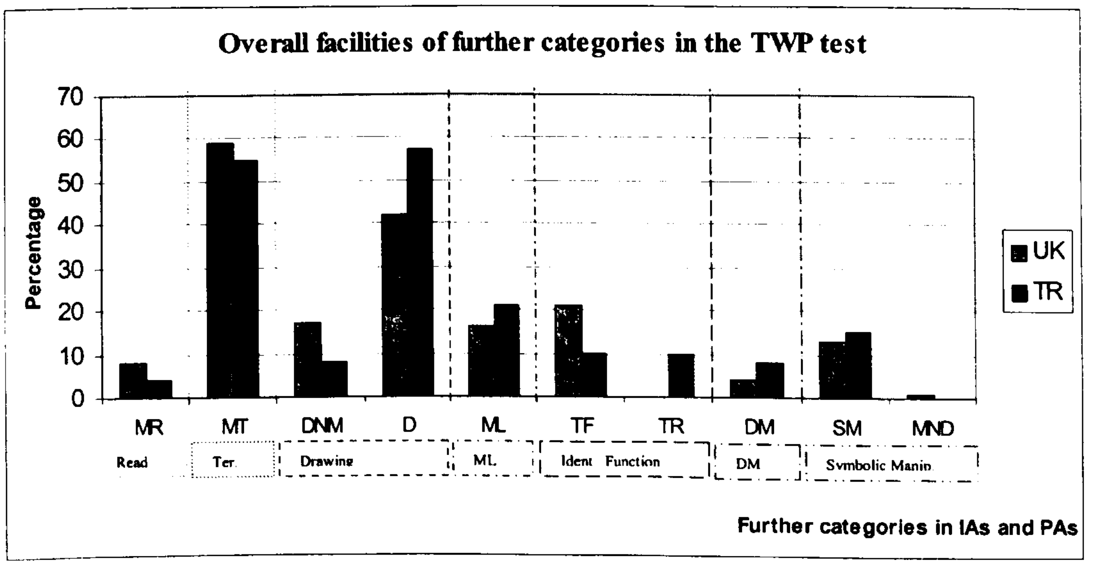
Figure 4.16. The UK and theTR students' performances in the TWP test.



Further categorisation of the incorrect and partial answers

The breakdown of incorrect and partial answers is displayed in Figure R.C 10-B and C. The results are arguably less significant than similar breakdowns from the trigonometry and algebra tests due to the low percentage of incorrect and partial answers in the TWP test. This low percentage also results in accentuated difference in Figure 4.17.

Figure 4.17. The UK and the TR students' further categories in incorrect and partial answers in the trigonometry word problems test.



χ^2 test statistics, df=8, were produced for all flaws seen in answering the TWP. In each case the table had the countries as rows and flaw categories as columns. p<0.001 for all flaws provides strong evidence that differences in the performance of students from the two countries did not arise by chance. [7]

The most common flaws were misuse of terminology and drawing. Both the UK and the TR students committed flaws of terminology in more than half of their answers. The UK students committed this flaw in slightly more answers than the TR students. Both countries' students committed flaws in using terminology in every question except 6. This flaw was most common in questions 2, 3, 4 and 5. The TR students committed flaws in drawing diagram in more answers than the UK students. Both the UK and the TR students committed flaws in drawing in every question but this was most common in questions 2, 3 and 4. The TR students committed flaws in labelling the diagrams in more question than the UK students and this was most common in questions 2, 3, 5 and 6. The UK students committed flaws in identifying trigonometric functions more often than the TR students did. This flaw was most commonly seen in question 1 and 6 (UK students) and question 5 (UK and TR students). [8]

1.4.2. Interviews with students

Interviews were conducted as described on page 79 with the same students who were interviewed following up the TWP test to understand the reasons for their responses. The questions used in the interview were either from the TWP test or very similar to the questions in the TWP test. I report below on the following: reading, terminology, drawing, matching and labelling, identifying function, mnemonics, developing mathematics, symbolic manipulations and procedure-how to solve TWP.

1.4.2.1. Reading

All UK and TR students explicitly or implicitly read the trigonometry word problems, some of them were very careful with their readings:

> $_{\text{UK\_}}$...Because that says an observer at the top of the tower find the distance between the men that could be a female...
>
> $_{\text{TR\_}}$... I am reading the question very carefully until I can visualise and draw the diagram...

1.4.2.2.Terminology

Almost all UK students and some TR students had problems with using angle of depression. They thought of it not as the angle below the horizontal but the angle next to vertical below the horizontal:

> $_{\text{UK\_}}$...I don't know if it's going down, depression...no I just don't understand whether it is right, angle of depression is due south...well it's due south, don't know if where the depressions is that side or that side...

---

[7] SM and MND were collapsed to meet χ^2 test conditions on minimum expected values.

[8] As mentioned in the footnotes 2 and 3 all results in this paragraph come from data not displayed in this thesis.

> TR_... I could not visualise south so I drew it like this ...the angle of depression is between the line and ground...

Some TR students also had difficulties with angles to the vertical and/or horizontal:

> ...I did not understand what is the angle of 10 below the horizon...

1.4.2.3. Drawing

All UK and TR students drew diagram to answer trigonometry word problems. They drew diagrams to make the problems understandable and clear:

> UK_...I'll draw it out to make it clearer...I drew the vertical wall first of all and then the point which was 25 meters from it and then I just from the point I did the angles of elevation from the top and the bottom of the wall to the statue...I usually draw diagrams...to picture it...
>
> TR_...I always draw diagrams because it makes thing easier...it is very helpful...after the reading I am trying to visualise and draw diagram...if I can have a draft diagram which represents the situation in the problem then I continue to solution...however I usually draw abstract diagrams...

Almost all UK students drew abstract diagrams whereas the TR students mostly drew realistic-abstract diagrams as well as abstract diagrams, the reason behind their drawing was explained as:

> UK_I prefer to draw abstract diagrams...because I just find it easier.
>
> TR_...I visualised it (rocket word problem, see Appendix C) mathematically and then drew a geometrical figure...I just wanted it to be helpful...I could draw this in other way around... diagram should make me approaching to the answer I do not care if it is realistic or abstract...
>
> TR_... most of my friend drew abstract diagrams but I cannot...to draw realistic diagrams helps me visually to understand and grasp what the problem wants...you do not need to draw perfectly it is important to have a picture which helps you...

Only, one of the UK students drew a realistic abstract diagram but then transformed it into an abstract one:

> ...I have transformed it into an abstract diagram...I have to do that because here what is talking that I cannot...Yeah I always draw a real picture then I can see what it is talking about and see exactly what I'm trying to find so that I can understand the question and then I think ..well I cannot solve it from this real picture because the triangles are all the wrong shape and things and then I have to draw another one maybe triangles....

When both the UK and the TR students were asked whether they could answer trigonometry word problems without drawing a diagram, the majority of them said 'no'. The students who said 'yes' emphasised that they would still draw a diagram:

> UK_You could but it just makes it easier. You look at your diagram and work it out...I usually draw diagram a in trigonometry word problems...because helps you...it is easy to use trigonometry with them...

TR_...no, no I first could not understand what is going on in the problem, but then I drew diagram and it helped me a lot...so diagrams are important...and they help you to construct mathematics as well...you combine diagram and symbolic manipulation part for the solution...

TR_...yeah I think I can solve without drawing a diagram but then the probability to reach an incorrect answer is going to be extremely high...

Some UK and TR students had difficulties with 'visualising/imagining' the situation in the problem, however they still drew a diagram:

UK_...(tower problem see Appendix C) I cannot think it in 3-dimension...

UK_...I was just thinking how I could draw it to get the south and west but I don't know...My mind above the towers here one line west there and one south there...

TR_...I am using my imagination to visualise how a yacht can be seen from the top of the cliff...but I cannot do it. I can imagine but I cannot draw it...I could not visualise the angle below the horizon...but I could not draw this...

TR_...It is very rare that I completely imagine the real picture in my mind it is always abstract-realistic in my mind but I draw abstract diagram on paper...for example boy and kite...I think a straight line in my mind and build everything on it...

Most UK and TR students stated that the difficult and important part of drawing diagrams is visualising but they think drawing is also important in answering trigonometry word problems:

UK_...difficult...It is not drawing it down it's visualising the problem...

TR_...difficult part is to visualise and draw then the remaining part is very easy...visualising is very important sometimes I cannot visualise well but I start to draw a diagram and my drawing helps me to visualise whole picture...

Some TR students stated that visualising and drawing are interconnected throughout their experiences:

...after reading to visualise problem is the most important part...I use my life experiences in visualising and drawing, for example with the rocket question I have read an article so I know how they move then I easily drew how rocket moves with the angle given in the question...

...first I thought since it said 15 degree to the vertical the angle should be in that way...but I have realised that rocket is going to be below the ground but it is ambiguous...there I have changed the angle other the way so that rocket can go up...

Although none of the interviewed UK and TR students had difficulties with 2-D drawings, some of them had problems with the 2-D drawing of 3-D situations and they always preferred to draw 2-D diagrams rather than 2-D representations of 3-D situations:

UK_...I find it hard to picture 3-dimensional word problems and I get confused a lot...I try to draw but I don't always get them right. I always try 2-D cause it makes it easier for me I think.

TR_...I drew it 2-D because drawing it piece by piece makes drawing and solution easier than 2-D of 3-D diagram can make...at the end when you combine the pieces you find the result...

1.4.2.4. Matching and labelling

All students in the interview matched the sides and angles of the diagrams they drew in answering TWP with the information given in the TWP and then they labelled the diagrams with the items given in the problem. Although almost none of the students verbalised during the matching and labelling, they did match and label and when they were asked 'what do you do after drawing a diagram?' they highlighted the importance of matching and labelling. All labelling was correctly done by both the UK and the TR students:

> $_{UK\_}$...I always label the sides to make the diagram and the situation in problem more clearer...

> $_{UK\_}$...diagram makes me see the values given in the question sort of, what they should be like so I can just look at it and sort it out...so I need to match my diagram with problem and identify the given values on diagram.

> $_{TR\_}$...after drawing the diagram the important thing is to match and label diagrams by the givens in the question correctly...I need this to do the mathematics correctly.

1.4.2.5. Identifying functions

Both the UK and the TR students identified trigonometric functions or other properties such as Pythagoras theorem or the sine rule to develop mathematics from the diagram. With some UK students, a correct diagram and labelling did not lead them to the correct answer because the function they identified was not correct, i.e some of them used sine instead of tangent.

1.4.2.6. Mnemonics

No TR but some UK students explicitly or implicitly used mnemonics, particularly SOHCAHTOA, in their answers:

> ...We were taught to remember it SOHCAHTOA so we remember the sine which is opposite over height...I did not write it on the paper, because I keep it in my mind...I looked which angles I had which sides I had then wrote it down...

1.4.2.7. Developing mathematics

None of the UK and the TR students interviewed had any difficulty developing the mathematics in either incorrect or correct answer in the interviews. After they identified the function they usually developed an equation with an unknown. Depending on what the question asked, the students developed different equations. Developing mathematics depended on the diagram and labelling the diagrams.

1.4.2.8. Symbolic manipulation

In symbolic manipulations, all UK but not TR students used calculators whereas some TR but no UK students used trigonometric tables. UK students did not make any mistakes in these calculations. Interestingly the TR students did not make considerable use of tables because:

...I did not use trigonometric table...I used to work with 30, 60 and 90...they are in my memory and result are more accurate with them...even when I wrote tan52-tan40 I thought whether I should apply formula or not instead of thinking their value...

1.4.2.9. Procedure-how to solve TWP

At the end of the interview, students from both countries were asked how they usually answered trigonometry word problems. Both the UK and the TR students highlighted the importance of reading, drawing diagrams and constructing the mathematical part to find the answer:

> $_{\text{UK\_}}$...Read through them, and draw a diagram and label it, and then find the side of the angle I've got to figure out and then just decide how I'd get that angle...
>
> $_{\text{TR\_}}$...first I have read it overall then I read it carefully part by part...I visualised the problem and then I drew it on paper...I must draw a diagram...then I use trig functions on the diagram and then I do manipulation to find the side or angle...

1.4.3. Verbal protocols with students

Concurrent verbal protocols of TWP were conducted, as described on page 82, to gain insight into students' thinking. In this section first the questions which are used in the protocol are introduced, then the coding of protocols is detailed and then the results of the verbal protocol are presented.

The problems used in the concurrent verbal protocol

Four trigonometric word problems were used in the concurrent verbal protocol sessions (see Appendix O). All problems were taken from the TWP test. The third and fourth questions were exactly the same as questions 3 and 4 in the TWP and the first and second questions were the same as the TWP questions but had different numerical values. The first and second questions were basic word problems, which asked students to find the length and an angle respectively. In the third problem, two right-angled triangles needed to be drawn. This problem was not as basic as the first and second one but was not considered particularly complex. The fourth problem involved three-dimensions. This was less familiar to the students, in relation to the other problems, but was considered a good example to observe and compare how students visualise a problem in three dimensions, and use trigonometric functions on it.

The coding of the protocols

Students' protocols were transcribed and then coded in terms of cognitive or observable actions described on page 83. The coding categories are presented in Table 4.14.

Table 4.14. Coding categories of the concurrent verbal protocols for the trigonometry word problems test.

Coding category	Description
Read	Reads the problem completely or partially
Recognise	Verbal response clearly indicate that students see a property or relationship
Visualise	Student verbalises that they have a picture of the situation (which could be partial or as a whole; abstract or realistic or neither of these) in the trigonometry word problems
Draw	Draws diagrams
Match	Matches the givens (known and unknowns) in the trigonometry word problem and the diagram constructed
Label	Names the sides of the constructed diagram
Identify function	Identifies the trigonometric function
Develop mathematics	Constructs the mathematical equations/expressions
Symbolic manipulation	Performs mathematical operations on numerical literal expressions
Result	Finds results or unknowns to reach the result
Review	Reviews the whole or partial answer

NB read, recognise, visualise, match, label, identifying function, develop mathematics and symbolic manipulations are coded 'implicit' if they are not verbalised but it is clear from subsequent verbal or written protocol data that read, recognise, visualise, match, label, identifying function, develop mathematics and symbolic manipulations had taken the place.

The results of the protocol analysis

As for algebra and trigonometry protocols, students' answers were first categorised using the initial categorisations. The results are presented in Table 4.15.

Table 4.15. The results of the UK ad TR students' performance on verbal protocol task.

Students	Trigonometry word problems in protocol			
	TWP1	**TWP2**	**TWP3**	**TWP4**
UKhs	CA	CA	CA	CA
UKjl	CA	CA	CA	IA
TRck	IA	CA	CA	IA
TRme	IA	CA	CA	IA
TRso	IA	CA	CA	IA
TRtfo	CA	CA	CA	CA

The protocols of students' answers were then divided into segments and coded.

The results of the coding categories in concurrent verbal protocol

Due to space restrictions and the length of this analysis and its diagrammatic representation, only one, either CA or IA, example of segment analysis of the protocols is presented for each problem.

UKjl and the second TWP in the protocol

UKjl solved the problem correctly. Figure 4.18. shows her diagram and calculations.

Figure 4.18. Scan of UKjl's answer to the second TWP of the protocol.

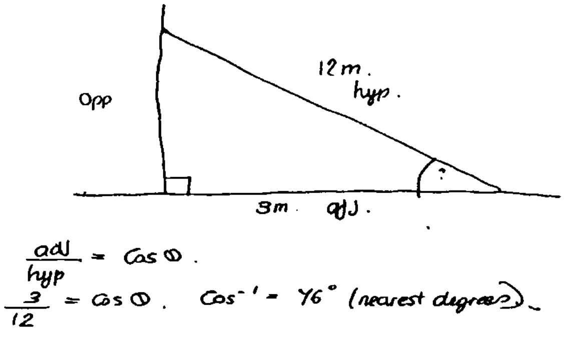


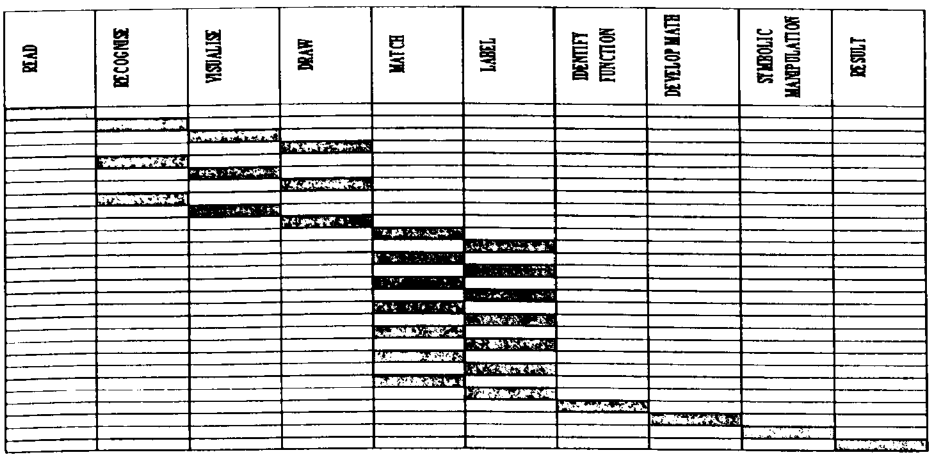
Table 4.16. shows the protocol segment analysis for this protocol and Figure 4.19. shows my schematic of this protocol.

Table 4.16. The segment analysis of UKjl's verbal protocol.

Stages in problem solving	***Segments of verbal protocol***
read	Ladder 12 metre long meets against a house and its lower end is 3 m from the bottom of the house wall
recognise	
visualise	so *vertical line for the house wall
draw	
recognise	
visualise	and a *straight horizontal line to the ground
draw	
recognise	
visualise	and the *ladder is leaning against the house
draw	
match	so I know that line 12 m long
label	so 12 m long
match	and distance from the ground is 3m
label	so 3m on horizontal line
match	and angle between the ladder and ground
label	so label angle
match	and then right angle
label	between house and the ground
match	so the hypotenuse
label	opposite to right angle
match	and the opposite is
label	the house wall
match	and the adjacent is
label	the ground
identify function	so use adjacent over hypotenuse to find the angle and the adjacent over hypotenuse up to is cos the angle
develop mathematics	and that is 3 over 12 equals $\cos\theta$
symbolic manipulation	so inverse of $\cos\theta$
result	is 76^0 is to the nearest degree.

** the moment student starts to draw*

Figure 4.19. The schematic of the protocol UKjl for the second TWP.



UKjl read the problem aloud, focused on the vertical (recognise, visual) and drew the vertical. She then repeated this for the horizontal and for the ladder (the hypotenuse). She then repeatedly matched and labelled the ladder, the horizontal, the unknown angle, the right angle, the hypotenuse, the opposite length and the adjacent length. She then identified the function as cosine and developed the mathematics, manipulated the expression to obtain the inverse cosine and calculated the angle (the result) to the nearest degree.

Figure 4.19 simply presents a schematic of these stages. A diagonal pattern is clearly visible. Similar diagonal patterns were produced for all protocol segment analyses with the same variation on iterative sub-patterns, e.g. recognise-visualise-draw, depending on the question and individual students' approaches.

UKhs and the third TWP in the protocol

UKhs solved the problem correctly. Figure 4.20. shows her diagram and calculations.

Figure 4.20. Scan of UKhs's answer to the second TWP of the protocol.

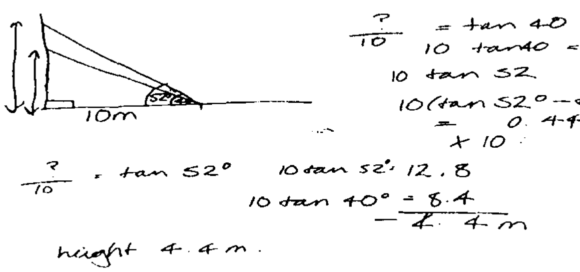


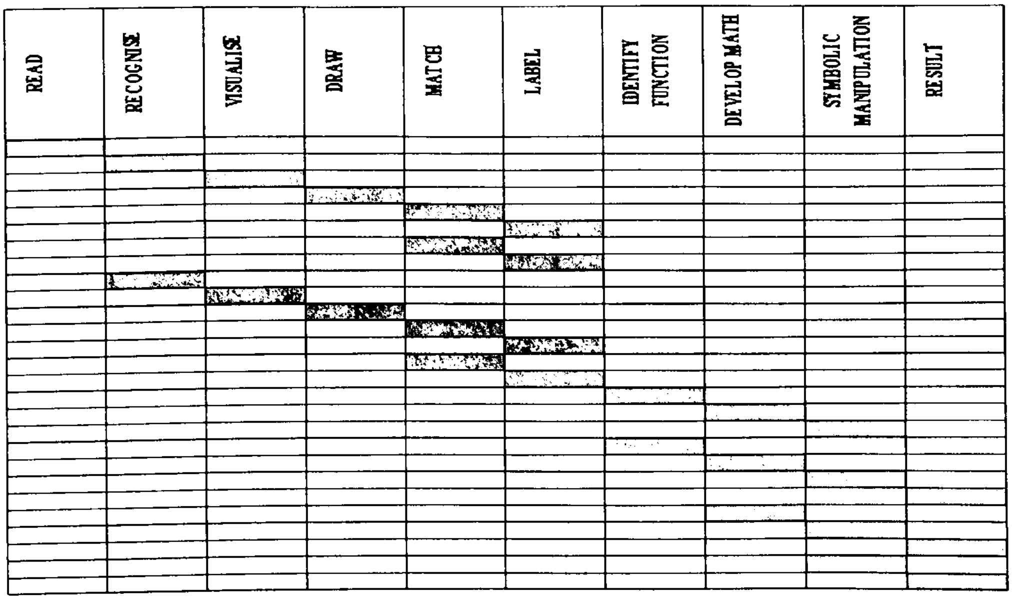
Table 4.17 shows the protocol segment analysis and Figure 4.20. shows my schematic of this protocol.

Table 4.17. The segment analysis of UKhs's verbal protocol.

Stages in problem solving	***Segments of verbal protocol***
read	Next question
recognise	
visualise	*vertical wall
draw	
match	so there will be a right angle
label	between the wall and the ground
match	
label	and these two angles
recognise	so I'm gonna have to make *the diagram
visualise	into two triangles which has got 40^0 and 52^0 and got elevation
draw	
match	so I'm gonna gave to work out and the *whole length of the side and then take away the side of the shortest triangle so get the answer so start with large triangle
label	(drawing the arrows next to the corresponding sides)
match	and that is bottom length
label	10 m I think yeah and
identify function	so it's going to be the adjacent and the opposite
develop mathematics	so it's gonna be something over 10 is the tan of 40
symbolic manipulation	so need to find 10 times the tan of 40 and then I need to take that away from the 10 times the tan of 52 so…it is gonna be 10 out as a common factor and put it in brackets and work out tan 52-tan 40 on the calculator (C) get point 44 that times that by 10 I don't think that is right
identify function	I'm gonna start again and tan bit
develop mathematics	right I'm gonna work out 10 time 52
symbolic manipulation	which is (using calculator)
result	12 no it doesn't sound right yeah does 12.8
develop mathematics	and then work out 10 tan 40
symbolic manipulation	(using calculator)
result	which is 8.4
symbolic manipulation	and take away yeah I was right in the first place
result	it is 4.4 m it is right now.

* *the moment student starts to draw*

Figure 4.21. The schematic of the protocol UKhs for the third TWP.

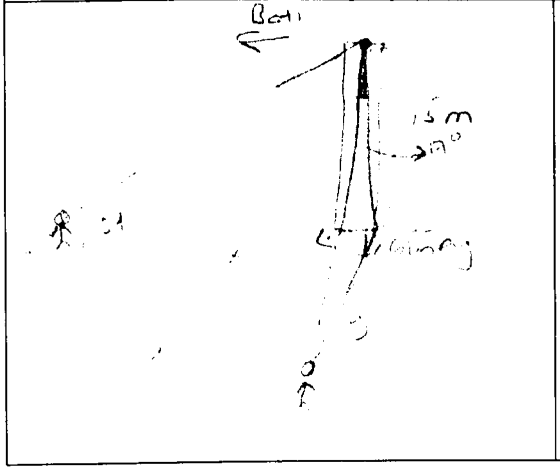
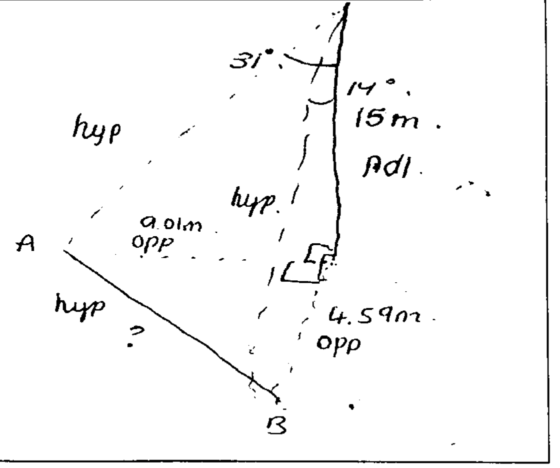


UKhs did not read the problem aloud. She focused on the vertical wall (recognise, visualise) and drew the vertical. She then repeatedly matched and labelled the right angle and known angle. She then focused on the two angles of elevations (recognise, visualise) and drew the line which formed two right-angled triangles. She then repeatedly matched and labelled the length of the wall opposite to the two given angles and the length of the adjacent side of the given angles. She then identified the function as a tangent for the short length of the wall and developed the mathematics, then manipulated the symbolic mathematics to obtain the unknown side and calculated the length of the statue (the result). She was not sure about this result and stated that she was going to start again. She identified the function implicitly as a tangent for the long length of the wall, developed the mathematics, manipulated the expression to obtain the long length of the wall and calculated it. She then developed the mathematics of the already identified tangent function for the short length of the wall and manipulated the expression to obtain the short length of the wall and calculated it. She then manipulated these lengths to obtain the unknown length and calculated the length of the statue (the result).

As explained for the Figure UKjl, a diagonal pattern is clearly visible. And since this is a different question and a different individual the diagonal pattern includes different iterative sub-patterns, e.g. recognise-visualise-draw-match-label, in Figure 4.21.

The fourth question in the protocol was the hardest one for students from both countries. Two sorts of drawing were seen in students' protocols 2-D representation of 3-D problem (see Figure 4.22) and 2-D drawing as in the next example of protocol (see Figure 4.23.).

Figure 4.22. An example of 2-D representation of 3-D problem.

TR student in TWP verbal protocol question 4	UK student in TWP verbal protocol question 4
	

The patterns in the schematic of the students' protocol, which included a 3-D diagram, included some variations on iterative sub-patterns, e.g. read-recognise-visualise-draw and matching-labelling and identifying functions-develop mathematics-symbolic manipulations-(result), depending on individual student's approaches, such as drawing the diagram part-by-part or as a whole.

The previous examples of protocols were correct answers, however, there were incorrect answers in protocols. These answers were investigated in the same way as the correct answers. The following protocol is an example of an incorrect answer.

TRck and the fourth TWP in the protocol

TRck solved the problem incorrectly. Figure 4.23. shows her diagram and calculations.

Figure 4.23. Scan of TRck's answer to the fourth TWP of the protocol.

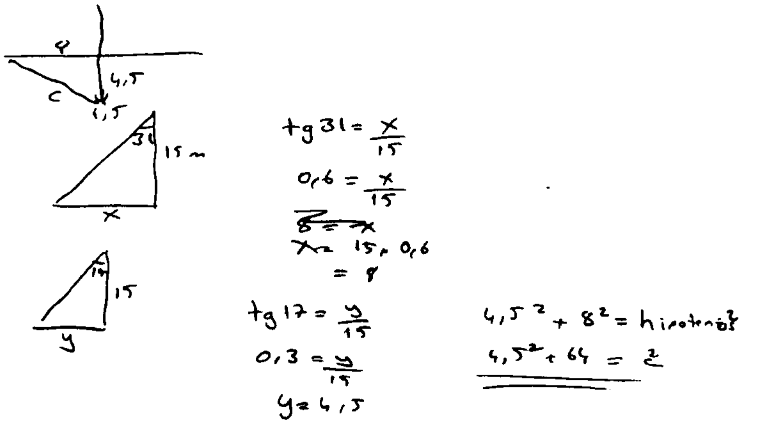


Figure 4.24. shows my schematic of this protocol and Table 4.18. shows the protocol segment analysis for TRck.

Figure 4.24. The schematic of the protocol TRck for the fourth TWP.

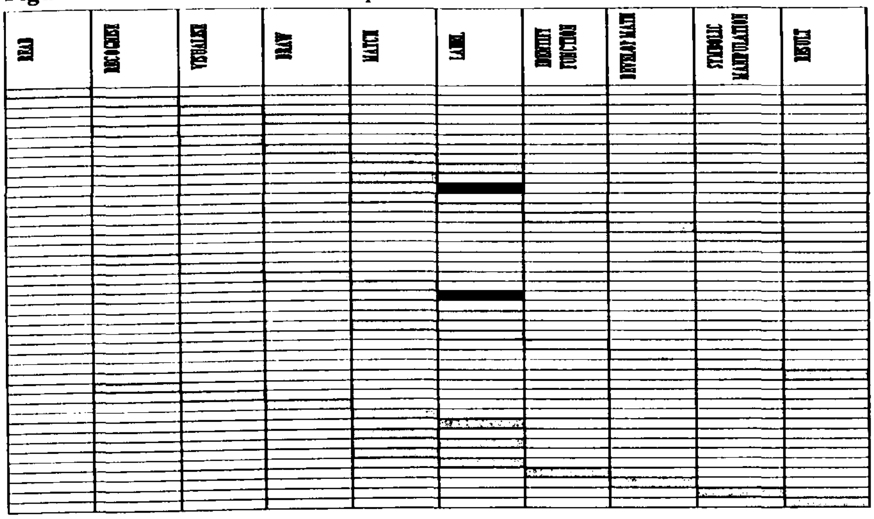


Table 4.18. The segment analysis of TRck's verbal protocol

Stages in problem solving	***Segments of verbal protocol***
read	An observer at the top of a tower of height 15 m. sees a man due west of him at an angle of depression 31^0. He sees another man due south at an angle of depression 17^0. Find the distance between the men?
recognise	Directions are..directions are 90^0 perpendicular to each other
visualise	
draw	*
recognise	The man at west…
visualise	First 31^0
draw	*
match	the height is
label	15 meters
match	Since it makes an angle of
label	31^0
match	if the opposite side
label	is named as x
identify function	I can find x by tan 31
develop mathematics	tan 31 is x over 15
symbolic manipulation	and since the nearer value of tan 31 is 0.6..15..8..15..6.. x is 15 times 0.6
result	x is going to be 8
recognise	The man at south with the angle of depression
visualise	
draw	*
match	Since the angle of depression for the other man is
label	17^0 and
match	the height is
label	15 meters
match	to find the distance with the other men
label	
identify function	I can use the tan 17
develop mathematics	..y over 15..
symbolic manipulation	the nearer value of tan 17 is 0.3
result	so y is 4.5
recognise	..eeh..since the angle between the directions is 90^0
visualisation	
draw	
match	the distance to the man due south
label	is 4.5 meters
match	and the distance to the man due west
label	is 8 meters
match	The distance between two men
label	is the hypotenuse
Identify function	and can be found from the sum of the square of two sides
develop mathematics	4.5 squared plus 8 squared
symbolic manipulation	
result	is the length of the length of the hypotenuse which is the distance between two men. The result is that

* *the moment student starts to draw*

TRck read the problem aloud, and focused on the directions south and west as perpendicular (recognise, visualise) and drew a cross representing the directions south, west, east and north. He then focused on the man due west to the observer (recognise, visualise) and drew a right-angled triangle. He then repeatedly matched and labelled the height of the tower, angle of depression (which was where he made a mistake) and the unknown side. He then identified the function as a tangent and developed the mathematics, manipulated the expression to obtain the distance between the man due west and the tower. He then focused on the man due south to the observer (recognise, visualise) and drew a right-angled triangle. He then repeatedly matched and labelled the angle of depression (which was where he made another mistake), the height of the tower, and the unknown side. He then identified the function as a tangent and developed the mathematics, manipulated the expression to obtain the distance between the man to the south and the tower. He then focused on the cross he drew and then drew a hypotenuse by combining south and west, which formed a right-angled triangle. He then repeatedly matched and labelled the length he found between the tower and the man due west and the tower and the man due south and the hypotenuse, which is unknown. He then focused on the Pythagorean property (identify function) and developed the mathematics, manipulated the expression to obtain the square of the hypotenuse and left the result as a numerical expression.

As in the previous two examples of protocols (Figure 4.19. and 4.21.), a diagonal pattern is clearly visible. Since this is a different question and a different individual, the diagonal pattern includes different iterative sub-patterns, e.g. identify function-develop mathematics-symbolic manipulation-result, in Figure 4.24.

In this protocol, the student made two flaws in labelling the diagrams. The flaws are shown by black cells in the schematics of the protocol (Figure 4.24.) corresponding to the grey highlighted cells in Table 4.18. However, it did not affect the pattern in the schematics of the protocol. Even though the student committed flaws in labelling. This student, as for other students who made mistakes in protocol sessions, did not realise that he had made a mistake.

1.5. The use of trigonometry in the context free questions

In this section, results of the trigonometric functions on right-angled triangles (TORT) test and subsequent interviews and verbal protocols are analysed.

1.5.1. Trigonometric functions on right-angled triangles test

The trigonometric functions on right-angled triangles test (see Appendix D) had parallel questions to the trigonometry word problem test, i.e TORT included context free forms of questions in the TWP. TORT had 11 items (5 questions but questions 1, 4 and 5 were subdivided into three items each). Questions 1 and 5 contained rotated triangles and the trigonometric functions sine and tangent (or cotangent) could be used to obtain answers respectively. Question 1 concerned finding the length of a side of a right-angled triangle and question 5 asked students to find an angle. There were more than two right-angled triangles in the geometric figures in questions 2 and 3. Question 4 presented a 2-D diagram of a 3-D rectengular block. The TORT test aimed:

- At revealing students' errors, misconceptions and difficulties in items.
- To investigate how students' visual and symbolic representations interact in the solution process.
- To explore the effect of context-free questions on students' performance of trigonometry word problems.

The trigonometric functions on right-angled triangles test was conducted with 55 UK and 65 TR students.

1.5.1.1. Categorisations of students' responses

Students' answers were categorised as described on pg. 74 with the exception that I work with trigonometric functions on right-angled triangles (see Table 4.19.).

Table 4.19. Initial categorisation of student responses to the items in the TORT test.

Groups of Answers	Abbr.	Description
Correct answers	**CA**	Appropriate trigonometric functions, ratios and algebraic properties are applied to yield a valid solution; manipulations resulted in students writing the expected response.
Incorrect answers	**IA**	Inappropriate use of trigonometric functions, ratios and/or miscalculations
Partial answer	**PA**	The student approached the question in a correct manner but stopped short of the expected answer
Non-attempted questions	**NAQ**	No response or simply a repetition of the question.

After this first iteration, all IAs and PAs were classified. The classification of IAs and PAs began with reading through each student's answers and analysing the flaws/reasons behind the results. In the exploration of the IAs and PAs, a 'working model' of students' flaws in the TORT tasks was constructed based on an analysis and categorisation of the types of flaws in incorrect and partial answers students made in the TORT test. The model has five basic quasi-hierarchical levels, which were also utilised in coding the flaws seen in incorrect and partial answers of trigonometric functions on the right-angled triangles test, reading, terminology, identify function, develop mathematics and symbolic manipulation (Table 4.20.).

Table 4.20. The categories seen in the incorrect and partial answers in the TORT test.

Levels	Description	Code
Reading	Includes flaws that occurs in reading	**MR**
Misuse of Terminology	Concern purely with the misuse of the definitions, e.g. the sum of interior angles of a triangle is 180^0	**MT**
Identify function	concerned with the misuse of trigonometric functions (TR), trigonometric ratios (TR) and the Pythagorean property(P)	**TF**
		TR
		P
Develop mathematics	Concerned with the construction of equations or other mathematical operations and equations depending on the diagram	**DM**
Symbolic manipulation	Concerned with flaws which occurs throughout the application of the operations and manipulations, e.g. incorrect calculation.	**SM**
		DA*

* *DA is direct answer (without any operation on the paper)*

1.5.1.2. Students' performance in the TORT test

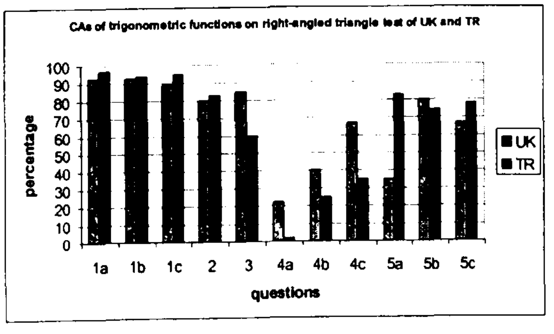
Overall, both the UK and the TR students performed well in the TORT test. They produced very few incorrect and/or partial answers, but to retain uniformity of data presentation, I will briefly present the results as I did with the other three tests. First, the results of initial categorisations and then the flaws committed by students in incorrect and partial answers are presented in this section.

Initial categorisation of the all answers in using trigonometric functions on right-angled triangles tasks

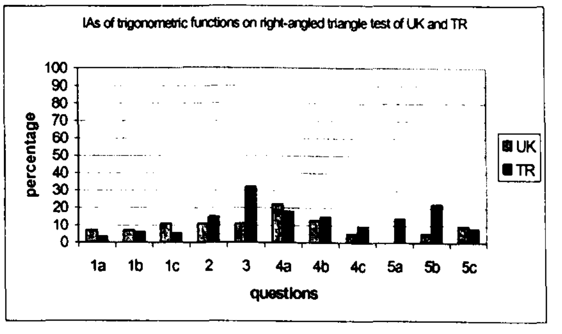
The percentages of the initial categorisations for each question are illustrated in graphs A-D in Figure 4.25.

Figure 4.25. The percentages of the UK and the TR students' initial categorisations in the TORT test.

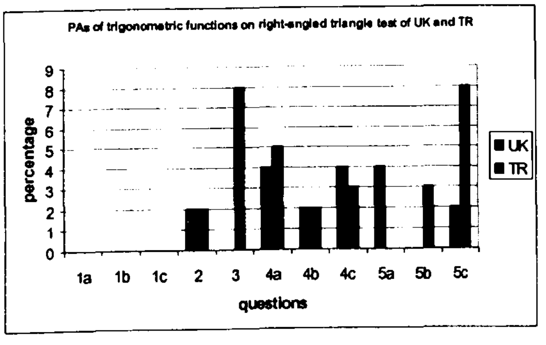
A



B



C



D

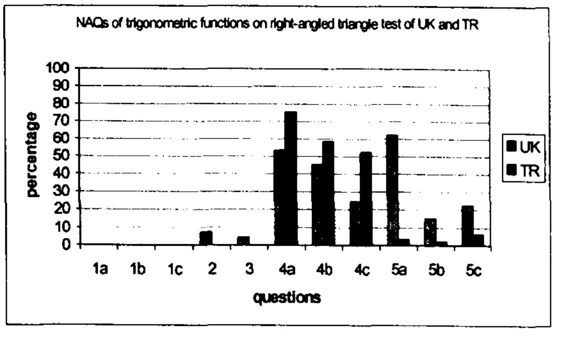


Figure 4.25.-A shows that with the exception of questions 3-5a, most questions were answered correctly by more than 65% of both the UK and the TR students. The difference between the percentage of the UK and the TR students was very small in questions 1-2 and 5b-5c. The percentage of UK students giving correct answers to questions 3-4c was greater than that of the TR students. In question 4c the percentage of UK students giving correct answers was nearly twice that of the TR students. On the other hand, 83% of the TR students answered the question 5a correctly, which is more than two times that of the UK students.

Figure 4.25.-B shows that in the majority of questions, less than 15% of students from both countries gave incorrect answers. The UK students did not give any incorrect answers to question 5a. Question 4a was answered incorrectly by more than 15% of both the UK and the

TR students. 32% of the TR students answered question 3 incorrectly, the highest percentage across any of the questions and approximately three times that of the UK students. Similarly, in question 5b, the percentage of the TR students giving incorrect answers was more than four times that of the UK students.

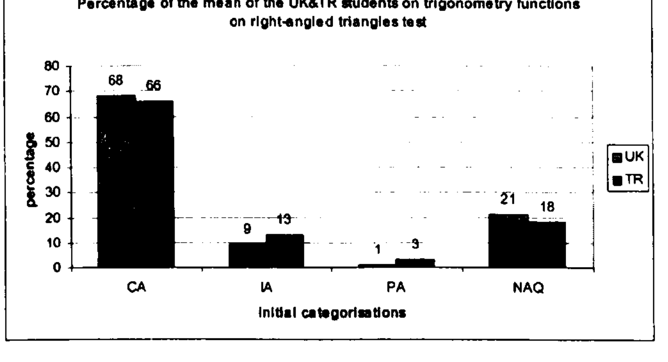
The results in Figure 4.25.-C shows that the percentage of questions answered partially by the UK and the TR students was less than 5% for the majority of the questions. Both the UK and the TR students gave no partial answers to questions 1a-1c. Questions 3 and 5c were answered partially by more than 5% of the TR students.

The results in Figure 4.25.-D reveal that every UK and TR student attempted questions 1a-1c. The percentage of the TR students who did not attempt questions 4a-4c was greater than that of the UK students. More UK students than the TR students made no attempt at questions 5a-5c.In question 5a the percentage of UK students giving no answer was more than twenty times that of the TR students.

In general, both countries' students attempted most of the questions and answered a high percentage of them correctly. Both the UK and the TR students did much better in questions 1a-1c, in which the unknown to be found was an length of a side, compared with questions 5a-5c, in which the unknown to be found was the angle. Both countries' students did not perform well in question 4, which presented a 3-D problem. The UK and the TR students gave more incorrect answer to questions 2 and 3, which contained more than one right-angled triangle.

Figure 4.26. displays the mean overall results to these questions. There was no notable difference between the percentages of the UK and the TR students' means in each initial categorisation. A high percentage of the questions were answered correctly by both the UK and the TR students. Even though the UK students did not attempt 21% of the questions, which is more than that of the TR students, UK students gave more correct answers than the TR students. The percentage of incorrect answers given by the TR students was greater than that of the UK students. A low percentage of the questions were partially answered by the UK and the TR students.

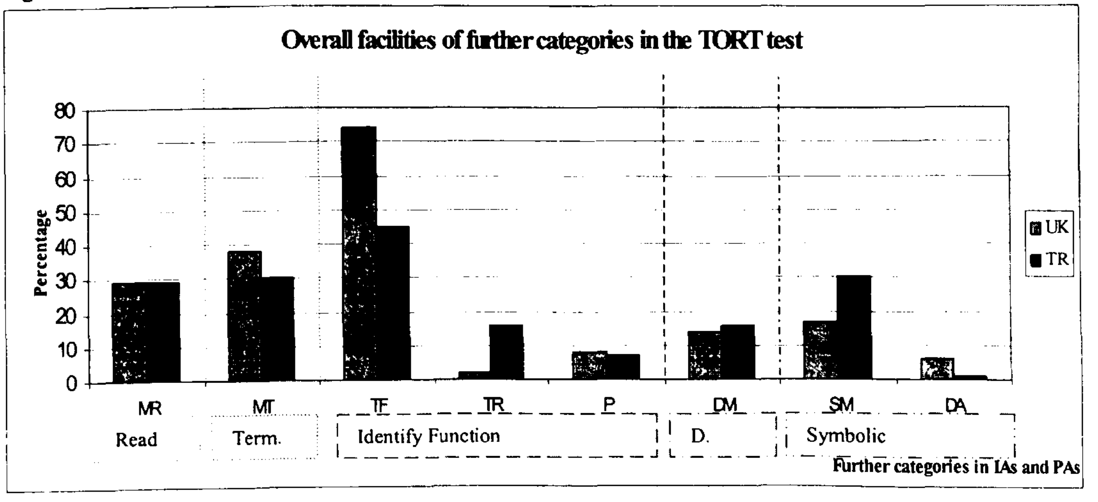
Figure 4.26. The UK and the TR students' overall performance in the trigonometric functions on right-angled triangles test.



Further categorisation of the incorrect and partial answers

The percentages of the further categories in incorrect and partial answers of trigonometric functions on right-angled triangles test are displayed in Figure 4.27.

Figure 4.27. The UK and the TR students' further categories in incorrect and partial answers in the trigonometric functions on right-angled triangles test



χ^2 tests statistics, df=2 and 6[9], were produced for identify function and all flaws seen in answering TWP respectively. In each case, the table had the countries as rows and flaw categories as columns. p<0.003 for identify function and p<0.004 for all flaws, which provides strong evidence that differences in the performance of students from the two countries did not arise by chance.

The most common flaw committed by both the UK and the TR students was identifying trigonometric functions. The percentages of the UK and TR students who committed flaws in identifying the trigonometric functions were more than 40%. The UK students committed this flaw in more questions than the TR students did. Flaws in trigonometric functions were committed in more than 60% of the UK students' answers to every question which was much more than the TR students except question 3 in which almost 70% of students from both countries committed this flaw. Both the UK and the TR students made flaws at using terminology, which were the second most common flaws seen across all questions. The flaws of terminology were most common in question 4 and it was also committed by the UK students in question 2 and by both the UK and the TR students in question 3. Students from both countries also committed flaws at reading the questions, which was seen in almost 30% of answers and mostly seen in question 4. Finally, 30% of the TR students, that is twice that of the UK students, committed flaws at doing symbolic manipulations. Both the UK and the TR students committed flaws at doing symbolic manipulations in question 5c, interestingly, in the questions 1b, 1c, 5a and 5b solely the TR students made this flaw.[10]

[9] SM and DA were collapsed to meet χ^2 test conditions on minimum expected values.
[10] As mentioned in the opening of this chapter, space constraints prevent me displaying all data analysed. All results in this paragraph come from data not displayed in this thesis.

1.5.2. Interviews with students

Interviews were conducted as described on page 79 with the same students who were interviewed following up the TORT test to understand the reasons for their responses. Both the UK and the TR students were very successful in the TORT test in that they answered a low percentage of the questions incorrectly. Interviews confirmed the findings in the analysis of students' flaws in the TORT test that the most common flaw was in identifying the trigonometric function. Students from both countries stated that the 3-D question was the most difficult one and that the basic problems were easy:

> it is so easy...we used to solve these sort of questions...it is only finding trig function and doing manipulations...if I do mistake it is because of my carelessness.

The TR students used 30^0, 60^0, 90^0 right-angled triangles instead of trigonometric tables to find the value of the trigonometric functions of these angles. In the rest, they used trigonometric tables and calculated the values manually. They said:

> ...we do not use to use the angles like 48, 17...we used to use the angles 30, 45, 60, 90 and I know the trig functions' values at these angles...it is very easy to do calculations with these angles as well.

However UK students used calculators for their calculations and they did not have any difficulties, as TR students sometimes did in their manual manipulation calculations. The majority of UK students used the mnemonic SOHCAHTOA in the interviews. They sometimes showed it by re-labelling the diagrams using mnemonics. Some of them did not appear to use a mnemonic but when they were asked whether they used a mnemonic or not, they said they used SOHCAHTOA, which was in their mind and so they did not need to write it out on paper.

1.5.3. Verbal protocol with students

Concurrent verbal protocols of TORT problems were conducted, as described on page 82, to gain insight into students' thinking. In this section first the questions which are used in the protocol are introduced, then the coding of the protocols is detailed and the results of the protocol are presented.

The problems used in the concurrent verbal protocol

There were four questions in the TORT protocols (see Appendix P). The first question had three sub-questions (a, b and c). TORT sub-question 1a and questions 3 and 4 were context-free parallel questions to the first, third and the fourth TWP protocol questions respectively. All problems were taken from the TORT test. The third and fourth questions were identical to questions three and four in the TORT test. The first and second questions were the same as TORT questions one and two but had different numerical values. The first question was a basic 'find the length' question. All questions involved the sine function. All questions used rotated right-angled triangles. In two of the right-angled triangles students needed to find the length of one vertical side and the hypotenuse and in the other one, the angle. In the second and the third problem there were more than two right-angled triangles to be used and so trigonometric functions were to be used twice. This problem was not as easy for students as the first one, although the principles are the same. The fourth question contained a three-dimensional diagram and students were to find the lengths of three sides. This was more difficult for the students compared to the other problems but it was a useful example to observe and compare how students visualise problems three dimensionally and use trigonometric functions.

The coding of the protocols

Students' protocols were transcribed and coded in terms of cognitive or observable actions as described on the page 83. The coding categories are presented in Table 4.21.

Table 4.21. Coding categories of the concurrent verbal protocols for the trigonometric functions on right-angled triangles test.

Coding category	Description
Read	Reads the problem completely or partially
Recognise	Verbal response clearly indicate that students see a property or relationship
Match and Label	Students match the diagram to the SOHCAHTOA or opposite-adjacent-hypotenuse and label the sides with respect to their knowledge and/or findings in the answering process
Identify function	Students identify the trigonometric function
Develop math	Constructs the mathematical equations/expressions
Symbolic manipulation	Performs mathematical operations on numerical and literal expressions
Result	Finds results or unknowns to reach the result
Review	Reviews the whole or partial answer

NB read, recognise, match and label, identifying function, develop mathematics and symbolic manipulations are coded 'implicit' if they are not verbalised but it is clear from subsequent verbal or written protocol data that read, recognise, match and label, identifying function, develop mathematics and symbolic manipulations has taken the place.

The results of the protocol analysis

As done in analysis of other protocols, students' answers were first categorised using the initial categorisations. The results are presented in Table 4.22.

Table 4.22. The UK and the TR students' performances on verbal protocol task.

Students	**Trigonometry word problems in protocol**					
	TORT1a	**TORT1b**	**TORT1c**	**TORT2**	**TORT3**	**TORT4 (HB,HF,BF)***
UKhs	CA	CA	CA	CA	CA	CA,CA,CA
UKjl	CA	CA	CA	IA	CA	CA,CA,CA
TRck	IA	CA	CA	CA	CA	IA,IA,CA
TRme	CA	CA	CA	CA	CA	IA,IA,IA
TRso	CA	CA	CA	CA	CA	NAQ,CA,CA
TRtfo	CA	CA	CA	CA	CA	NAQ,CA,CA

** the letters in the brackets shows the length students should find*

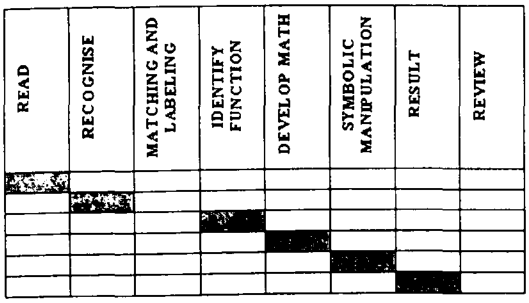
The protocols of every answer students gave was divided into segments and coded.

The results of the coding categories in concurrent verbal protocol

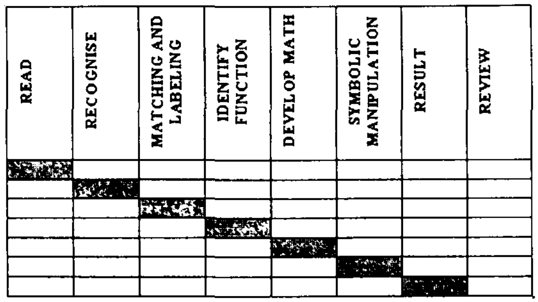
Figure 4.28. shows the coded segments of the UK and the TR students' protocols in answering the TORT sub-question 1a. Due to space restrictions, only the segment analysis of one student, TRme, is presented (see Table 4.23.). This student's work is also illustrated (see Figure 4.29.).

Figure 4.28. The schematic of the UK and the TR students' protocol for the first sub-question of question 1 in trigonometric functions on right-angled triangles protocol.

