A few problems on stochastic geometric wave equations

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Abstract

In this thesis we study three problems on stochastic geometric wave equations. First, we prove the existence of a unique local maximal solution to an energy critical stochastic wave equation with multiplicative noise on a smooth bounded domain $\mathcal{D} \subset \mathbb{R}^2$ with exponential nonlinearity. The main ingredients in the proof are appropriate deterministic and stochastic Strichartz inequalities which are derived in suitable spaces.

In the second part, we verify a large deviation principle for the small noise asymptotic of strong solutions to stochastic geometric wave equations. The method of proof relies on applying the weak convergence approach of Budhiraja and Dupuis to SPDEs where solutions are local Sobolev spaces valued stochastic processes.

The final result contained in this thesis concerns the local well-posedness theory for geometric wave equations, perturbed by a fractional Gaussian noise, on one dimensional Minkowski space $\mathbb{R}^{1+1}$ when the target manifold M is a compact Riemannian manifold and the initial data is rough. Here, to achieve the existence and the uniqueness of a local solution we extend the theory of pathwise stochastic integrals in Besov spaces to two dimensional case.

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To my parents Devendra Singh Rana and Madhu Rana

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Author's declaration

I declare that the work in this thesis was carried out in accordance with the requirements of the University's regulations for Research Degree Programmes and I am the sole author. This work was carried out under the supervision of Prof. Zdzisław Brzeźniak, and has not previously been presented for an award at this, or any other, University. All sources are acknowledged and have been listed in the Bibliography.

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CHAPTER 1

INTRODUCTION

This thesis consists of three parts about different problems on stochastic wave equations whose solutions take values in Euclidean space or in any compact Riemannian manifold. Since the motivation to study the stochastic non-linear wave equation and the geometric wave equation with random perturbation is different, we introduce, motivate, and state the main results, that we prove in this thesis, in the following two sections.

1.1 Stochastic non-linear wave equation

The nonlinear wave equations subject to random forcing, called the stochastic nonlinear wave equations (SNLWEs), have been thoroughly studied under the various sets of assumptions due to their numerous applications to physics, relativistic quantum mechanics and oceanography, see for example [22, 23, 36, 40, 41, 45, 51, 52, 59, 60, 62, 91, 102, 109, 112–114, 116, 119, 121–124, 127, 130] and references therein. The case that has attracted the most attention so far seems to be of the stochastic wave equation with initial data belonging to the energy space $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$. For such equations, the nonlinearities can be of polynomial type, for instance the following SNLWE

$$\partial_{tt} u - \Delta u = -u|u|^{p-1} + u|u|^{q-1}\dot{W}, \quad \text{s.t} \quad u(0) = u_0,\ \partial_t u(0) = u_1, \tag{1.1.1}$$

with the suitable exponents $p, q \in [1, \infty)$; see a series of papers by Ondreját [119, 121–124].

Another extensively studied important case is when the initial data is in $L^2(\mathbb{R}^d) \times H^{-1}(\mathbb{R}^d)$ (possibly with weights), see [127, 130] for more details. Similar problems on a bounded domain have been investigated in [27, 52, 119].

In the case of deterministic nonlinear wave equations, see for instance [151], the question of solvability of

$$\partial_{tt} u - \Delta u = -u|u|^{p-1}, \quad \text{s.t} \quad u(0) = u_0,\ \partial_t u(0) = u_1, \tag{1.1.2}$$

when the initial data belongs to $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^d)$, has been investigated in the following three cases:

(i) subcritical, i.e. $p < p_c$; (ii) critical, i.e. $p = p_c$; (iii) supercritical, i.e. $p > p_c$,

where

$$p_c = \frac{d+2}{d-2}. \tag{1.1.3}$$

In the subcritical and the critical cases, the existence and uniqueness of a global solution has been obtained, see for e.g. [74] and [142, 143], respectively. Notice that the proofs in the latter work are based on the so-called "Strichartz inequalities" for the solution of the linear inhomogeneous wave equation, see [75, 150]. Finally, in the supercritical case, the local and global well-posedness of solutions remains an important open problem except for some partial results, see for e.g. [32, 93, 94, 100] and references therein.

Let us note that for $d = 2$ any polynomial nonlinearity is subcritical. Thus, an exponential nonlinearity is a legitimate choice of a critical one. Nonlinearities of exponential type have been considered in many physical models, e.g. a model of self-trapped beams in plasma, see [102], and mathematically in [7, 45, 87, 88, 116]. With the help of suitable Strichartz estimates, the global well-posedness of the Cauchy problem in the energy space $H^1(\mathcal{D}) \times L^2(\mathcal{D})$, with Dirichlet boundary condition on a smooth bounded domain $\mathcal{D} \subset \mathbb{R}^2$, has been proved in [88], in the cases when the initial energy is strictly below or at the threshold given by the sharp Moser-Trudinger inequality. Moreover, an instability result has been shown when the energy of initial data is strictly above the threshold. The critical case on a 3-D smooth bounded domain has been considered in [34, 35] where the authors have proved the existence of a unique global solution to the problem (1.1.2) with Dirichlet and Neumann boundary conditions when the initial data is in the energy space. It is important to highlight that in all the works mentioned above regarding the semi-linear wave equation in domain, the most difficult part in the proof of the local existence result is to establish the required Strichartz type estimates for the solutions to the wave equation in a smooth bounded domain.

In Chapter 2 we extend the existing studies to the wave equation with exponential nonlinearity subject to randomness. In this way, we generalise the above mentioned results of Ondreját for two dimensional domain, by allowing the exponential nonlinearites, as well as the results of Ibrahim, Majdoub, and Masmoudi, see [87, 88], and others to allow randomness. To be precise, we prove the existence of a unique local maximal solution to the following stochastic nonlinear wave equation on a smooth bounded domain $\mathcal{D} \subset \mathbb{R}^2$,

$$\begin{cases} \partial_{tt}u + Au + F(u) = G(u)\dot{W} & \text{in } [0,\infty) \times \mathcal{D} \\ u(0) = u_0, \;\; u_t(0) = u_1 & \text{on } \mathcal{D}, \end{cases} \tag{1.1.4}$$

where A is either $-\Delta_D$ or $-\Delta_N$, i.e. $-A$ is the Laplace-Beltrami operator with Dirichlet or Neumann boundary conditions, respectively; $(u_0, u_1) \in \mathscr{D}(A^{1/2}) \times L^2(\mathcal{D})$; $W = \{W(t) : t \geq 0\}$ is a cylindrical Wiener process on some real separable Hilbert space K such that some orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$ of

K satisfies

$$\sum_{j\in\mathbb{N}} \|f_j\|^2_{L^\infty(\mathcal{D})} < \infty. \tag{1.1.5}$$

Let H, H_A and E be Hilbert and Banach spaces defined as

$$H := L^2(\mathcal{D});\;\; H_A := \mathscr{D}\left(A^{1/2}\right);\;\; E := \mathscr{D}\left(A_{B,q}^{(1-r)/2}\right),$$

where $q \in (1,\infty)$; $r \in [0,1]$; $B = D, N$; $A_{D,q}$ and $A_{N,q}$, respectively, stand for the Dirichlet or the Neumann–Laplacian on Banach space $L^q(\mathcal{D})$. In (1.1.4), for the nonlinearity F and the diffusion coefficient G we assume the following hypotheses.

H.1 Assume that

$$F : H_A \cap E \to H,$$

is a map such that for every $M \in (0,1)$ there exist a constant $C_F > 0$ and $\gamma \in (0,\infty)$ such that the following inequality holds

$$\|F(u) - F(v)\|_H \le C_F \left[1 + \frac{\|u\|_E}{M} + \frac{\|v\|_E}{M}\right]^\gamma \|u - v\|_{H_A},$$

provided

$$u, v \in H_A \cap E \text{ and } \|u\|_{H_A} \le M, \|v\|_{H_A} \le M. \tag{1.1.6}$$

H.2 Assume that

$$G : H_A \cap E \to \gamma(K,H),$$

is a map such that for every $M \in (0,1)$ there exist $\gamma \in (0,\infty)$ and a constant $C_G > 0$ such that

$$\|G(u) - G(v)\|_{\gamma(K,H)} \le C_G \left[1 + \frac{\|u\|_E}{M} + \frac{\|v\|_E}{M}\right]^\gamma \|u - v\|_{H_A},$$

provided u, v satisfy (1.1.6).

In particular, $F(u)$ and $G(u)$ are allowed to be of the form $u\left(e^{4\pi u^2} - 1\right)$, see Lemma 2.4.5, and hence our results cover the recent results obtained in [88].

The strategy to prove the existence of a unique local solution is first to derive the appropriate deterministic and stochastic Strichartz inequalities in suitable spaces and, then, apply the Banach Fixed Point Theorem to obtain the local well-posedness. To construct a maximal solution, by using the obtained local solutions, we rely on the standard methods, see e.g. [22, Theorem 5.4]. In particular, we prove the following result, refer Theorem 2.4.10 for the proof,

Theorem 1.1.1. *Let us assume that (γ, p, q, r) is a quadruple such that $0 < 2\gamma < p$ and (p, q, r) satisfy*

$$2 \le q \le p \le \infty, \quad \textit{and} \quad r = \begin{cases} \dfrac{5}{6} - \dfrac{1}{p} - \dfrac{2}{3q}, & \textit{if} \quad 2 \le q \le 8, \\ 1 - \dfrac{1}{p} - \dfrac{2}{q}, & \textit{if} \quad 8 \le q \le \infty. \end{cases}$$

Then for every $(u_0, u_1) \in \mathscr{D}(A^{1/2}) \times L^2(\mathcal{D})$ *satisfying*

$$\|u_0\|_{\mathscr{D}(A^{1/2})} < 1,$$

there exists a unique local maximal mild solution $u = \{u(t) : t \in [0, \tau)\}$*, to the Problem* (1.1.4)*, in the sense of Definition 2.4.8 for some accessible bounded stopping time* $\tau > 0$.

We would like to stress that, to the best of our knowledge, the present work is the first one to study the wave equations in two dimensional domain with an exponential nonlinearity and an additive or multiplicative noise. We emphasize that result on the stochastic Strichartz estimates for the wave equations generalises the corresponding results for the Schrödinger equation given in [22] and [82]. To underline the significance of the stochastic Strichartz estimates let us mention results by Brzeźniak, Hornung and Weis [17, 18], where such estimates were applied to the nonlinear Schrödinger equation (NLSE). Moreover, our fixed point argument is also similar to Hornung's paper [83] which on the one hand was also inspired by [22] but on the other hand was an improvement to several older NLSE results.

1.2 Geometric wave equation with random perturbation

Many problems in mathematical physics, for example a simplified model for the Einstein equation of general relativity [50], and the non-linear σ-models in particle systems [73], require the target space of the solutions to be a Riemannian manifold. Wave equations whose solutions take values in a Riemannian manifold are known as geometric wave equation (GWE) and the solutions to GWE are called wave maps. We ask the reader to refer [144] for a brilliant introduction of geometric wave equation with comprehensive references.

In brief, given an m-dimensional Riemannian manifold (M, g), a wave map $z : \mathbb{R}^{1+n} \to M$ is critical points of the Lagrangian

$$L(z) := \sum_{\mu=0}^{n} \int_{\mathbb{R}^{1+n}} \langle \partial_\mu z(t,x), \partial^\mu z(t,x) \rangle_{g(z)} \, dt \, dx,$$

where $\langle \cdot, \cdot \rangle_{g(u)}$ is the inner product on tangent space $T_u M$ induced by metric g, the domain $\mathbb{R}^{1+n}$ is the Minkowski space equipped with the flat metric $h = \operatorname{diag}[-1, 1, 1, \cdots, 1]$, and

$$(\partial_0, \partial_1, \cdots, \partial_n) := \left(\partial_t, \partial_{x_1}, \cdots, \partial_{x_n}\right) \text{ and } \left(\partial^0, \partial^1, \cdots, \partial^n\right) := \left(-\partial_t, \partial_{x_1}, \cdots, \partial_{x_n}\right).$$

An alternative description of the wave map is a function $z : \mathbb{R}^{1+n} \to M$ which satisfies the equation

$$\mathbf{D}_t \partial_t z(t,x) = \mathbf{D}_x \partial_x z(t,x), \tag{1.2.1}$$

where $\mathbf{D}_t$ and $\mathbf{D}_x$ are the covariant pull-back derivatives, induced by the Riemannian connection on M, in the bundle $z^{-1}TM$. To understand the operators $\mathbf{D}_t \partial_t$ and $\mathbf{D}_x \partial_x$, called also "acceleration" operators, in a friendly manner we ask a reader to see [25]. Since by the Nash Theorem, see [115], every Riemannian manifold can be embedded by an isometric embedding into some Euclidean space $\mathbb{R}^d$, one can identify M with its image in $\mathbb{R}^d$, i.e. we can assume that M is a submanifold of $\mathbb{R}^d$ with the Riemannian metric induced by the flat metric on $\mathbb{R}^d$. Consequently, it is now well known that the equation (1.2.1) is equivalent to the following classical second order PDE

$$\partial_{tt} z = \Delta_x z - A_z(\partial_x z, \partial_x z) + A_z(\partial_t z, \partial_t z),$$

where A is the second fundamental form of the submanifold $M \subseteq \mathbb{R}^d$. If we choose a local coordinate chart (U, ϕ) on M, then a smooth wave map satisfies the following system of equations, see Chapter 3 for a complete derivation, with $k = 1, \ldots, m$,

$$\Box Z^k(t,x) + \sum_{a,b=1}^{m} \sum_{\mu=0}^{n} \Gamma^k_{ab}(Z(t,x))\, \partial_\mu Z^a(t,x) \partial^\mu Z^b(t,x) = 0, \tag{1.2.2}$$

where $Z = \phi \circ z$, $\Box := \partial_{tt} - \Delta_x$ is the D'Alembertian operator, and $\Gamma^k_{ab} : \phi(U) \to \mathbb{R}$, $a, b, k = 1, \ldots, m$, are the Christoffel symbols on M in the chosen local coordinate (U, ϕ). It is important to observe that, for the purpose of well-posedness theory, the expression (1.2.2) can only be of use if we seek for continuous solution z but we will see later that it is in fact the case for us, see Sections 4.3 and 4.4.1 in Chapter 4, and Section 5.5 in Chapter 5.

The most natural problem to consider for GWEs (1.2.2) is the Cauchy problem with the initial data

$$Z(0,x) = Z_0(x), \quad \text{and} \quad \partial_t Z(0,x) = Z_1(x).$$

The question that has attracted the most attention of researchers so far is to find the minimum value of s such that if the initial data $(Z_0, Z_1) \in H^s(\mathbb{R}^n; \mathbb{R}^m) \times H^{s-1}(\mathbb{R}^n; \mathbb{R}^m)$, then there exist a number $T > 0$ and a unique Z such that

$$Z \in \mathcal{C}\big([0,T]; H^s(\mathbb{R}^n; \mathbb{R}^m)\big) \quad \text{and} \quad \partial_t Z \in \mathcal{C}\big([0,T]; H^{s-1}(\mathbb{R}^n; \mathbb{R}^m)\big).$$

The following theorem, which we will use for comparison purposes, summarizes the available local well-posedness results in the theory of deterministic geometric wave equations, see [92, 96, 98] for more details.

Theorem 1.2.1 (Local theory for GWE). *If $n \geq 2$ and $s > \frac{n}{2}$, then the GWE* (1.2.2) *is locally well-posed for the initial data in $H^s(\mathbb{R}^n; \mathbb{R}^m) \times H^{s-1}(\mathbb{R}^n; \mathbb{R}^m)$. For $n = 1$ case the result has only been proven if $s > \frac{3}{4}$.*

Except for some very special cases the ill-posedness of (1.2.2) has been shown for $s < \frac{n}{2}$ in [56] and [153]. We do not comment on the critical case $s = \frac{n}{2}$ here because it is much more complicated and not related to the problems considered in the thesis.

It also turns out that a solution to (1.2.2) can exhibit a very complex behaviour including blow ups, shrinking and expanding bubbles, see for e.g. [8, 101, 136]. In some cases it has also been proven that the global solutions to GWE eventually decouples into a solution to the associated linear equation and a part which does not disperse i.e. behaves like a soliton, see for e.g. [53, 54, 89]. This phenomena is known as "Soliton Resolution Conjecture" and it is one of the major open problems in the field of wave equation/map. Various concepts of stability of these phenomena, including the stability of soliton solutions, have also been intensely studied. It seems natural to investigate the stability of solutions to the wave maps by investigating the impact of random perturbations and this idea leads to the following stochastic geometric wave equation (SGWE)

$$\partial_{tt} z = \Delta_x z + A_z(\partial_t z, \partial_t z) - A_z(\partial_x z, \partial_x z) + Y_z(\partial_t z, \partial_x z)\,\dot{W}, \tag{1.2.3}$$

where Y is assumed to be of the following form, for any $p \in M$,

$$Y_p(\cdot,\cdot) : T_pM \times T_pM \ni (v_0, v_1) \mapsto Y_p(v_0, v_1) \in T_pM. \tag{1.2.4}$$

The precise definition of the considered noise and the assumptions on Y are provided in Chapters 4 and 5. Note that the equation (1.2.3) is a particular example of the so-called class "stochastic PDEs for manifold-valued processes" which has attracted a great deal of attention due to its wide range of applications in the kinetic theory of phase transitions and the theory of stochastic quantization, see e.g. [12, 15, 16, 23–26, 42, 71] and references therein.

Another motivation for studying equation (1.2.3) comes from the Hamiltonian structure of deterministic wave equation. Deterministic Hamiltonian systems may have infinite number of invariant measures and are not ergodic, see the discussion of this problem in [65]. Characterisation of such systems is a long standing problem. The main idea, which goes back to Kolmogorov-Eckmann-Ruelle, is to choose a suitable small random perturbation such that the solution to stochastic system is a Markov process with the unique invariant measure and then one can select a "physical" invariant measure of the deterministic system by taking the limit of vanishing noise, see for example [55], where this idea is applied to wave maps. A finite dimensional toy example was studied in [6].

To compare Theorem 1.2.1 with the existing results in the theory of stochastic wave equations with values in Riemannian manifolds, let us briefly outline the available results from the literature in the stochastic setting. To the best of our knowledge, SGWEs Cauchy problem have only been studied in a series of three papers by Z. Brzeźniak and M. Ondreját, see [23, 24, 26]. The first attempt to study a manifold valued wave equation with stochastic perturbation was made in [23] where authors proposed two rigorous formulations of the SGWE and proved the equivalence between them. They were also able to establish, under some technical assumptions on the coefficients, the existence and the uniqueness of global strong solutions for SGWEs on the one dimensional Minkowski space $\mathbb{R}^{1+1}$ for the initial data $(z_0, z_1) \in H^2_{\text{loc}} \times H^1_{\text{loc}}(\mathbb{R}; TM)$, when the target manifold M is an arbitrary compact Riemannian manifold and the random forcing was modelled by a spatially homogeneous Wiener process whose spectral measure has finite moment up to order 2.

In the subsequent paper [24] the above mentioned results from [23] were improved in the case when domain of considered SGWE is restricted to $\mathbb{R}^{1+1}$ but the target manifold is free of any restriction. Improvement was in the sense that assumptions on the spectral measure of the considered spatially homogeneous Wiener process were weaker than in [23] and the regularity of the initial data was lowered to $H^1_{\text{loc}} \times L^2_{\text{loc}}(\mathbb{R}; TM)$.

In the final paper [26] the authors proved the existence of a global weak solution of SGWEs, when the domain Minkowski space is of arbitrary dimension and the target manifold M is a compact Riemannian homogeneous space. From the stochastic point of view their method was not standard since it did not rely on any martingale representation theorem. Compared to [23], the assumptions on the spectral measure and on the space regularity of initial data were weakened but at the cost of the solution being only $H^1_{\text{loc}} \times L^2_{\text{loc}}(\mathbb{R}^{1+n}; TM)$-valued weakly continuous process.

Hence, by comparing the above paragraph and Theorem 1.2.1, it is quite visible that there is a huge gap between the optimal results with respect to well-posedness theory for the deterministic GWE and the results available for SGWE. In Chapter 5 we take a first modest step to fill this gap, in one dimensional case, to generalize the pioneering work of Keel and Tao [92] to the stochastic setting. With respect to [23, 24, 26] we extend the study to the GWEs with low regularity initial data and fractional (both in time and space) Gaussian noise. To be precise, we consider the Cauchy problem in the following form

$$\begin{cases} \partial_{tt} z = \partial_{xx} z - A_z(\partial_x z, \partial_x z) + A_z(\partial_t z, \partial_t z) + \kappa(z)\dot{\xi}, \\ z(0,x) = z_0(x), \quad \text{and} \quad \partial_t z(0,x) = z_1(x), \end{cases} \tag{1.2.5}$$

where $(z_0, z_1) \in H^s_{\text{loc}} \times H^{s-1}_{\text{loc}}(\mathbb{R}; TM)$, $\kappa : M \to TM$ is any sufficiently smooth vector field, and ξ is a suitable stochastic perturbation. In a given local coordinate chart (U, ϕ) on M, the SGWE Cauchy problem (1.2.5) takes the following form

$$\begin{cases} \Box Z(t,x) = -\sum_{a,b=1}^{m} \sum_{\mu=0}^{1} \Gamma_{ab}(Z)\partial_\mu Z^a \partial^\mu Z^b + \sigma(Z)\dot{\xi}, \\ Z(0,x) = Z_0(x) \in \mathbb{R}^m, \quad \text{and} \quad \partial_t Z(0,x) = Z_1(x) \in T\mathbb{R}^m, \end{cases} \tag{1.2.6}$$

where, $\sigma(\phi)$ is defined by

$$\sigma(\phi(p)) := (d_p\phi)(\kappa(p)) \in T_{\phi(p)}\mathbb{R}^m \simeq \mathbb{R}^m, \quad p \in U.$$

Before delving into more details about the stochastic generalization of [92] that we have achieved, we would like to highlight that, to the best of our knowledge, there is no literature available on the stability of wave maps under random excitations. In particular, the effect of a noise on the decoupling of global solutions to GWE is completely unknown. Hence, in Chapter 4, we take the opportunity to carry out the first step in this direction and establish the validity of a large deviation principle for the small noise asymptotic of solutions to SGWEs.

To introduce the model in a precise manner, that we consider in Chapter 4, let M be a m-dimensional compact Riemannain manifold and TM be the tangent bundle over M whose fibre at $p \in M$ is equal to the tangent space T_pM. Let us recall that, due to the celebrated Nash isometric embedding theorem [115], there exists $n \in \mathbb{N}$ such that M is a submanifold of $\mathbb{R}^n$. We are concerned with large deviations principles (LDP) of the solutions to the following one-dimensional stochastic wave equation, taking values in M,

$$\partial_{tt} u^\varepsilon = \partial_{xx} u^\varepsilon + A_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_t u^\varepsilon) - A_{u^\varepsilon}(\partial_x u^\varepsilon, \partial_x u^\varepsilon) + \sqrt{\varepsilon}\, Y(u^\varepsilon)\, \dot{W}, \tag{1.2.7}$$

with the parameter $\varepsilon \in (0,1]$ approaching zero. Here W is a spatially homogeneous Wiener process on $\mathbb{R}$ with a spectral measure μ satisfying

$$\int_{\mathbb{R}} (1+|x|^2)^2 \mu(dx) < \infty.$$

The diffusion coefficient Y in the equation (1.2.7) is a smooth vector field such that it has an smooth extension on $\mathbb{R}^n$, denote again by Y, which is defined by using [23, Propositon 3.9], and satisfies

D.1 there exists a compact set $K_Y \subset \mathbb{R}^n$ such that $Y(p) = 0$ if $p \notin K_Y$.

D.2 for $q \in O$, $Y(\Upsilon(q)) = \Upsilon'(q) Y(q)$, where Υ is a smooth compactly supported function $\Upsilon : \mathbb{R}^n \to \mathbb{R}^n$ which satisfies certain properties, see Lemma 4.2.5 for details.

D.3 for some $C_Y > 0$

$$|Y(p)| \le C_Y(1+|p|), \quad \left|\frac{\partial Y}{\partial p_i}(p)\right| \le C_Y, \text{ and } \left|\frac{\partial^2 Y}{\partial p_i \partial p_j}(p)\right| \le C_Y,$$

for $p \in K_Y, i, j = 1, \ldots, n$.

Let $(\Omega, \mathscr{F}, \mathbb{P})$ be a probability space with an increasing family $\mathbb{F} := \{\mathscr{F}_t, 0 \le t \le T\}$ of the sub-σ-fields of $\mathscr{F}$ satisfying the usual conditions. Let us set notation $\mathcal{X}_T$ for the following Polish space

$$\mathcal{X}_T := \mathcal{C}\left([0,T]; H^2_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)\right) \times \mathcal{C}\left([0,T]; H^1_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)\right).$$

The main result of Chapter 4 is as follows.

Theorem 1.2.2. *Let (u_0, v_0) be $\mathfrak{F}_0$-measurable random variable with values in $H^2_{loc} \times H^1_{loc}(\mathbb{R}, TM)$. The family of laws $\{\mathscr{L}(z^\varepsilon) : \varepsilon \in (0,1]\}$ on $\mathcal{X}_T$, where $z^\varepsilon := (u^\varepsilon, \partial_t u^\varepsilon)$ is the unique solution to* (1.2.7), *with initial data (u_0, v_0), satisfies the large deviation principle with rate function $\mathcal{I}$ defined in* (4.4.1).

Our proof of verifying the LDP relies on the weak convergence method introduced in [30], which is mainly based on a variational representation formula for certain functionals of the driving infinite dimensional Brownian motion, and have appeared in [16, 48, 63]. With respect to the available literature, Zhang's paper [163], on the LDP for stochastic beam equation, is the nearest to our work but instead of the weak convergence method he follows the classical approach based on the fundamental ideas from [2] and [131]. The main technical difficulty that arises here is to follow the local Sobolev spaces setup, which are only Fréchet spaces but are required given the model and to prove the conditions required to apply the result of [30].

It is relevant to emphasize that the method we follow here can be used to general beam equations, as in [21], and nonlinear wave equation with polynomial nonlinearity respectively, with spatially homogeneous noise in local Sobolev spaces which will generalize the result of [124, 163]. Moreover, such a generalization should lead to extend the work of Martirosyan [111], which is on bounded domain, to include the study of large deviations principle for the family of stationary measures generated by the flow of stochastic wave equation, with multiplicative white noise, in non-local Sobolev spaces over whole domain $\mathbb{R}^n$. It is important to mention that recently in [140] the authors have established a LDP for a general class of Banach space valued stochastic differential equations by a different approach but still based on Laplace principle. However, they do not cover our case because the wave operator does not imply the existence of a compact C_0-semigroup.

Coming back to the well-posedness theory for (1.2.6), in Chapter 5 we strive to study the existence and uniqueness of local (in time and space) solution to (1.2.6) with the initial data in

$$(Z_0, Z_1) \in H^s_{\text{loc}}(\mathbb{R};\mathbb{R}^m) \times H^{s-1}_{\text{loc}}(\mathbb{R};\mathbb{R}^m), \qquad s \in \left(\frac{3}{4}, 1\right).$$

What concerns the method of proving the existence of a solution to (1.2.6), as mentioned by Walsh in [159], an efficient way to simplify the computations of required estimates, in the use of the Banach Fixed Point Theorem, is to switch the coordinate-axes of (t,x)-variables to the null coordinates, see also [92] and [108] for the deterministic counterpart. However, it is not clear at all to us how to use the Walsh approach to stochastic PDEs (SPDEs), see also [59], in our setting with rough initial data (i.e. $s < 1$). On the other hand, from the theory of fractional integrals, see [160, 161], and rough paths, refer [107], it seems plausible to work with the pathwise approach to achieve our aim. In recent years, the pathwise approach has become extremely popular in SPDEs community as well due to the spectacular results of Gubinelli et al. and Hairer, see [70, 77, 78] for more details and recent advancements.

Moreover, in a few cases for the linear and non-linear wave equation driven by fractional Brownian motion, the pathwise appraoch has led to optimal results, see [4, 37, 133]. Hence in Chapter 5, motivated by the above discussion, we consider the fractional (both in time and space) Gaussian noise as a random perturbation and extend the recently developed theory of pathwise stochastic integrals in Besov spaces, refer [132], to two dimensional case which allows us to achieve the local well-posedness (in the sense of Definition 5.5.2) of (1.2.6).

To state assumptions and the main result of Chapter 5, we set the new coordinates to (α,β) and avoid writing ϕ for simplicity. We consider the following Cauchy problem, written in (α,β) is the following,

$$\begin{cases} \Diamond u = \mathcal{N}(u) + \sigma(u)\dot{\zeta}, \\ u(\alpha,-\alpha) = u_0(\alpha) \quad \text{and} \quad \dfrac{\partial u}{\partial \alpha}(\alpha,-\alpha) + \dfrac{\partial u}{\partial \beta}(\alpha,-\alpha) = u_1(\alpha). \end{cases} \tag{1.2.8}$$

Here $\Diamond u(\alpha,\beta) := 4\dfrac{\partial^2 u}{\partial\alpha\partial\beta}$ and

$$\mathcal{N}(u) := 4\sum_{a,b=1}^{m} \Gamma_{ab}(u)\frac{\partial u^a}{\partial \alpha}\frac{\partial u^b}{\partial \beta}.$$

The noise ζ is a fractional Brownian sheet (fBs), with Hurst indices greater than $\frac{1}{2}$, on $\mathbb{R}^2$, and $\sigma \in \mathcal{C}_b^3(\mathbb{R}^2)$. We prove the equivalence between (1.2.6) and (1.2.8) in Section 5.3.

As usual in the SPDE theory, the stochastic geometric wave equation (1.2.8) is understood in the following integral form

$$u = S(u_0, u_1) + \diamond^{-1}\mathcal{N}(u) + \diamond^{-1}\sigma(u)\dot{\zeta}, \tag{1.2.9}$$

where, for $(\alpha, \beta) \in \mathbb{R}^2$,

$$[S(u_0, u_1)](\alpha, \beta) := \frac{1}{2}\left[u_0(\alpha) + u_0(-\beta)\right] + \frac{1}{2}\int_{-\beta}^{\alpha} u_1(r)\, dr, \tag{1.2.10}$$

$$\left[\diamond^{-1}\mathcal{N}(u)\right](\alpha, \beta) := \frac{1}{4}\int_{-\beta}^{\alpha}\int_{-a}^{\beta} \mathcal{N}(u(a, b))\, db\, da, \tag{1.2.11}$$

and

$$\left[\diamond^{-1}\sigma(u)\dot{\zeta}\right](\alpha, \beta) := \frac{1}{4}\int_{-\beta}^{\alpha}\int_{-a}^{\beta} \sigma(u(a, b))\dot{\zeta}(a, b)\, db\, da. \tag{1.2.12}$$

In Section 5.4, we give a precise meaning (at least locally in some suitable space) to the above expressions in (1.2.10) - (1.2.12) which are merely some formal notation here. The proof of a local well-posedness result for a closely related problem to (1.2.9) is given in Section 5.5. Let $\eta \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$ be a cut-off function which satisfy

$$\eta(-x) = \eta(x), \qquad 0 \le \eta(x) \le 1, \qquad \eta(x) = \begin{cases} 1, & \text{if } |x| \le 2, \\ 0, & \text{if } |x| \ge 4. \end{cases} \tag{1.2.13}$$

Similarly, we define χ. Let us set

$$\mathbb{H}^{s,\delta} := H^s_\alpha H^\delta_\beta(\mathbb{R}^2) \cap H^s_\beta H^\delta_\alpha(\mathbb{R}^2),$$

where the product Sobolev space $H^s_\alpha H^\delta_\beta(\mathbb{R}^2)$ is defined in Section 5.1. The following theorem is the main result of Chapter 5 whose proof is based on the Banach Fixed Point Theorem, see the proof of Theorem 5.5.3 for details.

Theorem 1.2.3. *Let η, χ as defined in* (1.2.13) *and ψ be a bump function which is non zero on the support of χ, η and $\int_{\mathbb{R}} \psi(x)\, dx = 1$. Assume $s, \delta \in \left(\frac{3}{4}, 1\right)$ such that $\delta \le s$ and $(u_0, u_1) \in H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$. There exist a $R_0 \in (0,1)$ and a $\lambda_0 := \lambda_0(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}}, R_0) \gg 1$ such that for every $\lambda \ge \lambda_0$ there exists a unique $u := u(\lambda, R_0) \in \mathbb{B}_{R_0}$, where $\mathbb{B}_R := \left\{u \in \mathbb{H}^{s,\delta} : \|u\|_{\mathbb{H}^{s,\delta}} \le R\right\}$, which satisfies the following integral equation*

$$\begin{aligned} u(\alpha, \beta) = \eta(\lambda\alpha)\eta(\lambda\beta)\Big([S(\chi(\lambda)(u_0 - \bar{u}_0^\lambda), \chi(\lambda)u_1)](\alpha, \beta) + [\diamond^{-1}\mathcal{N}(u)](\alpha, \beta) \\ + [\diamond^{-1}\sigma(u)\dot{\zeta}](\alpha, \beta)\Big), \qquad (\alpha, \beta) \in \mathbb{R}^2. \end{aligned}$$

Here $\bar{u}_0^\lambda$ is defined as

$$\bar{u}_0^\lambda := \int_{\mathbb{R}} u_0\left(\frac{y}{\lambda}\right)\psi(y)\, dy.$$

CHAPTER 2

Energy critical 2-D stochastic wave equation with exponential nonlinearity in a bounded domain

We prove the existence and uniqueness of a local maximal solution to an energy critical stochastic wave equation with multiplicative noise on a smooth bounded domain $\mathcal{D} \subset \mathbb{R}^2$ with exponential nonlinearity. First, we derive the appropriate deterministic and stochastic Strichartz inequalities in suitable spaces and, then, we show the local well-posedness result for small initial data.

The organization of the present chapter is as follows. In Section 2.1, we introduce our notation and provide the required definitions. In Sections 2.2 and 2.3, we derive the required inhomogeneous and stochastic Strichartz estimates, respectively, by the methods introduced in [34, 35] and [22]. Section 2.4 is devoted to the estimates which are sufficient to apply the Banach Fixed Point Theorem in a suitable space and the proof of the existence and uniqueness of a local maximal solution is given. In Subsection 2.5.1, we provide a rigorous justification of our adopted definition of a local mild solution. We conclude the chapter with a brief Subsection 2.5.2, in which we state a relation, without proof, of two natural definitions of a mild solution for the considered SPDE (2.4.4).

2.1 Notation and conventions

In this section we introduce the notation and some basic estimates that we use throughout the chapter. For any two non-negative quantities a and b, we write $a \lesssim b$ if there exists a universal constant $c > 0$ such that $a \leq cb$, and we write $a \simeq b$ when $a \lesssim b$ and $b \lesssim a$. In case we want to emphasize the dependence of c on some parameters $a_1, \dots, a_k$, then we write, respectively, $\lesssim_{a_1,\dots,a_k}$ and $\simeq_{a_1,\dots,a_k}$. For any two Banach spaces X, Y, we denote by $\mathcal{L}(X, Y)$ the space of linear bounded operators $L : X \to Y$.

To state the definitions of required spaces here, we denote by E and H a separable Banach and Hilbert space, respectively. Let $T > 0$ be a positive real number.

2.1.1 Function spaces and interpolation theory

For the next few basic definitions and remarks, which are included here for the reader's convenience, from function spaces and interpolation theory we are borrowing the notation from [155]. We denote the set of natural numbers $\{1, 2, \dots\}$ by $\mathbb{N}$ and by $\mathbb{N}_0$ we mean $\mathbb{N} \cup \{0\}$.

By $L^q(\mathcal{D})$, for $q \in [1, \infty)$ and a bounded smooth domain $\mathcal{D}$ of $\mathbb{R}^2$, we denote the classical real Banach space of all (equivalence classes of) $\mathbb{R}$-valued q-integrable functions on $\mathcal{D}$. The norm in $L^q(\mathcal{D})$ is given by

$$\|u\|_{L^q(\mathcal{D})} := \left(\int_{\mathcal{D}} |u(x)|^q \, dx\right)^{\frac{1}{q}}, \qquad u \in L^q(\mathcal{D}).$$

By $L^\infty(\mathcal{D})$ we denote the real Banach space of all (equivalence classes of) Lebesgue measurable essentially bounded $\mathbb{R}$-valued functions defined on $\mathcal{D}$ with the norm

$$\|u\|_{L^\infty(\mathcal{D})} := \text{ess sup}\,\{|u(x)| : x \in \mathcal{D}\}, \qquad u \in L^\infty(\mathcal{D}).$$

We set, by $\mathcal{C}([0,T];H)$, the real Banach space of all H-valued continuous functions $u : [0,T] \to H$ endowed with the norm

$$\|u\|_{\mathcal{C}([0,T];H)} := \sup_{t \in [0,T]} \|u(t)\|_H, \qquad u \in \mathcal{C}([0,T];H).$$

We also define, for any $p \in [1, \infty)$, $L^p(0,T;E)$ as the real Banach space of all (equivalence classes of) E-valued measurable functions $u : [0,T] \to E$ with the norm

$$\|u\|_{L^p(0,T;E)} := \left(\int_0^T \|u(t)\|_E^p \, dt\right)^{\frac{1}{p}}, \qquad u \in L^p(0,T;E).$$

For any $s \in \mathbb{R}$ and $q \in (1, \infty)$, the Sobolev space $H^{s,q}(\mathbb{R}^2)$ is defined as

$$H^{s,q}(\mathbb{R}^2) := \left\{ f \in \mathcal{S}'(\mathbb{R}^2) : \|f\|_{H^{s,q}(\mathbb{R}^2)} := \left\| \mathcal{F}^{-1}\left(1 + |x|^2\right)^{\frac{s}{2}} \mathcal{F} f \right\|_{L^q(\mathbb{R}^2)} < \infty \right\},$$

where $\mathcal{F}$ stands for the Fourier transform and $\mathcal{S}'(\mathbb{R}^2)$ denotes the space of tempered distributions, which is dual to $\mathcal{S}(\mathbb{R}^2)$ (set of all real-valued rapidly decreasing infinitely differentiable functions defined on $\mathbb{R}^2$). The restriction, in the distributional sense, of $H^{s,q}(\mathbb{R}^2)$ to $\mathcal{D}$, is denoted by $H^{s,q}(\mathcal{D})$. With the following norm

$$\|f\|_{H^{s,q}(\mathcal{D})} := \inf_{\substack{g|_{\mathcal{D}} = f \\ g \in H^{s,q}(\mathbb{R}^2)}} \|g\|_{H^{s,q}(\mathbb{R}^2)}, \quad f \in H^{s,q}(\mathcal{D}),$$

$H^{s,q}(\mathcal{D})$ is a Banach space. We denote the completion of $\mathscr{C}_0^\infty(\mathcal{D})$ (set of smooth functions defined over $\mathcal{D}$ with compact support) in $H^{s,q}(\mathcal{D})$ by $\mathring{H}^{s,q}(\mathcal{D})$.

Throughout the whole chapter, we denote by A the Dirichlet or the Neumann–Laplacian on Hilbert space $L^2(\mathcal{D})$ with domains, respectively, defined by

$$\mathscr{D}(-\Delta_D) = H^{2,2}(\mathcal{D}) \cap \mathring{H}^{1,2}(\mathcal{D}),$$
$$\mathscr{D}(-\Delta_N) = \left\{ f \in H^{2,2}(\mathcal{D}) : \partial_\nu f\big|_{\partial\mathcal{D}} = 0 \right\}.$$

Here ν denotes the outward normal unit vector to $\partial\mathcal{D}$. It is well known, see for example [154], that the Dirichlet Laplacian $(-\Delta_D, \mathscr{D}(-\Delta_D))$ is a positive self-adjoint operator on $L^2(\mathcal{D})$ and there exists an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ of $L^2(\mathcal{D})$ which consist of eigenvectors of $-\Delta_D$. If we denote the corresponding eigenvalues by $\{\lambda_j^2\}_{j\in\mathbb{N}}$, then we have

$$-\Delta_D e_j = \lambda_j^2 e_j; \quad e_j \in \mathscr{D}(-\Delta_D), \forall j \geq 1; \quad 0 < \lambda_1^2 \leq \lambda_2^2 \leq \dots \quad \text{and} \quad \lambda_n^2 \xrightarrow[n\to\infty]{} \infty.$$

In the case of the Neumann Laplacian, $(-\Delta_N, \mathscr{D}(-\Delta_N))$ is a non-negative self-adjoint operator on $L^2(\mathcal{D})$ and there exists an orthonormal basis $\{e_j\}_{j\in\mathbb{N}}$ of $L^2(\mathcal{D})$ which consist of eigenvectors of $-\Delta_N$. Moreover, if we denote the corresponding eigenvalues by $\{\lambda_j^2\}_{j\in\mathbb{N}}$, then we have

$$-\Delta_N e_j = \lambda_j^2 e_j; \quad e_j \in \mathscr{D}(-\Delta_N), \forall j \geq 1; \quad \lambda_n^2 \xrightarrow[n\to\infty]{} \infty,$$
$$\text{and} \qquad 0 = \lambda_1^2 = \lambda_2^2 = \dots = \lambda_{m_0}^2 < \lambda_{m_0+1}^2 \leq \lambda_{m_0+2}^2 \leq \dots,$$

for some $m_0 \in \mathbb{N}$. Since we work with both the operators simultaneously, we denote the pair of operator and its domain by $(A, \mathscr{D}(A))$ and make the distinction wherever required.

From the functional calculus of self-adjoint operators, see for instance [162], it is known that, the power A^s of operator A, for every $s \in \mathbb{R}$, is well-defined and self-adjoint. It is also known that, for any $s \in \mathbb{R}$, $\mathscr{D}(A^{s/2})$, where $A = -\Delta_D$ or $A = -\Delta_N$, with the following norm

$$\|u\|_{\mathscr{D}(A^{s/2})} := \left(\sum_{j\in\mathbb{N}} (1+\lambda_j^2)^s |\langle u, e_j\rangle_{L^2(\mathcal{D})}|^2 \right)^{1/2},$$

is a Hilbert space. For $s \in (0,2)$ the space $\mathscr{D}(A^{s/2})$ is equal to the following complex interpolating space, refer [155, 2.5.3/(13)],

$$\mathscr{D}(A^{s/2}) = \left[L^2(\mathcal{D}), \mathscr{D}(A)\right]_{s/2}.$$

To derive the Strichartz estimate in a suitable space, we also need to consider the Dirichlet or the Neumann–Laplacian on Banach space $L^q(\mathcal{D})$, $q \in (1,\infty)$, denoted by $A_{D,q}$ and respectively $A_{N,q}$, with domains, respectively,

$$\mathscr{D}(A_{D,q}) = H^{2,q}(\mathcal{D}) \cap \mathring{H}^{1,q}(\mathcal{D}), \tag{2.1.1}$$
$$\mathscr{D}(A_{N,q}) = \left\{ f \in H^{2,q}(\mathcal{D}) : \partial_\nu f\big|_{\partial\mathcal{D}} = 0 \right\}. \tag{2.1.2}$$

Note that $A_{D,2} = -\Delta_D$ and $A_{N,2} = -\Delta_N$.

Under some reasonable assumptions on the regularity of the domain $\mathcal{D}$, one can show that both of these operators have very nice analytic properties. In particular both have bounded imaginary powers with exponent strictly less than $\frac{\pi}{2}$ (and thus both $-A_{D,q}$ and $-A_{N,q}$ generate analytic semigroups on the space $L^q(\mathcal{D})$). As in [155], one can define the fractional powers $(A_{B,q})^{r/2}$, where as below $B = D$ or $B = N$. For $q \in [2,\infty)$ and $\theta \geq 0$, we define domain $\mathscr{D}((A_{B,q})^{\theta})$ by

$$\mathscr{D}((A_{B,q})^{\theta}) = \left\{ u \in L^q(\mathcal{D}) : (\mathrm{id} + A_{B,q})^{\frac{\theta}{2}} u \in L^q(\mathcal{D}) \right\},$$

which is a Banach space with the norm $\|u\|_{\mathscr{D}((A_{B,q})^{\theta})} = \|(\mathrm{id} + A_{B,q})^{\frac{\theta}{2}} u\|_{L^q(\mathcal{D})}$.

Next, we fix the notation for the subspaces of $H^{s,q}(\mathcal{D})$ which are determined by differential operators. Fix $k \in \mathbb{N}$ and let

$$B_j f(x) = \sum_{|\alpha| \leq m_j} b_{j,\alpha}(x) D^{\alpha} f(x), \qquad b_{j,\alpha} \in \mathcal{C}^{\infty}(\partial\mathcal{D}),$$

for $j = 1, \dots, k$, be differential operators on $\partial\mathcal{D}$. Then $\{B_j\}_{j=1}^k$ is said to be a normal system iff

$$0 \leq m_1 < m_2 < \cdots < m_k,$$

and for every vector ν_x which is normal to $\partial\mathcal{D}$ at x the following holds

$$\sum_{|\alpha| = m_j} b_{j,\alpha}(x) \nu_x^{\alpha} \neq 0, \quad j = 1, \dots, k,$$

where for $\alpha \in \mathbb{N}_0^2$ and $y \in \mathbb{R}^2$, $y^{\alpha} = \Pi_i y_i^{\alpha_i}$.

Definition 2.1.1. Let $\{B_j\}_{j=1}^k$ be a normal system as defined above for some $k \in \mathbb{N}$. For $s > 0$, $q \in (1,\infty)$, we set

$$H^{s,q}_{\{B_j\}}(\mathcal{D}) := \left\{ f \in H^{s,q}(\mathcal{D}) : B_j f\big|_{\partial\mathcal{D}} = 0 \text{ whenever } m_j < s - \frac{1}{q} \right\}.$$

By taking the suitable choice of normal system $\{B_j\}$ in the Definition 2.1.1, for $s > 0$ and $q \in (1,\infty)$, we define

$$H^{s,q}_D(\mathcal{D}) := \left\{ f \in H^{s,q}(\mathcal{D}) : f\big|_{\partial\mathcal{D}} = 0 \text{ if } s > \frac{1}{q} \right\},$$

and

$$H^{s,q}_N(\mathcal{D}) := \left\{ f \in H^{s,q}(\mathcal{D}) : \nu_x \cdot \nabla f\big|_{\partial\mathcal{D}} = 0 \text{ if } s > 1 + \frac{1}{q} \right\}.$$

Since the $\mathring{H}^{1,q}(\mathcal{D})$ spaces can also be defined by using $f\big|_{\partial\mathcal{D}} = 0$ condition which appears in (2.1.1) and the Neumann boundary condition appearing in (2.1.2) can be written as $\nu_x \cdot \nabla f\big|_{\partial\mathcal{D}} = 0$, we expect to have some relation between the spaces $H^{s,q}_B(\mathcal{D})$ and $\mathscr{D}((A_{B,q})^{s/2})$ where $A = -\Delta_B$ with $B = D$ or $B = N$. The next stated result from the theory of functions spaces, see [155, Theorem 4.3.3], provides a suitable range of s for which the function spaces $H^{s,q}_B(\mathcal{D})$ and $\mathscr{D}((A_{B,q})^{s/2})$ are equivalent.

Lemma 2.1.2. *With our notation from this section, we have the following*

1. *For* $s \in (0,2) \setminus \left\{1 + \frac{1}{q}\right\}$,

$$H^{s,q}_N(\mathcal{D}) = \mathscr{D}((A_{N,q})^{s/2}).$$

2. *For* $s \in (0,2) \setminus \left\{\frac{1}{q}\right\}$,

$$H_D^{s,q}(\mathcal{D}) = \mathscr{D}((A_{D,q})^{s/2}).$$

We close this subsection with the following well known identity

$$\mathscr{D}(\sqrt{-\Delta_D}) = \mathring{H}^{1,2}(\mathcal{D}) \qquad \text{and} \qquad \mathscr{D}(\sqrt{-\Delta_N}) = H^{1,2}(\mathcal{D}).$$

2.1.2 Stochastic analysis

Now we state a few required definitions from the theory of stochastic analysis, refer [11] and [29] for more details. Let $(\Omega, \mathscr{F}, \mathbb{F}, \mathbb{P})$, where $\mathbb{F} := \{\mathscr{F}_t : t \geq 0\}$, be a filtered probability space which satisfies the usual assumptions, that is, the filtration $\mathbb{F}$ is right continuous and the σ-field $\mathscr{F}_0$ contains all $\mathbb{P}$-null sets of $\mathscr{F}$. As the noise we consider a cylindrical $\mathbb{F}$-Wiener process on a real separable Hilbert space K, see [29, Definition 4.1]. We denote by $L^p(\Omega, \mathscr{F}, \mathbb{P}; E)$, for $p \in [1, \infty)$, the Banach space of all (equivalence classes of) E-valued random variables equipped with the norm

$$\|X\|_{L^p(\Omega)} = \left(\mathbb{E}\left[\|X\|_E^p\right]\right)^{\frac{1}{p}}, \qquad X \in L^p(\Omega, \mathscr{F}, \mathbb{P}; E),$$

where $\mathbb{E}$ is the expectation operator w.r.t $\mathbb{P}$.

Definition 2.1.3. For any K, a separable Hilbert space, the set of γ-radonifying operators, denoted by $\gamma(K,E)$, consists of all bounded operators $\Lambda : K \to E$ such that the series $\sum_{j=1}^\infty \beta_j \Lambda(f_j)$ converges in $L^2(\Omega, \mathscr{F}, \mathbb{P}; E)$ for some (and then for every) orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$ of K and some (and then for every) sequence $\{\beta_j\}_{j\in\mathbb{N}}$ of i.i.d. $N(0,1)$ real random variables on probability space $(\Omega, \mathscr{F}, \mathbb{P})$. We set

$$\|\Lambda\|_{\gamma(K,E)} := \left(\mathbb{E}\left\| \sum_{j\in\mathbb{N}} \beta_j \Lambda(f_j)\right\|_E^2\right)^{\frac{1}{2}}.$$

One may prove that $\|\cdot\|_{\gamma(K,E)}$ is a norm, and $(\gamma(K,E), \|\cdot\|_{\gamma(K,E)})$ is a separable Banach space. Note that if $K = \mathbb{R}$, then $\gamma(\mathbb{R}, E)$ can be identified with E. Furthermore, it is known that, see for e.g. [117], $\Lambda \in \gamma(K,E)$ iff the cylindrical measure $\Lambda(\gamma_K)$ is σ-additive, where γ_K is the canonical cylindrical Gaussian measure on K.

Let $I = [0,T]$, for some $T > 0$, or $\mathbb{R}_+$. A stochastic process $\xi : I \times \Omega \to E$ is called progressively measurable (with respect to the filtration $\mathbb{F}$) if for every $t \in I$ the mapping

$$[0,t] \times \Omega \ni (s,\omega) \mapsto \xi(s,\omega) \in E,$$

is $\mathscr{B}([0,t]) \times \mathscr{F}_t$-measurable, where $\mathscr{B}([0,t])$ is the σ-algebra of Borel subsets of $[0,t]$.

A subset Γ of $I \times \Omega$ is progressive if the process $\xi = \mathbb{1}_\Gamma$ is progressively measurable. The family of progressive sets is a σ-algebra on $I \times \Omega$ which we will denote by $\mathscr{B}\mathbb{F}$. To introduce the notion of progressively measurable local process, it is useful to remember that ξ is progressive if and only if the map

$$I \times \Omega \ni (s,\omega) \mapsto \xi(s,\omega) \in E,$$

is measurable with respect to $\mathscr{B}\mathbb{F}$, see [135, Definition I.4.7].

An $\mathbb{F}$-stopping time τ is a map on Ω with values in $[0,\infty]$ such that for every t, $\{\tau \le t\} \in \mathscr{F}_t$. For any given $\mathbb{F}$-stopping time τ, we set

$$\Omega_t(\tau) := \{\omega \in \Omega : t < \tau(\omega)\} \qquad \text{and} \qquad [0,\tau) \times \Omega := \{(t,\omega) \in [0,\infty) \times \Omega : 0 \le t < \tau(\omega)\}.$$

Assume that $\tau : \Omega \to [0,\infty)$ is a random variable. Let us consider the process $\xi = \mathbb{1}_{[0,\tau)}$ defined by

$$\xi(t,\omega) := \begin{cases} 1, & \text{if } t < \tau(\omega), \\ 0, & \text{otherwise.} \end{cases}$$

Then it is easy to see that the process ξ is $\mathbb{F}$-adapted iff τ is a $\mathbb{F}$-stopping time. Hence, since for a given stopping time τ, every sample path of $\xi = \mathbb{1}_{[0,\tau)}$ is left continuous, ξ is progressively measurable with respect to the filtration $\mathbb{F}$. Motivated from above, a local stochastic process $\xi : [0,\tau) \times \Omega \to E$ is called $\mathbb{F}$-progressively measurable iff the process $\xi \mathbb{1}_{[0,\tau)}$ defined by

$$[\xi \mathbb{1}_{[0,\tau)}](t,\omega) := \begin{cases} \xi(t,\omega), & \text{if } t < \tau(\omega), \\ 0, & \text{otherwise,} \end{cases}$$

is $\mathbb{F}$-progressively measurable.

Stopping time τ (with respect to filtration $\mathbb{F}$) is called accessible iff there exists a sequence of $\mathbb{F}$-stopping times $\{\tau_n\}_{n\in\mathbb{N}}$ with the following properties:

1. $\lim_{n\to\infty} \tau_n = \tau$, $\mathbb{P}$-a.s.,
2. for every n, $\tau_n < \tau_{n+1}$, $\mathbb{P}$-a.s..

For such sequence we write $\tau_n \nearrow \tau$. Such a sequence $\{\tau_n\}_{n\in\mathbb{N}}$ is called an approximating sequence for τ.

To prove the uniqueness of a local solution we need the following criteria of equivalent processes.

Definition 2.1.4. Let $\tau_i, i = 1,2$ are stopping times. Two processes $\xi_i : [0,\tau_i) \times \Omega \to E$, $i = 1,2$ are called equivalent iff $\tau_1 = \tau_2$, $\mathbb{P}$-a.s. and for any $t > 0$ the following holds

$$\xi_1(\cdot,\omega) = \xi_2(\cdot,\omega) \quad \text{on } [0,t],$$

for almost all $\omega \in \Omega_t(\tau_1) \cap \Omega_t(\tau_2)$.

For an interval $I \subseteq \mathbb{R}$, we say that, an E-valued process $\{M_t\}_{t\in I}$ is an E-valued martingale iff $M_t \in L^1(\Omega,\mathscr{F},\mathbb{P};E)$ for $t \in I$ and

$$\mathbb{E}(M_t|\mathscr{F}_s) = M_s, \qquad \mathbb{P}-\text{a.s., for all } s \le t \in I.$$

To define the Itô type integrals for a Banach space valued stochastic process, we restrict ourselves to, so called, M-type 2 Banach spaces which are defined as follows.

Definition 2.1.5. A Banach space E is of M-type 2 iff there exists a constant $L := L(E) > 0$ such that for every E-valued martingale $\{M_n\}_{n=0}^N$ the following holds:

$$\sup_n \mathbb{E}(\|M_n\|_E^2) \le L \sum_{n=0}^{N} \mathbb{E}\big(\|M_n - M_{n-1}\|_E^2\big),$$

where $M_{-1} = 0$ as usual.

Assume that $p \in [1,\infty)$. If $T > 0$, then by $\mathcal{M}^p([0,T],E)$, we denote the space of all $\mathbb{F}$-progressively measurable E-valued processes $\xi : [0,T] \times \Omega \to E$ such that

$$\mathbb{E}\left[\int_0^T \|\xi(t)\|_E^p \, dt\right] < \infty.$$

As usual, see e.g. [135, Definition IV.2.1], by $M^p([0,T],E)$ we denote the space of equivalence classes of elements of $\mathcal{M}^p([0,T],E)$, which of course is a Banach space. Let us observe that $M^p([0,T],E)$ is the usual L^p space of E-valued $\mathscr{B}\mathbb{F}$-measurable functions defined on $[0,T] \times \Omega$ with respect to the measure $\mathrm{Leb} \otimes \mathbb{P}$, where Leb is the Lebesgue measure on $\mathbb{R}$.

We also need the following spaces in the remaining of the chapter. Assume that $p \in [1,\infty)$ and $T > 0$. If $q \in [1,\infty)$, by $\mathcal{M}^{q,p}([0,T],E)$, we denote the space of all $\mathbb{F}$-progressively measurable E-valued processes $\xi : [0,T] \times \Omega \to E$ such that

$$\mathbb{E}\left[\left(\int_0^T \|\xi(t)\|_E^q \, dt\right)^{p/q}\right] < \infty.$$

If $q = \infty$, then by $\mathcal{M}^{q,p}([0,T],E)$, we mean the space of all $\mathbb{F}$-progressively measurable E-valued continuous processes $\xi : [0,T] \times \Omega \to E$ such that

$$\mathbb{E}\left[\sup_{t \in [0,T]} \|\xi(t)\|_E^p\right] < \infty.$$

By $M^{q,p}([0,T],E)$ we denote the Banach space of equivalence classes of elements of $\mathcal{M}^{q,p}([0,T],E)$. We close our discussion of the conventions here by observing that, for $p \in [1,\infty)$,

$$\mathcal{M}^{p,p}([0,T],E) = \mathcal{M}^p([0,T],E) \qquad \text{and} \qquad M^{p,p}([0,T],E) = M^p([0,T],E).$$

2.2 Inhomogeneous Strichartz estimates

In this section we prove the deterministic Strichartz type estimate, see Theorem 2.2.2 below, which is a generalization of [88, Theorem 1.2] and sufficient to tackle, both, the Dirichlet and the Neumann boundary case.

Recall that in our setting, the operator $(A, \mathscr{D}(A))$ possesses a complete orthonormal system of eigenvectors $\{e_j\}_{j\in\mathbb{N}}$ in $L^2(\mathcal{D})$. We have denoted the corresponding eigenvalues by λ_j^2. From the functional calculus of self-adjoint operators, it is known that $\{(e_j, \lambda_j)\}_{j\in\mathbb{N}}$ is a sequence of the associated eigenvector and eigenvalue pair for $\sqrt{A}$. For any integer $\lambda \geq 0$, Π_λ is defined as the spectral projection of $L^2(\mathcal{D})$ onto the subspace spanned by $\{e_j\}_{j\in\mathbb{N}}$ for which $\lambda_j \in [\lambda, \lambda+1)$, i.e.

$$\Pi_\lambda u = \sum_{j=1}^{\infty} \mathbb{1}_{[\lambda,\lambda+1)}(\lambda_j)\langle u, e_j\rangle_{L^2(\mathcal{D})} e_j, \qquad u \in L^2(\mathcal{D}).$$

At this juncture, it is relevant to note that the proof of the Strichartz estimate in deterministic setting, see e.g. [34] and [35], is based on the following estimate in Lebesgue spaces of the spectral projector Π_λ, refer [147] for the proof.

Theorem 2.2.1. *For any smooth bounded domain $\mathcal{D} \subset \mathbb{R}^2$, the following holds, for all $u \in L^2(\mathcal{D})$,*

$$\|\Pi_\lambda u\|_{L^q(\mathcal{D})} \leq C\lambda^\rho \|u\|_{L^2(\mathcal{D})},$$

where

$$\rho := \begin{cases} \dfrac{2}{3}\left(\dfrac{1}{2} - \dfrac{1}{q}\right), & \text{if} \quad 2 \leq q \leq 8, \\ 2\left(\dfrac{1}{2} - \dfrac{1}{q}\right) - \dfrac{1}{2}, & \text{if} \quad 8 \leq q \leq \infty. \end{cases}$$

Since the below derived Strichartz estimate, for the inhomogeneous wave equation, holds for both the Dirichlet and the Neumann case, from now onwards, to shorten the notation, we denote $A_{B,q}$ and $A_{B,2}$, respectively, by A_q and A.

Theorem 2.2.2 (Inhomogeneous Strichartz Estimates)**.** *Fix any $T > 0$. Then there exists a positive constant C_T, which may also depend on p, q, r, such that the following holds: if u satisfy the following linear inhomogeneous wave equation*

$$\begin{cases} u_{tt} - \Delta u = F, & \text{on} \quad (0,T) \times \mathcal{D} \\ u(0,\cdot) = u_0(\cdot), \ u_t(0,\cdot) = u_1(\cdot), \end{cases}$$

with either boundary condition

$$\begin{aligned} &\textit{Dirichlet}: \quad && u|_{(0,T)\times\partial\mathcal{D}} = 0, \\ &\textit{Neumann}: \quad && \partial_\nu u|_{(0,T)\times\partial\mathcal{D}} = 0, \end{aligned}$$

where ν is the outward normal unit vector to $\partial\mathcal{D}$ and $F \in L^1(0,T;L^2(\mathcal{D}))$, then

$$\|u\|_{L^p(0,T;\mathscr{D}(A_q^{(1-r)/2}))} \leq C_T \left[\|u_0\|_{\mathscr{D}(A^{1/2})} + \|u_1\|_{L^2(\mathcal{D})} + \|F\|_{L^1(0,T;L^2(\mathcal{D}))}\right], \tag{2.2.1}$$

for all (p,q,r) which satisfy

$$2 \leq q \leq p \leq \infty, \qquad \textit{and} \qquad r = \begin{cases} \dfrac{5}{6} - \dfrac{1}{p} - \dfrac{2}{3q}, & \text{if} \quad 2 \leq q \leq 8, \\ 1 - \dfrac{1}{p} - \dfrac{2}{q}, & \text{if} \quad 8 \leq q \leq \infty. \end{cases} \tag{2.2.2}$$

Remark 2.2.3. Let us observe that if for $T > 0$, C_T denotes the smallest constant for which the inequality (2.2.1) holds for all data u_0, u_1 and F from appropriate spaces, then the function

$$(0,\infty) \ni T \mapsto C_T \in (0,\infty),$$

is non-decreasing (or weakly increasing as some people call).

Proof of Theorem 2.2.2 Without loss of generality we assume that $T = 2\pi$. The proof is divided into two cases. In the first case, we derive the Strichartz estimate for the homogeneous problem (i.e. $F = 0$) and then, in second case, we prove the inhomogeneous one (i.e. $F \neq 0$) by using the homogeneous estimate from first case.

First case : Estimate for the homogeneous problem. In this case, the Duhamel's formula gives

$$u(t) = \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 \tag{2.2.3}$$

where, from the functional calculus for self-adjoint operators, for each t, $\cos(t\sqrt{A})$ and $\frac{\sin(t\sqrt{A})}{\sqrt{A}}$ are well-defined bounded operators on $L^2(\mathcal{D})$. Moreover, we have

$$\cos(t\sqrt{A}) = \left(\frac{e^{it\sqrt{A}} + e^{-it\sqrt{A}}}{2}\right).$$

Let $\mathcal{L}_\pm(t)u_0 := e^{\pm it\sqrt{A}}u_0$ be the solution u of $\partial_t u = \pm i\sqrt{A}u$ such that $u(0) = u_0$. In other words, $\mathcal{L}_\pm = \big(\mathcal{L}_\pm(t)\big)_{t\geq 0}$ is C_0-group with the generator $\pm i\sqrt{A}$. Using the Minkowski's inequality we get

$$\begin{aligned}\|u\|_{L^p(0,T;\mathscr{D}(A_q^{(1-r)/2}))} \lesssim{}& \|e^{it\sqrt{A}}u_0\|_{L^p(0,T;\mathscr{D}(A_q^{(1-r)/2}))} \\ &+ \|e^{-it\sqrt{A}}u_0\|_{L^p(0,T;\mathscr{D}(A_q^{(1-r)/2}))} \quad + \left\|\frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1\right\|_{L^p(0,T;\mathscr{D}(A_q^{(1-r)/2}))}.\end{aligned} \tag{2.2.4}$$

Therefore, it is enough to estimate, as is done in the following Steps 1-4, the $L^p(0,T;\mathscr{D}(A_q^{(1-r)/2}))$-norm of $e^{it\sqrt{A}}u_0$ and $\frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1$. We will write the variables in subscript, wherever required, to avoid any confusion.

Step 1 : Here we show that

$$\|e^{itB}u_0\|_{L_t^p(0,2\pi;L_x^q(\mathcal{D}))} \leq C\|u_0\|_{\mathscr{D}(A^{r/2})}, \tag{2.2.5}$$

where B is the following "modification" of $\sqrt{A}$ operator by considering only the integer eigenvalues i.e.

$$B(e_j) = [\lambda_j]e_j, \qquad j \in \mathbb{N}.$$

The notation $[\cdot]$ stands for the integer part and e_j is an eigenfunction of A associated to the eigenvalue λ_j^2. Before moving further we prove the boundedness property of the operator $B - \sqrt{A}$.

Lemma 2.2.4. *The operator $B - \sqrt{A}$ is bounded on $\mathscr{D}(A^{1/2})$.*

Proof of Lemma 2.2.4 Indeed, observe that by definition of B we have for every $u \in \mathscr{D}(A^{1/2})$,

$$(B-\sqrt{A})u = \sum_{j\in\mathbb{N}} \{\lambda_j\}\langle u, e_j\rangle_{L^2(\mathcal{D})} e_j,$$

where $\{\lambda_j\} := \lambda_j - [\lambda_j]$ is the fractional part of λ_j. Then

$$\|(B-\sqrt{A})u\|^2_{L^2(\mathcal{D})} \le \sum_{j\in\mathbb{N}} \{\lambda_j\}^2 \,|\langle u, e_j\rangle_{L^2(\mathcal{D})}|^2 \le \sum_{j\in\mathbb{N}} |\langle u, e_j\rangle_{L^2(\mathcal{D})}|^2 = \|u\|^2_{L^2(\mathcal{D})}.$$

Moreover,

$$\|\sqrt{A}(B-\sqrt{A})u\|^2_{L^2(\mathcal{D})} = \sum_{j\in\mathbb{N}} \lambda_j^2\{\lambda_j\}^2 \,|\langle u, e_j\rangle_{L^2(\mathcal{D})}|^2 \le \sum_{j\in\mathbb{N}} \lambda_j^2|\langle u, e_j\rangle_{L^2(\mathcal{D})}|^2 = \|\sqrt{A}u\|^2_{L^2(\mathcal{D})}.$$

Hence, by the definition of norm in $\mathscr{D}(A^{1/2})$ we have

$$\|(B-\sqrt{A})u\|^2_{\mathscr{D}(A^{1/2})} = \|(B-\sqrt{A})u\|^2_{L^2(\mathcal{D})} + \|\sqrt{A}(B-\sqrt{A})u\|^2_{L^2(\mathcal{D})} \lesssim \|u\|^2_{\mathscr{D}(A^{1/2})}.$$

□

In continuation of the proof of (2.2.5), since $u_0 \in L^2(\mathcal{D})$, we can write

$$u_0 = \sum_{j\in\mathbb{N}} \langle u_0, e_j\rangle_{L^2(\mathcal{D})} e_j =: \sum_{j\in\mathbb{N}} u_{0,j} e_j.$$

By functional calculus for self-adjoint operators,

$$e^{itB}u_0(x) = \sum_{j\in\mathbb{N}} e^{it[\lambda_j]} u_{0,j} e_j(x) =: \sum_{k\in\mathbb{N}} u_k(t,x),$$

where,

$$u_k(t,x) = \sum_{j\in\mathbb{N}} \mathbb{1}_{[k,k+1)}(\lambda_j)\; e^{itk} u_{0,j} e_j(x) = e^{itk}\Pi_k u_0(x).$$

Thanks to the 1-D Sobolev embedding and Lemma 2.1.2, we have

$$H^{\frac{1}{2}-\frac{1}{p},2}(0,2\pi) \hookrightarrow L^p(0,2\pi) \qquad \text{for all} \quad p \ge 2,$$

and consequently we argue as follows:

$$\begin{aligned}
\|e^{itB}u_0\|^2_{L^q_x(\mathcal{D};L^p_t(0,2\pi))} &= \left(\int_{\mathcal{D}} \|e^{itB}u_0(x)\|^q_{L^p_t(0,2\pi)}\, dx\right)^{\frac{2}{p}} \\
&\lesssim \left(\int_{\mathcal{D}} \|e^{itB}u_0(x)\|^q_{H_t^{\frac{1}{2}-\frac{1}{p},2}(0,2\pi)}\, dx\right)^{\frac{2}{p}} = \left\| \|e^{itB}u_0(x)\|^2_{H_t^{\frac{1}{2}-\frac{1}{p},2}(0,2\pi)} \right\|_{L^{\frac{q}{2}}(\mathcal{D})}. \qquad (2.2.6)
\end{aligned}$$

Note that since the sequence $\{e^{itk}\}_{k\in\mathbb{N}}$ is an orthogonal system in $H^{\frac{1}{2}-\frac{1}{p},2}(0,2\pi)$ and

$$\left\| e^{itk} \right\|_{H_t^{\frac{1}{2}-\frac{1}{p},2}(0,2\pi)} \lesssim (1+k^2)^{\frac{1}{2}-\frac{1}{p}},$$

due to the Parseval formula we get, for fixed x,

$$\|[e^{itB}u_0](x)\|^2_{H_t^{\frac{1}{2}-\frac{1}{p},2}(0,2\pi)} \lesssim \sum_{k\in\mathbb{N}} (1+k)^{1-\frac{2}{p}} \|u_k(t,x)\|^2_{L^2(0,2\pi)}. \tag{2.2.7}$$

Combining the estimate (2.2.7) and (2.2.6) followed by Minkowski's inequality and Theorem 2.2.1 we obtain

$$\begin{aligned}
\|e^{itB}u_0\|^2_{L_x^q(\mathcal{D};L_t^p(0,2\pi))} &\lesssim \left\| \sum_{k\in\mathbb{N}} (1+k)^{1-\frac{2}{p}} \|u_k(t,x)\|^2_{L^2(0,2\pi)} \right\|_{L^{\frac{q}{2}}(\mathcal{D})} \le \sum_{k\in\mathbb{N}} (1+k)^{1-\frac{2}{p}} \|u_k(t,x)\|^2_{L_t^2(0,2\pi;L_x^q(\mathcal{D}))} \\
&\lesssim \sum_{k\in\mathbb{N}} (1+k)^{1-\frac{2}{p}} \|[\Pi_k u_0](x)\|^2_{L_x^q(\mathcal{D})} \lesssim \sum_{k\in\mathbb{N}} (1+k)^{1-\frac{2}{p}} k^{2\rho} \|\Pi_k u_0\|^2_{L_x^2(\mathcal{D})} \\
&= \sum_{k\in\mathbb{N}} (1+k)^{1-\frac{2}{p}+2\rho} \sum_{j\in\mathbb{N}} \mathbb{1}_{[k,k+1)}(\lambda_j) |\langle u_0, e_j\rangle_{L^2(\mathcal{D})}|^2 \\
&= \sum_{j\in\mathbb{N}} (1+[\lambda_j])^{1-\frac{2}{p}+2\rho} |\langle u_0, e_j\rangle_{L^2(\mathcal{D})}|^2 = \|u_0\|^2_{H_B^r(\mathcal{D})} \simeq \|u_0\|^2_{\mathscr{D}(A^{r/2})},
\end{aligned} \tag{2.2.8}$$

where, from ρ in Theorem 2.2.1, we have[1],

$$r := \frac{1}{2} - \frac{1}{p} + \rho = \begin{cases} \dfrac{5}{6} - \dfrac{1}{p} - \dfrac{2}{3q}, & \text{if} \quad 2 \le q \le 8, \\ 1 - \dfrac{1}{p} - \dfrac{2}{q}, & \text{if} \quad 8 \le q \le \infty. \end{cases}$$

Here it is important to highlight that, the equivalence $\|u_0\|_{H_B^r(\mathcal{D})} \simeq \|u_0\|_{\mathscr{D}(A^{r/2})}$ holds in the last step of (2.2.8), because $\mathscr{D}(A) = \mathscr{D}(B^2)$ and the function spaces H_B^r for $r \in [0,1]$ and $\mathscr{D}(A^{r/2})$, are equal to the complex interpolation spaces, between $L^2(\mathcal{D})$ and, respectively, $\mathscr{D}(B^2)$ and $\mathscr{D}(A)$, see for e.g. [155, Theorem 4.3.3].

Next, since $p \ge q$, by the Minkowski inequality we obtain the following desired result

$$\|e^{itB}u_0\|_{L_t^p(0,2\pi;L_x^q(\mathcal{D}))} \lesssim \|u_0\|_{\mathscr{D}(A^{r/2})},$$

which also implies that the operator e^{itB} is continuous from $\mathscr{D}(A^{r/2})$ to $L_t^p(0,2\pi;L_x^q(\mathcal{D}))$.

Step 2 : In this step we extend the inequality (2.2.5) to operator $\mathcal{L}_+$, i.e. we show that

$$\|\mathcal{L}_+(\cdot)u_0\|_{L_t^p(0,2\pi;L_x^q(\mathcal{D}))} \le C\|u_0\|_{\mathscr{D}(A^{r/2})}. \tag{2.2.9}$$

Let $v(t) = e^{it\sqrt{A}}u_0$. It is clear that v satisfies

$$\begin{cases} (\partial_t - iB)v = (-iB + i\sqrt{A})v, \\ v|_{t=0} = u_0, \end{cases}$$

and, therefore, according to the Duhamel's formula

$$v(t) = e^{itB}u_0 + \int_0^t e^{i(t-s)B}(-iB + i\sqrt{A})v(s)\,ds. \tag{2.2.10}$$

[1] Note that $r < \frac{3}{4}$ in the case $2 \le q \le 8$ and $r < 1$ in the complimentary case $8 \le q \le \infty$.

If we denote $e^{i(t-s)B}(-iB+i\sqrt{A})v(s,x)$ by $z(s,x)$ and $(-iB+i\sqrt{A})v(s,x)$ by $w(s,x)$, then using the Minkowski inequality, followed by estimate (2.2.5) and Lemma 2.2.4, we argue as follows:

$$\begin{aligned}\left\|\left(\int_0^{2\pi}\left[\int_0^t |z(s,x)|\,ds\right]^p dt\right)^{\frac{1}{p}}\right\|_{L^q_x(\mathcal{D})} &\le \left\|\int_0^{2\pi}\left[\int_0^{2\pi}|z(s,x)|^p\,dt\right]^{\frac{1}{p}} ds\right\|_{L^q_x(\mathcal{D})}\\ &\le \left(\int_{\mathcal{D}}\left(\int_0^{2\pi}\|z(s,x)\|_{L^p_t(0,2\pi)}\,ds\right)^q dx\right)^{\frac{1}{q}}\\ &\le \int_0^{2\pi}\left(\int_{\mathcal{D}}\|z(s,x)\|^q_{L^p_t(0,2\pi)}\,dx\right)^{\frac{1}{q}} ds\\ &\le \int_0^{2\pi}\|w(s,x)\|_{\mathscr{D}(A^{r/2})}\,ds \;\le\; \int_0^{2\pi}\|v(s,x)\|_{\mathscr{D}(A^{r/2})}\,ds. \end{aligned} \tag{2.2.11}$$

By putting together (2.2.10) and (2.2.11) we obtain

$$\begin{aligned}\|v(t,x)\|_{L^q_x(\mathcal{D};L^p_t(0,2\pi))} &\le \|e^{itB}u_0(x)\|_{L^q_x(\mathcal{D};L^p_t(0,2\pi))} + \int_0^{2\pi}\|v(s,x)\|_{\mathscr{D}(A^{r/2})}\,ds\\ &\le \|u_0\|_{\mathscr{D}(A^{r/2})} + \int_0^{2\pi}\|v(s,x)\|_{\mathscr{D}(A^{r/2})}\,ds.\end{aligned} \tag{2.2.12}$$

Now, from the boundedness of $e^{it\sqrt{A}}$ on $\mathscr{D}(A^{r/2})$, we infer that

$$\sup_{t\in[0,2\pi]}\left\|e^{it\sqrt{A}}u_0\right\|_{\mathscr{D}(A^{r/2})} \le C\|u_0\|_{\mathscr{D}(A^{r/2})}. \tag{2.2.13}$$

Combining (2.2.13) and (2.2.12) we get

$$\begin{aligned}\|v(t,x)\|_{L^q_x(\mathcal{D};L^p_t(0,2\pi))} &\le \|u_0\|_{\mathscr{D}(A^{r/2})} + \int_0^{2\pi}\|u_0\|_{\mathscr{D}(A^{r/2})}\,ds\\ &\lesssim \|u_0\|_{\mathscr{D}(A^{r/2})}.\end{aligned}$$

Hence, again, as an application of the Minkowski inequality we get (2.2.9) and finish with the proof of Step 2.

Step 3: Here, by using the well known consequence of Agmon-Douglis-Nirenberg regularity results for the elliptic operators, refer [1], we prove the required estimate of the first term in (2.2.4), in particular, we show that

$$\|\mathcal{L}_+(\cdot)u_0\|_{L^p_t(0,2\pi;\mathscr{D}(A_q^{(1-r)/2}))} \lesssim \|u_0\|_{\mathscr{D}(A^{1/2})}. \tag{2.2.14}$$

We start the proof by recalling the following consequence of the Agmon-Douglis-Nirenberg regularity results for the elliptic operators. The operators

$$-\Delta_D + I : H^{2,q}(\mathcal{D})\cap H^{1,q}_D(\mathcal{D}) = H^{2,q}(\mathcal{D})\cap \mathring{H}^{1,q}(\mathcal{D}) \to L^q(\mathcal{D}),$$

and

$$-\Delta_N + I : H^{2,q}(\mathcal{D})\cap H^{1,q}_N(\mathcal{D}) \to L^q(\mathcal{D}),$$

are isomorphisms. These operators will, respectively, be denoted by $A_{D,q}+I$ and $A_{N,q}+I$, or simply by A_q+I. Suppose that $u_0 \in D(A^k)$ for sufficiently large $k \in \mathbb{N}$ so that $Au_0 \in \mathscr{D}(A^{r/2})$. Then, since the operators A and $\mathcal{L}_+$ commute, we infer that for all $t \in [0,T]$,

$$\|\mathcal{L}_+(t)u_0\|_{H^{2,q}(\mathcal{D})} \simeq \|(A+I)\mathcal{L}_+(t)u_0\|_{L^q(\mathcal{D})} = \|\mathcal{L}_+(t)((A+I)u_0)\|_{L^q(\mathcal{D})}.$$

Consequently by (2.2.9) we get

$$\|\mathcal{L}_+(\cdot)u_0\|_{L^p_t(0,2\pi;H^{2,q}(\mathcal{D}))} \lesssim \|(A+I)u_0\|_{\mathscr{D}(A^{r/2})} \sim \|u_0\|_{\mathscr{D}(A^{(r+2)/2})}. \tag{2.2.15}$$

Thus, complex interpolation between (2.2.9) and (2.2.15) with $\theta = \frac{1-r}{2}$ gives the desired following estimate

$$\|\mathcal{L}_+(\cdot)u_0\|_{L^p(0,2\pi;\mathscr{D}(A_q^{(1-r)/2}))} \lesssim \|u_0\|_{\mathscr{D}(A^{1/2})}.$$

Hence we have completed the proof of Step 3.

Step 4: Here we incorporate the term with u_1, in (2.2.3), and complete the proof of the homogeneous Strichartz estimate.

Recall that $\lambda_1 = 0$ for the Neumann condition and $\lambda_1 > 0$ in the Dirichlet case. As mentioned before, we denote by m_0 the dimension of eigenspace corresponding to zero eigenvalue. It is known that $m_0 = 0$ for $A = -\Delta_D$ and a positive finite integer when $A = \Delta_N$. To proceed with the proof of this step, as in [35], we single out the contribution of zero eigenvalue and decompose $L^2(\mathcal{D})$ into the direct sum of a finite dimensional space $\ker A$ and the space orthogonal to $\ker A$, which we denote by $L^{2,+}(\mathcal{D})$. Let us observe that if $\mathcal{D}$ is connected, then $\ker A$ is a one dimensional vector space consisting of constant functions. Mathematically, it means, for all $u_1 \in L^2(\mathcal{D})$,

$$u_1 = \sum_{j=1}^{m_0} \langle u_1, e_j\rangle_{L^2(\mathcal{D})} e_j + \sum_{k>m_0} \langle u_1, e_k\rangle_{L^2(\mathcal{D})} e_k, =: \Pi u_1 + (\mathbb{1}-\Pi)u_1.$$

Note that the term Πu_1 does not exist in the Dirichlet condition. Then we argue as follows:

$$\begin{aligned}\frac{\sin(t\sqrt{A})}{\sqrt{A}} u_1 &= \frac{\sin(t\sqrt{A})}{\sqrt{A}}\Pi u_1 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}(\mathbb{1}-\Pi)u_1 \\ &= t\Pi u_1 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}(\mathbb{1}-\Pi)u_1,\end{aligned} \tag{2.2.16}$$

where the last step holds due to the following argument

$$\begin{aligned}\frac{\sin(t\sqrt{A})}{\sqrt{A}} u_1 &= \sum_{j\in\mathbb{N}} \frac{\sin(t\lambda_j)}{\lambda_j}\langle u, e_j\rangle_{L^2(\mathcal{D})} e_j \\ &= \sum_{j\in\mathbb{N}} t\mathbb{1}_{\{0\}}(\lambda_j)\frac{\sin(t\lambda_j)}{t\lambda_j}\langle u, e_j\rangle_{L^2(\mathcal{D})} e_j + \sum_{j\in\mathbb{N}} \mathbb{1}_{(0,\infty)}(\lambda_j)\frac{\sin(t\lambda_j)}{\lambda_j}\langle u, e_j\rangle_{L^2(\mathcal{D})} e_j \\ &= t\sum_{j\in\mathbb{N}} \mathbb{1}_{\{0\}}(\lambda_j)\langle u, e_j\rangle_{L^2(\mathcal{D})} e_j + \sum_{j\in\mathbb{N}} \mathbb{1}_{(0,\infty)}(\lambda_j)\frac{\sin(t\lambda_j)}{\lambda_j}\langle u, e_j\rangle_{L^2(\mathcal{D})} e_j\end{aligned}$$

$$= t\Pi u_1 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}(\mathbb{1}-\Pi)u_1. \tag{2.2.17}$$

Now, since $\left(\sqrt{A}\right)^{-1}$ is bounded from $L^{2,+}(\mathcal{D})$ into $\mathscr{D}(A^{1/2})$, we invoke (2.2.14) on $\left(\sqrt{A}\right)^{-1}((\mathbb{1}-\Pi)u_1)$ and get

$$\begin{aligned}\|\mathcal{L}_+(\cdot)\left(\sqrt{A}\right)^{-1}((\mathbb{1}-\Pi)u_1))\|_{L^p(0,2\pi;\mathscr{D}(A_q^{(1-r)/2}))} &\lesssim \|\left(\sqrt{A}\right)^{-1}((\mathbb{1}-\Pi)u_1)\|_{\mathscr{D}(A^{1/2})} \\ &\lesssim \|(\mathbb{1}-\Pi)u_1\|_{L^2(\mathcal{D})}.\end{aligned} \tag{2.2.18}$$

We mention that all the computations we have done so far in Steps 1-4 would work if we replace $\mathcal{L}_+$ by $\mathcal{L}_-$. Combining (2.2.16) and (2.2.18) we obtain

$$\begin{aligned}\left\|\frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1\right\|_{L^p(0,2\pi;\mathscr{D}(A_q^{(1-r)/2}))} &\lesssim \|t\Pi u_1\|_{L^p(0,2\pi;\mathscr{D}(A_q^{(1-r)/2})} \\ &\quad + \left\|\mathcal{L}_+(\cdot)\left(\sqrt{A}\right)^{-1}((\mathbb{1}-\Pi)u_1)\right\|_{L^p(0,2\pi;\mathscr{D}(A_q^{(1-r)/2})} \\ &\lesssim \|t\Pi u_1\|_{L^p(0,2\pi;\mathscr{D}(A_q^{(1-r)/2}))} + \|(\mathbb{1}-\Pi)u_1\|_{L^2(\mathcal{D})} \\ &\lesssim \|\Pi u_1\|_{\mathscr{D}(A_q^{(1-r)/2})} + \|u_1\|_{L^2(\mathcal{D})} \lesssim \|u_1\|_{L^2(\mathcal{D})}.\end{aligned}$$

This finishes the proof of Step 4 and, in particular, the first case.

Second case: when $L^1(0,2\pi;L^2(\mathcal{D})) \ni F \neq 0$: Due to the Duhamel's formula

$$u(t) = \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}F(s)\,ds.$$

Applying the case first and using the calculation of (2.2.12) and (2.2.17) we get

$$\begin{aligned}\|u\|_{L^p(0,2\pi;\mathscr{D}(A_q^{(1-r)/2}))} &\lesssim \|u_0\|_{\mathscr{D}(A^{1/2})} + \|u_1\|_{L^2(\mathcal{D})} \\ &\quad + \int_0^{2\pi} \left\|\frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}F(s)\right\|_{L^p(0,2\pi;\mathscr{D}(A_q^{(1-r)/2}))} ds \\ &\lesssim \|u_0\|_{\mathscr{D}(A^{1/2})} + \|u_1\|_{L^2(\mathcal{D})} + \int_0^{2\pi} \|F(s)\|_{L^2(\mathcal{D})}\,ds \\ &= \|u_0\|_{\mathscr{D}(A^{1/2})} + \|u_1\|_{L^2(\mathcal{D})} + \|F\|_{L^1(0,2\pi;L^2(\mathcal{D}))}.\end{aligned}$$

Hence we have proved the Theorem 2.2.2. ■

2.3 Stochastic Strichartz estimates

This section is devoted to prove a stochastic Strichartz inequality, which is sufficient to apply the Banach Fixed Point Theorem in the proof of a local well-posedness result, see Theorem 2.4.10 in Section 2.4.