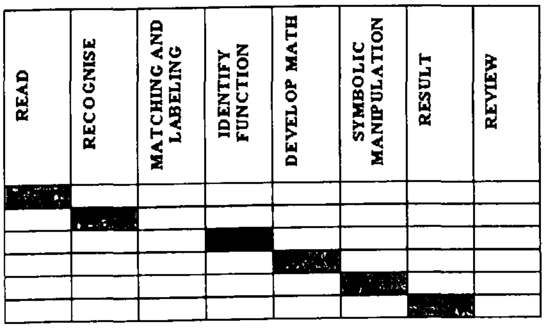
UKhs



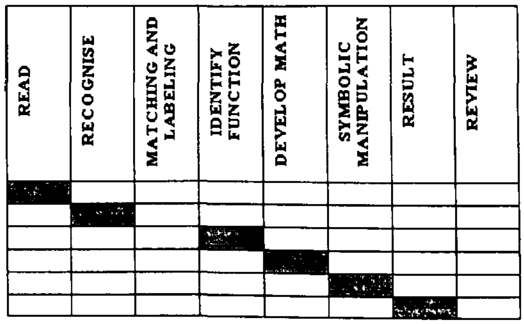
UKjl



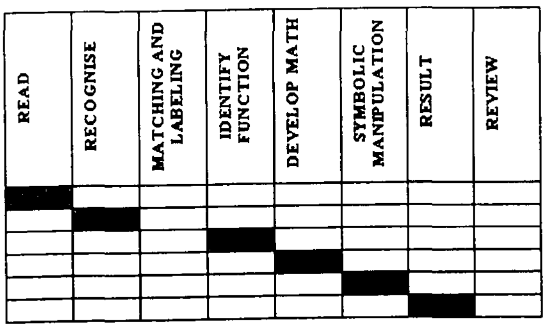
TRck*(dark cell shows the step where the student made a mistake)*



TRme



TRso



TRtfo

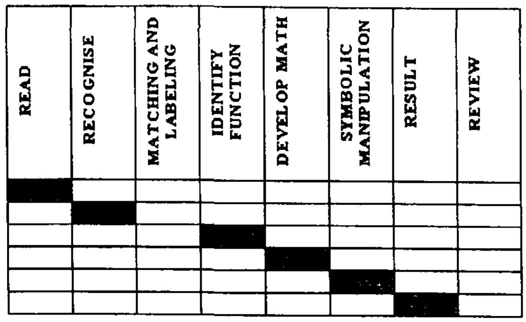


Table 4.23. The segment analysis of TRme's verbal protocol.

Stages in problem solving	***Segments of verbal protocol***
read	hypotenuse is asked.
recognise	there is a right-angled triangle
identify function	sin 60 is used here. Sin 60 is equal to opposite over hypotenuse, opposite side is 8 cm. over hypotenuse which is x.
develop mathematics	sin 60 is square root of 3 over 2 8 over x,
symbolic manipulation	then square root of 3 times x is equal to 16.
result	Then, hypotenuse, x, is 16 square root of 3 over 3.

Figure 4.29. Scan of TRme's answer to the first sub-question of question 1 in the trigonometric functions on right-angled triangles protocol.

$$\sin 60 = \frac{8cm}{x}$$

$$\frac{\sqrt{3}}{2} = \frac{8}{x}$$

$$\sqrt{3}x = 16$$

$$x = \frac{16\sqrt{3}}{3}$$

The student clearly read the statement and recognised the diagram. He then focused on the sin function and identified the sin 60^0 on the diagram by saying sin 60^0 is opposite over hypotenuse, namely, 8 over x. The student then developed the mathematical expression by constructing the equation $\frac{\sqrt{3}}{2}=\frac{8}{x}$ and then solved the equation to get the result $\frac{16\sqrt{3}}{3}$.

1.6. Comparison of students' performance in trigonometry word problems and trigonometric functions on right-angled triangles tests

In this section, students' performance on trigonometric functions in context and context-free questions are compared. The TWP and TORT tests are analysed for comparative purposes in terms of general comparison, common flaws made by students in the two tests and the parallel questions. Parallel questions will be discussed in detail later.

1.6.1. General comparison

Both countries' students' performances were better in the TORT test than in the TWP test. In both tests, the UK students' performance was better than the TR ones. Almost all of the UK and the TR students answered the questions in the TWP test. However, non-attempted questions account for almost 20% of responses in the TORT test. In both the TWP and the TORT tests, the percentage of partial answers given by the UK and the TR students was less than 5%, and virtually none of the the UK students answered the questions in the TWP and the TORT tests partially. The percentage of correct answers given in the TORT test by both the UK and the TR students was greater than the respective percentage in the TWP test. Although the percentage of the correct answers given by the UK students to questions in the TORT tests was slightly greater than the TR students, that difference was far much bigger in the TWP test. Interestingly, the percentage of incorrect answers in the TWP tests given by the TR students was slightly greater than the percentage of correct answers and, at the other extreme, the percentage of correct answers given by the UK students was almost twice that of incorrect answers. A striking feature of the overall results was that nearly one tenth of both the UK and the TR students answered the TORT questions incorrectly. The percentage of the UK and the TR students who partially answered the questions in both the TWP and the TORT tests were very low.

1.6.2. Comparison of the flaws

The flaw categories seen in both the UK and the TR students' incorrect and partial answers in the TWP and the TORT tests were presented in sections 1.4.1.1 and 1.5.1.1 respectively. There is some overlap in the flaw categories of the TWP and the TORT tests. The similarities and differences are shown in Table 4.24.

Table 4.24. The comparison of the flaw categories in the TWP and the TORT tests.

	Tests		
	TWP-TORT	**TWP**	**TORT**
Flaw categories	Read: MR Terminology: MT Identify function; TF, TR Develop mathematics: DM Symbolic manipulations: SM	Drawing: DNM, D Matching & Labelling: ML Symbolic manipulations: MND	Identify function: P Symbolic manipulations: DA

Reading, terminology, identifying functions and symbolic manipulations are the main flaw categories committed by both the UK and the TR students in both the TWP and the TORT tests. Students from both countries committed flaws in reading in the TORT test much more than they did in the TWP test. However, both the UK and the TR students committed flaws in terminology in the TWP test more than they did in the TORT test. The UK and the TR students committed flaws in identifying trigonometric functions in 75% and 45%, respectively, of the TORT questions (almost four times that of the TWP). In both tests students from both countries committed flaws in developing mathematics in a much lower percentage of the questions and there was not a big difference between them. Both the UK and the TR students committed flaws in symbolic manipulation in less than 30% of the questions in both the TWP and the TORT tests. However, they both committed more flaws in doing symbolic manipulations in the TWP test than they did in the TORT test.

1.6.3. Parallel questions in the TWP test and TORT test

1.6.3.1. Parallel questions

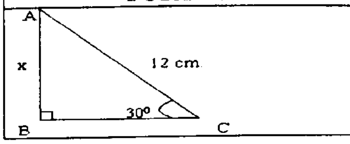
Every question in the TWP test had a parallel context-free form in the TORT tests (see Figure 4.30.). These questions were analysed in the same way as the TT and the AT tests were. Parallel questions and their initial categorisations of the UK and the TR students are presented in Figure 4.30. In Figure 4.30. and in the subsequent discussion I use TORTn and TWPn to refer, respectively, to question 'n' in the TORT and the TWP tests.

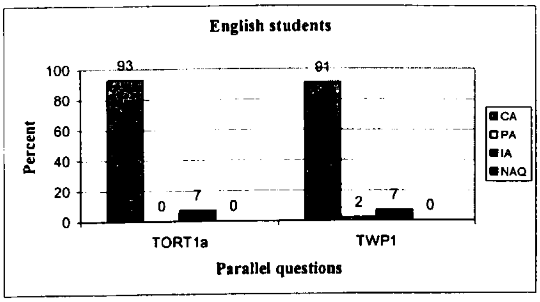
Parallel questions 1

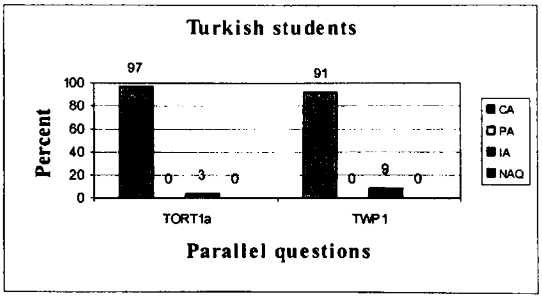
Both TWP1 and TORT1a were answered by all UK and TR students. With the exception of 2% of the UK students in TWP1, no partial answers were given by students from both countries in both questions. Both questions were answered correctly by more than 90% of the UK and the TR students. Both countries' students did slightly better in TORT1a than in TWP1. The percentage of TR students giving correct answers to TORT1a was slightly greater than the UK students, but both countries' students' performances were identical in TWP1 correctly.

Figure 4.30. The UK and the TR students' performance in parallel questions.

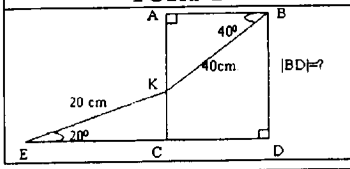
Parallel questions 1

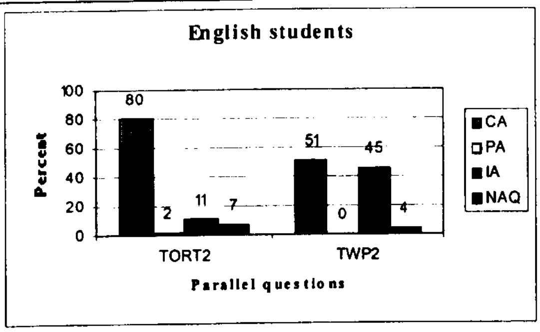
TORT-1a	TWP-1
	A boy is flying a kite from a string of length 55 m. If the string is taut and makes an angle of 60^0 with the horizontal, what is the height of the kite? Ignore the height of the boy.

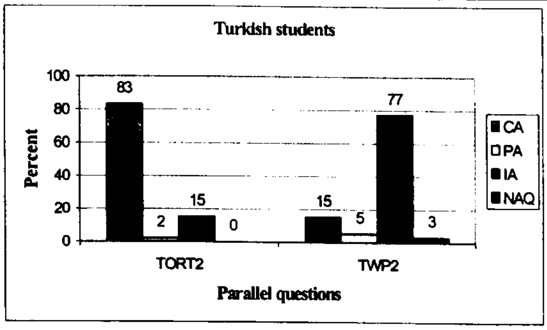




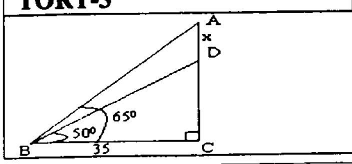
Parallel questions 2

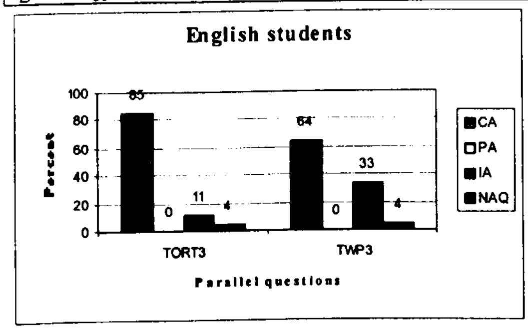
TORT-2	TWP-2
	A rocket flies 10 km. Vertically, then 20 km. at an angle of 15^0 to the vertical and finally 60 km. at an angle of 64^0 to the horizontal. Calculate the vertical height of the rocket at the end of the third stage.

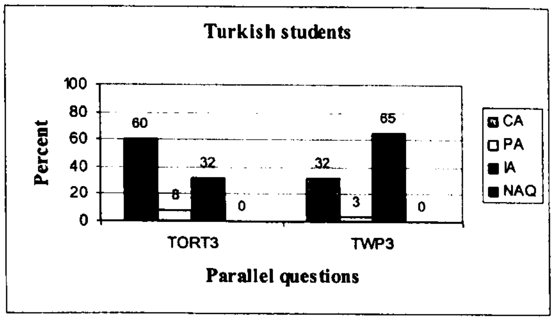




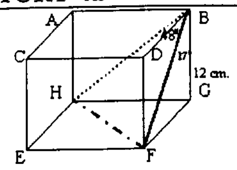
Parallel questions 3

TORT-3	TWP-3
	From a point 10 m. from a vertical wall, the angles of elevation of the bottom and the top of a statue of Isaac Newton, set in the wall, are 40^0 and 52^0. Calculate the length of the statue?

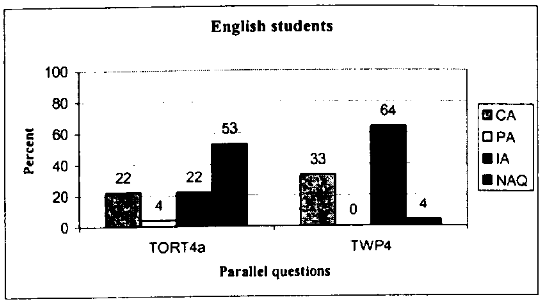


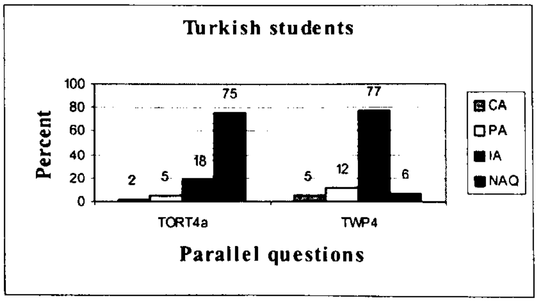


Parallel questions 4

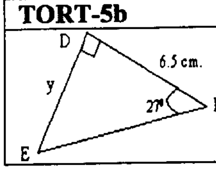
<table>
<tr><th>TORT-4a</th><th>TWP4</th></tr>
<tr><td>

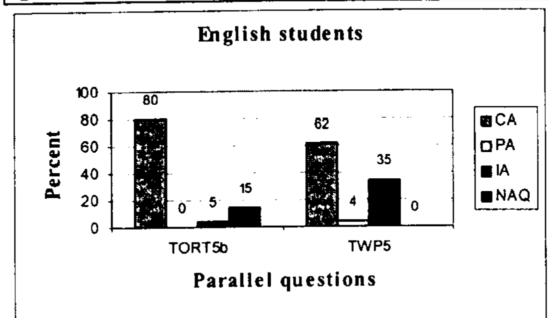
$\hat{FBG}=17^\circ$, $\hat{HBG}=48^\circ$ and |BG|=12 cm. are given in above rectangular block. Then find the following lengths |HF|=?</td><td>An observer at the top of a tower of height 15 m. sees a man due west of him at an angle of depression 31^0. He sees another man due south at an angle of depression 17^0. Find the distance between the men?</td></tr>
</table>

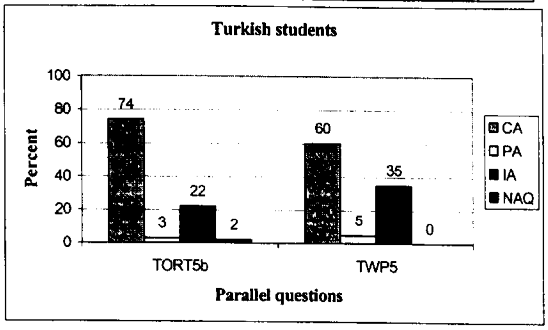




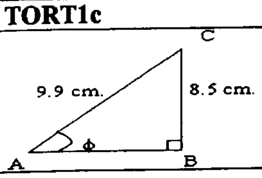
Parallel questions 5

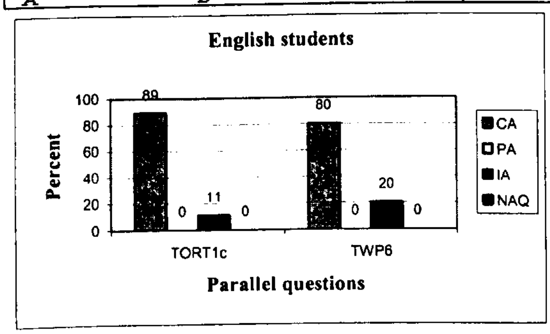
<table>
<tr><th>TORT-5b</th><th>TWP-5</th></tr>
<tr><td>
</td><td>The top of a cliff is 10 m. above sea level. A yacht is an angle of 10 below the horizon measured from the cliff top. How far is the yacht from the bottom of the cliff?</td></tr>
</table>

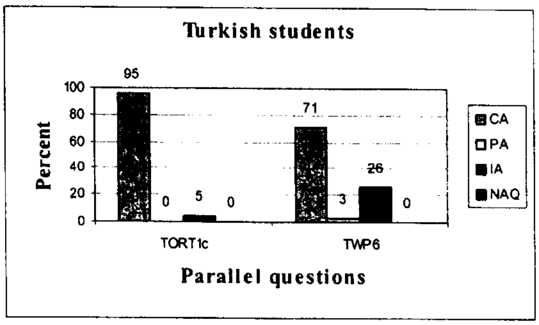




Parallel questions 6

<table>
<tr><th>TORT1c</th><th>TWP-6</th></tr>
<tr><td>
</td><td>A ladder of length 6 m. rests against a vertical wall so that the height of the point at which the ladder reaches the wall is 2.5 m. above the ground. What angle does the ladder make with the ground?</td></tr>
</table>





Parallel questions 2

The percentage of the UK and the TR students who answered TORT2 correctly was far greater than the percentage that answered TWP2 correctly. The difference between the percentages of the TR students answering TORT2 and TWP2 correctly was remarkable. TORT2 was answered correctly by 83% of the TR students, which was more than five times the percentage who answered TWP correctly. 77% of the TR students gave incorrect answers to TWP2 but only 45% of the UK students answered it incorrectly. Although the percentage of the TR students giving correct answers to TORT2 was slightly greater than the UK ones, the percentage of the UK students who answered TWP2 correctly was more than three times that of the TR students, 51% and 15% respectively. Less than 5% of the UK and the TR students gave partial answers and more than 95% of the students from both countries attempted both questions.

Parallel questions 3

Almost all of the UK and the TR students answered both TWP3 and TORT3. No partial answers were given by UK students and less than one tenth of the TR students gave partial answers. The percentages of both countries' students who answered TORT3 correctly were greater than those who answered TWP correctly. The percentage of UK students giving correct answers to both questions were greater than that of the TR students. In TWP3 the percentage of UK students giving correct answers was twice that of the TR ones. The percentage of TR students who answered TWP3 incorrectly was more than twice the percentage of the TR students who answered TORT3 incorrectly.

Parallel questions 4

In TWP4 and TORT4a, there are some notable results compared to the first three parallel questions. 75% of the TR and 53% of the UK students did not attempt TORT4a. However, 94%(96%) of the UK(the TR) students answered TWP4. Very low percentages of the TR students, 2% and 5% respectively, answered TORT4a and TWP4 correctly, but these were answered correctly by 22% and 33% respectively of the UK students. TWP4 was answered incorrectly by 64% and 77% of the UK and the TR students, even though almost all students attempted both questions. None of the UK students answered TWP4 partially, while 12% of the TR students did.

Parallel questions 5

In both TWP5 and TORT5b, the percentages of UK students giving correct answers were greater than those of the TR students. For example, the percentages of UK and TR students, who answered TORT5b correctly were 80% and 74% respectively but a greater percentage of the TR students attempted this question. Interestingly, both countries' students had approximately the same results in TWP5. All of them attempted this question and about 60% had correct answers and 35% had incorrect answers. The TR students incorrectly answered 22%

of TORT5b which was more than four times that which the UK students did. Since parallel questions 1 are very similar sorts of questions, this is a notable result.

Parallel questions 6

Both TW6 and TORT1c were answered by all UK and TR students. With the exception of 3% of the TR students for TWP6, no UK and TR students gave partial answers to these questions. The percentages of both the UK and the TR students who answered TORT1c correctly were far greater than the ones who answered TWP correctly: 9% more for the UK students and 24% more for the TR students. The TR students obtained a higher percentage of correct answers to TORT1c than the UK students, but the UK students obtained a higher percentage of correct answers in TWP6. Parallel questions 1 and 5 are similar sorts of questions to parallel questions 6. Results show that both countries' students' performance in parallel questions 6 was slightly poorer when compared to the parallel questions 1but slightly better when compared to the parallel questions 5.

1.6.3.2. The flaws in parallel questions

The flaws students committed in both the TWP and the TORT tests are presented in Table 4.24. In this section, the parallel questions are investigated in terms of flaws. Table 4.25. presents the percentage of the UK and TR students who committed the flaws in incorrect and partial answers in the parallel questions. Six pairs of the parallel questions in the TWP and the TORT tests are shown in the first column. The other columns show the flaws in both the TORT and the TWP tests.

Table 4.25. The percentage (and the raw number) of the UK and the TR students who committed the flaws in incorrect and partial answers in the parallel questions.

Parallel questions		Read		Terminology		Identify function				Develop mathematics		Symbolic manipulation	
		MR		MT		TF		TR		DM		SM	
		UK	TR	UK	TR	UK	TR	UK	TR	UK	TR	UK	TR
1	TORT1a	0 (0)	0 (0)	0 (0)	0 (0)	75 (3)	0 (0)	0 (0)	100 (2)	0 (0)	0 (0)	0 (0)	0 (0)
	TWP1	0 (0)	0 (0)	20 (1)	17 (1)	40 (2)	0 (0)	0 (0)	33 (2)	20 (1)	0 (0)	40 (2)	33 (2)
2	TORT2	0 (0)	27 (3)	43 (3)	0 (0)	71 (5)	18 (2)	0 (0)	0 (0)	29 (2)	9 (1)	14 (1)	45 (5)
	TWP2	4 (1)	15 (8)	56 (14)	60 (32)	4 (1)	6 (3)	0 (0)	13 (7)	0 (0)	9 (5)	20 (5)	15 (8)
3	TORT3	17 (1)	4 (1)	17 (1)	23 (6)	67 (4)	73 (19)	0 (0)	15 (4)	17 (1)	31 (8)	33 (2)	19 (5)
	TWP3	6 (1)	0 (0)	72 (13)	73 (32)	17 (3)	20 (9)	0 (0)	9 (4)	6 (1)	9 (4)	6 (1)	23 (10)
4	TORT4a	57 (8)	73 (11)	79 (11)	87 (13)	64 (9)	33 (5)	7 (1)	0 (0)	14 (2)	27 (4)	29 (4)	13 (2)
	TWP4	14 (5)	2 (1)	74 (26)	55 (32)	20 (7)	5 (3)	0 (0)	0 (0)	0 (0)	3 (2)	6 (2)	9 (5)
5	TORT5b	33 (1)	13 (2)	0 (0)	0 (0)	100 (3)	75 (12)	0 (0)	13 (2)	0 (0)	0 (0)	0 (0)	31 (5)
	TWP5	5 (1)	0 (0)	67 (14)	54 (14)	29 (6)	23 (6)	0 (0)	12 (3)	10 (2)	4 (1)	5 (1)	8 (2)
6	TORT1c	0 (0)	33 (1)	0 (0)	0 (0)	67 (4)	33 (1)	0 (0)	100 (2)	0 (0)	0 (0)	0 (0)	67 (2)
	TWP6	9 (1)	0 (0)	0 (0)	11 (2)	45 (5)	0 (0)	0 (0)	21 (4)	9 (1)	26 (5)	36 (4)	21 (4)

In the first parallel questions, TORT1a and TWP1, both countries' students committed more flaws in TWP1. Flaws in identify trigonometric functions was the most commonly committed flaw by the UK students in both TORT1a and TWP1, whereas this flaw was not committed by the TR students. The most common flaw committed by the TR students, however, were flaws in identifying trigonometric ratios which were not committed by UK students.

In TORT2, no TR student but almost half of the UK students committed flaws in using terminology. In TWP2, however, more than half of both the UK and the TR students committed flaws in using terminology. The percentage of UK students making flaws in identifying trigonometric functions was about four times that of the TR students. In TWP2, however, a very low percentage of both countries' students committed flaws in identifying trigonometric functions. Both the UK and the TR students committed flaws in symbolic manipulation in both TORT2 and TWP2.

The most common flaws in TORT3 and TWP3, were using terminology and identifying trigonometric functions. Flaws in terminology were the most common flaws committed by both the UK and the TR students in TWP3 (more than three times that of TORT3). In TORT3, on the other hand, identifying the function was the most common flaw made by both the UK and the TR students (more than three times that of TWP3). Both the UK and the TR students made flaws in symbolic manipulations. A high percentage of the UK students made this flaw in TORT3.

In TORT4a and TWP4, the most common flaws that both the UK and the TR students committed were reading and using terminology. More UK students than the TR students committed flaws in identifying function. The percentage of the UK students who made flaws in symbolic manipulations in TORT4a was nearly five times that of TWP4 and it is also greater than the TR students in both TORT4a and TWP4.

Identifying function was the only flaw that both the UK and the TR students committed in both parallel questions 5, TORT5b and TWP5. A very high percentage of the UK and the TR students made this flaw in TORT5b, more than three times that of their TWP5 flaws. Even though more than half of the UK and the TR students committed flaws in terminology in TWP5, this flaw was not committed by either the UK or the TR students in TORT5b.

Almost half of the UK students, but no TR students, committed identify function flaws in TWP6. Nevertheless, this flaw was committed by 67% of the UK students, which was approximately two times that of the TR students in TORT1c. The UK students made no flaws in symbolic manipulation in TORT1c whereas 67% of the TR students committed this flaw. However, nearly 35% of the UK students committed flaws in doing symbolic manipulations, which was 15% greater than that of the TR students in TWP6.

Overall, hardly any UK students committed flaws in identifying trigonometric ratios whereas the TR students committed this flaw in almost every question. In most parallel questions, the most commonly committed flaws by both the UK and the TR students were flaws in terminology and identifying trigonometric functions. Interestingly, even though UK students used a calculator they committed flaws in symbolic manipulation.

2. Possible factors on the UK and the TR students' performance

This section presents data pertaining to the second research question, which concerns possible factors affecting students' performance. Results and analysis are presented under two sub-sections, which mirror the sub-research questions: teachers' approaches to trigonometry and trigonometry in official documents.

2.1 The UK and the TR teachers' approaches to teaching trigonometry

In this section the results, which are presented descriptively, were obtained from the analysis of the data gathered by questionnaires, interviews and observations conducted with both the UK and the TR teachers. 10 UK and 65 TR teachers completed an extended questionnaire. I hoped for more UK returns but my attempts failed. I did not investigate the typicality or otherwise of the teachers. Their qualifications and teaching experience varied. No UK teacher graduated in Mathematics Education but half of the TR teachers graduated in Mathematics Education. Half of the UK teachers both graduated in mathematics and had a certificate to teach mathematics. More than half of the teachers from both countries had teaching experience of less than 20 years. Almost all of the UK and the TR teachers in the sample taught trigonometry during the data collection year. The majority of all teachers taught trigonometry to 16-17 year old students.

2.1.1 Teachers' questionnaire

The teachers' questionnaire (see Appendix E) yielded two sorts of data, free responses and closed responses to Likert scale questions. The results are presented under two sub-sections: teachers' expectation of students' strategies for solving trigonometry problems and trigonometry in teaching.

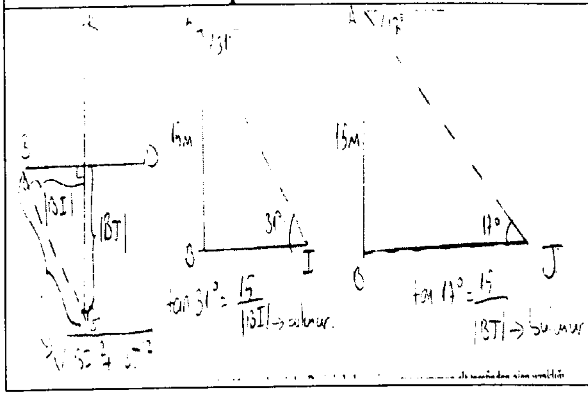
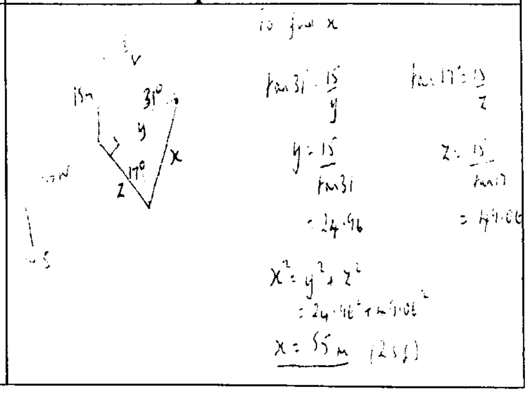
2.1.1.1. Teachers' expectation of students' strategies for solving trigonometry problems

This sub-section is divided into four parts. The first three parts mirror the first three parts on the questionnaire: trigonometry word problems, trigonometric functions on right-angled triangles and simplification of trigonometric expressions. The last part concerns teachers' thoughts on defining simplification.

Trigonometry word problems

Part-1, questions 1,2 and 3, concerns trigonometry word problems. Most UK and TR teachers showed all the steps they wanted to see on students' papers. Some of them wrote their expectations of students' answers as an outline rather than solving the problem. I categorised teachers' answers into the groups CA, PA, IA and NAQ (see p. 74) as for students' data. The UK teachers answered all attempted questions correctly and had a higher percentage of correct answers than the TR teachers did to all trigonometry word problems. No teacher gave partial answers. The only question some teachers did not attempt was the tower problem which required three-dimensional thinking and drawing. Approximately 10% of teachers from each country did not attempt this question. In their answers, every UK and TR teacher either drew a diagram or wrote down 'drawing diagram' as the first and important stage they wanted their students to do in answering trigonometry word problems. The UK teachers preferred more abstract diagrams compared with the TR teachers who drew realistic abstract diagrams. In the tower problem, the UK and the TR teachers drew diagrams in either a simple two-dimensional diagram or two-dimensional representation of the three-dimensional problem (Figure 4.31.).

Figure 4.31. Teachers' diagrams for TWP question 3.

2-D drawing for TWP question 3 in teacher questionnaire	3-D drawing for TWP question 3 in teacher questionnaire
	

The UK and the TR teachers, by and large, polarised in the symbolic forms they derived from right-angled triangles. The TR teachers tended to write expressions whereas the UK teachers tended to write numbers. An example may make this clear. In the statue problem the TR teachers wrote 10(tan 52-tan 40) whereas most UK teachers wrote 4.4. Table 4.26. shows the UK and the TR teachers' numerical answers and symbolic expressions for three trigonometry word problems in the questionnaire.

Table 4.26. Form of the answers in the UK and the TR teachers' trigonometry word problems.

CATEGORIES	TWP-1	TWP-2	TWP-3	FREQUENCIES(%)
Numerical answer	8	7	8	23 (88)
Symbolic expression	1	1	1	3 (12)

CATEGORIES	TWP-1	TWP-2	TWP-3	FREQUENCIES(%)
Numerical answer	16	9	14	39 (24)
Symbolic expression	37	42	43	122 (76)

Most UK teachers' symbolic manipulation, 88%, went towards a numerical answer, while the rest of the symbolic manipulations, 12%, went towards an expression. Quite the opposite was the case with the TR teachers. 76% of their symbolic manipulations went towards expressions and 24% went towards numerical answers. Some TR teachers highlighted an issue on their questionnaires which may help to explain why their answers go towards expressions:

> TR_...Note: we only teach the angles 30, 45 and 60 to the students so I ask questions of these angles if I ask some other angles like tan40 then I give them its equivalent form or value like tan40=a...

> TR_...we do not use calculators so the angles we use should be 30,45,60 and 75...

More than half of both the UK and the TR teachers, 69% and 85% respectively, marked a right-angle explicitly in their diagrams. However, although some UK and TR teachers did not mark a right-angle explicitly, they used it implicitly. Furthermore, in the statements they wrote on their papers they emphasised the importance of perpendicularity in drawing diagrams of trigonometric functions:

> UK_...I would be looking for the creation of right-angled triangles with 'other' angles – given in the question- marked clearly...

> TR_...Since the sum of the acute angles in a right-angled triangle is 90 degree, other angles are calculated after the right angle is allocated...

Trigonometric functions on right-angled triangles

Part-2, questions 4, 5 and 6, concern trigonometric functions on right-angled triangles. Most UK and TR teachers showed all the steps they wanted to see on students' papers. Some of them wrote their expectations of students answers as an outline rather than solving the problem. The geometric diagrams used in this part were context free forms of the trigonometry word problems in the first part. The fourth and sixth questions were 2-D, whereas the fifth one was a 2-D drawing of a 3-D rectangular block. I initially categorised responses into the groups CA, PA, IA and NAQ. Almost all UK and TR teachers attempted to answer the questions. There were no partial answers given by teachers from either country. Almost all UK and TR teachers answered the fourth and sixth questions correctly whereas a few teachers, 10% of the UK and 28% of the TR, gave an incorrect answer to the fifth question. The flaw teachers most commonly committed in the fifth question was assigning a right-angle on a triangle which was not right-angled.

Both the UK and the TR teachers developed the mathematical part from the diagram and then developed the symbolic manipulations. Most UK teachers' manipulations went towards a numerical answer while most TR teachers' manipulations went towards an expression. Interestingly, half of the UK teachers extracted right-angled triangles from the fifth question, before developing and manipulating the mathematics whilst almost no TR teachers did that.

Simplification of trigonometric expressions

Part-3, questions 7,8 and 9, is concerned with the simplification of trigonometric expressions. Most UK and TR teachers showed all the steps they wanted to see on students' papers. Some of them gave more than one way for the solution of the question and all of the ways went to the same answer. Moreover, some of them wrote their expectations of students answers as an outline rather than solving the problem. Responses were categorised into the groups CA, PA, IA and NAQ. All UK and TR teachers attempted all questions. There were almost no partial answers and almost all UK and TR teachers answered the question correctly, namely, they found the simplified form.

Teachers' answers showed that they used algebraic and trigonometric properties appropriately. Every answer used both algebraic and trigonometric properties but, overall, algebraic properties were used more than trigonometric properties.

Definition of simplification

Teachers were asked to define 'simplification' with regard to trigonometric expressions. Almost half of the UK teachers did not attempt this question whereas almost all TR teachers defined simplification. These definitions had similarities and differences. Almost half of the UK and most TR teachers defined simplification as reaching a single term or shortest expression as a final answer. Most of them supported their definitions by procedures, which could be used throughout simplification, e.g. trigonometric identities, algebraic manipulations such as factorising or cancellation, rather than saying merely finding a single term or shortest expression. However, some UK and TR teachers gave less clear definitions, giving very general expression like '*use identities, factorise, cancel*'" or only mentioned operations that should be used in simplification e.g algebraic manipulations and trigonometric identities. No UK but some TR teachers defined simplification as rewriting the expression or finding equivalent forms of the expression by using trigonometric identities and algebraic manipulations. Moreover, some TR teachers gave different explanations for simplification, e.g. it has the same meaning with the simplification of the numbers, it cannot be defined for trigonometry and it can carry more then only one meaning.

2.1.1.2. Trigonometry in teaching

This section reports on the Likert scale questionnaire data. The seven questions (see Appendix E) asked teachers to comment on desirable student traits. Descriptive statistics are used. All numbers in this section are percentages.

Trigonometry in the UK and the TR teachers' teaching

The questionnaire asked teachers to give their opinion of the importance of the statements in Table 4.27. The percentages of the UK and the TR teachers' responses are presented in the Table 4.27.

Table 4.27. How important for students.

	Not important UK TR	*Less important* UK TR	*Somewhat important* UK TR	*Very important* UK TR
1. remember formulae and procedures.	0 0	0 7	30 25	70 68
2. think in a sequential manner.	0 2	0 7	10 20	90 68
3. understand mathematical concepts, principles and strategies.	0 0	0 3	50 8	50 87
4. be able to think creatively.	1 0	30 5	40 23	20 72
5. understand how mathematics is used in the real world.	0 5	60 13	40 38	0 43
6. be able to provide reasons to support their solutions.	0 0	20 5	50 37	30 58
7. be able to manage using a calculator.	0 40	10 28	20 30	70 2

The majority of the UK and the TR teachers thought it is important for students to remember formulae and procedures. Being able to use a calculator is not important or less important for more than a half of the TR teachers whereas the majority of the UK teachers thought it is very important. None of the UK teachers thought understanding how mathematics is used in the real world is very important but 43% of the TR teachers did (see Table R.C 31).

The UK and the TR teachers' views of trigonometry

The questionnaire asked teachers to what extent they agree or disagree with statements on the teaching and learning of trigonometry. The percentages of the UK and the TR teachers' responses are presented in Table 4.28. Every item in this sub-section is referred to in this table.

Table 4.28. The UK and the TR teachers' view of trigonometry.

	Strongly disagree	*Disagree*	*Agree*	*Strongly agree*
	UK **TR**	**UK** **TR**	**UK** **TR**	**UK** **TR**
1. Trigonometry is primarily an abstract subject.	10 10	50 48	40 37	0 5
2. If students are having difficulty with trigonometric identities and trigonometric formulae, an effective approach is to give them more practice by themselves during the class.	10 2	50 8	20 45	20 45
3. More than one representation (picture, concrete material, symbol set, etc.) should be used in teaching trigonometry.	0 0	0 15	60 50	40 35
4. Trigonometry should be learned as sets of algorithms.	20 3	40 32	40 42	0 5
5. A student's success in trigonometry is strongly related to the student's algebra background.	0 0	10 7	30 58	60 35
6. I use printed resources other than the textbook in teaching trigonometry.	0 2	20 32	30 43	50 22
7. My expectations for what I want students to do in trigonometry lessons are clearly defined in the syllabus.	0 10	20 27	60 48	20 13
8. Students have difficulties with trigonometry word problems.	0 3	0 12	80 42	20 43
9. Some students cannot draw representations of situations in trigonometry word problems.	0 2	0 10	70 63	30 25
10. Picturing trigonometry word problems leads to solving the question.	0 0	10 5	40 57	50 35
11. I allow students to solve trigonometry problems on the blackboard.	0 0	20 3	70 43	0 53
12. I teach students some mnemonics in trigonometry lessons.	0 2	10 0	50 50	40 48
13. I give related algebra examples before simplification of trigonometric expressions.	0 0	10 7	60 55	30 38
14. I encourage students to memorise the trigonometric identities.	0 17	10 38	60 32	30 13
15. Mathematics teachers in this school regularly share ideas and materials related to trigonometry.	0 3	40 17	30 63	30 17
16. I enjoy teaching trigonometry.	0 3	0 12	30 32	70 53
17. I provide concrete experience in trigonometry word problems before abstract concepts are introducing.	0 0	10 15	50 58	40 25
18. I take students' prior understanding into account when planning the instruction.	0 5	0 8	30 42	70 43
19. I encourage the use of calculators in trigonometry.	0 33	0 47	20 17	80 2
20. I engage students in applications of trigonometry in a variety of contexts.	0 7	30 23	30 52	40 10
21. A student's ability in solving algebra problems affects solving trigonometry problems.	0 0	0 7	40 47	60 45
22. I follow the textbook as closely as possible in teaching trigonometry.	20 0	70 32	10 52	0 17
23. I let students use a variety of resources in addition to textbooks for trigonometry.	0 2	10 2	70 33	20 62
24. My trigonometry lessons follow a fixed pattern.	0 3	40 15	60 57	0 20
25. I set aside some time in trigonometry lesson in order to teach students how to use their calculators.	0 33	20 45	40 18	40 0

Trigonometry Word Problems

In items 3, 8, 9, 10 and 17, both countries' teachers show positive agreement that students have difficulties with trigonometry word problems: although picturing the TWP leads students to a solution, some students cannot draw the representations of situations described in the TWP. The responses to item 1 indicate that more than half of both the UK and the TR teachers do not see trigonometry as primarily abstract and their responses to question 3 support the view that more than one representation should be taught.

Trigonometry questions

In items 5, 13, and 21, teachers from both countries expressed a positive agreement that algebra and trigonometry were related and that students' algebraic competence has an effect on their performance in trigonometry, i.e. a student's success in trigonometry is strongly related to the student's algebra background. They expressed general agreement that algebra examples should be presented before the simplification of trigonometric expressions. Interestingly, almost all UK teachers agreed that they encourage students to memorise the trigonometric identities, item 14, whereas over half of the TR teachers disagreed with this.

Teachers approach to teaching trigonometry

In items 3 and 16, both the UK and the TR teachers expressed a positive agreement that they enjoy to teach trigonometry and they think more than one representation should be used in teaching trigonometry. In item 1 they expressed disagreement that trigonometry is primarily abstract. However, more than half of the UK teachers disagree that trigonometry should be learned as a set of algorithms whereas almost half of the TR teachers show agreement on this item

Teaching trigonometry to students in the class

Almost all TR teachers agreed that if students are having difficulty with trigonometric identities and trigonometric formulae, an effective approach is to give them more practice during the class, item 2, whereas more than half of the UK teachers expressed disagreement with this item. In items 11, 12 and 20, both the UK and the TR teachers showed positive agreement. The UK teachers agreed that they call students to the blackboard to solve trigonometry problems, item 11. Almost all UK and TR teachers agreed that they teach students some mnemonic ways, item 12. More than half of the teachers from both countries agreed that they engage students in applications of trigonometry in a variety of contexts, item 20. All UK teachers, 80% strongly agree and 20% agree, showed positive agreement for encouraging the use of calculators in trigonometry, item 19, whereas 81% of the TR teachers expressed disagreement on this item. Most of the UK teachers agreed to set aside some time in trigonometry lessons to teach students how to use calculators, item 25, whereas the majority of the TR teachers stated that they do not do this.

Textbooks

The item 22 suggests that most UK teachers do not follow textbooks closely, whereas most TR teachers do. The majority of the UK and the TR teachers, item 6, expressed positive agreement that they should use printed resources other than the textbook in teaching trigonometry. Moreover, almost all of both countries' teachers agreed on letting students use a variety of resources in addition to textbooks, item 23.

Scheme of works

Both the UK and the TR teachers expressed positive agreement, items 18, 15 and 24, that in planning instruction they take students' prior understanding into account, that in their schools they share ideas and materials related to trigonometry with their colleagues and that their trigonometry lessons follow a fixed pattern.

Curriculum

The majority of the UK and the TR teachers, 80% and 61% respectively, agreed that their expectations for what they want students to do in trigonometry lessons are clearly defined in the curriculum, item 7.

Calculators

The questionnaire asked teachers about the activities their students use calculators for in trigonometry lessons. The percentages of the UK and the TR teachers' responses are presented in the Table 4.29.

Table 4.29. Use of calculators in the activities.

	Not at all	*Rarely*	*Sometimes*	*Almost always*
	UK **TR**	**UK** **TR**	**UK** **TR**	**UK** **TR**
1. Checking answers	0 45	0 32	70 15	30 5
2. Tests and exams	0 75	0 7	20 12	80 3
3. Routine computation	0 42	0 20	10 28	90 7
4. Solving problems	0 42	0 22	40 27	60 5

No UK but most TR teachers said that their students do not use calculators in checking answer, tests and exams, routine computation and solving problems.

Planning trigonometry lessons

The questionnaire asked teachers to what extent they referred to official/printed documents in planning trigonometry lessons. The percentages of the UK and the TR teachers' responses are presented in Table 4.30.

Table 4.30. Use of official/printed documents in planning the trigonometry lesson.

	Not at all UK / TR	*Rarely* UK / TR	*Sometimes* UK / TR	*Almost always* UK / TR
1. National or Regional Examination Specifications	10 20	0 13	10 38	80 23
2. National or Regional Curriculum Guide	20 10	30 15	20 42	30 32
3. School Curriculum Guide	0 2	20 10	20 30	60 48
4. Teacher Edition of Textbook	20 3	20 18	40 47	20 28
5. Student Edition of Textbook	0 33	20 15	20 30	60 8
6. Other Resource Books	0 0	30 5	40 40	30 53

Overall result shows that the UK and the TR teachers usually use official resources and written documents to plan their lesson. The majority of the UK teachers almost always refer to National or Regional Examination Specifications in planning trigonometry lesson but only 23% of the TR teachers do this. Most UK teachers preferred to refer to student edition textbooks, but did use the teachers' edition of the textbook. However most TR teachers preferred the teachers' edition textbook and 33% of them did not refer to the student edition of the textbooks. No teachers never referred to other resource books.

2.1.2 Interviews with teachers

Interviews were conducted with five UK and nine TR teachers. As detailed on pp. 37 and 55 the interview schedule used a hierarchical focusing (Tomlinson, 1989) format. This allowed the teachers to introduce themes that they saw as important while simultaneously ensuring that my interests were addressed in the interviews. My interests may be expressed in four themes, which I use as sub-headings below:

- Curriculum resources and assessment,
- The development of trigonometry at GCSE and at A-level,
- The structure of trigonometry lessons,
- Simplification of trigonometric expressions.

Under each heading I introduce sub-themes which arose from, i.e. were grounded in, the teachers' responses. I use selected quotes from the interviews to illustrate arguments/opinions of the teachers on these themes and sub-themes.

2.1.2.1. Curriculum resources and assessment

This section is concerned with the main physical resources teachers typically use in their trigonometry lessons, e.g. whiteboards, textbooks, calculators, overhead projectors and computers. Government guidelines, e.g. the National Curriculum, Examination Board syllabii and past examination papers may also be used as resources by teachers. The section is divided into four sub-sections (themes): textbooks, curricula, lessons plans and schemes of work, and calculators. The first three sub-themes are not discrete and I attempt to bring out links between them in my commentary below.

Textbooks

The UK and the TR teachers generally use textbooks in teaching trigonometry. But they do not follow them strictly. Although the UK teachers express satisfaction with them, they supplement them with their own notes and worksheets:

> ...the textbooks provide two things. They provide the students with an explanation of basic ideas in trigonometry and then they provide the students with practice questions in trigonometry. The explanation I tend to use more as a back up, you know I've got my notes on trigonometry that's what I use with my students. I don't often refer to the textbook itself in terms of content.

However, almost all of the TR teachers do not think that the textbooks, especially the one which is published by the Ministry of Education of Turkey (MET), are sufficient to teach mathematics, so they supplement them with their own notes and worksheets like UK teachers:

> ...I have my own notes I have prepared in the light of my teaching experiences for lessons and I daily add new and various questions to these notes...only a textbook is not enough to teach...since I do not firmly follow a textbook I supplement my notes with worksheets...

UK teachers stated that they used their own notes (worksheets and handouts) when they were unhappy with the textbook or when they needed additional material:

> ...we have a variety [of textbooks] so I would pick and choose from different textbooks and perhaps produce my own handouts from those if I wasn't happy with the textbook that the students have at the time...
>
> ...We tend to just either have questions or reviews or questions from other textbooks or past exam questions just to supplement the resource that is in the textbook and they get a wider view of different types of trig questions.supplementary sheets are to give them extra practice at the basics, but also to stretch the better ones as well. You know to give them more practice with the more exam style questions so it's just there as a supplementary resource in case we run out of material in the textbook.

The TR teachers expressed similar views and practices:

> ...official MET resources are not satisfactory...in explanation of theory part and practical part...therefore I use some other resources with which I enrich the trigonometry lesson...because I collect various and different explanations and questions...
>
> ...textbooks are uniformly good, each has a good part, so I pick all good theory and practice parts of the various printed resources to prepare handouts...

This links with the second sub-theme is curricula.

Curricula

UK mathematics teachers placed great emphasis on syllabi and examinations. It would appear that UK teachers in general (see, e.g. Jenkins 2000) put a great deal of emphasis on syllabi and examinations but a discussion of this would distract from the argument here. The teachers interviewed expressed general satisfaction with the trigonometry syllabus but two of the five teachers thought it was introduced too late at GCSE. Teachers were steered by curricula and syllabus as to what should be taught but not how it should be taught:

> ...there's a syllabus produced by the board that tells us what we actually have to teach and a sample exam paper is produced by the board that tells us typical exam questions but they don't actually give any official way on how a particular topic should be taught...

On the other hand, although the TR mathematics teachers stated that they follow what the national curricula want them to do in teaching trigonometry, and that they are restrictedly tied to it, they agreed that they modify things in the curricula with their colleagues when some changes are needed. They did not talk about the examinations, they merely said they prepare students for the university entrance examination. In the interview, more than half of the TR teachers expressed unhappiness with the place of trigonometry in the curricula, they thought it should be introduced later than topics like geometry and analytical geometry. Moreover, they think trigonometry word problems are ignored in the curriculum:

> ...we should follow national curriculum...since trigonometry is a geometry lesson, it should be taught after geometry, unit circles...moreover you can not find many trigonometry word problem, you only may find few of them, it is the one of the missing parts in national curriculum...

Returning to the textbook sub-theme it is clear that curricula and syllabus influence UK teachers' choice of textbooks used:

> ...the textbook is endorsed by the exam body which means they have looked at it and said it's compatible with what we want so in that sense they're as close to official resources...because our textbook is written specifically for our syllabus.

However, in TR, the MET publishes an official magazine (Tebligler Dergisi) in which they list the textbooks which should be use in schools and those textbooks which fulfil the conditions for teaching mathematics as defined in the National Curriculum. So TR teachers have textbooks they must teach from. But, even though some TR teachers recommended MET textbooks to the students and want them to use those books in the classroom, they use many other textbooks and written resources in lessons:

> ...We recommend MET textbooks to the students and want them to buy, but I use the textbooks I want in the class...

Textbooks that are 'not in line' with syllabi are used 'with caution' by the UK and TR teachers:

> UK_...you can't just give them the textbook, the textbook is not perfectly in line to the syllabus that we have to teach...
>
> TR_...the teaching order of the trigonometry titles in the National Curriculum are presented in the textbooks, however sometimes me and my colleagues change the given order when it is really needed...

Lesson plan and schemes of work

Although teachers plan lessons they rarely, if ever, produce written lesson plans. In planning trigonometry lessons, teachers use syllabi, textbooks and schemes of works. The TR teachers especially are rigidly tied to the National Curriculum and schemes of work:

> $_{UK\_}$... we don't actually cover the national curriculum as such. There's a syllabus produced by the board that tells us what we actually have to teach and a sample exam paper is produced by the board that tells us typical exam questions ... the textbook is endorsed by the exam ...we have to adhere rigidly to the syllabus and because of constraints on time there isn't a lot of time to extend what the syllabus asks so we basically look at the syllabus see what's required to be taught and teach that so we stick fairly rigidly to the syllabus documentation...

> $_{TR\_}$...of course I stick to the National Curriculum to teach trigonometry and then we've written our scheme of work on the base of the National Curriculum. Moreover I use various textbooks and written resource.

Calculators

Calculators have an important place in trigonometry lessons in the UK. Their use is widespread. Almost all UK teachers interviewed endorsed the use of calculators:

> ...obviously they need to be familiar with the use of calculators...tables no longer exist...and calculators are the quickest, probably easiest way of finding a solution to many of the trig questions. So yeah certainly I do encourage them and use them heavily in my lessons...

They also stated that they only addressed issues of calculator use for special functions which the students might not be familiar with:

> ...Generally they know how to use a calculator...but sometimes they are unsure of certain functions and then I might just slip in a few minutes to make sure they can do it properly...

At the other extreme, calculators are not used in trigonometry lessons in TR:

> ...we never use calculators, because it is not in the National Curriculum, therefore it is never used in trigonometry lessons...

All TR teachers interviewed explained the reasons why they do not need calculators:

> ...in our education system we usually use the angles 30, 45, 60 and 90 so they do not need to use calculators...in university entrance examination no question is asked with angles like 48, so we do not use that kind of angles...

Some TR teachers stated their beliefs about not using calculators:

> ...in TR calculators are expensive and not practical...if I want students to buy calculator their parents might not be able to afford it...so we certainly do not use calculators in trigonometry lesson... and I do not think it is going to happen in TR...

> ...calculators are not used in trigonometry lesson...I do not believe that calculators will be helpful for student, even I think to use calculators pushes students to be lazy.