Let us set

$$H := L^2(\mathcal{D}); \qquad H_A := \mathscr{D}(A^{1/2}); \qquad E := \mathscr{D}(A_q^{(1-r)/2}),$$

where (p,q,r) is any suitable triple which satisfy (2.2.2) and K is any separable Hilbert space. Let us define the following two Banach spaces. For fix $T>0$, we put

$$Y_T := \mathcal{C}\left([0,T];\mathscr{D}(A^{1/2})\right)\cap L^p\left(0,T;\mathscr{D}(A_q^{(1-r)/2})\right),$$

with norm, which makes it a Banach space,

$$\|u\|_{Y_T}^p := \sup_{t\in[0,T]}\|u(t)\|_{\mathscr{D}(A^{1/2})}^p + \|u\|_{L^p(0,T;\mathscr{D}(A_q^{(1-r)/2}))}^p.$$

To prove the main result of this section we need the following consequence of the Kahane-Khintchin inequality and the Itô-Nisio Theorem, see [86]. For any $\Lambda\in\gamma(K,E)$, by the Itô-Nisio Theorem, the series $\sum_{j=1}^{\infty}\beta_j\Lambda(f_j)$ is $\mathbb{P}$-a.s. convergent in E, where $\{f_j\}_{j\in\mathbb{N}}$ and $\{\beta_j\}_{\mathbb{N}}$ are as in Definition 2.1.3, and then, by the Kahane-Khintchin inequality, for any $p\in[1,\infty)$, there exists a positive constant $C(p,E)$ such that

$$(C(p,E))^{-1}\,\|\Lambda\|_{\gamma(K,E)} \le \left(\mathbb{E}\Big\|\sum_{j\in\mathbb{N}}\beta_j\Lambda(f_j)\Big\|_E^p\right)^{\frac{1}{p}} \le C(p,E)\,\|\Lambda\|_{\gamma(K,E)}. \tag{2.3.1}$$

This inequality tells that the convergence in $L^2(\Omega,\mathscr{F},\mathbb{P};E)$ can be replaced by a condition of convergence in $L^p(\Omega,\mathscr{F},\mathbb{P};E)$ for some (or any) $p\in[1,\infty)$. Furthermore, we need the following version of Burkholder inequality which holds in our setting, refer [120] for the proof.

Theorem 2.3.1 (Burkholder inequality)**.** *Let E be a M-type 2 Banach space. Then for every $p\in(0,\infty)$ there exists a constant $B_p(E)>0$ such that for each accessible stopping time $\tau>0$ and $\gamma(K,E)$-valued progressively measurable processes ξ the following holds:*

$$\mathbb{E}\left(\sup_{t\in[0,\tau]}\Big\|\int_0^t\xi(s)\,dW(s)\Big\|_E^p\right) \le B_p(E)\,\mathbb{E}\left(\int_0^\tau\|\xi(t)\|_{\gamma(K,E)}^2\,dt\right)^{\frac{p}{2}}. \tag{2.3.2}$$

Moreover, the E-valued process $\int_0^t\xi(s)\,dW(s)$, $t\in[0,\tau]$, has a continuous modification.

Corollary 2.3.2. *Let E be a M-type 2 Banach space and $p\in(1,\infty)$. Then there exists a constant $\hat{B}_p(E)$ depending on E such that for every $T\in(0,\infty]$ and every $L^p(0,T;E)$-valued progressively measurable process $\{\zeta(s), s\in[0,T)\}$,*

$$\mathbb{E}\left(\Big\|\int_0^T\zeta(s)\,dW(s)\Big\|_{L^p(0,T;E)}^p\right) \le \hat{B}_p(E)\,\mathbb{E}\left(\int_0^T\|\zeta(s)\|_{\gamma(K,L^p(0,T;E))}^2\,ds\right)^{\frac{p}{2}}. \tag{2.3.3}$$

For a $\gamma(K,H)$-valued process ξ, let us define a $\gamma(K,L^p(0,T;E))$-valued process $\Xi=\{\Xi_r : r\in[0,T]\}$ as follows:

$$\Xi_r := \left\{[0,T]\ni t\mapsto \mathbb{1}_{[r,T]}(t)\frac{\sin((t-r)\sqrt{A})}{\sqrt{A}}\xi(r)\right\}\in\gamma(K,L^p(0,T;E)), \qquad r\in[0,T]. \tag{2.3.4}$$

Before proving the main result of this section, we prove the following auxiliary lemma.

Lemma 2.3.3. *Assume that* $T > 0$*. Let* $\xi \in M^{2,p}([0,T],\gamma(K,H))$*. Then the* $\gamma(K,L^p(0,T;E))$*-valued process* $\{\Xi_r : r \in [0,T]\}$ *defined by formula* (2.3.4)*, is progressively measurable, i.e.*

$$\Xi \in M^{2,p}([0,T],\gamma(K,L^p(0,T;E))), \tag{2.3.5}$$

and, for each $r \in [0,T]$*,*

$$\|\Xi_r\|_{\gamma(K,L^p(0,T;E))} \leq C(p,T,E,H)\ \|\xi(r)\|_{\gamma(K,H)}, \tag{2.3.6}$$

where $C(p,T,E,H) := C_T\ C(p,H)\ C(p,E)$*.*

Proof of Lemma 2.3.3 Let us consider $\{\beta_j\}_{j\in\mathbb{N}}$ of i.i.d. $N(0,1)$ random variables on probability space $(\Omega,\mathscr{F},\mathbb{P})$, and a sequence of orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$ of the separable Hilbert space K. In the proof first observe that the random variable Ξ_r is well-defined because by Theorem 2.2.2, for each $r \in [0,T]$ and $x \in H$, the solution of the following homogeneous wave equation

$$\begin{cases} u_{tt} - \Delta u = 0, & \text{on } \ [r, r+T], \\ u(r) = 0, \quad u_t(r) = x, \end{cases}$$

belongs to $L^p(r, r+T; E)$. In particular,

$$\mathbb{1}_{[r,T]}(\cdot)\frac{\sin((\cdot - r)\sqrt{A})}{\sqrt{A}}x \in L^p(0,T;E),$$

and the map

$$\Lambda_r : H \ni x \mapsto \mathbb{1}_{[r,T]}(\cdot)\frac{\sin((\cdot - r)\sqrt{A})}{\sqrt{A}}x \in L^p(0,T;E),$$

is linear and continuous. Moreover, $\sup_{r\in[0,T]} \|\Lambda_r\| < \infty$.

By the above argument and (2.3.4), we infer that

$$\Xi_r(\omega) = \Lambda_r \circ [\xi(r,\omega)], \quad (r,\omega) \in [0,T]\times\Omega.$$

Consequently, we deduce that the process Ξ is progressively measurable by [86, Proposition 1.1.28]. It only remains to prove inequality (2.3.6). For this aim let us fix any $r \in [0,T]$. Invoking the Strichartz estimates from Theorem 2.2.2 and (2.3.4) gives

$$\Xi_r(\omega) = \Lambda_r \circ \xi(r,\omega) : K \to L^p(0,T;E), \tag{2.3.7}$$

where $\Lambda_r \in \mathcal{L}(H, L^p(0,T;E))$ and $\xi(r) \in \gamma(K,H)$. Then, by using (2.3.1) we get

$$\begin{aligned} \|\Lambda_r \circ \xi\|_{\gamma(K,L^p(0,T;E))} &\leq C(p,E)\left(\mathbb{E}\left[\left\|\sum_{j\in\mathbb{N}} \beta_j\ \Lambda_r(\xi(r)(e_j))\right\|^p_{L^p(0,T;E)}\right]\right)^{\frac{1}{p}} \\ &= C(p,E)\left(\mathbb{E}\left[\left\|\Lambda_r\left(\sum_{j\in\mathbb{N}} \beta_j\ \xi(r)(e_j)\right)\right\|^p_{L^p(0,T;E)}\right]\right)^{\frac{1}{p}} \end{aligned}$$

$$
\begin{aligned}
&\leq C(p,E)\, \|\Lambda_r\|_{\mathcal{L}(H,L^p(0,T;E))} \left(\mathbb{E}\left[\Big\| \sum_{j\in\mathbb{N}} \beta_j\, \xi(r)(e_j) \Big\|_H^p \right] \right)^{\frac{1}{p}} \\
&\leq C(p,E)\, C(p,H)\, \|\Lambda_r\|_{\mathcal{L}(H,L^p(0,T;E))}\, \|\xi(r)\|_{\gamma(K,H)},
\end{aligned}
$$

where, by using the inhomogeneous Strichartz inequality (2.2.1), we have

$$
\begin{aligned}
\|\Lambda_r\|_{\mathcal{L}(H,L^p(0,T;E))} &= \sup_{\substack{h\in H \\ \|h\|_H \leq 1}} \|\Lambda_r h\|_{L^p(0,T;E)} = \sup_{\substack{h\in H \\ \|h\|_H \leq 1}} \left(\int_r^T \left\| \frac{\sin((t-r)\sqrt{A})}{\sqrt{A}} h \right\|_E^p dt \right)^{\frac{1}{p}} \\
&\leq \sup_{\substack{h\in H \\ \|h\|_H \leq 1}} C_T\, \|h\|_H = C_T.
\end{aligned}
$$

Consequently, since $\xi \in M^{2,p}([0,T],\gamma(K,H))$, we have $\Xi \in M^{2,p}([0,T],\gamma(K,L^p(0,T,E)))$ because

$$
\begin{aligned}
\mathbb{E}\left[\int_0^T \|\Xi_r\|^2_{\gamma(K,L^p(0,T;E))}\, dr \right]^{\frac{p}{2}} &= \int_\Omega \left[\int_0^T \|\Xi_r(\omega)\|^2_{\gamma(K,L^p(0,T;E))}\, dr \right]^{\frac{p}{2}} \mathbb{P}(d\omega) \\
&\leq (C(p,E))^p\, (C(p,H))^p\, C_T^p \int_\Omega \left[\int_0^T \|\xi(r,\omega)\|^2_{\gamma(K,H)}\, dr \right]^{\frac{p}{2}} \mathbb{P}(d\omega) < \infty.
\end{aligned}
$$

Hence the Lemma 2.3.3. ■

Remark 2.3.4. Results related to the previous lemma and the next theorem in the case of the Schrödinger group have been discussed in detail in the PhD thesis of Fabian Hornung [82], see Theorem 2.21 and Corollary 2.22.

The following main result of this section is one of the most important ingredient in the proof of the local existence theorem in Section 2.4. They are called the "stochastic Strichartz estimates".

Theorem 2.3.5 (Stochastic Strichartz Estimates)**.** *Let us assume that $T>0$ and $p\in(1,\infty)$. Then there exist constants*[1] $K(p,T,H)>0$ *and* $\tilde{C}(p,T,E,H)>0$ *such that if* ξ *is a progressively measurable process from the space* $M^{2,p}([0,T],\gamma(K,H))$*, then the following assertions hold.*

(I) There exists a separable and H_A-valued[2] *continuous and adapted modification $\tilde{u}$ of the process $u=\{u(t): t\in[0,T]\}$, defined by the following formula*

$$
u(t) := \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} \xi(s)\, dW(s), \quad t\in[0,T]. \tag{2.3.8}
$$

Moreover,

$$
\mathbb{E}\left[\sup_{t\in[0,T]} \|\tilde{u}(t)\|^p_{H_A} \right] \leq K(p,T,H)\, \mathbb{E}\left[\int_0^T \|\xi(t)\|^2_{\gamma(K,H)}\, dt \right]^{\frac{p}{2}}, \tag{2.3.9}
$$

where $K(p,T,H) \leq M_1 e^{mT} B_p(\mathcal{H})$ for some constants $m\geq 0$ and $M_1\geq 1$.

[1] The constant K depends on T only in the Neumann boundary conditions case.
[2] Let us recall that $H_A = \mathscr{D}(A^{1/2})$.

(II) There exists an E-valued progressively measurable process $\tilde{\tilde{u}}$ such that

$$j(\tilde{\tilde{u}}) = i(\tilde{u}) \quad \textit{for } \mathrm{Leb}\otimes\mathbb{P}\textit{-almost all} \quad (t,\omega)\in[0,T]\times\Omega, \tag{2.3.10}$$

where $i : H_A \hookrightarrow H$ and $j : E \hookrightarrow H$ are the natural embeddings. Moreover,

$$\mathbb{E}\left[\int_0^T \|\tilde{\tilde{u}}(t)\|_E^p \, dt\right] \le \tilde{C}(p,T,E,H)\,\mathbb{E}\left[\int_0^T \|\xi(t)\|^2_{\gamma(K,H)}\, dt\right]^{\frac{p}{2}}, \tag{2.3.11}$$

where $\tilde{C}(p,T,E,H) := C_T\, C(p,H)\, C(p,E)\, \hat{B}_p(E)$. In particular, the map

$$J : M^{2,p}([0,T],\gamma(K,H)) \to L^p(0,T;E),$$

is linear and bounded where $J\xi$ is a process defined by

$$J\xi := \tilde{\tilde{u}}. \tag{2.3.12}$$

Proof of Theorem 2.3.5 In what follows we fix the Dirichlet or the Neumann boundary conditions. To prove the first assertion, let us consider the following stochastic wave problem

$$\begin{cases} u_{tt} + Au = \xi\dot{W} & \text{in } [0,T]\times\mathcal{D} \\ (u,u_t)(0) = (0,0). \end{cases}$$

Then, see Subsection 2.5.2, by writing it as a first order system in space $\mathcal{H} := H_A \times H$, endowed with Hilbertian norm, we get

$$\begin{cases} d\mathfrak{u}(t) = \mathfrak{A}\mathfrak{u}(t)\,dt + \tilde{\xi}(t)dW(t) \\ \mathfrak{u}(0) = (0,0), \end{cases} \tag{2.3.13}$$

where

$$\mathfrak{u} = (u,u_t), \quad \mathfrak{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix} \quad \text{and} \quad \tilde{\xi}(t) = \begin{pmatrix} 0 \\ \xi(t) \end{pmatrix}.$$

Since A is non-negative and self adjoint in $L^2(\mathcal{D})$, one may prove that $\mathfrak{A}$ generate a C_0-group (of contraction in the Dirichlet case) on $\mathcal{H}$, which we denote by $\{S(t)\}_{t\ge 0}$ in the sequel. Moreover, one can write the concrete structure of $S(t)$ as

$$S(t) = \begin{pmatrix} \cos(t\sqrt{A}) & \sin(t\sqrt{A})/\sqrt{A} \\ -\sqrt{A}\sin(t\sqrt{A}) & \cos(t\sqrt{A}) \end{pmatrix}.$$

It is known that the solution of (2.3.13) exists, see e.g. [119], and has the following form

$$\mathfrak{u}(t) = \int_0^t S(t-s)\,\tilde{\xi}(s)\,dW(s), \quad t\in[0,T].$$

Next, we define the process $\tilde{\mathfrak{u}}$, by

$$\tilde{\mathfrak{u}}(t) := S(t)\int_0^t S(-s)\,\tilde{\xi}(s)\,dW(s), \quad t\in[0,T],$$

where by $\int_0^t S(-s)\,\tilde{\xi}(s)\,dW(s),\ t \in [0,T]$, we denote the separable, $\mathcal{H}$-valued continuous and adapted modification of the process denoted by the same symbol. Hence, since $\{S(t)\}_{t\geq 0}$ is a C_0-group, the process $\tilde{\mathfrak{u}}$ is separable, $\mathcal{H}$-valued continuous and adapted modification of the process $\mathfrak{u}$.

By defining a process $\tilde{u}$ by

$$\tilde{u}(t) := \pi_1(\tilde{\mathfrak{u}}(t)), \quad t \in [0,T], \tag{2.3.14}$$

where $\pi_1 : \mathcal{H} \to H_A$ is the natural projection, it follows that $\tilde{u}$ is separable H_A-valued continuous and adapted modification of u.

Moreover, using the Burkholder inequality (2.3.2) and the bound property of C_0-group, we argue as follows:

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\in[0,T]} \|\tilde{u}(t)\|^p_{\mathscr{D}(A^{1/2})}\right] &\leq \mathbb{E}\left[\sup_{t\in[0,T]} \|\tilde{\mathfrak{u}}(t)\|^p_{\mathcal{H}}\right] \\
&= \mathbb{E}\left[\sup_{t\in[0,T]} \left\| S(t)\int_0^t S(-s)\tilde{\xi}(s)\,dW(s)\right\|^p_{\mathcal{H}}\right] \\
&\leq K'_T\, B_p(\mathcal{H})\,\mathbb{E}\left[\int_0^T \|S(-s)\tilde{\xi}(s)\|^2_{\gamma(K,\mathcal{H})}ds\right]^{\frac{p}{2}} \\
&\leq K_T\, B_p(\mathcal{H})\,\mathbb{E}\left[\int_0^T \|\tilde{\xi}(s)\|^2_{\gamma(K,\mathcal{H})}ds\right]^{\frac{p}{2}} = K_T\, B_p(\mathcal{H})\,\mathbb{E}\left[\int_0^T \|\xi(s)\|^2_{\gamma(K,H)}ds\right]^{\frac{p}{2}},
\end{aligned}$$

where $K_T \leq M_1 e^{mT}$ for some constants $m \geq 0$ and $M_1 \geq 1$. By substituting $K(p,T,H) := K_T\, B_p(\mathcal{H})$ yields the inequality (2.3.9) and in particular, the assertion I.

We split the proof of assertion II in the following two steps. First we prove the theorem for a more regular process and then transfer the results to the concerned process by an argument of approximation.

Step 1: In this step we assume that ξ is a progressively measurable process from the space

$$M^{2,p}([0,T],\gamma(K,\mathscr{D}(A^k))),$$

where k is a chosen temporary auxiliary natural number such that the Hilbert space $\mathscr{D}(A^{k+1/2})$ is continuously embedded into the Banach space $E = \mathscr{D}(A_q^{(1-r)/2})$. By the classical Sobolev embedding, such a number exists. Thus, by assertion I, we infer that there exists a separable, $\mathscr{D}(A^{k+1/2})$-valued continuous and adapted modification $\tilde{u}$ of the process $u = \{u(t), t \in [0,T]\}$, defined by the formula (2.3.8). Moreover,

$$\mathbb{E}\Big[\sup_{t\in[0,T]} \|\tilde{u}(t)\|^p_{\mathscr{D}(A^{k+1/2})}\Big] \leq K(p,T,H)\,\mathbb{E}\left[\int_0^T \|\xi(t)\|^2_{\gamma(K,\mathscr{D}(A^k))}\,dt\right]^{\frac{p}{2}} < \infty.$$

Also, note that, because of our additional assumption in this step, the process $\tilde{u}$ is an E-valued continuous and adapted. Hence $\tilde{u}$ is an E-valued progressively measurable process. Furthermore,

$$\mathbb{E}\big[\|\tilde{u}\|^p_{L^\infty(0,T;E)}\big] < \infty.$$

Next, we define for each $r \in [0,T]$ an $L^p(0,T;E)$-valued random variable

$$\Xi_r(t,\omega) = \mathbb{1}_{[r,T]}(t)\frac{\sin((t-r)\sqrt{A})}{\sqrt{A}}\xi(r,\omega).$$

Then by the Burkholder inequality (2.3.3) with (2.3.6) we get

$$\begin{aligned}
\mathbb{E}\left[\int_0^T \left\|\left(\int_0^T \Xi_r\, dW(r)\right)(t)\right\|_E^p dt\right] &= \mathbb{E}\left[\left\|\int_0^T \Xi_r\, dW(r)\right\|_{L^p(0,T;E)}^p\right] \\
&\le \hat{B}_p(E)\, \mathbb{E}\left[\int_0^T \|\Xi_r\|^2_{\gamma(K,L^p(0,T;E))}\, dr\right]^{\frac{p}{2}} \\
&\le C_T\; C(p,H)\; C(p,E)\; \hat{B}_p(E)\, \mathbb{E}\left[\int_0^T \|\xi(r)\|^2_{\gamma(K,H)}\, dr\right]^{\frac{p}{2}}.
\end{aligned}$$

Let us define $\tilde{\tilde{u}}$ to be a representative of the $L^p(0,T;E)$-valued random variable $\int_0^T \Xi_r\, dW(r)$. Then we have (2.3.11) and $j\tilde{\tilde{u}}$ is an $L^2(0,T;H)$-valued random variable which is representative of an $L^2(0,T;H)$-valued Itô integral $\int_0^T j(\Xi_r)\, dW(r)$. Since the process $\tilde{u}$ has continuous H_A-valued trajectories, the process $i(\tilde{u}(t)), t \in [0,T]$ determines an $L^2(0,T;H)$-valued random variable denoted by $i(\tilde{u})$ which is a representative of the $L^2(0,T;H)$-valued Itô integral $\int_0^T j(\Xi_r) dW(r)$. Hence, the H-valued random variables $i(\tilde{u}(t))$ and $j(\tilde{\tilde{u}}(t))$, $t \in [0,T]$, are Leb $\otimes\, \mathbb{P}$ equal. Since, the former is H-valued progressively measurable, by the Kuratowski Theorem, see e.g. [126, Corollary I.3.3], we infer that process $\tilde{\tilde{u}}(t)$, $t \in [0,T]$, is E-valued progressively measurable. This concludes the proof of Step 1.

Step 2: Here we transfer the result of Step 1 to the concerned process. Let ξ be a progressively measurable process from the space $M^{2,p}([0,T],\gamma(K,H))$, where k is a temporary auxiliary natural number as in Step 1. We choose a sequence $\{\xi_n\}_{n\in\mathbb{N}}$ of processes from $M^{2,p}([0,T],\gamma(K,\mathscr{D}(A^k)))$ such that

$$\|\xi_n - \xi\|_{M^{2,p}([0,T],\gamma(K,H))} \to 0 \quad \text{sufficiently fast as} \quad n \to \infty. \tag{2.3.15}$$

We denote the corresponding processes for ξ_n, which are valid from previous step, as $\tilde{u}_n$ and $\tilde{\tilde{u}}_n$. By Step 1, for each n, the processes $\tilde{u}_n$ and $\tilde{\tilde{u}}_n$ satisfy the condition (2.3.10), the process $\tilde{u}_n$ satisfies inequality (2.3.9) and the process $\tilde{\tilde{u}}_n$ satisfies inequality (2.3.11). Thus, both sequences are Cauchy in the appropriate Banach spaces $M^{\infty,p}([0,T],H_A)$ and $M^p([0,T],E)$, respectively. Hence, there exist unique elements in those spaces, whose representatives, respectively, we denote by $\tilde{u}$ and $\tilde{\tilde{u}}$. Because the convergence (2.3.15) is sufficiently fast, we deduce that $\mathbb{P}$-a.s., $\tilde{\tilde{u}}_n \to \tilde{\tilde{u}}$ in $L^p(0,T;E)$ and $\tilde{u}_n \to \tilde{u}$ in $\mathcal{C}([0,T];H_A)$. Hence, we infer that $\tilde{u}$ is H_A-valued adapted and continuous process and $\tilde{\tilde{u}}$ is an E-valued progressively measurable process. Moreover, the processes $\tilde{u}$ and $\tilde{\tilde{u}}$ satisfy the condition (2.3.10). Hence we are done with the proof of Theorem 2.3.5. ■

2.4 Local well-posedness

The aim of this section is to formulate a theorem about the existence and uniqueness of solutions to the stochastic wave equation (1.1.4). Let us recall the notation

$$H = L^2(\mathcal{D}); \qquad H_A = \mathscr{D}(A^{1/2}); \qquad E = \mathscr{D}(A_q^{(1-r)/2}), \tag{2.4.1}$$

where $q \in (1,\infty)$ and $r \in [0,1]$. Let us also recall the definition of the spaces Y_T. For any $T>0$, we put

$$Y_T = \mathcal{C}\left([0,T];\mathscr{D}(A^{1/2})\right) \cap L^p\left(0,T;\mathscr{D}(A_q^{(1-r)/2})\right),$$

with norm

$$\|u\|_{Y_T}^p = \sup_{t\in[0,T]} \|u(t)\|_{\mathscr{D}(A^{1/2})}^p + \|u\|_{L^p(0,T;\mathscr{D}(A_q^{(1-r)/2}))}^p.$$

By $\mathbb{M}^p(Y_T)$ we denote the Banach space of $\mathbb{F}$-progressively measurable processes $\{u(t): t\in[0,T]\}$ which are E-valued and have a continuous $\mathscr{D}(A^{1/2})$-valued modification which satisfies

$$\|\xi\|_{\mathbb{M}^p(Y_T)}^p := \mathbb{E}\left[\|\xi\|_{\mathcal{C}([0,T];\mathscr{D}(A^{1/2}))}^p + \|\xi\|_{L^p(0,T;\mathscr{D}(A_q^{(1-r)/2}))}^p\right] < \infty. \tag{2.4.2}$$

We also put

$$X_T := L^p(0,T;\mathscr{D}(A_q^{(1-r)/2})) \quad \text{and} \quad Z_T := \mathcal{C}\left([0,T];\mathscr{D}(A^{1/2})\right), \tag{2.4.3}$$

to shorten the notation during computation.

If T is a bounded $\mathbb{F}$-stopping time, we write $\mathbb{M}^p(Y_T)$ to denote the Banach space of all E-valued $\mathbb{F}$-progressively measurable processes

$$\xi : \{(t,\omega) : \omega\in\Omega, 0\le t\le T(\omega)\} \to H_A \cap E,$$

which have a continuous $\mathscr{D}(A^{1/2})$-valued modification such that for each $\omega\in\Omega$, $\xi(\cdot,\omega)\in Y_{T(\omega)}$ and

$$\mathbb{E}\left[\|\xi\|_{\mathcal{C}([0,T(\omega)];\mathscr{D}(A^{1/2}))}^p + \|\xi\|_{L^p(0,T(\omega);\mathscr{D}(A_q^{(1-r)/2}))}^p\right] < \infty.$$

2.4.1 Considered SNLWE model with assumptions

Here we recall the considered SNLWE and state the assumptions on the nonlinear and diffusion terms. To be precise, we consider the following Cauchy problem of stochastic nonlinear wave equation with Dirichlet or Neumann boundary condition

$$\begin{cases} u_{tt} + Au + F(u) = G(u)\dot{W} & \text{in } [0,\infty)\times\mathcal{D} \\ u(0) = u_0, \;\; u_t(0) = u_1 & \text{on } \mathcal{D}, \end{cases} \tag{2.4.4}$$

where A is either $-\Delta_D$ or $-\Delta_N$; $(u_0,u_1)\in\mathscr{D}(A^{1/2})\times L^2(\mathcal{D})$ and $W=\{W(t): t\ge 0\}$ is a cylindrical Wiener process on some real separable Hilbert space K such that some orthonormal basis $\{f_j\}_{j\in\mathbb{N}}$ of K satisfies

$$\sum_{j\in\mathbb{N}} \|f_j\|_{L^\infty(\mathcal{D})}^2 < \infty. \tag{2.4.5}$$

In (2.4.4), for the nonlinearity F and the diffusion coefficient G we assume the following hypotheses.

A.1 Let H, H_A and E are Banach spaces. Assume that

$$F : H_A \cap E \to H,$$

is a map such that for every $M \in (0,1)$ there exist a constant $C_F > 0$ and $\gamma \in (0,\infty)$ such that the following inequality holds

$$\|F(u)-F(v)\|_H \le C_F\left[1+\frac{\|u\|_E}{M}+\frac{\|v\|_E}{M}\right]^\gamma \|u-v\|_{H_A},$$

provided

(2.4.6) $$u, v \in H_A \cap E \text{ and } \|u\|_{H_A} \le M, \|v\|_{H_A} \le M.$$

A.2 Let H_A and E are Banach spaces, and K and H are Hilbert spaces, moreover, K is separable. Assume that

$$G : H_A \cap E \to \gamma(K,H),$$

is a map such that for every $M \in (0,1)$ there exist $\gamma \in (0,\infty)$ and a constant $C_G > 0$ such that

$$\|G(u)-G(v)\|_{\gamma(K,H)} \le C_G\left[1+\frac{\|u\|_E}{M}+\frac{\|v\|_E}{M}\right]^\gamma \|u-v\|_{H_A},$$

provided u, v satisfy (2.4.6).

Next two lemmata are straightforward but necessary consequences of Assumptions **A.1** and **A.2**.

Lemma 2.4.1. *Let us assume that $T > 0$ and let $F : H_A \cap E \to H$ be a map satisfying Assumption **A.1** for Banach spaces H, H_A and E. If $M \in (0,1)$ and $\gamma \in (0,\infty)$ and C_F are as in Assumption **A.1**, then for $p > \gamma$, the following inequality holds*

$$\|F(u_1)-F(u_2)\|_{L^1(0,T;H)} \le C_F\left[T+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}\|u_1\|_{X_T}^\gamma+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}\|u_2\|_{X_T}^\gamma\right]\|u_1-u_2\|_{Z_T},$$

provided

$$u_1, u_2 \in \mathcal{C}([0,T];H_A)\cap L^p(0,T;E),$$

and

$$\sup_{t\in[0,T]}\|u_i(t)\|_{H_A} \le M, \;\; i = 1,2.$$

Proof of Lemma 2.4.1 Let us choose and fix $u_1, u_2 \in X_T \cap Z_T$. Then, by using Assumption **A.1**, followed by the Hölder inequality, we get

$$\begin{aligned}
\|F(u_1)-F(u_2)\|_{L^1(0,T;H)} &\le C_F\int_0^T\left[1+\frac{\|u_1(t)\|_E}{M}+\frac{\|u_2(t)\|_E}{M}\right]^\gamma\|u_1(t)-u_2(t)\|_{H_A}\,dt\\
&\le C_F\|u_1-u_2\|_{\mathcal{C}([0,T];H_A)}\left[T+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}\left(\int_0^T\|u_1(t)\|_E^p\,dt\right)^{\frac{\gamma}{p}}+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}\left(\int_0^T\|u_2(t)\|_E^p\,dt\right)^{\frac{\gamma}{p}}\right]\\
&\le C_F\left[T+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}\|u_1\|_{X_T}^\gamma+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}\|u_2\|_{X_T}^\gamma\right]\|u_1-u_2\|_{Z_T}.
\end{aligned}$$

Hence Lemma 2.4.1 follows. ∎

Lemma 2.4.2. *Assume that $T > 0$. Let $G : H_A \cap E \to \gamma(K,H)$ be a map satisfying Assumption **A.2** for Banach spaces H_A, E and Hilbert spaces K, H. If $M \in (0,1)$, $\gamma \in (0,\infty)$ and C_G are as in Assumption **A.2**, then for $p > 2\gamma$, the following inequality holds*

$$\|G(u_1) - G(u_2)\|^2_{L^2(0,T;\gamma(K,H))} \le C_G^2 \left[T + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}} \|u_1\|^{2\gamma}_{X_T} + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}} \|u_2\|^{2\gamma}_{X_T} \right] \|u_1 - u_2\|^2_{Z_T},$$

provided

$$u_1, u_2 \in \mathcal{C}([0,T]; H_A) \cap L^p(0,T;E),$$

and

$$\sup_{t\in[0,T]} \|u_i(t)\|_{H_A} \le M, \;\; i = 1,2.$$

Proof of Lemma 2.4.2 Let us use the notation X_T and Z_T introduced the previous proof. Let us choose and fix $u_1, u_2 \in X_T \cap Z_T$. Then, invoking Assumption **A.2** and the Hölder inequality, we obtain

$$\begin{aligned}
&\|G(u_1) - G(u_2)\|^2_{L^2(0,T;\gamma(K,H))} = \int_0^T \|G(u_1(t)) - G(u_2(t))\|^2_{\gamma(K,H)}\, dt \\
&\le C_G^2 \int_0^T \left[1 + \frac{\|u_1(t)\|_E}{M} + \frac{\|u_2(t)\|_E}{M} \right]^{2\gamma} \|u_1(t) - u_2(t)\|^2_{H_A}\, dt \\
&\le C_G^2 \|u_1 - u_2\|^2_{\mathcal{C}([0,T];H_A)} \left[T + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}} \left(\int_0^T \|u_1(t)\|^p_E\, dt \right)^{\frac{2\gamma}{p}} + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}} \left(\int_0^T \|u_2(t)\|^p_E\, dt \right)^{\frac{2\gamma}{p}} \right] \\
&\le C_G^2 \left[T + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}} \|u_1\|^{2\gamma}_{X_T} + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}} \|u_2\|^{2\gamma}_{X_T} \right] \|u_1 - u_2\|^2_{Z_T}.
\end{aligned}$$

Hence the proof of Lemma 2.4.2 is complete. ■

To prove the main result of this section we need the following known results. The first one is from [139].

Theorem 2.4.3. *[Moser-Trudinger Inequality]*
Let $\mathcal{D} \subseteq \mathbb{R}^2$ be a domain (bounded or unbounded), and $\alpha \le 4\pi$. Then

$$C(\alpha) = C(\alpha, \mathcal{D}) := \sup_{\substack{u\in H^{1,2}(\mathcal{D}),\\ \|u\|_{H^{1,2}(\mathcal{D})}\le 1}} \int_{\mathcal{D}} \left(e^{\alpha(u(x))^2} - 1 \right) dx < +\infty. \tag{2.4.7}$$

Moreover, this result is sharp in the sense that for any $\alpha > 4\pi$, the supremum in (2.4.7) *is infinite.*

The next required result is a well known Logarithmic inequality from [125].

Theorem 2.4.4. *Let $p, q, m \in \mathbb{R}$ satisfy $1 < p < \infty, 1 \le q < \infty$, and $m > n/q$. Then there exists a constant C such that for all $u \in H^{\frac{n}{p},p}(\mathcal{D}) \cap H^{m,q}(\mathcal{D})$, where $\mathcal{D}$ is any domain in $\mathbb{R}^n$, the following holds,*

$$\|u\|_{L^\infty(\mathcal{D})} \le C \|u\|_{H^{\frac{n}{p},p}(\mathcal{D})} \left[1 + \log\left(1 + \frac{\|u\|_{H^{m,q}(\mathcal{D})}}{\|u\|_{H^{\frac{n}{p},p}(\mathcal{D})}} \right) \right]^{1-\frac{1}{p}}. \tag{2.4.8}$$

In the next result we provide an example of functions f and g such that the corresponding maps F and G, respectively, satisfy the Assumptions **A.1** and **A.2**. The example below has been considered, in deterministic setting, by [87] and [88], but for the case when E is a suitable Hölder space. We prove the next result in detail because we need a slightly general version of the Moser-Trudinger inequality and the Logarithmic estimate, respectively, see Theorem 2.4.3 and 2.4.4, than used by [87] and [88].

Lemma 2.4.5. *Let $h : \mathbb{R} \to \mathbb{R}$ be a function defined by $h(x) = x\left(e^{4\pi x^2} - 1\right)$ for $x \in \mathbb{R}$. Then for every $M \in (0,1)$, there exist a number $\gamma \in (0,\infty)$, a pair (q,r) satisfying*

$$q > 2, \quad 0 < r < \min\left\{1, \frac{q-2}{2}\right\} \quad \textit{and} \quad r \neq 1 - \frac{1}{q}, \tag{2.4.9}$$

and a positive constant $C_{h,\gamma}$ such that

$$\|h \circ u - h \circ v\|_H \leq C_{h,\gamma} \left[1 + \frac{\|u\|_E}{M} + \frac{\|v\|_E}{M}\right]^\gamma \|u - v\|_{H_A},$$

provided u, v satisfy (2.4.6) *where the spaces H, H_A and E are defined in* (2.4.1).

Next result is about the Nemytskii operator G.

Lemma 2.4.6. *Assume that condition* (2.4.5) *holds. Assume that $g(x) = x\left(e^{4\pi x^2} - 1\right)$, $x \in \mathbb{R}$. Let G be the corresponding generalized Nemytskii operator defined by*

$$G(u) := \{K \ni k \mapsto (g \circ u) \cdot k \in H\}, \;\; u \in H_A \cap E.$$

Then G satisfies the following inequality

$$\|G(u) - G(v)\|_{\gamma(K,H)} \leq C_G \left[1 + \frac{\|u\|_E}{M} + \frac{\|v\|_E}{M}\right]^\gamma \|u - v\|_{H_A},$$

for all $u, v \in H_A \cap E$ such that u, v satisfy (2.4.6), *where the spaces H, H_A and E are defined in* (2.4.1) *and*

$$C_G := C_{h,\gamma} \left[\sum_{j \in \mathbb{N}} \|f_j\|^2_{L^\infty(\mathcal{D})}\right].$$

Proof of Lemma 2.4.6 By (2.4.5) and Lemma 2.4.5 (applied to $h = g$) we infer that

$$\begin{aligned}
\|G(u) - G(v)\|^2_{\gamma(K,L^2(\mathcal{D}))} &= \sum_{j \in \mathbb{N}} \|G(u)f_j - G(v)f_j\|^2_{L^2(\mathcal{D})} \\
&= \sum_{j \in \mathbb{N}} \|(g \circ u)f_j - (g \circ v)f_j\|^2_{L^2(\mathcal{D})} \leq \|g \circ u - g \circ v\|^2_{L^2(\mathcal{D})} \sum_{j \in \mathbb{N}} \|f_j\|^2_{L^\infty(\mathcal{D})} \\
&\leq \left[\sum_{j \in \mathbb{N}} \|f_j\|^2_{L^\infty(\mathcal{D})}\right] C_{h,\gamma} \left[1 + \frac{\|u\|_E}{M} + \frac{\|v\|_E}{M}\right]^\gamma \|u - v\|_{H_A},
\end{aligned}$$

as desired. ■

Proof of Lemma 2.4.5 Let $u, v \in H_A \cap E$. Then, by the Mean value theorem, for every x, there exists $\theta = \theta(x) \in (0,1)$ such that

$$u\left(e^{4\pi u^2}-1\right) - v\left(e^{4\pi v^2}-1\right) = (u-v)\left[\left(1+8\pi u_\theta^2\right)e^{4\pi u_\theta^2}-1\right], \tag{2.4.10}$$

with $u_\theta(x) = (1-\theta(x))u(x) + \theta(x)v(x)$. Thus, the triangle inequality and (2.4.6) gives

$$\|u_\theta\|_{H_A} \leq M. \tag{2.4.11}$$

Also, by (2.4.10) we get

$$\|h\circ u - h\circ v\|_{L^2(\mathfrak{D})} \leq \left\|(u-v)\left[\left(1+8\pi u_\theta^2\right)e^{4\pi u_\theta^2}-1\right]\right\|_{L^2(\mathfrak{D})}. \tag{2.4.12}$$

Applying the basic inequality,

$$(1+2a)e^a - 1 \leq 2\left(1+\frac{1}{\varepsilon}\right)\left(e^{(1+\varepsilon)a}-1\right), \quad \forall a, \varepsilon > 0,$$

followed by the Hölder inequality with Sobolev embedding, for any $\zeta \in (0,1)$ and $\varepsilon > 0$, we argue as follows:

$$\begin{aligned}
\left\|(u-v)\left[\left(1+8\pi u_\theta^2\right)e^{4\pi u_\theta^2}-1\right]\right\|_{L^2(\mathfrak{D})}^2 &\lesssim \left\|(u-v)\left(e^{4\pi(1+\varepsilon)u_\theta^2}-1\right)\right\|_{L^2(\mathfrak{D})}^2 \\
&\lesssim \|u-v\|_{L^{2+\frac{2}{\zeta}}(\mathfrak{D})}^2 \left\|\left(e^{4\pi(1+\varepsilon)u_\theta^2}-1\right)^2\right\|_{L^{1+\zeta}(\mathfrak{D})} \\
&\lesssim \|u-v\|_{\mathscr{D}(A^{1/2})}^2 \left\|\left(e^{4\pi(1+\varepsilon)u_\theta^2}-1\right)^2\right\|_{L^{1+\zeta}(\mathfrak{D})} \\
&\lesssim \|u-v\|_{\mathscr{D}(A^{1/2})}^2 \, e^{4\pi(1+\varepsilon)\|u_\theta^2\|_{L^\infty(\mathfrak{D})}} \left\|e^{4\pi(1+\varepsilon)u_\theta^2}-1\right\|_{L^{1+\zeta}(\mathfrak{D})}.
\end{aligned} \tag{2.4.13}$$

Moreover, since u_θ satisfy (2.4.11), the Moser-Trudinger inequality from Theorem 2.4.3 gives

$$\|e^{4\pi(1+\varepsilon)u_\theta^2}-1\|_{L^{1+\zeta}(\mathfrak{D})}^{1+\zeta} \leq \|e^{4\pi(1+\varepsilon)(1+\zeta)u_\theta^2}-1\|_{L^1(\mathfrak{D})} \leq C := C(4\pi, \mathfrak{D}), \tag{2.4.14}$$

provided that $\varepsilon > 0$ and $\zeta > 0$ are chosen such that

$$(1+\varepsilon)(1+\zeta)M^2 \leq 1.$$

Invoking the log estimate from Theorem 2.4.4, which is possible due to (2.4.9) and Lemma 2.1.2, we obtain

$$e^{4\pi(1+\varepsilon)\|u_\theta\|_{L^\infty(\mathfrak{D})}^2} \leq \exp\left[4\pi C^2(1+\varepsilon)\|u_\theta\|_{H^{1,2}(\mathfrak{D})}^2\left\{1+\log\left(1+\frac{\|u_\theta\|_{\mathscr{D}(A_q^{(1-r)/2})}}{\|u_\theta\|_{H^{1,2}(\mathfrak{D})}}\right)\right\}\right].$$

Using the fact that for any $b > 0$, the function $x \mapsto x^2\left(1+\log\left(1+\frac{b}{x}\right)\right)$ is non-decreasing, we deduce that,

$$e^{4\pi(1+\varepsilon)\|u_\theta\|_{L^\infty(\mathfrak{D})}^2} \leq \left[e\left(1+\frac{\|u_\theta\|_{\mathscr{D}(A_q^{(1-r)/2})}}{M}\right)\right]^{4\pi C^2(1+\varepsilon)M^2}. \tag{2.4.15}$$

By setting

$$\gamma := 2\pi C^2(1+\varepsilon)M^2,$$

from (2.4.12), (2.4.13), (2.4.14) and (2.4.15), we get

$$\begin{aligned}\|h\circ u - h\circ v\|_{L^2(\mathcal{D})} &\le \left\|(u-v)\left[\left(1+8\pi u_\theta^2\right)e^{4\pi u_\theta^2}-1\right]\right\|_{L^2(\mathcal{D})}\\ &\le e^\gamma\, C^{\frac{1}{(1+\zeta)}}\, \|u-v\|_{\mathscr{D}(A^{1/2})}\left(1+\frac{\|u_\theta\|_{\mathscr{D}(A_q^{(1-r)/2})}}{M}\right)^\gamma\\ &\le e^\gamma\, C^{\frac{1}{(1+\zeta)}}\, \|u-v\|_{\mathscr{D}(A^{1/2})}\left(1+\frac{\|u\|_{\mathscr{D}(A_q^{(1-r)/2})}}{M}+\frac{\|v\|_{\mathscr{D}(A_q^{(1-r)/2})}}{M}\right)^\gamma.\end{aligned}$$

Hence the Lemma 2.4.5 follows. ■

Remark 2.4.7. It is obvious, see e.g. [22], that the previous two lemmata hold for all polynomial functions.

2.4.2 Definition of a local mild solution

In this subsection we introduce the definitions of local and maximal local solutions that we adopt in this chapter; they are modifications of definitions used earlier, such as in [21].

Definition 2.4.8. A local mild solution to Problem (2.4.4) is a $\mathscr{D}(A^{1/2})$-valued continuous and adapted process $u=\{u(t): t\in[0,\tau)\}$, where

1. τ is an accessible $\mathbb{F}$-stopping time,
2. there exists an approximating sequence $\{\tau_n\}_{n\ge 1}$ of $\mathbb{F}$-stopping times for τ, such that

   $$u \text{ belongs to } \mathbb{M}^p(Y_{t\wedge\tau_n}) \text{ for all } t \text{ and every } n,$$

   and,

   $$\begin{aligned}u(t\wedge\tau_n) &= \cos((t\wedge\tau_n)\sqrt{A})u_0 + \frac{\sin((t\wedge\tau_n)\sqrt{A})}{\sqrt{A}}u_1\\ &\quad+\int_0^{t\wedge\tau_n}\frac{\sin((t\wedge\tau_n-s)\sqrt{A})}{\sqrt{A}}F(u(s))\,ds + I_{\tau_n}(G(u))(t\wedge\tau_n), \mathbb{P}\text{-a.s.},\end{aligned}$$

   for all $t\ge 0$ and $n\in\mathbb{N}$, where we define

   $$I_{\tau_n}(G(u))(t) := \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}\left(\mathbb{1}_{[0,\tau_n)}(s)G(u(s))\right)dW(s). \tag{2.4.16}$$

A local mild solution $u=\{u(t): t\in[0,\tau)\}$ to Problem (2.4.4) is called a local maximal mild solution to Problem (2.4.4) iff for any other local mild solution $\hat{u}=\{\hat{u}(t), t\in[0,\hat{\tau})\}$ to Problem (2.4.4) such that

$$\mathbb{P}(\hat{\tau}>\tau)>0,$$

there exists a measurable set $\hat{\Omega} \subset \{\hat{\tau} > \tau\}$ such that $\mathbb{P}(\hat{\Omega}) > 0$ and $u(\tau) \neq \hat{u}(\tau)$ on $\hat{\Omega}$.

In other words, a local mild solution $u = \{u(t), t \in [0,\tau)\}$ to Problem (2.4.4) is not a maximal local mild solution, iff there exists another local mild solution $\hat{u} = \{\hat{u}(t), t \in [0,\hat{\tau})\}$ to Problem (2.4.4) such that

$$\mathbb{P}\left(\hat{\tau} > \tau, u(\tau) = \hat{u}(\tau)\right) > 0.$$

If $u = \{u(t), t \in [0,\tau)\}$ is a local maximal solution to Problem (2.4.4), the stopping time τ is called the explosion time of u.

A local mild solution $u = \{u(t) : t \in [0,\tau)\}$ to problem (2.4.4) is unique iff for any other local solution $\hat{u} = \{\hat{u}(t) : t \in [0,\hat{\tau})\}$ to problem (2.4.4) the restricted processes $u|_{[0,\tau\wedge\hat{\tau})\times\Omega}$ and $\hat{u}|_{[0,\tau\wedge\hat{\tau})\times\Omega}$ are equivalent.

Remark 2.4.9. The definition of the process I_{τ_n} is explained in Lemma 2.5.1 of Subsection 2.5.1. The use of processes I_{τ_n} was first introduced for the SPDEs of parabolic type in [14] and [42] and in [21] for the hyperbolic SPDEs. The definition we use above is only in terms of the process u and thus it is different from the one used in [21] which is in terms of pair processes (u, u_t). In Subsection 2.5.2 we discuss an equivalence between these two approaches.

2.4.3 Existence and uniqueness result

The main result of the present chapter, i.e. the existence of an unique local maximal solution to the Problem (2.4.4), will be proved in this subsection.

Theorem 2.4.10. *Let us assume that (γ, p, q, r) is a quadruple such that*

$$0 < 2\gamma < p \qquad \textit{and} \qquad (p,q,r) \textit{ satisfy } (2.2.2).$$

Let H, H_A and E be Hilbert and Banach spaces defined in (2.4.1). *Let us assume that the maps*

$$F : E \cap H_A \to H \qquad \textit{and} \qquad G : E \cap H_A \to \gamma(K,H),$$

*where K is a separable Hilbert space, satisfy Assumptions **A.1** and **A.2**. Then for every*

$$(u_0, u_1) \in \mathscr{D}(A^{1/2}) \times L^2(\mathcal{D}) \qquad \textit{satisfying} \qquad \|u_0\|_{\mathscr{D}(A^{1/2})} < 1, \tag{2.4.17}$$

there exists a unique local maximal mild solution $u = \{u(t) : t \in [0,\tau)\}$, to the Problem (2.4.4)*, in the sense of Definition 2.4.8 for some accessible bounded stopping time $\tau > 0$.*

Remark 2.4.11. It is relevant to note that the solution $u = \{u(t) : t \in [0,\tau)\}$ we construct later on will satisfy the following,

$$\|u(t)\|_{\mathscr{D}(A^{1/2})} < 1, \quad \text{for } t \in [0,\tau), \quad \mathbb{P}\text{-a.s.}.$$

Proof of Theorem 2.4.10 We start the proof by remarking that it is enough to prove the existence of an unique local mild solution. Indeed, once we get such a result, the existence of a unique local maximal mild solution follows by methods which are standard now, see e.g. [22, Theorem 5.4] and references therein.

The proof is divided in four steps. First two steps are devoted to prove the existence and uniqueness of the solution of the truncated evolution equation. In Step III we prove the existence of a local mild solution, in the sense of Definition 2.4.8, to Problem (2.4.4). We complete the proof in Step IV by proving a local uniqueness result.

Step I: Here we define the truncated evolution equation, related to Problem (2.4.4), and prove a few required estimates which allow us to show local well-posedness of truncated equation in Step II.

Since the initial position u_0 is given and the norm $\|u_0\|_{\mathscr{D}(A^{1/2})}$ is less than 1, there exist M, M' in $(0,1)$ such that

$$\|u_0\|_{\mathscr{D}(A^{1/2})} < M' < M < 1.$$

To derive the truncated equation we introduce the following two auxiliary functions. Let $\theta : \mathbb{R}_+ \to [0,1]$ be a smooth function with compact support such that

$$\theta(x) = \begin{cases} 1, & \text{if } x \in [0,1], \\ 0, & \text{if } x \in [2,\infty), \end{cases}$$

and, for $n \geq 1$, set $\theta_n(\cdot) = \theta\left(\frac{\cdot}{n}\right)$. As another cut off function, we take $\hat{\theta} : \mathbb{R}_+ \to [0,1]$, a smooth function with compact support such that

$$\hat{\theta}(x) = \begin{cases} 1, & \text{if } x \in [0,M'], \\ 0, & \text{if } x \in [M,\infty). \end{cases}$$

We have the following lemmata about $\theta_n' s$ and $\hat{\theta}$ as a consequence of their description.

Lemma 2.4.12. *The maps* $\hat{\theta}$ *and*

$$\theta\hat{\theta} : \mathbb{R}_+ \ni x \to \theta(x)\hat{\theta}(x) \in [0,1],$$

are Lipschitz and bounded.

Lemma 2.4.13. *If* $h : \mathbb{R}_+ \to \mathbb{R}_+$ *is a non decreasing function, then for every* $x, y \in \mathbb{R}$,

$$\theta_n(x)h(x) \leq h(2n), \quad |\theta_n(x) - \theta_n(y)| \leq \frac{1}{n}|x-y|.$$

To achieve the aim of Step I, for each $n \in \mathbb{N}$ and $T > 0$, with the use of auxiliary functions $\theta, \hat{\theta}$ we define the map Ψ_T^n by

$$\Psi_T^n : \mathbb{M}^p(Y_T) \ni v \mapsto u \in \mathbb{M}^p(Y_T), \tag{2.4.18}$$

if and only if u satisfies the following equation, for all $t \in [0,T]$,

$$\begin{aligned} u(t) &= \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 \\ &+ \int_0^t \theta_n(\|v\|_{Y_s})\hat{\theta}(\|v\|_{Z_s})\frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}(F(v(s)))\,ds \\ &+ \int_0^t \theta_n(\|v\|_{Y_r})\,\hat{\theta}(\|v\|_{Z_r})\frac{\sin((t-r)\sqrt{A})}{\sqrt{A}}(G(v(r)))\,dW(r),\ \mathbb{P}\text{-a.s..} \end{aligned} \tag{2.4.19}$$

Now we show that, for each $n \in \mathbb{N}$, there exists $T_n > 0$ such that the right hand side of (2.4.19) is a strict contraction. We divide our argument in a couple of lemmata.

Lemma 2.4.14. *For any $T > 0$, the map*

$$\mathscr{I}_1^n : \mathscr{D}(A^{1/2}) \times L^2(\mathcal{D}) \ni (u_0,u_1) \mapsto \left\{[0,T] \ni t \mapsto \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1\right\} \in \mathbb{M}^p(Y_T),$$

is well-defined.

Proof of Lemma 2.4.14 It is known that, see e.g. [5], $w := \mathscr{I}_1^n(u_0,u_1)$ is the unique solution of the following homogeneous wave equation with the Dirichlet or the Neumann boundary condition

$$\begin{cases} \partial_{tt}w - \Delta w = 0 \\ w(0,\cdot) = u_0(\cdot),\ \ \partial_t w(0,\cdot) = u_1(\cdot), \end{cases}$$

and w belongs to $\mathcal{C}([0,T];H_A) = Z_T$. Moreover, due to Theorem 2.2.2, w belongs to X_T and satisfy

$$\|w\|_{X_T} \le C_T\left[\|u_0\|_{\mathscr{D}(A^{1/2})} + \|u_1\|_{L^2(\mathcal{D})}\right].$$

So, for every $\omega \in \Omega$, $w \in X_T \cap Z_T$ and (2.4.2) is satisfied. Furthermore, since w is an adapted and continuous process, it is progressively measurable and, hence, we have proved Lemma 2.4.14. □

Lemma 2.4.15. *For any $T > 0$, the following map*

$$\mathscr{I}_2^n : \mathbb{M}^p(Y_T) \ni v \mapsto \left\{[0,T] \ni t \mapsto \int_0^t \theta_n(\|v\|_{Y_s})\hat{\theta}(\|v\|_{Z_s})\frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}(F(v(s)))\,ds\right\} \in \mathbb{M}^p(Y_T)$$

is well-defined.

Proof of Lemma 2.4.15 Take any $v \in \mathbb{M}^p(Y_T)$ and $\tilde{v} := \mathscr{I}_2^n(v)$. Then, for fix $t \in [0,T]$, we have

$$\begin{aligned} \|\tilde{v}\|_{Z_T} \le \sup_{t\in[0,T]} \|\tilde{v}(t)\|_{H_A} &\le \sup_{t\in[0,T]} \left\| \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} \theta_n(\|v\|_{Y_s})\hat{\theta}(\|v\|_{Z_s})F(v(s))\,ds \right\|_{H_A} \\ &\le K_T \int_0^T \theta_n(\|v\|_{Y_s})\hat{\theta}(\|v\|_{Z_s})\|F(v(s))\|_H\,ds. \end{aligned} \tag{2.4.20}$$

Note that, in the last step above we have used the following consequence of the bound property of C_0-group $\{S(t)\}_{t\geq 0}$,

$$\left\| \frac{\sin(t\sqrt{A})}{\sqrt{A}} \right\| \leq K_T, \quad t \in [0,T], \tag{2.4.21}$$

where $K_T := M_1 e^{mT}$ for some constants $m \geq 0$ and $M_1 \geq 1$. Let T_1^* and T_2^* be the stopping times defined by

$$T_1^* := \inf\{t \in [0,T] : \|v\|_{Z_t} \geq M\}, \tag{2.4.22}$$

and

$$T_2^* := \inf\{t \in [0,T] : \|v\|_{Y_T} \geq 2n\}. \tag{2.4.23}$$

If the set in the definition of T_i^* is empty, then we set $T_i^* = T$. Now, we define the following $\mathbb{F}$-stopping time

$$T^* := \min\{T_1^*, T_2^*\}.$$

Returning back to (2.4.20) and by applying (2.4.23) we get

$$\int_0^T \theta_n(\|v\|_{Y_s})\, \hat{\theta}(\|v\|_{Z_s})\, \|F(v(s))\|_{L^2(\mathcal{D})}\, ds \leq \int_0^{T^*} \|F(v(s))\|_{L^2(\mathcal{D})}\, ds = \|F(v)\|_{L^1(0,T^*;L^2(\mathcal{D}))}. \tag{2.4.24}$$

In view of (2.4.22) and (2.4.23), we infer that $\mathbb{P}$-a.s. $\|v\|_{Z_{T^*}} \leq M$ and $\|v\|_{Y_{T^*}} \leq 2n$. Thus, since $F(0) = 0$, by Lemma 2.4.1 the following argument holds,

$$\|F(v)\|_{L^1(0,T^*,L^2(\mathcal{D}))} \leq C_F \left(T + \frac{T^{1-\frac{\gamma}{p}}}{M^\gamma} \|v\|_{Y_{T^*}}^\gamma \right) \|v\|_{Y_{T^*}} \leq 2n\, C_F \left(T + \frac{T^{1-\frac{\gamma}{p}}}{M^\gamma} (2n)^\gamma \right). \tag{2.4.25}$$

Combining (2.4.20), (2.4.24) and (2.4.25) we have

$$\mathbb{E}\left[\|\tilde{v}\|_{Z_T}^p \right] \leq (2n)^p\, C_F^p\, K_T^p \left(T + \frac{T^{1-\frac{\gamma}{p}}}{M^\gamma} (2n)^\gamma \right)^p. \tag{2.4.26}$$

Invoking, the inhomogeneous Strichartz estimates from Theorem 2.2.2 followed by (2.4.25) we get

$$\begin{aligned} \|\tilde{v}\|_{X_T} &\leq C_T \int_0^{T^*} \theta_n(\|v\|_{Y_s}) \hat{\theta}(\|v\|_{Z_s})\, \|F(v(s))\|_{L^2(\mathcal{D})}\, ds \\ &\leq C_T \|F(v)\|_{L^1(0,T^*;L^2(\mathcal{D}))} \leq 2n\, C_F\, C_T \left(T + \frac{T^{1-\frac{\gamma}{p}}}{M^\gamma} (2n)^\gamma \right), \end{aligned}$$

which consequently gives,

$$\mathbb{E}\left[\|\tilde{v}\|_{X_T}^p \right] \leq (2n)^p\, C_F^p\, C_T^p \left(T + \frac{T^{1-\frac{\gamma}{p}}}{M^\gamma} (2n)^\gamma \right)^p. \tag{2.4.27}$$

Finally by estimates (2.4.26) and (2.4.27) we have

$$\mathbb{E}\left[\|\tilde{v}\|_{Y_T}^p \right] \lesssim (2n)^p\, C_F^p\, (C_T^p + K_T^p) \left(T + \frac{T^{1-\frac{\gamma}{p}}}{M^\gamma} (2n)^\gamma \right)^p,$$

and hence we have Lemma 2.4.15. $\square$

The next result establishes the Lipschitz properties of $\mathscr{I}_2^n$ as a map acting on $\mathbb{M}^p(Y_T)$.

Lemma 2.4.16. *Fix any $T>0$. There exists a constant $L_2^n(T)>0$ such that the following assertions are true:*

- *$L_2^n(\cdot)$ is non decreasing;*
- *for every $n\in\mathbb{N}$, $\lim_{T\to 0} L_2^n(T)=0$;*
- *for every $v_1, v_2 \in \mathbb{M}^p(Y_T)$, $\mathscr{I}_2^n$ satisfy*

$$\|\mathscr{I}_2^n(v_1)-\mathscr{I}_2^n(v_2)\|_{\mathbb{M}^p(Y_T)} \lesssim_p \ L_2^n(T)\ \|v_1-v_2\|_{\mathbb{M}^p(Y_T)}.$$

Proof of Lemma 2.4.16 Let $v_1, v_2 \in \mathbb{M}^p(Y_T)$. Since $\mathscr{I}_2^n$ is well defined, we denote $\tilde v_1 := \mathscr{I}_2^n(v_1)$ and $\tilde v_2 := \mathscr{I}_2^n(v_2) \in \mathbb{M}^p(Y_T)$. As in Lemma 2.4.15, we define the following $\mathbb{F}$-stopping times

$$\begin{aligned}
&T_1^i := \inf\{t\in[0,T] : \|v_i\|_{Z_t} \geq M\}, \ \ i=1,2,\\
&T_2^i := \inf\{t\in[0,T] : \|v_i\|_{Y_t} \geq 2n\} \ \ i=1,2,\\
&T_1^* := \min\{T_1^1, T_2^1\} \text{ and } T_2^* := \min\{T_1^2, T_2^2\}.
\end{aligned}$$

Invoking the inhomogeneous Strichartz estimates from Theorem 2.2.2, followed by Lemma 2.4.1 and Lemma 2.4.13 with the above defined stopping times, we argue as follows:

$$\begin{aligned}
\mathbb{E}\left[\|\tilde v_1-\tilde v_2\|_{X_T}^p\right] &\leq C_T^p\, \mathbb{E}\left[\int_0^T \left\| \left[\theta_n(\|v_1\|_{Y_s})\hat\theta(\|v_1\|_{Z_s})\ F(v_1(s)) - \theta_n(\|v_2\|_{Y_s})\hat\theta(\|v_2\|_{Z_s})\ F(v_2(s))\right]\right\|_H ds\right]^p\\
&\lesssim_p C_T^p\, \mathbb{E}\left[\int_0^T \mathbb{1}_{T_1^*\leq T_2^*}(s) \left\|\theta_n(\|v_1\|_{Y_s})\hat\theta(\|v_1\|_{Z_s})\ F(v_1(s)) - \theta_n(\|v_2\|_{Y_s})\hat\theta(\|v_2\|_{Z_s})\ F(v_2(s))\right\|_H ds\right]^p\\
&\quad + C_T^p\, \mathbb{E}\left[\int_0^T \mathbb{1}_{T_2^*\leq T_1^*}(s) \left\|\theta_n(\|v_1\|_{Y_s})\hat\theta(\|v_1\|_{Z_s})\ F(v_1(s)) - \theta_n(\|v_2\|_{Y_s})\hat\theta(\|v_2\|_{Z_s})\ F(v_2(s))\right\|_H ds\right]^p\\
&\lesssim_p C_T^p\, \mathbb{E}\left[\int_0^T \mathbb{1}_{\{T_1^*\leq T_2^*\}}(t)\theta_n(\|v_1\|_{Y_t})\hat\theta(\|v_1\|_{Z_t})\|F(v_1(t))-F(v_2(t))\|_H\, dt\right]^p\\
&\quad + C_T^p\, \mathbb{E}\left[\int_0^T \mathbb{1}_{\{T_1^*\leq T_2^*\}}(t)|\theta_n(\|v_1\|_{Y_t})\hat\theta(\|v_1\|_{Z_t}) - \theta_n(\|v_2\|_{Y_t})\hat\theta(\|v_2\|_{Z_t})|\|F(v_2(t))\|_H\, dt\right]^p\\
&\quad + C_T^p\, \mathbb{E}\left[\int_0^T \mathbb{1}_{\{T_2^*\leq T_1^*\}}(t)\theta_n(\|v_2\|_{Y_t})\hat\theta(\|v_2\|_{Z_t})\|F(v_1(t))-F(v_2(t))\|_H\, dt\right]^p\\
&\quad + C_T^p\, \mathbb{E}\left[\int_0^T \mathbb{1}_{\{T_2^*\leq T_1^*\}}(t)|\theta_n(\|v_1\|_{Y_t})\hat\theta(\|v_1\|_{Z_t}) - \theta_n(\|v_2\|_{Y_t})\hat\theta(\|v_2\|_{Z_t})|\|F(v_1(t))\|_H\, dt\right]^p\\
&\leq C_T^p\, \mathbb{E}\left[\int_0^{T_1^*\wedge T_2^*} \theta_n(\|v_1\|_{Y_t})\hat\theta(\|v_1\|_{Z_t})\|F(v_1(t))-F(v_2(t))\|_H\, dt\right]^p\\
&\quad + C_T^p\, \mathbb{E}\left[\int_0^{T_1^*\wedge T_2^*} |\theta_n(\|v_1\|_{Y_t})\hat\theta(\|v_1\|_{Z_t}) - \theta_n(\|v_2\|_{Y_t})\hat\theta(\|v_2\|_{Z_t})|\|F(v_2(t))\|_H\, dt\right]^p\\
&\quad + C_T^p\, \mathbb{E}\left[\int_0^{T_1^*\wedge T_2^*} \theta_n(\|v_2\|_{Y_t})\hat\theta(\|v_2\|_{Z_t})\|F(v_1(t))-F(v_2(t))\|_H\, dt\right]^p\\
&\quad + C_T^p\, \mathbb{E}\left[\int_0^{T_1^*\wedge T_2^*} |\theta_n(\|v_1\|_{Y_t})\hat\theta(\|v_1\|_{Z_t}) - \theta_n(\|v_2\|_{Y_t})\hat\theta(\|v_2\|_{Z_t})|\|F(v_1(t))\|_H\, dt\right]^p
\end{aligned}$$

$$
\begin{aligned}
&\lesssim_p C_F^p C_T^p \mathbb{E}\left[\sup_{t\in[0,T]} \|v_1-v_2\|_{Y_t}\left(T+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}\|v_1\|^\gamma_{Y_{T_1^*\wedge T_2^*}}+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}\|v_2\|^\gamma_{Y_{T_1^*\wedge T_2^*}}\right)\right]^p\\
&+C_F^p C_T^p \mathbb{E}\left[\|v_1-v_2\|_{Z_T}\int_0^{T_1^*\wedge T_2^*}\|F(v_1(t))\|_H\,dt\right]^p+\frac{1}{n}\mathbb{E}\left[\|v_1-v_2\|_{Y_T}\int_0^{T_1^*\wedge T_2^*}\|F(v_1(t))\|_H\,dt\right]^p\\
&+C_F^p C_T^p \mathbb{E}\left[\|v_1-v_2\|_{Z_T}\int_0^{T_1^*\wedge T_2^*}\|F(v_2(t))\|_H\,dt\right]^p+\frac{1}{n}\mathbb{E}\left[\|v_1-v_2\|_{Y_T}\int_0^{T_1^*\wedge T_2^*}\|F(v_2(t))\|_H\,dt\right]^p\\
&\lesssim n^p\, C_F^p\, C_T^p\, \|v_1-v_2\|^p_{\mathbb{M}^p(Y_T)}\left(T+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}(2n)^\gamma\right)^p.
\end{aligned}
$$

Next, using the bound property (2.4.21), followed by repeating the calculations as above, we obtain

$$
\begin{aligned}
\mathbb{E}\left[\|\tilde v_1-\tilde v_2\|^p_{Z_T}\right] &\lesssim \mathbb{E}\left[\int_0^T\left\|\left[\theta_n(\|v_1\|_{Y_s})\hat\theta(\|v_1\|_{Z_s})\,F(v_1(s))-\theta_n(\|v_2\|_{Y_s})\hat\theta(\|v_2\|_{Z_s})\,F(v_2(s))\right]\right\|_H ds\right]^p\\
&\lesssim n^p\, C_F^p\, K_T^p\, \|v_1-v_2\|^p_{\mathbb{M}^p(Y_T)}\left(T+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}(2n)^\gamma\right)^p.
\end{aligned}
$$

Consequently, we get

$$
\begin{aligned}
\|\tilde v_1-\tilde v_2\|^p_{\mathbb{M}^p(Y_T)} &\lesssim_p\; n^p\, C_F^p\,(C_T^p+K_T^p)\left(T+\frac{T^{1-\frac{\gamma}{p}}}{M^\gamma}(2n)^\gamma\right)^p\, \|v_1-v_2\|^p_{\mathbb{M}^p(Y_T)}\\
&=: (L_2^n(T))^p\;\|v_1-v_2\|^p_{\mathbb{M}^p(Y_T)}.
\end{aligned}
$$

Since $\gamma<p$, by definition of $L_2^n(T)$, it is clear that, for each $n\in\mathbb{N}$, $\lim_{T\to 0} L_2^n(T)=0$. Thus we have proved the Lemma 2.4.16. □

In continuation of the proof of Theorem 2.4.10, we set

$$
\xi^n(t) := \theta_n(\|v\|_{Y_t})\hat\theta(\|v\|_{Z_t})G(v(t)),\quad t\in[0,T],
$$

then by (2.3.12), we write

$$
\begin{aligned}
&\int_0^t \theta_n(\|v\|_{Y_r})\,\hat\theta(\|v\|_{Z_r})\frac{\sin((t-r)\sqrt{A})}{\sqrt{A}}(G(v(r)))\,dW(r)\\
&=: \int_0^t \frac{\sin((t-r)\sqrt{A})}{\sqrt{A}}\xi^n(r)\,dW(r) =: [J\xi^n](t),\qquad t\in[0,T]. \qquad (2.4.28)
\end{aligned}
$$

In the next result, we show that I_3^n maps $\mathbb{M}^p(Y_T)$ into itself.

Lemma 2.4.17. *For any $T>0$, the map*

$$
\mathscr{I}_3^n : \mathbb{M}^p(Y_T)\ni v\mapsto J\xi^n\in\mathbb{M}^p(Y_T), \qquad (2.4.29)
$$

where $J\xi^n$ is as (2.4.28)*, is well-defined.*

Proof of Lemma 2.4.17 First observe that from (2.3.11), we have

$$\mathbb{E}\left[\|[J\xi^n]\|^p_{L^p(0,T;E)}\right] = \mathbb{E}\left[\int_0^T \|[J\xi^n](t)\|^p_E\, dt\right]$$

$$\leq \tilde{C}(p,T,E,H)\,\mathbb{E}\left[\int_0^T \|\xi^n(t)\|^2_{\gamma(K,H)}\, dt\right]^{\frac{p}{2}}. \tag{2.4.30}$$

As in Lemma 2.4.16, define the $\mathbb{F}$-stopping times as

$$T_1^* := \inf\{t \in [0,T] : \|v\|_{Z_t} \geq M\}, \qquad T_2^* := \inf\{t \in [0,T] : \|v\|_{Y_t} \geq 2n\}$$

and set

$$T^* := \min\{T_1^*, T_2^*\}.$$

In view of the above definition of stopping times $\theta_n(\|v\|_{Y_t}) = 0$, $\hat{\theta}(\|v\|_{Z_t}) = 0$ for all $t \in [T^*, T]$, and

$$\|v\|_{Y_{T^*}} \leq 2n, \qquad \text{and} \qquad \|v\|_{Z_{T^*}} \leq M, \ \mathbb{P}\text{-a.s.}.$$

Invoking Lemma 2.4.2, followed by the Hölder inequality, we get

$$\int_0^T \|\xi^n(t)\|^2_{\gamma(K,H)}\, dt = \int_0^T \theta_n(\|v\|_{Y_t})\hat{\theta}(\|v\|_{Z_t})\ \|G(v(t))\|^2_{\gamma(K,H)}\, dt$$

$$\leq \int_0^{T^*} \|G(v(t))\|^2_{\gamma(K,H)}\, dt \leq C_G^2 \sup_{t\in[0,T^*]} \|v(t)\|^2_{H_A}\left[T + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}\|v\|^{2\gamma}_{X_{T^*}}\right]$$

$$\leq (2n)^2\, C_G^2\, \left[T + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}(2n)^{2\gamma}\right]. \tag{2.4.31}$$

Consequently, by putting (2.4.31) in (2.4.30) we obtain

$$\mathbb{E}\left[\int_0^T \|[J\xi^n](t)\|^p_E\, dt\right] \leq (2n)^p\, C_G^p\, \tilde{C}(p,T,E,H)\ \left[T + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}(2n)^{2\gamma}\right]^{\frac{p}{2}}. \tag{2.4.32}$$

Next, to estimate $\mathbb{E}\left[\|J\xi^n\|^p_{\mathcal{C}(0,T;H_A)}\right]$, using the stochastic Strichartz estimate from Theorem 2.3.5, followed by (2.4.31), we get

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|[J\xi^n](t)\|^p_{H_A}\right] \leq K(p,T,H)\,\mathbb{E}\left[\int_0^T \|\xi^n(t)\|^2_{\gamma(K,H)}\, dt\right]^{\frac{p}{2}}$$

$$\leq (2n)^p\, C_G^p\, K(p,T,H)\ \left[T + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}(2n)^{2\gamma}\right]^{\frac{p}{2}}. \tag{2.4.33}$$

Combining (2.4.32) and (2.4.33) we have

$$\mathbb{E}\left[\|J\xi^n\|^p_{\mathcal{C}(0,T;H_A)} + \|J\xi^n\|^p_{L^p(0,T;E)}\right] \leq (2n)^p\, C_G^p\, (K(p,T,H) + \tilde{C}(p,T,E,H))\ \left[T + \frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}(2n)^{2\gamma}\right]^{\frac{p}{2}},$$

and hence the Lemma 2.4.17. □

The next result establishes the Lipschitz properties of $\mathscr{I}_3^n$ as a map acting on $\mathbb{M}^p(Y_T)$.

Lemma 2.4.18. *Fix any $T>0$. There exists a constant $L_3^n(T)>0$ such that the following assertions are true:*

- *$L_3^n(\cdot)$ is non decreasing;*
- *for every $n\in\mathbb{N}$, $\lim_{T\to 0} L_3^n(T)=0$;*
- *for $v_1,v_2\in\mathbb{M}^p(Y_T)$, $\mathscr{I}_3^n$ satisfy,*

$$\|\mathscr{I}_3^n(v_1)-\mathscr{I}_3^n(v_2)\|_{\mathbb{M}^p(Y_T)} \lesssim_p L_3^n(T)\,\|v_1-v_2\|_{\mathbb{M}^p(Y_T)}. \tag{2.4.34}$$

Proof of Lemma 2.4.18 To prove the contraction property (2.4.18), for $i=1,2$, we set

$$\xi_i^n(t)=\theta_n(\|v_i\|_{Y_t})\hat{\theta}(\|v_i\|_{Z_t})G(v_i(t)).$$

Then, applying (2.3.11) from Theorem 2.3.5 we get

$$\begin{aligned}\mathbb{E}\left[\|J\xi_1^n-J\xi_2^n\|_{L^p(0,T;E)}^p\right] &= \mathbb{E}\left[\int_0^T \|[J\xi_1^n](t)-[J\xi_2^n](t)\|_E^p\,dt\right]\\ &\le \tilde{C}(p,T,E,H)\,\mathbb{E}\left[\int_0^T \|\xi_1^n(t)-\xi_2^n(t)\|_{\gamma(K,H)}^2\,dt\right]^{\frac{p}{2}}.\end{aligned}\tag{2.4.35}$$

Next, we define the following $\mathbb{F}$-stopping times by

$$T_1^i:=\inf\{t\in[0,T]:\|v_i\|_{Z_t}\ge M\},\qquad T_2^i:=\inf\{t\in[0,T]:\|v_i\|_{Y_t}\ge 2n\},\quad i=1,2$$

and set

$$T_1^*:=\min\{T_1^1,T_2^1\}\qquad\text{and}\qquad T_2^*:=\min\{T_1^2,T_2^2\}.$$

Applying the stochastic Strichartz estimate from Theorem 2.3.5, we get

$$\mathbb{E}\left[\sup_{t\in[0,T]}\|[J\xi_1^n](t)-[J\xi_2^n](t)\|_{H_A}^p\right]\le K(p,T,H)\,\mathbb{E}\left[\int_0^T \|\xi_1^n(t)-\xi_2^n(t)\|_{\gamma(K,H)}^2\,dt\right]^{\frac{p}{2}}. \tag{2.4.36}$$

Using Lemmata 2.4.2 and 2.4.13 with the above defined stopping times, we argue as follows:

$$\begin{aligned}&\mathbb{E}\left[\int_0^T \|\xi_1^n(t)-\xi_2^n(t)\|_{\gamma(K,H)}^2\,dt\right]^{\frac{p}{2}}\\ &\lesssim_p \mathbb{E}\left[\int_0^T \mathbb{1}_{\{T_1^*\le T_2^*\}}(t)\,\|\xi_1^n(t)-\xi_2^n(t)\|_{\gamma(K,H)}^2\,dt\right]^{\frac{p}{2}}\\ &\qquad+\mathbb{E}\left[\int_0^T \mathbb{1}_{\{T_2^*\le T_1^*\}}(t)\,\|\xi_1^n(t)-\xi_2^n(t)\|_{\gamma(K,H)}^2\,dt\right]^{\frac{p}{2}}\\ &\lesssim_p \mathbb{E}\left[\int_0^T \mathbb{1}_{\{T_1^*\le T_2^*\}}(t)\theta_n(\|v_1\|_{Y_t})\hat{\theta}(\|v_1\|_{Z_t})\|G(v_1(t))-G(v_2(t))\|_{\gamma(K,H)}^2\,dt\right]^{\frac{p}{2}}\end{aligned}\tag{2.4.37}$$

$$
\begin{aligned}
&+\mathbb{E}\left[\int_0^T \mathbb{1}_{\{T_1^*\le T_2^*\}}(t)|\theta_n(\|v_1\|_{Y_t})\hat{\theta}(\|v_1\|_{Z_t})-\theta_n(\|v_2\|_{Y_t})\hat{\theta}(\|v_2\|_{Z_t})|^2\|G(v_2(t))\|^2_{\gamma(K,H)}\,dt\right]^{\frac{p}{2}}\\
&+\mathbb{E}\left[\int_0^T \mathbb{1}_{\{T_2^*\le T_1^*\}}(t)\theta_n(\|v_2\|_{Y_t})\hat{\theta}(\|v_2\|_{Z_t})\|G(v_1(t))-G(v_2(t))\|^2_{\gamma(K,H)}\,dt\right]^{\frac{p}{2}}\\
&+\mathbb{E}\left[\int_0^T \mathbb{1}_{\{T_2^*\le T_1^*\}}(t)|\theta_n(\|v_1\|_{Y_t})\hat{\theta}(\|v_1\|_{Z_t})-\theta_n(\|v_2\|_{Y_t})\hat{\theta}(\|v_2\|_{Z_t})|^2\|G(v_1(t))\|^2_{\gamma(K,H)}\,dt\right]^{\frac{p}{2}}\\
&\lesssim C_G^p\,\mathbb{E}\left[\|v_1-v_2\|^2_{Z_T}\left(T+\frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}\|v_1\|^{2\gamma}_{X_{T_1^*\wedge T_2^*}}+\frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}\|v_2\|^{2\gamma}_{X_{T_1^*\wedge T_2^*}}\right)\right]^{\frac{p}{2}}\\
&+C_G^p\,\mathbb{E}\left[\|v_1-v_2\|^2_{Z_T}\int_0^{T_2^*}\mathbb{1}_{\{T_1^*\le T_2^*\}}(t)\|G(v_2(t))\|^2_{\gamma(K,H)}\,dt\right]^{\frac{p}{2}}\\
&+\frac{C_G^p}{n^p}\,\mathbb{E}\left[\|v_1-v_2\|^2_{Y_T}\int_0^{T_2^*}\mathbb{1}_{\{T_1^*\le T_2^*\}}(t)\|G(v_2(t))\|^2_{\gamma(K,H)}\,dt\right]^{\frac{p}{2}}\\
&+C_G^p\,\mathbb{E}\left[\|v_1-v_2\|^2_{Z_T}\int_0^{T_1^*}\mathbb{1}_{\{T_2^*\le T_1^*\}}(t)\|G(v_1(t))\|^2_{\gamma(K,H)}\,dt\right]^{\frac{p}{2}}\\
&+\frac{C_G^p}{n^p}\,\mathbb{E}\left[\|v_1-v_2\|^2_{Y_T}\int_0^{T_1^*}\mathbb{1}_{\{T_2^*\le T_1^*\}}(t)\|G(v_1(t))\|^2_{\gamma(K,H)}\,dt\right]^{\frac{p}{2}}\\
&\lesssim C_G^p\left(T+\frac{2T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}(2n)^{2\gamma}\right)^{\frac{p}{2}}\|v_1-v_2\|^p_{\mathbb{M}^p(Y_T)}\\
&+n^p\,C_G^p\left(T+\frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}(2n)^{2\gamma}\right)^{\frac{p}{2}}\|v_1-v_2\|^p_{\mathbb{M}^p(Y_T)}\\
&\lesssim\; n^p\,C_G^p\left(T+\frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}(2n)^{2\gamma}\right)^{\frac{p}{2}}\|v_1-v_2\|^p_{\mathbb{M}^p(Y_T)}.
\end{aligned}
$$

By substituting (2.4.37) into (2.4.35) and (2.4.36) we get,

$$
\begin{aligned}
\|J\xi_1^n-J\xi_2^n\|^p_{\mathbb{M}^p(Y_T)} &\lesssim_p\; n^p\,C_G^p\,(K(p,T,H)+\tilde{C}(p,T,E,H))\left(T+\frac{T^{1-\frac{2\gamma}{p}}}{M^{2\gamma}}(2n)^{2\gamma}\right)^{\frac{p}{2}}\|v_1-v_2\|^p_{\mathbb{M}^p(Y_T)}\\
&=:(L_3^n(T))^p\;\;\|v_1-v_2\|^p_{\mathbb{M}^p(Y_T)}.
\end{aligned}
$$

Since $2\gamma<p$, by definition of $L_3^n(T)$, it is clear that $\lim_{T\to 0}L_3^n(T)=0$ for every n. Thus we have finished the proof for (2.4.34) and, in particular, for Lemma 2.4.18. □

Step II: In this step, we prove that, for each $n\in\mathbb{N}$, there exists $T_n>0$ such that the map $\Psi^n_{T_n}$ defined by (2.4.18)-(2.4.19) has a unique fixed point in the space $\mathbb{M}^p(Y_{T_n})$.

Let us fix an $n\in\mathbb{N}$. From Lemmata 2.4.14 - 2.4.18, we infer that, for any $T>0$, the map Ψ^n_T is well defined on $\mathbb{M}^p(Y_T)$ and for every $v_1,v_2\in\mathbb{M}^p(Y_T)$, we have

$$\|\Psi^n_T(v_1)-\Psi^n_T(v_2)\|_{\mathbb{M}^p(Y_T)}$$

$$\lesssim_p \quad L_2^n(T)\,\|v_1-v_2\|_{\mathbb{M}^p(Y_T)} + L_3^n(T)\,\|v_1-v_2\|_{\mathbb{M}^p(Y_T)} =: L_n(T)\|v_1-v_2\|_{\mathbb{M}^p(Y_T)},$$

where $L_n(\cdot)$ is non decreasing and $\lim_{T\to 0} L_n(T) = 0$. Hence, we can choose $T_n > 0$ such that $\Psi^n_{T_n}$ is a strict contraction on $\mathbb{M}^p(Y_{T_n})$. Thus, by the Banach Fixed Point Theorem there exists a unique fixed point $u_n \in \mathbb{M}^p(Y_{T_n})$ of the map $\Psi^n_{T_n}$.

Step III: Here we prove the existence of a local mild solution, in the sense of Definition 2.4.8, to Problem (2.4.4).

Fix any $n\in\mathbb{N}$. Then, from Step II, there exists a $T_n>0$ and a unique fixed point u_n of map $\Psi^n_{T_n}$ in the space $\mathbb{M}^p(Y_{T_n})$. Using the process u_n, we define the following $\mathbb{F}$-stopping time,

$$\tau_n := \inf\{t\in[0,T_n] : \|u_n\|_{Z_t} \geq M'\} \wedge \inf\{t\in[0,T_n] : \|u_n\|_{Y_t} \geq n\}. \tag{2.4.38}$$

At this juncture it is important to mention that, since $\|u_n(0)\|_{H_A} < M'$ and the maps $t\mapsto \|u_n\|_{Y_t}$ and $t\mapsto \|u_n\|_{Z_t}$ are continuous, the stopping time τ_n is strictly positive $\mathbb{P}$-a.s.. Let $\{\tau_{n_k}\}_{k\in\mathbb{N}}$ denote a sequence of $\mathbb{F}$-stopping times defined by

$$\tau_{n_k} := \inf\{t\in[0,T_n] : \|u_n\|_{Z_t} \geq M'\} \wedge \inf\left\{t\in[0,T_n] : \|u_n\|_{Y_t} \geq n-\frac{1}{k}\right\}.$$

Then we deduce that τ_n is actually an accessible $\mathbb{F}$-stopping time with the approximating sequence $\{\tau_{n_k}\}_{k\in\mathbb{N}}$.

Next, to simplify the notation, we denote $u:=u_n$; $\tau:=\tau_n$ and $\tau_k:=\tau_{n_k}$ in the remaining proof of Theorem 2.4.10. Since u is the fixed point of map $\Psi^n_{T_n}$, u satisfies the following,

$$\begin{aligned} u(t) - \cos(t\sqrt{A})u_0 - \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 - \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}\theta_n(\|u\|_{Y_s})\hat{\theta}(\|u\|_{Z_s})\, F(u(s))\,ds \\ = \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}\theta_n(\|u\|_{Y_s})\hat{\theta}(\|u\|_{Z_s})\, G(u(s))\,dW(s), \ \ \mathbb{P}\text{-a.s.}, \end{aligned} \tag{2.4.39}$$

for $t\geq 0$. In moving further we set

$$I(t) := \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}\theta_n(\|u\|_{Y_s})\hat{\theta}(\|u\|_{Z_s})\, G(u(s))\,dW(s).$$

Observe that, from the definition of $\mathbb{M}^p(Y_{T_n})$, the processes on both sides of equality (2.4.39) are continuous and hence, the equality even holds when the fixed deterministic time is replaced by the random one, in particular, (2.4.39) holds for $t\wedge\tau_k$. Since by the definition of θ_n, $\hat{\theta}$, and τ_k the following holds

$$\theta_n(\|u\|_{Y_{t\wedge\tau_k}}) = 1, \qquad \hat{\theta}(\|u\|_{Z_{t\wedge\tau_k}}) = 1, \quad \forall n,k\in\mathbb{N},$$

we have

$$\int_0^{t\wedge\tau_k} \frac{\sin((t\wedge\tau_k - s)\sqrt{A})}{\sqrt{A}}\theta_n(\|u\|_{Y_s})\hat{\theta}(\|u\|_{Z_s})\, F(u(s))\,ds$$

$$= \int_0^{t\wedge\tau_k} \frac{\sin((t\wedge\tau_k - s)\sqrt{A})}{\sqrt{A}}\, F(u(s))\, ds, \quad \mathbb{P}\text{-a.s..}$$

Invoke Lemma 2.5.1 from Section 2.5.1, which is a generalization of [21, Lemma A.1], we obtain

$$\begin{aligned} I(t\wedge\tau_k) &= \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} \Big(\theta_n(\|u\|_{Y_{s\wedge\tau_k}})\hat{\theta}(\|u\|_{Z_{s\wedge\tau_k}})\, \mathbb{1}_{[0,\tau_k)}(s) G(u(s))\Big)\, dW(s) \\ &= I_{\tau_k}(G(u))(t\wedge\tau_k), \quad \mathbb{P}\text{-a.s.,} \end{aligned}$$

where $I_{\tau_k}(G(u))(t)$ is defined in (2.4.16). This concludes the existence part.

Step IV: In this step we complete the proof of Theorem 2.4.10, by showing the equivalence, in the sense of Definition 2.1.4, of $u_n|_{[0,\tau_n)\times\Omega}$ and $u_k|_{[0,\tau_n)\times\Omega}$ for all $k,n\in\mathbb{N}$ such that $n\le k$.

Let us fix any $k,n\in\mathbb{N}$ such that $n\le k$. Then obviously, by (2.4.38), $\tau_n\le\tau_k$, $\mathbb{P}$-a.s.. Moreover, due to Step III, corresponding to n and k, respectively, $\{u_n(t): t\in[0,\tau_n)\}$ and $\{u_k(t): t\in[0,\tau_k)\}$ denote the local mild solutions to (2.4.4), in the sense of Definition (2.4.8).

Applying the uniqueness part of Step III, for every $(t,\omega)\in[0,\tau_n)\times\Omega$, we argue as follows:

$$\begin{aligned} u_n(t,\omega) &= \cos((t\wedge\tau_n)\sqrt{A})u_0 + \frac{\sin((t\wedge\tau_n)\sqrt{A})}{\sqrt{A}}u_1 \\ &\quad + \int_0^{t\wedge\tau_n} \frac{\sin((t\wedge\tau_n - s)\sqrt{A})}{\sqrt{A}} F(u_n(s))\, ds + I_{\tau_n}(G)(t\wedge\tau_n) \\ &= \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} F(u_n(s))\, ds + I_{\tau_n}(G)(t) \\ &= \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} F(u_n(s))\, ds \\ &\quad + \int_0^t \mathbb{1}_{[0,\tau_n)}(s) \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} G(u_n(s))\, dW(s) \\ &= \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} F(u_k(s))\, ds \\ &= \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} G(u_k(s))\, dW(s) = u_k(t,\omega). \end{aligned}$$

This implies $u_n|_{[0,\tau_n)\times\Omega}$ and $u_k|_{[0,\tau_n)\times\Omega}$ are equivalent in the sense of Definition 2.1.4. Hence we have completed Step IV, in particular, the proof of Theorem 2.4.10. ■

Remark 2.4.19. The method of proof using the cutoff function is indeed standard nowadays and in addition to [22] it has been used for the deterministic and stochastic NLS by Burq, Gerard and Tzvetkov [33], de Bouard and Debussche [58] as well as for parabolic SPDE, see L Hornung [84] and J Hussain [85].

2.5 Auxiliary results

2.5.1 Stopped processes

In this subsection, we present a detailed justification for the choice of I_τ process we made in the Definition 2.4.8. The result below generalises [21, Lemma A.1].

Lemma 2.5.1. *Let* $\xi \in M^{2,p}([0,T],\gamma(K,H))$. *Set*

$$I(t) := \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}\xi(s)\,dW(s), \tag{2.5.1}$$

and

$$I_\tau(t) := \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}\left(\mathbb{1}_{[0,\tau)}(s)\xi(s)\right)dW(s). \tag{2.5.2}$$

For any stopping time τ *and for all* $t \geq 0$, *the following holds*

$$I(t\wedge\tau) = I_\tau(t\wedge\tau), \qquad \mathbb{P}\text{-a.s.}. \tag{2.5.3}$$

Proof of Lemma 2.5.1 By the choice of process ξ, both the stochastic convolutions are well defined. Let us start with a deterministic (stopping) time $\tau = t_0$. There are two cases, (1) If $t < t_0$, then

$$\begin{aligned}
I(t\wedge\tau) = I(t) &= \int_0^{t_0} \mathbb{1}_{[0,t)}(s)\frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}\xi(s)\,dW(s)\\
&= \int_0^{t_0} \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}\left(\mathbb{1}_{[0,t)}(s)\mathbb{1}_{[0,t_0)}(s)\xi(s)\right)dW(s)\\
&= \int_0^{t_0} \mathbb{1}_{[0,t)}(s)\frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}\left(\mathbb{1}_{[0,t_0)}(s)\xi(s)\right)dW(s)\\
&= I_{t_0}(t) = I_\tau(t\wedge\tau).
\end{aligned}$$

(2) If $t \geq t_0$, then

$$\begin{aligned}
I(t\wedge\tau) = I(t\wedge t_0) &= \int_0^{t} \mathbb{1}_{[0,t_0)}(s)\frac{\sin((t_0-s)\sqrt{A})}{\sqrt{A}}\left(\xi(s)\right)dW(s)\\
&= \int_0^{t} \mathbb{1}_{[0,t_0)}(s)\frac{\sin((t_0-s)\sqrt{A})}{\sqrt{A}}\left(\xi(s)\right)dW(s)\\
&\quad+\int_0^{t} \mathbb{1}_{[0,t_0)}(s)\mathbb{1}_{[t_0,t)}(s)\frac{\sin((t_0-s)\sqrt{A})}{\sqrt{A}}\left(\xi(s)\right)dW(s)\\
&= \int_0^{t_0} \mathbb{1}_{[0,t_0)}(s)\frac{\sin((t_0-s)\sqrt{A})}{\sqrt{A}}\left(\xi(s)\right)dW(s)\\
&\quad+\int_{t_0}^{t} \mathbb{1}_{[0,t_0)}(s)\frac{\sin((t_0-s)\sqrt{A})}{\sqrt{A}}\left(\xi(s)\right)dW(s)\\
&= \int_0^{t} \frac{\sin((t_0-s)\sqrt{A})}{\sqrt{A}}\left(\mathbb{1}_{[0,t_0)}(s)\xi(s)\right)dW(s)
\end{aligned}$$

$$
\begin{aligned}
&= \int_0^{t_0} \frac{\sin((t_0-s)\sqrt{A})}{\sqrt{A}} \left(\mathbb{1}_{[0,t_0)}(s)\xi(s)\right) dW(s) \\
&= I_{t_0}(t_0) = I_\tau(t\wedge\tau).
\end{aligned}
$$

Thus the equality (2.5.3) holds for any deterministic time. Now let τ be any arbitrary stopping time. Define

$$
\tau_n := \frac{[2^n\tau]+1}{2^n}, \text{ for each } n \in \mathbb{N}.
$$

That is, $\tau_n = \frac{k+1}{2}$ if $\frac{k}{2^n} \leq \tau < \frac{k+1}{2^n}$. Then by straightforward calculation we get that for each $\omega \in \Omega$, $\tau_n \searrow \tau$ as $n \to \infty$. Since equality (2.5.3) holds for deterministic time $\frac{k}{2^n}$, we have

$$
\begin{aligned}
I(t\wedge\tau_n) &= \sum_{k=0}^{\infty} \mathbb{1}_{k2^{-n}\leq\tau<(k+1)2^{-n}}\, I\left(t\wedge(k+1)2^{-n}\right) \\
&= \sum_{k=0}^{\infty} \mathbb{1}_{k2^{-n}\leq\tau<(k+1)2^{-n}}\, I_{(k+1)2^{-n}}\left(t\wedge(k+1)2^{-n}\right) \\
&= I_{\tau_n}(t\wedge\tau_n)\,.
\end{aligned} \tag{2.5.4}
$$

Since $\tau_n \searrow \tau$, we infer that, by continuity of trajectories of the process I, for all $t \geq 0$,

$$
I(t\wedge\tau_n) \to I(t\wedge\tau), \quad \mathbb{P}\text{-a.s. as} \quad n\to\infty. \tag{2.5.5}
$$

Furthermore observe that,

$$
\begin{aligned}
&\mathbb{E}\left\| I_{\tau_n}(t) - I_\tau(t)\right\|_{H_A}^2 \\
&= \mathbb{E}\left[\left\| \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} \left(\mathbb{1}_{[0,\tau_n)}(s)\xi(s) - \mathbb{1}_{[0,\tau)}(s)\xi(s)\right) dW(s)\right\|_{H_A}^2\right] \\
&= \mathbb{E}\left[\int_0^t \left\| \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} \left(\mathbb{1}_{[0,\tau_n)}(s) - \mathbb{1}_{[0,\tau)}(s)\right)\xi(s)\right\|_{\gamma(K,H)}^2 ds\right].
\end{aligned} \tag{2.5.6}
$$

Since $\tau_n \searrow \tau$, $\mathbb{P}$-a.s., as $n\to\infty$, $\mathbb{1}_{[0,\tau_n)} \to \mathbb{1}_{[0,\tau)}$, $\mathbb{P}$-a.s., as $n\to\infty$. Also, note that since the C_0-group $\{S(t)\}_{t\geq 0}$ on $H_A \times H$ is of contraction type, the integrand is bounded by some constant (depending upon t) multiply with $2\|\xi(s)\|_{\gamma(K,H)}^2$. Moreover, by the choice of ξ, we have

$$
\mathbb{E}\left[\int_0^t \|\xi(s)\|_{\gamma(K,H)}^2\, ds\right] < \infty.
$$

Thus, by using the Lebesgue dominated convergence theorem in (2.5.6), we get

$$
\lim_{n\to\infty} \mathbb{E}\left\| I_{\tau_n}(t) - I_\tau(t)\right\|_{H_A}^2 = 0.
$$

Hence, there exists a subsequence of $\{I_{\tau_n}(t)\}_{n\in\mathbb{N}}$, say $\{I_{\tau_{n_k}}(t)\}_{k\in\mathbb{N}}$, which converges to $I_\tau(t)$, $\mathbb{P}$-a.s. as $n\to\infty$. So for any fix $t\geq 0$, by (2.5.4) and (2.5.5) we have, $\mathbb{P}$-a.s.,

$$
\|I(t\wedge\tau) - I_\tau(t\wedge\tau)\|_{H_A}
$$

$$
\begin{aligned}
&= \| I(t\wedge\tau) - I(t\wedge\tau_{n_k}) + I_{\tau_{n_k}}(t\wedge\tau_{n_k}) - I_\tau(t\wedge\tau)\|_{H_A} \\
&\le \| I(t\wedge\tau) - I(t\wedge\tau_{n_k})\|_H + \| I_{\tau_{n_k}}(t\wedge\tau_{n_k}) - I_\tau(t\wedge\tau)\|_{H_A} \\
&\to 0 \text{ as } k\to\infty.
\end{aligned}
$$

Thus, we get (2.5.3) and this completes the proof of Lemma 2.5.1. ■

In particular, it follows that if $\xi = 0$ on $[0,\tau)$, then $I(t\wedge\tau) = 0$ for all $t \ge 0$, $\mathbb{P}$-a.s.. It is relevant to mention that the importance of such results goes back to [11], [14], and [42].

2.5.2 About the definition of a solution

Here we state a relation, without proof, between two natural definitions of a mild solution for SPDE (2.4.4). We begin by recalling the framework from Section 2.4. In particular, we set

$$H = L^2(\mathcal{D}); \qquad H_A = \mathscr{D}(A^{1/2}); \qquad E = \mathscr{D}(A_q^{(1-r)/2}),$$

where (p,q,r) is any suitable triple which satisfy (2.2.2).

We assume that the maps F and G satisfy Assumptions **A.1** and **A.2**, respectively. Let us also recall that the space $\mathbb{M}^p(Y_T)$ has been defined in (2.4.2).

Proposition 2.5.2. *Suppose that $u_0 \in \mathscr{D}(A^{1/2})$, $u_1 \in H$, and $T > 0$. If an $\mathscr{D}(A^{1/2}) \times H$-valued process*

$$\mathfrak{u}(t) = (u(t), v(t)), \;\; t \in [0,T],$$

such that $u \in \mathbb{M}^p(Y_T)$, is a solution to

$$\mathfrak{u}(t) = e^{\mathfrak{A}(t)}\mathfrak{u}(0) + \int_0^t e^{\mathfrak{A}(t-s)}\tilde{F}[\mathfrak{u}(s)]\,ds + \int_0^t e^{\mathfrak{A}(t-s)}\tilde{G}[\mathfrak{u}(s)]\,dW(s), \tag{2.5.7}$$

where

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \;\; \tilde{G}[\mathfrak{u}] = \begin{pmatrix} 0 \\ G(u) \end{pmatrix}, \;\; \tilde{F}[\mathfrak{u}] = \begin{pmatrix} 0 \\ F(u) \end{pmatrix}, \tag{2.5.8}$$

then the process u, is a mild solution to Problem (2.4.4)*, i.e. for all $t \in [0,T]$, $\mathbb{P}$-a.s.,*

$$
\begin{aligned}
u(t) = \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 \quad &+ \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}F(u(s))\,ds \\
+ \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}}G(u(s))\,dW(s).&
\end{aligned} \tag{2.5.9}
$$

The following is a convenient reformulation of the previous result.

Proposition 2.5.3. *Suppose that $u_0 \in \mathscr{D}(A^{1/2})$, $u_1 \in H$, $T > 0$. Let f be a progressively measurable process from the space $M^{1,p}([0,T], L^2(\mathcal{D}))$, and ξ be a progressively measurable process from the space $M^{2,p}([0,T], \gamma(K,H))$. If an $\mathscr{D}(A^{1/2}) \times H$-valued process*

$$\mathfrak{u}(t) = (u(t), v(t)), \;\; t \in [0,T],$$

such that $\mathfrak{u} \in \mathbb{M}^p(Y_T)$*, solves the following equation:*

$$\mathfrak{u}(t) = e^{\mathfrak{A}(t)}\mathfrak{u}(0) + \int_0^t e^{\mathfrak{A}(t-s)} F(s)\,ds + \int_0^t e^{\mathfrak{A}(t-s)} \Xi[s]\,dW(s), \tag{2.5.10}$$

where for $\mathfrak{u}(0) = (u_0, u_1) \in \mathscr{D}(A^{1/2}) \times H$*, we put*

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ -A & 0 \end{pmatrix}, \quad \Xi[s] = \begin{pmatrix} 0 \\ \xi(s) \end{pmatrix}, \quad F(s) = \begin{pmatrix} 0 \\ f(s) \end{pmatrix}, \tag{2.5.11}$$

then $u = \{u(t), t \in [0,T]\}$ *is a mild solution to Problem* (2.4.4)*, i.e. for all* $t \in [0,T]$, $\mathbb{P}$*-a.s.,*

$$\begin{aligned} u(t) &= \cos(t\sqrt{A})u_0 + \frac{\sin(t\sqrt{A})}{\sqrt{A}}u_1 + \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} f(s)\,ds \\ &\quad + \int_0^t \frac{\sin((t-s)\sqrt{A})}{\sqrt{A}} \xi(s)\,dW(s). \end{aligned} \tag{2.5.12}$$

CHAPTER 3

GEOMETRIC WAVE EQUATION

In this chapter we collect some basic notions from differential geometry required to derive the wave map equation. We assume that reader is familiar with definitions of a smooth manifold, tangent space and vector field. We mostly follow [103] and [118] here.

Unless otherwise stated, let M and N be smooth manifolds of dimensions m and n, respectively. The set of all smooth functions $F : M \to N$ is denoted by $\mathfrak{F}(M,N)$. In case $N = \mathbb{R}$, we just write $\mathfrak{F}(M)$. For any nonnegative integer j, by $\mathcal{C}^j(\mathbb{R}^m;\mathbb{R}^n)$ we denote the space of $\mathbb{R}^n$-valued continuous functions whose derivatives up to order j are continuous on $\mathbb{R}^m$. Let

$$\mathcal{C}^\infty(\mathbb{R}^m;\mathbb{R}^n) := \bigcap_{j\in\mathbb{N}} \mathcal{C}^j(\mathbb{R}^m;\mathbb{R}^n),$$

and by $\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}^m;\mathbb{R}^n)$ we denote the space of $\mathcal{C}^\infty(\mathbb{R}^m;\mathbb{R}^n)$ functions with compact support.

3.1 Basic definitions

Given a (U,ϕ) coordinate chart of M and i-th coordinate function on $\mathbb{R}^m$, which is defined as

$$r^i : \mathbb{R}^m \ni (a_1, a_2, \ldots, a_m) \mapsto a_i \in \mathbb{R},$$

we set

$$x^i := r^i \circ \phi : U \to \mathbb{R}, \qquad i = 1, \ldots, m,$$

as coordinate functions of (U,ϕ).

3.1.1 Tangent space and differential

By T_pM, $p \in M$, we denote the tangent space to M at point p. It turns out that T_pM is a $\mathbb{R}$-linear space of dimension m and, for given coordinate chart (U, ϕ), such that $p \in U$, the set

$$\left(\frac{\partial}{\partial x^1}\bigg|_p, \dots, \frac{\partial}{\partial x^m}\bigg|_p \right),$$

forms a basis of $T_p(M)$. Here

$$\frac{\partial}{\partial x^i}\bigg|_p : \mathfrak{F}(M) \ni f \mapsto \frac{\partial (f \circ \phi^{-1})}{\partial r^i}(\phi(p)) := \frac{\partial}{\partial x^i}\bigg|_p f \in \mathbb{R}, \qquad i = 1, 2, \dots, m. \tag{3.1.1}$$

Thus a tangent vector $\vartheta \in T_pM$ can be written uniquely as

$$\vartheta = \sum_{i=1}^{m} \vartheta^i \frac{\partial}{\partial x^i}\bigg|_p, \quad \text{where } \vartheta^i = \vartheta(x^i). \tag{3.1.2}$$

We call the m-tuple $(\vartheta^1, \dots, \vartheta^m)$ as the coordinate representation on ϑ under $\phi = (x^1, \dots x^m)$. Given $F \in \mathfrak{F}(M, N)$ and $p \in M$ we define a map $d_pF : T_p(M) \to T_{F(p)}(N)$, called the differential of F at p, by

$$d_pF : T_pM \ni \vartheta \mapsto (d_pF)(\vartheta) = \{\mathfrak{F}(N) \ni f \mapsto \vartheta(f \circ F) \in \mathbb{R}\} \in T_{F(p)}N. \tag{3.1.3}$$

In our calculation we need the following local expression for the differential d_pF.

Lemma 3.1.1. *[103, Chapter 3] Let $F \in \mathfrak{F}(M, N)$ and $p \in M$. Let us assume that $(U, \phi = (x^1, \dots, x^m))$ and $(V, \psi = (y^1, \dots, y^n))$ be coordinate charts about p in M and $F(p)$ in N, respectively. Then relative to bases $\left\{ \frac{\partial}{\partial x^i}\big|_p \right\}_{i=1}^m$ and $\left\{ \frac{\partial}{\partial y^j}\big|_{F(p)} \right\}_{j=1}^n$ for T_pM and $T_{F(p)}N$, respectively, the differential $d_pF : T_pM \to T_{F(p)}N$ is represented by the matrix $\left[\frac{\partial F^j}{\partial x^i}(p) \right]_{n \times m}$, where $F^j := y^j \circ F$ and*

$$\frac{\partial F^j}{\partial x^i}(p) = \frac{\partial (F^j \circ \phi^{-1})}{\partial r^i}(\phi(p)).$$

In particular,

$$(d_pF)\left(\frac{\partial}{\partial x^i}\bigg|_p \right) = \sum_{j=1}^{n} \frac{\partial F^j}{\partial x^i}(p) \frac{\partial}{\partial y^j}\bigg|_{F(p)}.$$

3.1.2 Tangent bundle

The tangent bundle of M, denoted by TM, is defined as the disjoint union of the tangent spaces at all points of M, i.e. $TM := \cup_{p \in M}\{p\} \times T_pM$. We write an element of this disjoint union as an ordered pair (p, ϑ) with $\vartheta \in T_pM$. The tangent bundle is equipped with a natural projection map $\pi : TM \ni (p, \vartheta) \mapsto p \in M$. Observe that given a coordinate chart $(U, \psi = (x^1, \dots, x^m))$ of M, the image set of the map

$$\pi^{-1}(U) \ni \left(p, \vartheta = \sum_{i=1}^{m} \vartheta^i \frac{\partial}{\partial x^i}\bigg|_p \right) \mapsto (x^1(p), \dots, x^m(p), \vartheta^1, \dots, \vartheta^m)) \in \mathbb{R}^{2m},$$

is open in $\phi(U) \times \mathbb{R}^{2m}$. This leads to the fact that TM is a $2m$-dimensional smooth manifold.

3.1.3 Vector field

A vector field on M is a section of $\pi : TM \to M$. That is, a vector field X on M is a map $M \mapsto TM$ that associates, to each point $p \in M$, a vector denoted by $X_p \in T_p(M)$ with the property that $\pi \circ X = Id_M$. The set of smooth (as map between two manifolds) vector fields on M is denoted by $\mathfrak{X}(M)$. It is easy to see that, given any coordinate chart $(U, \phi = (x^1, \dots, x^m))$ for M, the assignment $p \mapsto \frac{\partial}{\partial x^i}\Big|_p$ is a vector field on U. This special vector field is known as the i-th coordinate vector field. Moreover, we write the value of X at any point $p \in U$ in terms of the coordinate basis vectors as

$$X_p = \sum_{i=1}^{m} X^i(p) \frac{\partial}{\partial x^i}\bigg|_p ,$$

where $X^i : U \to \mathbb{R}$ is called i-th component function of X in the given chart.

For $F \in \mathfrak{F}(M, N)$, by putting together the differentials of $d_p F$ at all points p of M, we define a map $dF : TM \to TN$, called the global differential. In other words, dF is a map whose restriction to each tangent space $T_p M$ is $d_p F$.

3.1.4 Cotangent bundle

Given the tangent space $T_p M$ at $p \in M$, we denote its dual space by $T_p^* M$ and call it the cotangent space at p. Elements of $T_p^* M$ are called cotangent or covector at p. The union $T^* M := \cup_{p \in M} \{p\} \times T_p^* M$ is called the cotangent bundle of M and it has a natural projection $\lambda : T^* M \ni (p, \varphi) \mapsto p \in M$. Mimicking the construction of the tangent bundle, we can show that $T^* M$ is also a $2m$-dimensional smooth manifold.

Similar to vector field, we define a covector field ξ or 1-form on M as a map that assigns to each $p \in M$ an element of $T_p^* M$ such that $\lambda \circ \xi = Id_M$. Now observe that, since $x^i = r^i \circ \phi : U \to \mathbb{R}, i = 1 \dots m$, where $(U, \phi = (x^1 \dots x^m))$ is any given coordinate chart on M, belongs to $\mathfrak{F}(U)$ and one can identify $T_p U$ with $T_p M$, at each point p, the differential $d_p x^i$ is an element of $T_p^* M$ and the set $\{d_p x^i\}_{i=1}^m$ forms a basis for cotangent space $T_p^* M$ dual to the basis $\left\{ \frac{\partial}{\partial x^i}\Big|_p \right\}_{i=1}^m$ for the tangent space $T_p M$. To give the visibility to basis of contangent space we write $(dx^i)_p$ instead of $d_p x^i$.

3.1.5 Riemannian manifold

A metric g on M is a mapping which assigns to each point $p \in M$ a scalar product (i.e. symmetric, bilinear and non-degenerate) g_p on each tangent space $T_p M$, such that for every $X, Y \in \mathfrak{X}(M)$, $g : M \ni p \mapsto g_p(X_p, Y_p) \in \mathbb{R}$ belongs to $\mathfrak{F}(M)$. If the metric g is positive (indefinite) definite, then it is called a (semi-)Riemannian metric on M. A (semi-)Riemannian manifold is a smooth manifold, equipped with a (semi-)Riemannian metric.

Thus for a given coordinate map $\phi = (x^1, \dots, x^m) : M \supset U \to \mathbb{R}^m$, a Riemannian metric g is represented by a positive definite and symmetric $m \times m$-matrix $[g_{ij}(p)], i, j = 1, \dots, m$. Hence, for any given two tangent vectors $\vartheta_1, \vartheta_2 \in T_p M$, respectively, with coordinate representation $(\vartheta_1^1, \dots, \vartheta_1^m)$ and

$(\vartheta_2^1, \dots, \vartheta_2^m)$, the action of g_p is

$$g_p(\vartheta_1, \vartheta_2) = \sum_{i,j=1}^{m} g_{ij}(p) \vartheta_1^i \vartheta_2^j.$$

In particular, $g_{ij}(p) = g_p\left(\left.\frac{\partial}{\partial x^i}\right|_p, \left.\frac{\partial}{\partial x^j}\right|_p\right)$. In coordinate map $\phi = (x^1, \dots, x^m)$, a Riemannian metric can be written as

$$g_p = \sum_{i,j=1}^{m} g_{ij}(p)(dx^i)_p \otimes (dx^j)_p,$$

where $(dx^i)_p \otimes (dx^j)_p : T_pM \times T_pM \ni (\vartheta_1, \vartheta_2) \mapsto (dx^i)_p(\vartheta_1)(dx^j)_p(\vartheta_2) \in \mathbb{R}$ is a tensor product in the sense of functions. Moreover, due to the symmetry property of g we have

$$\begin{aligned} g_p &= \sum_{i,j=1}^{m} \frac{1}{2} \left[g_{ij}(p)(dx^i)_p \otimes (dx^j)_p + g_{ji}(p)(dx^j)_p \otimes (dx^i)_p \right] \\ &=: \sum_{i,j=1}^{m} g_{ij}(p)(dx^i)_p (dx^j)_p, \end{aligned}$$

and in short $g = \sum_{i,j=1}^{m} g_{ij} dx^i dx^j$.

One important feature of a (semi-)Riemannian metric is that it provides a natural isomorphism between the tangent and cotangent bundles.

Lemma 3.1.2. *[103, Chapter 13] Let (M, g) a (semi-)Riemannian manifold. If we define $\hat{g} : TM \to T^*M$ by*

$$\hat{g} : TM \ni (p, \vartheta) \mapsto \hat{g}_p(\vartheta) = \{T_pM \ni v \mapsto g_p(\vartheta, v) \in \mathbb{R}\} \in T_p^*M,$$

then $\hat{g}_p : T_pM \to T_p^*M$ is linear and bijective.

Therefore, in any coordinate chart $(U, \phi = (x^1, \dots, x^m))$, for $X, Y \in \mathfrak{X}(M)$ with coordinate functions $(X^1, \dots, X^m)$ and $(Y^1, \dots, Y^m)$, respectively, we can write the action of $\hat{g}_p, p \in U$ as

$$\hat{g}_p(X)(Y) = \sum_{i,j=1}^{m} g_{ij}(p) X^i(p) Y^j(p).$$

So the covector field $\hat{g}(X) : M \ni p \mapsto \hat{g}_p(X_p, \cdot) \in T_p^*M$ has the form $\hat{g}(X) = \sum_{i,j=1}^{m} g_{ij} X^i dx^j$. Thus, the matrix of $\hat{g} : TM \to T^*M$ at point p, in coordinate chart U, ϕ, is same as $[g_{ij}(p)]$. Hence, the matrix corresponds to map $\check{g}_p := (\hat{g}_p)^{-1} : T_p^*M \to T_pM$ is $[g_{ij}(p)]^{-1}$. It is customary to denote $[g_{ij}(p)]^{-1}$ by $[g^{ij}(p)]$ and thus we have, in local coordinates,

$$\sum_{j=1}^{m} g^{ij}(p) g_{jk}(p) = \sum_{j=1}^{m} g_{kj}(p) g^{ji}(p) = \delta_k^i,$$

where $\delta_k^i = 1$ if $k = i$ and $\delta_k^i = 0$ elsewhere.

3.1.6 Pullback metric

Suppose (M, h) be a semi-Riemannian manifold. Let (N, g) be a Riemannian manifold, which we call target manifold, and $F : M \to N$ is smooth. The pullback metric $F^* g$ on M is defined by

$$(F^* g)_p(X_p, Y_p) := g_{F(p)}((d_p F)(X_p), (d_p F)(Y_p)), \qquad p \in M,\ X, Y \in \mathfrak{X}(M).$$

Let us denote the isomorphism between $T^* M$ and TM by $\check{h}$, i.e. for each $p \in M$,

$$\check{h}_p : T_p^* M \to T_p M,$$

is a linear bijection. Thus, for each $p \in M$,

$$\check{h}_p \circ (F^* g)_p : T_p M \ni \vartheta \mapsto \check{h}_p \left(g_p((d_p F)(\vartheta), \cdot) \right) \in T_p M,$$

is a linear operator. We denote the trace of $\check{h}_p \circ (F^* g)_p$ by $(\mathrm{tr}_h F^* g)(p)$. By its definition $(\mathrm{tr}_h F^* g)(p)$ is a smooth $\mathbb{R}$-valued function on M.

3.2 Derivation of geometric wave equation

Here we derive the geometric wave equation (GWE) and define a wave map in terms of local coordinates. Since we are dealing with two dimensional domain in this thesis, we are only deriving GWE for this setup.

Let (N, g) be a Riemannian manifold of m-dimension. Consider the Euclidean space $M := \mathbb{R}^{1+n}$. We know that (M, id) is a $n+1$-dimensional smooth manifold with identity map as a global chart and tangent space $T_p M$ at point $p \in M$ is isomorphic to M itself. If we define the metric h on M such that

$$h_p(v, w) := -v^0 w^0 + \sum_{i=1}^{n} v^i w^i,$$

for $v = (v^0, \ldots, v^n)$, $w = (w^0, \ldots, w^n) \in \mathbb{R}^m$. Equipped with such metric h, $\mathbb{R}^{1+n}$ is called Minkowski $(1+n)$-space.

We define a functional $\mathscr{L}$ on the set $\mathfrak{F}(M, N)$ by

$$\mathscr{L}(z) := \frac{1}{2} \int_M \mathrm{tr}_h z^* g = \frac{1}{2} \int_{\mathbb{R}^m} \mathrm{tr}_h (z^* g)(x)\, dx,$$

and we are interested in critical points of $\mathscr{L}$., i.e. we mean to study compactly supported variations. For that we need to find a smooth compactly supported map from $M \to N$, say ξ such that

$$\frac{d}{d\varepsilon} \mathscr{L}(z_\varepsilon)|_{\varepsilon=0} = 0 \qquad \text{where} \qquad z_\varepsilon := z + \varepsilon \xi.$$

But $z + \varepsilon\varphi$ makes no sense as a map from $M \to N$. To overcome this difficulty we work with coordinate charts and this is sufficient for our work in this thesis because we seek for the solutions which are continuous.

Let (U,ϕ) as a coordinate chart in N. By definition of a local chart, ϕ is smooth and $\phi(U)$ is an open subset of $\mathbb{R}^m$. Thus, $(\phi(U),\mathrm{id})$ is a smooth manifold of m-dimension. Since U has a metric induced from g and $\phi^{-1}:\phi(U)\to U$ is smooth, by [103, Proposition 13.9] the pull back of $g\big|_U$ to $\phi(U)$, which we denote by $\phi^{-1*}g$, is a Riemannian metric on $\phi(U)$.

Now suppose we consider a smooth function $Z:V\to\phi(U)$ where V is domain of coordinate chart in M such that $Z(V)\subset\phi(U)$. Such functions are possible to find because for any $f\in\mathfrak{F}(M,N)$, $f^{-1}(U)$ is an open subset of M and, then by choosing a coordinate chart domain $V\subset M$ in such a way that $f(V)\subset U$, we can take $Z:=\phi\circ f\big|_V$. If we let $\varphi:M\to\mathbb{R}^m$ be a smooth function having compact support in V, then we can talk about $Z+\varepsilon\varphi$ because for small ε it takes values in $\phi(U)$, since Z,φ are continuous. Note that since $M=\mathbb{R}^{1+n}$, V and M are isomorphic and we can consider the functions $Z\in\mathcal{C}^\infty(\mathbb{R}^{1+n};\phi(U))$ for a given local chart (U,ϕ) on N.

With above reasoning in mind we define the wave maps as follows. First, we define the required notion of functional.

Definition 3.2.1. *Let $\mathbb{R}^{1+n}$ endowed with a Minkowski metric h, and (N,g) be a Riemannian manifold with a given local chart (U,ϕ). Define a functional $\mathscr{L}$ on $\mathcal{C}^\infty(\mathbb{R}^{1+n};\phi(U))$ by*

$$\mathscr{L}_U(Z):=\frac{1}{2}\int_{\mathbb{R}^{1+n}} tr_h(Z^*g)(x)\,dx,\quad Z\in\mathcal{C}^\infty(\mathbb{R}^{1+n};\phi(U)).$$

A function $Z\in\mathcal{C}^\infty(\mathbb{R}^{1+n};\phi(U))$ is said to be critical point of $\mathscr{L}_U$ iff, for every $\varphi\in\mathcal{C}^\infty_{comp}(\mathbb{R}^{1+n};\mathbb{R}^m)$,

$$\frac{d}{d\varepsilon}\mathscr{L}_U(Z+\varepsilon\varphi)\big|_{\varepsilon=0}=0. \tag{3.2.1}$$

Definition 3.2.2 (Wave Map)**.** *Let $\mathbb{R}^{1+n}$ endowed with a Minkowski metric h, and (N,g) be a Riemannian manifold . We define wave map as a mapping $z:(\mathbb{R}^{1+n},h)\to(N,g)$ such that, for every coordinate chart (U,ϕ), the function*

$$\phi\circ z|_{z^{-1}(U)}:z^{-1}(U)\ni p\mapsto\phi(z(p))\in\phi(U),$$

is a critical point of $\mathscr{L}_U$.

Now we move to write the Lagrangian density $\mathrm{tr}_h(z^*g)$ in local coordinates and derive the system of partial differential equations for z. To avoid the notation complexity, in the remaining chapter we will write M instead of $\mathbb{R}^{1+n}$. Fix any chart (U,ϕ) on N and we write

$$Z:=\phi\circ z|_{z^{-1}(U)}:M\to Z(U)\subset\mathbb{R}^m,$$

and its k-th component by Z^k, $k=1,\dots,m$.

Let $p\in z^{-1}(U)\subset\mathbb{R}^{1+n}$ and $\left\{\frac{\partial}{\partial r^i}\Big|_p\right\}_{i=0}^n$ for a basis of T_pM. Since $T_pM\simeq M$ and we only have identity map as coordinate map on M, we can consider $\left\{\frac{\partial}{\partial x^i}\Big|_p\right\}_{i=0}^n$ as standard coordinates for $\mathbb{R}^{1+n}$ as well but to avoid the confusion we do not take this and write $\left\{\frac{\partial}{\partial r^i}\Big|_p\right\}_{i=0}^n$ for standard coordinate.

Moreover, we set, $\left\{ \frac{\partial}{\partial y^i}\Big|_{Z(p)} \right\}_{j=1}^{m}$, where $Z(p) = \phi(z(p))$, as a basis of $T_{Z(p)}\phi(U) \simeq T_{Z(p)}\mathbb{R}^m \simeq \mathbb{R}^m$. Again to avoid confusion we write $y^j = q^j \circ \mathrm{id}_{\phi(U)}$, for some standard coordinate of $\mathbb{R}^m$. This implies for any smooth map $f : \phi(U) \to \mathbb{R}$,

$$\frac{\partial}{\partial y^i}\Big|_{Z(p)} f := \frac{\partial \left(f \circ \mathrm{id}_{\phi(U)}^{-1} \right)}{\partial q^i}\Bigg|_{\mathrm{id}_{\phi(U)}(Z(p))},$$

but to simplify the notation we set the right hand side to $\frac{\partial f}{\partial q^i}\Big|_{Z(p)}$.

Recall that, since $Z : M \to \phi(U)$, by Lemma 3.1.1,

$$(d_p Z)\left(\frac{\partial}{\partial x^i}\Big|_p \right) = \sum_{k=0}^{n} \frac{\partial Z^k}{\partial x^i}(p) \frac{\partial}{\partial y^k}\Big|_{Z(p)}.$$

Consequently, because $\phi^{-1*} g$ is a metric on $\phi(U)$, we pullback it on M and have

$$\begin{aligned}
\left[Z^* \left(\phi^{-1*} g \right) \right]_p \left(\frac{\partial}{\partial x^i}\Big|_p, \frac{\partial}{\partial x^j}\Big|_p \right) &= \left(\phi^{-1*} g \right)_{Z(p)} \left((d_p Z)\left(\frac{\partial}{\partial x^i}\Big|_p \right), (d_p Z)\left(\frac{\partial}{\partial x^j}\Big|_p \right) \right) \\
&= \sum_{k=1}^{m} \sum_{l=1}^{m} \frac{\partial Z^k}{\partial x^i}(p) \frac{\partial Z^l}{\partial x^j}(p) \ \left(\phi^{-1*} g \right)_{Z(p)} \left(\frac{\partial}{\partial y^k}\Big|_{Z(p)}, \frac{\partial}{\partial y^l}\Big|_{Z(p)} \right) \\
&=: z_{ij}(p), \qquad i, j = 0, \dots, n.
\end{aligned}$$

Here, since $\left[Z^* \left(\phi^{-1*} g \right) \right]_p (\cdot, \cdot) : T_p M \ni \vartheta \mapsto \left[Z^* \left(\phi^{-1*} g \right) \right]_p (\vartheta, \cdot) \in T_p^* M$, we denote its matrix form by $[z_{ij}(p)]_{(n+1)\times(n+1)}$.

Next, since h is a semi-Riemannian metric on M, by Lemma 3.1.2, $\check{h}_p : T_p^* M \to T_p M$ is an isomorphism and we write its matrix form as $[h^{ij}(p)]_{(n+1)\times(n+1)}$. Thus,

$$\check{h}_p \circ \left[Z^* \left(\phi^{-1*} g \right) \right]_p : T_p M \to T_p M,$$

is well defined and in the matrix form it is given by the following multiplication

$$\begin{aligned}
&\begin{bmatrix} h^{00}(p) & h^{01}(p) & \dots & h^{0n}(p) \\ h^{10}(p) & h^{11}(p) & \dots & h^{1n}(p) \\ \vdots & \vdots & \vdots & \vdots \\ h^{n0}(p) & h^{n1}(p) & \dots & h^{nn}(p) \end{bmatrix} \begin{bmatrix} z_{00}(p) & z_{01}(p) & \dots & z_{0n}(p) \\ z_{10}(p) & z_{11}(p) & \dots & z_{1n}(p) \\ \vdots & \vdots & \vdots & \vdots \\ z_{n0}(p) & z_{n1}(p) & \dots & z_{nn}(p) \end{bmatrix} \\
&= \begin{bmatrix} \sum_{j=0}^n h^{0j}(p) z_{j0}(p) & \sum_{j=0}^n h^{0j}(p) z_{j1}(p) & \dots & \sum_{j=0}^n h^{0j}(p) z_{jn}(p) \\ \sum_{j=0}^n h^{1j}(p) z_{j0}(p) & \sum_{j=0}^n h^{1j}(p) z_{j1}(p) & \dots & \sum_{j=0}^n h^{1j}(p) z_{jn}(p) \\ \vdots & \vdots & \dots & \vdots \\ \sum_{j=0}^n h^{nj}(p) z_{j0}(p) & \sum_{j=0}^n h^{nj}(p) z_{j1}(p) & \dots & \sum_{j=0}^n h^{nj}(p) z_{jn}(p) \end{bmatrix}.
\end{aligned}$$

Hence the trace of $\check{h}_p \circ \left[Z^* \left(\phi^{-1*} g \right) \right]_p$ is $\sum_{i=0}^n \sum_{j=0}^n h^{ij}(p) z_{ji}(p)$. That is,

$$\mathrm{tr}_h \left[Z^* \left(\phi^{-1*} g \right) \right](p) = \sum_{i,j=0}^{n} \sum_{k,l=1}^{m} h^{ij}(p) \frac{\partial Z^k}{\partial x^i}(p) \frac{\partial Z^l}{\partial x^j}(p) \ \left(\phi^{-1*} g \right)_{Z(p)} \left(\frac{\partial}{\partial y^k}\Big|_{Z(p)}, \frac{\partial}{\partial y^l}\Big|_{Z(p)} \right),$$

where we denote $\operatorname{tr}_h\left[Z^*\left(\phi^{-1*}g\right)\right](p)$ by $\check{h}_p \circ \left[Z^*\left(\phi^{-1*}g\right)\right]_p$. Therefore the actional functional $\mathscr{L}_U$ is

$$\mathscr{L}_U(Z) = \frac{1}{2}\int_{\mathbb{R}^m} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^l}{\partial x^j}(p)(g^*_{kl}\circ\phi)(p)\,dp.$$

Here we set $(g^*_{kl}\circ\phi)(p) := \left(\phi^{-1*}g\right)_{Z(p)}\left(\frac{\partial}{\partial y^k}\Big|_{Z(p)}, \frac{\partial}{\partial y^l}\Big|_{Z(p)}\right)$. In other words we write the matrix of $\left(\phi^{-1*}g\right)_{Z(p)}$ as $[(g^*_{kl}\circ\phi)(p)]_{m\times m}$. Then, for $\varphi \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^{1+n};\mathbb{R}^m)$,

$$\mathscr{L}_U(Z+\varepsilon\varphi) = \frac{1}{2}\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial(Z+\varepsilon\varphi)^k}{\partial x^i}(p)\frac{\partial(Z+\varepsilon\varphi)^l}{\partial x^j}(p)(g^*_{kl}\circ(Z+\varepsilon\varphi))(p)\,dp.$$

Consequently, by differentiating w.r.t. to ε, we get

$$\begin{aligned}
\frac{d}{d\varepsilon}\mathscr{L}_U(Z+\varepsilon\varphi) &= \frac{1}{2}\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial\varphi^k}{\partial x^i}(p)\frac{\partial(Z+\varepsilon\varphi)^l}{\partial x^j}(p)(g^*_{kl}\circ(Z+\varepsilon\varphi))(p)\,dp\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial(Z+\varepsilon\varphi)^k}{\partial x^i}(p)\frac{\partial\varphi^l}{\partial x^j}(p)(g^*_{kl}\circ(Z+\varepsilon\varphi))(p)\,dp\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial(Z+\varepsilon\varphi)^k}{\partial x^i}(p)\frac{\partial(Z+\varepsilon\varphi)^l}{\partial x^j}(p)\sum_{r=1}^{m}\frac{\partial g^*_{kl}}{\partial y^r}(Z+\varepsilon\varphi)(p)\varphi^r(p)\,dp.
\end{aligned}$$

Since the metric h on $\mathbb{R}^{1+n}$ is constant w.r.t. p and $g^*_{kl} = g^*_{lk}$, by evaluating above at $\varepsilon = 0$ followed by the integration by parts we obtain

$$\begin{aligned}
\frac{d}{d\varepsilon}\mathscr{L}_U(Z+\varepsilon\varphi)\Big|_{\varepsilon=0} &= \frac{1}{2}\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial\varphi^k}{\partial x^i}(p)\frac{\partial Z^l}{\partial x^j}(p)(g^*_{kl}\circ Z)(p)\,dp\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial\varphi^l}{\partial x^j}(p)(g^*_{kl}\circ Z)(p)\,dp\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^l}{\partial x^j}(p)\sum_{r=1}^{m}\frac{\partial g^*_{kl}}{\partial y^r}(Z(p))\varphi^r(p)\,dp\\
&= \int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial\varphi^l}{\partial x^j}(p)(g^*_{kl}\circ Z)(p)\,dp\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^l}{\partial x^j}(p)\sum_{r=1}^{m}\frac{\partial g^*_{kl}}{\partial y^r}(Z(p))\varphi^r(p)\,dp\\
&= -\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial}{\partial x^j}\left[\frac{\partial Z^k}{\partial x^i}(p)(g^*_{kl}\circ Z)(p)\right]\varphi^l(p)\,dp\\
&\quad+\frac{1}{2}\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^l}{\partial x^j}(p)\sum_{r=1}^{m}\frac{\partial g^*_{kl}}{\partial y^r}(Z(p))\varphi^r(p)\,dp\\
&= -\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial^2 Z^k}{\partial x^j\partial x^i}(p)(g^*_{kl}\circ Z)(p)Z^l(p)\,dp\\
&\quad-\int_{\mathbb{R}^{1+n}} \sum_{i,j=0}^{n}\sum_{k,l=1}^{m} h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\varphi^l(p)\sum_{r=1}^{m}\frac{\partial g^*_{kl}}{\partial y^r}(Z(p))\frac{\partial Z^r}{\partial x^j}(p)\,dp
\end{aligned}$$

$$+\frac{1}{2}\int_{\mathbb{R}^{1+n}}\sum_{i,j=0}^{n}\sum_{k,l=1}^{m}h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^l}{\partial x^j}(p)\sum_{r=1}^{m}\frac{\partial g^*_{kl}}{\partial y^r}(Z(p))\varphi^r(p)\,dp$$
$$=: \mathscr{L}_1+\mathscr{L}_2+\mathscr{L}_3.$$

By setting $\eta^k(p) := \sum_{l=1}^m (g^*_{kl}\circ Z)(p)\varphi^l(p)$, we write $\mathscr{L}_1$ as

$$\mathscr{L}_1 = -\int_{\mathbb{R}^{1+n}}\sum_{i,j=0}^{n}\sum_{k=1}^{m}h^{ij}(p)\frac{\partial^2 Z^k}{\partial x^j\partial x^i}(p)\eta^k(p)\,dp.$$

To deal with $\mathscr{L}_3$, first note that since $(\phi^{-1*}g)$ is a Riemannian metric on $\phi(U)$, the map $(\phi^{-1*}g)_{Z(p)}$ is invertible and we denote the matrix of inverse by $[(g^{*lk}\circ\phi)(p)]_{m\times m}$. Thus from above notation we have $\varphi^l(p) = \sum_{s=1}^m (g^{*ls}\circ Z)(p)\eta^s(p)$ and, with notation $g^*_{kl,r} =: \frac{\partial g^*_{kl}}{\partial y^r}$,

$$\begin{aligned}
\mathscr{L}_3 &= \frac{1}{2}\int_{\mathbb{R}^{1+n}}\sum_{i,j=0}^{n}\sum_{k,l=1}^{m}h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^l}{\partial x^j}(p)\left[\sum_{r=1}^{m}g^*_{kl,r}(Z(p))\right]\left[\sum_{q=1}^{m}(g^{*rq}\circ Z)(p)\eta^q(p)\right]dp\\
&= \frac{1}{2}\int_{\mathbb{R}^{1+n}}\sum_{q=1}^{m}\left[\sum_{i,j=0}^{n}\sum_{k,l,r=1}^{m}h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^r}{\partial x^j}(p)g^*_{kr,l}(Z(p))(g^{*lq}\circ Z)(p)\right]\eta^q(p)\,dp.
\end{aligned}$$

Now to deal with $\mathscr{L}_2$ we use the symmetricity of g^{*kl} and get

$$\begin{aligned}
\mathscr{L}_2 &= -\frac{1}{2}\int_{\mathbb{R}^{1+n}}\sum_{i,j=0}^{n}\sum_{k,l,r=1}^{m}h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^r}{\partial x^j}(p)g^*_{kl,r}(Z(p))\left[\sum_{q=1}^{m}(g^{*lq}\circ Z)(p)\eta^q(p)\right]dp\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^{1+n}}\sum_{i,j=0}^{n}\sum_{k,l,r=1}^{m}h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^r}{\partial x^j}(p)g^*_{kl,r}(Z(p))\left[\sum_{q=1}^{m}(g^{*lq}\circ Z)(p)\eta^q(p)\right]dp\\
&= -\frac{1}{2}\int_{\mathbb{R}^{1+n}}\sum_{q=1}^{m}\left[\sum_{i,j=0}^{n}\sum_{k,l,r=1}^{m}h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^r}{\partial x^j}(p)g^*_{kl,r}(Z(p))(g^{*ql}\circ Z)(p)\right]\eta^q(p)\,dp\\
&\quad -\frac{1}{2}\int_{\mathbb{R}^{1+n}}\sum_{q=1}^{m}\left[\sum_{i,j=0}^{n}\sum_{k,l,r=1}^{m}h^{ij}(p)\frac{\partial Z^r}{\partial x^i}(p)\frac{\partial Z^k}{\partial x^j}(p)g^*_{rl,k}(Z(p))(g^{*ql}\circ Z)(p)\right]\eta^q(p)\,dp.
\end{aligned}$$

Hence we get

$$\begin{aligned}
\frac{d}{d\varepsilon}\mathscr{L}_U(Z+\varepsilon\varphi)\big|_{\varepsilon=0} &= -\int_{\mathbb{R}^{1+n}}\sum_{i,j=0}^{n}\sum_{k=1}^{m}h^{ij}(p)\frac{\partial^2 Z^k}{\partial x^j\partial x^i}(p)\eta^k(p)\,dp-\frac{1}{2}\int_{\mathbb{R}^m}\sum_{q=1}^{m}\Bigg[\sum_{i,j=0}^{n}\sum_{k,l,r=1}^{m}\\
&\quad h^{ij}(p)\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^r}{\partial x^j}(p)(g^{*ql}\circ Z)(p)\left\{g^*_{kl,r}(Z(p))+g^*_{rl,k}(Z(p))-g^*_{kr,l}(Z(p))\right\}\Bigg]\eta^q(p)\,dp.
\end{aligned}$$

Therefore, since $\frac{d}{d\varepsilon}\mathscr{L}_U(Z+\varepsilon\varphi)\big|_{\varepsilon=0}=0$, by the Du Bois-Reymond Lemma, see [157, Chapter 4] we get the wave maps system as, for every $q=1,\dots,m$,

$$\sum_{i,j=0}^{n}h^{ij}(p)\frac{\partial^2 Z^q}{\partial x^j\partial x^i}(p)-\sum_{i,j=0}^{n}\sum_{k,r=1}^{m}h^{ij}(p)\Gamma^q_{kr}(Z(p))\frac{\partial Z^k}{\partial x^i}(p)\frac{\partial Z^r}{\partial x^j}(p)=0. \tag{3.2.2}$$

Here

$$\Gamma^q_{kr} := \frac{1}{2}\sum_{l=1}^{m}g^{*ql}\left\{g^*_{kl,r}+g^*_{rl,k}-g^*_{kr,l}\right\},$$

is a Christoffel symbol (of the second kind) associated to the metric g.

We finish this chapter by observing that since $(x^0, x^1, \dots, x^n) = (t, x) \in \mathbb{R}^{1+n}$ and

$$h^{0,0} = -1; \quad h^{i,j} = 0 \text{ for } i \neq j \in \{0, \dots, n\}; \quad \text{and } h^{j,j} = 1 \text{ for } j = 1, \dots n,$$

the system (3.2.2) gives (1.2.2) which a smooth wave map satisfies when consider in terms of local charts.

CHAPTER 4

Large Deviations for Stochastic Geometric Wave Equation

We establish here the validity of a large deviation principle for the small noise asymptotic of strong solutions to stochastic geometric wave equations with values in a compact Riemannian manifold. The main novelty of this chapter lies in to be the first ever result on large deviations for stochastic geometric wave equations. Our proof relies on applying the weak convergence approach of Budhiraja and Dupuis [30] to SPDEs where solutions are local Sobolev spaces valued stochastic processes. This is a new approach with respect to the existing literature on the second order in time stochastic PDEs, see e.g. Zhang's work [163] on the stochastic beam equation.

The chapter is organized as follows. In Section 4.1, we introduce our notation and state the required definitions. In Section 4.2 we write all the preliminaries about the nonlinearity and the diffusion coefficient which we need to use later in the current chapter. Section 4.3 is to prove the existence of a unique global strong solution, in PDE sense, to the skeleton equation associated to (1.2.7). The proof of a large deviations principle (LDP), based on weak convergence approach, is in Section 4.4. We conclude the chapter with two Auxiliary Subsections 4.5.1 and 4.5.2, respectively, where we state the slightly modified version of the existing results on global well-posedness of (1.2.7) and an energy inequality from [23] which we use frequently in the sequel.

4.1 Notation

For any two non-negative quantities a and b, we write $a \lesssim b$ if there exists a universal constant $c > 0$ such that $a \le cb$, and we write $a \simeq b$ when $a \lesssim b$ and $b \lesssim a$. In case we want to emphasize the dependence of c on some parameters $a_1, \dots, a_k$, then we write, respectively, $\lesssim_{a_1,\dots,a_k}$ and $\simeq_{a_1,\dots,a_k}$. We will denote by $B_R(a)$, for $a \in \mathbb{R}$ and $R > 0$, the open ball in $\mathbb{R}$ with center at a and we put $B_R = B_R(0)$. Now we list the notation that we are going to use throughout the whole chapter.

- $\mathbb{N} = \{0, 1, \cdots\}$ denotes the set of natural numbers, $\mathbb{R}_+ = [0, \infty)$, Leb denotes the Lebesgue measure.
- Let $I \subseteq \mathbb{R}$ be an open interval. By $L^p(I;\mathbb{R}^n)$, $p \in [1, \infty)$, we denote the classical real Banach space of all (equivalence classes of) $\mathbb{R}^n$-valued p-integrable maps on I. The norm on $L^p(I;\mathbb{R}^n)$ is given by

$$\|u\|_{L^p(I;\mathbb{R}^n)} := \left(\int_I |u(x)|^p \, dx\right)^{\frac{1}{p}}, \qquad u \in L^p(I;\mathbb{R}^n),$$

where $|\cdot|$ is the Euclidean norm on $\mathbb{R}^n$. For $p = \infty$, we consider the usual modification to essential supremum.

- For any $p \in [1, \infty]$, $L^p_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ stands for a metrizable topological vector space equipped with a natural countable family of seminorms $\{p_j\}_{j\in\mathbb{N}}$ defined by

$$p_j(u) := \|u\|_{L^p(B_j;\mathbb{R}^n)}, \qquad u \in L^2_{\text{loc}}(\mathbb{R};\mathbb{R}^n),\ j \in \mathbb{N}.$$

- By $H^{k,p}(I;\mathbb{R}^n)$, for $p \in [1, \infty]$ and $k \in \mathbb{N}$, we denote the Banach space of all $u \in L^p(I;\mathbb{R}^n)$ for which $D^j u \in L^p(I;\mathbb{R}^n)$, $j = 0, 1, \ldots, k$, where D^j is the weak derivative of order j. The norm here is given by

$$\|u\|_{H^{k,p}(I;\mathbb{R}^n)} := \left(\sum_{j=0}^{k} \|D^j u\|^p_{L^p(I;\mathbb{R}^n)}\right)^{\frac{1}{p}}, \qquad u \in H^{k,p}(I;\mathbb{R}^n).$$

- We write $H^{k,p}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$, for $p \in [1, \infty]$ and $k \in \mathbb{N}$, to denote the space of all elements $u \in L^p_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ whose weak derivatives up to order k belong to $L^p_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$. It is relevant to note that $H^{k,p}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ is a metrizable topological vector space equipped with the following natural countable family of seminorms $\{q_j\}_{j\in\mathbb{N}}$,

$$q_j(u) := \|u\|_{H^{k,p}(B_j;\mathbb{R}^n)}, \qquad u \in H^{k,p}_{\text{loc}}(\mathbb{R};\mathbb{R}^n),\ j \in \mathbb{N}.$$

The spaces $H^{k,2}(I;\mathbb{R}^n)$ and $H^{k,2}_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ are usually denoted by $H^k(I;\mathbb{R}^n)$ and $H^k_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ respectively.

- We set $\mathcal{H} := H^2(\mathbb{R};\mathbb{R}^n) \times H^1(\mathbb{R};\mathbb{R}^n)$ and $\mathcal{H}_{\text{loc}} := H^2_{\text{loc}}(\mathbb{R};\mathbb{R}^n) \times H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$.
- To shorten the notation in calculation we set the following rules:
  - if the space where function is taking value, for example $\mathbb{R}^n$, is clear then to save the space we will omit $\mathbb{R}^n$, for example $H^k(I)$ instead $H^k(I;\mathbb{R}^n)$;
  - if $I = (0, T)$ or $(-R, R)$ or $B(x, R)$, for some $T, R > 0$ and $x \in \mathbb{R}$, then instead of $L^p(I;\mathbb{R}^n)$ we write, respectively, $L^p(0, T;\mathbb{R}^n)$, $L^p(B_R;\mathbb{R}^n)$, $L^p(B(x, R);\mathbb{R}^n)$. Similarly for H^k and H^k_{loc} spaces.
  - write $\mathcal{H}(B_R)$ or $\mathcal{H}_R$ for $H^2((-R, R);\mathbb{R}^n) \times H^1((-R, R);\mathbb{R}^n)$.
- For any nonnegative integer j, let $\mathcal{C}^j(\mathbb{R})$ be the space of real valued continuous functions whose derivatives up to order j are continuous on $\mathbb{R}$. We also need the family of spaces $\mathcal{C}^j_b(\mathbb{R})$ defined by

$$\mathcal{C}^j_b(\mathbb{R}) := \left\{u \in \mathcal{C}^j(\mathbb{R}); \forall\, \alpha \in \mathbb{N}, \alpha \le j, \exists K_\alpha, \|D^j u\|_{L^\infty(\mathbb{R})} < K_\alpha\right\}.$$

- Given $T > 0$ and Banach space E, we denote by $\mathcal{C}([0,T];E)$ the real Banach space of all E-valued continuous functions $u : [0,T] \to E$ endowed with the norm

$$\|u\|_{\mathcal{C}([0,T];E)} := \sup_{t\in[0,T]} \|u(t)\|_E, \qquad u \in \mathcal{C}([0,T];E).$$

By ${}_0\mathcal{C}([0,T],E)$ we mean the set of elements of $\mathcal{C}([0,T];E)$ vanishes at origin, that is,

$${}_0\mathcal{C}([0,T],E) := \left\{u \in \mathcal{C}([0,T],E) : u(0) = 0\right\}.$$

- For given metric space (X,ρ), by $\mathcal{C}(\mathbb{R};X)$ we mean the space of continuous functions from $\mathbb{R}$ to X which is equipped with the metric

$$(f,g) \mapsto \sum_{j=1}^{\infty} \frac{1}{2^j} \min\{1, \sup_{t\in[-j,j]} \rho(f(t),g(t))\}.$$

- We denote the tangent and the normal bundle of a smooth manifold M by TM and NM, respectively. Let $\mathfrak{F}(M)$ be the set of all smooth $\mathbb{R}$-valued functions on M.

- A map $u : \mathbb{R} \to M$ belongs to $H^k_{\text{loc}}(\mathbb{R};M)$ provided that $\theta \circ u \in H^k_{\text{loc}}(\mathbb{R};\mathbb{R})$ for every $\theta \in \mathfrak{F}(M)$. We equip $H^k_{\text{loc}}(\mathbb{R};M)$ with the topology induced by the mappings

$$H^k_{\text{loc}}(\mathbb{R};M) \ni u \mapsto \theta \circ u \in H^k_{\text{loc}}(\mathbb{R};\mathbb{R}), \quad \theta \in \mathfrak{F}(M).$$

Since the tangent bundle TM of a manifold M is also a manifold, this definition covers Sobolev spaces of TM-valued functions too.

- By $\mathscr{L}_2(H_1,H_2)$ we denote the class of Hilbert-Schmidt operators from a separable Hilbert space H_1 to another H_2. By $\mathcal{L}(X,Y)$ we denote the space of all linear continuous operators from a topological vector space X to Y.

- We denote by $\mathcal{S}(\mathbb{R})$ the space of Schwartz functions on $\mathbb{R}$ and write $\mathcal{S}'(\mathbb{R})$ for its dual, which is the space of tempered distributions on $\mathbb{R}$. By L^2_w we denote the weighted space $L^2(\mathbb{R},w,dx)$, where $w(x) := e^{-x^2}, x \in \mathbb{R}$, is an element of $\mathcal{S}(\mathbb{R})$. Let $H^s_w(\mathbb{R}), s \geq 0$, be the completion of $\mathcal{S}(\mathbb{R})$ with respect to the norm

$$\|u\|_{H^s_w(\mathbb{R})} := \left(\int_{\mathbb{R}} (1+|x|^2)^s |\mathcal{F}(w^{1/2}u)(x)|^2\, dx\right)^{\frac{1}{2}},$$

where $\mathcal{F}$ denotes the Fourier transform.

4.2 Preliminaries

In this section we discuss all the required preliminaries about the nonlinearity and the diffusion coefficient that we need in Section 4.3. We are following Sections 3 to 5 of [23] very closely here.

4.2.1 The Wiener process

The random forcing we consider is in the form of a spatially homogeneous Wiener process on $\mathbb{R}$ with a spectral measure μ satisfying

$$\int_{\mathbb{R}} (1+|x|^2)^2\,\mu(dx) < \infty. \tag{4.2.1}$$

Let μ be a finite and symmetric measure on $\mathbb{R}$. A $\mathcal{S}'(\mathbb{R})$-valued process $W = \{W(t), t \geq 0\}$, on a given stochastic basis $(\Omega, \mathscr{F}, (\mathscr{F}_t)_{t\geq 0}, \mathbb{P})$, is called a spatially homogeneous Wiener process with spectral measure μ provided that

1. for every $\varphi \in \mathcal{S}(\mathbb{R})$, $\{W(t)(\varphi), t \geq 0\}$ is a real-valued $\mathscr{F}_t$-Wiener process,
2. $\mathbb{E}\{W(t)(\varphi)W(t)(\psi)\} = t\langle \widehat{\varphi}, \widehat{\psi}\rangle_{L^2(\mu)}$ holds for every $t \geq 0$ and $\varphi, \psi \in \mathcal{S}(\mathbb{R})$.

It is shown in [129] that the Reproducing Kernel Hilbert Space (RKHS) H_μ of the Gaussian measure $W(1)$ is described as the subspace of tempered distributions

$$H_\mu := \left\{\widehat{\psi\mu} : \psi \in L^2(\mathbb{R}^n, \mu, \mathbb{C}), \psi(x) = \overline{\psi(-x)}, x \in \mathbb{R}\right\}$$

where $L^2(\mathbb{R}^n, \mu, \mathbb{C})$ is the classical Banach space of equivalence classes of complex-valued and square integrable functions with respect to measure μ. Note that H_μ endowed with inner-product

$$\left\langle \widehat{\psi_1\mu}, \widehat{\psi_2\mu}\right\rangle_{H_\mu} := \int_{\mathbb{R}} \psi_1(x)\overline{\psi_2(x)}\,\mu(dx),$$

is a Hilbert space.

Recall from [129, 130] that W can be regarded as a cylindrical Wiener process on H_μ and it takes values in any Hilbert space E such that the embedding $H_\mu \hookrightarrow E$ is Hilbert-Schmidt. Since we explicitly know the structure of H_μ, in the next result, whose proof is based on [127, Lemma 2.2] and discussion with Szymon Peszat [128], we provide an example of E such that the paths of W can be considered in $\mathcal{C}([0,T];E)$. Below we also use the notation $\mathcal{F}(\cdot)$, along with $\widehat{\cdot}$, to denote the Fourier transform.

Lemma 4.2.1. *Let us assume that the measure μ satisfies* (4.2.1). *Then the identity map from H_μ into $H^2_w(\mathbb{R})$ is a Hilbert-Schmidt operator.*

Proof of Lemma 4.2.1 To simplify the notation we set $L^2_{(s)}(\mathbb{R},\mu)$ to be the space of all $f \in L^2(\mathbb{R},\mu;\mathbb{C})$ such that $f(x) = \overline{f(-x)}$, $x \in \mathbb{R}$. Let $\{e_k\}_{k\in\mathbb{N}} \subset \mathcal{S}(\mathbb{R})$ be an orthonormal basis of $L^2_{(s)}(\mathbb{R},\mu)$. Then, by the definition of H_μ, $\{\mathcal{F}(e_k\mu)\}_{k\in\mathbb{N}}$ is an orthonormal basis of H_μ. Invoking the convolution theorem of Fourier transform and followed by the Bessel inequality, see [9], we obtain,

$$\begin{aligned}
\sum_{k=1}^{\infty} \|\widehat{e_k\mu}\|^2_{H^2_w} &= \sum_{k=1}^{\infty} \int_{\mathbb{R}} (1+|x|^2)|\mathcal{F}\left(w^{1/2}\mathcal{F}(e_k\mu)\right)(x)|^2\,dx \\
&= \int_{\mathbb{R}} (1+|x|^2)^2 \left(\sum_{k=1}^{\infty} |\mathcal{F}\left(w^{1/2}\mathcal{F}(e_k\mu)\right)(x)|^2\right) dx
\end{aligned}$$

$$
\begin{aligned}
&= \int_{\mathbb{R}} (1+|x|^2)^2 \left(\sum_{k=1}^{\infty} \left| \int_{\mathbb{R}} \mathcal{F}\left(w^{1/2}\right)(x-z) e_k(z)\, \mu(dz) \right|^2 \right) dx \\
&\leq \int_{\mathbb{R}^2} (1+|x|^2)^2 |\mathcal{F}\left(w^{1/2}\right)(x-z)|^2\, \mu(dz)\, dx \\
&= \int_{\mathbb{R}^2} (1+|x+z|^2)^2 |\mathcal{F}\left(w^{1/2}\right)(x)|^2\, \mu(dz)\, dx \\
&\lesssim \| w^{1/2} \|^2_{H^1_w(\mathbb{R})} \int_{\mathbb{R}} (1+|z|^2)^2\, \mu(dz).
\end{aligned}
$$

Hence the Lemma 4.2.1. ■

It is relevant to note here that $H^2_w(\mathbb{R})$ is a subset if $H^2_{\text{loc}}(\mathbb{R})$. The next result, whose detailed proof can be found in [119, Lemma 1], plays very important role in deriving the required estimates for the terms involving diffusion coefficient.

Lemma 4.2.2. *If the measure μ satisfies* (4.2.1) *then H_μ is continuously embedded in $\mathcal{C}^2_b(\mathbb{R})$. Moreover, for given any $g \in H^j(B(x,R);\mathbb{R}^n)$, where $x \in \mathbb{R}, R > 0$ and $j \in \{0,1,2\}$, the multiplication operator $H_\mu \ni \xi \mapsto g \cdot \xi \in H^j(B(x,R);\mathbb{R}^n)$ is Hilbert-Schmidt and $\exists\, c > 0$, independent of R, x, g, ξ and j, such that*

$$\|\xi \mapsto g \cdot \xi\|_{\mathcal{L}_2(H_\mu, H^j(B(x,R);\mathbb{R}^n))} \leq c \|g\|_{H^j(B(x,R);\mathbb{R}^n)}.$$

Remark 4.2.3. Note that the constant of inequality c in Lemma 4.2.2 does not depend on the size and position of the ball. However, if we consider a cylindrical Wiener process, then c will also depend on the centre x but will be bounded on bounded sets with respect to x.

4.2.2 Extensions of non-linearity

By definition $A_p : T_pM \times T_pM \to N_pM$, $p \in M$, where $T_pM \subseteq \mathbb{R}^n$ and $N_pM \subseteq \mathbb{R}^n$ are the tangent and the normal vector space at $p \in M$ respectively. It is well known, see e.g. [81], that A_p, $p \in M$, is symmetric bilinear.

Since we are following the approach of [12], [23], and [80], one of the main step in proof of the existence theorem is to consider the problem (1.2.7) in the ambient space $\mathbb{R}^n$ with an appropriate extension of A from their domain (product of tangent bundles) to $\mathbb{R}^n$. In this section we discuss two extensions of A which work fine in the context of stochastic wave map as displayed in [23].

Let us denote by $\mathcal{E}$ the exponential function

$$T\mathbb{R}^n \ni (p,\xi) \mapsto p + \xi \in \mathbb{R}^n,$$

relative to the Riemannian manifold $\mathbb{R}^n$ equipped with the standard Euclidean metric. The proof of the following proposition about the existence of an open set O containing M, which is called a tubular neighbourhood of M, can be found in [118, Proposition 7.26, p. 200].

Proposition 4.2.4. *There exists an $\mathbb{R}^n$-open neighbourhood O around M and an NM-open neighbourhood V around the set $\{(p,0) \in NM : p \in NM\}$ such that the restriction of the exponential map*

$\mathcal{E}|_V : V \to O$ is a diffeomorphism. Moreover, V can be chosen in such a way that $(p, t\xi) \in V$ whenever $-1 \le t \le 1$ and $(p,\xi) \in V$.

In case of no ambiguity, we will denote the diffeomorphism $\mathcal{E}|_V : V \to O$ by $\mathcal{E}$. By using the Proposition 4.2.4, diffeomorphism $i : NM \ni (p,\xi) \mapsto (p,-\xi) \in NM$ and the standard argument of partition of unity, one can obtain a function $\Upsilon : \mathbb{R}^n \to \mathbb{R}^n$ which identifies the manifold M as its fixed point set. In precise we have the following result.

Lemma 4.2.5. *[23, Corollary 3.4 and Remark 3.5] There exists a smooth compactly supported function $\Upsilon : \mathbb{R}^n \to \mathbb{R}^n$ which has the following properties:*

1. *restriction of Υ on O is a diffeomorpshim,*
2. *$\Upsilon|_O = \mathcal{E} \circ i \circ \mathcal{E}^{-1} : O \to O$ is an involution on the tubular neighborhood O of M,*
3. *$\Upsilon(\Upsilon(q)) = q$ for every $q \in O$,*
4. *if $q \in O$, then $\Upsilon(q) = q$ if and only if $q \in M$,*
5. *if $p \in M$, then*
$$\Upsilon'(p)\xi = \begin{cases} \xi, & \text{provided } \xi \in T_pM, \\ -\xi & \text{provided } \xi \in N_pM. \end{cases}$$

The following result is the first extension of the second fundamental form that we use in this chapter.

Proposition 4.2.6. *[23, Proposition 3.6] If we define*

$$B_q(a,b) = \sum_{i,j=1}^{n} \frac{\partial^2 \Upsilon}{\partial q_i \partial q_j}(q) a_i b_j = \Upsilon''_q(a,b), \qquad q \in \mathbb{R}^n, \quad a,b \in \mathbb{R}^n \tag{4.2.2}$$

and

$$\mathcal{A}_q(a,b) = \frac{1}{2} B_{\Upsilon(q)}(\Upsilon'(q)a, \Upsilon'(q)b), \qquad q \in \mathbb{R}^n, \quad a,b \in \mathbb{R}^n, \tag{4.2.3}$$

then, for every $p \in M$,

$$\mathcal{A}_p(\xi,\eta) = A_p(\xi,\eta),\ \xi,\eta \in T_pM,$$

and

$$\mathcal{A}_{\Upsilon(q)}(\Upsilon'(q)a, \Upsilon'(q)b) = \Upsilon'(q)\mathcal{A}_q(a,b) + B_q(a,b),\ q \in O,\ a,b \in \mathbb{R}^n. \tag{4.2.4}$$

Along with the extension $\mathcal{A}$, defined by formula (4.2.3), we also need the extension $\mathscr{A}$, defined by formula (4.2.5), of the second fundamental form tensor A which will be perpendicular to the tangent space.

Proposition 4.2.7. *[23, Proposition 3.7] Consider the function*

$$\mathscr{A} : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (q,a,b) \mapsto \mathscr{A}_q(a,b) \in \mathbb{R}^n,$$

defined by formula

$$\mathscr{A}_q(a,b) = \sum_{i,j=1}^{n} a_i v_{ij}(q) b_j = A_q(\pi_q(a), \pi_q(b)), \qquad q \in \mathbb{R}^n, \quad a \in \mathbb{R}^n, \quad b \in \mathbb{R}^n, \tag{4.2.5}$$

where π_p, $p \in M$ is the orthogonal projection of $\mathbb{R}^n$ to T_pM, and v_{ij}, for $i, j \in \{1,\dots,n\}$, are smooth and symmetric (i.e. $v_{ij} = v_{ji}$) extensions of $v_{ij}(p) := A_p(\pi_p e_i, \pi_p e_j)$ to ambient space $\mathbb{R}^n$. Then $\mathscr{A}$ satisfies the following:

1. *$\mathscr{A}$ is smooth in (q,a,b) and symmetric in (a,b) for every q,*
2. *$\mathscr{A}_p(\xi,\eta) = A_p(\xi,\eta)$ for every $p \in M$, $\xi,\eta \in T_pM$,*
3. *$\mathscr{A}_p(a,b)$ is perpendicular to T_pM for every $p \in M$, $a, b \in \mathbb{R}^n$.*

4.2.3 The C_0-group and the extension operators

Here we recall some facts on infinitesimal generators of the linear wave equation and on the extension operators in various Sobolev spaces. Refer [23, Section 5] for details.

Proposition 4.2.8. *Assume that $k, n \in \mathbb{N}$. The one parameter family of operators defined by*

$$S_t \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} \cos[t(-\Delta)^{1/2}]u^1 & + & (-\Delta)^{-1/2}\sin[t(-\Delta)^{1/2}]v^1 \\ & \vdots & \\ \cos[t(-\Delta)^{1/2}]u^n & + & (-\Delta)^{-1/2}\sin[t(-\Delta)^{1/2}]v^n \\ -(-\Delta)^{1/2}\sin[t(-\Delta)^{1/2}]u^1 & + & \cos[t(-\Delta)^{1/2}]v^1 \\ & \vdots & \\ -(-\Delta)^{1/2}\sin[t(-\Delta)^{1/2}]u^n & + & \cos[t(-\Delta)^{1/2}]v^n \end{pmatrix}$$

is a C_0-group on

$$\mathcal{H}^k := H^{k+1}(\mathbb{R};\mathbb{R}^n) \times H^k(\mathbb{R};\mathbb{R}^n),$$

and its infinitesimal generator is an operator $\mathcal{G}^k = \mathcal{G}$ defined by

$$\begin{aligned} D(\mathcal{G}^k) &= H^{k+2}(\mathbb{R};\mathbb{R}^n) \times H^{k+1}(\mathbb{R};\mathbb{R}^n), \\ \mathcal{G}\begin{pmatrix} u \\ v \end{pmatrix} &= \begin{pmatrix} v \\ \Delta u \end{pmatrix}. \end{aligned}$$

The following theorem is well known, see for example [104] and [66, Section II.5.4].

Proposition 4.2.9. *Let $k \in \mathbb{N}$. There exists a linear bounded operator*

$$E^k : H^k((-1,1);\mathbb{R}^n) \to H^k(\mathbb{R};\mathbb{R}^n),$$

such that

(i) $E^k f = f$ almost everywhere on $(-1,1)$ whenever $f \in H^k((-1,1);\mathbb{R}^n)$,

(ii) $E^k f$ vanishes outside of $(-2,2)$ whenever $f \in H^k((-1,1);\mathbb{R}^n)$,

(iii) $E^k f \in \mathcal{C}^k(\mathbb{R};\mathbb{R}^n))$, if $f \in \mathcal{C}^k([-1,1];\mathbb{R}^n))$,

(iv) if $j \in \mathbb{N}$ and $j < k$, then there exists a unique extension of E^k to a bounded linear operator from $H^j((-1,1);\mathbb{R}^n)$ to $H^j(\mathbb{R};\mathbb{R}^n)$.

Definition 4.2.10. For $k \in \mathbb{N}$, $r > 0$ we define the operators $E_r^k : H^j((-r,r);\mathbb{R}^n) \to H^j(\mathbb{R};\mathbb{R}^n)$, $j \in \mathbb{N}$, $j \leq k$, called as r-scaled E^k operators, by the following formula

$$(E_r^k f)(x) = \{E^k[y \mapsto f(yr)]\}\left(\frac{x}{r}\right), \qquad x \in \mathbb{R}, \tag{4.2.6}$$

for $r > 0$ and $f \in H^k((-r,r);\mathbb{R}^n)$.

The following remark will be useful in Lemma 4.3.4.

Remark 4.2.11. We can rewrite (4.2.6) as $(E_r^k f)(x) = (E^k f_r)(\frac{x}{r})$, $f \in H^k((-r,r);\mathbb{R}^n)$ where

$$f_r : (-1,1) \ni y \mapsto f(yr) \in \mathbb{R}^n.$$

Also, observe that for $f \in H^1((-r,r);\mathbb{R}^n)$

$$\|f_r\|^2_{H^1((-1,1);\mathbb{R}^n)} \leq (r^{-1} + r)\|f\|^2_{H^1((-r,r);\mathbb{R}^n)}.$$

4.2.4 Diffusion coefficient

In this subsection we discuss the assumptions on diffusion coefficient Y which we only need in Section 4.3. It is relevant to note that due to a technical issue, which is explained in Section 4.4, we need to consider stricter conditions on Y in establishing the large deviation principle for (1.2.7). Here $Y_p : T_pM \times T_pM \to T_pM$, for $p \in M$, is a mapping satisfying,

$$|Y_p(\xi,\eta)|_{T_pM} \leq C_Y(1 + |\xi|_{T_pM} + |\eta|_{T_pM}), \qquad p \in M, \quad \xi,\eta \in T_pM,$$

for some constant $C_Y > 0$ which is independent of p. Due to Lemma 4.2.5 and [23, Proposition 3.10], we can extend the noise coefficient to map $Y : \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \ni (p,a,b) \mapsto Y_p(a,b) \in \mathbb{R}^n$ which satisfies the following:

Y.1 for $q \in O$ and $a, b \in \mathbb{R}^n$,

$$Y_{\Upsilon(q)}\big(\Upsilon'(q)a, \Upsilon'(q)b\big) = \Upsilon'(q)Y_q(a,b), \tag{4.2.7}$$

Y.2 there exists an compact set $K_Y \subset \mathbb{R}^n$ containing M such that $Y_p(a,b) = 0$, for all $a, b \in \mathbb{R}^n$, whenever $p \notin K_Y$,

Y.3 Y is of $\mathcal{C}^2$-class and there exist positive constants $C_{Y_i}, i \in \{1,2,3\}$ such that, with notation $Y(p,a,b) := Y_p(a,b)$, for every $p, a, b \in \mathbb{R}^n$,

$$|Y_p(a,b)| \le C_{Y_0}(1+|a|+|b|), \tag{4.2.8}$$

$$\Big|\frac{\partial Y}{\partial p_i}(p,a,b)\Big| \le C_{Y_1}(1+|a|+|b|), \quad i = 1, \dots, n, \tag{4.2.9}$$

$$\Big|\frac{\partial Y}{\partial a_i}(p,a,b)\Big| + \Big|\frac{\partial Y}{\partial b_i}(p,a,b)\Big| \le C_{Y_2}, \quad i = 1, \dots, n, \tag{4.2.10}$$

$$\Big|\frac{\partial^2 Y}{\partial x_j \partial y_i}(p,a,b)\Big| \le C_{Y_3}, \quad x, y \in \{p,a,b\} \text{ and } i, j \in \{1, \dots, n\}. \tag{4.2.11}$$

4.3 The skeleton equation

The purpose of this section is to introduce and study the deterministic equation associated to (1.2.7). Define

$$_0H^{1,2}(0,T;H_\mu) := \left\{h \in {}_0\mathcal{C}([0,T],E) : \dot{h} \in L^2(0,T;H_\mu)\right\}.$$

Note that $_0H^{1,2}(0,T;H_\mu)$ is a Hilbert space with norm $\int_0^T \|\dot{h}(t)\|^2_{H_\mu}\,dt$ and the map

$$L^2(0,T;H_\mu) \ni \dot{h} \mapsto h = \left\{t \mapsto \int_0^t \dot{h}(s)\,ds\right\} \in {}_0H^{1,2}(0,T;H_\mu),$$

is an isometric isomorphism. For $h \in {}_0H^{1,2}(0,T;H_\mu)$, let us consider the so called "skeleton equation" associated to problem (1.2.7).

$$\begin{cases} \partial_{tt}u = \partial_{xx}u + A_u(\partial_t u, \partial_t u) - A_u(\partial_x u, \partial_x u) + Y_u(\partial_t u, \partial_x u)\,\dot{h}, \\ u(0,\cdot) = u_0, \partial_t u(0,\cdot) = v_0. \end{cases} \tag{4.3.1}$$

Recall that M is a compact Riemannian manifold which is embedded by an isometric embedding into some Euclidean space $\mathbb{R}^n$, and hence, we can assume that M is a submanifold of $\mathbb{R}^n$. The following main result of this section is the deterministic version of [23, Theorem 11.1].

Theorem 4.3.1. *Let $T > 0$, $h \in {}_0H^{1,2}(0,T;H_\mu)$ and $(u_0, v_0) \in H^2_{loc} \times H^1_{loc}(\mathbb{R};TM)$ are given. Then for every $R > T$, there exists a $u : [0,T) \times \mathbb{R} \to M \subset \mathbb{R}^n$ such that the following hold:*

1. $[0,T) \ni t \mapsto u(t,\cdot) \in H^2((-R,R);\mathbb{R}^n)$ *is continuous,*
2. $[0,T) \ni t \mapsto u(t,\cdot) \in H^1((-R,R);\mathbb{R}^n)$ *is continuously differentiable,*
3. $u(t,x) \in M$ *for every* $t \in [0,T), x \in \mathbb{R}$,
4. $u(0,x) = u_0(x)$ *and* $\partial_t u(0,x) = v_0(x)$ *holds for every* $x \in \mathbb{R}$,

5. *for every $t \in [0,T)$ the following will hold in $L^2((-R,R);\mathbb{R}^n)$,*

$$\begin{aligned}\partial_t u(t) = v_0 + \int_0^t \left[\partial_{xx} u(s) - A_{u(s)}(\partial_x u(s), \partial_x u(s)) + A_{u(s)}(\partial_t u(s), \partial_t u(s))\right] ds \\ + \int_0^t Y_{u(s)}(\partial_t u(s), \partial_x u(s)) \dot{h}(s)\, ds. \qquad (4.3.2)\end{aligned}$$

Moreover, if there exists another map $U : [0,T) \times \mathbb{R} \to M$ which also satisfy the above properties then

$$U(t,x) = u(t,x) \qquad \textit{for every} \quad |x| \le R - t \quad \textit{and} \quad t \in [0,T).$$

Proof of Theorem 4.3.1 The proof here is motivated from Sections 7-11 of [23] but presenting with more details. Since we expect that the solutions of the equation (4.3.1) take values on a compact Riemannian manifold M, we cannot expect them to belong to the Hilbert space $H^2(\mathbb{R};\mathbb{R}^n) \times H^1(\mathbb{R};\mathbb{R}^n)$. Indeed, suppose $M := \mathbb{S}^2$ and $u(t,x) \in \mathbb{S}^2$, then

$$\|u(t,\cdot)\|^2_{H^2(\mathbb{R};\mathbb{R}^n)} \ge \|u(t,\cdot)\|^2_{L^2(\mathbb{R};\mathbb{R}^n)} = \int_{\mathbb{R}} |u(t,x)|^2, dx = \int_{\mathbb{R}} 1\, dx = \infty.$$

Hence, in line with the PDE theory, we seek those solutions which will take values in the Fréchet space $H^2_{\text{loc}}(\mathbb{R};\mathbb{R}^n) \times H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$ but the theory of Bochner integration for integrand in such spaces is not available. To overcome this problem we localize the problem by a series of non-linear wave equations.

Let us fix $r > R + T$, and $k \in \mathbb{N}$. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a smooth compactly supported function such that $\varphi(x) = 1$ for $x \in (-r,r)$ and $\varphi(x) = 0$ for $x \notin (-2r,2r)$. Next, with the convention $z = (u,v) \in \mathcal{H}$, we define the following maps

$$\mathbf{F}_r \quad : \quad [0,T] \times \mathcal{H} \ni (t,z) \mapsto \begin{pmatrix} 0 \\ E^1_{r-t}[\mathcal{A}_u(v,v) - \mathcal{A}_u(u_x,u_x)] \end{pmatrix} \in \mathcal{H},$$

$$\mathbf{F}_{r,k} \quad : \quad [0,T] \times \mathcal{H} \ni (t,z) \mapsto \begin{cases} \mathbf{F}_r(t,z), & \text{if} \quad |z|_{\mathcal{H}_{r-t}} \le k \\ \left(2 - \frac{1}{k}|z|_{\mathcal{H}_{r-t}}\right)\mathbf{F}_r(t,z), & \text{if} \quad k \le |z|_{\mathcal{H}_{r-t}} \le 2k \\ 0, & \text{if} \quad 2k \le |z|_{\mathcal{H}_{r-t}} \end{cases} \in \mathcal{H},$$

$$\mathbf{G}_r \quad : \quad [0,T] \times \mathcal{H} \ni (t,z) \mapsto \begin{pmatrix} 0 \\ (E^1_{r-t} Y_u(v,u_x)) \cdot \end{pmatrix} \in \mathscr{L}_2(H_\mu, \mathcal{H}),$$

$$\mathbf{G}_{r,k} \quad : \quad [0,T] \times \mathcal{H} \ni (t,z) \mapsto \begin{cases} \mathbf{G}_r(t,z), & \text{if} \quad |z|_{\mathcal{H}_{r-t}} \le k \\ \left(2 - \frac{1}{k}|z|_{\mathcal{H}_{r-t}}\right)\mathbf{G}_r(t,z), & \text{if} \quad k \le |z|_{\mathcal{H}_{r-t}} \le 2k \\ 0, & \text{if} \quad 2k \le |z|_{\mathcal{H}_{r-t}} \end{cases} \in \mathscr{L}_2(H_\mu, \mathcal{H}),$$

$$\mathbf{Q}_r \quad : \quad \mathcal{H} \ni z \mapsto \begin{pmatrix} \varphi \cdot \Upsilon(u) \\ \varphi \cdot \Upsilon'(u) v \end{pmatrix} \in \mathcal{H},$$

where $(E^1_{r-t} Y_u(v,u_x))\cdot$ means that, for every $(u,v) \in \mathcal{H}$, $E^1_{r-t} Y_u(v,u_x) \in H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)$, and the multiplication operator defined by

$$(E^1_{r-t} Y_u(v,u_x)) \cdot : H_\mu \ni \xi \mapsto (E^1_{r-t} Y_u(v,u_x)) \cdot \xi \in H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n),$$

satisfies Lemma 4.2.2.