> ...in our education system we do not use calculators...I oppose the use of calculators in trigonometry lesson, because students become addicted to it and they cannot solve any

problem in their mind...moreover since we work with well known trigonometric ratios they do not need calculators and even trigonometric tables...

2.1.2.2. The development of trigonometry at GCSE/A-level in UK and at O3/L2* in TR

The development of trigonometry and the difference between introductory and further trigonometry

Trigonometry is taught at two age groups, in GCSE, secondary school (O3), and A-level, high school (L2) in the UK and TR respectively. Both countries' teachers think the development of trigonometry in secondary and high school years are reasonable in their National curriculum:

> $_{UK}$_...GCSE tends to be uh predominantly number based problems...whereas A Level tends to be manipulation of the sort of algebraic side ... with any starter problem they need to have the ground work at GCSE to be able to solve right angle triangles and non right angle triangles they also do sine and cosine with GCSE and they need that that key skill ... of manipulating trig functions before we can make the topic more abstract by going into things like trig identities so in that sense it's done in a sensible progression...

> $_{TR}$_... the trigonometry at O3 is merely the trigonometric ratios of acute angles on right-angled triangles...it is concrete...at L2 trigonometry comes more abstract and wider with trigonometric identities, formulae and equations...so trigonometry is concrete at O3 and abstract at L2...this is a sensible progression.

Both the UK and the TR teachers had different views on introducing trigonometry. The UK teachers interviewed agreed that introducing trigonometry is working with numbers. However, the TR teachers revealed three different views that introducing trigonometry concerns basic trigonometric functions on a right-angled triangles, figurative work and memorising trigonometric functions:

> $_{UK}$_...Trigonometry the first introduction I would see that being a more practical measure numerical aspect...

> $_{TR}$_...basic concepts, the trigonometric functions, sine, cosine, are given on a right-angled triangle at the first introduction...

> $_{TR}$_...I think it is more concrete at O3...you use right-angled triangles for introducing trigonometry...

> $_{TR}$_...introducing trigonometry is completely memorising the basic trigonometric functions... you present right-angled triangle and then this opposite side, this is adjacent side ... then they only memorise them...

The UK and the TR mathematics teachers stated two different opinions for further trigonometry and its difference from the introduction of trigonometry. The UK teachers highlighted the importance of algebra, which was seen as the main difference between introductory trigonometry and further trigonometry, for further trigonometry. On the other hand, none of the TR teachers mentioned algebra, they stated that further trigonometry is more abstract compared to the introducing trigonometry:

* O3 stands for year 8 in secondary school and L2 stands for year 10 in high school in TR.

> UK_...the major differences in the second bit is the fact they've got to be aware of the algebra and they've got to have a good algebraic background in order to cope with the further trigonometry...

> TR_... introduction of trigonometry is concrete... but it is more abstract and theoretical at L2 that students work with proofs and application of formulae they do not work with real world problems...

Some of the UK teachers interviewed also stated the important place of calculator use in trigonometry lesson in both phases:

> ...Trigonometry, the first introduction, I would see that being a more practical measure numerical aspect. Further trigonometry is the moving into algebraic ideas over that age range...but still working with calculator...

Introducing and building up trigonometry

All my contact with UK and TR teachers and students suggests that trigonometry is one of the most difficult subjects learnt at school. The development, from initial to further work, of a difficult subject is important. More than half of all teachers stated that students first experience of trigonometry should focus on right-angled triangles on which trigonometric functions like sine, cosine and the relations between sides and angles could be built up:

> UK_....looking at right angle triangles because it's something they've tried and they are familiar with so looking at it in a familiar context, and getting used to the idea of this is your angle, if you've got a particular angle which side is in relation to that angle is the opposite, which side is the adjacent, which side is the hypotenuse etc and establishing that idea first of all I think is probably the best way to start...then they have to be able to be confident with the idea that the opposite, adjacent and the hypotenuse should change depending on which angle you're looking at so making sure that they understand the idea the ratios are relative to whichever angle you're looking at in the right-angled triangle...

> TR_...first, they must particularly learn the right-angled triangles, because to understand the relation between the sides is difficult for them...then they should know the angles well...we want them to know the values of some trigonometric ratios of well known angle, like 30,45,60 and 90, by using equilateral and isosceles triangles without looking at the trigonometric tables.

Even though the rest of the UK and the TR teachers drew attention to right-angled triangles and teaching of trigonometric functions and the values of some trigonometric ratios their thoughts for the first experience of trigonometry included right-angled triangle work. No TR teacher thought real world applications of trigonometry should occur in the first place whereas all UK teachers did. One of the UK teachers interviewed highlighted the importance of triangles and applications of trigonometric functions to real life, which should be taught after the introduction:

> ...it depends on how it's developed but you've got to have an understanding...of similar triangles...they know that sine is opposite over hypotenuse but they don't know what that really means and unless they have an idea of similarity of triangles you know where the ratio of the different sines no matter how big the triangle is stays in the same proportion...then...once they've got the idea of trig functions they need to be able to apply those to real life problems....at GCSE level the aim I suppose is to get them to understand how appropriate the use of trig is in solving real life problems...

Another UK teacher thought the first experience of trigonometry should be practical, right-angled triangles and trigonometric function on them could come next:

> ...it could be practical stuff...It could be...just going out and measuring the height of something or...it doesn't have to be algebraic...it could be just trying to learn how you might measure how high a building is, measuring a few angles and doing stuff like that...then ....they need to know stuff like the right-angled triangle, sine or the cosine rule and they go on to some of the trig identities...

Although none of UK teachers interviewed mentioned the unit circle approach to trigonometry, one third of the TR teachers thought it should be used in introducing trigonometry:

> ...unit circle... after the unit circle, the trigonometric ratios of the angles like 30,45 and 90 should be taught on a right-angled triangle by using coordinate system...

The rest of the TR teachers interviewed stated that angles should be the initial focus:

> ...it should be angles, especially the corresponding angles, because trigonometric ratios are the relation between angles and the sides...then the use of them in a triangles...

Overall both countries' teachers agreed that right-angled triangles and then trigonometric functions on right-angled triangles should be taught first. This corresponds with how their students answered trigonometry word problems even though the UK teachers highlighted applications of trigonometry whereas the TR teachers did not. In answering trigonometry word problems, students from both countries first drew a diagram, a right-angled triangle or any diagram that could be transformed into a right-angled triangle.

Do trigonometry word problems help in the teaching of basic trigonometric functions?

More than half of the UK teachers believed that trigonometry word problems helped by practising trigonometric functions on right-angled triangles. All TR teachers but one agreed, however, they all complained about its place in the education system:

> ...yes I think it is helpful...trigonometry word problems are really nice and explanatory to see how trigonometry can be applicable...but we do not work with them so much...

Those who did not view them as useful stated:

> $_{UK\_}$...it has more than for the study of the trigonometry...I think word problems often distract can distract but it needs to be put in context so they can see why trigonometry is useful in that sense but I don't think it helps, I don't think, the words don't help them because the words tend to, the words don't help them to understand the trigonometry...
>
> $_{TR\_}$...I do not agree on that trigonometry word problems are helpful...I do not think that students can understand them well and so they can understand trigonometry...but maybe the simplest kind of trigonometry word problems could be helpful...

One of the UK teachers highlighted the difficulties students have in drawing diagrams and stated it is more than learning trigonometric functions on a right-angled triangle:

> ...I think initially they find word problems more difficult because they can't picture what's going on. I think once students learn how to construct a diagram to represent a word problem then they can apply something apply knowledge that they've acquired earlier...I

think without drawing a sketch I think they find problems of that kind difficult. So I think the thing with word problems is they need to learn how to adapt them to, uh, to be able to apply the sort of theoretical knowledge they've already learnt and then they can go on to do certain...In terms of helping their understanding yeah I guess it gives them a more awareness of how what real life style questions, how trigonometry might be applied... they use techniques that they'd have already learnt to solve their problems...

Do teachers follow an exact way to solve TWP?

In the light of students' data (the trigonometry test, verbal protocols and interviews with students on trigonometry word problems.) a model was developed (see p. 201) for solving trigonometry word problems. To observe the possible effect of teachers' approaches on how to teach solving the TWP, teachers were asked whether they have a method to solve the TWP or not. More than half of the UK teachers gave a method to solve a TWP:

> ...the first thing you do is to draw a diagram to represent the situation...adding all the features...like the length of the play ground, the height of the flag pole, angle of elevation put in all the information...then try and use your trig ratios to find the bits that are missing...

The majority of the TR teachers stated they follow a method to solve the TWP, but they merely gave an outline and emphasised the importance of the diagram, leaving students to develop their own ways to solve the TWP:

> TR_...the only and only thing that I want students to do is draw the diagram, I do not say anything about the rest of the solution...they can follow any way to find correct answer...
>
> UK_...probably asking them to draw a decent diagram in... if it is literately without a diagram and it's a bit more complicated obviously what you might do obviously is go for a decent diagram to start with and then see that you know stress the need for a decent diagram...

Overall, the UK and the TR teachers started to solve the TWP by drawing a diagram and emphasised how drawing diagrams is vital to solve a trigonometry word problem:

> UK_...diagram is the best place to start ...so you've got some sort of idea of what you've...
>
> TR_...you cannot teach trigonometry word problems without drawing diagram...the first thing to do in solving trigonometry word problems is to draw a diagram...

The way the UK and the TR teachers teach trigonometric functions on right-angled triangles is also important. How do students know which trigonometric function is which in a right-angled triangle was asked of teachers, whether they do this using mnemonic ways or not? All UK teachers stated that at the introducing trigonometry stage they used SOHCAHTOA as a mnemonic way to teach trigonometric ratios on right-angled triangles:

> ...The only example I can think of is the sort of learning of the basic trigonometric ratios of the first place of the SOHCAHTOA idea. But apart from that once you know yes I use SOHCAHTOA to try and help them to remember the three trigonometric ratios and encourage students perhaps to make up their own words for each letter or perhaps try to remember some story or something of that kind...

On the other hand the TR teachers do not have a common abbreviation for teaching trigonometric functions like SOHCATOA. They said they use their own methods like the capital letters and similarity of vowels and some sentences to help students remember the trigonometric functions:

> ...I sometimes make a humorous sentence that students like and do not forget it. Sometimes I play with words and sometimes I only use the first letters or vowels...my aim is try to make students memorise the trigonometric functions and keep them in their mind...

The majority of the TR teachers, but no UK teachers, interviewed stated that they use some shortcuts or abbreviations in further trigonometry as well in introductory trigonometry:

> ...I use some mnemonic ways for example to teach the trigonometric identity $\sin^2x+\cos^2x = 1$ I tell them to use some well known angles and their values like 30...

Memorising or not memorising trigonometric identities and formulae

As it is mentioned above, the UK teachers used SOHCAHTOA as a mnemonic for basic trigonometric ratios, whilst the TR teachers used a variety of ways. However, although the UK teachers did not use mnemonics in further trigonometry, such as teaching trigonometric identities and formulae, the TR teachers sometimes did this by using well known angles, e.g. $\sin^2 30^0+\cos^2 30^0=1$.

There are many trigonometric identities that students need to learn at A level. Although all trigonometric identities are written on formulae sheets some of the UK teachers think students should memorise some:

> ...They've certainly got to know the basics. In A level...they have had quite an extensive formula booklet which means they have no need to learn trig formulae. We have, however, encouraged them to learn at least the key ones, sort of when they're doing calculus...we do suggest that they sort of learn them to the extent that they recognise them even if they can't get them spot on...

The rest of the UK teachers interviewed thought it was not necessary to memorise them because trigonometric identities or formulae are written on the formula sheet. But the common opinion of all of the UK teachers was that students should recognise trigonometric identities and be familiar with them:

> ...now what students get is they get the identities on a formula sheet and I think that's probably more appropriate because I think the mathematical skill involved is being able to apply the identities and not actually learn to remember them. There is an argument to a certain extent that if you've learnt to remember them then you can see, you go through a problem more readily because you have them stowed away in your mind rather than actually looking at what's on a piece of paper. So to that extent I would agree that it would strengthen the situation but with the amount of demand on students in the current time I think it's something that perhaps at this stage they don't need to learn and they can just apply from a formula sheet...
>
> ...if they know the trig identities when they meet them in a question where they are asked to simplify things it can be recognised. Forms they might not know the exact form but I mean if they see $\sin\theta\cos\theta$ then they know that they met that somewhere you know they

might remember sin2θ and then they can approach the problem...I try to encourage them to know where the formulae have come from so that if they got a cos2θ they can use the $\cos^2$-$\sin^2$ then substitute into that...I am aware that they are not going to learn them all...but if they knew where they come from and they can recognise all of them it helps them an awful lot in their end of term exams...

Significantly, none of the TR teachers wanted students to memorise trigonometric identities and they highlighted that students should know where they are derived from:

...it is a very vital question...we have a system, which supports memorising, which I am always against...there is no need to memorise and spend your time memorising...I think the thing which shows a student's capability is knowing how to derive an identity and where to use it rather than memorising...

However, one third of the TR teachers interviewed also stated that memorising trigonometric identities is to the students' benefit in the university entrance examination because they have a very short time to solve questions:

...I think no need to memorise trigonometric identities, they can be derived from each other...but since the university examination system pushes students to be in rush they like to memorise them, so memorising them is for their advantages...

2.1.2.3. The structure of a trigonometry lesson

Interviews with teachers revealed that the structure of the UK and the TR teachers' trigonometry lessons have a similar pattern with slight differences. The majority of the UK teachers and almost half of the TR teachers interviewed stated they started the lesson by an introduction, an explanation of the topic:

$_{UK\_}$...basically there is part teaching where it's give and take between myself and the students, marks and note taking...

$_{TR\_}$... I introduce the title of the topic and the place it can be used... then if there are some theories and identities I prove them...

No UK, but almost half of the TR teachers interviewed stated that they remind students about the previous lesson by a brief summary:

... I summarise the last lesson or I remind them of the identities and equalities to be used in the lesson...so it is like first to remind them of the previous lesson, presentation of the new topic and after the new topic I work with examples to make the new topic well understood...

One each of the UK and TR teachers, however, started lessons with examples:

$_{UK\_}$...go over some examples on the board showing clearly how I wanted things set out, whether a diagram was involved so that they can see clearly the lay out and the development of the question...

$_{TR\_}$...I do not give the theory part of the topic of the lesson at first, I start lesson with a numerical example than I give an example with letters then I explain the topic...if I am going to teach sin2x=sinxcosx, I begin the lesson with saying let us write sin40 as 20+20...then to get sin2x=sinxcosx I work with letters...namely I, first, attract their attention on the topic by numerical examples and then give the theory part of it....then I do some examples again...

Explanation of the topics in the UK were followed by worked example. Then worked examples were followed by classroom practices:

> ...going through some material, going through some worked examples with some feed back from the students and then getting them to work through exercises...all of those three are very important...without the worked examples they are not sure what to do and without the practice it just does not sink in...

On the other hand, all trigonometry lessons in TR were followed by an explanation of the topic and examples repeatedly. In TR, there was no classroom practice session as the UK teachers stated:

> ...at the beginning of the lesson I give the some algebraic identities and properties, which I am going to use in the lesson like factorisation and difference of the square respectively...then I start to explain the topic of the lesson on the blackboard...if they do not understand I give some examples...namely the structure of my lessons is explanation of the subject, example, explanation of the subject, example...

The TR teachers used more written resources than tools whilst the UK teachers did it other way around. The resources the UK teachers used were computers, calculators, overhead projectors, textbooks, worksheets, whiteboards and sometimes geometrical equipment. The resources the TR teachers used were the MET's textbook, various textbooks, worksheets, own notes, question banks, private institutes' textbooks and tests, chalk, eraser, blackboard and geometrical equipment.

The UK teachers hardly ever call students to the blackboard to solve questions. The TR teachers, however, almost always call students to the blackboard:

> UK_...for some lessons yes, not very often...but only sort of very quick time at the board. I wouldn't ask students to come up and spend a lot of time at the board...
>
> TR_...of course! very very often...it is certainly one of the my trigonometry lessons characteristics...if you do not call students to the blackboard to solve questions you do not make them a part of lesson...it means that you explain everything to yourself...

All UK teachers stated that they walk around the classroom and check the students individually, especially in the worked examples and exercises part of the lesson:

> ...as long as time allows which it usually does I go around...as I said the structure is to teach them the theory, do some examples and then they work through questions. When they're working through questions the aim is to get round as many as you can...

Although almost half of the TR teachers interviewed stated that they walk around the classroom and check the students individually, other TR teachers said they cannot always do it:

> ...I think it should be done but I cannot...because the curriculum is overwhelming so I do not have any time to walk in the classroom and check students one by one...

The UK teachers usually solve the questions from old exam papers, worksheets or textbooks whilst the TR teachers use a wide range of written resources, e.g. a variety of textbooks, tests and textbooks of private institutes, which prepares students for the university entrance exam. All

UK and TR teachers stated that examples should cover as many aspects as possible and include a variety of questions and exam style questions. Almost all UK and TR teachers emphasised that the number of the questions solved in a trigonometry lesson depended on the topic and the level of the students, but on average, 2, 3 or 4 questions in the UK and 5 to 10 questions in the TR were covered:

> UK_...There is incredibly, uh basically I watch the pupils and according to the need of the group that I am teaching, I have a sort of sets a number that I want to get through to cover the main points but you can tell the students that are struggling that they have not quite understood some things so you'd like to sit in an extra while to cover that point...

> TR_...my questions are based on the important point of the topic...I believe in the benefit of solving many question...I trust myself on teaching well I solve 7 or 8 questions and when I observe that students understand I stop...I increase the number of the questions until everybody understands...

All UK teachers give homework to the students. Homework was either marked by teachers or left for students to check. Marked homework was always returned to the students. Homework was usually gone over in the class as well, particularly questions that students had difficulties with. Almost all TR teachers interviewed stated that they give homework but they do not check them or mark them. Both countries' teachers agreed that homework is good for practice.

2.1.2.4 Simplification of trigonometric expressions

In the simplification of trigonometric expressions, two important characteristics of manipulations were highlighted by both the UK and the TR teachers in interviews: using algebraic manipulations and using trigonometric identities. They agreed that students' algebraic skills should be developed, because trigonometry and algebra go hand in hand. Most UK teachers stated that they attempt to simplify algebraic expressions before trigonometric expressions, whereas the majority of the TR teachers stated that they do not attempt to simplify algebraic expressions before trigonometric expressions. The majority of both the UK and the TR teachers stated that they remind students of the required identities and properties:

> UK_...I probably would remind them about things like looking for difference of two squares but I wouldn't necessarily I mean hopefully by the time you come on to the sort of quite a lot trig identities, they sort of reasonably familiar with the sort of algebra that they might need but I might remind them about it...

> TR_...I remind them when I see it is necessary...for example what $\sin^2 x-\cos^2 x$ means $a^2 - b^2$ a-b, a+b... I only remind them of the identity or formula, I write them on the aside and I do not erase them...since they usually memorise the formula like $a^3 - b^3$, $a^3 + b^3$ they forget them, therefore I only remind them...

The majority of the UK teachers stated that they use substitution to convert trigonometric expressions into algebraic expressions, but they said that later on students, except the weaker ones, did not use substitution method in trigonometry. However, the TR teachers stated that they do not prefer to use the substitution method:

UK_...Once they become confident with identities some of them will be happy to factorise you know sine squared theta minus cos squared they can do it just by recognising the form from which it came but you know some of them particularly the weaker ones need it as something that with which they are more familiar and therefore substituting is the right way to do it...

TR_...I do not prefer solving algebraic questions before trigonometry, sometimes it could be complex...and I do not prefer to use substitution as well...because if students convert the expression $\sin^2x-\cos^2x$ into a^2-b^2, then they are going to miss that $\sin^2x-\cos^2x$ is $-\cos 2x$, so I solve the questions without algebraic substitution...

All UK and TR teachers agreed that simplifying algebraic expression and trigonometric expressions are similar but are not the same processes, some of the techniques are the same. All agreed that, in simplifying trigonometric expressions, algebraic manipulations could be used but trigonometric identities would be needed:

UK_...initially the algebraic helps me because we are using the ideas of algebraic factorisation but then a step further we need more identities to develop an extra simplification...to repeat we were saying the initial use of algebra...and then we have to look again and consider at that stage that the trig identities can help in getting further simplification...there's more rules they might have to be aware of if they're doing the trig...

TR_....they are so similar to each other...I can say that simplifying trigonometric expression is the application of algebraic manipulations and identities in trigonometric expressions...of course you use trigonometric identities when you need it...

TR_...our students are more successful at simplifying algebraic expressions...I think they should know more than algebraic manipulations and identities at simplifying trig expressions...they should know trigonometric identities...

Both countries' students were better at simplifying algebraic expressions than simplifying trigonometric expressions (see p. 94). Teachers were asked about the factors which might increase students' success in simplifying trigonometric expressions. The main factors the UK teachers listed were being competent and confident in algebraic manipulation, being familiar with the trigonometric identities and learning trigonometric identities by doing lots of practice and memorising. Only one of the UK teachers interviewed gave different factors citing experience, ability, determination and a competitive spirit. All TR teachers listed being able to use appropriate algebraic and trigonometric properties, familiarity with trigonometric identities and lots of practices:

UK_...So they need to be competent in manipulating algebra primarily. With the actual uh with trig expressions as well one of the difficult things that I think students find is that being able to spot or recognise what they need to do next. A familiarity with the identities will help and uh also I mean I think produce and practice because the more you practice the more times you've seen something a particular thing come up so you recognise it and think oh yeah the last time when I when that appeared this is what I did. So practice probably more than anything else and say that you've made yourself familiar with the different sort of forms that uh the expressions that come up...

TR_...using algebraic manipulations and identities with algebraic expressions are the most important factors...the other important factor is to know the most used trigonometric identities like tan=sin/cos, $\sin^2x+\cos^2x=1$ and their rewritten forms like $\cos^2x=1-\sin^2x$ but the most important one is the algebraic methods...in an expression there are really very few students who can recognise that '1' is $\sin^2x+\cos^2x$, however when you solve many questions they become familiar with it...

In the interview both the UK and the TR teachers were asked what they thought about the partial answers given by students to the question 'simplify $\frac{\sin\theta+\sin\theta\cos\theta}{\sin^2\theta}$' which were $\frac{1+\cos\theta}{\sin\theta}$, $\operatorname{cosec}\theta+\tan\theta$ and $\cot\theta/2$. The majority of the UK and almost half of the TR teachers agreed that the wording of the question, the way teachers teach and the exam style question are important:

> UK_...So if a question simply said simplify this expression each of these being correct would probably satisfy what the question was asking for. I don't think you would get a question that simply said simplify this expression. It would have to give them some indication of the format they wanted...because in the context of our syllabus they wouldn't get a question that just said simplify this without telling them in what manner they ought to simplify this...I think certainly our exam boards try and not to be too ambiguous and simply saying write down this in its simplest form...

> UK_...I suppose ideally for these three obviously you'd say the single term is perhaps the best way of leaving it but perhaps you'd have to you know if this was a style of the question that was cropping up regularly in a exam board you'd want to clarify with the exam board whether it always meant go down to a single term. Do you always have to go down to single thetas, and so, it is just something that I think the person who is setting the question has to decide in their own mind in advance, what quite what they wanted the person who is trying to do it. I mean obviously you can argue the best thing is going for the single term...

> TR_...it's up to me what I want students to find in the questions. I expect them to do the first one if I did not teach half angle formula, if I did then the last one...you should clearly express what you want, otherwise the students who solved in first way also think it is the correct answer...maybe you should give the third form and ask them to show the equality...otherwise they cannot go where to go...if there would be a definition of the simplifying in trigonometric expressions, it would be so helpful for the students...

> TR_...the simplified form is the third one...it is expected answer at university entrance examination...because the definition of the simplification at trigonometric expression is to get a single term...

Interestingly, one of the TR teachers interviewed could not define the simplification for trigonometric expressions:

> ...I think there is no clear definition of the simplification of a trigonometric expression as clear as there is for algebraic expressions...with trigonometric expressions you stop anywhere you think you have found the simplified form...

All UK and TR teachers agreed that none of the answers given to the expression $\frac{\sin\theta+\sin\theta\cos\theta}{\sin^2\theta}$ was incorrect. They were asked to give each of the three answers a mark out of 10, as an indicator of its aesthetic quality as a simplified form. The average marks the UK and the TR teachers gave to the partial answers ranged from 4 to 10 and 7 to 10 respectively. Although the majority of the UK teachers accepted the first expression as a simplified form, they gave the lowest mark to the first expression and the highest mark to third expression. Some of the UK teachers found the third expression very sophisticated. But if there was no specific instructions given, then teachers would accept any as a simplified form. Two UK teachers interviewed said that it could not be the second or the first one because:

> ...They are introduced to cosecant and cot but if they're asked to simplify an expression then just cancelling out common factors in a manner like this and leaving themselves with an expression that involves a cosine and sine only is really not much different to this. The problem being this involves a fraction and this doesn't but you can still argue this involves fractions because cosec and cot are not functions that they would naturally use on their calculator, you know, so it's a bit of an odd situation that in our syllabus they wouldn't be asked such an open-ended question because if they were they would have to give full marks for any of those answers because they're all simplified...
>
> ...So for this basically what they've done is they've taken out a factor and so this is just like they've treated it just like an algebraic manipulation isn't it, and they've taken it no further so they've not used any of the trig identities. So out of ten, ten marks seems quite generous for that type of calculation so I'd be reasonably generous but probably three or four so three out of ten for that...

On the other hand, although there was no clear single simplified form, all TR teachers accepted the third one as the simplified form:

> ...the third one is the simplified form of the expression because simplifying a trigonometric expression means reaching the single term...I do not give whole marks to the first one...

The definition of the simplification of trigonometric expressions was not clear to any teacher. When both teachers were asked to define the simplification of trigonometric expressions they did not all give the same definition, they all explained by example and they agreed that it was a very hard job to define it:

> ...it's just that through a feeling for whether you can take something any further or not. And one, yes one of the hardest things to do as you get to a point that's a simplified part where and knowing whether you can continue with it or not and I think that's one of the hardest things about trig identities...

Both the UK and the TR teachers expressed agreement that students do not like trigonometry and they had great difficulties handling trigonometry. Simplification is one of the topics students have difficulties with. In particular, the TR teachers underscored that students memorise trigonometric identities by heart and cannot properly use them in simplification. If students simplify a trigonometric expression by some inappropriate identities then they get stuck at some steps of simplification or they are led to incorrect answers. As the UK and the TR teachers said, it is the matter of being familiar with trigonometric identities and formulae, so doing lots of practice was the most important thing:

> UK_...you just need to know the identities very well and know having had practice at working through questions of a similar kind. So you start to recognise certain things so something comes up and you think yeah I can change that around into something else and that might and see a route through. So you've got to have the ability to be able to work algebraically and by looking at the trig identities and be able to perhaps see a few steps

> ahead to see whether it's going to simplify or not and even sometimes you know try something different and just see what happens and uh work through and just through practice then I think you learn and you get better at that particular skill...
>
> TR_... if they do more practice on the trigonometric identities then they can be familiar with them and recognise them in the expressions...

One of the UK teachers interviewed had a different view to the simplification of trigonometric expressions. He thought of it as an extension of algebra and stated that it rarely occurs in the context of a question:

> ...I am saying is it appears in the context of where we have to teach them...it's often just a question in its own right it's not something it's...probably get a bit too sophisticated if you're allowing to do much simplifying in the context of solving something else. So it's normally as a sort of an extension of kind of algebraic skills and techniques rather than in the context of solving problems from that point that point of view...

2.1.3. Observation of teachers in teaching environment

Five UK and nine TR teachers, with whom interviews and teachers questionnaire were conducted as well, were observed in lessons, to observe their teaching styles in a trigonometry context. Written accounts were used (see p. 63). The aim of the observation was to investigate teachers' teaching styles in the classroom and provide supportive qualitative data for the other data on teachers. Everything I considered relevant to my focus was recorded at regular 5 minutes time intervals. The results are presented in two-subsections: a general view of teaching in the UK and TR, which gives a picture of the UK and the TR teachers' physical teaching environment, and the UK and the TR teachers' actions in lessons, which gives a picture of teachers teaching.

2.1.3.1. A general view of teaching in the UK and TR

In the UK, the sixth form college mathematics teachers had their own department, including a computer cluster and teaching rooms. In their teaching rooms they had resources such as extra textbooks, calculators, worksheets, seats and tables for the teacher and students and a computer. Classrooms and even corridors were prepared for mathematics. Classrooms were small, their capacities were at most 20-25 students. There were single seats and tables for each student in the classrooms. There were computers in some of the classrooms and there was a computer cluster in the department as well. There was an overhead projector and the posters on the walls, which were about mathematics like sine, cosine and tangent on a right-angled triangle, the values of some well known angles such as 30^0, 45^0, 60^0 and 90^0 and some graphs.

In TR, the mathematics teachers in two of the schools had their own mathematics department room but there was no separate mathematics department room for teachers in the third school, all teachers used the same common room. The only facility in the classrooms, departmental or not, were draws (cupboards) to put their written resources in. There were no computers, calculators, textbooks and worksheets. So, unlike UK teachers' rooms, there were no additional resources. All classrooms in TR looked like each other and they were very different to UK classrooms. In TR, in contrast to the UK, there were no posters on the walls, including corridors, relevant to mathematics. There were no computers or overhead projectors in the classrooms, there was also no computer cluster in the schools. Classes were crowded, seating between 35-45 students. The resources in the classroom was limited to chalkboards.

2.1.3.2. The UK and the TR teachers' actions in the lessons.

It was so difficult to observe everything in only one lesson per teacher, I could not afford more than one observation lesson per teacher due to the existence of my total data collection. Teachers were observed in mathematics lessons some of which included trigonometry, including trigonometric identities, but some not. The topics, even in trigonometry lessons, were different from each other. This, of course may have biased my observations.

Overall, the instruction of the UK and the TR mathematics teachers' lessons is outlined in Table 4.31. The UK and the TR teachers started lessons in a similar style. They briefly reviewed the previous lesson and then introduced and explained a new topic or continued with the previous topic. After a theory part, they both worked on examples. The TR teachers sometimes finished the lesson with worked examples or, after a few worked examples, they explained another topic and then did more worked examples. After the worked example session, however, the UK teachers directed students to consolidation exercises, from various written resources, and they worked with students until a few minutes before the end of the lesson and then they gave homework. The TR teachers did not give any 'homework', only two of them left some questions to be completed because of time constraints.

Table 4.31. The UK and the TR teachers' stages of lessons.

UK teachers	**TR teachers**
- reviewing the previous lesson and introducing and explaining the topic of the lesson (5-10 minutes)	- reviewing the previous lesson and introducing and explaining the topic of the day (5-10 minutes)
- 3 or 4 (depending on the topic) worked examples (25-30 minutes)	- 3 or 4 (depending on the topic) worked examples (5-10 minutes)
- leaving some questions from worksheets or textbooks to students (the rest of the lesson until the last 2 minutes)	- explaining a new topic or carrying on with the previous topic (5-10 minutes)
- giving homework from worksheets or textbooks and sometimes telling students what is going to be done in next lesson	- 3 or 4 (depending on the topic) worked examples (5-10 minutes)

The UK teachers' teaching styles were similar but not identical. They never used the textbook during the explanation of the topic. Two of them used their own handouts. However, all of them used textbooks, worksheets and sometimes past exam questions for worked examples, classroom practices and homework. One of the teachers used an overhead projector during the theory part of the lesson. All teachers generally highlighted points students did not understand or points they considered important to the topic, e.g. "*...drawing a diagram is half of the battle...*" and algebraic manipulations. One of them underlined the importance of practice for recognising trigonometric identities in expressions. One of them recommended that students memorise basic trigonometric identities *"...you should repeat them every morning after breakfast and then draw a cross for each..."*

The TR teachers from the three different schools followed similar patterns. None of the TR teachers used textbooks in the explanation part of the lesson. Three of them used their own handouts throughout the lesson. The rest of the TR mathematics teachers used textbooks, university entrance examination questions, private institutes' textbooks and tests for worked examples. All of the TR teachers emphasised the key points and the points students did not understand: they asked the class whether they understood or not, if students did not then they repeated their explanations. Two of the TR teachers encouraged students to choose appropriate trigonometric identities in simplification questions *"...you should be careful in choosing trigonometric identities. You should anticipate what is going to happen in the next step and you should see whether the trigonometric identity you use really helps to find solution..."*, *"... you should recognise the appropriate trigonometric identities to use and move on in the solution ..."*. Another two emphasised techniques to learn trigonometric identities other than memorising *"...Do not try to memorise all trigonometric identities and formulae. Do not waste your time by memorising when you can derive them from each other as happens here, cosa.cosb can be derived from addition identities cos(a+b) and cos(a-b)..."*, *"...do not memorise...derive each trigonometry identity from each other... discover shortcuts for learning trigonometric identities and formulae..."*

Only two UK teachers called students to the whiteboard to solve examples and these two only did this one time each. Apart from these two instances, the UK teachers solved all examples on the whiteboard themselves. All TR teachers, however, encouraged students to come to the blackboard to solve questions throughout the worked examples part of the lesson. Both countries' teachers always worked on examples with students, and when needed, they revised prerequisites such as factorising trinomials or the difference of two squares.

From what I observed, there were two main exercise parts in the UK lessons (see Table 4.31.). These are a worked examples part and a classroom practice part, to which the last half of the lessons were dedicated. In TR, however, there were no classroom practice parts of lesson, but there were worked examples parts, among which some classroom practice took place. The UK teachers devoted more than half of the lesson to the classroom practice part, including worked examples, and they spent less than a quarter of the lesson on the theory part. The TR teachers, however, used their time to do worked examples. At the end of the lessons all the UK teachers gave homework. Only two TR teachers, however, left some worked examples as homework (to avoid spending more time on them in the lesson).

In the worked example part of the lessons, in co-operation with the students, UK teachers solved all questions at the whiteboard themselves. UK teachers never walked amongst the students during the worked example part but they did in the practice part of the lesson. In the practice part, they worked with the students, but they again wrote everything on the whiteboard themselves. They always first asked students about the solution and, if the students could not answer, they gave them some hints. They explained everything they wrote on the whiteboard and distributed their questions over all of the students. They tried to involve students in the lesson and share opinions on solutions. They answered all questions coming from students.

In contrast to UK teachers, the TR teachers left all answers, except the first one or two examples, to the students who solved them at the blackboard. When the student at the blackboard required help the teachers asked questions to the class and helped the student at the blackboard by underlining important points. Throughout the examples the TR teachers walked about the classroom and answered students' questions individually. The classes were crowded, so teachers insisted on quiet.

There were 9, 12, 13, 14 and 17 students in the UK classrooms and 33, 34, 34, 35, 35, 36, 36, 40, and 43 students in the TR classrooms. All UK students had their own calculators and textbooks and all TR students had their own textbook and notebooks.

2.2. Curriculum resources

As explained in the methodology chapter (see pp. 56 and 63), curricula, schemes of works, textbooks and high-stakes examinations in both the UK and TR were collected for document analysis to explore their possible influences on the UK and the TR students' performances. In this section I will present the result of the comparison of curricula, schemes of works, textbooks and high-stake exams in both the UK and TR in the following sub-sections: trigonometry in the curricula, trigonometry in teachers schemes of work, trigonometry in the textbooks and trigonometry questions in the high-stakes examinations respectively.

2.2.1. Trigonometry in the curricula

The trigonometry part of the national curriculum and syllabii in the UK and the national curriculum in TR are presented in Appendix Q. Trigonometry has a place in middle secondary and upper secondary school curricula in both the UK and TR. Trigonometry first appears in the middle secondary curricula for 14-16 year olds GCSE in the UK and year 8 of secondary school in TR. The second appearance of trigonometry occurs in upper secondary schools curricula for 16-18 year olds, A-level in the UK, and in year 10 of high schools in TR.

The first appearance of trigonometry is in the national curricula of both the UK and TR. The second appearance of trigonometry in the UK is at AS/A level syllabii whereas it is in the national curriculum in TR. In these curricula, trigonometry that has to be taught is presented in topic objectives. The trigonometry curricula of both the UK and TR were compared in terms of similarities and differences and briefly presented in Table 4.32. in terms of objectives.

Table 4.32. A comparison of the first and second appearance of trigonometry in the UK and the TR curricula.

Trigonometry in the middle secondary school (GCSE in the UK and year 8 in TR)			
Similarities	**Differences**		
	Subjects-Objectives	**UK**	**TR**
Trigonometrical relationships in right-angled triangles and use of these to solve problems.	Use of calculator	Y	N
	ICT	Y	N
	Real world application	Y	N
	Use of trigonometrical relationships in 3D contexts, including finding the angles between a line and a plane	Y	N
	Calculation of the area of a triangle using 1/2ab sin C	Y	N
	Drawing, sketching and describing the graphs of trigonometric functions for angles of any size, including transformations involving scalings in either or both the x and y directions	Y	N
	Use of sine and cosine rules to solve 2D and 3D problems	Y	N
	Surd forms of trigonometric ratios of 30, 45 and 60 (by using isosceles right-angled triangles and right-angled triangles with angles of 30 and 60)	N	Y
	Cotangent function and writing cotangent and tangent in terms of sine and cosine	N	Y
	Use of trigonometric tables	N	Y

Trigonometry in the upper secondary school (A-level in the UK and year 10 in TR) *Note: In the UK, trigonometry is presented under two separate headings 'plane trigonometry, and 'trigonometrical functions', whereas it is all presented under the heading 'trigonometry' in TR.*			
Similarities	**Differences**		
	Subjects-Objectives	**UK**	**TR**
Use of the sine and cosine rule and the formula 1/2absinC for the area of a triangle	Sector area of a circle	Y	N
	Defining unit circle, working with the angles in a unit circle, working with directed arcs on a unit circle and defining trigonometric functions on a unit circle	N	Y
	Trigonometric table	N	Y
	Calculator	Y	N
	Formula sheet	Y	N
	Existence of cotangent function in all objectives including trigonometric functions	N	Y
	Use of gradiends	N	Y
Note: There were no identical objectives in UK and TR curricula but there were some objectives, which partially coincided e.g. the expansion of sin(A±B), cos(A±B) and tan (A±B) in UK and the same functions with cot(A±B) in TR	Specific use of 'simplification'	Y	N
	Selection of appropriate identity	Y	N
	Half angle identity	N	Y
	Sum and product formulae	N	Y
	Tangent theory	N	Y
	Real world application	Y	N
	Periodicity and symmetries of the sine, cosine and tangent	Y	N
	Defining tangent and cotangent functions in the coordinate plane	N	Y
	Writing trigonometric ratios 3x in terms of the trigonometric ratios of x, e.g. $\cos 3x = 4\cos^3 x - 3\cos x$	N	Y
	Drawing graph of inverse trigonometric function	N	Y
	Writing the expressions 1+sinu, 1+cosu, 1+tanu, 1+cotu in product form	N	Y
	Finding the solutions of equations of the form t(x)=t(a) where t is a trigonometric function and a is any real number (i.e. cosx=cosa); t(f (x))=t(g (x)) where f and g functions from R to R (i.e. sin(f(x))=sin(g(x))); first and second degree homogenous equations for cosx and sinx; linear equations for sinx and cosx; factorable equations and solving a trigonometric equations by writing trigonometric function in terms of tan(x/2).	N	Y

2.2.2. Trigonometry in teachers schemes of works

All teachers who were observed produced schemes of works. Examples, one from each country, are shown in Appendix R. Schemes of works were analysed in terms of their format and content.

The TR teachers schemes of work had a common format with 12 columns. The columns from right to left were month, week, hour, subject (sub-objectives), core-objectives, method and techniques, resources, experiment/observation, cooperation of teachers, homework, examination and thoughts respectively. The TR teachers fill in the columns month, week, hour, subject (sub-objectives), core-objectives, method and techniques, resources and examinations. There was no specific format in the UK. The UK teachers annotated their copies of the syllabus.

The TR teacher's scheme of work in Appendix R has core subject objective (column 4) and topic objectives (column 5). Column 6, 7 and 10 concern questions & answers, textbooks & other resources and exercises respectively. Since the UK teacher used the original syllabus to prepare her scheme of work she automatically used all the objectives. She added two columns and wrote the corresponding pages of the textbooks and/or worksheets. Subsequently, the format of the schemes of works used in the UK and TR were different.

2.2.3. Trigonometry in the textbooks

To analyse trigonometry in textbooks, I chose the most used textbooks, according to teachers' questionnaire responses in the UK and TR. In the following, I look at teachers' choice of textbooks, topics presentation and types of the questions included in the textbooks.

2.2.3.1. Teachers' choice of textbooks

65 TR and 40 UK teachers completed the textbook questionnaire. All UK teachers used a textbook in teaching trigonometry. Half of the UK teachers' reasons for choosing their textbook were that it covered the syllabus. The remainder stated that they chose their textbook because of the explanation and the exercises. 23% of the TR teachers did not use any specific textbook, they prepared their own notes using various printed resources. All other TR teachers used a specific textbook, however almost half of them highlighted that they also use other printed resources to supplement the textbook used. A low percentage of the TR teachers used a specific textbook because it contains university entrance examination style test questions or it fitted in with the national curriculum. The majority of the TR teachers, nearly 75%, chose their textbook due to its explanations, quality of worked examples and exercises. Elsewhere in the interviews and the observations I saw teachers using worksheets to supplement textbooks but this did not come from the questionnaire.

As a result of the questionnaire the most used mathematics textbooks to teach trigonometry in both the UK and TR are analysed here. The most used textbooks in the first appearance of trigonometry, respectively, in the UK and TR are Rayner (1994) and Yildirim et. al. (1996). The most used textbooks in second appearance of trigonometry, respectively, in the UK and TR are Bostock and Chandler (1995) and Aydin and Asma (2000).

Before analysing the content of these textbooks the physical appearance of the textbooks are compared. The UK textbook is larger and heavier than the TR textbook. Trigonometry is presented under one heading in one chapter in the TR textbook. However, it is collected under three headings in two chapters in the UK textbook. The UK and the TR textbooks, which contain the second appearance of trigonometry, were almost the same size. However, there were two books, which included trigonometry, in the UK but only one in TR. The UK textbook presented trigonometry under several separate chapters distributed among other topics in the textbook whereas it was a single chapter in the TR textbook.

Before examining topics, presentation and high-stakes exams, I present the types of questions that arose in textbooks and examinations in either country. Table 4.33. presents the types of questions and introduces abbreviations which are used in the following five pages.

Table 4.33. Categories of the trigonometry questions in the UK and the TR textbooks.

Question categories appeared in the UK and the TR textbooks		
Categories	**Abbreviations**	**Explanations**
Geometric problems	GP	Questions in employing trigonometric functions on 2-D or 3-D geometric figures.
Exact value-numerical	EV-N	Calculating the value of a trigonometric expression when another trigonometric expression is given, e.g. given sin A=(3/4) find tan A.
Exact value-surd forms	EV-SF	Finding the value of a trigonometric expression using surd forms.
Exact value-algebraic	EV-A	Finding the value of a trigonometric expression in terms of variables.
Real world problems	TWP	Word problems in real world contexts.
Writing in terms of	W	Writing an expression in terms of another angle or function or variable or expression.
Solving equations	SE	Finding the solution of trigonometric equations.
Directed angles and arcs	DA	Questions regarding directed angles or arcs.
Simplification-algebraic	S-A	Simplifying expressions, e.g. $\frac{\sin x + \cos x}{\sin x - \cos x}$.
Simplification-numeric	S-N	Simplifying expressions, e.g. $\frac{\cos 80}{\cos 40 - \sin 40}$.
Elimination	E	Eliminating angles using trigonometric identities, e.g. eliminate θ from x=tan2θ y=tanθ.
Proof	PR	Proving identities.
Verification-algebraic	V-A	Verifying an expression in algebraic form.
Verification-geometric	V-G	Verifying an expression using a geometric figure.
Graph	G	Any questions relevant to graphs.
Periods	PE	Finding the period of a trigonometric function.
Inverse function	IF	Question solved by using 'arcsin', e.g. arccos(1/2).
Sign of functions	SoF	Finding the signs of trigonometric functions for angles of any magnitude.

Table 4.34. is the first part of a much larger table not included in this thesis due to its size. Table 4.34. focuses on part of the UK GCSE textbook and all of the TR year 8 textbooks. The table illustrates similarities and differences in named topics. This table is used in analysing topics, presentation of topics and types of the questions in the textbooks.

Table 4.34. Analysing the UK and the TR textbooks which contain the first and the second appearance of trigonometry.

UK textbook – GCSE		**TR textbook – year 8**	
Topic (page number)	**Category**	**Topic (page number)**	**Category**
Trigonometry (p.122)		Trigonometric ratios (p.155)	
Right-angled triangles; sine cosine, tangent (pp.122-123) - 2 examples (use of calculators) - [1-26] Exercises	 GP GP	a) Trigonometric ratios in right angle triangles; sin, cos, tan, cot (p.155) - [1a-1d] worked example - [2a-2b] worked example - 1 worked example - 3 worked example	 GP GP W EV-N, PR
Finding an unknown angle (p.125) - 1 worked example (calculator) [1-25] Exercise	 GP GP	b) The trigonometric ratios of the angles 30^0, 45^0 and 60^0 (p.162) - 2 worked examples (each of which with three sub-questions)	 EV-SF
Bearings (p.127) - Exercises; [1-2] 3 (Questions 4-21 not presented here) [22-28]	 TWP GP TWP	c) Table of trigonometric ratios (p.164) - 2 worked examples	 EV-N
Three-dimensional problems (p.132) - 2 examples - Exercises; [1a-1c] (Questions 2-7 not presented here) [8-9] Projections and planes (p.135) - [1a-1b] worked example - Exercises; [1a-1c]...7 (with sub- questions)	 GP (3-D) GP (3-D) TWP (3-D) GP (3-D) GP (3-D)	d) Problem solving by applying trigonometric ratios (p.166) - 3 worked example (first one has two sub-questions)	 GP, TWP
Sine, cosine, tangent for any angle (p.216) - Exercises; [1a-1b] (Questions 4-21 not presented here) [23a-23b]	 EV-N, G G, SE	Exercises; [1a-1f] [2a-2e] 3 (Questions 4-21 not presented here) 24	 GP GP EV-SF TWP

Key: GP-geometric representation, W-writing in terms of, EV-N-exact value-numerical, PR-proof, EV-SF-exact value-surd forms, TWP-trigonometry word problem, G- graph, SE-solving equations, 3-D-three-dimensional,

2.2.3.2. Topics

The topics of the textbooks were analysed in terms of their place in the curricula. In the first appearance of trigonometry, the TR textbook followed the same order as the objectives in the national curriculum and used each objective as the heading. The TR textbook also contained some extra topics such as cot A. tan A=1 and $\sin^2\theta+\cos^2\theta=1$. Likewise, the UK textbook also followed the same order of the objectives as in the national curriculum. However, the national curriculum objective, calculate the area of a triangle using $\frac{1}{2}ab\sin C$, was not in the textbook.

In the second appearance of trigonometry, the TR textbook included all curriculum objectives. Some of the curriculum objectives were included as worked example rather than a topic on its own in the textbook. There was also some variation in the order of the curriculum objectives. In the UK textbook there were two missing syllabus objectives, inverse trigonometric relations and the expression of asinθ+bsinθ in the forms Rsin(θ±α) and Rcos(θ±α). Moreover, working in 3-D context and trigonometry word problems were emphasised in the textbooks.

2.2.3.3. Presentation

For the purpose of the presentation structure of the textbooks, all topics, worked examples and exercises were counted. The averages of these counts are presented in brackets in Table 4.35.

In the first appearance of trigonometry, the TR textbook has a single set of exercises at the very end of the trigonometry section whereas in the UK textbook there was an exercise part after each topic/example pair. In the second appearance of trigonometry, there was an exercise part at the end of the each chapter in the UK textbook whereas there were a single set of exercises after each topic/topic/example triple and there were also 3 sets of exercises at the very end of trigonometry in the TR textbook.

Table 4.35. Structures of the UK and the TR textbooks in term of trigonometry.

Presentation structures of the trigonometry in the UK and the TR textbooks			
First appearance of trigonometry in the textbooks		Second appearance of trigonometry in the textbooks	
UK	TR	UK	TR
- Topic (1) - Worked examples (2) - Exercises (24)	- Topic (1) - Worked examples (3) - Exercises (43)	- Topic (1) - Worked examples (2) - Exercises (17)	- Topic (1) - Topic (1) - Worked examples (4) - Exercises (30)

2.2.3.4. Types of the question in textbooks

Tables 4.36. and Table 4.37. show the question types used in worked examples and exercises in each textbook analysed.

Table 4.36. Question types in first appearance of trigonometry in the textbooks.

Categories	Number of question types in				Total number of questions		Overall percentage	
	Worked examples		Exercises					
	UK	TR	UK	TR	UK	TR	UK	TR
GP	11	8	146	19	157	27	**68**	**43**
EV-N	0	4	4	9	4	13	**2**	**21**
EV-SF	0	6	0	3	0	9	**0**	**14**
TWP	1	2	33	4	34	6	**15**	**10**
W	0	1	16	5	16	6	**7**	**10**
SE	0	0	14	0	14	0	**6**	**0**
S-N	0	0	0	1	0	1	**0**	**2**
PR	0	1	0	0	0	1	**0**	**2**
G	0	0	7	0	7	0	**3**	**0**

Table 4.37. Question types in second appearance of trigonometry in the textbooks.

Categories	Number of question types in				Total number of questions		Overall percentage	
	Worked example		Exercises					
	UK	TR	UK	TR	UK	TR	UK	TR
GP	19	31	113	46	132	77	**24**	**13**
EV-N	5	19	55	37	60	56	**11**	**9**
EV-SF	4	27	37	39	41	66	**8**	**11**
EV-A	0	1	1	3	1	4	**0**	**1**
TWP	4	0	24	0	28	0	**5**	**0**
W	3	14	26	45	29	59	**5**	**10**
SE	12	32	106	73	118	105	**22**	**17**
DA	0	18	21	20	21	38	**4**	**6**
S-A	2	10	17	38	19	48	**3**	**8**
S-N	0	5	0	12	0	17	**0**	**3**
E	2	0	17	0	19	0	**3**	**0**
PR	3	0	20	0	23	0	4	**0**
V-A	0	8	5	33	5	41	**1**	**7**
V-G	0	2	2	23	2	25	**0**	**4**
G	1	2	45	9	46	11	**8**	**2**
PE	0	10	0	16	0	26	**0**	**4**
IF	0	9	0	16	0	25	**0**	**4**
SoF	0	1	0	5	0	6	**0**	**1**

2.2.4. Trigonometry questions in high-stakes examinations

To analyse the examination questions, the university entrance examination papers in TR and pure mathematics papers containing trigonometry questions in AS/A examination papers in the UK were collected for the years 1999, 2000 and 2001. As for the textbook analysis, examination papers are analysed as in Table 4.38. First, trigonometry questions were found and then categorised by questions types (as for the textbooks) and then their percentage (trigonometry question÷all questions×100) was calculated as a crude measure of the importance and place of trigonometry in high-stake examinations in the UK and TR. To save space, only questions from 1999 are presented in the UK part of the table.

Table 4.38. The comparison of the UK and the TR high-stakes examinations.

UK high-stake examinations			TR high-stake examinations		
- The marks are given for each question - Student must show their manipulation			- The number of marks are not given - Multiple-choice questions students do not need to show their manipulation		
Year	**Question number**	**Categories**	**Year**	**Question number**	**Categories**
1999 March1 PM1	4 6i 6ii	GP GP GP	1999 UEE	30 31 41	GP TWP GP
1999 November19 PM1	3 4 9iii	GP GP G	2000 UEE	-	-
1999 June9 PM3	1 [3a-3b] [4a-4b] 6 7ii [8i-8iii] [9a-9b]	C* 2 C V-G, C SE C C, W, GP 2 C	2001 UEE	33 35 38	GP GP GP

** C stands for calculus, which is the only category different than the ones used to categorised the question types in the textbooks, e.g. integration.*

In all the years considered, 1999-2000, there were, respectively 8 and 3 high-stakes examination papers from the UK and TR and a total of 80 (158 with sub-questions) and 134 questions in them respectively. The categories of trigonometry questions in UK and TR high-stakes examinations are presented in Table 39. There are more trigonometry questions in the UK than in the TR examinations. A surprising result is that there were no trigonometry word problem in any UK examinations but there was one in the 1999 TR examination.

Table 4.39. Trigonometry questions in high-stakes examination of the UK and TR.

Categories	**Questions in the UK**		**Questions in TR**	
	Number	**% Over all questions**	**Number**	**% Over all questions**
GP	12	8	5	4
EV-SF	1	1	0	0
TWP	0	0	1	1
W	2	1	0	0
SE	6	4	0	0
V-G	3	2	0	0
G	3	2	0	0
C	23	15	0	0
Total trigonometry questions	50	32	6	4

CHAPTER 5: DISCUSSION

The results in the last chapter generate many interesting discussion points but space does not allow me to discuss every issue. In this chapter I discuss two major theoretical constructs that emerged from data analysis: a model of students' manner of simplifying trigonometric expressions (§2) and students' methods of solving trigonometric word problems (§3). The chapter concludes with an examination of my research questions in the light of the data collected. I start, however, with a speculative thesis – that trigonometry in the two countries is, effectively, two different topics.

1. Global discussion of the results: are these two types of trigonometry?

Shortly after completing the data collection an interesting idea occurred to me – that the trigonometry students learnt in the two countries was sufficiently different to justify a claim that trigonometry in each country effectively forms two different topics. How might this claim be examined? My first thoughts were curricula – if the content curricula/syllabuses of the two countries have more differences than they have similarities then there is a sense in which the two topics are dissimilar or different. But there is more to learning and teaching than simply curriculum content. I observed substantial differences in the tools used, in the classroom organisation and activity and in the outputs of the students (what they did). The next four subsections (§1.1 - §1.4) examine arguments for and against my 'two different trigonometry' thesis under the headings: curriculum, teachers, tools used in trigonometry class and students' overall performance.

1.1. Curriculum

The results of the investigation of the trigonometry curricula/syllabii and analysis of the textbooks and examinations within the UK and TR are discussed in this subsection. Differences and similarities are presented under the headings: curricula/syllabii, textbooks and examinations. Differences and similarities in the each heading are examined in terms of middle secondary (first appearance) and upper secondary (second appearance) of trigonometry in the UK and TR (see also p. 153).

1.1.1. Curriculum/syllabi

The first appearance

In both the UK and the TR curriculum the first appearance of the trigonometry occurs at the 14-16 age group. In the light of the overall comparison of trigonometry in both curricula (see Table 4.32, p. 154) the UK curriculum provides considerably more content and objectives than the TR curriculum e.g. bearings and use of trigonometric relations in 3-D contexts. Both countries study trigonometric ratios in right-angled triangles and use these to solve problems. These were the only similarities between the two countries curricula and, as has been seen (see Table 4.32,

p. 154), the types of problems solved are quite different. Surd forms of trigonometric ratios of 30^0, 45^0 and 60^0 and the use of trigonometric tables were the TR but not the UK curriculum objectives. The UK but not the TR curriculum, however, includes sine and cosine rules and sketching graphs of functions for angles of any size. In brief, there appear to be more dissimilarities at this level than there are similarities.

The second appearance

The second appearance of trigonometry occurs in both curricula at the 16-18 age group. There were not exactly the same objectives in the UK and the TR curricula but there were some objectives which partially coincided. Namely, there were some similarities, e.g. sine and cosine rule, some use of sine, cosine and tangent functions, as well as dissimilarities between the curricula of both the UK and the TR. Dissimilarities in the trigonometry curricula of both countries are evident in the place trigonometry occupies in the curriculum organization of each country. In the UK, trigonometry is presented under two separate headings 'plane trigonometry, and 'trigonometrical functions', whereas it is all presented under the heading 'trigonometry' in the TR. The UK but not the TR, moreover, highlighted 'simplification', movement between 'mathematics and the real world' and 'mathematical models' prior to specifying topics and curriculum objectives.

In this second appearance of the trigonometry, contrary to the first appearance, the TR curriculum provides considerably more content and objectives information than the UK. There were 14 core curriculum objectives, which consisted of 77 sub-curriculum objectives whereas there were eight main curriculum objectives with no specifically given sub-curriculum objectives in the UK curriculum. In the TR, half of the objectives addressed 'understanding' of the topic and the other half addressed 'application' of these topics. Some of the TR applications were geometric but most were algebraic and none were real world problems. Although there were analogous objectives in both countries' curriculum, beyond the words there were some crucial peculiarities: more rules/theorems, formulae and identities were included in the TR curriculum than the UK curriculum which revealed considerable differences between the two countries (see Table 4.32, p. 154). In the rest of this sub-section these differences are presented.

- In TR, the sine rule and the cosine rule were named as the sine theorem and the cosine theorem respectively. Hereafter rule/theorem is used as a common word instead of saying merely rule or theorem. Although the sine and the cosine rules/theorems and the formula $\frac{1}{2}ab\sin C$ for the area of a triangle were common to both curricula the tangent rule/theorem, the relations between the area, sides and trigonometric ratios of angles of a triangle and other formulae for area of a triangle were additionally underscored in the TR curriculum. Further to this grad angle measures are included in the TR but not the UK curriculum.

- A remarkable dissimilarity was the place of the trigonometric function 'cotangent' in both countries' curriculum. This function is mostly neglected in the UK curriculum whereas sine, cosine, tangent and cotangent were collectively 'trigonometric functions', which rarely included secant and cosecant as well, in TR, and all reference to trigonometric functions in the TR includes cotangent. The UK but not the TR curriculum evidently highlighted the six trigonometric functions. The use of these functions for angles of any magnitude was in the UK curriculum. However the approach, which is either right-angled triangle or unit circle, to teach these functions was not defined in the UK curriculum. The TR curriculum, however, particularly used the 'unit circle' approach to define trigonometric functions.

- The TR curriculum emphasised 'unit circle' approaches although 'right-angled triangle' approaches were sometimes used for acute angles whereas the UK curriculum did not explicitly emphasise any of these approaches.

- In the UK, periodicity/symmetry were collected into an objective which focused in the periodicity and symmetry of sin, cosine, and tan functions and the forms of their graphs. On the other hand, periodicity/symmetry and the graphs of trigonometric functions were not collected into a single objective, but were, respectively, collected into the objectives for 'understanding the trigonometric functions' and 'drawing the graphs of trigonometric functions' in the TR curriculum.

- Another remarkable dissimilarity in the objectives of both countries' curricula concerned arcs of circles. The UK but not the TR curriculum emphasised the use of the formulae for the length of the arc and the sectors' area of a circle. The TR curriculum, however, includes finding initial and terminal points of a directed arc, finding the direction of an arc if the initial and terminal points are known, and finding the coordinates of terminal points of the directed arcs with the lengths $\pi/2$, π, $3\pi/2$, 2π, $\pi/4$ and $\pi/6$ on a unit circle. There was no specific objective for the length of an arc in the TR curriculum.

- With regard to inverse functions the UK curriculum specifically highlighted the notations $\sin^{-1}x$, $\cos^{-1}x$, $\tan^{-1}x$ and use of these notations to denote principle values of inverse trigonometric relations. The TR curriculum, on the other hand, emphasised the definitions of inverse trigonometric functions and their graphs. There was no explicitly given notation for inverse function in the TR.

- The same trigonometric expressions, e.g. sin $(A\pm B)$, were differently named in two countries' curricula: trigonometric identities in the UK and trigonometric formulae/ trigonometric relations in the TR. Trigonometric identities/formulae are used as common name hereafter. The UK but not the TR curriculum primarily articulated the aims of using trigonometric identities and selecting an appropriate trigonometric identity/formula to the context and some specific identities/formulae are given to be familiar with. The TR curriculum, however, provided considerably more trigonometric identities/formulae under different objectives. The trigonometric identities/formulae, which were particularly emphasised in the TR but not in the UK curriculum were writing trigonometric functions in terms of each other, half angle identities, sum and difference formulae, product formulae, writing the expression of 1+sinu, 1+cosu, 1+tanu, 1+cotu in the form of products, writing the expressions of sin3a, cos3a, tan3a, cot3a in terms of sina, cosa, tana, cota respectively (and same with replacing '3a' by a and 'a' by a/2). As was mentioned earlier the 'cotangent' function is used in every possible trigonometric identities/formulae in the TR but not the UK curriculum except $\cot\theta=\cos\theta/\sin\theta$. Writing the expression $a\sin\theta+b\cos\theta$ in the form of $R\sin(\theta\pm\alpha)$ and $R\cos(\theta\pm\alpha)$ was an objective in the UK but not the TR curriculum.

- There was a sharp difference between two countries' trigonometry curricula in terms of solving trigonometric equations. The UK curriculum had only one objective which was finding the solutions of the equations sin(kx)=c, cos(kx)=c, tan(kx)=c and of equations easily reducible to these forms within specified intervals. The TR curriculum, on the other hand, provided 22 sub-objectives, none of which coincided with the UK objectives. The TR objectives included: Finding the solutions of the equations of the form sinx=a, cosx=a, tanx=a and cotx=a within a specified interval; t (x)=t (a) where t is a trigonometric function and a is any real number (i.e. cosx=cosa); t (f (x))=t (g (x)) where f and g functions from R to R (i.e. sin (f (x))=sin (g (x))); first and second degree homogenous equations for cosx and sinx; linear equations for sinx and cosx; factorable equations and solving a trigonometric equations by writing trigonometric function in terms of tan(x/2).
- The understanding and use of trigonometric tables was a curriculum objective in TR but not in the UK. Interestingly, the UK but not the TR curriculum explicitly provided the recall and use of surd forms of the sin, cos, tan at the specific angles 30^0, 45^0 and 60^0. Moreover, in terms of notation, all angles are represented in radian form in the TR curriculum, e.g. $\pi/6$.

1.1.2. Textbooks

In this section a discussion of the UK and the TR mathematics textbooks in terms of the presentation of the trigonometry related to the framework of this research study is presented. The discussion mainly focuses on four features of the textbooks: layout and appearance, content in terms of the topics covered by both curricula and textbooks, worked examples and exercises.

The first appearance

Although nearly the same percentages of pages of textbooks (Rayner, 1994 in UK and Yildirim et. al. 1996 in TR) were occupied by trigonometry it should be noted that the UK textbook is larger and heavier than the TR textbook. Trigonometry is presented under one heading in one chapter in the TR textbook, however, it is collected under three headings in two chapters in the UK textbook.

Both countries' textbooks are tied to objectives in the curriculum. However, there were some differences between the curricula and textbooks of both countries. The TR textbook followed the curriculum objectives in order and nomenclature. However, some new concepts, which are not detailed in the curriculum, are introduced, such as quotients and Pythagorean identities in the TR. The UK textbook, however, included all the objectives of the curriculum, except one, to calculate the area of a triangle using ½ absinC, in the same order as the curriculum (see Table 4.34., p. 157).

The structure of the presentation of trigonometry in the two textbooks differed. Each topic in the trigonometry section provided explanatory material followed by worked example and exercises in the UK textbook. In the TR textbook, however, exercises were provided at the end of the section after a number of 'topic, example' pairs. The UK textbook provided many questions for each topic compared with the TR one, namely in total there was a notable difference between the number of questions provided in both countries' textbooks (see Table 4.35., p. 158).

There was a notable difference between the two countries' textbooks in terms of the questions, which were more arithmetical in the UK and algebraic in the TR textbooks. Nevertheless, there were observable similarities and differences between the question categories and styles (see Table 4.36., p. 158) that could be explained by the differences in the curriculum objectives. Interestingly, although they were not in the curriculum there were some totally different question categories in both textbooks: solving equations in the UK and exact value, proof and simplification in the TR. Most of the questions using trigonometry used geometrical figures in both textbooks. Interestingly in the UK but not in the TR textbook real world problems had a very important place. The TR textbooks, however, provided more exact value questions. The UK, but not the TR, textbook provided questions in bearings/real world problems, graphs, trigonometric functions in 2-D and 3-D context which were the main differences in terms of questions.

Some interesting points emerged from analyzing the questions in both countries' textbooks. I think it is important to highlight that the use of calculators is demonstrated throughout the worked examples in the UK textbook. In the TR, but not the UK, textbooks trigonometric table is presented and then used in solving questions. In the UK textbooks, the angles used in examples and exercises were often decimal whereas in the TR they were merely the integers, mainly the specific angles 30^0, 45^0 and 60^0. This might possibly be explained by the use of calculators in the UK but not in the TR and furthermore the non-comprehensive use of trigonometric table in TR.

The second appearance

The first noticeable similarity between two textbooks of 16-18 years old students was their size, which were almost same. However there were two books, which included trigonometry, in the UK but only one in the TR. the UK textbook presented trigonometry under several separate chapters distributed among other topics in the textbook whereas it was a single chapter in the TR one.

As was discussed earlier (see p. 162), the TR curriculum provided considerably more content and objectives information than the UK and this fact showed itself in terms of the pages devoted to trigonometry.

There was a notable difference between the percentages of the pages occupied by trigonometry. When the content was investigated it was observed that the TR textbook exactly covered each curriculum objective defined in the curriculum. The UK textbook, however, did not cover two objectives of the curriculum (see p. 158). Organization of the trigonometry in curricula and textbooks was different in both countries. Furthermore, both countries' textbooks contained some topics which were completely different than each other and not contained in both countries' curricula. The dissimilarities between the curricula of two countries were also seen in their textbooks.

At the second appearance of trigonometry, both the UK and the TR textbooks presented trigonometry in different structures, which were very similar to the ones in the textbooks in the first appearance. Each topic in the trigonometry section provided explanatory material followed by worked example and exercises in the UK textbook and there was an exercise at the end of the each chapter. On the other hand, exercises were provided after a number of 'topic, topic, and example' triples in the TR textbook and there were worked and non-worked tests at the end of the chapter. In average and in total there were more worked example and exercise questions in the TR textbook than the UK textbook.

Dissimilarities can be seen among the question categories incorporated by both countries' textbooks (see Table 4.37., p. 159). Use of trigonometry on geometrical figures and solving equations were the modal categories in both textbooks. However, questions in the same category can be very different in terms of difficulty. The weightings given to common topics in each textbook were different in terms of pages, information and styles and that produced different sorts of questions in solving equations, using trigonometry on geometrical figures, finding exact value-numerical and exact value-surd forms. There were some different question categories. This occurred either because of the differences in the curricula or the additional topics that emerged in textbooks (see p. 158). Although no proof question appeared in the worked examples and exercises in the TR textbook, all identities, which were treated as a topic, were proved or left as challenging questions to be completed by the student. In brief, the TR textbook was more algebraic compared with the UK one which was based on application questions such as trigonometric word problems. Interestingly, the TR textbook did not contain any trigonometry word problems.

Some interesting points emerged in investigating examples used. I think it is important to highlight these because these might exemplify reasons behind the dissimilarity between two textbooks. One is that the TR, unlike the UK textbook, the function 'cotangent' was treated equally with the other three basic functions in each topic such as graphs and inverse functions and this enriched the variety of the questions in the TR textbook. One of the other points is that the calculators (sometimes computers) appeared as the main tool to solve questions in the UK textbook, particularly for calculator exercises, however all questions must be done by paper and pencil method in the TR. This might be important in the development of different cognitive processes. An important point was that the TR, but not the UK, textbook provide the use of 'unit circle' as a main tool to explain trigonometry topics. The UK textbook gave an important part to working with graphs, particularly in solving trigonometric equations whereas it was only used at the application of drawing graph of the trigonometric functions in TR.

Given my research interests I was interested in the place of trigonometric word problems (TWP) and simplification of trigonometric expressions examples and questions in textbooks. Interestingly, although TWP was not explicitly given as a curriculum objective in the UK, developing understanding of the topics through the applications was explained before the curriculum objectives were given. By the application the use of word problems was highlighted in the UK. Subsequently, TWP and trigonometric functions and TWP in 3-D contexts had an important place in the UK textbook, as it was in the first appearance. In most of the TWP, diagrams were not given in the questions as they had to be to constructed by the solver. In TR, interestingly, even though there were explicitly given objectives of the 'application' of the topics (Appendix Q) TWP were not present in the TR textbook.

It can be observed from both textbooks that simplification was an operation which could be utilized in many trigonometry topics. Remarkably, simplification was not defined in both textbooks. However, the UK textbook, and curriculum as well, was clearly using it by saying the trigonometric identities can be used to simplify trigonometric expression (Bostock and Chandler, 1995 and see Appendix Q). In the TR textbook, there was only one use of simplification it was a warning note for the reader to 'simplify an expression in a trigonometric equation by using trigonometric identities and rules, if it is complicated' (Aydin and Asma, 2000). In some solving equation questions, simplification was required before solving the equation. With regard to other question categories neither textbook distinguished between simplification procedures in verifications or proofs. The textbooks, in all categories of questions, simply indicated that simplification was required.. Algebraic properties are also important in the simplification, but there was no specifically devoted section in either the UK or the TR textbook pertinent to use of algebraic properties. In solving equations part of the TR textbook, substitution is used to write trigonometric expressions as algebraic expression.

1.1.3. Examinations

After the curriculum and textbooks, the other important document source, with regard to my hypothesis that there are two types of trigonometry in the two countries is the examinations. By examinations I mean high-stakes examinations which are prepared by official sources and taken by all students. Internal school examinations were not considered, because they would be prepared by different teachers, who might emphasise different topics and even different question styles within the same topic and that is against the spirit of comparison. In view of the fact that there was no comparable high-stakes examinations in the UK and the TR at the 14-16 age group, the first appearance of the trigonometry is not considered. On the other hand, there were comparable high stakes- examinations at the 16-18 age group. These examinations were the A-level in the UK and the University Entrance Examination (UEE) in TR. For the sake of comparison, the most important common feature of the both examinations is that both were prepared by official sources for the same age group of students. It is worth noting that A-level at that time, was assessed by 6 to 12 modular examinations (depending on the A-level syllabus) spread out over a two year period in the UK. UEE occurs once at the end of the three year high school period in the TR.

The first noticeable dissimilarity between the high-stakes examinations in the UK and the TR is its presentation. The UK one has extended response questions where students need to show their work on the paper clearly and the number of the mark is given in the brackets at the end of each question. On the other hand, the UEE is a multiple-choice test with the final mark determined as a function of correct answers. An interesting difference was the tools used in the exams. In the UK but not in the TR examination the use of the calculators is particularly highlighted. This might be the reason that any angle in trigonometry questions in the UK examinations was used whereas only the specific angles 30^0, 45^0 and 60^0 were used in the TR.

Similarities and differences emerged in the comparison of both countries' high-stakes examinations in terms of trigonometry. There were fewer trigonometry questions in the TR examinations compared to the UK examinations. Please note that question types used below are those categorised in Table 4.33 on page 156. Geometric problems was a common question type in both countries' examinations. There was only one trigonometry word problem in the TR but none in the UK even though real world application problems of trigonometry are not emphasised in the TR and are emphasised in the UK (see Table 4.39, p. 160). The UK high-stakes examinations contained six types of questions: calculus, graph, verification-geometric, writing in terms of, exact value-surd forms and solving equations (see also page) which were not in the TR examinations. Calculus was the most common question category. After that solving equations, graphs and verifying were jointly the most common question types. The UK but not the TR high-stake examinations included algebraic questions. Consequently, even though the TR curriculum and textbook provided considerably more content and objectives information than the UK there was less emphasis on trigonometry in the TR high-stakes examination compared with the UK one (see Table 4.38., p.159).

1.2. Teachers

I am not interested in teachers per se but in how what they do affects students' understanding of trigonometry. In the early stages of design I simply expected to interview and observe them regarding their attitudes and teaching style. By the end of data collection, however, I was impressed by vast differences in what teachers did, where they did it and what they did it with. In my attempt to intellectualise my 'feelings' on these matters I focused on actions and motives (what is done and why that is done). I use Wertsch's (1998) discussion of Burke's 'pentad' approach to human actions and motives (ibid, 11-17):

> We shall use five terms as generating principles of our investigation. They are: Act, Scene, Agent, Agency, Purpose. In a rounded statement about motives, you must have some word that names the *act* (names what took place, in thought or deed), and another that names the *scene* (the background of the act, the situation in which it occurred); also, you must indicate what person or kind of person (*agent*) performed the act, what means or instruments he used (*agency*), and the *purpose*.

I use Burke's five terms (see 1969a and 1969b) in the subsections below but rename them with the more common words: what, where, who, how and why.

1.2.1. What: what do teachers do?

Teachers teach trigonometry in the classroom by using resources and tools. Lessons I observed consisted of two parts, teacher explanation and student practice. These two parts are discussed below.

Trigonometry lessons in the TR are more 'abstract' than in the UK. The TR trigonometry lessons centred on simplification, solving equation and inequalities, trigonometry on right-angled triangles and other geometrical figures. The TR teachers complained about the shortage of application problems in the curriculum and also said trigonometry word problems are not part of the university entrance examination, so they almost never do application problems. On the contrary, teachers and students in the UK solve more application problems than in TR. My observations suggested that the TR teachers use a wider variety of examples.

With regard to simplifying trigonometric expressions the TR teachers, by their verbal comments to students, placed a greater emphasis than the UK teachers on students memorizing trigonometric identities. Both the TR and the UK teachers emphasised the importance of getting the algebra correct but did this in different ways. Some of the UK, but none of the TR, teachers used algebra to explicitly illustrate what might be done to an expression, e.g. substitution to convert a trigonometric expression to an algebraic expression. The TR teachers, however, stuck with the trigonometry. They did not solve any algebra question before the trigonometry. When the TR teachers needed to remind students pertinent algebraic properties to use in trigonometry they wrote it on a side of the blackboard.

The procedures teachers followed in solving trigonometry questions were also different. In the UK questions were mostly based on the numerical calculations whereas in the TR they were all done by paper and pencil methods involving algebraic manipulations. There was a dissimilarity between what the UK and the TR teachers did in first appearance of the trigonometry. All the UK, but none of the TR, teachers used the same mnemonic way 'SOHCAHTOA' to teach basic trigonometric functions (and cotangent is not included) on a right-angled triangle (see p. 143, the use of the mnemonic 'SOHCAHTOA' is also appeared in the literature, e.g. Pritchard and Simpson (1999) and Kendal (1992)). On the other hand, every the TR teacher used their own acronyms to teach trigonometric function on right-angled triangles. Some the TR teachers also used some acronyms in the second appearance of trigonometry for the trigonometric identities whereas the UK teachers did not, they said it would be helpful but they are on a formulae sheet.

TWP was one of the topic students had difficulty with. The UK teachers gave a fixed set of detailed steps to follow for solving a TWP. The TR teachers, however, merely emphasised the importance of drawing diagrams and they wanted students to develop their own way, so they did not give fixed set of steps to follow for solving TWP (see p. 142). Subsequently the UK and the TR teachers had two different approaches for solving a TWP.

1.2.2. Where: the classroom

The classroom is where teaching occurs. Observation revealed interesting dissimilarities between the physical environments for teaching trigonometry. The UK teachers were working in a richer mathematics environment than the TR ones. The use of tools and resources differed (I will say more on this in section 1.3). In particular the UK classrooms have calculators, computers and overhead projectors. The UK school had dedicated mathematics rooms, whereas the TR school did not, and these had mathematical posters and resources. Backhouse et. al. (1994, p. 63) discusses the physical environment of mathematics classrooms and notes that different environments lead to different student activities. This is valid in my observations and supports my thesis that two types of trigonometry are being undertaken.

In UK, class sizes were, relative to TR, quite small. Ryan et al. (1989, p. 72) reported that class size is commonly believed to affect the nature of the teaching–learning process. In a very large class, for example, teachers may find it difficult to use small group practices or interact frequently with individual students. So, deliberate use of small groups encourages active participation. Sociological research in this field shows that participation lessens rapidly as groups grow in size (Brissenden, 1980, p. 76). In the UK teachers and students interact in trigonometry lessons more than in TR. That, particularly, appeared to be very important factor in teaching simplifying trigonometric expressions and solving real world problems, that students had difficulties with. The UK, but not the TR, classrooms had dedicated mathematics classrooms and the walls were full of posters on basic trigonometric functions, trigonometric ratios and geometrical figures. These posters summarise the essential points, which the class work aimed at, with the aim to fix the fundamental concepts. Every student had her/his own calculator. In TR, on the other hand, calculators are not allowed to be used in lessons. Overhead projectors and computers are not used in the lessons. Teachers mainly use chalkboards and students sit on benches.

I believe that it is also worth to mention about one more fact that might influence the classroom teaching of trigonometry. The UK teachers had their own (departmental) room which was full with textbooks, worksheets, calculators and a computer. Calculators and textbooks shortages might lead to problem in trigonometry teaching, especially when students get to keep them, so when students forgot to bring something like a textbook or calculator the teachers provided it. In TR, on the contrary, there was no departmental room, just a common staff room which did not contain mathematics resources.

1.2.3. Who: teachers' thoughts about trigonometry

Teachers have an important role in education systems. One of the duties is teaching of the topics designated by official sources. They interact with the students through the topics in the classroom by utilising official sanctioned resources and tools. So their thoughts of trigonometry reflects trigonometry in the UK and the TR contexts.

First and second appearance of trigonometry

First of all, both the UK and the TR teachers had different views and approaches with regard to the first and second appearance of trigonometry. The UK teachers saw the first appearance of trigonometry as working with numbers. The TR teachers, however, claimed that the first appearance of trigonometry was to provide experience with the basic trigonometric functions on right-angled triangles and that memorising trigonometric functions is important. The UK and the TR teachers differed in their views on the transition from the first to the second appearance of trigonometry. The UK teachers highlighted the importance of the algebra which, they declared, is the main difference between the first and second appearance of trigonometry. The TR teachers, on the other hand, drew a different picture. None of them mentioned algebra but claimed that the second appearance of trigonometry is more abstract and deeper compared to the first appearance.

Although both the UK and the TR teachers focused on right-angled triangles in the first appearance of trigonometry, they used different methods for teaching and building up students' experiences. The TR but not the UK teachers stated that the unit circle method should be taught in the first appearance of trigonometry. The UK but not the TR teachers highlighted the place of real world application of trigonometry at first appearance of trigonometry. Although the UK teachers highlighted the place of TWP in teaching trigonometry, interestingly the TR, but not the UK, teachers stated that understanding how mathematics is used in the real world is very important to be good at trigonometry.

In general, in both appearances, the TR teachers thought being able to manage using a calculator is not important to be good at trigonometry. On the contrary the UK teachers thought it is very important. The UK more than the TR teachers disagreed that trigonometry should be learnt as a set of algorithms.

Regarding the simplification of trigonometric expressions the UK teachers thought being competent and confident in algebraic manipulations, being familiar with the trigonometric identities and learning and memorising trigonometric identities by doing lots of lots of practice were important. The TR teachers focused on the use of algebraic methods and knowing trigonometric identities as the two foremost factors in being successful in simplifying trigonometric expressions. Although algebraic properties and trigonometric identities were highlighted by both the UK and the TR teachers, none of the TR teachers wanted students to memorise trigonometric identities, they said students should know how trigonometric identities are derived. However a small number of them also supported memorising because of the time limitation in the university entrance examination. On the contrary, although formula sheets are allowed the UK, teachers wanted their students to memorise trigonometric identities but also highlighted that students should recognise trigonometric identities. Despite these differences in teachers' views and approaches on students' simplification, procedures in the two countries were remarkably similar (I discuss this further in the section 2 of this chapter)

1.2.4. How: teaching styles and patterns of the lessons

Teaching is a complex and dynamic phenomenon. Every teacher has her/his own 'teaching style'. The UK and the TR teachers teaching style had dissimilarities as well as similarities in trigonometry lessons. In both countries teachers had different patterns to their lessons. Both the UK and the TR teachers said their trigonometry lesson follow a fixed pattern. Indeed, there was a fixed pattern of each observed in the TR and the UK teachers' lessons. These two patterns, however, were quite different from each other (see Table 5.1.).

Table 5.1. The pattern of the lessons in the UK and TR respectively.

England	**Turkey**
review of the last lesson introductory explanation worked examples students do exercises	review of the last lesson a) introductory explanation b) worked examples (sometimes by students at the board) (a) and (b) repeated several times

Just as the teachers in Pepin's (1999) study, both the TR and the UK teachers were concerned with covering the content of the curriculum. The UK teachers spent comparatively little time explaining the topic to the entire class. They introduced and explained a concept or skill to students, gave examples on the board and then expected students to practise on their own while they attended to individual students. The TR teachers, on the other hand, devoted the most of the lesson time to the explanation of the topic and there were more worked examples on the board. Brophy and Good (1986) reviewed the literature on teaching behaviours and found that students learn more in classes where their teachers spend most of their time actively teaching, rather than students working on their own without direct supervision. This could possibly be one of the reasons behind the students' performance in TT test (see p. 76). Interestingly, the UK and the TR lessons pattern were similar to the structure of the trigonometry presentation in the textbooks (see section 1.1.2, p. 164). Both the UK and the TR teachers had different observed teaching styles but differences in teaching styles do not necessarily mean a different trigonometry. However different teaching style might bring different approaches to teaching as Runesson (1999) showed, although teachers taught the same topic with similar classroom organizations they taught it in different ways by focussing on various aspects of the content. Moreover, she also found that different teachers exposed their students to variations on different dimensions according to what they focused on. This might affect students' performance in trigonometry because, as noted by Ling and Man (2000):

> the way teachers use variations on the object of learning, the number of dimensions of variation constituted on the same object of learning, and whether they are simultaneous, are significant to students learning in that they help students to discern the critical aspects of the object of learning, as well as the relationship between the critical aspects.

The topics both the UK and the TR teachers taught in trigonometry were strictly tied to the curricula. That means the topics in trigonometry they taught reflected curricula differences (I discussed curricula differences in section 1.1, p. 161). Teaching approaches to teach different topics or the same topics might change. Furthermore, teachers are a bridge between students and trigonometry and they teach trigonometry by using their teaching styles, so it is important to observe differences between two countries' teachers' styles too.

A majority of the TR teachers but very few of the UK teachers claimed they followed the textbook closely in teaching trigonometry, and this mirrors what was observed. Most of the TR teachers tied their lesson notes to the textbook and referred to these during the lesson. The UK teachers, however, used worksheets, their own printed explanations (handouts) and past examination papers as supplements for textbook work. The upshot of this with regard to teaching style was that the TR teachers wrote everything on the board from their notes whereas the UK teachers appeared to improvise more and interact with students' progress. The consequences with regard to student activity were listening and copying in the case of the TR students and a considerable amount of time working on problems in the case of the UK students.

In the worked example part of the lessons in co-operation with the students, the UK teachers solved all questions at the whiteboard by themselves. The UK teachers never walked among students during worked examples but they did do this in the practice part of the lesson. In the practice part, teachers worked with the students and led them to answer questions first, but they again wrote solutions on the whiteboard themselves rather than calling students to the whiteboard. This appeared to give an opportunity for some students to have ownership of the questions. The students who answer the questions correctly in the time teacher allocated might have the ownership over the questions because they found the answer before the teacher. The teachers' solutions on the board were for students who could not solve them. Teachers mostly used the students' answers on the board so that there was more room for students' methods. This is very important because solutions may vary from student to student especially in terms of the simplification process in which various trigonometric identities can be used. In this way students appear to increase their heuristic armoury for use on subsequent problems. The UK teachers did not focus on a student, their questioning was well distributed over students. All the UK teachers tried to involve each student in the lesson and share their solutions. They made good use of students' responses and opinions and answered all questions.

There was no specific seatwork exercise part in the TR lessons, but after the worked examples teachers sometimes gave questions for students to solve in a short time. A student then came to the board and solved the examples. These appeared to be some benefits for students learning to do this in that the student at the board was in control and the other students scrutinised their work. If students did not understand the example, the teacher solved a similar type of question or repeated the solution explaining steps more than once. The repetition, the novel situation of problem solving, and the ownership that students take over the problems all seem to aid the learning process (Ellerton and Clements, 1992). Throughout the examples some TR teachers walked about the classroom and answered students' questions individually but it was not common as in the UK.

1.2.5. Why: teachers' motives

Teachers' motives are complex and variable. I do not pretend to delve deeply into them here. However, all teachers I met during my data collection were clearly motivated that their students would understand trigonometry and do well in examinations. These were also motivated by a need to teach the prescribed curriculum. I sub-divide this section into three sub-sections: curricula, textbook and examinations.

Curricula

There was a national curriculum in TR and there were national curriculum and syllabus in the UK. The TR national curriculum incorporated the first and second appearance of trigonometry. In the UK, however, the first appearance of trigonometry was included in the national curriculum and the second appearance was included in the syllabus. The UK teachers stated general satisfaction with the trigonometry curriculum. Although the TR teachers stated that they follow what the national curriculum wants them to do for teaching trigonometry and that they were restrictedly tied to it, they agreed that they might modify something in the curriculum with their colleagues when some changes are needed, e.g. the teaching order of the topics. Even though teachers from both countries followed a national curriculum they viewed the trigonometry curriculum very differently. Evidence for this comes from their responses to a jigsaw curriculum task (not reported in this thesis). In this task teachers had to order trigonometry topics in the order that they would teach them. None of the UK and the TR teachers' rank of topics of teaching trigonometry coincided with each other, they were all different. Overall, both the UK and the TR teachers gave a different order of topics although there were some similarities. Further to this, one of the UK teachers emphasised the point that curriculum says what should be taught but not how it should be taught. Despite all different views, teachers from both countries follow their national curriculum and the differences in the curriculum might mirror in their teaching.

Textbooks

The main resources both countries' teachers used were printed resources. What appears in a mathematics textbook does not appear by chance. It is influenced by an educational culture. In this way mathematics textbook provide a window into the mathematics education world of a particular country (Harries and Sutherland, 2000). Therefore the UK and the TR textbooks reflected their countries' curriculum.

Although a majority of the TR teachers, but very few of the UK teachers, claimed they followed the textbook closely in teaching trigonometry, and this mirrors what was observed, they stated different views in their interview. The UK, but not the TR, teachers find textbooks satisfactory, however both provide their own notes. Further to this, worksheets and past examination papers in the UK and a variety of textbooks, tests, question banks and private institutes' textbooks in the TR are also provided by teachers. The UK teachers refer to the student edition of the textbooks whereas the TR teachers refer to other resource books and the teacher's edition textbooks in planning their trigonometry lessons.

Examinations

Lessons in both countries could be said to be 'driven' by high-stakes examinations, modular examinations in the UK and university entrance examinations in the TR. Examination boards, in the UK, also produced 'primers' booklets of examination type questions whereas private institutes in the TR prepare students for examinations with a wide range of questions.

In the TR questions banks, tests and textbooks from private institutes prepare students for the UEE. Furthermore, in the TR, author(s) and private institutes prepare the questions by themselves which are approved by the ministry of education. In the UK, however, examination boards prepare the questions and the primers. There are also commercially produced tests including examination sort of questions in the UK. The high-stakes examinations in both countries were compared (see p. 159-160) in terms of the trigonometry questions asked. The number of and types of questions in the examinations surely reflects on trigonometry in two countries. There were totally eight types of trigonometry questions in high-stakes examinations of both the UK and TR. Only trigonometry word problems and geometric problems appeared in the TR examinations, which was a very small percentage of the all questions on the UEE. In the UK, however, all types of questions, except trigonometry word problems, were seen in examinations, which accounted for almost one third of all the questions (see p. 160). This difference in types of questions may suggest two different trigonometries although this difference, with regard to TWP, was not expected from curricula considerations. Other examination differences were the angles used, any angle in the UK but 30^0, 45^0 and 60^0 in the TR and the use, or not, of calculators.

1.3. Tools used in trigonometry class

A tool is an object used to perform a task. Obvious mathematical tools are calculators and trigonometric tables. But formulae and algorithms are also mathematical tools. How tools are used by 'agents' (students and teachers) is at the heart of mathematics educational enquiry. A quite amazing difference in tool use in trigonometry classes was noted. I sub-divide my discussion below under two main titles: physical tools-calculators, formulae sheet, trigonometric tables; and conceptual tools, which are further sub-divided.

1.3.1. Physical tools

Physical tools are the concrete materials used in trigonometry classrooms. Apart from paper and pencil, which was common to both countries, a considerable difference appeared between the UK and the TR in terms of physical tools. In the UK, but not the TR, calculators and formula sheet were the physical tools mainly used. In the TR, but not the UK, classes, however, trigonometric table was the physical tool, which was mainly used, but this was not used extensively. Despite these differences there was a common tool extensively used in both the UK and the TR classrooms.

Calculators

Calculators were common place in the UK classrooms but were not allowed in the TR classrooms. This appears to partially explain at least two noted phenomena: little emphasis on secants, cosecants and cotangents in the UK; an emphasis on special angles, e.g. 30^0, 45^0, 60^0 and 90^0, and surd forms, e.g. $\sin 60 = \sqrt{3}/2$ in TR. Both the TR students and teachers complained about the range of angles considered. Because of this they did not make considerable use of trigonometric tables. The UK students used all kinds of angles.

The UK but not the TR teachers strongly agreed (see p. 133) that they encourage the use of calculators in trigonometry. Further to this they set aside some time in trigonometry lessons in order to teach students how to use their calculators. All the UK teachers also claimed that their students use calculators in trigonometry lessons for checking answers, routine computation and solving problems. As mentioned above, the UK teachers think being able to use calculators is important in trigonometry for students. The importance of using calculators is highlighted in both first and second appearance of trigonometry.

Official sources, national curriculum, syllabus, textbooks and high-stakes examinations stress calculator use in the UK. On the other hand, calculators are not used in the TR education system. The TR teachers' reasons for this include: it is not in official sources, e.g. curriculum and textbooks; the cost to students, family and school; and they make students lazy.

In the UK but not TR, decimal approximation method was used in angles and the values of trigonometric ratios such as 23^0, and 72.5^0 whereas the specific angles such as 30^0, 45^0 and 60^0 were almost always used in TR. This resulted in the UK teachers and students working with any angle (more realistic data) in trigonometry and the TR working with a limited range of angles. Calculators help the UK students make fewer mistakes on numerical manipulations. Calculators also saved the UK students' time in solving questions, because as it is frequently suggested that use of calculator frees students to focus on strategical issues when tackling problems (Ruthven 1996, p. 456). The development of trigonometry might become easier with calculators. It also becomes approachable earlier and to students of a wider range of ability than before (Noble-Nesbitt 1982, p. 150). In TR, on the other hand, students might spend a longer time on manipulations and that could distract students' focus. Undoubtedly, calculator makes trigonometry in both countries effectively different.

Formulae sheets

In the UK, but not in TR, formulae sheets were extensively used in trigonometry classrooms throughout explanations, worked examples, exercises and in exams. This was a big difference between the two countries' trigonometry classrooms because the UK teachers and students were concentrating on the formulae sheets to find the appropriate identity to use in simplifying trigonometric expressions whilst the TR teachers were proving or showing how to derive trigonometric identities and then expecting students to know them (see p. 144). I think it is important to note here that the number of trigonometric identities used in the TR is considerably more than in the UK. So, a possible formula sheet for the TR students would be much longer than the one the UK students use. Interestingly most of the TR teachers and students objected to the word 'memorising'. The TR teachers did not want their students to memorise trigonometric identities and the TR students advocated that they were learning trigonometric identities by logic or by solving lots of questions. Another difference that using a formulae sheet might make was in the cognitive actions. The UK teachers and students were learning to recognize and become familiar with the trigonometric identities in simplification questions by using a formulae sheet whereas the TR students had to correctly 'recall' trigonometric identities throughout their solutions. The UK students claimed that they cannot solve questions in their head without using a formula sheet. I observed the UK students solving all the questions using the formulae sheet. They spent as much time consulting the sheet as they were doing the solution. In every rewritten form of the initial expression in the simplification, they concentrated on the formula sheet.

Despite the formulae sheets in the UK and objection to formulae memorization in TR, both the UK and the TR teachers, particularly all of the UK ones, endorsed remembering formulae and procedures is important for students to be good at trigonometry. So they implied that students memorise the trigonometric identities although as noted above, the TR teachers and students objected to the term 'memorization'. The TR students may have some anxiety to whether they could remember (all) the identities or not. The UK students, however, would not have such a problem because they would have all identities on the formula sheet. Subsequently, in the TR, students were expected to memorise lots of trigonometric identities whereas the UK students were not. The effectiveness of the using and not using formula sheet might be discussed more deeply, but this thesis is not the place. Nor do I wish to judge on this matter. My point is simply that there is a sense in which trig with formula sheet and trig without formula sheet is somewhat different.

Trigonometric table

Trigonometric tables were a tool used in the TR, but not in the UK, trigonometry classrooms. The reason for their non use in the UK is the use of calculators. The use of trigonometric table was a curriculum objective in TR. However, it was not comprehensively used in trigonometry classrooms in the TR as calculators and formula sheets were used in the UK. It was only used when the use of trigonometric table was the focus of the lesson. Furthermore, the TR teachers claimed that since angles other than the special ones are not used in questions in the UEE they do not work with a wide range of angles, so they did not really use trigonometric table comprehensively. Consequently, it is a little used tool and is not a contributory factor for two types of trigonometry in the two countries.

Other physical tools

There were other physical tools but these were not used as often as the above ones. One is the overhead projector (OHP), which is used in the UK but not the TR classrooms. One of the UK teachers used the OHP in teaching the graph of the trigonometric functions. It allowed the teacher to present and explain many aspects of the graphs effectively and fluently. She used more graphs with the important parts highlighted than the other teachers. She wrote little at the board, but used the OHPs and worked orally with the class. Furthermore, the documents for the OHP were prepared carefully previously. The OHP helped the teacher to present trigonometry to students visually. Some students understand a relation better when it is visual. Interpreting graphs and using them to make other predictions are important skills for students to gain. Visual aspects, working with more and accurate examples contributes to the case for two trigonometries.

The computer is a tool in the UK mathematics lessons but is hardly ever used in the TR mathematics lessons. There was neither computer in the classroom nor computer cluster in the department of the observed school in the TR. In each observed UK class, however, there was a computer in the room. The department also had a computer cluster. Although they were not used throughout my data collection, they were used by both the UK teachers and students. The UK teachers stated that the computer could be used to aid students' understanding of mathematical ideas, for instance working with trigonometric graphs can support and encourage visual reasoning drawing on graphic representation and understanding some trigonometric identities. There was accessible and available software in the department for both teachers and students. One of them was used by teachers to prepare worksheets for the students. Students could play with numbers and expressions in a given equation and see the changes in graphs, so visually they discover the properties of the trigonometric identities and graphs.

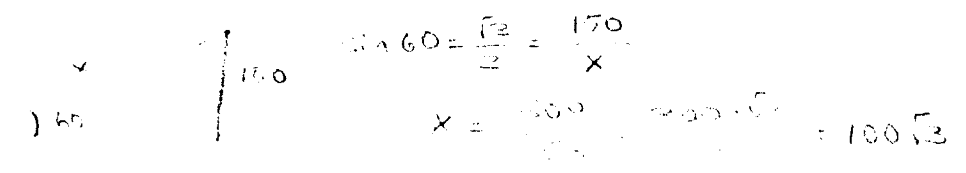
1.3.2. Conceptual tool

I use the term 'conceptual tool' in the sense of Douady (1991, p 115). She uses the term tool in a wide sense, which goes beyond physical tools. To Douady concepts, which are used for solving problems, are tools. These include signs, symbols, texts, formulae, graphic-symbolic devices that help individuals mater their perception, memory, attention, etc. Algebraic properties (Douady 1997, p 386), trigonometric properties and (mental) representations are conceptual tools. Conceptual tools are used throughout the cognitive processes and actions in solving trigonometry questions. A conceptual tool can be implicit or explicit. If it is implicit then the concept is elaborated, if it is explicit then there is an intentional implementation of a piece of knowledge (ibid.). Although conceptual tools are used in both the UK and the TR trigonometry classroom, there were differences or variations beyond these similarities. Conceptual tools are discussed in terms of algebraic, trigonometric and iconic tools. These tools appeared to be reasons for students' test performances.

Algebraic tools

Algebraic tools are techniques made by using algebraic properties such as factorisation, difference of two squares, distributivity and basic operations with fractions. They have an important place in the simplification of trigonometric expressions. Algebraic tools are emphasised in the two countries in different ways. Since paper and pencil was the main tool in TR and the TR trigonometry curriculum was more algebraic, the TR students used more algebraic tools in trigonometry lessons than the UK students who almost always used calculators in all activities. Calculators generally use decimal notation leading students away from fractional forms which may assist algebraic development. The TR students preferred to do manipulations using algebraic tools whereas the UK students did numerical calculations using calculators. Furthermore, in TR more implicit algebraic tools (i.e. $\sin x+\cos x=1/3$ find $\cot 2x$) are used in questions in the trigonometry lessons whereas more explicit algebraic tools are used in the UK (if tanA=3/4 find tan2A, in Bostock and Chandler 1991 p 102). In some application problems, the UK students were able to use their calculator to get an answer whereas the TR students had to do manipulations by using the algebraic tools to solve the questions (see . 5.1 for TR student).

Figure 5.1. The Turkish student's answer to trigonometry word problems test question 1



That means the TR students could develop their skills such as manipulating complex algebraic/trigonometric expressions by using algebraic tools in trigonometry context. Different algebraic tools arguably make the trigonometry different. For example if only decimals are used in teaching trigonometry in the UK whereas fractions are used in the TR, then it means the ones who work with fractions (with no calculator) are likely to develop their manipulation skills with (complex) fractions, but the ones who use calculators to do operations with decimals or use calculators to do operations with fractions may miss consolidating important techniques.

Trigonometric tools

Trigonometric functions have relationships between them such as $\cos 2\theta=1-2\sin^2\theta$. However performing the same operation to both sides of the equations or replacing $\sin^2\theta$ by $1-\cos^2\theta$ or replacing 1 by $\sin^2\theta+\cos^2\theta$ provides another expression/identity, e.g. $\cos^2\theta- \sin^2\theta$, which may initially be conceived of as a separate expression/identity. All these activities construct a relation between the rewritten items such as $\cos 2\theta=1-2\sin^2\theta= \cos^2\theta-\sin^2\theta$ or $2\sin^2\theta=1-\cos 2\theta$. Subsequently, each of these forms is called a trigonometric tool. These tools can be explained in terms of Barnard and Tall's cognitive units (1997), that is, in the course of simplification, students have to replace a trigonometric expression with its cognitive unit which I call as relation (e.g., replacing 1 with $\sin^2\theta+\cos^2\theta$). Trigonometric identities have an important place in both countries' education system. However the TR teachers and students used more trigonometric tools in trigonometry lessons than the UK teachers and students did. For instance, in the UK emphasis has been placed on the basic trigonometric functions, that is sine, cosine, and tangent. However, in TR cotangent, secant, and cosecant functions are almost emphasised as much as basic trigonometric functions. Therefore in the TR context, more trigonometric functions are used than in the UK. But the big difference in trigonometric tools occurred because of curricula differences (see section 1.1., p. 161). More trigonometric identities and expressions, which were not taught and used in the UK system, were used in the TR. This fact allowed the TR teachers and students to utilize more and different trigonometric tools compared with the UK ones in the second appearance of the trigonometry. At the first appearance of the trigonometry, however, in the UK but not in the TR, a common mnemonic way

'SOHCAHTOA' is used as a trigonometric tool, which shows the relation of the trigonometric ratios and the length of the sides on a right-angled triangle. Almost all of the UK students used this tool in their tests. In the TR, but not in the UK, however, Pythagoras and quotient identities were trigonometric tools, trigonometric identities in the first appearance of trigonometry.

Both the UK and the TR teachers explained the importance of the choosing appropriate identities, namely, being able to use trigonometric tools in trigonometry and especially in simplifying the trigonometric expressions. However, results showed that students had problems with using these tools.

Iconic tools

An interesting point was revealed in the data of TWP. My data showed that there were two main parts in solving a TWP: the diagrammatic part and mathematical (symbolic) part. The difficult part was the drawing diagram (see pp. 101-102). I called that visualizing and having a mental representation of the situation in the trigonometry word problems, an iconic tool, which is inspired from Bruner (1966). These tools are used to construct the diagrams in TWP solving. The central task of a word problem is to build a representation that will allow an effective search for a solution (Noddings 1989, p 253). So iconic tools have an imperative place in solving TWP. As was discussed before, because of the curriculum emphasis the UK lessons involved 'applications' and real world problems. Furthermore, trigonometry word problems occurred in both the first and the second appearance of trigonometry in the UK in both 2-D and 3-D contexts. In TR, however it is only used in the first appearance of trigonometry and there were no emphasis on working in 3-D. Therefore iconic tools were used in the UK more than TR. The TR students were less able to represent the situation in the TWP, because these tools were not emphasised in the TR curriculum. The TR students used these tools less than the UK students did.

1.4. Students' overall performance

The student is of obvious importance in all aspects of classroom teaching and learning for any topic, including trigonometry, in the education system of every country in the world. This section mainly focuses on the tests (see pp. 73, 85, 98 and 113), as a means of observing students' understanding of trigonometry. In terms of what students do, if their performances are different, then independent of curricula, there is an argument that there are two types of trigonometry in the UK and the TR. Nevertheless, overall performances of the UK and the TR students in the tests also contributed to argument for the existence of two types of trigonometry. Global performance highlighted that the UK and the TR students generally did what they were taught in the manner of 'simplifying' trigonometric expressions and real world applications of trigonometry.

The sub-sections below address: areas of relative strength (where students from each country did well); correct and partial answers; and how students went about aspects of solving trigonometry problems

1.4.1. Areas of relative strength

There were a number of differences in the overall performances of students from the two countries in all tests. An important difference was that the TR students were 'better' at symbolic (algebraic) aspects of trigonometry whilst the UK students were 'better' at trigonometry word problems (see pp. 76 and 100) which seemed to reflect what the curricula, and so teaching, in the two countries privileged (see global discussion 1.1 and 1.2).

Accordingly, student performance was strongly related to what curricula emphasised: 'context' word problems in the UK and 'algebra' in the TR. The TR students got used to working with symbolic (algebraic) expressions and as a result their manipulation skills were better than the UK students. Moreover, they worked with more and different trigonometric identities and expressions than the UK students. So they had more opportunity to develop their algebraic and trigonometric manipulation skills. The UK students committed many flaws in operating with symbolic expressions such as doing basic operations with fractions, cancellations, common factors (see pp. 77-78). This might be explained by less emphasis on working with symbolism in the UK, they often worked with application questions either numerical ones or in real world context.

In the UK, working with the real world context questions was emphasised in both the first and the second appearance of trigonometry. Not only trigonometry but in general terms, national curriculum assessment incorporated mainly 'realistic' test items, or as Cooper and Dunne (2000, p 3) asserted tests contained predominantly of items embedding mathematical operations in textually represented 'real life' situations. The UK students worked with TWP in both appearances of trigonometry including TWP of bearings, sine and cosine rules in both 2-D and 3-D (see also GD 1.1, 1.2 and 1.3) whereas the TR students worked with TWP merely in the first appearance of trigonometry with less emphasis. Minimal exposure to TWP in the TR might be the reason behind their poorer performance. The TR students had difficulties with the diagram parts of the solution. They had great difficulties transferring their mental representations onto paper properly, or, although they had the representation of the situation in the mind, they could not draw it on the paper (see p. 104). That also shows the important place of the visualization in education. Visualization is emphasised in the UK curricula where it is not in the TR one. Visual education is needed for effective and correct interaction with shapes, relationships between shapes and other entities (Hershkowitz et al. 1996, p 165). Moreover tools like calculators and computers allow the UK students to engage with visual thinking in order to understand, analyse and predict.