The following two properties, which we state without proof, of $\mathbf{Q}_r$ are taken from [23, Section 7].

Lemma 4.3.2. *If $z = (u, v) \in \mathcal{H}$ is such that $u(x) \in M$ and $v(x) \in T_{u(x)}M$ for $|x| < r$, then $\mathbf{Q}_r(z) = z$ on $(-r, r)$.*

Lemma 4.3.3. *The mapping $\mathbf{Q}_r$ is of $\mathcal{C}^1$-class and its derivative, with $z = (u, v) \in \mathcal{H}$, satisfy*

$$\mathbf{Q}_r'(z)w = \begin{pmatrix} \varphi \cdot \Upsilon'(u)w^1 \\ \varphi \cdot [\Upsilon''(u)(v, w^1) + \Upsilon'(u)w^2] \end{pmatrix}, \; w = (w^1, w^2) \in \mathcal{H}.$$

The next lemma is about the locally Lipschitz properties of the localized maps defined above.

Lemma 4.3.4. *For each $k \in \mathbb{N}$ the functions $\mathbf{F}_r$, $\mathbf{F}_{r,k}$, $\mathbf{G}_r$, $\mathbf{G}_{r,k}$ are continuous and there exists a constant $C_{r,k}$ such that*

$$\|\mathbf{F}_{r,k}(t,z) - \mathbf{F}_{r,k}(t,w)\|_{\mathcal{H}} + \|\mathbf{G}_{r,k}(t,z) - \mathbf{G}_{r,k}(t,w)\|_{\mathcal{L}_2(H_\mu,\mathcal{H})} \le C_{r,k}\|z - w\|_{\mathcal{H}_{r-t}}, \tag{4.3.3}$$

holds for every $t \in [0, T]$ and every $z, w \in \mathcal{H}$.

Proof of Lemma 4.3.4 Let us fix $t \in [0, T]$ and $z = (u, v), w = (\tilde{u}, \tilde{v}) \in \mathcal{H}$. First, note that due to the definitions of $\mathbf{F}_{r,k}$ and $\mathbf{G}_{r,k}$, it is sufficient to prove (4.3.3) in the case $\|z\|_{\mathcal{H}_{r-t}}, \|w\|_{\mathcal{H}_{r-t}} \le k$.

Let us set $I_{rt} := (t - r, r - t)$. Since in the chosen case $\mathbf{F}_{r,k}(t,z) = \mathbf{F}_r(t,z)$ and $\mathbf{F}_{r,k}(t,w) = \mathbf{F}_r(t,w)$, by Proposition 4.2.9 and Remark 4.2.11, there exists $C_E(r,t) > 0$ such that

$$\begin{aligned} \|\mathbf{F}_{r,k}(t,z) - \mathbf{F}_{r,k}(t,w)\|_{\mathcal{H}} \le C_E(r,t) \big[& \|\mathcal{A}_u(v,v) - \mathcal{A}_{\tilde{u}}(\tilde{v},\tilde{v})\|_{H^1(I_{rt})} \\ & + \|\mathcal{A}_u(u_x,u_x) - \mathcal{A}_{\tilde{u}}(\tilde{u}_x,\tilde{u}_x)\|_{H^1(I_{rt})} \big]. \end{aligned} \tag{4.3.4}$$

Since Υ is smooth and has compact support, see Lemma 4.2.5, from (4.2.3) observe that

$$\mathcal{A} : \mathbb{R}^n \ni q \mapsto \mathcal{A}_q \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n),$$

is smooth, compactly supported (in particular bounded) and globally Lipschitz. Recall the following well-known interpolation inequality, refer [16, (2.12)],

$$\|u\|^2_{L^\infty(I)} \le k_e^2 \|u\|_{L^2(I)} \|u\|_{H^1(I)}, \quad u \in H^1(I), \tag{4.3.5}$$

where I is any open interval in $\mathbb{R}$ and $k_e = 2\max\left\{1, \frac{1}{\sqrt{|I|}}\right\}$. Note that since $r > R + T$ and $t \in [0, T]$, $|I_{rt}| = 2(r - t) > 2R$ and we can choose $k_e = 2\max\left\{1, \frac{1}{\sqrt{|R|}}\right\}$. Then by using the above mentioned properties of $\mathcal{A}$ and the interpolation inequality (4.3.5) we get

$$\begin{aligned} \|\mathcal{A}_u(v,v) - \mathcal{A}_{\tilde{u}}(\tilde{v},\tilde{v})\|_{L^2(I_{rt})} \le & \|\mathcal{A}_u(v,v) - \mathcal{A}_{\tilde{u}}(v,v)\|_{L^2(I_{rt})} \\ & + \|\mathcal{A}_{\tilde{u}}(v,v) - \mathcal{A}_{\tilde{u}}(\tilde{v},v)\|_{L^2(I_{rt})} + \|\mathcal{A}_{\tilde{u}}(\tilde{v},v) - \mathcal{A}_{\tilde{u}}(\tilde{v},\tilde{v})\|_{L^2(I_{rt})} \\ \le & L_{\mathcal{A}} \|v\|^2_{L^\infty(I_{rt})} \|u - \tilde{u}\|_{L^2(I_{rt})} + B_{\mathcal{A}} \left[\|v\|_{L^\infty(I_{rt})} + \|\tilde{v}\|_{L^\infty(I_{rt})} \right] \|v - \tilde{v}\|_{L^2(I_{rt})} \end{aligned}$$

$$\leq C(L_{\mathcal{A}}, B_{\mathcal{A}}, R, k, k_e)\|z-w\|_{\mathcal{H}_{r-t}}, \tag{4.3.6}$$

where $L_{\mathcal{A}}$ and $B_{\mathcal{A}}$ are the Lipschitz constants and bound of $\mathcal{A}$, respectively. Next, since $\mathcal{A}$ is smooth and have compact support, if we set $L_{\mathcal{A}'}$ and $B_{\mathcal{A}'}$ are the Lipschitz constants and bound of

$$\mathcal{A}' : \mathbb{R}^n \ni q \mapsto d_q\mathcal{A} \in \mathcal{L}(\mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n; \mathbb{R}^n),$$

then by adding and subtracting the terms as we did to get (4.3.6) followed by the properties of $\mathcal{A}'$ and the interpolation inequality (4.3.5) we have

$$\begin{aligned}
&\|d_x[\mathcal{A}_u(v,v)-\mathcal{A}_{\tilde{u}}(\tilde{v},\tilde{v})]\|_{L^2(I_{rt})} \\
&\quad\leq \|d_u\mathcal{A}(v,v)(u_x)-d_{\tilde{u}}\mathcal{A}(\tilde{v},\tilde{v})(\tilde{u}_x)\|_{L^2(I_{rt})}+2\|\mathcal{A}_u(v_x,v)-\mathcal{A}_{\tilde{u}}(\tilde{v}_x,\tilde{v})\|_{L^2(I_{rt})} \\
&\quad\leq L_{\mathcal{A}'}\|u_x\|_{L^\infty(I_{rt})}\|v\|^2_{L^\infty(I_{rt})}\|u-\tilde{u}\|_{L^2(I_{rt})}+B_{\mathcal{A}'}\|v\|^2_{L^\infty(I_{rt})}\|u_x-\tilde{u}_x\|_{L^2(I_{rt})} \\
&\qquad+B_{\mathcal{A}'}\left[\|v\|_{L^\infty(I_{rt})}+\|\tilde{v}\|_{L^\infty(I_{rt})}\right]\|v-\tilde{v}\|_{L^2(I_{rt})}\|\tilde{u}_x\|_{L^\infty(I_{rt})} \\
&\qquad+2\left[L_{\mathcal{A}}\|u-\tilde{u}\|_{L^\infty(I_{rt})}\|v\|_{L^\infty(I_{rt})}\|v_x\|_{L^2(I_{rt})}+B_{\mathcal{A}}\|v_x-\tilde{v}_x\|_{L^2(I_{rt})}\|v\|_{L^\infty(I_{rt})}\right. \\
&\qquad\left.+B_{\mathcal{A}}\|v-\tilde{v}\|_{L^\infty(I_{rt})}\|\tilde{v}_x\|_{L^2(I_{rt})}\right] \\
&\quad\lesssim_{L_{\mathcal{A}},B_{\mathcal{A}},L_{\mathcal{A}'},B_{\mathcal{A}'},k_e}\Big[\|u-\tilde{u}\|_{H^2(I_{rt})}\|u\|_{H^2(I_{rt})}\|v\|^2_{H^1(I_{rt})}+\|u-\tilde{u}\|_{H^2(I_{rt})}\|v\|^2_{H^1(I_{rt})} \\
&\qquad+\|v-\tilde{v}\|_{H^1(I_{rt})}\left[\|v\|_{H^1(I_{rt})}+\|\tilde{v}\|_{H^1(I_{rt})}\right]\|\tilde{u}\|_{H^2(I_{rt})}+\|u-\tilde{u}\|_{H^2(I_{rt})}\|v\|^2_{H^1(I_{rt})} \\
&\qquad+\|v-\tilde{v}\|_{H^1(I_{rt})}\left(\|v\|_{H^1(I_{rt})}+\|\tilde{v}\|_{H^1(I_{rt})}\right)\Big] \\
&\quad\lesssim_k \|z-w\|_{\mathcal{H}_{r-t}},
\end{aligned} \tag{4.3.7}$$

where the last step is due to the case $\|z\|_{\mathcal{H}_{r-t}}, \|w\|_{\mathcal{H}_{r-t}} \leq k$. By following similar procedure of (4.3.6) and (4.3.7) we also get

$$\|\mathcal{A}_u(u_x,u_x)-\mathcal{A}_{\tilde{u}}(\tilde{u}_x,\tilde{u}_x)\|_{H^1(I_{rt})} \lesssim_{L_{\mathcal{A}},B_{\mathcal{A}},L_{\mathcal{A}'},B_{\mathcal{A}'},k_e,k} \|z-w\|_{\mathcal{H}_{r-t}}.$$

Hence by substituting the estimates back in (4.3.4) we are done with (4.3.3) for $F_{r,k}$-term.

Next, we move to the terms of $G_{r,k}$. As for $F_{r,k}$, it is sufficient to perform the calculations for the case $\|z\|_{\mathcal{H}_{r-t}}, \|w\|_{\mathcal{H}_{r-t}} \leq k$. By invoking Lemma 4.2.2 followed by Remark 4.2.11 we have

$$\begin{aligned}
\|\mathbf{G}_{r,k}(t,z)-\mathbf{G}_{r,k}(t,w)\|^2_{\mathscr{L}_2(H_\mu,\mathcal{H})} &\leq \|(E^1_{r-t}Y_u(v,u_x))\cdot-(E^1_{r-t}Y_{\tilde{u}}(\tilde{v},\tilde{u}_x))\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(\mathbb{R}))} \\
&\leq c_{r,t}\, C_E(r,t)\, \|Y_u(v,u_x)-Y_{\tilde{u}}(\tilde{v},\tilde{u}_x)\|^2_{H^1(I_{rt})}.
\end{aligned}$$

Recall that the 1-D Sobolev embedding gives $H^1(\mathbb{R}) \hookrightarrow L^\infty(\mathbb{R})$. Consequently, by the Taylor formula [43, Theorem 5.6.1] and inequalities (4.2.9)-(4.2.10) we have

$$\begin{aligned}
\|Y_u(v,\partial_x u)-Y_{\tilde{u}}(\tilde{v},\tilde{u}_x)\|^2_{L^2(I_{rt})} &\leq \int_{I_{rt}} |Y_{u(x)}(v(x),u_x(x))-Y_{\tilde{u}(x)}(v(x),u_x(x))|^2\,dx \\
&\quad+\int_{I_{rt}} |Y_{\tilde{u}(x)}(v(x),u_x(x))-Y_{\tilde{u}(x)}(v(x),\tilde{u}_x(x))|^2\,dx \\
&\quad+\int_{I_{rt}} |Y_{\tilde{u}(x)}(v(x),\tilde{u}_x(x))-Y_{\tilde{u}(x)}(\tilde{v}(x),\tilde{u}_x(x))|^2\,dx
\end{aligned}$$

$$\leq C_Y^2 \left[1 + \|v\|^2_{H^1(I_{rt})} + \|u\|^2_{H^1(I_{rt})}\right] \|u - \tilde{u}\|^2_{H^2(I_{rt})}$$
$$+ C_{Y_2}^2 \left[\|u_x - \tilde{u}_x\|^2_{H^1(I_{rt})} + \|v - \tilde{v}\|^2_{H^1(I_{rt})}\right]$$
$$\lesssim_{k,C_Y,C_{Y_2}} \|z - w\|^2_{\mathcal{H}_{r-t}}. \tag{4.3.8}$$

For homogeneous part of norm, that is L^2-norm of the derivative, we have

$$\|d_x\left[Y_u(v,u_x) - Y_{\tilde{u}}(\tilde{v},\tilde{u}_x)\right]\|^2_{L^2(I_{rt})}$$
$$\lesssim \int_{I_{rt}} \sum_{i=1}^n \left\{ \Big| \frac{\partial Y}{\partial p_i}(u(x), v(x), u_x(x)) \frac{du^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{u}^i}{dx}(x) \Big|^2 \right.$$
$$+ \Big| \frac{\partial Y}{\partial a_i}(u(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{v}^i}{dx}(x) \Big|^2$$
$$\left. + \Big| \frac{\partial Y}{\partial b_i}(u(x), v(x), u_x(x)) \frac{du_x^i}{dx}(x) - \frac{\partial Y}{\partial b_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\partial_x\tilde{u}^i}{dx}(x) \Big|^2 \right\} dx$$
$$=: Y_1 + Y_2 + Y_3. \tag{4.3.9}$$

We will estimate each term separately by using the 1-D Sobolev embedding, the Taylor formula and inequalities (4.2.9)-(4.2.11) as follows:

$$Y_1 \lesssim \int_{I_{rt}} \sum_{i=1}^n \Big| \frac{\partial Y}{\partial p_i}(u(x), v(x), u_x(x)) \frac{du^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{u}^i}{dx}(x) \Big|^2 dx$$
$$\lesssim \int_{I_{rt}} \sum_{i=1}^n \left\{ \Big| \frac{\partial Y}{\partial p_i}(u(x), v(x), u_x(x)) \frac{du^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{du^i}{dx}(x) \Big|^2 \right.$$
$$+ \Big| \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{du^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) \Big|^2$$
$$+ \Big| \frac{\partial Y}{\partial p_i}(\tilde{u}(x), v(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) \Big|^2$$
$$\left. + \Big| \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), u_x(x)) \frac{d\tilde{u}^i}{dx}(x) - \frac{\partial Y}{\partial p_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{u}^i}{dx}(x) \Big|^2 \right\} dx$$
$$\lesssim C_{Y_3}^2 \|u - \tilde{u}\|^2_{L^2(I_{rt})} \|u_x\|^2_{H^1(I_{rt})} + C_{Y_1}^2 \left[1 + \|v\|^2_{H^1(I_{rt})} + \|u_x\|^2_{H^1(I_{rt})}\right] \|u_x - \tilde{u}_x\|^2_{L^2(I_{rt})}$$
$$+ C_{Y_3}^2 \|v - \tilde{v}\|^2_{L^2(I_{rt})} \|\tilde{u}_x\|^2_{H^1(I_{rt})} + C_{Y_3}^2 \|u_x - \tilde{u}_x\|^2_{L^2(I_{rt})} \|\tilde{u}_x\|^2_{H^1(I_{rt})}$$
$$\lesssim_{k,C_{Y_2},C_{Y_3},C_{Y_1}} \|z - w\|^2_{\mathcal{H}_{r-t}}. \tag{4.3.10}$$

Terms Y_2 and Y_3 are quite similar so it is enough to estimate only one. For Y_2 we have the following calculation

$$Y_2 = \int_{I_{rt}} \sum_{i=1}^n \Big| \frac{\partial Y}{\partial a_i}(u(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), \tilde{v}(x), \tilde{u}_x(x)) \frac{d\tilde{v}^i}{dx}(x) \Big|^2 dx$$
$$\lesssim \int_{I_{rt}} \sum_{i=1}^n \left\{ \Big| \frac{\partial Y}{\partial a_i}(u(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) \Big|^2 dx \right.$$
$$+ \Big| \frac{\partial Y}{\partial a_i}(\tilde{u}(x), v(x), u_x(x)) \frac{dv^i}{dx}(x) - \frac{\partial Y}{\partial a_i}(\tilde{u}(x), \tilde{v}(x), u_x(x)) \frac{dv^i}{dx}(x) \Big|^2 dx$$

$$+\Big|\frac{\partial Y}{\partial a_i}(\tilde{u}(x),\tilde{v}(x),u_x(x))\frac{dv^i}{dx}(x)-\frac{\partial Y}{\partial a_i}(\tilde{u}(x),\tilde{v}(x),\tilde{u}_x(x))\frac{dv^i}{dx}(x)\Big|^2\,dx$$
$$+\Big|\frac{\partial Y}{\partial a_i}(\tilde{u}(x),\tilde{v}(x),\tilde{u}_x(x))\frac{dv^i}{dx}(x)-\frac{\partial Y}{\partial a_i}(\tilde{u}(x),\tilde{v}(x),\tilde{u}_x(x))\frac{d\tilde{v}^i}{dx}(x)\Big|^2\,dx\Big\}$$
$$\lesssim C_{Y_3}^2\|u-\tilde{u}\|_{H^1(I_{rt})}^2\|v_x\|_{L^2(I_{rt})}^2+C_{Y_3}^2\|v-\tilde{v}\|_{H^1(I_{rt})}^2\|v_x\|_{L^2(I_{rt})}^2$$
$$+C_{Y_3}^2\|u_x-\tilde{u}_x\|_{H^1(I_{rt})}^2\|v_x\|_{L^2(I_{rt})}^2+C_{Y_3}^2C_{r,t}\|v_x-\tilde{v}_x\|_{L^2(I_{rt})}^2$$
$$\lesssim_{k,C_{r,t}C_{Y_3}}\|z-w\|_{\mathcal{H}_{r-t}}^2. \tag{4.3.11}$$

Hence by substituting (4.3.10)-(4.3.11) into (4.3.9) we get

$$\|d_x\left[Y_u(v,u_x)-Y_{\tilde{u}}(\tilde{v},\tilde{u}_x)\right]\|_{L^2(I_{rt})}^2\lesssim_{k,C_{r,t},C_{Y_2},C_{Y_3},C_{Y_1}}\|z-w\|_{\mathcal{H}_{r-t}}^2,$$

which together with (4.3.8) gives $G_{r,k}$ part of (4.3.3). Hence the Lipschitz property Lemma 4.3.4. □

The following result follows directly from Lemma 4.3.4 and the standard theory of PDE via semigroup approach, refer [5] and [99] for detailed proof.

Corollary 4.3.5. *Given any $\xi\in\mathcal{H}$ and $h\in{}_0H^{1,2}(0,T;H_\mu)$, there exists a unique z in $\mathcal{C}([0,T];\mathcal{H})$ such that for all $t\in[0,T]$*

$$z(t)=S_t\xi+\int_0^t S_{t-s}\mathbf{F}_{r,k}(s,z(s))\,ds+\int_0^t S_{t-s}(\mathbf{G}_{r,k}(s,z(s))\dot{h}(s))\,ds.$$

Remark 4.3.6. Here by $\mathbf{G}_{r,k}(s,z(s))\dot{h}(s)$ we understand that both components of $\mathbf{G}_{r,k}(s,z(s))$ are acting on $\dot{h}(s)$.

From now on, for each $r>R+T$ and $k\in\mathbb{N}$, the solution from Corollary 4.3.5 will be denoted by $z_{r,k}$ and called the *approximate solution*. To proceed further we define the following two auxiliary functions

$$\widetilde{F}_{r,k}\;:\;[0,T]\times\mathcal{H}\ni(t,z)\mapsto\begin{pmatrix}0\\ \varphi\cdot\Upsilon'(u)\mathbf{F}_{r,k}^2(t,z)+\varphi B_u(v,v)-\varphi B_u(u_x,u_x)\end{pmatrix}-\begin{pmatrix}0\\ \Delta\varphi\cdot h(u)+2\varphi_x\cdot h'(u)u_x\end{pmatrix}\in\mathcal{H},$$

and

$$\widetilde{G}_{r,k}\;:\;[0,T]\times\mathcal{H}\ni(t,z)\mapsto\begin{pmatrix}0\\ \varphi\cdot\Upsilon'(u)\mathbf{G}_{r,k}^2(t,z)\end{pmatrix}\in\mathcal{H}.$$

Here $\mathbf{F}_{r,k}^2(s,z_{r,k}(s))$ and $\mathbf{G}_{r,k}^2(s,z_{r,k}(s))$ denote the second components of the vectors $\mathbf{F}_{r,k}(s,z_{r,k}(s))$ and $\mathbf{G}_{r,k}(s,z_{r,k}(s))$, respectively. The following corollary relates the solution $z_{r,k}$ with its transformation under the map $\mathbf{Q}_r$ and allow to understand the need of the functions $\widetilde{F}_{r,k}$ and $\widetilde{G}_{r,k}$.

Corollary 4.3.7. *Let us assume that $\xi:=(E_r^2u_0,E_r^1v_0)$ and that $z_{r,k}\in\mathcal{C}([0,T];\mathcal{H})$ satisfies*

$$z_{r,k}(t)=S_t\xi+\int_0^t S_{t-s}\mathbf{F}_{r,k}(s,z_{r,k}(s))\,ds+\int_0^t S_{t-s}(\mathbf{G}_{r,k}(s,z_{r,k}(s))\dot{h}(s))\,ds,\qquad t\in[0,T]. \tag{4.3.12}$$

Then $\widetilde{z}_{r,k} = \mathbf{Q}_r(z_{r,k})$ *satisfies, for each* $t \in [0,T]$,

$$\widetilde{z}_{r,k}(t) = S_t \mathbf{Q}_r(\xi) + \int_0^t S_{t-s} \widetilde{F}_{r,k}(s, z_{r,k}(s))\, ds + \int_0^t S_{t-s}(\widetilde{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s))\, ds.$$

Proof of Corollary 4.3.7 First observe that by the action of $\mathbf{Q}_r'$ and $\mathcal{G}$ on the elements of $\mathcal{H}$ from Lemma 4.3.3 and (4.2.8), respectively, we get

$$\mathbf{Q}_r'(z_{r,k}(s))\left(\mathbf{F}_{r,k}(s, z_{r,k}(s)) + \mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s)\right)$$

$$= \begin{pmatrix} 0 \\ \varphi \cdot \left\{[\Upsilon'(u_{r,k}(s))](\mathbf{F}^2_{r,k}(s, z_{r,k}(s))) + [\Upsilon'(u_{r,k}(s))](\mathbf{G}^2_{r,k}(s, z_{r,k}(s))\dot{h}(s)\right\} \end{pmatrix}. \tag{4.3.13}$$

Moreover, since by applying Lemma 4.3.3 and (4.2.8) to $z = (u, v) \in \mathcal{H}$ we have

$$F(z) := \mathbf{Q}_r'\mathcal{G}z - \mathcal{G}\mathbf{Q}_r z = \begin{pmatrix} \varphi \cdot [\Upsilon'(u)](v) \\ \varphi \cdot \left\{[\Upsilon''(u)](v, v) + [\Upsilon'(u)](u'')\right\} \end{pmatrix}$$

$$- \begin{pmatrix} \varphi \cdot [\Upsilon'(u)](v) \\ \varphi'' \cdot \Upsilon(u) + 2\varphi' \cdot [\Upsilon'(u)](u') + \varphi \cdot [\Upsilon'(u)](u'') + \varphi \cdot [\Upsilon''(u)](u', u') \end{pmatrix}, \tag{4.3.14}$$

substitution of $z = z_{r,k}(s) = (u_{r,k}(s), v_{r,k}(s)) \in \mathcal{H}$ in (4.3.14) with (4.3.13) followed by (4.2.2) gives, for $s \in [0,T]$,

$$\mathbf{Q}_r'(z_{r,k}(s))\left(\mathbf{F}_{r,k}(s, z_{r,k}(s)) + \mathbf{G}_{r,k}(s, z_{r,k}(s))\right) + F(z_{r,k}(s))$$

$$= \begin{pmatrix} 0 \\ \varphi \cdot [\Upsilon'(u_{r,k}(s))](\mathbf{F}^2_{r,k}(s, z_{r,k}(s))) + \varphi \cdot [\Upsilon''(u_{r,k}(s))](v_{r,k}(s), v_{r,k}(s)) \\ -\varphi \cdot [\Upsilon''(u_{r,k}(s))](\partial_x u_{r,k}(s), \partial_x u_{r,k}(s)) \end{pmatrix}$$

$$- \begin{pmatrix} 0 \\ -\varphi'' \cdot \Upsilon(u_{r,k}(s)) + 2\varphi' \cdot [\Upsilon'(u_{r,k}(s))](\partial_x u_{r,k}(s)) + \varphi \cdot [\Upsilon'(u_{r,k}(s))](\mathbf{G}^2_{r,k}(s, z_{r,k}(s))) \end{pmatrix}$$

$$= \widetilde{F}_{r,k}(s, z_{r,k}(s)) + \widetilde{G}_{r,k}(s, z_{r,k}(s)).$$

Hence, if we have

$$\int_0^T \left[\|\mathbf{F}_{r,k}(s, z_{r,k}(s))\|_{\mathcal{H}} + \|\mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s)\|_{\mathcal{H}}\right] ds < \infty, \tag{4.3.15}$$

then by invoking [23, Lemma 6.3] with

$$L = \mathbf{Q}_r,\ K = U = \mathcal{H},\ A = B = \mathcal{G},\ g(s) = 0,\ f(s) = \mathbf{F}_{r,k}(s, z_{r,k}(s)) + \mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s),$$

we are done with the proof here. But (4.3.15) follows by Lemma 4.3.4, because $h \in {}_0H^{1,2}(0,T; H_\mu)$ and the following holds, due to the Hölder inequality with the abuse of notation as mentioned in Remark 4.3.6,

$$\int_0^T \|\mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s)\|_{\mathcal{H}}\, ds = \int_0^T \|\mathbf{G}^2_{r,k}(s, z_{r,k}(s))\dot{h}(s)\|_{H^1(\mathbb{R})}\, ds$$

$$\leq \left(\int_0^T \|(\mathbf{G}^2_{r,k}(s, z_{r,k}(s))) \cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(\mathbb{R}))}\, ds\right)^{\frac{1}{2}} \left(\int_0^T \|\dot{h}(s)\|^2_{H_\mu}\, ds\right)^{\frac{1}{2}}.$$

□

Next we prove that the approximate solution $z_{r,k}$ stays on the manifold. Define the following three positive reals: for each $r > R+T$ and $k \in \mathbb{N}$,

$$\begin{cases} \tau_k^1 := \inf\{t \in [0,T] : \|z_{r,k}(t)\|_{\mathcal{H}_{r-t}} \geq k\}, \\ \tau_k^2 := \inf\{t \in [0,T] : \|\tilde{z}_{r,k}(t)\|_{\mathcal{H}_{r-t}} \geq k\}, \\ \tau_k^3 := \inf\{t \in [0,T] : \exists x, |x| \leq r-t, \, u_{r,k}(t,x) \notin O\}, \\ \tau_k := \tau_k^1 \wedge \tau_k^2 \wedge \tau_k^3. \end{cases} \tag{4.3.16}$$

Also, define the following $\mathcal{H}$-valued functions of time $t \in [0,T]$

$$a_k(t) = S_t\xi + \int_0^t S_{t-s}\mathbb{1}_{[0,\tau_k)}(s)\mathbf{F}_{r,k}(s,z_{r,k}(s))\,ds + \int_0^t S_{t-s}(\mathbb{1}_{[0,\tau_k)}(s)\mathbf{G}_{r,k}(s,z_{r,k}(s))\dot{h}(s))\,ds,$$

$$\tilde{a}_k(t) = S_t\mathbf{Q}_r(\xi) + \int_0^t S_{t-s}\mathbb{1}_{[0,\tau_k)}(s)\tilde{F}_{r,k}(s,z_{r,k}(s))\,ds + \int_0^t S_{t-s}(\mathbb{1}_{[0,\tau_k)}(s)\tilde{G}_{r,k}(s,z_{r,k}(s))\dot{h}(s))\,ds.$$

Proposition 4.3.8. *For each $k \in \mathbb{N}$ and $\xi := (E_r^2 u_0, E_r^1 v_0)$, the functions a_k, $\tilde{a}_k$, $z_{r,k}$ and $\tilde{z}_{r,k}$ coincide on $[0,\tau_k)$. In particular, $u_{r,k}(t,x) \in M$ for $|x| \leq r-t$ and $t \leq \tau_k$. Consequently, $\tau_k = \tau_k^1 = \tau_k^2 \leq \tau_k^3$.*

Proof of Proposition 4.3.8 Let us fix k. First note that, due to indicator function,

$$a_k = z_{r,k} \qquad \text{and} \qquad \tilde{a}_k = \tilde{z}_{r,k} \quad \text{on } [0,\tau_k). \tag{4.3.17}$$

Next, since $E_{r-s}^1 f = f$ on $|x| \leq r-s$, see Proposition 4.2.9, and $\varphi = 1$ on $(-r,r)$, by Lemma 4.3.2 followed by (4.2.4) we infer that

$$\begin{cases} \mathbb{1}_{[0,\tau_k)}(s)[\tilde{F}_{r,k}(s,z_{r,k}(s))](x) = \mathbb{1}_{[0,\tau_k)}(s)[\mathbf{F}_{r,k}(s,\tilde{z}_{r,k}(s))](x), \\ \mathbb{1}_{[0,\tau_k)}(s)[\tilde{G}_{r,k}(s,z_{r,k}(s))e](x) = \mathbb{1}_{[0,\tau_k)}(s)[\mathbf{G}_{r,k}(s,\tilde{z}_{r,k}(s))e](x), \quad e \in K, \end{cases} \tag{4.3.18}$$

holds for every $|x| \leq r-s$, $0 \leq s \leq T$. Now we claim that if we denote

$$p(t) := \frac{1}{2}\|a_k(t) - \tilde{a}_k(t)\|_{\mathcal{H}_{r-t}}^2,$$

then the map $s \mapsto p(s \wedge \tau_k)$ is continuous and uniformly bounded. Indeed, since, by Proposition 4.2.9, $\xi(x) = (u_0(x), v_0(x)) \in TM$ for $|x| \leq r$, the uniform boundedness is an easy consequence of bound property of C_0-group, Lemmata 4.3.2 and 4.3.4. Continuity of $s \mapsto p(s \wedge \tau_k)$ follows from the following:

1. for every $z \in \mathcal{H}$, the map $t \mapsto \|z\|_{\mathcal{H}_{r-t}}^2$ is continuous;
2. for each t, the map $L^2(\mathbb{R}) \ni u \mapsto \int_0^t |u(s)|^2\,ds \in \mathbb{R}$ is locally Lipschitz.

Now observe that by applying Proposition 4.5.2 for

$$k = 1,\ L = I,\ T = r,\ x = 0 \quad \text{and} \quad z(t) = (u(t), v(t)) := a_k(t) - \tilde{a}_k(t),$$

we get $\mathbf{e}(t, z(t)) = p(t)$, and the following

$$\mathbf{e}(t,z(t)) \leq \mathbf{e}(0,z_0) + \int_0^t V(r,z(r))\,dr. \tag{4.3.19}$$

Here

$$\begin{aligned}V(t,z(t)) &:= \langle u(t), v(t)\rangle_{L^2(B_{r-t})} + \langle v(t), f(t)\rangle_{L^2(B_{r-t})} + \langle \partial_x v(t), \partial_x f(t)\rangle_{L^2(B_{r-t})}\\ &\quad + \langle v(t), g(t)\rangle_{L^2(B_{r-t})} + \langle \partial_x v(t), \partial_x g(t)\rangle_{L^2(B_{r-t})},\end{aligned}$$

and

$$\begin{aligned}\begin{pmatrix} 0 \\ f(t)\end{pmatrix} &:= \mathbb{1}_{[0,\tau_k)}(t)[\mathbf{F}_{r,k}(s, z_{r,k}(t)) - \widetilde{F}_{r,k}(s, z_{r,k}(t))],\\ \begin{pmatrix} 0 \\ g(t)\end{pmatrix} &:= \mathbb{1}_{[0,\tau_k)}(t)[\mathbf{G}_{r,k}(s, z_{r,k}(t))\dot{h}(t) - \widetilde{G}_{r,k}(s, z_{r,k}(t))\dot{h}(t)].\end{aligned}$$

Since due to operators E_r^2 and E_r^1 the initial data ξ satisfies the assumption of Lemma 4.3.2,

$$S_t\mathbf{Q}_r(\xi) = S_t\xi,$$

and so $\mathbf{e}(0, z(0)) = p(0) = 0$. Next observe that by the Cauchy-Schwarz inequality we have

$$\begin{aligned}V(t,z(t)) &\leq \frac{1}{2}\|u(t)\|^2_{L^2(B_{r-t})} + \frac{3}{2}\|v(t)\|^2_{L^2(B_{r-t})} + \frac{1}{2}\|f(t)\|^2_{L^2(B_{r-t})}\\ &\quad + \|\partial_x v(t)\|^2_{L^2(B_{r-t})} + \frac{1}{2}\|\partial_x f(t)\|^2_{L^2(B_{r-t})} + \frac{1}{2}\|g(t)\|^2_{L^2(B_{r-t})} + \frac{1}{2}\|\partial_x g(t)\|^2_{L^2(B_{r-t})}\\ &\leq 3p(t) + \frac{1}{2}\|f(t)\|^2_{H^1(B_{r-t})} + \frac{1}{2}\|g(t)\|^2_{H^1(B_{r-t})}.\end{aligned}$$

By using above into (4.3.19) and, then, by invoking equalities (4.3.18) and (4.3.17), definition (4.3.16), Lemma 4.2.2 and Lemma 4.3.4 we have the following calculation, for every $t \in [0,T]$,

$$\begin{aligned}p(t) &\leq \int_0^t 3p(s)\,ds + \frac{1}{2}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|\mathbf{F}^2_{r,k}(s, z_{r,k}(s)) - \mathbf{F}^2_{r,k}(s, \tilde{z}_{r,k}(s))\|^2_{H^1(B_{r-s})}\,ds\\ &\quad + \frac{1}{2}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|\mathbf{G}^2_{r,k}(s, z_{r,k}(s)) - \mathbf{G}^2_{r,k}(s, \tilde{z}_{r,k}(s))\|^2_{\mathscr{L}_2(H_\mu, H^1(B_{r-s}))}\|\dot{h}(s)\|^2_{H_\mu}\,ds\\ &\leq 3\int_0^t p(s)\,ds + \frac{1}{2}C^2_{r,k}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|z_{r,k}(s) - \tilde{z}_{r,k}(s)\|^2_{\mathcal{H}_{r-s}}\,ds\\ &\quad + \frac{1}{2}C^2_{r,k}\int_0^t \mathbb{1}_{[0,\tau_k)}(s)\|z_{r,k}(s) - \tilde{z}_{r,k}(s)\|^2_{\mathcal{H}_{r-s}}\|\dot{h}(s)\|^2_{H_\mu}\,ds\\ &\leq (3 + C^2_{r,k})\int_0^t p(s)(1 + \|\dot{h}(s)\|^2_{H_\mu})\,ds. \qquad (4.3.20)\end{aligned}$$

Consequently by the Gronwall Lemma, for $t \in [0,\tau_k]$,

$$p(t) \lesssim_{C_{r,k}} p(0)\exp\left[\int_0^t (1 + \|\dot{h}(s)\|^2_{H_\mu})\,ds\right]. \qquad (4.3.21)$$

Note that the right hand side in (4.3.21) is finite because $h \in {}_0H^{1,2}(0,T;H_\mu)$. Since we know that $p(0) = 0$ we arrive to $p(t) = 0$ on $t \in [0,\tau_k]$. This further implies that $a_k(t,x) = \tilde{a}_k(t,x)$ hold for $|x| \leq r - t$ and $t \leq \tau_k$. Consequently, $z_{r,k}(t,x) = \tilde{z}_{r,k}(t,x)$ hold for $|x| \leq r - t$ and $t \leq \tau_k$. So, because $\tilde{z}_{r,k}(t,x) = \mathbf{Q}_r(z_{r,k}(t))$ and $\varphi = 1$ on $(-r,r)$,

$$u_{r,k}(t,x) = \Upsilon(u_{r,k}(t,x)), \qquad \text{for } |x| \leq r - t, \quad t \leq \tau_k. \qquad (4.3.22)$$

Since, by definition (4.3.16) of τ_k, $u_{r,k}(t,x) \in O$, equality (4.3.22) and Lemma 4.2.5, gives $u_{r,k}(t,x) \in M$ for $|x| \leq r-t$ and $t \leq \tau_k$. This suggests that $\tau_k \leq \tau_k^3$ and hence $\tau_k = \tau_k^1 \wedge \tau_k^2$. It remains to show that $\tau_k^1 = \tau_k^2$. But suppose it does not hold and without loss of generality we assume that $\tau_k^1 > \tau_k^2$. Then by definition (4.3.16) and the continuity of $z_{r,k}$ and $\tilde{z}_{r,k}$ in time we have

$$\|z_{r,k}(\tau_k^2,\cdot)\|_{\mathcal{H}_{r-\tau_k^2}} < k \quad \text{but} \quad \|\tilde{z}_{r,k}(\tau_k^2,\cdot)\|_{\mathcal{H}_{r-\tau_k^2}} \geq k,$$

which contradicts the above mentioned consequence of $p=0$ on $[0,\tau_k]$. Hence we conclude that $\tau_k^1 = \tau_k^2$ and this finishes the proof of Proposition 4.3.8. $\square$

Next in the ongoing proof of Theorem 4.3.1 we show that the approximate solutions extend each other. Recall that $r > R+T$ is fixed for given $T>0$.

Lemma 4.3.9. *Let $k \in \mathbb{N}$ and $\xi = (E_r^2 u_0, E_r^1 v_0)$. Then $z_{r,k+1}(t,x) = z_{r,k}(t,x)$ on $|x| \leq r-t$, $t \leq \tau_k$, and $\tau_k \leq \tau_{k+1}$.*

Proof of Lemma 4.3.9 Define

$$p(t) := \frac{1}{2}\|a_{k+1}(t) - a_k(t)\|^2_{H^1(B_{r-t})\times L^2(B_{r-t})}.$$

As an application of Proposition 4.5.2, by performing the computation based on (4.3.19) - (4.3.20), with $k=0$ and rest variables the same, we obtain

$$\begin{aligned} p(t) \leq{}& 2\int_0^t p(s)\,ds + \frac{1}{2}\int_0^t \|\mathbb{1}_{[0,\tau_{k+1})}(s)\mathbf{F}_r^2(s,z_{r,k+1}(s)) - \mathbb{1}_{[0,\tau_k)}(s)\mathbf{F}_r^2(s,z_{r,k}(s))\|^2_{L^2(B_{r-s})}\,ds \\ &+ \frac{1}{2}\int_0^t \|\mathbb{1}_{[0,\tau_{k+1})}(s)\mathbf{G}_r^2(s,z_{r,k+1}(s))\dot{h}(s) - \mathbb{1}_{[0,\tau_k)}(s)\mathbf{G}_r^2(s,z_{r,k}(s))\dot{h}(s)\|^2_{L^2(B_{r-s})}\,ds. \end{aligned} \tag{4.3.23}$$

Then, since F_r and G_r depends on $u_{r,k}(s)$, $u_{r,k+1}(s)$ and their first partial derivatives, with respect to time t and space x, which are actually bounded on the interval $(-(r-s), r-s)$ by some constant C_r for every $s < \tau_{k+1} \wedge \tau_k$, by evaluating (4.3.23) on $t \wedge \tau_{k+1} \wedge \tau_k$ following the use of Lemmata 4.3.4 and 4.2.2 we get

$$\begin{aligned} p(t\wedge\tau_{k+1}\wedge\tau_k) &\leq 2\int_0^t p(s\wedge\tau_{k+1}\wedge\tau_k)\,ds \\ &\quad + \frac{1}{2}\int_0^{t\wedge\tau_{k+1}\wedge\tau_k} \|\mathbf{F}_r^2(s,z_{r,k+1}(s)) - \mathbf{F}_r^2(s,z_{r,k}(s))\|^2_{L^2(B_{r-s})}\,ds \\ &\quad + \frac{1}{2}\int_0^{t\wedge\tau_{k+1}\wedge\tau_k} \|\mathbf{G}_r^2(s,z_{r,k+1}(s))\zeta(s) - \mathbf{G}_r^2(s,z_{r,k}(s))\dot{h}(s)\|^2_{L^2(B_{r-s})}\,ds \\ &\lesssim_k \int_0^t p(s\wedge\tau_{k+1}\wedge\tau_k)(1+\|\dot{h}(s)\|^2_{H_\mu})\,ds. \end{aligned}$$

Hence by the Gronwall Lemma we infer that $p=0$ on $[0,\tau_{k+1}\wedge\tau_k]$.

Consequently, we claim that $\tau_k \leq \tau_{k+1}$. We divide the proof of our claim in the following three exhaustive subcases. Due to (4.3.16), the subcases when $\|\xi\|_{\mathcal{H}_r} > k+1$ and $k < \|\xi\|_{\mathcal{H}_r} \leq k+1$ are trivial. In the last subcase when $\|\xi\|_{\mathcal{H}_r} \leq k$ we prove the claim $\tau_k \leq \tau_{k+1}$ by the method of contradiction, and so assume that $\tau_k > \tau_{k+1}$ is true. Then, because of continuity in time of $z_{r,k}$ and $z_{r,k+1}$, by (4.3.16) we have

$$\|z_{r,k}(\tau_{k+1})\|_{\mathcal{H}_{r-\tau_{k+1}}} < k \quad \text{and} \quad \|z_{r,k+1}(\tau_{k+1})\|_{\mathcal{H}_{r-\tau_{k+1}}} \geq k. \tag{4.3.24}$$

However, since $p(t) = 0$ for $t \in [0, \tau_{k+1} \wedge \tau_k]$ and $(u_0(x), v_0(x)) \in TM$ for $|x| < r$, by argument based on the one made after (4.3.21), in the Proposition 4.3.8, we get $z_{r,k}(t,x) = z_{r,k+1}(t,x)$ for every $t \in [0, \tau_{k+1}]$ and $|x| \leq r - t$. But this contradicts (4.3.24) and we finish the proof of our claim and, in result, the proof of Lemma 4.3.9. □

Since by definition (4.3.16) and Lemma 4.3.9 the sequence of stopping times $\{\tau_k\}_{k\geq 1}$ is bounded and non-decreasing, it makes sense to denote by τ the limit of $\{\tau_k\}_{k\geq 1}$. Now by using [23, Lemma 10.1] we prove that the approximate solutions do not explode which is same as the following in terms of τ.

Proposition 4.3.10. *For* τ_k *defined in* (4.3.16), $\tau := \lim_{k\to\infty} \tau_k = T$.

Proof of Proposition 4.3.10 We first notice that by a particular case of the Chojnowska-Michalik Theorem [49], when the diffusion coefficient is absent, we have that for each k the approximate solution $z_{r,k}$, as a function of time t, is $H^1(\mathbb{R};\mathbb{R}^n) \times L^2(\mathbb{R};\mathbb{R}^n)$-valued and satisfies

$$z_{r,k}(t) = \xi + \int_0^t \mathcal{G} z_{r,k}(s)\, ds + \int_0^t \mathbf{F}_{r,k}(s, z_{r,k}(s))\, ds + \int_0^t \mathbf{G}_{r,k}(s, z_{r,k}(s))\dot{h}(s)\, ds, \tag{4.3.25}$$

for $t \leq T$. In particular,

$$u_{r,k}(t) = \xi_1 + \int_0^t v_{r,k}(s)\, ds,$$

for $t \leq T$, where $\xi_1 = E_r^2 u_0$ and the integral converges in $H^1(\mathbb{R};\mathbb{R}^n)$. Hence

$$\partial_t u_{r,k}(s,x) = v_{r,k}(s,x), \qquad \text{for all} \quad s \in [0,T], \ x \in \mathbb{R}.$$

Next, by keeping in mind the Proposition 4.3.8, we set

$$l(t) := \|a_k(t)\|^2_{H^1(B_{r-t})\times L^2(B_{r-t})} \quad \text{and} \quad q(t) := \log\left(1 + \|a_k(t)\|^2_{\mathcal{H}_{r-t}}\right).$$

By applying Proposition 4.5.2, respectively, with $k = 0, 1$ and $L(x) = x, \log(1+x)$, followed by the use of Lemma 4.3.4 we get

$$\begin{aligned} l(t) \leq l(0) &+ \int_0^t l(s)\, ds + \int_0^t \mathbb{1}_{[0,\tau_k]}(s)\langle v_{r,k}(s), \varphi(s)\rangle_{L^2(B_{r-s})}\, ds \\ &+ \int_0^t \mathbb{1}_{[0,\tau_k]}(s)\langle v_{r,k}(s), \psi(s)\rangle_{L^2(B_{r-s})}\, ds, \end{aligned} \tag{4.3.26}$$

and

$$q(t) \leq q(0) + \int_0^t \frac{\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}{1 + \|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\, ds \tag{4.3.27}$$

$$
\begin{aligned}
&+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle v_{r,k}(s),\varphi(s)\rangle_{L^2(B_{r-s})}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle \partial_x v_{r,k}(s),\partial_x[\varphi(s)]\rangle_{L^2(B_{r-s})}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds\\
&+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle v_{r,k}(s),\psi(s)\rangle_{L^2(B_{r-s})}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds+\int_0^t \mathbb{1}_{[0,\tau_k]}(s)\frac{\langle \partial_x v_{r,k}(s),\partial_x[\psi(s)]\rangle_{L^2(B_{r-s})}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\,ds.
\end{aligned}
$$

Here

$$
\begin{aligned}
\varphi(s) &:= \mathcal{A}_{u_{r,k}(s)}(v_{r,k}(s),v_{r,k}(s))-\mathcal{A}_{u_{r,k}(s)}(\partial_x u_{r,k}(s),\partial_x u_{r,k}(s)),\\
\psi(s) &:= Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s).
\end{aligned}
$$

Since by Proposition 4.3.8 $u_{r,k}(s,x)\in M$ for $|x|\le r-s$ and $s\le\tau_k$, we have

$$
u_{r,k}(s,x)\in M \qquad \text{and} \qquad \partial_t u_{r,k}(s,x)=v_{r,k}(s,x)\in T_{u_{r,k}(s,x)}M,
$$

on the mentioned domains of s and x. Consequently, by Proposition 4.2.6, we get

$$
\begin{aligned}
&\mathcal{A}_{u_{r,k}(s,x)}(v_{r,k}(s,x),v_{r,k}(s,x)) = A_{u_{r,k}(s,x)}(v_{r,k}(s,x),v_{r,k}(s,x)),\\
&\mathcal{A}_{u_{r,k}(s,x)}(\partial_x u_{r,k}(s,x),\partial_x u_{r,k}(s,x)) = A_{u_{r,k}(s,x)}(\partial_x u_{r,k}(s,x),\partial_x u_{r,k}(s,x)),
\end{aligned}
\tag{4.3.28}
$$

on $|x|\le r-s$ and $s\le\tau_k$. Hence, since $v_{r,k}(s,x)\in T_{u_{r,k}(s,x)}M$, and by definition, $A_{u_{r,k}(s,x)}\in N_{u_{r,k}(s,x)}M$, the L^2-inner product on domain B_{r-s} vanishes and, in result, the second integrals in (4.3.26) and (4.3.27) are equal to zero.

Next, to deal with the integral containing terms ψ, we follow Lemma 4.3.4, we invoke Lemma 4.2.2, estimate (4.2.8), and Proposition 4.3.8 to get

$$
\begin{aligned}
&\langle v_{r,k}(s), Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\rangle_{L^2(B_{r-s})}\\
&\qquad\lesssim \|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\|^2_{L^2(B_{r-s})}\\
&\qquad\le \|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+C^2_{Y_0}C^2_r\left(1+\|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{L^2(B_{r-s})}\right)\|\dot{h}(s)\|^2_{H_\mu}\\
&\qquad\lesssim (1+l(s))(1+\|\dot{h}(s)\|^2_{H_\mu}),
\end{aligned}
\tag{4.3.29}
$$

for some $C_r>0$, and estimates (4.2.9)-(4.2.10) yields

$$
\begin{aligned}
&\langle v_{r,k}(s), Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\rangle_{L^2(B_{r-s})}\\
&\quad+\langle \partial_x v_{r,k}(s), \partial_x[Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)]\rangle_{L^2(B_{r-s})}\\
&\qquad\lesssim \|v_{r,k}(s)\|^2_{H^1(B_{r-s})}+\|Y_{u_{r,k}(s)}(\partial_t u_{r,k}(s),\partial_x u_{r,k}(s))\dot{h}(s)\|^2_{H^1(B_{r-s})}\\
&\qquad\le \|v_{r,k}(s)\|^2_{H^1(B_{r-s})}+\|\dot{h}(s)\|^2_{H_\mu}\Big[C^2_{Y_0}C^2_r\left(1+\|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{L^2(B_{r-s})}\right)\\
&\qquad\quad+C^2_{Y_1}\left(1+\|v_{r,k}(s)\|^2_{H^1(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{H^1(B_{r-s})}\right)\|u_{r,k}(s)\|^2_{H^1(B_{r-s})}\\
&\qquad\quad+C^2_{Y_2}\left(\|v_{r,k}(s)\|^2_{L^2(B_{r-s})}+\|\partial_x u_{r,k}(s)\|^2_{L^2(B_{r-s})}\right)\Big]\\
&\qquad\lesssim_{C_r,C_{Y_i}}\ (1+l(s))\ (1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}})(1+\|\dot{h}(s)\|^2_{H_\mu}),\quad i=0,1,2.
\end{aligned}
\tag{4.3.30}
$$

By substituting the estimates (4.3.28) and (4.3.29) in the inequality (4.3.26) we get

$$l(t) \lesssim l(0) + \int_0^t \mathbb{1}_{[0,\tau_k]}(s)(1+l(s))\,(1+\|\dot{h}(s)\|^2_{H_\mu})\,ds. \tag{4.3.31}$$

Now we define S_j as the set of initial data whose norm under extension is bounded by j, in precise,

$$S_j := \{(u_0, v_0) \in \mathcal{H}_{\text{loc}} : \|\xi\|_{\mathcal{H}_r} \le j \text{ where } \xi := (E_r^2 u_0, E_r^1 v_0)\}.$$

Then, for the initial data belonging to S_j, the Gronwall Lemma on (4.3.31) yields

$$1 + l_j(t \wedge \tau_k) \le K_{r,j}, \qquad t \le T, \quad j \in \mathbb{N}, \tag{4.3.32}$$

where the constant $K_{r,j}$ also depends on $\|\dot{h}\|_{L^2(0,T;H_\mu)}$ and l_j stands to show that (4.3.32) holds under S_j only.

Next to deal with the third integral in (4.3.27), denote by O its integrand, we recall the following celebrated Gagliardo-Nirenberg inequalities, see e.g. [69],

$$|\psi|^2_{L^\infty(r-s)} \le |\psi|^2_{L^2(B_{r-s})} + 2|\psi|_{L^2(B_{r-s})}|\dot{\psi}|_{L^2(B_{r-s})}, \qquad \psi \in H^1(B_{r-s}). \tag{4.3.33}$$

Thus by applying [23, Lemma 10.1] followed by the generalized Hölder inequality and (4.3.33) we infer

$$\begin{aligned}|O(s)| &\lesssim \mathbb{1}_{[0,\tau_k)}(s)\frac{\int_{B_{r-s}}\{|\partial_x v_{r,k}||\partial_x u_{r,k}||v_{r,k}|^2 + |\partial_{xx}u_{r,k}||\partial_x u_{r,k}|^2|v_{r,k}| + |\partial_x v_{r,k}||\partial_x u_{r,k}|^3\}\,dx}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}\\ &\lesssim \mathbb{1}_{[0,\tau_k)}(s)\frac{l(s)\|a_k(s)\|^2_{\mathcal{H}_{r-s}}}{1+\|a_k(s)\|^2_{\mathcal{H}_{r-s}}} \le \mathbb{1}_{[0,\tau_k)}(s)(1+l(s)).\end{aligned} \tag{4.3.34}$$

So, by substituting (4.3.28), (4.3.29) and (4.3.34) in (4.3.27) we get

$$q(t) \lesssim 1 + q(0) + \int_0^t \mathbb{1}_{[0,\tau_k)}(s)(1+l(s))\,(1+\|\dot{h}(s)\|^2_{H_\mu})\,ds.$$

Consequently, by applying (4.3.32), we obtain on S_j,

$$\begin{aligned}q_j(t\wedge\tau_k) &\lesssim 1 + q_j(0) + \int_0^t [1 + l_j(s\wedge\tau_k)]\,(1+\|\dot{h}(s)\|^2_{H_\mu})\,ds\\ &\le C_{r,j}\,\|\dot{h}\|_{L^2(0,T;H_\mu)}, \qquad j\in\mathbb{N}, t\in[0,T],\end{aligned} \tag{4.3.35}$$

for some $C_{r,j} > 0$, where in the last step we have used that $r > T$ and on set S_j the quantity $q_j(0)$ is bounded by $\log(1+j)$.

To complete the proof let us fix $t < T$. Then, by Proposition 4.3.8,

$$|a_k(\tau_k)|_{\mathcal{H}_{r-\tau_k}} = |z_{r,k}(\tau_k)|_{\mathcal{H}_{r-\tau_k}} \ge k \quad \text{whenever} \quad \tau_k \le t.$$

So for every k such that $\tau_k \le t$ we have

$$\log(1+k^2) \le q(\tau_k) = q(t\wedge\tau_k).$$

Thus by restricting us to S_j and using inequality (4.3.35), we obtain

$$\log(1+k^2) \leq q_j(t\wedge\tau_k) \lesssim C_{r,j}\|\dot{h}\|_{L^2(0,T;H_\mu)}. \tag{4.3.36}$$

In this way, if $\lim_{k\to\infty}\tau_k = t_0$ for any $t_0 < T$, then by taking $k\to\infty$ in (4.3.36) we get $C_{r,j}\|\dot{h}\|_{L^2(0,T;H_\mu)} \geq \infty$ which is absurd. Since this holds for every $j\in\mathbb{N}$ and $t_0 < T$, we infer that $\tau = T$. Hence, we are done with the proof of Proposition 4.3.10. $\square$

Now we have all the machinery required to finish the proof of Theorem 4.3.1 which is for the skeleton Cauchy problem (4.3.1). Define

$$w_{r,k}(t) := \begin{pmatrix} E^2_{r-t}u_{r,k}(t) \\ E^1_{r-t}v_{r,k}(t) \end{pmatrix},$$

and observe that $w_{r,k}:[0,T)\to\mathcal{H}$ is continuous. If we set

$$z_r(t) := \lim_{k\to\infty} w_{r,k}(t), \qquad t<T, \tag{4.3.37}$$

then by Lemma 4.3.9 and Proposition 4.3.10 it is straightforward to verify that, for every $t<T$, the sequence $\{w_{r,k}(t)\}_{k\in\mathbb{N}}$ is Cauchy in $\mathcal{H}$. But, since $\mathcal{H}$ is complete, the limit in (4.3.37) converges in $\mathcal{H}$. Moreover, since by Proposition 4.3.10 $z_{r,k}(t) = z_{r,k_1}(t)$ for every $k_1\geq k$ and $t\leq\tau_k$, we have that $z_r(t) = w_{r,k}(t)$ for $t\leq\tau_k$. In particular, $[0,T)\ni t\mapsto z_r(t)\in\mathcal{H}$ is continuous and $z_r(t,x) = z_{r,k}(t,x)$ for $|x|\leq r-t$ if $t\leq\tau_k$.

Hence, if we write $z_r(t) = (u_r(t), v_r(t))$, then we have shown that u_r satisfy the first conclusion of the Theorem 4.5.1. In the remaining proof of the existence part we will show that the z_r, defined in (4.3.37), will satisfy all the remaining conclusions. Evaluation of (4.3.25) at $t\wedge\tau_k$ together applying the result from previous paragraph gives

$$z_{r,k}(t\wedge\tau_k) = \xi + \int_0^{t\wedge\tau_k}\mathcal{G}z_{r,k}(s)\,ds + \int_0^{t\wedge\tau_k}\mathbf{F}_r(s,z_{r,k}(s))\,ds + \int_0^{t\wedge\tau_k}\mathbf{G}_r(s,z_{r,k}(s))\dot{h}(s)\,ds, \tag{4.3.38}$$

and this equality holds in $H^1(\mathbb{R};\mathbb{R}^n)\times L^2(\mathbb{R};\mathbb{R}^n)$. Restricting to the interval $(-R,R)$, (4.3.38) becomes

$$z_r(t\wedge\tau_k) = \xi + \int_0^{t\wedge\tau_k}\mathcal{G}z_r(s)\,ds + \int_0^{t\wedge\tau_k}\mathbf{F}_r(s,z_r(s))\,ds + \int_0^{t\wedge\tau_k}\mathbf{G}_r(s,z_r(s))\dot{h}(s)\,ds,$$

under the action of natural projection from $H^1(\mathbb{R};\mathbb{R}^n)\times L^2(\mathbb{R};\mathbb{R}^n)$ to $H^1((-R,R);\mathbb{R}^n)\times L^2((-R,R);\mathbb{R}^n)$. Here the integrals converge in $H^1((-R,R);\mathbb{R}^n)\times L^2((-R,R);\mathbb{R}^n)$. Taking the limit $k\to\infty$ on both the sides, the dominated convergence theorem yields

$$z_r(t) = \xi + \int_0^{t}\mathcal{G}z_r(s)\,ds + \int_0^{t}\mathbf{F}_r(s,z_r(s))\,ds + \int_0^{t}\mathbf{G}_r(s,z_r(s))\dot{h}(s)\,ds, \qquad t<T$$

in $H^1((-R,R);\mathbb{R}^n)\times L^2((-R,R);\mathbb{R}^n)$. In particular, by looking to each component separately we have, for every $t<T$,

$$u_r(t) = u_0 + \int_0^t v_r(s)\,ds, \tag{4.3.39}$$

in $H^1((-R,R);\mathbb{R}^n)$, and

$$\begin{aligned} v_r(t) = v_0 + \int_0^t \left[\partial_{xx} u_r(s) + A_{u_r(s)}(v_r(s), v_r(s)) - A_{u_r(s)}(\partial_x u_r(s), \partial_x u_r(s))\right] ds \\ + \int_0^t Y_{u_r(s)}(v_r(s), \partial_x u_r(s))\dot{h}(s)\, ds, \end{aligned} \tag{4.3.40}$$

holds in $L^2((-R,R);\mathbb{R}^n)$. It is relevant to note that in the formula above, we have replaced $\mathcal{A}$ by A which makes sense because due to Proposition 4.3.8 and Proposition 4.3.10, $u_r(t,x) = u_{r,k}(t,x) \in M$ for $|x| \leq r-t$ and $t < T$. Hence we are done with the proof of existence part.

Concerning the uniqueness, define

$$Z(t) := \begin{pmatrix} E_R^2 U(t) \\ E_R^1 \partial_t U(t) \end{pmatrix}, \qquad t < T,$$

and observe that it is a $\mathcal{H}$-valued continuous function of $t \in [0,T)$. Define also

$$\sigma_k := \tau_k \wedge \inf\{t < T : \|Z(t)\|_{\mathcal{H}_{r-t}} \geq k\},$$

and the $\mathcal{H}$-valued function, for $t < T$,

$$\beta(t) := S_t \xi + \int_0^t S_{t-s} \mathbb{1}_{[0,\sigma_k)}(s) \mathbf{F}_{r,k}(s, Z(s))\, ds + \int_0^t S_{t-s} \mathbb{1}_{[0,\sigma_k)}(s) \mathbf{G}_{r,k}(s, Z(s)) \dot{h}(s)\, ds.$$

In the same vein as in the existence part of the proof, as an application of the Chojnowska-Michalik Theorem and projection operator, the restriction of β on $\mathcal{H}_R$, which we denote by b, satisfies

$$\begin{aligned} b(t) = \xi + \int_0^t \mathcal{G} b(s)\, ds + \int_0^t \begin{pmatrix} 0 \\ \mathcal{A}_{U(s)}(\partial_t U(s), \partial_t U(s)) - \mathcal{A}_{U(s)}(\partial_x U(s), \partial_x U(s)) \end{pmatrix} ds \\ + \int_0^t \begin{pmatrix} 0 \\ Y_{U(s)}(\partial_t U(s), \partial_x U(s)) \dot{h}(s) \end{pmatrix} ds, \qquad t \leq \sigma_k, \end{aligned}$$

where the integrals converge in $H^1((-R,R);\mathbb{R}^n) \times L^2((-R,R);\mathbb{R}^n)$. Then since $U(t)$ and $\partial_t U(t)$ have similar form, respectively to (4.3.39) and (4.3.40), by direct computation we deduce that function p defined by

$$p(t) := b(t) - \begin{pmatrix} U(t) \\ \partial_t U(t) \end{pmatrix},$$

satisfies

$$p(t) = \int_0^t \mathcal{G} p(s)\, ds, \qquad t \leq \sigma_k.$$

Since above implies that p satisfies the linear homogeneous wave equation with null initial data, by [23, Remark 6.2], $p(t,x) = 0$ for $|x| \leq R - t$, $t \leq \sigma_k$.

Next, we set

$$q(t) := \|\beta(t) - a_k(t)\|^2_{\mathcal{H}_{R-t}},$$

and apply Proposition 4.5.2, with $k=1, T=r, L=I$, to obtain

$$\begin{aligned} q(t\wedge\sigma_k) \leq 2\int_0^{t\wedge\sigma_k} q(s)\,ds + \int_0^{t\wedge\sigma_k} \|\mathbf{F}_{r,k}(s,Z(s)) - \mathbf{F}_{r,k}(s,a_k(s))\|_{\mathcal{H}}^2\,ds \\ + \int_0^{t\wedge\sigma_k} \|\mathbf{G}_{r,k}(s,Z(s))\dot{h}(s) - \mathbf{G}_{r,k}(s,a_k(s))\dot{h}(s)\|_{\mathcal{H}}^2\,ds. \end{aligned} \tag{4.3.41}$$

But we know that $r-t > R-t$, and by definition $\sigma_k \leq \tau_k$ which implies

$$F_{r,k}(t,z) = F_{R,k}(t,z), \qquad G_{r,k}(t,z) = G_{R,k}(t,z) \qquad \text{on} \quad (t-R, R-t),$$

whenever $\|z\|_{\mathcal{H}_{r-t}} \leq k$. Consequently, the estimate (4.3.41) becomes

$$\begin{aligned} q(t\wedge\sigma_k) \leq 2\int_0^{t\wedge\sigma_k} q(s)\,ds + \int_0^{t\wedge\sigma_k} \|\mathbf{F}_{R,k}(s,Z(s)) - \mathbf{F}_{R,k}(s,a_k(s))\|_{\mathcal{H}}^2]\,ds \\ + \int_0^{t\wedge\sigma_k} \|\mathbf{G}_{R,k}(s,Z(s))\dot{h}(s) - \mathbf{G}_{R,k}(s,a_k(s))\dot{h}(s)\|_{\mathcal{H}}^2\,ds. \end{aligned}$$

Invoking Lemmata 4.3.4 and 4.2.2 yields

$$q(t\wedge\sigma_k) \leq C_R \int_0^{t\wedge\sigma_k} q(s)(1+\|\dot{h}(s)\|_{H_\mu}^2)\,ds.$$

Therefore, we get $q=0$ on $[0,\sigma_k)$ by the Gronwall Lemma. Since in the limit $k\to\infty$, σ_k goes to T as τ_k, by taking k to infinity, by Proposition 4.3.8 we obtain that $u_r(t,x) = U(t,x)$ for each $t<T$ and $|x| \leq R-t$. The proof of Theorem 4.3.1 completes here. ■

4.4 Large deviation principle

In this section we establish a large deviation principle (LDP) for system (1.2.7) via a weak convergence approach developed in [30] and [31] which is based on variational representations of infinite-dimensional Wiener processes.

First, let us recall the general criteria of LDP obtained in [30]. Let $(\Omega,\mathscr{F},\mathbb{P})$ be a probability space with an increasing family $\mathbb{F} := \{\mathscr{F}_t, 0\leq t\leq T\}$ of the sub-σ-fields of $\mathscr{F}$ satisfying the usual conditions. Let $\mathscr{B}(E)$ denotes the Borel σ-field of the Polish space E (i.e. complete separable metric space). Since we are interested in the large deviations of continuous stochastic processes, we follow [48] and consider the following definition of large deviations principle which is in terms of random variables.

Definition 4.4.1. The $(E,\mathscr{B}(E))$-valued random family $\{X^\varepsilon\}_{\varepsilon>0}$, defined on $(\Omega,\mathscr{F},\mathbb{P})$, is said to satisfy a large deviation principle on E with the good rate function $\mathcal{I}$ if the following conditions hold:

1. $\mathcal{I}$ **is a good rate function**: The function $\mathcal{I}: E\to[0,\infty]$ is such that for each $\mathcal{M}\in[0,\infty)$ the level set $\{\phi\in E : \mathcal{I}(\phi)\leq\mathcal{M}\}$ is a compact subset of E.

2. **Large deviation upper bound**: For each closed subset F of E

$$\limsup_{\varepsilon\to 0}\ \varepsilon\log\mathbb{P}\left[X^\varepsilon\in F\right] \leq -\inf_{u\in F}\mathcal{I}(u).$$

3. **Large deviation lower bound**: For each open subset G of E

$$\liminf_{\varepsilon\to 0} \varepsilon \log \mathbb{P}\left[X^\varepsilon \in G\right] \geq -\inf_{u\in G} \mathcal{I}(u),$$

   where by convention the infimum over an empty set is $+\infty$.

Assume that K, H are separable Hilbert spaces and the embedding $K \hookrightarrow H$ is Hilbert-Schmidt. Let $W := \{W(t), t \in [0,T]\}$ be a cylindrical Wiener process on K. Hence the paths of W take values in $\mathcal{C}([0,T];H)$. Note that the RKHS linked to W precisely is ${}_0H^{1,2}(0,T;K)$. Let $\mathscr{S}$ be the class of K-valued $\mathscr{F}_t$-predictable processes ϕ belonging to ${}_0H^{1,2}(0,T;K)$, $\mathbb{P}$-almost surely. For $\mathcal{M} > 0$, we set

$$S_{\mathcal{M}} := \left\{ h \in {}_0H^{1,2}(0,T;K) : \int_0^T \|\dot{h}(s)\|_K^2\, ds \leq \mathcal{M} \right\}.$$

The set $S_{\mathcal{M}}$ endowed with the weak topology obtained from the following metric

$$d_1(h,k) := \sum_{i=1}^{\infty} \frac{1}{2^i} \left| \int_0^T \langle \dot{h}(s) - \dot{k}(s), e_i\rangle_K\, ds \right|,$$

where $\{e_i\}_{i\in\mathbb{N}}$ is a complete orthonormal basis for $L^2(0,T;K)$, is a Polish space, see [31]. Define $\mathscr{S}_{\mathcal{M}}$ as the set of bounded stochastic controls by

$$\mathscr{S}_{\mathcal{M}} := \{\phi \in \mathscr{S} : \phi(\omega) \in S_{\mathcal{M}}, \mathbb{P}\text{-a.s.}\}.$$

Note that $\cup_{M>0}\mathscr{S}_{\mathcal{M}}$ is a proper subset of $\mathscr{S}$. Next, consider a family indexed by $\varepsilon \in (0,1]$ of Borel measurable maps

$$J^\varepsilon : {}_0\mathcal{C}([0,T],H) \to E.$$

We denote by μ^ε the "image" measure on E of $\mathbb{P}$ by J^ε, that is,

$$\mu^\varepsilon = J^\varepsilon(\mathbb{P}), \quad i.e. \quad \mu^\varepsilon(A) = \mathbb{P}\left((J^\varepsilon)^{-1}(A)\right), \quad A \in \mathscr{B}(E).$$

We have the following result.

Theorem 4.4.2. *[30, Theorem 4.4] Suppose that there exists a measurable map $J^0 : {}_0\mathcal{C}([0,T],H) \to E$ such that*

BD1 *: if $\mathcal{M} > 0$ and a family $\{h_\varepsilon\} \subset \mathscr{S}_{\mathcal{M}}$ converges in law as $S_{\mathcal{M}}$-valued random elements to $h \in \mathscr{S}_{\mathcal{M}}$ as $\varepsilon \to 0$, then the processes*

$${}_0\mathcal{C}([0,T],H) \ni \omega \mapsto J^\varepsilon\left(\omega + \frac{1}{\sqrt{\varepsilon}} \int_0^{\cdot} \dot{h}_\varepsilon(s)\, ds\right) \in E,$$

converges in law, as $\varepsilon \searrow 0$, to the process $J^0\left(\int_0^{\cdot} \dot{h}_\varepsilon(s)\, ds\right)$,

BD2 *: for every $\mathcal{M} >$, the set*

$$\left\{ J^0\left(\int_0^{\cdot} \dot{h}(s)\, ds\right) : h \in S_{\mathcal{M}} \right\},$$

is a compact subset of E.

Then the family of measures μ^ε satisfies the large deviation principle (LDP) with the rate function defined by

$$\mathcal{I}(u) := \inf\left\{\frac{1}{2}\int_0^T \|\dot{h}(s)\|_K^2\, ds : {}_0H^{1,2}(0,T;K) \text{ and } u = J^0\left(\int_0^{\cdot} \dot{h}(s)\, ds\right)\right\}, \tag{4.4.1}$$

with the convention $\inf\{\emptyset\} = +\infty$.

4.4.1 Main result

It is important to note that in transferring the general theory argument from Theorem 4.4.2 in our setting we require some information about the difference of solutions at two different times, hence we need to strengthen the assumptions on diffusion coefficient. In the remaining part of this chapter, we assume that $Y : M \ni p \mapsto Y(p) \in T_pM$ is a smooth vector field on compact Riemannian manifold M, which can be considered as a submanifold of $\mathbb{R}^n$, such that its extension, denote again by Y, on the ambient space $\mathbb{R}^n$ is smooth and satisfies

Y.4 there exists a compact set $K_Y \subset \mathbb{R}^n$ such that $Y(p) = 0$ if $p \notin K_Y$,

Y.5 for $q \in O$, $Y(\Upsilon(q)) = \Upsilon'(q)Y(q)$,

Y.6 for some $C_Y > 0$

$$|Y(p)| \le C_Y(1+|p|), \quad \left|\frac{\partial Y}{\partial p_i}(p)\right| \le C_Y, \text{ and } \left|\frac{\partial^2 Y}{\partial p_i \partial p_j}(p)\right| \le C_Y,$$

for $p \in K_Y, i, j = 1, \ldots, n$.

Remark 4.4.3. 1. Since K_Y is compact, there exists a C_K such that $|Y(p)| \le C_K$ for $p \in \mathbb{R}^n$.

2. For $M = \mathbb{S}^2$ case, $Y(p) = p \times e, p \in M$, for some fixed vector $e \in \mathbb{R}^3$ satisfies above assumptions.

Since, due to the above assumptions, Y and its first order partial derivatives are Lipschitz, by 1-D Sobolev embedding we easily get the next result.

Lemma 4.4.4. *There exists $C_{Y,R} > 0$ such that the extension Y defined above satisfy*

(1) $\|Y(u)\|_{H^j(B_R)} \le C_{Y,R}(1 + \|u\|_{H^j(B_R)}), \quad j = 0, 1, 2,$

(2) $\|Y(u) - Y(v)\|_{L^2(B_R)} \le C_{Y,R}\|u - v\|_{L^2(B_R)},$

(3) $\|Y(u) - Y(v)\|_{H^1(B_R)} \le C_{Y,R}\|u - v\|_{H^1(B_R)}\left(1 + \|u\|_{H^1(B_R)} + \|v\|_{H^1(B_R)}\right).$

Now we state the main result of this section for the following small noise Cauchy problem

$$\begin{cases} \partial_{tt}u^\varepsilon = \partial_{xx}u^\varepsilon + A_{u^\varepsilon}(\partial_t u^\varepsilon, \partial_t u^\varepsilon) - A_{u^\varepsilon}(\partial_x u^\varepsilon, \partial_x u^\varepsilon) + \sqrt{\varepsilon}\,Y(u^\varepsilon)\dot{W}, \\ \left(u^\varepsilon(0), \partial_t u^\varepsilon(0)\right) = (u_0, v_0), \end{cases} \tag{4.4.2}$$

with the hypothesis that (u_0, v_0) is $\mathscr{F}_0$-measurable $H^2_{\text{loc}} \times H^1_{\text{loc}}(\mathbb{R}, TM)$-valued random variable, such that $u_0(x,\omega) \in M$ and $v_0(x,\omega) \in T_{u_0(x,\omega)}M$ hold for every $\omega \in \Omega$ and $x \in \mathbb{R}$. Since the small noise problem (4.4.2), with initial data $(u_0, v_0) \in \mathscr{H}_{\text{loc}}(\mathbb{R}; M)$, is a particular case of Theorem 4.5.1, for given $\varepsilon > 0$ and $T > 0$, there exists a unique global strong solution to (4.4.2), which we denote by $z^\varepsilon := (u^\varepsilon, \partial_t u^\varepsilon)$, with values in the Polish space

$$\mathcal{X}_T := \mathcal{C}\left([0,T]; H^2_{\text{loc}}(\mathbb{R};\mathbb{R}^n)\right) \times \mathcal{C}\left([0,T]; H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)\right),$$

and satisfy the properties mentioned in Section 4.5.1. Thus, there exists a Borel measurable function, see for example [30] and [120, Theorems 12.1 and 13.2],

$$J^\varepsilon : {}_0\mathcal{C}([0,T],E) \to \mathcal{X}_T, \tag{4.4.3}$$

where space E can be taken as in Lemma 4.2.1, such that $z^\varepsilon(\cdot) = J^\varepsilon(W(\cdot))$, $\mathbb{P}$-almost surely.

Recall from Section 4.2 that the random perturbation W we consider is a cylindrical Wiener process on H_μ and there exists a separable Hilbert space H such that the embedding of H_μ in H is Hilbert-Schmidt. Hence we can apply the general theory from previous section with the notations defined by taking H_μ instead of K.

Let us define a Borel map

$$J^0 : {}_0\mathcal{C}([0,T],E) \to \mathcal{X}_T.$$

If $h \in {}_0\mathcal{C}([0,T],E) \setminus {}_0H^{1,2}(0,T;H_\mu)$, then we set $J^0(h) = 0$. If $h \in {}_0H^{1,2}(0,T;H_\mu)$ then by Theorem 4.3.1 there exists a function in $\mathcal{X}_T$, say z_h, that solves

$$\begin{cases} \partial_{tt}u = \partial_{xx}u + A_u(\partial_t u, \partial_t u) - A_u(\partial_x u, \partial_x u) + Y(u)\,\dot{h}, \\ u(0) = u_0, \partial_t u(0) = v_0, \end{cases} \tag{4.4.4}$$

uniquely and we set $J^0(h) = z_h$.

Remark 4.4.5. At some places in the chapter we denote $J^0(h)$ by $J^0\left(\int_0^\cdot \dot{h}(s)\,ds\right)$ to make it clear that in the differential equation we have control $\dot{h}$ not h.

The main result of this section is as follows.

Theorem 4.4.6. *The family of laws $\{\mathscr{L}(z^\varepsilon) : \varepsilon \in (0,1]\}$ on $\mathcal{X}_T$, where $z^\varepsilon := (u^\varepsilon, \partial_t u^\varepsilon)$ is the unique solution to* (4.4.2) *satisfies the large deviation principle with rate function $\mathcal{I}$ defined in* (4.4.1).

Note that, in light of Theorem 4.4.2, in order to prove the Theorem 4.4.6 it is sufficient to show the following two statements:

Statement 1 : For each $\mathcal{M} > 0$, the set

$$K_\mathcal{M} := \{J^0(h) : h \in S_\mathcal{M}\},$$

is a compact subset of $\mathcal{X}_T$, where $S_\mathcal{M} \subset {}_0H^{1,2}(0,T;H_\mu)$ is the centred closed ball of radius M endowed with the weak topology.

Statement 2 : Assume that $\mathcal{M} > 0$, that $\{\varepsilon_n\}_{n\in\mathbb{N}}$ is an $(0,1]$-valued sequence convergent to 0, that $\{h_n\}_{n\in\mathbb{N}} \subset \mathscr{S}_{\mathcal{M}}$ converges in law to $h \in \mathscr{S}_{\mathcal{M}}$ as $\varepsilon \to 0$. Then the processes

$$
{}_0\mathcal{C}([0,T],E) \ni \omega \mapsto J^{\varepsilon_n}\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon_n}}\int_0^{\cdot} \dot{h}_n(s)\,ds\right) \in \mathcal{X}_T, \tag{4.4.5}
$$

converges in law on $\mathcal{X}_T$ to $J^0\left(\int_0^{\cdot} \dot{h}(s)\,ds\right)$.

Remark 4.4.7. By combining the proofs of Theorems 4.5.1 and 4.3.1 we infer that the map (4.4.5) is well-defined and $J^{\varepsilon_n}\left(W(\cdot) + \frac{1}{\sqrt{\varepsilon_n}}\int_0^{\cdot} \dot{h}_n(s)\,ds\right)$ is the unique solution to the following stochastic control Cauchy problem

$$
\begin{cases} \partial_{tt} u^{\varepsilon_n} = \partial_{xx} u^{\varepsilon_n} + A_{u^{\varepsilon_n}}(\partial_t u^{\varepsilon_n}, \partial_t u^{\varepsilon_n}) - A_{u^{\varepsilon_n}}(\partial_x u^{\varepsilon_n}, \partial_x u^{\varepsilon_n}) + Y(u^{\varepsilon_n})\dot{h}_n + \sqrt{\varepsilon_n}\, Y(u^{\varepsilon_n})\dot{W}, \\ \left(u^{\varepsilon_n}(0), \partial_t u^{\varepsilon_n}(0)\right) = (u_0, v_0), \end{cases} \tag{4.4.6}
$$

where the initial data $(u_0, v_0) \in H^2_{\text{loc}} \times H^1_{\text{loc}}(\mathbb{R}; TM)$.

Remark 4.4.8. It is clear by now that verification of an LDP comes down to proving two convergence results, see [19, 20, 28, 48, 149]. As it was shown first in [16] the second convergence result follows from the first one via the Jakubowski version of the Skorokhod representation theorem. Therefore, establishing LDP, de facto, reduces to proving one convergence result for deterministic controlled problem called also the skeleton equation. This convergence result is specific to the stochastic PDE in question and require techniques related to the considered equation. Thus, for instance, the proof in [16, Lemma 6.3] for the stochastic Landau-Lifshitz-Gilbert equation, is different from the proof, for stochastic Navier-Stokes equation, of [48, Proposition 3.5]. On technical level, the proof of corresponding result, i.e. Statement 1, is the main contribution of our work.

4.4.2 Proof of Statement 1

Let $\{z_n = (u_n, v_n) := J^0(h_n)\}_{n\in\mathbb{N}}$ be a sequence in the set $K_{\mathcal{M}}$ corresponding to the sequence of controls $\{h_n\}_{n\in\mathbb{N}} \subset S_{\mathcal{M}}$. Since $S_{\mathcal{M}}$ is a bounded and closed subset of Hilbert space ${}_0H^{1,2}(0,T;H_\mu)$, $S_{\mathcal{M}}$ is weakly compact. Consequently, see [9], there exists a subsequence of $\{h_n\}_{n\in\mathbb{N}}$, still denoted this by $\{h_n\}_{n\in\mathbb{N}}$, which converges weakly to a limit $h \in {}_0H^{1,2}(0,T;H_\mu)$. But, since $S_{\mathcal{M}}$ is weakly closed, $h \in S_{\mathcal{M}}$. Hence to complete the proof of Statement 1 we need to show that the subsequence of solutions $\{z_n\}_{n\in\mathbb{N}}$ to (4.4.4), corresponding to the subsequence of controls $\{h_n\}_{n\in\mathbb{N}}$, converges to $z_h = (u_h, v_h)$ which solves the skeleton Cauchy problem (4.4.4) for control h. Before delving into the proof of this we establish the following a priori estimate which is a preliminary step required to prove, Proposition 4.4.14, the main result of this section.

Lemma 4.4.9. *Fix any $T > 0$, $x \in \mathbb{R}$. There exists a constant $\mathcal{B} := \mathcal{B}\left(\|(u_0, v_0)\|_{\mathcal{H}(B(x,T))}, \mathcal{M}, T\right) > 0$, such that*

$$
\sup_{h\in S_M}\ \sup_{t\in[0,T/2]} \boldsymbol{e}(t, z_h(t)) \le \mathcal{B}. \tag{4.4.7}
$$

Here z_h is the unique global strong solution to problem (4.4.4) *and*

$$\boldsymbol{e}(t,z) := \frac{1}{2}\|z\|^2_{\mathcal{H}_{B(x,T-t)}} = \frac{1}{2}\Big\{\|u\|^2_{L^2(B(x,T-t))} + \|\partial_x u\|^2_{L^2(B(x,T-t))} + \|v\|^2_{L^2(B(x,T-t))} + \|\partial_{xx} u\|^2_{L^2(B(x,T-t))} + \|\partial_x v\|^2_{L^2(B(x,T-t))}\Big\}, \qquad z=(u,v)\in\mathcal{H}_{loc}.$$

Moreover, if we restrict x on an interval $[-a,a]\subset\mathbb{R}$, then the constant $\mathcal{B}:=\mathcal{B}(\mathcal{M},T,a)$, which also depends on 'a' now, can be chosen such that

$$\sup_{x\in[-a,a]}\sup_{h\in S_M}\sup_{t\in[0,T/2]} \boldsymbol{e}(t,z_h(t)) \le \mathcal{B}.$$

Proof of Lemma 4.4.9 First note that the last part follows from the first one because by assumptions, $(u_0,v_0)\in\mathcal{H}_{\rm loc}$, in particular, $\|(u_0,v_0)\|_{\mathcal{H}(-a-T,a+T)}<\infty$ and therefore,

$$\sup_{x\in[-a,a]}\|(u_0,v_0)\|_{\mathcal{H}(B(x,T))} \le \|(u_0,v_0)\|_{\mathcal{H}(-a-T,a+T)}<\infty.$$

The procedure to prove (4.4.7) is based on the proof of Proposition 4.3.10. Let us fix h in $S_{\mathcal{M}}$ and denote the corresponding solution $z_h:=(u_h,v_h)$ which exists due to Theorem 4.3.1. Since x is fixed we will avoid writing it explicitly in the norm. Define

$$l(t) := \frac{1}{2}\|(u_h(t),v_h(t)\|^2_{H^1(B_{T-t})\times L^2(B_{T-t})}, \qquad t\in[0,T].$$

Thus, invoking Proposition 4.5.2, with $k=0$ and $L=I$, implies, for $t\in[0,T]$,

$$\begin{aligned} l(t) \le l(0) &+ \int_0^t \langle u_h(r), v_h(s)\rangle_{L^2(B_{T-s})}\,ds + \int_0^t \langle v_h(s), f_h(s)\rangle_{L^2(B_{T-s})}\,ds \\ &+ \int_0^t \langle v_h(s), Y(u_h(s))\dot{h}(s)\rangle_{L^2(B_{T-s})}\,ds, \end{aligned} \tag{4.4.8}$$

where

$$f_h(r) := A_{u_h(r)}(v_h(r), v_h(r) - A_{u_h(r)}(\partial_x u_h(r), \partial_x u_h(r).$$

Since $v_h(r)\in T_{u_h(r)}M$ and by definition $A_{u_h(r)}(\cdot,\cdot)\in N_{u_h(r)}M$, the second integral in (4.4.8) vanishes. Because $u_h(r)\in M$, invoking Cauchy-Schwartz inequality, Lemmata 4.2.2 and 4.4.4 implies

$$l(t) \le l(0) + \left(\frac{C_Y^2C_T^2}{2}+2\right)\int_0^t (1+l(s))(1+\|\dot{h}(s)\|^2_{H_\mu})\,ds.$$

Consequently, by appying the Gronwall Lemma and using $h\in S_{\mathcal{M}}$ we get

$$l(t) \lesssim_{C_Y,C_T} (1+l(0))\left[T+\|\dot{h}\|^2_{L^2(0,T;H_\mu)}\right] \le (T+\mathcal{M})(1+l(0)). \tag{4.4.9}$$

Next we define

$$q(t) := \log\left(1+\|z_h(t)\|^2_{\mathcal{H}_{T-t}}\right).$$

Then Proposition 4.5.2, with $k=1$ and $L(x)=\log(1+x)$, gives, for $t\in[0,T/2]$,

$$q(t) \le q(0) + \int_0^t \frac{\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}{1+\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\,ds$$

$$
\begin{aligned}
&+\int_0^t \frac{\langle v_h(s), f_h(s)\rangle_{L^2(B_{T-s})}}{1+\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\,ds+\int_0^t \frac{\langle \partial_x v_h(s), \partial_x[f_h(s)]\rangle_{L^2(B_{T-s})}}{1+\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\,ds \\
&+\int_0^t \frac{\langle v_h(s), Y(u_h(s))\dot{h}(s)\rangle_{L^2(B_{T-s})}}{1+\|z_k(s)\|^2_{\mathcal{H}_{T-s}}}\,ds+\int_0^t \frac{\langle \partial_x v_h(s), \partial_x[Y(u_h(s))\dot{h}(s)]\rangle_{L^2(B_{T-s})}}{1+\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\,ds.
\end{aligned}
$$

Since by perpendicularity the second integral in above vanishes, by doing the calculation based on (4.3.30) and (4.3.34) we deduce

$$
\begin{aligned}
q(t) &\lesssim_T 1+q(0)+\int_0^t \frac{l(s)\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}{1+\|z_h(s)\|^2_{\mathcal{H}_{T-s}}}\,ds \\
&\quad+\int_0^t \frac{(1+l(s))\ (1+\|z_h(s)\|^2_{\mathcal{H}_{T-s}})(1+\|\dot{h}(s)\|^2_{H_\mu})}{1+\|z_k(s)\|^2_{\mathcal{H}_{T-s}}}\,ds \\
&\le 1+q(0)+\int_0^t (1+l(s))(1+\|\dot{h}(s)\|^2_{H_\mu})\,ds,
\end{aligned}
$$

which further implies, due to (4.4.9) and $h\in S_{\mathcal{M}}$,

$$
q(t)\lesssim 1+q(0)+(T+\mathcal{M})^2(1+l(0)).
$$

In terms of z_h, that is, for each $x\in\mathbb{R}$ and $t\in[0,T/2]$,

$$
\|z_h(t)\|^2_{\mathcal{H}_{T-t}}\lesssim \exp\left[\|(u_0,v_0)\|^2_{\mathcal{H}_{T-t}}(T+\mathcal{M})^2\right].
$$

Since above holds for every $t\in[0,T/2]$, $h\in S_{\mathcal{M}}$, by taking supremum on t and h we get (4.4.7), and hence the proof of Lemma 4.4.9. ■

Remark 4.4.10. Since $B(x,T/2)\subseteq B(x,T-t)$ for every $t\in[0,T/2]$, Lemma 4.4.9 also implies

$$
\sup_{x\in[-a,a]}\sup_{h\in S_{\mathcal{M}}}\sup_{t\in[0,T/2]}\frac{1}{2}\left\{\|u_h(t)\|^2_{H^2(B(x,R))}+\|v_h(t)\|^2_{H^1(B(x,R))}\right\}\le \mathcal{B}(\mathcal{M},T,a),
$$

for $R=T/2$.