The UK and the TR students' relative performances varied over different question types. In the AT, the UK students displayed lower performance in questions which included fractional expressions. In the TT, there were more rules, properties and operations to use and interestingly both countries' students performance were lower than their performance in AT. In addition to the algebraic flaws the UK students did not do well on questions which included 'cotangent' function, fractions and implicit trigonometric identities[11]. The TR students, on the other hand, were not good at questions which required 3-D drawing or required drawing geometric diagrams containing more than one right-angled triangle in TWP. This provides some support for the claim that there are two types of trigonometry in the two countries.

1.4.2. Correct and partial answers

In both the algebra and trigonometry tests, the TR students gave more correct and partial answers than the UK students. How does this contribute to my thesis that these are two types of trigonometry in the two countries? Partial answers are neither correct nor incorrect answers. In partial answers, correct methods are used and some working is shown but students' answers fall short of the desired form. These answers were mostly seen in the tests AT/TT (there were more partial answers in TT than in AT) in both the UK and the TR (see Figure 4.2. on p. 77 and Figure 4.7. on p. 88). This is an argument in a way to say there are not two types of trigonometry, but there are variations. These variations could be explained in terms of dialectic differences in language. Trigonometry can be thought as the same language spoken in both the UK and the TR, however there are some regional (cultural) differences, which influence the 'pronunciation' of the some words ('simplification' in trigonometry). Most the TR students seemed to know what is absolutely correct in the manner of 'simplifying' trigonometric and algebraic expressions and the proper form to find as a simplified form of the given expression. It seemed that most of the UK students did not know what to do with the expressions to be simplified, so they mostly gave partial or incorrect answers.

The TR students gave less partial answers in AT which shows they knew what to get as a simplified algebraic form. The TR students had very high self-confidence with the manner of 'simplification' in AT, algebraic expressions were easier for them compared with simplifying trigonometric expressions. Some of them did not know how to get simplified forms of a trigonometric expression although they knew what the simplification process required in both algebraic and trigonometric contexts. The TR students gave more partial answers in TT compared with AT because they were less clear about the manner of 'simplifying' in trigonometry context and the simplified form. On the other hand, the UK students gave almost the same percentage of partial answers to questions in the TT but had many incorrect answers compared with the TR students. The UK students did not appear to be clear about finding the simplified form in trigonometric contexts as well as algebraic contexts and so they did not know

---

[11] The trigonometric identities which appear after some serial operations done on the initial expression, i.e. TT8.

where to go and gave more either partial or incorrect answers. This contradictory duality between the two countries may be explained in light of the following. First of all, although 'simplification' was the same 'concept' the UK and the TR students' performances in tests varied. There was no 'standard' definition of the manner of simplification in trigonometric contexts, as there is in algebra in both countries, that make the two trigonometries look different in terms of simplification.

Not surprisingly, the UK and the TR students were in the shadow of their teachers who were also not clear about the definition of 'simplification', even though the UK and the TR teachers provided very few partial answers, e.g. they knew how to simplify, even if they could not define it. This comparative problem for the UK students is surprising given that simplifying trigonometric expressions is highlighted in the UK but not in the TR curriculum as an objective (see Table 4.32., p. 154). However, it was only highlighted as a process, the product at the end of that process was not given or described. One factor was that in simplifying trigonometric expressions the appropriate trigonometric identities/formulae should be chosen, so both countries' students had a problem with that as well. Another simplification factor was the type of question asked. The UK teachers asked exam-centered questions, which were usually verification types, prepared by examination boards. In these sorts of questions the UK students were given an equation and asked to show that left hand side of the equation is equivalent to the right hand side, so students had a target. It might be said that, in the UK, 'simplification' questions were usually in the form where the expression's final form was given. The TR teachers, on the other hand, asked simplification questions in a wider range and style, verification was one of them. In contrast to the UK question types, the TR 'simplification' questions were usually of a kind where the final form is not given. There was no official resource to prepare examination style questions, and there were various printed resources that the TR teacher could use to prepare questions, so they might be called teacher-centred questions. So the TR students usually did not have any target to reach at the questions they had to discover the process and product by themselves.

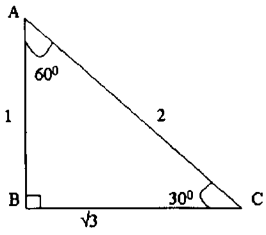
Despite all these facts, the TR students were more skilful in finding the simplified forms compared with the UK students. This implies that behind the simplification the TR and the UK students were doing different things. The TR students used more trigonometric identities and expressions and worked with algebraic aspects of trigonometry so they performed better in TT and their answers mostly went to either correct or partial answers. In contrast the UK students often worked with numerical aspects of trigonometry.

1.4.3. Actions in solving trigonometry word problems and trigonometric functions on right-angled triangles tests

By 'actions' I mean the 'things' students do in solving or simplifying which lead to their answers. Actions in solving questions are very important. They reflect the teaching as well as the understanding of the topic. In overall performances in the tests TWP/TORT, some interesting dissimilarities appeared in the UK and the TR students' actions throughout the answering process of the questions. Different processes and products were observed in students' answers. Computational procedures (algorithmic) applied by the UK and the TR students were dissimilar. The UK students performed the same process and product in all of their answers. All the UK students made one-step computations using calculators and working with certain values of trigonometric ratios as decimal numbers. They did not show any work on operations, they directly gave a numerical answer. The TR students, on the other hand, exhibited different actions in solving the TWP/TORT question. Most of them first found the value of the trigonometric functions (except 30^0 and 60^0 which are going to be discussed in the next paragraph) using trigonometric table. Then they rounded them to the nearest integer or one or two decimal places, even using the fraction forms of the rounded decimals (for instance 0.8=8/10). This action varied over the individuals but it seemed that they have tried to round the value in a way that it could be helpful in doing operations by using paper-and-pencil method. Because many TR students showed step-by-step manual computations on paper their papers were full of operations with decimals (particularly the four basic operations) compared with the UK ones. Sometimes they used some spare papers or the back of their test papers to do computations and then write the results on the test paper. Some of them used their test paper but after the computation they erased their work even though they were asked not to do it. Interestingly as a result of these processes answers were shaped in either of three forms which are numerical, as the UK students did, but mostly as algebraic or numerical expressions. Some TR students did not want to tackle decimals and operations so they left answers in algebraic form (e.g. a TR students left answer of the fourth TWP question as distance=tan31.15-tan17.15) or numerical expression form e.g. 1.2755x10+10x1.1918). It looked from TR students' papers that they spent considerable time on computations contrary to the UK ones.

An interesting action was seen in the TR students' answers of TWP and TORT questions which included the angles 60^0 and 30^0 respectively. In both questions the length of the hypotenuses were given and the length of the sides opposite to these angles were asked. Namely, sin 60^0 and sin 30^0 were required to answer the questions. For the values of these trigonometric functions the TR students preferred different ways which were using trigonometric table, using specific triangle or using rules. For the value of sin 60^0, some students preferred to use trigonometric table, they used either the decimal form (0.8 as nearest decimal place) or the fractional form (8/10) of it and then they did step-by-step paper-and-pencil manipulation. Most of the students, however, used a specific right-angled triangle (see Figure 5.2.) to find sin 60^0, so they used the surd form of sin 60^0 which is $\frac{\sqrt{3}}{2}$. Then they completed computations in forms of fractions and roots to get answer. An interesting action was observed in the use of sin 30^0. Almost none of the TR students used trigonometric table to find the sin 30^0, some of them used surd form as they did for sin 60^0. The majority of them, on the other hand, used a 'rule' to find the wanted side without using sin 30^0 nor doing any operation. Instead of doing operations they wrote the rule down and then gave the direct answer on the paper. The rule they wrote down was that in a right-angled triangle, the length of the side which is opposite to the angle 30, is equal to the half of the length of the hypotenuse.

Figure 5.2. Trigonometric ratios on the 30^0, 60^0, 90^0 right-angled triangle.



One of the other actions, which revealed the dissimilarity between the actions in answering the questions of TWP/TORT, was the use of common mnemonic ways by the UK but not the TR students. No mnemonic was observed in the TR students answers whereas SOHCAHTOA was used by most of the UK students by writing on the paper. Interestingly, the ones, who did not show it explicitly, said they used SOHCAHTOA but they did not write it on the paper because it is in their mind, so it was interiorised (see p. 105). Moreover, that mnemonic was the sort of evidence that shows the cotangent function is neglected in the UK context. This reflected itself in the UK students answers, none of them use the cotangent function whereas some of the TR students did. Furthermore, the difference also revealed that although both countries' teachers claimed that they teach mnemonic ways in trigonometry lesson they did not mean the same thing.

Another dissimilarity observed in the UK and the TR students' actions to answer TWP/TORT questions was seen in the trigonometric properties they used. A majority of the UK students used the basic definitions of the trigonometric functions on right-angled triangles whereas some the TR students used the sine rule throughout their answers. This might reflect the dissimilarity between the place of the sine and cosine rule in both countries' curricula, because the sine and cosine rule were in first and second appearance of the trigonometry in the UK and the TR curricula respectively, so it was more recent to the TR students.

Endnote: are these two types of trigonometry?

Everything seemed clear at the beginning of my data collection. Trigonometry was incorporated in both the UK and the TR curricula and appeared twice in middle secondary and upper secondary, namely, it was taught to same age group. At the end of the data collection, however, there were two different topics behind the 'trigonometry'. My first stop to examine my thesis was the UK and the TR curricula, which were prepared by official authorities and the core of the education system in the two countries. The first observable difference was the place of the trigonometric topics in the curricula organization. The UK curriculum provided considerably more content and objectives information than the TR in first appearance of the trigonometry. In this second appearance of the trigonometry, contrary to the first appearance, the TR curriculum provides considerably more content and objectives information than the UK. These also reflected an overall difference in the arrangement of the topics in both countries trigonometry curricula (see section 1.1.1, p. 161). These basic differences in both the UK and the TR curricula affected all other factors pertinent to teaching and learning trigonometry: curricula documents, teachers, tools and students' performance in trigonometry. the UK and the TR textbooks were tied to their corresponding curricula; however, there were some additional topics which appeared in textbooks which were different to the curricula and to each other. In addition to content, there were significant differences in terms of the questions in textbooks (see section 1.1.2, p. 164). Although the TR curriculum and textbook provided considerably more content and objectives information than the UK, in contrast, there was surprisingly less emphasis on trigonometry in the TR high-stakes examinations compared with the UK one and there were remarkable differences in terms of number and variety of the questions especially (see 1.1.3., p. 167). Subsequently, the differences between the UK and the TR curricula, textbooks and examinations provided evidence to my claim that there were two types of trigonometry.

Both the UK and the TR teachers were tied to their curriculum so their teaching reflected the differences seen in their curricula. Both countries' teachers' views of trigonometry were quite different which supports the thesis that there are two types of trigonometry in the two countries. The UK and the TR Teachers' motives, curricula, textbook and examinations had differences in terms of content (curricula) and questions. There were also differences in terms of what the UK and the TR teachers did in trigonometry lessons: trigonometry lessons in the TR were more 'abstract' than in the UK. In the UK, however, there were more application problems. There were also differences in terms of the tools they used in teaching trigonometry. All this evidence supports the thesis that there are two types of trigonometry. However, the physical environment and teachers' styles of teaching trigonometry in the UK and the TR, although effectively different, do not contribute considerably to this thesis.

A stark dissimilarity was observed in the tools used in the UK and the TR trigonometry classrooms. Calculators, formulae sheet and also OHPs and computers were physical tools used in the UK but not in the TR trigonometry classes. Non-comprehensive use of trigonometric table did not contribute to my thesis although it was a remarkably different tool used in the TR but not the UK. As was mentioned before, trigonometry was more 'abstract' in the TR and based more on 'application' in the UK. In the TR, considerably more content and objectives information were provided than the UK, this affected the variety of trigonometric identities (expressions) and questions used in trigonometry. Therefore more algebraic and trigonometric tools were used in TR. In the UK, on the other hand, there were more trigonometry word problems which appeared in both appearances of trigonometry. The diagrams were a vital part to solve these problems and so iconic tools were used more in the UK than the TR. Subsequently the differences in conceptual tools contributed my thesis that there were two types of trigonometry.

The UK and the TR students' performance in tests was strongly related to what curricula emphasised: 'context' word problems in the UK and 'algebra' in the TR. This emphasis can be observed in the correct and partial answers of the algebra and trigonometry tests with some other factors (see correct and partial answers in 1.4). Both the UK and the TR students' actions in answering the trigonometry word problem and trigonometric functions on right-angled triangle tests were effectively different. Subsequently both countries' students' performance also supports my thesis that there were two types of trigonometry.

Consequently, overall differences between the trigonometry in the UK and the TR were more than the similarities in terms of curricula, teachers, tools and students' performance. In other words, there are effectively two types of trigonometry in the UK and the TR.

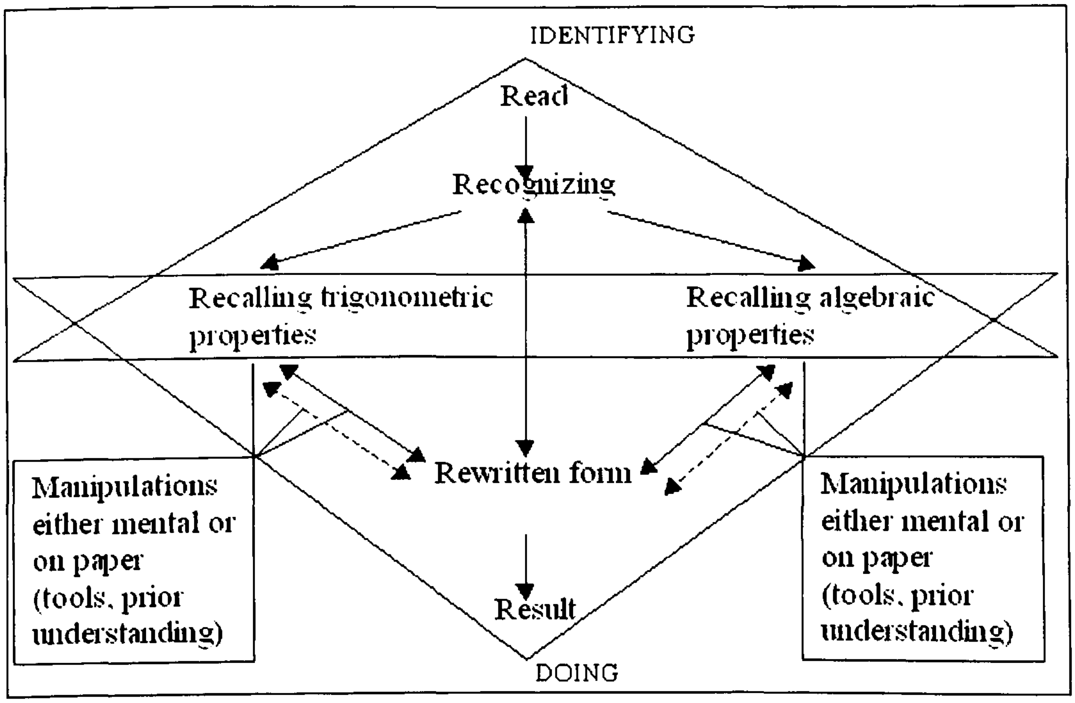
2. Identifying and doing in the 'simplification' of trigonometric expressions

In this section I present an operational model of how students simplify trigonometric expressions. I start by describing the model, which took shape in the period of the time I worked with students on think aloud verbal protocols. I then look at relevant data which may support or refute my model. Finally I compare and make links between this model and other models of doing mathematics.

2.1. An operational model of simplifying trigonometric expressions

Differences between trigonometric identities, trigonometric formulae and the notion of 'simplification' in the UK and the TR were illustrated and discussed in the first section of this chapter. Briefly there were more trigonometric identities and trigonometric formulae used in the TR than the UK in teaching trigonometry. The TR students were also presented with a greater number and a greater variety of questions in this area though no definition of 'simplification' in trigonometry was given. Moreover the TR students' performance in the Trigonometry (and Algebra) tests were better than the UK students. Despite this the protocols revealed a uniformity of approach with regard to approach to simplification. The operational model of simplifying trigonometric expressions presented below (Figure 5.3) thus holds, I believe, for the TR and the UK students. The components of this model are reading, recognising, recalling, manipulations, rewritten forms and result.

Figure 5.3. An operational model of simplifying trigonometric expressions.



Before explaining the components of the model I would like to note three observations. The first is that students use tools to carry out simplification actions. Tools take many forms and mediate mathematical actions. Trigonometric identities/formulae and algebraic properties may be regarded as tools and so may, say, computer algebra systems. Simplification using trigonometric identities/formulae, algebraic properties and pencil and paper should be regarded as distinct from simplification using a computer algebra system. My model should, then, properly speaking, be called 'an operational model of simplifying trigonometric expressions using trigonometric identities/formulae, algebraic properties and pencil and paper'. This is implicit in the remainder of this section.

My second observation is that cognitive activity is notoriously difficult (impossible) to observe and researchers make inferences on the external actions of students. The components of this model, except 'read' and 'result', are cognitive constructs based on what students said and wrote as I was working with them. Students' written symbolic manipulations on the paper are thus very important because they reflect individual cognitive actions and influence my interpretation of students' simplifications. They did not, however, write everything on paper, so whatever they wrote on paper I interpreted (rightly or wrongly) as that which they could not keep in their minds (or their working memory). For this reason the 'rewritten form' is important in this model.

My third observation is about actions in the verbal protocol. I am aware that protocols should not, strictly speaking, include implicit actions but implicit actions are important so I include them. Even though students in the protocols were well trained they still could have difficulties in uttering their thoughts. No verbalizing does not mean that no actions are occurring throughout the protocol task, these actions cannot be ignored. The actions are coded 'implicit' if they are not verbalised but it is clear from subsequent verbal or written protocol data that the actions has taking the place (see Table 4.14, p. 107).

In this paragraph I illustrate the components of the model with an example from the Trigonometry test. Students begin by reading the question, $\frac{\sin^2 2x - 4\sin^2 x}{(\cos 2x + 1)^2}$. This may be explicit (aloud) or implicit (unspoken). Students then focus on a subexpression or a form, e.g. sin2x or $\sin^2 2x$-$4\sin^2 x$. I call this 'recognising'. They then 'recall' trigonometric or algebraic properties, e.g. that sin2x=2sinxcosx or (sin2x+2sinx)(sin2x-2sinx). In the schematic of my model (see Figure 5.3.) read, recognise and recall are grouped together under the term 'identifying' because they all rely on sign association. Students then 'rewrite' the given expression with some form substituted for another, e.g. '2sinxcosx' substituted for sin2x or '(sin2x+2sinx)(sin2x-2sinx)' for $\sin^2 2x$-$4\sin^2 x$. Students may (explicit manipulation, a continuous line in Figure 5.3) or may not (implicit manipulation, a dotted line in Figure 5.3) write ancillary 'jottings', e.g. sin2x=2sinxcosx or $\sin^2 2x$-$4\sin^2 x$=(sin2x+2sinx)(sin2x-2sinx), prior to rewriting the expression to be simplified. What is important with regard to my model is that rewriting the expression always occurred with my students. The students then examine the rewritten form and recognise/recall another subexpression/ property and enter a further manipulate/rewrite phase or accept their rewritten form as the result, the simplification. Recall, manipulate and rewrite are grouped under the term 'doing' because they all rely on transforming signs. The model, in its current form, has 'recall' in both the 'identifying' and 'doing' groups. I do not see a contradiction here and would add that this duality appears to be a function of the dialectic between identifying and doing in this context.

I now consider the components of the model in greater detail. Students begin by reading the question. Is reading a cognitive action? I think so in general and especially here where, say, sinx is taken as a term and not a concatonation of letters. Please note that it does not matter, in this model, whether the student initially registers the whole expression or a part of it. Reading involves de-coding text (e.g. Perfetti, 1984), symbolic expressions in this case. These act as an external stimulus which evoke the information/ knowledge stored in students' memory. Memory has a vital place in learning. Memory and learning are interdependent processes and a student cannot remember something if it has not been learned (Lefton 1985, p. 104). Information/knowledge stored in the memory is retrieved and brought into consciousness (Ornstein and Cartensen 1991, p. 304). Retrieval is the process by which people use their memories to recognise something as familiar and recall something they previously learned (Wickelgren, 1977, p. 396).

Recognition in this model is the ability to identify an expression or form or procedure or property. This identification involves a match between the form etc. and the students' memory.

After recognition students employ the other function of the retrieval process, recall. In this context students may recall either algebraic or trigonometric properties. This algebra-trigonometry distinction is important to me because I observed students focusing on algebraic forms at times and on trigonometric forms at other times but it is possible to employ an undivided 'symbolic properties' component in the model. Recalling involves remembering the details of a property or procedure and placing them together in a meaningful framework to continue simplification actions usually without any cue or aid (see Lefton 1985, p 107).

I now further explore the relationship between recognition and recall. Recognition precedes recall in the schemata of the model. However, there were some cases in which students seemed to recall and recognise simultaneously. If this was indeed the case, then I believe that great familiarity with the expressions in the question was present, e.g. $\sin^2 x + \cos^2 x$ and 1. I have no definitive answer but psychology texts emphasise that there is no single and simple relationship between recall and recognition (i.e. Eysenck and Keane 1995, p. 154) though it is accepted that recognition is superior to, and easier than, recall. Watkins and Gardiner (1979) present a theory where recall requires an item to be retrieved and then recognised but this theory has attracted criticism. Tulving and Flexter (1992) note, for example, in various recognition-failure studies that recall performance depends much less on recognition than Watkins & Gardner's theory suggests. Wickelgren (1977, p. 266) sees the distinction as that between wh-questions (who, what, when, where, which and how) and yes-no questions. She/he also states that recognition and recall may use the same memory system, the same retrieval process, and sometimes even the same decision rules. Returning from this psychological debate to my model I proceed on the basis that recognition precedes recall.

After recalling, students 'rewrote' the expression with some trigonometric form substituted for another (or an algebraic manipulation). There was considerable variation in students' written manipulations (and verbalisations) and their apparent mental manipulations. I am, moreover, convinced that this reflects, by and large, students' mathematical traits rather than personality traits. 'Mental manipulations' pose a potential problem for my model simply because I am forced to speculate but when a student presented with $\tan^2 x \cos^2 x + \cot^2 x \sin^2 x$ simply writes down '1' (as happened) I feel on pretty safe grounds assuming that mental manipulations have taken place. My metaphor here (MacFarlane Smith, 1964, p. 132) is that some students have a mental blackboard and they start to answer the question on this blackboard first and then rewrite the expression so what they write is external representation of extractions of the internal representations of the solution. The internal and external representations are always interdependent and interrelated. There are, I am sure, issues here concerned with students' working memory and of memory overload but I do not pursue this.

The flow in this model is towards to the 'rewritten form'. This is intentional and, I believe, reflects students' intentions in simplifying to obtain a rewritten form which may be regarded as the simplification. An equivalent form of the initial expression is sought and this form is either an intermediate form, to be further simplified, or is judged to be the final result. If the rewritten form is not the final result, then further simplification is needed. The rewritten form is a product of the simplification process and dialectic between identification and doing groups of my model (see the model on p. 189) and are compressed forms of the expressions which have been manipulated. Students may present different rewritten forms as a result of their individual cognitive actions. Students may also vary in the number of rewritten forms they produce before declaring that they have a final result and their degree of certainty, that the final form is, indeed, the 'proper' simplified form, may vary. My experience in working with the students was that many did not know when to stop.

2.1.1. The potential of this model for understanding students' work

This model was grounded in students' work and sheds some light on the notions of 'simplification', 'simplified form' and 'the most simplified form'. Changing the focus for a moment away from students it is interesting to note that there no agreed definition of simplification was given by the UK or the TR teachers (see pp. 131 and 148). Nor did curriculum documents or textbooks from either country provide a definition, though the word itself was used quite freely. There appears to be a slight difference in the use of the term in the two countries. In the UK simplification was used with 'prove/show that' questions in textbooks, especially in examination questions. Simplification in the UK almost always had a 'target' form. The TR teachers, however, usually used the expression without a target from. So the TR students had to stop when they felt they found the simplified form. The model presents simplification as the sum of the actions from the reading component to result. Simplified forms are rewritten forms. 'The most simplified form' is problematic. It is again a rewritten form, but which one? This is not clear from model. But what is obvious is an expression might have different 'simplified forms' depending on the question, and individual cognitive actions of students.

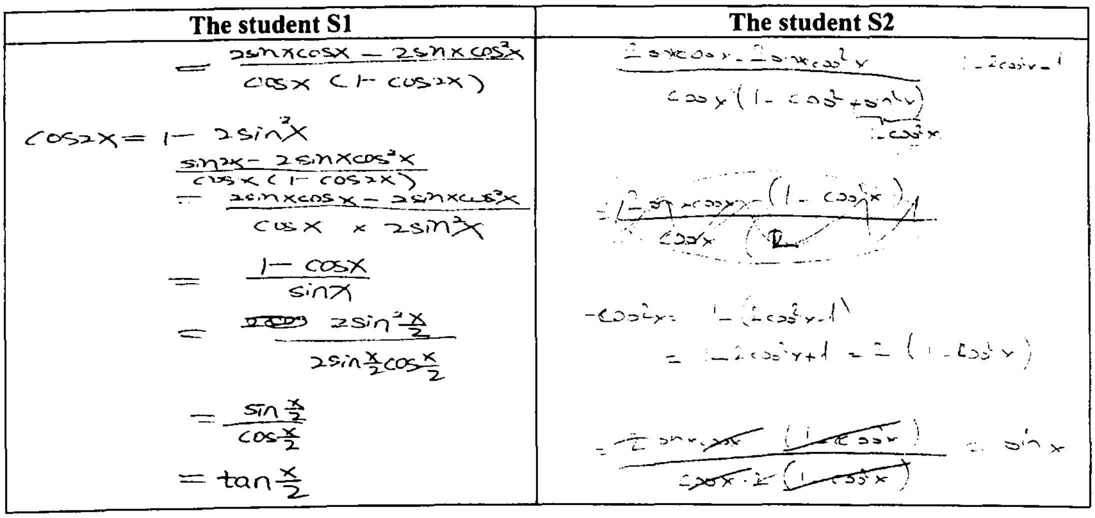
My model is operational in that it might be helpful to consider it as a framework to think about the processes involved in the simplification of trigonometric expressions. As Garofola and Lester (1985) suggest there are students who are unaware of the processes involved in problem solving and addressing this issue by my model may be useful way to make them conscious of the processes. It is important that students have knowledge of algebraic and trigonometric properties, choose the most appropriate ones and apply them to get a rewritten form and at the end of this process, find a simplified form. The model has potential as a heuristic in the teaching and learning of the simplification of trigonometric expressions and could be used as task specific heuristic which is arguably more effective than general heuristic instructions (Wilson et al., 1993). For example, in place of "find all possible trigonometric identities and algebraic identities and properties" teachers could suggest "substitute trigonometric identities with equivalent forms", "focus on rewritten forms"

2.1.2. Examples from the verbal protocols

Further to the outline of the operational model above, I discuss below the validity of the model with regard to my data. I take three examples. These examples are from three different students. The first two protocols provide supporting evidences for my model. One of them (S1) gave correct answer, other one (S2) gave incorrect answer to the task 'simplify the expression '$\frac{\sin 2x - 2\sin x\cos^2 x}{\cos x(1-\cos 2x)}$'. The third example which shows a possible weakness of my model, the student (S3) gave the incorrect answer to the task 'simplify the expression '$\frac{\sin x}{1-\cos x} - \frac{\cos x + 1}{\sin x}$'.

Even though the task was same for S1 and S2 the way students simplified the expressions and their manipulations on the paper was very different. I will discuss my model with the students. I will first take the students S1 and S2 into consideration then S3 (see Figure 5.4. and Table 5.2.). In Figure 5.4., S1, S2 and S3's solutions are presented, in Table 5.2. S1 and S2's part of segment analysis of their protocols and are presented to give a view to the reader.

Figure 5. 4. The answers of the students S1, S2 and S3.



The student S3

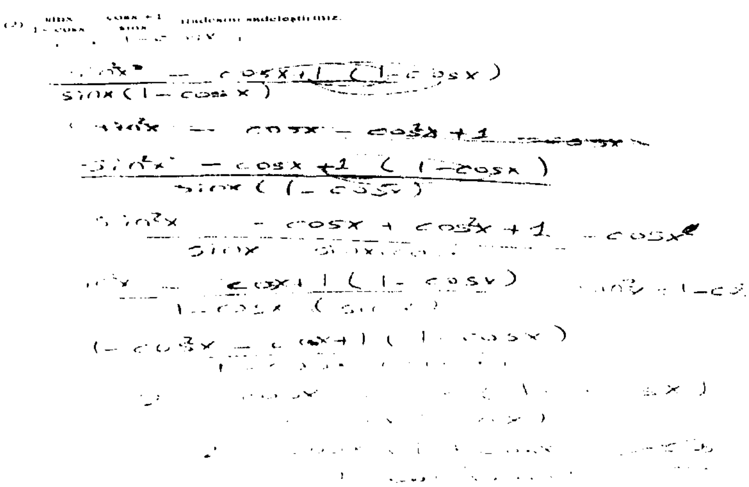


Table 5.2. A few lines of segment analysis of the students S1 and S2.

The student S1

Line number	Segments	Read ↓	Recognise ↓	Recall ↓	Manipulate ↓	Rewrite ↓	Result ↓
1							
2	..err.. equal to						
3	sin2x			TP*			
4	equal to						
5	2sinxcosx minus 2sinxcos squared x equal to over cosx and 1-cos2x						
6	cos2x			TP			
7	is equal to						
8	1-2sinx squared.						
9	So how can be equal to ..to.to..err..						
10	sinx cos squared x over cosx 1-cos2x						
11				ITP*			
12	is equal to						
13	2 sinx cosx minus 2 sinx cosx squared x over cosx times						
14							
15				IAP*			
16							
17	2 sin squared x						
18							
19	so cancelled out 2sinxcosx			AP*			
20							
21	so it is equal to 1-cosx over sinx						

The student S2

Line number	Segments	Read	Recognise	Recall	Manipulate	Rewrite	Result
1	Sin 2x minus 2 sinx cos squared x..						
2							
3	sin2x is			TP			
4							
5	2 sinx cos x minus 2 sinx cos squared x over cos x 1						
6				ITP			
7							
8							
9	minus						
10				IAP			
11							
12	cos squared x plus sin squared x..						
13							
14	by bringing common factors			AP			
15	2 sinx cosx outside of the top,						
16	**1 minus cos squared x is in the bracket**						
17	..eeh..the common factor			AP			
18	cos is taken out						
19	then 1 1 minus…						
20	sin squared x						
21				ITP			
22							
23	can be written as 1 minus cos squared x						

* *AP-Algebraic property, IAP-implicit algebraic property, TP-trigonometric property, ITP-implicit trigonometric property*

Both students clearly read the statement and recognised the terms. Note that it is irrelevant to my model whether this is implicit or not and that these two students' implicit/explicit actions were different. Both then focused on sin2x and rewrote the expression with 2sinxcosx in place of sin2x (the manipulation being a mental substitution). Both students focused on the cos2x term of the rewritten expression. S1 recognised that it is $1\text{-}2\sin^2x$ and wrote this down. Student 2 recalled that cos2x is $\cos^2x\text{-}\sin^2x$, mentally manipulates the minus sign and rewrites the denominator of the expression. Further rewritten forms accompany the rewritten expression. After they obtained the rewritten form of the initial expression they both focused on algebraic properties. S1 implicitly and correctly took common factors out and then, again implicitly, cancelled the same terms in numerator and denominator, then rewrote the expression on paper. On the other hand S2 followed a different and longer way to get the next rewritten form of the initial expression. S2 took a common factor out but made a manipulation mistake and wrote the incorrect rewritten form of the numerator down. S/he, next, focused on $\sin^2x$ in the denominator and recalled that $\sin^2x$ is $1\text{-}\cos^2x$ and rewrote this down as the previous rewritten form. S2 then focused on addition of the non-fractional expressions and then, after a manipulation on a side of the paper, rewrote the expression in the bracket, then s/he focused on taking common factor 2 out. However, S2 got stuck to simplify the expression but focused on the bracket in the initial expression and recalled that cos2x is $2\cos^2x\text{-}1$ (different from the first time). After recalling and applying taking the common factor out S2 rewrote the bracket as $2(1\text{-}\cos^2x)$. Then S2 rewrote the initial expression with rewritten forms. However S2 made the same mistake as before of taking a common factor out. As S1 did, implicitly, S2 focused on taking common factor out and then correctly cancelled the same terms in numerator and denominator by showing manipulations on the paper and found the simplified form. S1, on the other hand, first focused on cosx in numerator and sinx in denominator of the expression $\frac{1-\cos x}{\sin x}$ and rewrote the expression with $2\sin^2(x/2)$ in place of 1-cosx and $2\sin\frac{x}{2}\cos\frac{x}{2}$ in place of sinx. S1 then mentally focused on cancellation and cancelled the similar term without showing manipulations on the paper, then rewrote the expression in form of $\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}$. S/he then focused on this expression and recognised and recalled (most probably simultaneously) the tangent function and rewrote the expression as $\tan\frac{x}{2}$. S1 then stopped and regarded this as the final simplification.

Note the general diagonal pattern in the right-hand columns (Table 5.2., S1). These diagonals reflect the iterative 'recognise, recall, manipulate, rewrite' components of my model. It can be seen that 'rewrite' is imperative and central for this. It is the expression they produce by applying trigonometric or algebraic properties and procedures on the initial expressions.

Students focus on these expressions to continue the simplification process. In this component, terms of the initial expression and the initial expression itself are rewritten in their equivalent form throughout the component of the model. These forms then become either another expression to simplify or the final result.

Although both S1 and S2 focused on same sub-expression cos2x, they recalled different equivalent forms and did manipulation to find rewritten forms. Interestingly although that fact affected the way, steps, procedure and the length of the answer, this does not have any negative impact on the model, in other words the model worked with these two different answers. Moreover, even though S2 made a mistake, first in line 16 and then repeated it in the 34, the model still holds. That the model 'holds' for incorrect answers is, I believe, an important feature of the model. The model also 'holds' for partially simplified expressions.

So far my model provides a framework for a descriptive account of correct and incorrect answers in which different cognitive actions can be observed. There were, however, a few examples which were problematic for my model. For example student S3. S3 (Figure 5.4.) started to simplify trigonometric expression as the other two students did, read the statement and then recognised the terms/properties. S/he then focused on the two fractions in the expression and recalled subtraction of two fractions. Then S3 did the manipulations and rewrote the expression. However, it was observed on paper that s/he did not take 'cosx+1' under the bracket, S3 did not recognise that. S3 then again focused on numerator of the fraction and recalled the distributive law. Although S3 applied the distributive law as if 'cosx+1' is in bracket s/he rewrote the numerator incorrectly after manipulations on paper. Then before completing the manipulation S3 recognised something and went back to rewritten form (first rewritten form of initial expression), and applied the distributive law again. S3 then again did the same mistake but at the end of the manipulation found a new rewritten form of the initial expression. At this point in the protocol there is a link between symbolic manipulations and recognition which is not in my model. This link also appeared once for a UK student who used the formula sheet in the interview. In that protocol there was also a link from recognise to read which is not in my model. I do not see this example as a serious counter-example to my model but it suggests that further work could be done in refining the model.

In the light of these three protocols, it can be observed that a knowledge of algebraic property and trigonometric property, to know how to apply them and most importantly to decide which one to apply have a central role in the simplification of trigonometric expressions. This again underlines a need for a consideration of the place of metacognitive actions in my model.

2.2. Links to other models of doing mathematics

In this sub-section I will focus on the links between my model and other models of mathematical activity. My model has some similarities with other models of mathematical activity. These models are Saxe's (1991) four-parameter model, Dreyfus et al.'s (2001) operational model for abstraction and Greeno's stages of the problem solving (1973). The links to these models highlight the importance of rewritten forms and dialectic between identifying and doing groups in my model (see Figure 5.3.).

Rewritten forms in my model has links with Saxe's (1991, p.17) four-parameter model. The four parameters, which are implicated in the emergence of individuals' goals, are activity structures, prior understanding, conventions/artifacts and social interactions in this model. This model appeared in the first analytic component of Saxe's framework for the study of culture and development. This analytic component was related to the goals that emerged during participation in mathematical cultural practices. So what are the links between my model and Saxe's model? Although social interactions and activity structures are somewhat limited, to say the least, in the protocol items used in my research his inclusion of 'conventions' to the Vygotskian notion of mediational means seems particularly apt in the case of trigonometry, where symbolic manipulation conventions abound. In my model, by recalling cognitive functions, trigonometric or algebraic properties are used as tools during the symbolic manipulation. Moreover students use also their prior knowledge and experience in symbolic manipulations (and also between recognising and recalling components). That corresponds to the prior understanding parameter of Saxe's model in which individuals bring their prior understandings to bear on practices. The tools and prior knowledge/experience were utilised to get rewritten form(s). What is particularly interesting, however, is that Saxe's emergent goals appear to coincide with my rewritten forms – both appear and fade away in the course of the activity. My model, however, appears more relevant to an analysis of school mathematics symbolic manipulation whereas Saxe's model is relevant to out of school cultural practices.

My model also has direct links to Dreyfus et al.'s (2001) operational model for abstraction. Dreyfus et al.'s model has, at its centre, three epistemic actions: *constructing*, *recognising* and *building-with*:

> *Constructing* consists of assembling knowledge artifacts to produce a new structure to which the participants become acquainted. *Recognizing* a familiar mathematics structure occurs when a student realizes that the structure is inherent in a given mathematical situation. The process of recognizing involves appeal to an outcome of a previous action and expressing that it is similar (by analogy), or that it fits (by specialization). *Building-With* consists of combining existing artifacts in order to satisfy a goal such as solving a problem or justifying a statement.

Constructing new conceptual knowledge has no place in my protocol items because the tasks rely on students working with existing knowledge. However, Dreyfus et al.'s *recognising* and *building-with* are, I believe, parallel to the identifying and doing components of my model. In my model after reading the statements students focus on the forms/ sub-expressions/procedures/properties. So they realise the forms/sub-expressions/procedure/ properties inherent in the given expression and this occurs on the base of the existing knowledge not the constructed structure as it is in Dreyfus et al.'s model. After the recognition students recalled the properties, which they had already acquired, they needed to use in simplification process in my model. However recalling did not appear in Dreyfus et al.'s model. After recalling properties students apply them to get rewritten forms at the end of the symbolic manipulations. That component of my model is very similar to building-with of the Dreyfus et al.'s model, because at this action the goal is attained by using knowledge that was previously acquired or constructed. There is clearly scope for further work into the relationship between their work and mine but what my model emphasises, and theirs does not, is the central place of rewritten forms. In my model, the identifying and doing dialectic produced the rewritten form(s) each of which is an emergent goal. In Dreyfus et al.'s model actions are important, whereas in my model what students write down and the connection to their cognitive actions are important.

My model also had links with the stages of the problem solving given by Greeno (1973, p. 105). Greeno's stages were read text, interpret concepts, retrieve relevant information, construct a solution plan and carry out calculations or other operations. He emphasised that these stages were not to be carried out in a strictly sequential fashion. They could be overlapping. Since my model does not contain any links or components pertinent to metacognition, the 'interpreting concepts' and 'construct a solution plan' stages of Greeno's model did not appear in my model. However the remainder of the stages coincided with the components of my model. Both models have the 'read text' components. Then since the recognising and recalling are included in the retrieval process of memory these two components correspond to the stage 'retrieve relevant information' of Greeno's. After the relevant information has been retrieved, it is used to carry out the symbolic manipulations. This illustrates that the last stages of Greeno's model and the manipulation link of my model are very similar. Similar to Greeno's model, in manipulation links of my model, calculations or other operations are carried out to reach the rewritten form. Like the other models what my model emphasises, and Greeno's model does not, is the place of the rewritten form(s).

Although my model is grounded in the original data and resembles what really happens in simplification, there are two points which show potential weaknesses of the model. One is the use of formulae sheets in the interview and the Trigonometry test and the other one is the place of the metacognition in the model. First is the use of formula sheets. My model emerged from verbal protocols with students. Although formula sheets were given to the UK students they did not use it in the protocol. However most of the UK students in the TT and all the UK students in the interview used the formula sheet. The use of a formula sheet might affect the model. There were two types of use of a formula sheet observed in the interviews by the UK students. In one, after reading students focused on a terms/sub-expressions or procedure in the expression. They did not necessarily recognize the trigonometric property implicit in this sub-expression but thought, say, 'ah, what can I do with sin 2x?' They then referred to the formula sheet and 'recalled' the property. So, after recognising with the help of a formula sheet, they generated a trigonometric property. Since recalling usually occurs without any clue or aid then it is open to question whether the recalling component actually did not really happen with these students. In another case, a UK student focused on the formula sheet as soon as they read the question so they recognised the forms/subexpressions by the help of the formula sheet but again there was arguably no recall. So the use of a formula sheet creates another linkage between recognition and manipulation but complicates the link between recognition and recalling in terms of trigonometric properties not algebraic properties.

Another important consideration which did not appear on my model but, it seems is important effective throughout the simplification actions is metagognitive knowledge and skills. This might be inserted between the recognising and recalling components and also between the recalling and rewritten form components. After recognition students S1 and S2 above recalled the information to use in their simplification action. However for same question they did not choose the same trigonometric identities or trigonometric formulae or algebraic properties to simplify the expression. They chose different symbolic properties because they focused on the different terms/subexpressions or procedures in terms of their past experiences and knowledge. Actually after recognition students recall their strategic knowledge, i.e. metacognition, along with knowledge of symbolic properties. The decision students took directly affected their answering process: the length of the procedure (so the length of the answer), to get more complicated or simple expressions, to get the rewritten forms, to get stuck or to get the correct answer. So students' knowledge of algebraic and trigonometric properties (having the knowledge and knowing how to apply it) and students' strategic knowledge (knowing which to apply) are very important in simplifying actions and cannot be ignored (see Lewis 1981, p 87; Schoenfeld 1987, p. 191). Consequently it seems that metacognition has an important role in any refinement of my operational model.

The issue of mental manipulations and their link with written manipulations is important. A possible criticism of my hypothesis that rewritten forms are central to (paper and pencil) simplification is that many people can simplify expressions 'mentally' without rewriting. I agree and there was some evidence of this in this sample. First, the ability to mentally simplify without rewriting appears to be dependent on the simplifier's experience and the simplicity of the task. It might be expected that most advanced mathematics people can mentally substitute $\sin 2\theta$ for $2\sin\theta\cos\theta$ but I claim (admittedly without evidence) that there was a point in our development when this was not the case, when we had to perform a written substitution. Regarding the simplicity of the task there was a student who, it appeared, did not need to rewrite an expression (though he did so because he was asked to). This was an easier (for him, $\tan^2x\cos^2x+\cot^2x\sin^2x$) expression to the one in the protocols above. However, although he 'saw' that this was 1 I claim there was a point in his development when he would not have seen this. In Vygotskian terms this amounts to saying that external mediational means have been internalized (Vygotsky, 1978). That means a series of transformations occurred for the internalisation process (the use of trigonometric/algebraic properties that initially represents an external activity is reconstructed and begins to occur internally). Students then apply these properties in their practices in mathematics classes and, overtime and with practice, students become capable of using these properties without writing them down.

A second problematic issue with regard to mental manipulations is the use of memory in the absence of the internalization process. After retrieving information from semantic memory students use their working memory where coded information is temporarily stored for immediate use and where active processing of the information goes on (Greeno, 1973). The capacity of the working memory could affect mental manipulations. Written manipulations usually help working memory, they partially become external representation of the internal representation of problem solving. Furthermore, in trigonometry, lots of relations, identities and formulae exist in addition to algebraic rules and operations, namely rules and procedures, students need to know. In other words students have a vast amount of information to store in their memory. Students should have not only knowledge of rules but also knowledge of procedures to perform mental manipulations. To store this that much information might require to have (rich) cognitive units (Barnard and Tall, 2001) so that having a piece of information could lead students to lots of related information.

3. Answering trigonometry word problems

Despite all the variations seen in the topic of trigonometry in both the UK and the TR (see Global discussion of the results) and the big difference in the performances of the UK and the TR students in the trigonometry word problems (TWP) test, concurrent protocols revealed a uniformity of approach with regard to answering TWP. This approach will be discussed in the section 'the model of answering a TWP'. In the process of answering a TWP, two important phases were observed. These phases are constructing the diagram and doing the (symbolic) mathematics. By 'constructing' the diagram I mean drawing and labelling the diagram to represent the TWP. By 'doing' the mathematics I mean identifying the function, developing the calculations or algebra and doing symbolic manipulations to find the answer. The students used the diagrams in doing the mathematics. So these two phases and the relation between them are discussed under the titles; 'constructing' the diagram, 'doing' the mathematics and interaction between the diagrams and the mathematics.

3.1. The model of answering trigonometry word problems

The analysis of both the UK and the TR students' approaches to answering TWP in the protocols generated a model which consisted of the stages students progressed through. The model was not initially hypothesised, but is grounded in the actual data. The model accounts for the complete process of answering a TWP, from when the problem is presented through to completion. In this sub-section, the model is presented for two purposes: one is to provide the reader with the stages of the model and the other one is also to show the two main phases of the answering process of the model (see Figure 5.3.). Further analysis of the written answers of the students showed that there were two different mathematical characteristics to their answers, geometric and algebraic. Two different representations, diagrammatic and symbolic (algebraic/numerical), were used in these parts of the answers.

Many models of students doing mathematics have a review stage which can be the most important part of problem solving (Wilson et al., 1993). A 'review' stage of the model was not incorporated in either of these two phases as almost none of the students seemed to put their solution back into their pictorial or mental image on the paper to get a sense of whether the solution was of roughly the right size. Checking the diagram was not a natural procedure for them, if an answer was found students generally considered the answer as completed. This was the case in Pritchard and Simpson's (1999) study too, moreover Kantowski (1977) found little evidence among students of looking back even though instructions stressed this. It seems that developing the disposition to look back is very hard to accomplish with students. 'Review' stage is not developed in the students in my study as well even though it is emphasised as one of the important stages in problem solving strategy and heuristics (e.g. Polya, 1973).

Figure 5.5. The model of answering TWP.

(i) **Stages**

· read
· recognise
· visualise
· draw
· match
· label

(ii) **Phases**

'constructing' ***diagram***

· identify function
· develop math
· symbolic manipulation
· result

'doing' ***mathematics***

The model has similarities with the categories Pritchard and Simpson (1999) constructed of students exploring the use of pictorial images in solving TWP: the creation of diagrams, the use of diagrams in solving problems, the use of diagrams in checking and making meaning. Their first and second categories respectively correspond to the first and second phase of my model. However what my model emphasises and theirs does not is the cognitive perspective (actions and abilities) of answering a TWP. Moreover my model emphasises the central role of the 'diagrams' and the process of constructing diagrams in the process of answering TWP. It also reflects some commonalities with other models (see the elaborated theory of Davis 1984). Moreover, its stages also reflect some (cognitive) abilities needed to answer TWP (Lucangeli et al., 1998). Lucangeli et al. (ibid.) stated that the solution of a mathematical word problem requires a constellation of different cognitive and metacognitive abilities. Most of these abilities corresponded to the abilities with which some stages in the model could occur such as the capacity to have a good visual representation of the data.

Students always completed the 'constructing' diagram phase before the 'doing' mathematics phase. However how this model worked varied according to the type of TWP and over the students as problem solvers. So the main complexity in this model is that students do not merely cycle through these phases once, but more commonly will cycle through it more than once. Furthermore, the stages within each of the two phases were often revisited in the order I have listed these stages. However, in some answers only the stages 'recognition, visualising and drawing' and/or 'matching and labelling' in the constructing diagram phase were cycled (see Figure 4.19. on p. 108 and Figure 4.21 on p. 110). It seemed that progressing through the stages depended on the students' cognitive abilities as well as the complexity of the question. If students were able to construct the diagram at once and do the mathematics part to get an answer at once, then all stages in both phases were gone through in the order given in Figure 5.5. However, if students had difficulties in drawing diagrams or drew the diagram bit by bit then 'recognition, visualising and drawing' were cycled through. The 'matching and labelling' stages of the constructing the diagram phase usually cycled more than once (after the diagram was drawn). Another factor that caused variation in the answering model was the type of the TWP. If the TWP required a diagram including more than one right-angled triangle, in 2-D or 3-D, then the all of the stages or the stages 'recognition, visualising and drawing' and/or 'matching and labelling' in constructing diagram phase were repeated more than once. Moreover, the known and unknown values given in the TWP also affect the way my model works. If it was required to find some values first to get the answer then the 'doing' the mathematics part would be repeated (see Figure 4.21, p. 110). For example, for the statue problem students needed two right-angled triangles to find the length of the two sides of these right-angled triangles respectively and then find the length of the statue by applying algebraic/arithmetic processes, subtracting the smallest one from the longest one.

Pedagogic implications of the model may be drawn. Overall, what the model suggests for traditional types of the TWPs, which require one right-angled triangle to be drawn and only one unknown found, is that students' attention should be focused upon the particular activities occurring in each of the presented stages of the model. The stages in the order given in Figure 5.5. constituted to the students a pattern for their work (however the factors affecting the order of the stages should be emphasised by other examples). This pattern might not only help them to systematise their written work and their computations but also help them analyse TWPs and organise their thinking about them. Reading, recognising, visualising, drawing and labelling the diagram and indicating the given data gave the problem a concrete setting and facilitated the job of translating it into an equation. The selection and indication of a literal symbol to represent the unknown part of the diagram directed their attention to the fact that the object of the work is to determine the magnitude of this particular part. Writing the equation required analysis of the problem to determine which of the trigonometric functions is the appropriate one to use. The transition from the ratio concept to the numerical concept of a trigonometric function and the substitution of the numerical value for the ratio were of vital importance in understanding the use of the functions in indirect measurement. The actual solution of the equation for the unknown part and the reinterpretation of this in terms of the diagram or of the original problem situation brought a realisation of how the algebraic rules operate to give the required information by giving a explicit form of a relationship which was merely implicit before.

It is worth noting that two conflicting aspects of the UK and the TR education system were revealed in analysing students' TWP solutions. The UK students, 14-16 as well as 16-18, do a great number of TWP but the TR students do very little. The TWP in TR is not revisited by teachers and it is also almost never asked in schools or high-stake examinations (see pp. 141 and 160). Interestingly the way the UK teachers (see p. 142) wanted their students to answer TWP coincided with the model arising from students' answers. So teaching style might be reflected in the students' answers. On the contrary the TR teachers only emphasised the importance of the drawing of diagrams in answering TWP, they did not give any detailed steps to students for solving TWP. Inasmuch as TWPs are virtually ignored in the curriculum and in teaching in the TR (see section 1.1.1., p. 161) the same answering model appeared in the TR students' answers. Despite this contradiction, the stages both the UK and the TR students progressed through in the process of answering TWP were the same. Consequently, it seemed that teaching approaches, solving many/varied TWP questions or few/similar types of the questions as well as studying TWP did not have any direct influence on how they answered TWP. However, it seems that these factors made a big difference to their performance. Deeper investigation of the stages of the model, which were revealed in the students' answer, might help to discover the possible reasons behind students' performances and also might provide insights to understand the occurrence of each stage in their answers. Since the model shows the processes students employ to solve a TWP, that investigation and understanding would give some perspectives for teaching students to solve TWP.

This model provides a method to diagnose the obstacles that students faced when trying to solve TWP. This can be seen more clearly in the pattern presentation of the concurrent verbal protocol analysis (e.g. Figure 4.24, p. 112). It can be used to see what stage/phase students had difficulties (or flaws) either in drawing the diagram or doing symbolic manipulations. The model shows that some students encountered obstacles in the 'constructing' diagram phase and some the TR students made flaws in the 'doing' mathematics phase of the traditional TWP but the flaws students made did not make any change in the pattern of answering. Students did not recognise the flaws they made in the stages, they completed the answering process with the same approach. So detecting where the students do make flaws (or meet obstacles) might highlight the stages teachers should emphasise in their teaching.