Now we prove the main result of this subsection which will allow to complete the proof of Statement 1.

Proposition 4.4.11. *Fix $\mathcal{T}>0$. The sequence of solutions $\{z_n\}_{n\in\mathbb{N}}$ to the skeleton problem* (4.4.4) *converges to z_h in the $\mathcal{X}_{\mathcal{T}}$-norm (strong topology). In particular, for every $\mathcal{T},\mathcal{M}>0$, the mapping*

$$
S_M\in h\mapsto J^0(h)\in\mathcal{X}_{\mathcal{T}},
$$

is Borel.

Proof of Proposition 4.4.11 First note that the second conclusion follows from first immediately because continuous maps are Borel. Towards proving the first conclusion, let us fix any $n\in\mathbb{N}$. Recall that in our notation, by Theorem 4.3.1, $z_h = (u_h, v_h)$ and $z_n = (u_n, v_n)$, respectively, are the unique global strong solutions to

$$\begin{cases} \partial_{tt} u_h = \partial_{xx} u_h + A_{u_h}(\partial_t u_h, \partial_t u_h) - A_{u_h}(\partial_x u_h, \partial_x u_h) + Y(u_h)\dot{h}, \\ (u_h(0), v_h(0)) = (u_0, v_0), \quad \text{where } v_n h := \partial_t u_h, \end{cases} \tag{4.4.10}$$

and

$$\begin{cases} \partial_{tt} u_n = \partial_{xx} u_n + A_{u_n}(\partial_t u_n, \partial_t u_n) - A_{u_n}(\partial_x u_n, \partial_x u_n) + Y(u_n)\dot{h}_n, \\ (u_n(0), v_n(0)) = (u_0, v_0), \quad \text{where } v_n := \partial_t u_n. \end{cases} \tag{4.4.11}$$

Hence $\mathfrak{z}_n := (\mathfrak{u}_n, \mathfrak{v}_n) = z_h - z_n$ is the unique global strong solution to, with null initial data,

$$\begin{aligned} \partial_{tt}\mathfrak{u}_n = \partial_{xx}\mathfrak{u}_n - A_{u_h}(\partial_x u_h, \partial_x u_h) + A_{u_n}(\partial_x u_n, \partial_x u_n) + A_{u_h}(\partial_t u_h, \partial_t u_h) \\ - A_{u_n}(\partial_t u_n, \partial_t u_n) + Y(u_h)\dot{h} - Y(u_n)\dot{h}_n, \end{aligned} \tag{4.4.12}$$

where $\mathfrak{v}_n := \partial_t \mathfrak{u}_n$. This implies that

$$\mathfrak{z}_n(t) = \int_0^t S_{t-s} \begin{pmatrix} 0 \\ f_n(s) \end{pmatrix} ds + \int_0^t S_{t-s} \begin{pmatrix} 0 \\ g_n(s) \end{pmatrix} ds, \quad t \in [0, \mathfrak{T}].$$

Here

$$\begin{aligned} f_n(s) := -A_{u_h(s)}(\partial_x u_h(s), \partial_x u_h(s)) + A_{u_n(s)}(\partial_x u_n(s), \partial_x u_n(s)) + A_{u_h(s)}(\partial_t u_h(s), \partial_t u_h(s)) \\ - A_{u_n(s)}(\partial_t u_n(s), \partial_t u_n(s)), \end{aligned}$$

and

$$g_n(s) := Y(u_h(s))\dot{h}(s) - Y(u_n(s))\dot{h}_n(s).$$

We aim to show that

$$\mathfrak{z}_n \xrightarrow[n\to 0]{} 0 \quad \text{in} \quad \mathcal{C}\left([0,\mathfrak{T}], H^2_{\text{loc}}(\mathbb{R};\mathbb{R}^n)\right) \times \mathcal{C}\left([0,\mathfrak{T}], H^1_{\text{loc}}(\mathbb{R};\mathbb{R}^n)\right),$$

that is, for every $R > 0$ and $x \in \mathbb{R}$,

$$\sup_{t\in[0,\mathfrak{T}]} \left[\|\mathfrak{u}_n(t)\|^2_{H^2(B(x,R))} + \|\mathfrak{v}_n(t)\|^2_{H^1(B(x,R))} \right] \to 0 \text{ as } n \to \infty. \tag{4.4.13}$$

Without loss of generality we assume $x = 0$. Since a compact set in $\mathbb{R}$ can be convered by a finite number of any given closed intervals of any non-zero length, it is sufficient to prove above for a fixed $R > 0$ whose value will be set later. Let φ be a bump function which takes value 1 on B_R and vanishes outside $\overline{B_{2R}}$. Define $\bar{u}_n(t,x) := u_n(t,x)\varphi(x)$ and $\bar{u}_h(t,x) := u_h(t,x)\varphi(x)$, so

$$\bar{v}_n(t,x) = \varphi(x) v_n(t,x), \qquad \bar{v}_h(t,x) = \varphi(x) v_h(t,x),$$

and with notation $\bar{\mathfrak{u}}_n := \bar{u}_n - \bar{u}_h$,

$$\partial_{tt}\bar{\mathfrak{u}}_n - \partial_{xx}\bar{\mathfrak{u}}_n = \left[A_{u_n}(\partial_t u_n, \partial_t u_n) - A_{u_n}(\partial_x u_n, \partial_x u_n) - A_{u_h}(\partial_t u_h, \partial_t u_h) + A_{u_h}(\partial_x u_h, \partial_x u_h)\right]\varphi$$

$$
\begin{aligned}
&-(u_n-u_h)\partial_{xx}\varphi-2(\partial_x u_n-\partial_x u_h)\partial_x\varphi+\left[Y(u_n)\dot{h}_n-Y(u_h)\dot{h}\right]\varphi\\
=:&\ \bar{f}_n+\bar{g}_n.
\end{aligned}
$$

Here

$$
\begin{aligned}
\bar{f}_n(s):=&\big[A_{u_n(s)}(\partial_t u_n(s),\partial_t u_n(s))-A_{u_n(s)}(\partial_x u_n(s),\partial_x u_n(s))-A_{u_h(s)}(\partial_t u_h(s),\partial_t u_h(s))\\
&+A_{u_h(s)}(\partial_x u_h(s),\partial_x u_h(s))\big]\varphi-(u_n(s)-u_h(s))\partial_{xx}\varphi-2(\partial_x u_n(s)-\partial_x u_h(s))\partial_x\varphi,
\end{aligned}
$$

and

$$
\bar{g}_n(s):=\left[Y(u_n(s))\dot{h}_n(s)-Y(u_h(s))\dot{h}(s)\right]\varphi,\quad s\in[0,\mathcal{T}].
$$

Next, by direct computation we can find constants $C_\varphi,\bar{C}_\varphi>0$, depend upon $\varphi,\varphi',\varphi''$, such that, for $t\in[0,\mathcal{T}]$,

$$
\begin{aligned}
\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(-R,R)}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(-R,R)}&\leq C_\varphi\left[\|\mathfrak{u}_n(t)\|^2_{H^2(-R,R)}+\|\mathfrak{v}_n(t)\|^2_{H^1(-R,R)}\right]\\
&\leq \bar{C}_\varphi\left[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(-R,R)}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(-R,R)}\right].
\end{aligned}
\tag{4.4.14}
$$

Hence, instead of (4.4.13) it is enough to prove the following, for a fixed R,

$$
\sup_{t\in[0,\mathcal{T}]}\left[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(-R,R)}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(-R,R)}\right]\to 0 \text{ as } n\to\infty.
\tag{4.4.15}
$$

Let us set

$$
T:=4\mathcal{T}\qquad\text{and}\qquad R:=\frac{T}{4}=\mathcal{T}.
$$

The reason of such choice is due to the fact that (4.4.15) follows from

$$
\sup_{t\in[0,R]}\left[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_{T-t})}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_{T-t})}\right]\to 0\quad\text{as}\quad n\to\infty.
\tag{4.4.16}
$$

Indeed, because for every $t\in[0,R]$, $T-t>2R$, and we have

$$
\begin{aligned}
\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_R)}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_R)}&\leq\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_{2R})}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_{2R})}\\
&\leq\sup_{t\in[0,R]}\left[\|\bar{\mathfrak{u}}_n(t)\|^2_{H^2(B_{T-t})}+\|\bar{\mathfrak{v}}_n(t)\|^2_{H^1(B_{T-t})}\right].
\end{aligned}
$$

Next, we set $l(t,z):=\frac{1}{2}\|z\|^2_{\mathcal{H}_{T-t}}$, for $z=(u,v)\in\mathcal{H}_{\text{loc}}$ and $t\in[0,R]$. Invoking Proposition 4.5.2, with null diffusion part and $k=1,L=I,x=0$, gives, for every $t\in[0,R]$,

$$
l(t,\bar{\mathfrak{z}}_n(t))\leq\int_0^t\mathbb{V}(r,\bar{\mathfrak{z}}_n(r))\,dr,
\tag{4.4.17}
$$

where $\bar{\mathfrak{z}}_n(t)=(\bar{\mathfrak{u}}_n(t),\bar{\mathfrak{v}}_n(t))$ and

$$
\begin{aligned}
\mathbb{V}(t,\bar{\mathfrak{z}}_n(t))=&\langle\bar{\mathfrak{u}}_n(t),\bar{\mathfrak{v}}_n(t)\rangle_{L^2(B_{T-t})}+\langle\bar{\mathfrak{v}}_n(t),\bar{f}_n(t)\rangle_{L^2(B_{T-t})}\\
&+\langle\partial_x\bar{\mathfrak{v}}_n(t),\partial_x\bar{f}_n(t)\rangle_{L^2(B_{T-t})}+\langle\bar{\mathfrak{v}}_n(t),\bar{g}_n(t)\rangle_{L^2(B_{T-t})}\\
&+\langle\partial_x\bar{\mathfrak{v}}_n(t),\partial_x\bar{g}_n(t)\rangle_{L^2(B_{T-t})}
\end{aligned}
$$

$$=: \mathbb{V}_f(t,\bar{\mathfrak{z}}_n(t)) + \mathbb{V}_g(t,\bar{\mathfrak{z}}_n(t)).$$

We estimate $\mathbb{V}_f(t,\bar{\mathfrak{z}}_n(t))$ and $\mathbb{V}_g(t,\bar{\mathfrak{z}}_n(t))$ separately as follows. Since $T-t>2R$ for every $t\in[0,R]$ and $\varphi(y),\varphi'(y)=0$ for $y\notin\overline{B_{2R}}$, we have

$$\begin{aligned}\int_0^t \mathbb{V}_f(r,\bar{\mathfrak{z}}(r))\,dr &= \int_0^t \Big[\int_{B_{2R}} \{\varphi(y)\mathfrak{u}_n(r,y)\varphi(y)\mathfrak{v}_n(r,y) + \varphi(y)\mathfrak{v}_n(r,y)\bar{f}_n(r,y)\\ &\quad +\varphi'(y)\mathfrak{v}_n(r,y)\partial_x\bar{f}_n(r,y) + \varphi(y)\partial_x\mathfrak{v}_n(r,y)\partial_x\bar{f}_n(r,y)\}\,dy\Big]\,dr\\ &\lesssim_{\varphi,\varphi'} \int_0^t l(r,\bar{\mathfrak{z}}_n(r))\,dr + \int_0^t \|\bar{f}_n(r)\|^2_{H^1(B_{2R})}\,dr,\end{aligned}$$

and

$$\begin{aligned}&\int_0^t \big(\langle\bar{\mathfrak{v}}_n(r),\bar{g}_n(r)\rangle_{L^2(B_{T-r})} + \langle\partial_x\bar{\mathfrak{v}}_n(r),\partial_x\bar{g}_n(r)\rangle_{L^2(B_{T-r})}\big)\,dr\\ &\qquad = \int_0^t \big(\langle\bar{\mathfrak{v}}_n(r),\bar{g}_n(r)\rangle_{L^2(B_{2R})} + \langle\partial_x\bar{\mathfrak{v}}_n(r),\partial_x\bar{g}_n(r)\rangle_{L^2(B_{2R})}\big)\,dr.\end{aligned}$$

Let us estimate the terms involving $\bar{f}_n$ first. Since u_n, u_h takes values on manifold M, by using the properties of φ and invoking interpolation inequality (4.3.5), as pursued in Lemma 4.3.4, followed by Lemma 4.4.9 we deduce that

$$\begin{aligned}\|\bar{f}_n(r)\|^2_{L^2(B_{2R})} &\lesssim_{\varphi,\varphi',\varphi''} \|A_{u_n(r)}(v_n(r),v_n(r)) - A_{u_h(r)}(v_n(r),v_n(r))\|^2_{L^2(B_{2R})}\\ &\quad + \|A_{u_h(r)}(v_n(r),v_n(r)) - A_{u_h(r)}(v_n(r),v_h(r))\|^2_{L^2(B_{2R})}\\ &\quad + \|A_{u_h(r)}(v_n(r),v_h(r)) - A_{u_h(r)}(v_h(r),v_h(r))\|^2_{L^2(B_{2R})}\\ &\quad + \|A_{u_n(r)}(\partial_x u_n(r),\partial_x u_n(r)) - A_{u_h(r)}(\partial_x u_n(r),\partial_x u_n(r))\|^2_{L^2(B_{2R})}\\ &\quad + \|A_{u_h(r)}(\partial_x u_n(r),\partial_x u_n(r)) - A_{u_h(r)}(\partial_x u_n(r),\partial_x u_h(r))\|^2_{L^2(B_{2R})}\\ &\quad + \|A_{u_h(r)}(\partial_x u_n(r),\partial_x u_h(r)) - A_{u_h(r)}(\partial_x u_h(r),\partial_x u_h(r))\|^2_{L^2(B_{2R})}\\ &\quad + \|u_n(r)-u_h(r)\|^2_{L^2(B_{2R})} + 2\|\partial_x u_n(r) - \partial_x u_h(r)\|^2_{L^2(B_{2R})}\\ &\lesssim_{L_A,B_A,R} \|u_n(r)-u_h(r)\|_{L^2(B_{2R})}\|v_n(r)\|^2_{L^\infty(B_{2R})}\\ &\quad + \|v_n(r)-v_h(r)\|^2_{L^2(B_{2R})}\big(\|v_n(r)\|_{L^\infty(B_{2R})} + \|v_h(r)\|_{L^\infty(B_{2R})}\big)\\ &\quad + \|u_n(r)-u_h(r)\|_{L^2(B_{2R})}\|\partial_x u_n(r)\|^2_{L^\infty(B_{2R})}\\ &\quad + \|\partial_x u_n(r)-\partial_x u_h(r)\|^2_{L^2(B_{2R})}\big(\|\partial_x u_n(r)\|_{L^\infty(B_{2R})} + \|\partial_x u_h(r)\|_{L^\infty(B_{2R})}\big)\\ &\quad + \|u_n(r)-u_h(r)\|^2_{L^2(B_{2R})} + 2\|\partial_x u_n(r) - \partial_x u_h(r)\|^2_{L^2(B_{2R})}\\ &\lesssim_{L_A,B_A,R,k_e,\mathcal{B}} \|\mathfrak{z}_n(r)\|^2_{\mathcal{H}(B_{2R})} \lesssim l(r,\mathfrak{z}_n(r)).\end{aligned}\tag{4.4.18}$$

Similarly by using the interpolation inequality (4.3.5) and Lemma 4.4.9, based on the computation of (4.3.7), we get

$$\|\partial_x\bar{f}_n(r)\|^2_{L^2(B_{2R})} \lesssim_{L_A,B_A,R,k_e,\mathcal{B}} l(r,\mathfrak{z}_n(r)),$$

where constant of inequality is independent of n but depends on the properties of φ and its first two derivatives, consequently, we have, for some $C_{\bar{f}} > 0$,

$$\int_0^t \|\bar{f}_n(r)\|^2_{H^1(B_{2R})}\, dr \leq C_{\bar{f}} \int_0^t l(r, \mathfrak{z}_n(r))\, dr. \tag{4.4.19}$$

Now we move to the crucial estimate of integral involving $\bar{g}_n$. It is the part where we follow the idea of [48, Proposition 3.4] and [63, Proposition 4.4]. Let m be a natural number, whose value will be set later. Define the following partition of $[0, R]$,

$$\left\{0, \frac{1\cdot R}{2^m}, \frac{2\cdot R}{2^m}, \cdots, \frac{2^m\cdot R}{2^m}\right\},$$

and set

$$r_m := \frac{(k+1)\cdot R}{2^m} \quad \text{and} \quad t_{k+1} := \frac{(k+1)\cdot R}{2^m} \quad \text{if} \quad r \in \left[\frac{k\cdot R}{2^m}, \frac{(k+1)\cdot R}{2^m}\right).$$

Now observe that

$$\begin{aligned}
\int_0^t \langle \bar{\mathfrak{v}}_n(r), \bar{g}_n(r)\rangle_{H^1(B_{2R})}\, dr &= \int_0^t \langle \bar{\mathfrak{v}}_n(r), \varphi(Y(u_n(r)) - Y(u_h(r)))\dot{h}_n(r)\rangle_{H^1(B_{2R})}\, dr \\
&\quad + \int_0^t \langle \bar{\mathfrak{v}}_n(r) - \bar{\mathfrak{v}}_n(r_m), \varphi Y(u_h(r))(\dot{h}_n(r) - \dot{h}(r))\rangle_{H^1(B_{2R})}\, dr \\
&\quad + \int_0^t \langle \bar{\mathfrak{v}}_n(r_m), \varphi(Y(u_h(r)) - Y(u_h(r_m)))(\dot{h}_n(r) - \dot{h}(r))\rangle_{H^1(B_{2R})}\, dr \\
&\quad + \int_0^t \langle \bar{\mathfrak{v}}_n(r_m), \varphi Y(u_h(r_m))(\dot{h}_n(r) - \dot{h}(r))\rangle_{H^1(B_{2R})}\, dr \\
&=: G_1(t) + G_2(t) + G_3(t) + G_4(t).
\end{aligned} \tag{4.4.20}$$

For G_1, since $T - r > 2R$, Lemmata 4.2.2, 4.4.4 and 4.4.9 followed by (4.4.14) implies

$$\begin{aligned}
|G_1(t)| &\lesssim_\varphi \int_0^t \|\bar{\mathfrak{v}}_n(r)\|^2_{H^1(B_{2R})}\, dr + \int_0^t \|Y(u_n(r)) - Y(u_h(r))\|^2_{H^1(B_{2R})} \|\dot{h}_n(r)\|^2_{H_\mu}\, dr \\
&\lesssim_R \int_0^t \|\bar{\mathfrak{v}}_n(r)\|^2_{H^1(B_{2R})}\, dr \\
&\quad + \int_0^t \|u_n(r) - u_h(r)\|^2_{H^1(B_{2R})} \left(1 + \|u_n(r)\|^2_{H^1(B_{2R})} + \|u_h(r)\|^2_{H^1(B_{2R})}\right) \|\dot{h}_n(r)\|^2_{H_\mu}\, dr \\
&\lesssim_{\mathcal{B}} \int_0^t (1 + l(r, \mathfrak{z}_n(r)))\left(1 + \|\dot{h}_n(r)\|^2_{H_\mu}\right) dr.
\end{aligned} \tag{4.4.21}$$

To estimate $G_2(t)$ we invoke $\langle h, k\rangle_{H^1(B_{2R})} \leq \|h\|_{L^2(B_{2R})} \|k\|_{H^2(2R))}$ followed by the Hölder inequality and Lemmata 4.2.2, 4.4.4, 4.4.9 and 4.4.13 to get

$$\begin{aligned}
|G_2(t)| &\lesssim_{R,\varphi} \int_0^t \|\mathfrak{v}_n(r) - \mathfrak{v}_n(r_m)\|_{L^2(B_{2R})} \|Y(u_h(r))\|_{H^2(B_{2R})} \|\dot{h}_n(r) - \dot{h}(r)\|_{H_\mu}\, dr \\
&\lesssim_R \left(\int_0^t \|\mathfrak{v}_n(r) - \mathfrak{v}_n(r_m)\|^2_{L^2(B_{2R})}\, dr\right)^{\frac{1}{2}} \\
&\quad \times \left(\int_0^t \|u_h(r)\|^2_{H^2(B_{2R})}\left[1 + \|u_h(r)\|^2_{H^2(B_{2R})}\right] \|\dot{h}_n(r) - \dot{h}(r)\|^2_{H_\mu}\, dr\right)^{\frac{1}{2}}
\end{aligned}$$

$$\lesssim \sqrt{M_\mu}\left(\int_0^t |r-r_m|\,dr\right)^{\frac12} \sup_{r\in[0,T/2]} \|u_h(r)\|^2_{H^2(B_{T-r})}\left[1+\|u_h(r)\|^2_{H^2(B_{T-r})}\right]$$

$$\lesssim \frac{R\sqrt{M_\mu}}{2^{m/2}} \sup_{r\in[0,T/2]} l(r,z_h(r))\,[1+l(r,z_h(r))] \le \frac{R\sqrt{M_\mu}}{2^{m/2}}\,\mathcal{B}(1+\mathcal{B}), \tag{4.4.22}$$

where in the second last step we have used

$$\left(\int_0^t |r-r_m|\,dr\right)^{\frac12} \le \left(\int_0^R |r-r_m|\,dr\right)^{\frac12} = \left(\sum_{k=1}^{2^m}\int_{t_{k-1}}^{t_k}\left|r-\frac{kR}{2^m}\right|dr\right)^{\frac12} \le \frac{R}{2^{m/2}}.$$

Moreover, in the third last step we have also applied the following: since $\dot h_n \to \dot h$ weakly in $L^2(0,T;H_\mu)$, the sequence $\dot h_n - \dot h$ is bounded in $L^2(0,T;H_\mu)$ i.e. $\exists M_\mu>0$ such that

$$\int_0^t \|\dot h_n(r)-\dot h(r)\|^2_{H_\mu}\,dr \le M_\mu, \quad \forall n. \tag{4.4.23}$$

Before moving to $G_3(t)$ note that, since $2R=T/2$, due to Remark 4.4.10, for every $s,t\in[0,T/2]$,

$$\|u_h(t)-u_h(s)\|_{H^1(B_{2R})} \le \int_s^t \|v_h(r)\|_{H^1(B_{2R})}\,dr \lesssim \sqrt{\mathcal{B}}\,|t-s|.$$

Consequently, by the Hölder inequality followed by Lemmata 4.2.2, 4.4.13 and 4.4.4 we obtain

$$\begin{aligned}
|G_3(t)| &\lesssim_\varphi \left(\int_0^t \left[\|v_n(r_m)\|^2_{H^1(B_{2R})}+\|v_h(r_m)\|^2_{H^1(B_{2R})}\right]dr\right)^{\frac12}\\
&\qquad\times\left(\int_0^t \|Y(u_h(r))-Y(u_h(r_m))\|^2_{H^1(B_{2R})}\|\dot h_n(r)-\dot h(r))\|^2_{H_\mu}\,dr\right)^{\frac12}\\
&\lesssim_{T,\mathcal{B}} \left(\int_0^t \|u_h(r)-u_h(r_m)\|^2_{H^1(B_{2R})}\left[1+\|u_h(r)\|^2_{H^1(B_{2R})}+\|u_h(r_m)\|^2_{H^1(B_{2R})}\right]\|\dot h_n(r)-\dot h(r)\|^2_{H_\mu}\,dr\right)^{\frac12}\\
&\lesssim_{T,\mathcal{B}} \left(\int_0^t |r-r_m|\,\|\dot h_n(r)-\dot h(r)\|^2_{H_\mu}\,dr\right)^{\frac12}\left(\sum_{k=1}^{2^m}\int_{t_{k-1}}^{t_k}\left|r-\frac{kR}{2^m}\right|\,\|\dot h_n(r)-\dot h(r)\|^2_{H_\mu}\,dr\right)^{\frac12}\\
&\le \sqrt{\frac{R}{2^m}}\left(\int_0^t \|\dot h_n(r)-\dot h(r)\|^2_{H_\mu}\,dr\right)^{\frac12} \le \sqrt{T\frac{M_\mu}{2^m}}, \quad t\in[0,T].
\end{aligned}$$

Finally, we start estimating $G_4(t)$ by noting that for every $t\in[0,R]$,

$$\text{there exists}\quad k_t\le 2^m \qquad \text{such that} \qquad t\in\left[\frac{(k_t-1)\cdot R}{2^m},\frac{k_t\cdot R}{2^m}\right).$$

Note that on such interval $r_m=\frac{k_t\cdot R}{2^m}$. Then by Lemma 4.4.9 we have

$$\begin{aligned}
|G_4(t)| &\le \Big|\sum_{k=1}^{k_t-1}\int_{t_{k-1}}^{t_k}\left\langle \mathfrak{v}_n\left(\frac{k\cdot R}{2^m}\right),\varphi Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)(\dot h_n(r)-\dot h(r))\right\rangle_{H^1(B_{2R})}dr\\
&\qquad+\int_{t_{k_t-1}}^t\left\langle \mathfrak{v}_n\left(\frac{(k_t-1)\cdot R}{2^m}\right),\varphi Y\left(u_h\left(\frac{(k_t-1)\cdot R}{2^m}\right)\right)(\dot h_n(r)-\dot h(r))\right\rangle_{H^1(B_{2R})}dr\Big|\\
&\le \sum_{k=1}^{2^m}\Big|\left\langle \mathfrak{v}_n\left(\frac{k\cdot R}{2^m}\right),\varphi Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k}(\dot h_n(r)-\dot h(r))\,dr\right\rangle_{H^1(B_{2R})}\Big|
\end{aligned}$$

$$
\begin{aligned}
&+ \sup_{1\le k\le 2^m} \sup_{t_k\le t\le t_{k-1}} \left| \left\langle \bar{\mathfrak{v}}_n\left(\frac{(k-1)\cdot R}{2^m}\right), \varphi Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t} (\dot{h}_n(r)-\dot{h}(r))\,dr \right\rangle_{H^1(B_{2R})}\right| \\
&\le \sum_{k=1}^{2^m} \left\| \bar{\mathfrak{v}}_n\left(\frac{k\cdot R}{2^m}\right)\right\|_{H^1(B_{2R})} \left\| \varphi Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k} (\dot{h}_n(r)-\dot{h}(r))\,dr\right\|_{H^1(B_{2R})} \\
&+ \sup_{1\le k\le 2^m} \sup_{t_k\le t\le t_{k-1}} \left\| \bar{\mathfrak{v}}_n\left(\frac{(k-1)\cdot R}{2^m}\right)\right\|_{H^1(B_{2R})} \left\| \varphi Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t} (\dot{h}_n(r)-\dot{h}(r))\,dr\right\|_{H^1(B_{2R})} \\
&\lesssim_{\varphi,\mathcal{B}} \sum_{k=1}^{2^m} \left\| Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k} (\dot{h}_n(r)-\dot{h}(r))\,dr\right\|_{H^1(B_{2R})} \\
&+ \sup_{1\le k\le 2^m} \sup_{t_k\le t\le t_{k-1}} \left\| Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t} (\dot{h}_n(r)-\dot{h}(r))\,dr\right\|_{H^1(B_{2R})} \\
&=: G_4^1 + G_4^2,
\end{aligned}
$$

where the right hand side does not depend on t. By invoking Lemmata 4.2.2, 4.4.4, the Hölder inequality, and Lemma 4.4.9 we estimate G_4^1 as

$$
\begin{aligned}
G_4^1 &\lesssim_{R,T} \sup_{1\le k\le 2^m} \sup_{t_k\le t\le t_{k-1}} \left\| Y\left(u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right)\right\|_{H^1(B_{2R})} \left(\int_{t_{k-1}}^{t} \|\dot{h}_n(r)-\dot{h}(r)\|_{H_\mu}^2\,dr\right)^{\frac{1}{2}} \\
&\lesssim_{R,T} \sup_{1\le k\le 2^m} \sup_{t_k\le t\le t_{k-1}} \left[1+\left\| u_h\left(\frac{(k-1)\cdot R}{2^m}\right)\right\|_{H^1(B_{2R})}\right] \left(\int_{t_{k-1}}^{t} \|\dot{h}_n(r)-\dot{h}(r)\|_{H_\mu}^2\,dr\right)^{\frac{1}{2}} \\
&\lesssim_{R,T,\mathcal{B}} \sup_{1\le k\le 2^m} \sup_{t_k\le t\le t_{k-1}} \left(\int_{t_{k-1}}^{t} \|\dot{h}_n(r)-\dot{h}(r)\|_{H_\mu}^2\,dr\right)^{\frac{1}{2}} \\
&\le \sup_{1\le k\le 2^m} \left(\int_{t_{k-1}}^{t_k} \|\dot{h}_n(r)-\dot{h}(r)\|_{H_\mu}^2\,dr\right)^{\frac{1}{2}}. \qquad (4.4.24)
\end{aligned}
$$

For G_4^2 recall that, by Lemma 4.2.2, for every $\phi \in H^1(B(x,r))$ the multiplication operator

$$
Y(\phi)\cdot : K \ni k \mapsto Y(\phi)\cdot k \in H^1(B(x,r)),
$$

is γ-radonifying and hence compact. So Lemma 4.4.12 implies the following, for every k,

$$
\left\| Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k} (\dot{h}_n(r)-\dot{h}(r))\,dr\right\|_{H^1(B_{2R})} \to 0 \text{ as } n\to 0. \qquad (4.4.25)
$$

Hence, for fix m, each term of the sum in G_4^2 goes to 0 as $n\to\infty$. Consequently, by substituting the computation between (4.4.21) and (4.4.24) into (4.4.20) we obtain

$$
\begin{aligned}
\int_0^t \langle \bar{\mathfrak{v}}_n(r), \bar{g}_n(r)\rangle_{H^1(B_{2R})}\,dr &\lesssim_{R,L_A,B_A,\varphi,\mathcal{B}} \int_0^t (1+l(r,\mathfrak{z}_n(r)))\left(1+\|\dot{h}_n(r)\|_{H_\mu}^2\right)dr \\
&+ \sqrt{T\frac{M_\mu}{2^m}} + \sup_{1\le k\le 2^m}\left(\int_{t_{k-1}}^{t_k} \|\dot{h}_n(r)-\dot{h}(r)\|_{H_\mu}^2\,dr\right)^{\frac{1}{2}} \\
&+ \sum_{k=1}^{2^m} \left\| Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k} (\dot{h}_n(r)-\dot{h}(r))\,dr\right\|_{H^1(B_{2R})}, \quad t\in[0,T].
\end{aligned}
$$

Therefore, with (4.4.19) and (4.4.14), from (4.4.17) we have

$$
\begin{aligned}
l(t,\mathfrak{z}_n(t)) \lesssim & \int_0^t (1+l(r,\mathfrak{z}_n(r)))\left(1+\|\dot{h}_n(r)\|_{H_\mu}^2\right) dr \\
& + \sqrt{T\frac{M_\mu}{2^m}} + \sup_{1\le k\le 2^m}\left(\int_{t_{k-1}}^{t_k} \|\dot{h}_n(r)-\dot{h}(r)\|_{H_\mu}^2\, dr\right)^{\frac12} \\
& + \sum_{k=1}^{2^m}\left\| Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k} (\dot{h}_n(r)-\dot{h}(r))\, dr\right\|_{H^1(B_{2R})}, \quad t\in[0,T],
\end{aligned}
$$

and by the Gronwall Lemma, with the observation that all the terms in right hand side except the first are independent of t, and $h_n \in S_{\mathcal{M}}$ further we get

$$
\begin{aligned}
\sup_{t\in[0,R]} l(t,\mathfrak{z}_n(t)) \lesssim e^{T+\mathcal{M}} \Bigg\{ & \sqrt{T\frac{M_\mu}{2^m}} + \sup_{1\le k\le 2^m}\left(\int_{t_{k-1}}^{t_k} \|\dot{h}_n(r)-\dot{h}(r)\|_K^2\, dr\right)^{\frac12} \\
& + \sum_{k=1}^{2^m}\left\| Y\left(u_h\left(\frac{k\cdot R}{2^m}\right)\right)\int_{t_{k-1}}^{t_k} (\dot{h}_n(r)-\dot{h}(r))\, dr\right\|_{H^1(B_{2R})}\Bigg\}.
\end{aligned}
\tag{4.4.26}
$$

Now by [137, Theorem 6.11], for every $\alpha>0$ we can choose m such that

$$
\sup_{1\le k\le 2^m}\left(\int_{t_{k-1}}^{t_k} \|\dot{h}_n(r)-\dot{h}(r)\|_{H_\mu}^2\, dr\right)^{\frac12} + \sqrt{T\frac{M_\mu}{2^m}} < \alpha
$$

and for such chosen m, due to (4.4.25) by taking $n\to\infty$ in (4.4.26) we conclude that, for every $\alpha>0$,

$$
0 \le \lim_{n\to\infty}\sup_{t\in[0,R]} l(t,\mathfrak{z}_n(t)) < \alpha.
$$

Therefore, due to (4.4.14) we get (4.4.16) and hence the Proposition 4.4.11. ■

Now we come back to the proof of Statement 1. Previous proposition shows, for fix $T>0$, that every sequence in $K_{\mathcal{M}}$ has a convergent subsequence. Hence $K_{\mathcal{M}}$ is sequentially relatively compact subset of $\mathcal{X}_T$. Let $\{z_n\}_{n\in\mathbb{N}} \subset K_{\mathcal{M}}$ which converges to $z\in\mathcal{X}_T$. But Proposition 4.4.11 shows that there exists a subsequence $\{z_{n_k}\}_{k\in\mathbb{N}}$ which converges to some element z_h of $K_{\mathcal{M}}$ in the strong topology of $\mathcal{X}_T$. Hence $z=z_h$ and $K_{\mathcal{M}}$ is a closed subset of $\mathcal{X}_T$. This completes the proof of Statement 1.

Below is a basic result that we have used in the proof of previous proposition.

Lemma 4.4.12. *Let X, Y be separable Hilbert spaces such that the embedding $i: X\to Y$ is compact. If $g_n \to g$ weakly in $L^2(0,T;X)$, then*

$$
i\int_0^{\cdot} g_n(s)\, ds - i\int_0^{\cdot} g(s)\, ds \to 0 \text{ as } n\to\infty \quad \text{in} \quad \mathcal{C}([0,T];Y).
$$

Proof of Lemma 4.4.12 Define $G_n : [0,T] \ni t \mapsto \int_0^t g_n(s)\, ds \in X$. Then the sequence of functions $\{G_n\}_{n\in\mathbb{N}} \subset \mathcal{C}([0,T];X)$. Next, since weakly convergence sequence is bounded, the Hölder inequality gives

$$
\|G_n(t_2)-G_n(t_1)\|_X \le \int_{t_1}^{t_2} \|g_n(s)\|_X\, ds \le |t_2-t_1|^{\frac12}\left(\int_0^T \|g_n(s)\|_X^2\, ds\right) \le C_g|t_2-t_1|^{\frac12},
$$

for some $C_g > 0$. So the sequence $\{G_n\}_{n\in\mathbb{N}}$ is equicontinuous and uniformly bounded on $[0, T]$. Hence, $\{G_n\}_{n\in\mathbb{N}}$ is a bounded subset of $L^2(0, T; X)$ because $\mathcal{C}([0, T]; X) \subset L^2(0, T; X)$. Consequently, since the embedding $X \xhookrightarrow{i} Y$ is compact, due to Dubinsky Theorem [158, Theorem 4.1, p. 132], $\{iG_n\}_{n\in\mathbb{N}}$ is relatively compact in $\mathcal{C}([0, T]; Y)$, where $iG_n : [0, T] \ni t \mapsto i(G_n(t)) \in Y$. Therefore, there exists a subsequence, which we again indexed by $n \in \mathbb{N}$, $\{iG_n\}_{n\in\mathbb{N}}$ and $F \in \mathcal{C}([0, T]; Y)$ such that $iG_n \to F$, as $n \to \infty$, in $\mathcal{C}([0, T]; Y)$. This implies, for each $t \in [0, T]$, $G_n(t) \to F(t)$ in Y.

On the other hand, by weak convergence of g_n to g, we have, for every $x \in X$ and $t \in [0, T]$,

$$\begin{aligned}\langle G_n(t), x\rangle_X &= \int_0^T \langle g_n(s), x\mathbb{1}_{[0,t]}(s)\rangle_X \, ds = \langle g_n, x\mathbb{1}_{[0,t]}\rangle_{L^2(0,T;X)} \\ &\xrightarrow[n\to\infty]{} \langle g, x\mathbb{1}_{[0,t]}\rangle_{L^2(0,T;X)} = \langle G(t), x\rangle_X.\end{aligned}$$

Hence, for each $t \in [0, T]$, $\{G_n(t)\}_{n\in\mathbb{N}}$ is weakly convergent to $G(t)$ in X. Since $X \xhookrightarrow{i} Y$ is compact, $\{i(G_n(t))\}_{n\in\mathbb{N}}$ strongly converges to $i(G(t))$ in Y. So by the uniqueness of limit in Y, $i(G(t)) = F(t)$ for $t \in [0, T]$ and we have proved that every weakly convergent sequence $\{g_n\}_{n\in\mathbb{N}}$ has a subsequence, indexed again by $n \in \mathbb{N}$, such that $i\int_0^{\cdot} g_n(s)\, ds$ converges to $i\int_0^{\cdot} g(s)\, ds$ in $\mathcal{C}([0, T]; Y)$.

Since the same argument proves that from every weakly convergent subsequence in $L^2(0, T; X)$ we can extract a subsubsequence such that the last statement about convergence holds, we have proved the Lemma 4.4.12. ■

The following Lemma is about the Lipschitz property of the difference of solutions that we have used in proving Proposition 4.4.11.

Lemma 4.4.13. *Let $h_n, h \in S_{\mathcal{M}}$ and $I = [-a, a]$. There exists a positive constant $C := C(R, \mathcal{B}, \mathcal{M}, a)$ such that for $t, s \in [0, T/2]$ the following holds*

$$\sup_{x\in I} \|\mathfrak{v}_n(t) - \mathfrak{v}_n(s)\|_{L^2(B(x,R))} \lesssim C\,|t-s|^{\frac{1}{2}}, \tag{4.4.27}$$

for $R = T/2$, where $\mathfrak{v}_n$ is defined just after (4.4.11).

Proof of Lemma 4.4.13 Due to triangle inequality it is sufficient to show

$$\sup_{x\in I} \| v_h(t) - v_h(s)\|_{L^2(B(x,R))} \lesssim C|t-s|^{\frac{1}{2}}, \qquad t, s \in [0, T/2].$$

From the proof of existence part in Theorem 4.3.1 we have, for $t, s \in [0, T/2]$,

$$\begin{aligned}\| v_h(t) - v_h(s)\|_{L^2(B(x,R))} \leq & \int_s^t \|\partial_{xx} u_h(r)\|_{L^2(B(x,R))}\, dr \\ & + \int_s^t \left[\|f_h(r)\|_{L^2 B(x,R))} + \|g_h(r)\|_{L^2(B(x,R))}\right] dr,\end{aligned} \tag{4.4.28}$$

where

$$f_h(r) := A_{u_h(r)}(v_h(r), v_h(r)) - A_{u_h(r)}(\partial_x u_h(r), \partial_x u_h(r)), \quad \text{and} \quad g_h(r) := Y(u_h(r))\dot{h}(r).$$

But, since $h \in S_{\mathcal{M}}$, the Hölder inequality followed by Lemmata 4.2.2, 4.4.4 and 4.4.9 yields

$$\begin{aligned}\sup_{x\in I}\int_s^t \|g_h(r)\|_{L^2(B(x,R))}\,dr &\le |t-s|^{\frac{1}{2}}\left(\int_s^t \sup_{x\in I}\|Y(u_h(r))\|^2_{L^2(B(x,R))}\|\dot h(s)\|^2_{H_\mu}\,ds\right)^{\frac{1}{2}}\\ &\lesssim_{R,\mathcal{B},M} |t-s|^{\frac{1}{2}}, \qquad \text{for} \quad t,s\in[0,T/2],\end{aligned}$$

and, based on (4.4.18), we also have

$$\begin{aligned}\sup_{x\in I}\int_s^t \|f_h(r)\|_{L^2(B(x,R))}\,dr &\le |t-s|^{\frac{1}{2}}\left(\int_s^t \sup_{x\in I}\|A_{u_h(r)}(v_h(r),v_h(r))\|^2_{L^2(B(x,R))}\,dr\right)^{\frac{1}{2}}\\ &\quad + |t-s|^{\frac{1}{2}}\left(\int_s^t \sup_{x\in I}\|A_{u_h(s)}(\partial_x u_h(r),\partial_x u_h(r))\|^2_{L^2(B(x,R))}\,dr\right)^{\frac{1}{2}}\\ &\lesssim |t-s|^{\frac{1}{2}}\left(\int_s^t \sup_{x\in I}\|u_h(r)\|^2_{L^2(B(x,R))}\{\|v_h(s)\|^4_{L^2(B(x,R))}+\|\partial_x u_h(s)\|^4_{L^2(B(x,R))}\}\,ds\right)^{\frac{1}{2}}\\ &\lesssim |t-s|^{\frac{1}{2}}\,\mathcal{B}^{\frac{3}{2}}, \qquad \text{for} \quad t,s\in[0,T/2].\end{aligned}$$

Finally, by the Hölder inequality and Lemma 4.4.9, we obtain, for $t,s\in[0,T/2]$,

$$\begin{aligned}\sup_{x\in I}\int_s^t \|\partial_{xx}u_h(r)\|_{L^2(B(x,R))}\,dr &\le \left(\int_s^t 1\,dr\right)^{\frac{1}{2}}\left(\int_s^t \sup_{x\in I}\|u_h(r)\|^2_{H^2(B(x,R))}\,dr\right)^{\frac{1}{2}}\\ &\lesssim \sqrt{\mathcal{B}}\,|t-s|^{\frac{1}{2}}.\end{aligned}$$

Therefore, by collecting the estimates in (4.4.28) we get the required inequality (4.4.27) and we are done with the proof of Lemma 4.4.13. ∎

4.4.3 Proof of Statement 2

It will be useful to introduce the following notation for the processes

$$Z_n := (U_n,V_n) = J^{\varepsilon_n}\left(W+\frac{1}{\sqrt{\varepsilon_n}}h_n\right), \qquad z_n := (u_n,v_n) = J^0(h_n).$$

Let us fix any $a>0$ and $T>0$. Then set N a natural number such that

$$N > \|(u_0,v_0)\|_{\mathcal{H}(B(0,a+T))}.$$

For each $n\in\mathbb{N}$ we define an $\mathscr{F}_t$-stopping time

$$\tau_n(\omega) := \inf\{t>0 : \sup_{x\in[-a,a]}\|Z_n(t,\omega)\|_{\mathcal{H}(B(x,T-t))} \ge N\}\wedge T, \quad \omega\in\Omega. \tag{4.4.29}$$

Define, for $z=(u,v)\in\mathcal{H}_{\text{loc}}$,

$$\mathbf{e}(t,x,z) := \frac{1}{2}\left\{\|u\|^2_{H^2(B(x,T-t))}+\|v\|^2_{H^1(B(x,T-t))}\right\} = \frac{1}{2}\|z\|^2_{\mathcal{H}(B(x,T-t))}, \quad x\in\mathbb{R},\ t\in[0,T]. \tag{4.4.30}$$

In this framework we prove the following key result.

Proposition 4.4.14. *Let us define* $\mathcal{Z}_n := Z_n - z_n$*. For* τ_n *defined in* (4.4.29) *we have*

$$\lim_{n\to\infty} \sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{t\in[0,T/2]} \mathbf{e}(t\wedge\tau_n, x, \mathcal{Z}_n(t\wedge\tau_n))\right] = 0.$$

Proof of Proposition 4.4.14 Let us fix any $n \in \mathbb{N}$. To avoid complexity of notation we use an abuse of notation and write all the norms without reference of the centre of the ball x. First note that under our notation $Z_n = (U_n, V_n)$ and $z_n = (u_n, v_n)$, respectively, are the unique global strong solutions to the Cauchy problem

$$\begin{cases} \partial_{tt}U_n = \partial_{xx}U_n + A_{U_n}(\partial_t U_n, \partial_t U_n) - A_{U_n}(\partial_x U_n, \partial_x U_n) + Y(U_n)\dot{h}_n, \\ \qquad\qquad + \sqrt{\varepsilon_n}\, Y(U_n)\dot{W}, \\ (U_n(0), \partial_t U_n(0)) = (u_0, v_0), \quad \text{where } V_n := \partial_t U_n, \end{cases}$$

and

$$\begin{cases} \partial_{tt}u_n = \partial_{xx}u_n + A_{u_n}(\partial_t u_n, \partial_t u_n) - A_{u_n}(\partial_x u_n, \partial_x u_n) + Y(u_n)\dot{h}_n, \\ (u_n(0), \partial_t u_n(0)) = (u_0, v_0), \quad \text{where } v_n := \partial_t u_n. \end{cases}$$

Hence $\mathcal{Z}_n$ solves uniquely the Cauchy problem, with null initial data,

$$\begin{aligned} \partial_{tt}\mathcal{U}_n = \partial_{xx}\mathcal{U}_n &- A_{U_n}(\partial_x U_n, \partial_x U_n) + A_{u_n}(\partial_x u_n, \partial_x u_n) + A_{U_n}(\partial_t U_n, \partial_t U_n) \\ &- A_{u_n}(\partial_t u_n, \partial_t u_n) + Y(U_n)\dot{h}_n - Y(u_n)\dot{h}_n + \sqrt{\varepsilon_n}\, Y(U_n)\dot{W}, \end{aligned}$$

where $\mathcal{V}_n := \partial_t \mathcal{U}_n$. This is equivalent to say, for all $t \in [0, T/2]$,

$$\mathcal{Z}_n(t) = \int_0^t S_{t-s}\begin{pmatrix} 0 \\ f_n(s) \end{pmatrix} ds + \int_0^t S_{t-s}\begin{pmatrix} 0 \\ g_n(s) \end{pmatrix} dW(s). \tag{4.4.31}$$

Here

$$\begin{aligned} f_n(s) := &-A_{U_n(s)}(\partial_x U_n(s), \partial_x U_n(s)) + A_{u_n(s)}(\partial_x u_n(s), \partial_x u_n(s)) + A_{U_n(s)}(V_n(s), V_n(s)) \\ &- A_{u_n(s)}(v_n(s), v_n(s)) + Y(U_n(s))\dot{h}_n(s) - Y(u_n(s))\dot{h}_n(s), \end{aligned}$$

and

$$g_n(s) := \sqrt{\varepsilon_n}\, Y(U_n(s)).$$

Invoking Proposition 4.5.2, with $k = 1, L = I$, implies for every $t \in [0, T/2]$ and $x \in [-a, a]$,

$$\begin{aligned} \mathbf{e}(t, x, \mathcal{Z}_n(t)) \leq &\int_0^t \mathbb{V}(r, \mathcal{Z}_n(r))\, dr + \int_0^t \langle \mathcal{V}_n(r), g_n(r)\, dW(r)\rangle_{L^2(B_{T-r})} \\ &+ \int_0^t \langle \partial_x \mathcal{V}_n(r), \partial_x [g_n(r)\, dW(r)]\rangle_{L^2(B_{T-r})}, \end{aligned} \tag{4.4.32}$$

with

$$\mathbb{V}(t, \mathcal{Z}_n(t)) = \langle \mathcal{U}_n(t), \mathcal{V}_n(t)\rangle_{L^2(B_{T-t})} + \langle \mathcal{V}_n(t), f_n(t)\rangle_{L^2(B_{T-t})}$$

$$+\langle \partial_x \mathcal{V}_n(t), \partial_x f_n(t)\rangle_{L^2(B_{T-t})} + \frac{1}{2}\sum_{j=1}^{\infty} \|g_n(t)e_j\|^2_{L^2(B_{T-t})} + \frac{1}{2}\sum_{j=1}^{\infty} \|\partial_x[g_n(t)e_j]\|^2_{L^2(B_{T-t})},$$

for a given sequence $\{e_j\}_{j\in\mathbb{N}}$ of orthonormal basis of H_μ. Observe that, for any $\tau \in [0, T/2]$, by the Cauchy-Schwartz inequality

$$\begin{aligned}\sup_{0\le t\le \tau} \int_0^{t\wedge\tau_n} \mathbb{V}(r, \mathcal{Z}_n(r))\,dr \le{}& 2\int_0^{\tau\wedge\tau_n} \mathbf{e}(r, x, \mathcal{Z}_n(r))\,dr \\ &+ \frac{1}{2}\int_0^{\tau\wedge\tau_n} \left(\|f_n(r)\|^2_{H^1(B_{T-r})} + \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(B_{T-r}))}\right) dr,\end{aligned} \tag{4.4.33}$$

where $g_n(r)\cdot$ denotes the multiplication operator in the space $\mathscr{L}_2(H_\mu, H^1(B(x,R)))$, see Lemma 4.2.2.

Next, we define the process

$$\mathcal{Y}(t) := \int_0^t \langle \mathcal{V}_n(r), g_n(r)dW(r)\rangle_{H^1(B_{T-r})}. \tag{4.4.34}$$

By taking $\int_0^t \xi(r)\,dW(r)$ with

$$\xi(r): H_\mu \ni k \mapsto \langle \mathcal{V}_n(r), g_n(r)(k)\rangle_{H^1(B_{T-r})} \in \mathbb{R},$$

a Hilbert-Schmidt operator, note that

$$\mathcal{Q}(t) := \int_0^t \xi(r)\circ\xi(r)^\star\,dr,$$

is quadratic variation of $\mathbb{R}$-valued martingale $\mathcal{Y}$. Thus

$$\begin{aligned}\mathcal{Q}(t) &\le \int_0^t \|\xi(r)\|_{\mathscr{L}_2(H_\mu,\mathbb{R})}\|\xi(r)^\star\|_{\mathscr{L}_2(\mathbb{R},H_\mu)}\,dr = \int_0^t \|\xi(r)\|^2_{\mathscr{L}_2(H_\mu,\mathbb{R})}\,dr \\ &= \int_0^t \sum_{j=1}^{\infty} |\xi(r)(e_j)|^2\,dr = \int_0^t \sum_{j=1}^{\infty} |\langle \mathcal{V}_n(r), g_n(r)(e_j)\rangle_{H^1(B_{T-r})}|^2\,dr, \quad t\in[0,T/2].\end{aligned} \tag{4.4.35}$$

On the other hand by the Cauchy-Schwartz inequality

$$\sum_{j=1}^{\infty} |\langle \mathcal{V}_n(r), g_n(r)(e_j)\rangle_{H^1(B_{T-r})}|^2 \le \|\mathcal{V}_n(r)\|^2_{H^1(B_{T-r})}\|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))}.$$

Therefore,

$$\mathcal{Q}(t) \le \int_0^t \|\mathcal{V}_n(r)\|^2_{H^1(B_{T-r})}\|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))}\,dr, \quad t\in[0,T/2]. \tag{4.4.36}$$

Invoking the Davis inequality with (4.4.36) followed by the Young inequality gives

$$\begin{aligned}\mathbb{E}\left[\sup_{0\le t\le\tau} |\mathcal{Y}(t\wedge\tau_n)|\right] &\le 3\mathbb{E}\left[\sqrt{\mathcal{Q}(\tau\wedge\tau_n)}\right] \\ &\le 3\mathbb{E}\left[\sup_{0\le t\le\tau\wedge\tau_n} \|\mathcal{V}_n(t\wedge\tau_n)\|_{H^1(T-t)}\left\{\int_0^{\tau\wedge\tau_n} \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))}\,dr\right\}^{\frac{1}{2}}\right] \\ &\le 3\mathbb{E}\left[\varepsilon\sup_{0\le t\le\tau\wedge\tau_n} \|\mathcal{V}_n(t)\|^2_{H^1(T-t)} + \frac{1}{4\varepsilon}\int_0^{\tau\wedge\tau_n} \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu,H^1(B_{T-r}))}\,dr\right]\end{aligned}$$

$$\leq 6\varepsilon\, \mathbb{E}\left[\sup_{0\leq t\leq \tau\wedge\tau_n} \mathbf{e}(t,x,\mathcal{Z}_n(t))\right] + \frac{3}{4\varepsilon}\mathbb{E}\left[\int_0^{\tau\wedge\tau_n} \|g_n(r)\cdot\|^2_{\mathcal{L}_2(H_\mu,H^1(B_{T-r}))}\,dr\right]. \tag{4.4.37}$$

By choosing ε such that $6\varepsilon = \frac{1}{2}$ and taking $\sup_{0\leq s\leq t}$ followed by expectation $\mathbb{E}$ on the both sides of (4.4.32) after evaluating it at $\tau\wedge\tau_n$ we obtain

$$\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_n} \mathbf{e}(s,x,\mathcal{Z}_n(s))\right] \leq \mathbb{E}\left[\sup_{0\leq s\leq t}\int_0^{s\wedge\tau_n} \mathbb{V}(r,\mathcal{Z}_n(r))\,dr\right] + \mathbb{E}\left[\sup_{0\leq s\leq t}\mathcal{Y}(s\wedge\tau_n)\right].$$

Consequently, using (4.4.33) and (4.4.37) we infer that

$$\begin{aligned}\mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_n} \mathbf{e}(s,x,\mathcal{Z}_n(s))\right] &\leq 4\mathbb{E}\left[\int_0^{t\wedge\tau_n} \mathbf{e}(r,x,\mathcal{Z}_n(r))\,dr\right] + \mathbb{E}\left[\int_0^{t\wedge\tau_n}\|f_n(r)\|^2_{H^1(B_{T-r})}\,dr\right]\\ &\quad + 19\mathbb{E}\left[\int_0^{t\wedge\tau_n}\|g_n(r)\cdot\|^2_{\mathcal{L}_2(H_\mu,H^1(B_{T-r}))}\,dr\right].\end{aligned} \tag{4.4.38}$$

Now since the Hilbert-Schmidt operator $g_n(r)\cdot$ is defined as

$$H_\mu \ni k \mapsto g_n(r)\cdot k \in H^1(B_{T-r}),$$

Lemmata 4.2.2 and 4.4.4 gives,

$$\begin{aligned}\sup_{x\in[-a,a]}\mathbb{E}\left[\int_0^{t\wedge\tau_n}\|g_n(r)\cdot\|^2_{\mathcal{L}_2(H_\mu,H^1(B_{T-r}))}\,dr\right] &\lesssim_T \mathbb{E}\left[\int_0^{t\wedge\tau_n}\|\sqrt{\varepsilon_n}\,Y(U_n(r))\|^2_{H^1(B_{T-r})}\,dr\right]\\ &\lesssim_T \varepsilon_n\,\mathbb{E}\left[\int_0^{t\wedge\tau_n}\left(1+\|U_n(r)\|^2_{H^1(B_{T-r})}\right)dr\right]\\ &\leq \varepsilon_n\,\mathbb{E}\left[\int_0^{t\wedge\tau_n}\left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)dr\right]\\ &\lesssim_T \varepsilon_n\,(1+N^2).\end{aligned} \tag{4.4.39}$$

Here we observe that the constant in inequality (4.4.39) does not depend on a due to Lemma 4.2.2. To estimate the terms involving f_n we have

$$\begin{aligned}\|f_n(r)\|^2_{H^1(B_{T-r})} &\lesssim \|A_{U_n(r)}(\partial_x U_n(r),\partial_x U_n(r)) - A_{u_n(r)}(\partial_x u_n(r),\partial_x u_n(r))\|^2_{H^1(B_{T-r})}\\ &\quad + \|A_{U_n(r)}(V_n(r),V_n(r)) - A_{u_n(r)}(v_n(r),v_n(r))\|^2_{H^1(B_{T-r})}\\ &\quad + \|Y(U_n(r))\dot{h}_n(r) - Y(u_n(r))\dot{h}_n(r)\|^2_{H^1(B_{T-r})}\\ &=: f_n^1 + f_n^2 + f_n^3.\end{aligned} \tag{4.4.40}$$

By doing the computation based on Lemmata 4.3.4 and 4.4.4 we obtain,

$$\begin{aligned}f_n^1 &\lesssim \|A_{U_n(r)}(\partial_x U_n(r),\partial_x U_n(r)) - A_{u_n(r)}(\partial_x U_n(r),\partial_x U_n(r))\|^2_{H^1(B_{T-r})}\\ &\quad + \|A_{u_n(r)}(\partial_x U_n(r),\partial_x U_n(r)) - A_{u_n(r)}(\partial_x u_n(r),\partial_x U_n(r))\|^2_{H^1(B_{T-r})}\\ &\quad + \|A_{u_n(r)}(\partial_x u_n(r),\partial_x U_n(r)) - A_{u_n(r)}(\partial_x u_n(r),\partial_x u_n(r))\|^2_{H^1(B_{T-r})}\\ &\lesssim_{T,x} \|U_n(r)-u_n(r)\|^2_{H^2(B_{T-r})}\left(1+\|\partial_x U_n(r)\|^2_{H^1(B_{T-r})} + \|\partial_x U_n(r)\|^2_{H^1(B_{T-r})}\right)\times\end{aligned}$$

$$
\begin{aligned}
&\times \left(1+\|u_n(r)\|^2_{H^2(B_{T-r})}\right)\\
&+\|u_n(r)\|^2_{H^2(B_{T-r})}\|\partial_x[U_n(r)-u_n(r)]\|^2_{H^1(B_{T-r})}\|\partial_x[u_n(r)]\|^2_{H^1(B_{T-r})}\\
&\lesssim \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\ \left[\left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left(1+\|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)+\|z_n(r)\|^4_{\mathcal{H}_{T-r}}\right],
\end{aligned}\tag{4.4.41}
$$

and, by similar calculations,

$$
f_n^2 \lesssim_{T,x}\ \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\ \left[\left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left(1+\|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)+\|z_n(r)\|^4_{\mathcal{H}_{T-r}}\right].\tag{4.4.42}
$$

Furthermore, Lemmata 4.4.4 and 4.2.2 implies

$$
\begin{aligned}
f_n^3 &\lesssim_{T,x} \|U_n(r)-u_n(r)\|^2_{H^1(B_{T-r})}\left[1+\|U_n(r)|^2_{H^1(B_{T-r})}+\|u_n(r)|^2_{H^1(B_{T-r})}\right]\|\dot{h}_n(r)\|^2_{H_\mu}\\
&\lesssim \|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\ \left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}+\|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\|\dot{h}_n(r)\|^2_{H_\mu}.
\end{aligned}\tag{4.4.43}
$$

Hence by substituting (4.4.41)-(4.4.43) in (4.4.40) we get

$$
\begin{aligned}
\|f_n(r)\|^2_{H^1(B_{T-r})} \lesssim_{T,x} &\|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\ \left[\left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left(1+\|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)+\|z_n(r)\|^4_{\mathcal{H}_{T-r}}\right]\\
&+\|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\ \left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}+\|z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\|\dot{h}_n(r)\|^2_{H_\mu},
\end{aligned}
$$

consequently, the definition of τ_n and Lemma 4.4.9 suggest

$$
\begin{aligned}
\mathbb{E}\left[\int_0^{t\wedge\tau_n}\|f_n(r)\|^2_{H^1(B_{T-r})}\,dr\right] &\lesssim \mathbb{E}\left[\int_0^{t\wedge\tau_n}\Big\{\|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\ \left[\left(1+N^2\right)\left(1+\mathcal{B}^2\right)+\mathcal{B}^4\right]\right.\\
&\qquad\left.+\|\mathcal{Z}_n(r)\|^2_{\mathcal{H}_{T-r}}\ \left(1+N^2+\mathcal{B}^2\right)\ \left(1+\mathcal{B}^2\right)\|\dot{h}_n(r)\|^2_{H_\mu}\Big\}\,dr\right]\\
&\lesssim \mathbb{E}\left[\int_0^{t\wedge\tau_n}\mathbf{e}(r,x,\mathcal{Z}_n(r))\ C_{N,\mathcal{B}}\left(1+\|\dot{h}_n(r)\|^2_{H_\mu}\right)dr\right],
\end{aligned}\tag{4.4.44}
$$

for some constant $C_{N,\mathcal{B}}>0$ depending on $N,\mathcal{B}$, where $\mathcal{B}$ is a function of x which is bounded on compact sets. Thus substitution of (4.4.39) and (4.4.44) in (4.4.38) implies

$$
\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n}\mathbf{e}(s,x,\mathcal{Z}_n(s))\right]\lesssim_{T,x}\varepsilon_n\ (1+N^2)+C_{N,\mathcal{B}}\mathbb{E}\left[\int_0^{t\wedge\tau_n}[\sup_{0\le s\le r\wedge\tau_n}\mathbf{e}(s,x,\mathcal{Z}_n(s))]\left(1+\|\dot{h}_n(r)\|^2_{H_\mu}\right)dr\right].
$$

Therefore, invoking the stochastic Gronwall Lemma, see [63, Lemma 3.9], gives,

$$
\sup_{x\in[-a,a]}\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n}\mathbf{e}(s,x,\mathcal{Z}_n(s))\right]\lesssim_{T,a}\varepsilon_n\ (1+N^2)\exp\left[C_{N,\mathcal{B}}(T+M)\right].\tag{4.4.45}
$$

Since $\varepsilon_n\to 0$ as $n\to\infty$ and

$$
\mathbb{E}\left[\sup_{0\le s\le t\wedge\tau_n}\mathbf{e}(s,x,\mathcal{Z}_n(s))\right]=\mathbb{E}\left[\sup_{0\le s\le t}\mathbf{e}(s\wedge\tau_n,x,\mathcal{Z}_n(s\wedge\tau_n))\right],
$$

inequality (4.4.45) give $\lim_{n\to\infty}\sup_{x\in[-a,a]}\mathbb{E}\left[\sup_{0\le t\le T/2}\mathbf{e}(t\wedge\tau_n,x,\mathcal{Z}_n(t\wedge\tau_n))\right]=0$. Hence, we are done with the proof of Proposition 4.4.14. ■

To proceed further we also need the following stochastic analogue of Lemma 4.4.9.

Lemma 4.4.15. *There exists a constant* $\mathscr{B} := \mathscr{B}(N, T, \mathcal{M}) > 0$ *such that*

$$\limsup_{n\to\infty} \sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{t\in[0,T/2]} \mathbf{e}(t\wedge\tau_n, x, Z_n(t\wedge\tau_n))\right] \leq \mathscr{B}.$$

Proof of Lemma 4.4.15 Let us fix sequence $\{e_j\}_{j\in\mathbb{N}}$ of orthonormal basis of H_μ. Let us also fix any $n\in\mathbb{N}$. With the notation of this subsection, Proposition 4.5.2, with $k=1, L=I$, implies, for every $t\in[0,T/2]$ and $x\in[-a,a]$,

$$\mathbf{e}(t,x,Z_n(t)) \leq \int_0^t \mathbb{V}(r, Z_n(r))\,dr + \int_0^t \langle V_n(r), g_n(r)dW(r)\rangle_{H^1(B_{T-r})},$$

with

$$\mathbb{V}(r, Z_n(r)) = \langle U_n(r), V_n(r)\rangle_{L^2(B_{T-r})} + \langle V_n(r), f_n(r)\rangle_{H^1(B_{T-r})} + \frac{1}{2}\sum_{j=1}^{\infty} \|g_n(r)e_j\|^2_{H^1(B_{T-r})},$$

and

$$\begin{aligned} f_n(r) &:= A_{U_n(r)}(V_n(r), V_n(r)) - A_{U_n(r)}(\partial_x U_n(r), \partial_x U_n(r)) + Y(U_n(r))\dot{h}_n(r),\\ g_n(r) &:= \sqrt{\varepsilon_n}\, Y(U_n(r)). \end{aligned}$$

Next, we intent to follow the procedure of Proposition 4.4.14. By the Cauchy-Schwartz inequality, for $\tau\in[0,T/2]$ and $x\in[-a,a]$, we have

$$\begin{aligned} \sup_{0\leq t\leq\tau}\int_0^{t\wedge\tau_n} \mathbb{V}(r, Z_n(r))\,dr &\leq 2\int_0^{\tau\wedge\tau_n} \mathbf{e}(r, Z_n(r))\,dr \\ &\quad + \frac{1}{2}\int_0^{\tau\wedge\tau_n}\left(\|f_n(r)\|^2_{H^1(B_{T-r})} + \|g_n(r)\cdot\|^2_{\mathscr{L}_2(H_\mu, H^1(B_{T-r}))}\right)dr. \end{aligned}$$

Since the g_n here is same as in Proposition 4.4.14, the computation of (4.4.34)-(4.4.39) fits here too and we have

$$\begin{aligned} \mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_n} \mathbf{e}(s,x,Z_n(s))\right] &\lesssim_T \mathbb{E}\left[\int_0^{t\wedge\tau_n} \mathbf{e}(r,x,Z_n(r))\,dr\right] \\ &\quad + \mathbb{E}\left[\int_0^{t\wedge\tau_n} \|f_n(r)\|^2_{H^1(B_{T-r})}\,dr\right] + \varepsilon_n(1+N^2). \end{aligned} \tag{4.4.46}$$

Invoking Lemmata 4.2.2 and 4.4.4 implies

$$\begin{aligned} \|f_n(r)\|^2_{H^1(B_{T-r})} &\lesssim \|A_{U_n(r)}(\partial_x U_n(r), \partial_x U_n(r))\|^2_{H^1(B_{T-r})} + \|A_{U_n(r)}(V_n(r), V_n(r))\|^2_{H^1(B_{T-r})} \\ &\quad + \|Y(U_n(r))\dot{h}_n(r)\|^2_{H^1(B_{T-r})} \\ &\lesssim_{T,x} \left(1+\|U_n(r)\|^2_{H^1(B_{T-r})}\right)\left[1+\|\partial_x U_n(r)\|^2_{H^1(B_{T-r})} + \|V_n(r)\|^2_{H^1(B_{T-r})} + \|\dot{h}_n(r)\|^2_{H_\mu}\right] \\ &\lesssim \left(1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}}\right)\left[1+\|Z_n(r)\|^2_{\mathcal{H}_{T-r}} + \|\dot{h}_n(r)\|^2_{H_\mu}\right]. \end{aligned}$$

So from (4.4.46) and the definition (4.4.29) we get

$$\sup_{x\in[-a,a]} \mathbb{E}\left[\sup_{0\leq s\leq t\wedge\tau_n} \mathbf{e}(s,x,Z_n(s))\right] \lesssim_{T,a} N^2\mathbb{E}[t\wedge\tau_n] + \varepsilon_n(1+N^2)$$

$$+ (1+N^2)\mathbb{E}\left[\int_0^{t\wedge\tau_n}\left(1+N^2+\dot{h}_n(r)\|^2_{H_\mu}\right)dr\right]$$
$$\lesssim_T N^2T+(1+N^2)T+\mathcal{M}+\varepsilon_n(1+N^2).$$

Since $\lim\limits_{n\to\infty}\varepsilon_n=0$, taking $\limsup_{n\to\infty}$ on both the sides we get the required bound, and hence, the Lemma 4.4.15. ■

Lemma 4.4.16. *Given $\mathcal{T}>0$, the sequence of $\mathcal{X}_{\mathcal{T}}$-valued process $\{\mathcal{Z}_n\}_{n\in\mathbb{N}}$ converges in probability to* 0.

Proof of Lemma 4.4.16 Let us fix $T>0$ such that $\mathcal{T}=T/2$. We aim to show that for every $x\in\mathbb{R}$ and $R,\delta,\alpha>0$ there exists a natural number n_0 such that

$$\mathbb{P}\left[\omega\in\Omega:\sup_{t\in[0,\mathcal{T}]}\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,R)}}>\delta\right]<\alpha\quad\text{for all}\quad n\geq n_0. \tag{4.4.47}$$

Let us set $R=T$ and x,δ,α be any arbitrary. As a first step we show that, there exists $n_0\in\mathbb{N}$ (may depend on $x,\mathcal{T},\delta,\alpha$) such that

$$\mathbb{P}\left[\omega\in\Omega:\sup_{t\in[0,T/2]}\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,R)}}>\delta\right]<\alpha\quad\text{for all}\quad n\geq n_0. \tag{4.4.48}$$

Before moving further observe that, since $\|\cdot\|_{\mathcal{H}_{B(x,r)}}$ is increasing in r and for $t\in[0,T/2]$ we have $T-t\leq T=R$,

$$\{\omega\in\Omega:\sup_{t\in[0,T/2]}\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,R)}}>\delta\}\subseteq\{\omega\in\Omega:\sup_{t\in[0,T/2]}\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,T-t)}}>\delta\}. \tag{4.4.49}$$

Consequently

$$\mathbb{P}\left[\omega\in\Omega:\sup_{t\in[0,T/2]}\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,R)}}>\delta\right]\leq\mathbb{P}\left[\omega\in\Omega:\sup_{t\in[0,T/2]}\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,T-t)}}>\delta\right]. \tag{4.4.50}$$

Since x is fix in the argument now, there exists $a>0$ such that $x\in[-a,a]$. Further note that since $0\leq\mathbf{e}(t,\mathcal{Z}_n(t,\omega))=\frac{1}{2}\|\mathcal{Z}_n(t,\omega)\|^2_{\mathcal{H}_{B(x,T-t)}}$, due to (4.4.50) instead of showing (4.4.48) it is enough to show that there exists $n_0\in\mathbb{N}$ such that

$$\mathbb{P}\left[\omega\in\Omega:\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t,\omega))>\delta^2/2\right]<\alpha\quad\text{for all}\quad n\geq n_0. \tag{4.4.51}$$

But since $x\in[-a,a]$,

$$\mathbb{P}\left[\omega\in\Omega:\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t,\omega))>\delta^2/2\right]\leq\sup_{x\in[-a,a]}\mathbb{P}\left[\omega\in\Omega:\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t,\omega))>\delta^2/2\right].$$

Consequently instead of (4.4.51) it is sufficient to show that

$$\sup_{x\in[-a,a]}\mathbb{P}\left[\omega\in\Omega:\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t,\omega))>\delta^2/2\right]<\alpha\quad\text{for all}\quad n\geq n_0. \tag{4.4.52}$$

Now choose $N > \|(u_0, v_0)\|_{\mathcal{H}_{a+T}}$ such that, based on Lemma 4.4.15,

$$\frac{1}{N}\sup_{n\in\mathbb{N}}\sup_{x\in[-a,a]}\mathbb{E}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t\wedge\tau_n, x, Z_n(t\wedge\tau_n))\right] < \frac{\alpha}{2}, \tag{4.4.53}$$

and $n_0 \in \mathbb{N}$, due to Proposition 4.4.14,

$$\sup_{x\in[-a,a]}\mathbb{E}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t\wedge\tau_n, x, \mathcal{Z}_n(t\wedge\tau_n))\right] < \frac{\delta^2\alpha}{4} \text{ for all } n \geq n_0. \tag{4.4.54}$$

Then the Markov inequality followed by using of (4.4.53) and (4.4.54), for $n \geq n_0$, gives

$$\begin{aligned}
&\sup_{x\in[-a,a]}\mathbb{P}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t)) > \delta^2/2\right]\\
&\quad= \sup_{x\in[-a,a]}\mathbb{P}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t)) > \delta^2/2 \text{ and } \tau_n = T\right]\\
&\qquad+ \sup_{x\in[-a,a]}\mathbb{P}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t)) > \delta^2/2 \text{ and } \tau_n \neq T\right]\\
&\quad= \sup_{x\in[-a,a]}\mathbb{P}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t)) > \delta^2/2 \text{ and } \tau_n = T\right]\\
&\qquad+ \sup_{x\in[-a,a]}\mathbb{P}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t)) > \delta^2/2 \text{ and } \mathbf{e}(t, Z_n(t)) \geq N\right]\\
&\quad\leq \sup_{x\in[-a,a]}\mathbb{P}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t)) > \delta^2/2 \text{ and } \tau_n = T\right]\\
&\qquad+ \sup_{x\in[-a,a]}\mathbb{P}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,x,Z_n(t)) \geq N\right]\\
&\quad\leq \frac{2}{\delta^2}\sup_{x\in[-a,a]}\mathbb{E}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,x,\mathcal{Z}_n(t,\omega))\right]\\
&\qquad+ \frac{1}{N}\sup_{x\in[-a,a]}\mathbb{E}\left[\sup_{t\in[0,T/2]}\mathbf{e}(t,x,Z_n(t,\omega))\right] < \alpha.
\end{aligned} \tag{4.4.55}$$

Now we move to prove (4.4.47) when R is not set to T. Since the closure of $B(x,R)$ is compact and $B(x,R) \subset \cup_{y\in B(x,R)} B(y,T)$, we can find finitely many centre $\{x_i\}_{i=1}^m$ such that $B(x,R) \subset \cup_{i=1}^m B(x_i,T)$. Moreover, since $B(x,R)$ is bounded, there exists $a > 0$ such that $B(x,R) \in [-a,a]$. In particular, $x_i \in [-a,a]$ for all $i = 1,\dots,m$. Then since $\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,R)}} \leq \sum_{i=1}^m \|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x_i,T)}}$, we have

$$\{\omega\in\Omega : \sup_{t\in[0,T/2]}\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,R)}} \geq \delta\} \subset \{\omega\in\Omega : \sup_{t\in[0,T/2]}\sum_{i=1}^m \|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x_i,T)}} \geq \delta\} \tag{4.4.56}$$

Hence,

$$\sup_{x\in[-a,a]}\mathbb{P}\left[\omega\in\Omega : \sup_{t\in[0,T/2]}\|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,R)}} > \delta\right] \leq \sup_{x\in[-a,a]}\mathbb{P}\left[\omega\in\Omega : \sup_{t\in[0,T/2]}\sum_{i=1}^m \|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x_i,T)}} > \delta\right]$$

$$\leq \sum_{i=1}^{m} \sup_{x\in[-a,a]} \mathbb{P}\left[\omega\in\Omega: \sup_{t\in[0,T/2]} \|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,T)}} > \delta\right]$$

$$= m \sup_{x\in[-a,a]} \mathbb{P}\left[\omega\in\Omega: \sup_{t\in[0,T/2]} \|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,T)}} > \delta\right]$$

$$\leq m \sup_{x\in[-a,a]} \mathbb{P}\left[\omega\in\Omega: \sup_{t\in[0,T/2]} \mathbf{e}(t,x,\mathcal{Z}_n(t)) > \delta^2/2\right]. \tag{4.4.57}$$

Now by taking α as α/m in (4.4.55), of course with new a, we get that there exists an $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$,

$$\sup_{x\in[-a,a]} \mathbb{P}\left[\omega\in\Omega: \sup_{t\in[0,T/2]} \|\mathcal{Z}_n(t,\omega)\|_{\mathcal{H}_{B(x,R)}} > \delta\right] < \alpha. \tag{4.4.58}$$

Hence the Lemma 4.4.16. ■

Now we come back to the proof of Statement 2. Recall that S_M is a separable metric space. Since, by the assumptions, the sequence $\{\mathscr{L}(h_n)\}_{n\in\mathbb{N}}$ of laws on S_M converge weakly to the law $\mathscr{L}(h)$, due to the Skorokhod representation theorem, see for example [90, Theorem 3.30], there exists a probability space $(\tilde{\Omega},\tilde{\mathscr{F}},\tilde{\mathbb{P}})$, and on this probability space, one can construct processes $(\tilde{h}_n,\tilde{h},\tilde{W})$ such that the joint distribution of $(\tilde{h}_n,\tilde{W})$ is same as that of (h_n,W), the distribution of $\tilde{h}$ coincide with that of h, and $\tilde{h}_n \xrightarrow[n\to\infty]{} \tilde{h}$, $\tilde{\mathbb{P}}$-a.s. pointwise on $\tilde{\Omega}$, in the weak topology of S_M. By Proposition 4.4.11 this implies that

$$J^0\circ\tilde{h}_n \to J^0\circ\tilde{h} \quad \text{in} \quad \mathcal{X}_T \quad \tilde{\mathbb{P}}\text{-a.s. pointwise on } \tilde{\Omega}.$$

Next, we claim that

$$\mathscr{L}(z_n) = \mathscr{L}(\tilde{z}_n), \qquad \text{for all } n,$$

where

$$z_n := J^0\circ h : \Omega \to \mathcal{X}_T \qquad \text{and} \qquad \tilde{z}_n := J^0\circ\tilde{h}_n : \tilde{\Omega}\to\mathcal{X}_T.$$

To avoid complexity, we will write $J^0(h)$ for $J^0\circ h$. Let B be an arbitrary Borel subset of $\mathcal{X}_T$. Thus, since from Proposition 4.4.11, $J^0 : S_M \to \mathcal{X}_T$ is Borel, $(J^0)^{-1}(B)$ is Borel in S_M. So, we have

$$\mathscr{L}(z_n)(B) = \mathbb{P}\left[J^0(h_n)(\omega)\in B\right] = \mathbb{P}\left[h_n^{-1}\left((J^0)^{-1}(B)\right)\right] = \mathscr{L}(h_n)\left((J^0)^{-1}(B)\right).$$

But, since $\mathscr{L}(h_n) = \mathscr{L}(\tilde{h}_n)$ on $\mathcal{X}_T$, this implies $\mathscr{L}(z_n)(B) = \mathscr{L}(\tilde{z}_n)(B)$. Hence the claim and by a similar argument we also have $\mathscr{L}(z_h) = \mathscr{L}(z_{\tilde{h}})$.

Before moving forward, note that from Lemma 4.4.16, the sequence of $\mathcal{X}_T$-valued random variables, defined from Ω, $J^{\varepsilon_n}(h_n) - J^0(h_n)$ converges in measure $\mathbb{P}$ to 0. Consequently, because $\mathscr{L}(h_n) = \mathscr{L}(\tilde{h}_n)$ and $J^{\varepsilon_n} - J^0$ is measurable, we infer that $J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n) \xrightarrow{\tilde{\mathbb{P}}} 0$ as $n\to\infty$. Hence we can choose a subsequence $\{J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n)\}_{n\in\mathbb{N}}$, denoting by same, of $\mathcal{X}_T$-valued random variables converges to 0, $\tilde{\mathbb{P}}$-almost surely.

Now we claim to have the proof of Statement 2. Indeed, for any globally Lipschitz continuous and bounded function $\psi : \mathcal{X}_T \to \mathbb{R}$, see [64, Theorem 11.3.3], we have

$$\begin{aligned}
&\left|\int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^{\varepsilon_n}(h_n)) - \int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^0(h))\right| \\
&\qquad = \left|\int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^{\varepsilon_n}(\tilde{h}_n)) - \int_{\mathcal{X}_T} \psi(x)\, d\mathscr{L}(J^0(\tilde{h}))\right| \\
&\qquad = \left|\int_{\tilde{\Omega}} \psi\left(J^{\varepsilon_n}(\tilde{h}_n)\right) d\tilde{\mathbb{P}} - \int_{\tilde{\Omega}} \psi\left(J^0(\tilde{h})\right) d\tilde{\mathbb{P}}\right| \\
&\qquad \le \left|\int_{\tilde{\Omega}} \left\{\psi\left(J^{\varepsilon_n}(\tilde{h}_n)\right) - \psi\left(J^0(\tilde{h}_n)\right)\right\} d\tilde{\mathbb{P}}\right| \\
&\qquad\quad + \left|\int_{\tilde{\Omega}} \psi\left(J^0(\tilde{h}_n)\right) d\tilde{\mathbb{P}} - \int_{\tilde{\Omega}} \psi\left(J^0(\tilde{h})\right) d\tilde{\mathbb{P}}\right|.
\end{aligned}$$

Since $J^0(\tilde{h}_n) \xrightarrow[n\to\infty]{} J^0(\tilde{h})$, $\mathbb{P}$-a.s. and ψ is bounded and continuous, we deduce that the 2nd term in right hand side above converges to 0 as $n \to \infty$. Moreover we claim that the 1st term also goes to 0. Indeed, it follows from the dominated convergence theorem because the term is bounded by

$$L_\psi \int_{\tilde{\Omega}} |J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n)|\, d\tilde{\mathbb{P}},$$

where L_ψ is Lipschitz constant of ψ, and the sequence $\{J^{\varepsilon_n}(\tilde{h}_n) - J^0(\tilde{h}_n)\}_{n\in\mathbb{N}}$ converges to 0, $\tilde{\mathbb{P}}$-a.s. Therefore, Statement 2 holds true and we complete the proof of Theorem 4.4.6.

4.5 Auxiliary results

4.5.1 Existence and uniqueness result

In this part we recall a result about the existence of a uniqueness global solution, in strong sense, to problem

$$\begin{cases} \partial_{tt} u = \partial_{xx} u + A_u(\partial_t u, \partial_t u) - A_u(\partial_x u, \partial_x u) + Y_u(\partial_t u, \partial_x u)\, \dot{W}, \\ u(0) = u_0, \quad \partial_t u(0) = v_0. \end{cases} \tag{4.5.1}$$

In this framework, [23, Theorem 11.1] gives the following.

Theorem 4.5.1. *Fix $T > 0$ and $R > T$. For every $\mathscr{F}_0$-measurable random variable (u_0, v_0) with values in $H^2_{loc} \times H^1_{loc}(\mathbb{R}, TM)$, there exists a process $u : [0, T) \times \mathbb{R} \times \Omega \to M$, which we denote by $u = \{u(t), t < T\}$, such that the following hold:*

1. *$u(t, x, \cdot) : \Omega \to M$ is $\mathscr{F}_t$-measurable for every $t < T$ and $x \in \mathbb{R}$,*
2. *$[0, T) \ni t \mapsto u(t, \cdot, \omega) \in H^2((-R, R); \mathbb{R}^n)$ is continuous for almost every $\omega \in \Omega$,*
3. *$[0, T) \ni t \mapsto u(t, \cdot, \omega) \in H^1((-R, R); \mathbb{R}^n)$ is continuously differentiable for almost every $\omega \in \Omega$,*
4. *$u(t, x, \omega) \in M$, for every $t < T$, $x \in \mathbb{R}$, $\mathbb{P}$-almost surely,*

5. *$u(0,x,\omega) = u_0(x,\omega)$ and $\partial_t u(0,x,\omega) = v_0(x,\omega)$ holds, for every $x \in \mathbb{R}$, $\mathbb{P}$-almost surely,*

6. *for every $t \geq 0$ and $R > 0$,*

$$\partial_t u(t) = v_0 + \int_0^t \left[\partial_{xx} u(s) - A_{u(s)}(\partial_x u(s), \partial_x u(s)) + A_{u(s)}(\partial_t u(s), \partial_t u(s))\right] ds + \int_0^t Y_{u(s)}(\partial_t u(s), \partial_x u(s))\, dW(s),$$

holds in $L^2((-R,R);\mathbb{R}^n)$, $\mathbb{P}$-almost surely.

Moreover, if there exists another process $U = \{U(t); t \geq 0\}$ satisfing the above properties, then, for every $|x| < R - t$ and $t \in [0,T)$, $U(t,x,\omega) = u(t,x,\omega)$, $\mathbb{P}$-almost surely.

4.5.2 Energy inequality for stochastic wave equation

Recall the following slightly modified version of [23, Proposition 6.1] for a one (spatial) dimensional linear inhomogeneous stochastic wave equation. For $l \in \mathbb{N}$, we use the symbol $D^l h$ to denote the $\mathbb{R}^{n\times 1}$-vector $\left(\frac{d^l h^1}{dx^l}, \frac{d^l h^2}{dx^l}, \cdots, \frac{d^l h^n}{dx^l}\right)$.

Proposition 4.5.2. *Assume that $T > 0$ and $k \in \mathbb{N}$. Let W be a cylindrical Wiener process on a Hilbert space K. Let f and g be progressively measurable processes with values, respectively, in $H^k_{loc}(\mathbb{R};\mathbb{R}^n)$ and $\mathscr{L}_2(K, H^k_{loc}(\mathbb{R};\mathbb{R}^n))$ such that, for every $R > 0$,*

$$\int_0^T \left\{\|f(s)\|_{H^k((-R,R);\mathbb{R}^n)} + \|g(s)\|^2_{\mathscr{L}_2(K,H^k((-R,R);\mathbb{R}^n))}\right\} ds < \infty,$$

$\mathbb{P}$-almost surely. Let z_0 be an $\mathscr{F}_0$-measurable random variable with values in

$$\mathcal{H}^k_{loc} := H^{k+1}_{loc}(\mathbb{R};\mathbb{R}^n) \times H^k_{loc}(\mathbb{R};\mathbb{R}^n).$$

Assume that an $\mathcal{H}^k_{loc}$-valued process $z = z(t)$, $t \in [0,T]$, satisfies

$$z(t) = S_t z_0 + \int_0^t S_{t-s}\begin{pmatrix} 0 \\ f(s) \end{pmatrix} ds + \int_0^t S_{t-s}\begin{pmatrix} 0 \\ g(s) \end{pmatrix} dW(s), \qquad 0 \leq t \leq T.$$

Given $x \in \mathbb{R}$, we define the energy function $e : [0,T] \times \mathcal{H}^k_{loc} \to \mathbb{R}^+$ by, for $z = (u,v) \in \mathcal{H}^k_{loc}$,

$$\mathbf{e}(t,z) = \frac{1}{2}\left\{\|u\|^2_{L^2(B(x,T-t))} + \sum_{l=0}^k \left[\|D^{l+1}u\|^2_{L^2(B(x,T-t))} + \|D^l v\|^2_{L^2(B(x,T-t))}\right]\right\}.$$

Assume that $L : [0,\infty) \to \mathbb{R}$ is a non-decreasing $\mathcal{C}^2$-smooth function and define the second energy function $E : [0,T] \times \mathcal{H}^k_{loc} \to \mathbb{R}$, by

$$\mathbf{E}(t,z) = L(\mathbf{e}(t,z)),\ z = (u,v) \in \mathcal{H}^k_{loc}.$$

Let $\{e_j\}$ *be an orthonormal basis of* K. *We define a function* $V : [0,T] \times \mathcal{H}^k_{loc} \to \mathbb{R}$, *by*

$$\begin{aligned} V(t,z) &= L'(\mathbf{e}(t,z))\left[\langle u,v\rangle_{L^2(B(x,T-t))} + \sum_{l=0}^{k} \langle D^l v, D^l f(t)\rangle_{L^2(B(x,T-t))}\right] \\ &+ \frac{1}{2}L'(\mathbf{e}(t,z))\sum_j \sum_{l=0}^{k} |D^l[g(t)e_j]|^2_{L^2(B(x,T-t))} + \\ &+ \frac{1}{2}L''(\mathbf{e}(t,z))\sum_j \left[\sum_{l=0}^{k} \langle D^l v, D^l[g(t)e_j]\rangle_{L^2(B(x,T-t))}\right]^2, \ (t,z) \in [0,T] \times \mathcal{H}^k_{loc}. \end{aligned}$$

Then $\mathbf{E}$ *is continuous on* $[0,T] \times \mathcal{H}^k_{loc}$, *and for every* $0 \le t \le T$,

$$\begin{aligned} \mathbf{E}(t,z(t)) &\le \mathbf{E}(0,z_0) + \int_0^t V(r,z(r)\,dr \\ &+ \sum_{l=0}^{k} \int_0^t L'(\mathbf{e}(r,z(r)))\langle D^l v(r), D^l[g(r)\,dW(r)]\rangle_{L^2(B(x,T-r))}, \quad \mathbb{P}\text{-a.s.}. \end{aligned}$$

CHAPTER 5

STOCHASTIC GEOMETRIC WAVE EQUATION ON $\mathbb{R}^{1+1}$ WITH ROUGH DATA

In this chapter we establish the existence of a unique local (both time and space) solution to geometric wave equation, perturbed by a fractional (both in time and space) Gaussian noise, on one dimensional Minkowski space $\mathbb{R}^{1+1}$ when the target manifold M is an arbitrary compact Riemannian manifold and the initial data is rough in the sense that it belongs to a larger space than so-called the energy space $H^1_{\text{loc}} \times L^2_{\text{loc}}(\mathbb{R}, TM)$.

The chapter is structured as follows. In Section 5.1, we introduce our notation and provide the required definitions used later on. In Section 5.2 we rigorously justify that in order to prove the existence of a unique local solution to Cauchy problem (1.2.5) it is sensible to consider the problem (1.2.6). Section 5.3 is devoted to formulate the stochastic wave map Cauchy problem (1.2.6) in the null coordinates and to state all the necessary assumptions in detail. In Section 5.4 we derive the estimates needed in order to apply the Banach Fixed Point Theorem (in a suitable space). The complete proof of the existence and uniqueness of a local solution is given in Section 5.5. We conclude the chapter with two auxiliary results. First one is a useful result on the tensor product of Hilbert-Schmidt operators whose proof is in Subsection 5.6.1. In the second one, which is in Subsection 5.6.2, we show that the perturbed wave maps of sufficient regularity are invariant with respect to local charts.

5.1 Notation and function spaces setting

In this section we set the notation and define the function spaces that we use throughout the chapter. By symbol $\mathbb{N}$ we denote the set $\{0, 1, 2, \ldots\}$ of natural numbers. If x and y are two quantities (typically non-negative), we use $x \lesssim y$ or $y \gtrsim x$ to denote the statement that $x \leq Cy$ for some constant $C \geq 1$. More generally, given some parameters $a_1, \ldots, a_k$, we use $x \lesssim_{a_1,\ldots,a_k} y$ or $y \gtrsim_{a_1,\ldots,a_k} x$ to denote the statement that $x \leq C_{a_1,\ldots,a_k} y$ for some constant $C_{a_1,\ldots,a_k} \geq 1$ which can depend on the parameters $a_1,\dots,a_k$. We use $x \simeq y$ to denote the statement $x \lesssim y \lesssim x$, and similarly $x \simeq_{a_1,\dots,a_k} y$ denotes $x \lesssim_{a_1,\dots,a_k} y \lesssim_{a_1,\dots,a_k} x$.

Definition 5.1.1. *For $r \in \mathbb{N}$ we define $\mathcal{C}^r(\mathbb{R}^d)$ as the completion of $\mathcal{S}(\mathbb{R}^d)$ (class of Schwartz functions) in the norm*

$$\|f\|_{\mathcal{C}^r(\mathbb{R}^d)} := \sum_{|\alpha|\le r} \sup_{x\in\mathbb{R}^d} |D^\alpha f(x)|,$$

where $D^\alpha = \frac{\partial^{|\alpha|}}{\partial x_1^{\alpha_1}\cdots\partial x_d^{\alpha_d}}$, $\alpha = (\alpha_1,\dots,\alpha_d) \in \mathbb{N}^d$ is a multi-index, and $|\alpha| = \sum_{j=1}^d \alpha_j$. We define

$$\mathcal{C}^r_{comp}(\mathbb{R}^d) := \left\{u : u \in \mathcal{C}^r(\mathbb{R}^d), \exists \text{ a compact subset } K \text{ of } \mathbb{R}^d \text{ s.t supp } u \subseteq K\right\}.$$

If r is a positive real number with integer part $[r]$ and fractional part $\{r\} \in (0,1)$, then we set $\mathcal{C}^r(\mathbb{R}^d)$ as the completion of $\mathcal{S}(\mathbb{R}^d)$ w.r.t the norm

$$\|f\|_{\mathcal{C}^r(\mathbb{R}^d)} := \|f\|_{\mathcal{C}^{[r]}(\mathbb{R}^d)} + \sum_{|\alpha|=[r]} \sup_{x\neq y} \frac{|D^\alpha f(x) - D^\alpha f(y)|}{|x-y|^{\{r\}}}.$$

It is straightforward to check that the set $\mathcal{C}^r(\mathbb{R}^d)$, for every $r \ge 0$, is a separable Banach space. For simplicity we write $\mathcal{C}(\mathbb{R}^d)$ for $\mathcal{C}^0(\mathbb{R}^d)$. From now on, by a test function we mean an element of $\mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$.

Definition 5.1.2. *By $L^p(\mathbb{R}^d)$ for $p \in [1,\infty)$ we denote the classical real Banach space of all (equivalence classes of) $\mathbb{R}$-valued p-integrable functions on $\mathbb{R}^d$. The norm in $L^p(\mathbb{R}^d)$ is given by*

$$\|u\|_{L^p(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} |u(x)|^p \, dx\right)^{\frac{1}{p}}, \qquad u \in L^p(\mathbb{R}^d).$$

By $L^\infty(\mathbb{R}^d)$ we denote the real Banach space of all (equivalence classes of) Lebesgue measurable essentially bounded $\mathbb{R}$-valued functions defined on $\mathbb{R}^d$ with the norm

$$\|u\|_{L^\infty(\mathbb{R}^d)} := \text{ess sup}\,\{|u(x)| : x \in \mathbb{R}^d\}, \qquad u \in L^\infty(\mathbb{R}^d).$$

Definition 5.1.3. *For $s \in \mathbb{R}$, we define the Bessel-potential space $H^s(\mathbb{R}^d)$ by*

$$H^s(\mathbb{R}^d) := \left\{ u \in \mathcal{S}'(\mathbb{R}^d) : \|u\|_{H^s(\mathbb{R}^d)} := \left(\int_{\mathbb{R}^d} \langle\xi\rangle^{2s}\, |\hat{u}(\xi)|^2 \, d\xi\right)^{\frac{1}{2}} < \infty \right\},$$

where $\langle\xi\rangle := (1+|\xi|^2)^{\frac{1}{2}}$, $\mathcal{S}'(\mathbb{R}^d)$ is the set of all tempered distributions on $\mathbb{R}^d$, and $\hat{u}$ denotes the Fourier transform of u. Moreover, for any $u \in \mathcal{S}'(\mathbb{R}^d)$, we write $u \in H^s_{loc}(\mathbb{R}^d)$ if and only if $\varphi u \in H^s(\mathbb{R}^d)$ for all $\varphi \in \mathcal{C}^\infty_{comp}(\mathbb{R}^d)$.

Definition 5.1.4. *For $m \in \mathbb{N}$ and $p \in (1,\infty)$, the Sobolev space, denoted by $W^{m,p}(\mathbb{R}^d)$, is defined by,*

$$W^{m,p}(\mathbb{R}^d) := \{f \in L^p(\mathbb{R}^d) : \|f\|_{W^{m,p}(\mathbb{R}^d)} := \sum_{|\alpha|\le m} \|D^\alpha f\|_{L^p(\mathbb{R}^d)} < \infty\},$$

where the partial derivatives are understood in the sense of distributions.

Definition 5.1.5. *For $0 < s \neq$ integer and $p \in [1,\infty)$, we define the Slobodetski space $W^{s,p}(\mathbb{R}^d)$ by*

$$W^{s,p}(\mathbb{R}^d) = \Big\{ f \in W^{[s],p}(\mathbb{R}^d) : \|f\|_{W^{s,p}(\mathbb{R}^d)} := \|f\|_{W^{[s],p}(\mathbb{R}^d)} + \sum_{|\alpha|=[s]} \left(\int_{\mathbb{R}^d\times\mathbb{R}^d} \frac{|D^\alpha f(x) - D^\alpha f(y)|^p}{|x-y|^{d+p\{s\}}}\, dx\, dy \right)^{\frac{1}{p}} < \infty \Big\},$$

where $[s]$ and $\{s\}$ are the integral and fractional parts of s, respectively.

Next result is a well known equivalence in the theory of function spaces, see [155].

Lemma 5.1.6. *If $s \geq 0$, then $W^{s,2}(\mathbb{R}^d) = H^s(\mathbb{R}^d)$ with equivalent norms.*

In the next section we need the following Bessel-potential space of order $s \in \mathbb{R}$ on domain $\mathcal{O} \subset \mathbb{R}^d$ where we justify the consideration of SGWE Cauchy problem in terms of local coordinate charts on M.

Definition 5.1.7. *For any arbitrary bounded or unbounded domain $\mathcal{O} \subset \mathbb{R}^d$ we set*

$$H^s(\mathcal{O}) := \Big\{ f \in \mathcal{D}'(\mathcal{O}) : f = g \restriction \mathcal{O} \text{ for some } g \in H^s(\mathbb{R}^d) \Big\},$$

where $\mathcal{D}'(\mathcal{O})$ is the set of all distributions on $\mathcal{O}$ and $g \restriction \mathcal{O}$ denotes the restriction of $g \in \mathcal{D}'(\mathcal{O})$ in the sense of the theory of distributions.

Since $H^s(\mathcal{O})$ is a factor space, it is a Banach space (actually a separable Hilbert space) w.r.t. the following norm

$$\|f\|_{H^s(\mathcal{O})} := \inf_{\substack{g\in H^s(\mathbb{R}^d)\\ g\restriction\mathcal{O}=f}} \|g\|_{H^s(\mathbb{R}^d)}.$$

Moreover, for any closed set $F \subseteq \mathbb{R}^d$, the Sobolev space H^s_F is defined as

$$H^s_F := \Big\{ u \in H^s(\mathbb{R}^d) : \text{supp } u \subseteq F \Big\}.$$

Note that H^s_F is a closed subspace of $H^s(\mathbb{R}^d)$, and is therefore a Hilbert space when equipped with the restriction of the inner product of $H^s(\mathbb{R}^d)$.

For any class of functions $\mathscr{F}$ defined on $\mathbb{R}^d$, by $\mathscr{F}_{\text{comp}}$ we denote the set

$$\{ f \in \mathscr{F} : \text{supp } f \text{ is a compact subset of } \mathbb{R}^d \}.$$

To capture the dispersive smoothing effect of the nonlinear wave equation we need the following hyperbolic Sobolev spaces.

Definition 5.1.8. *The hyperbolic Sobolev space $H^{s,\delta}(\mathbb{R}^2)$ for $s, \delta \in \mathbb{R}$ is defined as the closure of $\mathcal{S}(\mathbb{R}^2)$ w.r.t. the norm :*

$$\|u\|_{H^{s,\delta}(\mathbb{R}^2)} := \left(\int_{\mathbb{R}^2} \langle |\tau| + |\xi| \rangle^{2s} \langle |\tau| - |\xi| \rangle^{2\delta}\, |[\mathcal{F}u](\tau,\xi)|^2\, d\xi\, d\tau \right)^{\frac{1}{2}},$$

where τ, ξ are the dual variables to t, x, respectively, and

$$[\mathcal{F}u](\tau,\xi) := \int_{\mathbb{R}^2} e^{-i(t\tau+x\xi)}\, u(t,x)\, dt\, dx, \tag{5.1.1}$$

is the space-time Fourier transform for $u = u(t,x) \in \mathcal{S}(\mathbb{R}^2)$.

To avoid many notations, from now on we will use the same notation $\mathcal{F}$ to denote the Fourier transform in the space variable, or in the time variable or in the space-time variables unless it is not clear from the expression. The next result concerns about the continuous embedding of the space $H^{s,\delta}(\mathbb{R}^2)$ into $\mathcal{C}(\mathbb{R}; H^s(\mathbb{R})) \cap \mathcal{C}^1\left(\mathbb{R}; H^{s-1}(\mathbb{R})\right)$, which in result provides a motivation to consider the $H^{s,\delta}(\mathbb{R}^2)$ space in the analysis of wave equation via an iteration procedure.

Theorem 5.1.9 (Trace Theorem)**.** *For every $s, \delta \in \left(\frac{3}{4}, 1\right)$, the following holds*

$$\|u(t)\|_{H^s_x(\mathbb{R})} + \|\partial_t u(t)\|_{H^{s-1}_x(\mathbb{R})} \lesssim \|u\|_{H^{s,\delta}(\mathbb{R}^2)}, \quad \textit{for every } t \in \mathbb{R}.$$

Proof of Theorem 5.1.9 Let us fix $u \in \mathcal{S}(\mathbb{R}^2)$. For any $t \in \mathbb{R}$, as an application of the Cauchy-Schwartz inequality, we get

$$\begin{aligned}
&\|u(t)\|^2_{H^s_x(\mathbb{R})} + \|(\partial_t u)(t)\|^2_{H^{s-1}_x(\mathbb{R})} \\
&\lesssim \int_{\mathbb{R}} \left[\int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s} + |\tau|^2\langle\xi\rangle^{2(s-1)}}{\langle|\tau|+|\xi|\rangle^{2s}\langle|\tau|-|\xi|\rangle^{2\delta}}\, d\tau\right]\left[\int_{\mathbb{R}} \langle|\tau_1|+|\xi|\rangle^{2s}\langle|\tau_1|-|\xi|\rangle^{2\delta}|(\mathcal{F}u)(\tau_1,\xi)|^2\, d\tau_1\right] d\xi.
\end{aligned} \tag{5.1.2}$$

To move forward we consider the integral $\int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s} + |\tau|^2\langle\xi\rangle^{2(s-1)}}{\langle|\tau|+|\xi|\rangle^{2s}\langle|\tau|-|\xi|\rangle^{2\delta}}\, d\tau$ and find a suitable estimate for this which holds uniformly w.r.t. ξ. On this path we divide the real line in two regions: $|\xi| < \frac{1}{2}$ and $|\xi| \geq \frac{1}{2}$ on which we estimate the integral separately. Due to the conditions $s \in (0,1)$ and $s + \delta > \frac{3}{2}$, for any $\xi \in \mathbb{R}$: $0 \leq |\xi| < \frac{1}{2}$, we deduce that

$$\int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s} + |\tau|^2\langle\xi\rangle^{2(s-1)}}{\langle|\tau|+|\xi|\rangle^{2s}\langle|\tau|-|\xi|\rangle^{2\delta}}\, d\tau \lesssim \int_{\{\tau\geq 0\}} \frac{\langle\tau\rangle^{2(1-s)}}{\langle|\tau|-|\xi|\rangle^{2\delta}}\, d\tau \lesssim \int_{\mathbb{R}} (1+\tau^2)^{-\frac{2s+2\delta-2}{2}}\, d\tau \simeq 1. \tag{5.1.3}$$

Next, we fix ξ in the other complement region such that $|\xi| \geq \frac{1}{2}$. We separate the estimate in the following two sub cases: (1) when the domain of τ is such that $|\tau| \leq 2|\xi|$. Here, by using $2\delta > 1$, we get

$$\begin{aligned}
&\int_{\{|\tau|\leq 2|\xi|\}} \frac{\langle\xi\rangle^{2s} + |\tau|^2\langle\xi\rangle^{2(s-1)}}{\langle|\tau|+|\xi|\rangle^{2s}\langle|\tau|-|\xi|\rangle^{2\delta}}\, d\tau \lesssim \int_{\{0\leq\tau\leq 2|\xi|\}} (1+|\tau-|\xi||^2)^{-\delta}\, d\tau \\
&\lesssim \int_{\mathbb{R}} (1+|\tau|^2)^{-2\delta/2}\, d\tau \simeq 1.
\end{aligned} \tag{5.1.4}$$

In the subcase (2), where the domain of τ is $\{\tau \in \mathbb{R} : |\tau| > 2|\xi|\}$, by invoking the relations $s < 1, s + \delta > \frac{3}{2}$ we obtain

$$\begin{aligned}
&\int_{\{|\tau|>2|\xi|\}} \frac{\langle\xi\rangle^{2s} + |\tau|^2\langle\xi\rangle^{2(s-1)}}{\langle|\tau|+|\xi|\rangle^{2s}\langle|\tau|-|\xi|\rangle^{2\delta}}\, d\tau \lesssim \int_{\{|\tau|>2|\xi|\}} \frac{\langle\xi\rangle^{2(s-1)}|\tau|^2}{(1+|\tau|^{2s})\langle|\tau|-|\xi|\rangle^{2\delta}}\, d\tau \\
&\lesssim \int_{\mathbb{R}} (1+|\tau|)^{-2(s+\delta-1)}\, d\tau \simeq 1.
\end{aligned} \tag{5.1.5}$$

Then by (5.1.3) - (5.1.5) we have

$$\int_{\mathbb{R}} \frac{\langle\xi\rangle^{2s} + |\tau|^2\langle\xi\rangle^{2(s-1)}}{\langle|\tau|+|\xi|\rangle^{2s}\langle|\tau|-|\xi|\rangle^{2\delta}}\, d\tau \lesssim 1,$$

which consequently finishes the proof of Theorem 5.1.9 because by using above in (5.1.2) we get

$$\|u(t)\|^2_{H^s_x(\mathbb{R})} + \|(\partial_t u)(t)\|^2_{H^{s-1}_x(\mathbb{R})} \lesssim \int_{\mathbb{R}^2} \langle|\tau_1|+|\xi|\rangle^{2s}\langle|\tau_1|-|\xi|\rangle^{2\delta}|(\mathcal{F}u)(\tau_1,\xi)|^2\, d\tau_1\, d\xi.$$

■

To define the integration with respect to paths of the fractional Brownian sheet we need to use the Besov spaces as well. Let us choose $\{\varphi_j\}_{j=0}^\infty \subset \mathcal{S}(\mathbb{R})$ an arbitrary dyadic partition of unity on $\mathbb{R}$, that is, a sequence $\{\varphi_j\}_{j=0}^\infty$ which has the following properties

1. Support property:

$$\begin{aligned} &\operatorname{supp}\varphi_0 \subset \{x : |x| \le 2\},\\ &\operatorname{supp}\varphi_j \subset \{x : 2^{j-1} \le |x| \le 2^{j+1}\}, \text{ if } j \in \mathbb{N}\setminus\{0\}. \end{aligned}$$

2. Bound property: for every $n \in \mathbb{N}$ there exists a number $C_n > 0$ such that

$$2^{jn}\varphi_j^{(n)}(x) \le C_n \text{ for all } j \in \mathbb{N} \text{ and all } x \in \mathbb{R},$$

   where $\varphi_j^{(n)}$ denotes the n-th derivative of φ_j.

3. Unity sum property:

$$\sum_{j=0}^\infty \varphi_j(x) = 1 \text{ for every } x \in \mathbb{R}.$$

We refer [156, Remark 2.3.1/1] and [3, Proposition 2.10] for the existence of such partitions. It is relevant to note that the Besov norms, in Definition 5.1.10, corresponds to any two dyadic partitions of unity are equivalent. Hence, without loss of generality, we fix the dyadic partition of unity, in the rest of the chapter, as the following system: let $\phi \in \mathcal{S}(\mathbb{R})$ be a non-negative function with

$$\operatorname{supp}\phi \subset \left\{x \in \mathbb{R} : \frac{1}{2} \le |x| \le 2\right\} \text{ and } \phi(x) > 0 \text{ if } \frac{1}{\sqrt{2}} \le |x| \le \sqrt{2}.$$

Let

$$\bar{\phi}(x) := \sum_{k=-\infty}^{\infty} \phi(2^{-k}x).$$

Due to support property of ϕ, the series in right hand side above is locally finite on the set $\mathbb{R}\setminus\{0\}$. Now with the function

$$\psi(x) := \phi(x)(\bar{\phi}(x))^{-1}, \tag{5.1.6}$$

we set

$$\varphi_j(x) := \psi(2^{-j}x), \quad \text{for } j = 1,2,\ldots, \quad \text{and} \quad \varphi_0(x) := 1 - \sum_{j=1}^{\infty} \varphi_j(x).$$

Given $h \in \mathcal{S}'(\mathbb{R})$, and $f \in \mathcal{S}'(\mathbb{R}^2)$, the 1-index and the 2-index, respectively, the Littlewood-Paley blocks are defined as

$$\Delta_j h := \begin{cases} 0, & \text{if } j \le -1, \\ \mathcal{F}^{-1}\left[\varphi_j(\xi)(\mathcal{F}h(\xi))\right], & \text{if } j \ge 0, \end{cases}$$

and

$$\Delta_{j,k} f := \begin{cases} 0, & \text{if } j \le -1 \text{ or } k \le -1, \\ \mathcal{F}^{-1}\left[\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}f)(\tau,\xi)\right], & \text{if } j,k \ge 0. \end{cases}$$

By the Paley-Wiener-Schwartz Theorem, see for example [68, Chapter 10], $\Delta_j h$ and $\Delta_{j,k} f$ are well-defined entire analytic functions. Based on such Littlewood-Paley blocks we define the Besov spaces as follows.

Definition 5.1.10. *Let $\{\varphi_j\}_{j=0}^{\infty}$ be a dyadic partition of unity. For $s \in \mathbb{R}$, $(s_1,s_2) \in \mathbb{R}^2$, $q \in (0,\infty]$ and $p \in (0,\infty]$, we define the Besov spaces, denoted by $B^s_{p,q}(\mathbb{R})$, as,*

$$B^s_{p,q}(\mathbb{R}) := \left\{ f \in \mathcal{S}'(\mathbb{R}) : \|f\|_{B^s_{p,q}(\mathbb{R})} := \left(\sum_{j\in\mathbb{Z}} 2^{sjq} \, \|\Delta_j f\|^q_{L^p(\mathbb{R})} \right)^{1/q} < \infty \right\},$$

and the Besov spaces of mixed smoothness, denoted by $S^{(s_1,s_2)}_{p,q}B(\mathbb{R}^2)$, as,

$$S^{(s_1,s_2)}_{p,q}B(\mathbb{R}^2) = \left\{ f \in \mathcal{S}'(\mathbb{R}^2) : \|f\|_{S^{(s_1,s_2)}_{p,q}B(\mathbb{R}^2)} < \infty \right\},$$

where

$$\|f\|_{S^{(s_1,s_2)}_{p,q}B(\mathbb{R}^2)} := \left(\sum_{j,k\in\mathbb{Z}} 2^{q(js_1+ks_2)} \|\Delta_{j,k} f\|^q_{L^p(\mathbb{R}^2)} \right)^{1/q}.$$

It is well known that $\left(B^s_{p,q}(\mathbb{R}), \|\cdot\|_{B^s_{p,q}(\mathbb{R})}\right)$ and $\left(S^{(s_1,s_2)}_{p,q}B(\mathbb{R}^2), \|\cdot\|_{S^{(s_1,s_2)}_{p,q}B(\mathbb{R}^2)}\right)$ are Banach spaces if we restrict to indices $p,q \in [1,\infty]$, see for example [141] and [156].

Next, we define the required Sobolev spaces of mixed smoothness which we will denote by $S^{(s_1,s_2)}_{2,2}H(\mathbb{R}^2)$.

Definition 5.1.11. *Let $(s_1,s_2) \in \mathbb{R}^2$. Then, the Sobolev space of mixed smoothness $S^{(s_1,s_2)}_{2,2}H(\mathbb{R}^2)$ is the set of all tempered distributions $\mathcal{S}'(\mathbb{R}^2)$ such that*

$$\|f\|_{S^{(s_1,s_2)}_{2,2}H(\mathbb{R}^2)} := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \langle\tau\rangle^{2s_1} \langle\xi\rangle^{2s_2} \, |[\mathcal{F}f](\tau,\xi)|^2 \, d\tau \, d\xi \right)^{\frac{1}{2}},$$

is finite.

Note that by definition the spaces $S^{(s_1,s_2)}_{2,2}H(\mathbb{R}^2)$ are nothing else but the product Sobolev space $H^{s_1}_t H^{s_2}_x(\mathbb{R}^2)$ defined, used in [92, 108], as

$$\left\{ f \in \mathcal{S}'(\mathbb{R}^2) : \|f\|_{H^{s_1}_t H^{s_2}_x(\mathbb{R}^2)} := \left(\int_{\mathbb{R}} \int_{\mathbb{R}} \langle \tau \rangle^{2s_1} \langle \xi \rangle^{2s_2} \, |[\mathcal{F} f](\tau,\xi)|^2 \, d\tau \, d\xi \right)^{\frac{1}{2}} < \infty \right\},$$

where $\mathcal{F}$ is the space-time Fourier transform defined in (5.1.1). Moreover, for any $f \in \mathcal{S}'(\mathbb{R}^2)$, we write $f \in H^{s_1}_{\text{loc}} H^{s_2}_{\text{loc}}(\mathbb{R}^2)$ if and only if $\varphi(t)\psi(x) f \in H^{s_1}_t H^{s_2}_x(\mathbb{R}^2)$ for all $\varphi, \psi \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$.

To do the required analysis we also need the vector-valued Lebesgue and Sobolev spaces. Let E be a separable Banach space.

Definition 5.1.12. *Let I be an either bounded or unbounded interval of $\mathbb{R}$, and $p \in [1,\infty]$, we define $L^p(I;E)$ as the set of all (equivalence classes of) strongly measurable E-valued functions such that*

$$\|u\|_{L^p(I;E)} := \left(\int_I \|u(x)\|^p_E \, dx \right)^{\frac{1}{p}} < \infty,$$

if $p < \infty$, and

$$\|u\|_{L^\infty(I;E)} := \text{ess sup } \{\|u(x)\|_E : x \in I\} < \infty,$$

if $p = \infty$.

For $k \in \mathbb{N}$ and $p \in [1,\infty]$, the Sobolev space $W^{k,p}(I;E)$ is the space of all $u \in L^p(I;E)$ whose all weak derivatives of orders $|\alpha| \le k$ exist and belong to $L^p(I;E)$. We set

$$\|u\|_{W^{k,p}(I;E)} := \sum_{|\alpha| \le K} \|D^\alpha u\|_{L^p(I;E)}.$$

Moreover, for $s \in (0,1)$, we define the E-valued Slobodetski space $W^{s,p}(I;E)$ as

$$W^{s,p}(I;E) := \left\{ u \in L^p(I;E) : \|u\|_{\dot{W}^{s,p}(I;E)} < \infty \right\},$$

where

$$\|u\|_{\dot{W}^{s,p}(I;E)} := \left(\int_{I\times I} \frac{\|u(x) - u(y)\|^p_E}{|x-y|^{1+ps}} \, dx \, dy \right)^{\frac{1}{p}}, \text{ if } p < \infty,$$

$$\text{and } \|u\|_{\dot{W}^{s,p}(I;E)} := \text{ess sup } \left\{ \frac{\|u(x) - u(y)\|_E}{|x-y|^s} : x, y \in I \right\}, \text{ if } p = \infty.$$

It is well known, in the theory of analysis of vector-valued functions, that the spaces $L^p(I;E)$, $W^{k,p}(I;E)$, and $W^{s,p}(I;E)$ are Banach spaces, respectively, with the norms $\|\cdot\|_{L^p(I;E)}$, $\|\cdot\|_{W^{k,p}(I;E)}$, and $\|\cdot\|_{L^p(I;E)} + \|\cdot\|_{\dot{W}^{s,p}(I;E)}$.

Definition 5.1.13. *Let $p \in (1,\infty)$ and $s \in \mathbb{R}$. For a Hilbert space E, we define the vector-valued Bessel-potential space $H^{s,p}(\mathbb{R};E)$ by*

$$H^{s,p}(\mathbb{R};E) := \left\{ u \in \mathcal{S}'(\mathbb{R};E) : \mathcal{F}^{-1}\left((1+|\xi|^2)^{\frac{s}{2}} \, (\mathcal{F}u)(\xi) \right) \in L^p(\mathbb{R};E) \right\},$$

where $\mathcal{S}'(\mathbb{R};E)$ is the space of E-valued tempered distributions, and put

$$\|u\|_{H^{s,p}(\mathbb{R};E)} := \left\| \mathcal{F}^{-1}\left((1+|\xi|^2)^{\frac{s}{2}} \, (\mathcal{F}u)(\xi) \right) \right\|_{L^p(\mathbb{R};E)}.$$

If $p=2$ and E a Hilbert space, we denote $H^{s,2}(\mathbb{R};E)$ by $H^s(\mathbb{R};E)$. Moreover, in this case, the vector-valued Plancherel Theorem holds and the norm defined above becomes

$$\|u\|_{H^{s,2}(\mathbb{R};E)} = \left(\int_{\mathbb{R}} (1+|\xi|^2)^s \, \|(\mathcal{F}u)(\xi)\|_E^2 \, d\xi\right)^{\frac{1}{2}}.$$

Furthermore, for any $s\geq 0$, $W^{s,2}(\mathbb{R};E)=H^s(\mathbb{R};E)$ with equivalent norms. For an arbitrary domain $\mathcal{O}$, we define $H^s(\mathcal{O})$ in the spirit of Definition 5.1.7.

To do the computation we also use the various embeddings and the equivalences between the defined functions spaces. Our next theorem provides the equivalence between spaces which are not included in the literature directly. We will use these equivalences without specifying it explicitly.

Theorem 5.1.14. *For $s,\delta\in\left(\frac{1}{2},1\right]$, the following hold in the sense of equivalent norms*

1. $W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R})) = W_x^{\delta,2}(\mathbb{R};W_t^{s,2}(\mathbb{R}))$.
2. $W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R})) = H_t^{s}(\mathbb{R};H_x^{\delta}(\mathbb{R}))$.
3. $H_t^{s}(\mathbb{R};H_x^{\delta}(\mathbb{R})) = H_x^{\delta}(\mathbb{R};H_t^{s}(\mathbb{R}))$.
4. $H_t^{s}(\mathbb{R};H_x^{\delta}(\mathbb{R})) = S_{2,2}^{s,\delta}H(\mathbb{R}^2)$.
5. $S_{2,2}^{s,\delta}B(\mathbb{R}^2) = S_{2,2}^{s,\delta}H(\mathbb{R}^2)$.

Proof of Theorem 5.1.14 Note that the Claim (5) of the theorem is Remark (ii) in Appendix A.2 of [145] and the Claim (3) is a direct consequence of points (1) and (2). Hence we only need to prove assertions (1), (2), and (4).

Before proving (1), recall from [86, Proposition 1.2.24] that the spaces $L_t^2(\mathbb{R};L_x^2(\mathbb{R}))$ and $L_x^2(\mathbb{R};L_t^2(\mathbb{R}))$ are isometrically isomorphic to $L^2(\mathbb{R}^2;\mathbb{R})$. Let us denote the corresponding isomorphisms as

$$\begin{cases} i_t : L_t^2(\mathbb{R};L_x^2(\mathbb{R})) \to L^2(\mathbb{R}^2;\mathbb{R}), \\ i_x : L_x^2(\mathbb{R};L_t^2(\mathbb{R})) \to L^2(\mathbb{R}^2;\mathbb{R}). \end{cases} \tag{5.1.7}$$

Let us fix $s,\delta\in(0,1)$ and take $f\in\mathcal{C}_{\mathrm{comp}}^{\infty}(\mathbb{R}_t;\mathcal{C}_{\mathrm{comp}}^{\infty}(\mathbb{R}_x))$, where we write $\mathbb{R}_t$ and $\mathbb{R}_x$ to show the variable explicitly. So, f belongs to $W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R}))$ and $L_t^{s,2}(\mathbb{R};L_x^{\delta,2}(\mathbb{R}))$. By Definition 5.1.12 and isomorphisms (5.1.7) followed by the Fubini Theorem we obtain

$$\begin{aligned}
\|(i_x^{-1}\circ i_t)(f)\|^2_{L_x^2(\mathbb{R};W_t^{s,2}(\mathbb{R}))} &= \|(i_x^{-1}\circ i_t)(f)\|^2_{L_x^2(\mathbb{R};L_t^2(\mathbb{R}))} \\
&\quad + \int_{\mathbb{R}} \left| \int_{\mathbb{R}^2} \frac{|\left([(i_x^{-1}\circ i_t)(f)](x)\right)(t_1) - \left([(i_x^{-1}\circ i_t)(f)](x)\right)(t_2)|^2}{|t_1-t_2|^{1+2s}} \, dt_1\, dt_2 \right| dx \\
&= \|f\|^2_{L_t^2(\mathbb{R};L_x^2(\mathbb{R}))} + \int_{\mathbb{R}^2}\int_{\mathbb{R}} \frac{|[i_t f](t_1,x) - [i_t f](t_2,x)|^2}{|t_1-t_2|^{1+2s}} \, dx\, dt_1\, dt_2 \\
&= \|f\|^2_{L_t^2(\mathbb{R};L_x^2(\mathbb{R}))} + \int_{\mathbb{R}^2} \frac{\|f(t_1)-f(t_2)\|^2_{L_x^2(\mathbb{R})}}{|t_1-t_2|^{1+2s}} \, dt_1\, dt_2 \leq \|f\|^2_{W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R}))}.
\end{aligned} \tag{5.1.8}$$

By similar calculations

$$\begin{aligned}
&\|(i_x^{-1}\circ i_t)(f)\|^2_{\dot W_x^{\delta,2}(\mathbb{R};W_t^{s,2}(\mathbb{R}))} \\
&\quad = \int_{\mathbb{R}^2} \frac{\|[(i_x^{-1}\circ i_t)(f)](x_1)-[(i_x^{-1}\circ i_t)(f)](x_2)\|^2_{L_t^2(\mathbb{R})}}{|x_1-x_2|^{1+2\delta}}\,dx_1\,dx_2 \\
&\qquad + \int_{\mathbb{R}^2} \frac{\|[(i_x^{-1}\circ i_t)(f)](x_1)-[(i_x^{-1}\circ i_t)(f)](x_2)\|^2_{\dot W_t^{s,2}(\mathbb{R})}}{|x_1-x_2|^{1+2\delta}}\,dx_1\,dx_2 \\
&\quad = \int_{\mathbb{R}}\int_{\mathbb{R}^2} \frac{|[i_tf](t,x_1)-[i_tf](t,x_2)|^2}{|x_1-x_2|^{1+2\delta}}\,dx_1\,dx_2\,dt \\
&\qquad + \int_{\mathbb{R}^4} \frac{|[i_tf](t_1,x_1)-[i_tf](t_2,x_1)-[i_tf](t_1,x_2)+[i_tf](t_2,x_2)|^2}{|t_1-t_2|^{1+2s}|x_1-x_2|^{1+2\delta}}\,dx_1\,dx_2\,dt_1\,dt_2 \\
&\quad = \int_{\mathbb{R}} \|f(t)\|^2_{\dot W_x^{\delta,2}(\mathbb{R})}\,dt + \int_{\mathbb{R}^2} \frac{\|f(t_1)-f(t_2)\|^2_{\dot W_x^{\delta,2}(\mathbb{R})}}{|t_1-t_2|^{1+2s}}\,dt_1\,dt_2 \\
&\quad = \|f\|^2_{W_t^{s,2}(\mathbb{R};\dot W_x^{\delta,2}(\mathbb{R}))} \le \|f\|^2_{W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R}))}.
\end{aligned} \tag{5.1.9}$$

Hence by (5.1.8) and (5.1.9) we proved that the map

$$\mathcal{I}_{tx} : \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_t;\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_x)) \ni f \mapsto (i_x^{-1}\circ i_t)f \in W_x^{\delta,2}(\mathbb{R};W_t^{s,2}(\mathbb{R})),$$

is continuous. The injectivity and linearity of $\mathcal{I}_{tx}$ is obvious. Since $\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_t;\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_x))$ is a dense subspace of $W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R}))$, there exists a unique continuous extension of this map, which we will also denote by $\mathcal{I}_{tx}$, from $W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R}))$ into $W_x^{\delta,2}(\mathbb{R};W_t^{s,2}(\mathbb{R}))$.

Similarly, we can prove that the map $\mathcal{I}_{xt} := i_t^{-1}\circ i_x$ defined by

$$\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_x;\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_t)) \ni f \mapsto (i_t^{-1}\circ i_x)f \in W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R})),$$

is continuous, linear and injective. Consequently, there exists a unique continuous extension of $\mathcal{I}_{xt}$, which we will also denote by $\mathcal{I}_{xt}$, from $W_x^{\delta,2}(\mathbb{R};W_t^{s,2}(\mathbb{R}))$ into $W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R}))$. Since

$$(i_x^{-1}\circ i_t)\circ(i_t^{-1}\circ i_x) = \mathrm{id}_{\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_x;\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_t))} \quad \text{and} \quad (i_t^{-1}\circ i_x)\circ(i_x^{-1}\circ i_t) = \mathrm{id}_{\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_t;\mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_x))},$$

we deduce that

$$\mathcal{I}_{tx}\circ\mathcal{I}_{xt} = \mathrm{id}_{W_x^{\delta,2}(\mathbb{R};W_t^{s,2}(\mathbb{R}))} \quad \text{and} \quad \mathcal{I}_{xt}\circ\mathcal{I}_{tx} = \mathrm{id}_{W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R}))}.$$

Hence, we are done with the proof of assertion (1) of the Theorem 5.1.14.

To prove the claim (2), let $j_x : W_x^{\delta,2}(\mathbb{R}) \to H_x^\delta(\mathbb{R})$ be an isomorphism, thanks to Lemma 5.1.6. So for any $f \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_t;W_x^{\delta,2}(\mathbb{R})) \subset W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R}))$ and $t\in\mathbb{R}$, $j_x(f(t)) \in H_x^\delta(\mathbb{R})$. Since for t outside a compact set $f(t)=0$ and the isomorphism j_x is linear, the following map

$$J : \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_t;W_x^{\delta,2}(\mathbb{R})) \ni f \mapsto j_xf \in \mathcal{C}^\infty_{\mathrm{comp}}(\mathbb{R}_t;H_x^\delta(\mathbb{R})) \subset H_t^s(\mathbb{R};H_x^\delta(\mathbb{R})),$$

is well-defined, where $j_xf : \mathbb{R}\ni t \mapsto j_x(f(t)) \in H_x^\delta(\mathbb{R})$. Observe that the injectivity and linearity of J is easy to see because j_x has both the properties. Invoking Lemma 5.1.6 followed by Sobolev embedding, since $s > \frac{1}{2}$, gives

$$\|Jf\|_{H_t^s(\mathbb{R};H_x^\delta(\mathbb{R}))} \lesssim \sup_{t\in\mathbb{R}} \|f(t)\|_{W_x^{\delta,2}(\mathbb{R})} = \|f\|_{L^\infty(\mathbb{R};W_x^{\delta,2}(\mathbb{R}))} \leq \|f\|_{W_t^{s,2}(\mathbb{R};W_x^{\delta,2}(\mathbb{R}))}. \tag{5.1.10}$$

Since $\mathcal{C}_{\text{comp}}^\infty(\mathbb{R}_t; W_x^{\delta,2}(\mathbb{R})) \hookrightarrow W_t^{s,2}(\mathbb{R}; W_x^{\delta,2}(\mathbb{R}))$ densely, there exists a unique continuous extension of the map J, which we again denote by J, from $W_t^{s,2}(\mathbb{R}; W_x^{\delta,2}(\mathbb{R}))$ into $H_t^s(\mathbb{R}; H_x^\delta(\mathbb{R}))$. This unique extension satisfies the bound (5.1.10) with the same constant of inequality. By following the similar steps we can prove that there exists a linear and continuous operator which maps $H_t^s(\mathbb{R}; H_x^\delta(\mathbb{R}))$ into $W_t^{s,2}(\mathbb{R}; W_x^{\delta,2}(\mathbb{R}))$. Then we finish the proof of second claim of Theorem 5.1.14 by a similar argument we made in the last for Assertion (1).

Assertion (3) follows from [145, Theorem 2.1] once we show that $H_t^s(\mathbb{R}; H_x^\delta(\mathbb{R}))$ is isomorphic to $H_t^s(\mathbb{R}) \otimes H_x^\delta(\mathbb{R})$ because all tensor products on Hilbert spaces are equivalent. It is well known that the Fourier transform $\mathcal{F} : L^2(\mathbb{R}) \to L^2(\mathbb{R})$ is a linear isomorphism. Let $I_H : H \to H$ be identity map for any separable Hilbert space H. Then [145, Lemma B.1] gives

$$\mathcal{F} \otimes I_H : L^2(\mathbb{R}) \otimes H \to L^2(\mathbb{R}) \otimes H,$$

a linear isomorphism. But since $L^2(\mathbb{R}) \otimes H$ is isomorphic to $L^2(\mathbb{R}; H)$, the map $\mathcal{F} \otimes I_H$ gives the following isomorphism

$$\mathcal{F}_H : L^2(\mathbb{R}; H) \to L^2(\mathbb{R}; H).$$

Now define, for $s > 0$, a closed subspace of $L^2(\mathbb{R}; H)$ by

$$L_s^2(\mathbb{R}; H) := \left\{ [f] : f : \mathbb{R} \to H \text{ and } \int_{\mathbb{R}} (1 + |\xi|^2)^s |f(\xi)|_H^2 \, d\xi < \infty \right\}.$$

By taking $(\mathbb{R}, (1 + |\xi|^2)^{\frac{s}{2}} d\xi)$ measure space in [152, Example E.12] we get that $L_s^2(\mathbb{R}) \otimes H$ is isomorphic to $L_s^2(\mathbb{R}; H)$. Next, let us set $E := H_x^\delta(\mathbb{R})$ and define

$$\tilde{H}_t^s(\mathbb{R}; E) := \mathcal{F}_E^{-1}(L_s^2(\mathbb{R}; E)) \qquad \text{with inner product}$$

$$\langle f, g \rangle_{\tilde{H}^s(\mathbb{R};E)} = \langle (1 + |\xi|^2)^{\frac{s}{2}} f, (1 + |\xi|^2)^{\frac{s}{2}} g \rangle_{L^2(\mathbb{R};E)} = \int_{\mathbb{R}} (1 + |\xi|^2)^s \langle f, g \rangle_E \, d\xi. \tag{5.1.11}$$

It is easy to show that there exists a linear bijection between $\tilde{H}_t^s(\mathbb{R}; E)$ and $H_t^s(\mathbb{R}; E)$. But by the proof of [145, Lemma B.1] we have

$$\tilde{H}_t^s(\mathbb{R}; E) \simeq (\mathcal{F} \otimes I_E)^{-1}(L_s^2(\mathbb{R}; E)) = \mathcal{F}^{-1}(L_s^2(\mathbb{R})) \otimes I_E^{-1}(E) = H^s(\mathbb{R}) \otimes E.$$

Hence we are done with Assertion (3). ■

We also need the following embeddings, whose proof can be found in [146], in our calculation.

Theorem 5.1.15. *For $s, \delta \in \left(\frac{1}{2}, 1\right)$ and a separable Banach space E, we have the following continuous embeddings*

(1) $W^{s,2}(\mathbb{R}; E) \hookrightarrow L^\infty(\mathbb{R}; E)$,

(2) $W^{s,2}(\mathbb{R}; W^{\delta,2}(\mathbb{R})) \hookrightarrow W^{s,2}(\mathbb{R}; L^\infty(\mathbb{R}))$,

(3) $W^{s,2}(\mathbb{R}; W^{\delta,2}(\mathbb{R})) \hookrightarrow L^\infty(\mathbb{R}; L^\infty(\mathbb{R}))$.

To make use of the $H^{s,\delta}$ space in proving the required estimates which allows one to apply the Banach fixed point theorem, authors in [92], [108] have taken advantage of the null coordinates. So let us define these first. We denote the null coordinates and their duals (the Fourier variables), respectively by,

$$(\alpha,\beta) := (t+x, t-x), \quad (\mu,\nu) := \left(\frac{\tau+\xi}{2}, \frac{\tau-\xi}{2}\right), \tag{5.1.12}$$

where (τ,ξ) denotes the Fourier variable of (t,x). To switch the coordinates, we use the following convention:

$$f^*(\alpha,\beta) := f\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right) = f(t,x) = f^*(t+x, t-x). \tag{5.1.13}$$

Usage of null coordinates is helpful due to the following isomorphism, which is stated in [92] without proof, between $H^{s,\delta}$ space and the defined (see below) product Sobolev space $\mathbb{H}^{s,\delta}_{\alpha,\beta}$ which allows one to solve the problem in the null coordinates.

Proposition 5.1.16. *If $s,\delta \in \mathbb{R}$ which satisfy $s \geq \delta$, then the following map*

$$H^{s,\delta}_{t,x} \ni u(t,x) \mapsto u^*(\alpha,\beta) \in H^s_\alpha H^\delta_\beta \cap H^s_\beta H^\delta_\alpha := \mathbb{H}^{s,\delta}_{\alpha,\beta},$$

is an isomorphism, where we take the Hilbertian norm on $H^s_\alpha H^\delta_\beta \cap H^s_\beta H^\delta_\alpha$, that is,

$$\|u^*\|_{H^s_\alpha H^\delta_\beta \cap H^s_\beta H^\delta_\alpha} := \sqrt{\|u^*\|^2_{H^s_\alpha H^\delta_\beta} + \|u^*\|^2_{H^s_\beta H^\delta_\alpha}}.$$

In particular,

$$\|u^*\|_{H^s_\alpha H^\delta_\beta \cap H^s_\beta H^\delta_\alpha} \lesssim \|u\|_{H^{s,\delta}_{t,x}} \lesssim \|u^*\|_{H^s_\alpha H^\delta_\beta \cap H^s_\beta H^\delta_\alpha}.$$

Proof of Proposition 5.1.16 By invoking the change of variable formula for the following map, with notation from (5.1.12) and (5.1.13),

$$(\alpha,\beta) \mapsto \left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right) = (t,x),$$

we observe that, for $u \in \mathcal{S}(\mathbb{R}^2)$,

$$\begin{aligned}(\mathcal{F}_{t,x}u)(\tau,\xi) &= \frac{1}{2}\int_{\mathbb{R}^2} e^{-i\frac{(\alpha-\beta)\xi}{2} - i\frac{(\alpha+\beta)\tau}{2}} u\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right) d\alpha\, d\beta \\ &= \frac{1}{2}\int_{\mathbb{R}^2} e^{-i\frac{(\alpha-\beta)(\mu-\nu)}{2} - i\frac{(\alpha+\beta)(\mu+\nu)}{2}} u^*(\alpha,\beta)\, d\alpha\, d\beta = \frac{1}{2}(\mathcal{F}_{\alpha,\beta}u^*)(\mu,\nu),\end{aligned} \tag{5.1.14}$$

where (μ,ν) and (τ,ξ) are related by (5.1.12). Which consequently gives

$$\begin{aligned}\|u\|^2_{H^{s,\delta}_{t,x}} \lesssim &\int_{\{(\mu,\nu):|\mu|\geq|\nu|\}} \left(1+2(|\mu|^2+|\nu|^2)+2(|\mu|+|\nu|)(|\mu|-|\nu|)\right)^s \times \\ &\quad \left(1+2(|\mu|^2+|\nu|^2)-2(|\mu|+|\nu|)(|\mu|-|\nu|)\right)^\delta |(\mathcal{F}_{\alpha,\beta}u^*)(\mu,\nu)|^2\, d\mu\, d\nu \\ &+\int_{\{(\mu,\nu):|\mu|\leq|\nu|\}} \left(1+2(|\mu|^2+|\nu|^2)+2(|\mu|+|\nu|)(|\nu|-|\mu|)\right)^s \times\end{aligned}$$

$$
\begin{aligned}
&\left(1+2(|\mu|^2+|\nu|^2)-2(|\mu|+|\nu|)(|\nu|-|\mu|)\right)^{\delta} |(\mathcal{F}_{\alpha,\beta}u^*)(\mu,\nu)|^2 \, d\mu \, d\nu \\
\lesssim &\int_{\mathbb{R}^2} \left(1+|\mu|^2\right)^s \left(1+|\nu|^2\right)^{\delta} |(\mathcal{F}_{\alpha,\beta}u^*)(\mu,\nu)|^2 \, d\mu \, d\nu \\
&+ \int_{\mathbb{R}^2} \left(1+|\nu|^2\right)^s \left(1+|\mu|^2\right)^{\delta} |(\mathcal{F}_{\alpha,\beta}u^*)(\mu,\nu)|^2 \, d\mu \, d\nu \\
= &\| u^* \|^2_{H^s_\alpha H^\delta_\beta \cap H^s_\beta H^\delta_\alpha}.
\end{aligned}
$$

To complete the proof we still need to show that $\| u^* \|_{H^s_\alpha H^\delta_\beta \cap H^s_\beta H^\delta_\alpha} \lesssim \| u \|_{H^{s,\delta}_{t,x}}$. To prove this note that by the relation (5.1.14) and by setting, to shorten the notation,

$$
|\hat{u}_{\tau,\xi}|^2 := \left(1+(|\xi|+|\tau|)^2\right)^s \left(1+(|\tau|-|\xi|)^2\right)^{\delta} |(\mathcal{F}_{t,x}u)(\tau,\xi)|^2,
$$

we have

$$
\begin{aligned}
\| u^* \|^2_{H^s_\alpha H^\delta_\beta \cap H^s_\beta H^\delta_\alpha} \lesssim &\int_{\mathbb{R}^2} \left(1+|\xi+\tau|^2\right)^s \left(1+|\tau-\xi|^2\right)^{\delta} |(\mathcal{F}_{t,x}u)(\tau,\xi)|^2 \, d\tau \, d\xi \\
&+ \int_{\mathbb{R}^2} \left(1+|\tau-\xi|^2\right)^s \left(1+|\xi+\tau|^2\right)^{\delta} |(\mathcal{F}_{t,x}u)(\tau,\xi)|^2 \, d\tau \, d\xi \\
= &\int_{\mathbb{R}^2} \frac{\left(1+|\xi+\tau|^2\right)^s \left(1+|\tau-\xi|^2\right)^{\delta}}{\left(1+(|\xi|+|\tau|)^2\right)^s \left(1+(|\tau|-|\xi|)^2\right)^{\delta}} \, |\hat{u}_{\tau,\xi}|^2 \, d\tau \, d\xi \\
&+ \int_{\mathbb{R}^2} \frac{\left(1+|\xi-\tau|^2\right)^s \left(1+|\tau+\xi|^2\right)^{\delta}}{\left(1+(|\xi|+|\tau|)^2\right)^s \left(1+(|\tau|-|\xi|)^2\right)^{\delta}} \, |\hat{u}_{\tau,\xi}|^2 \, d\tau \, d\xi \\
\lesssim &\| u \|^2_{H^{s,\delta}_{t,x}}.
\end{aligned}
$$

Here the last step is due to the following uniform estimates which are based on the relation $s \geq \delta$,

$$
\frac{\left(1+|\xi+\tau|^2\right)^s \left(1+|\tau-\xi|^2\right)^{\delta}}{\left(1+(|\xi|+|\tau|)^2\right)^s \left(1+(|\tau|-|\xi|)^2\right)^{\delta}} \lesssim 1,
$$

and

$$
\frac{\left(1+|\xi-\tau|^2\right)^s \left(1+|\tau+\xi|^2\right)^{\delta}}{\left(1+(|\xi|+|\tau|)^2\right)^s \left(1+(|\tau|-|\xi|)^2\right)^{\delta}} \lesssim 1,
$$

in each of the regions (1) $\{\tau, \xi \geq 0\}$; (2) $\{\tau, \xi < 0\}$; (3) $\{\tau < 0, \xi \geq 0\}$, and (4) $\{\tau \geq 0, \xi < 0\}$. ■

It is relevant to note here that the isomorphism proven in the above Proposition 5.1.16 preserves itself if we restrict ourselves to the Schwartz class in both the considered spaces.

To understand the construction of solution rigorously, we also need the following definitions of spaces as a subset of manifold valued functions. Let $\mathbb{R}^n$ be the Euclidean space such that $M \xhookrightarrow{e_M} \mathbb{R}^n$ and we can always find such $n \in \mathbb{N}$ due to the celebrated Nash isometric embedding theorem [115].

Definition 5.1.17. *For given $s \geq 0$, by $H^s(\mathbb{R}; M)$ we mean the set of $u \in H^s(\mathbb{R}; \mathbb{R}^n)$ such that $u(x) \in M$ almost surely. Similarly we define $H^s(I; M)$ for any open interval I.*

Remark 5.1.18. Let M be a m-dimensional smooth manifold. Then, the following two are equivalent:

1. $u \in H^s(\mathbb{R}; M)$ as per the above definition,
2. for every local chart (U, ϕ) on M, $(\phi \circ u)\big|_V \in H^s(V; \mathbb{R}^m)$, where $V := u^{-1}(U)$.

5.2 Justification of computation in local charts

Here we try to justify in a rigorous way that to prove the existence and uniqueness of a local solution to Cauchy problem (1.2.5) it is sensible to consider problem (1.2.6). Recall that M is an m-dimensional smooth manifold. First, let us observe that since we seek a solution of problem (1.2.5) that lives on the manifold M, we cannot expect it to belong to the Hilbert space $H^s(\mathbb{R}; \mathbb{R}^n)$, but, instead according to PDE theory, they will take values rather in the Fréchet space $H^s_{\text{loc}}(\mathbb{R}; \mathbb{R}^n)$. Hence it is reasonable to see our problem in local manner as below.

Before we see the explicit local formulation of problem, we understand the meaning of the initial data $(z_0, z_1) \in H^s_{\text{loc}} \times H^{s-1}_{\text{loc}}(\mathbb{R}; TM)$ for $s \in \left[\frac{1}{2}, 1\right)$. For the case $s \geq 1$, by $(z_0, z_1) \in H^s_{\text{loc}} \times H^{s-1}_{\text{loc}}(\mathbb{R}; TM)$, the meaning is clear, see for e.g. [23]-[26], we mean that for every local chart (U, ϕ) on M if $I \subset \mathbb{R}$ is an open and bounded interval such that $z_0(I) \subset U$, then

$$\left\{I \ni x \mapsto \phi(z_0(x)) \in \mathbb{R}^m\right\} \in H^s(I; \mathbb{R}^m), \tag{5.2.1}$$

and

$$\left\{I \ni x \mapsto \left(d_{z_0(x)}\phi\right)(z_1(x)) \in \mathbb{R}^m\right\} \in H^{s-1}(I; \mathbb{R}^m). \tag{5.2.2}$$

Definition 5.2.1. *For $s \in \left[\frac{1}{2}, 1\right)$, we say $(z_0, z_1) \in H^s_{loc} \times H^{s-1}_{loc}(\mathbb{R}; TM)$ if and only if* (5.2.1) *holds and whenever $I \subset \mathbb{R}$ is an open and bounded interval such that $z_0(I) \subset U_1 \cap U_2$, where $(U_i, \phi_i, i = 1, 2)$ are local charts on M, there exist $v_1, v_2 \in H^{s-1}(I; \mathbb{R}^m)$ such that*

$$_{H^{-s}}\langle v_2, f\rangle_{H^s} = {}_{H^{-s}}\langle v_1, \left(d_{\phi_1(z_0)}(\phi_2 \circ \phi_1^{-1})\right)^* f\rangle_{H^s}, \qquad f \in H^s(I; \mathbb{R}^m), \tag{5.2.3}$$

where, for each $x \in I$, $\left(d_{\phi_1(z_0(x))}(\phi_2 \circ \phi_1^{-1})\right)^*$ is the adjoint of $d_{\phi_1(z_0(x))}(\phi_2 \circ \phi_1^{-1}) : \mathbb{R}^m \to \mathbb{R}^m$.

Here note that by following the procedure from [13], we deduce that, for every $f \in H^s_{\text{loc}}(\mathbb{R}; \mathbb{R}^m)$, the map

$$\left\{x \mapsto \left(d_{\phi_1(z_0(x))}(\phi_2 \circ \phi_1^{-1})\right)^* f\right\} \in H^s_{\text{loc}}(\mathbb{R} : \mathbb{R}^m).$$

Now to move further we consider the following countable set $\{[n, n+2] : n \in \mathbb{Z}\}$ of compact intervals which covers $\mathbb{R}$. Since $z_0 \in H^s_{\text{loc}}(\mathbb{R}; M)$ and $s > \frac{1}{2}$, $z_0 \in \mathcal{C}(\mathbb{R}; M)$. Then for given n and any element $x \in [n, n+2]$ such that $z_0(x) \in M$, there exist $a_x, b_x \in \mathbb{R}$ such that $a_x < b_x$, $x = \frac{a_x + b_x}{2}$ and $z_0([a_x, b_x])$ lies in a single coordinate chart (U_x, ϕ_x) of M. Let

$$I_x := x + \left[-\frac{b_x - a_x}{2}, \frac{b_x - a_x}{2}\right] =: x + I_x^1, \quad \text{and} \quad J_x := x + \frac{1}{2}\mathring{I}_x^1 =: x + \mathring{J}_x^1,$$

where $\mathring{I}$ denotes the interior of the interval I. The collection $\{J_x : x \in \mathbb{R}\}$ forms an open cover of the compact interval $[n, n+2]$. Thus, we can find finitely many points $\{x_{n_i}\}_{i=1}^{k_n}$ in $[n, n+2]$ such that

$[n, n+2] \subset \cup_i J_{x_{n_i}}$. To make our method precise we choose the centres/points from $\{x_{n_i}\}_{i=1}^{n_k}$ in such a way that

1. at most two of the open intervals $J_{x_{n_i}}$ have non-empty overlap;
2. none of the open interval (i.e. an element of a cover) lies inside other completely;
3. image of z_0 on the closure of each $J_{x_{n_i}}$ is in one chart.

Since for every $n \in \mathbb{Z}$ we have a finite collection of open intervals which cover $[n, n+2]$ and the countable union of countable sets is countable, we have a countable open cover of $\mathbb{R}$, which we denote by $\mathbb{J} := \{J_{x_i}\}_{i\in\mathbb{N}}$ of $\mathbb{R}$ (after renumbering), such that the cover satisfies the above three conditions. In the remaining part of the justification we only need to know that a countable cover of open intervals exists, without knowledge of their midpoint, which satisfy the above mentioned assumptions, hence we set $\mathbb{J} := \{J_i\}_{i\in\mathbb{N}}$.

Next, note that for every $i \in \mathbb{N}$, $z_0 \in H^s(J_i; M)$ because $z_0 \in H^s_{\text{loc}}(\mathbb{R}; M)$. Thus, based on $\mathbb{J}$, there exists a sequence of coordinate charts (U_i, ϕ_i), $i \in \mathbb{N}$ on M such that $z_0(\bar{J}_i) \subset U_i$ and

$$^{\phi_i}z_0 := \phi_i \circ z_0 \in H^s(J_i; \mathbb{R}^m).$$

Moreover, since $z_1 \in H^{s-1}_{\text{loc}}(\mathbb{R}; TM)$, there exists an $^{\phi_i}z_1 \in H^{s-1}(J_i; \mathbb{R}^m)$ which satisfy the condition of Definition 5.2.1. Hence in order to talk about the solution, which are continuous in time and space, of SGWE (1.2.5) it is reasonable to work with the following sequence of local Cauchy problems, for $i \in \mathbb{N}$,

$$\begin{cases} \Box\, ^{\phi_i}z = -\sum_{a,b=1}^{m} \sum_{\mu=0}^{1} {}^{\phi_i}\Gamma_{ab}(^{\phi_i}z)\partial_\mu\, ^{\phi_i}z^a \partial^\mu\, ^{\phi_i}z^b + {}^{\phi_i}\sigma(^{\phi_i}z)\dot{\xi}, \\ \left(^{\phi_i}z(0), \partial_t\, ^{\phi_i}z(0)\right) = \left(^{\phi_i}z_0, {}^{\phi_i}z_1\right) \in H^s(J_i; \mathbb{R}^m) \times H^{s-1}(J_i; \mathbb{R}^m). \end{cases} \tag{5.2.4}$$

Here $^{\phi_i}z := \phi_i \circ z$, $^{\phi_i}\Gamma_{ab} : \phi_i(U_i) \to \mathbb{R}$, $^{\phi_i}\sigma : \phi_i(U_i) \to \mathbb{R}^m$. Thus, by Definition 5.1.7, for each $i \in \mathbb{N}$, there exist

$$^{\phi_i}Z_0 \in H^s_{\text{comp}}(\mathbb{R}; \mathbb{R}^m) \quad \text{and} \quad ^{\phi_i}Z_1 \in H^{s-1}_{\text{comp}}(\mathbb{R}; \mathbb{R}^m),$$

such that $^{\phi_i}Z_k\big|_{J_i} = {}^{\phi_i}Z_k$, $k = 0, 1$, in the sense of distributions. Therefore, instead of sequence of problems (5.2.4), in the current chapter we consider the following sequence of Cauchy problems,

$$\begin{cases} \Box\, ^{\phi_i}Z = -\sum_{a,b=1}^{m} \sum_{\mu=0}^{1} {}^{\phi_i}\Gamma_{ab}(^{\phi_i}Z)\partial_\mu\, ^{\phi_i}Z^a \partial^\mu\, ^{\phi_i}Z^b + {}^{\phi_i}\sigma(^{\phi_i}Z)\dot{\xi}, \\ \left(^{\phi_i}Z(0), \partial_t\, ^{\phi_i}Z(0)\right) = \left(^{\phi_i}Z_0, {}^{\phi_i}Z_1\right) \in H^s(\mathbb{R}; \mathbb{R}^m) \times H^{s-1}(\mathbb{R}; \mathbb{R}^m), \end{cases} \tag{5.2.5}$$

with some appropriate extensions, which we denote by the same, $^{\phi_i}\Gamma_{ab} : \mathbb{R}^m \to \mathbb{R}$ and $^{\phi_i}\sigma : \mathbb{R}^m \to \mathbb{R}^m$.

To simplify the exposition follow [92] and assume that the Christoffel symbols Γ_{ab} depends polynomially on u, that is,

$$\Gamma_{ab}(u(\cdot)) = \sum_{|l|=0}^{r} A^l_{ab} \left[u^1(\cdot)\right]^{l_1} \left[u^2(\cdot)\right]^{l_2}, \tag{5.2.6}$$

for some $r \in \mathbb{N}$ where $l = (l_1, l_2, \dots, l_m) \in \mathbb{N}^m$ is multi index variable; $A^l_{ab} \in \mathbb{R}^m$; and $u^k(\cdot) \in \mathbb{R}$ is the kth component of u. This is not a significant constraint on the target manifold M because it covers the most interesting cases, for example a unit sphere $\mathbb{S}^m$, the Euclidean space $\mathbb{R}^m$, or any compact analytic manifold. Moreover, we assume that, which surely holds in the case of compact manifolds, the value of r is fixed i.e. the degree of polynomial in (5.2.6) does not vary with the choice of local charts on M. To avoid much more notation complexity, just for the convenience, from now on we assume that the target manifold M is 2-dimensional. However, the calculation can be directly extended to an arbitrary $m \in \mathbb{N}$.

5.3 SGWE problem in rotated coordinates

In this section we formulate the stochastic wave map Cauchy problem (1.2.6) under the rotation of the (t,x)-coordinate axes by $-\frac{\pi}{4}$. Recall that, from (5.1.13),

$$u^*(\alpha,\beta) := u\left(\frac{\alpha+\beta}{2}, \frac{\alpha-\beta}{2}\right) = u(t,x) \quad \text{and} \quad u(t,x) = u^*(t+x, t-x).$$

Since $(\alpha,\beta) = (t+x, t-x)$, for each $k = 1, \dots, n$, at a formal level we have

$$\begin{aligned}
\frac{\partial u}{\partial x} &= \frac{\partial u^*}{\partial \alpha} - \frac{\partial u^*}{\partial \beta}, & \frac{\partial^2 u}{\partial x^2} &= \frac{\partial^2 u^*}{\partial \alpha^2} - 2\frac{\partial^2 u^*}{\partial \alpha \partial \beta} + \frac{\partial^2 u^*}{\partial \beta^2},\\
\frac{\partial u}{\partial t} &= \frac{\partial u^*}{\partial \alpha} + \frac{\partial u^*}{\partial \beta}, & \frac{\partial^2 u}{\partial t^2} &= \frac{\partial^2 u^*}{\partial \alpha^2} + 2\frac{\partial^2 u^*}{\partial \alpha \partial \beta} + \frac{\partial^2 u^*}{\partial \beta^2},
\end{aligned}$$

where to simplify the notation we do not write the superscript k. In particular,

$$\Box u = 4\frac{\partial^2 u^*}{\partial \alpha \partial \beta} =: \Diamond u^* \qquad \text{and}$$

$$\sum_{a,b=1}^{m} \sum_{\mu=0}^{n} \Gamma_{ab}(u) \partial_\mu u^a \partial^\mu u^b = -4 \sum_{a,b=1}^{m} \Gamma^*_{ab}(u^*) \frac{\partial u^{a*}}{\partial \alpha} \frac{\partial u^{b*}}{\partial \beta} =: -\mathcal{N}^*(u^*), \tag{5.3.1}$$

for some Γ^*_{ab} having the same regularity of Γ_{ab}. So by following [133] and [159] with notation $\zeta(\alpha,\beta,\omega) := \xi(t,x,\omega), \omega \in \Omega$ a.e., the stochastic wave map Cauchy problem (1.2.6) in (α,β)-coordinate that we consider is the following

$$\begin{cases} \Diamond u^* = \mathcal{N}^*(u^*) + \sigma(u^*)\dot{\zeta}, \\ u^*(\alpha, -\alpha) = u_0(\alpha) \quad \text{and} \quad \partial_\alpha u^*(\alpha,-\alpha) + \partial_\beta u^*(\alpha,-\alpha) = u_1(\alpha), \end{cases} \tag{5.3.2}$$

where $\sigma \in \mathcal{C}^3_b(\mathbb{R}^2)$, that is, σ is bounded and belongs to the space $\mathcal{C}^3(\mathbb{R}^2)$ and has bounded derivatives up to order 3. The noise ζ is a fractional Brownian sheet (fBs), with Hurst indices greater than $\frac{1}{2}$, on $\mathbb{R}^2$, i.e. for $H_1, H_2 \in \left(\frac{1}{2}, 1\right)$. That is, refer [159, Chapter 1], ζ is a centred Gaussian process defined on a given complete probability spaces $(\Omega, \mathfrak{F}, \mathbb{P})$, whose covariance function is given by, for $(\alpha_1,\beta_1), (\alpha_2,\beta_2) \in \mathbb{R}^2$,

$$\mathbb{E}\left[\zeta(\alpha_1,\beta_1)\, \zeta(\alpha_2,\beta_2)\right] = R_{H_1}(|\alpha_1|, |\alpha_2|)\, R_{H_2}(|\beta_1|, |\beta_2|),$$

where $R_H(a,b), a,b \in \mathbb{R}$, is the covariance function of a standard fractional Brownian motion (fBm) with Hurst parameter H:

$$R_H(a,b) = \frac{1}{2}\left(a^{2H} + b^{2H} - |a-b|^{2H}\right), \qquad a,b \in \mathbb{R}.$$

Since we will mostly work with the wave map problem (5.3.2) which is in (α,β)-coordinates and with asterisk $(^*)$ notation it becomes very clumsy, we will write the problem (5.3.2) without asterisk $(^*)$ mark in remaining of the chapter unless there is any confusion.

As usual in the SPDE theory, we understand the stochastic geometric wave equation (5.3.2) in the following integral (called often mild) form

$$u = S(u_0,u_1) + \diamond^{-1}\mathcal{N}(u) + \diamond^{-1}\sigma(u)\dot{\zeta}, \tag{5.3.3}$$

where, for $(\alpha,\beta) \in \mathbb{R}^2$,

$$[S(u_0,u_1)](\alpha,\beta) := \frac{1}{2}\left[u_0(\alpha) + u_0(-\beta)\right] + \frac{1}{2}\int_{-\beta}^{\alpha} u_1(r)\,dr, \tag{5.3.4}$$

$$\left[\diamond^{-1}\mathcal{N}(u)\right](\alpha,\beta) := \frac{1}{4}\int_{-\beta}^{\alpha}\int_{-a}^{\beta} \mathcal{N}(u(a,b))\,db\,da, \tag{5.3.5}$$

and

$$\begin{aligned}\left[\diamond^{-1}\sigma(u)\dot{\zeta}\right](\alpha,\beta) &:= \frac{1}{4}\int_{-\beta}^{\alpha}\int_{-a}^{\beta} \sigma(u(a,b))\dot{\zeta}(a,b)\,db\,da \\ &=: \frac{1}{4}\int_{-\beta}^{\alpha}\int_{-a}^{\beta} \sigma(u(a,b))\,\zeta(da,db).\end{aligned} \tag{5.3.6}$$

Note that, at present, the expressions in (5.3.4) - (5.3.6) are nothing more than some formal notation which we write in this manner because in the case of sufficient regular initial data it is of D'Alembert form in (α,β)-coordinates, see [66, Section 2.4]. In the forthcoming sections we will show that, with the assumptions we have made on the non-linearity and the noise, each term is well-defined locally and belongs to a suitable space.

5.4 Estimates

We define the following relation between any three real numbers a,b,c. We say that

$$c \prec \{a,b\}$$

if and only if

$$a+b \geq 0, \;\; c \leq a, \;\; c \leq b, \;\; c \leq a+b-\frac{1}{2},$$

and

$$c = a+b-\frac{1}{2} \Rightarrow a+b > 0, \;\; c < a, \;\; c < b.$$

Let $\eta \in \mathcal{C}^{\infty}_{\mathrm{comp}}(\mathbb{R})$ be a cut-off function which satisfy

$$\eta(-x) = \eta(x), \qquad 0 \le \eta(x) \le 1, \qquad \eta(x) = \begin{cases} 1, & \text{if } |x| \le 2, \\ 0, & \text{if } |x| \ge 4. \end{cases} \tag{5.4.1}$$

We put $\eta_T(x) := \eta\left(\frac{x}{T}\right)$ for any $T > 0$.

5.4.1 Some useful known facts

In this subsection we state a few important results from the point of view to our analysis but some of the results are already proven in the literature. The first result is the following lemma to control the localized norm of primitive of a function with respect to the function itself.

Lemma 5.4.1. *[132, Lemma 2.2] For every $s > \frac{1}{2}$, there exists a continuous linear map*

$$\mathcal{P} : H^{s-1}(\mathbb{R}) \ni f \mapsto \int_0^{\cdot} f(y)\, dy \in H^s_{loc}(\mathbb{R}),$$

such that $[\mathcal{P}(f)](x) = \int_0^x f(y)\, dy$ is the Riemann integral for every $f \in \mathcal{C}^{\infty}_{comp}(\mathbb{R})$. Moreover, for any smooth function χ with supp $\chi \subset [-T, T]$ for some $T > 0$, we have

$$\left\| \chi(x) \int_0^x f(y)\, dy \right\|_{H^s} \lesssim (1 \vee T^2)\, \|\chi\|_{\mathcal{C}^1}\, \|f\|_{H^{s-1}}.$$

The next two results are standard ones regarding the multiplication of one-dimensional Sobolev spaces and its extension to the product Sobolev spaces. We ask the reader to refer Lemma 3.2 and Lemma 3.3 of [92], respectively, for the proof.

Lemma 5.4.2. *If $s, \bar{s} \in \mathbb{R}$ such that $s > \frac{1}{2}$ and $\bar{s} \in [-s, s]$, then*

$$\|fg\|_{H^{\bar{s}}} \lesssim \|f\|_{H^s} \|g\|_{H^{\bar{s}}}.$$

Lemma 5.4.3. *If $s_1, s_2 > \frac{1}{2}$ and $\bar{s}_i \in [-s_i, s_i]$, $i = 1, 2$, then*

$$\|fg\|_{H^{\bar{s}_1}_{\alpha} H^{\bar{s}_2}_{\beta}} \lesssim \|f\|_{H^{s_1}_{\alpha} H^{s_2}_{\beta}} \|g\|_{H^{\bar{s}_1}_{\alpha} H^{\bar{s}_2}_{\beta}}, \tag{5.4.2}$$

and

$$\|fg\|_{H^{\bar{s}_1}_{\alpha} H^{\bar{s}_2}_{\beta}} \lesssim \|f\|_{H^{\bar{s}_1}_{\alpha} H^{s_2}_{\beta}} \|g\|_{H^{s_1}_{\alpha} H^{\bar{s}_2}_{\beta}}. \tag{5.4.3}$$

One straightforward consequence of Lemmata 5.4.2 and 5.4.3 is that the one dimensional and the product Sobolev spaces are stable under the multiplication by bump functions. We also need the following estimate which can be easily derived from [108, Lemma 2.4].

Lemma 5.4.4. *Let $a, b, c \in \mathbb{R}$ such that $c \prec \{a, b\}$ and $a + b > \frac{1}{2}$. Then the following linear estimate holds*

$$\|f(\alpha, -\alpha)\|_{H^c_{\alpha}} \lesssim \|f(\alpha, \beta)\|_{H^a_{\alpha} H^b_{\beta}}.$$

5.4.2 Scaling in inhomogeneous Sobolev and Besov spaces

The following lemma shows the action of $\mathcal{P}$ on a scaled function.

Lemma 5.4.5. *Given $\lambda \geq 1$, let $X_\lambda : \mathcal{S}'(\mathbb{R}) \to \mathcal{S}'(\mathbb{R})$ defined by duality as*

$$\langle X_\lambda f, \varphi \rangle := \left\langle f, \frac{1}{\lambda} Y_{\frac{1}{\lambda}} \varphi \right\rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}),$$

where $[Y_\lambda \varphi](x) := \frac{1}{\lambda}\varphi\left(\frac{x}{\lambda}\right), \varphi \in \mathcal{S}(\mathbb{R})$. Then for every $f \in H^{s-1}(\mathbb{R})$, the following holds

$$[\mathcal{P}(X_\lambda f)](\lambda x) = [\mathcal{P}(f)](x), \qquad x \in \mathbb{R}.$$

Proof of Lemma 5.4.5 It is easy to see that if we restrict to $f \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$, then $X_\lambda f = Y_\lambda f$ on $\mathbb{R}$ and, consequently, the change of variable gives

$$\mathcal{P}(X_\lambda f)(\lambda x) = \int_0^{\lambda x} \frac{1}{\lambda} f\left(\frac{y}{\lambda}\right) dy = \int_0^x f(z)\, dz = \mathcal{P}(f)(x). \tag{5.4.4}$$

But this concludes the proof because $\mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$ is dense in $H^{s-1}(\mathbb{R})$ and both sides of (5.4.4) are continuous w.r.t. $f \in H^{s-1}(\mathbb{R})$. However, we also provide a direct proof as follows.

Let us define

$$\mathcal{J} : H^{s-1}(\mathbb{R}) \ni f \mapsto [M_\lambda \mathcal{P} X_\lambda] f \in H^s_{\text{loc}}(\mathbb{R}),$$

where $[M_\lambda g](\cdot) := g(\lambda \cdot)$. Observe that for $f \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$,

$$[M_\lambda \mathcal{P} X_\lambda] f(x) = \mathcal{P}(f)(x). \tag{5.4.5}$$

By (5.4.12), $X_\lambda : H^{s-1}(\mathbb{R}) \to H^{s-1}(\mathbb{R})$ is well-defined and satisfy

$$\|X_\lambda g\|_{H^{s-1}(\mathbb{R})} \leq \lambda^{\frac{1}{2}-s} \|g\|_{H^{s-1}(\mathbb{R})}. \tag{5.4.6}$$

Next we claim that $M_\lambda : H^s_{\text{loc}}(\mathbb{R}) \to H^s_{\text{loc}}(\mathbb{R})$ is well-defined. Indeed, let $\psi \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$, then by using Lemma 5.1.6, for every $g \in H^s_{\text{loc}}(\mathbb{R})$ we have

$$\begin{aligned}
\|\psi(x)[M_\lambda g](x)\|^2_{H^s(\mathbb{R})} &\simeq \int_{\mathbb{R}} |\psi(x) g(\lambda x)|^2\, dx + \int_{\mathbb{R}^2} \frac{|\psi(x) g(\lambda x) - \psi(y) g(\lambda y)|^2}{|x-y|^{1+2s}}\, dx\, dy \\
&= \frac{1}{\lambda} \int_{\mathbb{R}} |\psi(x/\lambda) g(x)|^2\, dx + \frac{\lambda^{1+2s}}{\lambda^2} \int_{\mathbb{R}^2} \frac{|\psi(x/\lambda) g(x) - \psi(y/\lambda) g(y)|^2}{|x-y|^{1+2s}}\, dx\, dy \\
&= \frac{1}{\lambda} \|\psi_\lambda(x) g(x)\|^2_{L^2(\mathbb{R})} + \frac{1}{\lambda^{1-2s}} \|\psi_\lambda(x) g(x)\|^2_{\dot{H}^s(\mathbb{R})} < \infty.
\end{aligned} \tag{5.4.7}$$

Since $\psi \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$ and λ is fixed, there exists an $m \in \mathbb{N}$ such that $\operatorname{supp} \psi \subset [-m\lambda, m\lambda]$. Then invoking (5.4.7) followed by Lemma 5.4.1 and estimate (5.4.6) gives, for any $f \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$,

$$\begin{aligned}
\|\psi(x)[\mathcal{J} f](x)\|^2_{H^s(\mathbb{R})} &= \frac{1}{\lambda} \|\psi_\lambda(x)\ [\mathcal{P} X_\lambda f](x)\|^2_{L^2(\mathbb{R})} + \frac{1}{\lambda^{1-2s}} \|\psi_\lambda(x)\ [\mathcal{P} X_\lambda f](x)\|^2_{\dot{H}^s(\mathbb{R})} \\
&\lesssim (1 \vee m^2)^2 \frac{1}{\lambda^{1-2s}} \|\psi_\lambda\|^2_{\mathcal{C}^1} \|X_\lambda f\|^2_{H^{s-1}(\mathbb{R})} = (1 \vee m^2)^2 \|\psi_\lambda\|^2_{\mathcal{C}^1} \|f\|^2_{H^{s-1}(\mathbb{R})}.
\end{aligned}$$

So we have proved that the map $\tilde{\mathcal{J}} : \mathcal{C}^{\infty}_{\text{comp}}(\mathbb{R}) \ni f \mapsto M_\lambda \mathcal{P} X_\lambda f \in H^s_{\text{loc}}(\mathbb{R})$ is well-defined and (5.4.5) holds. In particular,

$$[\tilde{\mathcal{J}} f](x) = [\mathcal{P}(X_\lambda f)](\lambda x) = [\mathcal{P} f](x), \quad f \in \mathcal{C}^{\infty}_{\text{comp}}(\mathbb{R}). \tag{5.4.8}$$

Next, since $\mathcal{C}^{\infty}_{\text{comp}}(\mathbb{R})$ is dense in $\hookrightarrow H^{s-1}(\mathbb{R})$ and $H^s_{\text{loc}}(\mathbb{R})$ is a Fréchet space, there exists a unique continuous linear extension $\mathcal{J} : H^{s-1}(\mathbb{R}) \to H^s_{\text{loc}}(\mathbb{R})$ and (5.4.8) holds as well. Hence we have proved Lemma 5.4.5. ■

In our well-posedness result Theorem 5.5.3 we need the following lemma which allows us to scale the initial data in such a way that we can make their norm as small as we please. Let χ be defined similar to (5.4.1). Let ψ be a bump function which is non zero on the support of χ and $\int_{\mathbb{R}} \psi(x)\, dx = 1$. Then, for $\lambda \geq 1$ and $f, g \in L^1_{\text{loc}}(\mathbb{R})$, we define a scaling,

$$\begin{cases} T_\lambda : f \mapsto \chi(\cdot) \left[f\left(\frac{\cdot}{\lambda}\right) - \bar{f}^\lambda \right], \\ S_\lambda : g \mapsto \frac{1}{\lambda} \chi(\cdot) g\left(\frac{\cdot}{\lambda}\right), \end{cases} \tag{5.4.9}$$

where $\bar{f}^\lambda$ is defined as follows

$$\bar{f}^\lambda := \int_{\mathbb{R}} f\left(\frac{y}{\lambda}\right) \psi(y)\, dy.$$

Lemma 5.4.6 (Scaling Lemma)**.** *For every $s \in \left(\frac{1}{2}, 1\right)$ and $\lambda \geq 1$, the following inequality holds,*

$$\|S_\lambda g\|_{H^{s-1}(\mathbb{R})} \lesssim_\chi \lambda^{\frac{1}{2}-s} \|g\|_{H^{s-1}(\mathbb{R})}. \tag{5.4.10}$$

Moreover, there exists a constant $\varepsilon(\lambda) > 0$ such that

$$\|T_\lambda f\|_{H^s(\mathbb{R})} \lesssim_{\chi,\psi} \lambda^{-\varepsilon} \|f\|_{H^s(\mathbb{R})}, \qquad 0 < \varepsilon \leq \varepsilon(\lambda). \tag{5.4.11}$$

Proof of Lemma 5.4.6 Fix any arbitrary $s \in \left(\frac{1}{2}, 1\right)$ and $\lambda \geq 1$. The estimate (5.4.10) is straightforward due to the Lemma 5.4.2 and properties of Fourier transform. Indeed,

$$\begin{aligned} \|S_\lambda g\|_{H^{s-1}(\mathbb{R})} &\lesssim \|\chi\|_{H^s(\mathbb{R})} \left\| \lambda^{-1} g\left(\frac{\cdot}{\lambda}\right) \right\|_{H^{s-1}(\mathbb{R})} \\ &= \|\chi\|_{H^s(\mathbb{R})} \left(\int_{\mathbb{R}} (1+|\xi|^2)^{s-1} |[\mathcal{F} g](\lambda \xi)|^2 \, d\xi \right)^{\frac{1}{2}} \lesssim_\chi \lambda^{\frac{1}{2}-s} \|g\|_{H^{s-1}(\mathbb{R})}. \end{aligned} \tag{5.4.12}$$

To prove (5.4.11) we need to work a bit as follows. First, for a fixed $\varepsilon > 0$ (value to be set later), define a map

$$T : H^{\frac{3}{2}+\varepsilon}(\mathbb{R}) \ni f \mapsto T_\lambda f \in H^s(\mathbb{R}).$$

Clearly, T is linear. By using the algebra property of $H^{3/2+\varepsilon}(\mathbb{R})$ it is easy to check that T is well-defined. Next, since

$$\mathcal{F} f = \mathbb{1}_{\{\xi : |\xi| > \lambda\}} \mathcal{F} f + \mathbb{1}_{\{\xi : |\xi| \leq \lambda\}} \mathcal{F} f,$$

we can define

$$f_1 := \mathcal{F}^{-1}(\mathbb{1}_{\{\xi:|\xi|>\lambda\}}\mathcal{F}f) \qquad \text{and} \qquad f_2 := \mathcal{F}^{-1}(\mathbb{1}_{\{\xi:|\xi|\leq\lambda\}}\mathcal{F}f).$$

Observe that $f_1, f_2 \in H^{\frac{3}{2}+\varepsilon}(\mathbb{R})$, since f. Next, due to the linearity of $T_\lambda \in H^{\frac{3}{2}+s}(\mathbb{R})$,

$$\|T_\lambda f\|_{H^s(\mathbb{R})} \leq \|T_\lambda f_1\|_{H^s(\mathbb{R})} + \|T_\lambda f_2\|_{H^s(\mathbb{R})}, \tag{5.4.13}$$

it is enough to estimate $\|T_\lambda f_i\|_{H^s(\mathbb{R})}, i = 1, 2$.