Finding students' obstacles and flaws by the help of the model throughout the answering process is very important in terms of the understanding TWP (Novotna, 2000). The stages students have difficulties with could be highlighted in the teaching of the TWP and the stages in which students met obstacles and made flaws may be overcome by effective teaching approaches. Subsequently, to understand the model well and to use it effectively in teaching and learning the two phases 'constructing' the diagram and 'doing' the mathematics should be investigated more deeply.

3.1.1. *'Constructing'* the diagram

Both the UK and the TR students performed very well at TORT compared with TWP. This was an interesting result, because questions in TORT were context-free forms (with modified numerical values) of the questions in TWP. That means the context (mathematical terms and students' real life experience) is likely to affect student performance. Students, furthermore, needed to draw their own diagrams in TWP, so this might be another reason behind variation in performances. The comparison of the two countries' students' performance of TWP and TORT brought the importance of 'constructing' the diagrams to view. Results revealed that visualisation and drawing were the essential part of the 'constructing' the diagram phase (see p. 101 and section 1.4.2.3., p. 103).

Although no diagram was provided, (neither were the students asked or encouraged to draw diagrams throughout the data collection) all the TR and almost all the UK students drew their own diagrams at the very beginning of their answering process. This data highlights the place of the diagram in the process of answering TWP. 'Constructing' the diagram phase also contained the stages where students of both countries made the most common flaws. So discussing how diagrams are constructed and why they are needed should be enlightening concerning the cognitive and pedagogical aspects of the phase in the model.

3.1.1.1. How the diagrams are constructed

I want to understand how students constructed the diagrams because most of the flaws were in this phase, e.g. misuse of terminology, drawing and mislabelling. Moreover, even in correct answers students had problems with constructing diagrams. It seems that drawing diagrams was not a simple process for students. My data indicates that there might be some semantic and cognitive reasons behind their drawing: mathematical terms such as angle of elevation/depression and the angle to the vertical/horizontal, students' experiences (real life or classroom practices), and visual abilities.

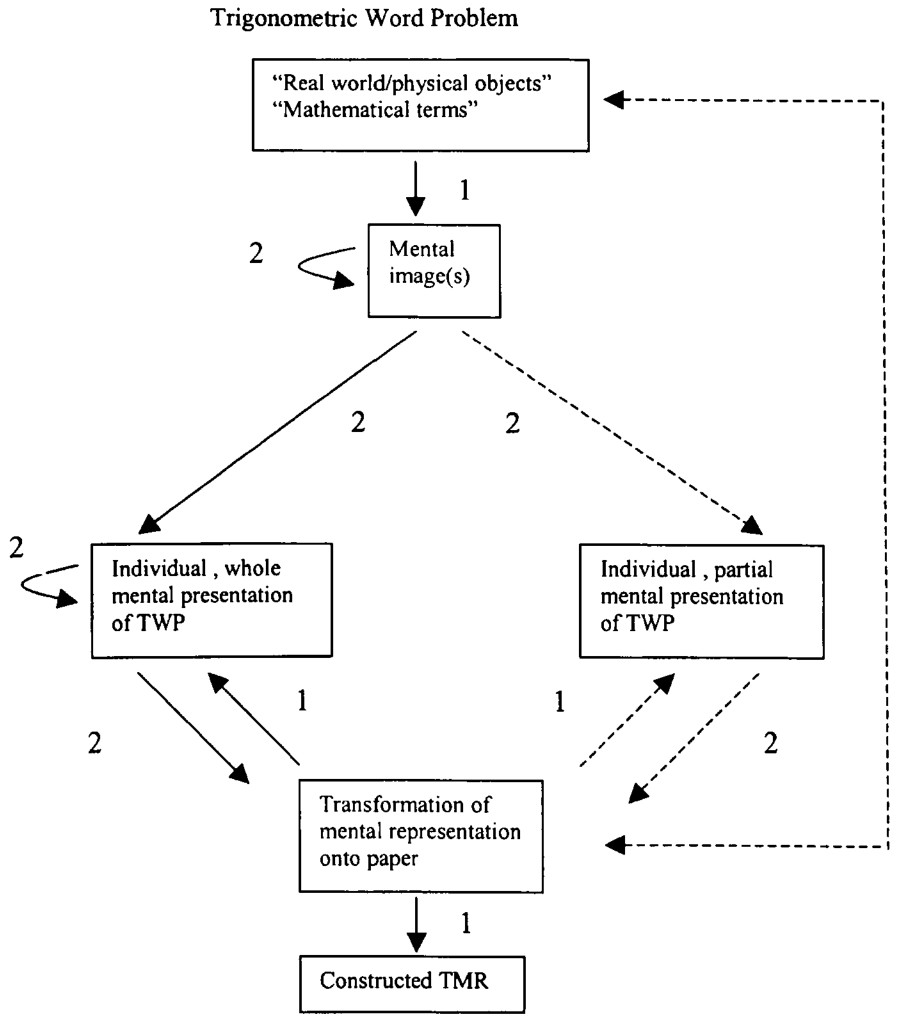
The semantic/text comprehension seemed to be important in the first stage in the process of answering TWP. The importance of the semantic/text comprehension is highlighted in other studies as well (Low et al. 1994; Mayer et al. 1984 and Lucangeli et al. 1998). When students read the word problem they clearly need to understand the words used in TWP. So they should understand the wording of the TWP first. In the interviews and concurrent verbal protocols most of the students emphasised the first stage of the model saying that *'to read carefully'* or *'to read more than one time till I understand'* was important (see p. 101). In TWP there are two types of knowledge used: real world and mathematical knowledge. In answering a TWP both forms of knowledge play an important role (see p. 102). Words such as kite, cliff, rocket etc. draw on students' real world/classroom practice experience and terms such as the angle of elevation/depression or angle to vertical/horizontal require mathematical knowledge.

From the reading stage to the drawing stage the processes seemed complicated but my data may be able to explain what happened through these stages. Important cognitive actions such as visualisation and mental representation occurred throughout these processes. The role that visualisation plays in mathematics learning is still unclear in spite of considerable research. My data shows that it is essential in the process of answering TWP. Working with visualisation and images is hard and complex work. Visualisation is a cognitive process. In my study, it is a kind of process or reasoning activity based on the use of mental images (in the sense of Kosslyn 1980), which are probably already created in the mind, and the manipulation of them to form mental representations of TWP and using that mental representations effectively for mathematical discovery and understanding (Zimmermann and Cunningham, 1991) in the course of answering TWP. Visualisation is, I hold, the integration of four elements: mental images, process of visualisation, spatial abilities and external representation (inspired from Gutierrez 1996). I will now explain what I mean by these four elements.

An example with respect to my model may be useful. After the reading and recognising stages, the words apparently evoke mental images of the physical objects (in terms of Kosslyn, 1980) or concept images of mathematical terms (in terms of Tall and Vinner 1981) (see Table 4.16., p. 108). So these objects could be a kite, cliff, etc. from real life/class practices or mathematical terms such as angle of elevation/depression and angle to the vertical/horizontal (see p. 104). In other words, students interpret the information in TWP visually using their spatial abilities (see pp. 109 and 112). This is the beginning of visualisation. Then in the process of visualisation, students manipulate images, such as generating connections between them using their spatial ability. Then they either construct a complete or partial mental representation of the situation given in TWP by interpreting the mental images and using their spatial abilities (see Figure 4.24., p. 112). After this first mental representation of TWP is in their minds, students may spatially transform it into another form. They may do this more than once. After getting a complete or partial mental representation of the situation of the TWP in their minds, students transform it onto paper. If the diagram is partially drawn then students go through the process again until they complete the diagram on paper. I call the diagram on paper the 'transformation of mental representation' of the situation in TWP. What I call a transformation is an imperfect form of the representation in the mind. As Brousseau *et al.* (1986, p 226) state, if a person tries to build a mental representation of her/his own home there will be many actual features which are coded incorrectly. The diagram on the paper becomes a self-constructed external representation of the mental representation of the situation in TWP. After the diagram is drawn, students label the sides or angles with the givens of the question (see p. 105). Then they use the diagram to complete the process of answering TWP. Visualisation, in my sense, is the total process I have described, from the beginning until the diagram is totally constructed. The diagram in Figure 5.6. shows my model of how students used visualisation to answer TWP. The TWP is interpreted by the students to generate a mental image. The first image is a starting step for visualisation depending on the TWP and students' abilities. The students use some of their abilities to perform different processes. Other mental images/representations and/or external representations may be generated before the students reach the answer.

Some key issues emerge in the processes of 'constructing' diagram phase. I now discuss them in the order of their appearance in the process. First is the mental/concept images. The importance of the representation and visualisation of the word problem structure for its understanding is mentioned in the literature (Novotna, 2000; Lucangeli, 1998; Cox, 1999). They appear important in the model of answering TWP as well. Mental images seem to be 'units' in the 'constructing' diagram phase of the model. Images are the basic operative units of visualisation (Gutierrez, 1996). Visualisation occurs in the absence of the objects, so students needed to use their memory to recall mental images of the objects. Any information in the short-term memory can be transferred to long term memory in some ways such as using the information more often and the information may remain in the long term memory if it is used repeatedly. If it is not, then it could vanish. It might be said that studying TWP for a long time and solving many and varied TWP keeps the UK students in touch with real life and enriches their experiences of TWP respectively. That may enable the UK students to use their long-term memory well in terms of using mental images. These two factors, studying TWP over a longer period of time and practice, may also develop the UK students' spatial ability to manipulate images mentally, because spatial abilities can be improved through training (Gorgorio, 1998). On the contrary, the TR students worked with TWP for a limited time period in the 14-16 age range and solved only a few similar types of the questions. Subsequently, the teaching and learning of the TWP should continue over several years (Semadeni, 1995) and many and varied questions should be used in teaching and learning TWP to develop students' ability in managing to manipulate and construct mental images of the objects. The students probably have experiences with the physical objects themselves such as flying a kite or using a ladder or watching rockets on TV or looking at pictures, e.g. cliffs and yachts. Furthermore for the mathematical terms, concept images are constructed in school and build up over years through all kinds of experiences. However, as well as acquisition of the mental images of the objects, knowing how to use them in problem solving is essential. This, once more, underlines the importance of the classroom practice/training in teaching and learning TWP.

Figure 5.6. Visualisation in 'constructing' the diagram phase of the answering model.



* The arrow represented by 1 shows the process in which students use their interpretation of visual information, what they see, and spatial abilities. The arrow represented by 2 shows the process in which students use their interpretation of mental images, what they construct in their mind, and spatial abilities.

* Interpretations are made in terms of past experiences of real life experience/classroom practice and mathematical knowledge

In my data, the number of the TR students who struggled with the drawing was larger than the UK students. Some of the TR students said they could not visualise the TWP, they could not form a picture of the problem to find the answer (see p. 104). Some possible reasons for this might be: students might not be skilful enough to interpret the information or mental images; as the mathematics literature (Eisenberg and Dreyfus, 1991) suggests, students may be reluctant to visualise in mathematics. So inability to visualise might be the reason behind partial answers and non-attempted questions. Despite this, it is believed that visualisation skills can be taught (Bishop, 1989). So the practice the UK students did (and maybe the way other topics are taught as well) might help them to develop their visualisation skills. In visualisation, having spatial abilities is also imperative, because students need to manipulate mental images to get a complete mental representation of TWP. Another reason for that is, as one of the students said (see p. 103), they could transform their mental representation into another form in their mind. So it might be said that the UK students' visualisation skills and spatial abilities are more developed than the TR ones. That might explain the difference between the performance of the UK and the TR students in TWP test, because a number of studies (e.g. Guay and McDaniel, 1977) have reported a positive relationship between spatial ability and mathematical performance. If both the UK and the TR students could not visualise, then they could not draw the diagram, then the answering process could not be completed. However it does not mean that if students visualise then they can draw the diagram, they first need to get a mental representation of the objects in TWP or a complete mental representation of TWP. Some students had difficulties in getting a mental representation of the TWP. Some of them said even though they could visualise TWP they could not get a whole picture of it. That may again be a spatial ability problem, i.e. they cannot manipulate with the picture in their mind. If they had the picture then does this mean that they could draw the diagram? The answer is 'not necessarily'. Some students said although they had the picture of TWP in their mind they could not completely draw it on to paper. Therefore it might be that students' ability to manipulate mental images and concrete (on paper) image vary. Furthermore, having mental images is not enough to 'construct' the diagram, they need to be manipulated through a visualisation process to get a complete or partial representation of TWP.

An interesting result was the characteristics of the diagrams. The UK students drew more abstract (mathematical) diagrams than the TR students who drew realistic-abstract diagrams, which is the mixture of picture of some objects from the problem and geometrical figures, e.g. line or triangle. Interestingly most of the TR students did not transform these diagrams into abstract diagrams. As some of them put it, they could see the picture of TWP better in that way. Visualisation and spatial abilities of the UK students allowed them to manipulate the images in their minds and then transform the mental representation of TWP into a more abstract diagram on paper. So the reason behind the more realistic-abstract drawing of the diagrams on the paper of the TR students might imply that the TR students are less able at using visualising and spatial abilities. However it is very difficult to say whether the characteristics of their drawing affected their performance. It seemed their abilities took them to one of three realistic, abstract, realistic-abstract forms of the diagrams and that one was the diagram they felt confident to work with. They emphasised that the important thing is that diagrams should help to answer the problem. So the form of the diagram, whether it was realistic or abstract or realistic-abstract, was generally not important in terms of performance.

Furthermore students' difficulties in drawing diagrams were connected to the questions itself. Students did well in TWPs which required them to draw only one right-angled triangle. However they did less well in TWPs which required them to draw more than one right-angled triangle. It might be said that it is more manageable to visualise and draw diagrams for students if only one right-angled triangle is needed. When the number of the required right-angled triangles increases, visualisation and drawing become more complicated. Familiarity with questions requiring more than one right-angled triangle is also likely to be important. Another reason might be working memory. If it is overloaded with the complexity of the question, then students might not be able to manage to answer TWP. Furthermore students also had difficulties with 2-D drawing of 3-D situations. Both the UK and the TR students' performances in TWP, which required 2-D drawing of 3-D situations, were lower than the TWP which required 2-D diagrams. They drew 2-D diagrams to represent separate segments of the 3-D situations. As some students put it, they cannot think in 3-D. This supports the claim that students' spatial ability and skills of visualisation are very important in answering TWP. Students could visualise and have a mental representation in their mind but transforming that onto paper requires another skill which, I believe, could be taught.

Consequently, the ways the students from both countries transform their mental representation of TWP onto paper were qualitatively different. My data, and results obtained by Pritchard and Simpson (1999), suggests that diagrams could be drawn in two ways. One way is to get the whole mental representation of the TWP and then effectively transform that representation onto paper in one of the realistic, realistic-abstract or abstract forms. Another way is constructing the diagram piece by piece taking one phrase from the question at a time. The use of visualisation seems to play an important role in both way of drawing diagrams.

In the sense of Solano and Presmeg (1995), which is embedded into my definition of visualisation, Pritchard and Simpson(1999) stated that the genuine use of visualisation played a part in the first way but there was much less visualisation in the second way. They said:

> most of the students transferred the information, a piece at a time, from the word problem to their diagram without, it seems, constructing a whole mental image.

This was the case with a small number of students (see Figure 4.19., p. 108 and Figure 4.21., p. 110) in my research too. However, in my data, most of the students seemed to have a whole mental image first and then they transferred the information, a piece at a time, from the word problem to their diagram on the paper. These students in interview and verbal protocol first read the whole questions and then they said they visualised the TWP and then they drew it piece by piece taking one phrase from the question at a time. There might be two possible explanations for that: first initial whole reading evoked a draft mental representation of TWP in the mind. Students, however, transformed that representation on paper piece by piece instead of drawing the whole at once. That is one of the ways of transforming a mental representation on to paper. Second, some students read the TWP more than once until they understand what is the situation in TWP. That understanding, it seemed, helped them to have a whole mental representation of TWP. However some drew the diagram piece by piece (see Figure 4.27., p. 117). This was particularly so in the 3-D question. So transforming the whole mental image onto paper could be in two ways, as a whole at once or part by part. To draw diagram in one of these ways depends on the cognitive, visualisation skills and spatial ability of the students (Lean and Clement, 198; Yakimanskaya, 1991). These factors are essential components of visualisation. A student might not be able to draw a complete diagram without them. This might be the reason behind the students who, it seemed, did not have a whole mental representation, drew the diagram piece by piece. Pritchard and Simpson (1999) state that there was 'less visualisation' with these students. But, in the light of my data, I think it is difficult to say whether the visualisation is less. So the term 'less visualisation', I believe, corresponds to having less visualisation and spatial skills for these students. So visualisation and its main components play an important role behind the two ways of drawing diagrams. All the aforementioned ways of drawing can be observed in the patterns of concurrent verbal protocol analysis (e.g. Figure 4.27., p. 117). For example recognise, visualise, draw, (match and label) stages of the model maybe repeated more than once in the pattern representation when a diagram is drawn piece by piece. Moreover, if students draw the diagram in one go, then the pattern representation will be a diagonal line from the reading stage to the labelling stage.

The diagrams students drew were classified as correct, incorrect or partial. In all correct answers the diagrams were also correct. However, not all correct diagrams lead to correct answers (see p. 111). In partial answers students did not complete the diagram, this might arise from either not having an appropriate mental representation of TWP or not being able to transform the mental representation of TWP onto paper. There were some interesting aspects of the incorrect diagrams of TWP. All incorrect diagrams led to incorrect answers.

After the diagram was completely drawn on paper the mental representation in the mind became a concrete representation on paper. A diagram should convince students that it is an appropriate representation of the TWP. However, in most incorrect answers inappropriate and unreasonable diagrams did not warn students that their diagram was incorrect. Two possible reasons for this are: students did not know the terminology or how to use the terminology in the diagrams; students did not have enough experience with the objects included in TWP, such as the movements of rockets. In addition to have a correct diagram students should also have knowledge of both the terminology and real world experience. If they have difficulties in using both of these, then the diagram is likely to be incorrect. However, sometimes, to be able to use one of these aspects may help students to discover their flaws and allow them to put diagram right. There were examples of correct use of real world knowledge which helped students to use mathematical knowledge correctly. For example, in some incorrect answers of the rocket TWP question 2, although students used the correct terminology they did not draw the correct diagram because they could not interpret how the rocket moved, so their rocket went under the ground. But one of the students recognised that the rocket moves up, as she put it "I first drew it incorrectly but then I remembered the scientific magazine I read about rockets", so her experience helped her to draw a correct diagram. However, the completed diagrams did not help students see that they had flaws in using the terminology, e.g. angle of depression, to draw the diagram (p. 111). Consequently students should know how to use both terminology and their real-life experiences in their diagrams. That also shows that visualisation continues to be important throughout the constructing phase of the answering model and also when the diagram is drawn on the paper (whole or partially). Visualisation acts as the means for travelling between internal and external mental representation of TWP (see Nemirovsky and Noble, 1997).

After diagrams are completely drawn on paper students then do matching and labelling which was one of the most common flaws in incorrect and partial answers (see p. 101). These are important stages of the 'constructing' the diagram phase of the model and students read the questions again when they are matching their diagram to the TWP and labelling. Matching and labelling involves a comparison and a sort of review of the mental representation of TWP in the mind and on the paper. However it was difficult to say to what extent these factors helped students to see what was wrong about the diagram in incorrect or partial answers. But it might be said that, at these stages, students did not modify the diagram by observing they did not erase the labelling on the diagram. Misuse of terminology also caused mislabelling in the diagrams. So terminology seems to have an important role in matching and labelling.

After all the stages mentioned so far diagrams were completely drawn with labels on them. At the end of the 'constructing' the diagram phase it might be said that TWP was transformed into a TORT question, a geometric figure of right-angled triangles with labelling on, at which students from both countries were very successful. Students did the mathematics part of the answering model of the TWP after they drew the diagrams, which will be discussed latter, but before discussing the 'doing' mathematics phase I want to discuss why diagrams are needed in answering TWP.

3.1.1.2. Why diagrams are needed

All students answered almost all TWP. In their answers they always drew diagrams even though some of them said they could answer TWP without drawing the diagram, whether they were familiar with the TWP or not (see p. 103). Then, is drawing a diagram needed in the process of answering TWP? Both the UK and the TR teachers and students highlighted the importance of diagrams in the interviews. My data supports the result found in the literature (i.e. Pritchard and Simpson, 1999). Simpson and Pritchard's (1999) findings also emphasise the important place of the diagram in answering TWP. They found a general flow to the methods of the students' solutions' of TWP: construction of a diagram, identification and extraction of data, choice of ratio and symbolic manipulations, moreover they also found that all of the students in all answers drew a diagram. Nickerson et al. (1985) claimed that if a graph or diagram is drawn, then the problem solver can bring perceptual process to bear on it. Diagrams give a picture of the TWP in which certain relations among the parts become apparent. Diagrams act as tools created by students to use in answering TWP and help students to grasp and understand the

TWP (see p. 103). By using diagrams students identify the function(s), develop the mathematics and do the symbolic manipulations to get an answer.

In answering TWP, students' comments suggest that diagrams play the following three complementary roles

- they provided greater clarity and detail *"I will draw it out to make it clearer...I drew the vertical wall first of all and then the point which was 25 meters from it and then I just from the point I did the angles of elevation from the top and the bottom of the wall to the statue...I usually draw diagrams...to picture it more helps"*
- they overcame the limitations of working memory *"I just want to draw it so that I can actually label it and think of other things"* (see Lucangeli, 1998)
- helped to understand problem *"no, no I first could not understand what is going on in the problem, but then I drew diagram and it helped me a lot... so diagrams are important...and they help you to construct mathematics as well...you combine diagram and symbolic manipulation part for the solution"*

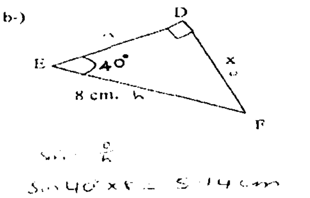
There were two UK students who did not draw any diagrams in answering some TWPs, but did the mathematical part. One of them answered one question in TWP and the other one answered two questions in TWP without constructing any diagrams. Two of the answers were correct but one was incorrect. So a correct answer can be obtained without drawing a diagram. So the reason why they did not draw a diagram might be explained by them having high spatial ability and good problem solving skills. They might be geometric thinkers (Krutetskii, 1976 p 18) but this, I admit, begs many deeper questions.

3.1.2. *'Doing'* the mathematics

The 'doing' the mathematics phase is the second phase model of the answering TWP. In this phase students used the diagrams they drew for identifying the function, constructing the equation and solving the equation to get an answer. These stages are almost the same, except for the matching and labelling stage as in the answering model of TORT (see p. 119). So it might be said that a TWP become a TORT question in the second phase of the model. That may have helped students to be more confident, because they were good at TORT. Students committed less flaws in this phase compared with the first phase because, as they succinctly put it, "after the diagram the rest is easy". That fact highlighted the importance of the first phase again. If diagrams are accessible and drawn correctly then the probability of getting an incorrect answer is be low.

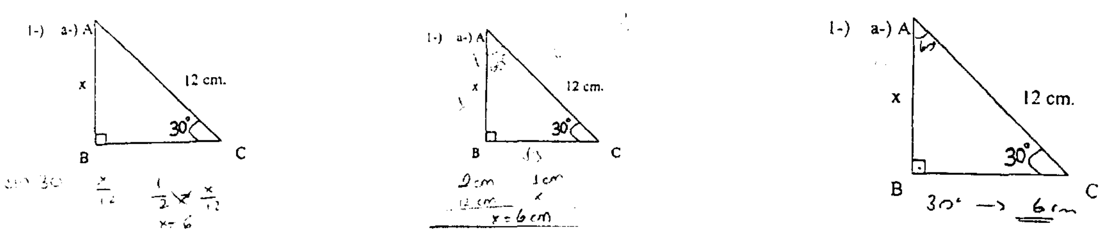
The four stages of the 'doing' mathematics phase, identifying the function, developing mathematics, doing symbolic manipulation and finding the result were seen in all answers of the students of both countries mentally and on the paper (Figure 5.7.). As a result of the analysis it might be said that they

Figure 5.7. One of the UK students answer to TORT sub-question 1b.



should occur in the order given in Figure 5.5. mentally or on the paper. Although both the UK and the TR students used the same process for answering TWP they used different styles. The UK students recalled the mnemonic ways they had been taught to enable them to recall the basic ratio definitions (in shortened form like "sine=opposite/hypothenuse", see also Pritchard and Simpson, 1999). They made a second matching and labelling on the diagram by using the letters in a mnemonic way (SOHCAHTOA) or by using the words "opposite" and "hypotenuse". The TR students, however, did not use any mnemonic to identify the function and so they did not do any matching and labelling on the diagram as the UK students did. So the answering model of TORT does not apply to the TR students. TR students used specific right-angled triangles such as 30^0, 60^0, 90^0 to find the value of the trigonometric function or they treated it as a rule and, without doing any manipulation, they answered TWP (Figure 5.8.).

Figure 5.8. Some TR students' answers to TORT sub-question 1a.



Both the UK and the TR students first learnt the definitions of trigonometric ratios on a right-angled triangle. However the way students of both countries used to remember trigonometric functions were different, as mentioned above. These two different approaches to remembering trigonometric function might be the reason between the UK and the TR students' performances in TWP and in TORT. Overall, the UK performed better than the TR in both tests. They performed well as they did in the TORT test. Interestingly most of them looked for right-angled triangles to identify functions even though it was not asked for. It might be said that the right-angled triangles with two known sides and one unknown angle or one known side and angle and one unknown side led students to use the definitions of the trigonometric functions. That might explain the close links between the concept definition of trigonometric functions and the concept images the students have. When the students first met the trigonometry, they were given definitions of trigonometric functions before any significant concept image of them. However, students always looked for a right-angled triangle to apply the concept definition of trigonometric function. Then since the concept image is the total cognitive structure associated with concepts which includes all mental pictures, associated properties and processes, it could be said that a right-angled triangle with its properties and process became the strongest aspect of concept image for students.

As well as the definitions, the given values/unknown(s) in TWP helped students to identify functions. The known and unknown sides or angles led students to the trigonometric function to be used. Replacement of the relevant information into the formula form of the definitions of trigonometric functions developed the (symbolic) mathematics part of the phase. It was observed in the data that sometimes these two stages, identifying the function and developing the mathematics, overlapped. Therefore, simultaneously, identifying trigonometric functions led to developing mathematics with the appropriate use of the algebraic skills they have.

After the development of the mathematics, in constructing the equation, students did the symbolic manipulations to get the result. Since the UK students used a calculator, they found the answer directly after achieving an algebraic/numeric expression. The TR students continued to work with the algebraic expressions, substituting into it relevant expressions, until a numerical expression was achieved. This expression was not reduced to a single number or value. In the protocols, the TR students declined to use trigonometric tables. Using a calculator or not did not significantly affect the UK and the TR students' performances in symbolic manipulation significantly, but the TR students did more symbolic manipulations than the UK students.

Consequently the four stages in 'doing' mathematics phase occur in the order given in Figure 5.5. and if there is more than one unknown to find by using the trigonometric functions then these stages would be repeated in the same order as many times as the number of the unknowns required to be found. The key stage is identifying function. If it is defined incorrectly than students produced an incorrect answers, it did not matter they did other stages correctly. 'Doing' the mathematics phase is symbolic representations of the relations among the parts in the diagram so it might be said that it is a transformation of the relations on the diagram. That also implies the existence of direct interaction between the diagram and the symbolic part of the answering TWP.

3.1.3. Interaction between the diagram and symbolic part of the answering TWP

In all answers given to TWP there were only two observably physical phases: diagrams and numeric/symbolic parts. It seemed that these two phases have the strict order given in Figure 5.5.. As can be seen in the previous sub-section, students always used the diagrams to construct the symbolic part of the answering TWP. All the TR and almost all the UK students, first of all, drew a diagram in the process of answering TWP. Even if the three answers which did not have a diagram, are taken into consideration, it could be argued that they constructed the diagrams in their mind and manipulated diagrams mentally. These students could have their own internal blackboard (MacFarlane-Smith 1964, p 132).

There were no correct answers which included a diagram but no manipulation. Subsequently my data highlighted that the symbolic part cannot be constructed without constructing the diagram (mentally or on the paper). The inevitable consequence is 'constructing' the diagram and 'doing' the mathematics are inseparable in the order given in Figure 5.5.

Diagrams seemed to become external stimuli to evoke concept definitions or thoughts/knowledge that students have and recall them to use in the process of answering TWP. Since students first learned the definition of trigonometric functions on right-angled triangles, the diagrams in that form help students to recall the concept definitions they learned.

Mnemonic ways were used by the UK students but not the TR students. However, both the UK and the TR students moved between the visual and symbolic representations, developing the symbolic whilst the visual remain unchanged. So the students did not modify the diagram as happened in Nunokawa's (1994) case. The students in my study used their diagram to do the mathematics phase of the answering model, namely they used the diagrams to organise and extract the information they needed to construct the mathematical part. So the students in my study used their diagrams in an organisational way (Simpson and Tall, 1998).

4. Revisiting the Research Questions

My study is a comparison of the two countries from the students' position (what they do and understand). I think it is important to say a few introductory words on 'doing' and 'understanding'. Student understanding is central to all cognitive based mathematics education studies. However, the term 'understanding' has become a problematic term over the last decade and people seem afraid to use it because we do not really understand exactly what understanding is. Given this I focused on student performance. This does not mean that I have behaviourist tendencies. I remained concerned with understanding but report on observable outcomes. Students do not do/perform in a vacuum. They have learning histories shaped by the curriculum and the culture of education of their countries. My central research focus is students' understanding (performance) but because of the importance of learning histories I have a second research focus concerned with the culture of learning.

There were two main research questions (RQs), which have guided and focused the study from the commencement of the study. My first research question concerns student performance on tasks concerned with trigonometric identities and formulae, their manner of 'simplifying' trigonometric expressions and their performance in solving trigonometry word problems. My second research question concerns the influence of teaching, the curriculum, examinations and resources on students' performance in this area. To save the reader referring back, I repeat my RQs before discussing them.

RQ1-i

The focus here is on students' performance of trigonometric identities, trigonometric formulae and their use in 'simplifying' trigonometric expressions:

a- What difficulties do they experience, what errors do they make?

b- How do they use their knowledge of trigonometric identities in their simplifications of trigonometric expressions?

c- How do these performances interact with their knowledge and use of algebraic conventions?

RQ1-ii

The focus here is on students' performance of trigonometric word problems:

a- What 'mental models' do students follow in solving trigonometric word problems?

b- What difficulties do they experience, what errors do they make and what conceptions do they hold?

c- To what extent do the context and the terminology affect the solution of trigonometric word problems?

d- How do visual and symbolic representations interact in the solution process?

RQ2-i

The focus here is on teachers in both countries:

a- How do they teach trigonometry, what resources do they use and not use, how does the curriculum affect their teaching of trigonometry?

b- What emphasis do they place on the foci of the first research question , e.g. how and in what order do they teach these?

RQ2-ii

The focus here is on the curriculum in both countries:

a- "What is it, as in written documents?"

b- How do teachers implement this in terms of classroom activities?

c- What aspects to textbooks 'privilege'? What is examined and how important are these examinations?

Throughout the study these research questions remained unchanged. However, as has been seen in this chapter, some new issues emerged. I arrange my discussion below in three areas; *Simplifying trigonometric expressions* addresses first part of the first research question, *trigonometry word problems*, focuses on the second part of the first research question and *the influence of teaching, the curriculum, examinations and resources on students' performance*, focuses on both parts of the second research question.

4.1. Simplifying trigonometric expressions

Simplifying a trigonometric expression is important, because a given initial trigonometric expression may need to be transformed to solve an equation or inequality, draw a graph, do integration, derivation and prove/verify equivalences.

- **What difficulties do they experience, what errors do they make?**

Overall the TR students did better in simplifying trigonometric expressions than the UK students did. However the flaws, the difficulties students experienced and errors they made, performed by students from both countries were the same. Students mainly performed two sorts of flaws: algebraic flaws, which are the basic manipulations and algebraic prerequisites and trigonometric flaws, which are the title recognition of trigonometric identities. The flaws included in these categories were explored in detail in the results chapter (see p. 78). The most common flaws in both the UK and the TR students' answers were recognising/recalling/using of the double angle identity and then use (or non-use) of the Pythagorean identities. These flaws in simplifying trigonometric expressions, which were confirmed by the qualitative data, occurred because students did not recognise either an algebraic/trigonometric property in the expression (see pp. 81-82) or an equivalent form of an identity. Further to this, since they did not know exactly what simplification meant in trigonometry context they did not know where to stop (see pp. 81-82) and often had difficulty knowing which identity to use in a simplification.

After the trigonometric flaws the most performed flaws by students from both countries were basic manipulations, which were addition and subtraction of non-fractional trigonometric expressions, multiplication and division of non-fractional trigonometric expressions, expanding brackets in non-fractional trigonometric expressions and addition and subtraction of fractional trigonometric expressions. Addition and subtraction of fractional trigonometric expressions was a particular obstacle for the UK students more than the TR students. The difference of the percentage of this flaw performed by the UK and the TR students were significantly high (see p. 78). With regard to algebraic prerequisites, students from both countries commonly performed flaws in cancellation in fractional trigonometric expressions, difference of two squares, common factors and taking squares of brackets. Both the UK and the TR students experienced considerable difficulty in applying algebraic rules in trigonometric contexts (see p. 78-79).

- **How do they use their knowledge of trigonometric identities/formula in their simplifications of trigonometric expressions?**

The model of simplifying trigonometric expressions (see p. 189) implies that trigonometric properties which incorporate trigonometric identities should be recognised and recalled in the simplification process. To do that, students require a knowledge of trigonometric identities, which is implicitly highlighted in the model. The UK and the TR students mainly have knowledge of trigonometric identities in two forms. In one form, identities are stored in their memory; in the other form it is written on the formula sheet. The TR students need to recognise and recall trigonometric identities in the simplification process whilst the UK students merely need to recognise. So being familiar with trigonometric identities and being able to use the appropriate identity is imperative, as well as knowing the trigonometric identities to be used in the simplification process. Students from both countries have difficulties in applying their knowledge of trigonometric identities when the identity in the expression is implicit By implicit, I mean students need to do some manipulations to explicitly see what identity to use, recognising and recalling very important aspects of knowledge of trigonometry. Practice, working with many and varied questions, helps students to use their knowledge of trigonometric identities appropriately in the simplification of trigonometric expressions. This helps students become familiar with the identities and procedures required in different types of questions. A knowledge of trigonometric identities, recognising and recalling them and using them properly, is sometimes not enough on its own, students may need to use algebraic manipulations to get the simplified form.

- **How do these performances interact with their knowledge and use of algebraic conventions?**

It is seen in the curriculum and textbooks and it is also emphasised by teachers that algebra and trigonometry are two disciplines which go hand in hand in mathematics (see pp. 146-147). That fact is reflected in the simplification of trigonometric expressions (see p. 189). Facility in algebraic manipulation is important in the simplification of trigonometric functions. Test results show (see p. 94) that students from both countries performed better in their algebra test compared with their trigonometry test. On the other hand when students' performance are compared, the TR students did better than the UK students in both tests, especially in the algebra test.

The parallel questions (see pp. 95-97) neatly illustrate how their algebra knowledge affected their performance. Although they performed very well in algebra questions they did not show the same success in the trigonometric questions because they met various trigonometric identities as well as doing algebraic manipulations and using algebraic properties in simplifying trigonometric expressions. Students' are aware of the differences between algebra and trigonometry. In students' eyes, algebra seems easy compared with trigonometry (see p. 91). They are used to working with 'a' s and 'b's for a long time but sin, cos, tan as terms to be manipulated, not just found, are less familiar and may appear 'strange'. Furthermore, both the UK and the TR students are more comfortable, confident and knew where to stop with simplifying algebraic expressions comparing with trigonometric ones. Surprisingly, virtually none of the students, from either country, used a substitution method to convert trigonometric expressions into algebraic ones to manipulate them in a way they were familiar with.

The reason behind most of the flaws students made in simplifying trigonometric expression was algebraic. The percentage of algebraic flaws, including basic manipulations and algebraic prerequisites, was twice the percentage of the trigonometric flaws in both countries' students' answers in the trigonometry test. Almost half of the flaws both countries' students made were basic manipulations in the trigonometry test. They, however, committed fewer of these flaws in simplifying algebraic expressions. Students' performance in using basic manipulations and algebraic prerequisites in simplifying expressions in trigonometric and algebraic contexts differed. Students coped better with algebra in algebraic contexts compared with trigonometry contexts. Students particularly had difficulties with addition and subtraction of fractional expressions, multiplication and division of non-fractional expressions and expanding brackets in the trigonometry test as well as in the algebra test (see p. 78). In a trigonometric context, basic algebraic manipulations seemed to be more difficult for students in that they did not have as many difficulties with them in an algebraic context.

With regard to algebraic prerequisites, the types of problems students mostly had difficulties with were cancellation in fractions, the difference of two squares and common factors. These difficulties were also the most common ones with squares of brackets and linear functions in the trigonometry test. Students from both countries met the same difficulties in their algebra and trigonometry test in terms of algebraic prerequisites. In the trigonometry context, students had difficulties applying algebraic properties.

Being successful in simplifying algebraic expression did not affect student performance in simplifying trigonometric expressions, contrary to what teachers and students from both countries thought. Applying algebraic properties in a trigonometric context is difficult for students (see p. 78). They must cope with trigonometric identities in the simplification as well as using algebraic properties.

4.2. Answering trigonometry word problems

Trigonometry word problems are 'real world' applications of trigonometry. TWP were solved as an application of the basic trigonometric functions. This is a different aspect of trigonometry from the aspect concerned with simplifying trigonometric expressions. Initial expectations were that students would use different strategies and abilities to solve trigonometry word problems compared with the simplification of trig expressions. The following sub-sections address the research questions concerned with the trigonometry word problems in the light of the data.

- **What 'mental models' do students follow in solving trigonometric word problems?**

The term 'mental model' has various meanings in the literature (Schwamb, 1990). When I framed this research question several years ago I did not appreciate the complexity of the term. The term "mental model" is occasionally used as a synonym for "mental representation" (Johnson-Laird and Byrne, 2000), but that was not the meaning I wanted to use although 'mental representations' occupied an important place in answering TWP. Throughout the analysis an operational model emerged and which I called an 'answering model' of trigonometry word problems, that basically corresponded to what I meant by 'mental model'. That model was not complicated and was grounded in my data.

The model has been explained in detail in the section 3 of this chapter. Students' answers always included two physical parts, diagrams and symbolic manipulations. In more detail (see Figure 5.5., p. 201), students go through the following steps when they are solving trigonometry word problems: reading, recognising, visualising, drawing, matching, labelling, identifying function, developing mathematics, symbolic manipulation, result and review. Depending on factors such as their ability, experience, the question type and characteristic of the drawings, some steps are repeated. The model provides a means to monitor students' actions throughout the answering of trigonometry word problem.

- **What difficulties do they experience, what errors do they make and what conceptions do they hold?**

The flaws students performed in answering TWP corresponded with the stages of the model of answering TWP (see pp. 99 and 201), i.e. reading, drawing, matching and labelling, identifying functions, developing the mathematics and doing symbolic manipulation. 'Terminology' was a flaw which was not a stage in the model. 'Terminology' was the most common flaw in the drawing stage of answering TWP. That provides evidence that students experienced substantial difficulties or, at least, made errors in the 'constructing diagram' phase of the model. Clearly students require knowledge of mathematical terminology if they are to visualise and draw the diagram, to understand the TWP. The TR students are, particularly, not good at the drawing correct diagram. The obstacles in drawing diagrams were: the number of right-angled triangles to be drawn and 2-D drawing of 3-D situations. Both the UK and the TR students made very few flaws in the 'doing mathematics' phase of the model. Students coped with trigonometric functions on geometrical figures, namely, they managed to identify the required function, develop the mathematics from the diagram they drew and perform the symbolic manipulations to get the result. The most common flaws in the 'doing mathematics' part of the TWP were identifying the functions (including finding the ratio of trigonometric functions by using special right-angled triangles, i.e. 30^0, 60^0, 90^0 right-angled triangle) and performing symbolic manipulations. The 'constructing diagram' phase of the answering TWP is more difficult and problematic for students than the 'doing' mathematics part.

One of the difficulties, which came out in the qualitative data, is visualizing the situation in the given TWP to construct the diagram. Being able to visualize or not visualize affected (see pp. 103 and 104) students' drawing and so affected their answer, because if they cannot visualize they cannot draw the diagram representing the situation in the TWP.

- **To what extent do the context and the terminology affect the solution of trigonometry word problems**

Trigonometry word problems are embedded into contexts, which are 'real world' situations for students to demonstrate their understanding of basic trigonometric functions. To observe the effect of context in solving word problems another test including context free questions of the TWP was used. The difference between students' performance was astonishing, both the UK and the TR students answered a very high percentage of the context free questions correctly compared with TWP (see pp. 121-122). Particularly the TR students' performance were very poor in TWP. Students did not have the diagrams in TWP, they had to draw their own diagram. That means they had to understand the words and the context in the problems to grasp the situation and then to draw the diagram correctly and then they should label the diagrams they drew correctly as well. Therefore constructing diagrams depends on understanding the context of TWP. It helps students to visualise and then draw the diagram. Students' performance and utterances underlined that the 'doing' mathematics part of the answering TWP is easier (see p. 104). These facts highlight the importance of context in answering TWP.

Working with trigonometry in real world context may help to improve students' ability to demonstrate their understanding of basic trigonometric functions, and facilitate the development of greater understanding of trigonometry principles. Context can also obscure the mathematics and divert the intended direction of the development of mathematical understanding (cela.albany.edu). So, as one of the teachers said, it is useful to show how trigonometry is used in 'the real world' but it does not help students to understand because they must draw the diagram first to apply their trigonometry knowledge. That, however, is the part students have difficulties with. So context can be an obstacle.

To understand TWP, students should know the words, which are either the words used in real life or mathematics terminology embedded into a context. So terminology is an important part of the context of TWP. The misuse of terminology was the most common flaw in the UK and the TR students' incorrect and partial answers (see p. 101). They particularly had difficulties with the use of angle of depression, angle of elevation, horizontal and vertical (e.g. p. 112). The terminology seems to be essential for constructing the diagram representing the situation in the TWP and also labelling the diagram correctly (see p. 112). This implies that the correct use of terminology helps students to construct correct diagrams and to label the drawn diagrams accurately.

- **How do visual and symbolic representations interact in the solution process?**

In the section 3 of this chapter I elaborated on the interaction between the diagram part and symbolic part of the answering process of TWP. These two parts were in almost all answers, so they seemed to be the main characteristics of answering a TWP. The part which was created firstly was the diagram part, because students in their tests always started by constructing a diagram at the very left of the answer sheet and in interview and verbal protocol they also started by constructing diagram then they did the mathematics part (e.g. 108-109). Constructed diagram acted as an external stimulus for students to do the mathematics part. There was only one way from the 'constructing' diagram to the 'doing' mathematics part. In answering TWP, constructed diagrams had a direct influence on 'doing' mathematics phase that students created the 'doing' mathematics phase from the 'constructing' diagram phase. On the other hand any changes or actions that was done by students in 'doing mathematics' phase never changed the diagram. The symbolic part was hardly ever created without a constructed diagram. 'Constructing' a diagram was the key in the interaction between the two phases. For a complete answer, 'constructing' diagram and 'doing' mathematics phases seem to be complementary to each other for answering TWP. That interaction highlights that students should rely on the diagram they construct to answer TWP. The UK, but not the TR, students used mnemonic ways to pass from the constructed diagram to the symbolic part (see Figure 5.7., p. 213). The TR students, however, used specific right-angled, i.e. 30^0, 60^0, 90^0, triangles to find the ratio of trigonometric functions for symbolic manipulations (see Figure 5.8., p. 213). An interesting point is that after a diagram is constructed a TWP becomes a TORT question.

4.3. The influence of teaching, the curriculum, examinations and resources on students' performance

Students do not do/perform in a vacuum. They have learning histories shaped by curricula and the culture of education of their countries. Possible influences on their performance in 'simplifying' trigonometric expression and answering trigonometry word problems are teachers and the curriculum of their countries. In the first section of this chapter, I have discussed these two factors in some detail so here I will just briefly address the relevant research questions;

The focus in the following is on teachers in both countries

- **How do teachers teach trigonometry, what resources do they use and not use, and does the curriculum affect their teaching of trigonometry?**

Teaching the simplification of expressions and solving trigonometry word problems occurred in classrooms in both the UK and the TR and various tools/resources are utilized throughout this teaching.

The UK lessons take a different form to those of the TR. The UK teachers first explain the topic, then do some worked examples and devote the remaining time for seat work. The TR teachers start with a very brief revision of the previous lesson. Then there is a cycle of explaining the topic and going over worked examples. Both the UK and the TR teachers think that trigonometry is one of the most difficult subjects learnt at school (see p. 140). The UK teachers gave homework which they regularly checked and marked. On the other hand, the TR teachers did not give homework regularly; if they did, they did not check them. The TR teachers used questions from private institutes which are competitive, challenging, and varied. On the other hand, the UK teachers worked with the questions provided by examination boards.

The TR teachers taught that more abstract aspects of trigonometry while the UK teachers taught more application aspects of trigonometry. TR teachers worked with a greater number and variety of questions whereas the UK teachers worked with examination style questions. All calculations in the UK were done by using calculators while in the TR paper and pencil methods were used. The UK teachers used specific mnemonic ways to teach basic trigonometric functions. The TR teachers used a variety of mnemonic ways.

Regarding answering TWP, the UK teachers stated the steps students should follow which virtually coincided with the stages in the answering model (see p. 142). On the other hand, the TR teachers did not state any specific ways. They only highlighted that students should do the drawing, then do the mathematics (see p. 142-143), but at the end both teachers emphasised the diagram and then doing mathematics part. Interestingly, teachers from both countries were not clear about what simplification means in trigonometry context. Although they answered and found the simplified form they were not clear about the definition and that was reflected in students' data as well. Students from both countries, particularly the UK students, did not know how to get a simplified form so that there was a high percentage of incorrect and partial answers in both countries' trigonometry test results (see Figure 4.1., p. 76). The UK teachers appeared to follow the examination board expected methods in simplifying trigonometric expressions while the TR teachers seemed more independent with regard to method of simplification.

The UK teachers used more varied tools than the TR teachers (see section 1.3., p. 176). The TR teachers used printed documents whereas, in addition to printed documents, the UK teachers used calculators, formula sheets, overhead projectors and computers.

Although teachers from both countries had different approaches to teaching trigonometry and used different tools they were all tied to their curriculum (see p. 136-137). A point made by a UK teacher was the *"The Curriculum tells us what to teach not how to teach"*. Further to this some the TR teachers said that they might play with the teaching order given in the curriculum if it was not suitable for their students. So although both the UK and the TR teachers follow the curriculum they have a critical attitude towards it.

- **What emphasis do they place on the foci of the first research question, e.g. how and in what order do they teach these?**

The UK teachers highlighted the use of calculators in what I have called the first and second appearance of trigonometry. The use of calculators restricts the UK teachers in some ways e.g. not using trigonometric functions like cotangent, cosecant and secant. However the UK teachers worked with all sort of angles. Calculators were not allowed in classrooms and examinations in the TR. the TR teachers made a number of negative points about the use of calculator (see

p. 138) and did all manipulations manually in the classroom. They almost always worked with the angles 30^0, 45^0 and 60^0 and rarely used other angles. Trigonometric table was rarely used.

Teachers from both countries were asked to help to put trigonometric topics, which are already in the curriculum of both countries, in a teaching order. The results suggest that there is no dominant ordering or even a set of orderings – teachers orderings were remarkable by their variation, but they all wanted to teach TWP just after the trigonometric functions and trigonometric values of general angles and before further trigonometry. The TR teachers complained about the absence of the TWP in the curriculum.

The focus in the following is on the curriculum in both countries in terms of trigonometry

- **What is the curriculum (in written documents)?**

How trigonometry was emphasised in the curriculum in the two countries was different. Although applications are mentioned in the TR curriculum, they were not real world applications; they were abstract algebraic or geometric applications, not real world problems as in the UK. One of the most interesting differences observed was the number of trigonometric identities or formula given in the TR compared to only a limited number of in the UK curriculum. The angles worked with in trigonometry in curriculum were also different. A wide range of angles were used in the UK curriculum while mostly 30^0, 45^0, 60^0 were used in TR. Interestingly, although it was not defined anywhere 'simplification' was highlighted in the UK curriculum whereas it is not in TR. One interesting difference between the curricula of the UK and the TR is the place of the four trigonometric functions in TR but not in the UK. Sine, cosine, tangent and cotangent were treated in an equal way whereas only sine, cosine and tangent were highlighted in the UK. Moreover, working with 3-D applications and graphs was highlighted in the UK but not in the TR. In summary the TR curriculum is more algebraic than the UK curriculum and, in the curricula of both countries, objectives were merely telling what to teach not how to teach.

- **How do teachers implement this in terms of classroom activities?**

Delving into the implemented curriculum needs more research and work, but my data allows me to comment on aspects on teachers' implementation of the curriculum in classroom activities.

Both the UK and the TR teachers were tied to the curriculum in their countries respectively. Although they rarely prepared lesson plans they prepared schemes of work according to the demand of the curriculum. The UK teachers appeared to follow the curriculum precisely. The TR teachers said they might change the teaching order of trigonometric topics if they could saw a problem but they mostly followed the order of the objectives in the curriculum as discussed on (see section 1.2., p. 169). The UK students were given more examples to do in the practice part of the lesson. The details of the topic and the examples used in teaching trigonometry were published in the textbooks.

- **What aspects do textbooks 'privilege', what is examined and how important are these examinations?**

There are many mathematics textbooks, each of which has its own style. Teachers from both countries follow one of them as the main textbook. Although they, particularly the UK teachers, found the textbook satisfactory they utilised other textbooks (or resources) in teaching trigonometry. The reason behind this seems to be that they did not really think just one textbook is satisfactory because it could be weak in terms of explanations or exercises. Both countries' teachers prepare their own written documents as well. Moreover, students' editions of textbooks are used more comprehensively in the UK than in the TR where teachers' editions of textbooks are mostly used (see p. 135).

Both the UK and the TR teachers prepared the students for high stake examinations. A-levels in the UK and UEE in the TR. These examinations occur just before University and strongly affect students' choice of university. So lessons in both countries are driven by examinations. Two notable results were observed in high stake examinations of both countries. First, although TWP was not emphasised in the TR but was in the UK, there was a TWP in the TR examination whereas there was not one in the UK examinations (see p. 160). Secondly, there was a greater variety of the type of trigonometry questions in the UK high stake examinations (see p. 160). These two results were not expected in the sense that although the TR worked more with algebraic aspect of trigonometry, e.g. use of more identities/formulae, it was the UK who made more use of a variety of algebraic aspect of trigonometry in their high-stake examinations.

CHAPTER 6: CONCLUSION OF STUDY

This study was a long personal journey full of adventure and pitfalls. It started from a love of trigonometry and a desire to find out how students understand trigonometry. Then came the preparation for the trek: stating and refining research questions, reading relevant literature, deciding on the data collection tools, samples and methods of data analysis. Collecting data proved to be a very interesting stage of the journey for students always seem to introduce something unexpected and, indeed, I ended up examining issues I never dreamt of at the outset. Now I must find a way to summarise this journey. I do this in three parts: an overview, educational implications and suggestions for further research. It is not uncommon for thesis conclusions to state the limitations of the study. I have not done this as I feel I have done this throughout the study.

1. Overview of the findings of the study

The main finding is that the TR students performed well in algebraic aspects of trigonometry whilst UK students performed well in the application aspect of trigonometry in the written tests. But why this is so is not so easy to state. I attempt an answer by looking at five related issues: are there two types of trigonometry?; students' manner of simplifying trigonometric expressions; students' manner of answering trigonometry word problems; students' performance in context and context-free questions; students' performance in simplifying algebraic and trigonometric expressions

1.1. Two types of trigonometry

I argued on page 186 that the results may suggest that there are two types of trigonometry in the UK and TR and that evidence for this exists in the curriculum, in teachers' 'content privileging', in the tools used in teaching and in students' performances. The following illustrate these differences:

- Differences start with the curriculum of both countries. There were different objectives in the trigonometry curriculum at the same age group in the UK and the TR, e.g. 3-D, graphs and real world applications were highlighted in the UK whilst algebraic aspects and more trigonometric identities were emphasised in the TR. A particular difference was the place of cotangent function which is treated as one of sine, cosine and tangent in the TR. Both unit circle and right-angled triangle techniques in teaching trigonometry are explicitly underlined in the TR curriculum but only the right-angled triangle technique in the UK.

- Lessons in the UK and TR could be said to be 'driven' by examination foci. The preparation for the examinations was very different in that the UK used examination board style questions where the TR used any written resources and especially the private institute question banks, which included many and varied questions compared with the UK The dissimilarities of the questions in the UK and TR might be a possible reason behind students' performance. However, in high-stakes examinations there were proportionally less trigonometry questions in TR than the UK. In examinations, calculators and formulae sheets were allowed in the UK but not in TR. The UK students showed their manipulations on examination papers but, on the other hand, in TR, examination was by multiple choice. Most of the time the angles 30^0, 45^0 and 60^0 were used in trigonometry questions in TR whereas any angle was used in the UK.

- More physical tools were effectively used by the UK teachers in the trigonometry classes such as calculators, over head projectors, computers and formulae sheets which are not used in the TR. Calculators especially were almost always used in every activity in the UK whereas paper and pencil methods were used in TR for all sorts of manipulations. Besides the differences in the curriculum, this appears to partially explain at least two noted phenomena: little emphasis on secants, cosecants and cotangents in the UK; an emphasis on special angles, e.g. 30^0, 45^0, 60^0 and 90^0, and surd forms, e.g. $\sin 60^0 = \frac{\sqrt{3}}{2}$ in the TR.

- Use of conceptual tools; because of the calculator more algebraic tools (see p. 179) and because of the curriculum differences more trigonometric tools (see p. 180) are used in trigonometry in TR than the UK. However, the mnemonic way, 'SOHCAHTOA', is the sole relational tool which is used in the UK but not in TR. However the TR students used 30^0, 60^0 ,90^0 (45^0, 45^0 ,90^0) right-angled triangle to find the ratio of the trigonometric functions. Furthermore, because of the place of the TWP in the UK, more iconic tools (see p. 181) are used in the UK comparing with the TR.

- The TR teachers teach more abstract aspect of trigonometry with many identities. On the other hand the UK teachers teach more application, real world and numerical aspects of trigonometry.

- In teaching trigonometry, both the UK and the TR teachers keep strictly to the curriculum, they usually use more than one textbook, they also usually use some written resources, e.g. past examination papers and question banks, and they also prepare their own handouts. All of these facts affect their teaching approach, teaching tools and their thoughts and beliefs about trigonometry. Teachers had different views on trigonometry (see p. 139). The first appearance of trigonometry was working with numbers for the UK teachers whilst it was working with basic trigonometric functions on a right-angled triangle and memorising the basic trigonometric functions for the TR teachers. The UK teachers see the algebra factor as a difference between the first and second appearance (see pp. 147-148) of trigonometry whereas the TR teachers did not mention about algebra. The TR teachers stated that the second appearance of trigonometry is more abstract.

- The UK and the TR teachers' motives, which are curriculum, textbooks and examinations in teaching trigonometry, also illustrate the two types of trigonometry in the two countries (see p. 174).

- The UK students performed better in trigonometry word problems than the TR students did. However, the TR students performed better than the UK students in both trigonometry and algebra tests. Their performances were almost the same in the trigonometric functions on the right-angled triangles test.

- Interviews and the curricula resources showed that procedures in answering trigonometry questions, particularly in the first appearance of trigonometry, are based on numerical calculations in the UK whereas it is mostly done by paper and pencil method through algebraic manipulations in the TR. These procedures were seen in the students' manipulations in the trigonometry word problems test and the trigonometric functions on right-angled triangles test. In these tests, furthermore, the UK students' symbolic manipulations went to numerical answer whereas the TR students mostly ended with algebraic or numerical expressions as they had rarely seen numerical answers as a result of the procedures.

Interestingly, despite the students' performance and all other dissimilarities between trigonometry in the UK and the TR, a uniformity of approach has been exhibited in both the UK and the TR students' verbal protocol to simplifying trigonometric expressions and answering trigonometric word problems. The model of answering trigonometry word problems was also helpful to answer the research question RQ-1-ii-a (see pp. 1-2).

1.2. A model of 'simplifying' trigonometric expression

Even though there was no definition of how to 'simplify' a trigonometric expression and to find the 'simplified form' in both the UK and the TR official resources, e.g. curriculum, textbook, the UK and the TR teachers gave some factors necessary to 'simplify' trigonometric expressions; being competent and confident in algebraic manipulation, being familiar with the trigonometric identities and learning trigonometric identities by doing lots of lots of practice and memorising. The UK teachers consistently underline that the important thing for students in simplification is recognising the trigonometric identities on the paper. The TR teachers give more superficial and common reasons such as algebraic and trigonometric properties, which should be known by students. But both the UK and the TR teachers are not clear about the 'simplification' of trigonometric expression neither are their students (see pp. 147-148 and 81-82). In the UK usually the final form of the expression is given in simplification questions whereas the final form is not given in the TR. Subsequently how the two counties' students performed revealed a model showing their manner of simplifying trigonometric expressions (see Figure 5.3., p. 189). The model discussed in detail in the discussion chapter reveals some important points which are summarised below.

Important points relevant to the model of 'simplifying' trigonometric functions are:

- Its stages show the obstacles students met.
- It highlights the importance of the algebraic and trigonometric properties as have been emphasised by teachers and students of both countries.
- It emphasises the importance of the memory in recognising and recalling stages which seems to correspond with teachers and students saying, "being familiar with", as well having the knowledge.
- It shows the existence of mental manipulations as well as manipulations on the paper which are very important in terms of the individual cognitive actions.
- It highlights the characteristic of the "rewritten forms" in the simplification of a trigonometric expression: rewritten forms are the product of the simplification process, rewritten forms are produced as a result of dialectic between identification and doing groups of the model and rewritten forms are the compressed form of the expressions (see p. 191).
- The model suggests definitions for simplification, simplified form and the most simplified form of the expression (see p. 192).

1.3. Answering model of trigonometry word problem

Despite the fact that trigonometry word problems have an important place highlighted in the UK and are almost ignored in the TR, students from both countries performed a uniformity of approach to answering these questions (see p. 201). The model discussed in detail in the discussion chapter reveals some important points which are briefly presented here:

- There were two important phases in answering a TWP, constructing a diagram and doing mathematics, which occur in this order and there is almost always no correct answer in the absence of one of them.
- In answering a TWP, constructing a diagram[12] and doing the mathematics seemed to be complimentary and there seemed to be one way from 'constructing the diagram' phase of the model to 'doing mathematics' phase of the model that is, doing the mathematics is always created from the constructed diagram. Mistakes and obstacles students have throughout the 'doing mathematics' phase did not affect the constructed diagram.

[12] All TR and UK students explicitely drew diagrams on the paper with two exceptions from UK, one of whom answered one trigonometry word problem without drawing any diagram and the other one did so for two trigonometry word problems.

- Diagrams acted as an external stimulus to help students construct the mathematical part.
- Students used their diagrams as organisational ways to answer trigonometry word problems.
- All stages except 'recognise', 'visualise' and 'result' in the model show the flaws students committed, revealing the difficulties students have in answering a TWP.
- The model highlights the important place of the diagrams in answering the TWP.
- In order to draw diagrams for answering TWP, both the UK and the TR students pass through the following three stages; read the TWP, recognise the TWP and visualisation. In the stage of visualisation, students have their mental representation of the situation. Then, the students manipulate their mental representations by using their spatial abilities and then they draw the diagram onto paper. The students' mental representations and the diagrams on the paper can interact until they completely construct the diagram. The model, therefore, highlights a very important factor in answering a TWP, which cannot be seen on the paper directly, which is visualisation.
- All correct answers had correct diagrams and all incorrect diagrams led to incorrect answers.
- A review stage rarely happened in the model.
- Stages of doing mathematics are akin to stages of answering the question of trigonometric functions on right-angled triangles.

1.4. Context-context free

The questions in trigonometry word problems test and trigonometric functions on right-angled triangle test are the same questions in context and context-free form respectively. The UK and the TR students' performances are very high with the context-free one in which diagrams with the values on are given, compared with the contextual one which contains written problems without any diagram. It seems that the success in the context-free test did not affect their performance in contextual test. However, it did in the second phase of answering contextual problems, which are trigonometry word problems, which is very similar to the students' answering model of the TORT questions. Students construct diagrams for the contextual questions and then treat it as a context-free question. As the results show (see p. 101), drawing the diagram and using terminology are the most common difficulties students have in answering a TWP. An interesting finding is that any trigonometry word problem becomes a question in trigonometric functions on right-angled triangles test after the constructing diagram part. After the diagram is constructed, the doing part seems easy for both countries' students. It becomes a matter of finding the unknown on the diagram and that means it is a question of trigonometric functions on right-angled triangle test, in which both countries' students performed well. Context plays an important role in answering trigonometry word problems, it was the reason why students performed poorly compared with the context-free form of the questions in the test. Students need to understand the words in a trigonometry word problem, which could be from real life and trigonometry terminology. Then they need to draw a diagram by visualising. Then they need to label the diagram using the givens in the problem. The students have difficulties and obstacles in the contextual problem until they completely draw the diagram. After 'constructing diagram' phase the trigonometry word problem becomes a question in trigonometric functions on right-angled triangles test which seems the easy part for students.

1.5. Algebra and trigonometry

Both the UK and the TR students were better at simplifying algebraic expressions than trigonometric expressions. The UK and particularly the TR students were more comfortable with simplification of algebraic expressions. However, in many answers students did not treat trigonometric expressions as algebraic expressions. They did not, in general, convert trigonometric expressions into algebraic expressions. Both the UK and the TR teachers and students are aware that, after some steps some trigonometric identities are needed in 'simplifying' trigonometric expressions. It is thus not automatic that, being successful in and having a good knowledge of algebra equates to success in trigonometry. Students (especially the TR students) could compare the difference between algebra and trigonometry and said algebra is very easy. They were used to working with the letters. However, they cannot work with algebra in trigonometry because they should know trigonometric identities and formulae as well. Students generally know where to stop in algebra but they often do not know where and when to stop in trigonometry, so simplification is more clear in algebraic contexts than trigonometric contexts.

So, consequently what is meant by 'trigonometry' in the UK and the TR varies greatly. Student performance is, not surprisingly, strongly related to what curricula emphasise: 'context' word problems in the UK and 'algebra' in the TR. So, in brief, the main finding is that you get what you teach.

2. Educational implications

In this section educational implications will be presented in terms of the curriculum and teaching and learning.

2.1. Curriculum

The quantitative and qualitative data showed that the weightings given to different aspects of trigonometry in the UK and TR are different, so that the TR students seemed better at algebraic aspects and the UK students seemed better at application aspects of trigonometry. The curriculum of the UK and TR had many dissimilarities. For example, there are more identities in TR than the UK and important differences in the place of cotangent, secant and cosecant functions in the TR curriculum. Although half of the objectives are relevant to application they are algebraic applications, not the real world applications which have got a very important place in the UK curriculum. So, different ways of presenting trigonometry produces different teaching and learning. That implies that if only one aspect of trigonometry is taken into account then students will be successful in the aspect that is emphasised. Therefore, curriculum designers should find a way to balance the topics in the curriculum to make students successful. If there is no balance between the topics then students will find some questions much easier then the ones they meet less and do less practice on.

Textbooks have an important place in the teaching and learning of trigonometry. Textbooks were 'in line' with the curriculum and teachers were very careful to provide supplementary documents if the textbooks were not in line with the curriculum. Both countries' teachers follow textbooks because they reflect the curricula objectives. However, both the UK and the TR teachers do not strictly follow textbooks: they provide their own notes, other textbooks and written resources, because they believe textbooks do not cover some aspects well. Although textbooks were supposed to follow the curriculum, there were some objectives they did not include and there were some topics, which were not objectives in the curriculum, they did include. An interesting finding was that the lesson structure of teachers from both countries were very similar to the textbook structures respectively. Subsequently, in terms of content and lesson plan, textbooks have a place in the teaching and learning of trigonometry. But it seemed that teachers found them unsatisfactory. Another aspect of the textbooks was the examples/questions they contain. The question styles in the UK and the TR textbooks were different. Questions in the UK textbooks were based on numerical and real world application problems and use of calculators was stressed whilst in the TR textbook questions were algebraic, even the applications were not real world but algebraic. Consequently, textbooks are, to some extent, used in the teaching of trigonometry so authors of the textbooks should take the opinions of the teachers into account and should make modifications for the next edition.

Teachers are the important part of the teaching system; they are involved with curriculum, syllabii, examination boards, textbooks. They teach children, they know the advantages or disadvantages of the instructional tools and the most important thing is that they know the students, the difficulties they meet and they are well positioned to observe the teaching system, so if the teachers could be a part of the community working on all resources, such as curriculum, textbooks etc., productive and effective resources, plans and instructional tools could be prepared.

2.2. Teaching and learning

Although the UK students worked with many questions through homework and worksheets (as did the TR students), they were not as successful as the TR students. However it is observed that the TR students worked with a variety of the questions whereas the UK students worked with the questions almost always close to each other in form and, in the textbooks, examples usually follow on from a worked example nearly identical to questions in the exercise (Haggarty and Pepin, 2002). The reasons behind the TR students' success thus may be the variety of the questions they work with and this variety is increased by the use of trigonometric identities which do not appear to be taught in the UK. The TR teachers use private institutes' question banks or textbooks which provide challenging and varied questions (for practice especially), therefore providing many and varied questions that may help students' performance. Furthermore, working with more trigonometric identities also provides a rich store for questions so maybe curriculum designers and examination boards should modify the curriculum in that respect.

In the simplification of trigonometric expressions, although the UK and the TR teachers highlighted the place of the algebra they should be more aware of it. The important place of algebra was observed in students' manipulations in the trigonometry tests, interviews and verbal protocols that in almost every answer students used algebraic properties, which influenced their performances. Teachers may prepare questions in which the difference between the use of algebra and trigonometry properties can be illustrated and highlighted. Algebra was not the only obstacle in the simplification of trigonometric expressions. The most important obstacles teachers should be more aware of and take into account were definition of simplification, simplified form and the most simplified form which might be defined with the help of the model.

Simplification appears to be a tool whose use is determined in practice – there is no definition of 'simplification'. Students' 'apprenticeship' into simplification practices clearly varies in the two countries. The UK students, more than the TR students, appear not to know where to stop or where to go with many expressions. The final form in simplification questions is given in the UK but not in the TR. The meaning of 'simplification' with regard to trigonometric expressions does not appear to be as clear to students as it does with regard to algebraic expressions. So simplification, simplified form and the most simplified form in trigonometry should be clearly defined.

Teachers should be aware of the importance of the constructing diagrams in the TWP. As well as the past and background knowledge in trigonometry, students need real world experience and knowledge to visualise and understand the situation in the TWP to construct diagram. So the TWP will be a practice to show how trigonometry is used in real life rather than a pure use of mathematical knowledge. Students may have enough trigonometry knowledge but they may not have an experience of the situation in the TWP which will help them to construct the diagram to apply mathematical knowledge. Therefore it seems that contextual problems in trigonometry are not easy as practice questions to teach trigonometric functions on right-angled triangles, because students first should construct diagrams (visually or/and on the paper) depending on some facts that are more than mathematical knowledge, e.g real life knowledge, visualisation and drawing abilities, then they apply their trigonometry knowledge to do the mathematical part of the answering model of the TWP. In other words, as well as trigonometry students use real life knowledge and some abilities such as spatial ability. Moreover, they have to use both geometrical and algebraic background knowledge.

The UK and the TR teachers appeared to enjoy teaching trigonometry but students often complained about it in recorded and unrecorded conversations. Teachers confessed that trigonometry was a difficult discipline in mathematics for students. As students said they used to work with numbers and variables like *a* and *b* but sine and cosine 'are different'. There are many formulae or identities to learn. In the UK, a formulae sheet is used for trigonometric identities although the majority of UK teachers say students do not need to memorise them some of them say they should know the basic ones. On the other hand, the TR teachers made strong objections to memorisation of the formulae. They want their students to derive identities from each other.

Teachers said students had difficulties with the TWP, although picturing the situation in the TWP can lead students to answering the TWP, students had difficulties in drawing diagrams in the TWP (see pp. 132-133). It seemed that visualisation (see p. 103) is an important factor for constructing diagrams so teachers should aim to enrich students' abilities in visualisation and constructing diagrams, i.e. by giving them more practice. Although all teachers give a way of answering trigonometry questions they want students to follow on their examination paper they did not give a precise model for it (especially for 'simplifying' trigonometric expressions). However their ways were very close to each other and coincided with the models that emerged in section 3.1. Accordingly teachers' way of answering the TT and the TWP were impossible to tell apart from the answering models in teaching trigonometry. So teachers implicitly use the model in answering the TWP and 'simplifying' trigonometric expressions.

Teachers should be aware that constructing diagrams is very important in answering the TWP, because diagrams play three complementary roles (see p. 212):

- Providing greater clarity and detail.
- Overcoming limitations of working memory.
- Helping to understand problem.

The important factors in constructing diagrams are not only mathematical knowledge but also real life experience and knowledge. These two may help each other in that incorrect use of terminology was corrected by correct knowledge of life (see p. 103). Other factors are; spatial abilities for visualisation and having a mental representation, manipulating it and then drawing it onto paper, the number of the unknowns and expected right-angled triangles and 2-D/3-D situation. Teachers should teach students these factors, particularly that the 2-D representation of 3-D problems/figures was an obstacle for most of the students (see p. 104), teachers can do more practice or do some activities to develop students' visualisation and spatial abilities. So, in addition to trigonometry knowledge the teacher should teach students to visualise, enrich their spatial ability and construct diagrams by using their real life experience.

The study highlighted an important point about the place of the trigonometry word problem in TR. Although the TWP is not emphasised in the curriculum and almost ignored in trigonometry lessons, the TR teachers complained about the system. They were happy to work with real world problem but they said the system pushes students to memorise everything just to 'do' the mathematics. In unrecorded conversations, a majority of the TR students came to me throughout my data collection time in their school and said they were very happy to work with the TWP and they did not understand why they did not usually work with these sort of questions. So both the TR teachers and students seemed to want to work with the TWP. That implies that the teaching of real world application of the trigonometry, namely the TWP, should be taken into account in trigonometry contexts.

3. Suggestion for Further Research

In this section suggestions for the further research are presented in terms of curriculum and teaching and learning.

3.1. Curriculum

In both 'simplifying' a trigonometric expression and answering a TWP the use of memory had an important place. In addition to the algebraic properties there were many trigonometric identities and formulae to recall throughout the simplification. The UK students used a formulae sheet but the TR students used their memory to recall these identities and formulae. Interestingly, the TR teachers and students made objection to the word memorisation, they always mentioned about logical learning, learning by solving many questions. So what this logical learning is, to what extent do they use their memory in simplifying trigonometric expressions, how can trigonometric identities appear in the mind and do they have any image? In what form are identities/formulae stored in the memory? Comparing it with the UK students who used formulae sheet would be very interesting. Likewise, the use of memory in answering the TWP seemed to have a more interesting place. Both the UK and the TR students used diagrams to help their working memory and some used mnemonic ways to remember the definition of basic trigonometric functions. Consequently, memory has an important role in trigonometry, to see how it is used and to what extent it is used would be helpful in teaching and learning approaches so that the curriculum can be redesigned, teachers modify their teaching approaches and textbooks could introduce new techniques.

In the jigsaw curriculum task (see pp. 174-175), when the UK and the TR teachers were asked to order trigonometry topics in the order that they would teach them, none of the UK and the TR teachers' rank of topics of teaching trigonometry coincided with each other, they were all different even though they have been teaching trigonometry for a long time. Overall, although both the UK and the TR teachers gave different order, the order of the topics which are my focus in this study were usually: trigonometric functions, trigonometry word problems and trigonometric identities. It seems trigonometry needs to be delving into a wider work starting from curriculum to textbook to teaching approaches to reveal whether there should be an order of teaching trigonometry topics or not.

One of the teachers (see pp. 137) from interviews highlighted an important point that the curriculum tells them what to teach but not how to teach. This is an important issue, which touches upon the area of teaching and learning styles. The question, however, of whether or not teachers should be directed to teach in specific ways is beyond the scope of my research.

Delving into the implemented curriculum of the UK and the TR revealed differences and similarities between the trigonometry in two countries. Even though some data were collected for implemented and attained curricula of both countries, the data were not strong enough to say more general things about the two curricula so another study could be conducted, focusing on the intended, implemented and attained curriculum.

3.2. Teaching and learning

In the study, two operational models of 'simplifying' trigonometric expressions and answering trigonometry word problems, emerged (see pp. 189 and 201). Their importance and implications have been presented. These models could be utilised in many perspectives in teaching and learning trigonometry such as task-based heuristics (see p. 192) or as a framework. Furthermore, by using these models, obstacles and errors students commit can be detected and so the models can be used to help students overcome their flaws and increase their performance by using these models. So these models should be explored more deeply for validity and reliability.

One of the interesting points that appeared in both of the models is that the 'review' stage did not really occur in this study as it did in other studies in the literature (see p. 201). In the model of simplifying trigonometric expressions, when students were stuck or their answer got longer they did not check what they did, or whether they used appropriate algebraic/trigonometric properties or not. Furthermore, when the answer seemed to have no problem students did not check their answer. Likewise, in answering TWP, students did not check either the diagram or mathematical part of their answers. To develop a 'Review' stage in the model, e.g. making that stage an automatic action, might be effective on the students' performances.

In the simplification of trigonometric expressions the main feature of the identification phase is using algebraic or trigonometric properties. There are many identities in trigonometry that can be connected to each other in a way that double angle identity or half angle identity can be derived from the addition identities, by using Pythagorean identity other two equivalent forms of cos2x can be found and that web, which includes all relevant information, can be extended. Barnard and Tall (2001) said rich, compact cognitive units allow the thinker to manipulate these ideas in efficient, insightful ways, whereas students with diffuse structures will not find it so easy to make connections between concepts that are themselves diffuse and vague. The reason behind the TR and the UK students' performances could be having rich, cognitive units or not. This could be researched and if students do have rich, cognitive unit, then what could be the reason affecting their performance? If they do not have them, could this be rectified? Furthermore the rewritten forms in the model seems to correspond to the procepts of Gray and Tall (1991). The notion of procept becoming richer (in interiority, to use Skemp's terminology) as different symbols and processes represented the same object, for instance, $\frac{1+\cos\theta}{\sin\theta}$ as $\frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$ or $\cot\frac{\theta}{2}$. In their terminology, a procept is a special case of a cognitive unit that grows with interiority as the cognitive structure of the individual gets more sophisticated. Rewritten forms and procept connections could be explored more deeply.

In the model of simplifying trigonometric expressions, it is seen that students' decisions throughout the simplification process are very important. They should be taught how to use appropriate algebraic or trigonometric properties and also they should be taught how to choose the appropriate trigonometric identities to obtain an answer in a short and simple way, otherwise their answer can get confused or can get longer or they can get stuck. So metacognition seems to have very important role in the simplification of trigonometric expressions. In my model metacognition has not been explored. Further research needs to be done to find out to what extent the effects of metacognitive knowledge, experience and skill in simplification process would make the proposed model stronger. To what extent does metacognition have an influence on simplifying trigonometric expressions?

References

Arcavi, A (1996). Symbol Sense: Informal Sense-Making in Formal Mathematics. *For the Learning of Mathematics*, 14(3): 24-35.

Arnold, M. (1960). On the Modern Element in the Literature. In R. H. Super (Ed.) *The Complete Prose Works of Matthew Arnold* in eleven volumes, v 1, p. 21. Ann Arbor: University of Michigan Press 1960.

Assessment of Performance Unit (APU) (1988). *Mathematical Development; A Review of Monitoring in Mathematics.* 1978 - 1982, Parts 1 and 2.

Aydin, N. and Asma, N. (2000). *Mathematics-3 for High Schools*. Aydin Yayinlari: Ankara, Turkey.

Backhouse, J., Haggarty, L., Pirie, S. and Stratton, J. (1992). *Improving the Learning of Mathematics*. London: Cassell.

Barnard, A. D. and Tall, D. O. (2001). A Comparative Study of Cognitive Units in Mathematical Thinking, *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, 2: 89–96. Utrecht, The Netherlands.

Barnard, T. & Tall, D.O. (1997). Cognitive Units, Connections, and Mathematical Proof. In E. Pehkonen, (Ed.), *Proceedings of the 21st Annual Conference for the Psychology of Mathematics Education*, 2: 41-48. Lahti, Finland.

Barraclough M. (1990). *Trigonometry from the Unit Circle.* 13th Biennial Conference AAMT: 112-115.

Behrens, J. and Smith, M. (1996). Data and Data Analysis. In D. C. Berliner, and R. C. Calfee, (Eds.) *Handbook of Educational Psychology* (p. 945-989). New York: MacMillan Library Reference.

Bell, J. (1987). *Doing Your Research Project: A Guide for First Time Researchers in Education and Social Science.* Open University Press, Milton Keynes.

Bishop, A. J. (1989). Review of Research on Visualisation in Mathematics Education. *Focus on Learning Problems in Mathematics*, 11: 257-269.

Blackett, N. and Tall, D. (1991). Gender and the Versatile Learning of Trigonometry Using Computer Software. *Proceedings of the 15th Conference of the International Group for the Psychology of Mathematics Education*, 1: 144–151. Assisi, Italy.

Blaikie, N. (1993). *Approaches to Social Enquiry*, Polity Press, UK.

Blum, W. and Niss, M. (1991). Applied Mathematical Problem Solving, Modelling, Applications, and Links to Other Subjects-State, Trends and Issues in Mathematics Instruction. *Educational Studies in Mathematics,* 22: 37-68.

Boaler, J. (1993). The Role of Context in the Mathematics Classroom: Do They Mathematics More Well? *For the Learning of Mathematics*, 13(2): 12-17.

Bogdan, R. and Taylor S. (1984). *Introduction to Qualitative Research Method*, J. Wiley, USA.

Booth, L. R. (1988). Children's Difficulties in Beginning Algebra. In *NCTM 1988 yearbook,* The ideas of Algebra, K-12 (p. 20-33). Reston, VA: NCTM.

Bostock, L. and Chandler, S. (1995). *Modular Mathematics, Pure Maths 1, Module A* (2nd edition). Stanley Thornes Publishers: Cheltenham, UK.

Branch, J. L. (2000). The Trouble with Think Alouds: Generating Data Using Concurrent Verbal Protocols. *CAIS 2000: Dimensions of a Global Information Science.* Canadian Association for Information Science Proceedings of the 28th Annual Conference.

Brissenden, T. H. F. (1980). *Mathematics Teaching: Theory in Practice.* London: Harper & Row.

Brophy, J. and Good, T. L. (1986). Teacher Behaviour and Student Achievement. In M. Wittrock (Ed.) *Handbook of Research Teaching*, 3rd edition, New York: Macmillan.

Brousseau, G., Davis, R. B. and Werner, T. (1986). Observing students at work. In B. Christiansen, A. G. Howson and M. Otte (Eds.) Perspectives on Mathematics Education, 205-241. Dordrecht: D. Reidel Publishing Company.

Bruhn, J. (1995). Mathematics Education and Comparative Studies: Two Examples. In W. Bos, and H. Lehmann (Eds.) *Reflections on Educational Achievement* (69-74). Germany: Waxmann.

Bruner ,J. S. (1966). *Towards a Theory of Instruction.* New York: Norton.

Bryman, A and Cramer, D. (1997). *Quantitative Data Analysis with SPSS for Windows: A Guide for Social Scientists.* London: Routledge.

Bryman, A. (1992). *Quantity an Quality in Social Research.* London: Routledge.

Burke, K. (1969a) A grammar of motives. Berkeley: University of California Press.

Burke, K. (1969b) A rhetoric of motives. Berkeley: University of California Press.

Butler, C. H. (1965). *The Teaching of Secondary Mathematics.* London: McGraw-Hill Book Company.

Campbell K. J., Collis, K. F. and Watson, J. M. (1995). Visual Processing During Mathematical Problem Solving. *Educational Studies in Mathematics,* 28: 177-194.

Carpenter, T. P. (1985). How Children Solve Simple Word Problems. *Education and Urban society*, 17: 417-425.

Chazan, D. (1996). Algebra for All Students? *Journal of Mathematical Behaviour,* 15: 455-477.

Chinnappan, M. (1998). Decomposing Trigonometry Problems for Schema Acquisition. *Journal of Applied Research in Education*, 2(1): 85-98.

Chinnappan, M. and Lawson, M. J. (1996). The Effects of Training in the Use of Executive Strategies in Geometry Problem Solving. *Learning and Instruction*, 6(1): 1-17.

Christiansen, B., Howson, A. G. and Otte, M. (1986). *Perspectives on Mathematics Education.* Papers submitted by members of the Bacomet group, pp 205-241. Dordrecht: Reidel.

Churchill, G. A. (1987). *Marketing Research: Methodological Foundation* (4th Ed.). London: Dryden Press.

Cockcroft, W. H. (Ed.) (1982). *Mathematics Counts.* Report of the Committee of Inquiry into the Teaching of Mathematics in Schools, London: Her Majesty's Stationery Office.

Coffey, A. and Atkinson, P. (1996). *Making Sense of Qualitative Data: Complementary Research Strategise.* London: SAGE Publication.

Cohen, L. and Manion, L. (1994). *Research Methods in Education*, (4th edition). London: Routledge.

Cohen, L., Manion, L. and Morrison, K. (2000). *Research Methods in Education*, (5th edition). London: Routledge.

Collier, C. C. and Lerch, H. H. (1969). *Teaching Mathematics in the Elementary School.* Canada: Collier-Macmillan.

Cooper, B. and Dunne, M. (2000). *Assessing Children's Mathematical Knowledge: Social Class, Sex and Problem Solving.* Open University Press.

Cornelius, M. (1985). From Jeffery Syllabus to Cockcroft Report. Part 1. *Mathematics in School*, 14(4): 31-34.

Cox, R. (1999). Representation Construction, Externalised Cognition and Individual Differences. *Learning and Instruction*, 9: 343-363.

Creswell, J. W. (1994). *Research design: Qualitative and quantitative approaches.* Thousand Oaks, CA: Sage Publications.

Crutcher, R. J. (1994). Telling What We Know: The Use of Verbal Report Methodologies in Psychological Research. *Psychological Science*, 5(5): 241-244.

Davis, R. B. (1984). *Learning Mathematics: The Cognitive Science Approach to Mathematics Education*, Norwood NJ: Ablex.

Davis, R. B. (1992). Understanding "Understanding". *Journal of Mathematical Behaviour*, 11: 225-241.

Denscombe, M. (1998). *The Good Research Guide for Small-Scale Social Research Projects.* Buckhingam: Open University Press.

Denzin, N. K. & Lincoln, Y. S. (1994). *Handbook of Qualitative Research.* Thousand Oaks, CA: Sage.

Denzin, N. K. and Lincoln, Y. S. (1994). Entering the Field of Qualitative Research. In N. K. Denzin and Y. S. Lincoln (Eds.), *Handbook of Qualitative Research*, (pp.1-16). Thousand Oaks, CA: Sage.

Doishita, A., Shimizu, H., Ucoka, T. and Ichizaki, M. (1986). Teaching Strategies in the Problem Solving: Conducting a Survey of the Pupils on Pictures and Figures and Putting It into Practice. *Journal of Japan Society of Mathematics Education*, 68 (4): 76-80.

Douady, R. (1991). Tool, Object, Setting, Window: Elements for Analyzing and Constructing Didactical Situations in Mathematics. In A. J. Bishop, S. Mellin-Olsen and J. van Dormolen (Eds.) *Mathematical Knowledge: Its Growth Through Teaching*, pp 109-130. London: Kluwer Academic Publishers.

Douady, R. (1997). Didactic Engineering. In T. Nunes, P. Bryant (Eds.) *Learning and Teaching Mathematics: An International Perspective* (pp 373-401). Hove: Psychology Press.

Douglis, A. (1970). *Ideas in Mathematics.* London: W. B. Saunders Company.

Drever, E. (1995). *Using Semi-Structured Interviews in Small-Scale Research.* Edinburgh: The Scottish Council for Research in Education

Dreyfus, T (1991). On the Status of Visual Reasoning in Mathematics and Mathematics Education. In: F Furinghetti (Ed), *Proceedings of the 15th Annual Conference of the International Group for the Psychology of Mathematics Education*. Assissi. Vol 1: 33-48.

Dreyfus, T., Hershkowitz R. and Schwarz B. (2001). The construction of Abstract Knowledge in Interaction. In Marja van den Heuvel-Panhuizen (Editor). *Proceedings of the 25th Conference of the International Group for the Psychology of Mathematics Education*, 2, 377-384. Utrecht, The Netherlands.

Driver, R. and Erickson, G. (1983). Theories in Action: Some Theoretical and Empirical Issues in The Study of Students' Conceptual Frameworks in Science. *Studies in Science Education*, 10: 37-60.

Dyer, C. (1995). *Beginning Research in Psychology: A Practical Guide to Research Methods and Statistics.* Blackwell, Oxford.

Eisenberg, T. and Dreyfus, T. (1991). On the Reluctance to Visualise in Mathematics. In W. Zimmermann and S. Cunningham (Eds.), *Visualisation in Teaching and Learning Mathematics*. (pp.25-38). MAA Notes no.19.

Ekmekci, S., Ayhan, K., Yildirim, H. and Yildirim, U. (1996). *Mathematics 8 for Secondary Schools.* Turkey: Yildirim Yayinlari.

Ellerton, N. and Clements, M. A. (1992). Some Pluses and Minuses of Radical Constructivism in Mathematics Education. *Mathematics Education Research Journal,* 4(2): 1-22.

Ellery, P. (1980). A Problem Solving Approach to Trigonometry. *Sigma*, 106: 15-19.

Ellis, J. (1990). *Toward Better Trigonometry Teaching. Whither Mathematics?* 27th Annual Conference Papers, pp. 452-454.

Ely, M., Anzul, M., Friedman, T., Garner, D. and Steinmetz, A. (1991). *Doing Qualitative Research: Circles within Circles*. London: The Falmer Press.

Ely, M., Vinz, R., Downing, M. and Anzul, M. (1997). *On Writing Qualitative Research: Living by Words*. London: The Falmer Press.

Emory, C. W. (1980). *Business Research Methods.* Hoemwood, IL: Richard D. Irwin.

Ericsson, K. A. and Simon, H. A. (1993). *Protocol Analysis: Verbal Reports as Data.* Oxford: Oxford University Press.

Ernest, P (1994). *An Introduction to Research Methodology and Paradigms*. Educational Research Monograph Series; 1 Exeter: School of Education, Research Support Unit, UK.

Ernest, P. (1994). *Educational Research, Its Philosophy and Purpose: An Introduction to Research Methodology and Paradigms*. Educational Research Monograph Series Exeter, The University, England.

Eysenck, M. W. and Keane, M. T. (1995). *Cognitive Psychology: A Student's Handbook* (3rd edition). Hove: Erlbaum, Psychology Press.

Fife-Schaw, (1995). Questionnaire Design. In Breakwell, G. M., Hammond S. and Fife-Schaw C. (Eds.) *Research Methods in Psychology*. London: Sage.

Fisher-Hoch, H. and Hughes, S. (1996). *What Makes Mathematics Exam Questions Difficult?* British Educational Research Association Annual Conference, University of Lancaster.

Gadanidis, G. (1988). Problem Solving: the Third Dimension in Mathematics Teaching. *The Mathematics Teacher*, 81: 16-20.

Gallimore, R. (1996). Classrooms Are Just Another Cultural Activity. In Speece, D. L. and Keogh, B. K. (eds.) *Research on classroom Ecologies: Implications of Inclusion of Children with Learning Disabilities*. Hillsdale, NJ: Lawrence Erlbaum.

Garfola, J. & Lester, F. K. (1985). Metacognition, Cognitive Monitoring, and Mathematical Performance. *Journal for Research in Mathematics Education*, 16: 163-176.

Gay, L. R., and Airasian, P. (1999). *Educational Research: Competencies for Analysis and Application*, (6th Edition), Upper Saddle River, NJ: Merrill.

Gold, R. L. (1958). Roles in Sociological Field Observation. *Social Forces*, 36: 217-223.

Gorgorio, N. (1998). Exploring the Functionality of Visual and Non-Visual Strategies in Solving Rotation Problems. *Educational Studies in Mathematics*, 35: 207-231.

Graf K. D., and Leung F. K. S. (2000). The Thirteenth ICMI Study on Mathematics Education in Different Cultural Traditions. *Educational Studies in Mathematics*, 43(1): 95-116.

Gray, E. and Tall, D. (1991). Duality, Ambiguity And Flexibility in Successful Mathematical Thinking. In Fluvia Furinghetti (Ed.), *Proceedings of the Fifteenth International Conference for the Psychology of Mathematics Education* (PME) (Vol. II, pp. 72-79). Assisi, Italy.

Green, A. (1998). *Verbal Protocol Analysis in Language Testing Research: A Handbook.* Cambridge: Cambridge University Press.

Green, C. and Gilhooly, K. (1996). Protocol Analysis: Practical Implementation. In J. T. E. Richardson (Ed.). *Handbook of Qualitative Research Methods for Psychology and the Social Sciences.* Leicester: British Psychological Society.

Greeno, J. G. (1973). The Structure of Memory and the Process of Solving Problems. In R. L. Solso (Ed.) *Contemporary Issues in Cognitive Psychology: The Loyola Symposium* (pp 103-134). Washington, D. C.: V. H. Winston and Sons.

Greer, B. (1997). Modelling Reality in Mathematics Classrooms: The Case of Word Problems. *Learning and Instruction*, 7(4): 293-307.

Grouws, D. A. (ed. (1992) *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan Publishing Company,.

Guay, R. and E. McDaniel. (1977). The Relationship between Mathematics Achievement and Spatial Abilities among Elementary School Children. *Journal for Research in Mathematics Education* 8(3): 211-215.

Guba E. G. and Lincoln Y. S. (1994). Competing Paradigms in Qualitative Research. In N. Denzin and Y. Lincoln (Eds.) *Handbook of Qualitative Research*, Sage Publications.

Guttierez, A. (1996). Visualization In 3-Dimensional Geometry: In Search Of A Framework. In L. Puig and A. Gutierrez (Eds.) *Proceedings of the XX Conference of the International Group for the Psychology of Mathematics Education* (PME), 2: 3-19.

Haggarty, L. and Pepin, B. (2002). An Investigation of Mathematics Textbooks and Their Use in English, French and German Classrooms: Who Gets an Opportunity to Learn What? *British Educational Research Journal*, 28(4): 567-590.

Hall, R. (2002). An Analysis of Errors Made in the Solution of Simple Linear Equations. *Philosophy of Mathematics Education Journal*, No 15.

Hammersley, M. and Atkinson, P. (1983) Ethnography: Principles on Practice. London: Methuen.

Harries, T. and Sutherland, R. (2000). *The Representation of Mathematical Concepts in Primary Mathematics Textbooks: A Focus on Multiplication*. Paper is available on the net: http://math.unipa.it/~grim/Jharries (Last checked out on 25th February 2003).

Harris, S. (1997). The Third International Mathematics and Science Study (TIMSS) Summary of National Results. *British Educational Research Association Annual Conference*. University of York, UK.

Her Majesty's Inspectorate (HMI): (1982). *Mathematics in the Sixth Form*, DES, London.

Hershkowitz, R., Parzysz, B. and van Dormolen, J. (1996). Space and Shape. In A. J. Bishop, K. Clements, J. Kilpatrick, and C. Laborde (Eds.) *International Handbook of Mathematics Education*, Part 1, pp 161-204. Dordrecht; London: Kluwer Academic.

Hiebert, J., and Lefevre, P. (1986). Conceptual and Procedural Knowledge in Mathematics: An Introductory Analysis. In J. Hiebert (Ed.) *Conceptual and Procedural Knowledge: The Case of Mathematics* (pp. 1-27). Hillsdale: Erlbaum.

Hoepfl, M., (1997). Choosing Qualitative Research: A Primer for Technology Education Researchers. *Journal of Technology Education*,9(1)

Howson, G. (1991). *National Curricula in Mathematics*. Leicester: Mathematical Association.

http://cela.albany.edu/publication/article/context.htm

Hughes, J. (1990). *The Philosophy of Social Research*. Longman.

Hughes, S., Fisher-Hoch, H., and Bramley, T. (1997). *What Is Involved in Answering a GCSE Math Question?* BERA Annual Conference, University of York.

Huysamen, G. K. (1997). Parallels Between Qualitative Research and Sequentially Performed Quantitative Research. *South African Journal of Psychology*, 27: 1-8.

Hycner, R. H. (1985) Some Guidelines for the Phenomenological Analysis of Interview Data. *Human Studies*, 8: 279-303.

Jenkins, E. W. (2000). The Impact of the National Curriculum on Secondary School Science Teaching in England and Wales. *International Journal of Science Education*, 22(3): 325-336.

Johnson, G. I. and Briggs, P. (1994). Question-Asking and Verbal Protocol Techniques In C. Cassell and G. Symon (Eds.) *Qualitative Methods in Organizational Research: A Practical Guide*. London: Sage

K. B. Schwamb (1990). Mental Models: A Survey, unpublished. (http://www.isi.edu/soar/schwamb/pubs.html)

Kantowski, M. G. (1977). Processes Involved in Mathematical Problem Solving. *Journal for Research in Mathematics Education*, 8: 163-180.

Kaput, J. (1999). Teaching and Learning a New Algebra. In E. Fennema and T. Romberg (Eds.) *Mathematics Classrooms That Promote Understanding*. Mahwah, NJ: Lawrence Erlbaum.

Kawanaka, T., Stigler, J. W. and Hiebert, J. (1999). *Studying Mathematics Classrooms in Germany, Japan and the United States: Lessons from the TIMSS Videotape Study.*

Kendal, M. (1992). *Trigonometry: A Comparison of Two Teaching Methods*. Unpublished Master of Education Thesis. University of Melbourne, Australia.

Kerlinger, F. N. (1986). *Foundations of Behavioural Research* (3rd edition). New York: Holt, Rinehart & Winston.

Kieran, C. (1992). The Learning and Teaching of School Algebra. In D. A. Grouws (ed.) *The Handbook of Research on Mathematics Teaching and Learning* (pp. 390-419). New York: Macmillan Publishing Company.

Kieran, C. (1997). Mathematical Concepts at the Secondary School Level: The Learning of Algebra and Functions. In T. Nunes, and P. Bryant (Eds.) *Learning and Teaching Mathematics: An International Perspective*, (pp: 133-158). Hove: Psychology Press.

Kilpatrick, J. (1988). Editorial. *Journal for Research in Mathematics Education*, 19(2): 98.

King, E. (1996). The Use of the Self in Qualitative Research. In J. T. E. Richardson (Ed.) *Handbook of Qualitative Research Methods for Psychology and the Social Sciences.* Leicester: British Psychological Society.

King, N. (1994). The Qualitative Research Interview. In C. Cassell and G. Symon (Eds.) *Qualitative Methods in Organisational Research: A practical Guide*. London: Sage.

Koedinger, K. R. and Tabachneck, H. J. M. (1994). *Two Strategies Are Better Than One: Multiple Strategy Use in Word Problem Solving*. Presented at the annual meeting of the American Educational Research Association, New Orleans, LA.

Kosslyn, S. M. (1980). *Image and Mind*. Harvard U.P.: London, G. B.

Krutetskii, V. A. (1976). *The Psychology of Mathematical Abilities in Schoolchildren*. Chicago: Univeristy of Chicago Press.

Lave, J. (1988). *Cognition in Practice.* Cambridge: Cambridge University Press.

Lean, G. and Clements, M. A. (1981). Spatial Ability, Visual Imagery and Mathematical Performance. *Educational Studies in Mathematics*, 12(3): 267-299.

Lefton, L. A. (1985). *Psychology* (3rd edition). London: Allyn and Bacon, Inc.

Lesh, R. (1981). Applied Mathematical Problem Solving. *Educational Studies in Mathematics*, 12(2): 235-265.

Lester, Frank K. (1985). Methodological Consideration in Research on Mathematical Problem-Solving Instruction. In E. A. Silver (Ed.) *Teaching and Learning Mathematical Problem Solving: Multiple Research Perspectives*. Hillsdale, NJ:Erlbaum.

Lewis, A. B. (1989). Training Students to Represent Arithmetic Word Problems. *Journal of Educational Psychology*, 81(4): 521-531.

Lewis, C. (1981). Skill in Algebra. In J. R. Anderson (Ed.) *Cognitive Skills and Their Acquisition*. New Jersey: Lawrence Erlbaum Associates Publishers, pp 85-110.

Lincoln, Y. S. and Guba, E. G. (1985). *Naturalistic inquiry*. Thousand Oaks, CA: Sage.

Ling, L. M. and Man, C. P. (2000). *What Matters? Progressive Versus Traditional Teaching Styles*. Paper presented at the 13th International Congress for School Effectiveness and Improvement.

Love, E. (1995). The Functions of Visualisation in Learning Geometry. In R. Sutherland and J. Mason (Eds.) *Exploiting Mental Imagery with Computers in Mathematics Education*. Berlin: Springer.

Low, R., Over, R., Doolan, L., and Michell, S. (1994). Solution of Algebraic Word Problems Following Training in Identifying Necessary and Sufficient Information within Problems. *American Journal of Psychology*, 107 (3): 423-439.

Lucangeli, D., Tressoldi, P. E. and Cendron, M. (1998). Cognitive and Metacognitive Abilities Involved in the Solution of Mathematical Word Problems: Validation of a Comprehensive Model. *Contemporary Educational Psychology*, 23: 257–275.

MacFarlane-Smith, I. M. (1964). *Spatial Ability, its Educational and Social Significance*. London: University of London Press.

Maeroff, G. (1991). The public's Expectations for Assessment of National Educational Goals. *The Assessment of National Goals: Proceedings of the 1990 ETS Invitational Conference*, (pp. 87–95).

Marjoram, D. T. E. (1974). *Teaching Mathematics*. London: Heinemann Educational Books.

Mason, J., Burton, L. and Stacy, K. (1982). *Thinking Mathematically*. Reading, MA: Addison-Wesley Publishing Company.

Mayer, R. E. (1982). The Psychology of Mathematical Problem Solving. In F. K. Lester and J. Garofalo (Eds.), *Mathematical Problem Solving: Issues in Research* (pp. 1-13). Philadelphia: Franklin Institute Press.

Mayer, R. E. (1985). Mathematical Ability. In R. J. Sternberg (Ed.), *Human Abilities: In Information Processing Approach* (pp 127-150). New York: Freeman.

Mayer, R. E. (1987). *Educational Psychology: A Cognitive Approach*. Boston: Little, Brown.

Mayer, R. E., Larkin, J. H. and Kadane, J. (1984). A Cognitive Analysis of Mathematical Problem Solving Ability. In R. Sternberg (Ed.), *Advances in the Psychology of Human Intelligence*. Hillsdale: Erlbaum.

Menghini, M. (1994). Form in Algebra: Reflecting, with Peacock, on Upper Secondary School Teaching. *For the Learning of Mathematics*, 14(3): 9-15.

Miles, M. B. and Huberman, A. M. (1984). *Qualitative Data Analysis: A Source Book of New Methods*. Newbury Park and London: SAGE.

Miles, M. B. and Huberman, A. M. (1998). Data Management and Analysis Methods. In L. Denzin and Y. Lincoln (Eds.) *Collecting and Interpreting Qualitative Materials* (pp. 179-210). Beverly Hills: Sage Publications.

Morrison, K. R. B. (1993). Planning and Accomplishing School-Centred Evaluation. Norfolk: Peter Francis Publishers.

Munby, H. (1997). Issues of Validity in Science Attitude Measurement. *Journal of Research in Science Teaching*, 34: 337-341.

National Council of Teachers of Mathematics. (1989). *Curriculum and Evaluation Standards for School Mathematics*. Reston, VA: The Author.

Nemirovsky, R., and Noble, T. (1997). On Mathematical Visualisation and the Place Where We Live. *Educational Studies in Mathematics* 33(2): 595-610

Nickerson, R., Perkins, D. and Smith, E. (1985). *The Teaching of Thinking*. Hillsdale NJ: Lawrence Erlbaum Associates.

Nickson, M. (1992). The Culture of the Mathematics Classroom: An Unknown Quantity. In D. A. Grouws (Ed.) *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan Publishing Company.

Nickson, M. and Green, S. (1996). *A Study of the Effects of Context in the Assessment of the Mathematical Learning of 10/11-Year Olds*. British Educational Research Association Annual Conference, University of Lancaster, September.

Noble-Nesbitt, C. (1982). Calculators in the Classroom. In M. Cornelius (Ed.) *Teaching Mathematics*. London: Croom Helm.

Noddings, N. (1989). Preparing Teachers to Teach Mathematical Problem Solving. In R. I. Charles and E. A. Silver (Eds.) *The Teaching and Assessing of Mathematical Problem Solving*, volume 3. National Council of Teachers of Mathematics.Virginia. Lawrence Erlbaum Associates.

Noss, R., Healy, L. & Hoyles, C. (1997). The Construction of Mathematical Meanings: Connecting the Visual with the Symbolic. *Educational Studies in Mathematics*, 33(2): 203-233.

Novak, J., Godwin, D. and Gowin, B. (1984). *Learning How to Learn*. Cambridge: University Press.

Novotna, J. (2000). *Students' Level of Understanding of Word Problems*. ICME-9: International Congress on Mathematics Education, Tokyo, Japan.

Nunokawa, K. (1994). Improving Diagrams Gradually: One Approach to Using Diagrams in Problem Solving. *For the Learning of Mathematics*, 14(1): 34-38.

Odvárko, O. (1990). *Methods of Solving Mathematical Problems*. Praha, SPN. (Translated from Czech.)

Oppenheim, A. N. (1992). *Questionnaire Design, Interviewing and Attitude Measurement*. London: Pinter.

Ornstein, R. and Cartensen, L. (1991). *Psychology: The Study of Human Experience* (3rd edition). London: Harcourt Brace Jovanovich Publishers.

Owen E. and Sweller J. (1985). What Do Students Learn While Solving Mathematics Problems. *Journal of Educational Psychology*, 77(3): 272-284.

Partington, G. (1998). *Validity as an Issue in Student Interview Data*. Paper presented at the 1998 Annual Conference of the Australian Association for Research in Education, Adelaide, 30 November - 3 December. Oaks, CA: Sage Publications.

Patton, M. Q. (1990). *Qualitative Evaluation and Research Methods* (2nd ed.). Newbury Park, Calif: Sage Publication.

Pepin, B. (1999). *Epistemologies, Beliefs and Conceptions of Mathematics Teaching and Learning: The Theory, and What Is Manifested in Mathematics Teachers' Work in England, France and Germany.* Thematic Network of Teacher Education in Europe (TNTEE) Conference, Lisbon, May 1999.

Perfetti, C. (1984). Reading Acquisition and Beyond: Decoding Includes Cognition. *American Journal of Education*, 93: 40-60.