To deal with the term involving f_2, first note that by [155, Theorem 2.8.1], [155, Remark 2.8.1/3] and the Definition 5.1.1,

$$H^{\frac{3}{2}+\varepsilon}(\mathbb{R}) \hookrightarrow \mathcal{C}^1(\mathbb{R}).$$

Moreover, since χ has compact support, $T_\lambda f \in \mathcal{C}^1_{\text{comp}}(\mathbb{R})$ whenever $f \in \mathcal{C}^1(\mathbb{R})$. Now since the support of $T_\lambda f_2$ is subset of support of χ and $\mathcal{C}^1_{\text{comp}}(\mathbb{R}) \hookrightarrow H^s(\mathbb{R})$, to estimate the $H^s(\mathbb{R})$-norm of $T_\lambda f_2$ it is enough to bound the $\mathcal{C}^1_{\text{comp}}(\mathbb{R})$-norm of $T_\lambda f_2$. In this line observe that, since $\mathcal{C}^1_{\text{comp}}(\mathbb{R})$ function is Lipschitz on compact sets with bounded derivative, if we denote the Lipschitz constant and the L^∞-norm of f_2' by L_{f_2} and $B_{f_2'}$, respectively, the embedding $\mathcal{C}^1_{\text{comp}}(\mathbb{R}) \hookrightarrow H^s(\mathbb{R})$ and the product rule gives

$$\begin{aligned}
\|T_\lambda f_2\|_{\mathcal{C}^1_{\text{comp}}(\mathbb{R})} &\leq \sup_{x\in\text{supp }\chi} \left|\chi(x)\left[f_2\left(\frac{x}{\lambda}\right) - \bar{f}_2^\lambda\right]\right| + \sup_{x\in\text{supp }\chi} \left|\chi'(x)\left[f_2\left(\frac{x}{\lambda}\right) - \bar{f}_2^\lambda\right]\right| \\
&\quad + \sup_{x\in\text{supp }\chi} \left|\chi(x)\lambda^{-1}(f_2)'\left(\frac{x}{\lambda}\right)\right| \\
&\leq (1 + \|\chi'\|_{L^\infty}) \sup_{x\in\text{supp }\chi} \chi(x) \int_{\mathbb{R}} \left|f_2\left(\frac{x}{\lambda}\right) - f_2\left(\frac{y}{\lambda}\right)\right| \psi(y)\, dy + \lambda^{-1} B_{f_2'} \\
&\lesssim L_{f_2} \sup_{x\in\text{supp }\chi} \int_{\text{supp }\psi} \left|\frac{x}{\lambda} - \frac{y}{\lambda}\right| \psi(y)\, dy + \lambda^{-1} B_{f_2'} \\
&\lesssim_{\chi,\psi} \lambda^{-1}\left[L_{f_2} + B_{f_2'}\right] \lesssim \lambda^{-1}\|f_2\|_{C^1(\mathbb{R})} \leq \lambda^{-1}\|f_2\|_{H^{\frac{3}{2}+\varepsilon}(\mathbb{R})} \\
&\lesssim \lambda^{\frac{1+2\varepsilon-2s}{2}} \left(\int_{\{\xi:|\xi\leq\lambda\}} (1 + |\xi|^2)^s\, |(\mathcal{F}f)(\xi)|^2\, d\xi\right)^{\frac{1}{2}} \leq \lambda^{\frac{1+2\varepsilon-2s}{2}}\, \|f\|_{H^s(\mathbb{R})}.
\end{aligned}$$

This consequently gives

$$\|T_\lambda f_2\|_{H^s(\mathbb{R})} \lesssim_{\chi,\psi} \lambda^{\frac{1}{2}+\varepsilon-s}\, \|f\|_{H^s(\mathbb{R})}. \tag{5.4.14}$$

To complete the proof we still need to deal with the term involving f_1. By invoking definition of f_1, Lemma 5.4.2 and the Plancherel Theorem followed by the Cauchy-Schwartz inequality we get

$$\begin{aligned}
\|T_\lambda f_1\|_{H^s(\mathbb{R})} &\lesssim_\chi \left\|f_1\left(\frac{\cdot}{\lambda}\right)\right\|_{H^s(\mathbb{R})} + |\bar{f}_1^\lambda| = \lambda\left(\int_{\mathbb{R}} (1 + |\xi|^2)^s\, |[\mathcal{F}f_1](\lambda\xi)|^2\, d\xi\right)^{\frac{1}{2}} \\
&\quad + \left|\int_{\mathbb{R}} \left[\mathcal{F}f_1\left(\frac{x}{\lambda}\right)\right](\xi)\, [\mathcal{F}\psi(x)](\xi)\, d\xi\right| \\
&\lesssim \lambda^{\frac{1}{2}-s}\left(\int_{\{\xi:|\xi|\gtrsim\lambda\}} \langle\xi\rangle^{2s} |[\mathcal{F}f](\xi)|^2\, d\xi\right)^{\frac{1}{2}} + \left[\int_{\{\xi:|\xi|>\lambda\}} \langle\xi\rangle^{2s} |(\mathcal{F}f)(\xi)|^2\, d\xi\right]^{\frac{1}{2}} \times
\end{aligned}$$

$$\times \left[\int_{\{\xi:|\xi|>\lambda\}} \langle \xi \rangle^{-2s} \left| (\mathcal{F}\psi)\left(\frac{\xi}{\lambda}\right) \right|^2 d\xi \right]^{\frac{1}{2}}$$
$$\lesssim_\psi \lambda^{\frac{1}{2}-s} \|f\|_{H^s(\mathbb{R})}. \tag{5.4.15}$$

Finally by substituting the estimates of $T_\lambda f_2$ and $T_\lambda f_1$ from, respectively, (5.4.14) and (5.4.15) into (5.4.13), we get, for every $f \in H^{\frac{3}{2}+\varepsilon}$,

$$\|T_\lambda f\|_{H^s(\mathbb{R})} \lesssim_{\chi,\psi} \lambda^{\frac{1}{2}-s+\varepsilon} \|f\|_{H^s(\mathbb{R})} + \lambda^{\frac{1}{2}-s} \|f\|_{H^s(\mathbb{R})} \lesssim \lambda^{\frac{1}{2}-s+\varepsilon} \|f\|_{H^s}. \tag{5.4.16}$$

Since $s > \frac{1}{2}, \lambda \geq 1$ are fixed, we can choose positive ε such that $\varepsilon(\lambda) := s - \frac{1}{2} - \varepsilon$ is still positive and this $\varepsilon(\lambda)$ gives (5.4.11).

Lastly, since $H^{\frac{3}{2}+\varepsilon}(\mathbb{R}) \hookrightarrow H^s(\mathbb{R})$ densely, there exists a unique continuous extension of T, denoting again by T, such that (5.4.16) holds with the same constant and $\varepsilon(\lambda)$ as in (5.4.16), for every $f \in H^s(\mathbb{R})$. Hence we are done with proof of the Lemma 5.4.6. ■

Next, to see the required scaling for Besov spaces on $\mathbb{R}^2$ we define the following scaling operator $\Pi_\lambda : \mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$, for any $\lambda \geq 1$,

$$\langle \Pi_\lambda f, \varphi \rangle := \left\langle f, \frac{1}{\lambda^2} \mathcal{Y}_{\frac{1}{\lambda}} \varphi \right\rangle, \qquad \varphi \in \mathcal{S}(\mathbb{R}^2), \tag{5.4.17}$$

where $[\mathcal{Y}_\lambda \varphi](x,y) := \frac{1}{\lambda^2} \varphi\left(\frac{x}{\lambda}, \frac{y}{\lambda}\right), \varphi \in \mathcal{S}(\mathbb{R}^2)$. Our next result allows us to scale the considered noise, see Theorem 5.5.3.

Lemma 5.4.7. *For every $r, s > 0$, $\lambda \geq 1$ and $f \in S^{-r,-s}_{2,2}H(\mathbb{R}^2)$ we have*

$$\|\Pi_\lambda f\|_{S^{-r,-s}_{2,2}H} \leq \lambda^{r+s-1} \|f\|_{S^{-r,-s}_{2,2}H}.$$

Proof of Lemma 5.4.7 The proof is by replicating the steps of proof of the Lemma 5.4.5. Another simple proof, which uses the properties of Fourier transform, is by substituting $\Pi_\lambda(\mathcal{F}f) = \mathcal{F}(\Pi_{\lambda^{-1}} f)$ in the definition of $\|\Pi_\lambda f\|_{S^{-r,-s}_{2,2}H}$. ■

5.4.3 The homogeneous solution term

Recall that, for given $\alpha, \beta \in \mathbb{R}$, by $S(u_0, u_1)$ we denote the image of initial data (u_0, u_1) under the following map

$$S : (u_0, u_1) \mapsto \frac{1}{2}\left[u_0(\alpha) + u_0(-\beta)\right] + \frac{1}{2}\int_{-\beta}^{\alpha} u_1(\gamma)\, d\gamma.$$

Next we show that the map S is locally well-defined and continuous from $H^s(\mathbb{R}) \times H^{s-1}(\mathbb{R})$ into $\mathbb{H}^{s,\delta}$.

Lemma 5.4.8. *For every $s, \delta > \frac{1}{2}$ which satisfy $\delta \leq s$,*

$$\|\eta(\alpha)\chi(\beta)S(u_0,u_1)\|_{\mathbb{H}^{s,\delta}} \lesssim_{\eta,\chi} \|u_0\|_{H^s} + \|u_1\|_{H^{s-1}}. \tag{5.4.18}$$

Proof of Lemma 5.4.8 Since $\mathcal{C}^\infty_{\text{comp}}(\mathbb{R})$ is dense in $H^s(\mathbb{R})$ and $H^{s-1}(\mathbb{R})$, it is sufficient to prove (5.4.18) for the following map,

$$S : \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}) \times \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}) \ni (u_0, u_1) \mapsto \frac{1}{2}\left[u_0(\alpha) + u_0(-\beta)\right] + \frac{1}{2}\int_{-\beta}^{\alpha} u_1(\gamma)\, d\gamma \in \mathbb{H}^{s,\delta}_{\text{loc}}.$$

First observe that, since we are in the one dimensional setting, the image of $S(u_0, u_1)$ is the unique solution to the following rotated version, i.e. in (α, β)-coordinate system, of the linear homogeneous wave equation Cauchy problem

$$\begin{cases} \diamond u(\alpha,\beta) = 0, \quad (\alpha,\beta) \in \mathbb{R}^2 \\ u(\alpha,-\alpha) = u_0(\alpha) \quad \text{and} \quad \dfrac{\partial u}{\partial \alpha}(\alpha,-\alpha) + \dfrac{\partial u}{\partial \beta}(\alpha,-\alpha) = u_1(\alpha),\ \alpha \in \mathbb{R}. \end{cases}$$

By Lemma 5.4.2 and the embedding $H^s(\mathbb{R}) \hookrightarrow H^\delta(\mathbb{R})$, the terms involving u_0 can be estimated locally as

$$\begin{aligned} \|\eta(\alpha)\chi(\beta)u_0(\alpha)\|^2_{\mathbb{H}^{s,\delta}} &= \|\eta(\alpha)u_0(\alpha)\|^2_{H^s_\alpha}\|\chi(\beta)\|^2_{H^\delta_\beta} + \|\eta(\alpha)u_0(\alpha)\|^2_{H^\delta_\alpha}\|\chi(\beta)\|^2_{H^s_\beta} \\ &\lesssim_\chi \|\eta(\alpha)u_0(\alpha)\|^2_{H^s} \lesssim_{\eta,\chi} \|u_0\|^2_{H^s}. \end{aligned}$$

Similarly, we estimate the norm $\|\eta(\alpha)\chi(\beta)u_0(-\beta)\|_{\mathbb{H}^{s,\delta}}$, up to a factor, by $\|u_0\|_{H^s}$. Next, since

$$\int_{-\beta}^{\alpha} u_1(s)\, ds = \int_0^\alpha u_1(s)\, ds + \int_{-\beta}^0 u_1(s)\, ds,$$

and η, χ are symmetric functions, Lemma 5.4.1 followed by the continuous embedding of $H^s(\mathbb{R})$ into $H^\delta(\mathbb{R})$ gives

$$\begin{aligned} \left\|\eta(\alpha)\chi(\beta)\int_{-\beta}^{\alpha} u_1(s)\, ds\right\|^2_{\mathbb{H}^{s,\delta}} &= \|\chi(\beta)\|^2_{H^\delta_\beta}\left\|\eta(\alpha)\int_0^\alpha u_1(s)\, ds\right\|^2_{H^s_\alpha} + \|\chi(\beta)\|^2_{H^s_\beta}\left\|\eta(\alpha)\int_0^\alpha u_1(s)\, ds\right\|^2_{H^\delta_\alpha} \\ &\quad + \|\eta(\alpha)\|^2_{H^s_\alpha}\left\|\chi(\beta)\int_{-\beta}^0 u_1(s)\, ds\right\|^2_{H^\delta_\beta} + \|\eta(\alpha)\|^2_{H^\delta_\alpha}\left\|\chi(\beta)\int_{-\beta}^0 u_1(s)\, ds\right\|^2_{H^s_\beta} \\ &\lesssim_{\eta,\chi} \|u_1\|^2_{H^{s-1}}. \end{aligned}$$

Hence we are done with the proof of Lemma 5.4.8. ■

5.4.4 The noise term

Recall that for a given $f \in \mathcal{S}'(\mathbb{R}^2)$ the 2-index Littlewood-Paley blocks are defined by

$$\Delta_{j,k} f = \begin{cases} 0, \quad \text{if } j \le -1 \text{ or } k \le -1, \\ \mathcal{F}^{-1}[\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}f)(\tau,\xi)], \quad \text{if } j,k \ge 0. \end{cases}$$

Here $\{\varphi_j\}_{j=0}^\infty$ is the dyadic partition of unity on $\mathbb{R}$ constructed in Section 5.1. The first result of this subsection is a generalization of [132, Lemma 2.2] but very close to [108, Lemma 2.5]. The proof presented here is based on the Littlewood-Paley theory which can be generalized to handle the less regular noise case where one needs to deal with Besov spaces. It is also relevant to remark that the method of proof below can be modified to include the noise of lower regularity than we are considering here.

Proposition 5.4.9. *Assume that* $s,\delta \in \left(\frac{3}{4},1\right)$. *For every* $f \in \mathbb{H}^{s-1,\delta-1}$, *there exists a unique* $F \in \mathbb{H}^{s,\delta}_{loc}$ *such that* $\frac{\partial^2 F}{\partial\alpha\partial\beta} = f$ *and which satisfies the following homogeneous boundary conditions*

$$F(\alpha,-\alpha) = 0 \quad \text{and} \quad \frac{\partial F}{\partial\alpha}(\alpha,-\alpha) + \frac{\partial F}{\partial\beta}(\alpha,-\alpha) = 0, \quad \alpha \in \mathbb{R}.$$

Moreover, for every η,χ *and* $T>0$, *there exists a positive constant* $C(\eta,\chi,T)$, *which is an increasing function of* T, *such that*

$$\|\eta_T(\alpha)\chi_T(\beta)F(\alpha,\beta)\|_{\mathbb{H}^{s,\delta}} \le C(\eta,\chi,T)\ \|f\|_{\mathbb{H}^{s-1,\delta-1}}.$$

Remark 5.4.10. To maintain the analogy with sufficient regularity cases, in the remaining part of the chapter, we will denote F by $\diamond^{-1} f$ or

$$F(\alpha,\beta) := \int_{-\beta}^{\alpha}\int_{-a}^{\beta} f(a,b)\,db\,da, \qquad (\alpha,\beta) \in \mathbb{R}^2.$$

Proof of Proposition 5.4.9 Define

$$H(\alpha,\beta) := \int_{-\beta}^{\alpha}\int_{-\gamma}^{\beta} (\Delta_{0,0} f)(\gamma,\tau)\,d\tau\,d\gamma, \tag{5.4.19}$$

and

$$\begin{aligned}
G(\alpha,\beta) :=& \sum_{j,k=1}^{\infty}\left[\mathcal{F}^{-1}[\frac{1}{(i\tau)(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\right](\alpha,\beta)\\
&-\frac{1}{2}\sum_{j,k=1}^{\infty}\left[\mathcal{F}^{-1}[\frac{1}{(i\tau)(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\right](\alpha,-\alpha)\\
&-\frac{1}{2}\sum_{j,k=1}^{\infty}\left[\mathcal{F}^{-1}[\frac{1}{(i\tau)(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\right](-\beta,\beta)\\
&-\frac{1}{2}\int_{-\beta}^{\alpha}\sum_{j,k=1}^{\infty}\left[\mathcal{F}^{-1}[\frac{1}{(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\right](\gamma,-\gamma)\,d\gamma\\
&-\frac{1}{2}\int_{-\beta}^{\alpha}\sum_{j,k=1}^{\infty}\left[\mathcal{F}^{-1}[\frac{1}{(i\tau)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\right](\gamma,-\gamma)\,d\gamma\\
=:& \sum_{i=1}^{5} G^i(\alpha,\beta),
\end{aligned} \tag{5.4.20}$$

and,

$$I(\alpha,\beta) := \int_{-\beta}^{\alpha}\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{i\xi}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right](\gamma,\beta)\,d\gamma$$

$$-\int_{-\beta}^{\alpha}\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{i\xi}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right](\gamma,-\gamma)\,d\gamma, \tag{5.4.21}$$

and,

$$\begin{aligned} J(\alpha,\beta) &:= \int_{-\alpha}^{\beta}\sum_{m=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{i\tau}\varphi_0(\xi)\varphi_m(\tau)(\mathcal{F}f)(\tau,\xi)\right](\alpha,\gamma)\,d\gamma \\ &\quad -\int_{-\alpha}^{\beta}\sum_{m=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{i\tau}\varphi_0(\xi)\varphi_m(\tau)(\mathcal{F}f)(\tau,\xi)\right](-\gamma,\gamma)\,d\gamma. \end{aligned} \tag{5.4.22}$$

We will prove that

$$F := H + I + J + G,$$

is the one which satisfy all the claims of the Proposition 5.4.9. We begin the proof with a few comments on the term $\Delta_{0,0}f$. By [138, Theorem 7.23], since

$$(\tau,\xi)\mapsto \varphi_0(\tau)\varphi_0(\xi)(\mathcal{F}f)(\tau,\xi)$$

is a distribution of compact support, $\Delta_{0,0}f := \mathcal{F}^{-1}\left[\varphi_0(\tau)\varphi_0(\xi)(\mathcal{F}f)(\tau,\xi)\right]$ is an analytic function of polynomial growth. Therefore, H is also an analytic function (also of polynomial growth) and the integral to define H is in the Riemann sense. In particular, H is a tempered distribution.

Next step is to find the bound for H. Let us fix f,η,χ,T. Then since $H^1_\alpha H^1_\beta(\mathbb{R}^2)$ is continuously embedded in $H^s_\alpha H^\delta_\beta(\mathbb{R}^2)$ for $s,\delta\le 1$, we have

$$\begin{aligned} \|\eta_T(\alpha)\chi_T(\beta)H(\alpha,\beta)\|_{H^s_\alpha H^\delta_\beta} &\le \|\eta_T(\alpha)\chi_T(\beta)H(\alpha,\beta)\|_{H^1_\alpha H^1_\beta} \\ &= \|\tilde{H}\|_{L^2_\alpha L^2_\beta} + \left\|\frac{\partial\tilde{H}}{\partial\alpha}\right\|_{L^2_\alpha L^2_\beta} + \left\|\frac{\partial\tilde{H}}{\partial\beta}\right\|_{L^2_\alpha L^2_\beta} + \left\|\frac{\partial^2\tilde{H}}{\partial\alpha\partial\beta}\right\|_{L^2_\alpha L^2_\beta}, \end{aligned} \tag{5.4.23}$$

where we write $\tilde{H}(\alpha,\beta) := \eta_T(\alpha)\chi_T(\beta)H(\alpha,\beta)$. We estimate each term in r.h.s above separately as follows: by the Hölder inequality and the support property of η,χ, the first term satisfies

$$\begin{aligned} \|\tilde{H}\|_{L^2_\alpha L^2_\beta} &\le \left[\int_{\mathbb{R}^2}|\eta_T(\alpha)\chi_T(\beta)|^2(\alpha+\beta)^2\left(\int_{-\beta}^{\alpha}\int_{-\gamma}^{\beta}|(\Delta_{0,0}f)(\gamma,\delta)|^2\,d\delta\,d\gamma\right)d\alpha\,d\beta\right]^{\frac12} \\ &\lesssim T\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta}\left(\int_{\mathbb{R}}\left|\eta\left(\frac{\alpha}{T}\right)\right|^2 d\alpha\right)^{\frac12}\left(\int_{\mathbb{R}}\left|\chi\left(\frac{\beta}{T}\right)\right|^2 d\beta\right)^{\frac12} \\ &= T^2\|\eta\|_{L^2_\alpha}\|\chi\|_{L^2_\beta}\,\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta}. \end{aligned} \tag{5.4.24}$$

For the second term we apply (5.4.24) and the Hölder inequality to obtain

$$\begin{aligned} \left\|\frac{\partial\tilde{H}}{\partial\alpha}\right\|_{L^2_\alpha L^2_\beta} &= \frac{1}{T}\left\|(\eta')_T(\alpha)\left(\frac{\alpha}{T}\right)\chi_T(\beta)H(\alpha,\beta)\right\|_{L^2_\alpha L^2_\beta} + \left\|\eta_T(\alpha)\chi_T(\beta)\int_{-\alpha}^{\beta}(\Delta_{0,0}f)(\alpha,\delta)\,d\delta\right\|_{L^2_\alpha L^2_\beta} \\ &\lesssim T\|\eta'\|_{L^2_\alpha}\|\chi\|_{L^2_\beta}\,\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta} + T^{\frac12}\left[\int_{\mathbb{R}^2}|\eta_T(\alpha)\chi_T(\beta)|^2\|(\Delta_{0,0}f)(\alpha,\cdot)\|^2_{L^2_\beta}\,d\alpha\,d\beta\right]^{\frac12} \\ &\le T\|\eta'\|_{L^2_\alpha}\|\chi\|_{L^2_\beta}\,\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta} + T\|\eta\|_{L^\infty_\alpha}\|\chi\|_{L^2_\beta}\,\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta}. \end{aligned} \tag{5.4.25}$$

Here in the last step we also use $\|\eta_T\|_{L^\infty_\alpha} = \|\eta\|_{L^\infty_\alpha}$. Similarly by using

$$\int_{-\beta}^{\alpha}\int_{-\gamma}^{\beta}(\Delta_{0,0}f)(\gamma,\delta)\,d\delta\,d\gamma = \int_{-\alpha}^{\beta}\int_{-\delta}^{\alpha}(\Delta_{0,0}f)(\gamma,\delta)\,d\gamma\,d\delta,$$

we have

$$\left\|\frac{\partial \tilde{H}}{\partial\beta}\right\|_{L^2_\alpha L^2_\beta} \lesssim T\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta}\left(\|\chi'\|_{L^2_\beta}+\|\chi\|_{L^\infty_\beta}\right)\|\eta\|_{L^2_\alpha}. \tag{5.4.26}$$

For the final term by using (5.4.25) we get

$$\begin{aligned}\left\|\frac{\partial^2\tilde{H}}{\partial\alpha\partial\beta}\right\|_{L^2_\alpha L^2_\beta} &\le \frac{1}{T^2}\left\|(\eta')_T(\alpha)(\chi')_T(\beta)H(\alpha,\beta)\right\|_{L^2_\alpha L^2_\beta} + \frac{1}{T}\left\|(\eta')_T(\alpha)\chi_T(\beta)\int_{-\beta}^{\alpha}(\Delta_{0,0}f)(\gamma,\beta)\,d\gamma\right\|_{L^2_\alpha L^2_\beta}\\ &\quad + \frac{1}{T}\left\|\eta_T(\alpha)(\chi')_T(\beta)\int_{-\alpha}^{\beta}(\Delta_{0,0}f)(\alpha,\delta)\,d\delta\right\|_{L^2_\alpha L^2_\beta} + \left\|\eta_T(\alpha)\chi_T(\beta)(\Delta_{0,0}f)(\alpha,\beta)\right\|_{L^2_\alpha L^2_\beta}\\ &\lesssim \|\eta'\|_{L^2_\alpha}\|\chi'\|_{L^2_\beta}\,\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta} + \|\chi\|_{L^\infty_\beta}\|\eta'\|_{L^2_\alpha}\,\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta}\\ &\quad + \|\eta\|_{L^\infty_\alpha}\|\chi'\|_{L^2_\beta}\,\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta} + T^2\|\eta\|_{L^2_\alpha}\|\chi\|_{L^2_\beta}\,\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta}.\end{aligned} \tag{5.4.27}$$

Hence, substitution of the estimates (5.4.24), (5.4.25), (5.4.26) and (5.4.27) into (5.4.23) gives

$$\begin{aligned}\|\eta_T(\alpha)\chi_T(\beta)H\|_{H^1_\alpha H^1_\beta} &\lesssim (1\vee T^2)\|\Delta_{0,0}f\|_{L^2_\alpha L^2_\beta}\Big[\|\eta\|_{L^2_\alpha}\|\chi\|_{L^2_\beta}\\ &\quad+\left(\|\eta'\|_{L^2_\alpha}+\|\eta\|_{L^\infty_\alpha}\right)\|\chi\|_{L^2_\beta}+\left(\|\chi'\|_{L^2_\beta}+\|\chi\|_{L^\infty_\beta}\right)\|\eta\|_{L^2_\alpha}\\ &\quad+\left(\|\eta'\|_{L^2_\alpha}\|\chi'\|_{L^2_\beta}+\|\eta\|_{L^\infty_\alpha}\|\chi'\|_{L^2_\beta}+\|\chi\|_{L^\infty_\beta}\|\eta'\|_{L^2_\alpha}+\|\eta\|_{L^2_\alpha}\|\chi\|_{L^2_\beta}\right)\Big].\end{aligned} \tag{5.4.28}$$

But, due to the Theorem 5.1.14, we have

$$\begin{aligned}\|f\|^2_{H^{s-1}_\alpha H^{\delta-1}_\beta} &\simeq \|\mathcal{F}^{-1}(\varphi_0\varphi_0\mathcal{F}f)\|^2_{L^2(\mathbb{R}^2)} + \sum_{(j,k)\in\mathbb{N}^2_0} 2^{2((s-1)j+(\delta-1)k)}\|\mathcal{F}^{-1}(\varphi_j\varphi_k\mathcal{F}f)\|^2_{L^2(\mathbb{R}^2)}\\ &\ge \|\mathcal{F}^{-1}(\varphi_0\varphi_0\mathcal{F}f)\|^2_{L^2(\mathbb{R}^2)} = \|\Delta_{0,0}f\|^2_{L^2_\alpha L^2_\beta},\end{aligned}$$

where $\mathbb{N}^2_0 = \{(j,k)\in\mathbb{N}^2 : (j,k)\neq(0,0)\}$. Consequently from (5.4.23) and (5.4.28) we proved that there exists $C(\eta,\chi,T)>0$ such that

$$\|\eta_T(\alpha)\chi_T(\beta)H(\alpha,\beta)\|_{H^s_\alpha H^\delta_\beta} \le C(\eta,\chi,T)\|f\|_{H^{s-1}_\alpha H^{\delta-1}_\beta}. \tag{5.4.29}$$

Similarly we can prove that

$$\|\eta_T(\alpha)\chi_T(\beta)H(\alpha,\beta)\|_{H^s_\beta H^\delta_\alpha} \le C(\eta,\chi,T)\|f\|_{H^{s-1}_\beta H^{\delta-1}_\alpha},$$

and hence, jointly with (5.4.29) we obtain,

$$\|\eta_T(\alpha)\chi_T(\beta)H\|_{\mathbb{H}^{s,\delta}} \le C_H(\eta,\chi,T)\|f\|_{\mathbb{H}^{s-1,\delta-1}}, \tag{5.4.30}$$

for some $C_H(\eta,\chi,T)>0$.

Next, we see how the other terms G, I, J are well-defined and satisfy the suitable estimates. We divide the proof in a sequence of Lemmata and to eliminate the frequent reference of Theorem 5.1.14 we will use it without specifying unless there is any confusion. For each $j, k \geq 1$, we observe that,

$$\begin{aligned} G^1_{j,k} &:= \mathcal{F}^{-1}\left[\varphi_j(\tau)\varphi_k(\xi)\left(\sum_{m,n\geq 1}\frac{1}{(i\tau)(i\xi)}\varphi_m(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right)\right] \\ &= \mathcal{F}^{-1}\left[\left(\sum_{n=(k-1)\vee 1}^{k+1}\sum_{m=(j-1)\vee 1}^{j+1}\frac{1}{(i\tau)(i\xi)}\varphi_j(\tau)\varphi_m(\tau)\varphi_k(\xi)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right)\right]. \end{aligned} \tag{5.4.31}$$

Lemma 5.4.11. *For all $j, k \geq 1$ and $f \in \mathbb{H}^{s-1,\delta-1}$, $G^1_{j,k}$ belongs to $L^2(\mathbb{R}^2)$.*

Proof of Lemma 5.4.11 Let us choose and fix $f \in \mathbb{H}^{s-1,\delta-1}$. Due to finite sum in (5.4.31), it is sufficient to prove the following. For any fix $j, k \geq 0$ and $m, n \geq 1$ in such a way that $n \in \{(k-1)\vee 1, k, k+1\}$ and $m \in \{(j-1)\vee 1, j, j+1\}$,

$$\mathcal{F}^{-1}\left[\frac{1}{(i\tau)(i\xi)}\varphi_j(\tau)\varphi_m(\tau)\varphi_k(\xi)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right] \in \mathcal{S}'(\mathbb{R}^2)\cap L^2(\mathbb{R}^2).$$

Since $f \in \mathbb{H}^{s-1,\delta-1}$, $\mathcal{F}f \in L^2_{\text{loc}}(\mathbb{R}^2)$. Indeed, since $s-1, \delta-1 < 0$, for every $R > 0$ we get

$$\begin{aligned} \iint_{|\xi|,|\tau|<R} |[\mathcal{F}f](\tau,\xi)|^2\, d\tau\, d\xi &\leq \frac{1}{(1+R^2)^{s+\delta-2}}\iint_{|\xi|,|\tau|<R}(1+|\tau|^2)^{s-1}(1+|\xi|^2)^{\delta-1}|[\mathcal{F}f](\tau,\xi)|^2\, d\tau\, d\xi \\ &\leq \frac{1}{(1+R^2)^{s+\delta-2}}\|f\|^2_{H^{s-1}_\alpha H^{\delta-1}_\beta} < \infty. \end{aligned}$$

So $f \in \mathbb{H}^{s-1,\delta-1}$ implies that $\varphi_j(\tau)\varphi_k(\xi)\mathcal{F}f \in L^2_{\text{comp}}(\mathbb{R}^2)$. Since φ_j and φ_k vanish at the origin,

$$\frac{1}{(i\tau)(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}f)(\tau,\xi) \in L^2_{\text{comp}}(\mathbb{R}^2),$$

and consequently,

$$\mathcal{F}^{-1}\left[\frac{1}{(i\tau)(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}f)(\tau,\xi)\right] \in L^2(\mathbb{R}^2) \subset \mathcal{S}'(\mathbb{R}^2).$$

By a similar argument for φ_m, φ_n with $m, n \geq 1$, we get

$$\mathcal{F}^{-1}\left[\frac{1}{(i\tau)(i\xi)}\varphi_j(\tau)\varphi_k(\xi)\varphi_m(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right] \in L^2(\mathbb{R}^2),$$

and we finish the proof of Lemma 5.4.11. Note that by [138, Theorem 7.19] we can write

$$G^1_{j,k} = \sum_{n=(k-1)\vee 1}^{k+1}\sum_{m=(j-1)\vee 1}^{j+1}\mathcal{F}^{-1}\left[\frac{1}{(i\tau)(i\xi)}\varphi_m(\tau)\varphi_n(\xi)\right] * \Delta_{j,k}f.$$

□

Next we show that $\sum_{j,k=1}^{\infty} G^1_{j,k}$ is a well-defined element of $L^2(\mathbb{R}^2)$.

Lemma 5.4.12. $\sum\limits_{j,k=1}^{\infty} G^1_{j,k}$ *converges in* $L^2(\mathbb{R}^2)$.

Proof of Lemma 5.4.12 Observe that by invoking the Cauchy-Schwartz and the Young inequalities we have

$$\sum_{j,k=1}^{\infty}\|G^1_{j,k}\|_{L^2(\mathbb{R}^2)} = \sum_{j,k=1}^{\infty}\frac{1}{2^{sj+\delta k}}\|2^{sj+\delta k}G^1_{j,k}\|_{L^2(\mathbb{R}^2)}$$

$$\lesssim \left(\sum_{j,k=1}^{\infty}2^{2(sj+\delta k)}\|\Delta_{j,k}f\|^2_{L^2(\mathbb{R}^2)}\sum_{n=(k-1)\vee 1}^{k+1}\sum_{m=(j-1)\vee 1}^{j+1}\left\|\mathcal{F}^{-1}\left[\frac{1}{(i\tau)(i\xi)}\varphi_m(\tau)\varphi_n(\xi)\right]\right\|^2_{L^1(\mathbb{R}^2)}\right)^{\frac{1}{2}}. \tag{5.4.32}$$

Consequently, since

$$\left\|\mathcal{F}^{-1}\left[\frac{1}{(i\tau)(i\xi)}\varphi_m(\tau)\varphi_n(\xi)\right]\right\|^2_{L^1} = 2^{-2m-2n}\left\|\mathcal{F}^{-1}\left[\frac{1}{(i\tau)(i\xi)}\psi(\tau)\psi(\xi)\right]\right\|^2_{L^1(\mathbb{R}^2)},$$

where ψ is defined in (5.1.6), we have

$$\begin{aligned}\sum_{j,k=1}^{\infty}\|G^1_{j,k}\|_{L^2(\mathbb{R}^2)} &\le \left(\sum_{j,k=1}^{\infty}2^{2(s-1)j+2(\delta-1)k}\|\Delta_{j,k}f\|^2_{L^2(\mathbb{R}^2)}\left\|\mathcal{F}^{-1}\left[\frac{1}{(i\tau)(i\xi)}\psi(\tau)\psi(\xi)\right]\right\|^2_{L^1(\mathbb{R}^2)}\right.\\ &\qquad\left.\times\sum_{n=(k-1)\vee 1}^{k+1}\sum_{m=(j-1)\vee 1}^{j+1}2^{-2m+2j+2k-2n}\right)^{\frac{1}{2}}\\ &\lesssim \left(\sum_{j,k=1}^{\infty}2^{2(s-1)j+2(\delta-1)k}\|\Delta_{j,k}f\|^2_{L^2(\mathbb{R}^2)}\left\|\mathcal{F}^{-1}\left[\frac{1}{(i\tau)(i\xi)}\psi(\tau)\psi(\xi)\right]\right\|^2_{L^1(\mathbb{R}^2)}\right)^{\frac{1}{2}}\\ &\lesssim_{\psi}\|f\|_{S^{s-1,\delta-1}_{2,2}B} = \|f\|_{H^{s-1}_{\alpha}H^{\delta-1}_{\beta}}.\end{aligned} \tag{5.4.33}$$

So $\sum_{j,k=1}^{\infty}G^1_{j,k}$ converges absolutely in $L^2(\mathbb{R}^2)$. Hence, since $L^2(\mathbb{R}^2)$ is a Banach space, $G^1 := \sum_{j,k=1}^{\infty}G^1_{j,k}$ is a well-defined element of $L^2(\mathbb{R}^2)$ and we are done with the proof of Lemma 5.4.12. □

Next we will prove that G^1 belongs to $\mathbb{H}^{s,\delta}$ and satisfy the following estimate,

$$\|\eta_T(\alpha)\chi_T(\beta)G^1(\alpha,\beta)\|_{\mathbb{H}^{s,\delta}} \lesssim_{\eta,\chi,T}\|G^1\|_{\mathbb{H}^{s,\delta}} \lesssim_{\psi}\|f\|_{\mathbb{H}^{s-1,\delta-1}}. \tag{5.4.34}$$

Proof of the estimate 5.4.34 Note that the first inequality in (5.4.34) follows from Lemma 5.4.3. To prove the second inequality, first observe that since G^1 is in $L^2(\mathbb{R}^2)$, the terms $\Delta_{j,k}G^1$ for $j,k\ge 0$ makes sense and by using the properties of dyadic partition, see Section 5.1, a straightforward calculation gives

$$\begin{aligned}&\varphi_j(\tau)\varphi_k(\xi)\left(\sum_{m,n=1}^{\infty}\frac{1}{(i\tau)(i\xi)}\varphi_m(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right)\\ &\qquad=\frac{\varphi_j(\tau)\varphi_k(\xi)}{(i\tau)(i\xi)}(1-\varphi_0(\tau))(1-\varphi_0(\xi))(\mathcal{F}f)(\tau,\xi)=0,\end{aligned}$$

in the cases (1) $j=k=0$; (2) $j=1,k=0$; and (3) $j=0,k=1$. Consequently,

$$\sum_{j,k=1}^{\infty}\varphi_j(\tau)\varphi_k(\xi)\left(\sum_{m,n=1}^{\infty}\frac{1}{(i\tau)(i\xi)}\varphi_m(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right)$$

$$= \sum_{j,k=0}^{\infty} \varphi_j(\tau)\varphi_k(\xi)\left(\sum_{m,n=1}^{\infty} \frac{1}{(i\tau)(i\xi)}\varphi_m(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right). \tag{5.4.35}$$

Next, since by Lemma 5.4.12 we know $G^1 \in L^2(\mathbb{R}^2)$, invoking the Fourier-Plancherel Theorem gives, for all $j,k \geq 1$,

$$\left(\Delta_{j,k}G^1\right)(\alpha,\beta) = G_{j,k}(\alpha,\beta).$$

Hence, using (5.4.35) followed by the calculation of (5.4.32) and (5.4.33) we get

$$\begin{aligned}\|G^1\|_{H^s_\alpha H^\delta_\beta} &= \Big(2^{2(0s+0\delta)}\|\Delta_{0,0}G^1\|^2_{L^2} + 2^{2(1s+0\delta)}\|\Delta_{1,0}G^1\|^2_{L^2} + 2^{2(0s+1\delta)}\|\Delta_{0,1}G^1\|^2_{L^2} \\ &\quad + \sum_{j,k=1}^{\infty} 2^{2(sj+\delta k)}\|\Delta_{j,k}G^1\|^2_{L^2}\Big)^{1/2} \\ &= \left(\sum_{j,k=1}^{\infty} 2^{2(sj+\delta k)}\|G^1_{j,k}\|^2_{L^2}\right)^{1/2} \lesssim_\psi \|f\|_{H^{s-1}_\alpha H^{\delta-1}_\beta}.\end{aligned} \tag{5.4.36}$$

By interchanging the roles of α,β in the computation of (5.4.36) we get

$$\|G^1\|_{H^\delta_\alpha H^s_\beta} \lesssim_\psi \|f\|_{H^{\delta-1}_\alpha H^{s-1}_\beta},$$

and hence, the estimate (5.4.34). □

Now note that since $G^1 \in \mathbb{H}^{s,\delta}$, $G^2(\alpha) = G^1(\alpha,-\alpha)$ is a well-defined function of α. Thus, since η is an even bump function, invoking Theorem 5.1.14 and Lemma 5.4.4 gives

$$\begin{aligned}\|\eta_T(\alpha)\chi_T(\beta)G^2(\alpha)\|_{H^s_\alpha H^\delta_\beta} &\lesssim_{\chi,T} \|\eta_T(\alpha)\eta_T(-\alpha)G^1(\alpha,-\alpha)\|_{H^s_\alpha} \\ &\lesssim \|\eta_T(\alpha)\eta_T(\beta)G^1(\alpha,\beta)\|_{H^s_\alpha H^\delta_\beta}.\end{aligned}$$

But this has been estimated in (5.4.34) and consequently, G^2 satisfies

$$\|\eta_T(\alpha)\chi_T(\beta)G^2(\alpha,\beta)\|_{\mathbb{H}^{s,\delta}} \lesssim_{\eta,\chi,T,\psi} \|f\|_{\mathbb{H}^{s-1,\delta-1}}. \tag{5.4.37}$$

Similarly we can bound the term G^3 locally which only depends on β. Now we find the bound of terms G^4 and G^5. Since they have similar structures we only work with G^4. First note that

$$\begin{aligned}G^4(\alpha\,\beta) &= \int_{-\beta}^{\alpha}\sum_{j,k=1}^{\infty}\left[\mathcal{F}^{-1}[\frac{1}{(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\right](\gamma,-\gamma)\,d\gamma \\ &= \int_{-\beta}^{\alpha}\left[\mathcal{F}^{-1}[\frac{1}{(i\xi)}(1-\varphi_0(\tau))(1-\varphi_0(\xi))(\mathcal{F}\phi)(\tau,\xi)]\right](\gamma,-\gamma)\,d\gamma,\end{aligned}$$

and since $\frac{1}{(i\xi)}(1-\varphi_0(\xi))$ has removable singularity at the origin,

$$\frac{1}{(i\xi)}(1-\varphi_0(\tau))(1-\varphi_0(\xi))(\mathcal{F}\phi)(\tau,\xi) \quad \text{is a tempered distribution.}$$

Thus, we use Lemmata 5.4.1 and 5.4.4 to obtain

$$\begin{aligned}
&\Big\| \eta_T(\alpha)\chi_T(\beta)\int_{-\beta}^{\alpha}\sum_{j,k=1}^{\infty}\Big[\mathcal{F}^{-1}[\frac{1}{(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\Big](\gamma,-\gamma)\,d\gamma\Big\|_{H^s_\alpha H^\delta_\beta}\\
&\quad\lesssim_{\eta,\chi,T}\Big\|\sum_{j,k=1}^{\infty}\Big[\mathcal{F}^{-1}[\frac{1}{(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\Big](\alpha,-\alpha)\Big\|_{H^{s-1}_\alpha}\\
&\qquad+\Big\|\sum_{j,k=1}^{\infty}\Big[\mathcal{F}^{-1}[\frac{1}{(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\Big](-\beta,\beta)\Big\|_{H^{\delta-1}_\beta}\\
&\quad\lesssim_{\eta,\chi,T}\Big\|\sum_{j,k=1}^{\infty}\Big[\mathcal{F}^{-1}[\frac{1}{(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\Big](\alpha,\beta)\Big\|_{H^{s-1}_\alpha H^\delta_\beta}\\
&\qquad+\Big\|\sum_{j,k=1}^{\infty}\Big[\mathcal{F}^{-1}[\frac{1}{(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\Big](\alpha,\beta)\Big\|_{H^s_\alpha H^{\delta-1}_\beta}.
\end{aligned}$$

So we conclude that to find the local estimate of G^4, G^5, it is sufficient to estimate

$$\sum_{j,k=1}^{\infty}\Big[\mathcal{F}^{-1}[\frac{1}{(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\Big](\alpha,\beta),$$

which is as follows: for fix $j,k\geq 0$, the support property of the dyadic partition we fixed, by replicating the calculation based on the proof of (5.4.34), we deduce

$$\begin{aligned}
&\Big\|\sum_{j,k=1}^{\infty}\Big[\mathcal{F}^{-1}[\frac{1}{(i\xi)}\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}\phi)(\tau,\xi)]\Big](\alpha,\beta)\Big\|_{H^{s-1}_\alpha H^\delta_\beta}\\
&\quad\lesssim\Bigg(\sum_{j,k=1}^{\infty}2^{2(s-2)j+2(\delta-1)k}\|\Delta_{j,k}f\|^2_{L^2(\mathbb{R}^2)}\Big\|\mathcal{F}^{-1}\Big[\frac{1}{(i\tau)(i\xi)}\psi(\tau)\psi(\xi)\Big]\Big\|^2_{L^1(\mathbb{R}^2)}\\
&\qquad\times\sum_{n=(k-1)\vee 1}^{k+1}\;\sum_{m=(j-1)\vee 1}^{j+1}2^{-2m+2j+2k-2n}\Bigg)^{\frac12}\\
&\quad\lesssim\Bigg(\sum_{j,k=1}^{\infty}2^{2(s-2)j+2(\delta-1)k}\|\Delta_{j,k}f\|^2_{L^2(\mathbb{R}^2)}\Big\|\mathcal{F}^{-1}\Big[\frac{1}{(i\tau)(i\xi)}\psi(\tau)\psi(\xi)\Big]\Big\|^2_{L^1(\mathbb{R}^2)}\Bigg)^{\frac12}\\
&\quad\lesssim_\psi\|f\|_{S^{s-2,\delta-1}_{2,2}B}\leq\|f\|_{S^{s-1,\delta-1}_{2,2}B}.
\end{aligned}\tag{5.4.38}$$

Hence by combining the estimates (5.4.34), (5.4.37) and (5.4.38) in the definition of G we have

$$\Big\|\eta_T(\alpha)\chi_T(\beta)G(\alpha,\beta)\Big\|_{\mathbb{H}^{s,\delta}}\lesssim_{\eta,\chi,T,\psi}\|f\|_{\mathbb{H}^{s-1,\delta-1}}.\tag{5.4.39}$$

Recall that

$$\begin{aligned}
I(\alpha,\beta):=&\int_{-\beta}^{\alpha}\sum_{n=1}^{\infty}\mathcal{F}^{-1}\Big[\frac{1}{i\xi}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\Big](\gamma,\beta)\,d\gamma\\
&-\int_{-\beta}^{\alpha}\sum_{n=1}^{\infty}\mathcal{F}^{-1}\Big[\frac{1}{i\xi}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\Big](\gamma,-\gamma)\,d\gamma.
\end{aligned}$$

First we see that due to the choice of dyadic partition

$$\sum_{n=1}^{\infty}\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)=\frac{1}{(i\xi)}\varphi_0(\tau)(1-\varphi_0(\xi))(\mathcal{F}f)(\tau,\xi).$$

Thus, since φ_0 is a smooth function and $\mathcal{F}f \in \mathcal{S}'(\mathbb{R}^2)$, we get

$$\sum_{n=1}^{\infty} \frac{1}{(i\xi)} \varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi) \quad \text{is a well-defined tempered distribution.}$$

Similarly, we can conclude for the corresponding part coming in the definition of J. Our next result provides the estimate for I and J.

Lemma 5.4.13. *The terms I and J satisfy, respectively, the following estimate*

$$\|\eta_T(\alpha)\chi_T(\beta)I\|_{\mathbb{H}^{s,\delta}} \lesssim_{\eta,\chi,T,\psi} \|f\|_{\mathbb{H}^{s-1,\delta-1}}, \tag{5.4.40}$$

and

$$\|\eta_T(\alpha)\chi_T(\beta)J\|_{\mathbb{H}^{s,\delta}} \lesssim_{\eta,\chi,T,\psi} \|f\|_{\mathbb{H}^{s-1,\delta-1}}. \tag{5.4.41}$$

Proof of Lemma 5.4.13 Due to the similarity in the definition of I and J, it is enough to show the estimate of I only. Observe that by invoking Lemma 5.4.1 we obtain

$$\begin{aligned}
\|\eta_T(\alpha)\chi_T(\beta)I\|_{H^s_\alpha H^\delta_\beta} \lesssim_{\eta,\chi,T} & \left\|\chi_T(\beta)\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right](\alpha,\beta)\right\|_{H^{s-1}_\alpha H^\delta_\beta} \\
&+\left\|\eta_T(\alpha)\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right](-\beta,\beta)\right\|_{H^s_\alpha H^{\delta-1}_\beta} \\
&+\left\|\chi_T(\beta)\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right](\alpha,-\alpha)\right\|_{H^{s-1}_\alpha H^\delta_\beta} \\
&+\left\|\eta_T(\alpha)\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right](-\beta,\beta)\right\|_{H^s_\alpha H^{\delta-1}_\beta}.
\end{aligned} \tag{5.4.42}$$

To handle the first term in the right hand side of (5.4.42), for fix $j,k \geq 0$, the choice of dyadic partition, continuity of $\mathcal{F}: \mathcal{S}'(\mathbb{R}^2) \to \mathcal{S}'(\mathbb{R}^2)$ and convolution theorem for Fourier transform implies

$$\begin{aligned}
&\mathcal{F}^{-1}\left[\varphi_j(\tau)\varphi_k(\xi)\mathcal{F}\left(\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right]\right)\right] \\
&\qquad = \sum_{n=(k-1)\vee 1}^{k+1}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)\right] * \Delta_{j,k}f.
\end{aligned}$$

Consequently, by invoking Lemmata 5.1.14, 5.4.2 and the Young inequality we get

$$\begin{aligned}
&\left\|\chi_T(\beta)\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right](\alpha,\beta)\right\|_{H^{s-1}_\alpha H^\delta_\beta} \\
&\lesssim_{\chi,T}\left(\sum_{j,k=0}^{\infty}2^{2(s-1)j+2\delta k}\left\|\sum_{n=(k-1)\vee 1}^{k+1}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)\right] * \Delta_{j,k}f\right\|^2_{L^2}\right)^{1/2} \\
&\lesssim\left(\sum_{j,k=0}^{\infty}\|\Delta_{j,k}f\|^2_{L^2}2^{2(s-1)j+2(\delta-1)k}\left\|\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\psi(\xi)\right]\right\|^2_{L^1(\mathbb{R}^2)}\sum_{n=(k-1)\vee 1}^{k+1}2^{2k-2n}\right)^{1/2} \\
&\lesssim_\psi \|f\|_{S^{s-1,\delta-1}_{2,2}B}.
\end{aligned} \tag{5.4.43}$$

Now for trace terms in (5.4.42), the tensor product argument used in Theorem 5.1.14 followed by Lemma 5.4.4 and the computation based on (5.4.43) gives

$$
\begin{aligned}
&\left\|\eta_T(\alpha)\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right](-\beta,\beta)\right\|^2_{H^s_\alpha H^{\delta-1}_\beta}\\
&\qquad\lesssim_{\eta,T}\left\|\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right](-\beta,\beta)\right\|^2_{H^{\delta-1}_\beta}\\
&\qquad\lesssim\left\|\sum_{n=1}^{\infty}\mathcal{F}^{-1}\left[\frac{1}{(i\xi)}\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi)\right](\alpha,\beta)\right\|^2_{H^s_\alpha H^{\delta-1}_\beta}\\
&\qquad\lesssim_\psi \|f\|_{S^{s-1,\delta-1}_{2,2}B}.
\end{aligned}
\tag{5.4.44}
$$

Due to the similarity of Terms 3 and 4 with Term 2 in the right hand side of (5.4.42), it is easy to see that they also follow above estimate. Hence, by using (5.4.43)-(5.4.44) into (5.4.42) we get

$$\|\eta_T(\alpha)\chi_T(\beta)I\|_{H^s_\alpha H^\delta_\beta}\lesssim_\psi \|f\|_{S^{\delta-1,s-1}_{2,2}B}.$$

By interchange of the roles of α and β we get the estimate as the same local estimate for $H^s_\beta H^\delta_\alpha$-norm of I. Hence we finish the proof of this Lemma 5.4.13. □

Now, since $s,\delta>\frac{1}{2}$, $S^{s,\delta}_{2,2}B\hookrightarrow\mathcal{C}_{bu}(\mathbb{R}^2)$, where $\mathcal{C}_{bu}(\mathbb{R}^2)$ is the space of uniformly continuous and bounded functions from $\mathbb{R}^2$ to $\mathbb{R}$. So by (5.4.30), estimate (5.4.39), and Lemma 5.4.13, locally G,H,I and J belongs to $\mathcal{C}_{bu}(\mathbb{R}^2)$. Since $F=H+G+I+J$,

$$\eta_T(\alpha)\chi_T(\beta)F\in\mathcal{C}_{\mathrm{comp}}(\mathbb{R}^2).$$

Consequently, by [141, Proposition 2.2.3/4] we have

$$\eta_T(\alpha)\chi_T(\beta)F\in\mathcal{C}_u(\mathbb{R}^2)\subset L^\infty(\mathbb{R}^2)\subset S^{0,0}_{\infty,\infty}B(\mathbb{R}^2).$$

Hence $\eta_T(\alpha)\chi_T(\beta)F\in\mathcal{S}'(\mathbb{R}^2)$ and we can calculate the $S^{s,\delta}_{2,2}B(\mathbb{R}^2)$ norm. By Lemma 5.4.30, estimate (5.4.39), and Lemma 5.4.13 we have

$$\|\eta_T(\alpha)\chi_T(\beta)F\|_{\mathbb{H}^{s,\delta}}\lesssim_{\eta,\chi,T,\psi}\|f\|_{\mathbb{H}^{s-1,\delta-1}}.\tag{5.4.45}$$

Finally by the properties of Fourier transform, for e.g. [138, Theorem 7.15], and since $f\in\mathcal{S}'(\mathbb{R}^2)$, we obtain

$$
\begin{aligned}
\frac{\partial^2 F}{\partial\alpha\partial\beta}&=\mathcal{F}^{-1}(\varphi_0(\tau)\varphi_0(\xi)(\mathcal{F}f)(\tau,\xi))+\sum_{j,k=1}^{\infty}\mathcal{F}^{-1}(\varphi_j(\tau)\varphi_k(\xi)(\mathcal{F}f)(\tau,\xi))\\
&\quad+\sum_{n=1}^{\infty}\mathcal{F}^{-1}(\varphi_0(\tau)\varphi_n(\xi)(\mathcal{F}f)(\tau,\xi))+\sum_{m=1}^{\infty}\mathcal{F}^{-1}(\varphi_m(\tau)\varphi_0(\xi)(\mathcal{F}f)(\tau,\xi))\\
&=\sum_{j,k=0}^{\infty}\Delta_{j,k}f=f.
\end{aligned}
$$

Hence we finish the proof of Proposition 5.4.9 since the uniqueness follows from the expression of F. Indeed, because the constructed F solves the inhomogeneous wave equation with null initial data in (α,β)-coordinates. ■

The following lemma shows the behaviour of $\diamond^{-1}$, see Proposition 5.4.9 for its definition, with two parameter scaled operator Π_λ.

Lemma 5.4.14. *For every $\lambda \geq 1$ the following holds*

$$\left(\diamond^{-1}[\Pi_\lambda(f)]\right)(\lambda\alpha,\lambda\beta) = (\diamond^{-1}f)(\alpha,\beta), \quad f \in \mathbb{H}^{s-1,\delta-1}, (\alpha,\beta) \in \mathbb{R}^2, \tag{5.4.46}$$

where the action of Π_λ on f is defined in (5.4.17).

Proof of Lemma 5.4.14 First notice that In this proof we use Theorem 5.1.14 recursively without specifying it. Suppose we take $f \in \mathcal{C}^\infty_{\rm comp}(\mathbb{R}^2)$, then an easy computation based on the change of variable, as done in (5.4.4), gives (5.4.46). For any fix $\lambda \geq 1$, let us define

$$\widetilde{\mathfrak{J}}_\lambda : \mathbb{H}^{s-1,\delta-1} \ni f \mapsto [\widetilde{M}_\lambda \diamond^{-1} \Pi_\lambda] f \in \mathbb{H}^{s,\delta}_{\rm loc},$$

where $[\widetilde{M}_\lambda g](\cdot,\cdot) := g(\lambda\cdot,\lambda\cdot)$. Observe that for $f \in \mathcal{C}^\infty_{\rm comp}(\mathbb{R}^2)$,

$$[\widetilde{M}_\lambda \diamond^{-1}\Pi_\lambda] f(\alpha,\beta) = (\diamond^{-1} f)(\alpha,\beta). \tag{5.4.47}$$

By Lemma 5.4.7, $\Pi_\lambda : \mathbb{H}^{s-1,\delta-1} \to \mathbb{H}^{s-1,\delta-1}$ is well-defined and satisfies

$$\|\Pi_\lambda g\|_{\mathbb{H}^{s-1,\delta-1}} \leq \lambda^{1-s-\delta} \|g\|_{\mathbb{H}^{s-1,\delta-1}}. \tag{5.4.48}$$

Next we claim that $\widetilde{M}_\lambda : \mathbb{H}^{s,\delta}_{\rm loc} \to \mathbb{H}^{s,\delta}_{\rm loc}$ is well-defined. Indeed, let $\varphi,\psi \in \mathcal{C}^\infty_{\rm comp}(\mathbb{R})$, then for every $g \in H^s_{\rm loc} H^\delta_{\rm loc}(\mathbb{R}^2)$ we have

$$\begin{aligned}
\|\varphi(\alpha)\psi(\beta)[\widetilde{M}_\lambda g](\alpha,\beta)\|^2_{H^s_\alpha H^\delta_\beta(\mathbb{R}^2)} &\simeq \int_{\mathbb{R}^2} |\varphi(\alpha)\psi(\beta) g(\lambda\alpha,\lambda\beta)|^2\, d\alpha\, d\beta \\
&\quad + \int_{\mathbb{R}^4} \frac{|\varphi(\alpha)\psi(\beta)g(\lambda\alpha,\lambda\beta) - \varphi(\alpha)\psi(\beta)g(\lambda a,\lambda b)|^2}{|\alpha-a|^{1+2s}|\beta-b|^{1+2\delta}}\, d\alpha\, da\, d\beta\, db \\
&= \frac{1}{\lambda^2}\int_{\mathbb{R}^2} |\varphi(\alpha/\lambda)\psi(\beta/\lambda)g(\alpha,\beta)|^2\, d\alpha\, d\beta \\
&\quad + \frac{\lambda^{2+2s+2\delta}}{\lambda^4}\int_{\mathbb{R}^4} \frac{|\varphi(\alpha/\lambda)\psi(\beta/\lambda)g(\alpha,\beta) - \varphi(\alpha/\lambda)\psi(\beta/\lambda)g(a,b)|^2}{|\alpha-a|^{1+2s}|\beta-b|^{1+2\delta}}\, d\alpha\, da\, d\beta\, db \\
&= \frac{1}{\lambda^2}\|\varphi_\lambda(\alpha)\psi_\lambda(\beta)g(\alpha,\beta)\|^2_{L^2(\mathbb{R}^2)} + \frac{1}{\lambda^{2-2s-2\delta}}\|\varphi_\lambda(\alpha)\psi_\lambda(\beta)g(\alpha,\beta)\|^2_{\dot{H}^s_\alpha \dot{H}^\delta_\beta(\mathbb{R}^2)} < \infty.
\end{aligned} \tag{5.4.49}$$

Similarly we can show the finiteness of $H^\delta_\alpha H^s_\beta(\mathbb{R}^2)$-norm. Since $\varphi,\psi \in \mathcal{C}^\infty_{\rm comp}(\mathbb{R})$ and λ is fixed, there exists an $m \in \mathbb{N}$ such that supp $\varphi \times$ supp $\psi \subset [-m\lambda, m\lambda]^2$. Thus (5.4.49) with Proposition 5.4.9 and estimate (5.4.48) gives, for any $f \in \mathcal{C}^\infty_{\rm comp}(\mathbb{R}^2)$,

$$\begin{aligned}
\|\varphi(\alpha)\psi(\beta)[\widetilde{\mathfrak{J}}_\lambda f](\alpha,\beta)\|^2_{\mathbb{H}^{s,\delta}} &= \frac{1}{\lambda^2}\|\varphi_\lambda(\alpha)\psi_\lambda(\beta)[\diamond^{-1}\Pi_\lambda](\alpha,\beta)\|^2_{L^2(\mathbb{R}^2)} \\
&\quad + \frac{1}{\lambda^{2-2s-2\delta}}\|\varphi_\lambda(\alpha)\psi_\lambda(\beta)[\diamond^{-1}\Pi_\lambda](\alpha,\beta)\|^2_{\dot{\mathbb{H}}^{s,\delta}} \\
&\leq C(\varphi,\psi,m,\lambda)\frac{1}{\lambda^{2-2s-2\delta}}\ \|\Pi_\lambda f\|^2_{\mathbb{H}^{s-1,\delta-1}} = C(\varphi,\psi,m,\lambda)\ \|f\|^2_{\mathbb{H}^{s-1,\delta-1}(\mathbb{R})}.
\end{aligned}$$

So we have proved that the map $\widetilde{\mathcal{J}}_\lambda : \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}) \ni f \mapsto M_\lambda \mathcal{P}\Lambda_\lambda f \in \mathbb{H}^{s,\delta}_{\text{loc}}(\mathbb{R})$ is well defined and (5.4.47) holds. In particular,

$$[\widetilde{\mathcal{J}}_\lambda f](\alpha,\beta) = [\diamond^{-1} f](x), \qquad f \in \mathcal{C}^\infty_{\text{comp}}(\mathbb{R}^2). \tag{5.4.50}$$

Next, since $\mathcal{C}^\infty_{\text{comp}}(\mathbb{R}) \hookrightarrow \mathbb{H}^{s-1,\delta-1}(\mathbb{R})$ densely and $\mathbb{H}^{s,\delta}_{\text{loc}}(\mathbb{R})$ is a Fréchet space, there exists a unique continuous linear extension $\widetilde{\mathcal{J}}_\lambda : \mathbb{H}^{s-1,\delta-1} \to \mathbb{H}^{s,\delta}_{\text{loc}}$ and (5.4.50) holds as well. Hence we have proved Lemma 5.4.14. ■

5.4.5 The non-linearity

Now we move to find the estimate for the non-linear term. Recall that, from (5.3.1),

$$\mathcal{N}(u) = -\sum_{a,b=1}^{2} \Gamma_{ab}(u) \frac{\partial u^a}{\partial \alpha} \frac{\partial u^b}{\partial \beta},$$

we avoid writing asterisk (*) mark for simplicity. Note that the above implies $\mathcal{N}(0) = 0$. Our next result gives the required growth and the Lipschitz property for (5.3.5) which involves $\mathcal{N}(u)$ with the assumption (5.2.6) we have made on the structure of Γ.

Lemma 5.4.15. *For $s,\delta \in \left(\frac{3}{4},1\right)$ such that $\delta \le s$, there exists a natural number $\gamma \ge 2$ such that*

$$\|\eta(\alpha)\chi(\beta)\diamond^{-1}(\mathcal{N}(\phi)\|_{\mathbb{H}^{s,\delta}} \lesssim_{\eta,\chi} \|\phi\|^{\gamma+1}_{\mathbb{H}^{s,\delta}}, \tag{5.4.51}$$

and

$$\|\eta(\alpha)\chi(\beta)\diamond^{-1}(\mathcal{N}(\phi)-\mathcal{N}(\psi))\|_{\mathbb{H}^{s,\delta}} \lesssim_{\eta,\chi} \|\phi-\psi\|_{\mathbb{H}^{s,\delta}} \left[\|\phi\|_{\mathbb{H}^{s,\delta}} + \|\psi\|_{\mathbb{H}^{s,\delta}}\right]^\gamma. \tag{5.4.52}$$

Proof of Lemma 5.4.15 Observe that in our notation, for some $r \in \mathbb{N}$ with the index of summation $l = (l_1, l_2)$, we have

$$\mathcal{N}(\phi) - \mathcal{N}(\psi) = \sum_{a,b=1}^{2} \sum_{|l|=0}^{r} A^l_{ab} \Big(\left[\psi^1(\cdot)\right]^{l_1} \left[\psi^2(\cdot)\right]^{l_2} \partial_\alpha \psi^a \partial_\beta \psi^b - \left[\phi^1(\cdot)\right]^{l_1} \left[\phi^2(\cdot)\right]^{l_2} \partial_\alpha \phi^a \partial_\beta \phi^b \Big).$$

For fix a,b and (l_1,l_2), by adding and subtracting the mixed term we get

$$\begin{aligned}
&\left[\psi^1(\cdot)\right]^{l_1} \left[\psi^2(\cdot)\right]^{l_2} \partial_\alpha \psi^a \partial_\beta \psi^b - \left[\phi^1(\cdot)\right]^{l_1} \left[\phi^2(\cdot)\right]^{l_2} \partial_\alpha \phi^a \partial_\beta \phi^b \\
&= \Big(\left[\psi^1(\cdot)\right]^{l_1} - \left[\phi^1(\cdot)\right]^{l_1} \Big) \left[\psi^2(\cdot)\right]^{l_2} \partial_\alpha \psi^a \partial_\beta \psi^b + \left[\phi^1(\cdot)\right]^{l_1} \Big\{ \partial_\alpha \psi^a \partial_\beta \psi^b - \partial_\alpha \phi^a \partial_\beta \phi^b \Big\} \left[\psi^2(\cdot)\right]^{l_2} \\
&\quad + \left[\phi^1(\cdot)\right]^{l_1} \Big\{ \left[\psi^2(\cdot)\right]^{l_2} - \left[\phi^2(\cdot)\right]^{l_2} \Big\} \partial_\alpha \phi^a \partial_\beta \phi^b.
\end{aligned} \tag{5.4.53}$$

So for fixed a,b and $l = (l_1,l_2)$, we get three terms inside summation and due to the linearity of $\diamond^{-1}$ it is enough to estimate each separately. Note that to avoid the complexity in notation we use the symbol $\|\cdot\|_{H^{s-1}_\beta H^{\delta-1}_\alpha}$ for both $\mathbb{R}^2$ and $\mathbb{R}$ valued functions in the remaining calculations of this proof.

Invoking Proposition 5.4.9 followed by inequalities (5.4.2) and (5.4.3) with appropriate exponents yields

$$
\begin{aligned}
&\|\eta(\alpha)\chi(\beta)\diamond^{-1}\left(\left(\left[\psi^1(\cdot)\right]^{l_1}-\left[\phi^1(\cdot)\right]^{l_1}\right)\left[\psi^2(\cdot)\right]^{l_2}\partial_\alpha\psi^a\partial_\beta\psi^b\right)\|^2_{\mathbb{H}^{s,\delta}}\\
&\quad\lesssim_{\eta,\chi}\|\left(\left[\psi^1(\cdot)\right]^{l_1}-\left[\phi^1(\cdot)\right]^{l_1}\right)\left[\psi^2(\cdot)\right]^{l_2}\partial_\alpha\psi^a\partial_\beta\psi^b\|^2_{H^{s-1}_\alpha H^{\delta-1}_\beta}\\
&\qquad+\|\left(\left[\psi^1(\cdot)\right]^{l_1}-\left[\phi^1(\cdot)\right]^{l_1}\right)\left[\psi^2(\cdot)\right]^{l_2}\partial_\alpha\psi^a\partial_\beta\psi^b\|^2_{H^{s-1}_\beta H^{\delta-1}_\alpha}\\
&\quad\lesssim\|\left[\psi^1(\cdot)\right]^{l_1}-\left[\phi^1(\cdot)\right]^{l_1}\|^2_{H^s_\alpha H^\delta_\beta}\|\left[\psi^2(\cdot)\right]^{l_2}\|^2_{H^s_\alpha H^\delta_\beta}\|\partial_\alpha\psi^a\|^2_{H^{s-1}_\alpha H^\delta_\beta}\|\partial_\beta\psi^b\|^2_{H^s_\alpha H^{\delta-1}_\beta}\\
&\qquad+\|\left[\psi^1(\cdot)\right]^{l_1}-\left[\phi^1(\cdot)\right]^{l_1}\|^2_{H^\delta_\alpha H^s_\beta}\|\left[\psi^2(\cdot)\right]^{l_2}\|^2_{H^\delta_\alpha H^s_\beta}\|\partial_\alpha\psi^a\|^2_{H^{\delta-1}_\alpha H^s_\beta}\|\partial_\beta\psi^b\|^2_{H^\delta_\alpha H^{s-1}_\beta}\\
&\quad\lesssim\|\psi-\phi\|^2_{H^s_\alpha H^\delta_\beta}\left(\|\psi\|_{H^s_\alpha H^\delta_\beta}+\|\phi\|_{H^s_\alpha H^\delta_\beta}\right)^{2\bar{\gamma}}\|\psi\|^{2\bar{\gamma}+4}_{H^s_\alpha H^\delta_\beta}\\
&\qquad+\|\psi-\phi\|^2_{H^\delta_\alpha H^s_\beta}\left(\|\psi\|_{H^\delta_\alpha H^s_\beta}+\|\phi\|_{H^\delta_\alpha H^s_\beta}\right)^{2\bar{\gamma}}\|\psi\|^{2\bar{\gamma}+4}_{H^\delta_\alpha H^s_\beta}\\
&\quad\lesssim\|\psi-\phi\|^2_{\mathbb{H}^{s,\delta}}\left(\|\psi\|_{\mathbb{H}^{s,\delta}}+\|\phi\|_{\mathbb{H}^{s,\delta}}\right)^{2\gamma_1},
\end{aligned}
\tag{5.4.54}
$$

for some suitable $\gamma_1\geq 2$. Note that the existence of $\bar{\gamma},\gamma_1$ is possible because

$$\|\psi^b\|_{H^s_\alpha H^\delta_\beta},\|\psi^b\|_{H^s_\alpha H^\delta_\beta}\leq\|\psi\|_{H^s_\alpha H^\delta_\beta}\qquad\text{and}\qquad l_i\leq|l|\leq r, i=1,2.$$

The third term in (5.4.53) can be estimated similarly for some suitable γ_2. By writing the expression $\partial_\alpha\psi^a\partial_\beta\psi^b-\partial_\alpha\phi^a\partial_\beta\phi^b$ equivalently as

$$\partial_\alpha\psi^a\partial_\beta\psi^b-\partial_\alpha\psi^a\partial_\beta\phi^b+\partial_\alpha\psi^a\partial_\beta\phi^b-\partial_\alpha\phi^a\partial_\beta\phi^b,$$

followed by the calculations similar to (5.4.54), the second term of (5.4.53) can be estimated similarly for some γ_3. Hence we get (5.4.52) since a,b and (l_1,l_2) take only finitely many values. Moreover, we get (5.4.51) by substituting 0 instead of ψ in (5.4.52), since $\mathcal{N}(0)=0$. Hence the Lemma 5.4.15. ■

Remark 5.4.16. By repeating the steps of the proof of Lemma 5.4.15 we infer that the map $\mathcal{N}$ defined by

$$\mathcal{N}:\mathbb{H}^{s,\delta}\ni u\mapsto\sum_{a,b=1}^{2}\Gamma_{ab}(u)\partial_\alpha u^a\partial_\beta u^b\in\mathbb{H}^{s-1,\delta-1},$$

is well-defined.

Before going into the local well-posedness theory part, let us prove the following generalization of [133, Lemma 3.1] which specify the property we have on diffusion coefficient. Recall that we are dealing in the range $s,\delta\in\left(\frac{3}{4},1\right)$ and $s\leq\delta$.

Proposition 5.4.17. *Assume that $\sigma\in\mathcal{C}^3_b(\mathbb{R}^2)$. Then $\sigma\circ u\in\mathbb{H}^{s,\delta}$ for every $u\in\mathbb{H}^{s,\delta}$ and there exist constants $C_i(\sigma):=C_i(\|\sigma\|_{\mathcal{C}^{i+1}_b})$, $i=1,2$ such that for $u,u_1,u_2\in\mathbb{H}^{s,\delta}$,*

$$\|\sigma\circ u\|^2_{\mathbb{H}^{s,\delta}}\leq C_1(\sigma)\|u\|^2_{\mathbb{H}^{s,\delta}}\left[1+\|u\|^2_{\mathbb{H}^{s,\delta}}\right],\tag{5.4.55}$$

$$\|\sigma\circ u_1-\sigma\circ u_2\|^2_{\mathbb{H}^{s,\delta}}\le C_2(\sigma)\|u_2-u_1\|^2_{\mathbb{H}^{s,\delta}}\left[1+\sum_{i,k=1}^{2}\|u_i\|^{2k}_{\mathbb{H}^{s,\delta}}\right]. \tag{5.4.56}$$

Proof of Proposition 5.4.17 To shorten the notation we avoid writing explicitly that the Euclidean norms separately for $\mathbb{R}^2$ and $\mathbb{R}$ valued functions, unless any confusion. Recall that by Definition 5.1.12

$$\|\sigma\circ u\|_{H^s_\alpha(\mathbb{R};H^\delta_\beta(\mathbb{R};\mathbb{R}^2))}\simeq\|\sigma\circ u\|_{L^2_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R};\mathbb{R}^2))}+\|\sigma\circ u\|_{\dot W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R};\mathbb{R}^2))}, \tag{5.4.57}$$

where for any separable Banach space E,

$$\|u\|_{\dot W^s(\mathbb{R};E)}=\left(\int_{\mathbb{R}^2}\frac{\|u(x)-u(y)\|^2_E}{|x-y|^{1+2s}}\,dx\,dy\right)^{\frac12}.$$

By expanding each term of right hand side of (5.4.57) we get

$$\begin{aligned}\|\sigma\circ u\|^2_{H^s_\alpha(\mathbb{R};H^\delta_\beta(\mathbb{R}))}&\lesssim\int_{\mathbb{R}^2}|[\sigma\circ u](\alpha,\beta)|^2\,d\beta\,d\alpha+\int_{\mathbb{R}^3}\frac{|[\sigma\circ u](\alpha,\beta_1)-[\sigma\circ u](\alpha,\beta_2)|^2}{|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha\\&+\int_{\mathbb{R}^3}\frac{|[\sigma\circ u](\alpha_1,\beta)-[\sigma\circ u](\alpha_2,\beta)|^2}{|\alpha_1-\alpha_2|^{1+2s}}\,d\beta\,d\alpha_1\,d\alpha_2\\&+\int_{\mathbb{R}^4}\frac{|[\sigma\circ u](\alpha_1,\beta_1)-[\sigma\circ u](\alpha_2,\beta_1)-\{[\sigma\circ u](\alpha_1,\beta_2)-[\sigma\circ u](\alpha_2,\beta_2)\}|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2\end{aligned}$$

$$=:A_1+A_2+A_3+A_4. \tag{5.4.58}$$

Since $\sigma:\mathbb{R}^2\to\mathbb{R}^2$ is continuous and bounded, we estimate the term A_1 as

$$A_1\le\|\sigma\|^2_{L^\infty(\mathbb{R}^2)}\ \|u\|^2_{L^2_\alpha L^2_\beta(\mathbb{R}^2)}\lesssim\|\sigma\|^2_{L^\infty(\mathbb{R}^2)}\ \|u\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}. \tag{5.4.59}$$

Since $\sigma\in\mathcal{C}^3_b(\mathbb{R}^2)\subset\mathcal{C}^1_b(\mathbb{R}^2)$, it is Lipschitz and by denoting the Lipschitz constant as L_σ we have the following estimate for A_2

$$A_2\le L^2_\sigma\int_{\mathbb{R}^3}\frac{|u(\alpha,\beta_1)-u(\alpha,\beta_2)|^2}{|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha\lesssim L^2_\sigma\ \|u\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}. \tag{5.4.60}$$

By interchanging the roles of variables we see that the term A_3 satisfies the same estimate as A_2. Now we move to estimate the term A_4. For fix $\alpha_1,\alpha_2,\beta_1,\beta_2\in\mathbb{R}$, let us denote the rectangle by

$$Q:=[\alpha_1,\alpha_2]\times[\beta_1,\beta_2]\subset\mathbb{R}^2.$$

Our first two claims are elementary which we write without proof.

Lemma 5.4.18. *With our notation if we define*

$$\Delta_Q[\sigma\circ u]:=[\sigma\circ u](\alpha_1,\beta_1)+[\sigma\circ u](\alpha_2,\beta_2)-[\sigma\circ u](\alpha_2,\beta_1)-[\sigma\circ u](\alpha_1,\beta_2),$$

then

$$\Delta_Q[\sigma\circ u]=\int_0^1\int_0^1\frac{\partial^2[\sigma\circ a]}{\partial\tau\partial\xi}(\tau,\xi)\,d\tau\,d\xi,$$

where $a : [0,1]^2 \to \mathbb{R}^2$ *is defined by, for* $(\tau,\xi) \in [0,1]^2$,

$$a(\tau,\xi) := u(\alpha_1,\beta_1) + \tau[u(\alpha_2,\beta_1) - u(\alpha_1,\beta_1)] + \xi[u(\alpha_1,\beta_2) - u(\alpha_1,\beta_1)] + \tau\xi\Delta_Q u, \tag{5.4.61}$$

and

$$\Delta_Q u := u(\alpha_1,\beta_1) + u(\alpha_2,\beta_2) - u(\alpha_1,\beta_2) - u(\alpha_2,\beta_1).$$

Next result is about writing $\frac{\partial^2[\sigma\circ a]}{\partial\xi\partial\tau}$ in terms of partial derivatives of σ.

Lemma 5.4.19. *In our notation for all* $\tau,\xi \in \mathbb{R}$, *we have*

$$\frac{\partial^2[\sigma\circ a]}{\partial\xi\partial\tau}(\tau,\xi) = \sum_{i=1}^{2} \frac{\partial\sigma}{\partial x_i}(a(\tau,\xi))\, \frac{\partial^2 a^i}{\partial\xi\partial\tau}(\tau,\xi) + \sum_{i,j=1}^{2} \frac{\partial^2\sigma}{\partial x_j \partial x_i}(a(\tau,\xi))\, \frac{\partial a^j}{\partial\xi}(\tau,\xi)\, \frac{\partial a^i}{\partial\tau}(\tau,\xi).$$

Thus by invoking Lemmata 5.4.18 and 5.4.19 in the expression of A_4 we get

$$\begin{aligned} A_4 &\lesssim \sum_{i=1}^{2} \int_{\mathbb{R}^4} \frac{\left|\int_0^1\int_0^1 \left[\frac{\partial\sigma}{\partial x_i}(a(\tau,\xi))\, \frac{\partial^2 a^i}{\partial\xi\partial\tau}(\tau,\xi)\right] d\tau\, d\xi\right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \\ &\quad + \sum_{i,j=1}^{2} \int_{\mathbb{R}^4} \frac{\left|\int_0^1\int_0^1 \left[\frac{\partial^2\sigma}{\partial x_j\partial x_i}(a(\tau,\xi))\, \frac{\partial a^j}{\partial\xi}(\tau,\xi)\, \frac{\partial a^i}{\partial\tau}(\tau,\xi)\right] d\tau\, d\xi\right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \\ &=: \sum_{i=1}^{2} A_4^i + \sum_{i,j=1}^{2} A_4^{ij}. \end{aligned} \tag{5.4.62}$$

We will estimate each term in the right hand side of (5.4.62) separately but first note that for fixed rectangle $Q = [\alpha_1,\alpha_2]\times[\beta_1,\beta_2]$ by using (5.4.61), we have $\frac{\partial^2 a^i}{\partial\xi\partial\tau} = \Delta_Q u^i$ and consequently we obtain

$$\begin{aligned} &\left|\int_0^1\int_0^1 \left[\frac{\partial\sigma}{\partial x_i}(a(\tau,\xi))\, \frac{\partial^2 a^i}{\partial\xi\partial\tau}(\tau,\xi)\right] d\tau\, d\xi\right| \\ &\qquad \le \left\|\frac{\partial\sigma}{\partial x_i}\right\|_{L^\infty(\mathbb{R}^2)} |u^i(\alpha_1,\beta_1) - u^i(\alpha_1,\beta_2) - u^i(\alpha_2,\beta_1) + u^i(\alpha_2,\beta_2)|. \end{aligned}$$

Using above A_4^i can be estimated as

$$\begin{aligned} A_4^i &\le \left\|\frac{\partial\sigma}{\partial x_i}\right\|_{L^\infty(\mathbb{R}^2)}^2 \int_{\mathbb{R}^4} \frac{|u^i(\alpha_1,\beta_1) - u^i(\alpha_1,\beta_2) - u^i(\alpha_2,\beta_1) + u^i(\alpha_2,\beta_2)|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \\ &\le \left\|\frac{\partial\sigma}{\partial x_i}\right\|_{L^\infty(\mathbb{R}^2)}^2 \|u\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}. \end{aligned} \tag{5.4.63}$$

Next, since for fixed $Q = [\alpha_1,\alpha_2]\times[\beta_1,\beta_2]$ by (5.4.61) we have

$$\begin{aligned} \frac{\partial a^i}{\partial\tau}(\tau,\xi) &= u^i(\alpha_2,\beta_1) - u^i(\alpha_1,\beta_1) + \xi\Delta_Q u^i, \\ \frac{\partial a^j}{\partial\xi}(\tau,\xi) &= u^j(\alpha_1,\beta_2) - u^j(\alpha_1,\beta_1) + \tau\Delta_Q u^j, \end{aligned}$$

the double integral term in A_4^{ij} satisfy the following,

$$\left|\int_0^1\int_0^1\left[\frac{\partial^2\sigma}{\partial x_j\partial x_i}(a(\tau,\xi))\,\frac{\partial a^j}{\partial\xi}(\tau,\xi)\,\frac{\partial a^i}{\partial\tau}(\tau,\xi)\right]d\tau\,d\xi\right|$$

$$\leq\left\|\frac{\partial^2\sigma}{\partial x_j\partial x_i}\right\|_{L^\infty(\mathbb{R}^2)}\int_0^1\int_0^1\Big[|u^i(\alpha_2,\beta_1)-u^i(\alpha_1,\beta_1)|\times|u^j(\alpha_1,\beta_2)-u^j(\alpha_1,\beta_1)|$$

$$+\xi|\Delta_Q u^i|\times|u^j(\alpha_1,\beta_2)-u^j(\alpha_1,\beta_1)|+\tau|\Delta_Q u^j|\times|u^i(\alpha_2,\beta_1)-u^i(\alpha_1,\beta_1)|$$

$$+\tau\xi|\Delta_Q u^i|\times|\Delta_Q u^j|\Big]\,d\tau\,d\xi$$

$$=:\left\|\frac{\partial^2\sigma}{\partial x_j\partial x_i}\right\|_{L^\infty(\mathbb{R}^2)}\{B_1+B_2+B_3+B_4\}. \tag{5.4.64}$$

By substituting (5.4.64) in terms A_4^{ij} from (5.4.62) we obtain

$$A_4^{ij}\lesssim\left\|\frac{\partial^2\sigma}{\partial x_j\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}\left[\int_{\mathbb{R}^4}\frac{B_1^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2\right.$$

$$+\int_{\mathbb{R}^4}\frac{B_2^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2$$

$$+\int_{\mathbb{R}^4}\frac{B_3^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2$$

$$\left.+\int_{\mathbb{R}^4}\frac{B_4^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2\right]$$

$$=:\left\|\frac{\partial^2\sigma}{\partial x_j\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}\left[B_1^{ij}+B_2^{ij}+B_3^{ij}+B_4^{ij}\right]. \tag{5.4.65}$$

By substituting B_1 from (5.4.64) and using Theorem 5.1.15, since $s,\delta>\frac{1}{2}$, term B_1^{ij} satisfies

$$B_1^{ij}\leq\int_{\mathbb{R}^4}\left[\text{ess sup}_{\beta_1\in\mathbb{R}}\frac{|u^i(\alpha_2,\beta_1)-u^i(\alpha_1,\beta_1)|^2}{|\alpha_1-\alpha_2|^{1+2s}}\right]$$

$$\times\left[\text{ess sup}_{\alpha_1\in\mathbb{R}}\frac{|u^j(\alpha_1,\beta_2)-u^j(\alpha_1,\beta_1)|^2}{|\beta_1-\beta_2|^{1+2\delta}}\right]d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2$$

$$\lesssim\|u^i\|^2_{W^\delta_\beta(\mathbb{R};W^s_\alpha(\mathbb{R}))}\,\|u^j\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}. \tag{5.4.66}$$

Similarly, we estimate B_2^{ij} as follows,

$$B_2^{ij}\lesssim\text{ess sup}_{\alpha,\beta\in\mathbb{R}}|u^j(\alpha,\beta)|^2\int_{\mathbb{R}^4}\left[\frac{|\Delta_Q u^i|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\right]d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2$$

$$\lesssim\|u^i\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\,\|u^j\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}. \tag{5.4.67}$$

Interchanging the roles of u^i with u^j and ξ by τ, we deduce that the term B_3^{ij} is bounded by the right hand side of (5.4.67). Similar computation gives

$$B_4^{ij}\lesssim\text{ess sup}_{\alpha,\beta\in\mathbb{R}}|u^j(\alpha,\beta)|^2\int_{\mathbb{R}^4}\left[\frac{|\Delta_Q u^i|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\right]d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2$$

$$\lesssim \|u^i\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} \, \|u^j\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}. \tag{5.4.68}$$

By substituting the estimates of $B^{ij}_k, k=1,\dots,4$ from (5.4.66)-(5.4.68) into (5.4.65) we get

$$A^{ij}_4 \lesssim \left\| \frac{\partial^2\sigma}{\partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \, \|u^i\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} \, \|u^j\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))},$$

and consequently with estimates of $A^i_4, i=1,\dots,4$ from (5.4.63) into (5.4.62) we have

$$A_4 \lesssim \left[\sum_{i=1}^2 \left\| \frac{\partial\sigma}{\partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} + \sum_{i,j=1}^2 \left\| \frac{\partial^2\sigma}{\partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \right] \left[\|u\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} + \|u\|^4_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} \right]. \tag{5.4.69}$$

Hence by substituting estimates from (5.4.59), (5.4.60), and (5.4.69) into (5.4.58) we obtain

$$\begin{aligned}
\|\sigma\circ u\|^2_{H^s_\alpha(\mathbb{R};H^\delta_\beta(\mathbb{R}))} &\lesssim \left[\|u\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} + \|u\|^4_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} \right] \\
&\quad \times \left[\|\sigma\|^2_{L^\infty(\mathbb{R}^2)} + L^2_\sigma + \sum_{i=1}^2 \left\| \frac{\partial\sigma}{\partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} + \sum_{i,j=1}^2 \left\| \frac{\partial^2\sigma}{\partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \right] \\
&=: C_1(\|\sigma\|_{\mathcal{C}^2_b(\mathbb{R}^2)}) \, \|u\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} \left[1 + \|u\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} \right].
\end{aligned} \tag{5.4.70}$$

Since we have not used any relationship between s and δ, by repeating the procedure of (5.4.70), we also get

$$\|\sigma\circ u\|^2_{H^s_\beta(\mathbb{R};H^\delta_\alpha(\mathbb{R}))} \lesssim C_1(\|\sigma\|_{\mathcal{C}^2_b(\mathbb{R}^2)}) \, \|u\|^2_{W^s_\beta(\mathbb{R};W^\delta_\alpha(\mathbb{R}))} \left[1 + \|u\|^2_{W^s_\beta(\mathbb{R};W^\delta_\alpha(\mathbb{R}))} \right],$$

which consequently allows us to conclude that $\sigma\circ u \in \mathbb{H}^{s,\delta}$ and the result (5.4.55) follows.

Now we move to a proof of (5.4.56). As in the first part of the proof it is enough to prove the local Lipschitz property w.r.t. the $\|\cdot\|_{H^s_\alpha(\mathbb{R};H^\delta_\beta(\mathbb{R}))}$-norm. Fix $u_1, u_2 \in \mathbb{H}^{s,\delta}$. Equivalence of H^s and W^s spaces, as (5.4.58), implies

$$\begin{aligned}
\|\sigma\circ u_1 - \sigma\circ u_2\|^2_{H^s_\alpha(\mathbb{R};H^\delta_\beta(\mathbb{R}))} &\lesssim \int_{\mathbb{R}^2} |[\sigma\circ u_1](\alpha,\beta) - [\sigma\circ u_2](\alpha,\beta)|^2 \, d\beta \, d\alpha \\
&+ \int_{\mathbb{R}^3} \frac{|[\sigma\circ u_1](\alpha,\beta_1) - [\sigma\circ u_2](\alpha,\beta_1) - \{[\sigma\circ u_1](\alpha,\beta_2) - [\sigma\circ u_2](\alpha,\beta_2)\}|^2}{|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1 \, d\beta_2 \, d\alpha \\
&+ \int_{\mathbb{R}^3} \frac{|[\sigma\circ u_1](\alpha_1,\beta) - [\sigma\circ u_2](\alpha_1,\beta) - \{[\sigma\circ u_1](\alpha_2,\beta) - [\sigma\circ u_2](\alpha_2,\beta)\}|^2}{|\alpha_1-\alpha_2|^{1+2s}} \, d\beta \, d\alpha_1 \, d\alpha_2 \\
&+ \int_{\mathbb{R}^4} \frac{\left| \begin{array}{l} [\sigma\circ u_1](\alpha_1,\beta_1) - [\sigma\circ u_2](\alpha_1,\beta_1) - \{[\sigma\circ u_1](\alpha_2,\beta_1) - [\sigma\circ u_2](\alpha_2,\beta_1)\} \\ -\left([\sigma\circ u_1](\alpha_1,\beta_2) - [\sigma\circ u_2](\alpha_1,\beta_2) - \{[\sigma\circ u_1](\alpha_2,\beta_2) - [\sigma\circ u_2](\alpha_2,\beta_2)\}\right) \end{array} \right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1 \, d\beta_2 \, d\alpha_1 \, d\alpha_2 \\
&=: D_1 + D_2 + D_3 + D_4.
\end{aligned} \tag{5.4.71}$$

Using the Lipschitz property of σ, term D_1, on the similar lines of (5.4.60), is estimated as

$$D_1 \leq L^2_\sigma \int_{\mathbb{R}^2} |u_1(\alpha,\beta) - u_2(\alpha,\beta)|^2 \, d\beta \, d\alpha \leq L^2_\sigma \, \|u_1 - u_2\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}. \tag{5.4.72}$$

To estimate D_2 term we need the following two elementary results whose proofs are straightforward.

Lemma 5.4.20. *For fix $\alpha,\beta_1,\beta_2\in\mathbb{R}$ and $u_1,u_2\in\mathbb{H}^{s,\delta}$ we have*

$$[\sigma\circ u_1](\alpha,\beta_1)-[\sigma\circ u_2](\alpha,\beta_1)-\{[\sigma\circ u_1](\alpha_1,\beta_2)-[\sigma\circ u_2](\alpha,\beta_2)\}=\int_0^1\int_0^1\frac{\partial^2[\sigma\circ b]}{\partial r\partial\xi}(r,\xi)\,d\xi\,dr,$$

where

$$b:[0,1]\times[0,1]\ni(r,\xi)\mapsto b_1(\xi)+r(b_2(\xi)-b_1(\xi))\in\mathbb{R}^2,$$

and for $i=1,2$

$$b_i(\xi):=u_i(\alpha,\beta_1)+\xi[u_i(\alpha,\beta_2)-u_i(\alpha,\beta_1)].$$

Lemma 5.4.21. *In our notation, we have*

$$\frac{\partial^2[\sigma\circ b]}{\partial r\partial\xi}(r,\xi)=\sum_{i=1}^2\frac{\partial\sigma}{\partial x_i}(b(r,\xi))\,\frac{\partial^2 b^i}{\partial\xi\partial r}(r,\xi)+\sum_{i,j=1}^2\frac{\partial^2\sigma}{\partial x_j\partial x_i}(b(r,\xi))\,\frac{\partial b^j}{\partial\xi}(r,\xi)\,\frac{\partial b^i}{\partial r}(r,\xi),$$

where

$$\begin{cases}\frac{\partial b^j}{\partial\xi}(r,\xi)=(1-r)[u_1^j(\alpha,\beta_2)-u_1^j(\alpha,\beta_1)]+r[u_2^j(\alpha,\beta_2)-u_2^j(\alpha,\beta_1)];\\ \frac{\partial b^i}{\partial r}(r,\xi)=u_2^i(\alpha,\beta_1)-u_1^i(\alpha,\beta_1)+\xi[u_2^i(\alpha,\beta_2)-u_1^i(\alpha,\beta_2)+u_1^i(\alpha,\beta_1)-u_2^i(\alpha,\beta_1)];\\ \frac{\partial^2 b^i}{\partial\xi\partial r}(r,\xi)=u_2^i(\alpha,\beta_2)-u_1^i(\alpha,\beta_2)+u_1^i(\alpha,\beta_1)-u_2^i(\alpha,\beta_1).\end{cases}$$

Invoking Lemmata 5.4.20 and 5.4.21, gives

$$\begin{aligned}D_2\lesssim&\sum_{i=1}^2\int_{\mathbb{R}^3}\frac{\left|\int_0^1\int_0^1[u_2^i(\alpha,\beta_2)-u_1^i(\alpha,\beta_2)+u_1^i(\alpha,\beta_1)-u_2^i(\alpha,\beta_1)]\,\frac{\partial\sigma}{\partial x_i}(b(r,\xi))\,d\xi\,dr\right|^2}{|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha\\ &+\sum_{i,j=1}^2\int_{\mathbb{R}^3}\frac{\left|\int_0^1\int_0^1\begin{bmatrix}\{(1-r)[u_1^j(\alpha,\beta_2)-u_1^j(\alpha,\beta_1)]+r[u_2^j(\alpha,\beta_2)-u_2^j(\alpha,\beta_1)]\}\times\\ \{u_2^i(\alpha,\beta_1)-u_1^i(\alpha,\beta_1)+\xi[u_2^i(\alpha,\beta_2)-u_1^i(\alpha,\beta_2)+u_1^i(\alpha,\beta_1)-u_2^i(\alpha,\beta_1)]\}\end{bmatrix}\frac{\partial^2\sigma}{\partial x_j\partial x_i}(b(r,\xi))\,d\xi\,dr\right|^2}{|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha\\ =:&\sum_{i=1}^2 D_2^i+\sum_{i,j=1}^2 D_2^{ij}.\end{aligned}$$

The term D_2^i satisfies

$$\begin{aligned}D_2^i&\le\left\|\frac{\partial\sigma}{\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}\int_{\mathbb{R}^3}\frac{\left|u_2^i(\alpha,\beta_2)-u_1^i(\alpha,\beta_2)+u_1^i(\alpha,\beta_1)-u_2^i(\alpha,\beta_1)\right|^2}{|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha\\ &\le\left\|\frac{\partial\sigma}{\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}\|u_2^i-u_1^i\|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))}.\end{aligned}\tag{5.4.73}$$

To estimate D_2^{ij} we need to work as follows. First note that by fixing the notation

$$\Delta_{Q_1}u^i:=u_2^i(\alpha,\beta_2)-u_1^i(\alpha,\beta_2)+u_1^i(\alpha,\beta_1)-u_2^i(\alpha,\beta_1),$$

we have

$$\{(1-r)[u_1^j(\alpha,\beta_2)-u_1^j(\alpha,\beta_1)]+r[u_2^j(\alpha,\beta_2)-u_2^j(\alpha,\beta_1)]\}\times$$

$$\begin{aligned}&\times\{u_2^i(\alpha,\beta_1)-u_1^i(\alpha,\beta_1)+\xi[u_2^i(\alpha,\beta_2)-u_1^i(\alpha,\beta_2)+u_1^i(\alpha,\beta_1)-u_2^i(\alpha,\beta_1)]\}\\&\quad=\{u_1^j(\alpha,\beta_2)-u_1^j(\alpha,\beta_1)\}\times\{u_2^i(\alpha,\beta_1)-u_1^i(\alpha,\beta_1)\}+r\xi\{\Delta_{Q_1}u^i\}\times\{\Delta_{Q_1}u^j\}\\&\qquad+\xi\{u_1^j(\alpha,\beta_2)-u_1^j(\alpha,\beta_1)\}\times\{\Delta_{Q_1}u^i\}+r\{\Delta_{Q_1}u^j\}\times\{u_2^i(\alpha,\beta_1)-u_1^i(\alpha,\beta_1)\},\end{aligned}$$

consequently from the expression of D_2^{ij} we get

$$\begin{aligned}D_2^{ij}\lesssim\left\|\frac{\partial^2\sigma}{\partial x_j\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}&\Bigg[\int_{\mathbb{R}^3}\frac{\left|\{u_1^j(\alpha,\beta_2)-u_1^j(\alpha,\beta_1)\}\times\{u_2^i(\alpha,\beta_1)-u_1^i(\alpha,\beta_1)\}\right|^2}{|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha\\&+\int_{\mathbb{R}^3}\frac{\left|\{u_1^j(\alpha,\beta_2)-u_1^j(\alpha,\beta_1)\}\times\{\Delta_{Q_1}u^i\}\right|^2}{|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha\\&+\int_{\mathbb{R}^3}\frac{\left|\{\Delta_{Q_1}u^j\}\times\{u_2^i(\alpha,\beta_1)-u_1^i(\alpha,\beta_1)\}\right|^2}{|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha\\&+\int_{\mathbb{R}^3}\frac{\left|\{\Delta_{Q_1}u^i\}\times\{\Delta_{Q_1}u^j\}\right|^2}{|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha\Bigg]\\\lesssim\left\|\frac{\partial^2\sigma}{\partial x_j\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}&\Bigg[\|u_2^i-u_1^i\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\,\|u_1^j\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\\&\qquad+\|u_2^i-u_1^i\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\,\|u_2^j-u_1^j\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\Bigg].\end{aligned}\tag{5.4.74}$$

By substituting D_2^i and D_2^{ij} from (5.4.73) and (5.4.74), respectively, back into the expression for D_2 we obtain

$$\begin{aligned}D_2\lesssim&\left[\sum_{i=1}^2\left\|\frac{\partial\sigma}{\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}+\sum_{i,j=1}^2\left\|\frac{\partial^2\sigma}{\partial x_j\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}\right]\|u_2-u_1\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\\&\times\left[1+\|u_1\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}+\|u_2\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\right].\end{aligned}\tag{5.4.75}$$

Interchanging the roles of α, ξ, s by β, τ, and δ, respectively, we bound D_3 by the right hand side of (5.4.75). Hence, the only term remaining to estimate is D_4. Recall that

$$D_4=\int_{\mathbb{R}^4}\frac{\left|\begin{array}{l}[\sigma\circ u_1](\alpha_1,\beta_1)-[\sigma\circ u_2](\alpha_1,\beta_1)-\{[\sigma\circ u_1](\alpha_2,\beta_1)-[\sigma\circ u_2](\alpha_2,\beta_1)\}\\-\big([\sigma\circ u_1](\alpha_1,\beta_2)-[\sigma\circ u_2](\alpha_1,\beta_2)-\{[\sigma\circ u_1](\alpha_2,\beta_2)-[\sigma\circ u_2](\alpha_2,\beta_2)\}\big)\end{array}\right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2.$$

Before proceeding further, by direct computation we infer the following two results.

Lemma 5.4.22. *In our notation with $Q=[\alpha_1,\alpha_2]\times[\beta_1,\beta_2]\subset\mathbb{R}^2$, and $u_1,u_2\in\mathbb{H}^{s,\delta}$ we have*

$$\begin{aligned}\Delta_Q[\sigma\circ u_2-\sigma\circ u_1]:=&[\sigma\circ u_2](\alpha_1,\beta_1)-[\sigma\circ u_1](\alpha_1,\beta_1)-[\sigma\circ u_2](\alpha_2,\beta_1)\\&+[\sigma\circ u_1](\alpha_2,\beta_1)-[\sigma\circ u_2](\alpha_1,\beta_2)+[\sigma\circ u_1](\alpha_1,\beta_2)\\&+[\sigma\circ u_2](\alpha_2,\beta_2)-[\sigma\circ u_1](\alpha_2,\beta_2)\\=&\int_0^1\int_0^1\int_0^1\big(\partial_r\partial_\tau\partial_\xi[\sigma\circ a]\big)(r,\tau,\xi)\,d\xi\,d\tau\,dr,\end{aligned}$$

where $\left(\partial_r\partial_\tau\partial_\xi[\sigma\circ a]\right) := \frac{\partial^3[\sigma\circ a]}{\partial r\partial\tau\partial\xi}$,

$$a(r,\cdot,\cdot) = a_1(\cdot,\cdot) + r\left(a_2(\cdot,\cdot) - a_1(\cdot,\cdot)\right) \in \mathbb{R}^2$$

and, for $i = 1,2$,

$$a_i(\tau,\xi) := u_i(\alpha_1,\beta_1) + \tau[u_i(\alpha_2,\beta_1) - u_i(\alpha_1,\beta_1)] + \xi[u_i(\alpha_1,\beta_2) - u_i(\alpha_1,\beta_1)] + \tau\xi\Delta_Q u_i.$$

Moreover, in our notation the following holds

$$\begin{aligned}\frac{\partial^3[\sigma\circ a]}{\partial\tau\partial\xi\partial r}(r,\tau,\xi) = \sum_{i=1}^{2}\frac{\partial\sigma}{\partial x_i}(a(r,\tau,\xi))\,\frac{\partial^3 a^i}{\partial\tau\partial\xi\partial r}(r,\tau,\xi)&\\ + \sum_{i,k=1}^{2}\frac{\partial^2\sigma}{\partial x_k\partial x_i}(a(r,\tau,\xi))\,\frac{\partial a^k}{\partial r}(r,\tau,\xi)\,\frac{\partial^2 a^i}{\partial\tau\partial\xi}(r,\tau,\xi)&\\ + \sum_{i,j=1}^{2}\frac{\partial^2\sigma}{\partial x_j\partial x_i}(a(r,\tau,\xi))\,\frac{\partial^2 a^j}{\partial\tau\partial r}(r,\tau,\xi)\,\frac{\partial a^i}{\partial\xi}(r,\tau,\xi)&\\ + \sum_{i,j=1}^{2}\frac{\partial^2\sigma}{\partial x_j\partial x_i}(a(r,\tau,\xi))\,\frac{\partial a^j}{\partial\tau}(r,\tau,\xi)\,\frac{\partial^2 a^i}{\partial\xi\partial r}(r,\tau,\xi)&\\ + \sum_{i,j,k=1}^{2}\frac{\partial^3\sigma}{\partial x_k\partial x_j\partial x_i}(a(r,\tau,\xi))\,\frac{\partial a^k}{\partial r}(r,\tau,\xi)\,\frac{\partial a^j}{\partial\tau}(r,\tau,\xi)\,\frac{\partial a^i}{\partial\xi}(r,\tau,\xi).&\end{aligned}\tag{5.4.76}$$

Using Lemma 5.4.22 and substituting (5.4.76) in the expression of D_4 we obtain

$$\begin{aligned}D_4 \lesssim &\sum_{i=1}^{2}\int_{\mathbb{R}^4}\frac{\left|\int_0^1\int_0^1\int_0^1\left[\frac{\partial\sigma}{\partial x_i}(a(r,\tau,\xi))\,\frac{\partial^3 a^i}{\partial\tau\partial\xi\partial r}(r,\tau,\xi)\right]d\xi\,d\tau\,dr\right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2\\ &+\sum_{i,k=1}^{2}\int_{\mathbb{R}^4}\frac{\left|\int_0^1\int_0^1\int_0^1\left[\frac{\partial^2\sigma}{\partial x_k\partial x_i}(a(r,\tau,\xi))\,\frac{\partial a^k}{\partial r}(r,\tau,\xi)\,\frac{\partial^2 a^i}{\partial\tau\partial\xi}(r,\tau,\xi)\right]d\xi\,d\tau\,dr\right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2\\ &+\sum_{i,j=1}^{2}\int_{\mathbb{R}^4}\frac{\left|\int_0^1\int_0^1\int_0^1\left[\frac{\partial^2\sigma}{\partial x_j\partial x_i}(a(r,\tau,\xi))\,\frac{\partial^2 a^j}{\partial\tau\partial r}(r,\tau,\xi)\,\frac{\partial a^i}{\partial\xi}(r,\tau,\xi)\right]d\xi\,d\tau\,dr\right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2\\ &+\sum_{i,j=1}^{2}\int_{\mathbb{R}^4}\frac{\left|\int_0^1\int_0^1\int_0^1\left[\frac{\partial^2\sigma}{\partial x_j\partial x_i}(a(r,\tau,\xi))\,\frac{\partial a^j}{\partial\tau}(r,\tau,\xi)\,\frac{\partial^2 a^i}{\partial\xi\partial r}(r,\tau,\xi)\right]d\xi\,d\tau\,dr\right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2\\ &+\sum_{i,j,k=1}^{2}\int_{\mathbb{R}^4}\frac{\left|\int_0^1\int_0^1\int_0^1\left[\frac{\partial^3\sigma}{\partial x_k\partial x_j\partial x_i}(a(r,\tau,\xi))\,\frac{\partial a^k}{\partial r}(r,\tau,\xi)\,\frac{\partial a^j}{\partial\tau}(r,\tau,\xi)\,\frac{\partial a^i}{\partial\xi}(r,\tau,\xi)\right]d\xi\,d\tau\,dr\right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\,d\beta_1\,d\beta_2\,d\alpha_1\,d\alpha_2\\ &=: \sum_{i=1}^{2}D_i^{I} + \sum_{i,k=1}^{2}D_{ik}^{II} + \sum_{i,j=1}^{2}D_{ij}^{III} + \sum_{i,j=1}^{2}D_{ij}^{IV} + \sum_{i,j,k=1}^{2}D_{ijk}^{V}.\end{aligned}$$

To estimate the right hand side terms in above, we observe that the partial derivative terms, by using the short notation as in Lemma 5.4.22, satisfy

$$
\begin{aligned}
&[\partial_\xi \partial_\tau \partial_r a](r,\tau,\xi) = \Delta_Q(u_2 - u_1);\\
&[\partial_\xi a](r,\tau,\xi) = [u_1(\alpha_1,\beta_2) - u_1(\alpha_1,\beta_1)] + r[[u_2-u_1](\alpha_1,\beta_2) - [u_2-u_1](\alpha_1,\beta_1)]\\
&\qquad\qquad + \tau \Delta_Q u_1 + r\tau \Delta_Q(u_2-u_1);\\
&[\partial_\tau a](r,\tau,\xi) = u_1(\alpha_2,\beta_1) - u_1(\alpha_1,\beta_1) + r[[u_2-u_1](\alpha_2,\beta_1) - [u_2-u_1](\alpha_1,\beta_1)]\\
&\qquad\qquad + \xi \Delta_Q u_1 + r\xi \Delta_Q(u_2-u_1);\\
&[\partial_r a](r,\tau,\xi) = [u_2-u_1](\alpha_1,\beta_1) + \tau[[u_2-u_1](\alpha_2,\beta_1) - [u_2-u_1](\alpha_1,\beta_1)]\\
&\qquad\qquad + \xi[[u_2-u_1](\alpha_1,\beta_2) - [u_2-u_1](\alpha_1,\beta_1)] + \tau\xi\Delta_Q(u_2-u_1);\\
&[\partial_\tau \partial_r a](r,\tau,\xi) = [u_2-u_1](\alpha_2,\beta_1) - [u_2-u_1](\alpha_1,\beta_1) + \xi\Delta_Q(u_2-u_1);\\
&[\partial_\xi \partial_r a](r,\tau,\xi) = [u_2(\alpha_1,\beta_2) - u_2(\alpha_1,\beta_1) - \{u_1(\alpha_1,\beta_2) - u_1(\alpha_1,\beta_1)\}] + \tau\Delta_Q(u_2-u_1);\\
&[\partial_\tau \partial_\xi a](r,\tau,\xi) = \Delta_Q u_1 + r\Delta_Q(u_2-u_1).
\end{aligned}
\tag{5.4.77}
$$

Next, by working on similar lines to (5.4.58), as an application of (5.4.77), we have

$$
\begin{aligned}
D_i^I &\leq \left\| \frac{\partial \sigma}{\partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} \int_{\mathbb{R}^4} \frac{|\int_0^1\int_0^1\int_0^1 \left[\Delta_Q(u_2^i - u_1^i)\right] d\xi\, d\tau\, dr|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2\\
&\leq \left\| \frac{\partial \sigma}{\partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} \| u_2 - u_1 \|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}.
\end{aligned}
\tag{5.4.78}
$$

Invoking the Theorem 5.1.15 and Definition 5.1.5 with (5.4.77) and following the last few steps of (5.4.75), we estimate D_{ik}^{II} as

$$
\begin{aligned}
D_{ik}^{II} \lesssim \left\| \frac{\partial^2 \sigma}{\partial x_k \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \Bigg[&\int_{\mathbb{R}^4} \frac{|[u_2^k - u_1^k](\alpha_1,\beta_1) \times \Delta_Q u_1^i|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2\\
&+ \int_{\mathbb{R}^4} \frac{|[[u_2^k - u_1^k](\alpha_2,\beta_1) - [u_2^k - u_1^k](\alpha_1,\beta_1)] \times \Delta_Q u_1^i|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2\\
&+ \int_{\mathbb{R}^4} \frac{|[[u_2^k - u_1^k](\alpha_1,\beta_2) - [u_2^k - u_1^k](\alpha_1,\beta_1)] \times \Delta_Q u_1^i|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2\\
&+ \int_{\mathbb{R}^4} \frac{|\Delta_Q(u_2^k - u_1^k) \times \Delta_Q u_1^i|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2\\
&+ \int_{\mathbb{R}^4} \frac{|[u_2^k - u_1^k](\alpha_1,\beta_1) \times \left(\Delta_Q(u_2^i - u_1^i)\right)|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2\\
&+ \int_{\mathbb{R}^4} \frac{|[[u_2^k - u_1^k](\alpha_2,\beta_1) - [u_2^k - u_1^k](\alpha_1,\beta_1)] \times \left(\Delta_Q(u_2^i - u_1^i)\right)|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2\\
&+ \int_{\mathbb{R}^4} \frac{|[[u_2^k - u_1^k](\alpha_1,\beta_2) - [u_2^k - u_1^k](\alpha_1,\beta_1)] \times \left(\Delta_Q(u_2^i - u_1^i)\right)|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2\\
&+ \int_{\mathbb{R}^4} \frac{|\Delta_Q(u_2^k - u_1^k) \times \left(\Delta_Q(u_2^i - u_1^i)\right)|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}}\, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \Bigg]
\end{aligned}
$$

$$\lesssim \left\| \frac{\partial^2 \sigma}{\partial x_k \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2;\mathbb{R}^2)} \| u_2 - u_1 \|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} \left[\| u_1 \|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} + \| u_2 \|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} \right], \tag{5.4.79}$$

and D^{III}_{ij} as

$$\begin{aligned}
D^{III}_{ij} \lesssim \left\| \frac{\partial^2 \sigma}{\partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \Bigg[& \int_{\mathbb{R}^4} \frac{\left| \begin{smallmatrix} \{[u_2^j - u_1^j](\alpha_2,\beta_1) - [u_2^j - u_1^j](\alpha_1,\beta_1)\} \\ \times \{u_1^i(\alpha_1,\beta_2) - u_1^i(\alpha_1,\beta_1)\} \end{smallmatrix} \right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \\
& + \int_{\mathbb{R}^4} \frac{\left| \{[u_2^j - u_1^j](\alpha_2,\beta_1) - [u_2^j - u_1^j](\alpha_1,\beta_1)\} \times \{\Delta_Q u_1^i\} \right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \\
& + \int_{\mathbb{R}^4} \frac{\left| \begin{smallmatrix} \{[u_2^j - u_1^j](\alpha_2,\beta_1) - [u_2^j - u_1^j](\alpha_1,\beta_1)\} \\ \times \{[[u_2^i - u_1^i](\alpha_1,\beta_2) - [u_2^i - u_1^i](\alpha_1,\beta_1)]\} \end{smallmatrix} \right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \\
& + \int_{\mathbb{R}^4} \frac{\left| \{[u_2^j - u_1^j](\alpha_2,\beta_1) - [u_2^j - u_1^j](\alpha_1,\beta_1)\} \times \{\Delta_Q (u_2^i - u_1^i)\} \right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \\
& + \int_{\mathbb{R}^4} \frac{\left| \{\Delta_Q (u_2^j - u_1^j)\} \times \{u_1^i(\alpha_1,\beta_2) - u_1^i(\alpha_1,\beta_1)\} \right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \\
& + \int_{\mathbb{R}^4} \frac{\left| \{\Delta_Q (u_2^j - u_1^j)\} \times \{\Delta_Q u_1^i\} \right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \\
& + \int_{\mathbb{R}^4} \frac{\left| \{\Delta_Q (u_2^j - u_1^j)\} \times \{[[u_2^i - u_1^i](\alpha_1,\beta_2) - [u_2^i - u_1^i](\alpha_1,\beta_1)]\} \right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \\
& + \int_{\mathbb{R}^4} \frac{\left| \{\Delta_Q (u_2^j - u_1^j)\} \times \{\Delta_Q (u_2^i - u_1^i)\} \right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2 \Bigg]
\end{aligned}$$

$$\lesssim \left\| \frac{\partial^2 \sigma}{\partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \| u_2 - u_1 \|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} \left[\| u_2 \|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} + \| u_1 \|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))} \right]. \tag{5.4.80}$$

By interchanging the role of variables, we deduce that D^{IV}_{ij} is bounded as above. Now we proceed to estimate the final term D^V_{ijk} which by using the notation from (5.4.77) satisfy

$$D^V_{ijk} \lesssim \left\| \frac{\partial^3 \sigma}{\partial x_k \partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \int_{\mathbb{R}^4} \frac{\left| \Pi_{l=1}^3 D^{V,l}_{ijk} \right|^2}{|\alpha_1-\alpha_2|^{1+2s}|\beta_1-\beta_2|^{1+2\delta}} \, d\beta_1\, d\beta_2\, d\alpha_1\, d\alpha_2, \tag{5.4.81}$$

where

$$\begin{aligned}
D^{V,1}_{ijk} &:= [u_2^k - u_1^k](\alpha_1,\beta_1) + [[u_2^k - u_1^k](\alpha_2,\beta_1) - [u_2^k - u_1^k](\alpha_1,\beta_1)] \\
&\quad + [[u_2^k - u_1^k](\alpha_1,\beta_2) - [u_2^k - u_1^k](\alpha_1,\beta_1)] + \Delta_Q (u_2^k - u_1^k), \\
D^{V,2}_{ijk} &:= u_1^j(\alpha_2,\beta_1) - u_1^j(\alpha_1,\beta_1) + \Delta_Q u_1^j
\end{aligned}$$

$$
\begin{aligned}
&\quad + [[u_2^j - u_1^j](\alpha_2,\beta_1) - [u_2^j - u_1^j](\alpha_1,\beta_1)] + \Delta_Q (u_2^j - u_1^j),\\
D_{ijk}^{V,3} &:= [u_1^i(\alpha_1,\beta_2) - u_1^i(\alpha_1,\beta_1)] + \Delta_Q u_1^i\\
&\quad + [[u_2^i - u_1^i](\alpha_1,\beta_2) - [u_2^i - u_1^i](\alpha_1,\beta_1)] + \Delta_Q (u_2^i - u_1^i).
\end{aligned}
$$

So the integrand consist of 64 terms because it is a mutiplication of 3 brackets and each bracket of has 4 terms which comes from (5.4.77). To be precise, these 64 terms consist of the terms which can be estimated as follows in a similar fashion to (5.4.80):

1. first 16 terms which will be bounded from above by some constant multiply with

$$
\left\| \frac{\partial^3 \sigma}{\partial x_k \partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \| u_2^k - u_1^k \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \| u_2^j - u_1^j \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \| u_2^i - u_1^i \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))}. \tag{5.4.82}
$$

2. other 16 terms which will be bounded from above by some constant multiply with

$$
\left\| \frac{\partial^3 \sigma}{\partial x_k \partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \| u_2^k - u_1^k \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \| u_2^j - u_1^j \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \| u_1^i \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))}. \tag{5.4.83}
$$

3. next 16 terms which will be bounded from above by some constant multiply with

$$
\left\| \frac{\partial^3 \sigma}{\partial x_k \partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \| u_2^k - u_1^k \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \| u_1^j \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \| u_2^i - u_1^i \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))}. \tag{5.4.84}
$$

4. last 16 terms which will be bounded from above by some constant multiply with

$$
\left\| \frac{\partial^3 \sigma}{\partial x_k \partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \| u_2^k - u_1^k \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \| u_1^j \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \| u_1^i \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))}. \tag{5.4.85}
$$

By using the estimates from (5.4.82) to (5.4.85) into (5.4.81), with the last few steps followed in (5.4.75), we get

$$
\begin{aligned}
D_{ijk}^V &\lesssim \left\| \frac{\partial^3 \sigma}{\partial x_k \partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \| u_2^k - u_1^k \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \times \Big[\| u_2^j - u_1^j \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \times \| u_2^i - u_1^i \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))}\\
&\quad + \| u_2^j - u_1^j \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \times \| u_1^i \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} + \| u_2^i - u_1^i \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \times \| u_1^j \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))}\\
&\quad + \| u_1^i \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \times \| u_1^j \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \Big]\\
&\lesssim \left\| \frac{\partial^3 \sigma}{\partial x_k \partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \| u_2 - u_1 \|^2_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \times \Big[\| u_2 \|^4_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} + \| u_1 \|^4_{W_\alpha^s(\mathbb{R};W_\beta^\delta(\mathbb{R}))} \Big].
\end{aligned} \tag{5.4.86}
$$

Hence by combining the estimates from (5.4.78), (5.4.79), (5.4.80), and (5.4.86) we see that D_4 satisfies the following bound

$$
D_4 \lesssim \left[\sum_{i=1}^{2} \left\| \frac{\partial \sigma}{\partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} + \sum_{i,j=1}^{2} \left\| \frac{\partial^2 \sigma}{\partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} + \sum_{i,j,k=1}^{2} \left\| \frac{\partial^3 \sigma}{\partial x_k \partial x_j \partial x_i} \right\|^2_{L^\infty(\mathbb{R}^2)} \right] \times
$$

(5.4.87)

$$\|u_2-u_1\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\left[1+\|u_2\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}+\|u_1\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}+\|u_2\|^4_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}+\|u_1\|^4_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\right].$$

So by putting the estimates of D_1 from (5.4.72), D_2 and D_3 from (5.4.75), and D_4 from (5.4.87) into (5.4.71) we have

$$\begin{aligned}
&\|\sigma\circ u_1-\sigma\circ u_2\|^2_{H^s_\alpha(\mathbb{R};H^\delta_\beta(\mathbb{R}))}\\
&\lesssim\left[L^2_\sigma+\sum_{i=1}^2\left\|\frac{\partial\sigma}{\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}+\sum_{i,j=1}^2\left\|\frac{\partial^2\sigma}{\partial x_j\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}+\sum_{i,j,k=1}^2\left\|\frac{\partial^3\sigma}{\partial x_k\partial x_j\partial x_i}\right\|^2_{L^\infty(\mathbb{R}^2)}\right]\|u_2-u_1\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\\
&\quad\times\left[1+\|u_2\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}+\|u_1\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}+\|u_2\|^4_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}+\|u_1\|^4_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\right]\\
&=:C_2(\|\sigma\|_{\mathcal{C}^3_b(\mathbb{R}^2)})\,\|u_2-u_1\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\times\\
&\qquad\times\left[1+\|u_2\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}+\|u_1\|^2_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}+\|u_2\|^4_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}+\|u_1\|^4_{W^s_\alpha(\mathbb{R};W^\delta_\beta(\mathbb{R}))}\right].
\end{aligned}$$

By interchange the roles of s,δ, similar to above, we get

$$\begin{aligned}
&\|\sigma\circ u_1-\sigma\circ u_2\|^2_{H^\delta_\alpha(\mathbb{R};H^s_\beta(\mathbb{R}))}\\
&\quad\lesssim C_2(\|\sigma\|_{\mathcal{C}^3_b(\mathbb{R}^2)})\,\|u_2-u_1\|^2_{W^\delta_\alpha(\mathbb{R};W^s_\beta(\mathbb{R}))}\times\\
&\qquad\times\left[1+\|u_2\|^2_{W^\delta_\alpha(\mathbb{R};W^s_\beta(\mathbb{R}))}+\|u_1\|^2_{W^\delta_\alpha(\mathbb{R};W^s_\beta(\mathbb{R}))}+\|u_2\|^4_{W^\delta_\alpha(\mathbb{R};W^s_\beta(\mathbb{R}))}+\|u_1\|^4_{W^\delta_\alpha(\mathbb{R};W^s_\beta(\mathbb{R}))}\right].
\end{aligned}$$

Hence, we obtain (5.4.56) and we finish the proof of Proposition 5.4.17. ■

5.5 Local well-posedness theory

In this section we present the main result related to local theory. Before stating the main theorem we will prove the following result which allows us to prove later that the localized version of $\diamond^{-1}\sigma(u)\dot{\zeta}$ belongs to $\mathbb{H}^{s,\delta}$.

Lemma 5.5.1. *Assume that $H_1,H_2\in(0,1)$ and $H_i'\in(0,H_1\wedge H_2)$, $i=1,2$. Then there exists a complete filtered probability space $(\Omega,\mathfrak{F},\mathbb{P})$ and a map,*

$$\zeta:\mathbb{R}^2_+\times\Omega\to\mathbb{R},$$

such that $\mathbb{P}$-a.s. $\zeta(\cdot,\cdot,\omega)\in\mathbb{H}^{H_1',H_2'}$ locally, i.e. for every bump function η,

$$\eta(\alpha)\eta(\beta)\zeta(\alpha,\beta,\omega)\in\mathbb{H}^{H_1',H_2'}.$$

Moreover, for $(\alpha_1,\beta_1),(\alpha_2,\beta_2)\in\mathbb{R}^2$,

$$\mathbb{E}\left[\zeta(\alpha_1,\beta_1)\,\zeta(\alpha_2,\beta_2)\right]=R_{H_1}(|\alpha_1|,|\alpha_2|)\,R_{H_2}(|\beta_1|,|\beta_2|).\tag{5.5.1}$$

Here $\mathbb{E}$ is the Expectation operator w.r.t. $\mathbb{P}$ and

$$R_H(\alpha,\beta)=\frac{1}{2}\left(\alpha^{2H}+\beta^{2H}-|\alpha-\beta|^{2H}\right),\qquad\alpha,\beta\in\mathbb{R}.$$

Proof of Lemma 5.5.1 Let us choose and fix $H_1, H_2 \in \big(0,1\big)$ and $H_i' \in (0, H_1 \wedge H_2)$, $i = 1,2$. We will prove the result only in the more difficult case $H_1, H_2 \in \big(0, \frac{1}{2}\big)$ as the other case $H_1, H_2 \in \big(\frac{1}{2}, 1\big)$ can be proved analogously but in a simpler manner.

For the time being let us also fix bump function η. To move forward define, for $a, b \in \big(\frac{1}{2}, 1\big)$,

$$I_\eta^a f(x) := \frac{\eta(x)}{\Gamma(a)} \int_0^x t^{a-1} f(x-t)\, dt,$$

and I_η^b similarly. It is known that, see e.g. [10] or [67, Section 2], the image of $L^2(\mathbb{R})$ by I_η^a is a subset of $H^a(\mathbb{R})$. Moreover, by [10] or [67, Theorem 11], if

$$0 < a' < a - \frac{1}{2} \qquad \text{and} \qquad 0 < b' < b - \frac{1}{2},$$

then the map

$$I_\eta^a : L^2(\mathbb{R}) \to H^{a'}(\mathbb{R}) \quad \text{and} \quad I_\eta^b : L^2(\mathbb{R}) \to H^{b'}(\mathbb{R}),$$

are Hilbert-Schmidt operators. Therefore, the tensor product of these maps is also a Hilbert-Schmidt operator, see Lemma 5.6.1 and for a more general result refer [39]. In other words, the map

$$I_\eta^a \otimes I_\eta^b : L^2(\mathbb{R}) \otimes L^2(\mathbb{R}) \to H^{a'}(\mathbb{R}) \otimes H^{a'}(\mathbb{R}),$$

is Hilbert-Schmidt whenever the relationship $0 < a' < a - \frac{1}{2}$ and $0 < b' < b - \frac{1}{2}$ hold.

Recall that, by a classical result, see e.g. [134], $L^2(\mathbb{R}) \otimes L^2(\mathbb{R})$ is isometrically isomorphic to $L^2(\mathbb{R}^2)$. Since by [145, Theorem 2.1], the space $H^{a'}(\mathbb{R}) \otimes H^{b'}(\mathbb{R})$ is isometrically isomorphic to $S_{2,2}^{a',b'}(\mathbb{R}^2)$, and $I_\eta^a \otimes I_\eta^b$ is equivalent to $I_\eta^{a,b}$ and by the ideal properties of the space of all Hilbert-Schmidt operators, we infer that the map

$$I_\eta^{a,b} : L^2(\mathbb{R}^2) \to S_{2,2}^{a',b'}(\mathbb{R}^2),$$

defined as,

$$I_\eta^{a,b} f(x,y) = \frac{\eta(x)\eta(y)}{\Gamma(a)\Gamma(b)} \int_0^x \int_0^y t^{a-1} s^{b-1} f(x-t, y-s)\, ds\, dt,$$

is Hilbert-Schmidt. Analogously, if $0 < a' < b - \frac{1}{2}$ and $0 < b' < a - \frac{1}{2}$, then

$$I_\eta^{a,b} : L^2(\mathbb{R}^2) \to S_{2,2}^{b',a'}(\mathbb{R}^2),$$

is Hilbert-Schmidt. Hence,

$$I_\eta^{a,b} : L^2(\mathbb{R}^2) \to S_{2,2}^{a',b'}(\mathbb{R}^2) \cap S_{2,2}^{b',a'}(\mathbb{R}^2) = \mathbb{H}^{a',b'},$$

is Hilbert-Schmidt whenever

$$a', b' \in \left(0, a - \frac{1}{2}\right) \wedge \left(0, b - \frac{1}{2}\right).$$

In particular, by taking $a = H_1 + \frac{1}{2}, b = H_2 + \frac{1}{2}$ and $a' = H_1', b' = H_2'$, we have that the map

$$I_\eta^{H_1 + \frac{1}{2}, H_2 + \frac{1}{2}} : L^2(\mathbb{R}^2) \to \mathbb{H}^{H_1', H_2'}(\mathbb{R}^2),$$

is Hilbert-Schmidt.

Let $\{e_n\}_{n=1}^{\infty}$ be an orthonormal basis (ONB) of $L^2(\mathbb{R}^2)$ and let $\{\beta_n\}_{n=1}^{\infty}$ be an i.i.d sequence of $N(0,1)$ random variables defined on some probability space $(\Omega,\mathfrak{F},\mathbb{P})$. In fact we consider an ONB of $L^2(\mathbb{R}^2)$ of the form $\tilde{e}_i\otimes\tilde{e}_j$, where $\{\tilde{e}_i\}_{i=1}^{\infty}$ is an ONB of $L^2(\mathbb{R})$ and we consider i.i.d sequence $\{\beta_{i,j}\}_{i,j\in\mathbb{N}^2}$ of $N(0,1)$ random variables but we use the simple index notation.

Consider the random series

$$S(\omega) := \sum_{n=1}^{\infty} \beta_n(\omega) I_{\eta}^{H_1+\frac{1}{2},H_2+\frac{1}{2}}(e_n). \tag{5.5.2}$$

Because the map $I_{\eta}^{H_1+\frac{1}{2},H_2+\frac{1}{2}}$ is Hilbert-Schmidt, the above series is convergent $\mathbb{P}$-a.s.

Next we choose a sequence of real numbers $\{R_i\}_{i=1}^{\infty}$ such that $R_i\nearrow\infty$ and a countable family of bump functions η^k. We put

$$\eta_{ik} = \eta^k\left(\frac{\cdot}{R_i}\right);\ (i,k)\in\mathbb{N}_0^2.$$

Because the family η_{ik} is countable, we infer that the series

$$S(\omega) := \sum_{n=1}^{\infty} \beta_n(\omega) I_{\eta_{ik}}^{H_1+\frac{1}{2},H_2+\frac{1}{2}}(e_n). \tag{5.5.3}$$

is $\mathbb{P}$-a.s. convergent for every (i,k). Hence the series

$$S(\omega) := \sum_{n=1}^{\infty} \beta_n(\omega) I^{H_1+\frac{1}{2},H_2+\frac{1}{2}}(e_n), \tag{5.5.4}$$

where

$$I^{H_1+\frac{1}{2},H_2+\frac{1}{2}}f = \frac{1}{\Gamma\left(H_1+\frac{1}{2}\right)\Gamma\left(H_2+\frac{1}{2}\right)}\int_0^x\int_0^y t^{H_1-\frac{1}{2}}s^{H_2-\frac{1}{2}}f(x-t,y-s)\,ds\,dt,$$

is $\mathbb{P}$-a.s. convergent in $\mathbb{H}_{\text{loc}}^{H_1',H_2'}$. Indeed, since $u_n\to u$ in $\mathbb{H}_{\text{loc}}^{H_1',H_2'}$ if and only if for all (i,k)

$$\eta_{ik}(x)\eta_{ik}(y)u_n \to \eta_{ik}(x)\eta_{ik}(y)u \quad \text{in} \quad \mathbb{H}^{H_1',H_2'},$$

we have, $\mathbb{P}$-a.s.

$$\begin{aligned}\eta_{ik}(x)\eta_{ik}(y)\sum_{n=1}^{\infty}\beta_n(\omega)[I^{H_1+\frac{1}{2},H_2+\frac{1}{2}}(e_n)](x,y) &= \sum_{n=1}^{\infty}\eta_{ik}(x)\eta_{ik}(y)\beta_n(\omega)[I^{H_1+\frac{1}{2},H_2+\frac{1}{2}}(e_n)](x,y)\\ &= \sum_{n=1}^{\infty}\beta_n(\omega)[I_{\eta_{ik}}^{H_1+\frac{1}{2},H_2+\frac{1}{2}}(e_n)](x,y).\end{aligned}$$

But, by (5.5.3), for $\omega\in\Omega$, $\mathbb{P}$-a.s. the r.h.s of above converges in $\mathbb{H}^{H_1',H_2'}$. Hence

$$\zeta(\omega) := \sum_{n=1}^{\infty}\beta_n(\omega) I^{H_1+\frac{1}{2},H_2+\frac{1}{2}}(e_n) \in \mathbb{H}_{\text{loc}}^{H_1',H_2'},$$

will give the Lemma 5.5.1.

Finally, we can prove that the condition (5.5.1) is satisfied by repeating the argument from [67] and using the special form of the ONB of $L^2(\mathbb{R}^2)$. ∎

Let us define the (pathwise) local solution that we consider.

Definition 5.5.2. *Assume that* $(u_0,u_1)\in H^s_{loc}(\mathbb{R};\mathbb{R}^m)\times H^{s-1}_{loc}(\mathbb{R};\mathbb{R}^m)$ *for some* $s\in\left(\frac{3}{4},1\right)$. *Let* $\sigma\in\mathcal{C}^3_b(\mathbb{R}^2;\mathbb{R}^2)$ *and* ζ *be a fractional Brownian sheet of Hurst indices* $H_1,H_2\in\left(\frac{1}{2},1\right)$ *defined on a probability space* $(\Omega,\mathfrak{F},\mathbb{P})$. *A process* u*, whose paths are almost surely in* $\mathbb{H}^{s,\delta}_{loc}$*, is said to be a unique local solution to the stochastic wave map Cauchy problem*

$$\begin{cases}\diamond u=\mathcal{N}(u)+\sigma(u)\dot{\zeta},\\ u(\alpha,-\alpha)=u_0(\alpha)\quad\textit{and}\quad\partial_\alpha u(\alpha,-\alpha)+\partial_\beta u(\alpha,-\alpha)=u_1(\alpha),\end{cases}$$

if and only if, for $\omega\in\Omega$, $\mathbb{P}$*-a.s., there exist an open set* $\mathcal{O}(\omega)$*, containing the diagonal*

$$\mathcal{D}:=\{(\alpha,-\alpha):\alpha\in\mathbb{R}\},$$

and a function $u(\cdot,\cdot,\omega):\mathcal{O}\to\mathbb{R}^2$ *such that* $u(\cdot,\cdot,\omega)$ *satisfies the integral equation* (5.3.3) *uniquely in* $\mathcal{O}$*, and for every* $(\alpha_0,-\alpha_0)\in\mathcal{D}$*, there exists* $r(\omega)>0$*, depending on the point* $(\alpha_0,-\alpha_0)$*, such that*

$$(\alpha_0-2r(\omega),\alpha_0+2r(\omega))\times(-\alpha_0-2r(\omega),-\alpha_0+2r(\omega))\subset\mathcal{O}(\omega),$$

and, for every bump function χ *which satisfy* $\mathbb{1}_{[-r(\omega),r(\omega)]}\le\chi\le\mathbb{1}_{[-2r(\omega),2r(\omega)]}$*, the following holds*

$$\chi(\alpha-\alpha_0)\chi(\beta+\alpha_0)u(\alpha,\beta)\in\mathbb{H}^{s,\delta}.$$

Next result is the main theorem of the current chapter.

Theorem 5.5.3. *Let* η,χ *as defined in* (5.4.1) *and* ψ *be a bump function which is non zero on the support of* χ,η *and* $\int_{\mathbb{R}}\psi(x)\,dx=1$. *Assume* $s,\delta\in\left(\frac{3}{4},1\right)$ *such that* $\delta\le s$ *and* $(u_0,u_1)\in H^s(\mathbb{R})\times H^{s-1}(\mathbb{R})$. *There exist a* $R_0\in(0,1)$ *and a* $\lambda_0:=\lambda_0(\|u_0\|_{H^s},\|u_1\|_{H^{s-1}},R_0)\gg1$ *such that for every* $\lambda\ge\lambda_0$ *there exists a unique* $u:=u(\lambda,R_0)\in\mathbb{B}_{R_0}$*, where* $\mathbb{B}_R:=\left\{u\in\mathbb{H}^{s,\delta}:\|u\|_{\mathbb{H}^{s,\delta}}\le R\right\}$*, which satisfies the following integral equation*

$$\begin{aligned}u(\alpha,\beta)=\eta(\lambda\alpha)\eta(\lambda\beta)\Big([S(\chi(\lambda)(u_0-\bar{u}_0^\lambda),\chi(\lambda)u_1)](\alpha,\beta)+[\diamond^{-1}\mathcal{N}(u)](\alpha,\beta)\\ +[\diamond^{-1}\sigma(u)\dot{\zeta}](\alpha,\beta)\Big),\qquad(\alpha,\beta)\in\mathbb{R}^2.\end{aligned}\tag{5.5.5}$$

Here the right hand side terms are, respectively, defined in (5.3.4)*-*(5.3.6) *and* $\bar{u}_0^\lambda$ *is*

$$\bar{u}_0^\lambda:=\int_{\mathbb{R}}u_0\left(\frac{y}{\lambda}\right)\psi(y)\,dy.$$

Proof of Theorem 5.5.3 Let us fix s,δ satisfying the assumption of the theorem. Since the dependency of constants on the variables in the estimates below plays an important role in proving the contraction property, (which in result allows us to apply the Banach Fixed Point Theorem), we write the proven estimates precisely as follows:

1. from Lemma 5.4.18 there exists $C_S:=C_S(\eta)>0$ such that, for every $u_0\in H^s(\mathbb{R},u_1\in H^{s-1}(\mathbb{R}))$,

$$\|\eta(\alpha)\eta(\beta)S(u_0,u_1)\|_{\mathbb{H}^{s,\delta}}\le C_S\left(\|u_0\|_{H^s}+\|u_1\|_{H^{s-1}}\right),\tag{5.5.6}$$

2. for the integral term involving the non-linearity of wave map equation in (5.5.5), by Lemma 5.4.15 we know that there exist a natural number $\gamma \geq 2$ and a positive constant $C_{\mathcal{N}} := C_{\mathcal{N}}(\eta)$ such that

$$\|\eta(\alpha)\eta(\beta)\diamond^{-1}[\mathcal{N}(u)]\|_{\mathbb{H}^{s,\delta}} \leq C_{\mathcal{N}}\ \|u\|_{\mathbb{H}^{s,\delta}}^{\gamma+1}, \tag{5.5.7}$$

and

$$\|\eta(\alpha)\eta(\beta)\diamond^{-1}[\mathcal{N}(u_1)-\mathcal{N}(u_2)]\|_{\mathbb{H}^{s,\delta}} \leq C_{\mathcal{N}}\|u_1-u_2\|_{\mathbb{H}^{s,\delta}}\left[\sum_{i=1}^{2}\|u_i\|_{\mathbb{H}^{s,\delta}}\right]^{\gamma}, \tag{5.5.8}$$

for every $u, u_1, u_2 \in \mathbb{H}^{s,\delta}$;

3. to estimate the integral w.r.t. the noise term in (5.5.5), note that due to Proposition 5.4.17, $\sigma(u) \in \mathbb{H}^{s,\delta}$ for any $u \in \mathbb{H}^{s,\delta}$. Next, since by Lemma 5.4.3 $\mathbb{H}^{s,\delta}\cdot\mathbb{H}^{s-1,\delta-1} \subset \mathbb{H}^{s-1,\delta-1}$ and $\dot{\zeta} \in \mathbb{H}^{s-1,\delta-1}_{\text{loc}}$, Proposition 5.4.9 (with $T=1$) tells that locally $\diamond^{-1}[\sigma(u)\dot{\zeta}]$ belongs to $\mathbb{H}^{s,\delta}$. In particular,

$$\begin{aligned}\|\eta(\alpha)\eta(\beta)\diamond^{-1}[\sigma(u)\dot{\zeta}]\|_{\mathbb{H}^{s,\delta}} &\leq C_{\zeta}\ \|\sigma(u)\|_{\mathbb{H}^{s,\delta}}\ \|\dot{\zeta}\|_{\mathbb{H}^{s-1,\delta-1}}\\ &\leq C_{\zeta}\ C_1(\sigma)\ \|u\|_{\mathbb{H}^{s,\delta}}\left[1+\|u\|_{\mathbb{H}^{s,\delta}}\right]\ \|\dot{\zeta}\|_{\mathbb{H}^{s-1,\delta-1}},\end{aligned} \tag{5.5.9}$$

and

$$\begin{aligned}\|\eta(\alpha)\eta(\beta)\diamond^{-1}[(\sigma(u_1)-\sigma(u_2))\dot{\zeta}]\|_{\mathbb{H}^{s,\delta}}^2 &\leq C_{\zeta}\ \|\sigma(u_1)-\sigma(u_2)\|_{\mathbb{H}^{s,\delta}}\ \|\dot{\zeta}\|_{\mathbb{H}^{s-1,\delta-1}},\\ &\leq C_{\zeta}\ C_2(\sigma)\ \|u_2-u_1\|_{\mathbb{H}^{s,\delta}}\left[1+\sum_{i,k=1}^{2}\|u_i\|_{\mathbb{H}^{s,\delta}}^{k}\right]\ \|\dot{\zeta}\|_{\mathbb{H}^{s-1,\delta-1}},\end{aligned} \tag{5.5.10}$$

for some positive constants $C_{\zeta} := C_{\zeta}(\eta,\chi)$ and $C_i(\sigma) := C_i(\|\sigma\|_{\mathcal{C}_b^{i+1}})$.

To move ahead in the proof we consider a map $\Theta^{\lambda} : \mathbb{H}^{s,\delta} \ni u \mapsto u_{\Theta^{\lambda}} \in \mathbb{H}^{s,\delta}$ defined by

$$u_{\Theta^{\lambda}} = \eta(\alpha)\eta(\beta)\left(S(u_0^{\lambda}, u_1^{\lambda}) + \diamond^{-1}[\mathcal{N}(u)] + \diamond^{-1}[\sigma(u)\dot{\zeta}^{\lambda}]\right), \tag{5.5.11}$$

where

$$u_0^{\lambda}(\alpha) := \chi(\alpha)\left[u_0\left(\frac{\alpha}{\lambda}\right) - \bar{u}_0^{\lambda}\right], \quad u_1^{\lambda}(\alpha) := \chi(\alpha)\lambda^{-1}u_1\left(\frac{\alpha}{\lambda}\right), \quad \text{and} \quad \dot{\zeta}^{\lambda} := \lambda^{-2}\Pi_{\lambda}\dot{\zeta}. \tag{5.5.12}$$

From (5.5.6) to (5.5.9), we infer that the map Θ^{λ} is well-defined. In order to prove that Θ^{λ} is a strict contraction we will use the concept of scaling together with the restriction map method. Let us take $R \in (0,1)$ (to be set later). By invoking Lemma 5.4.1 followed by the scaling Lemma 5.4.6, since $\delta \leq s$ we get

$$\|\eta(\alpha)\eta(\beta)S\left(u_0^{\lambda}, u_1^{\lambda}\right)\|_{\mathbb{H}^{s,\delta}} \leq C_S(\eta,\chi)\left(\lambda^{-\varepsilon}\|u_0\|_{H^s} + \lambda^{\frac{1}{2}-s}\|u_1\|_{H^{s-1}}\right), \tag{5.5.13}$$

for some $\varepsilon > 0$. Hence, by Lemma 5.4.7 we have

$$\|\dot{\zeta}^{\lambda}\|_{\mathbb{H}^{s-1,\delta-1}} \leq \lambda^{1-(s+\delta)}\|\dot{\zeta}\|_{\mathbb{H}^{s-1,\delta-1}}. \tag{5.5.14}$$

Next, observe that since $R^{\gamma+1} < R$ for $R \in (0,1)$, $\varepsilon > 0$ and for the considered range of s, δ, the exponents $\frac{1}{2} - s, -2+3-(s+\delta) < 0$, we can find $R \in (0,1)$ and $\lambda := \lambda(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}}, R)$ such that

$$\begin{aligned} R \geq\ & C_S(\eta,\chi)\left(\lambda^{-\varepsilon}\|u_0\|_{H^s} + \lambda^{\frac{1}{2}-s}\|u_1\|_{H^{s-1}}\right) + C_{\mathcal{N}} R^{\gamma+1} \\ & + \lambda^{-1-(s+\delta)} C_\zeta C_1(\sigma) R(1+R)\|\dot{\zeta}\|_{\mathbb{H}^{s-1,\delta-1}}. \end{aligned} \tag{5.5.15}$$

Consequently, for every pair (λ, R) such that (5.5.15) holds, the map $\Theta^\lambda : \mathbb{B}_R \to \mathbb{B}_R$ is well-defined because for any $u \in \mathbb{B}_R$, by triangle inequality in (5.5.11) followed by estimates (5.5.7), (5.5.9), (5.5.13), and (5.5.14) give

$$\begin{aligned} \|u_{\Theta^\lambda}\|_{\mathbb{H}^{s,\delta}} \leq\ & C_S(\eta,\chi)\left(\lambda^{-\varepsilon}\|u_0\|_{H^s} + \lambda^{\frac{1}{2}-s}\|u_1\|_{H^{s-1}}\right) + C_{\mathcal{N}} R^{\gamma+1} \\ & + \lambda^{1-(s+\delta)} C_\zeta C_1(\sigma) R(1+R)\|\dot{\zeta}\|_{\mathbb{H}^{s-1,\delta-1}}. \end{aligned} \tag{5.5.16}$$

To have the contraction, observe that by using the estimates (5.5.8), (5.5.10), and (5.5.14) in (5.5.11), for any $u, v \in \mathbb{B}_R$, we obtain

$$\begin{aligned} \|u_{\Theta^\lambda} - v_{\Theta^\lambda}\|_{\mathbb{H}^{s,\delta}} &\lesssim C_{\mathcal{N}}\|u-v\|_{\mathbb{H}^{s,\delta}} R^\gamma + C_\zeta\ C_2(\sigma)\ \|u-v\|_{\mathbb{H}^{s,\delta}}\left(1+R+R^2\right)\ \|\dot{\zeta}\|_{\mathbb{H}^{s-1,\delta-1}} \\ &\leq \left[C_{\mathcal{N}} R + \lambda^{1-(s+\delta)} C_\zeta\ C_2(\sigma)\ (1+R)\ \|\dot{\zeta}\|_{\mathbb{H}^{s-1,\delta-1}}\right]\ \|u-v\|_{\mathbb{H}^{s,\delta}}. \end{aligned} \tag{5.5.17}$$

Hence we can choose $R_0 \in (0,1), \lambda_0 := \lambda_0(\|u_0\|_{H^s}, \|u_1\|_{H^{s-1}}, R_0)$ in such a way that (5.5.15) is satisfied and the right hand side of (5.5.17) is bounded by $\frac{1}{2}\|u-v\|_{\mathbb{H}^{s,\delta}}$, i.e. Θ^λ is $\frac{1}{2}$-contraction as a map from $\mathbb{B}_{R_0}$ into itself. Thus, since $\mathbb{B}_{R_0}$ is a closed subset of $\mathbb{H}^{s,\delta}$, by the Banach Fixed Point Theorem there exists a unique $u^\lambda \in \mathbb{B}_{R_0}$ such that $u^\lambda = \Theta^\lambda(u^\lambda)$.

It is relevant to note that because λ in the r.h.s. of (5.5.15) and (5.5.17) appear with negative exponents we infer that (5.5.15) - (5.5.17) holds for the chosen R_0 and every $\lambda \geq \lambda_0$.

To complete the proof we do the inverse-scaling as below and reach back to the map Θ. In this direction, first we see that for any given suitable λ, R and the corresponding fixed point u^λ of the map Θ^λ, by defining

$$u(\alpha,\beta) := \frac{1}{\lambda^2}\Pi_{\lambda^{-1}} u^\lambda(\alpha,\beta) = u^\lambda(\lambda\alpha,\lambda\beta), \tag{5.5.18}$$

due to the special structure of the null form $\mathcal{N}$ we have

$$\mathcal{N}(u^\lambda(\alpha,\beta)) = \frac{1}{\lambda^2}\mathcal{N}\left(u\left(\frac{\alpha}{\lambda},\frac{\beta}{\lambda}\right)\right).$$

Consequently due to the choice of exponent -2 to scale the noise in (5.5.12) and since u^λ is fixed point of Θ^λ, from (5.5.11), (5.5.18) followed by Lemmata 5.4.1 and 5.4.7 we deduce

(5.5.19)

$$\begin{aligned} u(\alpha,\beta) &= \Theta^\lambda\left(u^\lambda\right)(\lambda\alpha,\lambda\beta) \\ &= \eta(\lambda\alpha)\eta(\lambda\beta)\left[\frac{1}{2}\left(u_0^\lambda(\lambda\alpha) + u_0^\lambda(-\lambda\beta)\right) + \frac{1}{2}\int_{-\lambda\beta}^{\lambda\alpha} u_1^\lambda(r)\,dr\right. \end{aligned}$$

$$
\begin{aligned}
&+\frac{1}{4}\int_{-\lambda\beta}^{\lambda\alpha}\int_{-a}^{\lambda\beta}\mathcal{N}(u^{\lambda}(a,b))\,db\,da+\frac{1}{4}\int_{-\lambda\beta}^{\lambda\alpha}\int_{-a}^{\lambda\beta}\sigma(u^{\lambda}(a,b))\,\dot{\zeta}^{\lambda}(da,db)\Big]\\
&=\eta(\lambda\alpha)\eta(\lambda\beta)\left[\frac{1}{2}\left\{\chi(\lambda\alpha)u_0(\alpha)-\chi(\lambda\alpha)\overline{u_0}^{\lambda}+\chi(-\lambda\beta)u_0(-\beta)-\chi(-\lambda\beta)\overline{u_0}^{\lambda}\right\}\right.\\
&\left.+\frac{1}{2}\int_{-\beta}^{\alpha}\chi(\lambda r)u_1(r)\,dr+\frac{1}{4}\int_{-\beta}^{\alpha}\int_{-a}^{\beta}\mathcal{N}(u(a,b))\,db\,da+\frac{1}{4}\int_{-\beta}^{\alpha}\int_{-a}^{\beta}\sigma(u(a,b))\dot{\zeta}(a,b)\,db\,da\right]\\
&=\eta(\lambda\alpha)\eta(\lambda\beta)\Big([S(\chi(\lambda)(u_0-\overline{u_0}^{\lambda}),\chi(\lambda)u_1)](\alpha,\beta)+[\diamond^{-1}\mathcal{N}(u)](\alpha,\beta)+[\diamond^{-1}\sigma(u)\dot{\zeta}](\alpha,\beta)\Big).
\end{aligned}
$$

Hence the Theorem 5.5.3. ■

To remove the dependence on $\overline{u_0}^{\lambda}$ we consider the stochastic wave map system work with another coordinate chart which is the translation of original one by the value $\overline{u_0}^{\lambda}$. In precise manner, since the wave map (in the integral form) is well-defined and λ is already fixed now due to the Theorem 5.5.3, by choosing the local chart (U,ϕ') where ϕ' differs from ϕ by the constant $\overline{u_0}^{\lambda}$ precisely, $\phi'(p):=\phi(p)-\overline{u_0}^{\lambda}$, $\forall p\in U$, the obtained localized solution u from Theorem 5.5.3 satisfies the following integral equation

$$
u(\alpha,\beta)=\eta(\lambda\alpha)\eta(\lambda\beta)\big([S(\chi(\lambda)u_0,\chi(\lambda)u_1)](\alpha,\beta)+[\diamond^{-1}\mathcal{N}(u)](\alpha,\beta)+[\diamond^{-1}\sigma(u)\dot{\zeta}](\alpha,\beta)\big). \tag{5.5.20}
$$

Now to obtain the pathwise local solution in the sense of Definition 5.5.2, let $(\alpha_0,-\alpha_0)$ on the negative diagonal $D_{\alpha\beta}:=\{(r,-r):r\in\mathbb{R}\}$. Next, by setting

$$
u_{0_{\alpha_0}}(\alpha):=u_0(\alpha-\alpha_0)\quad\text{and }0\quad u_{1_{\alpha_0}}(\alpha):=u_1(\alpha-\alpha_0),
$$

we get, by change of variable

$$
2[S(u_0,u_1)](\alpha-\alpha_0,\beta+\alpha_0)=u_{0_{\alpha_0}}(\alpha)+u_{0_{\alpha_0}}(-\beta)+\int_{-\beta}^{\alpha}u_{1_{\alpha_0}}(s)\,ds.
$$

Similarly by defining

$$
u_{\alpha_0}(\alpha,\beta):=u(\alpha-\alpha_0,\beta+\alpha_0),
$$

we obtain

$$
\left[\diamond^{-1}\mathcal{N}(u)\right](\alpha-\alpha_0,\beta+\alpha_0)=\left[\diamond^{-1}\mathcal{N}(u_{\alpha_0})\right](\alpha,\beta),
$$

and

$$
\left[\diamond^{-1}\sigma(u)\dot{\zeta}\right](\alpha-\alpha_0,\beta+\alpha_0)=\left[\diamond^{-1}\sigma(u_{\alpha_0})\dot{\zeta}_{\alpha_0}\right](\alpha,\beta),
$$

where $\zeta_{\alpha_0}(\alpha,\beta):=\zeta(\alpha-\alpha_0,\beta+\alpha_0)$.

Consider the following integral equation

$$
\begin{aligned}
u(\alpha-\alpha_0,\beta+\alpha_0)=\eta(\lambda\alpha)\eta(\lambda\beta)\big[&S(\chi(\lambda)u_0,\chi(\lambda)u_1)(\alpha-\alpha_0,\beta+\alpha_0)+\left(\diamond^{-1}\mathcal{N}(u)\right)(\alpha-\alpha_0,\beta+\alpha_0)\\
&+\left(\diamond^{-1}\sigma(u)\dot{\zeta}\right)(\alpha-\alpha_0,\beta+\alpha_0)\big]
\end{aligned} \tag{5.5.21}
$$

Since the Sobolev spaces $H^s(\mathbb{R}^d)$ are translation invariant and the constants of inequalities in the estimates (5.5.6) to (5.5.10) depend only on η and χ, we infer that by repeating the procedure followed in the proof of Theorem 5.5.3, with the same constants of estimates, and we get a unique $u_{\alpha_0} \in \mathbb{H}^{s,\delta}$ which satisfies (5.5.21) in some neighbourhood of $(\alpha_0, -\alpha_0)$. Hence, by using the uniqueness of localized solution we can glue "local" solutions to get a unique pathwise solution u as required in the sense of Definition 5.5.2.

5.6 Auxiliary results

5.6.1 Tensor product of Hilbert-Schmidt operators

The following lemma is a special case of [39].

Lemma 5.6.1. *Let E_i, F_i, $i = 1,2$, are given separable Hilbert spaces. Assume that $A \in \mathscr{L}_2(E_1, E_2)$ and $B \in \mathscr{L}_2(F_1, F_2)$, i.e. they are Hilbert-Schmidt operators. Then the tensor product $A \otimes B$, which is defined by,*

$$A \otimes B : E_1 \otimes F_1 \ni M \mapsto B \circ M \circ A^* \in E_2 \otimes F_2,$$

is a Hilbert-Schmidt operator from $E_1 \otimes F_1$ into $E_2 \otimes F_2$.

Proof of Lemma 5.6.1 Recall that by definition $E_i \otimes F_i = \mathscr{L}_2(E_i^*, F_i)$ for $i = 1,2$. Let $\{e_j^1\}_{j\in\mathbb{N}}$, $\{e_i^2\}_{i\in\mathbb{N}}$ and $\{f_k^1\}_{\mathbb{N}}$, respectively, are orthonormal basis (ONB) of E_1, E_2, and F_1. It is known that

$$e_j^1 \otimes f_k^1 : E_1^* \ni \phi \mapsto {}_{E_1^*}\langle \phi, e_j^1\rangle_{E_1} f_k^1 \in F_1,$$

and the sequence $\{e_j^1 \otimes f_k^1\}_{j,k\in\mathbb{N}}$ forms an ONB of $\mathscr{L}_2(E_1^*, F_1)$. Thus, since $\{e_i^{2*}\}$ are ONB of E_2^*,

$$\begin{aligned}\|(A\otimes B)(e_j^1 \otimes f_k^1)\|^2_{\mathscr{L}_2(E_2^*,F_2)} &= \sum_{i=1}^{\infty} \| {}_{E_1^*}\langle A^* e_i^{2*}, e_j^1\rangle_{E_1} B(f_k^1)\|^2_{F_2} \\ &= \sum_{i=1}^{\infty} | {}_{E_2^*}\langle e_i^{2*}, Ae_j^1\rangle_{E_2}|^2 \, \|B(f_k^1)\|^2_{F_2} = \|B(f_k^1)\|^2_{F_2} \|Ae_j^1\|^2_{E_2}.\end{aligned}$$

Hence

$$\sum_{j,k=1}^{\infty} \|(A\otimes B)(e_j^1 \otimes f_k^1)\|^2_{\mathscr{L}_2(E_2^*,F_2)} = \|A\|^2_{\mathscr{L}_2(E_1,E_2)} \|B\|^2_{\mathscr{L}_2(F_1,F_2)},$$

and we are done with the proof of Lemma 5.6.1. ■

5.6.2 Invariance of wave map under local charts

In this section we show that the wave maps, under perturbation, of sufficient regularity are invariant with respect to local charts. Here again, for simplicity, we restrict the computation to the case when M is 2-dimensional manifold. Let $(U, \phi = (x^1, x^2))$ and $(U, \psi = (y^1, y^2))$ be two local coordinate charts on M with a common domain U. We will also denote the standard coordinates of $\mathbb{R}^2$ by the same notation in the corresponding cases. Recall that the SGWE (1.2.6) has the following form in local coordinate (U,ϕ), for each $k=1,\dots,n$, we write ϕ to show the dependency explicitly,

$$\begin{cases} \Box\,{}^{\phi}Z^k(t,x) = \sum_{a,b=1}^{2}\sum_{\mu=0}^{1} {}^{\phi}\Gamma^k_{ab}({}^{\phi}Z)\partial_\mu {}^{\phi}Z^a\partial^\mu {}^{\phi}Z^b + {}^{\phi}\sigma^k({}^{\phi}Z)\dot{\xi}, \\ {}^{\phi}Z(0,x) = {}^{\phi}Z_0(x) \in \mathbb{R}^2, \quad \text{and} \quad \partial_t {}^{\phi}Z(0,x) = {}^{\phi}Z_1(x) \in \mathbb{R}^2, \end{cases} \tag{5.6.1}$$

where ${}^{\phi}Z = \phi\circ z : \mathbb{R}^2 \to \mathbb{R}^2$, ${}^{\phi}\Gamma^k_{ab}$ denote the Christoffel symbols on M in the chosen local coordinate (U,ϕ) and

$${}^{\phi}\sigma(\phi(p)) := (d_p\phi)(\kappa(p)) \in \mathbb{R}^2, \quad p \in U.$$

We also assume that ξ is sufficiently smooth so that the equation (5.6.1) makes sense in the differential form.

Theorem 5.6.2. *Suppose that (U,ϕ) and (U,ψ) are two coordinate charts on M. Let V be an open subset of $\mathbb{R}^2$ and $z : V \to U$ is of class $\mathcal{C}^2$. Define*

$${}^{\phi}Z = \phi\circ z \quad \textit{and} \quad {}^{\psi}Z = (\psi\circ\phi^{-1})({}^{\phi}Z) \quad \textit{on} \quad V.$$

Then

$${}^{\phi}Z_0(x) = [\psi\circ\phi^{-1}]\left({}^{\psi}u_0(x)\right), \tag{5.6.2}$$

and ${}^{\phi}Z(t,x)$ satisfies

$$\begin{aligned} &\frac{\partial^2 {}^{\phi}Z^k}{\partial t^2}(t,x) - \frac{\partial^2 {}^{\phi}Z^k}{\partial x^2}(t,x) - {}^{\phi}\sigma^k({}^{\phi}Z(t,x))\dot{\xi} \\ &\quad + \sum_{a,b=1}^{m} {}^{\phi_1}\Gamma^k_{ab}(\Phi_1(t,x))\left[\frac{\partial {}^{\phi}Z^a}{\partial t}(t,x)\frac{\partial {}^{\phi}Z^b}{\partial t}(t,x) - \frac{\partial {}^{\phi}Z^a}{\partial x}(t,x)\frac{\partial {}^{\phi}Z^b}{\partial x}(t,x)\right] = 0, \end{aligned} \tag{5.6.3}$$

if and only if ${}^{\psi}Z(t,x)$ satisfies

$$\begin{aligned} &\frac{\partial^2 {}^{\psi}Z^k}{\partial t^2}(t,x) - \frac{\partial^2 {}^{\psi}Z^k}{\partial x^2}(t,x) - {}^{\psi}\sigma^k({}^{\psi}Z(t,x))\dot{\xi} \\ &\quad + \sum_{\alpha,\beta=1}^{m} {}^{\psi}\Gamma^k_{\alpha\beta}(\Phi_2(t,x))\left[\frac{\partial {}^{\psi}Z^\alpha}{\partial t}(t,x)\frac{\partial {}^{\psi}Z^\beta}{\partial t}(t,x) - \frac{\partial {}^{\psi}Z^\alpha}{\partial x}(t,x)\frac{\partial {}^{\psi}Z^\beta}{\partial x}(t,x)\right] = 0, \end{aligned} \tag{5.6.4}$$

for $k=1,\dots,n$.

Proof of Theorem 5.6.2 Since $\psi\circ\phi^{-1} : \mathbb{R}^2 \ni \phi(p) \mapsto \psi(p) \in \mathbb{R}^2$ and $\phi\circ\psi^{-1} : \mathbb{R}^2 \ni \psi(p) \mapsto \phi(p) \in \mathbb{R}^2$ are diffeomorphisms, the equality (5.6.2) holds true because

$${}^{\phi}Z_0(x) = \phi(z_0(x)) = (\phi\circ\psi^{-1})(\psi(z_0(x))) = [\psi\circ\phi^{-1}]\left({}^{\psi}z_0(x)\right).$$

Now we move to the proof of the second claim in the theorem. Suppose ${}^{\phi}Z$ satisfies (5.6.3). Observe that by the chain rule for Jacobi matrices we have

$$\frac{\partial {}^{\phi}Z^a}{\partial t}(t,x) = \sum_{\lambda=1}^{2} \frac{\partial(\phi^a\circ\psi^{-1})}{\partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial(\psi^\lambda\circ z)}{\partial t}(t,x), \tag{5.6.5}$$

and

$$\frac{\partial\,^{\phi}Z^b}{\partial x}(t,x) = \sum_{\delta=1}^{2} \frac{\partial(\phi^b \circ \psi^{-1})}{\partial y^\delta}(\psi(u(t,x)))\,\frac{\partial(\psi^\delta \circ z)}{\partial x}(t,x). \tag{5.6.6}$$

For second derivative terms by similar calculations we get

$$\begin{aligned}\frac{\partial^2\,^{\phi}Z^a}{\partial t^2}(t,x) = &\sum_{\lambda,\delta=1}^{2} \frac{\partial^2(\phi^a \circ \psi^{-1})}{\partial y^\delta \partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial(\psi^\delta \circ z)}{\partial t}(t,x)\,\frac{\partial(\psi^\lambda \circ z)}{\partial t}(t,x)\\ &+ \sum_{\lambda=1}^{2} \frac{\partial(\phi^a \circ \psi^{-1})}{\partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial^2(\psi^\lambda \circ z)}{\partial t^2}(t,x),\end{aligned}$$

and

$$\begin{aligned}\frac{\partial^2\,^{\phi}Z^b}{\partial x^2}(t,x) = &\sum_{\lambda,\delta=1}^{2} \frac{\partial^2(\phi^b \circ \psi^{-1})}{\partial y^\delta \partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial(\psi^\delta \circ z)}{\partial x}(t,x)\,\frac{\partial(\psi^\lambda \circ z)}{\partial x}(t,x)\\ &+ \sum_{\lambda=1}^{2} \frac{\partial(\phi^b \circ \psi^{-1})}{\partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial^2(\psi^\lambda \circ z)}{\partial x^2}(t,x).\end{aligned} \tag{5.6.7}$$

It is well known, see [118, Chapter 3], that the Christoffel symbols transform under the change of coordinate chart as, for each $\mu = 1,2$,

$$\begin{aligned}\sum_{\kappa=1}^{2} {}^{\psi}\Gamma^{\kappa}_{\alpha\beta}(^{\psi}Z(t,x))\frac{\partial(\phi^\mu \circ \psi^{-1})}{\partial y^\kappa}(^{\psi}Z(t,x)) = &\sum_{\lambda,\delta=1}^{2}\Bigg[\frac{\partial^2(\phi^\mu \circ \psi)}{\partial y^\lambda \partial y^\delta}(^{\psi}Z(t,x))\\ &+\frac{\partial(\phi^\lambda \circ \psi)}{\partial y^\lambda}(^{\psi}Z(t,x))\,\frac{\partial(\phi^\delta \circ \psi)}{\partial y^\delta}(^{\psi}Z(t,x))\,{}^{\phi}\Gamma^{\mu}_{ab}(^{\phi}Z(t,x))\Bigg].\end{aligned} \tag{5.6.8}$$

Since $^{\phi}\sigma(^{\phi}Z(t,x)) = (d_{z(t,x)}\phi)(\kappa(z(t,x)))$, we infer that, for each $k = 1,2$,

$$^{\phi}\sigma^k(^{\phi}Z(t,x)) = \sum_{\gamma=1}^{2} \frac{\partial(\phi^k \circ \psi^{-1})}{\partial y^\gamma}(^{\psi}Z(t,x))\,{}^{\psi}\sigma^k(^{\psi}Z(t,x)). \tag{5.6.9}$$

Then, since $^{\phi}Z(t,x)$ satisfy (5.6.3), by substituting partial derivatives from (5.6.5)-(5.6.7) followed by combining the terms to apply the transformation laws (5.6.8) and (5.6.9) we get, for each $k = 1,2$,

$$\begin{aligned}0 = &\sum_{\lambda,\delta=1}^{2} \frac{\partial^2(\phi^k \circ \psi^{-1})}{\partial y^\delta \partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial(\psi^\delta \circ z)}{\partial t}(t,x)\,\frac{\partial(\psi^\lambda \circ z)}{\partial t}(t,x)\\ &+ \sum_{\lambda=1}^{2} \frac{\partial(\phi^k \circ \psi^{-1})}{\partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial^2(\psi^\lambda \circ z)}{\partial t^2}(t,x)\\ &- \sum_{\lambda,\delta=1}^{2} \frac{\partial^2(\phi^k \circ \psi^{-1})}{\partial y^\delta \partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial(\psi^\delta \circ z)}{\partial x}(t,x)\,\frac{\partial(\psi^\lambda \circ z)}{\partial x}(t,x)\\ &- \sum_{\lambda=1}^{2} \frac{\partial(\phi^k \circ \psi^{-1})}{\partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial^2(\psi^\lambda \circ z)}{\partial x^2}(t,x) - {}^{\phi}\sigma^k(\phi(z(t,x)))\dot{\xi}\\ &+ \sum_{a,b=1}^{m} {}^{\phi}\Gamma^k_{ab}(^{\phi}Z(t,x))\left[\left(\sum_{\lambda=1}^{2} \frac{\partial(\phi^a \circ \psi^{-1})}{\partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial(\psi^\lambda \circ z)}{\partial t}(t,x)\right)\right.\end{aligned}$$

$$
\begin{aligned}
&\times\left(\sum_{\delta=1}^{m}\frac{\partial(\phi^b\circ\psi^{-1})}{\partial y^\delta}(\psi(z(t,x)))\,\frac{\partial(\psi^\delta\circ z)}{\partial t}(t,x)\right)\\
&-\left(\sum_{\mu=1}^{m}\frac{\partial(\phi^a\circ\psi^{-1})}{\partial y^\mu}(\psi(z(t,x)))\,\frac{\partial(\psi^\mu\circ u)}{\partial x}(t,x)\right)\\
&\times\left(\sum_{\nu=1}^{m}\frac{\partial(\phi^b\circ\psi^{-1})}{\partial y^\nu}(\psi(z(t,x)))\,\frac{\partial(\psi^\nu\circ u)}{\partial x}(t,x)\right)\Bigg]\\
=&\sum_{\lambda=1}^{2}\frac{\partial(\phi^k\circ\psi^{-1})}{\partial y^\lambda}(\psi(z(t,x)))\left[\frac{\partial^2(\psi^\lambda\circ z)}{\partial t^2}(t,x)-\frac{\partial^2(\psi^\lambda\circ z)}{\partial x^2}(t,x)\right]\\
&+\sum_{\lambda,\delta=1}^{2}\frac{\partial(\psi^\delta\circ z)}{\partial t}(t,x)\,\frac{\partial(\psi^\lambda\circ z)}{\partial t}(t,x)\Bigg[\frac{\partial^2(\phi^k\circ\psi^{-1})}{\partial y^\delta\partial y^\lambda}(\psi(z(t,x)))\\
&+\sum_{a,b=1}^{m}{}^{\phi}\Gamma^k_{ab}({}^{\phi}Z(t,x))\left(\frac{\partial(\phi^a\circ\psi^{-1})}{\partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial(\phi^b\circ\psi^{-1})}{\partial y^\delta}(\psi(z(t,x)))\right)\Bigg]\\
&-\sum_{\lambda,\delta=1}^{2}\frac{\partial(\psi^\delta\circ z)}{\partial x}(t,x)\,\frac{\partial(\psi^\lambda\circ z)}{\partial x}(t,x)\Bigg[\frac{\partial^2(\phi^k\circ\psi^{-1})}{\partial y^\delta\partial y^\lambda}(\psi(z(t,x)))\\
&+\sum_{a,b=1}^{m}{}^{\phi}\Gamma^k_{ab}({}^{\phi}Z(t,x))\left(\frac{\partial(\phi^a\circ\psi^{-1})}{\partial y^\lambda}(\psi(z(t,x)))\,\frac{\partial(\phi^b\circ\psi^{-1})}{\partial y^\delta}(\psi(z(t,x)))\right)\Bigg]\\
&-\sum_{\gamma=1}^{2}\frac{\partial(\phi^k\circ\psi^{-1})}{\partial y^\gamma}(\psi(z(t,x)))\,{}^{\psi}\sigma^k(\psi(z(t,x)))\dot{\xi}\\
=&\sum_{\lambda=1}^{2}\frac{\partial(\phi^k\circ\psi^{-1})}{\partial y^\lambda}(\psi(z(t,x)))\left[\frac{\partial^2(\psi^\lambda\circ z)}{\partial t^2}(t,x)-\frac{\partial^2(\psi^\lambda\circ z)}{\partial x^2}(t,x)\right]\\
&+\sum_{\lambda,\delta=1}^{2}\Bigg[\left(\frac{\partial(\psi^\delta\circ z)}{\partial t}(t,x)\,\frac{\partial(\psi^\lambda\circ z)}{\partial t}(t,x)-\frac{\partial(\psi^\delta\circ z)}{\partial x}(t,x)\,\frac{\partial(\psi^\lambda\circ z)}{\partial x}(t,x)\right)\\
&\times\left(\sum_{\kappa=1}^{2}{}^{\psi}\Gamma^\kappa_{\lambda\delta}({}^{\psi}Z(t,x))\frac{\partial(\phi^k\circ\psi^{-1})}{\partial y^\kappa}(\psi(z(t,x)))\right)\Bigg]\\
&-\sum_{\gamma=1}^{2}\frac{\partial(\phi^k\circ\psi^{-1})}{\partial y^\gamma}(\psi(z(t,x)))\,{}^{\psi}\sigma^k(\psi(z(t,x)))\dot{\xi}\\
=&\sum_{\gamma=1}^{2}\frac{\partial(\phi^k\circ\psi^{-1})}{\partial y^\gamma}(\psi(z(t,x)))\left[\frac{\partial^2\,{}^{\psi}Z^\gamma}{\partial t^2}(t,x)-\frac{\partial^2\,{}^{\psi}Z^\gamma}{\partial x^2}(t,x)-{}^{\psi}\sigma^k({}^{\psi}Z(t,x))\dot{\xi}\right.\\
&\left.+\sum_{\lambda,\delta=1}^{2}{}^{\psi}\Gamma^\gamma_{\lambda\delta}({}^{\psi}Z(t,x))\left(\frac{\partial\,{}^{\psi}Z^\delta}{\partial t}(t,x)\,\frac{\partial\,{}^{\psi}Z^\lambda}{\partial t}(t,x)-\frac{\partial\,{}^{\psi}Z^\delta}{\partial x}(t,x)\,\frac{\partial\,{}^{\psi}Z^\lambda}{\partial x}(t,x)\right)\right]\\
=:&\sum_{\gamma=1}^{2}\frac{\partial(\phi^k\circ\psi^{-1})}{\partial y^\gamma}(\psi(z(t,x)))\;A(\gamma).
\end{aligned}
$$

This implies in matrix form

$$
\begin{pmatrix}0\\0\end{pmatrix}_{2\times1}=\begin{pmatrix}\frac{\partial(\phi^1\circ\psi^{-1})}{\partial y^1}(\psi(z(t,x))) & \frac{\partial(\phi^1\circ\psi^{-1})}{\partial y^2}(\psi(z(t,x)))\\ \frac{\partial(\phi^2\circ\psi^{-1})}{\partial y^1}(\psi(z(t,x))) & \frac{\partial(\phi^2\circ\psi^{-1})}{\partial y^2}(\psi(z(t,x)))\end{pmatrix}_{2\times2}\begin{pmatrix}A(1)\\A(2)\end{pmatrix}_{2\times1}. \tag{5.6.10}
$$

Since 2×2 matrix in (5.6.10) is invertible, we infer that $A(\gamma)=0$ for each $\gamma=1,2$. Hence proof of the

Theorem 5.6.2 is complete.

$\blacksquare$

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