Phil Johnson-Laird and Ruth Byrne (2000). Mental Models Website, *A Gentle Introduction*. (http://www.tcd.ie/Psychology/Ruth_Byrne/mental_models/)

Polya, G. (1945). *How to Solve It.* Princeton, NJ: Princeton University Press.

Polya, G. (1973). *How to Solve It.* Princeton, NJ: Princeton University Press. (Originally copyrighted in 1945).

Polya, G. (1987). On Learning, Teaching and Learning Teaching. In F.R. Curcio (Ed.), *Teaching and Learning: A problem Solving Focus* (pp 1-15). Reston, VA: National Council of Teachers of Mathematics.

Pritchard, L. and Simpson, A. (1999). The Role of Pictorial Images in Trigonometry Problems. *Proceedings of the 23rd Conference of the International Group for the Psychology of Mathematics Education,* 4: 81-88.

Rapple, B. (1989). Matthew Arnold and Comparative Education. *British Journal of Educational Studies:* 37(1): 54-71.

Ratcliff, D. (1995). *An Elementary School Hallway: Social Formations and Meanings Outside the Classroom.* Doctoral Dissertation, The University of Georgia. (http://www.don.ratcliff.net/qual/validity.html).

Rayner, D. (1994). *Higher GCSE Mathematics: Revision and Practice.* Oxford: Oxford University Press, UK.

Reed, S. K., Willis, D. and Guarino, J. (1994). Selecting Examples for Solving Word Problems. *Journal of Educational Psychology*, 86(3): 380-388.

Reusser, K. (1995). *The Suspension of Reality and of Sense-Making in the Culture of School Mathematics: The Case of Word Problems.* Paper Presented at the Sixth EARLI Conference, Nijmegen.

Reynolds, D. and Farrell, S. (1996). *Worlds Apart? A review of International Surveys of Educational Achievement Involving England.* London: HMSO.

Risnes, M. (1998). Self-efficacy as Mediators in Math Learning: A structural Model. *Proceedings of Twenty-second Annual Conference of the International Group for he Psychology of Mathematics Education* (PME 22, vol. 24). University of Steellenbosch, South Africa.

Robson, C. (1993). *Real World Research: A Resource for Social Scientists and Practitioner-Researchers*, (1st edition). Oxford: Blackwell.

Robson, C. (2002) *Real World Research: A Resource for Social Scientists and Practitioner-Researchers*, (2nd edition). Oxford: Blackwell.

Romberg, T. A. (1992). Perspectives on Scholarship and Research Method. In Grouws, D. A. (ed.) *Handbook of Research on Mathematics Teaching and Learning*. New York: Macmillan Publishing Company.

Runesson, U. (1999). *Teaching as Constituting a Space of Variation*. Paper presented at the 8th EARLI conference, Göteborg, Sweden, Aug24-25.

Ruthven, K. (1996). Calculators in the Mathematics Curriculum: the Scope of Personal Computational Technology. In A. J. Bishop, K. Clements, J. Kilpatrick, and C. Laborde (Eds.) *International Handbook of Mathematics Education, (*Part 1, pp 435-468). Dordrecht; London: Kluwer Academic.

Ryan, D. W., Hildyard, A. and Bourke, S. (1989). Description of Classroom Teaching and School Learning. In L. W. Anderson, D. W. Ryan and B. J. Shapiro (Eds.) *The IEA classroom environment study*. Oxford : Published for the International Association for the Evaluation of Educational Achievement by Pergamon.

Santos-Trigo, M. (1996). An Exploration of Strategies Used by Students to Solve Problems with Multiple Ways of Solution. *Journal of Mathematical Behavior*, 15: 263-284.

Saxe, G. B. (1991). *Culture and Cognitive Development: Studies in Mathematical Understanding*, Hillsdale, N.J: L. Erlbaum Associates.

Schmidt, H. W., McKnight, C. C., Valverde, G. A., Houang, R. T. and Wiley, D. E. (1997). *Many Visions, Many Aims:, A Cross-national Investigation of Curricular Intentions in School Mathematics,* Volume 1. Dordrecht London: Kluwer Academic Publisher.

Schmidt, W. H., Jakwerth P. M. and McKnight C. C. (1998). Curriculum Sensitive Assessment: Content Does Make a Difference. *International Journal of Educational Research:* 29: 503-527.

Schmidt, W. H., Jorde, D., Cogan, L. S., Barrier, E., Gonzalo, I., Moser, U., Shimizu, K., Sawada, T., Valverde, G. A., McKnight, C., Prawat, R. S., Wiley, D. E., Raizen, S. A., Britton, E. D. and Wolfe, R. (1996). *Characterizing Pedagogical Flow*, Dordrecht, Kluwer.

Schoenfeld, A. H. (1982). Some Thoughts on Problem-Solving Research and Mathematics Education. In F. K. Lester and J. Garofalo (Eds.) *Mathematical Problem Solving: Issues in Research* (pp. 27-37). Philadelphia: Franklin Institute Press.

Schoenfeld, A. H. (1983). Episodes and Executive Decisions in Mathematical Problem Solving. In R. Lesh and M. Landau (Eds.) Acquisition of Math Concepts and Processes. New York: Academic Press.

Schoenfeld, A. H. (1987). What's All the Fuss about Metacognition? In A. H. Schoenfeld (Ed.) *Cognitive Science and Mathematics Education* (pp. 189-216). New Jersey: Lawrence Erlbaum Associates Publishers.

Schoenfeld, A. H. (1989). Ideas in the Air: Speculations on Small Group Learning, Environmental and Cultural Influences on Cognition, and Epistemology. *International Journal of Educational Research,* 13(l), 71-88.

Schoenfeld, A. H. and Herrmann D.J. (1982). Problem Perception and Knowledge Structure in Expert and Novice Mathematical Problem Solvers. *Journal of Experimental Psychology: Learning, Memory and Cognition,* 8 (5): 484-494.

Schoenfled, A. H. (1985). *Mathematical Problem Solving*. San Diego: Academic Press.

Schofield, H. (1972). *Assessment and Testing: An introduction* . London: George Allen & Unwin Ltd.

Seale, C. (2002). Quality Issues in Qualitative Inquiry. *Qualitative Social Work Research and Practice*, 1(1): 97-110.

Semadeni, Z. (1995). Developing Children's Understanding of Verbal Arithmetical Problems. In M. Hejný and J. Novotná (Eds.) *Proceedings of the International Symposium on Elementary Math Teaching* (pp. 27-33). Praha, Faculty of Education, Charles University.

Sfard, A. (1987). Two Conceptions of Mathematical Notions: Operational and Structural. In J. C. Bergeron, Nicolas Herscovics and Carolyn Kieran (Eds.) *Proceedings of the Eleventh International Conference for the Psychology of Mathematics Education* (Vol. III, pp. 162-169). Montreal: University de Montreal.

Sfard, A. (1991). On the Dual Nature of Mathematical Conceptions: Reflections on Processes and Objects as Different Sides of the Same Coin. *Educational Studies in Mathematics*, 22: 1-36.

Sigurdson, S. E., Olson, A. T. and Mason, R. (1994). Problem Solving and Mathematics Learning. *Journal of Mathematical Behavior*, 13: 361-388.

Silver, E. A. (1982). Knowledge Organisation and Mathematical Problem Solving. In F. K. Lester and J. Garofalo (Eds.) *Mathematical problem solving: Issues in Research* (pp. 15-25). Philadelphia: Franklin Institute Press.

Silverman, D. (1993). *Interpreting Qualitative Data: Method for Analysing Talk, Text and Interaction.* London: SAGE Publications.

Silverman, D. (2000). *Doing Qualitative Research: A Practical Handbook.* London: SAGE Publications.

Simpson, A. and Tall, D. (1998). Computers and the Link Between Intuition and Formalism. In *Proceedings of the 18th International Conference for Technology in the teaching of Collegiate Mathematics, Addison-Wesley.*

Smith, J.A. (1996). Evolving Issues for Qualitative Psychology. In J. T. E. Richardson (Ed.) *Handbook of qualitative research methods for psychology and the social sciences.* Leicester: British Psychological Society.

Solano, A. and Presmeg, N. (1995). Visualisation as a Relation of Images. In L. Meira and D. Carraher (Eds.) *Proceedings of the 19th Conference of the International Group for the Psychology of Mathematics Education*, 3: 66-73. Recife, Brazil: Universidade Federal de Pernambuco.

Spradley, J. P. (1980). *Participant-observation.* New York: Holt, Rinehart & Winston.

Stanic, G. and Kilpatrick, J. (1989). Historical Perspectives on Problem Solving in the Mathematics Curriculum. In R. Charles and E. Silver (Eds.), *The Teaching and Assessing of Mathematical Problem Solving* (pp. 1-22). Reston, VA: National Council of Teachers of Mathematics.

Swanson, H. L., Cooney, J. B. and Brock, S. (1993). The Influence of Working Memory and Classification Ability on Children's Word Problem Solution. *Journal of Educational Psychology*, 3: 374-395.

Tall, D. and Thomas, M. (1991). Encouraging Versatile Thinking in Algebra Using the Computer. *Educational Studies in Mathematics*, 2: 125-147.

Tall, D. and Vinner, S. (1981). Concept Image and Concept Definition in Mathematics, with Special Reference to Limits and Continuity. *Educational Studies in Mathematics*, 12:151–169.

Tomlinson, P. (1989). Having It Both Ways: Hierarchical Focusing as Research Interview Method. *British Educational Research Journal*, 15(2): 155-175.

Tomlinson, P. (1994) Hierarchical Focusing A Step-By-Step Guide. Working paper. School of Education, University of Leeds.

Toom, A. (1999). Word Problems: Applications or Mental Manipulatives. *For the Learning of Mathematics*, 19(1).

Tulving, E. and Flexser, A. J. (1992). On the Nature of the Tulving-Wiseman Function. *Psychological Review*, 99: 543-546.

Usiskin, Z. (1999) Which Curriculum Is Best? *University of Chicago School Mathematics Project Newsletter*, No 24, p 8.

Verschaffel, L. and De Corte, E. (1997). Word Problems: A Vehicle for Promoting Authentic Mathematical Understanding and Problem Solving in the Primary School? In T. Nunes, and P. Bryant (Eds.) *Learning and Teaching Mathematics: An International Perspective*, (pp. 69-97). Hove: Psychology Press.

Verschaffel, L., De Corte, E. and Borghart I. (1997). Pre-Service Teachers' Conceptions and Beliefs about the Role of Real-World Knowledge in Mathematical Modelling of School Word Problems. *Learning and Instruction*, 7(4): 339-359.

Villarreal, M. (2000). Mathematical Thinking and Intellectual Technologies: The Visual and the Algebraic. *For the learning of Mathematics*, 20(2): 2-7.

Vithal, R. (1999). *In Search of a Pedagogy of Conflict and Dialogue for Mathematics Education*. Doctoral dissertation submitted to Aalborg University, Denmark.

Vulliamy, G. and Nikki M. L. (1997). The Comparative Context for Educational Reform in England and Finland. *British Educational Research Association Annual Conference*. University of York, UK.

Vygotsky, L. S. (1978). *Mind in Society: The Development of Higher Psychological Processes* (edited by M. Cole, V. John-Steiner, S. Scribner and E. Souberman). Cambridge, Mass.: Harvard University Press.

Wagner, S. and C. Kieran (1989). *Research in the Learning and Teaching of Algebra* (pp. 11-24). Hillsdale, NJ: Lawrence Erlbaum.

Watkins, M. J. and Gardiner, J. M. (1979). An Appreciation of Generate-Recognise Theory of Recall. *Journal of Verbal Learning and Verbal Behaviour*, 18: 687-704.

Weaver III, C. A. and Kintsch, W. (1992). Enhancing Students' Comprehension of the Conceptual Structure of Algebra Word Problems. *Journal of Educational Psychology*, 84(4): 419-428.

Wenger, R. H. (1987) Cognitive science and algebra learning. In A. H. Schoenfeld (Ed.) *Cognitive Science and Mathematics Education* (pp. 217-251). Hillsdale, NJ: L. Erlbaum Associates, Inc.

Wertsch, J. V. (1998). *Mind as action*. New York ; Oxford : Oxford University Press.

Wheatly, G (1991). Enhancing Mathematics Learning through Imagery. *Arithmetic Teacher*. 39(1): 34-36.

Wickelgren, W. A. (1977). *Learning and Memory*. New Jersey: Prentice-Hall, Inc.

Willis, G. B. and Fuson, K. C. (1988). Teaching Children to Use Schematic Drawings to Solve Addition and Subtraction Word Problems. *Journal of Educational Psychology*, 80(2): 192-201.

Wilson, J. W., Fernandez, M. L. and Hadaway, N. (1993). Mathematical Problem Solving. In P. S. Wilson (Ed.) *Research Ideas for the Classroom: High School Mathematics*. New York: MacMillan.

Wolf, A. and Steedman, H. (1998). Basic Competence in Mathematics: Swedish and English 16-Year Olds. *Comparative Education*, 34(3): 241-259.

Woodrow, D. (1997). *Democratic Education-Does It Exist-Especially for Mathematics Education?* Paper presented at the British Educational Research Association Annual Conference, September 11-14, University of York.

Yakimanskaya, I. S. (1991). The Development of Spatial Thinking in Schoolchildren. *Soviet Studies in Mathematics Education vol. 3*. NCTM: Reston, USA.

Yin, R. K. (1994). *Case Study Research: Design and Methods*, (2nd edition). Thousand Oaks, Calif.:Sage.

Zazkis, R., Dubinsky, E., and Dautermann, J. (1996). Using Visual and Analytic Strategies: A Study of Students' Understanding of Permutation and Symmetry Groups. *Journal for Research in Mathematics Education*, 27(4): 435–457.

Zikmund, W. G. (1991). *Business Research Methods* (2nd Edition). Chicago: Dryden Press.

Zimmermann, W, and Cunningham, S. (1991). *Visualisation in Teaching and Learning Mathematics*. Volume 19, USA: Mathematics Association of America.

(TS1) 1999 Mathematics Teacher Questionnaire, http://utenti.tripod.it/learning_paths/Questionnaires/Teachstylequest.htm

(TS2) http://utenti.tripod.it/learning_paths/Questionnaires/SupChquest.htm

(TS3) http://www.fcrc.indstate.edu/tstyles3.html

Appendix A: Trigonometry test

NAME:................................
CLASS:................................

Please show ALL working out and attempt every question

Simplify the following

(1) $\cos(A+B)-\cos(A-B)$

(2) $\dfrac{1}{\cos^2 x}-\dfrac{1}{\cot^2 x}$

(3) $\dfrac{\sin^2 x-5}{\sin x-1}-\dfrac{4\sin x}{1-\sin x}$

(4) $\dfrac{1}{1-\tan x}+\dfrac{1}{1-\cot x}$

(5) $\dfrac{\sin x-\cos x}{\sin x+\cos x}-\dfrac{\sin x+\cos x}{\sin x-\cos x}$

(6) if $\left.\begin{array}{l}\sin x+\cos y=\frac{3}{2}\\ \cos x+\sin y=\frac{3}{4}\end{array}\right\}$, find $\sin(x+y)$

(7) $\sin(\dfrac{A+B}{2})-\sin(\dfrac{A-B}{2})$

(8) $\dfrac{\sin^4 x-\cos^4 x}{\sin x-\cos x}$

Please turn over

(9) $\dfrac{\sin 2x - 2\sin x\cos^2 x}{\cos x(1-\cos 2x)}$

(10) $\dfrac{\tan^2 A - \tan^2 B}{1-\tan^2 A\tan^2 B}$

(11) $\cot x + \tan x$

(12) $\dfrac{\sin^2 2x - 4\sin^2 x}{(\cos 2x+1)^2}$

(13) $\dfrac{\cos^3 x\sin x - \cos x\sin^3 x}{\cos^3 x\sin x + 2\cos^2 x\sin^2 x + \cos x\sin^3 x}$

(14) if $x=\sin\theta, y=\cos\theta$, find $x^4 - y^4 + 3x^2 + 5y^2$

(15) $\dfrac{1+2\sin\theta\cos\theta}{\sin\theta+\cos\theta}$

(16) $\dfrac{\tan x + \cot x}{\operatorname{cosec} 2x}$

Appendix B: Algebra test

Please show ALL working out and attempt every question

NAME:..........................

CLASS:..........................

1-) Simplify $3(2a+1)-2(2a+3)-5(a-3)$

2-) Simplify $5a^2-(2a+1)^2$

3-) if $x = -2$, $y = 4$, $t = -6$ and $k = 8$, then find
$3x-2y+8t-5k$

4-) Solve the simultaneous equations
$2x = 3y+11$, $5x+2y = 18$

5-) Given that $a-b = 5$, simplify

a-) $(b-a)^2$ b-) (a^2-b^2)

6-) Simplify $\frac{a}{a-b}+\frac{b}{b-a}$

7-) Simplify $\frac{x^4-y^4}{x-y}$

8-) if $\left.\begin{array}{l}\sqrt{x-y}=3\\ \sqrt{x+y}=5\end{array}\right\}$, then $x=?$

Please turn over

9-) if m+n = a and y-x = b, write down (m+x) – (y-n) in terms of a and b.

10-) Simplify $\dfrac{(3xy)^2 - 9x^2}{(3xy+3x)^2}$

11-) Simplify $\dfrac{x^2 - xy}{y^2 - yx} \cdot \dfrac{y}{x}$

12-) Simplify $\dfrac{x(x+1)+6(x+1)+6}{x(x+2)+7(x+2)+6}$

13-) Simplify $\dfrac{x^2 - y^2}{1 - x^2y^2} \div \dfrac{x+y}{yx - x^3y^2}$

14-) Simplify $\left[\dfrac{a-b}{a+b} - \dfrac{a+b}{a-b}\right] \cdot \dfrac{a-b}{ab}$

15-) Simplify $\left[\dfrac{x}{y} + \dfrac{y}{x} + 2\right] \div \left[\dfrac{y}{x} - \dfrac{x}{y}\right]$

16-) Simplify $\dfrac{a^3b - ab^3}{a^3b + 2a^2b^2 + ab^3}$

Appendix C: Trigonometry word problems test

NAME:..CLASS: ....

Please show ALL working out and attempt every question

TWP-1-) A boy is flying a kite from a string of length 55 m. If the string is taut and makes an angle of 60^0 with the horizontal, what is the height of the kite? Ignore the height of the boy.

TWP-2-) A rocket flies 10 km. vertically, then 20 km. at an angle of 15^0 to the vertical and finally 60 km. at an angle of 64^0 to the horizontal. Calculate the vertical height of the rocket at the end of the third stage.

TWP-3) From a point 10 m. from a vertical wall, the angles of elevation of the bottom and the top of a statue of Isaac Newton, set in the wall, are 40^0 and 52^0. Calculate the length of the statue?

Please turn over

TWP-4-) An observer at the top of a tower of height 15 m. sees a man due west of him at an angle of depression 31^0. He sees another man due south at an angle of depression 17^0. Find the distance between the men?

TWP-5-) The top of a cliff is 10 m. above sea level. A yacht is an angle of 10 below the horizon measured from the cliff top. How far is the yacht from the bottom of the cliff?

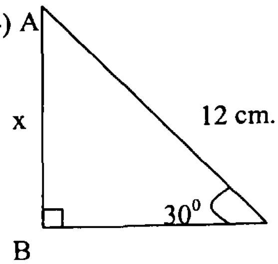
TWP-6-) A ladder of length 6 m. rests against a vertical wall so that the height of the point at which the ladder reaches the wall is 2.5 m. above the ground. What angle does the ladder make with the ground?

Appendix D: Trigonometric functions on right-angled triangles test

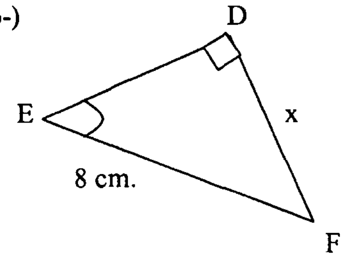
NAME:....................................... CLASS:...

Find the angle and the sides marked with a letter in the followings. Show <u>ALL</u> working out. If you need the use more place, you can use the back page and please show your question number. <u>ANSWER ALL QUESTIONS</u>.

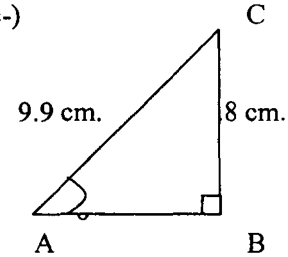
1-) a-)



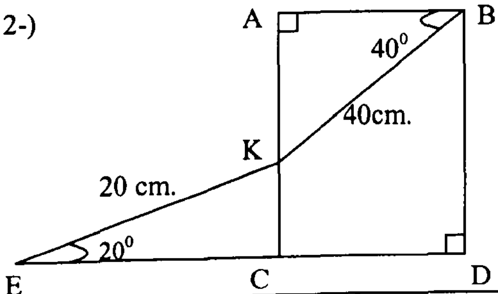
b-)



c-)

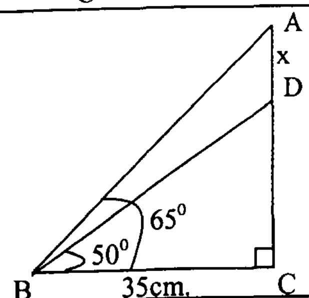


2-)

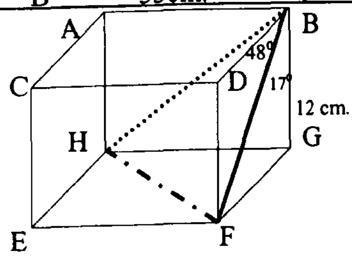


|BD|=?

3-)

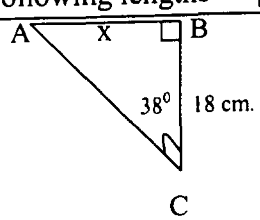


4-)

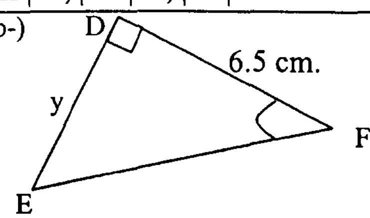


$\hat{FBG} = 17°, \hat{HBG} = 48°$ and |BG|=12 cm. are given in above rectangular block. Then find the following lengths |HF|=?, |HB|=?, |BF|=?.

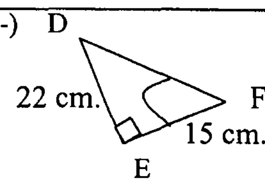
5-) a-)



b-)



c-)



Appendix E: Teachers Questionnaire

TEACHERS' QUESTIONNAIRE

SECTION I

What type of solution would you suggest to students as one of the better ways to solve each of the problems below?

NAME:.......................... **SCHOOL:.............................**

PLEASE DETAIL ALL THE STEPS YOU WOULD EXPECT TO SEE ON A STUDENT PAPER.

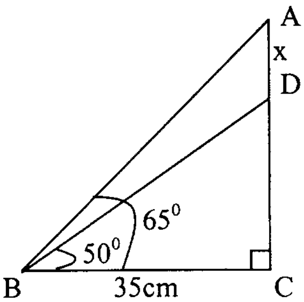
1-) From a point 10 m. from a vertical wall, the angles of elevation of the bottom and the top of a statue of Isaac Newton, set in the wall, are 40^0 and 52^0. Calculate the length of the statue?

2-) An observer at the top of a tower of height 15 m. sees a man due west of him at an angle of depression 31^0. He sees another man due south at an angle of depression 17^0. Find the distance between the men?

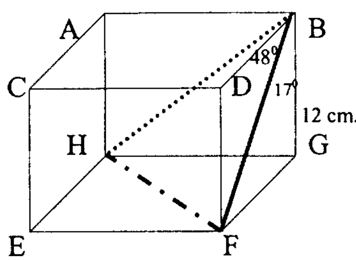
3-) The top of a cliff is 10 m. above sea level. A yacht is 57 m. away from the bottom of the cliff. What is the angle of depression measured from the cliff top?

Find the angle and the sides marked with a letter in the following.

4-)

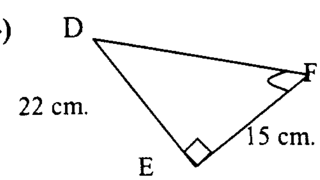


5-)



$\widehat{FBG} = 17°, \widehat{HBG} = 48°$ and $|BG|=12$ cm. are given in above rectangular block. Then find the following lengths $|HF|=?$, $|HB|=?$, $|BF|=?$.

6-)



, $D\hat{F}E = ?$

Simplify the followings

7-) $\dfrac{\sin^2 2x - 4\sin^2 x}{(\cos 2x + 1)^2}$

8-) $\dfrac{1}{1 - \tan x} + \dfrac{1}{1 - \cot x}$

9-) $\cos(A+B) - \cos(A-B)$

10-) Can you give the meaning of ***"simplification"*** in the questions of simplifying trigonometric expressions?

SECTION II

1-Describe your educational background?

Graduate in mathematics education	☐	Graduate in mathematics	☐
Certification to teach math	☐	Others:	☐

2-By the end of this school year how many years will you have been teaching altogether?........................

3-At which grade levels are you teaching trigonometry during this school year?...................................

4-To be good at trigonometry in mathematics at school, how important do you think it is for students to...

1. remember formulas and procedures.
2. think in a sequential manner.
3. understand mathematical concepts, principles and strategies.
4. be able to think creatively.
5. understand how mathematics is used in the real world.
6. be able to provide reasons to support their solutions.
7. be able to manage using a calculator.

A very four way Likert scale was presented here, from not important to very important. Given thesis margin requirements this has been removed.

5-To what extent do you agree or disagree with each of the following statements?

1. Trigonometry is primarily an abstract subject.
2. If students are having difficulty with trigonometric identities and trigonometric formulae, an effective approach is to give them more practice by themselves during the class.
3. More than one representation (picture, concrete material, symbol set, etc.) should be used in teaching trigonometry.
4. Trigonometry should be learned as sets of algorithms.
5. A student's success in trigonometry is strongly related to the student's algebra background.
6. I use printed resources other than the textbook in teaching trigonometry.
7. My expectations for what I want students to do in trigonometry lessons are clearly defined in the syllabus.
8. Students have difficulties with trigonometry word problems.
9. Some students cannot draw representations of situations in trigonometry word problems.
10. Picturing trigonometry word problems leads to solving the question.
11. I allow students to solve trigonometry problems on the blackboard.
12. I teach students some mnemonics in trigonometry lessons.

A very four way Likert scale was presented here, from strongly disagree to strongly agree. Given thesis margin requirements this has been removed.

13. I give related algebra examples before simplification of trigonometric expressions.

14. I encourage students to memorize the trigonometric identities.

15. Mathematics teachers in this school regularly share ideas and materials related to trigonometry.

16. I enjoy teaching trigonometry.

17. I provide concrete experience in trigonometry word problems before abstract concepts are introducing.

18. I take students' prior understanding into account when planning the instruction.

19. I encourage the use of calculators in trigonometry.

20. I engage students in applications of trigonometry in a variety of contexts.

21. A student's ability in solving algebra problems affects solving trigonometry problems.

22. I follow the textbook as closely as possible in teaching trigonometry.

23. I let students use a variety of resources in addition to textbooks for trigonometry.

24. My trigonometry lessons follow a fixed pattern.

25. I set aside some time in trigonometry lesson in order to teach students how to use their calculators.

6- ***my students in trigonometry lesson use calculators for the following activities***

1. Checking answers
2. Tests and exams
3. Routine computation
4. Solving problems

A very four way Likert scale was presented here, from not at all to almost always. Given thesis margin requirements this has been removed.

7-In planning trigonometry lessons, to what extent do you refer to each of the following

1. National or Regional Examination Specifications
2. National or Regional Curriculum Guide
3. School Curriculum Guide
4. Teacher Edition of Textbook
5. Student Edition of Textbook
6. Other Resource Books

A very four way Likert scale was presented here, from not at all to almost always. Given thesis margin requirements this has been removed.

Appendix F: Textbook questionnaire

QUESTIONNAIRE ON TEXTBOOKS USED IN TEACHING MATHEMATICS

I am involved in research on the teaching of trigonometry in the UK and in Turkey. A small part of this work will be comparing textbook approaches in the two countries. To this aim I need to find representative textbooks for both GCSE and A-level courses. I would be most grateful if you would give five minutes of your time to complete the questionnaire below.

Name:.................................... .... School: ....

1. Do you use a textbook in teaching mathematics to your class? Yes ☐ No ☐

If YES, which of the textbooks do you use most?

With GCSE classes	With A-level classes
a)	a) ....
b)	b) ....
c)	c) ....

Which of these (a, b, c) do you make the most use of?

With GCSE classes		With A-level classes	
Title	....	**Title**	....
Author	....	**Author**	....
Publisher	....	**Publisher**	....
How long have you used it?	________	**How long have you used it?**	________

Please state your reasons for choosing this book?

With GCSE classes	With A-level classes
______________________________	______________________________
______________________________	______________________________
______________________________	______________________________
______________________________	______________________________

If NO, then what resources do you use?

With GCSE classes	With A-level classes
______________________________	______________________________
______________________________	______________________________
______________________________	______________________________
______________________________	______________________________

Please write anything else which may be useful to me overleaf

Thank you very much for your co-operation

Appendix G: Teachers interview

INTERVIEW QUESTIONS WITH A TEACHER WHO TEACHES TRIGONOMETRY

1.) 1.1) What resources do you typically use in a trigonometry lesson?

1.1.1) What do you think about the use of calculators in trigonometry lessons? Do you encourage the students to use calculators in trigonometry lessons?
1.1.2) Do you think the textbooks gives the trigonometry satisfactorily?
1.1.3) Why do you use these documents?

1.2) What do you think about official resources from the view of trigonometry?

1.2.1) Is the time/place of trigonometry in the National Curriculum ——too early?
——just right?
——too late?

1.2.2) What extent do you follow official resources, such as national curriculum, school curriculum guide, teacher and student edition textbooks, to plan your trigonometry lesson?
1.2.3) Do you think National curriculum and textbooks are on the same line?

2-) How do you see the development of the trigonometry through the school age range 14-19? What is the difference between introducing trigonometry and further trigonometry?

2.1) What should students' earliest experience of trigonometry be?
2.2) What should they learn after these early experiences?
2.3) Do you think trigonometry word problems help students to understand trigonometric functions on right angled triangle well?
2.4) Do you follow an exact way to solve trigonometry word problems?
2.5) Do you think that trigonometric identities and sum formulae should be memorized?
2.6) What extent do you use mnemonic ways to teach trigonometry word problems, trigonometric identities and sum formulae?

3-) How would you describe the structure of a typical trigonometry lesson for pupils?

3.1) Do you call them to blackboard to solve questions?
3.2) Do you walk in classroom and check students individually?
3.3) When teaching trigonometry word problem, trigonometric identities and trigonometric formulae, how many example do you work through with the pupils?
3.4) On what basis do you select the examples which you work through with the pupils?
3.5) Do you give them homework?

4-) Do you solve simplification problems of algebra before the simplification problems of trigonometric expressions?

4.1) What extent do you use substitutions into algebraic forms for the solution of simplification problems of trigonometric expressions? e.g. using $\frac{a^2 - b^2}{a + b}$ for $\frac{\sin^2\theta - \cos^2\theta}{\sin\theta + \cos\theta}$
4.2) How do you define the difference between the simplification of an algebraic expression and a trigonometric expression? (Give example if it is needed $\frac{\sin\theta + \sin\theta\cos\theta}{\sin^2\theta}$)
4.3) What do you think are the factors which increase the success of students in simplifying the trigonometric expressions?

Appendix H: Construal interview agenda

CONSTRUAL INTERVIEW AGENDA

	Spontaneous	Prompted
1.) 1.1) Resources		
1.1.1) Chalk/blackboard/whiteboard	S	P
1.1.2) Calculations	S	P
1.1.3) Textbooks	S	P
1.1.4) Others	S	P
1.2) Official resources		
1.2.1) Time/place in National Curriculum	S	P
1.2.2) Plan, national curriculum, school curriculum guide student, teacher textbook	S	P
1.2.3) National curriculum, textbook same line	S	P
2.) Development of trigonometry in the lesson (int. trig. -further trig.)		
2.1) Earliest experiences	S	P
2.2) + Earliest experiences	S	P
2.3) TWP helps student to understand TF	S	P
2.4) Solution pattern of TWP	S	P
2.5) Memorisation of TI and TF	S	P
2.6) Mnemonic ways TWP, TI and TF	S	P
3.) Structure of the lesson		
3.1) Call students to black board	S	P
3.2) Walk in classroom and check students individually	S	P
3.3) Number of the problems to teach TWP, TI and TF	S	P
3.4) Variety (basis) of the problems to teach TWP, TI and TF	S	P
3.5) Homework	S	P
4.) Inter- relation between trigonometry and algebra		
4.1) Substitutions into algebraic form	S	P
4.2) Simplification of AEX-TEX	S	P
4.3) Factors for successful simplification in TEX	S	P

Abbrevations
TWP…trigonometry word problem
TI…trigonometric identity
TF…trigonometric sum formula
AEX…algebraic expression
TEX…trigonometric expression

Appendix I: Jigsaw - Steps for teaching trigonometry

STEPS FOR TEACHING TRIGONOMETRY

*****REVISION OF PREREQUISITIES***

For example;
-substution
-Equations
-Triangles
-Simplification

***** FUNDAMENTAL DEFINITIONS***

For example;
-Radian measure
-Angles

*****TRIGONOMETRIC FUNCTIONS***

For example;
Trigonometric functions on right-angled triangle
-sin, cos, tan, cot

*****PERIOD AND PERIODIC FUNCTIONS***

For example;
-Even and odd functions

*****TRIGONOMETRIC IDENTITIES***

For example;
$\sin^2 A + \cos^2 A = 1$
$\sin 2A = 2 \sin A \cos A$
$\tan (A+B) = \dfrac{\tan A + \tan B}{1 - \tan A \tan B}$

*****TRIGONOMETRIC VALUES OF GENERAL ANGLES AND TRIGONOMETRIC TABLE***

*****GRAPHS OF TRIGONOMETRIC FUNCTIONS***

For example;
$f(x)=\sin x \qquad 0 \leq x \leq 360$

*****INVERSE OF TRIGONOMETRIC FUNCTIONS***

For example;
$\sin^{-1} x$, arcsin x

*****TRIGONOMETRIC EQUATIONS AND INEQUALITIES***

For example;
$\tan x = 1$
$\cos x < 1$

*****TRIGONOMETRIC FORMULAE***

For example;
-Sine formula
$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}$$

-Cosine formula
$a^2=b^2+c^2-2bc \cos A$

***** FURTHER TRIGONOMETRY***

For example;
-Integration of trigonometric expressions
$$\int \sin x dx = -\cos x + c$$
-Differentiation of trigonometric expressions
$$\frac{d\cos x}{dx} = -\sin x$$

*****TRIGONOMETRY WORD PROBLEMS***

For example;

-Angle of Elevation and Depression
-Real world examples
-Trigonometric functions on right-angled triangle

Appendix J: Teacher observation instrument

CLASSROOM OBSERVATION OF TESACHERS IN TRIGONOMETRY LESSON

Time minute	**Activity**
5m-)	
10m-)	
15m-)	
20m-)	
25m-)	
30m-)	
35m-)	
40m-)	
45m-)	

Appendix K: Difference of think aloud and talk aloud instructions

DIFFERENCE OF THINK ALOUD AND TALK ALOUD INSTRUCTIONS

Talk aloud

In this study I am interested in what you say to yourself as you carry out the tasks I am going to give you. To do this, I am going to ask you to talk aloud as you work through the tasks. By 'talk aloud' I mean that I want you to say out loud everything that you say to yourself silently aloud as you work through the tasks. It may help if you imagine that you are in the room by yourself. If you are silent for any period of time, I shall remind you to keep talking.

Do you understand what I am asking you to do? Do you have any questions?
We shall start with a few practice problems.

Think aloud

In this study I am interested in what you think about as you carry out the tasks I am going to give you. To do this, I am going to ask you to think aloud as you work through the tasks. By 'think aloud' I mean that I want you to say out loud everything that you are thinking from the time you start the task until you complete it. I would like to talk constantly from the time you commence the task until you have completed it. It is important that you do not plan out or try to explain to me what you are thinking. It may help if you imagine that you are in the room by yourself. It is very important that you keep talking. If you are silent for any period of time, I shall remind you to keep talking.

Do you understand what I am asking you to do? Do you have any questions?
We shall start with a few practice problems.

Appendix L: Trigonometry verbal protocol instruction

TRIGONOMETRY VERBAL PROTOCOL INSTRUCTION

In this study I am interested in what you think about as you carry out the tasks I am going to give you. To do this, I am going to ask you to think aloud as you work through the tasks. By 'think aloud' I mean that I want you to say out loud everything that you are thinking from the time you start the task until you complete it. I would like to talk constantly from the time you commence the task until you have completed it. It is important that you do not plan out or try to explain to me what you are thinking. It may help if you imagine that you are in the room by yourself. It is very important that you keep talking. If you are silent for any period of time, I shall remind you to keep talking.

Do you understand what I am asking you to do? Do you have any questions?

We shall start with a few practice problems.

First, I would like you to think aloud as you simplify a trigonometric expression in your head. The expression is $\sin\theta.\cos\theta + \cos\theta.\tan\theta$

(Same instruction is used for algebra verbal protocol, trigonometry word problems protocol and trigonometric functions on right-angled triangles protocol)

Appendix M : Trigonometry verbal protocol questions

TRIGONOMETRY VERBAL PROTOCOL QUESTIONS

(1) Simplify $\tan^2 x\cos^2 x + \cot^2 x\sin^2 x$

(2) Simplify $\frac{\sin x}{1-\cos x} - \frac{\cos x + 1}{\sin x}$

(3) Simplify $\sec^2\theta + \operatorname{cosec}^2\theta$

(4) Simplify $\sin(\frac{A+B}{2}) - \sin(\frac{A-B}{2})$

(5) Simplify $\frac{\sin 2x - 2\sin x\cos^2 x}{\cos x(1-\cos 2x)}$

(In the protocols these were presented on separate pages)

Appendix N: Algebra verbal protocol questions

ALGEBRA VERBAL PROTOCOL QUESTIONS

1-) Simplify 12(a+3)-4(3a-2)

2-) Given that a-b = 5, simplify

a-) (a-b)(b-a)   b-) (1-a+b)

3-) Simplify $\dfrac{x(x-2)-2(x-2)-25}{x(x-3)-2(x-3)-20}$

4-) Simplify $\dfrac{a}{a-b}+\dfrac{b}{b-a}$

5-) Simplify $\dfrac{a^3b-ab^3}{a^3b+2a^2b^2+ab^3}$

(In the protocols these were presented on separate pages)

Appendix O: Trigonometry word problems verbal protocol questions

TRIGONOMETRY WORD PROBLEMS VERBAL PROTOCOL QUESTIONS

1-) A kite flying at a height of 150 m is attached to a string which makes an angle of 67^0 with the horizontal. What is the length of the string?

2-) A ladder 12 m. long rests against a house so that its lower ends is 3 m. from the bottom of the house wall. Find the angle between the ladder and the ground?

3-) From a point 10 m. from a vertical wall, the angles of elevation of the bottom and the top of a statue of Isaac Newton, set in the wall, are 40^0 and 52^0. Calculate the length of the statue?

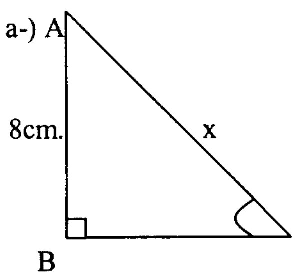
4-) An observer at the top of a tower of height 15 m. sees a man due west of him at an angle of depression 31^0. He sees another man due south at an angle of depression 17^0. Find the distance between the men?

(In the protocols these were presented on separate pages)

Appendix P: Trigonometric functions on right-angled triangles verbal protocol questions

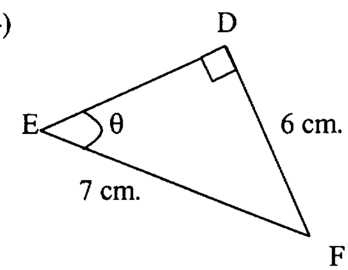
Trigonometric functions on right-angled triangles verbal protocol questions

1-) a-)



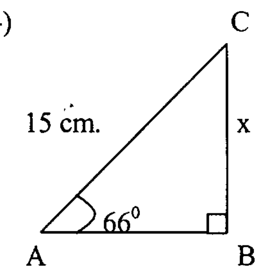
Find the length x.

b-)



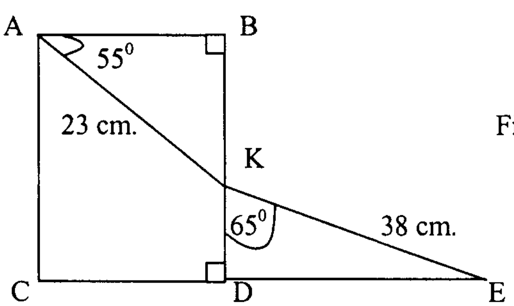
Find the angle θ.

c-)



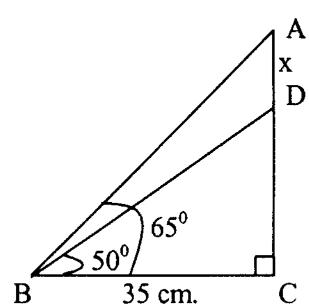
Find the length x.

2-)



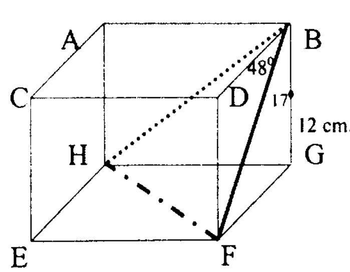
Find the |AC|

3-)



Find the length x.

4-)



$\hat{FBG} = 17°, \hat{HBG} = 48°$ and |BG|=12 cm. are given in above rectangular block. Then find the following lengths
|HF|=?, |HB|=?, |BF|=?.

(In the protocols these were presented on separate pages)

Appendix Q: Trigonometry in the English and Turkish curricula

Trigonometry for 14-16 year old UK and TR students

The National Curriculum in UK-Key Stage 4 Higher-Ma3 Shape, space and measures, p. 66, 2g.

"understand similarity of triangles and of other plane figures, and use this to make geometric inferences; understand, recall and use trigonometrical relationships in right-angled triangles, and use these to solve problems, including those involving bearings, then use these relationships in 3-D contexts, including finding the angles between a line and a plane (but not the angle between two planes or between two skew lines); calculate the area of a triangle using half $\frac{1}{2}ab\sin C$; draw, sketch and describe the graphs of trigonometric functions for angles of any size, including transformations involving scalings in either or both the x and y directions; use the sine and cosine rules to solve 2-D and 3-D problems"

The National Curriculum in TR* -Year 8

1- To understand the trigonometric ratios of the acute angles.
2- To calculate the trigonometric ratios of the angles 30^0, 60^0 and 45^0 on the right-angled triangles.
3- To use the trigonometric table.
4- To apply trigonometric ratios in various problems.

***I did not translate all sub-objectives as this would add more than a page to this Appendix.**

Trigonometry for 16-18 year old students

AS/A level syllabus in the England, P1

THEME OR TOPIC	CURRICULUM OBJECTIVES
	Candidates should be able to:
6. Plane trigonometry	- relate the periodicity and symmetries of the sine, cosine and tangent functions to the form of their graphs; - use the sine and cosine rules, and the formula $\Delta = \frac{1}{2}ab\sin C$ for the area of a triangle; - understand the definition of a radian, and recall and use the relationship between degrees and radians; - use the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ for the arc length and sector area of a circle.

AS/A level syllabus in the England, P3

2. Trigonometrical functions	- use the six trigonometric functions for angles of any magnitude; - recall and use the exact values of the sine, cosine and tangent of 30°, 45°, 60°, e.g. $\cos 30° = \frac{\sqrt{3}}{2}$; - use the notations $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ to denote the principal values of the inverse trigonometric relations; - use trigonometrical identities for the simplification and exact evaluation of expressions, and select an identity or identities appropriate to the context, showing familiarity in particular with the use of $\frac{\sin\theta}{\cos\theta} = \tan\theta$ and $\frac{\cos\theta}{\sin\theta} = \cot\theta$, $\sin^2\theta + \cos^2\theta = 1$ and equivalent statements, the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$, the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$; the expression of $a\sin\theta + b\cos\theta$ in the forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$; - find all the solutions, within a specified interval, of the equations $\sin(kx) = c$, $\cos(kx) = c$, $\tan(kx) = c$, and of equations easily reducible to these forms.

The TR National Curriculum* -Year 10

1- To comprehend directed angles and measurement of the angles.
2- To gain the ability of the application of the basic concepts of the directed angles and measurement of the angles.
3- To comprehend trigonometric functions.
4- To gain the ability of the application of the trigonometric functions.
5- To comprehend the trigonometric table.
6- To gain the ability of the application of the trigonometric table.
7- To be able to draw the graph of the trigonometric functions.
8- To gain the ability of the application of the graph of the trigonometric functions.
9- To comprehend the sine and cosine theories.
10- To be able to apply the sine and cosine theories.
11- To comprehend the trigonometric ratios of the addition and subtraction of two real numbers.
12- To be able to perform operations by using the trigonometric ratios of the addition and subtraction of two real numbers.
13- To comprehend the trigonometric equations.
14- To gain the ability of the application of trigonometric equations.

I did not translate all sub-objectives as this would add a further 12 pages to this Appendix.

Appendix R: English and Turkish teachers' schemes of works

UK teacher

A-Level, P1

Plane trigonometry

relate the periodicity and symmetries of the sine, cosine and tangent functions to the form of their graphs; P 7.1 nos 1–6, P 7.2 nos 1–3 } Using graphs To solve
B 4.1A+B nos. 1–4

use the sine and cosine rules, and the formula $\Delta = \frac{1}{2}ab\sin C$ for the area of a triangle; B 4.4 P 8.2

understand the definition of a radian, and recall and use the relationship between degrees and radians;
use the formulae $s = r\theta$ and $A = \frac{1}{2}r^2\theta$ for the arc length and sector area of a circle. } P. 14.1 B. 10.3

A-level, P3

Trigonometrical functions		1)-6) 8) 10) 11).
- use the (*) six trigonometric functions for angles of any magnitude;	(*) sin cos tan cosec sec cot	P1 Theory first on Pg 13.
- recall and use the exact values of the sine, cosine and tangent of 30°, 45°, 60°, e.g. $\cos 30° = \frac{\sqrt{3}}{2}$;	B1 Pg 45/46	30°, 45°, 60° Pg 131/2 $\sin^{-1}$ $\cos^{-1}$ $\tan^{-1}$ Pg 133/4 Defns tan, cot Pg 129
- use the notations $\sin^{-1} x$, $\cos^{-1} x$, $\tan^{-1} x$ to denote the principal values of the inverse trigonometric relations;	Pg 36 B1	Then Pg 135 Ex 7.2. 1)-7)
- use trigonometrical identities for the simplification and exact evaluation of expressions, and select an identity or identities appropriate to the context, showing familiarity in particular with the use of $\frac{\sin\theta}{\cos\theta} = \tan\theta$ and $\frac{\cos\theta}{\sin\theta} = \cot\theta$, $\sin^2\theta + \cos^2\theta = 1$ and equivalent statements,	B1 Pg 146 Ch10. 4.3A/4.3B.	P1. Theory first on Pg 138 $\sin^2 + \cos^2 = 1$ Then Pg 140 Ex 7.3 Selectively 4)-7) 8)-10) 13)-22)
the expansions of $\sin(A \pm B)$, $\cos(A \pm B)$ and $\tan(A \pm B)$,	B1 Pg 156 Ex 10.1A/10.1B	Theory on $\sin(A \pm B)$ etc. $\sin 2A$ etc
the formulae for $\sin 2A$, $\cos 2A$ and $\tan 2A$;	B1 Pg 161 Ex 10.2A/10.2B (Not $t = \tan\frac{\theta}{2}$)	Pg 143 Ex 7.4 1)-6) 8)-11)
(*) the expression of $a\sin\theta + b\cos\theta$ in the forms $R\sin(\theta \pm \alpha)$ and $R\cos(\theta \pm \alpha)$;		
- find all the solutions, within a specified interval, of the equations $\sin(kx) = c$, $\cos(kx) = c$, $\tan(kx) = c$, and of equations easily reducible to these forms.	Pg 174 10.4A 1)-3) 10.4B 1) 2) Not general soln.	Pg 146 Ex 7.5. 1)-4) 6)-11) 16)
3. Differentiation then	(*) B2 Pg 72 19.1A/B (using sin/cos graphs)	Theory $R\cos(\theta \pm \alpha)$ Pg 151 Ex 7.7.

TR teacher

HALİDE EDİP ADIVAR LİSESİ MÜDÜRLÜĞÜ

2000 / 2001 DERS YILI MATEMATİK DERSİ LİSE II SINIFI YILLIK DERS PLANI

AY	HAFTA	SAAT	KONULAR	AMAÇLAR	YÖNTEM VE TEKNİKLER	KAYNAK ARAÇ VE GEREÇ	DENEY GEZİ GÖZLEM	ZÜMRE ÖĞR. İŞBİRLİĞİ	ÖDEV KONUSU	YAZILI YOKLAMA	DÜŞÜNCELER
EYLÜL	2 HAFTA	2	Açıyı tanımlama ve açı ölçme birimlerinin kavranması. Birim çemberin tanımlaması.	Yönlü açılar ve açı ölçü birimlerinin kavranması ve bu temel kavramlarla ilgili uygulamalar yapabilme.	Tüme varım Soru-Cevap ve Anlatım	Ders Kitabı ve Diğer Çalışma Kitapları			Amaçlar doğrultusunda konuların pekiştirilmesi için alıştırmalar.		
		2	Açı ölçü birimlerinin birbirine çevrilmesi, çemberde açı ölçme ve uygulamalarla pekiştirme								
		1	Yönlü açı kavramının açıklanması ve uygulanması								
	3 HAFTA	2	Açıların esas ölçülerinin tanımlanması. Dik üçgende dar açı-kenar bağıntıları ile ilgili trigonometrik bağıntıları verme ve açıların trigonometrik değerlerini bulma.	Trigonometrik fonk. kavranması ve bu fonk. ilgili uygulamalar yapabilme							
		2	Trigonometrik fonk. tanımlama. Birim çemberde $0, \frac{\pi}{2}, \pi, \frac{3\pi}{2}, 2\pi$ uzunluğundaki yayların bitim noktalarının koordinatlarını bulma								
		1	Açıların bölgelere göre durumlarının ve işaretlerinin incelenmesi.								
	4 HAFTA	2	Periyot kavramının tanımı ve trigonometrik fonk. periyot bulmayı öğrenme ve uygulama.								
		2	Analitik düzlemde tan ve cotg fonksiyonlarını tanımlama. Trigonometrik fonksiyonları birbiri cinsinden bulup yazma.								
		1	Ölçüsü verilen açıların trigonometrik oranlarını [illegible] bulup yazma								
	5 HAFTA	2	Sinüs fonksiyonlarının grafik çizimlerini yapma ve uygulamalarla pekiştirme.	Trigonometrik fonksiyonların grafiklerinin çizilmesi ve bu grafiklerle ilgili uygulamaların yapılması							
		2	Cos fonksiyonlarının grafik çizimlerini yapma ve uygulamalarla pekiştirme								
		1	Tan, cotg fonk. grafik çizimlerini yapma ve uygulamalarla pekiştirme.								
EKİM	2 HAFTA	2	Sinüs, cos, tan ve cotg fonksiyonlarının [illegible] olduklarını [illegible] ve uygulama.	Cos ve sin teoremlerinin kavranması ve bu teoremlerle ilgili uygulamaların yapılması.							
		2	Sin, cos ters fonk. tanımlarını ve grafiklerini çizme uygulamalarla pekiştirme								
		1	Verilen bir fonksiyonun belirtilen aralıkta grafiğini çizme.								